The Theta Correspondence

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MT 2023

Abstract

These are the notes from a study group on the theta correspondence, following a set of lecture notes by Wee Teck Gan titled "The Shimura Correspondence à la Waldspurger".

Contents

1 Introduction & motivation

Speaker: George Robinson

1.1 Half-integer weight modular forms

We start classically: let $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ where $q = e^{2\pi i\tau}$, which is natural to study if we want to understand representations of integers as sums of squares. For example, n-th coefficient of θ^4 is the number of ways in which n can be written as a sum of four squares. This clearly satisfies invariance in $\tau \mapsto \tau + 1$, but less obviously, the Poisson summation formula implies that

$$\theta(\frac{-1}{4\tau}) = \sqrt{-2i\tau}\theta(\tau). \tag{1.1}$$

We recognise this as the action of the Möbius transformation associated to $\begin{pmatrix} 0 & 1/2 \\ -2 & 0 \end{pmatrix}$. One can check that $\left\langle \begin{pmatrix} 0 & 1/2 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(4)$, and it follows that θ is "almost a modular form", i.e. satisfies $\theta(\gamma\tau) = j(\gamma,\tau)\theta(\tau)$ for any $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(4)$, where

$$j(\gamma,\tau) = \epsilon_d^{-1}\left(\frac{c}{d}\right)(c\tau + d)^{1/2} \quad \text{and} \quad \epsilon_d = \begin{cases} 1 \text{ if } d \equiv 1 \bmod 4, \\ i \text{ if } d \equiv -1 \bmod 4. \end{cases} \tag{1.2}$$

Note that this is a bit different from the usual $j(\gamma,\tau)$, and that $\theta^4 \in M_2(\Gamma_0(4))$. We can take this as a rudimentary definition of half-integer weight modular form:

Definition 1.1. Fix an odd integer κ . A holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ is a weight $\kappa/2$ modular form of level $\Gamma \subset \Gamma_0(4)$ if for any $\gamma \in \Gamma$, $f(\gamma\tau) = j(\gamma,\tau)^\kappa f(\tau)$.

A better perspective was given by Weil: define a "toric cover" $\mathscr{M}$ of $\mathrm{SL}_2(\mathbb{R})$ by

$$\mathscr{M} := \{(\gamma,\phi) : \gamma \in \mathrm{SL}_2(\mathbb{R}),\ \phi : \mathfrak{h} \to \mathbb{C} \text{ s.t. } \phi(z)^2 = (cz + d)t \text{ for some } t \in S^1\}, \tag{1.3}$$

where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. This is naturally an algebraic group under the operation $(\gamma,\phi)\cdot(\delta,\psi) = (\gamma\delta, z \mapsto \phi(\delta z)\psi(z))$, and a cover of $\mathrm{SL}_2(\mathbb{R})$ via $(\gamma,\phi) \mapsto \gamma$, but there is no splitting. For $\Gamma \le \mathscr{M}$ discrete, we can define modular forms to be functions on the upper half plane f such that $f|_\kappa(\gamma,\phi)$, where the slash action is defined in terms of $j(\gamma,\tau)$. Let $M_{\kappa/2}(\Gamma)$ denote the vector space of all such functions, and $S_{\kappa/2}(\Gamma)$ be the space of cusp forms, i.e. those whose first Fourier coefficient vanishes.

If $\Gamma \subset \Gamma_0(4)$, then there is a natural splitting $\gamma \mapsto (\gamma, j(\gamma,\cdot))$ which recovers the above notion. In fact, it suffices to pass to a subgroup of $\mathscr{M}$ which is a double cover of $\mathrm{SL}_2(\mathbb{R})$ (as opposed to a toric extension), denoted $\widetilde{\mathrm{Sp}}_2(\mathbb{R})$, and half-integer weight modular forms are the natural automorphic forms on this group.

Remark. One can show that $\widetilde{\mathrm{Sp}}_2(\mathbb{R})$ is not a linear algebraic group, by arguing that any finite-dimensional faithful representation (i.e. injection) $\widetilde{\mathrm{Sp}}_2(\mathbb{R}) \to \mathrm{GL}(V)$ factors through $\mathrm{SL}_2(\mathbb{R})$, hence cannot be injective.

1.2 Fourier coefficients

To understand the Fourier coefficients of modular forms, we usually introduce Hecke operators, and this works in the half-integer weight case as well. Let $\Gamma \subset \widetilde{\mathrm{Sp}}_2(\mathbb{Z})$ be a subgroup of finite index. Given $\zeta \in \mathrm{GL}_2(\mathbb{Q})$, we can decompose the double cosets as $\Gamma\zeta\Gamma = \bigsqcup_i \Gamma\zeta_i$ for some $\zeta_i \in \widetilde{\mathrm{Sp}}_2(\mathbb{R})$, and define

$$f|_\kappa[\Gamma\zeta\Gamma] := \sum_i f|_\kappa\zeta_i. \tag{1.4}$$

Just like for classical modular forms, this gives an endomorphism of $M_{\kappa/2}(\Gamma)$.

Theorem 1.2 (Shimura). *Let N be a positive integer divisible by 4, and let $\Gamma = \Gamma_0(N)$. Then for each $m \in \mathbb{N}$ there is a Hecke operator $T(m) = T_{\kappa,\chi}^N(m)$ acting on $M_{\kappa/2}(\Gamma,\chi)$ which maps cusp forms to cusp forms. Furthermore, if $(m,m') = 1$, then $T(m)$ and $T(m')$ commute.*

However, something strange happens in this theory:

$$\text{if } m \text{ is not a perfect square, then } T(m) = 0.$$

Let us look at a concrete example:

Example 1.3. Let $\kappa = 9$ and $N = 4$. The space $S_{9/2}(\Gamma_0(4))$ is one-dimensional, spanned by

$$f(\tau) = \frac{\eta(2\tau)^{12}}{\theta(\tau)^3} = q - 6q^2 + 12q^3 - 8q^4 + 12q^6 - 48q^7 + 48q^8 - 15q^9 + 60q^{10} - 12q^{11} + \dots \tag{1.5}$$

where η denotes the Dedekind eta function. By Shimura's theorem, f is an eigenfunction of all the Hecke operators, and the computation of all its Fourier coefficients reduces to computing

(i) The eigenvalues of $T(p^2)$ for p prime, and
(ii) the coefficients $a_n(f)$ for squarefree integers n.

1.3 The Shimura correspondence

Fix an eigenform $f = \sum_{n=1}^\infty a_n q^n \in S_{\kappa/2}(\Gamma_0(N),\chi)$ with $T(p^2)$-eigenvalue ω_p, and for $t \in \mathbb{N}$ fixed, one computes

$$\sum_{n=1}^\infty \frac{a_{tn^2}}{n^s} = a_t \prod_p \left(1 - \chi(p)\left(\frac{-1}{p}\right)^{(\kappa-1)/2} \cdot \left(\frac{t}{p}\right) \cdot p^{(\kappa-3)/2}\right) \cdot \left(1 - \omega_p p^{-s} + \chi(p)^2 p^{\kappa-2-2s}\right)^{-1}. \tag{1.6}$$

Shimura observed that the last factor in the product is independent of t, and therefore defined

$$L(s,f) := \prod_p \left(1 - \omega_p p^{-s} + \chi(p)^2 p^{\kappa-2-2s}\right)^{-1} \tag{1.7}$$

to be the L-function of f. But the shape of this corresponds exactly to the L-function of a *classical* modular form F of weight $\kappa - 1$, Nebentype χ^2 and $T(p)$-eigenvalue ω_p. Using Weil's converse theorem, Shimura then proved:

Theorem 1.4 (Shimura Correspondence). *Let $\kappa \geq 3$, and let $f \in S_{\kappa/2}(\Gamma_0(N),\chi)$ be as above. If $L(s,f) = \sum_{n=1}^\infty \frac{A_n}{n^s}$, then the formal power series $F(\tau) = \sum_{n=1}^\infty A_n q^n$ defines an element of $M_{\kappa-1}(\Gamma_0(N/2),\chi^2)$.*

The form F is called the **Shimura lift** of f.

Example 1.5. For $\kappa = 9$ as above, one computes that $S_8(\Gamma_0(2))$ is one-dimensional, spanned by the form F with q-expansion

$$q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 - 512q^8 - 2043q^9 + 1680q^{10} + 1092q^{11} + \dots \tag{1.8}$$

which is the Shimura lift of the form in example 1.3.

In a sense, Shimura's proof is unsatisfying because it appeals to a global result (Weil's converse theorem, which roughly says that modular forms are determined by twists of its L-functions, which are global objects) to prove a local results, namely the equality of the Hecke eigenvalues $T(p^2)f = T(p)F$, which can be defined purely locally.

The goal of the next few lectures is to develop the machinery to understand what is going on locally in Shimura's correspondence. In particular, we will pass to the language of automorphic forms and representations, details of which can be found in the notes from last term's study group, or a book like Gelbart's, [Gel73].

Later on, we will define an adelic group $\widetilde{\mathrm{Sp}}_2(\mathbb{A})$, and upgrade a half-integer weight modular form f to a function $\phi_f\colon \mathrm{Sp}_2(\mathbb{Q})\backslash\widetilde{\mathrm{Sp}}_2(\mathbb{A}) \to \mathbb{C}$ spanning an automorphic representation π_f. By a version of Flath's theorem, $\pi_f = \otimes' \pi_{f,v}$ where each $\pi_{f,v}$ is an irreducible admissible representation of $\widetilde{\mathrm{Sp}}_2(\mathbb{Q}_v)$ Our goal is to understand the relationship between the local representations $\pi_{f,v}$ and $\pi_{F,v}$ coming from the Shimura lift F. This is essentially what Waldspurger did.

To motivate Waldspurger's (or perhaps Weil's) idea, it is helpful to go forward in time to Niwa [Niw75], who proved:

Theorem 1.6 (Niwa). *Shimura's correspondence can be realised as integrating against a kernel function:*

$$F(\tau) = \int_{\Gamma_0(N)\backslash\mathfrak{h}} \Theta_\Phi(z,\tau)\overline{f(z)}dz, \tag{1.9}$$

where $\Theta(z,\tau)$ is some explicit theta function, roughly of the shape $\sum_{(a,b,c)\in\mathbb{Z}^3} q^{ac-b^2}$.

What is this theta function? A key idea, perhaps due to Weil, is that theta functions such as Θ_Φ, can be "explained" in terms of the *Weil representation*, a distinguished infinite-dimensional representation of groups like $\widetilde{\mathrm{Sp}}_2(\mathbb{R})$.

Example 1.7. Let V be an orthogonal space of dimension m, and let W be a symplectic space of dimension $2n$.[1] Then there is a natural map $O(V)\times\widetilde{\mathrm{Sp}}(W) \hookrightarrow \widetilde{\mathrm{Sp}}(V\otimes W)$, and $\widetilde{\mathrm{Sp}}$ has a "nice" representation[2] $\omega_\psi = \omega_{V,W,\psi}\colon \widetilde{\mathrm{Sp}}(V\otimes W) \to \mathcal{S}(V^n)$. This gives an explicit map $\Theta\colon \mathcal{S}(V^n) \to \mathcal{A}(\widetilde{\mathrm{Sp}}(V\otimes W))$, by $\Theta(\Phi)(g) = \sum_{v\in V^n(\mathbb{Q})}(\omega_\psi(g)\cdot\Phi)(v)$.

For Niwa, V is the 3-dimensional orthogonal space with quadratic form $ac-b^2$, and W is a 2-dimensional symplectic space, so $\widetilde{\mathrm{Sp}}(W) \cong \widetilde{\mathrm{Sp}}_2(\mathbb{R})$. Integrating against $\overline{f(z)}$ can be thought of as picking out the f-isotypic component[3] inside $L^2(O(V)\times\widetilde{\mathrm{Sp}}(W))$. The connection with classical modular forms comes from noting that $O(V)(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})$, by an accidental isomorphism.

In general, the goal will be to use the Weil representation to understand irreducible representations of $\widetilde{\mathrm{Sp}}(W)$ via decomposition

$$\mathrm{Irr}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}(\mathrm{SO}(V^-)) \leftrightarrow \mathrm{Irr}(\widetilde{\mathrm{Sp}}(W)), \tag{1.10}$$

for suitable subspaces $V^\pm$, and hope that this is some kind of bijection.

Theorem 1.8 (Waldspurger). *We have a decomposition*

$$L^2_{\mathrm{disc}}(\widetilde{\mathrm{Sp}}(W)) = (\textit{easy theta functions}) \oplus \left(\bigoplus_{\pi \text{ on } \mathrm{SO}(V^+)} L^2_\pi\right) \oplus \left(\bigoplus_{\pi \text{ on } \mathrm{SO}(V^-)} L^2_\pi\right). \tag{1.11}$$

I'm not quite sure if this is specific to the V and W we chose already, or if it's a general thing.

A natural question to ask is, when is $\Theta(f)$ non-zero? Often it suffices to compute a Fourier coefficient, or more generally, a *period*. Let F be as above, and $C \subset \widetilde{\mathrm{Sp}}(W)$. Then by a change of variables,

$$\int_C F(\tau)d\tau = \int_C\int_{\Gamma_0\backslash\mathfrak{h}} \Theta_\Phi(z,\tau)\overline{f(z)}dzd\tau = \int_{\Gamma_0(N)\backslash\mathfrak{h}} \overline{f(z)}\int_C \Theta_\Phi(z,\tau)dzd\tau, \tag{1.12}$$

[1] These terms, and the example, will be studied more carefully in the next lecture.

[2] This will be the Weil representation.

[3] I.e. f-eigenspace

and with a good choice of C we might realise the inner integral as a theta lift of the constant function. Traditionally, the theta lift of a constant function becomes an Eisenstein series, and a theorem proving such a result is referred to as a *Siegel–Weil theorem*.

If we are in such a situation, then

$$(1.12) = \int_{Y_0(N)} \overline{f(z)} E(z,s)\,dz = \text{Rankin–Selberg } L\text{-value}. \tag{1.13}$$

If we know a non-vanishing result for such L-values, then we can deduce that $\Theta(f) \neq 0$.

2 The metaplectic group and the Weil representation

Speaker: James Newton

We start out locally: let k be a local field of characteristic 0. The origins of the Weil representation, or the "Segal–Shale–Weil representation", lie in quantum mechanics, see for example these notes by Garrett.

2.1 Symplectic vector spaces

Definition 2.1. A **symplectic space** is a k-vector space W equipped with a non-degenerate alternating bilinear form $\langle -,- \rangle\ W \times W \to k$, also known as a *symplectic form*.

Example 2.2. The easiest example of a symplectic space is $k \oplus k$ with standard basis e_1, e_2, and the standard symplectic form $\langle e_1, e_2 \rangle = 1$, $\langle e_i, e_i \rangle = 0$ for $i = 1, 2$. Alternatively, write $\langle x, y \rangle = x^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$. More generally, k^{2n} can be equipped with a standard symplectic form

$$\langle x, y \rangle = x^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y. \tag{2.1}$$

Exercise. Show that the dimension of a symplectic space is always even.

Definition 2.3. Let W be a symplectic space. The **symplectic group** associated to W is the set of linear transformations of W preserving $\langle -,- \rangle$:

$$\mathrm{Sp}(W) := \{g \in \mathrm{GL}(W) : \langle gx, gy \rangle = \langle x, y \rangle \text{ for all } x, y \in W\}. \tag{2.2}$$

Exercise. Show that $\mathrm{Sp}(k \oplus k) \cong \mathrm{SL}_2(k)$, but that $\mathrm{Sp}(W) \neq \mathrm{SL}(W)$ in general.

Definition 2.4. A **Lagrangian subspace** of a symplectic space W is a maximal linear subspace $V \subset W$ such that $\langle v, v' \rangle = 0$ for all $v, v' \in V$. A **Lagrangian decomposition**, or **(principal) polarisation** is a decomposition $W = V \oplus V'$ where V and V' are Lagrangian subspaces of W.

For example, $W = k \oplus k$ has the rather trivial Lagrangian decomposition $e_1 k \oplus e_2 k$.

Proposition 2.5. *Let W be a symplectic space of dimension $2n$. Every Lagrangian subspace $V \subset W$ has dimension n, and gives rise to a Lagrangian decomposition $W = V \oplus V^*$, where V^* is the linear dual of V.*

Exercise. Find a polarisation of k^{2n} with the standard symplectic form.

2.2 The Heisenberg group

Definition 2.6. Let W be a symplectic space. The **Heisenberg group of** W is the nontrivial central extension of W with underlying set $W \oplus k$ and group operation

$$(w_1, t_2) \cdot (w_2, t_2) := (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle).^{4} \tag{2.3}$$

In other words, $H(W)$ is generally not a direct product of the underlying groups, but fits into a short exact sequence

$$0 \to k \to H(W) \to W \to 0, \tag{2.4}$$

which is central: the image of k is precisely the centre of $H(W)$. If $k = \mathbb{R}$, $H(W)$ is naturally a Lie group, and if $k = \mathbb{Q}_p$, a locally profinite group. Furthermore, it is non-commutative with $H(W)^{\mathrm{ab}} = W$.

Exercise. Describe the Lie algebra of $H(W)$.

Our next goal is to understand the representations of $H(W)$. Let $\psi: k \to \mathbb{C}^\times$ be an additive character. A representation $\pi: H(W) \to \mathrm{GL}(S)$ has *central character* ψ if $\pi(t)v = \psi(t)v$ for all $t \in Z(H(W))$ and $v \in S$. If π has trivial central character, then it factors through W.

Theorem 2.7 (Stone–von Neumann)**.** *Let $\psi: k \to \mathbb{C}^\times$ be a nontrivial unitary character. Then there exists a unique irreducible representation $\omega_\psi: H(W) \to \mathrm{GL}(S)$ with central character ψ.*

This has an explicit description in terms of the *Schrödinger model*: fix a polarisation $W = V \oplus V'$, and let $S = \mathcal{S}(V)$ be the set of Schwartz–Bruhat functions on V. Then ω_ψ is realised by

$$\omega_\psi(0,t)f(x) = \psi(t)f(x), \tag{2.5a}$$

$$\omega_\psi(v,0)f(x) = f(x+v), \tag{2.5b}$$

$$\omega_\psi(0,v')f(x) = \psi(\langle x, v'\rangle)f(x). \tag{2.5c}$$

Exercise. (i) Check that ω_ψ respects the group operation in $H(W)$,

(ii) Think about what happens if you change ψ to $\psi_a(x) := \psi(ax)$,

(iii) Work out the isomorphism of representations obtained by interchanging the roles of V and V'.

What does this have to do with the symplectic group? For each $g \in \mathrm{Sp}(W)$, consider the automorphism of $H(W)$ given by $g \cdot (w,t) = (gw, t)$. Precomposing ω_ψ with this automorphism gives a new irreducible representation of $H(W)$, hence by theorem 2.7 these are isomorphic: we can find $M_g \in \mathrm{GL}(S)$ such that $\omega_\psi \circ g = M_g \omega_\psi M_g^{-1}$, and by a version of Schur's lemma[5] M_g is unique up to rescaling by an element of $\mathbb{C}^\times$. This gives rise to a so-called *projective representation* of $\mathrm{Sp}\, W$, $A_\psi \to \mathrm{Sp}\, W \to PGL(S)$.

A well-studied question in representation theory asks "when can we lift a projective representation to a genuine representation"? One way to do so is to pass to a cover: let $\widetilde{\mathrm{Sp}}W$ be the closed subgroup of $\mathrm{Sp}\, W \times \mathrm{GL}\, S$ defined by

$$\widetilde{\mathrm{Sp}}_\psi(W) := \{(g, M) : \omega_\psi \circ g = M\omega_\psi M^{-1}\}. \tag{2.6}$$

By what we just discussed, $\widetilde{\mathrm{Sp}}_\psi(W)$ is a covering of $\mathrm{Sp}(W)$, fitting into a diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{C}^\times \arrow[r] \arrow[d] & \widetilde{\mathrm{Sp}}(W) \arrow[r] \arrow[d] & \mathrm{Sp}(W) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{C}^\times \arrow[r] & \mathrm{GL}(S) \arrow[r] & \mathrm{GL}(S)/\mathbb{C}^\times \arrow[r] & 1 \end{tikzcd} \tag{2.7}$$

There are other variants of this, for example, taking a toric extension, as in section 1.

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[4] The factor of $1/2$ is a normalisation thing.

[5] whose proof we agreed is a bit subtle, because we are dealing with infinite-dimensional spaces. There is a proof in these notes by Conrad, says James.

Proposition 2.8. (i) $[\widetilde{\mathrm{Sp}}_\psi(W), \widetilde{\mathrm{Sp}}_\psi(W)] =: \mathrm{Mp}_\psi(W)$ *surjects onto* $\mathrm{Sp}(W)$ *with kernel* $\{\pm 1\}$.

(ii) *This extension is non-split, and is isomorphic to* $\mathrm{Mp}(W)$ *which is the unique double cover of* $\mathrm{Sp}(W)$.

This corresponds to the unique nontrivial cocycle in $H^2(\mathrm{Sp}(W), \{\pm 1\})$. By restricting ω_ψ to $\mathrm{Mp}_\psi(W)$, we get a genuine representation.

Definition 2.9. The representation $\omega_\psi \colon \mathrm{Mp}(W) \to \mathrm{GL}(S)$ is called the **Weil representation** of W.

Note that the Weil representation is unique, at least after fixing the additive character ψ.

Exercise. Describe the action of $\mathrm{Sp}\, W$ on $\mathcal{S}(W_1)$ modulo constants.

The Weil representation is not irreducible, but splits as $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$, where the two summands can be described as odd and even functions, respectively, in the Schrödinger model.

2.3 What is it good for?

According to Dan Ciubotaru, a consequence of the double centraliser theorem is that one should expect simple modules for a pair of subgroups which are mutual centralisers in a fixed supergroup, to match up in a certain way. I think this is made precise in [How89]. This motivates:

Definition 2.10. Let G and G' be algebraic subgroups of $\mathrm{Sp}\, W$, for some symplectic space W. We say (G, G') is a **reductive dual pair** if $Z_{\mathrm{Sp}\, W}(G) = G'$ and vice versa, and G and G' are reductive groups.

Example 2.11. Let U be a symplectic space, and V an orthogonal space. Then $W := U \otimes V$ is naturally a symplectic space with the form defined by

$$\left\langle u \otimes v, u' \otimes v' \right\rangle_W := \left\langle u, u' \right\rangle_U \otimes \left\langle v, v' \right\rangle_V, \tag{2.8}$$

and this gives a natural embedding $\mathrm{Sp}\, U \times O(V) \to \mathrm{Sp}\, W$. Furthermore, it is a good exercise to check that the two factors are mutual centralisers in $\mathrm{Sp}\, W$; in other words, $\mathrm{Sp}\, U$ and $O(V)$ form a reductive dual pair.

For finite groups, if $G_1 \times G_2 \hookrightarrow G$ and π is a representation of G, then $\pi|_{G_1 \otimes G_2} \cong \bigoplus_i \pi_1^i \otimes \pi_2^i$, and the hope is that one might be able to understand π_1^i in terms of π_2^i or vice versa. The hope is that a similar thing happens for algebraic groups (I believe this is the content of the Howe duality conjecture, but I'm not quite sure).

For a dual pair (G, G'), let $\widetilde{G}$ and $\widetilde{G}'$ denote the preimages in the metaplectic group. It is then a fact that $\widetilde{G}$ and $\widetilde{G}'$ commute, and we can consider $\omega_\psi|_{G \times G'}$.

Conjecture (Howe)**.** *The Weil representation gives a bijection between certain representations of G and G':* $R_\psi(\widetilde{G}) \leftrightarrow R_\psi(\widetilde{G}')$.

We will see a concrete example of this in the next lecture, in the context of the previous example.

2.4 The global metaplectic group

Now let F be a number field, and consider an additive character $\psi \colon F \backslash \mathbb{A}_F \to \mathbb{C}^\times$, $\psi = \otimes \psi_v$. If W is a symplectic space over F, then localising at v gives a corresponding map of symplectic spaces $W \to W_v / F_v$ by extension of scalars. In particular, for each v we get a metaplectic group $\mathrm{Mp}\, W_v$, and it is natural to ask if we can glue this to a "global cover" of $\mathrm{Sp}\, W_{\mathbb{A}_F}$.

This is indeed possible. Since we want it to be a double cover globally as well, let $Z := \{(z_v) \in \{\pm 1\} : \prod_v z_v = 1\}$. We define $\mathrm{Mp}\, W_{\mathbb{A}_F} := \prod_v' \mathrm{Mp}(W_v)/Z$ with Z coming from the kernel of each map to $\mathrm{Sp}\, W_v$.

Then the fact that Z acts trivially under $\otimes' \omega_{\psi_v}$ implies that the local representations glue to a big global representation $\operatorname{Mp} W_{\mathbb{A}_F} : \omega_\psi \to \mathrm{GL}(S)$ where S is an adelic function space obtained by gluing; one reference is [Wei64, §29].

Here's a reference I found helpful! In particular, what really matters is not that G and G' are mutual centralisers, but that the von Neumann algebras generated by the images $\omega(\widetilde{G})$ and $\omega(\widetilde{G}')$ are mutual commutators. This is where the connection with Schur–Weyl duality arises, apparently, but I don't know what that is.

3 The local Shimura correspondence

Speaker: Alex Horawa

3.1 Genuine representations

Let k be a local field, and suppose $W = ke_1 \oplus ke_2$ is the 2-dimensional symplectic space from example 2.2.

Definition 3.1. An irreducible representation of $\operatorname{Mp} W$ is called **genuine** if it does not factor through $\mathrm{Sp}_2(k)$.

We denote by $\operatorname{Irr} \operatorname{Mp} W$ the set of such representations.

Remark. One might think that the local Langlands correspondence is the right tool to understand $\operatorname{Irr} \operatorname{Mp} W$. However, $\operatorname{Mp} W$ is not linear algebraic, and so falls outside the domain of the standard Langlands conjectures. See however the work of Marty Weissman and others, for example [Wei18].

We have already seen some elements of $\operatorname{Irr} \operatorname{Mp} W$: $\omega_\psi^\pm$ which is the even or odd functions in the Schrödinger model of the Weil representation, is a pair of irreducible representations.

Example 3.2. Let $a \in k^\times$, and consider $\psi_a : x \mapsto \psi(ax)$ where ψ is the usual fixed additive character of k. This gives a new representation ω_{ψ_a}, and one can check that $\omega_{\psi_a} \cong \omega_{\psi_b}$ if and only if $a/b \in (k^\times)^2$. We can also think of these as ω_ψ twisted by a quadratic character $\chi : k^\times \to \{\pm 1\}$, as we will soon see. Explicitly, $\omega^\pm_{\psi,\chi} = \omega^\pm_{\psi_a}$ if $a = \prod b$ where the product runs over $b \in k^\times/(k^\times)^2$ such that $\chi(b) = -1$.

To understand representations of algebraic groups, one often tries to first understand characters of embedded tori. So, let $T \subset \mathrm{SL}_2(k)$ be the diagonal torus $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, and consider the diagram

$$\begin{tikzcd} 1 \arrow[r] & \{\pm 1\} \arrow[r] & \operatorname{Mp} W \arrow[r] & \mathrm{SL}_2(k) \arrow[r] & 1 \\ 1 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[u, equal] & \widetilde{T} \arrow[r] \arrow[u, hook] & T \arrow[r] \arrow[u, hook] & 1 \end{tikzcd} \tag{3.1}$$

Here $\widetilde{T}$ is the preimage of T in $\operatorname{Mp} W$ as usual.

Definition 3.3. We denote by $\chi_\psi : \widetilde{T} \to \mathbb{C}^\times$ the character sending $(t(a), \epsilon)$ to $\epsilon \cdot \gamma(a, \psi)^{-1}$, where $\gamma(a, \psi)$ is the epsilon-factor $\epsilon(1/2, \eta_a, \overline{\psi})$, η_a being the quadratic character associated to the extension $k(\sqrt{a})/k$.

One can also write $\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi)$ where the latter γ is the so-called *Weil index*. One reference for this equality is [Szp18].

Proposition 3.4. *The map $\mu \mapsto \chi_\psi \cdot \mu$ gives a bijection between characters of T and genuine characters of $\widetilde{T}$.*

Note that this bijection depends on a choice of character ψ. Now let $Z \subset T$ be the centre of $\mathrm{SL}_2(k)$ and consider its lift $\widetilde{Z}$. Depending on whether or not -1 is a square in k, this is either isomorphic to $C_2 \times C_2$ or C_4. If $\sigma \in \operatorname{Irr} \operatorname{Mp} W$, then the central character is either $\chi_\psi|_{\widetilde{Z}}$ or $\chi_\psi|_{\widetilde{Z}} \cdot \operatorname{sgn}$. Let $z_\psi(\sigma)$ be 1 or -1 in the corresponding cases.

Now we want to describe $\operatorname{Irr} \operatorname{Mp} W$ using the theta correspondence. Let (V, q) be a 3-dimensional orthogonal space. As before, we have a dual reductive pair $\operatorname{Mp} W \times O(V) \subset \operatorname{Mp} W \otimes V$. The Bruhat–Schwartz space $\mathcal{S}(ke_2 \otimes V)$ can naturally be identified with $\mathcal{S} := \mathcal{S}(V)$, and we have the following explicit formulas for the Weil representation:[6]

$$\omega_\psi(\epsilon, h) \cdot \phi(v) = \phi(h^{-1}v) \quad \text{for} \quad h \in O(V), \tag{3.2a}$$

$$\omega_\psi(\epsilon, t(a)) \cdot \phi(v) = |a|^{3/2}\chi_\psi(a)\phi(v) \quad \text{for} \quad t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \tag{3.2b}$$

$$\omega_\psi(\epsilon, n(x)) \cdot \phi(v) = \psi(x \cdot q(v)) \cdot \phi(v) \quad \text{for} \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{3.2c}$$

$$\omega_\psi(\epsilon, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\phi(v) = \gamma(\psi) \int_V \psi(-\langle v, z\rangle\, \phi(z))dz. \tag{3.2d}$$

Next we define theta lifts on the level of representations:

Definition 3.5. The **big Θ-lift** of a smooth $O(V)$ representation π is the π-isotypic component of $\omega_{\psi,V,W}|_{O(V)\times \operatorname{Mp} W}$. In other words, it is the largest smooth representation $\Theta(\pi) = \Theta_{\psi,V,W}(\pi)$ of $\operatorname{Mp} W$ such that $\pi \boxtimes \Theta(\pi) \subset \omega_{\psi,V,W}$.

This gives a map $\operatorname{Rep} O(V) \to \operatorname{Rep} \operatorname{Mp} W$. Completely analogously, we define a big Θ-lift in the reverse direction. To understand this map explicitly, we first need an concrete understanding of orthogonal spaces of dimension 3.

Proposition 3.6. *Let V/k be a 3-dimensional orthogonal space. Then V is either V^+ satisfying $\mathrm{SO}(V^+) \cong \mathrm{PGL}_2(k)$, or V^- satisfying $\mathrm{SO}(V^-) \cong PD^\times$ where D/k is the unique nonsplit quaternion algebra over k.*

We distinguish these by the invariant $\epsilon(V^\pm) := \pm 1$.

3.2 Waldspurger's theorem

Theorem 3.7 (Waldspurger)**.**

(i) *If $\pi \in \operatorname{Irr} \mathrm{SO}(V)$, then*
   - *(a) there exists a unique character ϵ such that $\Theta_\psi(\pi^\epsilon) \neq \emptyset$,*
   - *(b) For this ϵ, $\Theta_\psi(\pi^\epsilon)$ has a unique irreducible quotient $\vartheta_\psi(\pi^\epsilon) \in \operatorname{Irr} \operatorname{Mp} W$,*
   - *(c) $\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi, \psi)$.*

(ii) *If $\sigma \in \operatorname{Irr} \operatorname{Mp} W$, then*
   - *(a) there exists a unique 3-dimensional orthogonal space V such that $\Theta_\psi(\sigma) \neq 0$,*
   - *(b) $\Theta_\psi(\sigma)$ has a unique irreducible quotient $\vartheta_\psi(\sigma)$,*
   - *(c) $\epsilon(V) = z_\psi(\sigma) \cdot \epsilon(1/2, \sigma, \psi)$, where the last factor is some local ϵ-factor for $\operatorname{Mp} W$ which we have yet to define.*

(iii) *The map*

$$\operatorname{Irr} \operatorname{Mp} W \to \operatorname{Irr} \mathrm{SO}(V^+) \sqcup \operatorname{Irr} \mathrm{SO}(V^-) \quad \textit{given by} \quad \sigma \mapsto \vartheta_\psi(\sigma) \tag{3.3}$$

is a bijection preserving L, ϵ and γ-factors, suitably defined for $\operatorname{Mp} W$.

[6] NB: there was some confusion during the lecture, and these formulas might not be correct. One paper that's often cited with presumably correct formulas is [Rao93].

The bijection in eq. (3.3) is called the *Shimura correspondence*, and the maps ϑ are called *small theta lifts*.

There is an alternative way to package these results: by the Jacquet–Langlands correspondence, we can match up representations of $\mathrm{SO}(V^-)$ with discrete series representations of $\mathrm{SO}(V^+)$. If D denotes the associated quaternion algebra over k, let π_D be a representation of $PD^\times$, and $\pi := \mathrm{JL}(\pi_D)$ be the Jacquet–Langlands lift. Then we can rewrite Waldspurger's theorem as an equality

$$\mathrm{Irr}\,\mathrm{Mp}\,W = \bigsqcup_{\pi\in\mathrm{Irr}\,\mathrm{PGL}_2(k)} A_\psi(\pi) \quad \text{where} \quad A_\psi(\pi) = \{\sigma^+ := \theta_{\psi,W,V^+}(\pi),\ \sigma^- := \theta_{\psi,W,V^-}(\pi_D)\}, \tag{3.4}$$

where π_D is assumed trivial if π is not discrete series. The set $A_\psi(\pi)$ is called a *Waldspurger packet*.

Remark. Analogous ideas were used by Howe and Piatetski-Shapiro to cook up a counterexample to the naive generalisation of the Ramanujan conjecture to arbitrary automorphic forms. For them V is an orthogonal space of real signature $(2,2)$, giving a lift to $\mathrm{Sp}\,4$.

3.3 An explicit local Shimura correspondence

Let's try to make this even more explicit, using the classification of representations of $\mathrm{PGL}_2(k)$: Let $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ be the upper-triangular Borel subgroup, and $\mu\colon k^\times \to \mathbb{C}^\times$ a character. Then $\mu\times\mu^-$ is evaluated at the diagonal entries is naturally a character of B, and define $\pi(\mu,\mu^{-1}) := \mathrm{Ind}_B^G\, \mu\times\mu^{-1}$.

Proposition 3.8. *The following is a classification of irreducible representations of* $\mathrm{PGL}_2(k)$*:*

(i) *The representation $\pi(\mu,\mu^{-1})$ is irreducible if and only if $\mu^2 \neq |\cdot|^{\pm 1}$.*
(ii) *If $\mu = \chi\cdot|\cdot|^{1/2}$ for some quadratic character χ, then there is a short exact sequence*

$$0 \to \mathrm{St}_\chi \to \pi(\mu,\mu^{-1}) \to \chi\circ\det \to 0, \tag{3.5}$$

where St_χ is an irreducible representation of $\mathrm{PGL}_2(k)$ called the twist of the Steinberg representation. *Similarly, if $\mu = \chi\cdot|\cdot|^{-1/2}$, we have a short exact sequence*

$$0 \to \chi\circ\det \to \pi(\mu,\mu^{-1}) \to \mathrm{St}_\chi \to 0 \tag{3.6}$$

(is this also twist of Steinberg? Confused.)
(iii) *The representations not covered by this construction are known as* supercuspidal representations.

The representation theory of $PD^\times$ is simpler. The numbering is meant to match up with the previous proposition under Jacquet–Langlands.

Proposition 3.9. *The irreducible representations of $PD^\times$ are as follows:*

(i) *$\emptyset$ — there are no principal series representations.*
(ii) *$\chi\circ N_D$ where $N_D\colon D^\times \to k^\times$ is the norm character.*
(iii) *irreducible representations of dimension* > 1.

Finally, we need to describe the representations of the metaplectic group.

Proposition 3.10. *The irreducible representations of* $\mathrm{Mp}\,W$ *for* $\dim W = 2$ *are given by:*

(i) *let μ be a character of T so that $\mu\cdot\chi_\psi$ is a character of $\widetilde{T}$, giving the* principal series representation $\sigma_\psi(\mu) = \mathrm{Ind}_{\widetilde{B}}^{\mathrm{Mp}\,W}\, \mu\cdot\chi_\psi$[7]. *This is irreducible iff $\mu^2 \neq |\cdot|^{\pm 1}$.*

[7] Not obvious how this gives a character of the Borel?

(ii) *If* $\mu = \chi \cdot |\cdot|^{1/2}$ *with* χ *quadratic, then we get a* twisted Steinberg representation *by*

$$0 \to \mathrm{St}_{\psi,\chi} \to \sigma_{\psi,\mu} \to \omega^{+}_{\psi,\chi} \to 0, \tag{3.7}$$

and similar for $\mu = \chi \cdot |\cdot|^{-1/2}$.

(iii) *Supercuspidal representations, for example* ω^{-}_{ψ}.

We can now match up the representations explicitly:

Irr Mp W	Irr $\mathrm{PGL}_2(k)$	Irr $PD^\times$
$\sigma_\psi(\mu)$	$\pi(\mu, \mu^{-1})$	
$\mathrm{St}_{\psi,\mathbb{1}}$		$\mathbb{1}$
$\mathrm{St}_{\psi,\chi}, \chi \neq \mathbb{1}$	St_χ	
$\omega^{+}_{\psi,\chi}$	$\chi \circ \det$	
ω^{-}_{ψ}	$\mathrm{St}_{\mathbb{1}}$	
$\omega^{-}_{\psi,\chi}$		$\chi \circ N_D$
Other sc.	Other sc.	Other sc.

Table 1: Explicit Shimura correspondence

Here we have marked in similar colours the representations appearing in the same Waldspurger packet.

4 The global Shimura correspondence

Speaker: Alex Horawa

4.1 Global Waldspurger packets

Today we will describe irreducible discrete series representations of Mp $W_{\mathbb{A}}$ in terms of those of $\mathrm{PGL}_2\,\mathbb{A}$. Suppose σ is an automorphic representation of Mp $W_{\mathbb{A}}$. A version of Flath's theorem for the metaplectic group then implies that $\sigma = \otimes_v \sigma_v$, where each σ_v lies in the Waldspurger packet of some representation π_v of $\mathrm{PGL}_2\, F_v$: $\sigma_v \in A_{\psi_v}(\pi_v)$

We want to construct *global* Waldspurger packet, and for this we need to understand which σ_v "glue together".

Definition 4.1. Let π be an automorphic representation of $\mathrm{PGL}_2\,\mathbb{A}$. The **global Waldspurger packet** attached to π is the set

$$\{\sigma^\epsilon := \otimes'_v \sigma_v^{\epsilon_v} : \epsilon \in \otimes_v \{\pm 1\} \text{ and } \sigma_v^{\epsilon_v} \in A_{\psi_v}(\pi_v)\}, \tag{4.1}$$

where the sign ϵ_v attached to $\sigma_v^{\epsilon_v}$ matches the sign in the local packet $A_{\psi_v}(\pi_v)$.

Now σ^ϵ is an abstract representation of Mp $W_{\mathbb{A}}$, and we ask:

(i) Is σ^ϵ always an automorphic representation?
(ii) Does every automorphic representation of Mp $W_{\mathbb{A}}$ arise in this way?

It turns out that the answer to both of these questions is "no". For example, we recall from section 3.3 that $\omega^{-}_{\psi_v\chi_v} \in A_{\psi_v}(\chi_v \circ \det)$ for χ_v nontrivial. But $\chi \circ \det$ is a one-dimensional representation, so the local components do not arise from an irreducible representation of the metaplectic group.

Now construct $\omega_\psi = \otimes'_v \omega_{\psi_v} \colon \operatorname{Mp} W_{\mathbb{A}} \to \mathcal{S}(V_{\mathbb{A}})$. Since $\omega_{\psi_v} = \omega^-_{\psi_v} \oplus \omega^+_{\psi_v}$, this is "highly reducible" in the sense that each local component is reducible. For a fixed finite set of places S of F we define

$$\omega_{\psi,S} \coloneqq \big(\otimes_{v\in S} \omega^-_{\psi_v} \big) \otimes \big(\otimes_{v\notin S} \omega^+_{\psi_v} \big), \tag{4.2}$$

and we claim that these have a better chance of being automorphic.

Proposition 4.2. *Let Θ_ψ be the big theta lift from definition 3.5, and identify its domain with $\mathcal{S}(\mathbb{A})$:*

$$\Theta_\psi \colon \phi \mapsto \Big(g \mapsto \sum_{x\in F} (\omega_\psi(g)\phi)(x) \Big). \tag{4.3}$$

(i) *For any ϕ, $\Theta_\psi(\phi)$ is left $\operatorname{Sp} W_F$-invariant,*
(ii) $\ker \Theta_\psi = \bigoplus_{\#S \text{ odd}} \omega_{\psi,S}$, *so that* $\operatorname{Im}\Theta_\psi \cong \bigoplus_{\#S \text{ even}} \omega_{\psi,S}$,
(iii) $\operatorname{Im}\Theta_{\psi,S} \subset L^2_{\mathrm{disc}}(\operatorname{Mp} W_{\mathbb{A}})$, *and the image is cuspidal unless $S = \emptyset$.*

We can also obtain a version of this after twisting by a quadratic character $\chi\colon F^\times\backslash\mathbb{A}^\times \to \mathbb{C}$; this amounts to replacing ψ with ψ_a. We define $\Theta_{\psi,\chi}$ and $\omega_{\psi,\chi}$ in the natural way. Pick a nontrivial set of places S, and consider $\omega_{\psi,\chi,S}$, which one shows is an automorphic representation not coming from a global Waldspurger packet. However, we have the following:

$$L^2_{\mathrm{disc}}(\operatorname{Mp} W_{\mathbb{A}}) \subset \Big(\bigcup_S \omega_{\psi,\chi,S} \Big) \cup \Big(\bigcup_\pi A_\psi(\pi) \Big), \tag{4.4}$$

so $\omega_{\psi,\chi,S}$ is essentially the only counterexample.

Proposition 4.3. *Let σ^ϵ be as above. If σ^ϵ is nontrivial, then $\prod_v \epsilon_v = \epsilon(1/2,\pi,\psi)$.*

In fact, the converse is also true, as we will show in the remainder of the seminar.

Proof. By the discussion after proposition 3.4, the central character of $\sigma_v^{\epsilon_v}$ is given by $\omega_{\sigma_v^{\epsilon_v}}/\chi_{\psi_v} = \epsilon_v \cdot \epsilon(1/2,\pi_v,\psi_v)$, viewed as characters of $Z(F_v) = \{\pm 1\}$. But $\chi_\psi = \oplus_v \chi_{\psi_v}$ and ω_{σ^ϵ} global automorphic characters restricted from $\widetilde{T}$ to $Z(\mathbb{A})$. Their product is trivial, so $1 = (\prod_v \epsilon_v)\cdot \epsilon(1/2,\pi,\psi)$, and this proves the claim. ☐

4.2 The main global theorem

Theorem 4.4 (Main global theorem)**.** *We have a decomposition*

$$L^2_{\mathrm{disc}}(\operatorname{Mp} W_{\mathbb{A}}) = \underbrace{\Bigg[\bigoplus_{\chi \text{quad.}} \bigoplus_{\#S\in 2\mathbb{Z}_{\geq 0}} \omega_{\psi,\chi,S} \Bigg]}_{\text{elementary theta functions}} \oplus \bigoplus_{\pi\in\mathcal{A}_0(\mathrm{PGL}_2(\mathbb{A}))} L^2_\pi, \tag{4.5}$$

where $L^2_\pi = \bigoplus_{\sigma^\epsilon} m_\psi(\sigma^\epsilon)\cdot\sigma^\epsilon$, the sum running over $\sigma^\epsilon \in \mathcal{A}_\psi(\pi)$, and

$$m_\psi(\sigma^\epsilon) \coloneqq \begin{cases} 1 & \text{if} \quad \prod_v \epsilon_v = \epsilon(1/2,\pi,\psi), \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Furthermore, each L^2_π is a "full near equivalence class". We will spend the rest of the course trying to understand the proof of this theorem. First, we need to construct σ^ϵ explicitly. To that end, we revert to the language of a orthosymplectic dual pair $O(V)\times \operatorname{Mp} W \subset \operatorname{Mp} W\otimes V$, and use the Weil representation:

Definition 4.5. Let π be an automorphic representation of $O(V)$. For any algebraic group G over F, we write $[G] := G(F)\backslash G(\mathbb{A})$. The **theta lift of** π, $\Theta_\psi(\pi)$ is the representation spanned by

$$\theta_{\psi,W,V}(\phi,f)\colon \operatorname{Sp} W_F\backslash \operatorname{Mp} W_{\mathbb{A}} \to \mathbb{C}, \qquad \theta_{\psi,W,V}(\phi,f)\colon g \mapsto \int_{[O(V)]} \theta_\psi(\phi)(g,h)\overline{f(h)}dh, \tag{4.7}$$

as ϕ runs through elements of $\mathcal{S}(V_{\mathbb{A}})$, and $f \in \pi$. Here θ_ψ is the global theta lift on $\operatorname{Mp} W_{\mathbb{A}} \times O(V_{\mathbb{A}})$,

$$\theta(\phi)(g,h) = \sum_{x\in V_F} (\omega_\psi(g,h)\phi)(x), \tag{4.8}$$

for ω_ψ the global Weil representation on $\operatorname{Mp} W_{\mathbb{A}} \times O(V_{\mathbb{A}})$.

As usual, we suppress subscripts which are clear from the context. One should think of $\theta(\phi,f)$ as the "f-isotypic component of $\theta(\phi)(g,h)$", by orthogonality of the inner product. This one could plausible hope to be an automorphic representation — assuming convergence — and which could be cuspidal when π is. This will be shown in the next lecture.

We will take the following as a black box:

Proposition 4.6. *The representation $\Theta_{\psi,W,V}(\pi)$ has an irreducible quotient isomorphic to $\otimes'_v\theta_{\psi,W_v,V_v}(\pi_v)$, with local components defined as in section 3.2.*

5 Periods of global theta lifts

Speaker: Håvard Damm-Johnsen

In this lecture, we will try to understand the representation $\Theta_\psi(\pi)$ by computing its associated *period functionals*. This will allow us to show prove non-vanishing and cuspidality.

5.1 $O(V)$ vs $\mathrm{SO}(V)$

In calculations, it is often easier to work with $\mathrm{SO}(V)$ than $O(V)$. We claim that little is lost in doing so: first, note that $O(V) = SO(V) \times \mu_2$, where μ_2 is the algebraic group with underlying set $\{\pm 1\}$. An automorphic representation of μ_2 is of the form

$$\operatorname{sgn}_S\colon [\mu_2] \to \mathbb{C} \qquad \operatorname{sgn}_S = \left(\bigotimes_{v\in S} \operatorname{sgn}_v\right) \otimes \left(\bigotimes_{v\notin S} \mathbb{1}_v\right), \tag{5.1}$$

where S is some finite set of places of F, and any representation π of $\mathrm{SO}(V)$ can be extended to $O(V)$ by fixing some S and taking $\pi \boxtimes \operatorname{sgn}_S$. Note that S should be even for sgn_S to factor through the adelic quotient. We can extend the definition of θ_ψ by setting $\Theta_\psi(\pi) = \sum_{\#S\in 2\mathbb{Z}_{\geq 0}} \Theta_\psi(\pi \boxtimes \operatorname{sgn}_S)$. However, the following proposition shows that it is not necessary when π is cuspidal:

Proposition 5.1. *Let π be a cuspidal automorphic representation of $\mathrm{SO}(V)$.*

(i) *There is at most one set S such that $\Theta_\psi(\pi \boxtimes \operatorname{sgn}_S)$ is nonzero.*
(ii) *This set S is characterised by the property*

$$v \in S \quad \text{if and only if} \quad \epsilon(1/2,\pi_v,\psi_v) = -1. \tag{5.2}$$

This is a consequence of the main local theorem as stated in Gan's lectures, Thm. 2.1. A consequence is that if $\#S$ is odd, then $\Theta_\psi(\pi) = 0$. The content of the proposition is therefore both a vanishing criterion, *and* the statement that passing to $\mathrm{SO}(V)$ gives no loss of generality.

5.2 Periods

Modular forms gives rise to numerical invariants known as periods by integrating against suitable sets, and these generalise to the adelic setting:

Definition 5.2. Let G be a reductive algebraic group, $H \leq G$ a reductive subgroup, and $\chi\colon [H] \to \mathbb{C}$ a character of H. If π is an automorphic representation of G and $f \in \pi$, define the **(H,χ)-period of f** to be the *function*

$$f_{H,\chi}(g) := \int_H f(hg)\overline{\chi(h)}dh. \tag{5.3}$$

The **(H,χ)-period functional** $\mathscr{P}_{H,\chi}$ is the linear map $\pi \to \mathbb{C}$ sending f to $f_{H,\chi}(1)$.

Note that $\mathscr{P}_{H,\chi}(\pi) \neq 0$ implies that π is nontrivial; if so, we call π "(H,χ)-distinguished".

Example 5.3. Let $\tilde{f}$ be a modular form of weight k and level $\Gamma_0(N)$, and f its associated automorphic form: $\tilde{f}(z) = f(g_z)\cdot y^{-k/2}$, where $g_z = (\mathrm{Id}_2, \dots, (\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix})) \in \mathrm{GL}_2(\mathbb{A})$. Let $H = \begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}$, and for $n \in \mathbb{Z}$ let $\chi = \otimes_v \chi_v$ where $\chi_\infty(x) = e^{2\pi i n x}$ and $\chi_p(x) = e^{-2\pi i n\{x\}}$.[8] Here and below we identify H with $[\mathbb{A}] = \mathbb{Q}\backslash\mathbb{A}$ via the map $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}$. Then we compute

$$f_{H,\chi}(g_z) = \int_{[\mathbb{A}]} f(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g_z)\chi(x)dx \tag{5.4a}$$

$$= \prod_p \left(\int_{\mathbb{Z}_p} f(\begin{pmatrix} 1 & x_p \\ 0 & 1\end{pmatrix})e^{2\pi i n\{x_p\}}dx_p\right)\cdot \int_{\mathbb{R}/\mathbb{Z}} f(g_{z+x_\infty})e^{-2\pi i n x_\infty}dx_\infty \tag{5.4b}$$

$$= 1 \cdot \int_0^1 \tilde{f}(z+x)y^{k/2}e^{-2\pi i n x_\infty}dx_\infty = a_n(\tilde{f})y^{k/2}. \tag{5.4c}$$

In other words, the (H,χ)-period computes exactly the n-th Fourier coefficient of f, and being (H,χ) distinguished for this choice of H simply means that the corresponding Fourier coefficient does not vanish.

In the example, we see that when χ is trivial and $N = \begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}$, then $f_{N,\chi}$ is precisely the constant term of $\tilde{f}$. Now we want to compute this for automorphic forms on $\mathrm{Mp}\, W_{\mathbb{A}}$, recalling that N splits over $\mathrm{Mp}\, W_{\mathbb{A}}$:

$$\mathscr{P}_N(\theta_\psi(\phi,f)) = \int_{[N]} \theta_\psi(\phi,f)(n)dn = \int_{[N]}\int_{[\mathrm{SO}(V)]} \theta(\phi)(n,h)\overline{f(h)}dhdn \tag{5.5a}$$

$$= \int_{[N]}\int_{[\mathrm{SO}(V)]} \overline{f(h)}\left(\sum_{x\in V_F}(\omega_\psi(n,h)\phi)(x)\right)dhdn =: (*) \tag{5.5b}$$

Now, a suitably altered version of eq. (3.2c) implies that $\omega_\psi(n,h)\phi(x) = \psi(nq(x))\phi(h^{-1}x)$, and this gives

$$(*) = \int_{[SO(V)]} \overline{f(h)}\left(\sum_{x\in V_F}\phi(h^{-1}x)\int_{[N]}\psi(nq(x))dn\right)dh = \int_{[SO(V)]}\overline{f(h)}\left(\sum_{\substack{x\in V_F\\ q(x)=0}}\phi(h^{-1}x)\right)dh \tag{5.6}$$

where the last equality comes from orthogonality of characters.

We now split into two cases: either V is a *definite* orthogonal space, meaning $q(x) = 0$ if and only if $x = 0$, or not. If it is, then $\mathscr{P}_N(\phi,f) = \phi(0)\int_{[SO(V)]}\overline{f(h)}dh$, which we recognise as a multiple of the constant term of $\overline{f}$. In particular, this is zero iff f is a cusp form.

[8] Curly braces means "p-adic fractional part".

The case of V indefinite is a bit more complicated. Recall that we call a nontrivial element $x \in V_F$ *isotropic* if $q(x) = 0$. It is a fact that $\mathrm{SO}(V)$ acts transitively on isotropic vectors[9], and the stabiliser U of a fixed isotropic vector x_0 is the unipotent radical of a Borel of $\mathrm{SO}(V)$. [Find reference to Conrad]

Example 5.4. Let $B = M_2(\mathbb{Q})$ and $V = B^{\mathrm{tr}=0}$ so that $\mathrm{SO}(V) = \mathrm{PGL}_2(\mathbb{Q})$, acting on V by conjugation. The vector $x_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is isotropic, and we have $\mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Q})}\, x_0 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, which is indeed a unipotent subgroup of the upper triangular Borel.

It follows that $X_0 \coloneqq \{x \in V_F : q(x) = 0\} = \{0\} \sqcup \mathrm{SO}(V)/U \cdot x_0$. Now

$$\mathscr{P}_N(\theta_\psi(\phi, f)) = \int_{[\mathrm{SO}(V)]} \overline{f(h)} \left(\sum_{\gamma \in U(F) \backslash \mathrm{SO}(V_F)} \phi(h^{-1}\gamma^{-1}x_0) \right) dh \tag{5.7a}$$

$$= \int_{U(F) \backslash \mathrm{SO}(V_{\mathbb{A}})} \overline{f(h)} \phi(h^{-1}x_0)\, dh \tag{5.7b}$$

$$= \int_{(U \backslash \mathrm{SO}(V))(\mathbb{A})} \int_{[U]} \overline{f(uh)} \phi(h^{-1}u^{-1}x_0)\, du\, dh \tag{5.7c}$$

$$= \int_{(U \backslash \mathrm{SO}(V))(\mathbb{A})} \phi(h^{-1}x_0) \underbrace{\left(\int_{[U]} \overline{f(uh)}\, du \right)}_{\text{period of } \overline{f}} dh \tag{5.7d}$$

The inner integral in the last equation is more or less the constant term of f, hence we have:

Proposition 5.5. *Let π be a cuspidal automorphic representation of $\mathrm{SO}(V)$. The representation $\Theta_\psi(\pi)$ is cuspidal unless V is anistropic and π trivial.*

While we have found a criterion for when $\Theta_\psi(\pi)$ is cuspidal, we still don't know if it is non-zero. To characterise this, we will exhibit non-zero periods, or Fourier coefficients. Fix a character ψ_a of N with $a \in F^\times$. By an identical argument as above, for any $f \in \pi$ we have

$$\mathscr{P}_{N,\psi_a}(f) = \int_{[\mathrm{SO}(V)]} \overline{f(h)} \sum_{x \in X_a(F)} \phi(h^{-1}x)\, dh \tag{5.8}$$

where $X_a(F) = \{x \in V_F : q(x) = a\}$. We fix $x_a \in X_a(F)$, and by Witt's extension theorem, the decomposition $V = Fx_a \oplus V_a$ makes $V_a = (Fx_a)^\perp$ a quadratic space; furthermore, $\mathrm{Stab}_{\mathrm{SO}(V)}\, x_a = \mathrm{SO}(V_a)$, which is an embedded torus in $\mathrm{SO}(V)$.

Example 5.6. Take $x_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $M_2(\mathbb{Q})$, with $q(x_a) = -\det x_a = 1$. Then $V_a = \mathbb{Q}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{Q}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, which is an isotropic subspace, and one checks that $\mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Q})}\, x_a$ is the diagonal torus. This can be identified with $\mathrm{SO}(1,1) = \mathrm{SO}(V_a)$.

With the same manipulations as for $a = 0$, we get

$$\mathscr{P}_{N,\psi_a}(\theta_\psi(\phi, f)) = \int_{(\mathrm{SO}(V_a) \backslash \mathrm{SO}(V))(\mathbb{A})} \phi(h^{-1}x_a) \underbrace{\int_{[\mathrm{SO}(V_a)]} \overline{f(th)}\, dt}_{=:\mathscr{P}_{V_a}(h \cdot \overline{f})} dh. \tag{5.9}$$

So the nonvanishing of the the period $\mathscr{P}_{N,\psi_a}(\theta_\psi(\phi, f))$ implies the nonvanishing of the torus period $\mathscr{P}_{V_a}(f)$. With a little more work, one can show the converse.

Proposition 5.7. *Let π be a cuspidal automorphic representation of $\mathrm{SO}(V)$. The (N, ψ_a)-period of $\Theta_\psi(\pi)$ is trivial if and only if $\mathscr{P}_{V_a}$ is non-zero on π.*

[9] I think one way to see this is to use [OMe63, Thm. 42:17]: argue that the subspaces spanned by two isotropic vectors are isometric, and that this isometry extends to all of V. In fact, this argument shows that $O(V)$ acts transitively on d-dimensional isotropic subspaces for each d. Why the isometry has determinant 1 requires some more thought.

5.3 Split torus periods

Now we specialise further, to the case where $B = M_2(F)$, and $T_a \subset PB^\times$ is the split diagonal torus, $T_a = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$. Then we can relate the toric period functional $\mathscr{P}_{V_a}$ from the previous section, to central L-values of twists of π using the following theorem:

Proposition 5.8 (Hecke–Jacquet–Langlands). *Let*

$$Z(s, f) := \int_{[\mathbb{A}^\times]} f\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} \, da, \tag{5.10}$$

and let S be the set of places of F which are ramified for π. Then for any decomposable $f = \otimes_v f_v \in \pi$ we have $Z(s, f) = L(s, \pi) \cdot \prod_{v \in S} Z_v^\#(s, f_v)$, where $Z_v^\#(s, f_v)$ is a local zeta integral. Furthermore, for any s_0 there exists $f_v \in \pi_v$ such that $Z_v^\#(s_0, f_v) \neq 0$.

Corollary 5.9. *The period functional $\mathscr{P}_{T_a}$ is non-zero on π if and only if $L(1/2, \pi) \neq 0$. In particular, nonvanishing implies $\Theta_\psi(\pi) \neq 0$, independently of ψ.*

In fact, there's no loss of generality in studying split torus periods as opposed to all of them:

Theorem 5.10. *Let π be a cuspidal automorphic representation of $\mathrm{SO}(V) \cong \mathrm{PGL}_2(F)$. Then the following are equivalent:*

(i) $L(1/2, \pi) \neq 0$,
(ii) *π has a non-zero split torus period,*
(iii) *π has a non-zero torus period,*
(iv) $\Theta_\psi(\pi) \neq 0$.

Now that we have two invariants depending on π with equal "support", it is natural to ask whether they are proportional. Waldspurger proved that this is indeed the case:

Theorem 5.11 (Waldspurger). *Let π be a cuspidal automorphic representation of $\mathrm{PGL}_2(F)$, and let $\pi^\vee$ denote its contragredient. For any $f_1 \in \pi$ and $f_2 \in \pi^\vee$ and any torus $T \subset \mathrm{PGL}_2(F)$, we have*

$$\mathscr{P}_{T,\chi}(f_1)\mathscr{P}_{T,\overline{\chi}}(f_2) = c\frac{L(1/2, \pi \otimes \chi)}{L(1, \pi, \mathrm{Ad})} \cdot (\textit{local factors}), \tag{5.11}$$

where c is some simple, explicit constant not depending on π.

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