Diagonal Restrictions of Hilbert Modular Forms

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Abstract

In this thesis, we study certain questions arising from the recent theory of *real quadratic singular moduli*, developed by Darmon and Vonk. The starting point is two results of Darmon–Pozzi–Vonk regarding diagonal restrictions of Eisenstein series on the Hilbert modular group. In the first part, we extend the results of [DPV21] to ring class characters, or equivalently, to values of analytic theta cocycles at non-fundamental RM points. The main contribution is an explicit adelic construction of Hilbert Eisenstein series transforming with respect to $\mathrm{SL}_2(\mathcal{O})$ where $\mathcal{O}$ is a non-maximal order in a real quadratic field.

In the second part, we describe algorithms based on [DPV23] to compute Gross–Stark units, which are generators of class fields of real quadratic fields, and Stark–Heegner points, which are conjecturally algebraic points on elliptic curves. As an application, we compute a fair amount of data and point out some statistical behaviour.

Contents

Introduction

A highlight of 19th and 20th century number theory is the theory of Complex Multiplication, or *CM theory*, for short. Among other things, this provides an explicit description of class fields of imaginary quadratic fields by evaluating Klein's j-function at imaginary quadratic irrationalities in the complex upper half plane $\mathfrak{h}$, so-called *singular moduli.* The 12th of Hilbert's 23 problems posed at the International Congress of Mathematics in 1900 asks for a similar analytic construction of abelian extensions of general fields. Beyond the rational numbers and CM fields, little is known, though Stark's conjectures, and notably the work of Dasgupta and Kakde [DK23], give one possible resolution over totally real fields.

Recently, Darmon and Vonk proposed a new approach similar in spirit to CM theory. Building on the pioneering work of [Dar01], they propose that certain cocycles on the *p-adic upper half plane* should play a role analogous to Klein's j-function on the complex upper half plane.

Intersections of modular geodesics

The starting point for CM theory is to consider imaginary quadratic roots in the complex upper half plane. In the context of real quadratic fields, it is natural to instead consider *modular geodesics*:

If τ is a generator of a real quadratic field F, there is a geodesic $\mathrm{geo}(\tau) \subset \mathfrak{h}$ connecting τ and its conjugate, τ'. The subgroup of $\mathrm{SL}_2(\mathbb{Z})$ preserving $\mathrm{geo}(\tau)$ is of rank 1, and we fix a generator γ_τ. Then the line segment $(z, \gamma_\tau z)$ for any $z \in \mathrm{geo}(\tau)$ defines a closed loop C_τ in the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. If $\gamma_\tau \in \Gamma_0(p)$, then C_τ similarly defines a class in $H_1(Y_0(p), \mathbb{Z})$, which can be acted upon by the Hecke correspondences T_n.

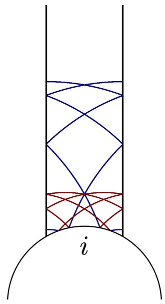

Figure 1: *The geodesics C_τ in the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ for $\tau = \pm\frac{3+\sqrt{33}}{4}$*

The duality pairing

$$H_1(X_0(p), \{0, \infty\}; \mathbb{Z}) \times H_1(Y_0(p), \mathbb{Z}) \to \mathbb{Z} \tag{0.1}$$

is compatible with the Hecke action, and we can pair $T_n C_\tau$ with the cycle $I = (0, \infty)$ in $H_1(X_0(p), \{0, \infty\}; \mathbb{Z})$. This pairing is simply the topological oriented intersection number between the two geodesics.

Fix an order $\mathcal{O} \subset F$ of discriminant D, which might be non-maximal. There is a natural bijection between equivalence classes of τ modulo $\mathrm{SL}_2(\mathbb{Z})$ of discriminant D and the narrow class group $\mathrm{Cl}^+ \mathcal{O}$. If $\psi : \mathrm{Cl}^+ \mathcal{O} \to \mathbb{C}$ is a totally odd ring class character, let Δ_ψ be the formal combination of modular geodesics $\Delta_\psi = \sum_{A \in \mathrm{Cl}^+ \mathcal{O}} \psi(A) C_{\tau_A}$.

Theorem 0.1 ([DPV21, Theorem A])**:** *Suppose $\mathcal{O}$ is maximal, and that p splits in $\mathcal{O} \subset F$. Then*

$$(\text{constant term}) - 2\sum_{n=1}^{\infty} \langle I, T_n \Delta_\psi \rangle q^n = E_\psi^{(p)}(z, z), \tag{0.2}$$

where $E_\psi^{(p)}(z_1, z_2)$ is a Hilbert Eisenstein series of weight $(1, 1)$ and level $\Gamma_0(p\mathcal{O}_F)$.

In Theorem 3.16 we prove the corresponding statement when ψ is a ring class character, or equivalently, when $\mathcal{O}$ is non-maximal. A generalization of Theorem 0.1 in the maximal case to totally real fields was given by Branchereau [Bra22], and we expect that his methods could be adapted to yield a proof for non-maximal orders.

A p-adic analogue: rigid meromorphic cocycles

When p is inert in $\mathbb{Q}(\tau)$, the diagonal restriction in Theorem 0.1 vanishes because of the triviality of $M_2(\mathrm{SL}_2(\mathbb{Z}))$, and the geodesic associated to τ does not give a cycle on $X_0(p)$. A striking programme initiated by Darmon and his collaborators suggests that one should instead consider intersections on "$\mathfrak{h}_p \times \mathfrak{h}$", where $\mathfrak{h}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ is the *p-adic upper half plane*. On this we consider $\mathcal{A}$, the ring of p-adic analytic functions on $\mathfrak{h}_p$.

Darmon and Vonk's theory of *rigid meromorphic cocycles* ([DV21], [DV22a]) proposes that values of cocycles in $H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times)$ play a similar role to that of modular units appearing in CM theory, when evaluated at real quadratic points $\tau \in \mathfrak{h}_p$. One of these, the *winding cocycle* $J_w \in H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/\mathbb{C}_p^\times)$, can be interpreted as a higher p-adic winding number with respect to I. It has an explicit expression as an infinite product of elements of $\mathbb{C}_p$,

$$J_w[\tau] := J_w(\gamma_\tau)(\tau) = \prod_{(r,s)\in \mathrm{SL}_2(\mathbb{Z}[1/p])\cdot(0,\infty)} c(r,s;\xi,\tau)^{[r,s]\cdot[\xi,\gamma\xi]} \tag{0.3}$$

where c denotes the modular cross-ratio, ξ is a suitable base point embedding simultaneously into $\mathfrak{h}$ and $\mathfrak{h}_p$, and $[r,s]\cdot[\xi,\gamma\xi]$ is the oriented intersection number on $\mathfrak{h}$. The construction turns out to be independent on the choice of ξ. The value $J_w[\tau]$ can also be thought of as a type of Massey product on $\mathfrak{h} \times \mathfrak{h}_p$, as described in [DV22b].

The following theorem should be thought of as a p-adic analogue of Theorem 0.1.

Theorem 0.2 ([DPV21, Theorem B]): *Suppose p is inert in $\mathcal{O}_F$, let ψ be a totally odd unramified character and fix Δ_ψ as above. Then*

$$(\text{constant term}) - 2\sum_{n=1}^{\infty} \log_p \mathrm{Nm}(T_n J_w[\Delta_\psi]) q^n = e^{\mathrm{ord}}\left(\frac{\mathrm{d}}{\mathrm{d}k}\mathcal{E}_k \mid_{k=1} (z,z)\right), \tag{0.4}$$

where $\mathcal{E}_k$ is the standard p-adic family of Hilbert Eisenstein series specialising to $E_\psi^{(p)}$.

The second main result in this thesis is a generalization to non-maximal orders $\mathcal{O}$. A different generalization of the theorem was proved in [DPV23], in which the modularity of a corresponding generating series *with the norms removed* is proved. Explicitly, they prove:

Theorem 0.3 ([DPV23, Theorem C]): *Suppose p is inert in the maximal order $\mathcal{O}_F \subset F$, let $\psi : \mathrm{Cl}^+ \mathcal{O}_F \to \mathbb{C}^\times$ be a totally odd character, and let Δ_ψ be as above. Then there exists a classical modular form $G_\psi \in M_2(\Gamma_0(p))$ with q-expansion*

$$G_\psi(z) = \log_p(u_\psi) - \sum_{n=1}^{\infty} \log_p(T_n J_w[\Delta_\psi]) q^n. \tag{0.5}$$

Here u_ψ denotes a certain *Gross–Stark unit*, a p-unit in the narrow Hilbert class field H of F. This is proved by an elaborate calculation of a *cuspidal* p-adic deformation of the parallel weight 1 Eisenstein series.

If we believe the analogy between the rigid cocycles J and modular units, then we expect the value of J at τ generating a non-maximal order $\mathcal{O}$ to be an invariant defined over a suitable ring class field. In [DPV23, Remark on p. 16], it was suggested that one might use Hilbert Eisenstein series with $\Gamma_1(N\mathcal{O}_F)$-level structure, where N is the conductor of $\mathcal{O}$, as considered in [DDP11], to generalize

Theorem 0.3 in this direction. Two complications then arise: firstly, the diagonal restriction of these Eisenstein series has level $M_2(\Gamma_0(Np))$, which suggests the spectral expansion gives information about cocycles with additional level structure. Furthermore, even when p is inert, the diagonal restriction does not vanish. To bypass these complications, the authors propose applying a degeneracy map $\mathrm{Tr}_p^{Np} : M_2(\Gamma_0(Np)) \to M_2(\Gamma_0(p))$. However, this introduces several challenges in understanding the corresponding p-adic family of Hilbert modular forms, for which the $\mathrm{GL}_2(\mathbb{Q})$-degeneracy map has no obvious natural interpretation.

The main conceptual contribution of this thesis is to suggest an alternative to this. Instead of the usual Hilbert Eisenstein series with $\Gamma_1(N\mathcal{O}_F)$-structure, we propose using Hilbert modular forms transforming with respect to $\mathrm{SL}_2(\mathcal{O})$, which is a congruence subgroup of level $N\mathcal{O}_F$. We refer to these as *Hilbert modular forms with order level structure*. This was motivated by the observation that in the natural generalization of Theorem 0.1, Theorem 3.16, one naturally finds sums of $\mathcal{O}$-ideals.

To define Eisenstein series with this unusual order structure, we use a very general construction of Eisenstein series which goes back to [Jac72], and specialize to the situation of interest. With a view towards future generalization, we give a bit more generality than necessary, treating Eisenstein series with a pair of characters instead of one. As an application, we extend some of the theorems above. The main results in the first half of the thesis are extensions of Theorem 0.1 and Theorem 0.2, which appear as Theorem 3.16 and Theorem 3.23, respectively.

Extending results about maximal orders to non-maximal orders is not a new invention. For example, the somewhat recent work of Longo, Martin and Hu [LMH20] extends work of Bertolini and Darmon about rationality of Stark–Heegner point associated to genus characters of non-maximal orders. Notably, their approach is also adelic.

Computational aspects

When the modular form G_ψ is expanded in terms of a Hecke eigenform basis, the coefficients are shown to give Gross–Stark units and *Stark-Heegner points*, or *Darmon points*, on modular abelian varieties, conjecturally defined over H. We can make the construction in Theorem 0.3 completely explicit, and hence give an algorithm for computing these invariants numerically. This is done in Chapter II, the content of which is published in [Dam24]. Much inspiration here comes from [LV22], where Lauder and Vonk compute p-adic L-functions over totally real fields using diagonal restrictions of classical Hilbert Eisenstein series. The main novelty in our work is a detailed description of how one might recover invariants such as u_ψ from a numerical approximation to $\log_p u_\psi$. A key tool is a theorem due to C. Meyer, which gives an explicit formula for the p-valuation of u_ψ. The proof of this is difficult to find in the literature, so we give a modern exposition in Appendix A.

Example 0.4: Let $p = 11$ and consider $E : y^2 + y = x^3 - x^2 - 10x - 20$, a model for $X_0(11)$. Using Algorithm 6 we find the points on E described in Table 1:

D	X	Y	P
21	$x^2 + 3x + 4$	$x^2 + 3x + 4$	$11x^2 - 6x + 11$
24	$x^2 + 8$	$x^2 + 10x + 57$	$11x^2 - 14x + 11$
28	$x^2 + \frac{71}{16}x + \frac{23}{4}$	$x^2 - \frac{101}{64}x + \frac{599}{64}$	$11x^2 - 6x + 11$
57	$x + \frac{1065}{304}$	$x^2 + x + \frac{1130412905}{28094464}$	$11x^2 - 3x + 11$
76	$x + \frac{1065}{304}$	$x^2 + x + \frac{1130412905}{28094464}$	$11x^2 - 3x + 11$

Table 1: *Table of Stark–Heegner points on* $E : y^2 + y = x^3 - x^2 - 10x - 20$*, for* $D < 100$.

For each row, the polynomials in columns X and Y are the minimal polynomials of the x- and y-coordinates, respectively, of a Stark–Heegner point on E defined over the narrow Hilbert class field of $\mathbb{Q}\left(\sqrt{D}\right)$. This field is generated over $\mathbb{Q}\left(\sqrt{D}\right)$ by a root of the polynomial P in the final column, which is a Gross–Stark unit. For example, for $D = 24$, $\left(2\sqrt{-2}, 5 + 4\sqrt{-2}\right)$ is a Stark–Heegner point on $X_0(11)$ defined over $\mathbb{Q}\left(\sqrt{24}, \sqrt{-2}\right)$, which is the splitting field over $\mathbb{Q}\left(\sqrt{24}\right)$ of $11x^2 - 14x + 11$.

The algorithms are implemented in both `magma` and `sage`, and can be found in the repositories https://github.com/havarddj/drd and https://github.com/havarddj/hilbert-eisenstein respectively.

While our algorithms simultaneously give Gross–Stark units and Stark–Heegner points, it is much faster to just compute the Gross–Stark unit. The following example shows the extent to which we pushed the Gross–Stark unit computations:

Example 0.5: Let $D = 8441 = 23 \cdot 367$. Then $F := \mathbb{Q}\left(\sqrt{D}\right)$ has narrow class number 26, and combining Algorithm 2 and Algorithm 5 gives the polynomial

$$\begin{aligned}
3^{43}x^{26} - 3^{28} \cdot 74700593x^{25} \quad & + 3^{21} \cdot 413213377697x^{24} \\
-3^{14} \cdot 1491793680346193x^{23} \quad & + 3^{11} \cdot 48103058975883121x^{22} \\
-3^{8} \cdot 117695071995350183 0x^{21} \quad & + 3^{8} \cdot 841442767734656470x^{20} \\
-3^{6} \cdot 5230173358710191479x^{19} \quad & + 3^{7} \cdot 1983729129037937219x^{18} \\
-3^{5} \cdot 28800297384178354201x^{17} \quad & + 3^{6} \cdot 13798304822142405250x^{16} \\
-3^{2} \cdot 131401208998818663 3625x^{15} \quad & + 3^{2} \cdot 135008529703506577 8356x^{14} \\
-1207461049666092903 0725x^{13} \quad & + 3^{2} \cdot 135008529703506577 8356x^{12} \\
-3^{2} \cdot 131401208998818663 3625x^{11} \quad & + 3^{6} \cdot 13798304822142405250x^{10} \\
-3^{5} \cdot 28800297384178354201x^{9} \quad & + 3^{7} \cdot 1983729129037937219x^{8} \\
-3^{6} \cdot 5230173358710191479x^{7} \quad & + 3^{8} \cdot 841442767734656470x^{6} \\
-3^{8} \cdot 117695071995350183 0x^{5} \quad & + 3^{11} \cdot 48103058975883121x^{4} \\
-3^{14} \cdot 1491793680346193x^{3} \quad & + 3^{21} \cdot 413213377697x^{2} \\
-3^{28} \cdot 74700593x \quad & + 3^{43}.
\end{aligned} \tag{0.6}$$

The roots of this polynomial are 3-units generating the narrow Hilbert class field of F, a degree 52 extension of $\mathbb{Q}$, and their square roots are Gross–Stark units attached to narrow ideal classes in F, as defined in Section 6.

Future directions

An obvious next step is to attempt to generalize Theorem 0.3 to ring class characters. Conceptually, this should be quite simple; the deformation theory arguments in [DPV23] are not sensitive to ramification away from p, and the main difficulty would be setting up the Hida theory for Hilbert modular forms with order level structure. We are optimistic that this might work in a manner similar to the unramified case, given a suitable definition of the full Hecke algebra of level $\mathrm{GL}_2(\mathcal{O})$.

Another interesting avenue of exploration is that of higher degree totally real fields. Theorem 0.1 was generalized in [Bra22], and a generalization of the cocycles J_w, J_{DR} and J_f^- to SL_n are defined in ongoing work of Xu and Roset Julià.

Finally, a version of Theorem 0.1 on compact Shimura curves, where the intersections in question are $\langle C_{\tau_1}, T_n C_{\tau_2} \rangle$, is the subject of ongoing work of the author. The modularity of the generating series is proved in [Ric22], and we compare with the diagonal restriction of a certain Hilbert theta series using the methods in [Bra22]. Its p-adic analogue is treated in forthcoming work of Darmon and Vonk.

In hope of finding a common framework for real quadratic singular moduli and CM theory, a potentially interesting line of inquiry involves the so-called *fake real quadratic orders* introduced by Cohen (see [Oh14] and [Wan17]). Let K be an imaginary quadratic field, and let $\ell = \lambda\lambda'$ be a split prime. Then the order $\mathcal{O}_K[\lambda^{-1}]$ behaves in several ways like the order in a real quadratic field; in particular, its unit group has rank 1. It would be interesting to understand whether there are natural analogues of our results for these orders, and in particular, if there is a natural construction of rigid meromorphic cocycles in this setting.

Structure of thesis

The thesis is divided into three parts. Chapter I is concerned with the generalization of the results of [DPV21]: Theorems A, B and C therein correspond to Theorem 3.16, Theorem 3.23 and Theorem 3.25, respectively. The brunt of the work lies in defining the correct Eisenstein series, done in Section 1, where the main result is Proposition 2.17.

The second part of the thesis, Chapter II, is a lightly modified version of the paper [Dam24]. Starting with Section 5, we quote relevant results from [DPV23] and explain how to make them explicit enough for a computer to understand. The subsequent Section 6 describes how to recover algebraic invariants from the output of the preceding algorithms.

Chapter III is an appendix which contains an exposition of Meyer's theorem (Appendix A), and tables of numerical data computed using the algorithms in part II (Appendix B).

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This thesis is dedicated to my grandparents, Arild and Berit.

Chapter I. Hilbert modular Eisenstein series

1 Hilbert Eisenstein series

In this section, we describe an adelic construction of certain Eisenstein series over real quadratic fields.

1.1 Notation and preliminaries

We first collect some notation and technical results which will be used in the remainder of the thesis. The casual reader may want to skip this and proceed directly to Section 1.2.

Local fields and adeles

Throughout the thesis, F will denote a real quadratic field, although many of the results we quote and prove extend to totally real fields of arbitrary degree. We usually denote the discriminant $\Delta_{F/\mathbb{Q}}$ by D_0.

Let v be a place of F. If v is an infinite place, written $v \mid \infty$, we define $F_\infty := \mathbb{R} \otimes_{\mathbb{Q}} F \cong \prod_{v|\infty} F_v$, and fix the usual Lebesgue measure $\mathrm{d}x$ on $F_v \cong \mathbb{R}$. This is a Haar measure for the topological group $(F_v, +)$. Our choice of multiplicative Haar measure on $F_v^\times$ is given by $\mathrm{d}^\times x := |x|^{-1}\,\mathrm{d}x$. The function $x \mapsto e^{2\pi i x}$ defines an additive character $\psi_v : F_v \to \mathbb{C}^\times$.

When v is a finite place, we write $\mathcal{O}_{F_v}$ for the ring of integers of the completion F_v, $\mathfrak{p}_v \leq \mathcal{O}_{F_v}$ for the maximal ideal of $\mathcal{O}_{F_v}$, and we fix a choice of uniformiser $\varpi_v \in \mathcal{O}_{F_v}$ generating $\mathfrak{p}_v$. The size of the residue field $\mathbb{F}_v := \mathcal{O}_{F_v}/\mathfrak{p}_v$ is denoted by q_v. We write $v(x)$ for the valuation of an element $x \in F_v$, so that $|x|_v = q_v^{-v(x)}$.

We say an additive character $\psi : F_v \to \mathbb{C}^\times$ has *conductor* $\delta \in \mathbb{Z}$ if δ is the largest integer such that $\psi\left(\varpi^{-\delta}\mathcal{O}_{F_v}\right) = \{1\}$. Given a choice of ψ, we normalize the Haar measure $\mathrm{d}x$ on F_v by

$$\operatorname{Vol}\mathcal{O}_{F_v} := \int_{\mathcal{O}_{F_v}} \mathrm{d}x = q_v^{-\delta/2}, \tag{1.1.1}$$

and normalize the multiplicative Haar measure on $F_v^\times$ by

$$\mathrm{d}^\times x := \frac{q_v}{q_v - 1}|x|_v^{-1}\,\mathrm{d}x. \tag{1.1.2}$$

This gives

$$\operatorname{Vol}^\times \mathcal{O}_{F_v}^\times := \int_{\mathcal{O}_{F_v}^\times} \mathrm{d}^\times x = q_v^{-\delta/2}, \tag{1.1.3}$$

so in particular $\operatorname{Vol}^\times \mathcal{O}_{F_v}^\times = 1$ when $\delta = 0$. When v lies above a rational prime p, a standard choice of additive character ψ_v is given by the formula

$$\psi_v(x) = e^{-2\pi i \left\{\operatorname{Tr}_{F_v/\mathbb{Q}_p}(x)\right\}}, \tag{1.1.4}$$

where the curly brackets denote the p-adic "fractional part" sending $\sum_{n=-N}^{\infty} a_n p^n$ to $\sum_{n=-N}^{0} a_n p^n$. The sign is chosen so that $\prod_v \psi_v(x) = 1$ for any $x \in F$. Then ψ_v has conductor $\delta_v = v(\mathfrak{d})$, where $\mathfrak{d}$ is the different ideal of F. Note that $\delta_v = 0$ when $F_v/\mathbb{Q}_p$ is unramified.

Confluent hypergeometric functions

Here we collect some analytic results which underlie the analytic continuation of the various Eisenstein series we consider.

Definition 1.1: The *confluent hypergeometric function* is defined by

$$\Xi(y;\alpha,\beta;t)=\int_{\mathbb{R}}\frac{e^{-2\pi itu}}{(u+iy)^{\alpha}(u-iy)^{\beta}}\,\mathrm{d}u,\tag{1.1.5}$$

for $y>0$, $t\in\mathbb{R}$ and $\Re(\alpha+\beta)>1$.

Note that our Ξ is $i^{\alpha-\beta}\xi$, where ξ is the function appearing in [Miy89].

Proposition 1.2 ([Miy89, Theorem 7.2.5]):

(i) *For any $y>0$, Ξ has a meromorphic continuation in α and β.*

(ii) *For fixed $t\neq 0$, Ξ is holomorphic in $(\alpha,\beta)\in\mathbb{C}^2$.*

(iii) *When $t=0$, we have*

$$\Xi(y;\alpha,\beta;t)=(2\pi)^{\alpha+\beta}\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}(4\pi y)^{1-\alpha-\beta}.\tag{1.1.6}$$

Consequently, $\Gamma(\alpha+\beta-1)^{-1}\Xi(y,\alpha,\beta,0)$ is holomorphic in $(\alpha,\beta)\in\mathbb{C}^2$.

We will need the following identities, which may easily be extracted from [Miy89, p.281]:

Proposition 1.3:

(i) *For $t<0$ and $\beta=0$, we have $\Xi(y;\alpha,0;t)=0$.*

(ii) *When $t>0$ and $\beta=0$,*

$$\Xi(y;\alpha,0;t)=(2\pi)^{\alpha}t^{\alpha-1}\frac{e^{-2\pi ty}}{\Gamma(\alpha)}.\tag{1.1.7}$$

Proof: (i): This follows from noting the explicit expression in [Miy89, Theorem 7.2.5], as the function ω therein is holomorphic, and $\Gamma(\beta)^{-1}=0$ when $\beta=0$. To prove (ii), combine the explicit formula for ξ with [Miy89, Lemma 7.2.6]. □

Local and global orders

Fix a real quadratic field F. Let $\mathcal{O}\subset F$ be an order of conductor N, and let $\hat{\mathcal{O}}:=\mathcal{O}\otimes\hat{\mathbb{Z}}$. For a rational prime ℓ dividing N, $\mathcal{O}_\ell:=\mathcal{O}\otimes\mathbb{Z}_\ell$ does not split even if ℓ splits in F. However, $\mathcal{O}_\ell$ is a local order in the étale algebra $F_\ell:=\mathbb{Q}_\ell\otimes F$, and we have $\hat{\mathcal{O}}=\prod'_{v\nmid N}\mathcal{O}_{F_v}\times\prod_{\ell\mid N}\mathcal{O}_\ell$. It is convenient to write $|\cdot|_\ell$ for the function on F_ℓ defined by $|(\alpha_v)|_\ell:=\prod_{v\mid\ell}|\alpha_v|_v$. Similarly, if $\rho:\mathbb{A}_F\to\mathbb{C}^\times$ is a character, we denote by ρ_ℓ the restriction to F_ℓ. The trace dual of $\mathcal{O}_\ell$ is

$$\mathfrak{d}_\ell^{-1}:=\{x\in F_\ell:\operatorname{Tr}(x\cdot y)\in\mathbb{Z}_\ell\text{ for all }y\in\mathcal{O}_\ell\}.\tag{1.1.8}$$

This is principal, generated by $\left(\sqrt{D}\right)^{-1}$, and we denote its inverse by $\mathfrak{d}_\ell=\sqrt{D}\mathcal{O}_\ell$.

The norm of an $\mathcal{O}$-ideal $\mathfrak{a}$ is by definition $\operatorname{Nm}(\mathfrak{a}):=\#(\mathcal{O}/\mathfrak{a})$, while the norm of an element a is $\operatorname{Nm}(a):=a\cdot a'$, where a' is the Galois conjugate of a. With this convention, we have $\operatorname{Nm}(a\mathcal{O}_F)=|\operatorname{Nm}(a)|$, although we note that the equality $\operatorname{Nm}(a\mathcal{O})=|\operatorname{Nm}(a)|$ does not hold for general orders $\mathcal{O}$, unless $a\mathcal{O}$ is coprime to the conductor of $\mathcal{O}$:

Proposition 1.4 ([Cox11] Proposition 7.20): *The map $\mathfrak{a}\mapsto\mathfrak{a}\mathcal{O}_F$ is a norm-preserving bijection between integral $\mathcal{O}$-ideals coprime to N and $\mathcal{O}_F$-ideals coprime to N.*

Accordingly, the places of F not dividing N are in natural bijection with the primes of $\mathcal{O}$ coprime to N. The following gives a local-to-global principle for $\mathcal{O}$-ideals:

Proposition 1.5 ([Voi21, Theorem 9.4.9]): *Fix a $\mathbb{Q}$-vector space V and a $\mathbb{Z}$-lattice $M \subset V$. There is a bijection between $\mathbb{Z}$-lattices $N \subset V$ and collections $\{N_\ell\}$, $N_\ell \subset V \otimes \mathbb{Q}_\ell$, where ℓ runs over primes of $\mathbb{Q}$, such that $N_\ell = M_\ell := M \otimes \mathbb{Z}_\ell$ for all but finitely many primes ℓ.*

Note that fractional $\mathcal{O}$-ideals are precisely the rank 2 $\mathbb{Z}$-lattices which are isomorphic to $\mathcal{O}_\ell$ for all but finitely many primes ℓ.

Ring class groups, classically and adelically

The *narrow ring class group* $\mathrm{Cl}^+ \mathcal{O}$ of an order $\mathcal{O}$ is the quotient of the group of invertible $\mathcal{O}$-ideals by the subgroup consisting of totally positive principal $\mathcal{O}$-ideals, $\{\alpha\mathcal{O} : \alpha \in F^\times_{\gg 0}\}$. Since every class in $\mathrm{Cl}^+ \mathcal{O}$ can be represented by an integral ideal coprime to N, we may describe $\mathrm{Cl}^+ \mathcal{O}$ in terms of the maximal order:

Proposition 1.6: *The narrow ring class group $\mathrm{Cl}^+ \mathcal{O}$ is isomorphic to the group of $\mathcal{O}_F$-ideals coprime to N modulo principal ideals with a totally positive generator congruent to an integer modulo N.*

Let F_∞^+ be the set of totally positive elements of $F_\infty = F \otimes \mathbb{R}$. The narrow ring class group has the following adelic description:

Proposition 1.7: *Let $\mathcal{O}$ be an order in a real quadratic field F. Then*

$$\mathrm{Cl}^+ \mathcal{O} \cong F^\times \setminus \mathbb{A}_F^\times / \hat{\mathcal{O}} F_\infty^+. \tag{1.1.9}$$

For a more detailed discussion, see [Cox11, §15E].

The inclusion $1 + N\hat{\mathcal{O}}_F \hookrightarrow \hat{\mathcal{O}}$ realises $\mathrm{Cl}^+ \mathcal{O}$ as a quotient of the narrow ray class group $\mathrm{Cl}_N^+ F$ of modulus N. As explained in [LMH20], we can describe this isomorphism on the level of ideals. Recall that the *ray class group* $\mathrm{Cl}_N^+ F$ is generated by $\mathcal{O}_F$-ideals relatively prime to N modulo $P_N^1 := \{\alpha\mathcal{O}_F : (\alpha, N) = 1, \alpha \gg 0, \alpha \equiv 1 \bmod N\}$, while $\mathrm{Cl}^+ \mathcal{O}$ is generated by the same ideals modulo $P_N^{\mathbb{Z}} := \{\alpha\mathcal{O}_F : (\alpha, N) = 1, \alpha \gg 0, \alpha \equiv \mathbb{Z} \bmod N\}$, by [Cox11] Proposition 7.22. Since $P_N^{\mathbb{Z}} = \sqcup_{c \in (\mathbb{Z}/N\mathbb{Z})^\times} cP_N^1$, one finds:

Proposition 1.8: *The ring class group $\mathrm{Cl}^+ \mathcal{O}$ is naturally a quotient of the ray class group $\mathrm{Cl}_N^+ F$,*

$$\mathrm{Cl}^+ \mathcal{O} \cong \frac{\mathrm{Cl}_N^+ F}{(\mathbb{Z}/N\mathbb{Z})^\times}. \tag{1.1.10}$$

In particular, any ring class character $\chi : \mathrm{Cl}^+ \mathcal{O} \to \mathbb{C}$ may be identified with a ray class character trivial on $(\mathbb{Z}/N\mathbb{Z})^\times$, and hence a unitary Hecke character on $\mathbb{A}_F^\times$. This is characterised by the property that for any place v not dividing N, we have then $\chi(\mathfrak{p}_v) = \chi(\varpi_v)$.

L-functions

Given a ray class character χ of conductor $\mathfrak{n} \leq \mathcal{O}_F$, define the associated Hecke L-function

$$L(s, \chi) := \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F \\ (\mathfrak{a}, \mathfrak{n}) = 1}} \frac{\chi(\mathfrak{a})}{\mathrm{Nm}(\mathfrak{a})^s}, \quad \Re(s) > 1. \tag{1.1.11}$$

We say χ is *totally odd* if $\chi(x) = \mathrm{sgn}(\mathrm{Nm}(x))$ for all $x \in F^\times$ satisfying $x \equiv 1 \bmod \mathfrak{n}$.

Proposition 1.9 ([Miy89, Theorem 3.3.1]): *For a totally odd Hecke character χ of conductor $\mathfrak{n}$, the function $L(s, \chi)$ has an Euler product*

$$L(s,\chi) = \prod_{(\mathfrak{p},\mathfrak{n})=1} \frac{1}{1-\chi(\mathfrak{p})\mathrm{Nm}(\mathfrak{p})^{-s}}. \tag{1.1.12}$$

The completed L-function

$$\Lambda(s,\chi) := \left(\Delta_{F/\mathbb{Q}}\mathrm{Nm}(\mathfrak{n})\right)^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s+1)^2 L(s,\chi), \quad \text{where} \quad \Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \tag{1.1.13}$$

satisfies the functional equation $\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-s,\overline{\chi})$ for some $\varepsilon(\chi) \in \mathbb{C}$ with $|\varepsilon(\chi)| = 1$.

This gives a meromorphic continuation of $L(s,\chi)$ to all of $\mathbb{C}$, and if χ is not the trivial character, $L(s,\chi)$ is holomorphic. Now assume that χ is a totally odd ring class character, interpreted as a ray class character of conductor $N\mathcal{O}_F$. Then

$$\chi_f(x\mathcal{O}_F) := \chi(x)\cdot \mathrm{sgn}(\mathrm{Nm}(x)) \tag{1.1.14}$$

defines a character $\chi_f : (\mathcal{O}_F/\mathfrak{n})^\times \to \mathbb{C}^\times$, and $\varepsilon(\chi)$ is given by

$$\varepsilon(\chi) = -\frac{\tau(\chi)}{N}, \quad \text{where} \quad \tau(\chi) := \sum_{x\in(\mathcal{O}_F/N)^\times} \chi_f(x)e^{2\pi i\frac{\mathrm{Tr}(x)}{N}}. \tag{1.1.15}$$

To relate $L(s,\chi)$ to $\mathcal{O}$, we use the bijection from Proposition 1.6 to rewrite $L(s,\chi)$ as a sum over $\mathcal{O}$ -ideals. When χ is primitive, that is, does not factor through $\mathrm{Cl}^+\mathcal{O}'$ for any order $\mathcal{O}' \supset \mathcal{O}$, we can extend it to a sum over elements of $\mathcal{O}$ not necessarily coprime to N, using the following argument from [Mey57] which goes back to Dedekind, [Ded00, §10].

Lemma 1.10: *Suppose χ is a primitive ring class character, and let ε be a fundamental unit of $\mathcal{O}$. Then*

$$L(s,\chi) = \sum_{A\in \mathrm{Cl}^+\mathcal{O}} \chi(A)\zeta(s,A), \tag{1.1.16}$$

where for any fixed ideal $\mathcal{O}$-ideal $\mathfrak{b}$ such that $[\mathfrak{b}] = A^{-1}$,

$$\zeta(s,A) = \mathrm{Nm}\,\frac{(\mathfrak{b})^s}{2} \sum_{\gamma\in\mathfrak{b}/\varepsilon} \frac{1}{|\mathrm{Nm}(\gamma)|^s}. \tag{1.1.17}$$

Proof: Fix an integral ideal $\mathfrak{a} \in A$. Then $\mathfrak{a}\mathfrak{b} = (\gamma)$ for some $\gamma \in \mathfrak{b}$. Note that $|\mathrm{Nm}(\gamma)| = \mathrm{Nm}(\mathfrak{a}\mathfrak{b})$, and since γ is uniquely determined modulo totally positive units, we get a 2-to-1 map between elements $\gamma \in \mathfrak{b}/\varepsilon$ coprime to N and $\mathcal{O}$-ideals prime to N with class A, taking signs into account. Let (γ, N) denote the $\mathcal{O}_F$-ideal generated by γ and N. We have

$$\sum_{\mathfrak{a}\in A} \frac{1}{\mathrm{Nm}(\mathfrak{a})^s} = \frac{\mathrm{Nm}(\mathfrak{b})^s}{2} \sum_{\substack{\gamma\in\mathfrak{b}/\varepsilon \\ (\gamma,N)=\mathcal{O}_F}} \frac{1}{|\mathrm{Nm}(\gamma)|^s}, \tag{1.1.18}$$

and it remains to show that we can remove the condition $(\gamma,N) = \mathcal{O}_F$. Note any element $\gamma \in \mathcal{O}$ may be written $\gamma = x + N\alpha$ for some $\alpha \in \mathcal{O}_F$, so the $\mathcal{O}_F$-ideal $(\gamma, N) = (x, N)$ is principal and generated by the integer $\gcd(x,N)$. With slight abuse of notation, we write $(\gamma,N) = \gcd(x,N)$ in this case. Now,

$$\zeta_d(s,A) := \sum_{\substack{\gamma\in\mathfrak{b}/\varepsilon \\ (\gamma,N)=d}} \frac{1}{|\mathrm{Nm}(\gamma)|^s} = d^{-s} \sum_{\substack{\gamma'\in\mathfrak{b}/\varepsilon \\ (\gamma',N)=1}} \frac{1}{|\mathrm{Nm}(\gamma')|^s}. \tag{1.1.19}$$

Note that γ' naturally runs over elements of the order $\mathfrak{b}$ viewed as an ideal of $\mathcal{O}'$, the order of conductor N/d. Let $\mathrm{pr} : \mathrm{Cl}^+\mathcal{O} \to \mathrm{Cl}^+\mathcal{O}'$ denote the natural projection map. Since χ is primitive,

$$\sum_{\mathrm{pr}(A)=A'} \chi(A) = 0 \quad \text{so} \quad \sum_{\mathrm{pr}(A')=A} \chi(A)\zeta_d(s,A) = 0, \tag{1.1.20}$$

for any fixed $A' \in \mathrm{Cl}^+ \mathcal{O}'$, as the value of $\zeta_d(s,A)$ depends only on $\mathrm{pr}(A)$. It follows that the contribution from terms corresponding to $(\gamma, N) = d$ is 0 unless $d = 1$. □

1.2 Example: A classical Eisenstein series

In this section, we give an illustrative computation of a classical Hilbert modular Eisenstein series with order level structure. This is independent of the main argument, but serves to indicate that one might in principle define the p-adic Eisenstein family without adelic machinery, at the cost of making the Hecke theory rather more complicated.

Let $j : \mathrm{GL}_2(\mathbb{R}) \times \mathfrak{h} \to \mathbb{C}$ be the function

$$j(g,z) = \det(g)^{-\frac{1}{2}}(cz+d), \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}). \tag{1.2.1}$$

Note that

$$j(gg'z) = j(g, g'z)j(g', z), \quad \text{for all } g, g' \in \mathrm{GL}_2(\mathbb{R}), \tag{1.2.2}$$

and

$$j(r(\theta), i) = e^{i\theta}, \quad \text{for } r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}). \tag{1.2.3}$$

Let $F/\mathbb{Q}$ be a totally real field of degree $d := [F : \mathbb{Q}]$, and let $\Sigma_F := \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$. For $F_\infty := F \otimes \mathbb{R} \cong \prod_{\sigma\in\Sigma_F} F_\sigma$, there is a natural embedding $\mathrm{GL}_2(F) \to \mathrm{GL}_2(F_\infty) \cong \prod_{\sigma\in\Sigma_F} \mathrm{GL}_2(\mathbb{R})$ given by $g \mapsto (\sigma(g))_\sigma$, and hence a natural action of $\mathrm{GL}_2(F)$ on $\mathfrak{h}_F^\Sigma \cong \mathfrak{h}^d$. If $z = (z_\sigma) \in \mathfrak{h}_F^\Sigma$ and $g \in \mathrm{GL}_2(F)$, then we write

$$gz := \frac{az+b}{cz+d} = \prod_{\sigma\in\Sigma_F} \frac{\sigma(a)z_\sigma + \sigma(b)}{\sigma(c)z_\sigma + \sigma(d)} \tag{1.2.4}$$

We also use the symbol j to denote the function $j : \mathrm{GL}_2(\mathbb{R})^{\Sigma_F} \times \mathfrak{h}^{\Sigma_F} \to \mathbb{C}$ defined by

$$j(g,z) := j((g_\sigma),(z_\sigma)) = \prod_{\sigma\in\Sigma_F} \det(g_\sigma)^{-\frac{1}{2}} j(g_\sigma, z_\sigma). \tag{1.2.5}$$

A *classical Hilbert modular form of weight* $k = (k_\sigma)$ is a holomorphic function

$$f : \mathfrak{h}^{\Sigma_F} \to \mathbb{C} \tag{1.2.6}$$

which satisfies

$$(f|_k\, \gamma)(z) := j(\gamma, z)^{-k} f(\gamma z) = f(z) \tag{1.2.7}$$

for all $\gamma \in \Gamma$, where $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_F)$ is some finite index subgroup. If $k_\sigma = k_{\sigma'}$ for all $\sigma, \sigma' \in \Sigma_F$, we identify k with k_σ and say that f has *parallel weight* k.

Now let $\mathcal{O} \subset F$ be an order of conductor N in a real quadratic field F. We define an associated Hilbert Eisenstein series classically by the formula

$$G_\chi(z,s) := \sum_{\mathfrak{a}\in \mathrm{Cl}^+\mathcal{O}} \chi(\mathfrak{a}) G_{\mathfrak{a}}(z,s), \quad \text{where } G_{\mathfrak{a}}(z,s) := \sum_{c,d\in\mathfrak{a}}{}' \frac{\mathrm{Nm}(\mathfrak{a})^k}{(cz+d)^k|cz+d|^{2s}}. \tag{1.2.8}$$

Here the symbol $\sum'$ means that the sum should be interpreted modulo the diagonal action of $\mathcal{O}_F^\times$ given by $\varepsilon \cdot (c, d) = (\varepsilon c, \varepsilon d)$, and leave out the term $(0, 0)$. By $\mathfrak{a} \in \mathrm{Cl}^+ \mathcal{O}$ we mean an implicitly chosen integral representative $\mathfrak{a}$ for its corresponding class.

Lemma 1.11: *The series defining $G_\chi(z, s)$ converges absolutely for $\Re(s) > 1 - \frac{k}{2}$, and the function $\Im(z)^s G_\chi(z, s)$ is invariant under the weight k action of $\mathrm{SL}_2(\mathcal{O})$ in the variable z.*

This is easily proved using the same argument as for Eisenstein series over $\mathbb{Q}$.

Theorem 1.12: *Fix $k \in \mathbb{N}$, and let $\chi : \mathrm{Cl}^+ \mathcal{O} \to \mathbb{C}^\times$ be a primitive character such that $\chi(\alpha) = \operatorname{sgn}(\mathrm{Nm}(\alpha))^k$ for any $\alpha \in \mathcal{O}$. Then the function $G_\chi(z, s)$ has meromorphic continuation to all $s \in \mathbb{C}$, with Fourier expansion at $s = 0$ given by*

$$\begin{aligned} G_\chi(z, 0) = L(k, \overline{\chi}) &- \delta_{k=1} \frac{(2\pi)^{2k}}{\sqrt{D}^{2k-1} \cdot \Gamma(k)^2} L(1-k, \overline{\chi}) \\ &+ \frac{(2\pi)^{2k}}{\sqrt{D}^{2k-1} \Gamma(k)^2} \sum_{\nu \in \mathfrak{d}_\mathcal{O}^{-1}} \left(\sum_{\substack{\mathfrak{a} \subset \nu \mathfrak{d}_\mathcal{O} \\ \text{proper}}} \chi(\mathfrak{a}) \mathrm{Nm}(\mathfrak{a})^{k-1} \right) e(\mathrm{Tr}(\nu z)), \end{aligned} \tag{1.2.9}$$

where $e(\mathrm{Tr}(\nu z)) = e^{2\pi i (\nu z_1 + \nu' z_2)}$. Here $\delta_{k=1}$ is 1 if $k = 1$ and 0 otherwise.

The proof is classical, and follows [Hec24].

Proof: We split $G_\mathfrak{a}(z, s)$ into sums $c = 0$ and $c \neq 0$. The former gives the series

$$\sum_{d \in \mathfrak{a}/\mathcal{O}^\times} \frac{1}{\mathrm{Nm}(d)^k \ |\mathrm{Nm}(d)|^{2s}} = \sum_{d \in \mathfrak{a}/\mathcal{O}^\times} \frac{\operatorname{sgn}(\mathrm{Nm}(d))^k}{|\mathrm{Nm}(d)|^{2s+k}}, \tag{1.2.10}$$

which equals $\zeta(2s + k, [\mathfrak{a}]^{-1})$. The corresponding contribution to $G_\chi(z, 0)$ is therefore $L(k, \overline{\chi})$.

For fixed $c \neq 0$, we apply Poisson summation with the lattice $\mathfrak{a} \subset F \otimes \mathbb{R}$ to obtain

$$\sum_{d \in \mathfrak{a}} \frac{1}{(cz + d)^k \ |cz + d|^{2s}} = \frac{1}{\mathrm{Nm}(\mathfrak{a}) \sqrt{D}} \sum_{\mu \in \mathfrak{a}^\vee} \int_{F \otimes \mathbb{R}} \frac{e(-\mathrm{Tr}(\mu u))}{(cz + u)^k |cz + u|^{2s}} \, \mathrm{d}u. \tag{1.2.11}$$

The integral splits into a product of two integrals corresponding to each embedding $\sigma_i : F \to \mathbb{R}$. Writing c for $\sigma_1(c)$, $c' = \sigma_2(c)$, $z' = z_2$ and so on, one finds by a change of variables that

$$I(\mu, c, s) := \int_\mathbb{R} \frac{e(-\mu u)}{(cz + u)^k |cz + d|^{2s}} \, \mathrm{d}u = \frac{\operatorname{sgn}(c)}{|c|^{2s} c^{k-1}} \int_\mathbb{R} \frac{e(-\mu c u)}{(z + u)^{k+s} (\overline{z} + u)^s} \, \mathrm{d}u. \tag{1.2.12}$$

This integral can be described in terms of the so-called *confluent hypergeometric function* in Definition 1.1: replacing u with $u - \Re(z)$ shows that

$$I(\mu, c, s) = \operatorname{sgn}(c) \frac{e(\mu c \Re(z))}{|c|^{2s} c^{k-1}} \Xi(\Im(z), k + s, s, \mu c). \tag{1.2.13}$$

Now Proposition 1.2 implies that the higher-order coefficients have analytic continuation to all of $\mathbb{C}$ in the variable s. We first consider the case $\mu = 0$. Then

$$\Xi(\Im(z), k + s, s, 0) = (2\pi)^{2s-k} \frac{\Gamma(2s + k - 1)}{\Gamma(s) \Gamma(s + k)} (4\pi \Im(z))^{1-(2s+k)}, \tag{1.2.14}$$

by Proposition 1.2 (iii), and summing over $c \in \mathfrak{a}/\mathcal{O}^\times$ shows that $\sum_c I(0, c, s) \cdot I(0, c', s)$ equals

$$(2\pi)^{2s+k}\left(\frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)}\right)^2(4\pi\Im(z))^{1-(2s+k)}\sum_c\frac{\operatorname{sgn}(\mathrm{Nm}(c))^{k-1}}{|\mathrm{Nm}(c)|^{2s+k-1}}. \tag{1.2.15}$$

Consequently, the contribution from $\mu = 0$ to the constant term of $G_\chi(z,s)$ is

$$\frac{\mathrm{Nm}(\mathfrak{a})^{k-1}}{\sqrt{D}}(2\pi)^{2s+k}\left(\frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)}\right)^2(4\pi\Im(z))^{-(2s+k-1)}2L(2s+k-1,\overline{\chi}). \tag{1.2.16}$$

When $k \geq 2$, this vanishes at $s = 0$ due to the pole of $\Gamma(s)$ in the denominator. For $k = 1$, $s = 0$, the poles of the gamma factors cancel, leaving a residue of $\frac{1}{2}$, and the remaining contribution is

$$-\frac{(2\pi)^2}{\sqrt{D}\cdot\Gamma(k)^2}\frac{L(0,\overline{\chi})}{4}. \tag{1.2.17}$$

For μ non-zero, we evaluate Ξ using Proposition 1.3. If μc is not totally positive, $I(\mu,c,0)\cdot I(\mu',c',0) = 0$ due to the pole of $\Gamma(s)$ at $s = 0$. In the totally positive case, the formulae give

$$I(\mu,c,0)\cdot I(\mu',c',0) = \operatorname{sgn}(\mathrm{Nm}(c))\frac{(2\pi i)^{2k}}{\Gamma(k)^2}\mathrm{Nm}(\mu)^{k-1} \tag{1.2.18}$$

for $s = 0$.

Let $\nu = \mu c$, and write $\mu = \sqrt{D}\alpha$ for some $\alpha \in \mathfrak{a}^{-1}$. Then $\alpha\mathfrak{a}$ is a proper integral $\mathcal{O}$-ideal which divides $\nu\mathfrak{d}$, since $\alpha\mathfrak{a} = \mu\sqrt{D}\mathfrak{a} \supset \mu\sqrt{D}c = \nu\mathfrak{d}_{\mathcal{O}}$. If we fix ν and vary μ and c such that $\mu c = \nu$, then $\mathfrak{b} := \alpha\mathfrak{a}$ runs through the proper integral $\mathcal{O}$-ideals dividing $\nu\mathfrak{d}$ with class $[\mathfrak{b}] = [\mathfrak{a}]$ exactly twice each. This accounts for all the proper ideal divisors of $\nu\mathfrak{d}_{\mathcal{O}}$: if $\nu \in \mathfrak{d}_{\mathcal{O}}$ and a proper ideal $\mathfrak{b} \subset \nu\mathfrak{d}_{\mathcal{O}}$ in A, then $\mathfrak{b}\mathfrak{a}^{-1}$ is principal and generated by some $\alpha \in \mathfrak{a}^{-1}$.

Since ν is totally positive, $\chi(\alpha) = \operatorname{sgn}(\mathrm{Nm}(\alpha))^k = -\operatorname{sgn}(\mathrm{Nm}(c))^k$, and a short computation shows that

$$\mathrm{Nm}(\mathfrak{b})^{k-1}\chi(\mathfrak{b}) = \mathrm{Nm}(\mu)^{k-1}(-D)^{k-1}\mathrm{Nm}(\mathfrak{a})^{k-1}\chi(\mathfrak{a})\operatorname{sgn}(\mathrm{Nm}(c)). \tag{1.2.19}$$

Collecting terms, we find that the ν-th coefficient of G_χ is given by

$$\frac{(2\pi)^{2k}}{\sqrt{D}^{2k-1}\Gamma(k)^2}\sum_{\substack{\mathfrak{a}\subset\nu\mathfrak{d}_{\mathcal{O}}\\ \text{proper}}}\chi(\mathfrak{a})\mathrm{Nm}(\mathfrak{a})^{k-1}, \tag{1.2.20}$$

which gives the result. □

Corollary 1.13: *Let*

$$E_\chi(z) := \Gamma(k)^2\frac{\sqrt{D}^{2k-1}}{(2\pi)^{2k}}\cdot G_\chi(z,0). \tag{1.2.21}$$

Then $E_\chi(z)$ has Fourier expansion

$$\begin{aligned} E_\chi(z) &= \varepsilon(\chi)\frac{L(1-k,\chi)}{4} - \delta_{k=1}\frac{L(1-k,\overline{\chi})}{4} \\ &+ \sum_{\nu\in\mathfrak{d}_{\mathcal{O}}^{-1}}\left(\sum_{\substack{\mathfrak{a}\subset\nu\mathfrak{d}_{\mathcal{O}}\\ \text{proper}}}\chi(\mathfrak{a})\mathrm{Nm}(\mathfrak{a})^{k-1}\right)e(\mathrm{Tr}(\nu z)), \end{aligned} \tag{1.2.22}$$

and is a holomorphic Hilbert modular form of parallel weight k.

Proof: By the functional equation for $L(s,\chi)$,

$$L(1-k,\chi)=\overline{\varepsilon(\chi)}L(k,\overline{\chi})\sqrt{D}^{2k-1}\frac{\Gamma_{\mathbb{R}}(k+1)^2}{\Gamma_{\mathbb{R}}(2-k)^2}. \tag{1.2.23}$$

The standard identities for the Γ-function give

$$\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}=2^{1-s}\sin\left(\frac{\pi s}{2}\right)\frac{\Gamma(s)}{\sqrt{\pi}}, \tag{1.2.24}$$

so

$$L(1-k,\chi)=\overline{\varepsilon(\chi)}L(k,\overline{\chi})\sqrt{D}^{2k-1}4\frac{\Gamma(k)^2}{(2\pi)^{2k}}. \tag{1.2.25}$$

This proves the first claim. For a proof of the second, which is equivalent to understanding the behaviour of Eisenstein series under certain lowering operators, see [Gar90, §4.7]. □

Note that this is very similar to the Fourier expansion of the Hilbert Eisenstein series of level $\mathrm{SL}_2(\mathcal{O}_F)$ with *unramified* character χ, compare [DDP11, Prop. 2.1].

1.3 Adelic Hilbert modular forms

The theory of Hecke operators for Hilbert modular forms is best understood using the adelic language. This is due to complications arising when the ground field has non-trivial class number. While good references for adelic Hilbert modular forms such as [RT11], [Gar90], [DV13] and [BH21] exist, we give some details for completeness.

Recall from [GH19, §6] the definition of an automorphic form: an *automorphic form for* $\mathrm{GL}_2(\mathbb{A}_F)$ is a smooth function $\Phi:\mathrm{GL}_2(\mathbb{A}_F)\to\mathbb{C}$ satisfying:

(i) $\Phi(\gamma g)=\Phi(g)$ for all $\gamma\in\mathrm{GL}_2(F)$;
(ii) The span of $\Phi(gk)$ for $k\in K_fK_\infty$, where $K_f:=\mathrm{GL}_2\left(\hat{\mathcal{O}}_F\right)\subset\mathrm{GL}_2(\mathbb{A}_F)$ and $K_\infty:=\mathrm{SO}_2(\mathbb{R})^d\subset\mathrm{GL}_2(F_\infty)$, is finite-dimensional;
(iii) Φ is $Z(\mathfrak{g})$-finite, where $\mathfrak{g}$ is the Lie algebra of $\mathrm{GL}_2(F_\infty)$;
(iv) There exists an adelic Hecke character $\omega:\mathbb{A}_F^\times\to\mathbb{C}$ such that $\Phi(zg)=\omega(z)\Phi(g)$ for any $z\in Z(G)$.

We denote the space of such functions by $\mathcal{A}_k(K,\omega)$. To interpret an automorphic form as a Hilbert modular form, first note that by [PRR93] Proposition 8.1, the determinant map induces a bijection

$$\mathrm{GL}_2(F)\setminus\mathrm{GL}_2(\mathbb{A}_F)/\mathrm{GL}_2^+(\mathbb{R})K_f\to F^\times\setminus\mathbb{A}_F^\times/F_\infty^+\hat{\mathcal{O}}_F\cong\mathrm{Cl}^+\mathcal{O}_F. \tag{1.3.1}$$

Thus we can pick representatives $t_\lambda=\left(\begin{smallmatrix}\lambda&0\\0&1\end{smallmatrix}\right)$, where $\lambda\in\mathbb{A}_F^\times$ has infinite components 1, and corresponds to an ideal $\mathfrak{t}_\lambda$. For the representative corresponding to $\mathcal{O}$, we always pick $\lambda=1$. This gives a decomposition

$$\mathrm{GL}_2(\mathbb{A}_F)=\bigsqcup_\lambda\mathrm{GL}_2(F)t_\lambda K_f\,\mathrm{GL}_2^+(\mathbb{R}\otimes F). \tag{1.3.2}$$

Since $\mathfrak{h}^d=\mathrm{GL}_2^+(F_\infty)/Z(\mathbb{R})K_\infty$, we have

$$\mathrm{GL}_2(F)\setminus\mathrm{GL}_2(\mathbb{A}_F)/Z(\mathbb{R})K_fK_\infty=\bigsqcup_\lambda^h\Gamma_\lambda\setminus\mathfrak{h}^d, \tag{1.3.3}$$

with $\Gamma_\lambda:=\mathrm{GL}_2^+(F)\cap t_\lambda K_ft_\lambda^{-1}$. An automorphic form on $\mathrm{GL}_2(\mathbb{A}_F)$ therefore gives rise to a tuple of functions $\{\Phi_i:\mathfrak{h}^d\to\mathbb{C}\}$ defined by

$$\Phi_\lambda(z) = \Phi(t_\lambda g_z), \tag{1.3.4}$$

where $g_z \in \mathrm{GL}_2(\mathbb{A}_F)$ is 1 at finite places, and for $v \mid \infty$,

$$g_{z_v} = \begin{pmatrix} 1 & x_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_v^{\frac{1}{2}} & 0 \\ 0 & y_v^{-\frac{1}{2}} \end{pmatrix} \quad \text{for } z_v = x_v + iy_v \in \mathfrak{h}. \tag{1.3.5}$$

Note that for any $\lambda \in \mathrm{Cl}^+ F$ and $\gamma \in \Gamma_\lambda$, $\Phi_\lambda \mid_k \gamma = \Phi_\lambda$. If Φ_λ is holomorphic, then it is a classical Hilbert modular form of level Γ_λ as described in Section 1.2.

Fourier–Whittaker expansions

Let $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ be an adelic Hilbert modular form. By $\mathrm{GL}_2(F)$-invariance,

$$\Phi\left(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} g\right) = \Phi(g) \quad \text{for all } \alpha \in F, \tag{1.3.6}$$

so we can write Φ in terms of its Fourier–Whittaker expansion,

$$\begin{aligned} \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) &= a_0(\Phi, g) + \sum_{\nu \in F^\times} W_\nu(\Phi, g) \cdot \psi(\nu x) \quad \text{for } x \in \mathbb{A}_F, \\ W_\nu(\Phi, g) &:= \int_{F \backslash \mathbb{A}_F} \Phi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-\nu u)\, \mathrm{d}u, \\ a_0(\Phi, g) &:= \int_{F \backslash \mathbb{A}_F} \Phi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}u, \end{aligned} \tag{1.3.7}$$

where ψ is the standard additive adelic character. By [GG12, Theorem 5.8], when $g = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have

$$\Phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbb{A}_F} \left(c(y) + \sum_{\xi \in F_{\gg 0}^\times} b(\xi y_f) W_k(\xi x_\infty) \psi(\xi y) \right) \tag{1.3.8}$$

for some $c(y) \in \mathbb{C}$, $b : \mathbb{A}_{F,f} \to \mathbb{C}$, and some function $W_k : F \otimes \mathbb{R} \to \mathbb{C}$. Furthermore, $b(\alpha)$ depends only on the fractional ideal $\mathfrak{a} = F \cap \alpha \cdot \hat{\mathcal{O}}_F$, and vanishes unless $\mathfrak{a}$ is integral. To emphasize the dependence on $\mathfrak{a}$ and Φ, we write

$$C(\mathfrak{a}, \Phi) := b(\alpha). \tag{1.3.9}$$

Similarly, we let $C_\lambda(0, \Phi) := a_0(\Phi, t_\lambda)$, where $\lambda \in \mathbb{A}_F^\times$ is an idele representing a given class in $\mathrm{Cl}^+ \mathcal{O}$, and $t_\lambda = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By the local-global principle for lattices, any integral ideal $\mathfrak{a}$ has an associated number $C(\mathfrak{a}, \Phi)$.

Hecke actions and p-refinement

Let v be a place of F, and fix a compact open subgroup $K_v \subset G_v := \mathrm{GL}_2(F_v)$. The *Hecke algebra at v* is the space $\mathcal{H}_v(G_v, K_v)$ of locally constant K_v-bi-invariant functions $G_v \to \mathbb{C}$. This is a $\mathbb{C}$-algebra with respect to the convolution product

$$(\varphi_1 * \varphi_2)(g) := \int_{G_v} \varphi_1(gh^{-1}) \varphi_2(h) dh, \tag{1.3.10}$$

where we equip G_v with the usual Haar measure, normalised so that $\mathrm{Vol}(K_v) = 1$.

A basis for this space is given by functions of the form $\mathbb{1}_{K_v g K_v}$ for $g \in \mathrm{GL}_2(F_v)$. If K_v is maximal, then $\mathcal{H}_v$ is said to be *spherical*, and is commutative by [GH19, Theorem 5.5.1]. A distinguished function is given by

$$T_{\mathfrak{p}_v} := \mathbb{1}_{K_v \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) K_v}. \tag{1.3.11}$$

Globalizing, we let $K_f \subset \mathrm{GL}_2(\mathbb{A}_{F,f})$ be a compact open subgroup, and fix $\alpha \in \mathrm{GL}_2(\mathbb{A}_{F,f})$. The double coset $K_f \alpha K_f$ can be decomposed into a finite disjoint union of right K_f-cosets, $K_f \alpha K_f = \bigsqcup_j \beta_j K_f$. For any K_f-invariant automorphic form Φ, we then define

$$([K_f \alpha K_f] \Phi)(g) := \sum_{\beta} \Phi(g\beta). \tag{1.3.12}$$

More formally, this is the convolution of Φ with the function $\mathbb{1}_{K_f \alpha K_f}$. For almost all places v, K_v is spherical and $\alpha_v \in K_v$, so the action at v is trivial.

If $\mathfrak{m} = \prod_v \mathfrak{p}_v^{m_v} \leq \mathcal{O}_F$ is an integral ideal, write $\varpi_{\mathfrak{m}} := \prod_v \varpi_v^{m_v}$ and define

$$T_{\mathfrak{m}} \Phi := \left[K \begin{pmatrix} \varpi_{\mathfrak{m}} & 0 \\ 0 & 1 \end{pmatrix} K \right] \Phi. \tag{1.3.13}$$

Write $\mathbb{T}_R(K)$ for the R-subalgebra of $\mathrm{End}(\mathcal{A}_k(K, \omega))$ generated by $\{T_{\mathfrak{m}}\}$.

An *eigenform* is an automorphic form Φ which is an eigenvector for all the elements of $\mathbb{T}_{\mathbb{C}}$, and it is *normalised* if $C(\mathcal{O}, \Phi) = 1$. In this case, it is known that $C(\mathfrak{m}, \Phi)$ is the $T_{\mathfrak{m}}$-eigenvalue of Φ, see [BH21, Proposition 3.2.8].

We recall from [BH21, §3.4] the notion of a p-refinement. For a finite place v of F, let $V_v^- := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi_v \end{smallmatrix}\right)$.

Lemma 1.14: *Suppose Φ has level $K_f \subset \mathrm{GL}_2(\mathbb{A}_{F,f})$, and suppose K_v is spherical. Then*

(i) $V_v^- \Phi$ has level $K_f^{(v)} \cdot K_1(\mathfrak{p}_v)$, and is independent of the choice of ϖ_v.

(ii) For any $c \in \mathbb{C}$, $a_1((1 - cV_v^-)\Phi) = a_1(\Phi)$.

(iii) For any integral ideal $\mathfrak{m}$ such that $v \nmid \mathfrak{m}$, $T_{\mathfrak{m}} V_v^- \Phi = V_v^- T_{\mathfrak{m}} \Phi$.

Proof: See [BH21, Lemma 3.4.1]. □

Let Φ be a normalised eigenform with central character ω, and fix a place v of F such that Φ is invariant under K_v. Given a root α_v of $T^2 - a(\mathfrak{p}_v, \Phi)T + \omega(\mathfrak{p}_v) q_v$, the *$\mathfrak{p}_v$-refinement of Φ* (with respect to α_v) is the automorphic form

$$\Phi^{(\mathfrak{p}_v)} := (1 - \alpha_v V_v^-)\Phi. \tag{1.3.14}$$

More generally, if p is a rational prime, a *p-refinement of Φ* is given by

$$\Phi^{(p)} := \left(\prod_{v \mid p} (1 - \alpha_v V_v^-) \right) \Phi, \tag{1.3.15}$$

given a choice of $(\alpha_v)_{v|p}$ as above.

Proposition 1.15: *$\Phi^{(p)}$ is a normalised eigenform of level $K_f^{(p)} K_1(p)$, with Fourier coefficients given by*

$$C(\mathfrak{p}_v^j, \Phi^{(p)}) = \begin{cases} C(\mathfrak{p}_v^j, \Phi) & \text{if } v \nmid p, \\ \alpha_v^j & \text{if } v \mid p. \end{cases} \tag{1.3.16}$$

Proof: See [BH21, Proposition 3.4.4]. □

2 Adelic Eisenstein series

References: [Che15], [Shi88], [Shi83]

In this section, we adopt the shorthands $G = \mathrm{GL}_2$, $K_f := \mathrm{GL}_2\left(\hat{\mathcal{O}}\right) \subset G\left(\mathbb{A}_{F,f}\right)$, and write K_∞ for the maximal compact subgroup of $G(F_\infty)$, isomorphic to $\prod_{\sigma \in \Sigma_F} \mathrm{SO}_2(\mathbb{R})$. Let $K := K_f K_\infty \subset G(\mathbb{A}_F)$. We also let $Z \subset G$ denote the standard diagonal torus.

2.1 Principal series and Godement sections

Definition 2.1: Let χ_1 and χ_2 be (not necessarily unitary) Hecke characters on $F^\times \setminus \mathbb{A}^\times$. We define $\mathcal{B}(\chi_1, \chi_2) := \mathrm{nInd}(\chi_1 \times \chi_2)$ to be the space of functions $f \in C^\infty(G(\mathbb{A}_F), \mathbb{C})$ such that

- for all $a, b \in \mathbb{A}_F^\times$ and $x \in \mathbb{A}_F$,

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \left|\frac{a}{b}\right|^{\frac{1}{2}} \chi_1(a)\chi_2(b) f(g); \tag{2.1.1}$$

- f is right K-finite: the span of $\{f(gk) : k \in K\}$ is finite-dimensional.

The vector space $\mathcal{B}(\chi_1, \chi_2)$ is naturally a right $G(\mathbb{A}_F)$-representation with action $(g \cdot f)(h) = f(gh)$, and decomposes as a restricted tensor product of similarly defined local spaces, $\mathcal{B}(\chi_1, \chi_2) = \bigotimes_v \mathcal{B}_v\left(\chi_{1,v}, \chi_{2,v}\right)$. One may naturally construct elements of $\mathcal{B}(\chi_1, \chi_2)$ via so-called *Tate integrals*: let $\varphi \in \mathcal{S}\left(\mathbb{A}_F^2\right)$ be a Schwartz function, and consider

$$\chi_1(\det g)|\det g|^{\frac{1}{2}} \int_{\mathbb{A}_F^\times} |t| \cdot \chi_1\chi_2^{-1}(t)\varphi((0,t)g)\,\mathrm{d}^\times t. \tag{2.1.2}$$

A straightforward computation shows that this defines an element of $\mathcal{B}(\chi_1, \chi_2)$ whenever the integral converges. This gives a $G(\mathbb{A}_F)$-equivariant map $\mathcal{S}(\mathbb{A}_F^2) \to \mathcal{B}(\chi_1, \chi_2)$. It is proved in [Shi88, Lemma 5] that this map is in fact surjective.

It is convenient to separate out the unitary part of χ_1 and χ_2 by twisting by suitable powers of the norm character. This motivates the following definition:

Definition 2.2: Let $\varphi \in \mathcal{S}\left(\mathbb{A}_F^2\right)$ be a Schwartz function, fix $s \in \mathbb{C}$ and a pair of unitary Hecke characters $\chi_1, \chi_2 : \mathbb{A}_F^\times \to \mathbb{C}^\times$. Then the associated *Godement section* is the element

$$f(g) := \chi_1(\det g)|\det g|^s \int_{\mathbb{A}_F^\times} |t|^{2s} \cdot \chi_1\chi_2^{-1}(t)\varphi((0,t)g)\,\mathrm{d}^\times t, \tag{2.1.3}$$

which defines an element of $\mathcal{B}\left(\chi_1 \,|\cdot|^{s-\frac{1}{2}}, \chi_2 \,|\cdot|^{-s+\frac{1}{2}}\right)$.

One can show that this converges for $\mathfrak{R}(s) > 1$. From now on, f will always denote a Godement section of this form. If χ_1, χ_2 and φ factor as restricted tensor products over places of F, then f does as well:

$$f = \bigotimes_v f_v = f_{\mathrm{fin}} \otimes f_\infty. \tag{2.1.4}$$

We can construct Schwartz functions $\varphi \in \mathcal{S}\left(\mathbb{A}_F^2\right)$ by taking products of local Schwartz functions at each place. For almost all places these are given as follows:

Definition 2.3: Fix an integer $k \geq 0$.

(i) The *standard weight k Schwartz function* $\varphi_k \in \mathcal{S}(\mathbb{R}^2)$ is given by the formula

$$\varphi_k(u, v) := (-iu + v)^k e^{-\pi(u^2+v^2)}. \tag{2.1.5}$$

(ii) Fix a place $v \nmid \infty$. We say φ_v is *spherical* if

$\varphi_v = \mathbb{1}_{\mathcal{O}_{F_v} \times \mathcal{O}_{F_v}}$.

The function f almost defines automorphic form on $G(\mathbb{A}_F)$, but fails to be left invariant under multiplication by elements of $G(F)$. To remedy this, we simply average; note that f is trivial on $B(F)$, so the function

$$\sum_{\gamma \in B(F) \backslash G(F)} f(\gamma g) \tag{2.1.6}$$

is well-defined and $B(F)$-invariant with central character $\chi_1 \chi_2$ whenever it converges.

Definition 2.4: Let f be a Godement section. The *Eisenstein series associated to f* is given by the formula

$$E(g, f) := \sum_{\gamma \in B(F) \backslash G(F)} f(\gamma g). \tag{2.1.7}$$

This is determined by the choice of characters χ_1 and χ_2, the Schwartz function φ and the complex parameter s.

Proposition 2.5 ([Gar90, pp. 110-111]): *When $\Re(s) > 1$, the Eisenstein series $E(g, f)$ converges absolutely, and uniformly for $g \in \mathrm{GL}_2(\mathbb{A}_F)$ in compact sets. Furthermore, it has a meromorphic continuation in $s \in \mathbb{C}$.*

Just like the classical Eisenstein series in Section 1.2, the meromorphic continuation of $E(g, f)$ follows from the computation of the Fourier expansion which we will see later. The following proposition implies that E is an eigenform with respect to the Hecke operators T_v for which K_v is spherical.

Proposition 2.6: *Suppose $E(g, f)$ is right $\mathrm{GL}_2\left(\mathcal{O}_{F_v}\right)$-invariant, and that*

$$f\begin{pmatrix} uu' & 0 \\ 0 & 1 \end{pmatrix} = f\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} f\begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1.8}$$

Then

$$T_v E(g, f) = E(g, f) \cdot \int_{K_v \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right) K_v} f(h^{-1}) \, \mathrm{d}h. \tag{2.1.9}$$

Proof: See [Gar90, p. 119]. □

Note that this condition is satisfied if χ_1 and χ_2 are unramified at v and φ_v is right $\mathrm{GL}_2\left(\mathcal{O}_{F_v}\right)$-invariant.

To see the relationship with the more usual expression for Eisenstein series, we compute the associated classical Hilbert modular form in the unramified case. When $F = \mathbb{Q}$, this is well-known and appears in for example [Che15, Corollary 2.4.12] and [Gar90, Chapter 4], but for totally real fields there does not seem to be any explicit passage from adelic to classical in the literature. Notably, the classical reference [Shi78, §3] introduces the Eisenstein series in terms of their classical Poincaré series.

Since f is decomposable, this boils down to one computation for each of the places of F. We first gather a few lemmas for the archimedean places.

Lemma 2.7: *Let φ_k be the standard weight k Schwartz function. If $r(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then $\varphi_k((u,v)r(\theta)) = e^{-ik\theta}\varphi_k(u,v)$. Consequently, for any $v \mid \infty$, we have $f_v(gr(\theta)) = f_v(g)e^{-ik\theta}$.*

Proof: First, note that u^2+v^2 is preserved under multiplication by $r(\theta)$. Combining this with the calculation $(u,v)r(\theta) = (-iu+v)(\cos\theta - i\sin\theta)$ finishes the proof. □

Lemma 2.8: *Suppose $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, and let $g_z := \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ whenever $z = x+iy$. Then*

$$\gamma g_z = g_{\gamma z} \cdot r(\theta) \cdot |cz+d|, \quad \text{where } e^{i\theta} = \frac{cz+d}{|cz+d|}. \tag{2.1.10}$$

Proof: Since $\gamma g_z \cdot i = g_{\gamma z} \cdot i$, we have $\gamma g_z = g_{\gamma z} r(\theta) \cdot C$ for some $r(\theta) \in \mathrm{SO}_2(\mathbb{R})$ $C \in \mathbb{R}$. To find θ and C, we compare the bottom rows:

$$cy = C\sin\theta \quad \text{and} \quad cx+d = C\cos(\theta). \tag{2.1.11}$$

It follows that $e^{i\theta} = C^{-1}(cz+d)$, and since $\theta \in \mathbb{R}$, $C = |cz+d|$. □

Lemma 2.9: *Suppose $v \mid \infty$. For any $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and $z = x+iy \in \mathfrak{h}$,*

$$f_v(\gamma g_z) = \Gamma_{\mathbb{R}}(2s+k)\frac{y^s}{|cz+d|^{2s-k}} j(\gamma,z)^{-k}. \tag{2.1.12}$$

Proof: Since $f \in B\left(\chi_1 \mid\cdot\mid^{s-\frac{1}{2}}, \chi_2 \mid\cdot\mid^{-s+\frac{1}{2}}\right)$, $f(g_z) = \chi_{1_v}(y)|y|_v^s f(1)$, and by [Che15, 2.4.8], $f(1) = \Gamma_{\mathbb{R}}(2s+k)$. It follows that

$$\begin{aligned} f_v(\gamma g_z) &= f_v\big(g_{\gamma z} r(\theta)|cz+d|\big) \\ &= f_v\big(g_{\gamma z}\big)e^{-ik\theta} \\ &= \Gamma_{\mathbb{R}}(2s+k)\mathfrak{I}(\gamma z)^s e^{-ik\theta} \\ &= \Gamma_{\mathbb{R}}(2s+k)\mathfrak{I}(\gamma z)^s \left(\frac{j(\gamma,z)}{|cz+d|}\right)^{-k}. \end{aligned} \tag{2.1.13}$$

Finally, since $\mathfrak{I}(\gamma z) = y|cz+d|^{-2}$, this gives the claim. □

For the finite places, we also require a few lemmas.

Lemma 2.10: *Fix a set of representatives $\mathfrak{r}$ for the wide ideal classes in $\mathrm{Cl}\,F$, and let $\delta_{\mathfrak{r}}$ be any matrix $\delta_r = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ such that $c\mathcal{O}_F + d\mathcal{O}_F = \mathfrak{r}$. Then we have a decomposition*

$$\mathrm{GL}_2(F) = \bigsqcup_{[\mathfrak{r}] \in \mathrm{Cl}\,F} B(F)\delta_{\mathfrak{r}}\,\mathrm{GL}_2(\mathcal{O}_F), \tag{2.1.14}$$

and consequently, a bijection

$$B(F) \setminus \mathrm{GL}_2(F) \to \bigsqcup_{[\mathfrak{r}] \in \mathrm{Cl}\,F} \Gamma_{\delta_{\mathfrak{r}}} \cap B(F) \setminus \Gamma_{\delta_{\mathfrak{r}}}, \tag{2.1.15}$$

where $\Gamma_\delta := \delta\,\mathrm{GL}_2(\mathcal{O}_F)\delta^{-1}$.

Proof: It is easily checked that $B(F) \setminus \mathrm{GL}_2(F) \cong \mathbb{P}^1(F)$ via the map $g \mapsto (0,1)g$. By [Gee88, Proposition 1.1], $\mathbb{P}^1(F)/\mathrm{GL}_2(\mathcal{O}_F)$ is in bijection with wide ideal classes via $[c:d] \mapsto c\mathcal{O}_F + d\mathcal{O}_F$, and picking a set of representatives proves the first claim. The second map is defined as follows: an element $g \in \mathrm{GL}_2(F)$ lies in a double coset corresponding to some $\delta_{\mathfrak{r}}$, so $g = b\delta_{\mathfrak{r}}\gamma$ for some $\gamma \in \mathrm{GL}_2(\mathcal{O}_F)$ and $b \in B(F)$. Then $g\delta_{\mathfrak{r}}^{-1} \in B(F)\Gamma_{\delta_{\mathfrak{r}}}$, and it is straightforward to check that the map $B(F)g \mapsto B(F)g\delta_{\mathfrak{r}}^{-1}$ is bijective. □

Note that we may freely adjust $\delta_{\mathfrak{r}}$ to have determinant 1.

Proposition 2.11: *Fix a representative $\delta = \delta_{\mathfrak{r}} \in \mathrm{SL}_2(F)$ as above. Then*

$$\prod_{v \nmid \infty} f_v(\delta) = \frac{1}{\sqrt{D}} \mathrm{Nm}(\mathfrak{r})^{2s} \rho(\mathfrak{r}) L(2s, \rho). \tag{2.1.16}$$

Proof: We compute:

$$f_v(\delta) = \int_{F_v^\times} |t|_v^{2s} \rho(t) \mathbb{1}_{\mathcal{O}_{F_v} \times \mathcal{O}_{F_v}}(ct, dt) \mathrm{d}^\times t. \tag{2.1.17}$$

Note that $ct, dt \in \mathcal{O}_{F_v}$ if and only if $t \in \left(\mathfrak{r}\mathcal{O}_{F_v}\right)^{-1}$. Thus

$$\begin{aligned} f_v(\delta) &= \int_{\left(\mathfrak{r}\mathcal{O}_{F_v}\right)^{-1}} |t|_v^{2s} \rho(t) \mathrm{d}^\times t \\ &= q_v^{-\frac{\delta_v}{2}} \sum_{k = -v\left(\mathfrak{r}\mathcal{O}_{F_v}\right)}^{\infty} q_v^{-2sk} \rho(\varpi)^k \\ &= q_v^{-\frac{\delta_v}{2}} \cdot \frac{q_v^{2sv\left(\mathfrak{r}\mathcal{O}_{F_v}\right)} \rho(\varpi_v)^{-v\left(\mathfrak{r}\mathcal{O}_{F_v}\right)}}{1 - q_v^{-2s} \rho(\varpi)}. \end{aligned} \tag{2.1.18}$$

Here the last equality comes from summing the geometric series. Taking the product over all finite places gives our result. □

Corollary 2.12: *Suppose f is the Godement section associated to a pair of unramified characters χ_1 and χ_2, and a Schwartz function φ which is spherical at all finite places. Fix $k \in \mathbb{N}$ and assume $\varphi_v = \varphi_k$ for all $v \mid \infty$.*

Then

$$E(g_z, f) = L(2s, \rho) \Gamma_{\mathbb{R}}(2s + k) \sum_{[\mathfrak{r}] \in \mathrm{Cl}\, F} \frac{\mathrm{Nm}(\mathfrak{r})^{2s}}{\sqrt{D}} \rho(\mathfrak{r}) \sum_{(c,d) = \mathfrak{r}}{}' \frac{y^s}{|cz + d|^{2s-k}} \frac{1}{(cz + d)^k} \tag{2.1.19}$$

for any $z \in \mathfrak{h}$.

As before, the symbol $\sum'$ means that the sum should be interpreted modulo the diagonal action of $\mathcal{O}_F^\times$, $\varepsilon \cdot (c, d) = (\varepsilon c, \varepsilon d)$.

Proof: First, note that if $\gamma \in \mathrm{GL}_2(F)$ corresponds to $\delta_{\mathfrak{r}}$ in the bijection in Equation (2.1.14), then

$$\prod_v f_v(\gamma) = \prod_v f_v(\delta_{\mathfrak{r}} k) = \prod_{v \nmid \infty} f_v(\delta_{\mathfrak{r}}) \prod_{v \mid \infty} f_v(\delta_r k), \tag{2.1.20}$$

as f_v are right $\mathrm{GL}_2\left(\mathcal{O}_{F_v}\right)$-invariant when v is a finite place. Each matrix $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ in $B(F) \cap \mathrm{GL}_2(\mathcal{O}_F) \cdot \delta_{\mathfrak{r}}$ corresponds to a pair of elements $c, d \in \mathfrak{r}$ modulo units such that $c\mathcal{O}_f + d\mathcal{O}_F = \mathfrak{r}$.

Now Proposition 2.11 and Lemma 2.9 give the claim. □

Fourier expansion of adelic Eisenstein series

We now return to the general setup where f is an arbitrary Godement section associated to a Schwartz function φ and a pair of Hecke characters χ_1, χ_2.

Proposition 2.13: *The Fourier–Whittaker expansion of $E(g, f)$ is given by*

$$E(g,f) = f(g) + \mathcal{M}(f)(g) + \sum_{\nu \in F^\times} W\left(\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} g\right), \tag{2.1.21}$$

where

$$\begin{aligned} \mathcal{M}(f)(g) &= \int_{\mathbb{A}_F} f\left(w\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}u, \\ W(g) &= \int_{\mathbb{A}_F} f\left(w\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-u) \, \mathrm{d}u, \end{aligned} \tag{2.1.22}$$

for $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof: The Bruhat decomposition of $\mathrm{GL}_2(F)$ gives

$$B(F) \setminus G(F) = \{1\} \sqcup \left\{ w\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} : \xi \in F \right\} \quad \text{for } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.1.23}$$

hence

$$E(g,f) = f(g) + \sum_{\xi \in F} f\left(w\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g\right). \tag{2.1.24}$$

Now plug into the Whittaker expansion to obtain

$$\begin{aligned} c_\nu(g) &= \int_{F \backslash \mathbb{A}_F} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-\nu u) \, \mathrm{d}u + \int_{F \backslash \mathbb{A}_F} \sum_{\xi \in F} f\left(w\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-\nu u) \, \mathrm{d}u \\ &= f(g) \int_{F \backslash \mathbb{A}_F} \psi(-\nu u) \, \mathrm{d}u + \int_{\mathbb{A}_F} f\left(w\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-\nu u) \, \mathrm{d}u, \end{aligned} \tag{2.1.25}$$

by unfolding the integral. From character orthogonality we get

$$\int_{F \backslash \mathbb{A}_F} \psi(-\nu u) \, \mathrm{d}u = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu \neq 0, \end{cases} \tag{2.1.26}$$

so the first term only contributes to $c_\nu(g)$ for $\nu = 0$. Suppose now $\nu \neq 0$ and perform the change of variables $u' = \nu u$. Since $|\nu|_{\mathbb{A}_F} = 1$,

$$\begin{aligned} c_\nu(g) &= \int_{\mathbb{A}_F} f\left(w\begin{pmatrix} 1 & u'\nu^{-1} \\ 0 & 1 \end{pmatrix} g\right) \psi(-u') \, \mathrm{d}u' \\ &= \int_{\mathbb{A}_F} f\left(w\begin{pmatrix} 1 & u'\nu^{-1} \\ 0 & 1 \end{pmatrix} g\right) \psi(-u') \, \mathrm{d}u' \\ &= \int_{\mathbb{A}_F} f\left(\begin{pmatrix} 1 & 0 \\ 0 & \nu^{-1} \end{pmatrix} w \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} g\right) \psi(-u') \, \mathrm{d}u' \\ &= \int_{\mathbb{A}_F} f\left(w\begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} g\right) \psi(-u') \, \mathrm{d}u' = W\left(\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} g\right), \end{aligned} \tag{2.1.27}$$

proving our claim. □

The adelic integrals defining the higher-order Fourier–Whittaker coefficients of $E(g, f)$ decompose as products over the places of F whenever φ does:

Proposition 2.14: *Let $E(g, f) = \sum_\nu c_\nu(g)$, where f is the Godement section associated to a pair of Hecke characters χ_1, χ_2 and $\varphi = \varphi_f \otimes \varphi_\infty$ where $\varphi_f(v_1, v_2) = \alpha(v_1)\beta(v_2)$ for some $\alpha, \beta : \mathbb{A}_{F,f} \to \mathbb{C}$. For $\rho = \chi_1\chi_2^{-1}$, we have $E(g, f) = 0$ unless $\rho_\infty(\alpha) = (\operatorname{sgn}\operatorname{Nm}(\alpha))^k$, in which case we have:*

(i) If $\nu \neq 0$, then

$$\begin{aligned} c_\nu\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\0 & 1\end{pmatrix}\right) &= \psi(\nu x)\chi_1(\nu y)|\nu y|^s\left(\int_{\mathbb{A}_{F,f}^\times}|t|^{2s-1}\rho(t)\alpha(ty\nu)\hat{\beta}(t^{-1})\,\mathrm{d}^\times t\right) \\ &\times \Gamma_\mathbb{R}(2s+k)\Xi\left(\nu y; s+\frac{k}{2}, s-\frac{k}{2}; 1\right), \end{aligned} \tag{2.1.28}$$

for all $y \in \mathbb{A}_F^\times$ and $x \in \mathbb{A}_F$, with Ξ given by Definition 1.1.

(ii) The constant term of E satisfies

$$\begin{aligned} c_0\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\0 & 1\end{pmatrix}\right) &= \chi_1(y)|y|^s\alpha(0)\Gamma_\mathbb{R}(2s+k)^d\int_{\mathbb{A}_{F,f}^\times}|t|^{2s}\rho(t)\beta(t)\,\mathrm{d}^\times t \\ &+\chi_2(y)|y|^{1-2s}\hat{\beta}(0)\int_{\mathbb{A}_{F,f}^\times}|t|^{2s-1}\rho(t)\alpha(t)\,\mathrm{d}^\times t\cdot C_\infty^d \end{aligned} \tag{2.1.29}$$

where

$$C_\infty = \pi\cdot 2^{2-2s}\frac{\Gamma_\mathbb{R}(2s+k)\Gamma(2s-1)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right)}. \tag{2.1.30}$$

This is similar to the statement of [Gar90, second Corollary on p. 118], though this contains some typographical errors.

Proof: A change of variables shows that $W\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}g\right) = \psi(x)W(g)$ for any $g \in \mathrm{GL}_2(\mathbb{A}_F)$ and $x \in \mathbb{A}_F$. Since

$$\begin{pmatrix}\alpha & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & \beta\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & \alpha\beta\\0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\0 & 1\end{pmatrix}, \tag{2.1.31}$$

for any $\alpha, \beta \in \mathbb{A}_F$, we obtain

$$c_\nu\left(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\0 & 1\end{pmatrix}\right) = W\left(\begin{pmatrix}\nu & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\0 & 1\end{pmatrix}\right) = \psi(x\nu)W\left(\begin{pmatrix}\nu y & 0\\0 & 1\end{pmatrix}\right), \tag{2.1.32}$$

and so it suffices to compute $W\left(\begin{pmatrix}y & 0\\0 & 1\end{pmatrix}\right)$. Note that

$$\begin{aligned} W\left(\begin{pmatrix}y & 0\\0 & 1\end{pmatrix}\right) &= \int_{\mathbb{A}_F} f\left(w\begin{pmatrix}1 & u\\0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\0 & 1\end{pmatrix}\right)\psi(-u)\,\mathrm{d}u \\ &= \chi_1(y)|y|^s\int_{\mathbb{A}_F}\int_{\mathbb{A}_F^\times}|t|^{2s}\rho(t)\varphi(ty, tu)\,\mathrm{d}^\times t\,\psi(-u)\,\mathrm{d}u. \end{aligned} \tag{2.1.33}$$

We first compute the archimedean part, and fix a place $v \mid \infty$. Since $\varphi_v = \varphi_k$,

$$\begin{aligned} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= \chi_{1,v}(y)|y|_v^s \int_{F_v} \psi(-u) \int_{F_v^\times} |t|_v^{2s} \rho_v(t) t^k (-iy+u)^k e^{-\pi t^2(y^2+u^2)} \mathrm{d}^\times t \,\mathrm{d}u \\ &= \chi_{1,v}(y)|y|_v^s \int_{F_v} \psi(-u)(-iy+u)^k \left(\int_{F_v^\times} |t|_v^{2s} \rho_v(t) t^k e^{-\pi t^2(y^2+u^2)} \mathrm{d}^\times t \right) \mathrm{d}u. \end{aligned} \tag{2.1.34}$$

If we split the domain of the inner integral into $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$, we see that the two terms cancel unless $\rho_v(t) = (\operatorname{sgn} t)^k$, in which case the inner integral equals

$$2 \int_0^\infty t^{2s+k} e^{-\pi t^2(u^2+y^2)} \mathrm{d}^\times t. \tag{2.1.35}$$

When $r := \pi t^2(u^2+y^2)$ we have $\frac{\mathrm{d}r}{r} = 2\frac{\mathrm{d}t}{t}$, which gives

$$\begin{aligned} &(u^2+y^2)^{-(s+\frac{k}{2})} \pi^{-(s+\frac{k}{2})} \int_0^\infty r^{s+\frac{k}{2}} e^{-r} \frac{\mathrm{d}r}{r} \\ &= (u^2+y^2)^{-(s+\frac{k}{2})} \pi^{-(s+\frac{k}{2})} \Gamma\left(s+\frac{k}{2}\right) \\ &= (u^2+y^2)^{-(s+\frac{k}{2})} \Gamma_{\mathbb{R}}(2s+k). \end{aligned} \tag{2.1.36}$$

Thus

$$\begin{aligned} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= \chi_{1,v}(y)|y|_v^s \Gamma_{\mathbb{R}}(2s+k) \int_{\mathbb{R}} \psi(-u)(u-iy)^k (u^2+y^2)^{-(s+\frac{k}{2})} \,\mathrm{d}u \\ &= \chi_{1,v}(y)|y|_v^s \Gamma_{\mathbb{R}}(2s+k) \int_{\mathbb{R}} \frac{\psi(-u)}{(u+iy)^{s+\frac{k}{2}}(u-iy)^{s-\frac{k}{2}}} \,\mathrm{d}u \\ &= \chi_{1,v}(y)|y|_v^s \Gamma_{\mathbb{R}}(2s+k) \Xi\left(y; s+\frac{k}{2}, s-\frac{k}{2}; 1\right), \end{aligned} \tag{2.1.37}$$

where Ξ is the confluent hypergeometric function from Definition 1.1. As a consequence, we note that $c_{\nu,v}$ has an analytic continuation to $s \in \mathbb{C}$.

We now turn to the non-archimedean component. Fix a finite place v of F and suppose $\varphi_v = \alpha_v \otimes \beta_v$. Then

$$W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \chi_{1,v}(y)|y|_v^s \int_{F_v^\times} |t|_v^{2s} \rho_v(t) \alpha_v(ty) \left(\int_{F_v} \psi_v(-u) \beta_v(tu) \,\mathrm{d}u \right) \mathrm{d}^\times t, \tag{2.1.38}$$

by Fubini–Tonelli. The inner integral can be written in terms of the local Fourier transform of β,

$$\int_{F_v} \psi_v(-u) \beta_v(tu) \,\mathrm{d}u = |t|_v^{-1} \int_{F_v} \psi_v(-u/t) \beta_v(u) \,\mathrm{d}u = |t|_v^{-1} \hat{\beta}_v(1/t), \tag{2.1.39}$$

by a change of variables. Replacing y with νy then finishes the proof of (i).

For (ii), recall that the constant term of E is given by $f + \mathcal{M}(f)$, and that

$$f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \chi(y)|y|^s \Gamma_{\mathbb{R}}(2s+k) \int_{\mathbb{A}_{F,f}} |t|^{2s} \rho(t) \varphi_f(0,t) \,\mathrm{d}t. \tag{2.1.40}$$

Since $\varphi_f = \alpha \otimes \beta$, this matches the first term of c_0. For $\mathcal{M}(f)$, a change of variables shows that

$$\mathcal{M}(f)\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \mathcal{M}(f)\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{2.1.41}$$

To evaluate

$$\mathcal{M}(f)\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \chi_1(y)|y|^s \int_{\mathbb{A}_F} \int_{\mathbb{A}_F^\times} |t|^{2s} \rho(t) \varphi(ty, tu) \,\mathrm{d}^\times t \,\mathrm{d}u, \tag{2.1.42}$$

perform a pair of substitutions $t' = ty$ and $v = uy^{-1}$ to obtain

$$\mathcal{M}(f)\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \chi_2(y)|y|^{1-2s} \int_{\mathbb{A}_F} \int_{\mathbb{A}_F^\times} |t'|^{2s} \rho(t') \varphi(t', vt') \,\mathrm{d}^\times t' \,\mathrm{d}v. \tag{2.1.43}$$

The non-archimedean factor of the integral is easily evaluated as before, and equals

$$\hat{\beta}(0) \int_{\mathbb{A}_{F,f}^\times} |t|^{2s-1} \rho(t) \alpha(t) \,\mathrm{d}^\times t. \tag{2.1.44}$$

On the other hand, the double integral at an archimedean place is

$$\begin{aligned} &\Gamma_{\mathbb{R}}(2s+k) \int_{\mathbb{R}} (-i+u)^k \left(1+u^2\right)^{-s+\frac{k}{2}} \,\mathrm{d}u \\ &= \Gamma_{\mathbb{R}}(2s+k) \Xi\left(1; s+\frac{k}{2}, s-\frac{k}{2}; 0\right). \end{aligned} \tag{2.1.45}$$

By Proposition 1.2, we have

$$\Xi\left(1; s+\frac{k}{2}, s-\frac{k}{2}; 0\right) = (2\pi)^{2s} \frac{\Gamma(2s-1)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right)} (4\pi)^{1-2s}, \tag{2.1.46}$$

so the contribution to $\mathcal{M}(f)$ from each infinite place is

$$C_\infty := \pi \cdot 2^{2-2s} \frac{\Gamma_{\mathbb{R}}(2s+k)\Gamma(2s-1)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right)}, \tag{2.1.47}$$

and this gives the result. □

2.2 Hilbert modular forms with order level structure

The subgroup $K_{\mathcal{O}} := \mathrm{GL}_2\left(\hat{\mathcal{O}}\right) \subset \mathrm{GL}_2\left(\hat{\mathcal{O}}_F\right)$ is compact and open, and contains $K(N\mathcal{O}_F) := \left\{g \in \mathrm{GL}_2\left(\hat{\mathcal{O}}_F\right) : g \equiv 1 \bmod N\hat{\mathcal{O}}_F\right\}$. The congruence subgroup $\Gamma_{\mathcal{O}} := \mathrm{GL}_2^+(F) \cap K_{\mathcal{O}}$ generalizes the one considered in Section 1.2, where we assumed F was real quadratic.

Definition 2.15: An adelic Hilbert modular form is said to have *order level structure* if it is invariant under $K_{\mathcal{O}}$ for some order $\mathcal{O} \subset F$.

Order level Hilbert modular forms do not seem to appear in the literature, with one notable exception: A proof of a conjecture in [Hut98] extending the work of Gross and Zagier on factorizations of norms of singular moduli [GZ85] has been announced by Yoshinori Mizuno [Miz24]. In the announcement, a certain Hilbert Eisenstein series with order level structure and genus character makes an appearance.

Remark 2.16: Varying the order $\mathcal{O}$,

$$\mathrm{GL}_2(F) \setminus \mathrm{GL}_2(\mathbb{A}_F)/Z(\mathbb{R})K_\infty K_{\mathcal{O}} \tag{2.2.1}$$

gives an explicit class of Hilbert modular surfaces. It would be interesting to compute their invariants systematically along the lines of [Ass+24].

Eisenstein series with order level structure

Next we construct a certain adelic Eisenstein series which generalizes that of Section 1.2. As before, let $\mathcal{O}$ be an order of conductor N in a real quadratic field F.

Fix a Godement section f as in Equation (2.1.2), where χ_1 and χ_2 are ring class characters, $\rho = \chi_1\chi_2^{-1}$, and $\varphi = \varphi_f \otimes \varphi_\infty$, where φ_∞ is the standard weight k Schwartz function, and

$$\varphi_f = \frac{1}{\mathrm{Vol}^\times\left(\hat{\mathcal{O}}^\times\right)} \cdot \mathbb{1}_{\hat{\mathcal{O}}\times\hat{\mathcal{O}}}. \tag{2.2.2}$$

Note that φ_f is invariant under multiplication by $K_{\mathcal{O}}$ on the right. Define

$$E_k(g, \mathcal{O}) = \frac{1}{\Gamma_{\mathbb{R}}(2s+k)} \cdot \frac{\left(\sqrt{D}\right)^{2k-1}\Gamma(k)^2}{(2\pi i)^{2k}} \cdot E(g, f) \tag{2.2.3}$$

to be the renormalised Eisenstein series at $s = \frac{k}{2}$. The associated classical modular form is then a Hilbert modular form of level $\mathrm{SL}_2(\mathcal{O})$ and parallel weight k. The function $E_k(g_z, \mathcal{O})$ has a Fourier expansion which may be written in terms of the *generalized divisor sums*

$$\sigma_{k-1}(\alpha; \chi_1, \chi_2) := \sum_{\substack{\mathfrak{a}\subset\mathcal{O}\text{ proper,}\\ \alpha\hat{\mathcal{O}}\subset\mathfrak{a}\hat{\mathcal{O}}}} \chi_1\left(\mathfrak{a}^{-1}\alpha\hat{\mathcal{O}} \cap F\right)\chi_2(\mathfrak{a})\mathrm{Nm}(\mathfrak{a})^{k-1}, \tag{2.2.4}$$

given as follows:

Proposition 2.17: *The Fourier expansion of $E_k(g_z, \mathcal{O})$ is given by*

$$y^{-\frac{k}{2}}E_k(g_z, \mathcal{O}) = \varepsilon(\rho)\frac{L(1-k, \overline{\rho})}{4} - \delta_{k=1}\frac{L(1-k, \rho)}{4} + \sum_{\substack{\nu\in\mathfrak{d}_{\mathcal{O}}^{-1}\\ \nu\gg 0}} \sigma_{k-1}(\nu; \chi_1, \chi_2)e^{2\pi i\nu z}. \tag{2.2.5}$$

In particular, for $\chi_1 = 1$ and $\chi = \chi_2$, we recover the Eisenstein series in Section 1.2.

Proof: First we consider the higher order coefficients c_ν in the Fourier expansion from Proposition 2.14. For the archimedean factors, note that

$$\Xi(y\nu; k, 0; 1) = \frac{(2\pi)^k}{\Gamma(k)}e^{-2\pi y\nu}, \tag{2.2.6}$$

by Proposition 1.3. As $\psi_\infty(\nu x)e^{-2\pi y\nu} = e^{2\pi\nu(ix-y)} = e^{2\pi iz\nu}$, the archimedean contribution to c_ν when $\nu \neq 0$ is the product over the infinite places,

$$y^{\frac{k}{2}}\left(\frac{(2\pi)^k}{\Gamma(k)}\right)^2 e^{2\pi i\nu z} \tag{2.2.7}$$

where we recall the conventional multi-index notation

$$\nu z := \sum_{\sigma\in\Sigma_F} \sigma(\nu)z_\sigma. \tag{2.2.8}$$

We now turn to the non-archimedean contribution. Recall that $\mathfrak{d}_{\mathcal{O}}^{-1}$ denotes the inverse different of $\mathcal{O}$, given by $\mathfrak{d}_{\mathcal{O}}^{-1} := \{y \in F : \mathrm{Tr}(xy) \in \mathbb{Z} \text{ for all } x \in \mathcal{O}\}$. Similarly, we define $\mathfrak{d}_\ell^{-1}$ in terms of $\mathcal{O}_\ell$, and set $\hat{\mathfrak{d}} = \prod_\ell \mathfrak{d}_\ell$. By character orthogonality,

$$\int_{\mathcal{O}_\ell} \psi(x\xi)\,\mathrm{d}x = \begin{cases} \mathrm{Vol}(\mathcal{O}_\ell) \text{ if } \xi \in \mathfrak{d}_\ell^{-1}, \\ 0 \text{ otherwise,} \end{cases} \tag{2.2.9}$$

so

$$\hat{\mathbb{1}}_{\hat{\mathcal{O}}}(\xi) = \prod_\ell \int_{F\otimes\mathbb{Q}_\ell} \mathbb{1}_{\mathcal{O}_\ell}(x)\psi(x\xi)\,\mathrm{d}x = \mathbb{1}_{\hat{\mathfrak{d}}_{\mathcal{O}}^{-1}}(\xi)\cdot \mathrm{Vol}\Big(\hat{\mathcal{O}}\Big). \tag{2.2.10}$$

Since $[F:\mathbb{Q}] = 2$, the different $\mathfrak{d}$ is principal, and generated by $\left(\sqrt{D}\right)$, where D is the discriminant of $\mathcal{O}$. The finite-place contribution

$$\int_{\mathbb{A}_{F,f}^\times} |t|^{k-1}\rho_f(t)\mathbb{1}_{\hat{\mathcal{O}}}(t\nu)\mathbb{1}_{\hat{\mathfrak{d}}_{\mathcal{O}}^{-1}}(t^{-1})\mathrm{d}^\times t \tag{2.2.11}$$

splits as a product over places of $\mathbb{Q}$ since $\hat{\mathcal{O}}$ does, and in fact over places of F not dividing N. The factor corresponding to a prime ℓ is given by

$$\int_{F_\ell^\times} |t|^{k-1}\rho_\ell(t)\mathbb{1}_{\mathcal{O}_\ell}(t\nu)\mathbb{1}_{\mathfrak{d}_\ell^{-1}}(t^{-1})\mathrm{d}^\times t, \tag{2.2.12}$$

and we perform the change of variables $t \mapsto t^{-1}$ which results in

$$\int_{F_\ell^\times} |t|^{1-k}\rho_\ell(t)\mathbb{1}_{\mathcal{O}_\ell}(t^{-1}\nu)\mathbb{1}_{\mathfrak{d}_\ell^{-1}}(t)\mathrm{d}^\times t. \tag{2.2.13}$$

Since the integrand is trivial on $\mathcal{O}_\ell^\times$ and has valuation bounded above and below, we may rewrite it as a finite sum

$$\mathrm{Vol}^\times\, \mathcal{O}_\ell^\times \cdot \sum_{\substack{\alpha_\ell \in \mathfrak{d}_\ell^{-1}/\mathcal{O}_\ell^\times \\ \nu/\alpha_\ell \in \mathcal{O}_\ell}} |\alpha|_\ell^{1-k}\overline{\rho}(\alpha). \tag{2.2.14}$$

Note that we have normalised φ_f precisely to cancel out the volume factor. Now, since $|\alpha|_\ell^{-1} = \mathrm{Nm}\Big(\alpha\hat{\mathcal{O}} \cap F\Big)$, viewing α as the finite idele concentrated at places above ℓ, we may rewrite this in terms of the adelic divisor sum,

$$\sum_{\substack{\alpha_\ell \in \mathfrak{d}_\ell^{-1}/\mathcal{O}_\ell^\times \\ \nu/\alpha_\ell \in \mathcal{O}_\ell}} |\alpha|^{1-k}\overline{\rho}(\alpha) \sum_{\substack{\mathfrak{a}\subset \hat{\mathfrak{d}}_{\mathcal{O}}^{-1} \text{ proper,} \\ \nu\hat{\mathcal{O}}\subset\mathfrak{a}\hat{\mathcal{O}}}} \overline{\rho}(\mathfrak{a})\mathrm{Nm}(\mathfrak{a})^{k-1} =: \tilde{\sigma}_{k-1}(\nu,\overline{\rho}) \tag{2.2.15}$$

appearing in [Gar90, p. 125]. Note that this expression is multiplicative in the finite idele α, and by a straightforward computation is related to σ by the formula

$$\sigma_{k-1}\Big(\alpha\sqrt{D};\chi_1,\chi_2\Big) = -(-D)^{k-1}\chi_1(\alpha)\tilde{\sigma}_{k-1}(\alpha,\overline{\rho}). \tag{2.2.16}$$

For $\alpha = \left(\alpha_f, \alpha_\infty\right) = \nu$, we note that $\chi_{1,f}(\nu) = \overline{\chi}_{1,\infty}(\nu) = 1$ as $\nu \gg 0$. Thus we get the following expression for the non-archimedean factor,

$$(2.2.11) = (-1)(-D)^{1-k}\chi_{1,f}(\alpha)\sigma_{k-1}\Big(\nu\sqrt{D};\chi_1,\chi_2\Big). \tag{2.2.17}$$

All proper $\mathcal{O}$-ideals $\mathfrak{a}$ dividing $\nu\sqrt{D}$ appear in the sum by the bijection in Proposition 1.5 between invertible $\mathcal{O}$-ideals and tuples of locally principal $\mathcal{O}_\ell$-lattices. As the local volume factors from the Fourier transform of β multiply to $\mathrm{Vol}\Big(\hat{\mathcal{O}}\Big) = D^{-1/2}$, we conclude that ν-th coefficent of $E_k(g_z,\mathcal{O})$ is given by

$$c_\nu(E_k(g_z,\mathcal{O}))\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix} = y^{\frac{k}{2}}\frac{(2\pi i)^{2k}}{\left(\sqrt{D}\right)^{2k-1}\Gamma(k)^2}\sigma_{k-1}(\nu,\overline{\rho})e^{2\pi i\nu z}. \tag{2.2.18}$$

Now we compute the constant term. By Proposition 2.14 it is a sum of two terms, each of which is a product of Γ-factors and local integrals. The first integral is

$$\frac{1}{\operatorname{Vol}^\times \hat{\mathcal{O}}^\times}\int_{\mathbb{A}_{F,f}^\times}|t|^{2s}\rho(t)\mathbb{1}_{\hat{\mathcal{O}}}(t)\mathrm{d}^\times t. \tag{2.2.19}$$

This factors over rational primes ℓ, and if $\ell\nmid N$, further over places of v. We first suppose $\ell\nmid N$, and fix a place $v\mid\ell$ of F. Then ρ_v is unramified at v, and the corresponding local integral is

$$\int_{\mathcal{O}_{F_v}}|t|_v^{2s}\rho_v(t)\mathrm{d}^\times t = \sum_{j=0}^\infty q_v^{-2sj}\rho_v(\varpi_v)^j\int_{\mathcal{O}_{F_v^\times}}\mathrm{d}^\times t = q_v^{-\frac{v(\mathfrak{d}_{\mathcal{O}})}{2}}\cdot\frac{1}{1-\rho(\varpi_v)q_v^{-2s}}, \tag{2.2.20}$$

by our normalization of the Haar measure. If $\ell\mid N$, then the computation is more complicated: recall from Section 1.1 that the étale algebra $F_\ell = F\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ contains the non-maximal order $\mathcal{O}_\ell = \mathcal{O}\otimes\mathbb{Z}_\ell$. Suppose first ℓ splits in F, and fix places v and v' above ℓ. For $j,j'\in\mathbb{N}$, define

$$D(j,j') := \mathcal{O}_\ell\cap\left(\varpi_v^j\mathcal{O}_{F_v^\times}\oplus\varpi_{v'}^{j'}\mathcal{O}_{F_{v'}^\times}\right)\subset F_\ell \tag{2.2.21}$$

so that

$$\int_{\mathcal{O}_\ell}|t|_\ell^{2s}\rho_\ell(t)\mathrm{d}^\times t = \sum_{j,j'=0}^\infty q_v^{-2s}\int_{D(j,j')}\rho_\ell(t)\mathrm{d}^\times t. \tag{2.2.22}$$

We note the following:

(i) If $j=j'=0$, then $D(j,j')=\mathcal{O}_\ell^\times$.

(ii) If $j\geq v(N)$ and $j'\geq v'(N)$, then $D(j,j')=\varpi_v^j\mathcal{O}_{F_v^\times}\oplus\varpi_{v'}^{j'}\mathcal{O}_{F_{v'}^\times}$, and so

$$\int_{D(j,j')}\rho_\ell(t)\mathrm{d}^\times = \rho\left(\varpi_v^j\varpi_{v'}^{j'}\right)\int_{\mathcal{O}_{F_v^\times}\oplus\mathcal{O}_{F_{v'}^\times}}\rho_\ell(t)\mathrm{d}^\times t. \tag{2.2.23}$$

By character orthogonality, the integral is identically 0.

(iii) If $j\geq v(N)>j'$, then as $\mathcal{O}_\ell=\mathbb{Z}_\ell+N\left(\mathcal{O}_{F_v}\oplus\mathcal{O}_{F_{v'}}\right)$, we may write $x\in D(j,j')$ as $x=a+N(b,b')$ for some $a\in\mathbb{Z}_\ell, b\in\mathcal{O}_{F_v}$ and $b'\in\mathcal{O}_{F_{v'}}$. Then $v(a)\geq v(N)$ so N divides x, hence $v'(x)\geq v(N)$. It follows that $D(j,j')=\emptyset$.

(iv) If $0<j\leq j'<v(N)$, then by similar reasoning we have $j=j'$. Hence we may write $x=\ell^j y$ for some $y\in\mathcal{O}_\ell'$, where $\tilde{\mathcal{O}}_\ell$ denotes the local order of conductor N/ℓ^j. Then a change of variables gives

$$\int_{D(j,j')}\rho_\ell(t)\mathrm{d}^\times t = \rho(\ell^j)\int_{\tilde{\mathcal{O}}_\ell^\times}\rho_\ell(t)\mathrm{d}^\times t, \tag{2.2.24}$$

which again vanishes by character orthogonality, as we assume ρ is primitive.

In summary, all the terms except $j=j'=0$ vanish, and so

$$(2.2.22) = \operatorname{Vol}^\times(\mathcal{O}_\ell^\times), \tag{2.2.25}$$

since ρ is trivial on $\mathcal{O}_\ell^\times$.

If ℓ is inert or ramified in F, then write $\ell=\varpi_\ell^j$ and note that for $j\geq v(N)$,

$$\int_{\mathcal{O}_\ell \cap \varpi_v^j \mathcal{O}_{F_v^\times}} \rho(t) \mathrm{d}^\times t = 0, \tag{2.2.26}$$

as in the split case. When ℓ is ramified, say $\ell = \varpi_v^2$, then for odd $0 < j \le v(N)$, $\mathcal{O}_\ell \cap \varpi_v^j \mathcal{O}_{F_v^\times} = \emptyset$. On the other hand, when j is even, $\varpi_v^j = \ell^{j/2}$, and we get vanishing by the argument in (iv). Finally, the term $j = 0$ gives $\mathrm{Vol}^\times\ \mathcal{O}_\ell^\times$ as before. Therefore

$$(2.2.22) = \mathrm{Vol}^\times(\mathcal{O}_\ell^\times), \tag{2.2.27}$$

regardless of the splitting behaviour of ℓ, and so the first part of the constant term of $E_k(g_z, \mathcal{O})$ equals

$$y^s \cdot L(2s, \rho) \cdot \Gamma_{\mathbb{R}}(2s+k)^2. \tag{2.2.28}$$

Taking $s = \frac{k}{2}$, multiplying by the normalizing factor and applying the functional equation in Equation (1.2.25) then gives

$$L(1-k, \overline{\rho}) \cdot \frac{\varepsilon(\rho)}{4}, \tag{2.2.29}$$

as claimed. The second part is computed similarly; since $\hat{\beta}(0)$ contributes $\mathrm{Vol}\big(\hat{\mathcal{O}}\big) = D^{-1}$, this equals

$$y^{1-2s} D^{-\frac{1}{2}} L(2s-1, \rho) \cdot C_\infty. \tag{2.2.30}$$

At $s = \frac{k}{2}$, this vanishes unless $k = 1$ as in the proof of Corollary 1.13. When $k = 1$ we find

$$\frac{C_\infty^2}{\Gamma_{\mathbb{R}}(2s+k)^2} \cdot \frac{\left(\sqrt{D}\right)^{2k-1} \Gamma(k)^2}{(2\pi i)^{2k}} = -\frac{1}{4}\sqrt{D}, \tag{2.2.31}$$

and so the second part of the constant term equals $-\delta_{k=1} \frac{L(0,\rho)}{4}$ as required.

□

Fix an unramified prime $p \nmid N$. Since $K_{\mathcal{O}}$ is spherical at all places above p, Proposition 2.6 implies that $E_k(g, \mathcal{O})$ is an eigenfunction of $T_{\mathfrak{p}}$ for any prime $\mathfrak{p} \mid p\mathcal{O}_F$, with eigenvalue $a_{\mathfrak{p}} := \chi_1(\mathfrak{p}) + \chi_2(\mathfrak{p})\mathrm{Nm}(\mathfrak{p})^{k-1}$. Since

$$X^2 - a_{\mathfrak{p}} X + \chi_1\chi_2(\mathfrak{p})\mathrm{Nm}(\mathfrak{p})^{k-1} = (X - \chi_2(\mathfrak{p}))\big(X - \chi_1(\mathfrak{p})\mathrm{Nm}(\mathfrak{p})^{k-1}\big), \tag{2.2.32}$$

the two possible $\mathfrak{p}$-refinements correspond to $\alpha_{\mathfrak{p}} := \chi_1(\mathfrak{p})$ and $\beta_{\mathfrak{p}} := \chi_2(\mathfrak{p})\mathrm{Nm}(\mathfrak{p})^{k-1}$. We then define

$$E_k^{(p)}(g, \mathcal{O}) := \prod_{\mathfrak{p} \mid (p)} \left(1 - \alpha_{\mathfrak{p}} V_{\mathfrak{p}}^-\right) E_k(g, \mathcal{O}), \tag{2.2.33}$$

as in Equation (1.3.15).

Proposition 2.18: *The higher-order Fourier coefficients of $E_k^{(p)}(g, \mathcal{O})$ are given by*

$$C\left(\mathfrak{a}, E_k^{(p)}\right) = \sum_{\substack{\mathfrak{b} \mid \mathfrak{a} \\ (\mathfrak{a}, p) = 1}} \chi_1(\mathfrak{b}/\mathfrak{a})\chi_2(\mathfrak{b})\mathrm{Nm}(\mathfrak{b})^{k-1}. \tag{2.2.34}$$

Proof: Since the Fourier coefficients of $E_k^{(p)}$ factor, it suffices to show that

$$C\left(E_k^{(p)}, \mathfrak{p}^j\right) = \chi_1(\mathfrak{p})^j \tag{2.2.35}$$

whenever $\mathfrak{p} \mid p$. But this follows from [BH21, Proposition 3.4.4]. □

The constant term will be computed later by identifying $E_{\mathcal{O}}^{(p)}$ with the weight 1 specialisation of a p-adic Eisenstein family.

Diagonal restrictions

In this section, we consider the natural inclusion $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \to \mathrm{GL}_2(\mathbb{A}_F)$, which in particular induces the diagonal map $\mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(F \otimes \mathbb{R})$, and hence a map $\Delta : \mathfrak{h} \to \mathfrak{h}^{\Sigma_F}$. Pulling back automorphic forms under the first inclusion amounts to evaluating the $\lambda = 1$-component of a classical Hilbert modular form at the diagonal argument $(z, ..., z)$. If Φ is an adelic Hilbert modular form, we write $\Delta^*\Phi$ for the corresponding automorphic form of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. If Φ has weight $k = (k_\sigma)_\sigma$ and level $K_f \subset \mathrm{GL}_2(\hat{\mathcal{O}}_F)$, then $\Delta^*\Phi$ has weight $\sum_\sigma k_\sigma$ and level $K_f \cap \mathrm{GL}_2(\hat{\mathbb{Z}})$.

Remark 2.19: From an automorphic point of view, there is no particular choice involved in this restriction. However, for a *classical* Hilbert modular form (Φ_λ), one can consider the diagonal restrictions of any of the components Φ_λ. If $\Phi_\lambda(g) = \Phi(t_\lambda g)$ for some adelic Hilbert modular form Φ, then one can view $\Delta^*\Phi_\lambda$ as Φ restricted to the image of the twisted embedding $\Delta_\lambda : \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) \to \mathrm{GL}_2(\mathbb{A}_F)$ given by $g \mapsto t_\lambda g$. While this can often be interesting, we will *restrict* our attention to the case $\lambda = 1$.

Returning to our p-refined Eisenstein series, there is a qualitative difference between the cases p inert and p split:

Proposition 2.20: *Let $k = (1, 1)$, and suppose p is inert in F. Then the diagonal restriction $\Delta^* E_k^{(p)}(g_z, \mathcal{O})$ vanishes identically.*

Proof: Note that $\Delta^* E_k^{(p)}$ defines a modular form for $\mathrm{GL}_2(\mathbb{Q})$ which is right invariant under $\mathrm{GL}_2(\hat{\mathbb{Z}})$ and weight 2. Since p is inert, the operator $(1 - V_p^-)$ agrees with p-refinement on $\mathrm{GL}_2(\mathbb{Q})$, and commutes with diagonal restriction. In other words, $\Delta^* E_k$ is the p-refinement of a modular form of level 1 and weight 2, hence 0. □

On the other hand, when p is split, the diagonal restriction of the p-refinement does not automatically vanish as V_v^- is not an operator on $\mathrm{GL}_2(\mathbb{Q})$-automorphic forms. In fact, as we show in Theorem 3.16, in this case $\Delta^* E_k^{(p)}$ has an interesting topological interpretation.

2.3 A Λ-adic Eisenstein family

References: [DDP11, §3], [Wil88, §1]

Our next goal is to show that the weight $(1, 1)$ Eisenstein series, suitably modified, naturally lives in a p-adic family. This family is a crucial ingredient in the proof of Theorem 3.23.

In this section, we assume that the rational prime p is inert in the real quadratic field F. Let E be a finite extension of $\mathbb{Q}_p$, and define the *Iwasawa algebra*

$$\Lambda := \mathcal{O}_E[\![\mathbb{Z}_p^\times]\!] := \varprojlim_n \mathbb{Z}_p\left[\mathbb{Z}_p^\times / (1 + p^n\mathbb{Z}_p)\right] \tag{2.3.1}$$

where each term in the limit is a standard group algebra.

The p-adic Teichmüller character ω is the $\mathbb{Q}_p$-valued character satisfying $\omega(\mathfrak{a}) \equiv \mathrm{Nm}(\mathfrak{a}) \bmod q$, where $q = p$ when p is odd and $q = 4$ otherwise. Furthermore, the function $\langle\cdot\rangle : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$ defined by $\langle x \rangle := x\omega(x)^{-1}$ induces a map $\mathbb{Z}_p^\times \to \Lambda$. For each $(k, p) = 1$, the map $x \mapsto x^k$ extends by continuity to a map $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$, and we denote its linear extension to Λ by $\nu_k : \Lambda \to \mathbb{Z}_p^\times$. We call this the *weight k specialisation.*

The p-adic L-function associated to a Hecke character χ, constructed by Deligne and Ribet [DR80], is a p-adic analytic function interpolating classical L-values in the sense that

$$L_p(1-k,\chi) = L\big(1-k,\chi\omega^{-k}\big)\big(1-\chi\omega^{-k}(p)\mathrm{Nm}(p)^{k-1}\big), \tag{2.3.2}$$

for all $k \geq 1$. Formally, this gives rise to a pseudo-measure on $\mathbb{Z}_p^\times$, and hence an element of $\mathrm{Frac}(\Lambda)$, the fraction field of Λ.

Recall from Section 1.3 the notion of Fourier coefficients $C(\mathfrak{a},\Phi)$.

Definition 2.21: A *Λ-adic family $\mathcal{F}$ of Hilbert modular forms with level* $\mathrm{SL}_2\big(\hat{\mathcal{O}}\big)$ is a collection of elements

$$\begin{cases} C(\mathfrak{a},\mathcal{F}) \in \Lambda \text{ for all } \mathfrak{a} \leq \mathcal{O} \text{ proper}, \\ C_\lambda(0,\mathcal{F}) \in \Lambda \text{ for all } \lambda \in \mathrm{Cl}^+\,\mathcal{O}, \end{cases} \tag{2.3.3}$$

such that for all but finitely many integers k, the collection $\nu_k(C(\mathfrak{a},\mathcal{F}))$ and $\nu_k(C_\lambda(0,\mathcal{F}))$ coincides with the coefficients of a component of a Hilbert modular form of level $\mathrm{SL}_2(\mathcal{O})$ and parallel weight k. We write $\mathcal{M}(\mathrm{SL}_2(\mathcal{O}))$ for the Λ-module of all such forms.

To construct a Λ-adic Eisenstein family, we will modify the Eisenstein series considered above with a p-adic character. Suppose ω is a ring class character unramified away from p, with conductor $c_v > 0$ at each place dividing p. To accommodate the level, we change the Schwartz function at each place v dividing p: set $\varphi_v^{(p)}(x_1,x_2) := \alpha_v^{(p)}(x_1) \times \beta_v^{(p)}(x_2)$ where

$$\alpha_v^{(p)}(x_1) := \mathbb{1}_{\mathcal{O}_v}(x_1) \quad \text{and} \quad \beta_v^{(p)}(x_2) := \frac{1}{\mathrm{Vol}^\times(1+\mathfrak{p}_v^{c_v})}\mathbb{1}_{\varpi^{-c_v}\mathcal{O}_v}(x_2)\psi_v(x_2), \tag{2.3.4}$$

where ψ_v is the fixed local additive character. As before, we set

$$\varphi_f^{(p)} := \bigotimes_{v\mid p} \varphi_v^{(p)} \otimes {\bigotimes_{v\nmid p}}' \mathbb{1}_{\mathcal{O}_v\times\mathcal{O}_v}, \tag{2.3.5}$$

and $\varphi^{(p)} := \varphi_\infty \otimes \varphi_f^{(p)}$.

Lemma 2.22: *Suppose v is a place of F dividing p. Then the local Fourier transform of $\beta_v^{(p)}$ is given by*

$$\hat{\beta}_v^{(p)}(t) = \frac{q_v^{c_v}}{\mathrm{Vol}^\times(1+\mathfrak{p}_v^{c_v})}\mathbb{1}_{1+\varpi^{-c_v}}(t). \tag{2.3.6}$$

Proof: We compute

$$\begin{aligned} \mathrm{Vol}^\times(1+\mathfrak{p}_v^{c_v}) \cdot \hat{\beta}_v^{(p)}(t) &= \int_{\varpi_v^{-c_v}\mathcal{O}_v} \psi_v(u(1-t))\,\mathrm{d}u \\ &= q_v^{c_v}\int_{\mathcal{O}_v} \psi_v(u(1-t)\varpi^{-c})\,\mathrm{d}u. \end{aligned} \tag{2.3.7}$$

If $z \in F_v^\times$, then $\psi_z(u) := \psi_v(zu)$ is also an additive character of F. For any non-zero $\alpha \in \mathcal{O}_v$,

$$(\psi_z(\alpha)-1)\int_{\mathcal{O}_v} \psi_z(u)\,\mathrm{d}u = 0, \tag{2.3.8}$$

by the change of variables $u \mapsto u + \alpha$. Therefore

$$\int_{\mathcal{O}_v} \psi_z(u)\,\mathrm{d}u = 0 \tag{2.3.9}$$

unless $\psi_z(\alpha) = 0$ for all $\alpha \in \mathcal{O}_v - \{0\}$, in which case

$$\int_{\mathcal{O}_v} \psi_z(u)\,\mathrm{d}u = \mathrm{Vol}^+(\mathcal{O}_v) = 1, \tag{2.3.10}$$

as v is unramified by assumption. Applying this with $z = (1-t)\varpi^{-c}$, and noting that $(1-t)\varpi^{-c_v} \in \mathcal{O}_v$ if and only if $t \in 1 + \varpi^v \mathcal{O}_v$, we obtain the result. □

Given a pair of ring class characters χ_1 and χ_2, let f be the Godement section associated with $\varphi^{(p)}$, χ_1 and $\chi_2\omega^{k-1}$, where ω is the p-adic Teichmüller character. As before, write $\rho := \chi_1\chi_2^{-1}\omega^{k-1}$. The associated classical Eisenstein series we denote by $E_k(\chi_1, \chi_2\omega^{k-1})$; here we interpret $\chi_2\omega^{k-1}$ to have conductor p even when $k = 1$. Finally, define the p-depleted divisor sums

$$\sigma_{k-1}^{(p)}(\alpha; \chi_1, \chi_2) := \sum_{\substack{\mathfrak{a} \subset \mathcal{O} \text{ proper}, \\ \alpha\hat{\mathcal{O}} \subset \mathfrak{a}\hat{\mathcal{O}} \\ (\mathfrak{a}, p) = 1}} \chi_1\left(\mathfrak{a}^{-1}\alpha\hat{\mathcal{O}} \cap F\right)\chi_2(\mathfrak{a})\mathrm{Nm}(\mathfrak{a})^{k-1}. \tag{2.3.11}$$

Proposition 2.23: *The Eisenstein series $E_k(\chi_1, \chi_2\omega^{1-k})$ has Fourier expansion*

$$E_k(\chi_1, \chi_2\omega^{k-1}) = \varepsilon \cdot \frac{L_p(1-k, \chi_1^{-1}\chi_2)}{4} + \sum_{\substack{\nu \in \mathfrak{d}_{\mathcal{O}}^{-1} \\ \nu \gg 0}} \sigma_{k-1}^{(p)}(\nu; \chi_1, \chi_2\omega^{1-k}), \tag{2.3.12}$$

where $\varepsilon = \varepsilon(\chi_1\chi_2^{-1})$.

Proof: We begin with the higher-order coefficients. The computation for $v \nmid p$ runs as in the proof of Proposition 2.17. Suppose $v \mid p$. The local contribution at v to the ν-th Fourier coefficient is given by

$$\int_{F_v^\times} |t|^{2s-1}\rho(t)\alpha(t\nu)\hat{\beta}(t^{-1})\mathrm{d}^\times t = \frac{1}{\mathrm{Vol}^\times(1 + \mathfrak{p}_v^{c_v})} \int_{1+\mathfrak{p}_v^c} \rho(t)\mathbb{1}_{\mathcal{O}_v}(t\nu)\mathrm{d}^\times t = \mathbb{1}_{\mathcal{O}_v}(\nu), \tag{2.3.13}$$

since ρ is trivial on $1 + \mathfrak{p}_v^{c_v}$. By the argument in the proof of Proposition 2.17, we find that ν-th coefficient is given by

$$\sigma_{k-1}^{(p)}(\nu; \chi_1, \chi_2\omega^{k-1}), \tag{2.3.14}$$

as required.

Next we turn to the constant term. At places $v \nmid p$, this proceeds as before. When $v \mid p$, we have

$$f_v(1) = \frac{1}{\mathrm{Vol}^\times(1 + \mathfrak{p}_v^{c_v})}\alpha(0) \int_{\mathfrak{p}_v^{-c_v}} |t|^{2s}\rho(t)\psi_v(t)\mathrm{d}^\times t. \tag{2.3.15}$$

Using [Sch02, Lemma 1.1.1], this equals

$$\frac{1}{\mathrm{Vol}^\times(1 + \mathfrak{p}_v^{c_v})} \sum_{j=-c}^{\infty} q_v^{-2js} \int_{\varpi_v^j \times \mathcal{O}_{F_v^\times}} \rho(t)\psi(t)\mathrm{d}^\times t = \varepsilon(0, \overline{\rho}, \psi), \tag{2.3.16}$$

where ε is the constant in the functional equation of $L(s, \overline{\rho})$. Since $\hat{\beta}(0) = 0$, the contribution from $\mathcal{M}(f)$ to the constant term is identically 0. We conclude that

$$C_1(0, E_k(\chi_1, \chi_2\omega^{1-k})) = \varepsilon(\chi_1\chi_2^{-1})\frac{L_p(1-k, \chi_1^{-1}, \chi_2)}{4}. \tag{2.3.17}$$

□

Now let E be a finite extension of $\mathbb{Q}_p$ containing all the coefficients of E, and let $\Lambda = \mathcal{O}_E\left[\left[\mathbb{Z}_p^\times\right]\right]$ be the Iwasawa algebra over $\mathcal{O}_E$.

Corollary 2.24: *There is a family* $\mathcal{E}(\chi_1, \chi_2) \in \mathcal{M}(\mathrm{SL}_2(\mathcal{O})) \otimes_\Lambda \mathrm{Frac}(\Lambda)$ *whose weight* k *specialisation is given by*

$$\nu_k(\mathcal{E}(\chi_1, \chi_2)) = E_k\left(\chi_1, \chi_2 \omega^{1-k}\right). \tag{2.3.18}$$

Remark 2.25: The obstruction to $\mathcal{E}$ defining a Λ-adic form in $\mathcal{M}(\mathrm{SL}_2(\mathcal{O}))$ comes from the constant term. One can find an integrality criterion in the case of *even* characters in [Wil88, Proposition 1.3.1].

We adopt the shorthand $\mathcal{E}_k(\chi_1, \chi_2) := \nu_k(\mathcal{E}(\chi_1, \chi_2))$ for the weight k specialisation.

Proof: For any $k \in \mathbb{Z}_p^\times$ and $a \in \mathbb{Z}_p^\times$, the number $\langle a \rangle^k$ is an element of Λ. Since $(\mathrm{Nm}(\mathfrak{a}), p) = 1$ when $(\mathfrak{a}, p) = 1$, it follows that the coefficients

$$c\left(\mathfrak{m}, E_k\left(\chi_1, \chi_2 \omega^{1-k}\right)\right) = \sum_{\substack{\mathfrak{a} \subset \mathfrak{m} \text{ proper} \\ (\mathfrak{a}, p) = 1}} \chi_1\left(\frac{\mathfrak{m}}{\mathfrak{a}}\right) \chi_2(\mathfrak{a}) \langle \mathrm{Nm}(\mathfrak{a}) \rangle^{k-1} \tag{2.3.19}$$

also lie in Λ, being finite $\mathcal{O}_E$-linear combinations of such elements. Fix a class $\lambda \in \mathrm{Cl}^+ \mathcal{O}$, and identify it with a finite idele $\lambda \in \mathbb{A}_F$ which is 1 at infinity. To find the constant term of the λ-component of the classical Eisenstein series associated to E, note that

$$E\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} g_z, f\right) = \chi_1(\lambda) |\lambda|^{\frac{k}{2}} E(g_z, f). \tag{2.3.20}$$

By our normalization of constant terms, we conclude that the constant term of the λ-component of $E_k\left(\chi_1, \chi_2 \omega^{1-k}\right)$ is

$$\varepsilon \cdot \chi_1(\lambda) \frac{L_p(1-k, \chi_1^{-1} \chi_2)}{4}. \tag{2.3.21}$$

These have p-adic interpolation by the construction of p-adic L-functions. □

By comparing the higher-order coefficients, it is clear that for the weight 1 specialisation, we have $\nu_1(\mathcal{E}(\chi_1, \chi_2)) = E_1^{(p)}$, and in particular, the constant terms agree.

Remark 2.26: It is also possible to construct Eisenstein families attached to characters with additional ramification away from p, by choosing suitable Schwartz functions at primes dividing their conductors. Examples of such may be found in [HY22, §3.2]. For our purposes, ring class characters suffice.

3 Intersections of real quadratic geodesics

In this section, we give a brief overview of indefinite binary quadratic forms, and describe how to compute the generating series $\sum_{n=1}^{\infty} \langle I, T_n C_\tau \rangle q^n$ explicitly. Recall our setup: F is a real quadratic field, and $\mathcal{O} \subset F$ is an order of conductor $N \in \mathbb{N}$. When D_0 is the discriminant of F, the discriminant of $\mathcal{O}$ equals $D := N^2 D_0$.

3.1 Preliminaries on indefinite binary quadratic forms

References: [Cox11, §7], [BV07].

Let $Q(x, y) = Ax^2 + Bxy + Cy^2$ be an indefinite binary quadratic form with discriminant $D = B^2 - 4AC$. We can order the roots of $Q(x, 1)$ by

$$\tau = -\frac{B + \sqrt{D}}{2A}, \quad \tau' = -\frac{B - \sqrt{D}}{2A}. \tag{3.1.1}$$

Conversely, given a real quadratic point $\tau \in \mathbb{R}$, let Q_τ be the primitive binary quadratic form whose first root is τ. Recall that Q is *primitive* if $(A, B, C) = 1$.

Quadratic forms admit a natural right action of the monoid $\mathrm{Mat}_2(\mathbb{Z}) - \{0\}$, the set of 2-by-2-matrices with integer coefficients, defined by

$$Q \circ \gamma := Q(ax + by, cx + dy) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}). \tag{3.1.2}$$

If we view Q as a function on row vectors $\binom{x}{y}$, then $(Q \circ \gamma)\binom{x}{y} = Q\left(\gamma \cdot \binom{x}{y}\right)$. When $\det \gamma = 1$, the discriminant of Q is preserved.

Proposition 3.1 ([BV07] Corollary 8.4.7): *The map*

$$Q(x, y) := ax^2 + bxy + cy^2 \quad \mapsto \quad \mathfrak{a}_Q := \mathbb{Z}a + \mathbb{Z}\frac{b + \sqrt{D}}{2} \tag{3.1.3}$$

induces a bijection between $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$*-orbits of quadratic forms of discriminant* D *and proper integral* $\mathcal{O}$*-ideals.*

More generally, any fractional $\mathcal{O}$-ideal can be written as $q \cdot \mathfrak{a}_Q$ for some $q \in \mathbb{Q}$ and some indefinite binary quadratic form Q of discriminant D. An explicit inverse for the above map is given in [BV07, Proposition 8.4.8].

For a fixed discriminant D, the number of orbits of quadratic forms of discriminant D under $\mathrm{SL}_2(\mathbb{Z})$ is finite, and these form a group with respect to Gauss composition. This is the *form class group*, denoted by $\mathrm{Cl}^+ D$.

Proposition 3.2 ([BV07, §9.3.3]): *The map in Equation (3.1.3) induces an isomorphism* $\mathrm{Cl}^+ D \xrightarrow{\sim} \mathrm{Cl}^+ \mathcal{O}$.

Fix a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, and let $Y_\Gamma := \Gamma \setminus \mathfrak{h}$ be the associated open modular curve. We also fix an indefinite quadratic form Q with roots τ and τ'. These determine a geodesic in $\mathfrak{h}$ which is preserved by the set $\mathrm{SL}_2(\mathbb{Z})[\tau]$ consisting of elements which fix Q. More generally, set $\Gamma[\tau] := \{\gamma \in \Gamma : Q \circ \gamma = Q\}$.

Lemma 3.3: *The group* $\Gamma[\tau]$ *is abelian of rank at most* 1*, and its torsion subgroup is* $\{\pm 1\} \cap \Gamma$.

Proof: As $\Gamma[\tau] \subset \mathrm{SL}_2(\mathbb{Z})[\tau]$, the first claim follows from showing that $\mathrm{SL}_2(\mathbb{Z})[\tau]$ has rank 1. Consider the lattice $\mathbb{Z}^2$ with quadratic form Q, and let $V = \mathbb{Z}^2 \otimes \mathbb{Q}$ be the associated rational quadratic space.

Since $V \otimes \mathbb{R}$ is isometric to $\mathbb{R}^2$ with the quadratic form $Q'(x, y) = xy$, we have that $\mathrm{SO}_V(\mathbb{R}) \cong \mathbb{R}^\times$, and $\mathrm{SL}_2(\mathbb{Z})[\tau]$ can be identified with the stabiliser of the lattice $\mathbb{Z}^2$, which is a discrete subgroup. But the only discrete subgroups of $\mathbb{R}^\times$ are given by $\varepsilon^\mathbb{Z}$ and $\varepsilon^\mathbb{Z} \times \{\pm 1\}$ for some $\varepsilon \in \mathbb{R}^\times$.

To prove the second claim, one can check by hand that the only finite order matrices in $\mathrm{SL}_2(\mathbb{Z})$ stabilising Q are ± 1. □

A more explicit proof of the above lemma is found using the bijection in Proposition 3.1, in which one finds that ε corresponds to the fundamental unit of $\mathcal{O}_\tau$.

Definition 3.4: Let (u, v) be a fundamental solution to the Pell equation $u^2 - Dv^2 = 1$, and write $Q(x, y) = Ax^2 + Bxy + Cy^2$. Then the matrix

$$\gamma_\tau = \begin{pmatrix} u - Bv & -2Cv \\ 2Av & u + Bv \end{pmatrix} \tag{3.1.4}$$

is called *the automorph of* Q.

Given an element $\gamma \in \Gamma$ and a point $z \in \mathfrak{h}$, the hyperbolic geodesic between z and γz maps to a closed loop in Y_Γ. This defines a map $\Gamma \to H_1(Y_\Gamma, \mathbb{Z})$, which depends on z. Let $\mathrm{geo}(\tau) \subset \mathfrak{h}$ denote the oriented geodesic from τ to τ'.

Definition 3.5: Fix a real quadratic point τ such that $\gamma_\tau \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, and let $z \in \mathrm{geo}(\tau)$. The *geodesic cycle* C_τ *associated to* τ in Y_Γ is the image of $[z, \gamma_\tau z]$ in $H_1(Y_\Gamma, \mathbb{Z})$.

Lemma 3.6: *The cycle* C_τ *does not depend on the choice of* $z \in \mathrm{geo}(\tau)$.

Proof: Suppose $z, z' \in \mathrm{geo}(\tau)$, and fix $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma z = z'$. Then γ preserves the geodesic $\mathrm{geo}(\tau)$, so $[z', \gamma_\tau z'] = \gamma \cdot [z, \gamma_\tau z]$, as $\mathrm{SL}_2(\mathbb{R})[\tau]$ is abelian by the argument of Lemma 3.3. In the quotient, γ preserves the loop $C_\tau \subset Y_\Gamma$, and this proves the claim. □

Remark 3.7: It is often conceptually easier to prove these kinds of statements by viewing C_τ as the symmetric space of a torus $\mathrm{Res}_\mathbb{Q}^{\mathbb{Q}(\tau)} \mathbb{G}_m$ embedded inside that of GL_2.

From Definition 3.4 it is clear that when $M \mid A$, we have $\gamma_\tau \in \Gamma_0(M)$. In this case the $\mathrm{SL}_2(\mathbb{Z})$-orbit of Q breaks into two $\Gamma_0(M)$-orbits distinguished by $B \bmod 2M$: note that $b(Q \circ \gamma) \equiv B \bmod M$ for any $\gamma \in \Gamma_0(M)$.

Proposition 3.8 ([Dar94, Proposition 1.4]): *Suppose there exists* $B \in \mathbb{Z}$ *such that* $B^2 \equiv D \bmod M$. *Then any class of indefinite binary quadratic forms of discriminant* D *has a representative* Q *with* $M \mid a(Q)$ *and* $b(Q) \equiv B \bmod M$.

This shows that little is lost in considering quadratic forms modulo the action of $\Gamma_0(M)$ instead of $\mathrm{SL}_2(\mathbb{Z})$.

So far, we have studied the cycles C_τ, which correspond to ring embeddings $\mathbb{Z}[\tau] \hookrightarrow \mathrm{Mat}_2(\mathbb{Z})$. On the other hand, if 0 and ∞ are cusps for Γ, then the line $[0, \infty] \subset \mathfrak{h}$ maps to a cycle in homology relative to the cusps, denoted $I \in H_1(X_\Gamma, \mathrm{Cusps}; \mathbb{Z})$. This corresponds to the diagonally embedded split torus $\mathbb{Q}^\times \times \mathbb{Q}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q})$. For more on relative homology, see [MS74, Appendix A].

There is a natural intersection pairing

$$\langle \cdot, \cdot \rangle_\Gamma : H_1(X_\Gamma, \mathrm{Cusps}; \mathbb{Z}) \times H_1(Y_\Gamma, \mathbb{Z}) \to \mathbb{Z} \tag{3.1.5}$$

which coincides with the oriented intersection product of two cycles. Under Poincaré duality, this corresponds to the cup product.

3.2 Hecke action on homology

References: [Ste82], [DPV21].

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and for $n \in \mathbb{N}$ define

$$\mathrm{Mat}_2(\mathbb{Z})_n = \{\delta \in \mathrm{Mat}_2(\mathbb{Z}) : \det \delta = n\}. \tag{3.2.1}$$

For $\alpha \in \mathrm{Mat}_2(\mathbb{Z})_n$, one can find a finite set of elements $\beta \in M_2(\mathbb{Z})_n$ such that

$$\Gamma\alpha\Gamma = \bigcup_{\beta} \Gamma\beta. \tag{3.2.2}$$

This defines an action on modular forms $f \in M_k(\Gamma)$ by $f|_k\ [\Gamma\alpha\Gamma] \coloneqq \sum_\beta f\ |_k\ [\beta]$. When $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$, this gives the usual definition of the Hecke operator T_n.

Example 3.9: If $\Gamma = \Gamma_0(M)$ for some $M \in \mathbb{N}$ and $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$, then a complete set of elements β is given by

$$M_n \coloneqq \bigcup_{\substack{d \mid n \\ \gcd(d,M)=1}} \bigcup_{j=0}^{\frac{n}{d}-1} \begin{pmatrix} d & j \\ 0 & n/d \end{pmatrix}. \tag{3.2.3}$$

as described in [DS06], Exercise 5.3.1.

We can also act on the points of Y_Γ in a natural way, namely by

$$T_n \cdot z = \sum_{\beta \in M_n} \beta \cdot z. \tag{3.2.4}$$

On cycles $[z, \gamma z]$, this gives the action

$$T_n \cdot [z, \gamma z] = \sum_{\beta \in M_n} [\beta z, \beta\gamma z]. \tag{3.2.5}$$

Lemma 3.10: *Hecke operators are self-adjoint with respect to the intersection pairing in Equation (3.1.5).*

Note that we have not specified the Hecke action on $H^1(X_\Gamma, \mathrm{Cusps}; \mathbb{Z})$, but this is done in the following proof.

Proof: We use the notation of [Bel21, §4,5], where it is proved that the cup product gives rise to a Hecke-equivariant pairing

$$\mathrm{Symb}_\Gamma(\mathbb{Z}) \times H^1(Y_\Gamma, \underline{\mathbb{Z}}) \to \mathbb{Z}, \tag{3.2.6}$$

defined in [Bel21, Eq. (5.2.2)]. By Lefschetz duality, by which we mean Poincaré duality for manifolds with boundary, $H^1(Y_\Gamma, \mathbb{Z}) \cong H_1(X_\Gamma, \partial X_\Gamma; \mathbb{Z})$, and we note that this isomorphism *defines* the Hecke action on the relative cohomology group. On the other hand, by [Bel21, Theorem 4.4.2], there is a Hecke-equivariant isomorphism $\mathrm{Symb}_\Gamma(\mathbb{Z}) \cong H^1_c(Y_\Gamma, \underline{\mathbb{Z}})$, and another application of Lefschetz duality gives the result. □

The map $T_n \mapsto \langle I, T_n C\rangle_\Gamma$ then defines a linear functional on the Hecke algebra acting on $H_1(Y_\Gamma, \mathbb{Z})$. In our case of interest, namely $\Gamma = \Gamma_0(p)$, Hecke equivariance of the Eichler–Shimura isomorphism implies that the Hecke action factors through the Hecke algebra $\mathbb{T}_0(p)$, the Hecke algebra acting faithfully on $M_2(\Gamma_0(p))$. The following lemma, which seems to be well-known, shows that the functional gives rise to a modular form.

Lemma 3.11: *Let $\mathbb{T}_2(\Gamma_0(p))$ be the full Hecke algebra acting on $M := M_2(\Gamma_0(p), \mathbb{Q})$. Then $\mathbb{T}_2(\Gamma_0(p))^\vee \cong M$.*

This follows from [DI95, Proposition 12.4.13], which proves the analogous result for the cuspidal Hecke algebra, along with the fact that the 1-dimensional Eisenstein subspace is Hecke stable. Since the isomorphism in question satisfies $\lambda \mapsto a_0 + \sum_{n=1}^{\infty} \lambda(T_n) q^n$, we obtain the following:

Corollary 3.12: *For any $C \in H_1\left(Y_{\Gamma_0(p)}, \mathbb{Z}\right)$, there exists a modular form in $M_2(\Gamma_0(p))$ with q-expansion*

$$f(z) = a_0 + \sum_{n=1}^{\infty} \langle I, T_n C \rangle_{\Gamma_0(p)} q^n, \tag{3.2.7}$$

for some $a_0 \in \mathbb{Q}$.

To compute the intersection numbers $\langle I, T_n C_\tau \rangle_\Gamma$, it is convenient to lift the intersections to $\mathfrak{h}$. If $C \subset Y_\Gamma$ is the cycle corresponding to a segment $[z, \gamma_C z]$, then

$$\langle I, C \rangle_\Gamma = \sum_{\gamma \in \Gamma} [0, \infty] \cdot [\gamma z, \gamma \gamma_C z]. \tag{3.2.8}$$

Lemma 3.13: *For any $z \in \operatorname{geo}(\tau)$, we have $\operatorname{geo}(\tau) = \bigcup_{n \in \mathbb{Z}} [\gamma_\tau^n z, \gamma_\tau^{n+1} z]$.*

One may prove this by finding an explicit bijection $\operatorname{geo}(\tau) \to \mathbb{R}_{>0}$ under which γ_τ acts via multiplication by ε, the fundamental unit in $\mathcal{O}_\tau$.

Recall from Example 3.9 that the action of the Hecke operator T_n can be described explicitly using the set of coset representatives M_n. Let $M_n(\tau)$ be the subset of matrices β representing inequivalent cosets modulo right multiplication by $\Gamma_0(p)[\tau]$,

$$\bigsqcup_{\beta} \Gamma_0(p) \beta = \bigsqcup_{\delta} \Gamma_0(p) \delta \Gamma_0(p)[\tau]. \tag{3.2.9}$$

Note that we may choose the coset representatives for M_n to be $\{\beta \gamma_\tau^i : i = 0..j-1\}$ where $j = [\mathrm{SL}_2(\mathbb{Z})[\tau] : \Gamma_0(p)[\tau]]$.

Proposition 3.14 (Unfolding lemma [DPV21, Lemma 1.10]): *Let τ be a real quadratic point, and fix $n \in \mathbb{N}$. For any $z \in \mathfrak{h}$,*

$$\langle I, T_n C_\tau \rangle_{\Gamma_0(p)} = 2 \sum_{\delta \in M_n(\tau)} \sum_{\gamma \in \Gamma_0(p)/\Gamma_0(p)[\delta\tau]} [0, \infty] \cdot \gamma \operatorname{geo}(\delta \tau). \tag{3.2.10}$$

Proof: For any real quadratic point ρ, fix a generator $\gamma_\rho \in \Gamma_0(p)[\rho]$ whose stable fixed point is ρ. If $z \in \operatorname{geo}(\delta\tau)$, then

$$[0, \infty] \cdot \gamma \operatorname{geo}(\delta \tau) = \sum_{k \in \mathbb{Z}} [0, \infty] \cdot \left[\gamma \gamma_\rho^k z, \gamma \gamma_\rho^{k+1} z\right], \tag{3.2.11}$$

and so the right hand side of Equation (3.2.10) unfolds to

$$\sum_{\delta \in M_n(\tau)} \sum_{\gamma \in \Gamma_0(p)} [0, \infty] \cdot \left[\gamma \delta z, \gamma \gamma_\rho z\right], \tag{3.2.12}$$

Here the factor of 2 cancels with the torsion subgroup $\{\pm 1\} \subset \Gamma_0(p)[\delta\tau]$. Setting $z' = \delta^{-1}z$, and using that $\gamma_\rho = \delta\gamma_\tau^f\delta^{-1}$ for some $f \in \mathbb{N}$, we note that

$$[0,\infty] \cdot \left[\gamma z, \gamma\gamma_\rho z\right] = [0,\infty] \cdot \bigcup_{i=0}^{f-1} \left[\gamma\delta\gamma_\tau^i z', \gamma\delta\gamma_\tau^{i+1} z'\right]. \tag{3.2.13}$$

Since the matrices $\left\{\delta\gamma_\tau^i : \delta \in M_n(\tau), i = 0, ..., f-1\right\}$ form a complete set of representatives for M_n,

$$\begin{aligned}\sum_{\delta\in M_n(\tau)} \sum_{\gamma\in\Gamma_0(p)} [0,\infty] \cdot \left[\gamma\delta z, \gamma\gamma_\rho z\right] &= \sum_{\beta\in M_n} \sum_{\gamma\in\Gamma_0(p)} [0,\infty] \cdot (\gamma\beta z', \gamma\beta\gamma_\tau z') \\ &= \langle I, C_\tau\rangle_{\Gamma_0(p)},\end{aligned} \tag{3.2.14}$$

which finishes the proof. □

The correspondence between quadratic forms and ideals relates the intersection numbers to the Fourier coefficients of the diagonal restriction of Hilbert Eisenstein series. Given a fixed class $A \in \mathrm{Cl}^+ \mathcal{O}$, define

$$\mathbb{I}(n, A) := \left\{(\nu, \mathfrak{a}) : \nu \in \left(\sqrt{D}\right)^{-1}_{\gg 0}, \mathrm{Tr}\,\nu = n, \mathfrak{a} \subset \left(\nu\sqrt{D}\right) \text{ proper}, [\mathfrak{a}] = A\right\}, \tag{3.2.15}$$

and for a fixed quadratic form Q_0, let

$$\mathrm{QF}(n, Q_0) := \left\{(\delta, Q) : \delta \in M_n\left(r_{Q_0}\right), Q \in \mathrm{SL}_2(\mathbb{Z}) \cdot Q_0 \circ \delta, r_Q > 0 > r_Q'\right\}. \tag{3.2.16}$$

Proposition 3.15: *Suppose Q_0 is an indefinite quadratic form of discriminant D, and A is the associated class in $\mathrm{Cl}^+ \mathcal{O}$. For any $n \in \mathbb{N}$, there is a bijection between $\mathbb{I}(n, A)$ and $\mathrm{QF}(n, Q_0)$.*

Proof: The proof is analogous to [DPV21, Lemma 1.9], which proves the fundamental case. A pair $(\nu, \mathfrak{a})$ gives rise to a quadratic form $Q(x,y) = \mathrm{Nm}(\mathfrak{a})x^2 + bxy - \mathrm{Nm}\left(\mathfrak{a}^{-1}\nu\sqrt{D}\right)y^2$ where b is the unique integer such that

$$\nu = \frac{-b + n\sqrt{D}}{2\sqrt{D}}. \tag{3.2.17}$$

Fix the representative $\mathfrak{A} = a(Q_0)\mathbb{Z} + \tau a(Q_0)\mathbb{Z}$ for A, and note that $\mathfrak{a}'\mathfrak{A}$ is principal with totally positive generator λ. The lattice associated to $\mathfrak{A}$ contains $\Lambda := \mathbb{Z}\lambda + \mathbb{Z}\lambda r_Q$ with index n. The corresponding change of basis matrix is δ.

Conversely, given $(\delta, Q) \in \mathrm{QF}(n, Q_0)$ with $Q(x,y) = ax^2 + bxy + cy^2$ one defines ν by Equation (3.2.17). If one picks $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q = Q_0 \circ \gamma\delta$, then $\lambda := a(Q_0)(c\tau + d)$ for $\gamma\delta = \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)$ satisfies

$$\begin{pmatrix}\lambda w \\ \lambda\end{pmatrix} = \delta\gamma\begin{pmatrix} a(Q_0)\tau \\ a(Q_0)\end{pmatrix}. \tag{3.2.18}$$

Note that $\lambda \in \mathfrak{A}$, so the ideals $\mathfrak{a} := (\lambda')(\mathfrak{A}')^{-1}$ and $\mathfrak{b} := (\lambda w)\mathfrak{A}^{-1}$ are integral and satisfy $\mathfrak{a}\mathfrak{b} = \left(\nu\sqrt{D}\right)$, and this gives the required map. □

Note that by construction, the first coefficent of Q is the norm of the corresponding ideal $\mathfrak{a}$.

Fix a choice of $s \in \mathbb{Z}$ such that $s^2 \equiv D \bmod p$. Given a class $A \in \mathrm{Cl}^+ \mathcal{O}$, by Proposition 3.8 we may pick representatives $Q_{\pm s}$ with class A such that $p \mid a(Q_{\pm s})$ and $b(Q_{\pm s}) \equiv \pm s \bmod p$. We then set $g_A \in H_1(Y_0(p), \mathbb{Z})$ by $g_A := C_{\tau_s} + C_{\tau_{-s}}$ where $\tau_{\pm s}$ is a root of $Q_{\pm s}$.

Theorem 3.16: *Let ψ be a narrow ring class character associated to an order $\mathcal{O}$, and suppose p is a rational prime which splits in $\mathcal{O}$. Then*

$$\Delta^* E_{\mathcal{O}}^{(p)} = L_p(0,\psi) - 2 \sum_{A \in \mathrm{Cl}^+ \mathcal{O}} \psi(\mathfrak{a}) \sum_{n=1}^{\infty} \langle I, T_n g_A \rangle_{\Gamma_0(p)} q^n. \tag{3.2.19}$$

This extends [DPV21, Theorem A], where ψ is assumed to be unramified, or equivalently, $\mathcal{O}$ is maximal. The proof is essentially the same.

Proof: The diagonal restriction $\Delta^* E_{\mathcal{O}}^{(p)}$ and the intersection series are elements of $M_2(\Gamma_0(p))$. It is enough to show that $a_n := a_n\left(\Delta^* E_{\mathcal{O}}^{(p)}\right) = \langle I, T_n C_\tau \rangle_{\Gamma_0(p)}$ for all $n \in \mathbb{N}$ such that $(n,p) = 1$, since then the difference is an oldform of weight 2 and level $\Gamma_0(p)$, hence 0. By Proposition 2.18,

$$a_n = 4 \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \mathrm{Tr}(\nu) = n}} \sum_{\substack{\mathfrak{a} \subset \left(\nu\sqrt{D}\right) \text{ proper} \\ (\mathfrak{a},p)=1}} \psi(\mathfrak{a}) = 4 \sum_{A \in \mathrm{Cl}^+ \mathcal{O}} \psi(A) \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \mathrm{Tr}(\nu) = n}} \sum_{\substack{\mathfrak{a} \subset \left(\nu\sqrt{D}\right) \text{ proper} \\ (\mathfrak{a},p)=1 \\ [\mathfrak{a}]=A}} 1. \tag{3.2.20}$$

Since $\Delta^* E_{\mathcal{O}} = 0$,

$$\sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \mathrm{Tr}(\nu) = n}} \sum_{\substack{\mathfrak{a} \subset \left(\nu\sqrt{d}\right) \text{ proper} \\ (\mathfrak{a},p)=1 \\ [\mathfrak{a}]=a}} 1 = - \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \mathrm{Tr}(\nu) = n}} \sum_{\substack{\mathfrak{a} \subset \left(\nu\sqrt{D}\right) \text{ proper} \\ p \mid \mathrm{Nm}(a) \\ [\mathfrak{a}]=A}} 1 = -\#\mathbb{I}(n,A)_p, \tag{3.2.21}$$

where $\mathbb{I}(n,A)_p := \{(\nu,\mathfrak{a}) \in \mathbb{I}(n,A) : p \mid \mathrm{Nm}(a)\}$. By Proposition 3.15, this is in bijection with pairs $(Q,\delta) \in \mathrm{QF}\ (n, Q_0)$ such that $p \mid a(Q)$, and so we may write

$$a_n = 4 \sum_{A \in \mathrm{Cl}^+ \mathcal{O}} \psi(A) \sum_{\substack{(Q,\delta) \in \mathrm{QF}(n,Q_A) \\ p \mid a(Q)}} 1. \tag{3.2.22}$$

Note that $\psi\left(A\left[\sqrt{D}\right]\right) = -\psi(A)$ and $\left[\sqrt{D}\right]$ corresponds to the involution $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on quadratic forms, which interchanges the roles of r_Q and $r_{Q'}$. As

$$[0,\infty] \cdot \mathrm{geo}(Q) = \begin{cases} 1 & \text{if } r_Q > 0 > r_{Q'} \\ -1 & \text{if } r_{Q'} > 0 > r_Q \\ 0 & \text{otherwise,} \end{cases} \tag{3.2.23}$$

we find

$$\begin{aligned} a_n &= 2 \sum_{A \in \mathrm{Cl}^+ \mathcal{O}} \psi(A) \sum_{\delta \in M_n(\tau_A)} \sum_{\substack{Q \sim Q_0 \circ \delta \\ p \mid a(Q)}} \mathbb{1}_{r_Q > 0 > r_{Q'}} - \mathbb{1}_{r_{Q'} > 0 > r_Q} \\ &= 2 \sum_{A \in \mathrm{Cl}^+ \mathcal{O}} \psi(A) \sum_{\delta \in M_n(\tau)} \sum_{\substack{\gamma \in \mathrm{SL}_2(\mathbb{Z}) / \mathrm{SL}_2(\mathbb{Z})[\delta\tau_A] \\ p \mid a\left(Q_{\gamma\delta\tau_A}\right)}} [0,\infty] \cdot \gamma\, \mathrm{geo}(\delta\tau_A). \end{aligned} \tag{3.2.24}$$

We now aim to apply the unfolding lemma Proposition 3.14 by comparing $\mathrm{SL}_2(\mathbb{Z})$-orbits of quadratic forms with $\Gamma_0(p)$-orbits. Fix $s \in \mathbb{Z}$ such that $s^2 \equiv D \bmod p$, and fix matrices $A_{\pm s} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau_{\pm s} := A_{\pm s} \tau_A$ has $b(\tau_{\pm s}) = \pm s$. To see that such matrices always exist, write $Q_A(x,y) = ax^2 + bxy + cy^2$ with $(c,p) = 1$, and fix $r \in \mathbb{Z}$ with $rC \equiv \frac{1}{2}(s - b) \bmod p$. Then

$$\begin{aligned} Q\left(\begin{pmatrix}1 & 0\\ r & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right) &= x^2(a+cr^2+br)+xy(b+2rc)+Cy^2 \\ &\equiv x^2(r(rc+b))+xys+Cy^2 \bmod p \\ &\equiv x^2\left(a+\frac{s^2-b^2}{4c}\right)+xys+Cy^2 \bmod p \\ &\equiv x^2\left(a+\frac{-4ac}{4c}\right)+xys+Cy^2 \bmod p, \end{aligned} \tag{3.2.25}$$

since $s^2 \equiv D = b^2-4ac \bmod p$. Now define $\delta_{\pm s} := A_{\pm s}\delta(A^{\pm s})^{-1}$, so that

$$\mathrm{SL}_2(\mathbb{Z})\delta\tau_A = \Gamma_0(p)\delta^s\tau_s \sqcup \Gamma_0(p)\delta^{-s}\tau_{-s}, \tag{3.2.26}$$

and let $N_n^{\pm s} := A_{\pm s}M_n(\tau_A)A_{\pm s}^{-1}$ be the collection of all such $\delta_{\pm s}$. Note that

$$\{w \in \mathrm{SL}_2(\mathbb{Z})\delta\tau_A : p \mid a(w)\} = \Gamma_0(p)\delta_s\tau_s \sqcup \Gamma_0(p)\delta_{-s}\tau_{-s}, \tag{3.2.27}$$

as in [DPV21, Theorem 1.12]. Moreover, $\mathrm{SL}_2(\mathbb{Z})[\tau_{\pm s}] = \Gamma_0(p)[\tau_{\pm s}]$, and $\#N_n^{\pm s} = \#M_n(\tau_A)$, so

$$\bigsqcup_{\delta\in M_n(\tau_A)} \mathrm{SL}_2(\mathbb{Z})\delta\,\mathrm{SL}_2(\mathbb{Z})[\tau_A] = \bigsqcup_{\delta_{\pm s}\in N_n^{\pm s}} \Gamma_0(p)\delta_{\pm s}\Gamma_0(p)[\tau_{\pm s}], \tag{3.2.28}$$

because $\gamma\delta_{\pm s} = \delta'_{\pm s}\gamma^i_{\tau_{\pm s}}$ implies $\gamma \in \Gamma_0(p)$. Thus

$$\begin{aligned} a_n &= 2\sum_{A\in \mathrm{Cl}^+\mathcal{O}} \psi(A) \sum_{\delta\in M_n(\tau)} \sum_{\substack{\gamma\in \mathrm{SL}_2(\mathbb{Z})/\,\mathrm{SL}_2(\mathbb{Z})[\delta\tau_A] \\ p\mid a(\gamma Q_{\delta\tau})}} [0,\infty]\cdot\gamma\,\mathrm{geo}(\delta\tau_A) \\ &= 2\sum_{A\in \mathrm{Cl}^+\mathcal{O}} \psi(A) \sum_{\delta_s\in N_n^s(\tau_s)} \sum_{\gamma\in\Gamma_0(p)/\Gamma_0(p)[\delta_s\tau_s]} [0,\infty]\cdot\gamma\,\mathrm{geo}(\delta_s\tau_s) \\ &\qquad + \sum_{\delta_{-s}\in N_n^{-s}(\tau_{-s})} \sum_{\gamma\in\Gamma_0(p)/\Gamma_0(p)[\delta_{-s}\tau_{-s}]} [0,\infty]\cdot\gamma\,\mathrm{geo}(\delta_{-s}\tau_{-s}). \end{aligned} \tag{3.2.29}$$

Applying Proposition 3.14 to each of the sums then gives

$$a_n = \sum_{A\in \mathrm{Cl}^+\mathcal{O}} \psi(A)\big(\langle I, C_{\tau_s}\rangle_{\Gamma_0(p)} + \langle I, C_{\tau_{-s}}\rangle_{\Gamma_0(p)}\big), \tag{3.2.30}$$

as required. □

3.3 Rigid meromorphic cocycles

References: [DV21], [Fus+23, §3].

In this section, we give a quick overview of rigid meromorphic cocycles, following the excellent survey [Fus+23]. Let $\mathfrak{h}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ be the p-adic upper half plane. This is a rigid analytic space which can be described as an increasing limit of the open affinoid coverings

$$\mathfrak{h}_p^{\le n} := \left\{[z_0 : z_1] \in \mathbb{P}^1(\mathbb{C}_p) : \mathrm{ord}_p \det\begin{pmatrix}x_0 & z_0\\ x_1 & z_1\end{pmatrix} \le n \text{ for all } [x_0 : x_1] \in \mathbb{P}^1(\mathbb{Q}_p)\right\}, \tag{3.3.1}$$

where each projective coordinate is subject to the condition $\max(|z_0|, |z_1|) \le 1$. For further details on the p-adic upper half plane, see [BC91] or [FV04]. A *rigid analytic function on* $\mathfrak{h}_p$ can be described as a function $f : \mathfrak{h}_p \to \mathbb{C}_p$ whose restriction to $\mathfrak{h}_p^{\le n}$ is a limit of rational functions on $\mathbb{P}^1(\mathbb{C}_p)$ with no poles in $\mathfrak{h}_p^{\le n}$. We denote the space of rigid analytic functions on $\mathfrak{h}_p$ by $\mathcal{A}$. We define $\mathcal{M}$, the space of *rigid meromorphic functions*, to be the fraction field of $\mathcal{A}$.

There is a natural action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathfrak{h}_p$ by Möbius transformations. The subgroup $\Gamma_p := \mathrm{SL}_2(\mathbb{Z}[1/p])$, sometimes referred to as the *Ihara group*, is a p-arithmetic group which in the p-adic theory plays the role of $\mathrm{SL}_2(\mathbb{Z})$ in the real setting. Whereas CM-theory deals with the Klein j-invariant which one may view as an element of $H^0\left(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_{\mathfrak{h}}^\times\right)$, the lack of interesting elements of $H^0\left(\Gamma_p, \mathcal{M}^\times\right)$ suggests looking in higher cohomology groups.

A *rigid analytic (resp. meromorphic) cocycle* is a class in $H^1\left(\Gamma_p, \mathcal{A}^\times\right)$ (resp. $H^1\left(\Gamma_p, \mathcal{M}^\times\right)$), while a *rigid analytic theta cocycle* is a class in $H^1\left(\Gamma_p, \mathcal{A}^\times / \mathbb{C}_p^\times\right)$. Note that despite the suggestive name, these rigid cocycles are cohomology classes, not cocycles. Darmon and Vonk construct several explicit examples of rigid meromorphic cocycles in [DV21] and subsequent work. These cohomology groups have natural actions of Hecke operators, defined as follows: write

$$\bigcup_{\substack{\alpha \in \mathrm{Mat}_2(\mathbb{Z}) \\ \det(\alpha) = n}} \Gamma_p \alpha \Gamma_p = \bigsqcup_{\beta \in M(\Gamma_p)_n} \Gamma_p \beta, \tag{3.3.2}$$

where $M\left(\Gamma_p\right)_n$ is a fixed finite set of representatives in $\mathrm{Mat}_2(\mathbb{Z})$. For any 1-cocycle J for Γ_p, define

$$(T_n J)(\gamma) := \prod_\beta \det(\beta)\beta^{-1} \cdot J(\gamma'), \tag{3.3.3}$$

where γ' is defined by $\beta\gamma = \gamma'\beta'$ for some β'. By [Shi71, §8.3], changing the set $M\left(\Gamma_p\right)_n$ transforms $T_n J$ by a coboundary, and so $T_n J$ is well-defined as a cohomology class.

A large supply of rigid analytic cocycles comes from the *multiplicative Schneider–Teitelbaum lift*, $\mathrm{ST}^\times$, introduced in [DV22a, §3]. The annulus $U := \left\{z \in \mathbb{C}_p : 1 < |z|_p < p\right\} \subset \mathbb{P}^1\left(\mathbb{C}_p\right)$ has stabiliser $\Gamma_0(p)$ in Γ_p. The map

$$\mathrm{res}_U \circ \mathrm{d}\log : \mathcal{A}^\times \to \mathbb{C}_p \quad \text{given by} \quad f \mapsto \mathrm{res}_U\left(\frac{f'(z)}{f(z)}\,\mathrm{d}z\right) \tag{3.3.4}$$

is trivial on constants, and induces a map in cohomology $\delta_U : H^1\left(\Gamma_p, \mathcal{A}^\times / \mathbb{C}_p^\times\right) \to H^1(\Gamma_0(p), \mathbb{Z})$.

Proposition 3.17 ([DV22a]): *The induced map*

$$\delta_U : H^1\left(\Gamma_p, \mathcal{A}^\times / \mathbb{C}_p^\times\right) \to H^1(\Gamma_0(p), \mathbb{Q}) \tag{3.3.5}$$

is surjective, and the kernel is generated by

$$J_{\mathrm{triv}}(\gamma)(z) := \frac{z - \gamma\xi}{z - \xi}, \quad \xi \in \mathbb{P}^1\left(\mathbb{Q}_p\right). \tag{3.3.6}$$

Furthermore, δ_U has a Hecke-equivariant section

$$\mathrm{ST}^\times : H^1(\Gamma_0(p), \mathbb{Z}) \to H^1\left(\Gamma_p, \mathcal{A}^\times / \mathbb{C}_p^\times\right). \tag{3.3.7}$$

By Eichler–Shimura, this implies that the Hecke action on $H^1\left(\Gamma_p, \mathcal{A}^\times / \mathbb{C}_p^\times\right) \otimes \mathbb{Q}$ factors through the Hecke algebra of $M_2(\Gamma_0(p))$.

Definition 3.18: Let φ_w be the *Mazur winding element* $\varphi_w(\gamma) := \langle I, \gamma \rangle$, which is an element of $H^1(Y_0(p), \mathbb{Z})$. Then the *winding cocycle* is defined by $J_w := \mathrm{ST}^\times(2\varphi_w) \in H^1\left(\Gamma_p, \mathcal{A}^\times / \mathbb{C}_p^\times\right)$.

One can alternatively define J_w explicitly as in [DPV21], and prove the equivalence using [DPV21, Proposition 3.3].

Let $\tilde{\Gamma}_p := \mathrm{GL}_2^+(\mathbb{Z}[1/p]) = \mathrm{GL}_2^+(\mathbb{Q}) \cap \prod'_{\ell \neq p} \mathrm{GL}_2(\mathbb{Z}_\ell)$, which fits into the short exact sequence

$$1 \to \Gamma_p \to \tilde{\Gamma}_p \to p^{\mathbb{Z}} \to 1. \tag{3.3.8}$$

Proposition 3.19 ([DPV21, Theorem 2.10]): *Let $J_w \in H^1\left(\Gamma_p, \mathcal{A}^\times/\mathbb{C}_p^\times\right)$ be the winding cocycle, and fix a real quadratic point τ of discriminant D with $\left(\frac{p}{D}\right) = -1$. Then for any $n \in \mathbb{N}$ such that $(n,p) = 1$, we have*

$$T_n J_w[\tau] = \prod_{\delta \in M_n(\tau)} \prod_{\substack{w \in \tilde{\Gamma}\delta\tau \\ v_p(w)=0}} w^{[0,\infty] \cdot \operatorname{geo}(\tau)}. \tag{3.3.9}$$

This should be thought of as a multiplicative analogue of Proposition 3.14. Furthermore, since the Hecke algebra acting on $H^1\left(\Gamma_p, \mathcal{A}^\times/\mathbb{C}_p^\times\right)$ is a quotient of $\mathbb{T}_2(\Gamma_0(p))$, the expression $\sum_{n=1}^\infty \log_p(\operatorname{Nm} T_n J_w[\tau])q^n$ is the non-constant coefficients of some modular form in $M_2(\Gamma_0(p))$. By explicit computation, we will identify these with the coefficients of the ordinary projection of a p-adic modular form, and hence obtain the constant term. The strategy is parallel to that of Theorem 3.16.

Proposition 3.20 ([DPV21, Lemma 2.1]): *Let $\mathcal{F}_t$ be a family of overconvergent p-adic modular forms of weight $k(t)$ indexed by t in some closed rigid analytic disk D. If $t_0 \in D$ such that $\mathcal{F}_{t_0} = 0$ and $k(t_0) \in \mathbb{Z}$, then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_t\right)|_{t=t_0} \tag{3.3.10}$$

is an overconvergent modular form of weight $k(t_0)$.

To apply this to $\Delta^*\mathcal{E}_k$, we first need to check that it defines a family of overconvergent modular forms.

Lemma 3.21: *The power series $\Delta^*\mathcal{E}_k$ defines a family of overconvergent modular forms of weight 2 and tame level 1.*

Proof: By restricting to level

$$K(N) := \left\{g \in \mathrm{GL}_2\left(\hat{\mathcal{O}}_F\right) : g \equiv 1 \bmod N\right\} \subset \mathrm{GL}_2\left(\hat{\mathcal{O}}\right) \tag{3.3.11}$$

one sees that $\mathcal{E}_k$ is an overconvergent family of tame level $K(N)$ in the sense of [AIP16, §4][1]. It follows that $\Delta^*\mathcal{E}_k$ is an overconvergent family of tame level $\Gamma(N)$. Averaging over the action of $\Gamma_0(p)/(\Gamma_0(p) \cap \Gamma(N))$ gives a family with the same q-expansion and tame level 1, and so $\Delta^*\mathcal{E}_k$ is itself overconvergent of tame level 1. □

Applying Proposition 3.20 to the Eisenstein family $\Delta^*\mathcal{E}_k$ from Corollary 2.24 which vanishes at $k = 1$, we get that

$$\partial f := \left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta^*\mathcal{E}_k\right)|_{k=1} \tag{3.3.12}$$

is an overconvergent modular form in $M_2^\dagger(\Gamma_0(p))$. Now let $e^{\mathrm{ord}} : M_2^\dagger(\Gamma_0(p)) \to M_2(\Gamma_0(p))$ denote Hida's ordinary projector, defined by

$$e^{\mathrm{ord}}(f) := \lim_{n\to\infty} U_p^{n!} f. \tag{3.3.13}$$

[1]Technically, they work with μ_N-level structure, but this merely simplifies the underlying moduli problem, and the construction goes through in greater generality. Alternatively, one may simply pull back the overconvergent sheaves via the covering map on the Hilbert modular surfaces.

Lemma 3.22: *Let $n \in \mathbb{N}$ be coprime to p. Then*

$$a_n\left(e^{\mathrm{ord}}\partial f\right) = -2\log_p\left(\mathrm{Nm}\, T_n J_w\left[\Delta_\psi\right]\right), \tag{3.3.14}$$

where Δ_ψ is the divisor $\sum_{A\in \mathrm{Cl}^+\mathcal{O}} \psi(A)\tau_A \in \mathbb{Q}(\psi)[\mathrm{Cl}^+\mathcal{O}]$.

The proof is essentially the same as that of [DPV21, Theorem 2.11].

Proof: By differentiating the Fourier expansion of $\Delta^*\mathcal{E}_k$ termwise, we find that

$$a_n(\partial f) = 4 \sum_{\substack{\nu\in\mathfrak{d}_{\mathcal{O}}^{-1}\\ \nu\gg 0\\ \mathrm{Tr}(\nu)=n}} \sum_{\substack{\mathfrak{a}\subset\nu\sqrt{D}\text{ proper}\\ (\mathfrak{a},p)=1}} \psi(\mathfrak{a})\log_p(\mathrm{Nm}(\mathfrak{a})). \tag{3.3.15}$$

For a fixed class $A \in \mathrm{Cl}^+\mathcal{O}$, let τ be an associated real quadratic point. Using the norm-preserving bijection Proposition 3.15, we may rewrite this as a sum over elements of

$$\mathrm{QF}(n, Q_\tau)^{(p)} := \{(Q, \delta_n) : (a(Q), p) = 1\}. \tag{3.3.16}$$

Fix n coprime to p. Then $a_n\left(e^{\mathrm{ord}}\partial f\right) = \lim_{m\to\infty} a_{np^m}(\partial f)$, and $a_{np^m}(\partial f)$ is a weighted combination of expressions of the form

$$4 \sum_{(Q,\delta_{np^m})\in\mathrm{QF}(n,Q_\tau)^{(p)}} [0,\infty]\cdot \mathrm{geo}\left(w_Q\right)\log_p(a(Q)), \tag{3.3.17}$$

where w_Q denotes the first root of Q. For $\delta_n \in M_n(\tau)$, define

$$X_m(\delta_n) := \left\{w \in \tilde{\Gamma}_p\delta_n : v_p(w) = 0, v_p(\mathrm{disc}(w)) \le 2m\right\}. \tag{3.3.18}$$

Then there is a bijection

$$\bigsqcup_{\delta_n} X_m(\delta_n) \to \mathrm{QF}(np^m, Q_\tau)^{(p)} \tag{3.3.19}$$

given as follows: for $w \in X_m(\delta_n)$ with $\mathrm{disc}(w) = p^{2m-2k}$, $\tilde{w} := p^k w$ is a root of $ax^2 + bp^k x + cp^{2k}$ for some integers a, b, c, and $\tilde{w} \in \mathrm{SL}_2(\mathbb{Z})\delta_{p^m}\delta_n\tau$ for some $\delta_{p^m} \in M_{p^m}(\tau)$. Letting $Q_{\tilde{w}} = ax^2 + bxy + cy^2$, the map $w \mapsto \left(Q_{\tilde{w}}, \delta_{p^m}\right)$ defines a map $X_m(\delta_n) \to \mathrm{QF}(np^m, Q_\tau)^{(p)}$, with inverse $\tilde{w} \mapsto p^{-v_p(\tilde{w})}\tilde{w}$. It follows that

$$\lim_{m\to\infty} a_{np^m}(\partial f) = 4\sum_{A\in\mathrm{Cl}^+\mathcal{O}} \psi(A) \sum_{M_n(\tau_A)} \sum_{w\in\tilde{\Gamma}\delta_N\tau_A} [0,\infty]\cdot\mathrm{geo}\left(w_Q\right)\log_p\left(a\left(w_Q\right)\right). \tag{3.3.20}$$

A short computation shows that $\log_p \mathrm{Nm}(w) = -\log_p a(Q_w) + \log_p c(Q_w)$. Note that $c(Q_w) = a(Q_w \circ S)$ for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The map $Q \mapsto Q\circ S$ preserves the indexing set, and since $[0,\infty]\cdot \mathrm{geo}\left(w_Q\right) = -[0,\infty]\cdot\mathrm{geo}\left(w_{Q\circ S}\right)$, we conclude that

$$\begin{aligned}&\sum_{M_n(\tau_A)} \sum_{w\in\tilde{\Gamma}\delta_n\tau_A} [0,\infty]\cdot\mathrm{geo}\left(w_Q\right)\log_p\left(a\left(w_Q\right)\right)\\ &= -2\sum_{M_n(\tau_A)} \sum_{w\in\tilde{\Gamma}\delta_n\tau_A} [0,\infty]\cdot\mathrm{geo}\left(w_Q\right)\log_p\left(\mathrm{Nm}\left(w_Q\right)\right).\end{aligned} \tag{3.3.21}$$

The result now follows from Proposition 3.19. □

Since $e^{\mathrm{ord}}\partial f - \sum_{n=1}^{\infty} \log_p \mathrm{Nm}(T_n J_w[\tau])q^n$ is an oldform of level $\Gamma_0(p)$ and weight 2, it is identically 0. This proves the following:

Theorem 3.23: *Let p be inert in the order $\mathcal{O} \subset F$. For any totally odd primitive ring class character ψ : $\mathrm{Cl}^+ \mathcal{O} \to \mathbb{C}^\times$,*

$$e^{\mathrm{ord}}(\partial f) = \varepsilon(\psi)L_p'(0,\psi) - 2\sum_{n=1}^{\infty} \log_p \mathrm{Nm}_{F_p/Q_p}\left(J_w T_n\left[\Delta_\psi\right]\right)q^n. \tag{3.3.22}$$

This extends [DPV21, Theorem B], which is the corresponding statement for $\mathcal{O} = \mathcal{O}_F$.

3.4 Spectral decomposition

By [DPV21, Lemma 3.4], the Hecke decomposition of winding element is given by

$$\varphi_w = \frac{1}{p-1}\varphi_{\mathrm{DR}} + \sum_f L_{\mathrm{alg}}(f,1)\varphi_f^-, \tag{3.4.1}$$

where[2]

$$\varphi_{\mathrm{DR}}(\gamma) = \frac{1}{2\pi i}\int_{z_0}^{\gamma z_0} E_2^p \,\mathrm{d}z, \tag{3.4.2}$$

$L_{\mathrm{alg}} = L(1,f)/\Omega_f^- \in \mathbb{Q}$ is the "rational part" of the L-value $L(1,f)$, and

$$\varphi_f^-(\gamma) = \frac{1}{\Omega_f^-}\int_{z_0}^{\gamma z_0} \omega_f^-. \tag{3.4.3}$$

Applying the map $\mathrm{ST}^\times$, one obtains [DPV21, Lemma 3.6]:

$$J_w = \frac{2}{p-1}J_{\mathrm{DR}} \cdot \prod_f J_f^- \bmod J_{\mathrm{univ}}, \tag{3.4.4}$$

where $J_f^- = \mathrm{ST}^\times\left(\varphi_f^-\right)$ and $J_{\mathrm{DR}} = \mathrm{ST}^\times(\varphi_{\mathrm{DR}})$.

Remark 3.24: In [DPV23], J_{DR} is studied in greater detail. In particular, they show in [DPV23, Theorem A] that J_{DR} is naturally a cocycle in $Z^1\left(\Gamma_p, \mathcal{A}^\times/p^{\mathbb{Z}}\right)$ by constructing an explicit cochain representative using Siegel units. Furthermore, they prove algebraicity of $J_{\mathrm{DR}}[\tau]$ at a real quadratic point τ with fundamental discriminant, up to some torsion ambiguity.

Using a variant of the original Siegel unit construction, Sim [Sim23] showed that J_{DR} may be viewed as a natural level 1 analogue of cocycles constructed by Darmon and Dasgupta [DD06]. An alternative construction of J_{DR} is given in [Geh22, §3.3].

Theorem 3.25: *Fix an eigenbasis $E_2^{(p)}, \{f\}$ for $M_2(\Gamma_0(p))$, and a ring class character ψ : $\mathrm{Cl}^+ \mathcal{O} \to \mathbb{C}^\times$ of an order $\mathcal{O} \subset F$.*

(i) If p splits in $\mathcal{O}$, then

$$\Delta^* E_\psi^{(p)} = \lambda_0 E_2^{(p)} + \sum_f \lambda_f \cdot f \tag{3.4.5}$$

where

$$\lambda_0 = -2\frac{\varepsilon(\psi)}{p-1}\varphi_{\mathrm{DR}}(g_\psi) \quad \text{and} \quad \lambda_f = 2L_{\mathrm{alg}}(1,f) \cdot \varphi_f^-(g_\psi) \tag{3.4.6}$$

[2] Note that "DR" stands for "Dedekind–Rademacher", and not "de Rham".

.

(ii) If p is inert in $\mathcal{O}$, then

$$e^{\mathrm{ord}}\left(\frac{\mathrm{d}}{\mathrm{d}k}\Delta^*\mathcal{E}_k\ |_{k=1}\right) = \lambda_{0'}E_2^{(p)} + \sum_f \lambda'_f \cdot f \tag{3.4.7}$$

where

$$\lambda'_0 = -\frac{4\varepsilon(\psi)}{p-1}\log_p \mathrm{Nm}\, J_{\mathrm{DR}}\left[\Delta_\psi\right] \quad \text{and} \quad \lambda'_f = -4L_{\mathrm{alg}}(1,f)\cdot\log_p \mathrm{Nm}\, J_f^-\left[\Delta_\psi\right]. \tag{3.4.8}$$

This generalizes [DPV21, Theorem C], and the proof is completely identical: see [DPV21, §3.4-5].

Remark 3.26: It would be interesting to modify the algorithms from [LV22] and [Dam24] to compute explicitly the modular forms in Theorem 3.16, Theorem 3.23 and Theorem 3.25. The first could hypothetically give faster algorithms for computing p-adic L-functions associated with ring class characters than those in [LV22]. The reason for this is that they need to compute bases for $M_k(\Gamma_0(Np))$ for various k in order to solve for the constant term of a certain p-adic Eisenstein family. On the other hand, the classical specializations of our Eisenstein series are found in $M_k(\Gamma_0(p))$, which is a much smaller space. However, our attempts at generalizing the algorithms have so far been unsuccessful due to problems with enumerating non-invertible ideals.

Chapter II. Computation of Gross–Stark units and Stark–Heegner points

4 Background

Let F be a real quadratic field and p a rational prime. While there is no direct analogue of the construction of the elliptic units from CM theory over F, Gross [Gro81] constructed what are now known as *Gross–Stark units*, formal powers of p-units in class fields of F, and formulated a p-adic analogue of Stark's conjectures for these. His conjecture, which relates the value of derivatives of p-adic L-functions at $s = 0$ to local norms of Gross–Stark units, was proved in [DDP11]. This was refined to a statement with norms removed in [DKV18], and recently Dasgupta and Kakde proved an integral version where formal units are replaced with global units [DK23].

The computation of Gross–Stark units over real quadratic fields was studied in [TY13] when p splits in F, and [FL22] for p inert in F. In the real-analytic setting, in [CR00] Cohen and Roblot used Stark's conjectures to compute wide Hilbert class fields of real quadratic fields, and similar algorithms form the basis for general algorithms to compute ray class fields in `pari/GP`. By analogy with Heegner points, Darmon's work [Dar01] uses p-adic analysis to construct points on elliptic curves. These so-called *Stark–Heegner points* are conjectured to be defined over ring class fields of F. While this conjecture is still wide open in general, it is supported by extensive computational evidence. Analogous formulas for Gross–Stark units are given in [DD06].

In [DV21], Darmon and Vonk introduce rigid meromorphic cocycles which take the p-adic theory beyond Stark's conjectures. Their framework gives an analogue of singular moduli for real quadratic fields. As a by-product, they recover a common framework for Stark units and Stark–Heegner points: in subsequent work, Darmon, Pozzi and Vonk [DPV23] use p-adic families of Hilbert modular forms to give an explicitly computable modular form whose spectral expansion encodes both Gross–Stark units and Stark–Heegner points.

More specifically, the authors construct a classical modular form G from a parallel weight 1 Hilbert Eisenstein series $E_{1,1}$ over a real quadratic field F in which p is inert. First, they define the *anti-parallel weight deformation* of $E_{1,1}$, and modify by a linear combination of Eisenstein families. Then they restrict the argument to the diagonally embedded upper half plane $\mathfrak{h}$ in $\mathfrak{h} \times \mathfrak{h}$, and differentiate with respect to the weight. This is shown to be a p-adic modular form, to which they finally apply Hida's ordinary projector to get the modular form $G \in M_2(\Gamma_0(p))$. They also prove that the form is non-trivial exactly when F has no unit of negative norm.

The spectral expansion of G has great arithmetic significance: if $E_2^{(p)}$ denotes the Eisenstein series on $M_2(\Gamma_0(p))$, then

$$\langle G, E_2^{(p)} \rangle_{\Gamma_0(p)} = \frac{1}{p-1} \log_p(u), \tag{4.9}$$

where u is a Gross–Stark unit. When $f \in S_2(\Gamma_0(p))$ is a cuspidal eigenform with coefficients in $\mathbb{Q}$,

$$\langle G, f \rangle_{\Gamma_0(p)} = L_{\mathrm{alg}}(1, f) \log_{E_f}(P_f), \tag{4.10}$$

for $L_{\mathrm{alg}}(1, f)$ the algebraic part of the special value $L(1, f)$ of the L-function attached to f, E_f the elliptic curve associated to f via the Eichler–Shimura construction, $\log_{E_f}$ the formal logarithm on E_f, and P_f a Stark–Heegner point on E_f, conjecturally defined over the narrow Hilbert class field of F. A more precise statement may be found in Theorem 5.2.

The goal of this chapter is to show that the steps defining G can be made completely explicit in a computer algebra system such as `sage` [The22] or `magma` [BCP97], and in particular we can compute the spectral coefficients of G to arbitrary precision. A key tool is algorithms for overconvergent modular forms due to Lauder ([Lau11], [Lau14]), with necessary modifications for $p \in \{2, 3\}$ from [Von15]. As a proof of concept, we compute tables of Gross–Stark units over $\mathbb{Q}(\sqrt{D})$ for fundamental discriminants $D < 10000$ and $p < 20$, and Stark–Heegner points on elliptic curves for $D < 100$, $p < 20$. This can be viewed as a numerical verification of [DV22a, Conjecture 3.19]. For p equal to 2 or 3, these tables are virtually complete, with only a handful of omissions due to the large height of the polynomials. Some of the data is presented in Appendix B.

Previous computations of Gross-Stark units and Stark-Heegner points

Systematic computation of Gross–Stark units was introduced in Slavov's thesis [Sla07], based on Shintani's method for constructing L-functons and related work of Dasgupta [Das08]. In [TY13], similar computations are done using a certain p-adic gamma-function.

In the wake of Darmon's work on Gross–Stark units and Stark–Heegner points, there appeared a growing literature on computational aspects. The paper [Dar01, §11] includes several examples of Stark–Heegner points computed using p-adic multiplicative integrals. These computations were extended in [DG02], which systematically tabulates points on elliptic modular curves of class numbers 1 and 2, in addition to selected examples for larger class numbers. Furthermore, they ask the natural question of whether there exists a polynomial time (in p) algorithm for computing Stark–Heegner point. This question is answered affirmatively in [DP06], in which the authors describe an algorithm using Pollack–Stevens overconvergent modular symbols.[3] Their techniques form the basis for other implementations of Stark–Heegner point algorithms, see for example the `sage` package `DarmonPoints` at https://github.com/mmasdeu/darmonpoints. This package, created by Masdeu and based on [GM13], [GM14], [GMS15] and other work, computes generalizations of Stark–Heegner points (or Darmon points, as they call them), as well.

5 The modular algorithm

5.1 Notation

In this chapter, F will denote a real quadratic extension of $\mathbb{Q}$ of discriminant D, and $\mathcal{O}_F$ its ring of integers. Its different ideal, which is principal and generated by $\sqrt{D}$, will be denoted $\mathfrak{d}$. If $\alpha \in F$, let α' be its conjugate.

We let Cl^+ be the narrow ideal class group of F, so that $\mathrm{Cl}^+ \cong G := \mathrm{Gal}(H/F)$ where H is the narrow Hilbert class field of F, the maximal abelian extension of F unramified at all finite places, of degree h^+ over F. Given an integral ideal $\mathfrak{a}$ of F, let $[\mathfrak{a}]$ denote the class in Cl^+ to which $\mathfrak{a}$ belongs. For $\sigma \in G$, the corresponding class in Cl^+ is denoted A_σ, and conversely a class A in Cl^+ determines an automorphism $\sigma_A \in G$. The narrow ideal class group is strictly larger than the wide ideal class group if and only if F has no units of norm -1. We restrict our attention to this case, as otherwise the modular forms in question vanish identically.

[3] The associated code, found at https://www.math.mcgill.ca/darmon/programs/shp/shp.html, does not work out of the box due to changes in `magma`. Below is a quick-fix, which recovers some functionality:

```
## Change line 175 in stark-heegner.magma from
Phi := Isomorphism(ETate,EoverCp);
## to
EWeiers, f1, _ := WeierstrassModel(ETate);
psi := Isomorphism(EWeiers, EoverCp);
Phi := f1 *psi;
```

Under this assumption, the principal ideal $\mathfrak{d}$ defines an element $[\mathfrak{d}]$ of order 2 in Cl^+. Furthermore, H is a CM extension of the wide Hilbert class field of F, and the automorphism $\kappa = \sigma_{[\mathfrak{d}]}$ plays the role of complex conjugation in G. We frequently write $\overline{\alpha}$ instead of $\kappa(\alpha)$ if the meaning is clear from the context.

Let p be a rational prime inert in F. Then $(p) \subset \mathcal{O}_F$ splits completely in H, and we fix a prime $\mathfrak{P}$ of H above (p). This determines an isomorphism of completions $F_p \cong H_{\mathfrak{P}}$, where for brevity we set $F_p = F_{(p)}$. A function $f : \mathrm{Cl}^+ \to \mathbb{C}$ is odd if $f(A[\mathfrak{d}]) = -f(A)$ for all $A \in \mathrm{Cl}^+$. The field generated by the values of a character ψ of G is denoted by $\mathbb{Q}(\psi)$.

We say an element $\alpha \in F$ is totally positive if $\rho(\alpha) > 0$ for all embeddings $\rho : F \hookrightarrow \mathbb{R}$, and we write $\alpha \gg 0$. If $X \subset F$ is any subset, we set $X_+ := \{\alpha \in X : \alpha \gg 0\}$.

Given an integral ideal $\mathfrak{a}$ of F, let $\mathrm{Nm}(\mathfrak{a}) := \#(\mathcal{O}_F/\mathfrak{a})$, and this extends to fractional ideals by $\mathrm{Nm}(\mathfrak{a}/\mathfrak{b}) := \mathrm{Nm}(\mathfrak{a})/\mathrm{Nm}(\mathfrak{b})$, and to elements $\alpha \in F^\times$ by $\mathrm{Nm}(\alpha) = \mathrm{Nm}((\alpha))$, where (α) denotes the fractional ideal generated by α. By convention, we also set $\mathrm{Nm}(x) = x^2$ when x is an indeterminate. For any number field K, $\mu(K)$ denotes the set of all roots of unity in K.

If $\mathfrak{P}$ is a non-zero prime ideal of H and $\alpha \in H^\times$, then we set $|\alpha|_{\mathfrak{P}} = \mathrm{Nm}(\mathfrak{P})^{-\mathrm{ord}_{\mathfrak{P}}\alpha}$, where $\mathrm{ord}_{\mathfrak{P}}\alpha$ denotes the power of $\mathfrak{P}$ appearing in the prime ideal factorisation of (α). This is the so-called normalised absolute value with respect to $\mathfrak{P}$, and in particular $\mathrm{Nm}(\mathfrak{P}) = p^2$ in the present setting. All of our absolute values will be normalised, and we refer to [Gro81, p. 980] for a general definition which applies to both the finite and infinite places of H.

The p-units in H is the group $\mathcal{O}_H[1/p]^\times := \{\alpha \in H^\times : |\alpha|_v = 1 \ \text{ if } \ v \nmid p\}$, where v runs over all places of H. In particular, $\alpha \in \mathcal{O}_H[1/p]^\times$ has absolute value 1 under every embedding $H \hookrightarrow \mathbb{C}$. This is a finitely generated abelian group by a version of Dedekind's unit theorem, [Neu99, Cor. 11.7].

5.2 Gross-Stark units and Stark–Heegner points

Gross [Gro81, Prop. 3.8] proved the existence and uniqueness of a "formal power of a p-unit" $u \in \mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}$ characterised by the properties

$$\mathrm{ord}_{\mathfrak{P}}\,\sigma(u) = \zeta(0, A_\sigma) \quad \text{for all} \quad \sigma \in G \quad \text{and } \overline{u} = 1/u, \tag{5.2.1}$$

where the bar denotes complex conjugation, and $\zeta(s, A_\sigma)$ is the partial ζ-function defined by the Dirichlet series $\zeta(s, A_\sigma) = \sum_{\mathfrak{a} \leq \mathcal{O}_F,\, [\mathfrak{a}] = A_\sigma} \mathrm{Nm}\,(\mathfrak{a})^{-s}$, which admits a meromorphic continuation to $\mathbb{C}$ in the usual manner. This depends only on the choice of prime $\mathfrak{P}$ of H above p. In [DPV23, Eq. (4)], the authors twist by elements of G to get units $u_A := \sigma_A(\overline{u})$ indexed by $A \in \mathrm{Cl}^+$, equal to u_τ when $A = [\mathbb{Z} + \tau\mathbb{Z}]$ in their notation. It is therefore characterised by

$$\mathrm{ord}_{\mathfrak{P}^\sigma} u_A = -\zeta(0, AA_{\sigma^{-1}}) \quad \text{for all} \quad \sigma \in G \quad \text{and } \overline{u}_A = 1/u_A. \tag{5.2.2}$$

This is referred to as the *Gross–Stark unit attached to* A. Note that these are all G-conjugate: $\sigma(u_A) = u_{AA_\sigma}$.

The Brumer–Stark conjecture, proven for $p \neq 2$ in [DK23], and in general in [Das+23], implies that u_A^e, where $e = \#\mu(H)$, gives an element of $\mathcal{O}_H[1/p]^\times$. More precisely, there exists an element $\varepsilon \in \mathcal{O}_H[1/p]^\times$ satisfying $\varepsilon \otimes 1 = e \cdot u$ such that $H(\sqrt[e]{\varepsilon})/F$ is an abelian extension. We set $\varepsilon_A := \sigma_A(\overline{\varepsilon})$, which we refer to as the *Brumer–Stark unit attached to* A. These are the units we compute in Section 6. An immediate consequence of the second part of Equation (5.2.2) is that ε_A lies on the unit circle under any embedding $H \hookrightarrow \mathbb{C}$.

We also attach a Gross--Stark unit to a character $\psi : G \to \mathbb{C}^\times$ by setting

$$u_\psi := \prod_{A \in \mathrm{Cl}^+} u_A^{\psi(A)} = \prod_{\sigma \in G} \sigma(\overline{u})^{\psi(A_\sigma)}, \tag{5.2.3}$$

which lies in $\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}(\psi)$, and satisfies $\mathrm{ord}_{\mathfrak{P}}\, u_\psi = -L(0, \psi)$ and $\sigma(u_\psi) = \overline{\psi}(A_\sigma) u_\psi$ for all $\sigma \in G$. This is compatible with the notation in [DDP11].[4]

The Stark–Heegner points $P_{\psi,f}$ are defined in [Dar01] and [Das05], and for brevity we give a description of their properties instead of a strict definition. When f has rational coefficients, these are p-adic points on the elliptic curve E_f associated to f through the Eichler–Shimura construction. We henceforth assume that this is the case. The reader may find further details about the construction of Stark–Heegner points in [DV22a, §3.7].

Fix such an elliptic curve E_f. In this setting, $P_{\psi,f}$ comes from an element of F_p defined via p-adic analytic methods. By [Sil09, Thm. 14.1], $E_f(F_p)$ is isomorphic to $F_p^\times / q^{\mathbb{Z}}$ where q is the Tate parameter attached to E_f. We can find an explicit isomorphism $E_f(F_p) \to F_p^\times / q^{\mathbb{Z}}$ as follows: first find an isomorphism between E_f and the corresponding Tate curve E_q by computing their Weierstraß equations and using the command `IsIsomorphic` in `magma`. Then compute the isomorphism $E_q \to F_p^\times / q^{\mathbb{Z}}$ using the formulae in [Sil09, § C.14]. This gives a point $P_{\psi,f}$ in $E_f(F_p)$. However, it is conjectured in [Dar01] that it is actually defined over H via the embedding $H \hookrightarrow H_{\mathfrak{P}} \cong F_p$, and in Section 6.2 we verify this computationally.

5.3 Diagonal restriction derivatives

Let ψ be an odd character on Cl^+. Following [DPV23] we consider the Hilbert modular Eisenstein series $E_{1,1}(\psi)$ of parallel weight 1 whose q-expansion at the cusp $\mathfrak{d}$ is given by the series

$$E_{1,1}(\psi)_{\mathfrak{d}} = \sum_{\nu \in \mathfrak{d}_+^{-1}} \sigma_{0,\psi}(\nu\mathfrak{d}) q^{\mathrm{tr}\,\nu}, \tag{5.3.1}$$

where $\sigma_{0,\psi}(\nu\mathfrak{d})$ is the divisor sum

$$\sigma_{0,\psi}(\nu\mathfrak{d}) := \sum_{\mathfrak{a} | \nu\mathfrak{d}} \psi(\mathfrak{a}). \tag{5.3.2}$$

For p a rational prime inert in F, we also define the p-stabilisation of $E_{1,1}(\psi)$ by $E_{1,1}^{(p)}(\psi)(z_1, z_2) := E_{1,1}(\psi)(z_1, z_2) - pE_{1,1}(\psi)(pz_1, pz_2)$. There is a certain p-adic family of modular forms $\mathcal{F}^+$, a linear combination of two Eisenstein families along with the *anti-parallel weight deformation*, whose weight 1 specialisation equals $E_{1,1}^{(p)}(\psi)$. Note that $\mathcal{F}^+$ is different from the parallel weight Eisenstein family used in [DPV21], and computing its q-expansion requires a fairly delicate argument using Galois deformation theory, the details of which are in [DPV23, §3]. Since $E_{1,1}^{(p)}(\psi)(z, z)$ is the p-stabilization of a classical modular form of level 1 and weight 2 and therefore identically 0, $E_{1,1}^{(p)}(\psi)$ vanishes along the diagonally embedded copy of $\mathfrak{h}$ in its domain $\mathfrak{h} \times \mathfrak{h}$. Taking the derivative of $\mathcal{F}^+$ in the weight space and restricting to weight 1 then gives an overconvergent modular form in one variable, denoted by ∂f_ψ^+. We refer to this as the *diagonal restriction derivative*, and its q-expansion is given as follows:

Proposition 5.1 ([DPV23, Prop. 4.6]): *The diagonal restriction derivative is an overconvergent modular form of weight* 2 *and tame level* 1 *with q-expansion*

[4]However, it is different from the formula in [DPV23, Eq. (51)], in which u_ψ depends on τ, and the corresponding formula for $\mathrm{ord}_{\mathfrak{P}}\, u_\psi$ in the proof of Lemma 3.5 is off by a factor of $\psi(\sigma_A)$, or $\psi(\tau)$ in their notation.

$$\partial f_\psi^+(q) = \frac{1}{2}\log_p(u_\psi) - \sum_{n=1}^{\infty} \sum_{\substack{\nu \in \mathfrak{d}_+^{-1} \\ \mathrm{tr}\ \nu = n}} \sum_{\substack{\mathfrak{a} \mid (\nu)\mathfrak{d} \\ (\mathfrak{a},p)=1}} \psi(\mathfrak{a}) \log_p\left(\frac{\nu\sqrt{D}}{\mathrm{Nm}(\mathfrak{a})}\right) q^n. \tag{5.3.3}$$

It has rate of overconvergence r for any $r < p/(p+1)$.

The symbol $\log_p$ denotes the p-adic logarithm, defined by the power series $\log_p(1-x) = \sum_{n=1}^{\infty} x^n/n$ on its domain of convergence in $\mathcal{O}_{F_p}$, and extended by setting $\log_p(p) = \log_p(\zeta) = 0$ for any root of unity ζ in F_p. To evaluate this at elements of F, we identify F with its image in F_p.

Applying Hida's ordinary projection operator e^{ord} to ∂f_ψ^+ gives a classical modular form of level $\Gamma_0(p)$ and weight 2. The space of such forms is spanned by the Eisenstein series

$$E_2^{(p)}(z) = \frac{p-1}{24} + \sum_{n=1}^{\infty}\left(\sum_{\substack{d \mid n \\ (d,p)=1}} d\right) q^n, \tag{5.3.4}$$

along with eigenforms f, which we normalise so that $a_1(f) = 1$ in the q-expansion at ∞.

Theorem 5.2: *Set $F = \mathbb{Q}(\sqrt{D})$ and let p be a prime inert in F. Write*

$$e^{\mathrm{ord}}(\partial f_\psi^+) = \lambda_0 E_2^{(p)} + \sum_f \lambda_f f, \quad \text{where } \lambda_0, \lambda_f \in F_p. \tag{5.3.5}$$

Then $\lambda_0 = \frac{1}{p-1}\log_p(u_\psi)$, and if $a_n(f) \in \mathbb{Q}$ for all n, then $\lambda_f = L_{\mathrm{alg}}(1,f)\log_{E_f}(P_{\psi,f})$, where $P_{\psi,f}$ is a Stark–Heegner point in $E_f(\mathbb{C}_p)$, the elliptic curve attached to f by the Eichler–Shimura construction, and $L_{\mathrm{alg}}(1,f)$ is the algebraic part of the value $L(1,f)$.

Conjecture 3.19 in [DV22a] states that the points $P_{\psi,f}$ are in fact algebraic, defined over the narrow Hilbert class field of F.

Proof: By [DPV23, Prop. 4.7], $G := e^{\mathrm{ord}}(\partial f_\psi^+)$ can be written as a generating series[5]

$$2G(z) = \log_p(u_\psi) + \sum_{n=1}^{\infty} \log_p(T_n J_w[\psi]) q^n. \tag{5.3.6}$$

Meanwhile, by [DPV23, eq. 29] the cocycle J_w decomposes as follows:

$$J_w = \frac{2}{p-1} J_{\mathrm{DR}} + 2\sum_f L_{\mathrm{alg}}(1,f) J_f^- \bmod J_{\mathrm{univ}}^{\mathbb{Z}}. \tag{5.3.7}$$

Plugging the expression for J_w into the n-th Fourier coefficient for $n \geq 1$ coprime to p, we obtain

$$\begin{aligned} a_n(G) &= \frac{2}{p-1}\log_p(T_n J_{\mathrm{DR}}[\psi]) + 2\sum_f L_{\mathrm{alg}}(1,f)\log_p(T_n J_f^-[\psi]) \\ &= \frac{2}{p-1}\log_p(J_{\mathrm{DR}}[\psi]) \cdot a_n\left(E_2^{(p)}\right) + \sum_f L_{\mathrm{alg}}(1,f)\log_p(J_f^-[\psi]) \cdot a_n(f). \end{aligned} \tag{5.3.8}$$

Theorem B of [DPV23] combined with the proof of Theorem 4.8 in the same paper implies that $J_{\mathrm{DR}}[\psi] = u_\psi^{24}$, and conjecture 3.19 in [DV22a] implies that $J_f^-[\psi]$ maps to $P_{\psi,f} \in E_f(F_p)$ under the Tate uniformisation. Denoting the composite of the Tate map and $\log_p$ by $\log_{E_f}$, we get that

[5]There is a sign missing in the proof of Thm. 4.8 which propagates back to Prop. 4.7. As written, the constant term of the Eisenstein series in the spectral expansion is off by a factor of -1. We assume here that the statement of Thm. 4.8 is correct as written.

$$a_n(G) = \frac{24}{p-1}\log_p(u_\psi)\cdot a_n\left(E_2^{(p)}\right) + \sum_f L_{\mathrm{alg}}(1,f)\log_{E_f}(P_{\psi,f})\cdot a_n(f). \tag{5.3.9}$$

As in the proof of [DPV23, Prop. 4.7], there exists a modular form in $M_2(\Gamma_0(p))$ with prime to p coefficients $a_n(G)$, which we denote by g. Now $g - G$ is an oldform in $M_2(\Gamma_0(p))$ as all its coefficients of index coprime to p vanish, hence equals 0, and this completes the proof. □

This construction can be made completely explicit in a computer algebra system such as `magma` or `sage`, at least to finite p-adic precision:

(i) Compute the terms $\{a_n\}_{n=1}^M$ of the q-expansion of ∂f_ψ^+ in Equation (5.3.3) up to a certain bound M by enumerating the elements $\nu \in \mathfrak{d}_+^{-1}$ of trace n and factorising $\nu\mathfrak{d}$. Since $\log_p(xy) = \log_p(x) + \log_p(y)$ for any $x, y \in F_p$, we only need to evaluate this once per n.

(ii) Compute a basis for the space of overconvergent modular forms to sufficiently high precision using [Lau11, Algorithm 1].

(iii) Solve for ∂f_ψ^+ and its constant term in this basis.

(iv) Compute the ordinary projection as a matrix on the basis, and apply to the vector defining ∂f_ψ^+ to get $e^{\mathrm{ord}}\left(\partial f_\psi^+\right)$. This is described in detail in step (6) of [Lau14, Alg. 2.1].

(v) Solve for $e^{\mathrm{ord}}\left(\partial f_\psi^+\right)$ in an eigenbasis of $M_2(\Gamma_0(p))$, which can be found explicitly using built-in methods in `sage` and `magma`.

In practice, the first step is very slow due to the cost of evaluating $\psi(\mathfrak{a})$ for many $\mathfrak{a}$. Moreover, the coefficients of ∂f_ψ^+ lie in an extension of F_p generated by the values of ψ, which is of high degree if the narrow class number of F is large.

Improvements using quadratic forms

To get around these difficulties, we combine two observations: the first is that if we split the sum into a sum over classes $A \in \mathrm{Cl}^+$, then it suffices to compute sums corresponding to all pairs $(\nu, \mathfrak{a})$ where $\mathfrak{a} \mid \nu\mathfrak{d}$ and $\mathfrak{a}$ has class A in the narrow class group. Moreover, these partial sums all lie in F_p. The second is that by the correspondence between ideals of $\mathbb{Q}\left(\sqrt{D}\right)$ and indefinite binary quadratic forms of discriminant D, we can use reduction theory to enumerate all such ideals.

Proposition 5.3 ([Cox11, Ex. 7.21]): *There is a natural map from ideals of $\mathbb{Q}\left(\sqrt{D}\right)$ to indefinite binary quadratic forms of discriminant D given by $\mathfrak{a} = \alpha\mathbb{Z} + \beta\mathbb{Z} \mapsto \frac{\mathrm{Nm}(x\alpha - y\beta)}{\mathrm{Nm}(\mathfrak{a})}$. This map respects the class group structure in the following sense: two ideals are in the same narrow ideal class if and only if the corresponding quadratic forms are equivalent under the right action of $\mathrm{SL}_2(\mathbb{Z})$,*

$$Q \circ \begin{pmatrix} r & s \\ t & u \end{pmatrix} := Q(rx + sy, tx + uy). \tag{5.3.10}$$

Furthermore, the map induces a bijection between Cl^+ and $\mathrm{SL}_2(\mathbb{Z})$-orbits of indefinite binary quadratic forms of discriminant D.

We say that an indefinite quadratic form $Q(x,y) = ax^2 + bxy + cy^2$ is **reduced** if $0 < b < \sqrt{D}$ and $\left|\sqrt{D} - 2|a|\right| < b$. Any given form is equivalent to at most finitely many reduced forms.

Proposition 5.4: *Let $F = \mathbb{Q}\left(\sqrt{D}\right)$ be a real quadratic field and $A \in \mathrm{Cl}^+$ a fixed class with associated reduced quadratic form Q_0. Then there is a bijection between*

$$\mathbb{I}(n, A) := \left\{(\mathfrak{a}, \nu) : \nu \in \mathfrak{d}_+^{-1}, \mathrm{tr}\ \nu = n, \mathfrak{a} \mid (\nu)\mathfrak{d}, [\mathfrak{a}] = A\right\} \tag{5.3.11}$$

and

$$M(n,A) := \{(Q = ax^2 + bxy + cy^2, \gamma) : \gamma \in N_n, Q \sim Q_0^\gamma, a > 0 > c\}, \tag{5.3.12}$$

where N_n is a set of double coset representatives of

$$\mathrm{SL}_2(\mathbb{Z}) \setminus \{\gamma \in \mathrm{Mat}_2(\mathbb{Z}) : \det\gamma = n\} / \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(Q_0). \tag{5.3.13}$$

Proof: This is essentially Proposition 3.15 with conductor 1, where the bijection is given explicitly. □

We call an element $Q \in M(n,A)$ a *nearly reduced form* since although it might not be reduced in the strict sense, it is an element of the reduced cycle of Q_0, as defined in [BV07, Ch. 6]. Note that N_n can be found as a subset of the coset representatives of $\mathrm{SL}_2(\mathbb{Z}) \setminus \{\det\gamma = n\}$, which we can choose to be

$$\begin{pmatrix} n/m & j \\ 0 & m \end{pmatrix}, \quad m \mid n, 0 \le j \le m-1, (m, n/m) = 1. \tag{5.3.14}$$

The sets $M(n,A)$ and $M(d,A)$ for $d \mid n$ are not independent: if $Q \sim Q_0^{\gamma_n}$ for some $\gamma_n \in N_n$, then we can find corresponding elements γ_d and $\gamma_{n/d}$ such that $\gamma_n = \gamma_d\gamma_{n/d}$, and so we can generate it in $M(n,A)$ by applying suitable Hecke matrices to pairs in $M(d,A)$. This gives a recursive algorithm for computing $M(n,A)$, described in Algorithm 1.

Algorithm 1: Compute the set $M(n,A)$ of nearly reduced forms

Input: A fundamental discriminant D, a class $A \in \mathrm{Cl}^+$ represented by a reduced quadratic form Q, and a positive integer n.

Output: A set of sets $\{M(d,A)\}$ indexed by divisors $d \mid n$.

if $n = 1$ then
    return $\{(Q,1)\}$
$M_n \leftarrow \emptyset$
$p \leftarrow$ smallest prime dividing n
$d \leftarrow \frac{n}{p}$
$M_d \leftarrow M(d,A)$
$H_p \leftarrow \left\{ \begin{pmatrix} p/m & j \\ 0 & m \end{pmatrix} : m \in \{1,p\} \text{ and } 0 \le j < m \right\}.$
for $(Q_d, \gamma_d) \in M_d$ **do**
    for $\delta \in H_p$ **do**
        $Q' \leftarrow Q_d^\delta$
        if $Q' \underset{\mathrm{SL}_2(\mathbb{Z})}{\nsim} Q$ for all $(Q,\gamma) \in M_n$ **then**
            $Q_1, \ldots, Q_c \leftarrow$ `ReducedCycle`(Q')
            $M_n \leftarrow M_n \cup \{(Q_1, \delta\gamma_d), \ldots (Q_1, \delta\gamma_d)\}$
return $\{M_d : d \mid n\}$

It is convenient to work with so-called *odd indicator functions on* Cl^+, meaning functions of the form

$$\mathbf{1}_A^*(B) := \mathbf{1}_A(B) - \mathbf{1}_{A[\mathfrak{d}]}(B) = \begin{cases} 1 & \text{if } B = A, \\ -1 & \text{if } B = A[\mathfrak{d}], \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.15}$$

We can pass between odd characters and odd indicator functions via the change of basis formulae

$$\psi(A) = \frac{1}{2}\sum_{B\in \mathrm{Cl}^+} \psi(B)\mathbf{1}^*_B(A) \quad \text{and} \quad \mathbf{1}^*_A(B) = \frac{2}{h^+}\sum_{\psi \text{odd}} \psi(B)\overline{\psi}(A). \tag{5.3.16}$$

These are simple consequences of the orthogonality relations for characters, see [Ser77, §2.3]. By linearity, we obtain the following version of Proposition 5.1:

Corollary 5.5: *Fix an indefinite quadratic form Q corresponding to a class $A \in \mathrm{Cl}^+$. The series $\partial f^+_Q(q) = \log_p(u_A) + \sum_{n=1}^{\infty} a_n\left(\partial f^+_Q\right)$, where*

$$a_n\left(\partial f^+_Q\right) = \sum_{\substack{(Q,\gamma)\in M(n,A)\\ Q=\langle a,b,c\rangle \\ (a,p)=1}} \log_p\left(\frac{-b+n\sqrt{D}}{2a}\right) - \sum_{\substack{(Q,\gamma)\in M(n,A[\mathfrak{d}])\\ Q=\langle a,b,c\rangle \\ (a,p)=1}} \log_p\left(\frac{-b+n\sqrt{D}}{2a}\right) \tag{5.3.17}$$

defines an r-overconvergent modular form of weight 2 and tame level 1 for any $r < p/(p+1)$.

Proof: Define $\partial f^+_Q(q) := \frac{2}{h^+}\sum_{\psi \text{ odd}} \overline{\psi}(A)\partial f^+_\psi(q)$, which has the effect of replacing $\psi(\mathfrak{a})$ in Equation (5.3.3) with $\mathbb{1}^*_A([\mathfrak{a}])$. Being a linear combination of overconvergent modular forms, it is itself overconvergent of the same weight, level and rate of overconvergence.

Using Proposition 5.4, we can rewrite the series in terms of $M(n,A)$ and $M(n,A[\mathfrak{d}])$, showing that Equation (5.3.17) holds for the non-constant terms. To compute the constant term of $\partial f^+_Q(q)$, note that formally, $u_\psi = \sum_{A\in \mathrm{Cl}^+} \psi(A)\cdot u_A$, so

$$\begin{aligned} &\frac{2}{h^+}\sum_{\psi \text{ odd}} \overline{\psi}(A)\cdot u_\psi \\ &= \sum_{A\in \mathrm{Cl}^+} \frac{2}{h^+}\sum_{\psi \text{ odd}} \overline{\psi}(A)\psi(A)\cdot u_A \\ &= \sum_{A\in \mathrm{Cl}^+} \mathbf{1}^*_A\cdot u_A \\ &= u_A\cdot u^{-1}_{A[\mathfrak{d}]}. \end{aligned} \tag{5.3.18}$$

The condition $\overline{u}_A = 1/u_A$ is equivalent to $u_{A[\mathfrak{d}]} = u_A^{-1}$, so

$$\frac{2}{h^+}\sum_{\psi \text{ odd}} \frac{1}{2}\log_p(u_\psi) = \log_p(u_A), \tag{5.3.19}$$

finishing the proof. ☐

This gives a reasonably efficient algorithm for computing $\log_p(u_A)$, described in Algorithm 2.

Algorithm 2: Computing $\log_p(u_A)$

Input: A real quadratic field $\mathbb{Q}\left(\sqrt{D}\right)$ in which p is inert, a class $A \in \mathrm{Cl}^+$ represented by a quadratic form Q_0, and a positive integer N.

Output: $\log_p(u_A)$ as an element of F_p to p-adic precision N

1 $m \leftarrow p \cdot N$
2 Compute $\{M(n, A)\}_{n \leq m}$ using Algorithm 1
3 Compute $\left\{a_n\left(\partial f_Q^+\right)\right\}_{n \leq m}$ using Equation (5.3.17)
4 $B \leftarrow$ `KatzBasis` $\left(M_2^\dagger(\mathrm{SL}_2(\mathbb{Z}))\right) \bmod p^N, q^m)$
5 $\log_p(u_A) \leftarrow$ `FindConstantTerm` $\left(\{a_n\}_{n \leq m}, B\right)$

The step `KatzBasis` is described in step 3 of [Lau11, Algorithm 1]. Roughly speaking, a Katz basis form is the ratio of a classical modular form of weight $2 + (p-1)i$ and E_{p-1}^i. Computing finitely many of these to sufficiently high finite precision, we obtain a basis for a subspace of $M_2^\dagger(\mathrm{SL}_2(\mathbb{Z}))$ in which we can uniquely detect ∂f_Q^+. Further details and proofs can be found in [Kat73, Chap. 2].

The function `FindConstTerm` first solves a linear system obtained by solving for the higher order coefficients of ∂f_Q^+ in terms of those in B, so that the constant term of ∂f_Q^+ is a linear combination of the constant terms of the Katz basis forms. The number of terms m computed in the q-expansion of ∂f_Q^+ ensures that it can always be found in the Katz basis from [Lau11, Algorithm 1], although in practice smaller values of m are often sufficient.

With a little extra work we can compute the spectral expansion of $e^{\mathrm{ord}}\left(\partial f_Q^+\right)$. To compute the ordinary projection, we use a trick due to Lauder. The idea is to compute a matrix for the U_p-operator acting on the Katz basis B from 1, computed to precision $\dim M_{k'}(\mathrm{SL}_2(\mathbb{Z}))$, where $k' := 2 + (p-1)\lfloor N(p+1)/p \rfloor$. Since this approximate basis is finite, the matrix U_p has finite rank. Raising the matrix to the power $2m$ and applying to the vector defining ∂f_ψ^+ then gives the ordinary projection. We denote this step by `OrdinaryProjection` in Algorithm 3.

Algorithm 3: Computing the spectral expansion of $e^{\mathrm{ord}}\left(\partial f_Q^+\right)$

Input: A real quadratic field $\mathbb{Q}\left(\sqrt{D}\right)$ in which p is inert, a class $A \in \mathrm{Cl}^+$ represented by a quadratic form Q, and an integer N.

Output: The coefficients λ_0 and $\left\{\lambda_f\right\}$ of $e^{\mathrm{ord}}\left(\partial f_Q^+\right)$ as elements of F_p to p-adic precision N.

1 $m \leftarrow \dim M_{2+(p-1)\lfloor N(p+1)/p \rfloor}(\mathrm{SL}_2(\mathbb{Z}))$
2 Compute $\{M(n, A)\}_{n \leq m}$ using Algorithm 1
3 Compute $B \bmod (p^m, q^N)$ and $\left\{a_n\left(\partial f_\psi^+\right)\right\}_{n=0}^N$ as in Algorithm 2
4 $G \leftarrow$ `OrdinaryProjection` $\left(\left\{a_n\left(\partial f_\psi^+\right)\right\}_{n=0}^N, B\right)$
5 $M \leftarrow M_2(\Gamma_0(p)) \otimes F_p$
6 $\{\lambda_0\}, \left\{\lambda_f\right\}_f \leftarrow$ `FindInSpace`(G, M)

Here `FindInSpace`(G, M) solves for $G = e^{\mathrm{ord}}\left(\partial f_Q^+\right)$ in terms of the eigenbasis for $M_2(\Gamma_0(p))$ and returns the corresponding coefficients, which are precisely λ_0 and the λ_f for eigenforms f. The same algorithm works for $e^{\mathrm{ord}}\left(\partial f_\psi^+\right)$.

6 From logarithms to invariants

In this section we explain how to recover u_A from $\log_p(u_A)$ and $P_{\psi,f}$ from λ_f.

6.1 Recovering a Gross–Stark unit from its p-adic logarithm

The "virtual units" u_A are difficult to work with because they are formal powers of units in H, and thus do not have a unique minimal polynomial. Instead, we use the Brumer–Stark conjecture and look instead for the element $\varepsilon_A \in \mathcal{O}_H[1/p]^\times$ satisfying $e \cdot u_A = \varepsilon_A \otimes 1$, where $e := \#\mu(H)$. This property implies that $\log_p(u_A) = \frac{1}{e}\log_p(\varepsilon_A)$. Note that while u_A is determined uniquely by Equation (5.2.2) because $\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}$ is torsion-free, ε_A is only unique up to roots of unity in H. This ambiguity is natural for several reasons. First, the Brumer–Stark units over $\mathbb{Q}$ constructed in [Gro81] are Gauss sums, which by definition require a choice of a root of unity to determine the additive character. Second, ε_A being defined only up to torsion in $\mathcal{O}_H[1/p]^\times$ mirrors the fact that Stark–Heegner points are defined up to torsion in $E(H)$.

We can find the exact value of e without computing the unit group of $\mathcal{O}_H$ directly by noting that any root of unity in H will lie in the *genus field* of F, the largest subextension of H which is abelian over $\mathbb{Q}$. This has the following classical description:

Proposition 6.1 ([Lem00, Prop. 2.19]): *Let $F = \mathbb{Q}(\sqrt{D})$, and let $D = D_1 \cdots D_t$ be the factorisation of D into prime discriminants, meaning each D_i is either -4, -8, 8 or $(-1)^{(p-1)/2}p$ for an odd prime p. Then the genus field of F equals $\mathbb{Q}(\sqrt{D_1}, \ldots, \sqrt{D_t})$.*

Since the only quadratic extensions with other roots of unity than ± 1 are $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, we obtain the following:

Corollary 6.2: *The torsion subgroup of $\mathcal{O}_H^\times$ is given by*

$$\#\mu(H) = \begin{cases} 12 & \text{if } 12 \mid D \text{ and } \frac{D}{4} \equiv 3 \bmod 4, \\ 6 & \text{if } 3 \mid D \text{ and either } 4 \nmid D \text{ or } \frac{D}{4} \equiv 1 \bmod 4, \\ 4 & \text{if } 3 \nmid D, 4 \mid D \text{ and } \frac{D}{4} \equiv 3 \bmod 4, \\ 2 & \text{otherwise.} \end{cases} \tag{6.1.1}$$

The kernel of $\log_p$ is much larger than that of the archimedean log, containing powers of p as well as roots of unity. Passing from $\log_p(\varepsilon_A)$ to ε_A requires knowing both $\operatorname{ord}_{\mathfrak{P}} \varepsilon_A$ and $\varepsilon_A \bmod \mathfrak{P}$. We can deal with the latter by looping through all the roots of unity in $H_{\mathfrak{P}}$, of which there are $p^2 - 1$, and test each product separately. This, along with the computation of the Katz basis, are the main bottlenecks in the algorithm for large values of p. Certain Stark units modulo p appear in a recent conjecture of Harris–Venkatesh [HV19], and it would be interesting to see if an analogous conjecture could describe the mod $\mathfrak{P}$ reduction of u_A.

To find the $\mathfrak{P}$-valuation of ε_A, we use a classical theorem due to C. Meyer which we now describe. Let $A \in \mathrm{Cl}^+$ be a narrow ideal class, and recall that the corresponding partial ζ-function is given by

$$\zeta(s, A) := \sum_{\mathfrak{a} \leq \mathcal{O}_F,\, [\mathfrak{a}] = A} \frac{1}{\mathrm{Nm}(\mathfrak{a})^s}, \quad \Re(s) > 1. \tag{6.1.2}$$

Let $\zeta_-(s, A) := \frac{1}{2}(\zeta(s, A) - \zeta(s, A[\mathfrak{d}]))$. This is non-zero if and only if F has no unit of negative norm, which is our running assumption.

Let ε_F denote the fundamental unit of F, by assumption satisfying $\mathrm{Nm}(\varepsilon_F) = 1$, and fix a representative $\mathfrak{a} \leq \mathcal{O}_K$ for A. A choice of basis gives an identification of $\mathfrak{a}$ with $\mathbb{Z}^2$, and multiplication by ε_F

gives rise to an element $\gamma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{Z})$. Using the quadratic form $Q := Q_1x^2 + Q_2xy + Q_3y^2$ associated to $\mathfrak{a}$ by Proposition 5.3, we may describe this matrix explicitly. Suppose $\varepsilon_F = u + t\sqrt{D}$. Then

$$\gamma_{\mathfrak{a}} = \begin{pmatrix} t + Q_2u & 2Q_3u \\ -2Q_1u & t - Q_2u \end{pmatrix} \tag{6.1.3}$$

fixes Q. Replacing $\mathfrak{a}$ with $\mathfrak{a}'$ in the same class A has the effect of conjugating $\gamma_{\mathfrak{a}}$, so the conjugacy class $\gamma_A = [\gamma_{\mathfrak{a}}]$ is well-defined.

Let $\Phi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{R}$ denote the *Dedekind symbol* defined by

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} b/d & \text{if } \ c = 0, \\ \frac{a+d}{c} - 12 \ \mathrm{sgn}\ (c) \cdot s(a,c) & \text{if } \ c \neq 0, \end{cases} \tag{6.1.4}$$

where $s(a,c)$ is the *Dedekind sum*

$$s(a,c) := \sum_{k=1}^{|c|} \left(\left(\frac{ak}{c}\right)\right)\left(\left(\frac{k}{c}\right)\right) \quad \text{for } (a,c) = 1, c \neq 0, \tag{6.1.5}$$

with $((x)) = 0$ if $x \in \mathbb{Z}$ and $((x)) = x - \lfloor x \rfloor - 1/2$ otherwise.

By adding a correction term to Φ, Rademacher showed that the eponymous *Rademacher symbol*,

$$\Psi(\gamma) := \Phi(\gamma) - 3 \ \mathrm{sgn}\ (c(a+d)), \tag{6.1.6}$$

is integer-valued and depends only on the conjugacy class of γ.

Theorem 6.3 (Meyer): *Fix a class $A \in \mathrm{Cl}^+$, and let $\gamma_A \in \mathrm{SL}_2(\mathbb{Z})$ be the associated matrix. Then*

$$\zeta_-(0,A) = \frac{1}{12}\Psi(\gamma_A). \tag{6.1.7}$$

This follows from a version of Kronecker's limit formula for real quadratic fields, and we give an exposition of its proof in Appendix A.

Corollary 6.4: *Let u_A be a Gross–Stark unit attached to a narrow ideal class A. Then*

$$\mathrm{ord}_{\mathfrak{P}}\, u_A = -\frac{1}{12}\Psi(\gamma_A). \tag{6.1.8}$$

Similarly, for the associated Brumer–Stark unit ε_A,

$$\mathrm{ord}_{\mathfrak{P}}\, \varepsilon_A = -\frac{e}{12}\Psi(\gamma_A), \tag{6.1.9}$$

where $e = \#\mu(H)$.

Proof: By Equation (5.2.2),

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}}\, u_A &= \frac{1}{2}\left(\mathrm{ord}_{\mathfrak{P}}\, u_A - \mathrm{ord}_{\mathfrak{P}}\, u_{A[\mathfrak{d}]}\right) \\ &= -\frac{1}{2}(\zeta(0,A) - \zeta(0,A[\mathfrak{d}])) \\ &= -\zeta_-(0,A) = -\frac{1}{12}\Psi(\gamma_A). \end{aligned} \tag{6.1.10}$$

The second claim follows immediately from the identity $e \cdot u_A = \varepsilon_A \otimes 1$. □

Algorithm 4 describes how to efficiently compute $\mathrm{ord}_{\mathfrak{P}}\, \varepsilon_A$ using Meyer's theorem.

Algorithm 4: Compute $\operatorname{ord}_{\mathfrak{P}} \varepsilon_A$ using Meyer's formula.

Input: An indefinite binary quadratic form $Q(x, y) = Q_1x^2 + Q_2xy + Q_3y^2$ of square-free discriminant D, representing a narrow ideal class A of $F = \mathbb{Q}(\sqrt{D})$.

Output: $\operatorname{ord}_{\mathfrak{p}} \varepsilon_A$ for any $\mathfrak{p} \mid p$.

$t, u \leftarrow$ `PellSolution` (D)

$\gamma_A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftarrow \begin{pmatrix} t+Q_2u & 2Q_3u \\ -2Q_1u & t-Q_2u \end{pmatrix}$

if $c = 0$:

  $\Phi \leftarrow b/d$

else:

  $\Phi \leftarrow \frac{a+d}{c} - 12\operatorname{sgn}(c) \cdot$ `DedekindSum` (a, c)

$\Psi \leftarrow \Phi - 3 \operatorname{sgn}(c(a + d))$

return $-e \cdot \Psi/12$.

The fundamental solution of Pell's equation grows very quickly as D gets large, so computing Dedekind sums by evaluating Equation (6.1.5) directly can be very slow for large values of D. Instead we use a formula from [Apo90, Ex. 3.10]: By replacing c by $-c$ and a by $a \bmod c$, we can assume that $0 < a < c$. Let $r_0 := c$, $r_1 := a$ and define r_j recursively to be the remainders in the Euclidean algorithm applied to a and c, satisfying $r_{j+1} \equiv r_{j-1} \bmod r_j$ and $1 = r_{n+1} < \ldots r_{j+1} < r_j \ldots < r_0$ for all $1 \leq j \leq n-1$. Then

$$s(a, c) = \frac{1}{12} \sum_{j=1}^{n+1} \left(\frac{r_j^2 + r_{j-1}^2 + 1}{r_j r_{j-1}} \right) - \frac{(-1)^n + 1}{8}. \tag{6.1.11}$$

This is very efficient in practice.

Remark 6.5: It is also possible to compute the value of $\zeta_-(0, A)$ using a theorem due to Zagier, [Zag81, §14, Satz 2], which expresses $\zeta_-(0, A)$ as an elementary sum of numbers appearing in the reduction algorithm for indefinite quadratic forms. We thank an anonymous referee for pointing this out. Comparing Zagier reduction and Algorithm 4 in the `sage` implementation, it turns out that Zagier's formula is much faster in practice. However, if we compute the automorph using reduction theory instead of by solving Pell's equation, then the algorithms perform roughly equally well.

By the *minimal polynomial of* ε we mean the irreducible polynomial P of minimal degree satisfying $P(\varepsilon) = 0$ with coefficients in $\mathcal{O}_F$ not all divisible by the same prime, such that the leading term is a positive power of p.

Lemma 6.6: *Let ε be a Brumer–Stark unit in $\mathcal{O}_H[1/p]^\times$, and let $P(T) = \sum_{i=0}^d a_i T^i = a_d \prod_{\sigma \in G}(T - \sigma(\varepsilon))$ be its minimal polynomial. Then:*

(i) ε is a primitive element of H over F, $H = F(\varepsilon)$.

(ii) P is of degree h^+, and after possibly twisting ε by a root of unity in H, has rational integer coefficients.

(iii) P is reciprocal, $a_i = a_{d-i}$ for all $0 \leq i \leq d$.

Proof: (i) We follow the strategy of [Rob97, Théorème 2.3]. Suppose $\sigma(\varepsilon) = \varepsilon$ for some $\sigma \in G$. For any character $\chi : G \to \mathbb{C}^\times$, let $L_S(s, \chi)$ denote the L-function of χ with the Euler factor at $\mathfrak{p} = (p) \subset \mathcal{O}_F$ removed. Since $\sigma_{\mathfrak{p}} = 1$, $\chi(\sigma_{\mathfrak{p}}) = 1$, and so we have $L_S(0, \chi) = 0$. A consequence of the Brumer–Stark conjecture, see for example [Tat81, Prop. (5.5) and Conj. (4.2)], is that ε satisfies

$$L'_S(0,\chi) = -\frac{1}{e}\sum_{\sigma'\in G}\chi(\sigma')\log|\sigma'(\varepsilon)|_{\mathfrak{P}} \tag{6.1.12}$$

for all χ. It follows that

$$\begin{aligned} L'_S(0,\chi) &= -\frac{1}{e}\sum_{\sigma'\in G}\chi(\sigma')\log|\sigma'(\varepsilon)|_{\mathfrak{P}} \\ &= -\frac{1}{e}\sum_{\sigma'\in G}\chi(\sigma')\log|\sigma'\sigma(\varepsilon)|_{\mathfrak{P}} \\ &= -\frac{\overline{\chi}(\sigma)}{e}\sum_{\sigma''\in G}\chi(\sigma'')\log|\sigma''(\varepsilon)|_{\mathfrak{P}} \\ &= \overline{\chi}(\sigma)L'_S(0,\chi). \end{aligned} \tag{6.1.13}$$

If χ is odd, then $L'_S(0,\chi)\neq 0$ by [Gro81, Eq. 3.1], so $\sigma\in\bigcap_{\chi\ \text{odd}}\ker\chi$. Fix an odd character ψ, and note that there is a bijection between even characters χ and the set of characters $\psi\cdot\psi'$ where ψ' runs over all odd characters. Now

$$\sum_{\chi\in\hat{G}}\chi(\sigma) = \sum_{\chi\ \text{odd}}\chi(\sigma) + \sum_{\chi\ \text{even}}\chi(\sigma) = (1+\psi(\sigma))\sum_{\psi'\ \text{odd}}\psi'(\sigma) = 2\#\{\chi\ \text{odd}\} = h^+, \tag{6.1.14}$$

and so $\sigma = 1$.

(ii) The degree of P is h^+ since ε is primitive. Let τ be an RM-point in the sense of [DPV23]. As described in [DV21, §3.2], $\mathrm{Gal}(H/\mathbb{Q})\cong\mathrm{Gal}(H/F)\rtimes\mathrm{Gal}(F/\mathbb{Q})$, and we can identify the image of the generator of $\mathrm{Gal}(F/\mathbb{Q})$ with $\sigma_{\mathfrak{P}}$.

By the Shimura reciprocity conjecture [DV21, Conj. 3.14], $\sigma_{\mathfrak{P}}(J_{\mathrm{DR}}[\tau]) = J_{\mathrm{DR}}[\tau']$. If we let τ be the RM point corresponding to the identity class in Cl^+, then $J_{\mathrm{DR}}[\tau] = J_{\mathrm{DR}}[\tau']$, and so $J_{\mathrm{DR}}[\tau]$ is fixed by $\sigma_{\mathfrak{P}}$. Thus the minimal polynomial of $J_{\mathrm{DR}}[\tau]$ is fixed by $\sigma_{\mathfrak{P}}$, and as ε is a conjugate of $J_{\mathrm{DR}}[\tau]$ up to roots of unity in H, the result follows.

(iii) P being reciprocal is equivalent to $P(T) = T^dP(1/T)$, which is true if for any non-zero root v of P, $1/v$ is also a root of P. But with κ denoting complex conjugation in G, Equation (5.2.2) implies that $\kappa(\sigma(\varepsilon)) = 1/\sigma(\varepsilon)$. □

Knowing the $\mathfrak{P}$-valuations of all the conjugates of ε lets us bound the valuations of the coefficients of P using the following lemma.

Lemma 6.7: *Let $v_0\ldots,v_{d/2-1}$ be the $\mathfrak{P}$-valuations of the conjugates of ε which are positive, ordered so that $v_0\geq v_1\geq\ldots\geq v_{d/2-1}\geq 0$, and $v_{d/2}=0$. Then for any $i=0,\ldots,d/2$ we have $\mathrm{ord}_p(a_i)\geq\sum_{j=0}^{d/2-i}v_{d/2-j}$. In particular, $\mathrm{ord}_p(a_d)=\mathrm{ord}_p(a_0)=\sum_{j=0}^{d/2}v_j$.*

Proof: By Lemma 6.6 (iii), the Newton polygon of P is symmetric around the vertical line $x=d/2$, and its slopes are precisely equal to the p-valuations of the roots of P, the conjugates of u. Since P is normalised, we know that $\mathrm{ord}_p\,a_{d/2}=0$, so the Newton polygon of P intersects the x-axis at the point $(0,d/2)$. To estimate the remaining coefficients, note that the Newton polygon of P will always lie in the convex hull of the polygon determined as follows: the boundary is symmetric around the line $x=d/2$, and is determined by the points $\left(i,\sum_{j=0}^{d/2-i}v_j\right)$ for $0\leq i\leq d/2$. Since the y-coordinate of a point determining the Newton polygon of P is the $\mathfrak{P}$-valuation of the corresponding coefficient, this gives the required inequality. □

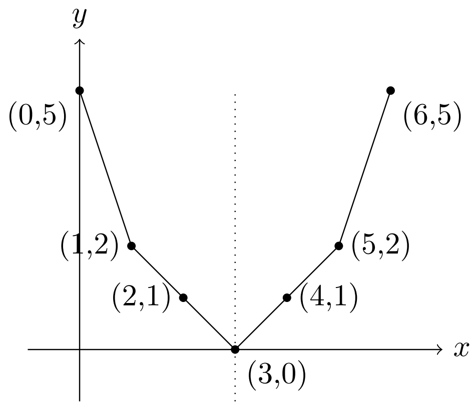

Figure 2: *The largest possible Newton polygon determined by the* $\mathfrak{P}$*-valuations of the conjugates of a Brumer–Stark unit over* $\mathbb{Q}\left(\sqrt{469}\right)$*, where the vector of valuations is given by* $(-3,-1,-1,1,1,3)$.

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}/p^m \times \mathbb{Z}/p^m$ be an approximation of $\exp_p\left(\log_p(\varepsilon_A)\right)$, where for a fixed generator s of $\mathbb{Q}_{p^2}$ over $\mathbb{Q}_p$ we define the natural map

$$\mathbb{Z}_{p^2} = \mathbb{Z}_p[s] \to \mathbb{Z}/p^m \times \mathbb{Z}/p^m \quad \text{by } a + bs \mapsto (a \bmod p^m, b \bmod p^m). \tag{6.1.15}$$

To find the minimal polynomial P of α, we apply the LLL algorithm to look for linear integral relations between powers of α. This is a standard application of lattice reduction algorithms, and a more detailed exposition can be found in [Coh93, § 2.7.2]. Roughly speaking, the LLL algorithm takes as input a basis $b_1, \ldots, b_d$ for a Euclidean lattice $\Lambda \subset \mathbb{R}^n$, and returns a "better" basis $b_1^*, \ldots, b_d^*$ for Λ, in the sense that b_1^* has relatively small norm and that the vectors are approximately orthogonal. Let $v_0, \ldots, v_{d/2-1}$ be the $\mathfrak{P}$-valuations of the conjugates of ε ordered as in Lemma 6.7, computed using Algorithm 4. We want to find a short nontrivial vector in the lattice spanned by the rows of the following $(d/2+3) \times (d/2+3)$-matrix:

$$\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & p^{v_0}\left(1+\alpha^d\right)_1 & p^{v_0}\left(1+\alpha^d\right)_2 \\
0 & 1 & 0 & \ldots & 0 & p^{v_1}\left(\alpha^1+\alpha^{d-1}\right)_1 & p^{v_1}\left(\alpha^1+\alpha^{d-1}\right)_2 \\
0 & 0 & 1 & \ldots & 0 & p^{v_2}\left(\alpha^2+\alpha^{d-2}\right)_1 & p^{v_2}\left(\alpha^2+\alpha^{d-2}\right)_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & \left(\alpha^{d/2}\right)_1 & \left(\alpha^{d/2}\right)_2 \\
0 & 0 & 0 & \ldots & 0 & p^m & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & p^m
\end{pmatrix} \tag{6.1.16}$$

A vector

$$\begin{aligned} w = \Bigg(& n_0, \ldots, n_{d/2}, n_{d/2}\alpha_1^{d/2} + \sum_{i=0}^{d/2-1} p^{v_i} n_i \left(\alpha^i + \alpha^{d-i} + p^m\right)_1, \\ & n_{d/2}\alpha_2^{d/2} + \sum_{i=0}^{d/2-1} p^{v_i} n_i \left(\alpha^i + \alpha^{d-i} + p^m\right)_2 \Bigg), \end{aligned} \tag{6.1.17}$$

in the lattice is small only if $n_{d/2}\alpha^{d/2} + \sum_{i=0}^{d/2-1} p^{v_i} n_i\left(\alpha^i + \alpha^{d-i}\right) \equiv 0 \bmod p^m$. Then the polynomial $\sum_{i=0}^{d/2} p^{v_i} n_i x^i + \sum_{i=d/2+1}^{d} p^{v_{d/2-i}} n_{d-i} x^i$ is a good candidate for the minimal polynomial of P over $\mathbb{Q}$. This suggests the following algorithm:

Algorithm 5: Find the minimal polynomial of ε_A from the p-adic approximation of $\log_p \varepsilon_A$

Input: $\alpha \in \mathbb{Q}_{p^2}$ an approximation to $\exp_p(\log_p \varepsilon_A)$ and $v_0, \dots v_{d/2}$ as in Lemma 6.7.
Output: The minimal polynomial $P \in \mathbb{Z}[x]$ of ε_A

$\zeta \leftarrow$ primitive (p^2-1)-st root of unity in $\mathbb{Q}_{p^2}$
$\alpha' \leftarrow \zeta^k \alpha$ $M \leftarrow$ matrix described in Equation (6.1.16) with α' in place of α
$v = (n_i) \leftarrow$ first vector returned by $\mathtt{LLL}(M)$
$P \leftarrow \sum_{i=0}^{d/2} n_i x^i + \sum_{i=d/2+1}^{d} n_{d-i} x^i$
$\mathtt{IsBSUnitCharPoly}(P)$.

In practice, it is convenient to pick $A \in \mathrm{Cl}^+$ so that $\mathrm{ord}_{\mathfrak{P}}\, \varepsilon_A$ is as close to 0 as possible. A similar algorithm for recognising an algebraic number from a p-adic approximation is given in [Gau+06, §4.2].

The function `IsBSUnitCharPoly` performs a series of tests in order, and returns `False` if any test fails:

(i) if P is irreducible over F, hence generates an extension of F of degree h^+,

(ii) if the absolute discriminant of $H' := F[x]/(P(x))$ is a power of D, which is equivalent to H'/F being unramified at all finite places,

(iii) if H'/F is abelian.
At this point we know that $H' \cong H$, but to ensure that P is the minimal polynomial of a Brumer–Stark unit and not just any generator of H, we perform a further test:

(iv) test if the extension generated by $P(x^e)$ is a central extension.

If all of these tests are passed, then it is quite likely, although not absolutely certain, that the polynomial P has a Brumer–Stark unit as a root. One should also test if $P(x^e)$ generates an abelian extension of F, but this is computationally unfeasible when both h^+ and e are large. Furthermore, we need to verify that $(\varepsilon) \subset \mathcal{O}_H$ is only divisible by primes of H above p, and that

$$\mathrm{ord}_{\mathfrak{P}}(\sigma_A(\varepsilon)) = -e \cdot \zeta_-(0, A) \quad \text{for all } A \in \mathrm{Cl}^+ . \tag{6.1.18}$$

To check the second condition, given a polynomial P with a chosen root ε, we do the following: for each prime $\mathfrak{P} \subset \mathcal{O}_H$ dividing p satisfying $\mathrm{ord}_{\mathfrak{P}}(\varepsilon) = -e \cdot \zeta_-(0, [\mathcal{O}_F])$, test whether all $A \in \mathrm{Cl}^+$ satisfy Equation (6.1.18) by computing $\sigma_A(\varepsilon)$ explicitly. If one such $\mathfrak{P}$ works, then the second condition is verified. The Artin map $A \mapsto \sigma_A$ is conveniently provided in `magma` by the function `ArtinMap`.

Remark 6.8: The requirement that the extension should be central was part of Stark's original conjecture, see [Sta80, Conj. 1], and in [PRS11, p. 40] Stark notes that this was sufficient for the factorisation of regulators which motivated it. The condition that the extension should in fact be abelian was observed by Tate, leading to the formulation of the Brumer–Stark conjecture. This is now known to be true, by the work of Dasgupta and Kakde [DK23].

It would be interesting to know whether "central implies abelian" in this situation, that is: if α is a p-unit which generates H with $\mathfrak{P}^\sigma$-valuations specified by Equation (6.1.9) and $\sqrt[e]{\alpha}$ generates a central extension of F, is the extension actually abelian?

To describe the test in (4), it is convenient to introduce some notation: Let $K := H(\sqrt[e]{\varepsilon_A})$ and $G_e := \mathrm{Gal}(K/H)$. By Kummer theory, $G_e \cong \mathbb{Z}/e\mathbb{Z}$. In this case $\Gamma := \mathrm{Gal}(K/F)$ is a group extension of G_e and G,

$$1 \to G_e \to \Gamma \to G \to 1. \tag{6.1.19}$$

The following lemma gives a simple criterion for deciding whether Γ is a central extension, that is, if G_e lies in the centre of Γ, without computing Γ directly:

Lemma 6.9: *Let F be a number field, H/F a Galois extension containing all e-th roots of unity, and $\alpha \in H^\times$. Define $\chi_{\mathrm{cyc}} : G := \mathrm{Gal}(H/F) \to (\mathbb{Z}/e\mathbb{Z})^\times$ by $\zeta^{\chi_{\mathrm{cyc}}(\sigma)} = \sigma(\zeta)$ for any $\zeta \in \mu_e(H)$. Then $K := H(\sqrt[e]{\alpha})/F$ is a central extension if and only if for all $\sigma \in G$ there exists some $\beta \in H^\times$ such that $\sigma(\alpha) = \alpha^{\chi_{\mathrm{cyc}}(\sigma)}\beta^e$.*

Proof: There is a natural action of G on $G_e := \mathrm{Gal}(K/H)$ by conjugation, $\sigma \cdot g := \sigma g \sigma^{-1}$, which is well-defined precisely because G_e is abelian. The extension K/F is central if and only if the action is trivial. Let Δ be a set of representatives of $H^\times/(H^\times)^e$, and note that this admits a natural action of G. The Kummer pairing ([Gra03, §I.6]) gives a G-equivariant isomorphism $G_e \cong \mathrm{Hom}(\Delta, \mu_e(K))$. The action of G_e on the right-hand side is given by $(\sigma \cdot \phi)(\alpha) = \phi(\sigma^{-1}(\alpha))^{\chi_{\mathrm{cyc}}(\sigma)}$, where $\chi_{\mathrm{cyc}}(\sigma)$ is defined by $\sigma \cdot \zeta_e = \zeta_e^{\chi_{\mathrm{cyc}}(\sigma)}$. The action of G on G_e is trivial if and only if the action on $\mathrm{Hom}(\Delta, \mu_e)$ is trivial. Each element of this group is given by $\psi_g : \delta \mapsto \langle \delta, g \rangle := \frac{g\sqrt[e]{\delta}}{\sqrt[e]{\delta}}$ for some $g \in G_e$, and so Γ is central if and only if $(\sigma \cdot \psi_g)(\delta) = \psi_g(\delta)$ for all $\delta \in \Delta$, $g \in G_e$ and $\sigma \in G$. Equivalently,

$$\left(\frac{g\sqrt[e]{\sigma^{-1}(\delta)}}{\sqrt[e]{\sigma^{-1}(\delta)}}\right)^{\chi_{\mathrm{cyc}}(\sigma)} = \frac{g\sqrt[e]{\delta}}{\sqrt[e]{\delta}} \quad \text{hence } g\left(\sqrt[e]{\frac{\alpha^{\chi_{\mathrm{cyc}}(\sigma)}}{\sigma(\alpha)}}\right) = \sqrt[e]{\frac{\alpha^{\chi_{\mathrm{cyc}}(\sigma)}}{\sigma(\alpha)}}, \tag{6.1.20}$$

where $\alpha := \sigma^{-1}(\delta)$. This being true for all g is equivalent to $\frac{\alpha^{\chi_{\mathrm{cyc}}(\sigma)}}{\sigma(\alpha)}$ being an e-th power for all σ. Finally, note that G acts transitively on Δ, so it suffices to check the criterion for a single α. □

This test can be implemented quite easily, and is mainly bottlenecked by the computation of $\mathrm{Gal}(H/F)$, at least when $[H : F]$ is reasonably large.

Remark 6.10: A test for whether an extension is abelian is found in [Coh12, Algorithm 4.4.6]. In short, the Takagi existence theorem gives a bijection between abelian extensions K/F and certain *Takagi subgroups* of a ray class group $\mathrm{Cl}_{\mathfrak{m}} F$, where $\mathfrak{m}$ is a sufficiently large modulus. However, this is very slow when e and h^+ are large, because it requires computing the ray class group of F of modulus equal to the relative discriminant of $H(\sqrt[e]{\alpha})/F$, which is relatively large.

6.2 Detecting Stark–Heegner points

Our method of finding Stark–Heegner points is much more primitive, because we don't have an equivalent of the Brumer–Stark conjecture.

Let $E/\mathbb{Q}$ be an elliptic curve with split multiplicative reduction at p. Recall from Theorem 5.2 that if E has associated eigenform $f \in M_2(\Gamma_0(p))$, then the corresponding spectral coefficient $\lambda_f = -L_{\mathrm{alg}}(1, f)\log_E(P_{\psi,f})$ involves a point $P_{\psi,f}$ conjecturally defined over H. To find this, we make use of the Tate curve E_q isomorphic to E, which is described explicitly with the formulae in [Sil09, § C.14]. From this we can find an explicit isomorphism $F_p^\times/q^{\mathbb{Z}} \xrightarrow{\phi} E_q(F_p)$, where q is an element satisfying $|q| < 1$ generating a discrete subgroup. An approximation to $\alpha := \exp_p(-\lambda_f/L_{\mathrm{alg}}(1, f))$ can then be mapped to a point on the Tate curve $E_q(F_p)$. Mapping further into $E(F_p)$, we may compute using descent a generating set $\{g\}$ for $E(H)$ and attempt to write the image of α as an integral combination of them. Since $P_{\psi,f}$ is only defined up to torsion, it is reasonable to look for a dependence between the formal logarithms of α and the generators $\{g\}$. To ensure convergence of the corresponding power series, we replace α by α^{p-1} and each g by $(p-1)g$. Then we look for an integer relation by applying the `LLL`-algorithm to a suitable lattice as in the previous section. Following the convention in `pari/gp`, we call this step `lindep`.

In summary, we have Algorithm 6.

Algorithm 6: Find Stark–Heegner point $P_{\psi,f}$ from λ_f

Input: A normalised eigenform $f \in M_2(\Gamma_0(p))$ with coefficients in $\mathbb{Q}$ and associated elliptic curve E, and $\lambda_f \in (\mathbb{Z}/p^m\mathbb{Z})^2$ an approximation to $-L_{\mathrm{alg}}(1,f)\log_{E_f} P_{\psi,f} \in F_p$
Output: The point $P_{\psi,f}$ on E

$E_q \leftarrow$ `TateCurve`(E) `// Using formulae in [Sil09, §C.14]`
$\phi \leftarrow$ `Isomorphism` $\left(F_p^\times/q^{\mathbb{Z}}, E_q\right)$ `// As in [Sil09, Thm. 14.1]`
$\beta \leftarrow \phi\left(-\lambda_f/L_{\mathrm{alg}}(1,f)\right)$
$H \leftarrow$ `NarrowHilbertClassField`(F)
$E(H) \leftarrow$ `MordellWeilGroup` (E/H)
$L \leftarrow \left[\log_{E_q}((p-1)\beta)\right]$
$\left(n_1, \left(n_g\right)\right) \leftarrow$ `lindep` (L) `// Find integer relation between formal logarithms using LLL`
$\sum_g n_g \cdot g/n_1 \in E(H)$

By linearity, the algorithm works equally well when λ_f comes from ∂f_Q^+, in which case the corresponding Stark–Heegner point is a weighted sum of points $P_{\psi,f}$. The algebraic part of the L-value can be computed either directly in `magma` using the intrinsic `LRatio`, or by using the BSD formula and the invariants of E since $L(s,f) = L(s,E)$, or even analytically by approximating $L(1,E)$ and computing the periods of E.

One limitation of Algorithm 6 is that computing $E(H)$ is very slow when $[H : \mathbb{Q}] > 4$. We hope to resolve this in the future by improving the algorithms for detecting polynomials from p-adic approximations to their roots.

In the table below we list the minimal polynomials of the X and Y coordinates of the Stark–Heegner points coming from ∂f_ψ^+ on the curve $E : y^2 + xy + y = x^3 - x^2 - x - 14$. This is a model for $X_0(17)$, for which we have $L_{\mathrm{alg}}(1,f) = 1/4$, so $\lambda_f = -\frac{1}{4}\log_E\left(P_{\psi,f}\right)$. Here ψ denotes the genus character associated with $\mathbb{Q}\left(\sqrt{D}\right)$: since all the fields $\mathbb{Q}\left(\sqrt{D}\right)$ for $D < 100$ with no fundamental unit of negative norm such that $\left(\frac{D}{17}\right) = -1$ have narrow class number 2, there is a unique nontrivial character. This satisfies $\partial f_\psi^+ = -\partial f_Q^+$, where Q is a quadratic form with class corresponding to the inverse different in Cl^+. Note that this matches the table on p. 545 of [DPV21].

D	X	Y
12	$x^2 - 6x + 10$	$x^2 - 2x + 10$
24	$x^2 + \frac{2}{9}x + \frac{89}{9}$	$x^2 + \frac{230}{27}x + 25$
28	$x^2 - 6x + 10$	$x^2 + 10x + 41$
44	$x^2 - 14x + 338$	$x^2 - 26x + 7394$
56	$x^2 + \frac{2}{9}x + \frac{89}{9}$	$x^2 + \frac{230}{27}x + 25$
57	$x^2 + \frac{2306}{1225}x + \frac{6521}{1225}$	$x^2 + \frac{111042}{42875}x + \frac{15319}{8575}$
88	$x^2 + \frac{2}{9}x + \frac{89}{9}$	$x^2 - \frac{182}{27}x + \frac{401}{9}$
92	$x^2 - 6x + 10$	$x^2 - 2x + 10$

Table 2: Table of Stark–Heegner points on $E : y^2 + xy + y = x^3 - x^2 - x - 14$, for $D < 100$.

6.3 Statistics of Brumer–Stark units

Below we show some tables of minimal polynomials of Brumer–Stark units in different ranges. Full tables are in the author's `github` repository, https://github.com/havarddj/drd.

D	P_D	D	P_D	D	P_D
44	$3x^2+5x+3$	152	$3x^2+2x+3$	236	$27x^2+5x+27$
56	$3x^2+2x+3$	161	$27x^2+38x+27$	248	$27x^2-46x+27$
77	$3x^2+5x+3$	188	$243x^2-298x+243$	284	$2187x^2-4090x+2187$
92	$27x^2+38x+27$	209	$3x^2+5x+3$	305	$9x^4+5x^3+17x^2+5x+9$
140	$81x^4+6x^3-149x^2+6x+81$	221	$9x^4-2x^3-5x^2-2x+9$	329	$243x^2-298x+243$

Table 3: Minimal polynomials of Brumer–Stark units for $p=3$, $D<330$.

D	P_D
2005	$2^{12}x^8+2^4\cdot 1055x^7+2^2\cdot 9419x^6+57995x^5+66831x^4+57995x^3+2^2\cdot 9419x^2+2^4\cdot 1055x+2^{12}$
2013	$2^{30}x^4-2^3\cdot 57677665x^3-1118365527x^2-2^3\cdot 57677665x+2^{30}$
2021	$2^9x^6+2^2\cdot 111x^5+2^1\cdot 123x^4-101x^3+2^1\cdot 123x^2+2^2\cdot 111x+2^9$
2037	$2^{18}x^4+2^3\cdot 16215x^3-263887x^2+2^3\cdot 16215x+2^{18}$
2045	$2^6x^4-9x^3-65x^2-9x+2^6$
2077	$2^3x^2+15x+2^3$
2085	$2^{24}x^4-2^3\cdot 6289393x^3+70333881x^2-2^3\cdot 6289393x+2^{24}$
2093	$2^8x^4-2^1\cdot 217x^3+645x^2-2^1\cdot 217x+2^8$
2101	$2^{13}x^6+2^6\cdot 79x^5-2^3\cdot 1009x^4-10161x^3-2^3\cdot 1009x^2+2^6\cdot 79x+2^{13}$

Table 4: Minimal polynomials of Brumer–Stark units for $p=2$, $2000\le D\le 2101$.

Given the data computed, it is natural to study the "horizontal properties" of Brumer–Stark units, meaning the behaviour of the p-units ε in $\overline{\mathbb{Q}}$ as D varies.

The coefficients of the polynomials are all of roughly the same magnitude, despite the strong conditions on the p-valuation of the constant terms. In particular, the logarithmic height of the middle coefficient is roughly $\operatorname{ord}_p a_0$, which is easily computed in terms of partial ζ-values using Equation (6.1.9). A classical result of Schur says that the coefficients of cyclotomic polynomials can be arbitrarily large. It would be interesting to know whether the same holds for our polynomials, normalised to be monic. The largest value we find is 822.637, across the tables for $p\in\{2,3,5,7,11\}$. Figure 3 shows the absolute value of the middle coefficient of the normalised polynomials against the discriminant for different p.

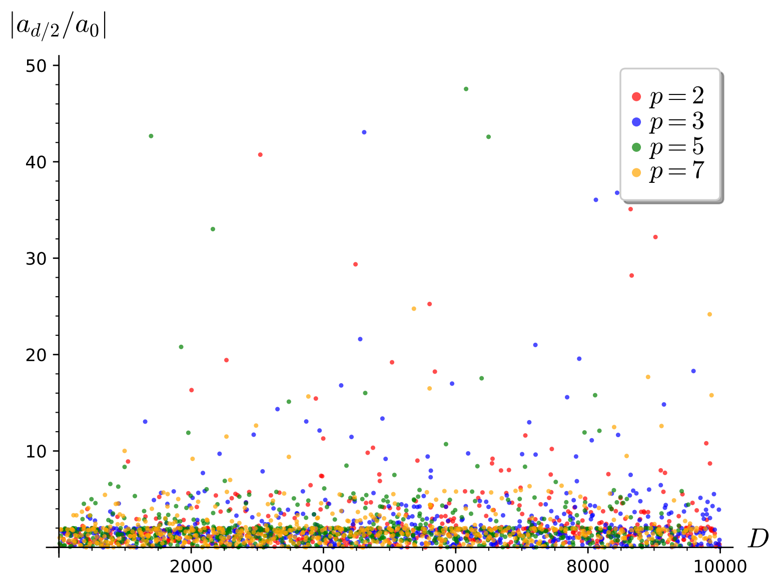

Figure 3: Normalised middle coefficients for various primes p.

If we plot the roots of the minimal polynomials on the unit circle as D varies, it is natural to ask how the Brumer–Stark units distribute. It is well-known that the set of Galois orbits of primitive N-th roots of unity becomes equidistributed with respect to the Haar measure as N tends to infinity. One might expect a similar thing to hold for a sequence of Brumer–Stark units, as the size of the corresponding orbits tends to infinity. A weaker statement is that the Brumer–Stark units, for p fixed, become dense in the unit circle as $D \to \infty$. In statistical experiements, the distribution of our data appears very close to a uniform distribution, both visually and by applying tools like the Kolmogorov–Smirnov and Kuiper tests. We expect that a type of equidistribution result may be inferred from standard circular equidistribution results, for which the current missing ingredient is an asymptotic bound for values of the Rademacher symbol Ψ.

Chapter III. Appendices

Appendix A: A proof of Meyer's theorem

In this section we give a self-contained proof of Meyer's theorem, Theorem 6.3. The proof is not new, but modifies the exposition of [Sie80] using ideas of [Zag75] and [DIT18].

The strategy of the proof is to evaluate the "period" $\int_{z_0}^{\gamma z_0} E_2(z)dz$ in two different ways. Recall from Section 6.1 that we chose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying

$$\varepsilon_F w = aw + b \quad \text{and } \varepsilon_F = cw + d, \tag{1.1}$$

where $\{1, w\}$ is a $\mathbb{Z}$-basis for the lattice $\mathfrak{b}$ with class $[\mathfrak{b}] = A^{-1}$ such that $w > 0$, and ε_F is the totally positive fundamental unit of f. The first manipulation is to apply Stokes' theorem to $E_2(z)dz \sim d \log \eta(z)$ and the transformation law for $\log \eta$. On the other hand, we can also perform a change of variables to map the contour onto a segment in $\mathbb{R}_{>0}$, then decompose this interval as a union of the intervals $(\varepsilon_F^{2n}, \varepsilon_F^{2n-2})$ for $n \in \mathbb{Z}$. After interchanging summation and integration, some straighforward calculations give the special value.

A technical point here is that $\zeta_-(0, A)$ is defined via its functional equation, so to avoid handling divergent sums we need to replace E_2 with the real-analytic Eisenstein series $E(z, s)$. The corresponding identity is called *Hecke's integral formula.* This also complicates the first step, which requires a version of Kronecker's first limit formula.

Recall the definition of the *weight* 0 *real analytic Eisenstein series,*

$$E(z, s) := \sum_{\Gamma_\infty \backslash \Gamma(1)} \mathfrak{I}(\gamma z)^s = \frac{1}{2\zeta(2s)} \sum_{n,m \in \mathbb{Z}}{}' \frac{y^s}{|m + nz|^{2s}}, \quad \mathfrak{R}(s) > 1. \tag{1.2}$$

As before, the symbol $\sum'$ means we omit the indices for which the summand is undefined, in this case $(m, n) = (0, 0)$. We also define the "completion" of $E(z, s)$,

$$E^*(z, s) := \pi^{-s}\Gamma(s)\zeta(2s)E(z, s), \tag{1.3}$$

which satisfies the functional equation $E^*(z, 1 - s) = E^*(z, s)$. This gives a meromorphic continutation in the s-variable to all of $\mathbb{C}$ with poles at $s = 0$ and $s = 1$. Finally, the partial ζ-function $\zeta_-(s, A) := \frac{1}{2}(\zeta(s, A) - \zeta(s, A[\mathfrak{d}]))$, may also be completed with archimedean Euler factors, by $\Lambda_-(s, A) := D^{s/2}\pi^{-s}\Gamma\left(\frac{s+1}{2}\right)^2 \zeta_-(s, A)$.

If we fix a non-zero ideal $\mathfrak{b} \leq \mathcal{O}_K$ with $[\mathfrak{b}] = A^{-1}$, then the map $\mathfrak{a} \mapsto \mathfrak{ab} = (\beta)$ sets up a bijection between integral ideals with class A and principal ideals in $\mathfrak{b}$ generated by a totally positive element. Then we compute

$$\begin{aligned} L(s, A) \operatorname{Nm}(\mathfrak{b})^{-s} &= \sum_{\mathfrak{a} \leq \mathcal{O}_K,\, [\mathfrak{a}] = A} \frac{1}{\operatorname{Nm}(\mathfrak{ab})^s} \\ &= \sum_{(\beta) \leq \mathfrak{b},\, \beta \gg 0} \frac{1}{|\operatorname{Nm}(\beta)|^s} \\ &= \sum_{\beta \in \mathfrak{b}/\pm\varepsilon_F,\, \beta \gg 0} \frac{1}{|\operatorname{Nm}(\beta)|^s} \\ &= \sum_{\substack{\beta \in \mathfrak{b}/\varepsilon_F, \\ \operatorname{Nm}(\beta) > 0}} \frac{1}{|\operatorname{Nm}(\beta)|^s}. \end{aligned} \tag{1.4}$$

A natural representative for $\left(A^*\right)^{-1}$ is $\left(\sqrt{D}\right)\mathfrak{b}$, and proceeding as before we find

$$
\begin{aligned}
L\left(s, A^*\right) &= \mathrm{Nm}\left(\sqrt{D}\right)^s \mathrm{Nm}\left(\mathfrak{b}\right)^{s\prime} \sum_{\substack{\alpha\in\left(\sqrt{D}\right)\mathfrak{b}/\pm\varepsilon_F,\\ \alpha\gg 0}} \frac{1}{|\mathrm{Nm}(\alpha)|^s} \\
&= \mathrm{Nm}\left(\sqrt{D}\right)^s \mathrm{Nm}\left(\mathfrak{b}\right)^s \sum_{\substack{\beta\in\mathfrak{b}/\pm\varepsilon_F,\\ \beta>0>\beta\prime}} \frac{1}{\left|\mathrm{Nm}\left(\sqrt{D}\beta\right)\right|^s} \qquad (1.5)\\
&= \mathrm{Nm}\left(\mathfrak{b}\right)^s \sum_{\substack{\beta\in\mathfrak{b}/\varepsilon_F,\\ \mathrm{Nm}(\beta)<0}} \frac{1}{|\mathrm{Nm}(\beta)|^s}.
\end{aligned}
$$

It follows that

$$
\zeta_-(s, A) = \frac{1}{2}\mathrm{Nm}(\mathfrak{b})^s \sum_{\beta\in\mathfrak{b}/\varepsilon_F} \frac{\operatorname{sgn}\mathrm{Nm}(\beta)}{|\mathrm{Nm}(\beta)|^s}. \qquad (1.6)
$$

Given an odd character $\chi : \mathrm{Cl}_K^+ \to \mathbb{C}^\times$ we have the relations

$$
L(s, \chi) = \sum_{A\in\,\mathrm{Cl}_K^+} \chi(A)\zeta_-(s, A) \qquad (1.7)
$$

and

$$
\zeta_-(s, A) = \frac{1}{2}\sum_\chi \left(\overline{\chi}(A) - \overline{\chi}\left(A^*\right)\right)L(s, \chi) = \sum_{\chi\text{ odd}} \overline{\chi}(A)L(s, \chi), \qquad (1.8)
$$

as for even characters $\chi(A) - \chi\left(A^*\right) = 0$.

Step 1

The proof of Meyer's formula hinges on the following:

Lemma 1.1 (Hecke's integral formula): *We have*

$$
\Lambda_-(s, A) = i\int_{z_0}^{\gamma z_0} \frac{\mathrm{d}}{\mathrm{d}z}E(z, s)\,\mathrm{d}z. \qquad (1.9)
$$

We prove this in a manner similar to [Sie80, §2.45], but with different notation and a different choice of parametrisation of the geodesic.

Proof (of Lemma 1.1):

It is a straightforward, albeit somewhat tedious, exercise to check that

$$
\frac{\mathrm{d}}{\mathrm{d}z}E(z, s) = \Gamma(s)\pi^{-s}\frac{s}{2i}\sum_{m,n} \frac{y^{s-1}}{(m+nz)^2|m+nz|^{2s-2}} \quad \text{for } \Re(s) > 1. \qquad (1.10)
$$

Note that the function $\phi(t) = \frac{t^2 iw + w\prime}{t^2 i + 1}$ maps the interval $(0, \infty)$ onto the geodesic in the upper-half plane connecting w and w'. Let $t_0 = \phi^{-1}(z_0)$, and note that $\phi(\varepsilon_F t) = \gamma\phi(t)$, so $\phi^{-1}(\gamma z_0) = \varepsilon_F t_0$. We will use this to evaluate the integral

$$
\int_{z_0}^{\gamma z_0} \frac{y^{s-1}\,\mathrm{d}z}{(m+nz)^2|m+nz|^{2s-2}}, \qquad (1.11)
$$

when $mn \neq 0$. One easily computes that

$$\phi'(t) = 2ti\frac{w-w'}{(t^2i+1)^2} \quad \text{and } \Im\phi(t) = \frac{t^2(w-w')}{|t^2i+1|^2}. \tag{1.12}$$

Let $\beta = m + nw$, and note that as (m, n) ranges through all pairs of integers, β runs through the elements of $\mathfrak{b}$. Then $m + nz = \frac{t^2 i\beta+\beta\prime}{t^2i+1}$, and so

$$\begin{aligned} (1.11) &= 2i(w-w')^s \int_{t_0}^{\varepsilon_F t_0} \frac{t^{2s}}{(t^2i\beta+\beta')^2|t^2i\beta+\beta'|^{2s-2}}\frac{dt}{t} \\ &= 2i(w-w')^s \int_{t_0}^{\varepsilon_F t_0} \frac{t^{2s}}{(t^2i\beta+\beta')^2(t^4i\beta^2+(\beta')^2)^{s-1}}\frac{dt}{t}. \end{aligned} \tag{1.13}$$

Summing over m, n, or equivalently $\beta \in \mathfrak{b}$, we find

$$\int_{z_0}^{\gamma z_0} \frac{\mathrm{d}}{\mathrm{d}z}E(z,s)dz = \Gamma(s)\pi^{-s}s(w-w')^s \sum_{\beta\in\mathfrak{b}} \int_{t_0}^{\varepsilon_F t_0} \frac{t^{2s}}{(t^2i\beta+\beta')^2(t^4i\beta^2+(\beta')^2)^{s-1}}\frac{\mathrm{d}t}{t} \tag{1.14}$$

Next we consider the integral

$$c(s) := \int_0^\infty \frac{q^s}{(qi+1)^2(q^2+1)^{s-1}}\frac{dq}{q}, \tag{1.15}$$

Integration by parts along with properties of the Beta-function, or alternatively an argument based on the poles and zeroes of c, show that $c(s) = -i\Gamma\left(\frac{s+1}{2}\right)^2/\Gamma(s+1)$. For any $a, b \in \mathbb{R}^\times$, rescaling by $|a|/|b|$ gives

$$\frac{c(s)}{|ab|^s} = \int_0^\infty \frac{q^s}{(|a|qi+|b|)^2(a^2q^2+b^2)^{s-1}}\frac{dq}{q}. \tag{1.16}$$

On the other hand, rescaling q by $\sigma \in \{\pm 1\}$ gives

$$\sigma c(s) = \int_0^\infty \frac{q^s}{(qi+\sigma)^2(q^2+1)^{s-1}}\frac{dq}{q}. \tag{1.17}$$

We put $a = \beta$, $b = \beta'$ and $\sigma = \sigma(\beta) := \operatorname{sgn}(\mathrm{Nm}(\beta))$. Then $(|\beta|qi+|\beta'|)^2 = (\beta qi+\sigma(\beta)\beta')^2$, and we obtain

$$c(s)\sum_{\beta\in\mathfrak{b}/\varepsilon_F} \frac{\sigma(\beta)}{|\mathrm{Nm}(\beta)|^s} = \sum_{\beta\in\mathfrak{b}/\varepsilon_F} \int_0^\infty \frac{q^s}{(\beta qi+\beta')^2(\beta^2q^2+(\beta')^2)^{s-1}}\frac{dq}{q}, \tag{1.18}$$

We can decompose the domain of integration $(0, \infty)$ as $\bigcup_{n\in\mathbb{Z}}\left(t_0^2\varepsilon_F^n, t_0^2\varepsilon_F^{n+2}\right)$, and the change of variables $q \mapsto \varepsilon_F^2 q$ corresponds to replacing β with $\varepsilon_F\beta$. Therefore we can rewrite Equation (1.18) as

$$(1.18) = \sum_{\beta\in\mathfrak{b}} \int_{t_0^2}^{t_0^2\varepsilon_F^2} \frac{q^s}{(\beta qi+\beta')^2(\beta^2q^2+(\beta')^2)^{s-1}}\frac{dq}{q} \tag{1.19}$$

Setting $q = t^2$ and combining with Equation (1.14) gives

$$\begin{aligned}\int_{z_0}^{\gamma z_0} \frac{\mathrm{d}}{\mathrm{d}z} E(z,s) dz &= \frac{\Gamma(s) s (w-w')^s}{\pi^s} \sum_{\beta \in \mathfrak{b}} \int_{t_0}^{\varepsilon_F t_0} \frac{t^{2s}}{(t^2 i\beta + \beta')^2 (t^4 i\beta^2 + (\beta')^2)^{s-1}} \frac{dt}{t} \\ &= \frac{\Gamma(s) s (w-w')^s}{\pi^s} \frac{c(s)}{2} \sum_{\beta \in \mathfrak{b}/\varepsilon_F} \frac{\sigma(\beta)}{|\mathrm{Nm}(\beta)|^s} \\ &= -i\pi^{-s} \Gamma\left(\frac{s+1}{2}\right)^2 \left(\sqrt{D}\right)^s \zeta_-(s,A) = -i\Lambda_-(s,A),\end{aligned} \tag{1.20}$$

since $(w - w') = \mathrm{Nm}(\mathfrak{b})\sqrt{D}$. Here we use the formula for Γ in [Art64, p. 24]: $\Gamma((s+1)/2)^2 = s/2 \cdot \Gamma(s/2)\Gamma((s+1)/2) = s/2 \cdot \Gamma(s) \cdot \sqrt{\pi}/2^{s-1}$. This finishes the proof of Hecke's formula. □

Step 2

Next we compute $\lim_{s\to 0} \int_{z_0}^{\gamma z_0} \frac{\partial}{\partial z} E(z,s)\, \mathrm{d}z$. By the usual argument we can pass the limit inside the integral and derivative, and apply the following identity:

Lemma 1.2: *We have*

$$\lim_{s\to 0} \frac{\partial}{\partial z} E^*(z,s) = -\frac{\partial}{\partial z} \log \eta(z) + \frac{i}{4y}. \tag{1.21}$$

Proof: Since E^* is invariant under $z \mapsto z+1$, it has a Fourier expansion, which according to [Kub73, §2.2] is given by

$$\begin{aligned} E^*(z,s) = \Lambda(2s) y^s &+ \Lambda(2-2s) y^{1-s} \\ &+ 4y^{1/2} \sum_{n=1}^{\infty} n^{1/2-s} \sigma_{2s-1}(n) K_{s-1/2}(2\pi n y) \cos(2\pi n x), \end{aligned} \tag{1.22}$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completion of the Riemann zeta function, $\sigma_{2s-1}(n) := \sum_{d|n} d^{2s-1}$ is the usual divisor sum, and $K_{s-1/2}$ is the modified Bessel function of the second kind. By absolute convergence for s sufficiently large, we can differentiate inside the summation sign, to which end we compute

$$\frac{\partial}{\partial x}\left(y^{1/2} K_{s-1/2}(2\pi n y) \cos(2\pi n x)\right) = -2\pi n y^{1/2} K_{s-1/2}(2\pi n y) \sin(2\pi n x) \tag{1.23}$$

and

$$\begin{aligned} \frac{\partial}{\partial y}\left(y^{1/2} K_{s-1/2}(2\pi n y) \cos(2\pi n x)\right) = \cos(2\pi n x) \Bigg(& \frac{1}{2y^{1/2}} K_{s-1/2}(2\pi n y) \\ & - y^{1/2} 2\pi n \left(K_{-s-1/2}(2\pi n y) + \frac{s-1/2}{2\pi n y} K_{s-1/2}(2\pi n y) \right) \Bigg), \end{aligned} \tag{1.24}$$

since $K_\nu(y) = K_{-\nu}(y)$, so that $\frac{\partial}{\partial y} K_\nu(y) = -K_{\nu-1}(y) - \frac{\nu}{y} K_\nu(y)$ and setting $\nu = 1/2 - s$. Note that when we set $s = 0$ in the expression above, the first and third summand cancel. Therefore, applying

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{1.25}$$

and letting s tend to 0 gives

$$\begin{aligned}\lim_{s\to 0}\frac{\partial}{\partial z}\Big(y^{1/2}K_{s-1/2}(2\pi ny)\cos(2\pi nx)\Big) &= \frac{iy^{1/2}}{2}K_{1/2}(2\pi ny)2\pi ne^{2\pi inx} \\ &= \frac{iy^{1/2}}{2}\sqrt{\frac{1}{4ny}}2\pi ne^{2\pi n(ix-y)} \\ &= \frac{i}{4\sqrt{n}}2\pi nq^n,\end{aligned} \tag{1.26}$$

where in the second equality we use the identity $K_{1/2}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}$. Next, the contribution from the constant term is $\frac{-i}{2}(\Lambda(2s)sy^{s-1} + \Lambda(2-2s)(1-s)y^{-s})$. The function $s \mapsto \Lambda(2s)$ has a simple pole at $s = 0$ with residue $-1/2$, and taking the limit we therefore get $\frac{i}{4y} - \frac{i\Lambda(2)}{2}$. Because $n\sigma_{-1}(n) = \sigma_1(n)$ and $\Lambda(2) = \frac{\pi}{6}$, we find

$$\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}z}E^*(z,s) = \frac{i}{4y} - \frac{\pi i}{12} + 2\pi i\sum_{n=1}^{\infty}\sigma_1(n)q^n. \tag{1.27}$$

On the other hand, it is an easy exercise (see for example [DS06, Prop. 1.2.5]) to show that $\frac{\mathrm{d}}{\mathrm{d}z}\log\eta(z) = \frac{\pi i}{12} - 2\pi i\sum_{n=1}^{\infty}\sigma_1(n)q^n$, which proves our claim. □

Step 3

Note first that the path of integration is a subset of the semicircle

$$\frac{z-w}{z-w'} + \frac{\overline{z}-w}{\overline{z}-w'} = 0 \quad \text{where } z \in \mathfrak{h}, \tag{1.28}$$

so for $p := -(w+w')/2$ and $q := ww'$ we have $z\overline{z} + p(z+\overline{z}) + q = 0$. Thus

$$-2\cdot\frac{i}{4y} = \frac{1}{2iy} = \frac{1}{z-\overline{z}} = \frac{z+p}{z^2+2pz+q} = \frac{\mathrm{d}}{\mathrm{d}z}\sqrt{\log(z-w)(z-w')}. \tag{1.29}$$

We now combine Lemma 1.1 with Lemma 1.2, and use the functional equation of η,

$$\log\eta\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}z\right) - \log\eta(z) = \frac{\pi i}{12}\Phi\begin{pmatrix} a & b \\ c & d\end{pmatrix} + \frac{1}{2}\log(-i\operatorname{sgn}(c)(cz+d)), \tag{1.30}$$

proved in [RG72, §4A], to obtain

$$\begin{aligned}\Lambda_-(0,A) = \pi\zeta_-(0,A) &= i\int_z^{\gamma_A z} -\frac{\mathrm{d}}{\mathrm{d}z}\log\eta(z) + \frac{i}{4y}dz \\ &= i\int_z^{\gamma_A z} -\frac{\mathrm{d}}{\mathrm{d}z}\log\eta(z) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\log\sqrt{(z-w)(z-w')}dz \\ &= \frac{\pi}{12}\Phi(\gamma_A) - \frac{i}{2}\log(-i\operatorname{sgn}(c)(cz+d)) - \frac{i}{2}\log\sqrt{\frac{(\gamma_A z-w)(\gamma_A z-w')}{(z-w)(z-w')}}.\end{aligned} \tag{1.31}$$

By noting that

$$\gamma_A z - w = \frac{z-w}{\varepsilon_F(cz+d)} \quad \text{and } \gamma_A z - w' = \frac{z-w'}{\varepsilon'_F(cz+d)} \tag{1.32}$$

we get that

$$\zeta_-(0,A) = \frac{1}{12}\Phi(\gamma_A) + \frac{1}{2\pi i}\log(-i\operatorname{sgn}(c)(cz+d)) + \frac{1}{2\pi i}\log\sqrt{\frac{1}{(cz+d)^2}}. \tag{1.33}$$

One then verifies that the correct choice of square root is given by $\operatorname{sgn}(a+d)(cz+d)^{-1}$, and so

$$\frac{1}{2\pi i}\log(-i\operatorname{sgn}(c(a+d))) = -\frac{1}{4}\operatorname{sgn}(c(a+d)) = \frac{1}{12}\cdot(-3\operatorname{sgn}c(a+d)), \tag{1.34}$$

and this concludes the proof of Meyer's theorem.

Remark 1.3: It would be interesting to have a "topological" proof of Meyer's theorem along the lines of [BCG20, Theorem 25].

Appendix B: Tables of Gross–Stark units

Table 5: A selection of minimal polynomials of Gross–Stark units for $p = 2$

D	P_D
21	$4x^2 - x + 4$
69	$8x^2 + 7x + 8$
77	$2x^2 + 3x + 2$
93	$8x^2 + 15x + 8$
133	$2x^2 + 3x + 2$
141	$32x^2 - 17x + 32$
165	$64x^4 - 66x^3 + 29x^2 - 66x + 64$
205	$4x^4 - 5x^3 + 7x^2 - 5x + 4$
213	$128x^2 - 185x + 128$
221	$4x^4 + 3x^3 - x^2 + 3x + 4$
237	$32x^2 + 15x + 32$
253	$8x^2 + 7x + 8$
285	$256x^4 + 286x^3 + 141x^2 + 286x + 256$
301	$2x^2 + 3x + 2$
309	$32x^2 + 39x + 32$
341	$8x^2 + 15x + 8$
357	$1024x^4 + 1980x^3 + 2273x^2 + 1980x + 1024$
381	$32x^2 + 63x + 32$
413	$8x^2 - 9x + 8$
429	$4096x^4 + 12190x^3 + 16269x^2 + 12190x + 4096$
437	$8x^2 + 7x + 8$
453	$128x^2 - 105x + 128$
469	$32x^6 - 20x^5 - 18x^4 + 19x^3 - 18x^2 - 20x + 32$
⋮	⋮
1005	$262144x^4 + 296384x^3 + 148569x^2 + 296384x + 262144$
1045	$256x^8 - 776x^7 + 1472x^6 - 2080x^5 + 2281x^4 - 2080x^3 + 1472x^2 - 776x + 256$
1077	$524288x^2 + 864535x + 524288$
1085	$64x^4 + 30x^3 + 101x^2 + 30x + 64$
1101	$8192x^6 + 1980x^5 - 4908x^4 - 10161x^3 - 4908x^2 + 1980x + 8192$
1133	$32x^2 + 39x + 32$
1141	$2x^2 + 3x + 2$
1149	$131072x^2 - 243377x + 131072$
1173	$262144x^4 + 540540x^3 + 769313x^2 + 540540x + 262144$
⋮	⋮
3565	$4096x^4 - 9480x^3 + 12449x^2 - 9480x + 4096$
3597	$274877906944x^4 + 279166820830x^3 + 35058555213x^2 + 279166820830x + 274877906944$
3605	$1024x^4 + 1404x^3 + 769x^2 + 1404x + 1024$
3621	$4398046511104x^4 - 2069928284160x^3 - 608386200559x^2 - 2069928284160x + 4398046511104$
3629	$8192x^2 - 7025x + 8192$
3661	$32x^2 - 57x + 32$
3669	$34359738368x^2 + 47565810487x + 34359738368$
3685	$262144x^4 + 161664x^3 - 199591x^2 + 161664x + 262144$
3693	$134217728x^2 + 130114335x + 134217728$
⋮	⋮

Table 6: A selection of minimal polynomials of Gross–Stark units for $p = 3$

D	P_D
44	$3x^2 + 5x + 3$
56	$3x^2 + 2x + 3$
77	$3x^2 + 5x + 3$
92	$27x^2 + 38x + 27$
140	$81x^4 + 6x^3 - 149x^2 + 6x + 81$
152	$3x^2 + 2x + 3$
161	$27x^2 + 38x + 27$
188	$243x^2 - 298x + 243$
209	$3x^2 + 5x + 3$
221	$9x^4 - 2x^3 - 5x^2 - 2x + 9$
236	$27x^2 + 5x + 27$
248	$27x^2 - 46x + 27$
284	$2187x^2 - 4090x + 2187$
305	$9x^4 + 5x^3 + 17x^2 + 5x + 9$
329	$243x^2 - 298x + 243$
332	$27x^2 + 29x + 27$
341	$27x^2 - 10x + 27$
344	$3x^2 + 2x + 3$
377	$9x^4 - 5x^3 + x^2 - 5x + 9$
380	$6561x^4 + 2556x^3 + 7366x^2 + 2556x + 6561$
413	$27x^2 + 5x + 27$
428	$27x^2 + 53x + 27$
437	$27x^2 + 38x + 27$

D	P_D
440	$81x^4 - 210x^3 + 259x^2 - 210x + 81$
473	$243x^6 + 360x^5 - 240x^4 - 725x^3 - 240x^2 + 360x + 243$
476	$59049x^4 - 60348x^3 + 7222x^2 - 60348x + 59049$
497	$2187x^2 - 4090x + 2187$
⋮	⋮
1001	$59049x^4 - 34830x^3 + 53323x^2 - 34830x + 59049$
1004	$2187x^2 - 3370x + 2187$
1016	$19683x^6 + 33786x^5 - 6099x^4 - 40916x^3 - 6099x^2 + 33786x + 19683$
1052	$1594323x^2 - 3061354x + 1594323$
1064	$729x^4 - 660x^3 - 74x^2 - 660x + 729$
1085	$729x^4 + 1014x^3 + 739x^2 + 1014x + 729$
1112	$27x^2 - 46x + 27$
1121	$27x^2 + 5x + 27$
1133	$243x^2 - 475x + 243$
1148	$43046721x^4 + 74977188x^3 + 83503558x^2 + 74977188x + 43046721$
1169	$177147x^2 - 922x + 177147$
1196	$531441x^4 - 17262x^3 + 457603x^2 - 17262x + 531441$
⋮	⋮
3512	$14348907x^2 + 26595314x + 14348907$
3521	$10460353203x^2 + 19406873942x + 10460353203$
3548	$68630377364883x^2 - 41296506721258x + 68630377364883$
3560	$43046721x^8 - 44891820x^7 + 24573418x^6 + 64530000x^5 - 73694957x^4 + 64530000x^3 + 24573418x^2 - 44891820x + 43046721$
3569	$14348907x^6 - 3493368x^5 - 3900744x^4 + 16070093x^3 - 3900744x^2 - 3493368x + 14348907$
3596	$150094635296999121x^{12} + 274893716979050274x^{11} + 127587088873030941x^{10} - 161570733623574 66x^9 + 15201693671781834x^8 - 268849681252954998x^7 - 530916879720144923x^6 - 268849681252954998x^5 + 15201693671781834x^4 - 16157073362357466x^3 + 127587088873030941x^2 + 274893716979050274x + 150094635296999121$
3605	$59049x^4 - 159201x^3 + 206704x^2 - 159201x + 59049$
3608	$729x^4 - 966x^3 + 1099x^2 - 966x + 729$
3629	$1594323x^2 - 2784490x + 1594323$
3641	$27x^2 - 10x + 27$
3644	$617673396283947x^2 + 1102302840524870x + 617673396283947$
3689	$205891132094649x^4 - 804129173807652x^3 + 1196724390302422x^2 - 804129173807652x + 205891132094649$
3692	$22876792454961x^4 + 61605128740302x^3 + 79528282968163x^2 + 61605128740302x + 22876792454961$
⋮	⋮

Table 7: A selection of minimal polynomials of Gross–Stark units for $p = 7$

D	P_D
517	$16807x^2 + 31922x + 16807$
524	$16807x^2 - 32435x + 16807$
537	$16807x^2 - 10907x + 16807$
545	$49x^4 + 101x^3 + 105x^2 + 101x + 49$
552	$1628413597910449x^4 + 3628866784725770x^3 + 4013798329579011x^2 + 3628866784725770x + 1628413597910449$
556	$343x^2 - 565x + 343$
572	$282475249x^4 + 24266564x^3 - 313639770x^2 + 24266564x + 282475249$
573	$96889010407x^2 + 44342429053x + 96889010407$
584	$49x^4 + 2x^3 - 21x^2 + 2x + 49$
633	$343x^2 + 286x + 343$
636	$2253934029069225808786324 9x^4 - 1201959078537463065681862 0x^3 - 2943405286722736658219802x^2 - 12019590785374630656818620x + 22539340290692258087863249$
649	$343x^2 - 155x + 343$
664	$343x^2 - 61x + 343$
668	$1977326743x^2 - 3946423058x + 1977326743$
689	$5764801x^8 + 10393929x^7 + 10384031x^6 + 8819352x^5 + 6609147x^4 + 8819352x^3 + 10384031x^2 + 10393929x + 5764801$
696	$1628413597910449x^4 - 6046474837894582x^3 + 8869060908086667x^2 - 6046474837894582x + 1628413597910449$

D	P_D
705	$22539340290692258087863249x^4 + 7429856301459255172342372x^3 - 13236353781307558431105498x^2 + 7429856301459255172342372x + 22539340290692258087863249$
712	$49x^4 - 50x^3 + 91x^2 - 50x + 49$
713	$40353607x^2 - 72260210x + 40353607$
717	$4747561509943x^2 + 529727692357x + 4747561509943$
741	$13841287201x^4 + 13420002596x^3 + 655430406x^2 + 13420002596x + 13841287201$
745	$49x^4 + 61x^3 + 25x^2 + 61x + 49$
748	$5764801x^4 - 726474x^3 - 9972325x^2 - 726474x + 5764801$
776	$117649x^4 + 241204x^3 + 252294x^2 + 241204x + 117649$
780	?
789	$96889010407x^2 + 44342429053x + 96889010407$
796	$40353607x^2 - 37493170x + 40353607$
817	$96889010407x^{10} - 105841746360x^9 + 178296252764x^8 - 136629231095x^7 + 122571582182x^6 - 69098758595x^5 + 122571582182x^4 - 136629231095x^3 + 178296252764x^2 - 105841746360x + 96889010407$
824	$16807x^2 - 242x + 16807$
860	$678223072849x^4 - 2184397520956x^3 + 3055123408614x^2 - 2184397520956x + 678223072849$
885	$1628413597910449x^4 + 204791874640430x^3 - 641434815054669x^2 + 204791874640430x + 1628413597910449$
913	$343x^2 - 61x + 343$
957	$1628413597910449x^4 + 2112866467719098x^3 + 3938491639741947x^2 + 2112866467719098x + 1628413597910449$
969	$117649x^4 - 303170x^3 + 383811x^2 - 303170x + 117649$
993	$4747561509943x^6 - 20012739381222x^5 + 39055304380281x^4 - 47510509885652x^3 + 39055304380281x^2 - 20012739381222x + 4747561509943$
888	$191581231380566414401x^4 - 24803008940944453109 8x^3 + 459995653260746005803x^2 - 248030089409444531098x + 191581231380566414401$
892	$11398895185373143x^6 + 38685113123717046x^5 + 53214797136063129x^4 + 51850173291453364x^3 + 53214797136063129x^2 + 38685113123717046x + 11398895185373143$
908	$16807x^2 + 15227x + 16807$
920	$117649x^4 - 293706x^3 + 402739x^2 - 293706x + 117649$
⋮	⋮
1501	$16807x^2 + 32525x + 16807$
1517	$117649x^4 - 173340x^3 + 268198x^2 - 173340x + 117649$
1529	$343x^2 - 565x + 343$
1545	$22539340290692258087863249x^4 - 8117634999172237388809996x^3 - 29366051403531281624524506x^2 - 8117634999172237388809996x + 22539340290692258087863249$
1581	$1628413597910449x^4 + 1402857765914330x^3 + 2341613101067931x^2 + 1402857765914330x + 1628413597910449$
1532	$232630513987207x^2 + 315805512934414x + 232630513987207$
1560	?
1564	$191581231380566414401x^4 + 213828124161526145796x^3 + 82618395064789769606x^2 + 213828124161526145796x + 191581231380566414401$
1580	$79792266297612001x^4 + 306806936641045414x^3 + 454160449942453851x^2 + 306806936641045414x + 79792266297612001$
1592	$40353607x^2 - 37493170x + 40353607$
⋮	⋮

Table 8: A selection of minimal polynomials of Gross–Stark units for $p = 13$

D	P_D
21	$13x^2 - x + 13$
24	$13x^2 - x + 13$
28	$13x^2 - 10x + 13$
33	$13x^2 - x + 13$
44	$13x^2 - 10x + 13$
57	$13x^2 - x + 13$
60	$4826809x^4 + 17830670x^3 + 26104443x^2 + 17830670x + 4826809$

D	P_D
76	$13x^2 - 10x + 13$
93	$2197x^2 + 506x + 2197$
124	$2197x^2 - 4070x + 2197$
136	$169x^4 - 508x^3 + 694x^2 - 508x + 169$
141	$371293x^2 - 141961x + 371293$
161	$2197x^2 + 1082x + 2197$
177	$2197x^2 + 506x + 2197$
184	$2197x^2 + 1082x + 2197$
188	$371293x^2 + 291336x + 371293$
201	$13x^2 - x + 13$
213	$62748517x^2 + 33388991x + 62748517$
236	$2197x^2 - 4070x + 2197$
249	$2197x^2 + 506x + 2197$
253	$2197x^2 + 1082x + 2197$
268	$13x^2 - 10x + 13$
280	$169x^4 - 114x^3 + 59x^2 - 114x + 169$
284	$62748517x^2 + 46322630x + 62748517$
301	$13x^2 - 17x + 13$
305	$169x^4 + 225x^3 + 337x^2 + 225x + 169$
332	$2197x^2 - 4070x + 2197$
344	$13x^2 - 17x + 13$
345	$112455406951957393129x^4 + 11747418581693703766x^3 - 16853755082908573101x^2 + 11747418581693703766x + 112455406951957393129$
357	$137858491849x^4 + 70774656550x^3 - 7011955677x^2 + 70774656550x + 137858491849$
385	$28561x^4 - 40014x^3 + 30131x^2 - 40014x + 28561$
408	$4826809x^4 - 12923950x^3 + 18242043x^2 - 12923950x + 4826809$
437	$2197x^2 + 1082x + 2197$
440	$28561x^4 - 12636x^3 - 24794x^2 - 12636x + 28561$
444	?
453	$62748517x^2 + 33388991x + 62748517$
460	$23298085122481x^4 + 31496391619420x^3 + 39149169555174x^2 + 31496391619420x + 23298085122481$
473	$371293x^6 + 776048x^5 + 35984x^4 - 741025x^3 + 35984x^2 + 776048x + 371293$

D	P_D
476	$137858491849x^4 + 67998405072x^3 - 136363825198x^2 + 67998405072x + 137858491849$
489	$13x^2 - x + 13$
492	$665416609183179841x^4 - 2349559047469661170x^3 + 3370189313691943107x^2 - 2349559047469661170x + 665416609183179841$
501	$1792160394037x^2 + 1471948630151x + 1792160394037$
505	$815730721x^8 - 5053583340x^7 + 14879614126x^6 - 26903413385x^5 + 32533743881x^4 - 26903413385x^3 + 14879614126x^2 - 5053583340x + 815730721$
552	$112455406951957393129x^4 - 304529235540829089934x^3 + 429644727055390799499x^2 - 304529235540829089934x + 112455406951957393129$
553	$371293x^2 + 731210x + 371293$
561	$4826809x^4 + 498550x^3 - 922557x^2 + 498550x + 4826809$
604	$62748517x^2 + 46322630x + 62748517$
609	$112455406951957393129x^4 + 347141196630288638734x^3 + 491744899221812341899x^2 + 347141196630288638734x + 112455406951957393129$
616	$28561x^4 + 72540x^3 + 93734x^2 + 72540x + 28561$
632	$371293x^2 + 731210x + 371293$
645	$3937376385699289x^4 + 3534126734050406x^3 + 4443673329407859x^2 + 3534126734050406x + 3937376385699289$
652	$13x^2 - 10x + 13$
665	$815730721x^4 - 2196600146x^3 + 2880091491x^2 - 2196600146x + 815730721$
668	$1792160394037x^2 - 1295098551624x + 1792160394037$
669	$62748517x^2 + 33388991x + 62748517$
681	$371293x^2 - 141961x + 371293$
696	$112455406951957393129x^4 - 133912034308419949966x^3 + 452464170835129692999x^2 - 133912034308419949966x + 112455406951957393129$
713	$10604499373x^2 + 14401098646x + 10604499373$

D	P_D
717	$51185893014090757x^2 + 55850658624240986x + 51185893014090757$
721	$371293x^2 + 740938x + 371293$
748	$815730721x^4 - 394449380x^3 - 826434426x^2 - 394449380x + 815730721$
749	$2197x^2 + 4273x + 2197$
760	$815730721x^4 - 693663204x^3 + 1311378566x^2 - 693663204x + 815730721$
812	$23298085122481x^4 - 10754121312480x^3 + 2851310014562x^2 - 10754121312480x + 23298085122481$
813	$1792160394037x^2 + 1471948630151x + 1792160394037$
817	$302875106592253x^{10} - 1226780694234784x^9 + 2524596373632892x^8 - 3241424962208251x^7 + 3094472579467074x^6 - 2822219265781367x^5 + 3094472579467074x^4 - 3241424962208251x^3 + 2524596373632892x^2 - 1226780694234784x + 302875106592253$
824	$371293x^2 + 740938x + 371293$
840	?
856	$2197x^2 + 4273x + 2197$
860	$3937376385699289x^4 + 2121981902601288x^3 + 6059964960211127x^2 + 2121981902601288x + 3937376385699289$
869	$371293x^2 + 731210x + 371293$
876	?
889	$371293x^2 - 738742x + 371293$
892	$1461920290375446110677x^6 - 3171414591452020554290x^5 + 2704915147614772002235x^4 - 1924719470449418361980x^3 + 2704915147614772002235x^2 - 3171414591452020554290x + 1461920290375446110677$
905	$815730721x^8 - 1243745867x^7 + 2124262231x^6 - 2631720992x^5 + 3118357939x^4 - 2631720992x^3 + 2124262231x^2 - 1243745867x + 815730721$
908	$371293x^2 + 291336x + 371293$
917	$371293x^2 + 116905x + 371293$
921	$2197x^2 + 506x + 2197$
956	$51185893014090757x^2 + 36189236900246650x + 51185893014090757$

D	P_D
957	$1124554069519573931 29x^4 - 66560296566848515366x^3 + 61020360939834420099x^2 - 66560296566848515366x + 112455406951957393129$
969	$4826809x^4 - 12923950x^3 + 18242043x^2 - 12923950x + 4826809$
973	$2197x^2 - 4255x + 2197$
993	$51185893014090757x^6 + 14775792045508299x^5 + 81257077474802502x^4 - 86121355581336 89x^3 + 81257077474802502x^2 + 14775792045508299x + 51185893014090757$
1016	$10604499373x^6 - 50373516674x^5 + 110013382115x^4 - 140427138524x^3 + 110013382115x^2 - 50373516674x + 10604499373$
1020	?
1032	$137858491849x^4 - 88912161754x^3 + 136705168419x^2 - 88912161754x + 137858491849$
1045	$815730721x^8 - 1557009506x^7 + 908583377x^6 + 1537560830x^5 - 2782729244x^4 + 1537560830x^3 + 908583377x^2 - 1557009506x + 815730721$
1048	$371293x^2 + 116905x + 371293$
1064	$4826809x^4 + 17401540x^3 + 25232406x^2 + 17401540x + 4826809$
1068	?
1077	$1461920290375446110677x^2 + 2861211170816759241913x + 1461920290375446110677$
1081	$51185893014090757x^2 + 96954465195037082x + 51185893014090757$
1084	$1792160394037x^2 - 1295098551624x + 1792160394037$
1085	$4826809x^4 + 5697796x^3 + 2260374x^2 + 5697796x + 4826809$
1112	$2197x^2 - 4255x + 2197$
1113	?
1133	$371293x^2 + 740938x + 371293$
1137	$2197x^2 + 506x + 2197$
1149	$8650415919381337933x^2 - 10525338489559169641x + 8650415919381337933$
1164	?
1177	$2197x^2 + 4273x + 2197$
1185	?

D	P_D
1228	$2197x^2 - 4070x + 2197$
1240	$4826809x^4 + 5804604x^3 + 2551574x^2 + 5804604x + 4826809$
1253	$371293x^2 - 612265x + 371293$
1272	?
1276	$23298085122481x^4 - 5941933482576x^3 - 29790623147614x^2 - 5941933482576x + 23298085122481$
1281	?
1292	?
1293	$247064529073450392704413x^2 - 357172201574584820855926x + 247064529073450392704413$
⋮	⋮

Table 9: A selection of minimal polynomials of Gross–Stark units for $p = 19$

D	P_D
12	$x^2 - x + 1$
21	$19x^2 + 37x + 19$
33	$19x^2 + 37x + 19$
56	$19x^2 + 34x + 19$
60	$47045881x^4 + 11517572x^3 + 80331798x^2 + 11517572x + 47045881$
69	$6859x^2 + 10582x + 6859$
88	$19x^2 + 34x + 19$
105	$47045881x^4 - 52329838x^3 + 95336763x^2 - 52329838x + 47045881$
124	$6859x^2 - 10618x + 6859$
129	$19x^2 + 37x + 19$
136	$361x^4 + 508x^3 + 310x^2 + 508x + 361$
141	$2476099x^2 + 2024677x + 2476099$
165	$47045881x^4 + 11517572x^3 + 80331798x^2 + 11517572x + 47045881$
184	$6859x^2 - 2482x + 6859$
204	$16983563041x^4 + 58364312708x^3 + 83835241158x^2 + 58364312708x + 16983563041$
205	$361x^4 + 1234x^3 + 1771x^2 + 1234x + 361$
217	$6859x^2 - 10618x + 6859$
221	$361x^4 - 751x^3 + 1104x^2 - 751x + 361$
236	$6859x^2 - 443x + 6859$
249	$6859x^2 + 10582x + 6859$
268	$19x^2 - 29x + 19$
280	$361x^4 + 68x^3 + 438x^2 + 68x + 361$
284	$893871739x^2 + 1681323622x + 893871739$
312	$2213314919066161x^4 + 7750717393892942x^3 + 11074850562225603x^2 + 7750717393892942x + 2213314919066161$
316	$322687697779x^6 + 329983416446x^5 + 166529437693x^4 - 2130331612x^3 + 166529437693x^2 + 329983416446x + 322687697779$
341	$6859x^2 - 10618x + 6859$
344	$19x^2 + 34x + 19$
345	$1041273502979112415328 41x^4 + 269313316354909668842066x^3 + 370298516319289803354411x^2 + 269313316354909668842066x + 104127350297911241532841$
357	$6131066257801x^4 - 21130670891902x^3 + 30404785917003x^2 - 21130670891902x + 6131066257801$
364	$130321x^4 + 274550x^3 + 299067x^2 + 274550x + 130321$
376	$2476099x^2 + 3353726x + 2476099$
393	$2476099x^2 + 2024677x + 2476099$
412	$2476099x^2 + 3880586x + 2476099$
413	$6859x^2 - 443x + 6859$
417	$6859x^2 + 10582x + 6859$
428	$6859x^2 - 5051x + 6859$
440	$130321x^4 + 16492x^3 - 70842x^2 + 16492x + 130321$
469	$2476099x^6 - 11834663x^5 + 25354930x^4 - 31990523x^3 + 25354930x^2 - 11834663x + 2476099$
489	$19x^2 + 37x + 19$
497	$893871739x^2 + 1681323622x + 893871739$
508	$2476099x^2 - 3241334x + 2476099$
545	$361x^4 + 661x^3 + 681x^2 + 661x + 361$
553	$2476099x^2 - 806902x + 2476099$
561	$47045881x^4 - 111524452x^3 + 135187158x^2 - 111524452x + 47045881$
572	$6131066257801x^4 - 19556071678588x^3 + 27787600238838x^2 - 19556071678588x + 6131066257801$
573	$42052983462257059x^2 - 83128476573258443x + 42052983462257059$
584	$361x^4 + 1143x^3 + 1600x^2 + 1143x + 361$
597	$322687697779x^2 - 308559680858x + 322687697779$
604	$893871739x^2 + 1787627878x + 893871739$
⋮	⋮

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