Attaching Galois representations to automorphic forms

Contents

1 Lecture 1: $\mathrm{GL}_2/\mathbb{Q}$

The goal is to prove the following theorem:

Theorem 1.1: Let ℓ be prime, $N > 5$, f a weight 2 newform, λ a place of K_f above ℓ. Then there exists a continuous representation $\rho_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{f,\lambda})$, such that
- $\rho_{f,\lambda}$ is unramified away from $N\ell$,
- $\rho_{f,\lambda}(\sigma_p)$ has min poly $X^2 - a_p(f)X$

1.1 The Eichler–Shimura relation

Let $Y_1(N)$ be the open modular curve, a scheme defined over $\mathbb{Z}[1/N]$. As a scheme it admits a finite flat map $Y_1(N) \to \mathbb{A}^1_{\mathbb{Z}[1/N]}$, and we can define $X_1(N)$ as the normalization of $Y_1(N)$. This has good reduction away from N.

We also define $Y_1(N,p)$ to be the cover of $Y_1(N,p)$ with additional data given by $C \subset E$ (for $(E,P) \in Y_1(N)$), and with this define geometric Hecke operators:

$$X_1(N) \leftarrow X_1(N,p) \to X_1(N), \tag{1}$$

Geometrically, this gives rise to endomorphisms T_p^*, $(T_p)_*$ of $\mathrm{Pic}^0\, X_1(N)$, which we can make sense of $\operatorname{mod} p$. This is not obvious because $X_1(N,p)$ is only defined over $\mathbb{Z}[1/Np]$.

We can also define diamond operators $\langle q\rangle$ for $(q,N) = 1$, which are also endomorphisms of $\mathrm{Pic}^0(X_1(N))$. We also define the Frobenius isogeny F in there.

Theorem 1.2 (Eichler–Shimura congruence relation): We have the following identity in $\mathrm{End}_{\mathbb{F}_p}(\mathrm{Pic}^0\, X_1(N))$,

$$F + \langle p\rangle_* F^v = (T_p)_* \tag{2}$$

The idea is to check that the two sides agree on the complement of the supersingular locus.

For $\ell \nmid N$, let $T = T_\ell\left(\mathrm{Pic}^0_{X_1}(N)\right)$ be the ℓ-adic Tate module, and let $V_\ell = T_\ell \otimes \mathbb{Q}_\ell$, and we denote this Galois representation by ρ_ℓ. By the Néron–Ogg–Shafarevich criterion, this is unramified away from $N\ell$. To find $\rho_\ell(\sigma_p)$, we use the diagram

$$\begin{tikzcd}
V_\ell(\operatorname{Pic}^0 X_1 N/\mathbb{Q}) \arrow[d] & V_\ell(\operatorname{Pic}^0 X_1 N/\mathbb{Z}[1/N]) \arrow[l] \arrow[r] & V_\ell\big(\operatorname{Pic}^0 X_1 N/\mathbb{F}_p\big) \arrow[d] \\
H^1_{\text{ét}}\Big(X_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell\Big)^\vee \arrow[rr] & & H^1_{\text{ét}}\Big(X_1(N)_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell\Big)^\vee
\end{tikzcd}$$

and note that $F^2 - \big(T_p\big)_* F + \langle p \rangle_* [p] = 0$ by Eichler–Shimura, since $FF^\vee = [p]$. By some magic, we can replace F with $\rho_\ell\big(\sigma_p\big)$.

To relate this to modular forms, we pass to $\mathbb{C}$ and work with $X^{\text{an}} := X_1(N)^{\text{an}}$. The exponential short exact sequence

$$0 \to 2\pi i \underline{\mathbb{Z}} \to \mathcal{O}_{X^{\text{an}}} \to \mathcal{O}_{X^{\text{an}}} \to 0, \tag{3}$$

gives a long exact sequence with which we identify $\operatorname{Pic}^0(X^{\text{an}}) \cong H^1(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})/H^1(X^{\text{an}}, \underline{\mathbb{Z}})$. We can set up the Hecke correspondences as before.

Apparently there's a discrepancy between $\big(T_p\big)_*$ and T_p^* where the former are used in the previous story, and the latter coincide with the ones related to modular forms. The way to pass between them is to use the Weil pairing on $V_\ell(X^{\text{an}})$, which "swaps the role of the two", apparently.

Now use the universal coefficient theorem to show that $T_\ell\big(\operatorname{Pic}^0 X^{\text{an}}\big) \cong H^1(X^{\text{an}}, \mathbb{Z}) \otimes \mathbb{Z}_\ell$. Using the Hodge decomposition, we can identify this with $H^0\big(X^{\text{an}}, \Omega^1\big) \oplus \overline{H^0(X^{\text{an}}, \Omega^1)}$. Technically, we should use Kodaira–Spencer or something. This is the weight 2 Eichler–Shimura isomorphism.

Maybe this is explained in Ribet–Stein?