Rigid meromorphic cocycles for orthogonal groups

Håvard Damm-Johnsen

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These are some rough notes from a study group at the Max Planck Institute in Bonn, autumn 2024. The topic discussed is rigid meromorphic cocycles for orthogonal groups, closely following the paper [DGL23].

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2 Lecture 2: Orthogonal groups and symmetric spaces

Let $V/\mathbb{Q}$ be a vector space with a non-degenerate symmetric bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}. \tag{2.1}$$

Then $q(v) := \frac{1}{2}\langle v, v \rangle$ is a quadratic form. We define the corresponding orthogonal groups

$$O_V := \{g \in \mathrm{GL}(V) : q(gv) = q(v)\} \quad \text{and} \quad \mathrm{SO}_V := O_V \cap \mathrm{SL}(V). \tag{2.2}$$

We may diagonalize the form q over $\mathbb{R}$, and we say V has (real) signature (r, s) if q is equivalent to

$$\sum_{j=1}^{r} x_j^2 - \sum_{j=1}^{s} x_{r+j}^2 \tag{2.3}$$

over $\mathbb{R}$. We let $n = r + s$ denote the dimension of V.

2.1 Archimedean symmetric spaces

Definition 2.1: The *archimedean symmetric space* of X_∞ is the set of maximal negative definite subspaces of $V_\mathbb{R} := V \otimes \mathbb{R}$.

One can prove that the dimension of X_∞ is $r \cdot s$.

Lemma 2.2: The group $O_V(\mathbb{R})$ acts transitively on X_∞.

Proof: Let z and z' be elements of X_∞, and view them as subspaces of $V_\mathbb{R}$ with the induced quadratic form. Since quadratic spaces over $\mathbb{R}$ are determined by their signature up to isometry, and both have signature $(0, s)$, there exists an isometry $z \to z'$. By Witt's extension theorem this extends to an isometry $V_\mathbb{R} \to V_\mathbb{R}$. □

Fix a point $z_0 \in X_\infty$. The lemma implies that we may identify X_∞ with $O_V(\mathbb{R}) / \operatorname{Stab}_{O_V(\mathbb{R})} z_0$.

Example 2.3:
- Suppose V has signature $(r, 0)$. Then X_∞ is simply a point.
- Suppose V has signature $(r, 1)$. Over $\mathbb{R}$, q is equivalent to $q_\mathbb{R}(x) := x_1^2 + \dots + x_r^2 - x_{r+1}^2$. If $q_\mathbb{R}(x) < 0$, then $x_1^2 + \dots + x_r^2 < x_{r+1}^2$. Since we are interested in the line spanned by x, we may rescale so that $x_{r+1} = 1$. Then the line corresponds to a unique point $(x_1, \dots, x_r) \in \mathbb{R}^r$ with

  $$x_1^2 + \dots + x_r^2 < 1. \tag{2.1}$$

  This implies that X_∞ can be identified with the unit ball in $\mathbb{R}^r$. Note that the topology is not the subspace topology, but rather the hyperbolic topology.

Example 2.4: Let V be of signature $(r, 2)$. For any field $K/\mathbb{Q}$, we define the *quadric of isotropic lines over K* to be

$$\mathcal{Q}(K) := \{v \in V_K - \{0\} : q_K(v) = 0\}/K^\times. \tag{2.2}$$

This is a closed subvariety of $\mathbb{P}^1(V)$. We now define

$$\tilde{X}_\infty := \{[v] \in \mathcal{Q}(\mathbb{C}) : \langle v, \overline{v} \rangle < 0\}, \tag{2.3}$$

which is an open subset of $\mathcal{Q}(\mathbb{C})$. The involution $x \mapsto \overline{x}$ exchanges the two connected components of $\tilde{X}_\infty$. Given a line $[v] \in \tilde{X}_\infty$, write $v = v_1 + iv_2$. Then one can check that $q_\mathbb{R}(v_1) = q_\mathbb{R}(v_2) = 0$, so $\mathbb{R}v_1 + \mathbb{R}v_2 \in X_\infty$. This gives a 2-to-1 cover $\tilde{X}_\infty \to X_\infty$. In particular, this gives X_∞ the structure of a complex manifold. This is specific to the signature $(r, 2)$ setting; in general signature there is no complex structure on X_∞.

We can also define

$$\tilde{X}'_\infty := \{[v] \in \mathcal{Q}(\mathbb{C}) : \langle v, w\rangle \neq 0 \text{ for all } [w] \in \mathcal{Q}(\mathbb{R})\}. \tag{2.4}$$

This natural contains $\tilde{X}_\infty$.

Exercise 2.5: Show that $\tilde{X}'_\infty = \tilde{X}_\infty$ unless $r = 2$.

2.2 p-adic symmetric spaces

In this section we will assume $n \geq 3$, and fix $p \geq 3$. We define $\mathbb{C}_p$ to be the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$.

Suppose $V_{\mathbb{Q}_p}$ contains a self-dual $\mathbb{Z}_p$-lattice Λ. Then q induces a non-degenerate $\mathbb{F}_p$-valued pairing on $\Lambda/p\Lambda$. By the Chevalley–Warning theorem, this form has a zero, which lifts to an isotropic vector in Λ by Hensel's lemma. It follows that $\mathcal{Q}(\mathbb{Q}_p)$ is non-empty. Inspired by the definition of $\tilde{X}'_\infty$, we have the following:

Definition 2.6: The *p-adic symmetric space of O_V* is

$$X_p := \left\{[v] \in \mathcal{Q}\left(\mathbb{C}_p\right) : \langle v, w\rangle \neq 0 \text{ for all } w \in \mathcal{Q}\left(\mathbb{Q}_p\right)\right\}. \tag{2.1}$$

Proposition 2.7: *The space X_p carries the structure of a rigid analytic variety.*

Proof: For any line $[w] \in \mathcal{Q}\left(\mathbb{Q}_p\right)$, we may find $w' \in \Lambda' := \Lambda - p\Lambda$ such that $[w] = [w']$. Similarly, $[v] \in \mathcal{Q}\left(\mathbb{C}_p\right)$, let v' be a corresponding vector in $\Lambda_{\mathbb{C}_p} - \mathfrak{m}_{\mathbb{C}_p}\Lambda_{\mathbb{C}_p}$. We extend the valuation on $\mathbb{Q}_p$ to $\mathbb{C}_p$, and so for $k \in \mathbb{N}$ the set

$$X_{p,\Lambda}^{\leq k} := \left\{v \in \mathcal{Q}\left(\mathbb{C}_p\right) : \operatorname{ord}_p\langle v', w'\rangle \leq k \text{ for all } [w] \in \mathcal{Q}\left(\mathbb{Q}_p\right)\right\} \tag{2.2}$$

is well-defined. Then $X_p = \bigcup_k X_{p,\Lambda}^{\leq k}$, and one can show that $X_p^{\leq k}$ is an affinoid open. □

Note that while the choice of basic affinoids depends on Λ, the space X_p itself is independent.

Example 2.8: Suppose V has real signature $(1, 2)$. We claim that

$$X_p \cong \mathfrak{h}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p). \tag{2.3}$$

3 Lecture 3: Special cycles

In this lecture, the goal is to construct certain divisors on X_∞ and X_p. We start with X_∞.

3.1 Archimedean divisors

Definition 3.1: Fix $v \in V_{\mathbb{R}}$ with $q(v) > 0$, and define

$$\Delta_{v,\infty} := \{z \in X_\infty : z \subset v^\perp\}, \tag{3.1}$$

where $v^\perp$ is the orthogonal complement of the span of v in V.

This can be identified with the symmetric space of the orthogonal group of $v^\perp \subset V_{\mathbb{R}}$.

Remark 3.1.1:

(i) Define $\mathcal{T} := \{(w,z) \in V_{\mathbb{R}} \times X_\infty : w \in z\}$. *This is naturally a vector bundle over* X_∞ *via projection onto the second factor. Taking* $\mathrm{pr}_z : V \to z$ *to be the orthogonal projection, we obtain a section* $s_v : X_\infty \to \mathcal{T}$ *given by* $z \mapsto (\mathrm{pr}_z(v), z)$. *Then* $\Delta_{v,\infty}$ *is the preimage of the* 0 *-section* $(0,z)$ *under* v.

(ii) The action of $G(\mathbb{R})$ *lifts to* $\mathcal{T}$, *and using this it is easy to verify that* $g \cdot \Delta_{v,\infty} = \Delta_{gv,\infty}$.

3.2 p-adic divisors

We now turn to the divisors on X_p.

Definition 3.1: Let $v \in V_{\mathbb{Q}_p}$ be a vector with $q(v) \neq 0$, i.e. v is anisotropic. By analogy with the archimedean setting, we define

$$\Delta_{v,p} := \{\xi \in X_p : \xi \subset v^\perp\}, \tag{3.1}$$

called a *special divisor* on X_p.

Then for any $g \in G(\mathbb{Q}_p)$ we have $g \cdot \Delta_{v,p} = \Delta_{gv,p}$. Recall that a *hyperbolic plane* is a 2-dimensional quadratic space with quadratic form $q(x,y) = x \cdot y$. A *hyperbolic space* $\mathbb{H}$ is a direct sum of hyperbolic planes.

Example 3.2: Suppose $V_{\mathbb{Q}_p} \cong \mathbb{Q}_p v \cdot \mathbb{H}$. Then $\Delta_{v,p}$ is trivial. More generally, if V is a quadratic space with

$$q(x) = x_1^2 + x_2^2 + x_3^2, \tag{3.2}$$

then $\Delta_{v,p}$ is trivial if and only if $q(v)$ is a square in $\mathbb{Q}_p^\times$.

Recall that we fixed a self-dual lattice Λ in $V_{\mathbb{Q}_p}$. To understand the intersections of $\Delta_{v,p}$ with the basic affinoids $X_{p,\Lambda}^{\leq k}$, we first relate v and Λ.

Definition 3.3: Let $v \in V_{\mathbb{Q}_p}$. Then we define the *order of v with respect to* Λ to be

$$\mathrm{ord}_\Lambda(v) := \sup\left\{\ell \in \mathbb{Z} : \frac{v}{p^\ell} \in \Lambda\right\} \in \mathbb{Z} \cup \{\infty\}. \tag{3.3}$$

We also define the *isotropy level*

$$\mathrm{iso}_\Lambda(v) := \mathrm{ord}_p\big(q(v)\big) - 2\,\mathrm{ord}_\Lambda(v). \tag{3.4}$$

In other words, $\mathrm{iso}_\Lambda(v) = \mathrm{ord}_p\big(q(v_0)\big)$ if $v = p^\ell v_0$ with $v_0 \in \Lambda' := \Lambda - p\Lambda$.

Lemma 3.4: Fix an anisotropic vector $v \in V_{\mathbb{Q}_p}$, and let $k_v = \mathrm{iso}_\Lambda(v)$. Then:
(i) for any $\varepsilon > 0$, the intersection $\Delta_{v,p} \cap X_{p,\Lambda}^{k_v - \varepsilon}$ is empty.
(ii) If $v^\perp$ is not a hyperbolic space, then

$$\Delta_{v,p} \cap X_{p,\Lambda}^{\leq \lceil 3k_v/2 \rceil} \neq \emptyset. \tag{3.5}$$

[TODO: Insert drawing]

Corollary 3.5: *Fix $m \in \mathbb{Q}_p^\times$ and $k > 0$. If $v \in V_{\mathbb{Q}_p}$ with $q(v) = m$ such that $\Delta_{v,p} \cap X^{\leq k}$, then $v \in p^{-\ell}\Lambda$ for $\ell \leq \frac{1}{2}\big(k - \mathrm{ord}_p(m)\big)$.*

Proof: TODO: fill in □

3.3 Locally finite divisors

In this subsection, we combine the two above constructions. Fix a $\mathbb{Z}[1/p]$-lattice L in V. Let Γ be a subgroup of SO_V which stabilises Λ. Such a group is called a p-arithmetic *p-arithmetic subgroup of* SO_V. This is a discrete subgroup of $\mathrm{SO}_V(\mathbb{R}) \times \mathrm{SO}_V\big(\mathbb{Q}_p\big)$.

The construction of mixed divisors relies on the following set of data:

(i) a compact subset $C \subset X_\infty$,
(ii) a finite subset $S \subset \mathbb{Q}_{>0}$,
(iii) if V^+ denotes the set of positive vectors, a set of integers $(a_v) \in \mathbb{Z}^{V^+}$ satisfying:
  - $a_{\gamma v} = a_v$ for all $\gamma \in \Gamma$,
  - $a_v = 0$ if $\Delta_{v,\infty} \cap C = \emptyset$ or $q(v) \notin S$.

Definition 3.6: The formal sum

$$\Delta := \sum_{v \in V^+} a_v \cdot \Delta_{v,p} \tag{3.1}$$

is called a *locally rational finite quadratic divisor* in X_p.

Note that for any basic affinoid $\mathscr{A}$,

$$\Delta \cap \mathscr{A} := \sum_{\substack{v \in V^+ \\ \Delta_{v,p} \cap \mathscr{A} \neq \emptyset}} a_v \Delta_{v,p} \tag{3.2}$$

is a finite formal sum. Indeed, Corollary 3.5, the set

$$\left\{ v \in V^+ : \Delta_{v,\infty} \cap C, q(v) \in S \text{ and } \Delta_{v,p} \cap X_{p,\Lambda}^{\leq k} \right\} \tag{3.3}$$

is both compact and discrete, hence finite.

4 Appendix: Non-split orthogonal groups

We first consider the situation over $\mathbb{R}$. Let (V, q) be a quadratic space over $\mathbb{R}$ of signature (r, s), with $n = r + s$. After diagonalizing, we can assume q has the shape

$$q(x) = \sum_{j=1}^{r} x_j^2 - \sum_{j=1}^{s} x_{r+j}^2. \tag{4.4}$$

The orthogonal group of V is then naturally identified with

$$O(m, n) := \left\{ g \in \mathrm{GL}_n(\mathbb{R}) : g \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} g^t = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} \right\}, \tag{4.5}$$

where I_r is the $r \times r$ identity matrix. Similarly, $\mathrm{SO}(m, n) := O(m, n) \cap \mathrm{SL}_n(\mathbb{R})$.

Proposition 4.1: *Suppose $r, s > 0$. Then the group $\mathrm{SO}(r, s)$ is not connected.*

Proof: The map

$$\mathrm{SO}(r,s) \to \mathbb{R}^\times \text{ given by } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \det(A), \tag{4.6}$$

where the blocks are as in the previous definition, is surjective and continuous, so $\mathrm{SO}(r,s)$ has at least two connected components. □

It turns out that $\mathrm{SO}(r,s)$ has two connected components, and we denote the component of the identity by $\mathrm{SO}^+(r,s)$. We then have the following table of groups and their maximal compact subgroups:[1]

G	$O(r,s)$	$\mathrm{SO}(r,s)$	$\mathrm{SO}^+(r,s)$
K	$O(r) \times O(S)$	$S(O(r) \times O(s))$	$\mathrm{SO}(r) \times \mathrm{SO}(s)$

We can decompose V as $V = V^+ \oplus V^-$, where V^+ is the span of $x_1, ..., x_r$, and V^- is the span of the remaining basis vectors. Then $\mathrm{SO}^+(r,s)$ is precisely the subgroup of $\mathrm{SO}(r,s)$ which preserves the individual orientations on V^+ and V^-.

Example 4.2: In this example, we consider $\mathrm{SO}(2,2)$, which can be described explicitly through its exceptional isogeny with $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. The first hint that such an isogeny might exist comes from the Dynkin diagrams: $\mathfrak{so}_4$ has Dynkin diagram D_2, which is simply two dots (and no lines). The Dynkin diagram of each $\mathfrak{sl}_2$ is a single dot. Of course, I think this can be made precise using Satake–Tits diagrams.

More concretely, fix the *vector space* $V = \mathrm{Mat}_2(\mathbb{R})$. One can check that det defines a quadratic form on V. Furthermore, since

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \frac{1}{4}\left((a+d)^2 - (a-d)^2 + (b-c)^2 + (b+c)^2\right), \tag{4.7}$$

the space V has signature $(2,2)$. An orthogonal basis for V is given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad X_+ = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}. \tag{4.8}$$

Then I and X_+ span V^+, and h and X_- span V^-.

Note that $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ acts naturally on V via $(g_1, g_2) \cdot v = g_1 v g_2^{-1}$. This gives an embedding $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_4(\mathbb{R})$, but I don't think it's the natural one. Let's find the elements which preserve det: if $\det(g_1 v g_2^{-1}) = \det(v)$ for all $v \in V$, then evidently $\det(g_1) = \det(g_2)$, and vice versa. Therefore, we obtain a map

[1] A reference for this is https://www.math.toronto.edu/mein/teaching/LieClifford/cl12.pdf.

$$\tilde{G} := \mathrm{GL}_2(\mathbb{R}) \times_{\det} \mathrm{GL}_2(\mathbb{R}) \to O_V \cong O(2,2). \tag{4.9}$$

Note that the diagonal matrices act trivially, so the kernel of this map is $\mathbb{R}^\times$. The map is not surjective; consider the map sending I to $-I$ and preserving the other basis vectors. This is certainly an orthogonal transformation, but it is straightforward to check by bashing matrix multiplication this is not encoded by an element of $\tilde{G}$.

We claim that the image of $\tilde{G}$ is in fact $\mathrm{SO}(2,2)$. One way to see this is by noting that both groups have two connected components, and then showing that the induced maps on Lie algebras is surjective. In other words, we have a short exact sequence

$$1 \to \mathbb{R}^\times \xrightarrow{\Delta} \mathrm{GL}_2(\mathbb{R}) \times_{\det} \mathrm{GL}_2(\mathbb{R}) \to \mathrm{SO}(2,2) \to 1. \tag{4.10}$$

In fact, this proves that $\tilde{G} = \mathrm{GSpin}_V$. It also gives an explicit description of $\mathrm{SO}^+(2,2)$; it is isomorphic to

$$\mathrm{GL}_2\,(\mathbb{R})^+ \times_{\det} \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times. \tag{4.11}$$

Example 4.3: In this example we find a convenient model for $\mathrm{SO}(3,1)$. The punchline is that it is naturally isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, viewed as a real Lie group. Intuitively, we can think $\mathrm{SL}_2(\mathbb{C})$ as a form of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, so this is compatible with the previous example. NB: I will take a slightly different model from the one in the paper! This seems simpler to me, but there might be a good reason why they chose the other one.

For $X \in \mathrm{Mat}_2(\mathbb{C})$, let $X^\dagger := \overline{X}^t$, the conjugate transpose. We set

$$V = \{X \in \mathrm{Mat}_2(\mathbb{C}) : X^\dagger = X\}. \tag{4.12}$$

Explicitly, it consists of matrices

$$X = \begin{pmatrix} a & x+iy \\ x-iy & d \end{pmatrix}, \tag{4.13}$$

so

$$\det(X) = ad - x^2 - y^2 = \frac{1}{4}\left((a+d)^2 - (a-d)^2\right) - x^2 - y^2. \tag{4.14}$$

From this we see that a convenient basis consists of I, h and X_-, as in the previous example, and $i \cdot X_+$. Moreover, the form $q(X) = -\det(X)$ has real signature $(3,1)$.

We act by $\mathrm{SL}_2(\mathbb{C})$ in a similar way as before: $g \cdot X \coloneqq gXg^\dagger$. Note that $\left(gXg^\dagger\right)^\dagger = gX^\dagger g^\dagger = gXg^\dagger$, so the action on V is well-defined.

Bibliography

[DGL23] H. Darmon, L. Gehrmann, and M. Lipnowski, "Rigid Meromorphic Cocycles for Orthogonal Groups." Accessed: Aug. 29, 2023. [Online]. Available: http://arxiv.org/abs/2308.14433