Automorphic representations study group

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Contents

1 Algebraic groups and adeles

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1.1 Adeles

Definition 1.1. **Global fields** are finite extensions of $\mathbb{Q}$ or $\mathbb{F}_q(x)$, that is, number fields or function fields.

Definition 1.2. A valuation on a field F is a map $v : F \to \mathbb{R} \sqcup \{\infty\}$ satisfying for all $a, b \in F$,

(i) $v(a) = \infty$ if and only if $a = 0$,

(ii) $v(ab) = v(a) + v(b)$,

(iii) $v(a + b) \geq \min(v(a), v(b))$.

Definition 1.3. An **absolute value** is a function $|\cdot| : F \to \mathbb{R}$ satisfying the usual axioms (see [Getz, def. 1.2], for example). If $0 < \alpha < 1$ and v is any valuation, then $|a|_v := \alpha^{v(a)}$ defines an absolute value on F.

Definition 1.4. Two absolute values are **equivalent** if they induce the same topology; a **place** is an equivalence class of absolute values.

Places correpsonding to non-archmedean absolute values are called **finite**, and the others **infinite**.

Proposition 1.5. *Let F be a global field.*

(i) *If F is a function field, then all places are finite.*

(ii) *If F is a number field, then the infinite places are in bijection with embeddings $F \hookrightarrow \mathbb{C}$ modulo conjugation, and finite places in bijection with prime ideals of $\mathcal{O}_F$. Explicitly, this is given by*

$$\iota : F \hookrightarrow \mathbb{C} \quad \text{goes to} \quad |x| := |\iota(x)|^{[\iota(F)\otimes\mathbb{R}:\mathbb{R}]}, \tag{1.1}$$

and

$$\mathfrak{p} \leq \mathcal{O}_F \quad \text{goes to} \quad |x|_{\mathfrak{p}} := q^{-v_{\mathfrak{p}}(x)} \text{ where } q = \#\mathcal{O}_F/\varpi\mathcal{O}_F \tag{1.2}$$

and $v_{\mathfrak{p}}(x) = \max\{x \in \mathbb{N} : x \in \varpi^n \mathcal{O}_F\}$.

We define completions in the usual way, as equivalence classes of Cauchy sequences with respect to the absolute value.

Definition 1.6. Let F be a global field. We define the **adeles over** F, $\mathbb{A}_F := \prod'_v F_v$, where $\prod'$ denotes the restricted product,

$$\mathbb{A}_F = \{(x_v)_v \in \prod_v F_v : x_n \in \mathcal{O}_{F_v} \text{ for almost all } v\}. \tag{1.3}$$

If v is infinite, we adopt the convention $\mathcal{O}_{F_v} = F_v$. The adeles $\mathbb{A}_F$ has a natural topology generated by fixing a finite set of places S, and for each $v \in S$ fixing $U_v \subset F_v$ open and taking $U = \prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_{F_v}$.

Proposition 1.7. *$\mathbb{A}_F$ is a locally compact Hausdorff topological ring.*

The diagonal image of F in $\mathbb{A}_F$ is discrete.

Definition 1.8. Let S be a finite set of places. Then

$$\mathbb{A}_F^S := \prod_{v \notin S}' F_v \quad \text{and} \quad \mathbb{A}_{F,S} := \prod_{v \in S} F_v. \tag{1.4}$$

We also set $F_\infty = \prod_{v | \infty} F_v$.

Proposition 1.9 (Approximation for adeles)**.** *We have a decomposition $\mathbb{A}_F = F_\infty + \prod_{v \nmid \infty} \mathcal{O}_{F_v} + F$, where we identify F with its diagonally embedded image.*

1.2 Algebraic groups

We are interested in studying algebraic groups like GL_n, SL_n, SO_n etc, which can all be viewed as locally closed subschemes of $\mathrm{Mat}_n \cong \mathbb{A}^{n^2}$ (affine n^2-space, not to be confused with the adeles.)

Example 1.10. We can realise the set $\mathrm{GL}_n(R)$ as the subset of $\mathbb{A}^{n^2+1} = \operatorname{Spec} A$ for $A = R[x_{11}, x_{12} \dots, x_{nn}, y]$ given by $\operatorname{Spec} A/(\det\left(x_{ij}\right)y - 1)$.

Definition 1.11. An **affine group scheme** is a functor $G : \mathbf{Alg}_F \to \mathbf{Grp}$ represented by an F-algebra, denoted $\mathcal{O}(G)$.

The goal is to use algebrogeometric methods to study matrix groups. A morphism of two affine group schemes is given by a natural transformation of functors, and so we have a category of affine group schemes over F, $\mathbf{AffGrpSch}_F$.

Remark 1.12. We define a morphism $H \to G$ to be injective if $\mathcal{O}(G) \to \mathcal{O}(H)$ is surjective. If F is a field, then this is equivalent to every induced map on F-algebras being injective, but not if F is any ring.

Definition 1.13. G is **linear** if there exists a faithful representation $G \hookrightarrow \mathrm{GL}_n$ for some n.

Definition 1.14. Suppose $F \hookrightarrow F'$ is a field embedding, and G a group scheme over F. Then we define the extension of scalars of G to F' by $G_{F'}(R) := G(R)$.

We can go back as well:

Definition 1.15. $\operatorname{Res}_F^{F'} G(R) := G(R \otimes_F F')$ is called the restriction of scalars.

If F'/F is finite and locally free (as an extension of rings), then the restriction is also linear when G is.

Definition 1.16. An affine algebraic group is a group scheme over F represented by a finitely generated F-algebra.

Proposition 1.17. *Let F be a topological field. Then there is a natural topology on $G(F)$ so that $G(F) \to X(F)$ is continuous for all schemes X/F. This is compatible with imeersions, fibre products etc.*

The following shows that we really only need to care about subgroups of GL_n.

Proposition 1.18. *If G is an algebraic group, then it is linear.*

An element $x \in \mathrm{Mat}_n(\overline{F})$ is *semisimple* if it is diagonalisable over F, *nilpotent* if $x^m = 0$ for some $m \in \mathbb{N}$, and *unipotent* if $x - 1$ is nilpotent.

Similarly, say $x \in G(\overline{F})$ is semisimple (nilpotent, unipotent) if $\phi(x)$ is semisimple (nilpotent, unipotent) for some faithful representation $\phi : G \to \mathrm{GL}_n$. One can check that this does not depend on ϕ.

Theorem 1.19 (Jordan decomposition). *If $x \in G(\overline{F})$, then there exist $x_s, x_u, \in G(\overline{F})$ where x_s is semisimple and x_u is unipotent such that $x = x_s x_u = x_u x_s$.*

Definition 1.20. The **Lie algebra** of G, $\mathrm{Lie}\, G$, is the kernel of the map

$$G(F[x]/x^2) \to G(F). \tag{1.5}$$

Example 1.21. Let $G = \mathrm{GL}_n$. Then we can find a bijection between $\mathrm{Lie}\, G$ and Mat_n by noting that $(1+\epsilon A)(1-\epsilon A) = 1$, where A is any matrix.

We define a bracket on $\mathrm{Lie}\, \mathrm{GL}_n$ by $[X, Y] := XY - YX$, and use this to get brackets on all other linear algebraic groups; note that $\mathrm{Lie}\, G \hookrightarrow \mathrm{Lie}\, \mathrm{GL}_n$.

There is natural action of G on $\mathrm{Lie}\, G$ via conjugation, giving a map $G \to \mathrm{GL}_n(\mathrm{Lie}\, G)$. This is called the **adjoint action**.

We also need the usual algebraic groups $\mathbb{G}_a(A) := (A, +)$ and $\mathbb{G}_m(A) := (A^\times, \times)$.

Definition 1.22. An algebraic group T is called a **torus** if $T_{F^{\mathrm{sep}}} \cong \mathbb{G}_m^r$ for some $r \in \mathbb{N}$, which is called the **rank** of T.

If $T \cong \mathbb{G}_m^r$ without passing to F^{sep}, then T is said to be *split*.

Definition 1.23. A character is an element of $X^*(G) := \mathrm{Hom}(G, \mathbb{G}_m)$.

If $G = T$ is a split torus, then $X^*(T) \cong \mathbb{Z}^r$, but in general it can be smaller. If $X^*(T) = \{0\}$, then T is called **anisotropic**. There is a decomposition $T = T^{\mathrm{anis}} T^{\mathrm{split}}$, where their intersection is finite.

Definition 1.24. The **unipotent radical** of G, $R_u(G)$ is the maximal connected (as scheme) unipotent (all elements are unipotent) normal (closed) subgroup of G.

The radical of a group H is the maximal connected normal solvable subgroup H.

Definition 1.25. If $R(G) = \{1\}$ then G is **semisimple**; if $R_u(G) = \{1\}$, then G is **reductive**.

Note that $R_u(G) \subset R(G)$ so semisimple implies reductive.

Remark 1.26. We are glossing over some details on *smoothness*, which won't be covered here.

Definition 1.27. A **Borel subgroup of** G is a subgroup B such that $B_{F^{\mathrm{sep}}} \subset G_{F^{\mathrm{sep}}}$ is maximal, connected and solvable.

These are nice because G/B is always represented by a projective scheme, and B is minimal with respect to this property.

Definition 1.28. A subgroup P of G is **parabolic** if it contains a Borel subgroup of G, so that G/P is also projective.

Definition 1.29. A torus $T \subset G$ is a **maximal torus** if $T_{F^{\mathrm{sep}}}$ is maximal with respect to inclusion.

Example 1.30. $G = \mathrm{GL}_n$, T = diagonal matrices; this forms a split maximal torus.

Proposition 1.31. *Reductive groups have maximal torii.*

Definition 1.32. We say G is split if a maximal torus is split. If G has a Borel subgroup, then it is quasi-split.

Example 1.33. GL_n has Borel subgroup given by upper triangular (or lower triangular) matrices.

Proposition 1.34 (Levi decomposition). *If $P \subset G$ is a parabolic subgroup, then $P = MN$ where $N = R_u(P)$ and $M \leq P$ is a reductive subgroup.*

2 Automorphic representations over non-Archimedean fields