TALK 6: Automorphic representations.

2/23/23

G = connective reductive group $/F = \#$ fld , Σ = places of F, $\Sigma_\infty = \{v \in \Sigma : v \mid \infty\}$

<u>Recall</u>: $L^2_\psi(G(F)\backslash G(\mathbb{A}_F))$ = admissible $(\mathfrak{g}, K) \times G(\mathbb{A}_F^\infty)$ – module

(Harish-Chandra)

> <u>Def.</u> An <u>automorphic representation of $G(\mathbb{A}_F)$</u> is an admissible $(\mathfrak{g}, K) \times G(\mathbb{A}_F^\infty)$ – module isomorphic to a subquotient of $L^2_\psi(G(F)\backslash G(\mathbb{A}_F))$.

<u>Goals.</u> (1) Admissible $G(\mathbb{A}_f^\infty)$ - modules & alternative definition.

(2) Flath's factorization theorem $\pi = \pi_\infty \times \bigotimes'_{v\nmid\infty} \pi_v$.

(3) Multiplicity.

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(1) Hecke algebra definition of aut. reps.

Assume G = affine gp scheme of f.t. $/\mathcal{O}_F$ s.t. G_F is a reductive gp.

E.g. $GL_{n,\mathbb{Z}} \rightsquigarrow GL_{n,\mathbb{Q}}$.

(1.1) Non-archimedean Hecke algebras.

$\mathbb{A}_F^\infty$ = finite adeles : $\mathbb{A}_F^S = \prod'_{v\notin S} F_v$, $F_S = \prod_{v\in S} F_v$, $S \subseteq$ non-arch. places

> <u>Def.</u> $C_c^\infty(G(\mathbb{A}_F^S)) = \{$ loc. constant fcn. on $G(\mathbb{A}_F^S)\}$
>
> $C_c^\infty(G(F_S)) = \{$ loc. constant fcn. on $G(F_S)\} =: \mathcal{H}_S$
>
> $\mathcal{H}^\infty := C_c^\infty(G(\mathbb{A}_F^\infty))$ w/ convolution $f * h(g) = \int_{G(\mathbb{A}_F)} f(x)h(x^{-1}g)\,dx$
>
> <u>Hecke algebra away from ∞</u>

Note: $C_c^\infty(G(\mathbb{A}_F^\infty)) = \varinjlim_{S\subseteq\{\text{NA places}\}} C_c^\infty(G(F_S)) \otimes \bigotimes_{v\notin S\cup\infty} \mathbb{1}_{G(\mathcal{O}_v)}$

"restricted tensor product"

Def. (π, V) = rep. of H^∞ is <u>admissible</u> if non-degenerate and $\forall K^\infty \leq G(\mathbb{A}_F^\infty)$ cpct open, $V^{K^\infty} := \pi(\mathbb{1}_{K^\infty}) V$ is finite-dim'l] "smooth"

M = A-module <u>non-degenerate</u> if $\forall m \in M$ $\quad m = a_1 m_1 + \cdots + a_k m_k$

(e.g. if $1 \in A$, then $m = 1 \cdot m$, but $1 \notin H^\infty$).

(Recall ...)

Key Lemma. G = locally cpct disconnected gp $\quad (G = G(F_v),\ v \nmid \infty)$

$$\left\{ \begin{matrix} \text{smooth reps} \\ \text{of } G \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{non-degenerate} \\ C_c^\infty(G)\text{-modules} \end{matrix} \right\}$$

$$(\pi, V) \longmapsto \pi(f)(v) := \mu(K) \cdot \sum_{i=1}^{m} c_i \pi(g_i) \cdot v$$

$$\exists K \text{ s.t. } v \in V^K \ \&\ f = \sum c_i \mathbb{1}_{g_i K}$$

Def. (π, V) = rep. of $G(\mathbb{A}_F^\infty)$ is <u>admissible</u> if the associated H^∞-rep. is admissible.

(1.2) Archimedean Hecke algebra.

$G := (R_{F/\mathbb{Q}} G_F)_\mathbb{R}$ real red. alg. gp $/\mathbb{R}$

$\rightsquigarrow K_\infty \leq G(\mathbb{R})$ max'l cpct

Def. • <u>Hecke algebra at ∞</u>: $\quad H_\infty := H(G(\mathbb{R}), K_\infty)$

convolution alg. of distributions on $G(\mathbb{R})$ supported on K_∞.

• $\sigma : K_\infty \longrightarrow \mathrm{Aut}(V)$ rep. of dim $d(\sigma) < \infty$, χ_σ = char., dk_∞ Haar measure

$\rightsquigarrow$ <u>fundamental idempotent</u> $\mathbb{1}_\sigma = \dfrac{1}{d(\sigma)\, dk_\infty(K_\infty)} \chi_\sigma \, dk_\infty \in H_\infty$

• A cts rep. π of $G(\mathbb{R})$ on Hilbert space V is <u>admissible</u> if $\forall$ irrep. σ of K_∞, $\pi(\mathbb{1}_\sigma) V$ is finite-dimensional.

(Equivalent to James's definition.)

(1.3) Global Hecke algebra. $\mathcal{H} := \mathcal{H}_\infty \otimes \mathcal{H}^\infty$

A rep. (π, V) of $\mathcal{H}$ decomp. as $\underset{\mathcal{H}_\infty\text{-rep.}}{(\pi_\infty, V_\infty)} \boxtimes \underset{\mathcal{H}^\infty\text{-rep.}}{(\pi^\infty \otimes V^\infty)}$

$\rightsquigarrow$ (π, V) admissible if both (π_∞, V_∞) & (π^∞, V^∞) admissible

The space $L^2(G(F)\backslash G(\mathbb{A}_F))$ has a natural $\mathcal{H}$-action:

$$(f * \phi)(g) = \int_{G(\mathbb{A}_F)} \phi(gh) f(h) \, dh \quad \text{for } f \in \mathcal{H}, \ \phi \in L^2.$$

(assume Z_G is anisotropic).

Def. An automorphic rep. of $G(\mathbb{A}_F)$ is an adm. rep. of $\mathcal{H}$ which is isom. to subquotient of $L^2(G(F)\backslash G(\mathbb{A}_F))$.

(Equivalent definition.)

(2) Flath's Factorization Theorem.

Goal. π = aut. rep. of $G(\mathbb{A}_F)$ $\Longrightarrow$

$\pi = \pi_\infty \otimes \pi^\infty$ and $\pi^\infty \cong \bigotimes'_{v \nmid \infty} \pi_v$, π_v = irrep. of $G(F_v)$.

Definitions:

- Restricted direct product of vector spaces:

I = countable index set $\supseteq I_0$ finite subset

$\rightsquigarrow$ $\{W_v : v \in I\}$ $\mathbb{C}$-vector spaces & $\forall v \in I \setminus I_0$, $\phi_v \in W_v \setminus \{0\}$ fixed vector.

$$\rightsquigarrow \quad W := \bigotimes_v' W_v := \{(w_v)_{v \in I} : w_v = \phi_v \text{ for a.a. } v \in I\}$$

$$= \varinjlim_S W_S \quad \text{for } W_S = \bigotimes_{v \in S} W_v,$$

$S \subseteq S' \rightsquigarrow W_S \to W_{S'}$

$\bigotimes_{v \in S} w_v \mapsto \bigotimes_{v \in S} w_v \otimes \bigotimes_{v \in S' \setminus S} \phi_v$

Note: $\mathrm{Hom}(W, W) = \{(f_v : W_v \to W_v)_{v \in I} : f_v(\phi_v) = \phi_v \text{ for a.a. } v \in I\}$

• Restricted direct product of algebras:

$$\{A_v : v \in I\} \ \mathbb{C}\text{-algebras} \ \& \ \forall v \in I \setminus I_0,\ e_v \in A_v \text{ idempotent}$$

$$\rightsquigarrow A := \bigotimes_v{}' A_v = \{(a_v)_{v\in I} : a_v = e_v \text{ for a.a. } v \in I\}$$

$$= \varinjlim A_S \quad \text{for } A_S = \bigotimes_{v\in S} A_v, \quad \begin{array}{l} S \subseteq S' \rightsquigarrow A_S \to A_{S'} \\ \bigotimes_{v\in S} a_v \mapsto \bigotimes_{v\in S} a_v \otimes \bigotimes_{v\in S'\setminus S} e_v \end{array}$$

Examples. (1) $I = \mathbb{N}$, $A_i := \mathbb{C}[X_i] \ni a_v := 1$

$$\Rightarrow \bigotimes_i{}' \mathbb{C}[X_i] = \mathbb{C}[X_1, X_2, \dots]$$

(2) $I = \Sigma \setminus \Sigma_\infty \rightsquigarrow C_c^\infty(G(\mathbb{A}_F^\infty)) \cong \bigotimes{}' C_c^\infty(G(F_v))$ w.r.t. $e_{K_v} = \frac{1}{\mathrm{vol}(K_v)} \mathbb{1}_{K_v}$

i.e. $\mathcal{H}^\infty \cong \bigotimes_v{}' \mathcal{H}_v$ $\qquad K_v \subseteq G(F_v)$ hyperspecial subgroup

Further, for $I = \Sigma \rightsquigarrow \mathcal{H} \cong \bigotimes_v{}' \mathcal{H}_v$.

• Actions. If $W_v = A_v$-module $\forall v \in I$ & $e_v \phi_v = \phi_v$ for a.a. $v \in I$

$\Rightarrow W := \bigotimes_v{}' W_v$ is an $A := \bigotimes_v{}' A_v$-module.

Note: As an A-module, the isom. class on W does not change under $\phi_v \mapsto \lambda_v \phi_v$.

Def. Let $A = \bigotimes_v{}' A_v$. An A-module W is factorizable if $W \cong \bigotimes_v{}' W_v$.

We consider $I := \Sigma \setminus \Sigma_\infty$

$$A := \mathcal{H}^\infty \cong \bigotimes_v{}' \mathcal{H}_v.$$

Suppose for a.a. v, $\dim W_v^{K_v} = 1$ & choose $\phi_v \in W_v^{K_v}$ "spherical vector"

& note that $e_{K_v} \phi_v = \phi_v$.

Note: $\bigotimes_v{}' W_v$ as an $\mathcal{H}^\infty$-module well-defined up to isom. b/c $\dim W_v^{K_v} = 1$.

Thm (Flath). W = adm. irred. $\mathcal{H}^\infty$-rep. $\Rightarrow$ W is factorizable

Step 1. "Weak version"

Thm 1. G_1, G_2, $G := G_1 \times G_2$

(1) V_i = adm. irrep. of G_i $\Rightarrow$ $V_1 \boxtimes V_2$ = adm. irrep. of G

(2) V = adm. irrep of G $\Rightarrow$ $\exists V_i$ adm. irrep. of G_i s.t. $V \cong V_1 \boxtimes V_2$ & isom. class of V_i is det. by V.

Pf. (1) We use:

Irreducibility criterion: V = smooth G-rep. irred. $\Leftrightarrow$ V^K = irred. $C_c^\infty(G/\!/K)$-module $\forall K \leq G$ open cpct

$\forall K_1 \times K_2 \leqslant G_1 \times G_2$ open cpct,

$$\underbrace{V_1^{K_1} \boxtimes V_2^{K_2}}_{\cong (V_1 \boxtimes V_2)^{K_1 \times K_2}} = \text{irred. } \underbrace{C_c^\infty(G_1/\!/K_1) \times C_c^\infty(G_2/\!/K_2)}_{\cong C_c^\infty(G_1 \times G_2/K_1 \times K_2)} \text{-module}$$

$\Rightarrow$ $V_1 \boxtimes V_2$ admissible & irreducible.

(2) W = adm. G-module, $K = K_1 \times K_2$ s.t. $W^K \neq 0$

$\rightsquigarrow \dim W^K < \infty$ $\Rightarrow$ $\exists W_i(K_i) = C_c^\infty(G_i/\!/K_i)$-modules s.t.

$W^K \xrightarrow{\cong} W_1(K_1) \boxtimes W_2(K_2)$ as $C_c^\infty(G/\!/K)$-mod.

Set: $W_1 := \varinjlim_{K_1} W_1(K_1)$, $W_2 := \varinjlim_{K_2} W_2(K_2)$. $\square$

Step 2. Theory of Gelfand pairs implies ...

Theorem 2. G = conn. red. gp $/F$ unram. outside S

$\rightsquigarrow$ $\forall v \notin S$ finite $K_v \leq G(F_v)$ hyperspecial subgp

$$\rightsquigarrow K^S := \prod_{v \notin S} K_v \leq G(\mathbb{A}_F^S)$$

If W^S = irred. adm. $G(\mathbb{A}_F^S)$-rep., then $\dim W^{K^S} = 1$.

Def. $H \leq G$ closed subgp $\rightsquigarrow$ (G,H) Gelfand pair if

$\forall V$ adm. irrep. of G $\dim_H(V,\mathbb{C}) \cdot \dim_H(V^\vee,\mathbb{C}) \leq 1$.

Examples. • $(G\times G, G)$ Gelfand pair $\Leftrightarrow$ Schur's lemma for G

• (G,K), K open cpct, Gelfand pair $\Leftrightarrow$ $C_c^\infty(G/\!\!/K)$ commutative.

• True for $K_v \leq G(F_v)$ hyperspecial.

Assume this & prove Flath's decomposition theorem.

W = irred. adm. rep. W of $C_c^\infty(G(\mathbb{A}_F^\infty))$ is factorizable, i.e.

$C_c^\infty(G(\mathbb{A}_F^\infty)) \cong \bigotimes_v' C_c^\infty(G(F_v))$ & $W \cong \bigotimes_v' W_v$ w.r.t. $\phi_v \in W_v^{K_v}$, $\dim W_v^{K_v} = 1$.

Let $A_S := C_c^\infty(G(F_S^\infty)) \otimes e_{K^S} \leq C_c^\infty(G(\mathbb{A}_F^\infty))$ subalgebra

$\Rightarrow$ $W^{K^S} = A_S$-rep. and

• Thm 1 $\Rightarrow$ $W^{K^S} \cong \bigotimes_{v\in S} W_v \otimes W^S$

• Thm 2 $\Rightarrow$ $\dim W^S = 1$.

By admissibility: $W = \varinjlim_S W^{K^S} \cong \varinjlim_S \bigotimes_{v\in S} W_v \otimes W^S$

$$C_c^\infty(G(\mathbb{A}_F^\infty)) \cong \varinjlim_S A_S$$

$\Rightarrow$ above isom. is equivariant. $\square$

(3) Automorphic multiplicity. Take $\pi_v = G(F_v)$-rep $\forall v \rightsquigarrow \pi := \bigotimes_v' \pi_v$. When is this rep. aut.? How many copies of it can you embed into L^2-space?

Def. $\varphi \in L^2(G(F)\backslash G(\mathbb{A}))$ cuspidal if $\forall P = MN$ parabolic

$$\int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\, dn = 0.$$

$\rightsquigarrow L^2_{cusp}(G(F)\backslash G(\mathbb{A}))$ cuspidal subspace.

Def. Suppose π = admissible irrep. of $G(\mathbb{A})$.

- The multiplicity of π is $m(\pi) := \dim \operatorname{Hom}_{G(\mathbb{A})}(\pi, L^2_{cusp}(G(F)\backslash G(\mathbb{A})))$.
- π equivalent to π' ($\pi \sim \pi'$) if $\pi' \cong \pi$ as $G(\mathbb{A})$-reps

  $\Rightarrow [\pi]_\sim := \{\pi' \sim \pi : \pi' \text{ automorphic}\}$ equivalence class

  & $m(\pi) = \#[\pi]_\sim$.
- π nearly equivalent to π' ($\pi \approx \pi'$) if $\pi'_v \cong \pi_v$ for a.a. v

  $\Rightarrow [\pi']_\approx := \{\pi' \approx \pi : \pi' \text{ automorphic}\}$ near-equiv. class.

Thm. (Piatetski-Shapiro). $G = GL_n$, π = aut. rep. of $GL_n(\mathbb{A}_F)$

- Multiplicity one: $m(\pi) = 1$ i.e. $[\pi]_\sim = \{\pi\}$
- Strong mult. one: $[\pi]_\approx = \{\pi\}$.

Note: modular form det'd by a_p for a.a. p.

Failure of strong mult. one for $G = GSp_4$ (Howe-Piatetski-Shapiro).

$$\begin{array}{llcll} GL_2 \times GL_2/\mathbb{G}_m & \pi_1 \otimes \pi_2 & \xrightarrow{\theta\text{-lift}} & \pi & GSp_4 \\ & \big\updownarrow \text{Jacquet-Langlands} & & & \\ D^\times \times D^\times/\mathbb{G}_m & \pi_1^{D^\times} \otimes \pi_2^{D^\times} & \xrightarrow[\theta\text{-lift}]{} & \pi_D & GSp_4 \end{array}$$

D = quat. algebra

Fact. $\pi_v \cong (\pi_D)_v$

$\Leftrightarrow D_v \cong GL_{2,F_v}$.

Goal of global Langlands program. Classify aut. reps of $G(\mathbb{A})$ in terms of parameters $\varphi : G_{\mathbb{Q}} \to {}^L G$.

Arthur multiplicity formula:

conjectural description of near-equivalence classes (of unitary reps) of automorphic reps & their multiplicities in aut. spectrum, including non-tempered representations.