Étale cohomology seminar: towards the Weil conjectures

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The Weil Conjectures

Recall the set up for the Weil conjectures: X is a smooth projective variety over a finite field $k = \mathbb{F}_q$. Define its zeta function as

$$Z(X,t) := \exp\left(\sum_{n>0} \#X(\mathbb{F}_{q^n})t^n/n\right).$$

Weil conjectures: $Z(X,t)$ is rational, satisfies a functional equation, relation with Betti numbers in $\mathbf{C}$, Riemann hypothesis. Rationality and the functional equation follow easily from a well behaved cohomology theory with coefficients in a field of characteristic 0. Grothendieck et al invented ℓ-adic cohomology, with coefficients in $\mathbb{Q}_\ell$.

We will prove Lefschetz trace formula:

Proposition

Let X be a smooth projective variety over $k = k^{sep}$, and $\phi : X \to X$ be an endomorphism. Then for $\ell \neq p$

$$(\Gamma_\phi \cdot \Delta) = \sum_{r=0}^{2d} (-1)^r \mathrm{Tr}(\phi^* | H^r(X, \mathbb{Q}_l))$$

Applying it for ϕ the Frobenius map, since $\Gamma_{\phi^n} \cdot \Delta = \#X(\mathbb{F}_{q^n})$, this gives $Z(X,t) \in \mathbb{Q}_l(t)$ (and thus $Z(X,t) \in \mathbb{Q}(t)$).
The trace formula is a purely topological result, once we have done the hard work of proving that ℓ-adic cohomology is well behaved.

Motivation for ℓ-adic cohomology

Étale cohomology only behaves well using torsion sheaves, e.g.

$$H^1(\mathrm{Spec}(k), \mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Z}) = 0,$$

since $\mathbb{Z}$ has no nontrivial finite subgroup. Similarly

$$H^1(\mathrm{Spec}(k), \mathbb{Z}_\ell) = H^1(\mathrm{Spec}(k), \varprojlim \mathbb{Z}/\ell^n) = \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Z}_\ell) = 0,$$

with $\mathbb{Z}_\ell$ having the discrete topology. Instead we should define $H^r(X, \mathbb{Z}_\ell) = \varprojlim H^r(X, \mathbb{Z}/\ell^n)$. Then

$$H^1(\mathrm{Spec}(k), \mathbb{Z}_\ell) = \varprojlim \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Z}/\ell^n) = \mathrm{Hom}_{\mathrm{cont}}(G_k, \mathbb{Z}_\ell),$$

but now $\mathbb{Z}_\ell$ has its standard topology as a profinite group.

Historical motivation

Let C/K be a curve, with $K \subset \mathbb{C}$. Using Kummer sequence $0 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ we proved that $H^1(\overline{C}, \mu_{\ell^n}) \cong \mathrm{Pic}(\overline{C})[\ell^n]$ and that $H^1(\overline{C}, \mu_{\ell^n})$ is naturally dual to $H^1(\overline{C}, \mathbb{Z}/\ell^n)$, where $\overline{C} = C \times \overline{K}$. Taking inverse limits

$$H^1(\overline{C}, \mathbb{Z}_\ell) \cong (\varprojlim \mathrm{Pic}(\overline{C})[\ell]^m)^v = T_\ell(J(C))^v$$

This is in fact an isomorphism of Galois representations: the isomorphism respects the action of G_K. Historically, the rank of the Tate module was known to be the same as the rank of $H^1(C \otimes \mathbb{C}, \mathbb{Z})$. We would hope to obtain similar ℓ-adic representations in higher dimensions.

ℓ-adic sheaves

A $\mathbb{Z}_\ell$-module M can be seen as a projective system $(M_n \in \mathbb{Z}/\ell^n - \mathrm{Mod})$ satisfying $M_{n+1}/\ell^n M_{n+1} \cong M_n$: given M take $M_n = M/\ell^n M$. Similarly:

Definition

Let ℓ be a prime number, a ℓ-adic sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is a projective system $(\mathcal{F}_n)_{n\geq 0}$ with $\mathcal{F}_n$ a $\mathbb{Z}/\ell^n$-sheaf such that that for all n

$$\mathcal{F}_{n+1}/\ell^n \mathcal{F}_{n+1} \cong \mathcal{F}_n.$$

We define $H^r(X, \mathcal{F}) := \varprojlim H^r(X, \mathcal{F}_n)$ (same for $H^r_c(X, -)$)

We say that $\mathcal{F}$ is constructible/locally constant if each $\mathcal{F}_n$. Lisse=locally constant + constructible. Also,

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') := \varprojlim_m \varinjlim_n \mathrm{Hom}(\mathcal{F}_n, \mathcal{F}'_m) = \varprojlim_m \mathrm{Hom}(\mathcal{F}_m, \mathcal{F}'_m)$$

Recall the Tate twist (for ℓ invertible in X)

$$\mathbb{Z}/\ell(r) = \begin{cases} \mu_\ell^{\otimes r} \text{ for } r > 0 \\ \mathbb{Z}/\ell \text{ for } r = 0 \\ \mathrm{Hom}(\mathbb{Z}/\ell(-r), \mathbb{Z}/\ell) \ r < 0 \end{cases}$$

Then we can define the ℓ-adic sheaf $\mathbb{Z}_\ell(r) = (\mathbb{Z}/\ell^n(r))_n$, and more generally $\mathcal{F}(r) = (\mathcal{F}(r))_n$ for a $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$. As in the abelian sheaf case, if X is a variety over k separably closed

$$H^r(X, \mathcal{F}(r)) \cong H^r(X, \mathcal{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(r)$$

Sheaves of $\mathbb{Z}/\ell^n$-modules are also ℓ-adic by taking $\mathcal{F}_n = \mathcal{F}_{n+1}...$ to be annihilated by ℓ^{n+1}.

Sheaves of $\mathbb{Q}_\ell$-modules

Definition

The category of (constructible) $\mathbb{Q}_\ell$-sheaves is defined as the quotient of all (constructible) $\mathbb{Z}_\ell$-sheaves by the subcategory of torsion sheaves (those killed by a power of ℓ). That is, the map

$$\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is bijective, and

$$\mathrm{Hom}(\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \mathcal{F}' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) := \mathrm{Hom}(\mathcal{F}, \mathcal{F}') \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

For a sheaf of $\mathbb{Q}_\ell$-modules $\mathcal{F}$ define

$$H^r(X, \mathcal{F}) = (\varprojlim H^r(X, \mathcal{F}_n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad H^r(X, \mathbb{Q}_\ell) = H^r(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

A $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ is flat if for all n, s, the sequence

$$0 \to \mathcal{F}_s \to \mathcal{F}_{n+s} \to \mathcal{F}_n \to 0$$

obtained from tensoring $0 \to \mathbb{Z}/\ell^s \to \mathbb{Z}/\ell^{n+s} \to \mathbb{Z}/\ell^n \to 0$ by $\mathcal{F}_{n+s}$ is exact.

Lemma

Let $\mathcal{F}$ be a flat $\mathbb{Z}_\ell$-sheaf such that $H^r(X, \mathcal{F}_n)$ is finite for all r, n. Then $H^r(X, \mathcal{F})$ is a finitely generated $\mathbb{Z}_\ell$-module and there is a SES

$$0 \to H^r(X, \mathcal{F})/\ell^n H^r(X, \mathcal{F}) \to H^r(X, \mathcal{F}_n) \to H^{r+1}(X, \mathcal{F})[\ell^n] \to 0$$

Proof

Fact: inverse limits of exact sequences of finite abelian groups are exact.

We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{s+1} \arrow[r] \arrow[d] & \mathcal{F}_{n+s+1} \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}_s \arrow[r] & \mathcal{F}_{n+s} \arrow[r] & \mathcal{F}_n \arrow[r] & 0
\end{tikzcd}$$

Taking cohomology and then inverse limits we obtain the SES

$$\to H^r(X, \mathcal{F}) \xrightarrow{\ell^n} H^r(X, \mathcal{F}) \to H^r(X, \mathcal{F}_n) \to H^{r+1}(X, \mathcal{F}) \to$$

which gives

$$0 \to H^r(X, \mathcal{F})/\ell^n H^r(X, \mathcal{F}) \to H^r(X, \mathcal{F}_n) \to H^{r+1}(X, \mathcal{F})[\ell^n] \to 0$$

Taking inverse limits again, and noting that $\varprojlim H^{r+1}(X, \mathcal{F})[\ell^n] = 0$, we have $\varprojlim H^r(X, \mathcal{F})/\ell^n \cong H^r(X, \mathcal{F})$, so as a $\mathbb{Z}_\ell$-module $H^r(X, \mathcal{F})$ is generated by generators of $H^r(X, \mathcal{F})/\ell$.

One consequence of the above, is that for a smooth projective variety X over $k = k^{\mathrm{sep}}$, all cohomology groups $H^r(X, \mathbb{Q}_l)$ are finite dimensional vector spaces, using finiteness results on $H^*(X, \mathbb{Z}/n)$. Thus we can define ℓ-adic Betti numbers

$$\beta_r(X, \ell) = \dim_{\mathbb{Q}_\ell} H^r(X, \mathbb{Q}_\ell)$$

and ℓ-adic Euler characteristic

$$\chi(X, \ell) = \sum_{r=0}^{2d} (-1)^r \beta_r(X, \ell).$$

As a consequence of the Lefschetz formula $\chi(X, \ell)$ is independent of ℓ and equal to $(\Delta \cdot \Delta)$ (for $\phi = \mathrm{id}$).

Properties of $H^r(-, \mathbb{Q}_\ell)$

$H^*(-, \mathbb{Q}_\ell)$ is a functor from the category of smooth projective varieties over k separably closed (char $k = p$) to the category of graded-commutative $\mathbb{Q}_\ell$-algebras (for $\ell \neq p$). It satisfies the axioms of a Weil cohomology theory:

- Each $H^r(X, \mathbb{Q}_\ell)$ is a finite dimensional $\mathbb{Q}_\ell$-vector space, and $H^r(X, \mathbb{Q}_\ell) = 0$ for $r > 2d$.
- For each closed irreducible subvariety $Z \subset X$ of codimension c there is a cohomology class $\mathrm{cl}_X(Z) \in H^{2c}(X, \mathbb{Q}_\ell)$ satisfying

$$\mathrm{cl}_X(Z \cdot W) = \mathrm{cl}_X(Z) \cup \mathrm{cl}_X(W).$$

- (Kunneth Formula) The map $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ sending $a \otimes b$ to $\pi_1^*(a) \cup \pi_2^*(b)$ is an isomorphism of graded rings.

Properties of $H^r(-,\mathbb{Q}_\ell)$

- (Poincaré duality) There is a perfect bilinear pairing

$$H^r(X) \times H^{2d-r}(X) \xrightarrow{\cup} H^{2d}(X,\mathbb{Q}_\ell) \xrightarrow{\eta} \mathbb{Q}_\ell$$

  and η (trace map) is an isomorphism. Moreover $\eta \circ \mathrm{cl}_X(P) = 1$ for any closed point P.

Most are easy consequences of the theorems we have proved. E.g. we have

$$\begin{array}{ccc} H^{2d}(X,\mathbb{Z}/\ell^{n+1}(d)) & \xrightarrow{\cong} & \mathbb{Z}/\ell^{n+1} \\ \downarrow & & \downarrow \\ H^{2d}(X,\mathbb{Z}/\ell^{n}(d)) & \xrightarrow{\cong} & \mathbb{Z}/\ell^{n} \end{array}$$

Taking inverse limits $H^{2d}(X,\mathbb{Z}_\ell(d)) \xrightarrow{\cong} \mathbb{Z}_\ell$, since both are finite abelian groups. As before we identify $\mathbb{Z}_\ell(1)$ and $\mathbb{Z}_\ell$.

Pushforward in cohomology

Let $\phi : X \to Y$ be a morphism of varieties ($d = \dim X$, $s = \dim Y$), it induces a map

$$\phi^* : H^{2d-r}(Y, \mathbb{Q}_\ell) \to H^r(X, \phi^* \mathbb{Q}_\ell) = H^{2d-r}(X, \mathbb{Q}_\ell)$$

by functoriality. Poincaré duality gives a pushforward map

$$\phi_* : H^r(X, \mathbb{Q}_\ell) \to H^{2s-2d+r}(Y, \mathbb{Q}_\ell)$$

satisfying $\eta_Y(\phi_*(x) \cup y) = \eta_X(x \cup \phi^*(y))$ for all $x \in H^r(X)$, $y \in H^{2d-r}(Y)$. It is easy to see that this pushforward is a covariant functor.

We will use the following properties of ϕ_*:

1. If $\phi : Z \to X$ is a closed immersion, then $\phi_* : H^r(Z, \mathbb{Q}_\ell) \to H^{r-2c}(X)$ is the Gysin map.
2. For ϕ proper $\phi_*(x \cup \phi^*(y)) = \phi_*(x) \cup y$: for any other y'

$$\begin{aligned}\eta_Y(\phi_*(x \cup \phi^*(y)) \cup y') &= \eta_X(x \cup \phi^*(y) \cup \phi^*(y')) \\ &= \eta_X(x \cup \phi^*(y \cup y')) = \eta_Y(\phi_*(x) \cup y \cup y')\end{aligned}$$

3. For $\pi_1, \pi_2 : X \times X \to X$ and $\alpha \in H^r(X)$

$$(\pi_1)_*(\pi_2)^*(\alpha) = \eta_X(\alpha)$$

if $r = 2d$, and 0 otherwise $((\pi_1)_* : H^r(X \times X) \to H^{r-2d}(X))$. For $\beta \in H^{2d}(X)$

$$\eta_X((\pi_1)_*(\pi_2)^*(\alpha) \cup \beta) = \eta_{X \times X}((\pi_2)^*\alpha \cup (\pi_1)^*\beta) = \eta_X(\beta)\eta_X(\alpha)$$

Lefschetz trace formula

Proposition

Let X be a smooth projective variety over $k = k^{sep}$, and $\phi : X \to X$ be an endomorphism. Then for $\ell \neq p$

$$(\Gamma_\phi \cdot \Delta) = \sum_{r=0}^{2d} (-1)^r \mathrm{Tr}(\phi^* | H^r(X, \mathbb{Q}_\ell))$$

Whenever Γ_ϕ and Δ intersect transversaly the formula computes $\#\{x \in X : \phi(x) = x\}$, as both are of codimension d in $X \times X$. The key lemma is the following computation:

Lemma

For any $b \in H^*(X, \mathbb{Q}_\ell)$

$$\phi^*(b) = (\pi_1)_*(\mathrm{cl}_{X\times X}(\Gamma_\phi) \cup (\pi_2)^* b)$$

$$
\begin{aligned}
(\pi_1)_*(\mathrm{cl}_{X\times X}(\Gamma_\phi) \cup (\pi_2)^* b) &= (\pi_1)_*((1,\phi)_*(1) \cup (\pi_2)^* b) \quad (1)\\
&= (\pi_1)_* \circ (1,\phi)_*((1) \cup (1,\phi)^*(\pi_2)^* b) \quad (2)\\
&= \mathrm{id}_*(1 \cup \phi^* b) = \phi^*(b)
\end{aligned}
$$

We identify $H^*(X) \otimes H^*(X)$ with $H^*(X \times X)$. Let (e_i^r) be a basis for $H^r(X)$. Poincaré duality gives a dual basis $f_j^{2d-r} \in H^{2d-r}$ such that $\eta(e_i^r \cup f_j^{2d-r}) = \delta_{ij}$. Then

$$
\mathrm{cl}_{X\times X}(\Gamma_\phi) = \sum_{r,i} a_i^r \otimes f_i^{2d-r}
$$

for unique elements a_i^r. Using the previous lemma

$$
\begin{aligned}
\phi^*(e_j^{r'}) &= (\pi_1)_*((\sum a_i^r \otimes f_i^{2d-r}) \cup (1 \otimes e_j^{r'})) \overset{(3)}{=} (\pi_1)_*(a_j^{r'} \otimes e^{2d})\\
&= (\pi_1)_*(\pi_2^*(e^{2d}) \cup \pi_1^*(a_j^{r'})) \overset{(2)}{=} (\pi_1)_*(\pi_2)^*(e^{2d}) \cup a_j^{r'} \overset{(3)}{=} a_j^{r'}
\end{aligned}
$$

Thus

$$\mathrm{cl}_{X\times X}(\Gamma_\phi) = \sum_{r,i} \phi^*(e_i^r) \otimes f_i^{2d-r},$$

and ($\phi = \mathrm{id}$)

$$\mathrm{cl}(\Delta) = \sum_{r,i} e_i^r \otimes f_i^{2d-r} = \sum_{r,i} (-1)^r f_i^{2d-r} \otimes e_i^r.$$

Hence

$$\begin{aligned}\mathrm{cl}(\Gamma_\phi \cdot \Delta) = \mathrm{cl}(\Gamma_\phi) \cup \mathrm{cl}(\Delta) &= \sum_{r,i} (-1)^r \phi^*(e_i^r) \cup f_i^{2d-r} \otimes e^{2d} \\ &= \sum_{r=0}^{2d} (-1)^r \mathrm{Tr}(\phi^*) e^{2d} \otimes e^{2d}\end{aligned}$$

Applying $\eta_{X\times X}$ on both sides we obtain Lefschetz trace formula.

Proof of the rationality conjecture

We use the following easy fact: if ϕ is an endomorphism on a f-d vector space V

$$\det(I - t\phi) = \exp\left(-\sum_{m\geq 1} \mathrm{Tr}(\phi^m|V)t^m/m\right)$$

Now let $\overline{X} = X \times \overline{k}$, and let $\phi : \overline{X} \to \overline{X}$ be the Frobenius endomorphism. Let $N_m = \#X(\mathbb{F}_{q^m}) = \#\{x \in \overline{X} : \phi^m(x) = x\}$. The graph of ϕ is transverse to Δ, so $N_m = \sum_0^{2d}(-1)^r \mathrm{Tr}((\phi^m)^*|H^r(\overline{X}))$. Thus

$$Z(X,t) = \exp\left(\sum_{m\geq 1}\sum_{0}^{2d}(-1)^r \mathrm{Tr}((\phi^m)^*|H^r(\overline{X}))t^m/m\right)$$

$$= \prod_{r=0}^{2d} \det(I - t\phi^*|H^r(\overline{X}))^{(-1)^{r+1}} \in \mathbb{Q}_l(t)$$

Weil cohomology theory with coefficients in $\mathbb{Q}$?

Could we have a Weil cohomology theory (for varieties over k of characteristic p) with coefficients in $\mathbb{Q}$?
Serre: Suppose we have such a theory. Let E be a supersingular ($\mathrm{Aut}(E) \otimes \mathbb{R} \cong \mathbb{H}$) elliptic curve over a finite field. By comparison theorems $H^1(E, \mathbb{Q})$ shoud be 2-dimensional, so that $H^1(X) \otimes \mathbb{R}$ would be 2-dimensional. But it is also a module over $\mathbb{H}$, so its real dimension should be divisible by 4.
A similar argument (considering class field theory) also rules out coefficients in $\mathbb{Q}_p$. However there are Weil cohomology theories with extensions of $\mathbb{Q}_p$ as coefficients, e.g. crystalline cohomology with coefficients in $W(k)$.