Computing p-adic L-functions

using Hilbert Eisenstein series

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Outline

- **Motivation**: why study L-functions?
- **Theory** behind the Klingen-Siegel method
- **Algorithm** for computing p-adic L-functions
- **Data**: some explicit L-functions
- **Further research**: coming up

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- Data is important!

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- Congruences between non-constant terms gives congruences between the constant terms, in fact, the Kummer congruences.

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- **Key idea:** knowing a basis for M_{dk} and enough coefficients of $\Delta_{k,\psi}(z)$ lets us solve for $L_p(\psi, 1-k)$!

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(IV) Interpolate the $\delta_m + 1$ values of $L_p(\psi, 1-k_j)$ to find $L_p(\psi, s)$.

Data, $F = \mathbb{Q}(\sqrt{7})$

p	$L_p(1,s) \in \mathscr{O}_F[s] \pmod{p^m}$
5	$O(5^8)s^{14} + O(5^8)s^{13} + O(5^8)s^{12} + O(5^8)s^{11} + O(5^8)s^{10} + O(5^8)s^9 + \left(5^7 \cdot 1 + O(5^8)\right) s^8 + \left(5^6 \cdot 2 + O(5^8)\right) s^7 + \left(5^5 \cdot 49 + O(5^8)\right) s^6 + \left(5^4 \cdot 568 + O(5^8)\right) s^5 + \left(5^4 \cdot 186 + O(5^8)\right) s^4 + \left(5^3 \cdot 2476 + O(5^8)\right) s^3 + \left(5^2 \cdot 12643 + O(5^8)\right) s^2 + \left(5 \cdot 116522 + O(5^9)\right) s + 1005394 + O(5^{10})$
11	$O(11^9)s^{12} + O(11^9)s^{11} + O(11^{10})s^{10} + \left(11^9 \cdot 9 + O(11^{10})\right) s^9 + \left(11^8 \cdot 45 + O(11^{10})\right) s^8 + \left(11^7 \cdot 538 + O(11^{10})\right) s^7 + \left(11^6 \cdot 5908 + O(11^{10})\right) s^6 + \left(11^5 \cdot 94233 + O(11^{10})\right) s^5 + \left(11^4 \cdot 653451 + O(11^{10})\right) s^4 + \left(11^3 \cdot 1368033 + O(11^{10})\right) s^3 + \left(11^3 \cdot 691404 + O(11^9)\right) s^2 + \left(11 \cdot 43622653 + O(11^9)\right) s + 25656523351 + O(11^{10})$
13	$O(13^{10})s^{11} + O(13^{10})s^{10} + \left(13^9 \cdot 9 + O(13^{10})\right) s^9 + \left(13^8 \cdot 167 + O(13^{10})\right) s^8 + \left(13^7 \cdot 825 + O(13^{10})\right) s^7 + \left(13^6 \cdot 20775 + O(13^{10})\right) s^6 + \left(13^5 \cdot 260717 + O(13^{10})\right) s^5 + \left(13^4 \cdot 3958931 + O(13^{10})\right) s^4 + \left(13^3 \cdot 10298345 + O(13^{10})\right) s^3 + \left(13^3 \cdot 37593275 + O(13^{10})\right) s^2 + \left(13 \cdot 10196962616 + O(13^{10})\right) s + 104887446825 + O(13^{10})$
17	$O(17^{10})s^{11} + O(17^{10})s^{10} + \left(17^9 \cdot 16 + O(17^{10})\right) s^9 + \left(17^9 \cdot 11 + O(17^{10})\right) s^8 + \left(17^7 \cdot 3442 + O(17^{10})\right) s^7 + \left(17^6 \cdot 43576 + O(17^{10})\right) s^6 + \left(17^5 \cdot 731121 + O(17^{10})\right) s^5 + \left(17^4 \cdot 19535454 + O(17^{10})\right) s^4 + \left(17^3 \cdot 2220157 + O(17^{10})\right) s^3 + \left(17^2 \cdot 311956925 + O(17^{10})\right) s^2 + \left(17 \cdot 12743287888 + O(17^{10})\right) s + 497360978290 + O(17^{10})$
19	$O(19^{10})s^{11} + O(19^{10})s^{10} + \left(19^9 \cdot 3 + O(19^{10})\right) s^9 + \left(19^8 \cdot 356 + O(19^{10})\right) s^8 + \left(19^7 \cdot 5512 + O(19^{10})\right) s^7 + \left(19^6 \cdot 86567 + O(19^{10})\right) s^6 + \left(19^5 \cdot 784303 + O(19^{10})\right) s^5 + \left(19^4 \cdot 35196026 + O(19^{10})\right) s^4 + \left(19^3 \cdot 755707686 + O(19^{10})\right) s^3 + \left(19^2 \cdot 13133906787 + O(19^{10})\right) s^2 + \left(19 \cdot 27470894456 + O(19^{10})\right) s + 226617386081 + O(19^{10})$

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- Roblot '15: Alternative algorithm based on Cassou-Noguès' construction of p-adic L-functions - less efficient in practice.

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- Can we generalise to p-adic L-functions of automorphic representations?

Thank you!

Questions?