Computing p-adic L-functions

of Hecke characters

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University of Oxford
Junior number theory seminar

Outline

- L-functions: a 5-min elevator pitch

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- p-adic L-functions: a quick intro

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- Data: some explicit p-adic L-functions
- Applications & extensions

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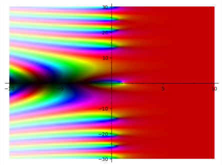
- **Selberg class L-function**: Dirichlet series $\sum_n \frac{a_n}{n^s}$ satisfying the above + growth condition on a_n.



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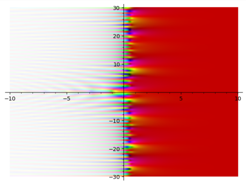
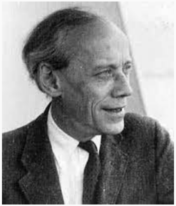
- By viewing χ as a 1-dim. rep. $\mathrm{Gal}(\mathbb{Q}(\zeta_N)) \to \mathbb{C}^\times = \mathrm{Gl}_1(\mathbb{C})$, this generalises to **Artin L-functions** attached to representations.





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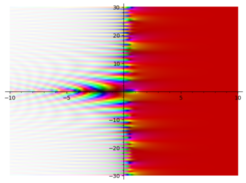
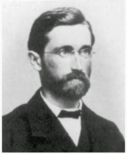
- Dedekind's conjecture: given F'/F, the function $\zeta_{F'}(s)/\zeta_F(s)$ is entire





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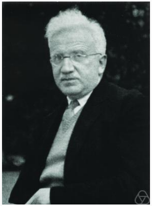

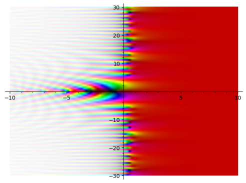

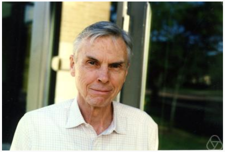

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$$\left(5^2 \cdot 4828 + O(5^8)\right) s^2 + \left(5 \cdot 60514 + O(5^8)\right) s + 51662 + O(5^8) \text{ `` } \approx \text{ '' } \zeta_5(s)$$

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- Congruences between non-constant terms gives congruences between the constant terms, in fact, the Kummer congruences.

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- **Key idea:** knowing a basis for M_{dk} and enough coefficients of $\Delta_{k,\psi}(z)$ lets us solve for $L_p(\psi, 1-k)$!

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Example $F = \mathbb{Q}(\sqrt{5})$

p	$L_p(\chi, s) \in \mathcal{O}_F[s] \pmod{p^m}$, $\chi : \mathrm{Cl}^+_{(4)} \to \mathbb{C}^\times$ unique totally odd char.
3	$\left(3^8 \cdot 1 + O(3^9)\right) s^{12} + \left(3^7 \cdot 19 + O(3^{10})\right) s^{11} + \left(3^7 \cdot 1 + O(3^9)\right) s^{10} + \left(3^5 \cdot 185 + O(3^{10})\right) s^9 + \left(3^7 \cdot 8 + O(3^{10})\right) s^8 + \left(3^5 \cdot 598 + O(3^{11})\right) s^7 + \left(3^6 \cdot 22 + O(3^9)\right) s^6 + \left(3^4 \cdot 182 + O(3^{10})\right) s^5 + \left(3^4 \cdot 194 + O(3^9)\right) s^4 + \left(3^2 \cdot 6025 + O(3^{10})\right) s^3 + \left(3^3 \cdot 5689 + O(3^{11})\right) s^2 + \left(3 \cdot 22742 + O(3^{12})\right) s + 3^{13} \cdot 2 + O(3^{14})$
5	$\left(5^8 \cdot 1 + O(5^9)\right) s^8 + \left(5^6 \cdot 13 + O(5^9)\right) s^7 + \left(5^5 \cdot 401 + O(5^9)\right) s^6 + \left(5^4 \cdot 3069 + O(5^9)\right) s^5 + \left(5^5 \cdot 329 + O(5^9)\right) s^4 + \left(5^3 \cdot 14164 + O(5^9)\right) s^3 + \left(5^2 \cdot 2202 + O(5^9)\right) s^2 + \left(5 \cdot 157656 + O(5^9)\right) s + O(5^{10})$
7	$\left(7^6 \cdot 1 + O(7^7)\right) s^7 + \left(7^6 \cdot 12 + O(7^8)\right) s^6 + \left(7^5 \cdot 220 + O(7^8)\right) s^5 + \left(7^4 \cdot 846 + O(7^8)\right) s^4 + \left(7^3 \cdot 13352 + O(7^8)\right) s^3 + \left(7^2 \cdot 4657 + O(7^8)\right) s^2 + \left(7 \cdot 40340 + O(7^7)\right) s + 3955624 + O(7^8)$
11	$O(11^7) s^6 + \left(11^5 \cdot 42 + O(11^7)\right) s^5 + \left(11^4 \cdot 959 + O(11^7)\right) s^4 + \left(11^3 \cdot 6328 + O(11^7)\right) s^3 + \left(11^2 \cdot 102789 + O(11^7)\right) s^2 + \left(11 \cdot 964668 + O(11^7)\right) s + 11390493 + O(11^7)$
13	$\left(13^6 \cdot 7 + O(13^7)\right) s^6 + \left(13^5 \cdot 62 + O(13^7)\right) s^5 + \left(13^4 \cdot 154 + O(13^7)\right) s^4 + \left(13^3 \cdot 3395 + O(13352^7)\right) s^3 + \left(13^2 \cdot 172247 + O(13^7)\right) s^2 + \left(13^2 \cdot 31667 + O(13^7)\right) s + 26080095 + O(13^7)$
17	$\left(17^6 \cdot 7 + O(17^7)\right) s^6 + \left(17^5 \cdot 159 + O(17^7)\right) s^5 + \left(17^4 \cdot 90 + O(17^7)\right) s^4 + \left(17^3 \cdot 34940 + O(17^7)\right) s^3 + \left(17^2 \cdot 461695 + O(17^7)\right) s^2 + \left(17 \cdot 21148809 + O(17^7)\right) s + 309732348 + O(17^7)$
19	$\left(19^6 \cdot 6 + O(19^7)\right) s^6 + \left(19^5 \cdot 345 + O(19^7)\right) s^5 + \left(19^4 \cdot 1579 + O(19^7)\right) s^4 + \left(19^3 \cdot 98406 + O(19^7)\right) s^3 + \left(19^2 \cdot 645519 + O(19^7)\right) s^2 + \left(19 \cdot 40194503 + O(19^7)\right) s + 354713675 + O(19^7)$

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- Roblot '15: Alternative algorithm based on Cassou-Noguès' construction of p-adic L-functions - less efficient in practice.

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  - Conjecture ([EJV11], inspired by p-adic random matrix heuristic):

$$\mathbb{P}(\mathrm{ord}_p \, \lambda(L_p(\chi, T)) = r) = p^{-r} \prod_{t>r} 1 - p^{-r},$$

as χ ranges over quadratic characters $\left(\frac{\bullet}{-D}\right)$ for which $\left(\frac{p}{-D}\right) = -1$.

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  - **Iwasawa invariants:** μ and $\lambda := \deg P(T) \in \mathbb{N} \sqcup \{0\}$.
  - Conjecture ([EJV11], inspired by p-adic random matrix heuristic):

$$\mathbb{P}(\operatorname{ord}_p \lambda(L_p(\chi, T)) = r) = p^{-r} \prod_{t>r} 1 - p^{-r},$$

  as χ ranges over quadratic characters $\left(\frac{\bullet}{-D}\right)$ for which $\left(\frac{p}{-D}\right) = -1$.
- Over $\mathbb{Q}(\sqrt{D})$ we can compute λ-invariants of $L(\chi, T)$ for $\chi\colon \mathrm{Cl}^+_{(N)} \to \mathbb{C}^\times$ quickly using reduction theory of indefinite binary QFs.

Applications

- Statistics of λ-invariants à la Ellenberg-Jain-Venkatesh.
  - $K/\mathbb{Q}_p$ a finite extension with uniformiser ϖ; fix $f \in \mathcal{O}_K[\![T]\!]$.
  - **Weierstraß preparation thm**: $f(T) = \varpi^\mu P(T)u(T)$ where $P \in \mathcal{O}_K[T]$, $u \in \mathcal{O}_K[\![T]\!]^\times$.
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- Some finicky details to work out still.

Thank you!

Questions?