Computing p-adic L-functions

using Hilbert Eisenstein series

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Outline

- **Motivation**: why study L-functions?
- **Theory** behind the Klingen-Siegel method
- **Algorithm** for computing p-adic L-functions
- **Data**: some explicit L-functions
- **Further research**: coming up

Motivation: why do we care about L-functions?

- *L-functions*: $L(X,s)$ - meromorphic functions encoding arithmetic data
  - $X = F$ number field: Dirichlet class number formula

$$\mathrm{Res}_{s=1}\,\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}\,\mathrm{Reg}_F\,h_F}{w_F\sqrt{|D_F|}} \quad \text{where} \quad \zeta_F(s) := \sum_{I\subset\mathcal{O}_F}\frac{1}{N(I)^s}.$$

  - $X =$ elliptic curve: BSD conjecture, Gross-Zagier formula
  - $X =$ algebraic variety: Conjectures of Bloch, Beilinson, Deligne, ...
- *p-adic L-functions*: $L_p(X,s)\colon \mathbb{Z}_p \to \mathbb{C}_p$ - interpolates special values of $L(X,s)$
  - Iwasawa main conjectures
  - p-adic BSD conjecture
- Data is important!

Theory I

- **Euler + functional equation:** for k even, $\zeta(1-k) = -B_k/k$.
- **Kummer:** if $k \equiv k' \mod p-1p^N$ are even, then

$$(1-p^{k-1})\frac{B_k}{k} \equiv (1-p^{k'-1})\frac{B_{k'}}{k'} \mod p^N.$$

- By p-adic analysis, if $k_0 \equiv k_1 \equiv \dots \mod p-1$, then $(1-p^{k_i-1})\zeta(1-k_i)$ interpolate to $\zeta_p \colon \mathbb{Z}_p \to \mathbb{C}_p$.
- different "branches" corresponding to $\overline{k}_0 \in \mathbb{Z}/(p-1)\mathbb{Z}$.
- We can approximate $\zeta_p(s)$ by computing $\zeta(1-k)$ for enough k and use polynomial interpolation:
- given pairs $(1-k_0, \zeta(1-k_0)), \dots, (1-k_d, \zeta(1-k_d))$, find a polynomial P of degree d such that $P(1-k_i) = \zeta(1-k_i)$.
- **Eg.** $p = 5, k_i \in \{2, 6, 10\}$, interpolate $\{(k_i, -(1-5^{k_i-1})B_{k_i}/k_i)\}$:

$$\left(5^2 \cdot 4828 + O(5^8)\right) s^2 + \left(5 \cdot 60514 + O(5^8)\right) s + 51662 + O(5^8) \text{ `` } \approx \text{ '' } \zeta_5(s)$$

Theory II

- Fix F a totally real number field of degree d, ψ a character of $\mathrm{Cl}_{\Updownarrow}^{+}$.
- Define $L(\psi, s) := \sum_{n>0} \psi(n) n^{-s}$; then $L(\psi, 1-k) = -B_{k,\psi}/k \in \overline{\mathbb{Q}}$.
- Interpolation: $L_p(\psi, 1-k) = (1-\psi(p)p^{k-1})L(\psi, 1-k)$.
- However, not obvious how to compute $B_{k,\psi}$ algebraically, so we need a new strategy!
- Serre (Antwerp III): The modular form

$$G_k^* = \frac{1}{2}\zeta_p(1-k) + \sum_{n=1}^{\infty}\left(\sum_{d|n,\,(d,p)=1} d^{k-1}\right)q^n$$

  on $\Gamma_0(p)$ is a p-adic limit of G_{k_i}.
- Congruences between non-constant terms gives congruences between the constant terms, in fact, the Kummer congruences.

Theory III

- **Idea:** look for modular forms with constant term $L(\psi, 1-k)$.
- There is a *Hilbert modular form*

$$G_{k,\psi}^{(p)}(z_1, \ldots, z_d) = L_p(\psi, 1-k) + 2^d \sum_{\nu \in \mathfrak{d}_+^{-1}} \Big(\sum_{\substack{\mathfrak{a} | (\nu)\mathfrak{d} \\ (\mathfrak{a}, p)=1}} \psi(\mathfrak{a}) N(\mathfrak{a})^{k-1} \Big) e^{2\pi i (\sigma_1(\nu) z_1 + \ldots \sigma_d(\nu) z_d)}.$$

- Setting $z_1 = \ldots = z_d = z$, we get the **diagonal restriction**

$$\Delta_{k,\psi}(z) = L_p(\psi, 1-k) + 2^d \sum_{n>0} \sum_{\substack{\nu \in \mathfrak{d}_+^{-1} \\ \mathrm{tr}(\nu)=n}} \Big(\sum_{\substack{\mathfrak{a} | (\nu)\mathfrak{d} \\ (\mathfrak{a}, p)=1}} \psi(\mathfrak{a}) N(\mathfrak{a})^{k-1} \Big) q^n$$

- This is a classical modular form of weight dk.
- **Key idea:** knowing a basis for M_{dk} and enough coefficients of $\Delta_{k,\psi}(z)$ lets us solve for $L_p(\psi, 1-k)$!

Algorithm

Input:

- F totally real number field,
- p odd prime,
- $k_0 \in \mathbb{Z}$ "starting weight", $m \in \mathbb{N}$ "precision",
- ψ a character of F with the same parity as k_0.

Output: $L_p(\psi, s)$ as an element of $\mathscr{O}_F[s]/(p^m)$.

(I) Set interpolation weights $k_j = k_0 + j(p-1)$, compute bases for $M_{dk_j}(\Gamma_0(M), \psi)$ for $j = 1, \ldots, \delta_m$.

(II) Compute non-constant coefficients of $\Delta_{k_j,\psi}$:

$$a_n = \sum_{\substack{\nu \in \mathfrak{d}_+^{-1} \\ \mathrm{tr}(\nu)=n}} \sum_{\substack{\mathfrak{a}|(\nu)\mathfrak{d} \\ (\mathfrak{a},p)=1}} \psi(\mathfrak{a}) N(\mathfrak{a})^{k_j-1}, \qquad n > 0.$$

(III) Write $\Delta_{k_j,\psi}$ as linear combination of modular forms in M_{dk_j} and solve for constant term, $L_p(\psi, 1-k_j)$.

(IV) Interpolate the $\delta_m + 1$ values of $L_p(\psi, 1-k_j)$ to find $L_p(\psi, s)$.

Data, $F = \mathbb{Q}(\sqrt{7})$

p	$L_p(1,s) \in \mathcal{O}_F[s] \pmod{p^m}$
5	$O(5^8)s^{14} + O(5^8)s^{13} + O(5^8)s^{12} + O(5^8)s^{11} + O(5^8)s^{10} + O(5^8)s^9 + \left(5^7 \cdot 1 + O(5^8)\right)s^8 + \left(5^6 \cdot 2 + O(5^8)\right)s^7 + \left(5^5 \cdot 49 + O(5^8)\right)s^6 + \left(5^4 \cdot 568 + O(5^8)\right)s^5 + \left(5^4 \cdot 186 + O(5^8)\right)s^4 + \left(5^3 \cdot 2476 + O(5^8)\right)s^3 + \left(5^2 \cdot 12643 + O(5^8)\right)s^2 + \left(5 \cdot 116522 + O(5^9)\right)s + 1005394 + O(5^{10})$
11	$O(11^9)s^{12} + O(11^9)s^{11} + O(11^{10})s^{10} + \left(11^9 \cdot 9 + O(11^{10})\right)s^9 + \left(11^8 \cdot 45 + O(11^{10})\right)s^8 + \left(11^7 \cdot 538 + O(11^{10})\right)s^7 + \left(11^6 \cdot 5908 + O(11^{10})\right)s^6 + \left(11^5 \cdot 94233 + O(11^{10})\right)s^5 + \left(11^4 \cdot 653451 + O(11^{10})\right)s^4 + \left(11^3 \cdot 1368033 + O(11^{10})\right)s^3 + \left(11^3 \cdot 691404 + O(11^9)\right)s^2 + \left(11 \cdot 43622653 + O(11^9)\right)s + 25656523351 + O(11^{10})$
13	$O(13^{10})s^{11} + O(13^{10})s^{10} + \left(13^9 \cdot 9 + O(13^{10})\right)s^9 + \left(13^8 \cdot 167 + O(13^{10})\right)s^8 + \left(13^7 \cdot 825 + O(13^{10})\right)s^7 + \left(13^6 \cdot 20775 + O(13^{10})\right)s^6 + \left(13^5 \cdot 260717 + O(13^{10})\right)s^5 + \left(13^4 \cdot 3958931 + O(13^{10})\right)s^4 + \left(13^3 \cdot 10298345 + O(13^{10})\right)s^3 + \left(13^3 \cdot 37593275 + O(13^{10})\right)s^2 + \left(13 \cdot 10196962616 + O(13^{10})\right)s + 104887446825 + O(13^{10})$
17	$O(17^{10})s^{11} + O(17^{10})s^{10} + \left(17^9 \cdot 16 + O(17^{10})\right)s^9 + \left(17^9 \cdot 11 + O(17^{10})\right)s^8 + \left(17^7 \cdot 3442 + O(17^{10})\right)s^7 + \left(17^6 \cdot 43576 + O(17^{10})\right)s^6 + \left(17^5 \cdot 731121 + O(17^{10})\right)s^5 + \left(17^4 \cdot 19535454 + O(17^{10})\right)s^4 + \left(17^3 \cdot 2220157 + O(17^{10})\right)s^3 + \left(17^2 \cdot 311956925 + O(17^{10})\right)s^2 + \left(17 \cdot 12743287888 + O(17^{10})\right)s + 497360978290 + O(17^{10})$
19	$O(19^{10})s^{11} + O(19^{10})s^{10} + \left(19^9 \cdot 3 + O(19^{10})\right)s^9 + \left(19^8 \cdot 356 + O(19^{10})\right)s^8 + \left(19^7 \cdot 5512 + O(19^{10})\right)s^7 + \left(19^6 \cdot 86567 + O(19^{10})\right)s^6 + \left(19^5 \cdot 784303 + O(19^{10})\right)s^5 + \left(19^4 \cdot 35196026 + O(19^{10})\right)s^4 + \left(19^3 \cdot 755707686 + O(19^{10})\right)s^3 + \left(19^2 \cdot 13133906787 + O(19^{10})\right)s^2 + \left(19 \cdot 27470894456 + O(19^{10})\right)s + 226617386081 + O(19^{10})$

Implementation

- Currently only implemented for trivial character, real quadratic fields.
- `sage` has experimental support for Hecke characters via `gp/pari`.
- Computationally heavy bits:
  - ◇ Computing diagonal restriction coefficients
  - -> For ring class characters, reduction theory of quadratic forms
  - ◇ Finding q-expansion bases of modular forms of high weight
  - -> Randomised algorithm due to Lauder
  - -> Is highly parallelisable
- Roblot '15: Alternative algorithm based on Cassou-Noguès' construction of p-adic L-functions - less efficient in practice.

Further venues

- Statistics of λ-invariants à la Ellenberg-Jain-Venkatesh.
- Darmon-Pozzi-Vonk: if $L(\psi, s)$ has an exceptional zero at 0, then $L'(\psi, 0)$ is the constant term of an overconvergent modular form
  - computed using a similar algorithm
  - gives log of "Gross-Stark units" of the Hilbert class field of F
- Can we generalise to p-adic L-functions of automorphic representations?

Thank you!

Questions?