Linear Algebra Formulas

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Operations on one matrix

Solving linear systems

Solving systems with substitution

1. Get a variable by itself in one of the equations.
2. Take the expression you got for the variable in step 1, and plug it (substitute it using parentheses) into the other equation.
3. Solve the equation in step 2 for the remaining variable.
4. Use the result from step 3 and plug it into the equation from step 1.

Solving systems with elimination

1. If necessary, rearrange both equations so that the x-terms are first, followed by the y-terms, the equals sign, and the constant term (in that order). If an equation appears to have not constant term, that means that the constant term is 0.
2. Multiply one (or both) equations by a constant that will allow either the x-terms or the y-terms to cancel when the equations are added or subtracted (when their left sides and their right sides are added separately, or when their left sides and their right sides are subtracted separately).
3. Add or subtract the equations.
4. Solve for the remaining variable.

5. Plug the result of step 4 into one of the original equations and solve for the other variable.

Graphing method

1. Solve for y in each equation.

2. Graph both equations on the same Cartesian coordinate system.

3. Find the point of intersection of the lines (the point where the lines cross).

Matrix dimensions and entries

"rows $\times$ columns"

$$K = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} \\ k_{2,1} & k_{2,2} & k_{2,3} \end{bmatrix}$$

Augmented matrix

$$M = \left[\begin{array}{cccc|c} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & C_1 \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & C_2 \end{array}\right]$$

Pivot entries and pivot columns

Pivot entry, leading entry, pivot: the first non-zero entry in each row.

Pivot column: Any column that houses a pivot.

Echelon forms

Row-echelon form (ref)

1. All the pivot entries are equal to 1.
2. Any row(s) that consist of only 0s are at the bottom of the matrix.
3. The pivot in each row sits in a column to the right of the column that houses the pivot in the row above it. In other words, the pivot entries sit in a staircase pattern, where they stair-step down from the upper left corner to the lower right corner of the matrix.

Reduced row-echelon form (rref)

1. All the pivot entries are equal to 1.
2. Any row(s) that consist of only 0s are at the bottom of the matrix.
3. The pivot in each row sits in a column to the right of the column that houses the pivot in the row above it. In other words, the pivot entries sit in a staircase pattern, where they stair-step down from the upper left corner to the lower right corner of the matrix.
4. Every pivot is the only non-zero entry in its column.

Gauss-Jordan elimination, Gaussian elimination

1. Optional: Pull out any scalars from each row in the matrix.

2. If the first entry in the first row is 0, swap it with another row that has a non-zero entry in its first column. Otherwise, move to step 3.

3. Multiply through the first row by a scalar to make the leading entry equal to 1.

4. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

5. Go back step 2 and repeat the process until the matrix is in reduced row-echelon form.

Number of solutions to the system

One unique solution: $\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$

No solutions: $\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 0 & | & c \end{bmatrix}$

Infinitely many solutions: $\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

Operations on two matrices

Matrix addition

- Matrix dimensions must be identical
- Add corresponding matrix entries to find the sum
- Matrix addition is commutative and associative

Matrix subtraction

- Matrix dimensions must be identical
- Subtract corresponding matrix entries to find the difference
- Matrix subtraction is not commutative and not associative

Scalar multiplication

Scalar: a constant that gets multiplied by every entry in the matrix

Opposite matrices

A and $-A$

Matrix multiplication

The number of columns in the first matrix must match the number of rows in the second matrix.

In the product of A and B, with

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$$

the entries of AB are

$(AB)_{1,1}$ is the product of the first row and first column

$(AB)_{2,1}$ is the product of the second row and first column

$(AB)_{1,2}$ is the product of the first row and second column

$(AB)_{2,2}$ is the product of the second row and second column

The dimensions of the product of any A and B are the rows of the first matrix by the columns of the second matrix.

Matrix multiplication is associative and distributive, but not commutative.

Zero and identity matrices

Zero matrix: A matrix with only 0 entries

Identity matrix: A matrix with 1s down the main diagonal; 0s otherwise

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of the identity matrix and a matrix A:

$IA = A$, but I must have the same number of columns as A has rows

$AI = A$, but I must have the same number of rows as A has columns

Matrices as vectors

Vectors

Vector: A vector has two pieces of information contained within it:

1. the direction in which the vector points, and
2. the magnitude of the vector, which is just the length of the vector.

Row and column vectors

Column vector: A vector expressed as a one-column matrix

Row vector: A vector expressed as a one-row matrix

Sketching vectors

Initial point: The point where the vector begins

Terminal point: the point where the vector ends

Standard position: A vector sketched in standard position has its initial point at the origin

Vector operations

Sum of vectors: $\vec{a} + \vec{b} = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$

Difference of vectors: $\vec{a} - \vec{b} = (a_1, a_2) - (b_1, b_2) = (a_1 - b_1, a_2 - b_2)$

Unit vectors and basis vectors

Unit vector: a vector with length 1

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$$

Vector length: $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2}$

Standard basis vectors

For $\mathbb{R}^2$: $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$

For $\mathbb{R}^3$: $\mathbf{i} = (1,0,0)$ and $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$

Linear combinations and span

Linear combination: The sum of scaled vectors

Span of a vector set: The collection of all vectors which can be represented by linear combinations of the set.

Linear dependence and independence

Linear dependence: A set of vectors are linearly dependent when one vector in the set can be represented by a linear combination of the other vectors in the set.

Collinear: Parallel vectors, or vectors that lie along the same line, are collinear.

Coplanar: Parallel vectors, or vectors that lie in the same plane, are coplanar.

Test for linear independence

If $(c_1, c_2, c_3, \dots c_n) = (0,0,0...0)$ is the only solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \dots + c_n\vec{v}_n = \vec{O}_n$, then the vector set $V = \{v_1, v_2, v_3, \dots v_n\}$ is linearly independent. If any other solution exists, then V is linearly dependent.

Linear subspaces

Subspace: A linear subspace always

1. includes the zero vector,
2. is closed under scalar multiplication, and
3. is closed under addition.

Possible subspaces

For $\mathbb{R}^n$:

1. $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
2. Any line through the origin is a subspace of $\mathbb{R}^n$.
3. Any plane through the origin, when the plane is defined in n or fewer dimensions, is a subspace of $\mathbb{R}^n$.
4. The zero vector in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Spans

Span: The span of a vector set is all the linear combinations of that set.

Span as a subspace: A span is always a subspace.

Basis

Basis: If you have a basis for a space, it means you have enough vectors to span the space, but not more than you need. So a vector set is a basis for a space if it

1. spans the space, and

2. is linearly independent.

Standard basis:

In $\mathbb{R}^2$, the standard basis vectors are $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In $\mathbb{R}^3$, the standard basis vectors are $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Dot products and cross products

Dot products

Dot product of two vectors:

$$\vec{u} \cdot \vec{v} = [u_1 \quad u_2] \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1v_1 + u_2v_2$$

Vector length and the dot product: The square of the length of a vector is equal to the vector dotted with itself:

$$||\vec{u}||^2 = \vec{u} \cdot \vec{u}$$

Properties of dot products:

Commutative

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

Distributive

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$(\vec{u} - \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} - \vec{v} \cdot \vec{w}$$

Associative

$$(c\vec{u}) \cdot \vec{v} = c(\vec{v} \cdot \vec{u})$$

Cauchy-Schwarz inequality

$$|\vec{u} \cdot \vec{v}| \le ||\vec{u}|| \, ||\vec{v}||$$

The two sides are equal when the vectors are linearly dependent

The two sides are unequal when the vectors are linearly independent

Vector triangle inequality

The length of the sum of two vectors will always be less than or equal to the sum of the lengths of the vectors.

$$||\vec{u}+\vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$$

Angle between vectors

Angle between vectors:

$$\vec{u}\cdot\vec{v} = ||\vec{u}||||\vec{v}||\cos\theta$$

Perpendicular vectors: When vectors are perpendicular (or orthogonal), their dot product is 0.

$$\vec{u}\cdot\vec{v} = 0$$

Angle between the zero vector:

1. the zero vector is orthogonal to every non-zero vector, and
2. the zero vector is orthogonal to itself.

Planes and normal vectors

Plane: A plane is a perfectly flat surface that goes on forever in every direction in three-dimensional space. It's the set of all vectors that are perpendicular (orthogonal) to one given normal vector, which is the vector that's perpendicular (orthogonal) to the plane.

Standard equations of a plane:

$$Ax + By + Cz = D$$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ when $\overrightarrow{n} = (A, B, C)$

Cross product

Cross product: The cross product $\overrightarrow{a} \times \overrightarrow{b}$ is orthogonal to the crossed vectors, $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$.

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

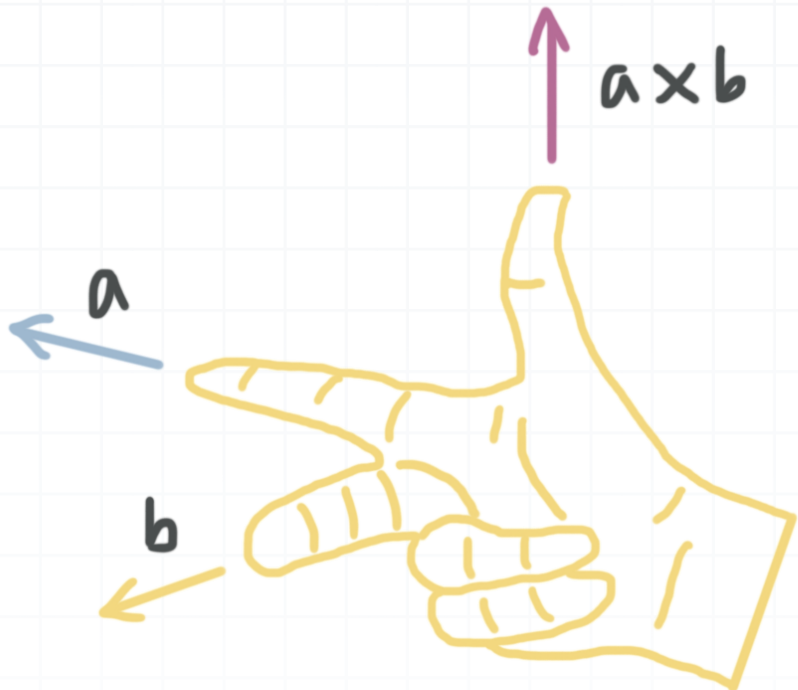
$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Length of the cross product:

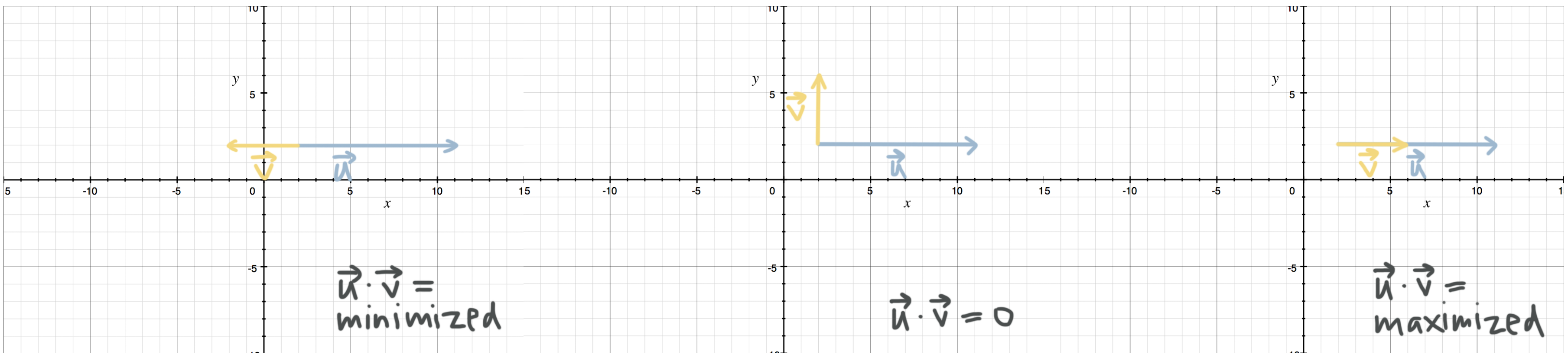
$$||\overrightarrow{u} \times \overrightarrow{v}|| = ||\overrightarrow{u}||\ ||\overrightarrow{v}|| \sin\theta$$

Right-hand rule

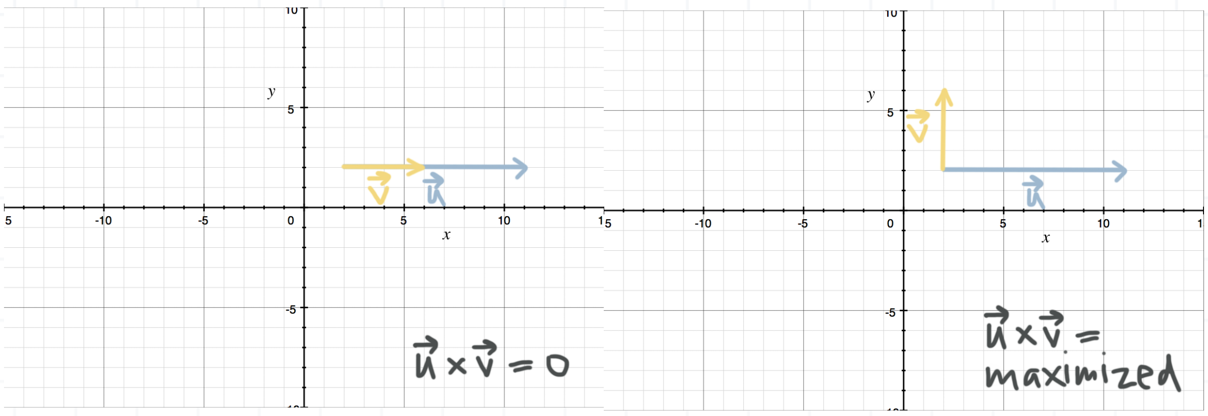


Dot product vs. cross product

The more two vectors point in the same direction, the larger the dot product.



The more two vectors point in opposite directions, the longer the length of the cross product vector.



Matrix-vector products

Matrix-vector products

In $A\vec{x}$, $\vec{x}$ must be a column vector.

In $\vec{x}A$, $\vec{x}$ must be a row vector.

Null space

The null space is always a subspace: It's closed under scalar multiplication and closed under addition

Null space of the matrix A: All the vectors $\vec{x}$ that satisfy $A\vec{x} = \vec{O}$, and $N(A) = N(\text{rref}(A))$.

Linear independence: The columns of A are linearly independent when the null space includes only the zero vector. The columns of A are linearly dependent when the null space includes any vector in addition to the zero vector.

Column space

For a matrix $A = [v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n]$ with column vectors $v_1, v_2, v_3, \dots v_n$, the column space is $C(A) =$ Span$(v_1, v_2, v_3, \dots v_n)$.

Solving $A\overrightarrow{x} = \overrightarrow{b}$

General solution: The general solution, also called the complete solution, is the sum of the complementary and particular solutions, $\overrightarrow{x} = \overrightarrow{x}_n + \overrightarrow{x}_p$.

Complementary solution: Any $\overrightarrow{x}$ that satisfies $A\overrightarrow{x} = \overrightarrow{O}$.

Particular solution: Any $\overrightarrow{x}$ that satisfies $A\overrightarrow{x} = \overrightarrow{b}$.

Dimensionality, nullity, and rank

Dimension of a vector space: The number of basis vectors required to span that space.

Nullity: The dimension of the null space of a matrix A is also called the nullity of A, and can written as either Dim$(N(A))$ or nullity(A). It's given by the number of free variables in the system.

Rank: The dimension of the column space of a matrix A is also called the rank of A, and can written as either Dim$(C(A))$ or rank(A). It's given by the number of pivot variables in the system.

Transformations

Functions and transformations

Function: A rule that maps one value to another.

Vector-valued function: A function defined in terms of vectors.

Transformation: Maps vectors from one space to another.

Domain: The space $\mathbb{R}^m$ that's being mapped *from*.

Codomain: The space $\mathbb{R}^n$ that's being mapped *to*.

Range: The specific vectors within $\mathbb{R}^n$ that are being mapped to, within the codomain.

Image of the subset

Subset: The vector set that's being transformed.

Preimage: The vector set before the transformation is applied.

Image: The vector set after the transformation is applied.

Kernel of the transformation: All of the vectors that result in the zero vector under the transformation T.

$$\text{Ker}(T) = \left\{ \vec{x} \in \mathbb{R}^2 \ \middle|\ T(\vec{x}) = \vec{O} \right\}$$

Linear transformations

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if, for any two vectors $\vec{u}$ and $\vec{v}$ that are both in $\mathbb{R}^n$, and for a c that's also in $\mathbb{R}$ (c is any real number), then

- the transformation of their sum is equivalent to the sum of their individual transformations, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, and
- the transformation of a scalar multiple the vector is equivalent to the product of the scalar and the transformation of the original vector, $T(c\vec{u}) = cT(\vec{u})$ and $T(c\vec{v}) = cT(\vec{v})$.

Rotation matrix

In $\mathbb{R}^2$:

$$\text{Rot}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In $\mathbb{R}^3$:

$$\text{Rot}_{\theta \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\text{Rot}_{\theta \text{ around } z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations are linear transformations:

$$\text{Rot}_\theta(\vec{u} + \vec{v}) = \text{Rot}_\theta(\vec{u}) + \text{Rot}_\theta(\vec{v})$$

$$\text{Rot}_\theta(c\vec{u}) = c\text{Rot}_\theta(\vec{u})$$

Modifying transformations

Given transformations $S(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^m$ as $S(\vec{x}) = A\vec{x}$ and $T(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^m$ as $T(\vec{x}) = B\vec{x}$, where A and B are $m \times n$ matrices,

- The sum of the transformations is $(S + T)(\vec{x}) = (A + B)\vec{x}$
- The scaled transformation is $cT(\vec{x}) = c(B\vec{x}) = (cB)\vec{x}$

Projections

The projection of $\vec{v}$ onto L, where L is given as scaled versions of $\vec{x}$, is

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}}\right)\vec{x}$$

or as

$$\text{Proj}_L(\vec{v}) = (\vec{v} \cdot \hat{u})\hat{u}$$

$$\text{Proj}_L(\vec{v}) = A\vec{v} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \vec{v}$$

when $\vec{x}$ is normalized to $\hat{u}$.

Projections are linear transformations

$$\text{Proj}_L(\vec{a} + \vec{b}) = \text{Proj}_L(\vec{a}) + \text{Proj}_L(\vec{b})$$

$$\text{Proj}_L(c\vec{a}) = c\text{Proj}_L(\vec{a})$$

Compositions of transformations

Compositions are linear transformations, which are closed under addition and closed under scalar multiplication.

$$T \circ S(\vec{x} + \vec{y}) = T \circ S(\vec{x}) + T \circ S(\vec{y})$$

$$T(S(c\vec{x})) = cT(S(\vec{x}))$$

Compositions as matrix-vector products:

$$T \circ S(\vec{x}) = T(S(\vec{x})) = T(A\vec{x}) = BA\vec{x} = C\vec{x}$$

Identity transformation

The identity transformation maps a vector to itself.

$$I_x(\vec{x}) = \vec{x}$$

Inverses

Inverse transformations

Surjective: If every vector $\vec{b}$ in B is being mapped to, then T is surjective, or onto.

Injective: If every $\vec{a}$ maps to a unique $\vec{b}$, then T is injective, or one-to-one.

Invertible transformations: A transformation is invertible if, for every $\vec{b}$ in B, there's a unique $\vec{a}$ in A, such that $T(\vec{a}) = \vec{b}$.

Inverse transformations are linear transformations, which are closed under addition and closed under scalar multiplication.

$$T^{-1}(\vec{u} + \vec{v}) = T^{-1}(\vec{u}) + T^{-1}(\vec{v})$$

$$T^{-1}(c\vec{u}) = cT^{-1}(\vec{u})$$

Inverse matrices

Only square matrices can be invertible. Given a transformation $T(\vec{x}) = M\vec{x}$, if you put the matrix M into reduced row-echelon form and get the identity matrix I_n, then you know that

- the matrix M was a square $n \times n$ matrix, and that

- the transformation T maps $\mathbb{R}^n \to \mathbb{R}^n$, and that

- T is invertible.

Determinant formula for the inverse matrix:

$$M^{-1} = \frac{1}{|M|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

When $\text{Det}(M) = |M| = ad - bc = 0$, the matrix is singular, which means it's not invertible. When $\text{Det}(M) = |M| = ad - bc \neq 0$, the matrix is invertible.

Determinants

Rule of Sarrus for 3×3 matrices

Given

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

the determinant is

$$|A| = aei + bfg + cdh - afh - bdi - ceg$$

Cramer's rule

For a system of two linear equations in two unknowns,

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

the solution is given by

$$x = \frac{D_x}{D}, \ y = \frac{D_y}{D}, \text{ with } D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \ D_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}, \ D_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

Determinant rules

Multiplying a row by a scalar: Multiplying a row of the matrix by a scalar requires that we multiply the determinant by the same scalar, Det$(B) = |B| = k|A|$.

Swapped rows: When two rows of a matrix are swapped, the determinant must be multiplied by -1.

Duplicate rows: When two rows in a matrix are identical, the determinant will be 0, which means the matrix is singular, and not invertible.

Upper and lower triangular matrices

Main diagonal: The main diagonal of a matrix is made of the entries that run from the upper left corner of the matrix down to the lower right corner of the matrix.

Upper triangular matrix: A matrix is upper triangular when all the entries below the main diagonal are 0.

Lower triangular matrix: A matrix is lower triangular when all the entries above the main diagonal are 0.

Determinant: The determinant of upper and lower triangular matrices is the product of the entries in the main diagonal.

Determinants to find area

The area of the parallelogram formed by $v_1 = (a, c)$ and $v_2 = (b, d)$ from

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by Area $= |\text{Det}(A)|$. If a figure f is transformed by T into g, then the area of g is Area$_g = |\text{Area}_f \cdot \text{Det}(T)|$.

Transposes

Transposes

The transpose: The transpose A^T of a matrix A is simply the matrix you get when you swap all the rows and columns.

The determinant of the transpose: $|A| = |A^T|$.

Transpose of the transpose: $(A^T)^T = A$

Transpose of a matrix product: $(XY)^T = Y^T X^T$

Transpose of a matrix sum: $(X+Y)^T = X^T + Y^T$

Transpose of a matrix inverse: $(X^T)^{-1} = (X^{-1})^T$

Invertibility of the product: $A^T A$ is invertible if the columns of A are linearly independent.

Row space and left null space

The row space is the span of the row vectors of A (the span of the column vectors of A^T), and the left null space is the vector set that satisfies $\vec{x}^T A = \vec{O}^T$.

Subspace	**Symbol**	**Space**	**Dimension**
Column space of A	$C(A)$	$\mathbb{R}^m$	$\text{Dim}(C(A)) = r$
Null space of A	$N(A)$	$\mathbb{R}^n$	$\text{Dim}(N(A)) = n - r$
Row space of A (column space of A^T)	$C(A^T)$	$\mathbb{R}^n$	$\text{Dim}(C(A^T)) = r$
Left null space of A (null space of A^T)	$N(A^T)$	$\mathbb{R}^m$	$\text{Dim}(N(A^T)) = m - r$

Orthogonality and change of basis

Orthogonal complements

If a set of vectors V is a subspace of $\mathbb{R}^n$, then the orthogonal complement of V, called $V^\perp$, is a set of vectors where every vector in $V^\perp$ is orthogonal to every vector in V.

$$V^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \quad \text{for every} \quad \vec{v} \in V\}$$

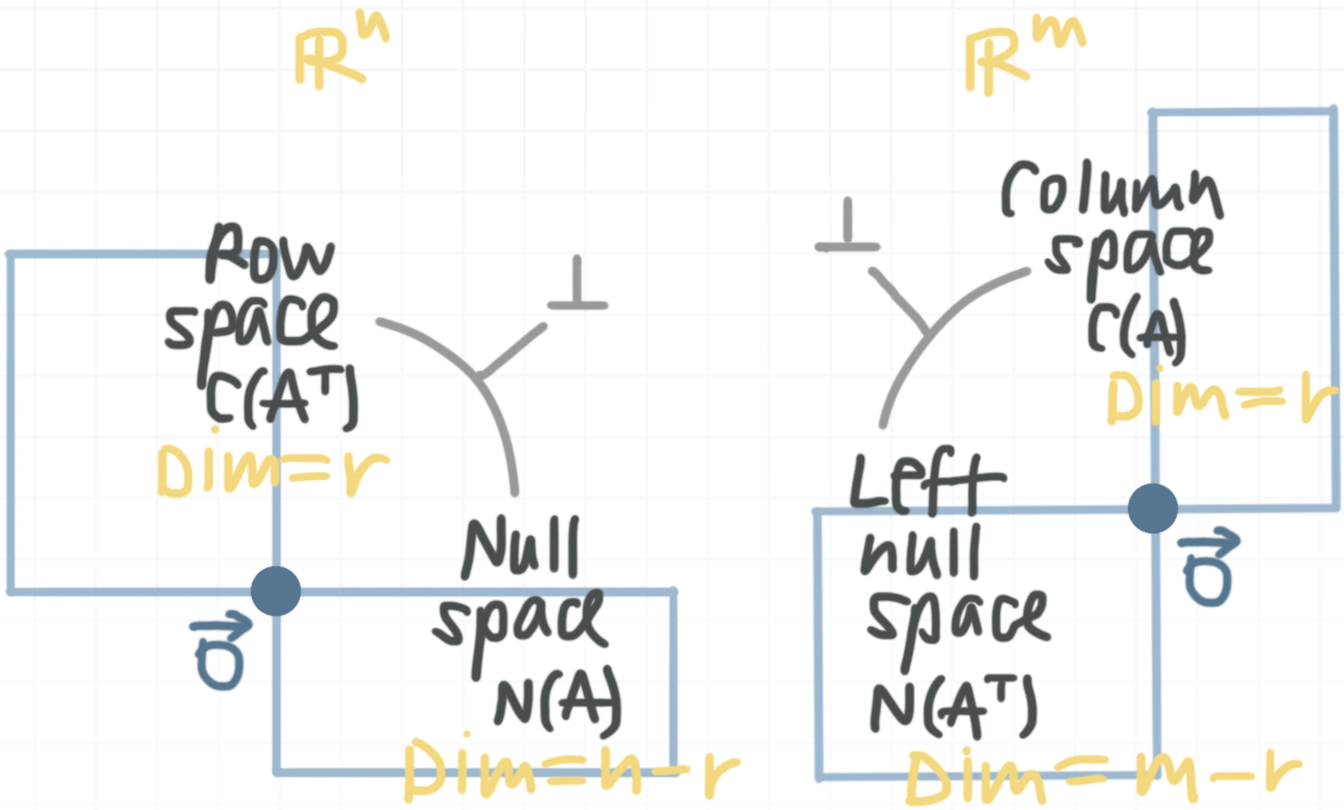
The orthogonal complement is a subspace, which means it's closed under addition and closed under scalar multiplication.

Complement of the complement: $(V^\perp)^\perp = V$

Orthogonality of the fundamental subspaces

The null space $N(A)$ and row space $C(A^T)$ are orthogonal complements, $N(A) = (C(A^T))^\perp$, or $(N(A))^\perp = C(A^T)$.

The left null space $N(A^T)$ and column space $C(A)$ are orthogonal complements, $N(A^T) = (C(A))^\perp$, or $(N(A^T))^\perp = C(A)$.



Projection onto a subspace

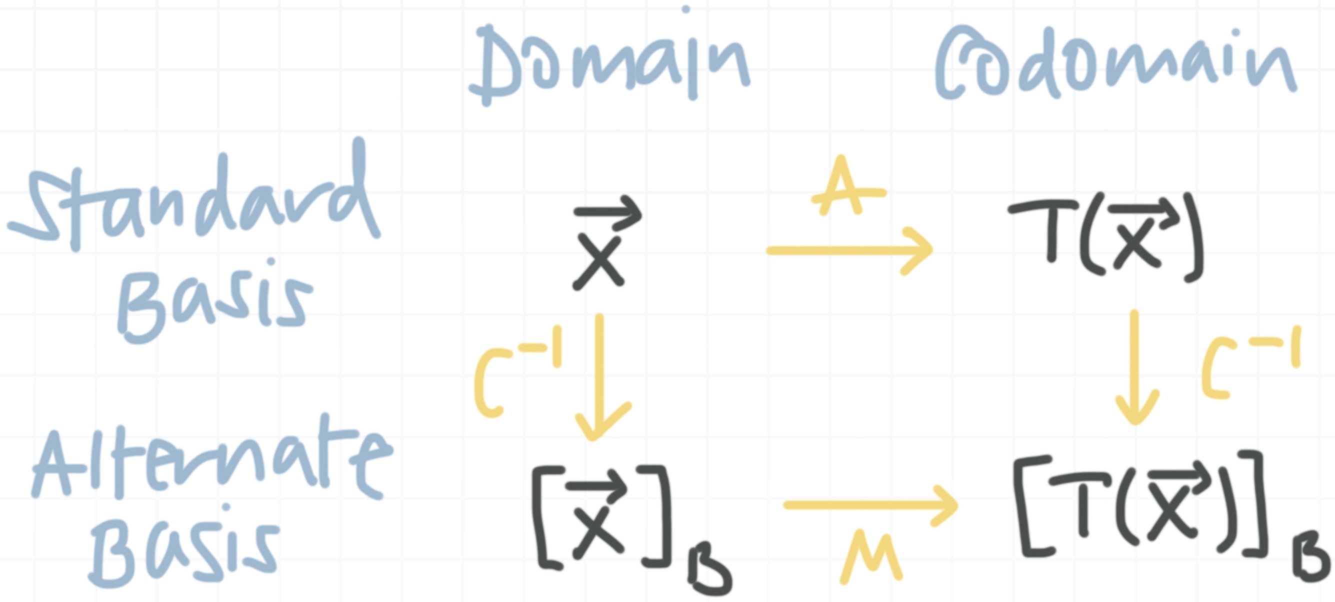
The projection of $\vec{x}$ onto a subspace V is a linear transformation that can be written as the matrix-vector product where A is a matrix of column vectors that form the basis for the subspace V.

$$\text{Proj}_V \vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

Least squares solution

$$A^TA\vec{x}^* = A^T\vec{b}$$

Transformation matrix for a basis



Orthonormal bases and Gram-Schmidt

Orthonormal bases

Orthonormal basis: An orthonormal basis is a basis in which every vector in the basis is both 1 unit in length and orthogonal to every other vector in the basis.

Orthogonal matrix: A square matrix whose columns form an orthonormal set.

Orthonormal matrix: A rectangular matrix whose columns form an orthonormal set.

Projection onto an orthonormal basis:

$$\text{Proj}_V \vec{x} = AA^T\vec{x}$$

Gram-Schmidt process

Given $V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_n)$,

1. Normalize $\vec{v}_1$ to make the basis $V = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_n)$

2. Find $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$, then normalize $\vec{w}_2$, to make the basis $V = \text{Span}(\vec{u}_1, \vec{u}_2, \vec{v}_3 \dots \vec{v}_n)$

3. Find $\vec{w}_3 = \vec{v}_3 - \left[(\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2\right]$, then normalize $\vec{w}_3$, to make the basis $V = \text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3 \dots \vec{v}_n)$

4. Continue this iterative process until every vector has been changed into an orthogonal, normalized vector.

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Any vector $\vec{v}$ that satisfies $T(\vec{v}) = \lambda\vec{v}$ is an eigenvector for the transformation T, and λ is the eigenvalue that's associated with the eigenvector $\vec{v}$.

$A\vec{v} = \lambda\vec{v}$ for nonzero vectors $\vec{v}$ when $|\lambda I_n - A| = 0.$

λ is an eigenvalue of A when $|\lambda I_n - A| = 0.$

Trace: sum of the entries along the main diagonal

Trace(A) = sum of A's eigenvalues

Det$|A|$ = product of A's eigenvalues

Eigenspace: $E_\lambda = N(\lambda I_n - A)$