Dot products and cross products

$\vec{a} \cdot \vec{b}$

Dot product:
How much $\vec{a}$ and $\vec{b}$ point in the same direction

$\vec{a} \times \vec{b}$

Cross product:
The length shows how much $\vec{a}$ and $\vec{b}$ point in different directions

Dot products

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 =$

$\vec{a} = (a_1, a_2)$ $\quad b_1a_1 + b_2a_2$

$\vec{b} = (b_1, b_2)$

$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\vec{a}\cdot\vec{b} = [a_1 \ \ a_2]\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$= [b_1 \ \ b_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$

$\boxed{\|\vec{a}\|^2} = a_1^2 + a_2^2 = \boxed{\vec{a}\cdot\vec{a}} = a_1a_1 + a_2a_2$

$a_1^2 + a_2^2$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$\vec{b} = (-1, 2, 3)$

$\|\vec{b}\|^2 = \vec{b}\cdot\vec{b}$

$= (-1)(-1) + (2)(2) + (3)(3)$

$= 14$

$\sqrt{\|\vec{b}\|^2} = \sqrt{14}$

$\|\vec{b}\| = \sqrt{14}$

1. Commutative
2. Distributive
   $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
3. Associative
   $(3\vec{r})\cdot\vec{s} = 3(\vec{r}\cdot\vec{s})$

BUT DOT PRODUCT OF TWO MATRICES ARE NOT COMMUTATIVE,

THEY ARE ONLY COMMUTATIVE WHEN BOTH MATRICES ARE ONE DIMENSIONAL (VECTORS)

Cauchy-Schwarz inequality

$$|\vec{a}\cdot\vec{b}| \le \|\vec{a}\|\,\|\vec{b}\|$$

$=$ $\quad \vec{a} = c\vec{b} \qquad c = 2$

$\vec{b} = c\vec{a} \qquad c = \frac{1}{2}$

parallel

collinear

$=$ $\boxed{L.D.}$ Linearly dependent

$<$ $\boxed{L.I.}$ Linearly independent

$15 \le \sqrt{5}\,(3\sqrt{5})$

$15 = 3\cdot 5$

$15 = 15$

$i = (1,0) \qquad j = (0,1)$

$\|i\| = \sqrt{1^2 + 0^2}$

$= \sqrt{1+0}$

$= \sqrt{1}$

$= 1$

$i\cdot j = (1)(0) + (0)(1)$

$= 0 + 0$

$= 0$

$|i\cdot j| = 0$

$0 \le 1\cdot 1$

$0 < 1 \quad L.I.$

$-3\vec{a} = \vec{b}$

$\vec{a} = (2,1) \qquad \vec{b} = (-6,-3)$

$\|\vec{a}\| = \sqrt{2^2 + 1^2}$

$= \sqrt{4+1}$

$= \sqrt{5}$

$\|\vec{b}\| = \sqrt{(-6)^2 + (-3)^2}$

$= \sqrt{36+9}$

$= \sqrt{45} = 9\cdot 5$

$= 3\sqrt{5}$

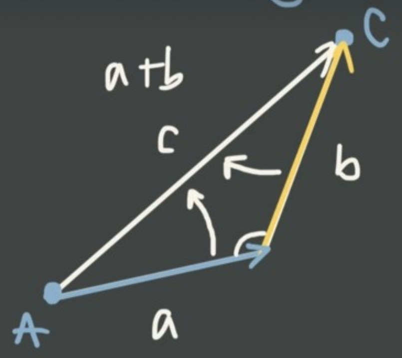
$|\vec{a}\cdot\vec{b}| = |(2)(-6) + (1)(-3)|$

$= |-12-3|$

$= |-15|$

$= 15$

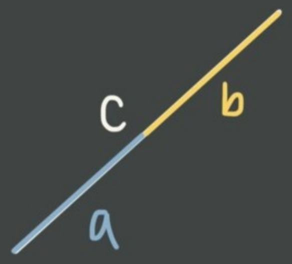
Vector triangle inequality



$$\mathbb{R}^2: \; c \le a+b$$

$$c = a+b$$

$$c < a+b$$



$$\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$$

$$\mathbb{R}^4: \left.\begin{aligned} \vec{a} &= (1,1,2,2) \\ \vec{b} &= (1,0,-3,1) \end{aligned}\right\} \text{L.}$$

$$\|\vec{a}\| = \sqrt{1^2+1^2+2^2+2^2}$$
$$= \sqrt{1+1+4+4}$$
$$= \sqrt{10}$$

$$\|\vec{b}\| = \sqrt{1^2+0^2+(-3)^2+1^2}$$
$$= \sqrt{1+9+1}$$
$$= \sqrt{11}$$

$$\|\vec{a}+\vec{b}\| = \sqrt{2^2+1^2+(-1)^2+3^2}$$
$$= \sqrt{4+1+1+9}$$
$$= \sqrt{15}$$

$$\vec{a}+\vec{b} = (2,1,-1,3)$$

$$\sqrt{15} \le \sqrt{10}+\sqrt{11}$$

$$3.87 < 3.16 + 3.32$$

Angle between vectors

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\cos(90^\circ)$$

$$0$$

$$\vec{a}\cdot\vec{b} = 0$$

$$0 = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\frac{0}{\|\vec{a}\|\|\vec{b}\|} = \cos\theta$$

$$0 = \cos\theta$$

$$\theta = 90^\circ$$

$$\left.\begin{aligned}\vec{a} &= (1,0,-2,0)\\ \vec{b} &= (0,3,0,1)\end{aligned}\right\}\mathbb{R}^4$$

$$\vec{a}\cdot\vec{b} = (1)(0)+(0)(3)+(-2)(0)+(0)(1)$$
$$= 0+0+0+0$$
$$= 0$$

$$\vec{a} = (0,3) \qquad \vec{b} = (-1,-4)$$

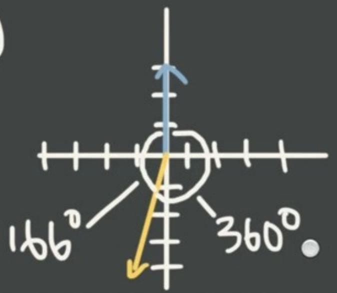
$$\vec{a}\cdot\vec{b} = (0)(-1)+(3)(-4)$$
$$= 0-12$$
$$= -12$$

$$\|\vec{a}\| = \sqrt{0^2+3^2}$$
$$= \sqrt{9}$$
$$= 3$$

$$\|\vec{b}\| = \sqrt{(-1)^2+(-4)^2}$$
$$= \sqrt{1+16}$$
$$= \sqrt{17}$$

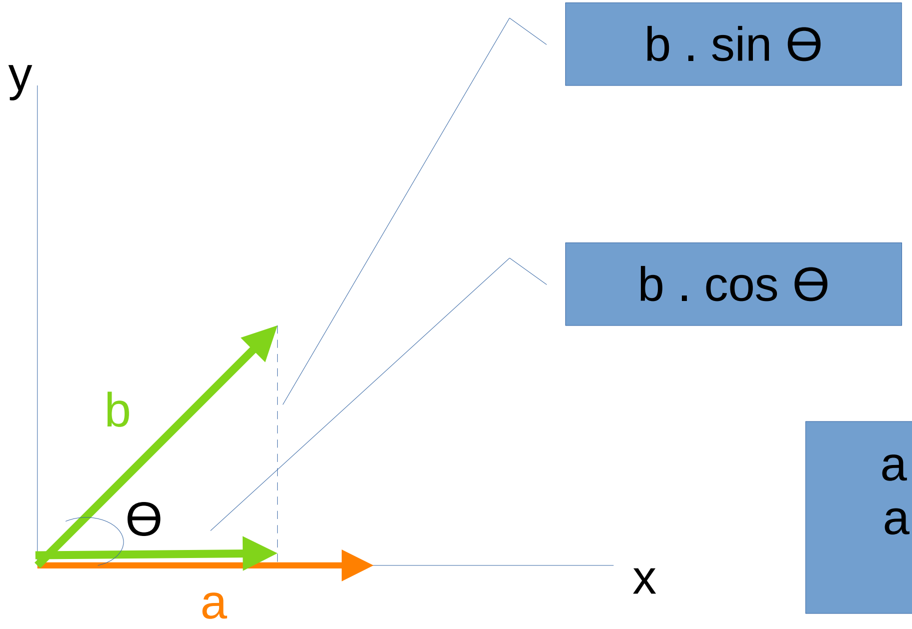
$$-12 = 3\sqrt{17}\cos\theta$$

$$\cos\theta = -\frac{12}{3\sqrt{17}} = -\frac{4}{\sqrt{17}}$$



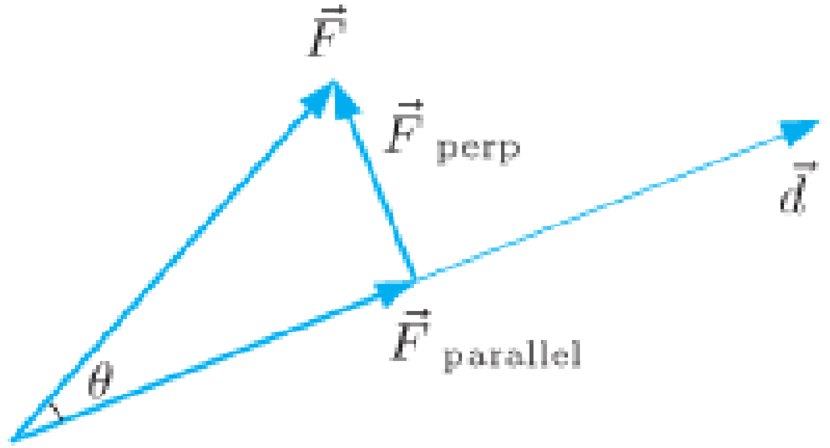
$$\cancel{\arccos}\,(\cancel{\cos\theta}) = \arccos\left(-\frac{4}{\sqrt{17}}\right)$$

$$\theta = 166^\circ$$



$$a \cdot b = a \cdot b \cdot \cos\theta + a \cdot b \cdot \sin\theta$$
$$a \cdot b = a \cdot b \cdot \cos\theta + 0$$

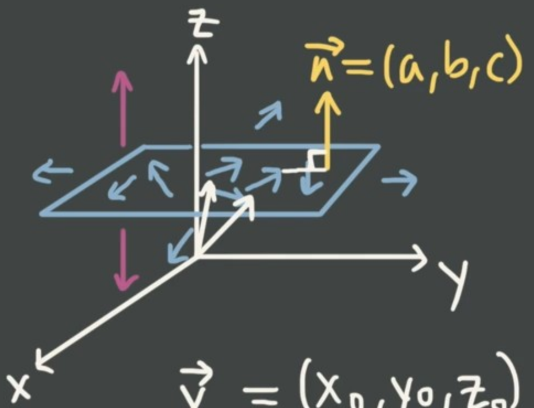
$$W = (\|\vec{F}\| \cos\theta)\|\vec{d}\| = \|\vec{F}\|\,\|\vec{d}\|\cos\theta = \vec{F}\cdot\vec{d}.$$



The *work*, W, done by a force $\vec{F}$ acting on an object through a displacement $\vec{d}$ is given by

$$W = \vec{F}\cdot\vec{d}.$$

Equation of a plane, and normal vectors



$\vec{n} = (a,b,c)$

$\vec{x}_0 = (x_0, y_0, z_0)$

$\vec{x} = (x, y, z)$

$$\vec{x} - \vec{x}_0 = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$$\left[a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \right]$$

$$\vec{n} = (\underset{a}{2}, \underset{b}{5}, \underset{c}{-3}) \qquad (\underset{x_0}{1}, \underset{y_0}{0}, \underset{z_0}{-2})$$

$$2(x-1) + 5(y-0) - 3(z+2) = 0$$

$$2x - 2 + 5y - 3z - 6 = 0$$

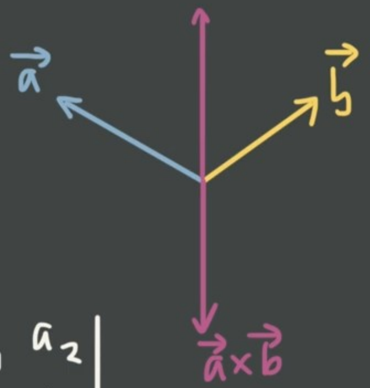
$$\boxed{2}x + \boxed{5}y \boxed{-3}z = 8 \qquad \vec{n} = (A, B, C)$$

$$Ax + By + Cz = D \qquad \vec{n} = (2, 5, -3)$$

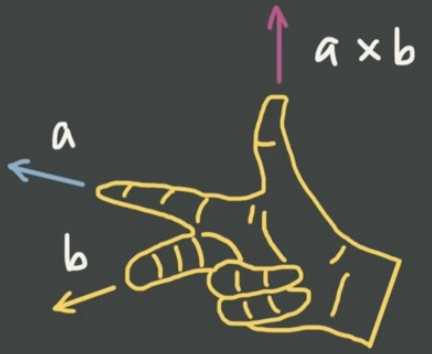
ûdemy

Cross products

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$



$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$



$$= i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 b_2 - a_2 b_1)$$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$$

$\vec{a} = (1, 0, 2)$

$\vec{b} = (-2, 1, 0)$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{vmatrix} = i \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}$$

$$= i(0 \cdot 0 - 2 \cdot 1) - j(1 \cdot 0 - 2(-2)) + k(1 \cdot 1 - 0(-2))$$

$$= i(-2) - j(4) + k(1)$$

$$= -2i - 4j + k$$

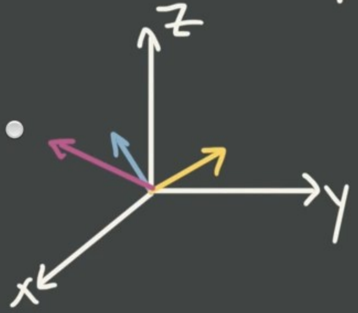
$$= (-2, -4, 1) = \vec{a} \times \vec{b}$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{(-2)^2 + (-4)^2 + 1^2}$$

$$= \sqrt{4 + 16 + 1}$$

$$= \sqrt{21}$$

$$\approx 4.6$$



udemy

Dot and cross products as opposite ideas

Dot product

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

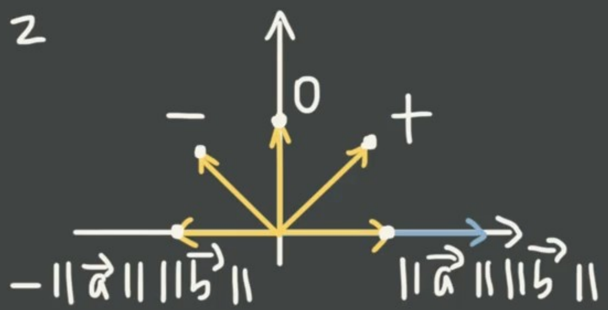
0 For $\theta = 90°$: $\boxed{\vec{a}\cdot\vec{b} = 0}$

10 For $\theta = 0°$: $\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|}$

−10 For $\theta = 180°$: $\boxed{\vec{a}\cdot\vec{b} = -\|\vec{a}\|\,\|\vec{b}\|}$

$\|\vec{a}\| = 5$

$\|\vec{b}\| = 2$



cross product

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$$

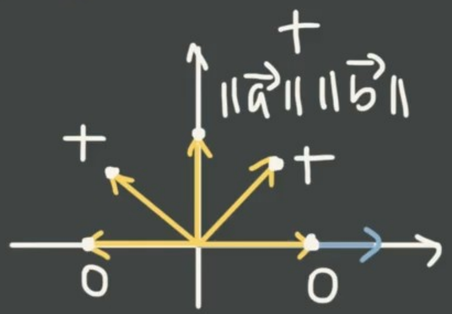
10 For $\theta = 90°$: $\boxed{\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|}$

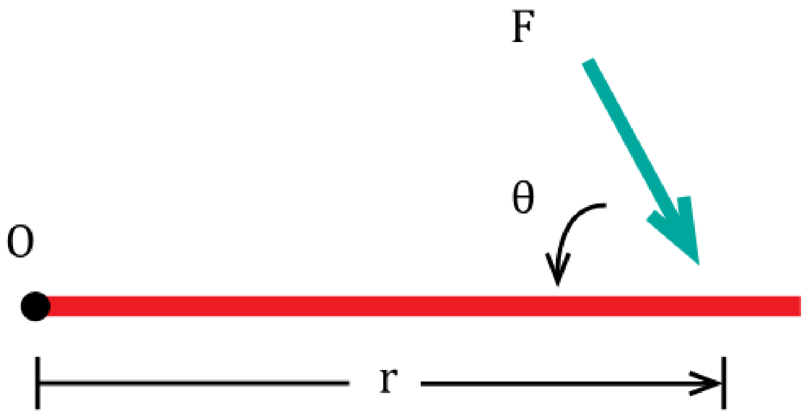
0 For $\theta = 0°$: $\boxed{\|\vec{a}\times\vec{b}\| = 0}$

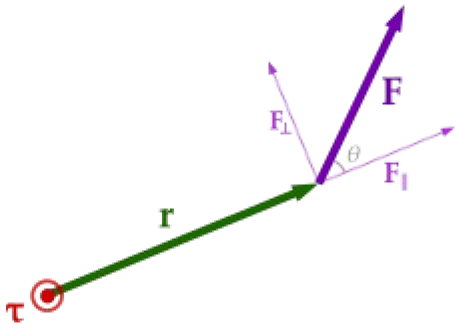
0 For $\theta = 180°$: $\boxed{\|\vec{a}\times\vec{b}\| = 0}$

$\|\vec{a}\| = 5$

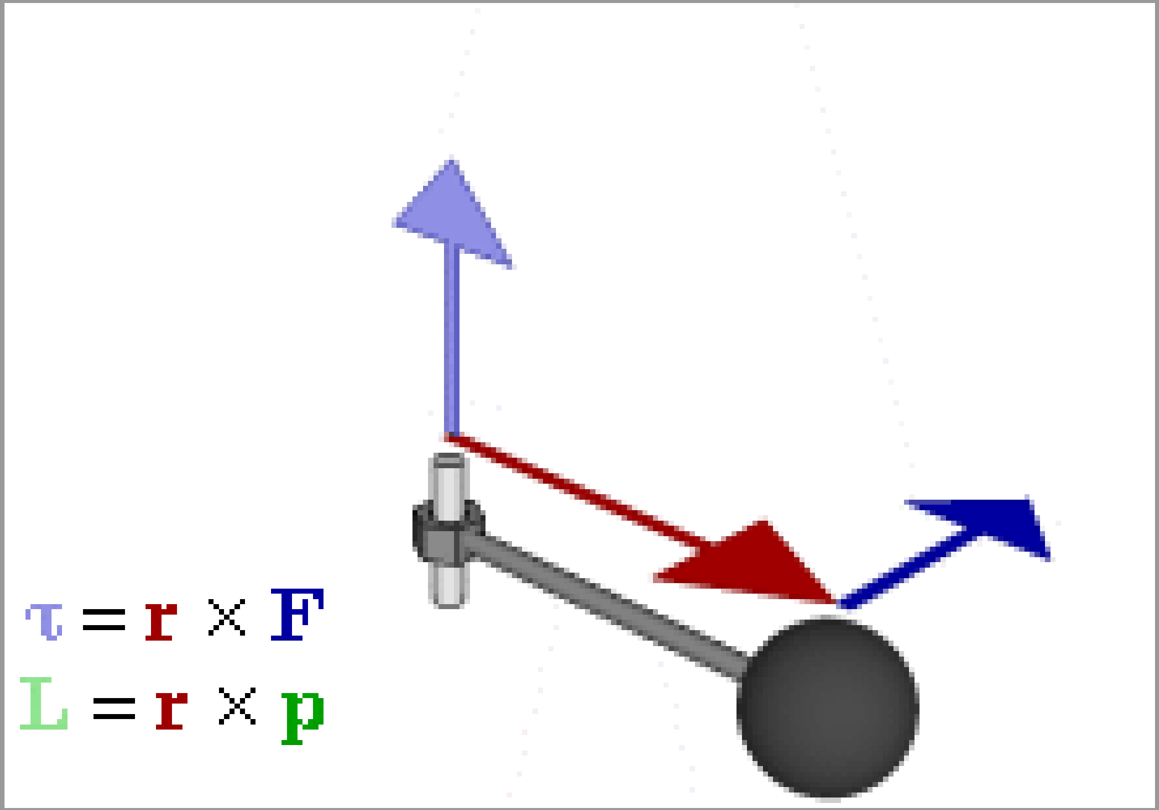
$\|\vec{b}\| = 2$







$$\Gamma = r \times F = rF\sin(\theta).$$



F kuvvetinin etkisinde dönen bir cisme, döndürme etkisini sadece kuvvetin konum vektörüne dik olan bileşeni uygular. $\tau = r \times F$, büyüklüğü $\tau = r F_{\perp} = r F \sin\theta$ olur.