Orthogonality and change of basis

$$V = \text{Span}\left(\begin{bmatrix}\vec{v}_1\end{bmatrix}, \begin{bmatrix}\vec{v}_2\end{bmatrix}\right)$$

1. Direction

~~2. Length~~

Pick better vectors

$$V = \text{Span}\left(\begin{bmatrix}\vec{u}_1\end{bmatrix}, \begin{bmatrix}\vec{u}_2\end{bmatrix}\right) \longleftarrow$$

Change of basis

Orthogonal complements

$$\vec{v}\cdot\vec{x}=0$$

$\mathbb{R}^n$ $\quad V=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\dots\vec{v}_n\}$

$\mathbb{R}^n$ $\quad V^\perp=\{\vec{x}_1,\vec{x}_2,\vec{x}_3,\dots\vec{x}_m\}$

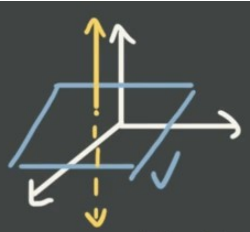
$$V^\perp=\{\vec{x}\in\mathbb{R}^n \mid \vec{x}\cdot\vec{v}=0 \text{ for every } \vec{v}\in V\}$$

$$\vec{x}_1\cdot\vec{v}=0$$
$$\vec{x}_2\cdot\vec{v}=0$$

$$\vec{x}_1\cdot\vec{v}+\vec{x}_2\cdot\vec{v}=0$$
$$(\vec{x}_1+\vec{x}_2)\cdot\vec{v}=0$$

$$c\vec{x}_1\cdot\vec{v}=0 \qquad c(\vec{x}_1\cdot\vec{v})=0 \qquad c(0)=0$$
$$0=0$$

$$V=\text{Span}\left(\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}-1\\1\\3\end{bmatrix}\right)$$



$$V^\perp=\left\{\vec{x}\in\mathbb{R}^3 \mid \vec{x}\cdot\begin{bmatrix}1\\0\\2\end{bmatrix}=0 \text{ and } \vec{x}\cdot\begin{bmatrix}-1\\1\\3\end{bmatrix}=0\right\}$$

$$\vec{x}=(x_1,x_2,x_3)$$

$$1x_1+0x_2+2x_3=0$$
$$-1x_1+1x_2+3x_3=0$$

$$\left[\begin{array}{ccc|c}1&0&2&0\\-1&1&3&0\end{array}\right]$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c}\boxed{\begin{matrix}1\\0\end{matrix}}&\boxed{\begin{matrix}0\\1\end{matrix}}&\boxed{\begin{matrix}2\\5\end{matrix}}&\begin{matrix}0\\0\end{matrix}\end{array}\right]$$

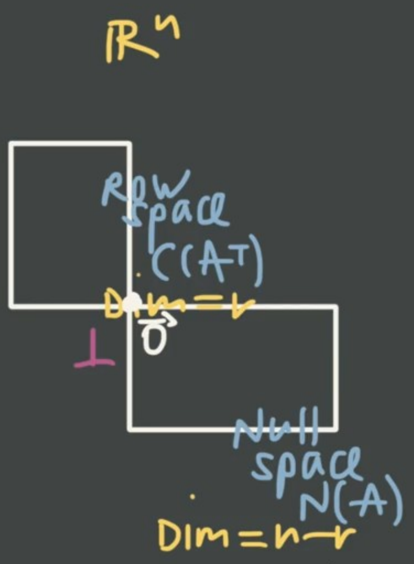
$$x_1+2x_3=0$$
$$x_2+5x_3=0$$

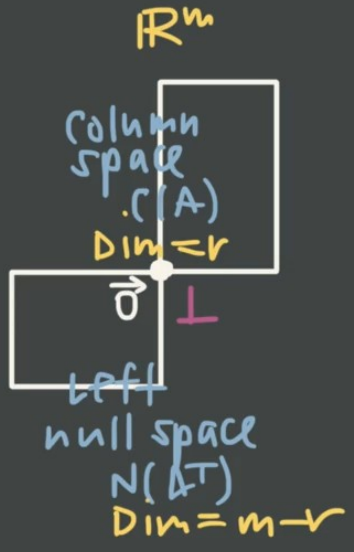
$$x_1=-2x_3$$
$$x_2=-5x_3$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}-2\\-5\\1\end{bmatrix}$$

$$V^\perp=\text{Span}\left(\begin{bmatrix}-2\\-5\\1\end{bmatrix}\right)$$

Orthogonal complements of the fundamental subspaces





$$N(A) = (C(A^T))^{\perp}$$
$$(N(A))^{\perp} = C(A^T)$$

$$C(A) = (N(A^T))^{\perp}$$
$$N(A^T) = (C(A))^{\perp}$$

$\mathbb{R}^n$: V $\quad V^{\perp}$

$$\mathrm{Dim}(V) + \mathrm{Dim}(V^{\perp}) = n$$

$n = 4$, $3 = m$

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} & \boxed{\begin{matrix}2\\-4\\3\end{matrix}} \end{bmatrix}$$

$$C(A) = \mathrm{span}\left(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}\right)$$

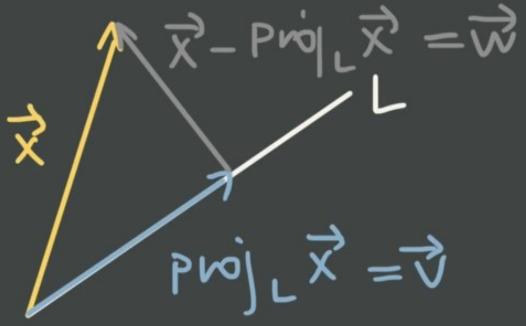
$$N(A) = \mathrm{Span}\left(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}\right)$$

$$C(A^T) = \mathrm{span}\left(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}\right)$$

$$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

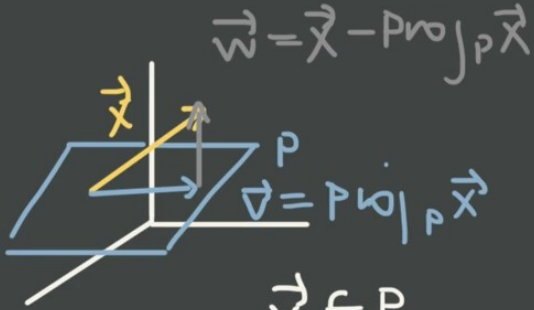
Space		Dim
$C(A)$	$\mathbb{R}^3$	3
$N(A)$	$\mathbb{R}^4$	1
$C(A^T)$	$\mathbb{R}^4$	3
$N(A^T)$	$\mathbb{R}^3$	0

Projection onto the subspace



$\vec{v} \in L$

$\vec{w} \in L^\perp$



$\vec{v} \in P$

$\vec{w} \in P^\perp$

V in $\mathbb{R}^n$

$$\text{proj}_V \vec{x} = \underline{A(A^TA)^{-1}A^T}\vec{x}$$

$$V = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{11} & -\frac{1}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{6}{11} \end{array}\right]$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{11} & -\frac{1}{11} \\ -\frac{1}{11} & \frac{6}{11} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{11} - \frac{1}{11} & -\frac{2}{11} + \frac{6}{11} \\ \frac{2}{11} + 0 & -\frac{1}{11} + 0 \\ \frac{2}{11} + \frac{1}{11} & -\frac{1}{11} - \frac{6}{11} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3/11 & 4/11 \\ 2/11 & -1/11 \\ 3/11 & -7/11 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{6}{11} + \frac{4}{11} & \frac{3}{11} + 0 & \frac{3}{11} - \frac{4}{11} \\ \frac{4}{11} - \frac{1}{11} & \frac{2}{11} - 0 & \frac{2}{11} + \frac{1}{11} \\ \frac{6}{11} - \frac{7}{11} & \frac{3}{11} - 0 & \frac{3}{11} + \frac{7}{11} \end{bmatrix} = \begin{bmatrix} \frac{10}{11} & \frac{3}{11} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{3}{11} & \frac{10}{11} \end{bmatrix} \vec{x} = \frac{1}{11} \begin{bmatrix} 10 & 3 & -1 \\ 3 & 2 & 3 \\ -1 & 3 & 10 \end{bmatrix} \vec{x}$$

Least squares solution

$\vec{x}^* = \text{Proj}_{C(A)} \vec{b}$

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

2 1 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

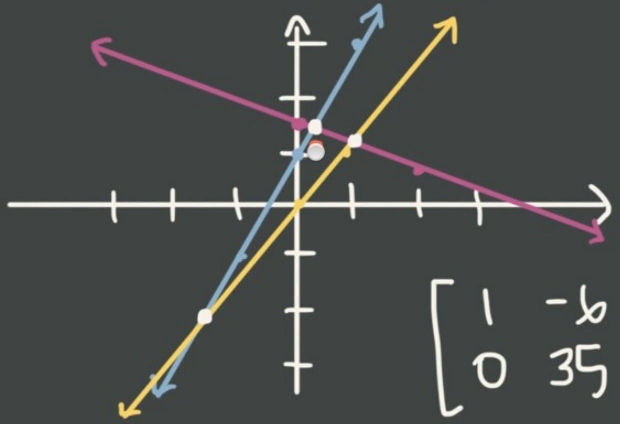
$$A^T A \vec{x}^* = A^T \vec{b}$$

$-2x + y = 1$ $\quad y = 2x + 1$

$-x + y = 0$ $\quad y = x$

$x + 2y = 3$ $\quad 2y = -x + 3$

$y = -\frac{1}{2}x + \frac{3}{2}$



$$\begin{array}{ccc} A & \vec{x} & \vec{b} \\ \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}\vec{x}^* = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & -1 & 1 \\ -1 & 6 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 6 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 35 & 43 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 1 & \frac{43}{35} \end{array}\right]$$

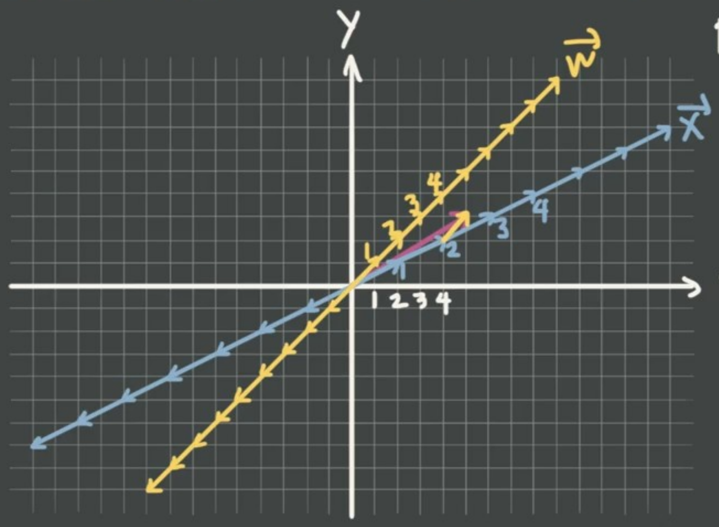
$$\left[\begin{array}{cc|c} 1 & 0 & 13/35 \\ 0 & 1 & 43/35 \end{array}\right] \begin{array}{l} \approx 0.37 \\ \approx 1.23 \end{array}$$

$$-7 + 6\left(\frac{43}{35}\right)$$

$$-\frac{245}{35} + \frac{258}{35} = \frac{13}{35}$$

$$\vec{x}^* = \begin{bmatrix} 13/35 \\ 43/35 \end{bmatrix}$$

coordinates in a new basis



$\mathbb{R}^2$

$i=(1,0)$
$j=(0,1)$

$B=\{i,j\}$

$\vec{v}=(5,3)$
$=5i+3j$
$=2\vec{x}+\vec{w}$

$\left\{\begin{array}{l}\vec{x}=(2,1)\\ \vec{w}=(1,1)\end{array}\right\} B$

$[\vec{v}]_B=\begin{bmatrix}2\\1\end{bmatrix}$

$$\overset{A}{\begin{bmatrix}2&1\\1&1\end{bmatrix}}[\vec{v}]_B=\begin{bmatrix}5\\3\end{bmatrix}$$

$$\left[\begin{array}{cc|c}2&1&5\\1&1&3\end{array}\right]\quad\left[\begin{array}{cc|c}1&1&3\\2&1&5\end{array}\right]\quad\left[\begin{array}{cc|c}1&1&3\\0&-1&-1\end{array}\right]\quad\left[\begin{array}{cc|c}1&1&3\\0&1&1\end{array}\right]\quad\left[\begin{array}{cc|c}1&0&2\\0&1&1\end{array}\right]$$

$$[\vec{v}]_B=\begin{bmatrix}2\\1\end{bmatrix}$$

$$A[\vec{v}]_B=\vec{v}$$
$$A^{-1}A[\vec{v}]_B=A^{-1}\vec{v}$$
$$I[\vec{v}]_B=A^{-1}\vec{v}$$
$$[\vec{v}]_B=A^{-1}\vec{v}$$

$$\left[\begin{array}{cc|cc}2&1&1&0\\1&1&0&1\end{array}\right]\quad\left[\begin{array}{cc|cc}1&1&0&1\\2&1&1&0\end{array}\right]$$

$$\left[\begin{array}{cc|cc}1&1&0&1\\0&-1&1&-2\end{array}\right]\quad\left[\begin{array}{cc|cc}1&1&0&1\\0&1&-1&2\end{array}\right]$$

$$\left[\begin{array}{cc|cc}1&0&1&-1\\0&1&-1&2\end{array}\right]\qquad A^{-1}=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}$$

$$[\vec{v}]_B=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}\qquad\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4+1\\2+1\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}$$

Transformation matrix for a basis

$$T(\vec{x}) = A\vec{x}$$

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

$$C[\vec{x}]_B = \vec{x}$$

$$M = C^{-1}AC$$

Domain   codomain

$$\begin{tikzcd}
\vec{x} \arrow[r, "T"', "A"] \arrow[d, "C^{-1}"'] & T(\vec{x}) \arrow[d, "C^{-1}"] \\
{[\vec{x}]_B} \arrow[r, "T"', "M"] & {[T(\vec{x})]_B}
\end{tikzcd}$$

$$T(\vec{x}) = \overset{A}{\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}}\vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$$

$$C = \overset{C}{\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}} \qquad C^{-1} = \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$$

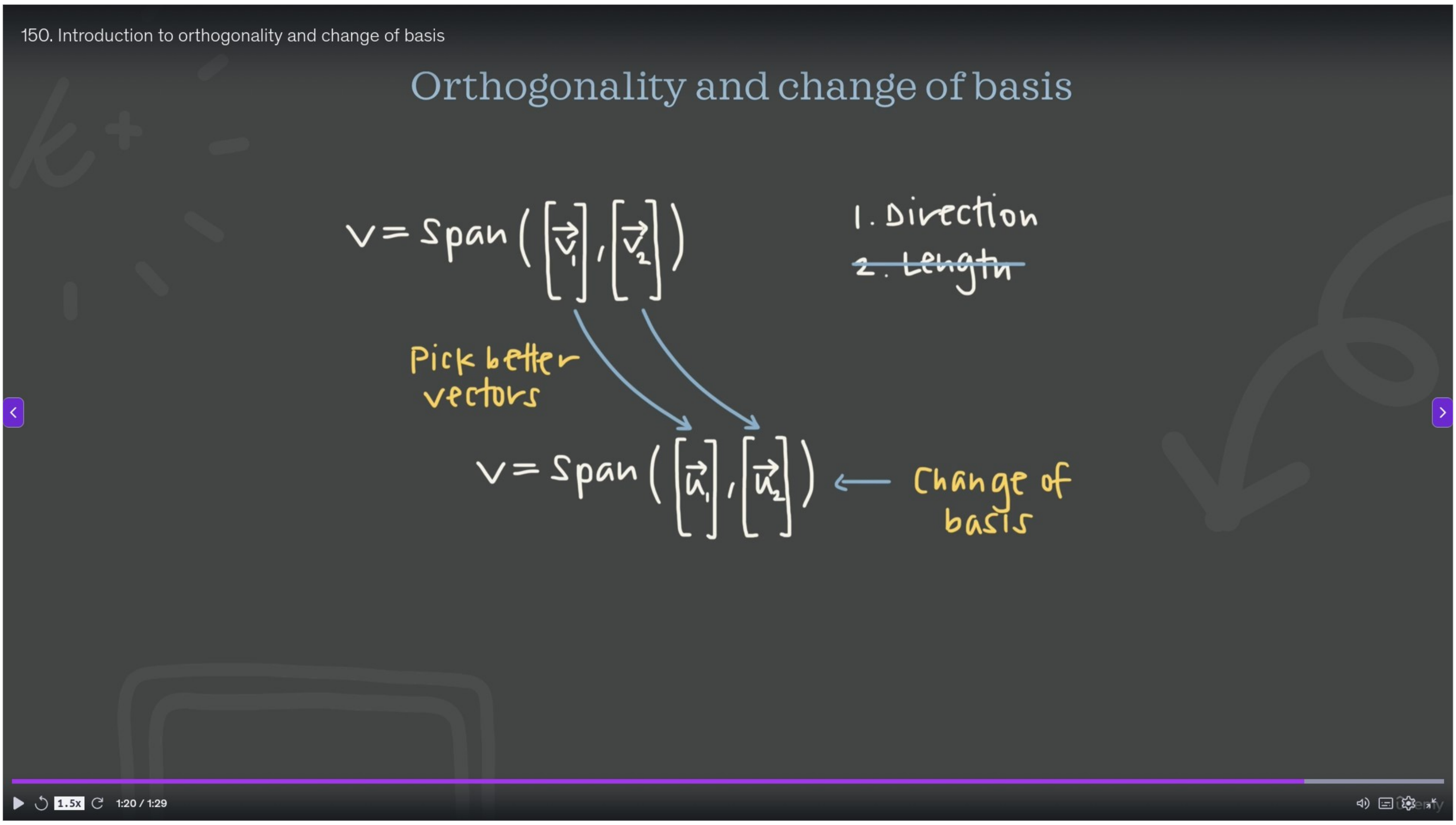
$$[\vec{x}]_B = (1,3)$$

$$M = \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 1+\frac{4}{3} & -2-\frac{20}{3} \\ 0+\frac{2}{3} & 0-\frac{10}{3} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & -\frac{26}{3} \\ \frac{2}{3} & -\frac{10}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix}$$

S.B. $$[T(\vec{x})]_B = \frac{1}{3}\begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

A.B. $$= \frac{1}{3}\begin{bmatrix} 7-78 \\ 2-30 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -71 \\ -28 \end{bmatrix} = \begin{bmatrix} -71/3 \\ -28/3 \end{bmatrix}$$


150. Introduction to orthogonality and change of basis
Orthogonality and change of basis
$V = \text{Span}\left(\left[\vec{v}_1\right], \left[\vec{v}_2\right]\right)$
1. Direction
~~2. Length~~
Pick better vectors
$V = \text{Span}\left(\left[\vec{u}_1\right], \left[\vec{u}_2\right]\right)$
Change of basis
1.5x
1:20 / 1:29

Orthogonal complements

$$\vec{v}\cdot\vec{x}=0$$

$\mathbb{R}^n$ $V=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\dots\vec{v}_n\}$

$\mathbb{R}^n$ $V^\perp=\{\vec{x}_1,\vec{x}_2,\vec{x}_3,\dots\vec{x}_m\}$

$V^\perp=\{\vec{x}\in\mathbb{R}^n \mid \vec{x}\cdot\vec{v}=0$ for every $\vec{v}\in V\}$

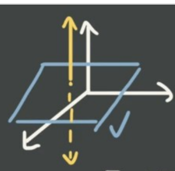
$\vec{x}_1\cdot\vec{v}=0$ $\qquad$ $\vec{x}_1\cdot\vec{v}+\vec{x}_2\cdot\vec{v}=0$

$\vec{x}_2\cdot\vec{v}=0$ $\qquad$ $(\vec{x}_1+\vec{x}_2)\cdot\vec{v}=0$

$c\vec{x}_1\cdot\vec{v}=0$ $\qquad$ $c(\vec{x}_1\cdot\vec{v})=0$ $\qquad$ $c(0)=0$

$0=0$

$$V=\mathrm{Span}\left(\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}-1\\1\\3\end{bmatrix}\right)$$



$$V^\perp=\left\{\vec{x}\in\mathbb{R}^3 \mid \vec{x}\cdot\begin{bmatrix}1\\0\\2\end{bmatrix}=0 \text{ and } \vec{x}\cdot\begin{bmatrix}-1\\1\\3\end{bmatrix}=0\right\}$$

$\vec{x}=(x_1,x_2,x_3)$ $\qquad$ $V^\perp=\mathrm{Span}\left(\begin{bmatrix}-2\\-5\\1\end{bmatrix}\right)$

$1x_1+0x_2+2x_3=0$

$-1x_1+1x_2+3x_3=0$

$$\left[\begin{array}{ccc|c}1&0&2&0\\-1&1&3&0\end{array}\right]$$

$x_1=-2x_3$

$x_2=-5x_3$

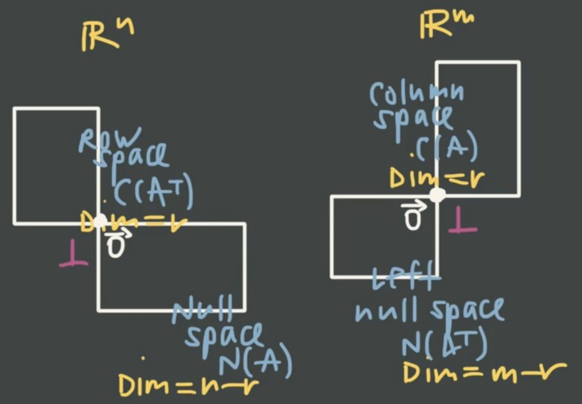
$$\begin{array}{cccc}x_1&x_2&x_3&\\ \hline \boxed{\begin{matrix}1\\0\end{matrix}}&\boxed{\begin{matrix}0\\1\end{matrix}}&\boxed{\begin{matrix}2\\5\end{matrix}}&\begin{matrix}0\\0\end{matrix}\end{array}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}-2\\-5\\1\end{bmatrix}$$

$x_1+2x_3=0$

$x_2+5x_3=0$

Orthogonal complements of the fundamental subspaces



$N(A) = (C(A^T))^\perp$ $\qquad$ $C(A) = (N(A^T))^\perp$

$(N(A))^\perp = C(A^T)$ $\qquad$ $N(A^T) = (C(A))^\perp$

$\mathbb{R}^n: \quad V \quad V^\perp$

$\text{Dim}(V) + \text{Dim}(V^\perp) = n$

$n = 4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \quad 3 = m$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \boxed{\begin{matrix}0\\0\\1\end{matrix}} & \boxed{\begin{matrix}2\\-4\\3\end{matrix}} \end{bmatrix}$$

$C(A) = \text{span}\left(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}\right)$

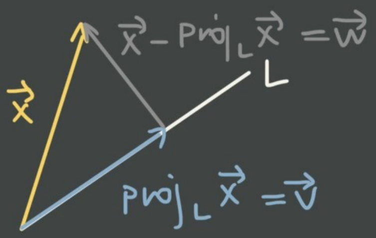
$N(A) = \text{Span}\left(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}\right)$

$C(A^T) = \text{Span}\left(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}\right)$

$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

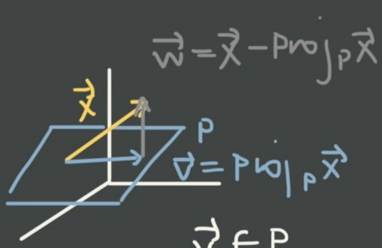
Space		Dim
$C(A)$	$\mathbb{R}^3$	3
$N(A)$	$\mathbb{R}^4$	1
$C(A^T)$	$\mathbb{R}^4$	3
$N(A^T)$	$\mathbb{R}^3$	0

Projection onto the subspace



$\vec{v} \in L$

$\vec{w} \in L^\perp$



$\vec{v} \in P$

$\vec{w} \in P^\perp$

V in $\mathbb{R}^n$

$$\text{proj}_V \vec{x} = \underline{A(A^TA)^{-1}A^T}\vec{x}$$

$$V = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{11} & -\frac{1}{11} \\ 0 & 1 & -\frac{1}{11} & \frac{6}{11} \end{array}\right]$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{2}{11} & -\frac{1}{11} \\ -\frac{1}{11} & \frac{6}{11} \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{11} - \frac{1}{11} & -\frac{2}{11} + \frac{6}{11} \\ \frac{2}{11} + 0 & -\frac{1}{11} + 0 \\ \frac{2}{11} + \frac{1}{11} & -\frac{1}{11} - \frac{6}{11} \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3/11 & 4/11 \\ 2/11 & -1/11 \\ 3/11 & -7/11 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{6}{11} + \frac{4}{11} & \frac{3}{11} + 0 & \frac{3}{11} - \frac{4}{11} \\ \frac{4}{11} - \frac{1}{11} & \frac{2}{11} - 0 & \frac{2}{11} + \frac{1}{11} \\ \frac{6}{11} - \frac{7}{11} & \frac{3}{11} - 0 & \frac{3}{11} + \frac{7}{11} \end{bmatrix} = \begin{bmatrix} \frac{10}{11} & \frac{3}{11} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{3}{11} & \frac{10}{11} \end{bmatrix}\vec{x} = \frac{1}{11}\begin{bmatrix} 10 & 3 & -1 \\ 3 & 2 & 3 \\ -1 & 3 & 10 \end{bmatrix}\vec{x}$$

Least squares solution

$\vec{x}^* = \text{Proj}_{C(A)} \vec{b}$

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$2 \qquad 1 \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

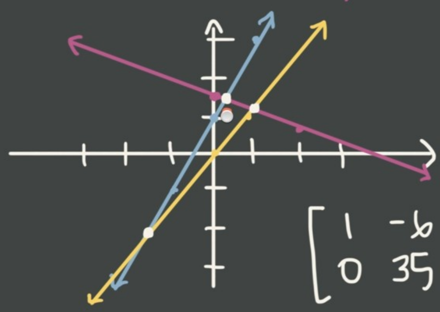
$$A^T A \vec{x}^* = A^T \vec{b}$$

$-2x + y = 1 \qquad y = 2x + 1$

$-x + y = 0 \qquad y = x$

$x + 2y = 3 \qquad 2y = -x + 3$

$\qquad y = -\frac{1}{2}x + \frac{3}{2}$



$$\begin{array}{ccc} A & \vec{x} & \vec{b} \\ \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & -1 & 1 \\ -1 & 6 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 6 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 35 & 43 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 1 & \frac{43}{35} \end{array}\right]$$

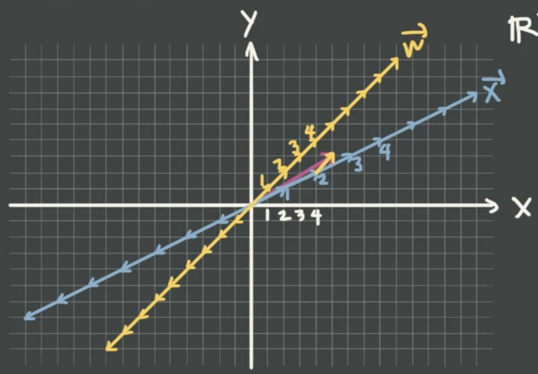
$$\left[\begin{array}{cc|c} 1 & 0 & 13/35 \\ 0 & 1 & 43/35 \end{array}\right] \begin{array}{l} \approx 0.37 \\ \approx 1.23 \end{array}$$

$-7 + 6\left(\frac{43}{35}\right)$

$-\frac{245}{35} + \frac{258}{35} = \frac{13}{35}$

$$\vec{x}^* = \begin{bmatrix} 13/35 \\ 43/35 \end{bmatrix}$$

coordinates in a new basis



$\mathbb{R}^2$

$i = (1,0)$

$j = (0,1)$

$B = \{i, j\}$

$\vec{v} = (5,3)$

$= 5i + 3j$

$= 2\vec{x} + \vec{w}$

$\left\{ \begin{array}{l} \vec{x} = (2,1) \\ \vec{w} = (1,1) \end{array} \right\} B$

$[\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

A

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} [\vec{v}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 1 & 3 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -1 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

$[\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A[\vec{v}]_B = \vec{v}$

$A^{-1}A[\vec{v}]_B = A^{-1}\vec{v}$

$I[\vec{v}]_B = A^{-1}\vec{v}$

$[\vec{v}]_B = A^{-1}\vec{v}$

$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$ $\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$ $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$[\vec{v}]_B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4+1 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Transformation matrix for a basis

$$T(\vec{x}) = A\vec{x}$$

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

$$C[\vec{x}]_B = \vec{x}$$

$$M = C^{-1}AC$$

Domain $\qquad$ codomain

$$\begin{array}{ccc} \vec{x} & \xrightarrow[A]{T} & T(\vec{x}) \\ C^{-1}\downarrow & & \downarrow C^{-1} \\ [\vec{x}]_B & \xrightarrow[M]{T} & [T(\vec{x})]_B \end{array}$$

$$T(\vec{x}) = \overset{A}{\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}}\vec{x} \qquad B = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$$

$$C = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$$

$$[\vec{x}]_B = (1,3)$$

$$M = \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 1+\frac{4}{3} & -2-\frac{20}{3} \\ 0+\frac{2}{3} & 0-\frac{10}{3} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} & -\frac{26}{3} \\ \frac{2}{3} & -\frac{10}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix}$$

S.B. $\quad [T(\vec{x})]_B = \frac{1}{3}\begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

A.B. $\quad = \frac{1}{3}\begin{bmatrix} 7-78 \\ 2-30 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -71 \\ -28 \end{bmatrix} = \begin{bmatrix} -71/3 \\ -28/3 \end{bmatrix}$