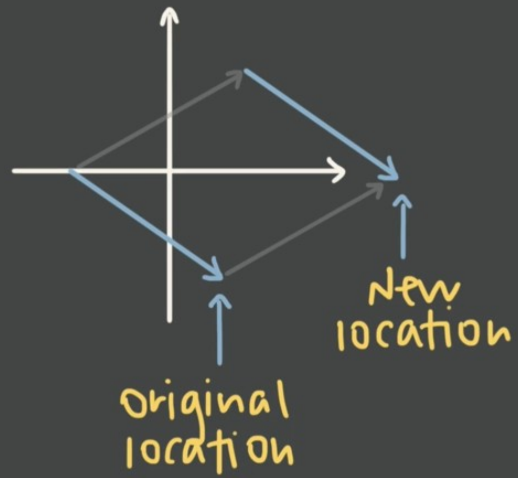
Transformations



Reflection
stretch horizontally
Rotate

↓

Transformation matrix

Functions and transformations

function $f(x) = x^2$

x	$f(x)$
0	0
1	1
2	4
3	9
⋮	

$f: x \mapsto x^2$

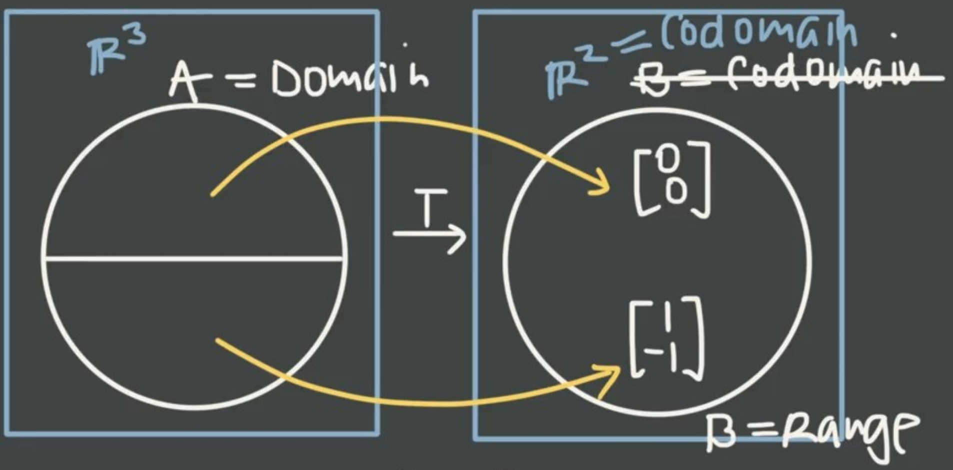
vector-valued function

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

transformation

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} -5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7 \\ -15 \end{bmatrix}$



$T: A \to B$

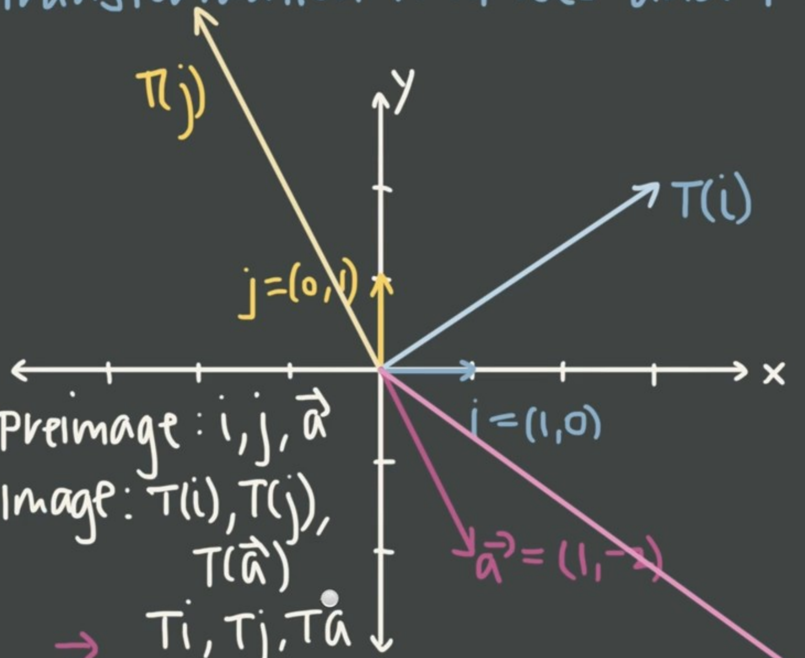
$T: \mathbb{R}^3 \to \mathbb{R}^2$

Domain : $\mathbb{R}^3$

Codomain : $\mathbb{R}^2$

Range : $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Transformation matrices and the image of the subset



$$T = \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} \end{array}$$

$$T = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$$

$$T = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \end{array}$$

$i : (1,0,0) \to (1,1,1)$

$j : (0,1,0) \to (2,2,2)$

$k : (0,0,1) \to (3,3,3)$

Preimage: $i, j, \vec{a}$

Image: $T(i), T(j), T(\vec{a})$

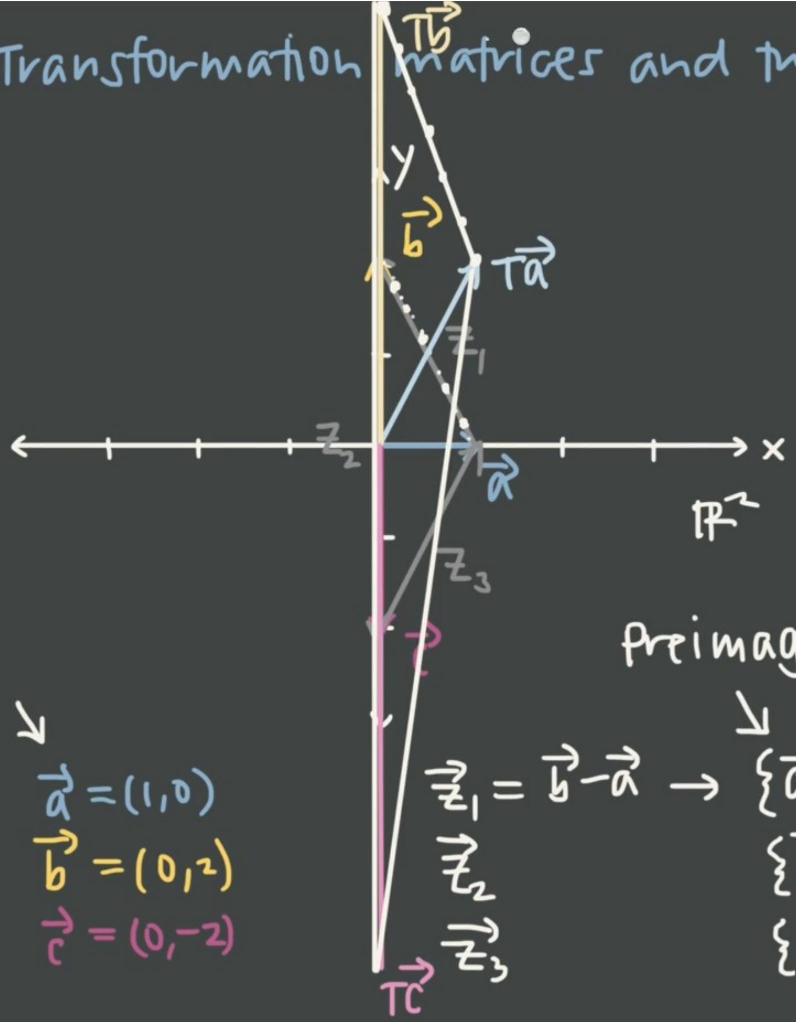
$Ti, Tj, T\vec{a}$

$\vec{a}$

$$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) : \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) - 2(-2) \\ 2(1) + 4(-2) \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

Transformation matrices and the image of the subset



$$T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$T\vec{a} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T\vec{b} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$T\vec{c} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$$

Image

$T(\vec{a} + t(\vec{b} - \vec{a}))$

$T\vec{a} + T(t(\vec{b} - \vec{a}))$

$T\vec{a} + tT(\vec{b} - \vec{a})$

$T\vec{a} + t(T\vec{b} - T\vec{a})$

Preimage

$\vec{a} = (1,0)$

$\vec{b} = (0,2)$

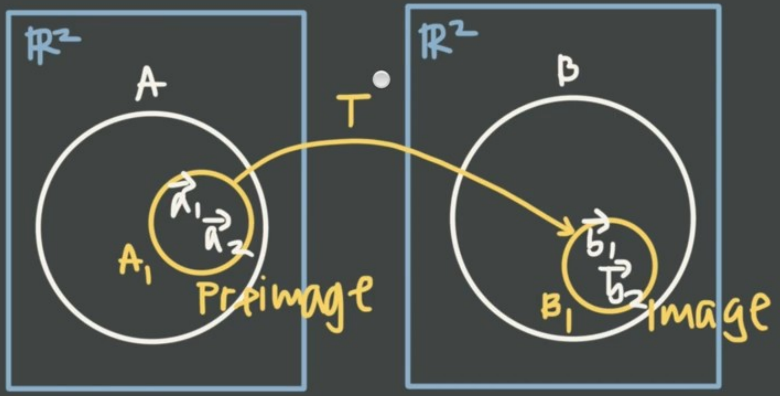
$\vec{c} = (0,-2)$

$\vec{z}_1 = \vec{b} - \vec{a} \rightarrow \{\vec{a} + t(\vec{b} - \vec{a}) \mid 0 \le t \le 1\}$

$\vec{z}_2 \quad \{\vec{b} + t(\vec{c} - \vec{b}) \mid 0 \le t \le 1\}$

$\vec{z}_3 \quad \{\vec{c} + t(\vec{a} - \vec{c}) \mid 0 \le t \le 1\}$

Preimage, image, and the kernel



$$\overset{T}{\begin{bmatrix}4&0\\-2&3\end{bmatrix}}\vec{a}_1=\vec{b}_1=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\parallel \quad \vec{a}_2=\vec{b}_2=\begin{bmatrix}4\\1\end{bmatrix}$$

$\boxed{\mathrm{Ker}(T)}$: all the vectors that map to $\vec{0}$

$T: A \to B$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\boxed{\begin{bmatrix}0\\0\end{bmatrix}\ \begin{bmatrix}3\\4\end{bmatrix}\ \begin{bmatrix}1\\2\end{bmatrix}} \to \begin{bmatrix}0\\0\end{bmatrix}$$

$$T:\ \vec{a}_1=\begin{bmatrix}0\\0\end{bmatrix}\to\vec{b}_1=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\vec{a}_2=\begin{bmatrix}1\\1\end{bmatrix}\to\vec{b}_2=\begin{bmatrix}4\\1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}4&0&0\\-2&3&0\end{array}\right] \qquad \left[\begin{array}{cc|c}4&0&4\\-2&3&1\end{array}\right]$$

$$\left[\begin{array}{cc|c}1&0&0\\-2&3&0\end{array}\right] \qquad \left[\begin{array}{cc|c}1&0&1\\-2&3&1\end{array}\right]$$

$$\left[\begin{array}{cc|c}1&0&0\\0&3&0\end{array}\right] \qquad \left[\begin{array}{cc|c}1&0&1\\0&3&3\end{array}\right]$$

$$\left[\begin{array}{cc|c}1&0&\boxed{0}\\0&1&\boxed{0}\end{array}\right] \qquad \left[\begin{array}{cc|c}1&0&\boxed{1}\\0&1&\boxed{1}\end{array}\right]$$

$\vec{a}_1=\begin{bmatrix}0\\0\end{bmatrix}$ $x=0$, $y=0$

$x=1$, $y=1$

$\vec{a}_2=\begin{bmatrix}1\\1\end{bmatrix}$

Linear transformations as matrix-vector products

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{a}+\vec{b}) = T(\vec{a})+T(\vec{b})$

$T(c\vec{a}) = cT(\vec{a})$

$$\overset{T}{\begin{bmatrix}1&0\\2&3\end{bmatrix}}\overset{\vec{a}}{\begin{bmatrix}a_1\\a_2\end{bmatrix}}=\overset{\vec{b}}{\begin{bmatrix}b_1\\b_2\end{bmatrix}}$$

$$T\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)=\begin{bmatrix}-a_1+2a_2\\a_2-3a_1\\a_1-a_2\end{bmatrix}$$

$i=(1,0)$

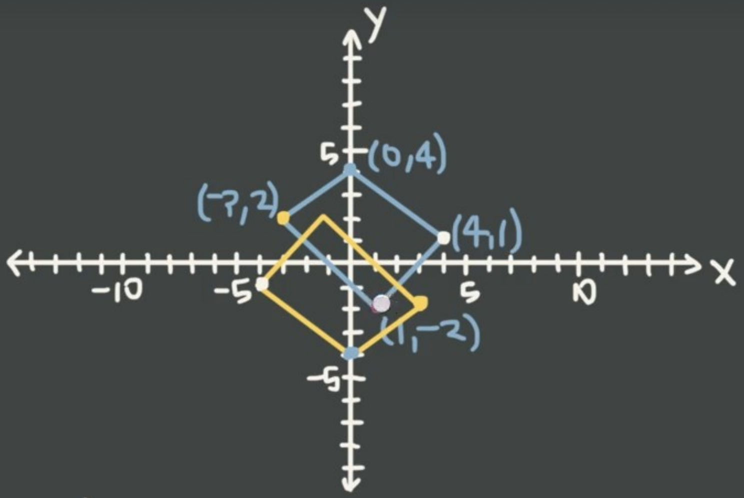
$j=(0,1)$

$$\begin{bmatrix}1\\2\end{bmatrix}\to\begin{bmatrix}3\\-1\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1+2(0)\\0-3(1)\\1-0\end{bmatrix}=\begin{bmatrix}-1\\-3\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0+2(1)\\1-3(0)\\0-1\end{bmatrix}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

$$T=\begin{bmatrix}-1&2\\-3&1\\1&-1\end{bmatrix}\quad\vec{a}=\vec{b}$$



Flip across y-axis

Flip across x-axis

$$\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}-x\\y\end{bmatrix}\to\begin{bmatrix}-x\\-y\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\-1\end{bmatrix}$$

$$\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}=\boxed{\begin{bmatrix}-4\\-1\end{bmatrix}}\qquad\begin{bmatrix}-3\\2\end{bmatrix}=\boxed{\begin{bmatrix}3\\-2\end{bmatrix}}$$

$$\begin{bmatrix}0\\4\end{bmatrix}=\boxed{\begin{bmatrix}0\\-4\end{bmatrix}}\qquad\begin{bmatrix}1\\-2\end{bmatrix}=\boxed{\begin{bmatrix}-1\\2\end{bmatrix}}$$

Linear transformations as rotations

$\mathbb{R}^2$

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Rot_\theta(\vec{a}+\vec{b}) = Rot_\theta(\vec{a}) + Rot_\theta(\vec{b})$$

$$Rot_\theta(c\vec{a}) = c\,Rot_\theta(\vec{a})$$

$\mathbb{R}^3$

$$Rot_{\theta\,\text{around}\,x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$Rot_{\theta\,\text{around}\,y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Rot_{\theta\,\text{around}\,z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\theta = 90^\circ$

$$Rot_{90^\circ} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Adding and scaling linear transformations

Adding

$S: \boxed{\begin{matrix}\mathbb{R}^n \\ \mathbb{R}^n\end{matrix}} \rightarrow \boxed{\begin{matrix}\mathbb{R}^m \\ \mathbb{R}^m\end{matrix}}$

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$A\vec{x}$ $\quad A = m \times n$

$B\vec{x}$ $\quad B = m \times n$

$$(S+T)(\vec{x}) = S(\vec{x}) + T(\vec{x})$$
$$= A\vec{x} + B\vec{x}$$
$$= (A+B)\vec{x}$$

$$S(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

scaling

$cT(\vec{x})$

$c(B\vec{x}) = c\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

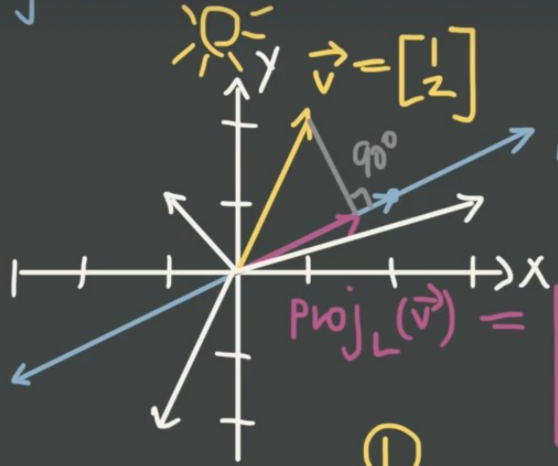
$(cB)\vec{x}$

$$= -2\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

Projections as linear transformations



$\vec v = \begin{bmatrix}1\\2\end{bmatrix}$

$L = c\begin{bmatrix}2\\1\end{bmatrix} = \vec x$

$Proj_L(\vec v) = \begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}$

$\|\vec x\| = \sqrt{2^2+1^2}$

$= \sqrt{4+1}$

$= \sqrt{5}$

③ $Proj_L(\vec a + \vec b)$

$= Proj_L(\vec a) + Proj_L(\vec b)$

$Proj_L(c\vec a) = c\,Proj_L(\vec a)$

② $$\hat u = \frac{1}{\sqrt5}\begin{bmatrix}2\\1\end{bmatrix} = \begin{array}{cc}\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{bmatrix} & \begin{matrix}u_1\\ u_2\end{matrix}\end{array}$$

$Proj_L(\vec v) = A\vec v$

$$A = \begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}$$

① $$Proj_L(\vec v) = c\vec x = \left(\frac{\vec v\cdot\vec x}{\vec x\cdot\vec x}\right)\vec x = (\vec v\cdot\vec x)\vec x = (\vec v\cdot\hat u)\hat u$$

$$c = \frac{\vec v\cdot\vec x}{\vec x\cdot\vec x} \qquad = \left(\frac{[1\ 2]\begin{bmatrix}2\\1\end{bmatrix}}{[2\ 1]\begin{bmatrix}2\\1\end{bmatrix}}\right)\begin{bmatrix}2\\1\end{bmatrix}$$

$$c = \frac{\vec v\cdot\vec x}{\|\vec x\|^2} \qquad = \left(\frac{4}{5}\right)\begin{bmatrix}2\\1\end{bmatrix} = \boxed{\begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}}$$

$$c = \vec v\cdot\vec x$$

$$= \left([1\ 2]\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{bmatrix}\right)\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{bmatrix}$$

$$= \left(\frac{2}{\sqrt5}+\frac{2}{\sqrt5}\right)\begin{bmatrix}\ \end{bmatrix}$$

$$\frac{4}{\sqrt5}\begin{bmatrix}\ \end{bmatrix} = \boxed{\begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}}$$

$$A = \begin{bmatrix}\frac45 & \frac25\\ \frac25 & \frac15\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix}\frac45+\frac45\\ \frac25+\frac25\end{bmatrix} = \boxed{\begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}}$$

Intuition [edit]

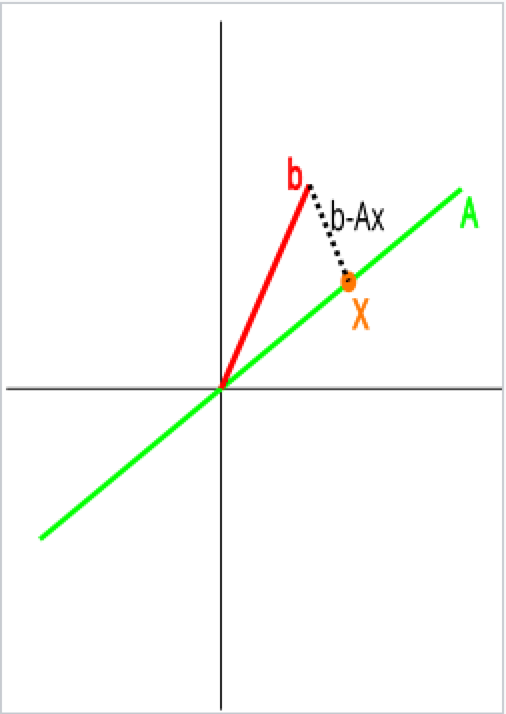
From the figure, it is clear that the closest point from the vector $\mathbf{b}$ onto the column space of $\mathbf{A}$, is $\mathbf{Ax}$, and is one where we can draw a line orthogonal to the column space of $\mathbf{A}$. A vector that is orthogonal to the column space of a matrix is in the nullspace of the matrix transpose, so

$$\mathbf{A}^{\mathsf{T}}(\mathbf{b} - \mathbf{Ax}) = 0.$$

From there, one rearranges, so

$$\begin{aligned} & \mathbf{A}^{\mathsf{T}}\mathbf{b} - \mathbf{A}^{\mathsf{T}}\mathbf{Ax} = 0 \\ \Rightarrow \quad & \mathbf{A}^{\mathsf{T}}\mathbf{b} = \mathbf{A}^{\mathsf{T}}\mathbf{Ax} \\ \Rightarrow \quad & \mathbf{x} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{b} \end{aligned}.$$

Therefore, since $\mathbf{Ax}$ is on the column space of $\mathbf{A}$, the projection matrix, which maps $\mathbf{b}$ onto $\mathbf{Ax}$, is $\mathbf{A}\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}$.

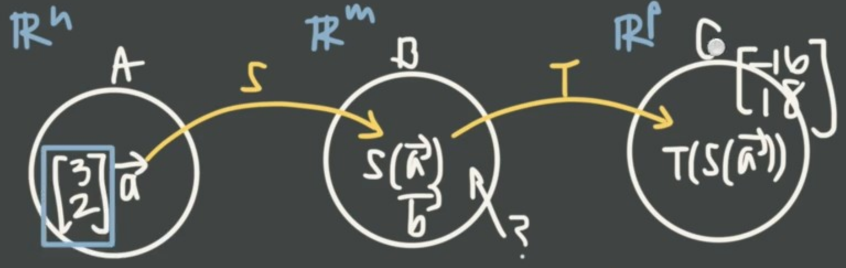


A matrix, $\mathbf{A}$ has its column space depicted as the green line. The projection of some vector $\mathbf{b}$ onto the column space of $\mathbf{A}$ is the vector $\mathbf{x}$

Linear model [edit]

Suppose that we wish to estimate a linear model using linear least squares. The model

Compositions of linear transformations



$S: A \to B$

$T: B \to C$

$\cancel{S(T(\vec{a}))}$

$\cancel{S \circ T(\vec{a})}$

$T(S(\vec{a}))$

$T \circ S(\vec{a})$

$m \times n$ $\quad S(\vec{a}) = A\vec{a}$

$p \times m$ $\quad T(\vec{b}) = B\vec{b}$

$$T \circ S(\vec{a}) = T(S(\vec{a}))$$
$$= T(A\vec{a})$$
$$= BA\vec{a} \qquad = C\vec{a}$$

$BA = C$

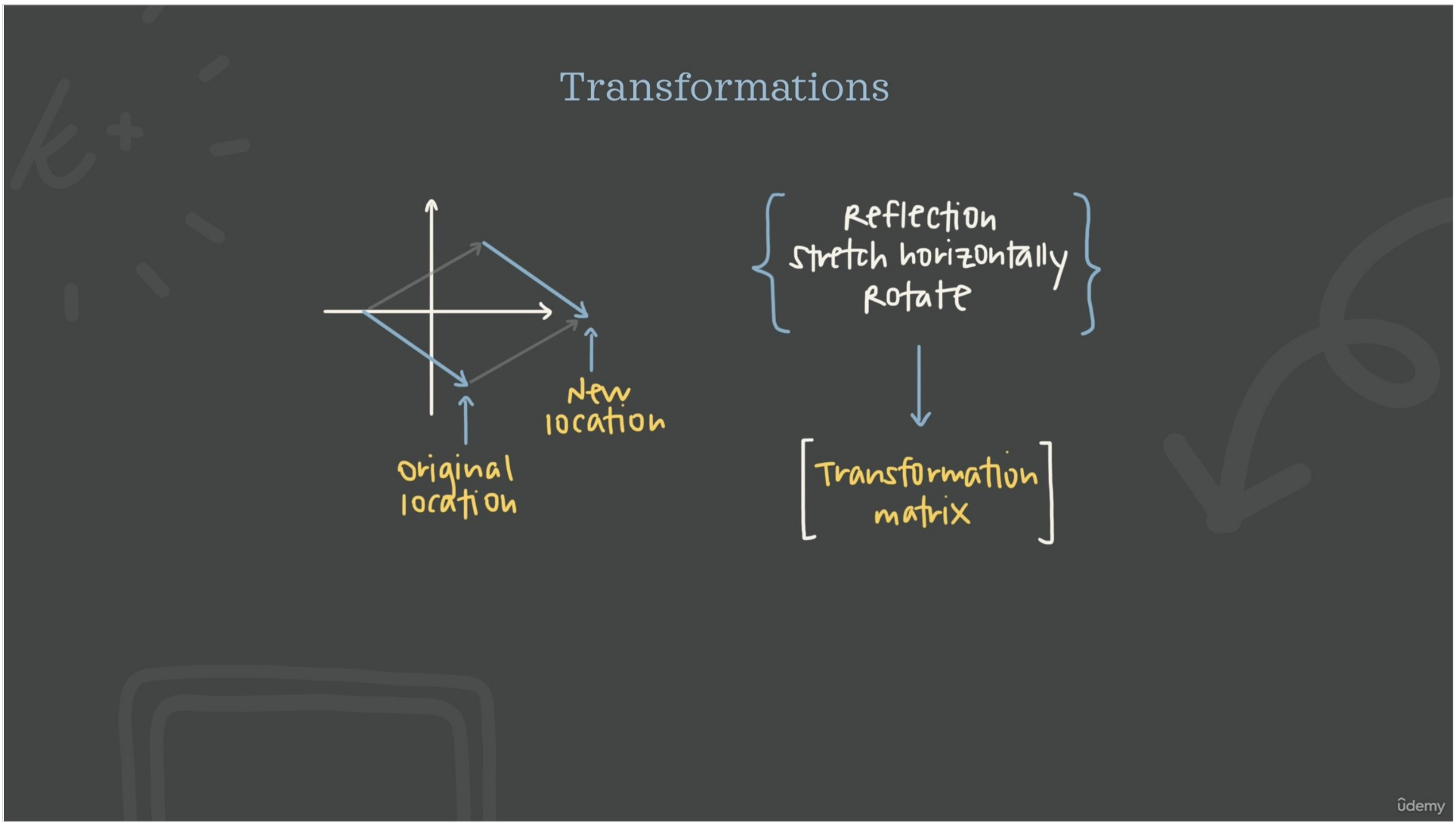
$$S(\vec{a}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix} \vec{a} \qquad \vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$T(\vec{b}) = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix} \vec{b}$$

$$T(S(\vec{a})) = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -8 \\ 6 & 0 \end{bmatrix}$$

$$C\vec{a} = \begin{bmatrix} 0 & -8 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 \\ 18 \end{bmatrix}$$


Transformations
new location
original location
Reflection
stretch horizontally
Rotate
Transformation matrix
ûdemy

Functions and transformations

function

$f(x) = x^2$

x	$f(x)$
0	0
1	1
2	4
3	9
$\vdots$	

$f: x \mapsto x^2$

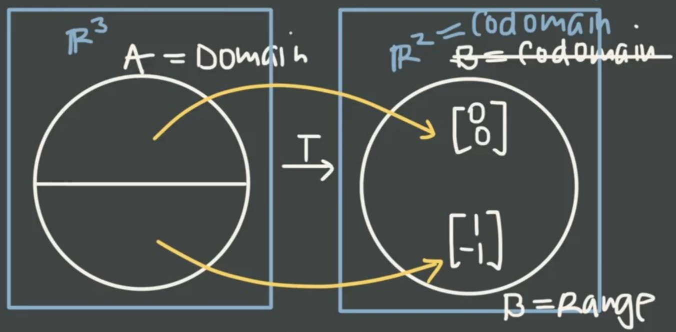
vector-valued function

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

transformation

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} -5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7 \\ -15 \end{bmatrix}$



$T: A \to B$

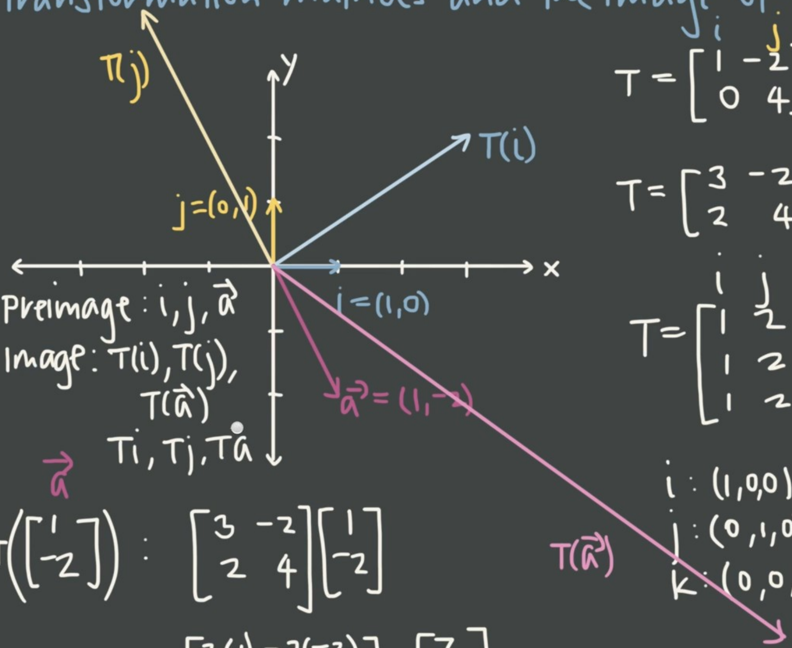
$T: \mathbb{R}^3 \to \mathbb{R}^2$

Domain: $\mathbb{R}^3$

Codomain: $\mathbb{R}^2$

Range: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Transformation matrices and the image of the subset



Preimage: $i, j, \vec{a}$

Image: $T(i), T(j), T(\vec{a})$

$Ti, Tj, T\vec{a}$

$$T = \begin{array}{cc} i & j \\ \end{array}$$
$$T = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$$

$$T = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \leftarrow$$

$$T = \begin{array}{ccc} i & j & k \end{array}$$
$$T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$i: (1,0,0) \rightarrow (1,1,1)$

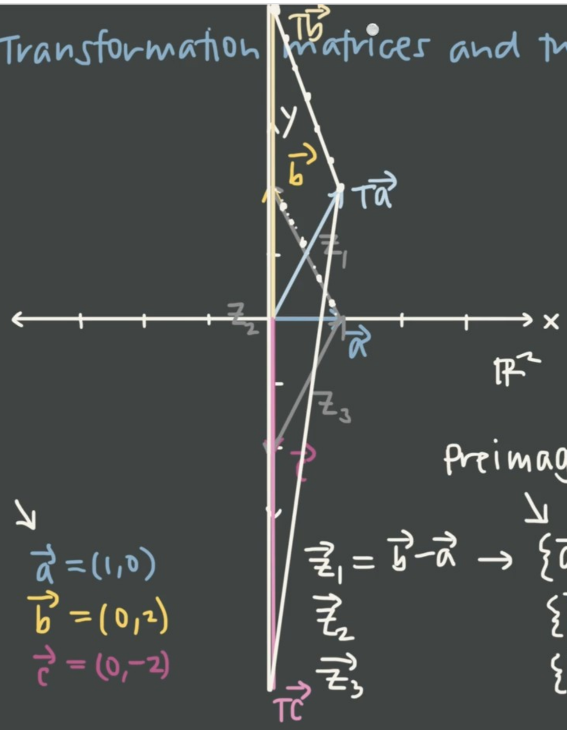
$j: (0,1,0) \rightarrow (2,2,2)$

$k: (0,0,1) \rightarrow (3,3,3)$

$$T\left(\underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\vec{a}}\right) : \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1) - 2(-2) \\ 2(1) + 4(-2) \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

Transformation matrices and the image of the subset



$$T=\begin{bmatrix}1&0\\2&3\end{bmatrix}$$

$$T\vec{a}=\begin{bmatrix}1&0\\2&3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

$$T\vec{b}=\begin{bmatrix}1&0\\2&3\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}0\\6\end{bmatrix}$$

$$T\vec{c}=\begin{bmatrix}1&0\\2&3\end{bmatrix}\begin{bmatrix}0\\-2\end{bmatrix}=\begin{bmatrix}0\\-6\end{bmatrix}$$

$\vec{a}=(1,0)$

$\vec{b}=(0,2)$

$\vec{c}=(0,-2)$

Preimage

$\vec{z}_1=\vec{b}-\vec{a} \rightarrow \{\vec{a}+t(\vec{b}-\vec{a}) \mid 0\le t\le 1\}$

$\vec{z}_2 \quad \{\vec{b}+t(\vec{c}-\vec{b}) \mid 0\le t\le 1\}$

$\vec{z}_3 \quad \{\vec{c}+t(\vec{a}-\vec{c}) \mid 0\le t\le 1\}$

Image

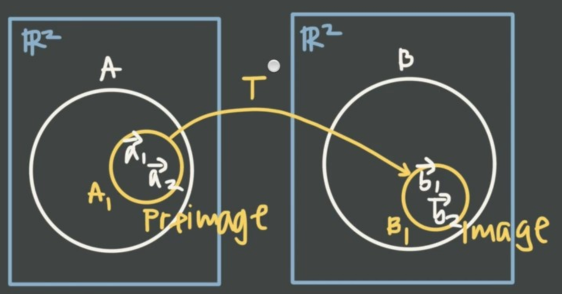
$T(\vec{a}+t(\vec{b}-\vec{a}))$

$T\vec{a}+T(t(\vec{b}-\vec{a}))$

$T\vec{a}+tT(\vec{b}-\vec{a})$

$T\vec{a}+t(T\vec{b}-T\vec{a})$

Preimage, image, and the kernel



$\boxed{\text{Ker}(T)}$: all the vectors that map to $\vec{0}$

$T: A \to B$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\boxed{\begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}3\\4\end{bmatrix}} \to \begin{bmatrix}0\\0\end{bmatrix}$

$$T: \begin{array}{l} \vec{a}_1 = \begin{bmatrix}0\\0\end{bmatrix} \to \vec{b}_1 = \begin{bmatrix}0\\0\end{bmatrix} \\ \vec{a}_2 = \begin{bmatrix}1\\1\end{bmatrix} \to \vec{b}_2 = \begin{bmatrix}4\\1\end{bmatrix} \end{array}$$

$\vec{a}_1 = \begin{bmatrix}0\\0\end{bmatrix}$ $x=0$, $y=0$

$$\overset{T}{\begin{bmatrix}4 & 0\\-2 & 3\end{bmatrix}} \vec{a}_1 = \vec{b}_1 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\vec{a}_2 = \vec{b}_2 = \begin{bmatrix}4\\1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}4 & 0 & 0\\-2 & 3 & 0\end{array}\right] \qquad \left[\begin{array}{cc|c}4 & 0 & 4\\-2 & 3 & 1\end{array}\right]$$

$$\left[\begin{array}{cc|c}1 & 0 & 0\\-2 & 3 & 0\end{array}\right] \qquad \left[\begin{array}{cc|c}1 & 0 & 1\\-2 & 3 & 1\end{array}\right]$$

$$\left[\begin{array}{cc|c}1 & 0 & 0\\0 & 3 & 0\end{array}\right] \qquad \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 3 & 3\end{array}\right]$$

$$\left[\begin{array}{cc|c}1 & 0 & \boxed{0}\\0 & 1 & \boxed{0}\end{array}\right] \qquad \left[\begin{array}{cc|c}1 & 0 & \boxed{1}\\0 & 1 & \boxed{1}\end{array}\right]$$

$x=1$

$y=1$

$\vec{a}_2 = \begin{bmatrix}1\\1\end{bmatrix}$

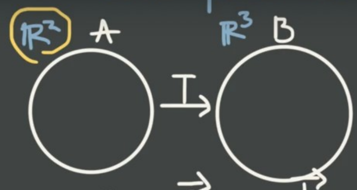
101. Linear transformations as matrix-vector products

Linear transformations as matrix-vector products

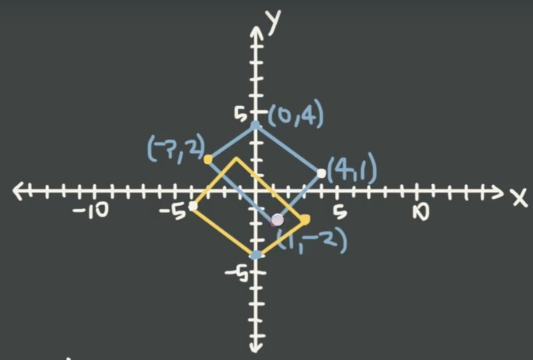
$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$

$T(c\vec{a}) = cT(\vec{a})$



$$\begin{array}{ccc} T & \vec{a} & \vec{b} \\ \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = & \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{array}$$



$$T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} -a_1 + 2a_2 \\ a_2 - 3a_1 \\ a_1 - a_2 \end{bmatrix}$$

$i = (1,0)$

$j = (0,1)$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

$$T = \begin{bmatrix} -1 & 2 \\ -3 & 1 \\ 1 & -1 \end{bmatrix} \vec{a} = \vec{b}$$

Flip across y-axis

Flip across x-axis

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -x \\ y \end{bmatrix} \to \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 + 2(0) \\ 0 - 3(1) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -0 + 2(1) \\ 1 - 3(0) \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ -1 \end{bmatrix}} \qquad \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ -2 \end{bmatrix}}$$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ -4 \end{bmatrix}} \qquad \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

2x 14:35 / 14:49

Linear transformations as rotations

$\mathbb{R}^2$

$$Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Rot_\theta(\vec{a}+\vec{b}) = Rot_\theta(\vec{a}) + Rot_\theta(\vec{b})$$

$$Rot_\theta(c\vec{a}) = c\,Rot_\theta(\vec{a})$$

$\mathbb{R}^3$

$$Rot_{\theta \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

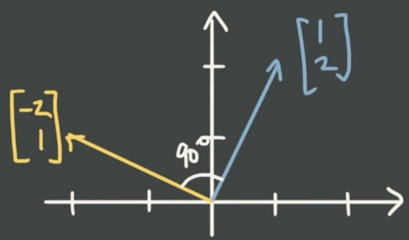
$$Rot_{\theta \text{ around } y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Rot_{\theta \text{ around } z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\theta = 90°$

$$Rot_{90°} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Adding and scaling linear transformations

Adding

$S: \boxed{\mathbb{R}^n} \to \boxed{\mathbb{R}^m}$

$T: \boxed{\mathbb{R}^n} \to \boxed{\mathbb{R}^m}$

$A\vec{x}$ $\quad A = m \times n$

$B\vec{x}$ $\quad B = m \times n$

$$(S+T)(\vec{x}) = S(\vec{x}) + T(\vec{x})$$
$$= A\vec{x} + B\vec{x}$$
$$= (A+B)\vec{x}$$

$$S(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Scaling

$cT(\vec{x})$

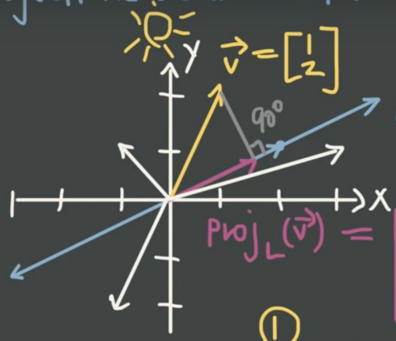
$c(B\vec{x})$

$(cB)\vec{x}$

$$= c\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= -2\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

Projections as linear transformations

107. Projections as linear transformations



$\vec v = \begin{bmatrix}1\\2\end{bmatrix}$

$L = c\begin{bmatrix}2\\1\end{bmatrix} = \vec x$

$\mathrm{Proj}_L(\vec v) = \begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}$

$$\|\vec x\| = \sqrt{2^2+1^2} = \sqrt{4+1} = \sqrt5$$

(2)

$$\hat u = \frac{1}{\sqrt5}\begin{bmatrix}2\\1\end{bmatrix} = \begin{array}{cc}\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{bmatrix} & \begin{array}{c}u_1\\ u_2\end{array}\end{array}$$

(3)

$$\mathrm{Proj}_L(\vec a+\vec b) = \mathrm{Proj}_L(\vec a) + \mathrm{Proj}_L(\vec b)$$

$$\mathrm{Proj}_L(c\vec a) = c\,\mathrm{Proj}_L(\vec a)$$

$$\mathrm{Proj}_L(\vec v) = A\vec v$$

$$A = \begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}$$

$$A = \begin{bmatrix}\frac{4}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{1}{5}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\frac{4}{5}+\frac{4}{5}\\ \frac{2}{5}+\frac{2}{5}\end{bmatrix} = \boxed{\begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}}$$

(1)

$$\mathrm{Proj}_L(\vec v) = c\vec x = \left(\frac{\vec v\cdot\vec x}{\vec x\cdot\vec x}\right)\vec x = (\vec v\cdot\vec x)\vec x = (\vec v\cdot\hat u)\hat u$$

$$c = \frac{\vec v\cdot\vec x}{\vec x\cdot\vec x} \qquad = \left(\frac{[1\ 2]\begin{bmatrix}2\\1\end{bmatrix}}{[2\ 1]\begin{bmatrix}2\\1\end{bmatrix}}\right)\begin{bmatrix}2\\1\end{bmatrix} \qquad = \left([1\ 2]\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{bmatrix}\right)\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{bmatrix}$$

$$c = \frac{\vec v\cdot\vec x}{\|\vec x\|^2} \qquad = \left(\frac{4}{5}\right)\begin{bmatrix}2\\1\end{bmatrix} = \boxed{\begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}} \qquad = \left(\frac{2}{\sqrt5}+\frac{2}{\sqrt5}\right)\begin{bmatrix} \ \end{bmatrix}$$

$$c = \vec v\cdot\vec x \qquad \frac{4}{\sqrt5}\begin{bmatrix} \ \end{bmatrix} = \boxed{\begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}}$$

1x 15:39 / 15:52

udemy

Intuition [edit]

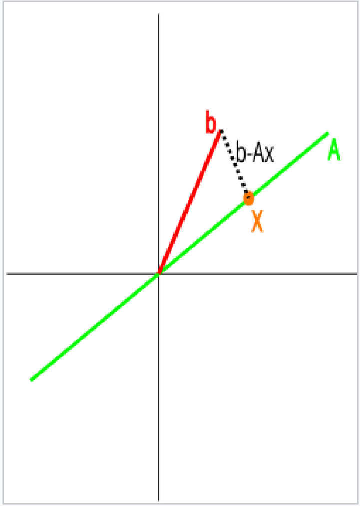
From the figure, it is clear that the closest point from the vector $\mathbf{b}$ onto the column space of $\mathbf{A}$, is $\mathbf{Ax}$, and is one where we can draw a line orthogonal to the column space of $\mathbf{A}$. A vector that is orthogonal to the column space of a matrix is in the nullspace of the matrix transpose, so

$$\mathbf{A}^{\mathsf{T}}(\mathbf{b}-\mathbf{Ax})=0.$$

From there, one rearranges, so

$$\begin{aligned} & \mathbf{A}^{\mathsf{T}}\mathbf{b}-\mathbf{A}^{\mathsf{T}}\mathbf{Ax}=0 \\ \Rightarrow \quad & \mathbf{A}^{\mathsf{T}}\mathbf{b}=\mathbf{A}^{\mathsf{T}}\mathbf{Ax} \\ \Rightarrow \quad & \mathbf{x}=\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{b} \end{aligned}.$$

Therefore, since $\mathbf{Ax}$ is on the column space of $\mathbf{A}$, the projection matrix, which maps $\mathbf{b}$ onto $\mathbf{Ax}$, is $\mathbf{A}\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}$.

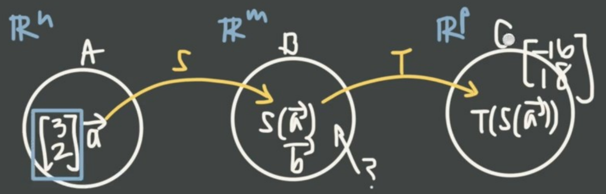


A matrix, $\mathbf{A}$ has its column space depicted as the green line. The projection of some vector $\mathbf{b}$ onto the column space of $\mathbf{A}$ is the vector $\mathbf{x}$

Linear model [edit]

Suppose that we wish to estimate a linear model using linear least squares. The model

Compositions of linear transformations



$S: A \to B$

$T: B \to C$

$\cancel{S(T(\vec{a}))}$

$\cancel{S \circ T(\vec{a})}$

$m\times n \quad S(\vec{a}) = A\vec{a}$

$p\times m \quad T(\vec{b}) = B\vec{b}$

$T(S(\vec{a}))$

$T \circ S(\vec{a})$

$$T\circ S(\vec{a}) = T(S(\vec{a})) = T(A\vec{a}) = BA\vec{a} = C\vec{a}$$

$BA = C$

$$S(\vec{a}) = \begin{bmatrix}2 & 0\\ -1 & 4\end{bmatrix}\vec{a} \qquad \vec{a} = \begin{bmatrix}3\\2\end{bmatrix}$$

$$T(\vec{b}) = \begin{bmatrix}-1 & -2\\ 3 & 0\end{bmatrix}\vec{b}$$

$$T(S(\vec{a})) = \begin{bmatrix}-1 & -2\\ 3 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\ -1 & 4\end{bmatrix}$$

$$C = \begin{bmatrix}0 & -8\\ 6 & 0\end{bmatrix}$$

$$C\vec{a} = \begin{bmatrix}0 & -8\\ 6 & 0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-16\\18\end{bmatrix}$$