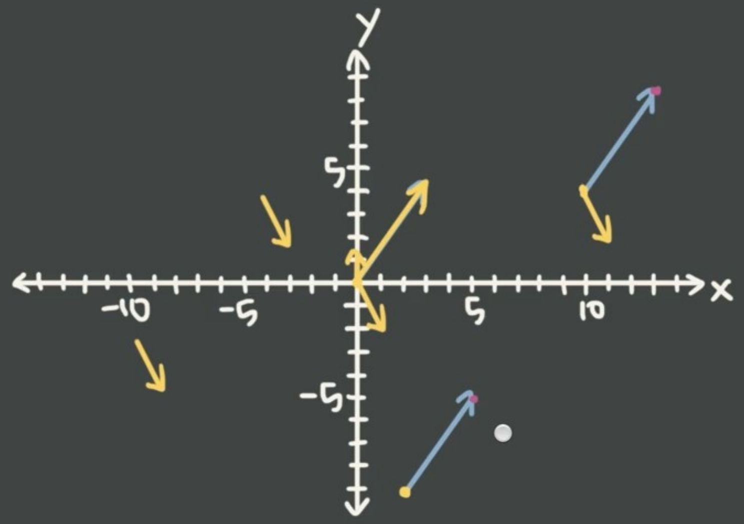
Vectors

1. direction
2. magnitude (length)



$\vec{a} = (3, 4)$

$\vec{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\vec{a} = \begin{bmatrix} 3 & 4 \end{bmatrix}$

$K = \begin{bmatrix} 1 & 0 & 0 & 3 \\ -2 & 1 & 1 & 4 \end{bmatrix}$

Col. vectors

$k_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $k_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $k_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$k_4 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\mathbb{R}^2$

Row vectors

$k_1 = \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$ $\mathbb{R}^4$

$k_2 = \begin{bmatrix} -2 & 1 & 1 & 4 \end{bmatrix}$

vector operations

$$\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix} \qquad \begin{bmatrix}-1\\2\end{bmatrix}-\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}-4\\-2\end{bmatrix}$$

$\vec{a}=(-1,2)=A$

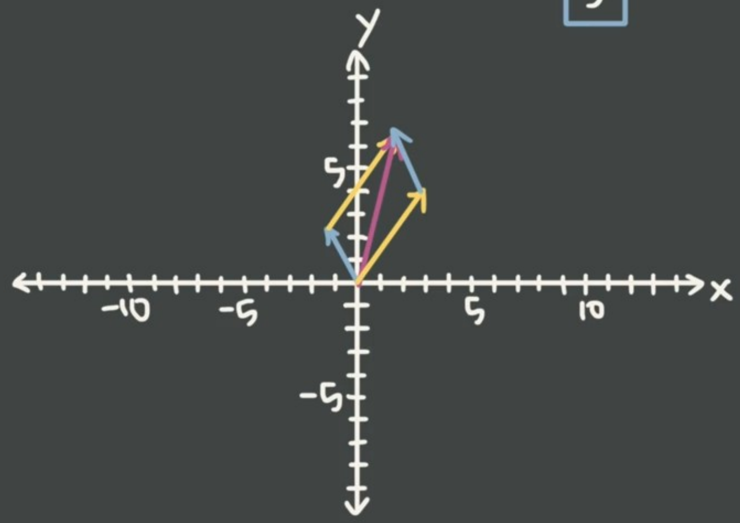
$\vec{b}=(3,4)=B$

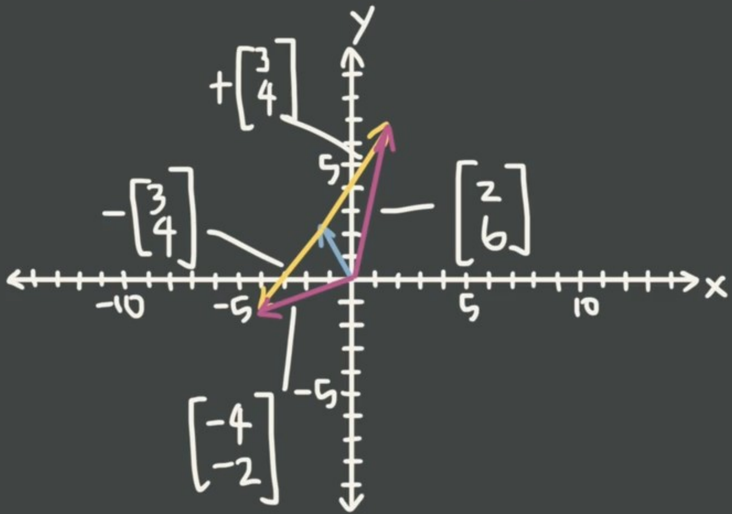
$$\vec{a}\cdot\vec{b}=(-1)(3)+(2)(4)$$
$$=-3+8$$
$$=\boxed{5}$$

$1\times2 \quad 2\times1$

$$AB=\begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}$$
$$=(-1)(3)+(2)(4)$$
$$=-3+8=\boxed{5}$$

$$BA=\begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$$
$$=(3)(-1)+(4)(2)$$
$$=-3+8$$
$$=\boxed{5}$$





Dot products and cross products

$\vec{a} \cdot \vec{b}$

Dot product:
How much $\vec{a}$ and $\vec{b}$
point in the same direction

$\vec{a} \times \vec{b}$

Cross product:
The length shows how
much $\vec{a}$ and $\vec{b}$ point
in different directions

Unit vectors and basis vectors

$\vec{u}$ $\hat{u}$

✓ direction
✗ length = 1

$\hat{i}$ $\hat{j}$ $\hat{k}$ = standard basis

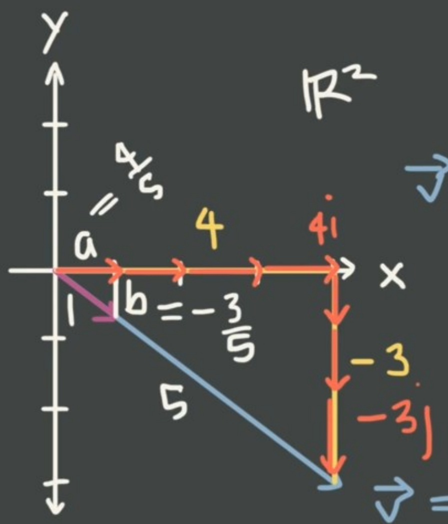
i j k

$$\|\vec{v}\| = \sqrt{4^2 + (-3)^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= 5$$

$\vec{v} = (4, -3)$

$\vec{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} = [4 \ \ -3]$

$\vec{v} = 4i - 3j$



$\vec{v} = 4i - 3j$

$\vec{v} = (4, -3)$

$\vec{u} = \left(\frac{4}{5}, -\frac{3}{5}\right)$

$$\frac{4}{5} = \frac{a}{1} \qquad -\frac{3}{5} = \frac{b}{1}$$

$$a = \frac{4}{5} \qquad b = -\frac{3}{5}$$

$\mathbb{R}^2$ $i = (1, 0)$
$j = (0, 1)$

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v}$$
$$= \frac{1}{5} \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$

$\vec{v} = i + 2j + 3k$ $\mathbb{R}^3$

$\vec{v} = (1, 2, 3)$

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2}$$
$$= \sqrt{1 + 4 + 9}$$
$$= \sqrt{14}$$

$$\hat{u} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

$\mathbb{R}^3$ $i = (1, 0, 0)$
$j = (0, 1, 0)$
$k = (0, 0, 1)$

Linear combinations and span

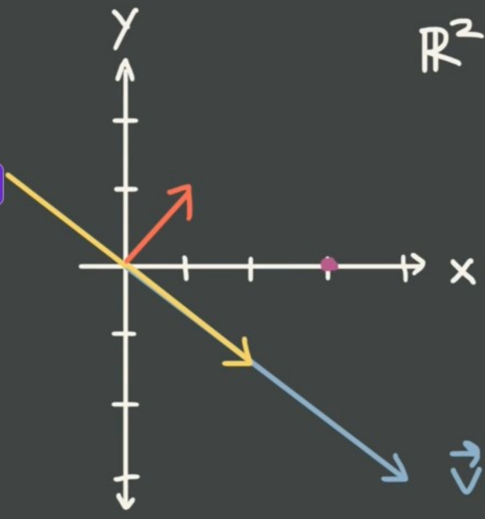
↳ sum of scaled vectors

$$\vec{v} = 4i - 3j$$
$$= 2\vec{a} + 4\vec{b} - \vec{c}$$

↳ all the linear combinations

$$\mathbb{R}^2 = \text{span}(\hat{i}, \hat{j})$$

$$\vec{a} = (7,-2) = 7i - 2j \qquad \vec{c} = (0,4) = 4j$$
$$\vec{b} = (-2,9) = -2i + 9j$$



$\mathbb{R}^2$

$$\vec{v} = (4,-3)$$
$$\vec{v} = 4i - 3j$$

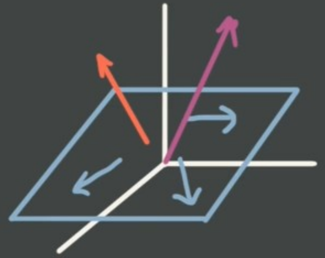
$$\vec{v} = 4\begin{bmatrix}1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathbb{R}^3 = \text{span}(\hat{i}, \hat{j}, \hat{k})$$

$$\vec{d} = (0,17,-4) = 17j - 4k$$

$\mathbb{R}^2$: 2, L.I., in $\mathbb{R}_2$

$\mathbb{R}^3$: 3, L.I., in $\mathbb{R}_3$

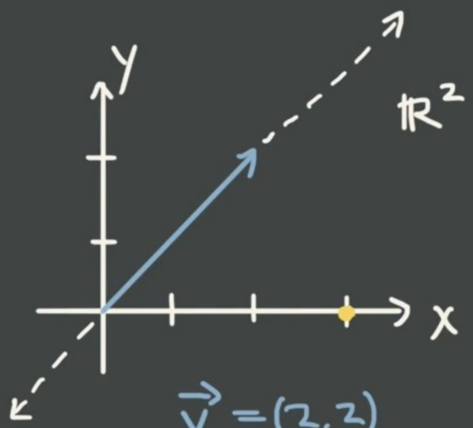


Linear independence in $\mathbb{R}^2$

$\mathbb{R}^2$: 2, L.I., in $\mathbb{R}^2$

$\mathbb{R}^3$: 3, L.I., in $\mathbb{R}^3$

$\mathbb{R}^n$: n, L.I., in $\mathbb{R}^n$



$\vec{v} = (2,2)$

$\vec{w} = (4,4)$

L.D.

$$2\vec{v} = 2\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix} = \vec{w}$$

$$2\vec{v} = \vec{w}$$

$$\vec{v} = \frac{1}{2}\vec{w}$$

$$c_1\begin{bmatrix}2\\2\end{bmatrix}$$

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\6\end{bmatrix} + c_3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}1\\1\end{bmatrix} \quad \begin{bmatrix}0\\6\end{bmatrix}}_{\text{L.I.}}$$

$$c_1 = -2c_3$$

$$c_1 = -6c_2 - 3c_3$$

$$-6c_2 - 3c_3 = -2c_3$$

$$-6c_2 = c_3$$

$$c_1 + 2c_3 = 0$$

$$c_1 + 6c_2 + 3c_3 = 0$$

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$c_1 + 0 = 0 \Rightarrow \boxed{c_1 = 0}$$

$$c_1 + 6c_2 = 0$$

$$0 + 6c_2 = 0$$

$$6c_2 = 0$$

$$\boxed{c_2 = 0}$$

$$c_1\begin{bmatrix}2\\2\end{bmatrix} + c_2\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$2c_1 + 4c_2 = 0 \rightarrow c_1 + 2c_2 = 0$$

$$2c_1 + 4c_2 = 0$$

$$c_1 = -2c_2$$

$c_2 = 1 \quad c_1 = -2$

$c_2 = 2 \quad c_1 = -4$

Linear independence in $\mathbb{R}^3$

- $\mathbb{R}^2$: 2, L.I., in $\mathbb{R}^2$
- $\mathbb{R}^3$: 3, L.I., in $\mathbb{R}^3$
- $\mathbb{R}^n$: n, L.I., in $\mathbb{R}^n$

$$\left.\begin{bmatrix}1\\1\\1\end{bmatrix} \begin{bmatrix}0\\-3\\2\end{bmatrix} \begin{bmatrix}1\\0\\3\end{bmatrix}\right\} \text{L.I.}$$

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\-3\\2\end{bmatrix} + c_3\begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 4/3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\boxed{\begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{array}}$$

Linear subspaces

$\mathbb{R}^2$: all 2D vectors $\vec{v} = (x, y)$

$\mathbb{R}^3$: all 3D vectors $\vec{v} = (x, y, z)$

$\vdots$

$\mathbb{R}^n$: all nD vectors $\vec{v} = (v_1, v_2, v_3, \ldots v_n)$

subspace of $\mathbb{R}^2$ → $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x + y = 0 \right\}$

$x = -y \quad y = -x$

$= \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ldots \right\}$

Subspace

✓1. include zero vector

✓2. closed under scalar multiplication

✓3. closed under addition

$V = \{ v_1, v_2, v_3, v_4 \ldots \}$

$0 = c v_3 \qquad v_2 + v_4$

✓Scalar multiplication

$V = \begin{bmatrix} x \\ -x \end{bmatrix} \quad \begin{bmatrix} -y \\ y \end{bmatrix}$

$c \begin{bmatrix} x \\ -x \end{bmatrix} = \begin{bmatrix} cx \\ -cx \end{bmatrix} \qquad cx + (-cx) = 0$

$cx - cx = 0$

$0 = 0$ ✓

$c \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} -cy \\ cy \end{bmatrix} \qquad -cy + cy = 0$

$0 = 0$ ✓

✓Addition

$\begin{bmatrix} x \\ -x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$

$x - \cancel{y} + \cancel{y} - x = 0 \qquad 0 = 0$ ✓

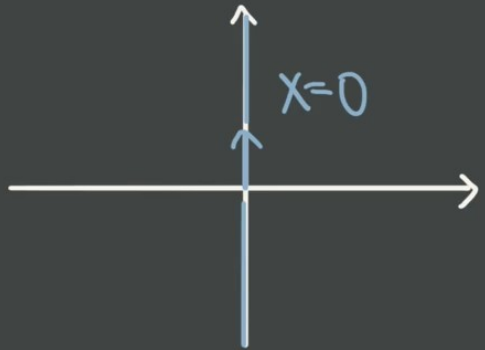
Spans as subspaces

1. Closed under scalar multiplication
2. Closed under addition

$V_2 = \text{span}\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$

$V_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}\right)$

$\mathbb{R}^2$	$\mathbb{R}^3$
$\mathbb{R}^2$	$\mathbb{R}^3$ V_3
$\vec{0} = (0,0)$	$\vec{0} = (0,0,0)$
Any line through $(0,0)$ V_2	Any plane through $(0,0,0)$ V_1



$V_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}\right) \rightarrow$ plane

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Basis

Subspace:

vector set that is

1. closed under multiplication
2. closed under addition

Basis:

1. span the subspace ✓
2. Linearly independent ✓

$\left[\begin{array}{l} \mathbb{R}^2: \ 2, \ \text{L.I.}, \ \text{in } \mathbb{R}^2 \\ \mathbb{R}^3: \ 3, \ \text{L.I.}, \ \text{in } \mathbb{R}^3 \end{array}\right]$

$$V = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$V = \left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\} \quad \text{Basis for } \mathbb{R}^2?$$

$$c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} x \\ y \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 5 & x \\ -3 & 1 & y \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 5/2 & x/2 \\ 0 & 1 & \frac{2}{17}y + \frac{3}{17}x \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 5/2 & x/2 \\ 0 & 17/2 & y + 3x/2 \end{array}\right] \qquad \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{17}x - \frac{5}{17}y \\ 0 & 1 & \frac{2}{17}y + \frac{3}{17}x \end{array}\right] \end{array}$$

$$c_1 = \frac{1}{17}x - \frac{5}{17}y \rightarrow c_1 = 0$$

$$c_2 = \frac{2}{17}y + \frac{3}{17}x \rightarrow c_2 = 0$$

$$V = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

$$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$