Multiplying matrices by vectors

AB

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix}$$

$$\vec{b} = (5, -1)$$

$A\vec{b}$

$m \times n \quad n \times 1$

$$A\vec{b} = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$3 \times \boxed{2 \quad 2} \times 1 = 3 \times 1$$

$$= \begin{bmatrix} 1(5) + 0(-1) \\ -2(5) + 4(-1) \\ 0(5) + 1(-1) \end{bmatrix}$$

$$A\vec{b} = \begin{bmatrix} 5 \\ -14 \\ -1 \end{bmatrix} \quad m \times \underline{\underline{1}}$$

m-row

C.V.

$\vec{w}B$

$1 \times m \quad m \times n$

$$\vec{w}B = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$1 \times \boxed{3 \quad 3} \times 2$$

$$= \begin{bmatrix} (1)(3) + (0)(1) - 2(0) & (1)(-4) + 0 + 4 \end{bmatrix}$$

$$\vec{w}B = \begin{bmatrix} 3 & 0 \end{bmatrix} \quad \underline{\underline{1}} \times n$$

n-column

R.V.

The null space and $A\vec{x}=\vec{0}$

Subspace

✓1. zero vector

$$\overset{A}{\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} a_{1,1}(0)+a_{1,2}(0) \\ a_{2,1}(0)+a_{2,2}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

✓2. under addition

$\vec{x}_1$

$\vec{x}_2$

$\vec{x}_1+\vec{x}_2$

$$\begin{bmatrix} A\vec{x}_1=\vec{0} \end{bmatrix}$$
$$A\vec{x}_2=\vec{0}$$

$$A\vec{x}_1+A\vec{x}_2=\vec{0}$$
$$A(\vec{x}_1+\vec{x}_2)=\vec{0}$$

✓3. Under multiplication

$\vec{x}_1$ $\quad c\vec{x}_1$

$$A(c\vec{x}_1)=\vec{0}$$
$$cA\vec{x}_1=\vec{0}$$

$$c\vec{0}=\vec{0}$$
$$\vec{0}=\vec{0}$$

$\vec{x}$ That satisfies

$A\vec{x}=\vec{0}$

is in the null space

$N(A)$

$\vec{0}$

$N(A)=\vec{0}$ L.I.

$N(A)=\vec{0}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ L.D.

$$x_1\begin{bmatrix} 4 \\ 2 \end{bmatrix}+x_2\begin{bmatrix} -2 \\ -1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1-2x_2=0$$
$$2x_1-x_2=0$$

$$A=\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right] \quad R_1-2R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$x_1-\frac{1}{2}x_2=0$$

$$x_1=\frac{1}{2}x_2 \qquad N(A)=$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\ldots$$

The null space of a matrix

$$A=\begin{bmatrix}2&0&-1&2\\-4&0&2&-4\\-6&0&-3&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$A \qquad \vec{x} \quad = \vec{0}$$

$$3\times 4 \qquad 4\times 1 \quad = 3\times 1$$

$R_2+2R_1\to R_2$

$$A=\begin{bmatrix}2&0&-1&2\\0&0&0&0\\-6&0&-3&-6\end{bmatrix}$$

$$=\begin{bmatrix}2&0&-1&2\\-6&0&-3&-6\\0&0&0&0\end{bmatrix}$$

$3R_1+R_2\to R_2$

$$=\begin{bmatrix}2&0&-1&2\\0&0&-6&0\\0&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&-1/2&1\\0&0&-6&0\\0&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&-1/2&1\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$R_1+\frac{1}{2}R_2\to R_1$

$$\mathrm{rref}(A)=\begin{array}{c}\begin{array}{cccc}x_1&x_2&x_3&x_4\end{array}\\\begin{bmatrix}1&0&0&1\\0&0&1&0\\0&0&0&0\end{bmatrix}\end{array}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

pivot

free

$$x_1+x_4=0 \;\to\; x_1=-x_4+0x_2$$

$$x_3=0 \qquad x_3=0+0x_4$$

$$x_4=-x_1+0x_2$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

$$N(A)=\mathrm{Span}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right)$$

Example: Robotics and Degrees of Freedom

Imagine a robotic arm with 7 joints trying to reach a point in 3D space. Because it has more joints than needed, there is a null space in its movement matrix.

This allows the robot to "wiggle" its elbow or adjust its posture (to avoid an obstacle) while keeping its hand perfectly still at the target coordinates.

Would you like to see the mathematical steps for finding the null space of a simple 2D mass-spring system?

The column space and $A\vec{x} = \vec{b}$

$N(A)$ $\quad A\vec{x} = \vec{0}$

$C(A)$ = linear combinations of columns of A
= span of columns of A

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \qquad C(A) = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)$$

$$\begin{array}{ccc} A & \vec{x} & \vec{b} \\ \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = & \begin{bmatrix} 6 \\ 9 \end{bmatrix} \end{array}$$

If $\vec{b} \in C(A)$, Then $\vec{x}$ exists as a solution

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$N(A) = \vec{0}$ columns of A L.I. → Basis

$N(A) = \vec{0}, \vec{v}_1$ " " " L.D. → Do not form a basis

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \rightarrow \overset{\text{rref}(A)}{\begin{array}{cc} x_1 & x_2 \\ \hline 1 & 2 \\ 0 & 0 \\ \text{pivot} & \text{free} \end{array}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\underline{x_2}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$N(A) = N(\text{rref}(A)) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$$

Basis

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \cancel{x_2} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$

Basis for $C(A) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$C(A) = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$

Solving $A\vec{x} = \vec{b}$

$A\vec{x} = \vec{0}$ ⟵ complementary

$A\underline{\vec{x}} = \vec{b}$ ⟵ particular

$$G = P + C$$

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\begin{aligned} & A\vec{x}_n = \vec{0} \\ + \; & A\vec{x}_p = \vec{b} \\ \hline & A\vec{x}_n + A\vec{x}_p = \vec{0} + \vec{b} \\ & A(\underline{\vec{x}_n + \vec{x}_p}) = \vec{b} \end{aligned}$$

$$A\vec{x} = \cancel{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 4 \\ 3 & 4 & 4 & 7 \end{bmatrix} \quad \text{with} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \checkmark \quad \begin{matrix} 2 \\ 5 \\ 7 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 5/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 1 \; \cancel{-b_1 + b_2} \\ 0 & 1 & 5/2 & 1 & 0 \; \cancel{-\tfrac{1}{2}b_2 + b_1} \\ 0 & 0 & 0 & 0 & 0 \; \cancel{b_3 - b_2 - b_1} \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_3 + x_4 &= 0 \\ x_2 + \tfrac{5}{2}x_3 + x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_1 - \cancel{2x_3} + \cancel{x_4} &= 1 \\ x_2 + \cancel{\tfrac{5}{2}x_3} + \cancel{x_4} &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= 2x_3 - x_4 \\ x_2 &= -\tfrac{5}{2}x_3 - x_4 \end{aligned} \qquad \begin{aligned} x_1 &= 1 \\ x_2 &= 0 \end{aligned}$$

$$\vec{x}_n = c_1 \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Dimensionality, nullity, and rank

$$A = \begin{bmatrix} \overbrace{\begin{bmatrix}2\\1\\0\end{bmatrix}\begin{bmatrix}0\\3\\-2\end{bmatrix}\begin{bmatrix}4\\-1\\1\end{bmatrix}}^{*} & \begin{matrix}-2\\0\\1\end{matrix}\end{bmatrix}$$

Dimension : # of basis vectors

$$\begin{matrix} 1 & 0 & 2 & -1 \\ 0 & 3 & -3 & 1 \\ 0 & -2 & 1 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 1 & -5/3 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -5/3 \end{matrix}$$

$$\begin{array}{cc} & \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ A = & \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 7/3 \\ -4/3 \\ -5/3 \end{matrix} \end{bmatrix} \end{array}$$

$m \times n$

3×4

$\mathrm{Dim}(C(A)) = r$

$\mathrm{Dim}(N(A)) = n - r$

Dimension : Null space

"Nullity"

$\mathrm{Dim}(N(A))$, $\mathrm{nullity}(A)$

$\mathrm{nullity}(A) = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -7/3 \\ 4/3 \\ 5/3 \\ 1 \end{bmatrix}$$

$N(A) = \mathrm{span}(\nearrow)$

Dimension : Column space

"Rank"

$\mathrm{Dim}(C(A))$, $\mathrm{rank}(A)$

$\mathrm{rank}(A) = 3$

$C(A) = \mathrm{Span}(*)$

Summary Table: Null Space vs. Column Space

Feature	Null Space (Kernel)	Column Space (Image)
Physical Meaning	Redundancy/Equilibrium	Possible Outcomes/Reachability
Question it Answers	"What moves can I make that change nothing?"	"Where can I actually go?"
System Health	If too large: The system is unstable or loose.	If too small: The system is restricted or "trapped."