

## Dot products and cross products

$$\vec{a} \cdot \vec{b}$$

Dot product:  
How much  $\vec{a}$  and  $\vec{b}$   
point in the same direction



$$\vec{a} \times \vec{b}$$

Cross product:  
The length shows how  
much  $\vec{a}$  and  $\vec{b}$  point  
in different directions



# 65. Dot products

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$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 =$$

$$\vec{a} = (a_1, a_2) \quad b_1 a_1 + b_2 a_2$$

$$\vec{b} = (b_1, b_2)$$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \end{aligned}$$

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

$$\boxed{\|\vec{a}\|^2} = a_1^2 + a_2^2 = \boxed{\vec{a} \cdot \vec{a}} = a_1 a_1 + a_2 a_2$$
$$a_1^2 + a_2^2$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{a} = (a_1, a_2, a_3)$$

$$\vec{b} = (b_1, b_2, b_3)$$

$$\vec{b} = (-1, 2, 3)$$

$$\begin{aligned} \|\vec{b}\|^2 &= \vec{b} \cdot \vec{b} \\ &= (-1)(-1) + (2)(2) + (3)(3) \\ &= 14 \end{aligned}$$

$$\sqrt{\|\vec{b}\|^2} = \sqrt{14}$$

$$\|\vec{b}\| = \sqrt{14}$$

1. Commutative

2. Distributive

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

3. Associative

$$(3\vec{r}) \cdot \vec{s} = 3(\vec{r} \cdot \vec{s})$$

BUT DOT PRODUCT OF TWO MATRICES ARE NOT COMMUTATIVE,  
THEY ARE ONLY COMMUTATIVE WHEN BOTH MATRICES ARE ONE  
DIMENSIONAL (VECTORS)

## Cauchy-Schwarz inequality

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

$$= \vec{a} = c\vec{b} \quad c=2$$

$$\vec{b} = c\vec{a} \quad c=\frac{1}{2}$$

parallel  
collinear

= **L.D.** Linearly dependent

< **L.I.** Linearly independent

$$15 \leq \sqrt{5} (3\sqrt{5})$$

$$15 = 3 \cdot 5$$

$$15 = 15$$

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$$\vec{i} = (1, 0) \quad \vec{j} = (0, 1)$$

$$\|\vec{i}\| = \sqrt{1^2 + 0^2}$$

$$= \sqrt{1+0}$$

$$= \sqrt{1}$$

$$= 1$$

$$\therefore \vec{j} = (1)(0) + (0)(1)$$

$$= 0 + 0$$

$$\vec{i} \cdot \vec{j} = 0$$

$$0 \leq 1 \cdot 1$$

$$0 < 1 \quad \text{L.I.}$$

$$-3\vec{a} = \vec{b}$$

$$\vec{a} = (2, 1) \quad \vec{b} = (-6, -3)$$

$$\|\vec{a}\| = \sqrt{2^2 + 1^2}$$

$$= \sqrt{4+1}$$

$$= \sqrt{5}$$

$$\|\vec{b}\| = \sqrt{(-6)^2 + (-3)^2}$$

$$= \sqrt{36+9}$$

$$= \sqrt{45} = 3\sqrt{5}$$

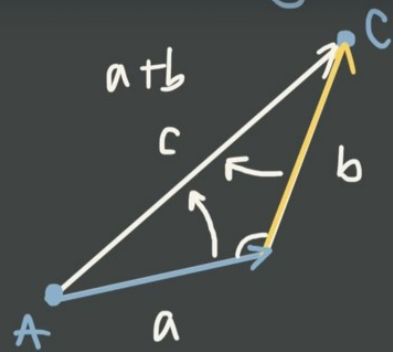
$$|\vec{a} \cdot \vec{b}| = |(2)(-6) + (1)(-3)|$$

$$= |-12-3|$$

$$= |-15|$$

$$= 15$$

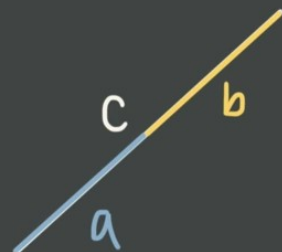
# 67. Vector triangle inequality



$$\mathbb{R}^2: c \leq a + b$$

$$c = a + b$$

$$c < a + b$$



$$\vec{a} + \vec{b} = (2, 1, -1, 3)$$

$$\sqrt{15} \leq \sqrt{10} + \sqrt{11}$$

$$3.87 < 3.16 + 3.32$$

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

$$\mathbb{R}^4: \left. \begin{array}{l} \vec{a} = (1, 1, 2, 2) \\ \vec{b} = (1, 0, -3, 1) \end{array} \right\} \text{L.}$$

$$\begin{aligned} \|\vec{a}\| &= \sqrt{1^2 + 1^2 + 2^2 + 2^2} \\ &= \sqrt{1 + 1 + 4 + 4} \\ &= \sqrt{10} \end{aligned}$$

$$\begin{aligned} \|\vec{b}\| &= \sqrt{1^2 + 0^2 + (-3)^2 + 1^2} \\ &= \sqrt{1 + 9 + 1} \\ &= \sqrt{11} \end{aligned}$$

$$\begin{aligned} \|\vec{a} + \vec{b}\| &= \sqrt{2^2 + 1^2 + (-1)^2 + 3^2} \\ &= \sqrt{4 + 1 + 1 + 9} \\ &= \sqrt{15} \end{aligned}$$

## Angle between vectors

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

$$\cos(90^\circ)$$

$$0$$

$$\vec{a} \cdot \vec{b} = 0$$

$$\left. \begin{array}{l} \vec{a} = (1, 0, -2, 0) \\ \vec{b} = (0, 3, 0, 1) \end{array} \right\} \mathbb{R}^4$$

$$\vec{a} \cdot \vec{b} = (1)(0) + (0)(3) + (-2)(0) + (0)(1)$$

$$= 0 + 0 + 0 + 0$$

$$= 0$$

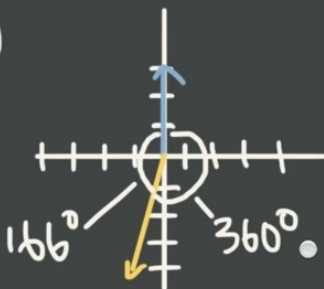


$$0 = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

$$\frac{0}{\|\vec{a}\| \|\vec{b}\|} = \cos \theta$$

$$0 = \cos \theta$$

$$\theta = 90^\circ$$



$$\vec{a} = (0, 3) \quad \vec{b} = (-1, -4)$$

$$\vec{a} \cdot \vec{b} = (0)(-1) + (3)(-4)$$

$$= 0 - 12$$

$$= -12$$

$$\|\vec{a}\| = \sqrt{0^2 + 3^2}$$

$$= \sqrt{9}$$

$$= 3$$

$$\|\vec{b}\| = \sqrt{(-1)^2 + (-4)^2}$$

$$= \sqrt{1 + 16}$$

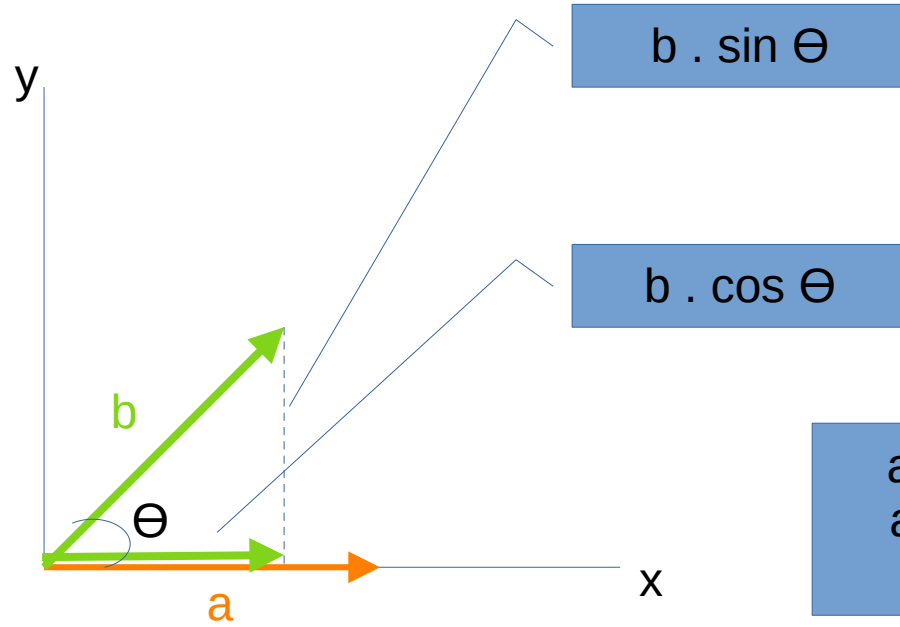
$$= \sqrt{17}$$

$$-12 = 3\sqrt{17} \cos \theta$$

$$\cos \theta = -\frac{12}{3\sqrt{17}} = -\frac{4}{\sqrt{17}}$$

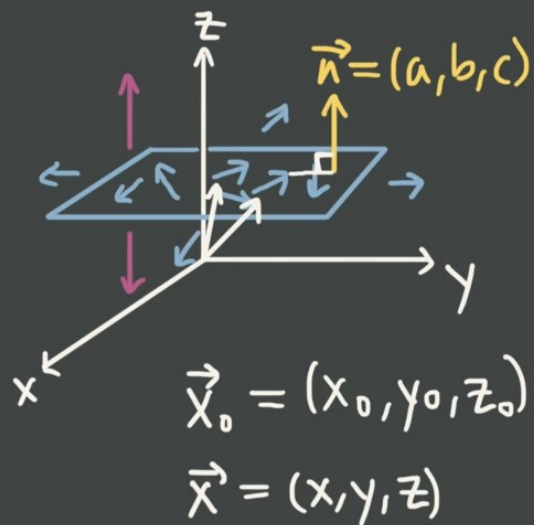
$$\arccos(\cos \theta) = \arccos\left(-\frac{4}{\sqrt{17}}\right)$$

$$\theta = 166^\circ$$



$$a \cdot b = a \cdot b \cdot \cos \Theta + a \cdot b \cdot \sin \Theta$$
$$a \cdot b = a \cdot b \cdot \cos \Theta + 0$$

# Equation of a plane, and normal vectors



$$\vec{n} \cdot (\vec{X} - \vec{X}_0) = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} = 0$$

$$\left[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \right]$$

$$\vec{X} - \vec{X}_0 = \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix}$$

$$\vec{n} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \quad \begin{matrix} (1, 0, -2) \\ x_0 \ y_0 \ z_0 \end{matrix}$$

$$2(x-1) + 5(y-0) - 3(z+2) = 0$$

$$2x - 2 + 5y - 3z - 6 = 0$$

$$\boxed{2}x + \boxed{5}y - \boxed{3}z = 8 \quad \vec{n} = (A, B, C)$$

$$Ax + By + Cz = D$$

$$\vec{n} = (2, 5, -3)$$

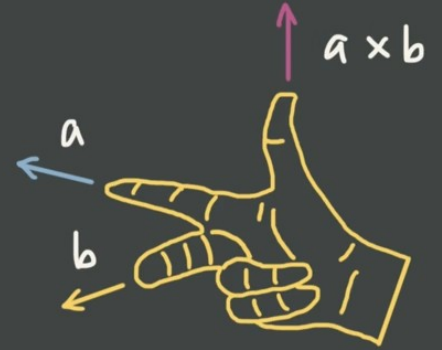
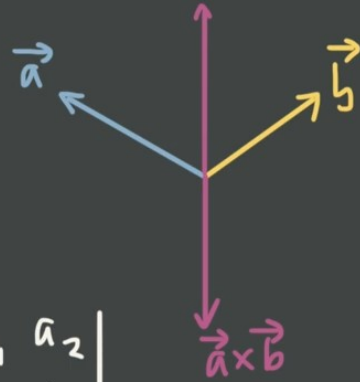


# Cross products

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \vec{i} (a_2 b_3 - a_3 b_2) - \vec{j} (a_1 b_3 - a_3 b_1) + \vec{k} (a_1 b_2 - a_2 b_1)$$



$$\begin{aligned} \vec{a} &= (1, 0, 2) \\ \vec{b} &= (-2, 1, 0) \end{aligned} \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}$$

$$= \vec{i} (0 \cdot 0 - 2 \cdot 1) - \vec{j} (1 \cdot 0 - 2(-2)) + \vec{k} (1 \cdot 1 - 0(-2))$$

$$= \vec{i} (-2) - \vec{j} (4) + \vec{k} (1)$$

$$= -2\vec{i} - 4\vec{j} + \vec{k}$$

$$= (-2, -4, 1) = \vec{a} \times \vec{b}$$

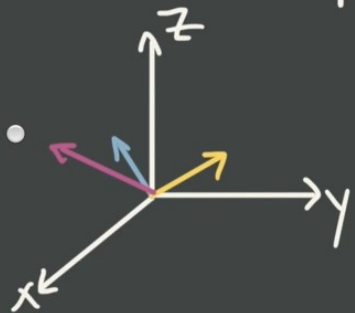
$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{(-2)^2 + (-4)^2 + 1^2}$$

$$= \sqrt{4 + 16 + 1}$$

$$= \sqrt{21}$$

$$\approx 4.6$$



# Dot and cross products as opposite ideas

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## Dot product

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

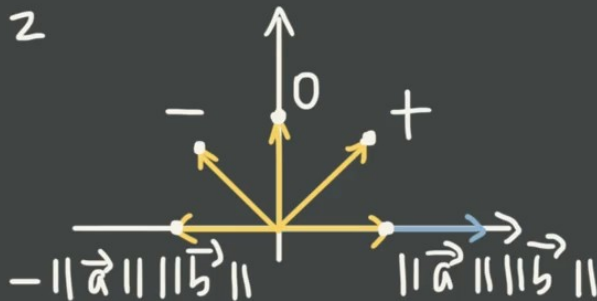
0  $\uparrow$  For  $\theta = 90^\circ$ :  $\vec{a} \cdot \vec{b} = 0$

10  $\rightarrow$  For  $\theta = 0^\circ$ :  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\|$

-10  $\leftarrow$  For  $\theta = 180^\circ$ :  $\vec{a} \cdot \vec{b} = -\|\vec{a}\| \|\vec{b}\|$

$$\|\vec{a}\| = 5$$

$$\|\vec{b}\| = 2$$



## Cross product

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

10 For  $\theta = 90^\circ$ :  $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\|$

0  $\rightarrow$  For  $\theta = 0^\circ$ :  $\|\vec{a} \times \vec{b}\| = 0$

0  $\leftarrow$  For  $\theta = 180^\circ$ :  $\|\vec{a} \times \vec{b}\| = 0$

$$\|\vec{a}\| = 5$$

$$\|\vec{b}\| = 2$$

