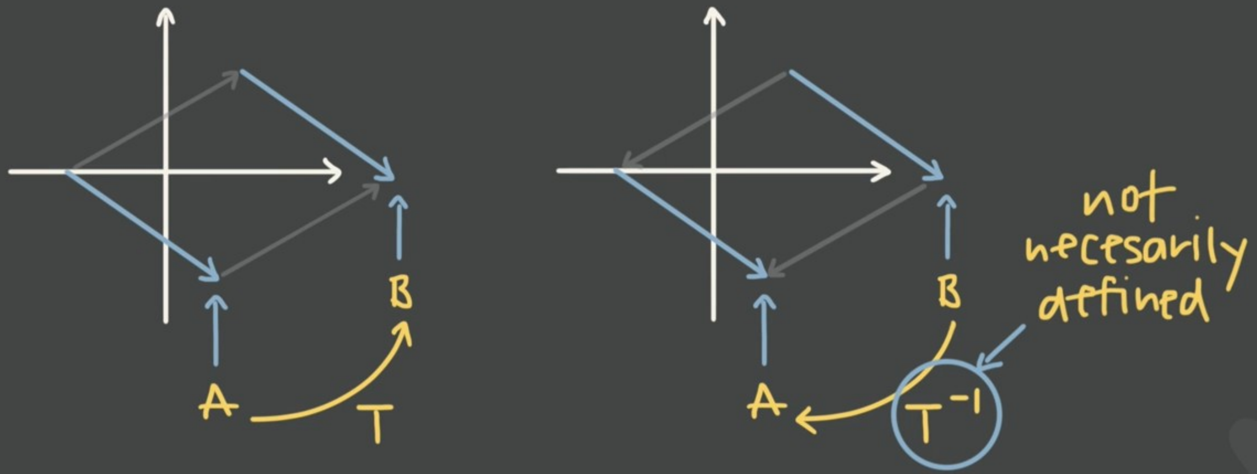
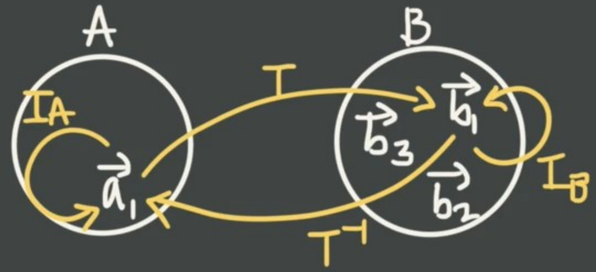
Inverses



$$\begin{bmatrix} A \end{bmatrix} \underbrace{\begin{bmatrix} A^{-1} \end{bmatrix}}_{\text{not necesarily defined}} = \begin{bmatrix} \text{Identity} \\ \text{matrix} \\ I_n \end{bmatrix}$$

Inverse of a transformation



1. T^{-1} is unique
2. $\vec{a}_1$ maps to only one $\vec{b}_1$
3. $\vec{b}_1$ maps to only one $\vec{a}_1$

$T(\vec{a}) = [\quad]\vec{a}$

$T^{-1}(\vec{b}) = \underline{[\quad]}\vec{b}$

$T: A \to B$

$I_A: A \to A \qquad I_A(\vec{a}_1) = \vec{a}_1$

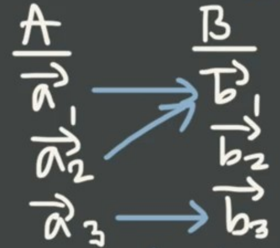
$I_B: B \to B \qquad I_B(\vec{b}_1) = \vec{b}_1$

T is surjective/onto:
every $\vec{b}$ is being mapped to

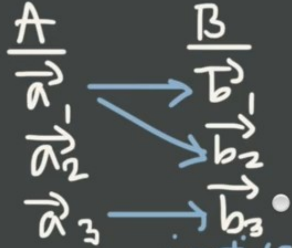
T is injective/one-to-one:
only one $\vec{a}$ is mapping to a given $\vec{b}$

$$\left.\begin{aligned} T^{-1}(T(\vec{a}_1)) &= I_A(\vec{a}_1) \\ T(T^{-1}(\vec{b}_1)) &= I_B(\vec{b}_1) \end{aligned}\right\}$$

T is invertible
T has an inverse



not surjective
not injective



surjective
not injective

Invertibility from the matrix-vector product

$T(\vec{x}) = A\vec{x}$ $\qquad T: \mathbb{R}^n \to \mathbb{R}^n$

T can only be invertible when A is square

$$\begin{bmatrix} 3 \times \boxed{3 & } \end{bmatrix}\begin{bmatrix} \boxed{3} \times 1 \end{bmatrix} = \begin{bmatrix} 3 \times 1 \end{bmatrix}$$

$\mathbb{R}^3 \qquad \mathbb{R}^3$

R×C
m×n

$\boxed{n > m}$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} \underline{1} & 0 & 1/3 \\ 0 & \underline{1} & -1 \end{bmatrix}\vec{x} = \vec{0}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/3 \\ 1 \\ 1 \end{bmatrix}$$

$$N(A) = \text{Span}\left(\begin{bmatrix} -1/3 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} -1/3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \vec{0}$$

Not injective

$$\underset{\substack{m \times n \\ R \times C}}{B} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & -2 \end{bmatrix}$$

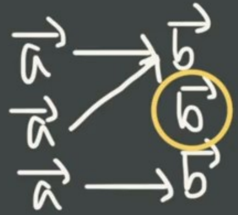
$\boxed{m > n}$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C(B) = \text{Span}\left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}\right)$$

$\mathbb{R}^3$

Not surjective



Inverse transformations are linear

1. Linear transformation T
2. T is invertible

$\Rightarrow T^{-1}$: linear transformation

$$T^{-1}(\vec{a}+\vec{b}) = T^{-1}(\vec{a}) + T^{-1}(\vec{b})$$

$$T^{-1}(c\vec{a}) = cT^{-1}(\vec{a})$$

$$T(\vec{x}) = \overset{B^{-1}}{A}\vec{x} \qquad T^{-1}(T(\vec{x})) = \boxed{A^{-1}A}\vec{x}$$

$$T^{-1}(\vec{x}) = \underset{B}{A^{-1}}\vec{x} \qquad T^{-1}(T(\vec{x})) = \boxed{I}\vec{x}$$

$$T(\vec{x}) = \overset{A}{\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}}\vec{x}$$

$$T^{-1}(\vec{x}) = \overset{A^{-1}}{\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}}\vec{x}$$

$$[A \mid I] \rightarrow [I \mid A^{-1}]$$

$$\begin{array}{c|c} A & I \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right]$$

$$\begin{array}{cc} I & A^{-1} \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right]$$

Matrix inverses, and invertible and singular matrices

$\frac{3}{3} = 1 \qquad 3 \cdot \frac{1}{3} = 1$

$X \cdot \frac{1}{X} = 1$

$X \cdot X^{-1} = 1$

$A \cdot A^{-1} = I$

$[A \mid I] \Rightarrow [I \mid A^{-1}]$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|A| = ad - bc$

$X = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix}$

$X^{-1} = \frac{1}{-2-0}\begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix}$

$X^{-1} = \begin{bmatrix} -1/2 & 0 \\ 2 & 1 \end{bmatrix}$

$ad - bc = 0$

$ad = bc$

$\frac{a}{b} = \frac{c}{d}$

$|A| \neq 0$

$Z = \begin{bmatrix} -2 & -4 \\ 4 & 8 \end{bmatrix}$

$-16 - (-16) = 0$

$-16 = -16$

$1/2 = 1/2$

$|A| \neq 0$: Invertible

$|A| = 0$: singular

$-2 - 0 = -2 \neq 0$

$-2 \neq 0$

und. $\cancel{-2/0} \neq \cancel{4/1}\ 4$

The "Division by Zero" Analog: In basic arithmetic, every number has a reciprocal (e.g., 5 becomes 1/5) except for zero.

A singular matrix is the matrix equivalent of the number zero.

It is "singular" because it represents a point where the standard rules of matrix algebra (like finding an inverse) break down.

solving systems with inverse matrices

$$7x + 5y = -4$$
$$-6x + 3y = -33$$

$$\begin{bmatrix} 7 & 5 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -33 \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

$$A^{-1} A \vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

$$A^{-1} = \frac{1}{21+30}\begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}$$

$$= \frac{1}{51}\begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \frac{1}{17} & -\frac{5}{51} \\ \frac{2}{17} & \frac{7}{51} \end{bmatrix}\begin{bmatrix} -4 \\ -33 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{17}(-4) - \frac{5}{\cancel{51}_{17}}(\cancel{-33}^{11}) \\ \frac{2}{17}(-4) + \frac{7}{\cancel{51}_{17}}(\cancel{-33}^{11}) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{17} + \frac{55}{17} \\ -\frac{8}{17} - \frac{77}{17} \end{bmatrix}$$

$$= \begin{bmatrix} 51/17 \\ -85/17 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

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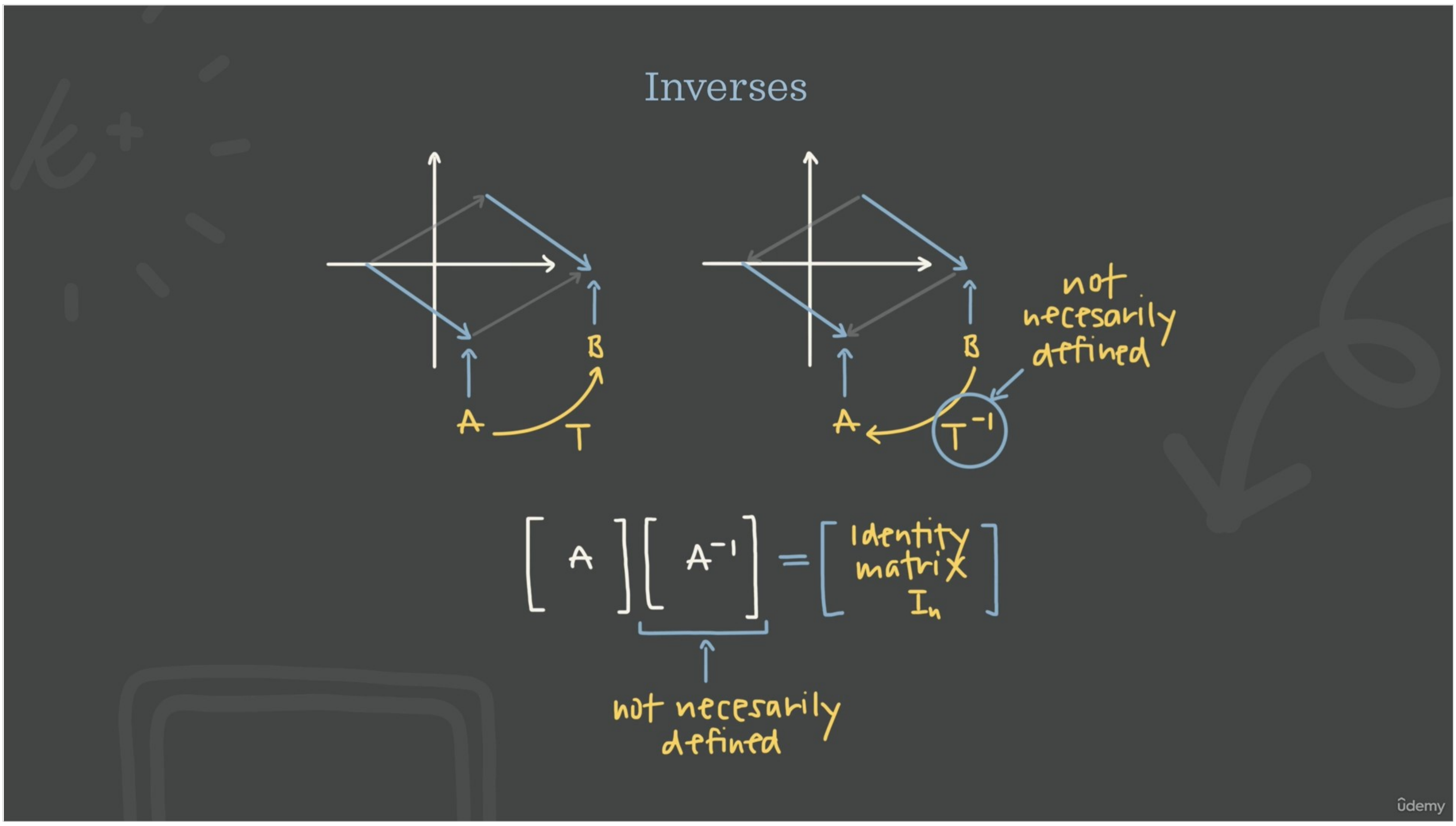
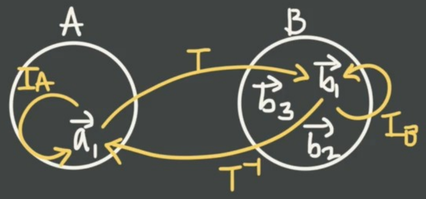


Inverses
A
B
T
A
B
T^{-1}
not necesarily defined
A
A^{-1}
=
Identity matrix I_n
not necesarily defined

Inverse of a transformation



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T has an inverse

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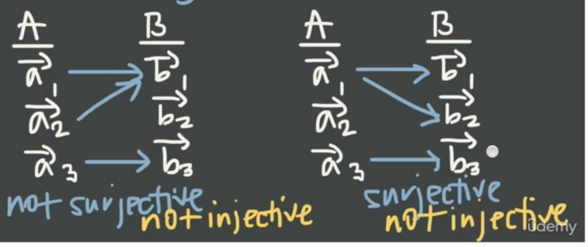

Invertibility from the matrix-vector product

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T can only be invertible when A is square

$$\left[\ 3 \times \boxed{3}\ \right]\left[\ \boxed{3} \times 1\ \right] = \left[\ 3 \times 1\ \right]$$

$\mathbb{R}^3 \qquad \mathbb{R}^3$

$R \times C$
$m \times n$ $\qquad \boxed{n > m}$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} \underline{1} & 0 & 1/3 \\ 0 & \underline{1} & -1 \end{bmatrix}\vec{x} = \vec{0}$$

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Not injective

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$m \times n$
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$\boxed{m > n}$

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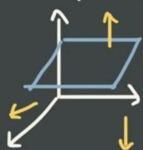
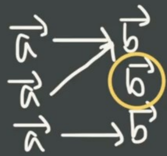
$\mathbb{R}^3$



Not surjective



udemy

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