

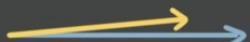
# Dot products and cross products

$$\vec{a} \cdot \vec{b}$$

$$\vec{a} \times \vec{b}$$

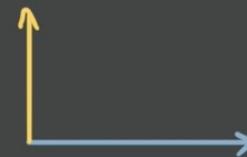
Dot product:

How much  $\vec{a}$  and  $\vec{b}$   
point in the same direction



Cross product:

The length shows how  
much  $\vec{a}$  and  $\vec{b}$  point  
in different directions



# 6. Dot products

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$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 =$$

$$\vec{a} = (a_1, a_2) \quad b_1 a_1 + b_2 a_2$$

$$\vec{b} = (b_1, b_2)$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{a} = (a_1, a_2, a_3)$$

$$\vec{b} = (b_1, b_2, b_3)$$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\vec{b} = (-1, 2, 3)$$

$$\vec{a} \cdot \vec{b} = [a_1 \ a_2] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\|\vec{b}\|^2 = \vec{b} \cdot \vec{b}$$
$$= (-1)(-1) + (2)(2) + (3)(3)$$

$$= [b_1 \ b_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\sqrt{\|\vec{b}\|^2} = \sqrt{14}$$

$$\|\vec{b}\| = \sqrt{14}$$

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$$

$$\|\vec{a}\|^2 = a_1^2 + a_2^2 = \boxed{\vec{a} \cdot \vec{a}} = a_1 a_1 + a_2 a_2$$
$$a_1^2 + a_2^2$$

1. Commutative

2. Distributive

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

3. Associative

$$(3\vec{r}) \cdot \vec{s} = 3(\vec{r} \cdot \vec{s})$$

BUT DOT PRODUCT OF TWO MATRICES ARE NOT COMMUTATIVE,  
THEY ARE ONLY COMMUTATIVE WHEN BOTH MATRICES ARE ONE  
DIMENSIONAL (VECTORS)

## Cauchy-Schwarz inequality

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$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

$$\begin{aligned} &= \vec{a} = c\vec{b} \quad c=2 \\ &\vec{b} = c\vec{a} \quad c=\frac{1}{2} \end{aligned}$$

parallel  
collinear

= L.D. Linearly dependent

< L.I. Linearly independent

$$15 \leq \sqrt{5} (3\sqrt{5})$$

$$15 = 3 \cdot 5$$

$$15 = 15$$

$$\begin{matrix} i = (1, 0) \\ j = (0, 1) \end{matrix}$$

$$\begin{aligned} \|i\| &= \sqrt{1^2 + 0^2} \\ &= \sqrt{1+0} \\ &= \sqrt{1} \end{aligned}$$

$$= 1$$

$$\begin{aligned} i \cdot j &= (1)(0) + (0)(1) \\ &= 0+0 \end{aligned}$$

$$|i \cdot j| = 0$$

$$0 \leq |i|$$

$$0 < |i| \text{ L.I.}$$

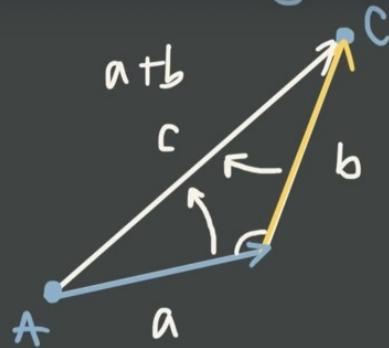
$$-3\vec{a} = \vec{b}$$

$$\vec{a} = (2, 1) \quad \vec{b} = (-6, -3)$$

$$\begin{aligned} \|\vec{a}\| &= \sqrt{2^2 + 1^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5} \end{aligned}$$

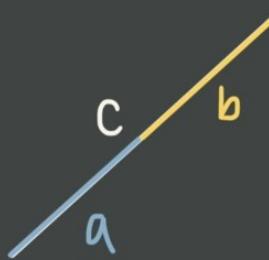
$$\begin{aligned} \|\vec{b}\| &= \sqrt{(-6)^2 + (-3)^2} \\ &= \sqrt{36+9} \\ &= \sqrt{45} = 3\sqrt{5} \\ &= 3\sqrt{5} \end{aligned}$$

$$\begin{aligned} |\vec{a} \cdot \vec{b}| &= |(2)(-6) + (1)(-3)| \\ &= |-12-3| \\ &= |-15| \\ &= 15 \end{aligned}$$



$$\mathbb{R}^2: c \leq a+b$$

$$\begin{aligned} c &= a+b \\ c &< a+b \end{aligned}$$



$$\vec{a} + \vec{b} = (2, 1, -1, 3)$$

$$\sqrt{15} \leq \sqrt{10} + \sqrt{11}$$

$$3.87 < 3.16 + 3.32$$

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

$$\begin{array}{l} \mathbb{R}^4: \quad \vec{a} = (1, 1, 2, 2) \\ \quad \vec{b} = (1, 0, -3, 1) \end{array} \left. \right\} \text{L.}$$

$$\|\vec{a}\| = \sqrt{1^2 + 1^2 + 2^2 + 2^2}$$

$$= \sqrt{1 + 1 + 4 + 4}$$

$$= \sqrt{10}$$

$$\|\vec{b}\| = \sqrt{1^2 + 0^2 + (-3)^2 + 1^2}$$

$$= \sqrt{1 + 9 + 1}$$

$$= \sqrt{11}$$

$$\|\vec{a} + \vec{b}\| = \sqrt{2^2 + 1^2 + (-1)^2 + 3^2}$$

$$= \sqrt{4 + 1 + 1 + 9}$$

$$= \sqrt{15}$$

## 69. Angle between vectors

$$\vec{a} \cdot \vec{b} = \| \vec{a} \| \| \vec{b} \| \cos \theta$$

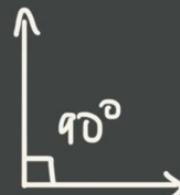
$$\cos(90^\circ)$$

0

$$\vec{a} \cdot \vec{b} = 0$$

$$\begin{aligned}\vec{a} &= (1, 0, -2, 0) \\ \vec{b} &= (0, 3, 0, 1)\end{aligned}\quad \left\{ \mathbb{R}^4$$

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (1)(0) + (0)(3) + (-2)(0) + (0)(1) \\ &= 0 + 0 + 0 + 0 \\ &= 0\end{aligned}$$



$$D = ||\vec{a}|| \cdot ||\vec{b}|| \cos\theta$$

$$\frac{0}{\|\vec{r}\| \|\vec{t}\|} = \cos \theta$$

$$\theta = \cos t$$

$$\theta = 90^\circ$$

$$\vec{a} = (0, 3) \quad \vec{b} = (-1, -4)$$

$$\vec{a} \cdot \vec{b} = (9)(-1) + (3)(-4) \\ = 0 - 12 \\ = -12$$

$$\begin{aligned}\|\vec{a}\| &= \sqrt{0^2 + 3^2} \\ &= \sqrt{9} \\ &= 3\end{aligned}$$

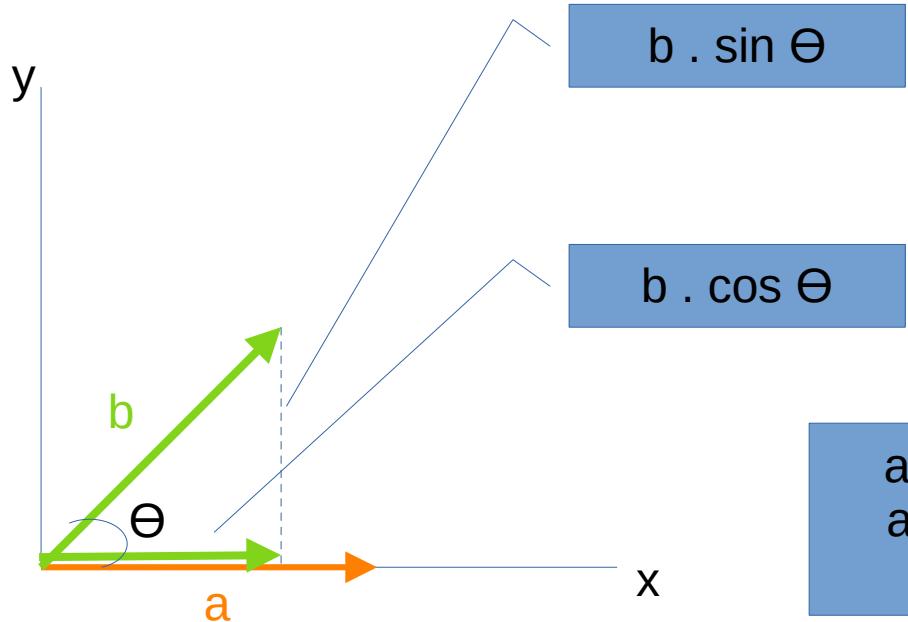
$$\begin{aligned}\|\vec{b}\| &= \sqrt{(-1)^2 + (-4)^2} \\ &= \sqrt{1 + 16} \\ &= \sqrt{17}\end{aligned}$$

$$-12 = 3\sqrt{17} \cos\theta$$

$$\cos \theta = -\frac{12}{3\sqrt{17}} = -\frac{4}{\sqrt{17}}$$

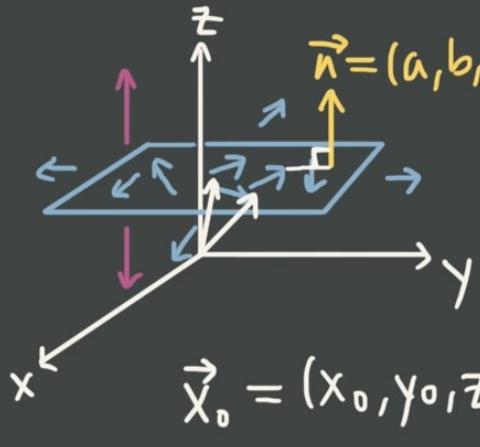
$$\arccos(\text{last}) = \arccos\left(-\frac{4}{\sqrt{17}}\right)$$

$\theta = 156^\circ$



$$a \cdot b = a \cdot b \cdot \cos \theta + a \cdot b \cdot \sin \theta$$
$$a \cdot b = a \cdot b \cdot \cos \theta + 0$$

# Equation of a plane, and normal vectors



$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

$$[a \ b \ c] \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$$\vec{x}_0 = (x_0, y_0, z_0)$$

$$\vec{x} = (x, y, z)$$

$$[a(x - x_0) + b(y - y_0) + c(z - z_0) = 0]$$

$$\vec{n} = (2, 5, -3)$$

$$a \ b \ c$$

$$x_0 \ y_0 \ z_0$$

$$2(x-1) + 5(y-0) - 3(z+2) = 0$$

$$2x - 2 + 5y - 3z - 6 = 0$$

$$2x + 5y - 3z = 8$$

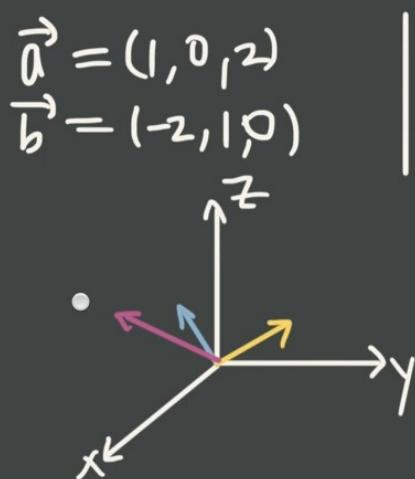
$$Ax + By + Cz = D$$

$$\vec{n} = (A, B, C)$$

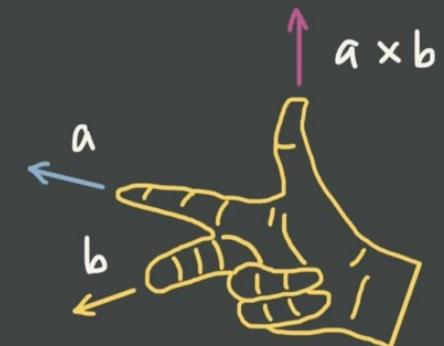
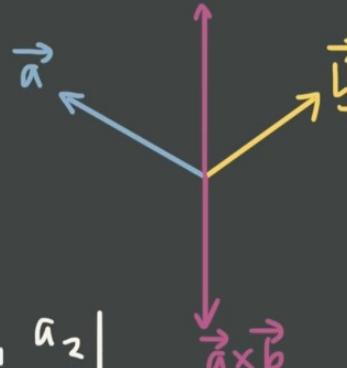
$$\vec{n} = (2, 5, -3)$$

## Cross products

$$\begin{aligned}
 \vec{a} \times \vec{b} &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
 &= i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 b_2 - a_2 b_1) \\
 &= i(0 \cdot 0 - 2 \cdot 1) - j(1 \cdot 0 - 2(-2)) + k(1 \cdot 1 - 0(-2))
 \end{aligned}$$



$$\begin{aligned}
 &i \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \\
 &= i(0 \cdot 0 - 2 \cdot 1) - j(1 \cdot 0 - 2(-2)) + k(1 \cdot 1 - 0(-2)) \\
 &= i(-2) - j(4) + k(1) \\
 &= -2i - 4j + k \\
 &= (-2, -4, 1) = \vec{a} \times \vec{b}
 \end{aligned}$$



$$\begin{aligned}
 \|\vec{a} \times \vec{b}\| &= \|\vec{a}\| \|\vec{b}\| \sin\theta \\
 \|\vec{a} \times \vec{b}\| &= \sqrt{(-2)^2 + (-4)^2 + 1^2} \\
 &= \sqrt{4 + 16 + 1} \\
 &= \sqrt{21} \\
 &\approx 4.6
 \end{aligned}$$

# Dot and cross products as opposite ideas

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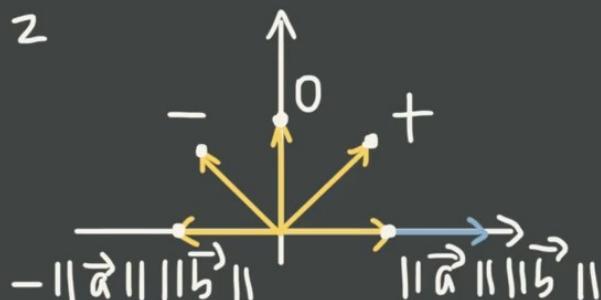
## Dot product

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

- 0  $\uparrow$  For  $\theta=90^\circ$ :  $\vec{a} \cdot \vec{b} = 0$
- 10  $\rightarrow$  For  $\theta=0^\circ$ :  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\|$
- 10  $\leftarrow$  For  $\theta=180^\circ$ :  $\vec{a} \cdot \vec{b} = -\|\vec{a}\| \|\vec{b}\|$

$$\|\vec{a}\| = 5$$

$$\|\vec{b}\| = 2$$



## Cross product

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

- 10 For  $\theta=90^\circ$ :  $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\|$
- 0  $\rightarrow$  For  $\theta=0^\circ$ :  $\|\vec{a} \times \vec{b}\| = 0$
- 0  $\leftarrow$  For  $\theta=180^\circ$ :  $\|\vec{a} \times \vec{b}\| = 0$

$$\|\vec{a}\| = 5$$

$$\|\vec{b}\| = 2$$

