

# Euclidean Geometry in Mathematical Olympiads Solutions

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## §1 Angle Chasing

**Problem** (1.51, IMO 1985/1)

A circle has center on the side  $\overline{AB}$  of the cyclic quadrilateral  $ABCD$ . The other three sides are tangent to the circle. Prove that  $AD + BC = AB$ .

*Proof.* **INCOMPLETE** Call the center of the circle point  $O$ . Let the point  $T$  be where  $AD = AT$ .  $\square$

## §2 Circles

### Lemma (2.19)

Prove that the A-exradius has length

$$r_a = \frac{s}{s-a}r,$$

where  $r$  is the inradius.

*Proof.* Drop perpendiculars from  $I$  and  $I_A$  to  $AB$ . Call the feet of these perpendiculars  $B_1$  and  $B_2$  respectively. Notice that  $IB_1 = r$  and  $I_AB_2 = r_a$  and that  $\triangle AB_1I \sim \triangle AB_2I_A$ . Therefore

$$\frac{r}{r_a} = \frac{AB_1}{AB_2},$$

but by Lemmas 2.15 and 2.17, we know that  $AB_1 = s - a$  and  $AB_2 = s$ , hence

$$r_a = \frac{s}{s-a}r,$$

and we are done. □

### Lemma (2.20)

Let  $ABC$  be a triangle. Suppose its incircle and A-excircle are tangent to  $BC$  at  $D$  and  $X$ , respectively. Show that  $BD = CX$  and  $BX = CD$ .

*Proof.* We will first show that  $BD = CX$ . Let the incircle be tangent to side  $AB$  at point  $F$  and let to side  $AC$  at point  $E$ . Let the A-excircle be tangent to the extension of line  $AC$  at  $C_1$  and to the extension of line  $AB$  at  $B_1$ . Then

$$\begin{aligned}
BD &= BF \\
&= AB_1 - AF - BB_1 \\
&= (AC_1 - AE) - BX \\
&= (CC_1 + CE) - (BC - CX) \\
&= CX + (CD - BC) + CX \\
&= 2CX - BD \\
2BD &= 2CX \rightarrow BD = CX.
\end{aligned}$$

It follows that  $BX = CD$  because

$$\begin{aligned}
BD &= CX \\
BD + DX &= DX + CX \\
BX &= CD.
\end{aligned}$$

□

**Lemma (2.24)**

Let  $ABC$  be a triangle with  $I_A, I_B$ , and  $I_C$  as excenters. Prove that triangle  $I_AI_BI_C$  has orthocenter  $I$  and that triangle  $ABC$  is its orthic triangle.

*Proof.* By the Incenter-Excenter Lemma, we know that  $AI_A, BI_B$ , and  $CI_C$  coincide at the incenter  $I$ . We also know from the Lemma that  $II_A$  is the diameter of circle  $BICI_A$ . Therefore we have that

$$\angle I_CCI_A = \angle ICI_A = 90^\circ \text{ and } \angle I_BBI_A = \angle IBI_A = 90^\circ.$$

This follows similarly for  $II_B$  and  $II_C$ . Now we know that  $AI_A, BI_B, CI_C$  are in fact the altitudes of triangle  $I_AI_BI_C$ , therefore  $I$  is the orthocenter of triangle  $I_AI_BI_C$ . Note that since  $A, B$ , and  $C$  are the feet of the altitudes,  $ABC$  is the orthic triangle of triangle  $I_AI_BI_C$ . □

**Theorem (2.25, The Pitot Theorem)**

Let  $ABCD$  be a quadrilateral. If a circle can be inscribed in it, prove that  $AB + CD = BC + DA$ .

*Proof.* Call the points where  $AB, BC, CD, DA$  are tangent to the circle  $E, F, G, H$ , respectively. Let  $AE = AH = a, BE = BF = b, CF = CG = c, DG = DH = d$ . Now note that our condition can be manipulated as follows:

$$\begin{aligned} AB + CD &= BC + DA \\ (AE + BE) + (CG + DG) &= (BF + CF) + (AH + DH) \\ a + b + c + d &= b + c + a + d. \end{aligned}$$

Hence, we are done.  $\square$

**Problem (2.26, USAMO 1990/5)**

An acute-angled triangle  $ABC$  is given in the plane. The circle with diameter  $\overline{AB}$  intersects altitude  $\overline{CC'}$  and its extension at points  $M$  and  $N$ , and the circle with diameter  $\overline{AC}$  intersects altitude  $\overline{BB'}$  and its extensions at  $P$  and  $Q$ . Prove that the points  $M, N, P, Q$  lie on a common circle.

*Proof.* Let the circle with diameter  $\overline{AB}$  be called  $\omega_1$  and the circle with diameter  $\overline{AC}$  be called  $\omega_2$ . By Theorem 2.9, it suffices to show that the intersection of  $\overline{MN}$  and  $\overline{PQ}$  lies on the radical axis of  $\omega_1$  and  $\omega_2$ . Since  $\overline{MN}$  and  $\overline{PQ}$  are altitudes of  $\triangle ABC$ , their intersection is the orthocenter of  $\triangle ABC$ . We will call this point  $H$ . Note that  $\overline{AH}$  is the third altitude of  $\triangle ABC$ . Call the foot of this altitude  $A'$ . Now note that  $\angle AA'B = \angle AA'C = 90^\circ$ , and since  $\overline{AB}$  and  $\overline{AC}$  are diameters of their respective circles,  $\omega_1$  and  $\omega_2$  must intersect at  $A'$ . Hence,  $\overline{AA'}$  is the radical axis of  $\omega_1$  and  $\omega_2$ , and since  $A, A', H$  are colinear,  $H$  lies on this line.  $\square$

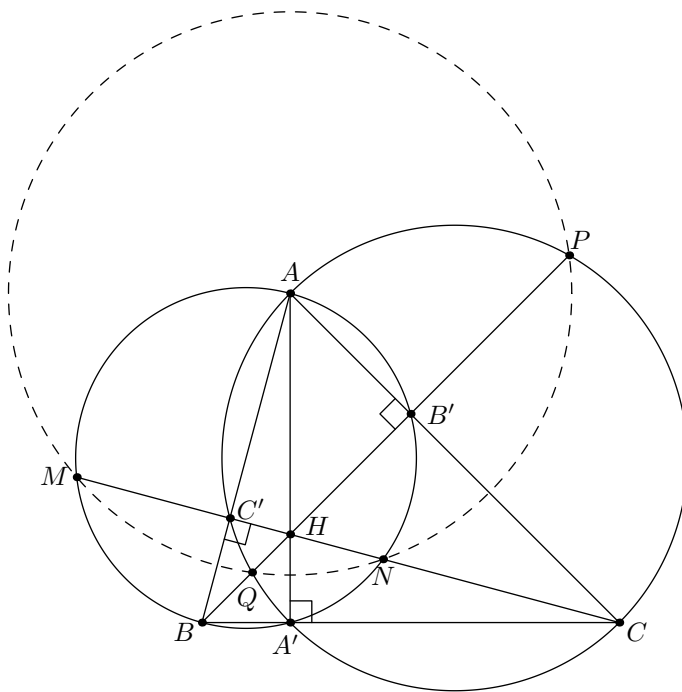


Figure 1: Problem 2.26

**Problem (2.27, BAMO 2012/4)**

Given a segment  $\overline{AB}$  in the plane, choose on it a point  $M$  different from  $A$  and  $B$ . Two equilateral triangles  $AMC$  and  $BMD$  in the plane are constructed on the same side of segment  $\overline{AB}$ . The circumcircles of the two triangles intersect in point  $M$  and another point  $N$ .

- Prove that  $\overline{AD}$  and  $\overline{BC}$  pass through point  $N$ .
- Prove that no matter where one chooses point  $M$  along segment  $\overline{AB}$ , all lines  $MN$  will pass through some fixed point  $K$  in the plane.

*Proof.* We will prove (a) by angle chasing. Notice that since  $ACNM$  and  $BDNM$  are cyclic, we have that

$$\angle AMC = \angle ANC = \angle ACM = \angle ANM = \angle MDB = \angle MNB = 60^\circ,$$

and since  $\angle ANC + \angle ANM + \angle MNB = 60^\circ + 60^\circ + 60^\circ = 180^\circ$ , we have

that  $BC$  is a straight line passing through  $N$ . A very similar argument follows for  $AD$ .

We will now prove (b) using radical axes. First, construct an equilateral triangle  $ABE$  on the same side as the other two equilateral triangles. Let the circumcircles around triangles  $AMC$ ,  $BMD$ , and  $ABE$  be  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$ , respectively. Note that  $MN$  is the radical axis of circles  $\omega_1$  and  $\omega_2$ , the line tangent to circles  $\omega_1$  and  $\omega_3$  at point  $A$  is the radical axis of circles  $\omega_1$  and  $\omega_3$ , and the line tangent to circles  $\omega_2$  and  $\omega_3$  at point  $B$  is the radical axis of circles  $\omega_2$  and  $\omega_3$ . Since the centers of  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  are not colinear, their radical axes (one of which is  $MN$ ) must coincide at the radical center  $K$ . Since changing the location of  $M$  on  $AB$  does not change the tangents at  $A$  and  $B$ , the point  $K$  does not move, hence all possible lines  $MN$  must pass through  $K$ .  $\square$

**Problem (2.28, JMO 2012/1)**

Given a triangle  $ABC$ , let  $P$  and  $Q$  be points on segments  $\overline{AB}$  and  $\overline{AC}$ , respectively, such that  $AP = AQ$ . Let  $S$  and  $R$  be distinct points on segment  $\overline{BC}$  such that  $S$  lies between  $B$  and  $R$ ,  $\angle BPS = \angle PRS$ , and  $\angle CQR = \angle QSR$ . Prove that  $P, Q, R, S$  are concyclic.

*Proof.* Since  $\angle BPS = \angle PRS$  by the Tangent Criterion,  $\overline{AB}$  is tangent to  $(PRS)$ . Likewise we have that  $\overline{AC}$  is tangent to  $(QRS)$ . Suppose  $(PRS)$  and  $(QRS)$  are not the same circle. Then since  $AP = AQ$  are both tangents to their respective circles,  $A$  must lie on the radical axis  $\overline{BC}$ , but since  $ABC$  is a triangle, this is obviously impossible. Hence  $P, Q, R, S$  are concyclic.  $\square$

**Problem (2.29, IMO 2008/1)**

Let  $H$  be the orthocenter of an acute-angled triangle  $ABC$ . The circle  $\Gamma_A$  centered at the midpoint of  $\overline{BC}$  and passing through  $H$  intersects the sideline  $BC$  at points  $A_1$  and  $A_2$ . Similarly, define the points  $B_1, B_2, C_1$ , and  $C_2$ . Prove that six points  $A_1, A_2, B_1, B_2, C_1$ , and  $C_2$  are concyclic.

*Proof.* We will first show that  $B_1, B_2, C_1, C_2$  are concyclic. Since  $\Gamma_A, \Gamma_B, \Gamma_C$  all intersect at  $H$ ,  $H$  is the radical center. We claim that  $\overline{AH}$  is the radical axis of  $\Gamma_B$  and  $\Gamma_C$ . By similar triangles,  $M_B M_C$  is parallel to  $BC$ , and

since  $\overline{AH} \perp BC$ ,  $\overline{AH} \perp M_B M_C$ . The centers of circles  $\Gamma_B$  and  $\Gamma_C$  are  $M_B$  and  $M_C$ , respectively, thus  $\overline{AH}$  is the radical axis of circles  $\Gamma_B$  and  $\Gamma_C$ . Since  $\overline{B_1 B_2}$  and  $\overline{C_1 C_2}$  intersect at  $A$ , by Theorem 2.9 we have shown that  $B_1, B_2, C_1, C_2$  are concyclic. Note that the circumcenter of  $(B_1 B_2 C_1 C_2)$  is the intersection of the perpendicular bisectors of  $B_1 B_2$  and  $C_1 C_2$ , which is the orthocenter  $O$  of triangle  $ABC$ . Thus what we have proven is that  $OB_1 = OB_2 = OC_1 = OC_2$ . A similar argument can be pursued for  $OA_1$  and  $OA_2$ , hence we are done.  $\square$

**Problem (2.30, USAMO 1997/2)**

Let  $ABC$  be a triangle. Take points  $D, E, F$  on the perpendicular bisectors of  $\overline{BC}, \overline{CA}, \overline{AB}$  respectively. Show that the lines through  $A, B, C$  perpendicular to  $\overline{EF}, \overline{FD}, \overline{DE}$  respectively are concurrent.

*Proof.* Consider the circles with centers  $D, E, F$  with chords  $BC, CA, AB$ , respectively. Note that the radical axes of these three circles are the lines through  $A, B, C$  perpendicular to  $\overline{EF}, \overline{FD}, \overline{DE}$ , and since the centers of these three circles are not colinear, their radical axes must intersect at a point.  $\square$

(These centers can be colinear, but we won't talk about that)

**Problem (2.31, IMO 1995/1)**

Let  $A, B, C, D$  be four distinct points on a line, in that order. The circles with diameters  $\overline{AC}$  and  $\overline{BD}$  intersect at  $X$  and  $Y$ . The line  $XY$  meets  $\overline{BC}$  at  $Z$ . Let  $P$  be a point on the line  $XY$  other than  $Z$ . The line  $CP$  intersects the circle with diameter  $AC$  at  $C$  and  $M$ , and the line  $BP$  intersects the circle with diameter  $BD$  at  $B$  and  $N$ . Prove that the lines  $AM, DN, XY$  are concurrent.

*Proof.* Since  $P$  lies on the radical axis of these two circles, and  $\overline{BN} \cap \overline{CM} = P$ ,  $MNBC$  is cyclic by Theorem 2.9. (Reminder that the symbol  $\angle$  denotes the directed angle.) Note that

$$\angle NMC = \angle NBC = \angle NBD = 90^\circ - \angle BDN = 90^\circ - \angle ADN,$$

so

$$\angle NMA = \angle NMC - 90^\circ = (90^\circ - \angle ADN) - 90^\circ = -\angle ADN = \angle NDA,$$

therefore quadrilateral  $DAMN$  is cyclic. The radical axes of the circles  $(DAMN)$ ,  $(AMC)$ , and  $(BND)$  are  $\overline{AM}$ ,  $\overline{DN}$ ,  $\overline{XY}$ , and since the centers of these circles are never colinear, they must intersect at the radical center.  $\square$

**Problem (2.32, USAMO 1998/2)**

Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be concentric circles, with  $\mathcal{C}_2$  in the interior of  $\mathcal{C}_1$ . From a point  $A$  on  $\mathcal{C}_1$  one draws the tangent  $\overline{AB}$  to  $\mathcal{C}_2$  ( $B \in \mathcal{C}_2$ ). Let  $C$  be the second point of intersection of ray  $AB$  and  $\mathcal{C}_1$ , and let  $D$  be the midpoint of  $\overline{AB}$ . A line passing through  $A$  intersects  $\mathcal{C}_2$  at  $E$  and  $F$  in such a way that the perpendicular bisectors of  $DE$  and  $CF$  intersect at a point  $M$  on  $AB$ . Find, with proof, the ratio  $AM/MC$ .

*Proof.* **INCOMPLETE** Note that  $CDEF$  is cyclic (need to prove).  $M$  is the center of circle  $(CDEF)$ . Thus  $CM = DM$ .  $\square$



### §3 Lengths and Ratios

**Theorem (3.2, Angle Bisector Theorem)**

Let  $ABC$  be a triangle and  $D$  a point on  $\overline{BC}$  so that  $\overline{AD}$  is the internal angle bisector of  $\angle BAC$ . Show that

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

*Proof.* Let  $\angle BAD = \alpha = \angle CAD$  and  $\angle ADB = \beta$ . Note that  $\angle ADC = 180^\circ - \beta$ . By Law of Sines, we have

$$\frac{DB}{\sin \alpha} = \frac{AB}{\sin \beta} \text{ and } \frac{DC}{\sin \alpha} = \frac{AC}{\sin(180^\circ - \beta)}.$$

Note that  $\sin(180^\circ - \beta) = \sin \beta$ . Rearranging terms, we have that

$$\frac{\sin \beta}{\sin \alpha} = \frac{AB}{BD} = \frac{AC}{CD}.$$

It follows that  $\frac{AB}{AC} = \frac{DB}{DC}$ . □

**Problem (3.5)**

Show the trigonometric form of Ceva holds.

*Proof.* Recall that the trigonometric form of Ceva's Theorem is as follows: Let  $\overline{AX}$ ,  $\overline{BY}$ ,  $\overline{CZ}$  be cevians of a triangle  $ABC$ . They concur if and only if

$$\frac{\sin \angle BAX \sin \angle CBY \sin \angle ACZ}{\sin \angle XAC \sin \angle YBA \sin \angle ZCB} = 1.$$

By the Law of Sines, we have that

$$\frac{\sin \angle BAX}{BX} = \frac{\sin B}{AX}$$

and

$$\frac{\sin \angle XAC}{XC} = \frac{\sin C}{AX}.$$

Combining these two equations gives us

$$AX = \frac{BX \sin B}{\sin \angle BAX} = \frac{XC \sin C}{\sin \angle XAC} \Rightarrow \frac{\sin \angle BAX}{\sin \angle XAC} = \frac{BX}{XC} \cdot \frac{\sin C}{\sin B}.$$

Similarly, we have that

$$\frac{\sin \angle CBY}{\sin \angle YBA} = \frac{CY}{YA} \cdot \frac{\sin A}{\sin C}$$

and

$$\frac{\sin \angle ACZ}{\sin \angle ZCB} = \frac{AZ}{ZB} \cdot \frac{\sin B}{\sin A}.$$

Plugging these values into the original equation, we have that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,$$

and we know this is true from the original statement of Ceva's Theorem.  $\square$

**Problem (3.6)**

Let  $\overline{AM}$ ,  $\overline{BE}$ , and  $\overline{CF}$  be concurrent cevians of a triangle  $ABC$ . Show that  $\overline{EF} \parallel \overline{BC}$  if and only if  $BM = MC$ .

*Proof.* Suppose  $\overline{EF} \parallel \overline{BC}$ . Call the point where  $\overline{AM}$  intersects  $\overline{EF}$  point  $Q$ . Notice that  $\triangle BPM \sim \triangle EPQ$  and  $\triangle CPM \sim \triangle FPQ$ . Thus we have the following relationship:

$$\frac{BM}{EQ} = \frac{MP}{QP} = \frac{CM}{FQ}.$$

Now also notice that  $\triangle BAM \sim \triangle FAQ$  and  $\triangle CAM \sim \triangle EAQ$ . Thus we have the following relationship:

$$\frac{BM}{FQ} = \frac{MA}{QA} = \frac{CM}{EQ}.$$

Putting these two relationships together, it follows that  $BM = CM$ . We will now prove the other direction. Suppose  $BM = MC$ . Then by Ceva's Theorem we have that

$$\begin{aligned}\frac{CE}{AE} &= \frac{BF}{AF} \\ \frac{CE}{BF} &= \frac{AE}{AF} = \frac{CE + AE}{BF + AF} = \frac{AC}{AB} \\ \frac{AE}{AC} &= \frac{AF}{AB}.\end{aligned}$$

Since  $\angle FAE = \angle BAC$ , we have that  $\triangle FAE \sim \triangle BAC$ . Thus  $\angle AEF = \angle ACB$ , therefore  $\overline{EF} \parallel \overline{BC}$ .  $\square$

**Problem (3.12)**

Give an alternative proof of Lemma 3.9 by taking a negative homothety.

*Proof.* Consider a homothety centered at  $G$  with  $M = h(A)$ ,  $N = h(B)$ ,  $L = h(C)$ . Note that  $\triangle ACB \sim \triangle NCM$  by midpoints and that  $\triangle ALG \sim \triangle Mh(L)G$  by homothety. Also notice that  $h(L)$  is the midpoint of  $NM$ . Since  $AB/NM = 2/1$ ,

$$\frac{AB}{NM} = \frac{AL}{Mh(L)} = \frac{AG}{MG} = \frac{2}{1}.$$

$\square$

**Lemma (3.13, Euler Line)**

In triangle  $ABC$ , prove that  $O, G, H$  (with their usual meanings) are collinear and that  $G$  divides  $\overline{OH}$  in a  $2 : 1$  ratio.

*Proof.* We will first show that  $O, G, H$  are collinear. Call the point where the perpendicular from  $O$  meets  $\overline{BC}, \overline{CA}, \overline{AB}$  points  $A', B', C'$ , respectively. Since  $\overline{BC}, \overline{CA}, \overline{AB}$  are chords of the circle  $(ABC)$ , points  $A', B', C'$  are in fact the midpoints of their respective line segments. Thus  $A'$  lies on  $\overline{AG}$ ,  $B'$  lies on  $\overline{BG}$ , and  $C'$  lies on  $\overline{CG}$ . Now notice that  $\overline{AH} \parallel \overline{OA'}, \overline{BH} \parallel \overline{OB'}, \overline{CH} \parallel \overline{OC'}$  since they are all perpendicular to some side of the triangle  $ABC$ . Thus, a homothety  $h$  centered at  $G$  exists such that  $h(A) = A', h(B) = B', h(C) = C'$ . Thus,  $h(O) = H$ , so  $O, G, H$  are collinear.

We will now show that  $G$  divides  $\overline{OH}$  in a  $2 : 1$  ratio. This is equivalent to showing that the homothety  $h$  must have a scale factor  $k = -2$ . From Lemma 3.9 (Centroid Division) we have that  $AG/GA' = 2/1$ . Since  $G$  lies in between  $A$  and  $A'$ , we have that  $k = -2$ , as desired. (!!!)  $\square$

**Problem (3.16)**

Let  $ABC$  be a triangle with contact triangle  $DEF$ . Prove that  $\overline{AD}, \overline{BE}, \overline{CF}$  concur. The point of concurrency is the Gergonne point of triangle  $ABC$ .

*Proof.* Notice by Lemma 2.15 we have that

$$\begin{aligned} AE &= AF = s - a \\ BD &= BF = s - b \\ CD &= CE = s - c. \end{aligned}$$

Thus, by Ceva's Theorem, we have that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1.$$

$\square$

**Lemma (3.17)**

In cyclic quadrilateral  $ABCD$ , points  $X$  and  $Y$  are the orthocenters of  $\triangle ABC$  and  $\triangle BCD$ . Show that  $AXYD$  is a parallelogram.

*Proof.* Reflect  $X$  and  $Y$  across  $\overline{BC}$  and call these points  $X'$  and  $Y'$  respectively. Notice that  $X'$  and  $Y'$  lie on  $(ABCD)$ . Thus  $ADX'Y'$  is a cyclic quadrilateral. Then we have that

$$\angle AXY = \angle X'XY = \angle Y'X'X = \angle Y'X'A = \angle Y'DA = \angle YDA.$$

Similarly, we have that  $\angle DAX = \angle XYD$ . Hence  $AXYD$  is a parallelogram.  $\square$

**Problem (3.18)**

Let  $\overline{AD}, \overline{BE}, \overline{CF}$  be concurrent cevians in a triangle, meeting at  $P$ . Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

*Proof.* By Area Ratios, we can transform each term in our desired equation as follows:

$$\begin{aligned} \frac{PD}{AD} &= \frac{[BPC]}{[BAC]}, \\ \frac{PE}{BE} &= \frac{[CPA]}{[CBA]}, \\ \frac{PF}{CF} &= \frac{[APB]}{[ACB]}. \end{aligned}$$

Therefore our desired equation turns into

$$\frac{[BPC]}{[BAC]} + \frac{[CPA]}{[CBA]} + \frac{[APB]}{[ACB]} = 1.$$

Notice that  $[BPC] + [CPA] + [APB] = [ABC]$ . Hence we are done.  $\square$

**Problem (3.19, Shortlist 2006/G3)**

Let  $ABCDE$  be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \text{ and } \angle ABC = \angle ACD = \angle ADE.$$

Diagonals  $BD$  and  $CE$  meet at  $P$ . Prove that ray  $AP$  bisects  $\overline{CD}$ .

*Proof.* Let  $B'$  be intersection of diagonals  $AC$  and  $BD$ , and let  $E'$  be the intersection of diagonals  $AD$  and  $CE$ . Also let  $A'$  be the intersection of ray  $AP$  with  $CD$ . Notice that the given angle conditions imply that  $\triangle ABC \sim \triangle ACD \sim \triangle ADE$ . From this it follows that quadrilaterals  $ABCD$  and  $ACDE$  are similar. Since  $B'$  and  $E'$  are the intersections of

the diagonals of their respective quadrilaterals, we have that  $\frac{CB'}{B'A} = \frac{DE'}{E'A}$ . By Ceva's on  $\triangle ACD$ , we have that

$$\frac{AE'}{E'D} \cdot \frac{DA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

Since  $\frac{CB'}{B'A} \cdot \frac{AE'}{E'D} = 1$ , we have that  $DA' = A'C$ .  $\square$

**Problem (3.20 (BAMO 2013/3))**

Let  $H$  be the orthocenter of an acute triangle  $ABC$ . Consider the circumcenters of triangles  $ABH$ ,  $BCH$ , and  $CAH$ . Prove that they are the vertices of a triangle that is congruent to  $ABC$ .

*Proof.* Let  $A', B', C'$  be the circumcenters of  $(BCH)$ ,  $(CAH)$ ,  $(ABH)$ , respectively. Note that  $H$  is the radical center of  $(ABH)$ ,  $(BCH)$ ,  $(CAH)$ . Thus  $\overline{AH} \perp \overline{B'C'}$ . Also notice by properties of circumcenters,  $A'$  is on the perpendicular bisector of  $\overline{BC}$ . Let  $O$  be where the perpendicular bisectors of  $\triangle ABC$  intersect (namely, the circumcenter of  $\triangle ABC$ ). Since  $\overline{A'O} \parallel \overline{AH}$ ,  $\overline{A'O} \perp \overline{B'C'}$ . This follows similarly for  $B'$  and  $C'$ , hence  $O$  is the orthocenter of  $\triangle A'B'C'$ . Also notice that, by construction,  $H$  is the circumcenter of  $\triangle A'B'C'$ . Therefore, a homothety of scale factor  $-1$  exists that sends  $H$  to  $O$ ,  $A$  to  $A'$ ,  $B$  to  $B'$ , and  $C$  to  $C'$ . Hence,  $\triangle ABC \cong \triangle A'B'C'$ .  $\square$

**Problem (3.21 (USAMO 2003/4))**

Let  $ABC$  be a triangle. A circle passing through  $A$  and  $B$  intersects segments  $AC$  and  $BC$  at  $D$  and  $E$ , respectively. Lines  $AB$  and  $DE$  intersect at  $F$ , while lines  $BD$  and  $CF$  intersect at  $M$ . Prove that  $MF = MC$  if and only if  $MB \cdot MD = MC^2$ .

*Proof.* By assuming  $MB \cdot MD = MC^2$ , we have that  $\frac{MB}{MC} = \frac{MC}{MD}$ , and since  $\angle BMC = \angle CMD$ , this implies that  $\triangle BMC \sim \triangle CMD$ . Since  $ABDE$  is a cyclic quadrilateral,  $\angle DAE = \angle DBE$ . Now we have that

$$\angle CAE = \angle DAE = \angle DBE = \angle MBC = \angle MCD = \angle FCA,$$

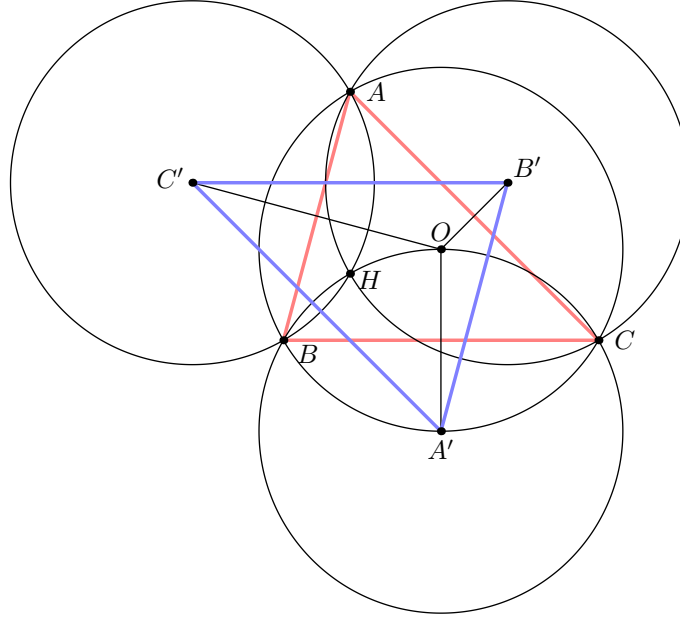


Figure 2: Problem 3.20

hence  $\overline{AE} \parallel \overline{CF}$ . Therefore  $\triangle ABE \sim \triangle FBC$  and  $\frac{FB}{AB} = \frac{CB}{EB}$ . Then

$$\begin{aligned} \frac{FB}{AB} &= \frac{CB}{EB} \\ \frac{FA + AB}{AB} &= \frac{CE + EB}{EB} \\ 1 + \frac{FA}{AB} &= 1 + \frac{CE}{EB} \\ \frac{FA}{AB} &= \frac{CE}{EB}. \end{aligned}$$

By Ceva's on  $\triangle BCF$ , we have that

$$\frac{FA}{AB} \cdot \frac{BE}{EC} \cdot \frac{CM}{MF} = 1.$$

Since  $\frac{FA}{AB} = \frac{CE}{EB}$ , we have that  $MF = MC$ .

We will now go in the reverse direction. We assume  $MF = MC$ . By Ceva's on  $\triangle BCF$ ,

$$\frac{FA}{AB} \cdot \frac{BE}{EC} \cdot \frac{CM}{MF} = 1.$$

and since  $MF = MC$ , we have that  $\frac{FA}{AB} \cdot \frac{BE}{EC} = 1$ . It follows that

$$\begin{aligned}\frac{FA}{AB} &= \frac{CE}{EB} \\ 1 + \frac{FA}{AB} &= 1 + \frac{CE}{EB} \\ \frac{AB}{AB} + \frac{FA}{AB} &= \frac{EB}{EB} + \frac{CE}{EB} \\ \frac{FB}{AB} &= \frac{CB}{EB}.\end{aligned}$$

Thus  $\triangle ABE \sim \triangle FBC$ . This implies that  $\overline{AE} \parallel \overline{CF}$ . Since  $ABDE$  is a cyclic quadrilateral, we have that  $\angle FCA = \angle DAE = \angle DBE$ , and since  $\angle BMC = \angle CMD$ , we have that  $\triangle BMC \sim \triangle CMD$  by  $AA \sim$ . Thus  $\frac{MB}{MC} = \frac{MC}{MD} \rightarrow MB \cdot MD = MC^2$ , as desired.  $\square$

**Theorem (3.22, Monge's Theorem)**

Consider disjoint circles  $\omega_1, \omega_2, \omega_3$  in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.

*Proof.* Let the points  $O_1, O_2, O_3$ , be the centers of  $\omega_1, \omega_2, \omega_3$ , respectively. Let the external tangents of  $\omega_1$  and  $\omega_2$  meet at  $X$ , and define  $Y$  and  $Z$  analogously. Note that  $X, Y, Z$  are each on an extension of a side of  $\triangle O_1 O_2 O_3$ . Let  $T_1$  and  $T_2$  be points of tangency of  $\omega_1$  and  $\omega_2$ , respectively, where  $T_1$  and  $T_2$  are on the same side of line  $XO_1 O_2$ . Note that it is impossible for  $X$  to be between  $O_1$  and  $O_2$ , since  $X$  is the intersection of external tangents. Since tangents are always perpendicular to their circles, we have that  $\triangle T_1 O_1 X \sim \triangle T_2 O_2 X$  by  $AA \sim$ , thus with directed lengths we have  $\frac{O_1 X}{X O_2} = -\frac{r_1}{r_2}$ , where  $r_1$  and  $r_2$  are the radii of  $\omega_1$  and  $\omega_2$ . Similar arguments can be applied to the other two pairs of circles to give  $\frac{O_2 Y}{Y O_3} = -\frac{r_2}{r_3}$  and  $\frac{O_3 Z}{Z O_1} = -\frac{r_3}{r_1}$ . Thus

$$\frac{O_1 X}{X O_2} \cdot \frac{O_2 Y}{Y O_3} \cdot \frac{O_3 Z}{Z O_1} = \left(-\frac{r_1}{r_2}\right) \left(-\frac{r_2}{r_3}\right) \left(-\frac{r_3}{r_1}\right) = -1.$$

By Menelaus's Theorem, this proves that  $X, Y, Z$  are collinear.  $\square$



**Theorem (3.23, Cevian Nest)**

Let  $\overline{AX}, \overline{BY}, \overline{CZ}$  be concurrent cevians of  $ABC$ . Let  $\overline{XD}, \overline{YE}, \overline{ZF}$  be concurrent cevians in triangle  $XYZ$ . Prove that rays  $AD, BE, CF$  concur.

*Proof.* By the Ratio Lemma on  $\triangle ZAY$ , we have that

$$\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{\sin \angle ZAD}{\sin \angle YAD} = \frac{AY}{YC} \cdot \frac{ZD}{DY}.$$

Similarly, for  $\triangle XBZ$  and  $\triangle YCX$  we have that

$$\frac{\sin \angle CBE}{\sin \angle ABE} = \frac{BZ}{XB} \cdot \frac{XE}{EZ}$$

and

$$\frac{\sin \angle ACF}{\sin \angle BCF} = \frac{CX}{YC} \cdot \frac{YF}{FX}.$$

Multiplying these three equations together gives us

$$\begin{aligned} \frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} &= \left( \frac{AY}{YC} \cdot \frac{CX}{XB} \cdot \frac{BZ}{ZA} \right) \cdot \left( \frac{ZD}{DY} \cdot \frac{YF}{FX} \cdot \frac{XE}{EZ} \right) \\ &= 1 \cdot 1 \\ &= 1. \end{aligned}$$

Note that each of the factors in parentheses on the RHS of the first equation are equal to 1 by Ceva's on  $\triangle ABC$  and  $\triangle XYZ$ , respectively. By the trigonometric form of Ceva's, this implies rays  $AD, BE, CF$  concur.  $\square$

**Problem (3.24)**

Let  $ABC$  be an acute triangle and suppose  $X$  is a point on  $(ABC)$  with  $\overline{AX} \parallel \overline{BC}$  and  $X \neq A$ . Denote by  $G$  the centroid of triangle  $ABC$ , and by  $K$  the foot of the altitude from  $A$  to  $BC$ . Prove that  $K, G, X$  are collinear.

*Proof.* Denote by  $A', B', C'$  the midpoints of sides  $\overline{BC}, \overline{CA}, \overline{AB}$ . Note that  $A', B', C', K$  are on the nine-point circle of  $\triangle ABC$ . Also note that each side of  $\triangle A'B'C'$  is parallel to a side of  $\triangle ABC$ . Therefore there exists a homothety  $h$  centered at  $G$  such that  $h(A) = A', h(B) = B', h(C) = C'$ . Since  $\overline{AX} \parallel \overline{BC} \parallel \overline{A'K}$ ,  $h$  sends  $K$  to  $X$ . Therefore  $K, G, X$  are collinear.  $\square$

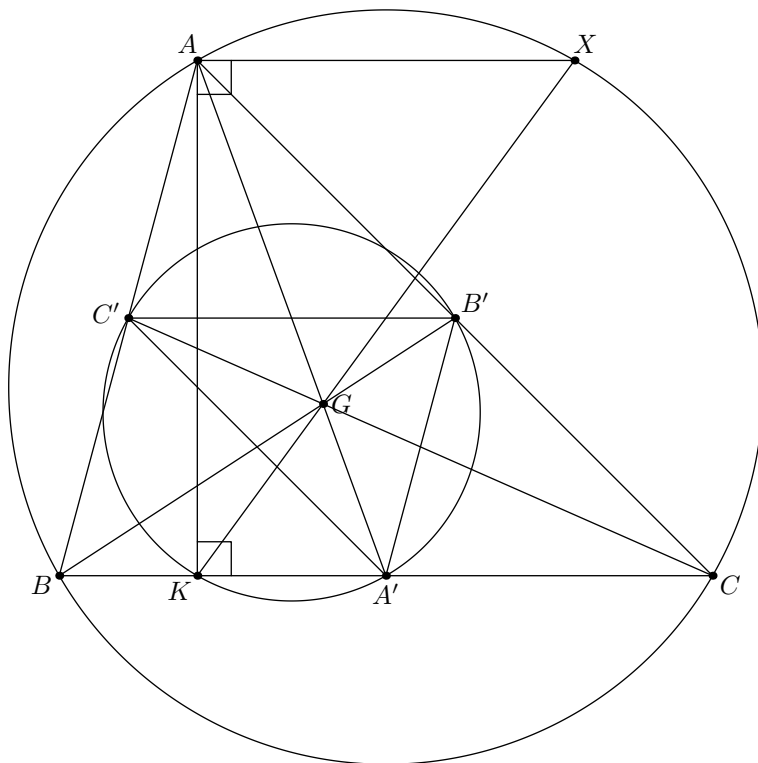


Figure 3: Problem 3.24

**Problem (3.25 (USAMO 1993/2))**

Let  $ABCD$  be a quadrilateral whose diagonals  $\overline{AC}$  and  $\overline{BD}$  are perpendicular and intersect at  $E$ . Prove that the reflections of  $E$  across  $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$  are concyclic.

*Proof.* Denote by  $P, Q, R, S$  the projections of  $E$  onto  $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ , respectively. Note that quadrilaterals  $APES, BQEP, CREQ, DSER$  are cyclic. We will show  $PQRS$  is cyclic by angle chase.

$$\begin{aligned}
 \angle SPQ &= \angle SPE + \angle EPQ \\
 &= \angle SAE + \angle EBQ \\
 &= 90^\circ - \angle EDS + 90^\circ - \angle QCE \\
 &= \angle SRE + \angle ERQ \\
 &= \angle SRQ.
 \end{aligned}$$

A homothety centered at  $E$  with a scale factor of 2 sends  $PQRS$  to the desired quadrilateral. Hence we are done.  $\square$

**Problem (3.26 (EGMO 2013/1))**

The side  $BC$  of the triangle  $ABC$  is extended beyond  $C$  to  $D$  so that  $CD = BC$ . The side  $CA$  is extended beyond  $A$  to  $E$  so that  $AE = 2CA$ . Prove that if  $AD = BE$  then the triangle  $ABC$  is right-angled.

*Proof.* Let  $BC = a, CA = b, AB = c, \angle BAC = \theta$ . Thus  $CD = a$  and  $AE = 2b$ . Let  $F$  be a point on  $\overline{AB}$  such that  $\overline{DF} \parallel \overline{AC}$ . By similar triangles,  $AF = c, DF = 2b$ , and  $\angle AFD = \theta$ . We also have that  $\angle EAC = 180^\circ - \theta$ . Thus by the Law of Cosines,

$$BE^2 = (2b)^2 + c^2 - 2 \cdot 2b \cdot c \cdot \cos(180^\circ - \theta)$$

and

$$AD^2 = (2b)^2 + c^2 - 2 \cdot 2b \cdot c \cdot \cos(\theta).$$

Since  $AD = BE$ , we have that  $\cos(180^\circ - \theta) = \cos(\theta)$ . Therefore  $\theta = 90^\circ$ , as desired.  $\square$

**Problem (3.27 (APMO 2004/2))**

Let  $O$  be the circumcenter and  $H$  the orthocenter of an acute triangle  $ABC$ . Prove that the area of one of the triangles  $AOH, BOH$ , and  $COH$  is equal to the sum of the areas of the other two.

*Proof.* (NOT ORIGINAL) WLOG let  $B$  and  $C$  be on the same side of line  $\overline{OH}$ . Let  $M$  be the midpoint of  $BC$ . Denote by  $A', B', C', M'$  the projections onto  $OH$  of  $A, B, C, M$ , respectively. Notice that a homothety centered at  $G$  with a ratio of  $-2$  sends  $\triangle MM'G$  to  $\triangle AA'G$ , so  $AA' = 2MM' = BB' + CC'$ , which implies the result.  $\square$

**Problem (3.28 (Shortlist 2001/G1))**

Let  $A_1$  be the center of the square inscribed in acute triangle  $ABC$  with two vertices of the square on side  $BC$ . Thus one of the two remaining vertices of the square is on side  $AB$  and the other is on  $AC$ . Points  $B_1$  and  $C_1$  are defined in a similar way for inscribed squares with two vertices on sides  $AC$  and  $AB$ , respectively. Prove that lines  $AA_1, BB_1, CC_1$  are concurrent.

*Proof.* (NOT ENTIRELY ORIGINAL) Denote by  $A_\square$  the square with center  $A_1$ . Denote by  $P, Q$  the vertices of  $A_\square$  on sides  $AB$  and  $AC$ , respectively. Consider a homothety centered at  $A$  that sends  $P$  to  $B$  and  $Q$  to  $C$ . The center  $A'_1$ , of the new square lies on  $AA_1$ . Similar arguments hold for homotheties centered at  $B$  and  $C$ . Thus we just need to show that  $AA'_1, BB'_1, CC'_1$  are concurrent. By the Law of Sines,

$$\frac{A'_1B}{\sin \angle A'_1AB} = \frac{AA'_1}{\sin(B + 45^\circ)} \text{ and } \frac{A'_1C}{\sin \angle A'_1AC} = \frac{AA'_1}{\sin(C + 45^\circ)}.$$

Since  $A'_1B = A'_1C$ , we have that  $\frac{\sin(B + 45^\circ)}{\sin(C + 45^\circ)} = \frac{\sin \angle A'_1AB}{\sin \angle A'_1AC}$ . Doing this for  $B'_1$  and  $C'_1$  then multiplying all three equations together proves the desired conclusion by the Trigonometric form of Ceva's Theorem.  $\square$

## §4 Assorted Configurations

### Proposition

Proposition [4.1] Prove that the Simson line is parallel to  $\overline{AK}$  in the notation of Figure 4.1A.

*Proof.* Notice that  $BXPZ$  is cyclic since  $\angle BZP = \angle BXP$ . Then

$$\angle YXP = \angle ZXP = \angle ZBP = \angle ABP = \angle AKP.$$

□

### Problem (4.2)

Let  $K'$  be the reflection of  $K$  across  $\overline{BC}$ . Show that  $K'$  is the orthocenter of  $\triangle PBC$ .

*Proof.* Note that  $K'$  is already on the altitude  $PX$ . Also note that  $K$  is on the circumcircle of  $\triangle PBC$ . Since  $K$  and  $K'$  are reflections of each other over  $BC$ , by Lemma 1.17 we have that  $K'$  is the orthocenter of  $\triangle PBC$ . □

### Problem (4.3)

Show that  $LHXP$  is a parallelogram.

*Proof.* Note that  $\overline{LH} \parallel \overline{XP}$  by construction. Thus it suffices to show that  $LH = XP$ . By Proposition 4.1 we have that  $LA = XK$ . Also, by the conclusion made after Problem 4.2 we have that  $AH = PK'$ . Thus

$$LH = LA + AH = XK + PK' = XK' + PK' = XP.$$

□

### Problem (4.5)

Check  $\angle IAI_B = 90^\circ$  and  $\angle IAI_C = 90^\circ$ .

*Proof.* By the Incenter-Excenter Lemma we know that  $II_B$  is the diameter of circle  $(AICI_B)$ . Therefore  $\angle IAI_B = 90^\circ$ . A similar argument holds to show  $\angle IAI_C = 90^\circ$ . □

**Problem (4.7)**

How are Lemma 1.18, Lemma 3.11, and Lemma 4.6 related?

Let  $L$  be the midpoint of  $II_A$ . Then in Figure 4.2A,  $(ABCL)$  is in fact the nine-point circle of  $\triangle I_AI_BI_C$  (Lemma 3.11). Moreover, by Lemma 1.18,  $I, B, C, I_A$  all lie on a circle centered at  $L$ . Finally, by Lemma 4.6, we know  $I$  is the orthocenter of  $\triangle I_AI_BI_C$ , but we also can derive this from Lemma 3.11. This means that any two lemmas can prove the third.

**Problem (4.8)**

Prove that  $A, E$ , and  $X$  are collinear and that  $\overline{DE}$  is a diameter of the incircle.

*Proof.* Consider the homothety bringing  $\triangle AB'C'$  to  $\triangle ABC$ . This homothety brings the circle with center  $I$  to the circle with center  $I_A$ , so it must bring point  $E$  to point  $X$ . Hence  $A, E, X$  are collinear. Now notice that  $\overline{BC} \perp \overline{ID}$  and  $\overline{B'C'} \perp \overline{IE}$  by tangency and  $\overline{B'C'} \parallel \overline{BC}$ . This is enough to show  $I, D, E$  are collinear, therefore  $\overline{DE}$  is a diameter of the incircle.  $\square$

**Lemma (4.10, Diameter of the Excircle)**

In the notation of Lemma 4.9, suppose  $\overline{XY}$  is a diameter of the  $A$ -excircle. Show that  $D$  lies on  $\overline{AY}$ .

*Proof.* Consider again the homothety bringing  $\triangle AB'C'$  to  $\triangle ABC$ . Since  $\overline{DE}$  and  $\overline{YX}$  are diameters of the incircle and  $A$ -excircle, respectively, the homothety brings  $D$  to  $Y$ . Hence  $D$  lies on  $\overline{AY}$ .  $\square$

**Problem (4.11)**

If  $M$  is the midpoint of  $\overline{BC}$ , prove that  $\overline{AE} \parallel \overline{IM}$ .

*Proof.* Since  $M$  is the midpoint of  $BC$ , we know that  $M$  is also the midpoint of  $DX$  since  $BD = XC$ . We also know that  $I$  is the midpoint of  $DE$ . Thus we find that  $\triangle DEX \sim \triangle DIM$ , implying the result.  $\square$

**Problem (4.12)**

Prove that points  $X, I, M$  are collinear.

*Proof.* Note that  $AK \parallel ED$ , so the homothety centered at  $X$  that sends  $\triangle AXK$  to  $\triangle EXD$  sends  $M$  to  $I$  because they are midpoints of  $AK$  and  $ED$ , respectively. This implies that  $X, I, M$  are collinear.  $\square$

**Problem (4.13)**

Show that  $D, I_A, M$  are collinear.

*Proof.* Note that  $AK \parallel XY$ , so the homothety centered at  $D$  that sends  $\triangle XDY$  to  $\triangle KDA$  sends  $I_A$  to  $M$  since they are midpoints of  $XY$  and  $AK$ , respectively. This implies that  $D, I_A, M$  are collinear.  $\square$