CNCM Problem of the Day Solutions

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Notice that the expected number of dollars Tommy expects to win is equivalent to the following infinite series:

$$\frac{1}{6}\sum_{n=0}^{\infty} n^2 \left(\frac{5}{6}\right)^n .$$

Define

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n$$

where we want to find the value of $f(5/6)$. Then

$$\begin{aligned}
f(x) &= x\sum_{n=0}^{\infty} n^2 x^{n-1} \\
&= x\frac{d}{dx}\left[\sum_{n=0}^{\infty} n x^n\right] \\
&= x\frac{d}{dx}\left[x\sum_{n=0}^{\infty} n x^{n-1}\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\frac{1}{1-x}\right]\right] \\
&= x\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] \\
&= x\left[\frac{(1-x)^2+2x(1-x)}{(1-x)^4}\right] \\
&= x\left[\frac{(1-x)+2x}{(1-x)^3}\right] \\
&= x\left[\frac{1+x}{(1-x)^3}\right] \\
f(5/6) &= \frac{5}{6}\left[\frac{1+5/6}{(1-5/6)^3}\right] \\
&= \frac{5}{6}\cdot\frac{11}{6}\cdot\frac{6^3}{1} \\
&= 330.
\end{aligned}$$

Then our final answer is $\frac{1}{6}f(\frac{5}{6}) = \frac{330}{6} = \boxed{55}$.