Random Problems

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Problem (OTIS Excerpts #7)
Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

Proof. We claim that $c = 10$. We know this value works because for $n \geq 1, c^n \in \{10, 100, 1000, \dots\}$, and since all digits from the 10s and to the left are less than 4, adding 1 to them will not violate our digit condition. We will now check that this is the smallest possible value for c.

- $c = 1$ fails at $n = 1$ since $1 + 4 = 5$.
- $c = 2$ fails at $n = 1$ since $2 + 4 = 6$.
- $c = 3$ fails at $n = 1$ since $3 + 4 = 7$.
- $c = 4$ fails at $n = 1$ since $4 + 4 = 8$.
- $c = 5$ fails at $n = 1$ since $5 + 4 = 9$.
- $c = 6$ fails at $n = 2$ since $36 + 2014 = 2050$.
- $c = 7$ fails at $n = 2$ since $49 + 2014 = 2063$.
- $c = 8$ fails at $n = 2$ since $64 + 2014 = 2078$.
- $c = 9$ fails at $n = 2$ since $81 + 2014 = 2095$.

Since every value of c less than 10 fails, we are done. $\square$

Problem (OTIS Excerpts #77, HMMT Februrary 2013)

Values $a_1, \ldots, a_{2013}$ are chosen independently and at random from the set $\{1, \ldots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \ldots, a_{2013}\}$?

Solution. Let P be the number of distinct values in $a_1, \ldots, a_{2013}$, and for each $i = 1, 2, \ldots, 2013$ let

$$P_i := \begin{cases} 1 & \text{if } a_i \neq a_j \text{ for all } j < i \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $P = P_1 + \cdots + P_{2013}$. Thus it follows that

$$\begin{aligned} E[P] &= E[P_1] + E[P_2] + \cdots + E[P_{2013}] \\ &= 1 + (1 - 1/2013) + \cdots + (1 - 1/2013)^{2012} \\ &= \frac{1 - (2012/2013)^{2013}}{1 - 2012/2013} \\ &= 2013 \left(1 - \left(\frac{2012}{2013}\right)^{2013}\right). \end{aligned}$$

$\square$

Problem (100 Geometry Problems # 8)

Let ABC be a triangle with $\angle CAB$ a right angle. The point L lies on the side BC between B and C. The circle BAL meets the line AC again at M and the circle CAL meets the line AB again at N. Prove that L, M, and N lie on a straight line.

Proof. Since $ANLC$ and $ALBM$ are cyclic quadrilaterals, $\angle CAN = \angle CLN = \angle 90^\circ = \angle BAM = \angle BLM$. Since $\angle CLN + \angle BLN = 180^\circ$, we have $\angle BLN = \angle BLM = 90^\circ$, as desired. $\square$