CNCM Problem of the Day Solutions

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6 August 2021

Notice that the expected number of dollars Tommy expects to win is equivalent to the following infinite series:

$$\frac{1}{6}\sum_{n=0}^{\infty} n^2 \left(\frac{5}{6}\right)^n .$$

Define

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n$$

where we want to find the value of $f(5/6)$. Then

$$\begin{aligned}
f(x) &= x \sum_{n=0}^{\infty} n^2 x^{n-1} \\
&= x \frac{d}{dx}\left[\sum_{n=0}^{\infty} n x^n\right] \\
&= x \frac{d}{dx}\left[x \sum_{n=0}^{\infty} n x^{n-1}\right] \\
&= x \frac{d}{dx}\left[x \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]\right] \\
&= x \frac{d}{dx}\left[x \frac{d}{dx}\left[\frac{1}{1-x}\right]\right] \\
&= x \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] \\
&= x \left[\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}\right] \\
&= x \left[\frac{(1-x) + 2x}{(1-x)^3}\right] \\
&= x \left[\frac{1+x}{(1-x)^3}\right] \\
f(5/6) &= \frac{5}{6}\left[\frac{1+5/6}{(1-5/6)^3}\right] \\
&= \frac{5}{6} \cdot \frac{11}{6} \cdot \frac{6^3}{1} \\
&= 330.
\end{aligned}$$

Then our final answer is $\frac{1}{6} f(\frac{5}{6}) = \frac{330}{6} = \boxed{55}$.

11 August 2021

Here let R_n denote the remaining water after the n-th pour.

$$\begin{aligned}
R_0 &= 1 \\
R_1 &= \left(1 - \frac{1}{2}\right) R_0 = \frac{1}{2} \\
R_2 &= \left(1 - \frac{1}{3}\right) R_1 = \frac{1}{3} \\
R_3 &= \left(1 - \frac{1}{4}\right) R_2 = \frac{1}{4}
\end{aligned}$$

Therefore we can assume by Engineer's Induction that $R_n = \frac{1}{n+1}$. Hence $R_9 = \frac{1}{10}$ for a final answer of $\boxed{9}$.

12 August 2021

For a two-game block, there is a probability of $\dfrac{3}{4} \cdot \dfrac{1}{4} + \dfrac{1}{4} \cdot \dfrac{3}{4} = \dfrac{3}{8}$ of the entire match ending then and there. The only other outcome after two games is a tie, since each game must declare a winner, and this happens with probability 5/8. Thus the expected number of games in a match is the following:

$$\begin{aligned}
\frac{3}{8} \cdot 2 + \frac{3}{8} \cdot \frac{5}{8} \cdot 4 + \frac{3}{8} \cdot \left(\frac{5}{8}\right)^2 \cdot 6 + \cdots &= \sum_{n=0}^{\infty} 2(n+1) \cdot \frac{3}{8} \cdot \left(\frac{5}{8}\right)^n \\
&= \frac{3}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{5}{8}\right)^n \\
&= \frac{3}{4} \sum_{n=0}^{\infty} n \left(\frac{5}{8}\right)^n + \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n.
\end{aligned}$$

Define $f(x) = \sum_{n=0}^{\infty} nx^n$. We would like to find the value of $f(5/8)$. Note that

$$\begin{aligned} f(x) &= x\sum_{n=0}^{\infty} nx^{n-1} \\ &= x\frac{d}{dx}\sum_{n=0}^{\infty} x^n \\ &= x\frac{d}{dx}\left[\frac{1}{1-x}\right] \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

Thus $f(5/8) = \dfrac{5/8}{(1-5/8)^2} = \dfrac{5}{8}\cdot\dfrac{8^2}{3^2} = \dfrac{40}{9}$. Then our original sum becomes

$$\begin{aligned} \frac{3}{4}\sum_{n=0}^{\infty} n\left(\frac{5}{8}\right)^n + \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n &= \frac{3}{4}\cdot\frac{40}{9} + \frac{3}{4}\cdot\frac{1}{1-5/8} \\ &= \frac{10}{3} + \frac{3}{4}\cdot\frac{8}{3} \\ &= \frac{10}{3} + 2 \\ &= \frac{16}{3}. \end{aligned}$$

Our final answer is $160 + 3 = \boxed{163}$.

26 August 2021

Our recurrence relation is

$$7a_n = -a_{n-1} + 8a_{n-2}.$$

By simple calulations we determine that $a_1 = 25$. Note that the recurrence is linear and homogenous. Its characteristic equation is

$$\begin{aligned} 7r^2 + r - 8 &= 0 \\ (7r+8)(r-1) &= 0 \\ r_{1,2} &= -\frac{8}{7}, 1. \end{aligned}$$

So by some theorem (idk) $a_n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2(1)^n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2$ is a solution. To find α_1, α_2 we must solve the following system:

$$\begin{cases} a_0 = \alpha_1 + \alpha_2 = 4 \\ a_1 = -\frac{8}{7}\alpha_1 + \alpha_2 = 25. \end{cases}$$

Solving this gets us $(\alpha_1, \alpha_2) = (-49/5, 69/5)$. Thus what we have left to evaluate is

$$\begin{aligned} a_7 &= -\frac{49}{5}\left(-\frac{8}{7}\right)^7 + \frac{69}{5} \\ &= \frac{49}{5}\left(\frac{8}{7}\right)^7 + \frac{69}{5} \\ &= \frac{1}{5} \cdot \frac{8^7}{7^5} + \frac{69}{5} \\ &= \frac{8^7 + 69 \cdot 7^5}{5 \cdot 7^5} \\ &= \frac{8^7 + 69 \cdot 16807}{5 \cdot 7^5} \\ &= \frac{2097152 + 69 \cdot 16807}{5 \cdot 7^5} \\ &= \frac{2097152 + 1159683}{5 \cdot 7^5} \\ &= \frac{3256835}{5 \cdot 7^5} \\ &= \frac{651367}{16807}. \end{aligned}$$

Therefore our final answer is $651367 + 16807 + 28795 = \boxed{696969}$.

2 September 2021

Let $BD = x$. By the Angle Bisector Theorem

$$\frac{8}{12} = \frac{x}{10-x}.$$

Solving for x gives us $x = 4$. Thus $BD = 4$ and $CD = 6$. By Stewart's Theorem on $\triangle ABC$ we have

$$\begin{aligned}
b^2m + c^2n &= a(d^2 + mn) \\
8^2 \cdot 6 + 12^2 \cdot 4 &= 10(d^2 + 6 \cdot 4) \\
384 + 576 &= 10d^2 + 240 \\
d^2 &= 72 \\
AD &= 6\sqrt{2}.
\end{aligned}$$

We will now find BD'. Applying Stewart's Theorem again on $\triangle ABD'$ gives us

$$\begin{aligned}
8^2 \cdot 2\sqrt{2} + c^2 \cdot 6\sqrt{2} &= 8\sqrt{2}(4^2 + 24) \\
128 + 6c^2 &= 128 + 192 \\
c^2 &= 32 \\
BD' &= 4\sqrt{2}.
\end{aligned}$$

Similarly to find CD' we apply Stewart's Theorem a third time on $\triangle ACD'$, which gives us

$$\begin{aligned}
b^2m + c^2n &= a(d^2 + mn) \\
12^2 \cdot 2\sqrt{2} + c^2 \cdot 6\sqrt{2} &= 8\sqrt{2}(6^2 + 24) \\
288 + 6c^2 &= 288 + 192 \\
c^2 &= 32 \\
CD' &= 4\sqrt{2}.
\end{aligned}$$

Thus $BD' \cdot CD' = (4\sqrt{2})^2 = \boxed{32}$.

6 September 2021

Let $PY = x$ and $QY = y$. Note that the area of $\triangle XYZ$ (henceforth denoted $[XYZ]$) is $\frac{1}{2} \cdot 12 \cdot 2004$. Also note that $[XYZ] = [XPY] + [YPZ]$. It follows that

$$\begin{aligned}
[XYZ] &= [XPY] + [YPZ] \\
\frac{1}{2} \cdot 12 \cdot 2004 &= \frac{1}{2} \cdot 12 \cdot \frac{x}{2} + \frac{1}{2} \cdot 2004 \cdot \frac{x\sqrt{3}}{2} \\
12 \cdot 2004 &= 6x + 1002\sqrt{3}x \\
x &= \frac{2 \cdot 2004}{1 + 167\sqrt{3}}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
[XYZ] &= [XQY] + [YQZ] \\
\frac{1}{2} \cdot 12 \cdot 2004 &= \frac{1}{2} \cdot 12 \cdot \frac{y\sqrt{3}}{2} + \frac{1}{2} \cdot 2004 \cdot \frac{y}{2} \\
12 \cdot 2004 &= 6\sqrt{3}y + 1002y \\
y &= \frac{2 \cdot 2004}{167 + \sqrt{3}}
\end{aligned}$$

Thus

$$\begin{aligned}
(PY + YZ)(QY + XY) &= \left(\frac{2 \cdot 12 \cdot 167}{1 + 167\sqrt{3}} + 12 \cdot 167\right)\left(\frac{2 \cdot 12 \cdot 167}{167 + \sqrt{3}} + 12\right) \\
&= 12 \cdot 167 \cdot 12 \left(\frac{2 + 1 + 167\sqrt{3}}{1 + 167\sqrt{3}}\right)\left(\frac{2 \cdot 167 + 167 + \sqrt{3}}{167 + \sqrt{3}}\right) \\
&= 12 \cdot 167 \cdot 12 \left(\frac{3 + 167\sqrt{3}}{1 + 167\sqrt{3}}\right)\left(\frac{3 \cdot 167 + \sqrt{3}}{167 + \sqrt{3}}\right) \\
&= 12 \cdot 167 \cdot 12 \left(\frac{167^2 \cdot 3\sqrt{3} + 12 \cdot 167 + 3\sqrt{3}}{167^2\sqrt{3} + 4 \cdot 167 + \sqrt{3}}\right) \\
&= 12 \cdot 167 \cdot 12 \cdot 3 \left(\frac{167^2\sqrt{3} + 4 \cdot 167 + \sqrt{3}}{167^2\sqrt{3} + 4 \cdot 167 + \sqrt{3}}\right) \\
&= \boxed{72144}.
\end{aligned}$$

8 September 2021

We have $f(x) = \log_2(x + \underbrace{\log_2(x + \log_2(x + \cdots))}_{f(x)}) = \log_2(x + f(x))$. Solving for x gives us $x = 2^{f(x)} - f(x)$. Thus

$$f^{-1}(x) = 2^x - x.$$

It follows that

$$\begin{aligned}
\sum_{k=2}^{10} \left[2^k - k\right] &= \sum_{k=2}^{10} 2^k - \sum_{k=2}^{10} k \\
&= \sum_{k=0}^{8} 2^{k+2} - \sum_{k=0}^{8} (k+2) \\
&= 4 \cdot \frac{2^9 - 1}{2 - 1} - \frac{8 \cdot 9}{2} - 2 \cdot 9 \\
&= 4(511) - 36 - 18 \\
&= \boxed{1990}.
\end{aligned}$$

17 September 2021

Right off the bat, we can ignore dividing by 1 since every number is divisible by 1.

Claim — All $n = 11(2k+1), k \in \mathbb{Z}$ fail.

Proof. Note that

$$\begin{aligned}
n &\equiv 22k + 11 \equiv 1 && \pmod 2 \\
n &\equiv 0 && \pmod{11}.
\end{aligned}$$

Since $1 > 0$, this is a decerasing sequence. □

Claim — The remainders of 11, 22, 121, and 242 always follow a non-decreasing sequence except for $n \in [121, 131]$.

Proof. We will start with $(11, 22)$. We can represent $n = 22q_1 + r_1$ with $0 \leq r_1 < 22$. Then

$$\begin{aligned} n &= 22q_1 + r_1 \\ &= 11(2q_1) + r_1 \\ &= 11(2q_1) + 11k + r_2 \\ &= 11(2q_1 + k) + r_2, \end{aligned}$$

where $0 \leq r_2 < 11$. If $r_1 \geq 11$, then $k = 1$ and $r_1 > r_2$. If $r_1 < 11$, then $k = 0$ and $r_1 = r_2$. Thus $n \pmod{22} > n \pmod{11}$. Very similar arguments apply between $(11, 121), (11, 242)$, and $(121, 242)$ since, in each pair, one is a multiple of the other. Additionally, since each pair cannot decrease the remainder of n, together they must form a non-decreasing sequence.
However, we are not done, since we must still consider $(22, 121)$ because 121 is not a multiple of 22. Write $n = 121q_1 + r_1$ with $0 \leq r_1 < 121$. Note that if q_1 is even, we are done since we can follow a very similar line of reasoning as above to show that $r_1 \geq r_2$. We will now deal with the case if q_1 is odd. Note that

$$\begin{aligned} n &= 121q_1 + r_1 \\ &= 121(2k + 1) + r_1 \\ &= 2 \cdot 121k + 121 + r_1 \\ &= 22(11k + 5) + 11 + r_1, \end{aligned}$$

where $k \geq 0$. If $r_1 \geq 11$, then $r_1 > r_2$, so we are good here. However, if $r_1 < 11$, then $r_2 = 11 + r_1 > r_1$, so this case fails here. The only possible odd value that q_1 can be and still have $n \leq 242$ is 1, so $n \in \{121, 122, ..., 131\}$ all fail. $\square$

Now, counting up how many n fail and subtracting from 242 we get $242 - \underbrace{11}_{\text{first claim}} - \underbrace{11}_{\text{second claim}} + \underbrace{1}_{\text{double counting 121}} = \boxed{221}$.