

CNCM Problem of the Day Solutions

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6 August 2021

Notice that the expected number of dollars Tommy expects to win is equivalent to the following infinite series:

$$\frac{1}{6} \sum_{n=0}^{\infty} n^2 \left(\frac{5}{6}\right)^n.$$

Define

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n$$

where we want to find the value of $f(5/6)$. Then

$$\begin{aligned} f(x) &= x \sum_{n=0}^{\infty} n^2 x^{n-1} \\ &= x \frac{d}{dx} \left[\sum_{n=0}^{\infty} n x^n \right] \\ &= x \frac{d}{dx} \left[x \sum_{n=0}^{\infty} n x^{n-1} \right] \\ &= x \frac{d}{dx} \left[x \frac{d}{dx} \left[\sum_{n=0}^{\infty} x^n \right] \right] \\ &= x \frac{d}{dx} \left[x \frac{d}{dx} \left[\frac{1}{1-x} \right] \right] \\ &= x \frac{d}{dx} \left[\frac{x}{(1-x)^2} \right] \\ &= x \left[\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} \right] \\ &= x \left[\frac{(1-x) + 2x}{(1-x)^3} \right] \\ &= x \left[\frac{1+x}{(1-x)^3} \right] \\ f(5/6) &= \frac{5}{6} \left[\frac{1+5/6}{(1-5/6)^3} \right] \\ &= \frac{5}{6} \cdot \frac{11}{6} \cdot \frac{6^3}{1} \\ &= 330. \end{aligned}$$

Then our final answer is $\frac{1}{6}f(\frac{5}{6}) = \frac{330}{6} = \boxed{55}$.

11 August 2021

Here let R_n denote the remaining water after the n -th pour.

$$\begin{aligned} R_0 &= 1 \\ R_1 &= \left(1 - \frac{1}{2}\right) R_0 = \frac{1}{2} \\ R_2 &= \left(1 - \frac{1}{3}\right) R_1 = \frac{1}{3} \\ R_3 &= \left(1 - \frac{1}{4}\right) R_2 = \frac{1}{4} \end{aligned}$$

Therefore we can assume by Engineer's Induction that $R_n = \frac{1}{n+1}$. Hence $R_9 = \frac{1}{10}$ for a final answer of 9.

12 August 2021

For a two-game block, there is a probability of $\frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{8}$ of the entire match ending then and there. The only other outcome after two games is a tie, since each game must declare a winner, and this happens with probability $5/8$. Thus the expected number of games in a match is the following:

$$\begin{aligned} \frac{3}{8} \cdot 2 + \frac{3}{8} \cdot \frac{5}{8} \cdot 4 + \frac{3}{8} \cdot \left(\frac{5}{8}\right)^2 \cdot 6 + \cdots &= \sum_{n=0}^{\infty} 2(n+1) \cdot \frac{3}{8} \cdot \left(\frac{5}{8}\right)^n \\ &= \frac{3}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{5}{8}\right)^n \\ &= \frac{3}{4} \sum_{n=0}^{\infty} n \left(\frac{5}{8}\right)^n + \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n. \end{aligned}$$

Define $f(x) = \sum_{n=0}^{\infty} nx^n$. We would like to find the value of $f(5/8)$. Note that

$$\begin{aligned} f(x) &= x \sum_{n=0}^{\infty} nx^{n-1} \\ &= x \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ &= x \frac{d}{dx} \left[\frac{1}{1-x} \right] \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

Thus $f(5/8) = \frac{5/8}{(1-5/8)^2} = \frac{5}{8} \cdot \frac{8^2}{3^2} = \frac{40}{9}$. Then our original sum becomes

$$\begin{aligned} \frac{3}{4} \sum_{n=0}^{\infty} n \left(\frac{5}{8}\right)^n + \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n &= \frac{3}{4} \cdot \frac{40}{9} + \frac{3}{4} \cdot \frac{1}{1-5/8} \\ &= \frac{10}{3} + \frac{3}{4} \cdot \frac{8}{3} \\ &= \frac{10}{3} + 2 \\ &= \frac{16}{3}. \end{aligned}$$

Our final answer is $160 + 3 = \boxed{163}$.

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Our recurrence relation is

$$7a_n = -a_{n-1} + 8a_{n-2}.$$

By simple calculations we determine that $a_1 = 25$. Note that the recurrence is linear and homogenous. Its characteristic equation is

$$\begin{aligned} 7r^2 + r - 8 &= 0 \\ (7r + 8)(r - 1) &= 0 \\ r_{1,2} &= -\frac{8}{7}, 1. \end{aligned}$$

So by some theorem (idk) $a_n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2 (1)^n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2$ is a solution. To find α_1, α_2 we must solve the following system:

$$\begin{cases} a_0 = \alpha_1 + \alpha_2 = 4 \\ a_1 = -\frac{8}{7}\alpha_1 + \alpha_2 = 25. \end{cases}$$

Solving this gets us $(\alpha_1, \alpha_2) = (-49/5, 69/5)$. Thus what we have left to evaluate is

$$\begin{aligned}
 a_7 &= -\frac{49}{5} \left(-\frac{8}{7} \right)^7 + \frac{69}{5} \\
 &= \frac{49}{5} \left(\frac{8}{7} \right)^7 + \frac{69}{5} \\
 &= \frac{1}{5} \cdot \frac{8^7}{7^5} + \frac{69}{5} \\
 &= \frac{8^7 + 69 \cdot 7^5}{5 \cdot 7^5} \\
 &= \frac{8^7 + 69 \cdot 16807}{5 \cdot 7^5} \\
 &= \frac{2097152 + 69 \cdot 16807}{5 \cdot 7^5} \\
 &= \frac{2097152 + 1159683}{5 \cdot 7^5} \\
 &= \frac{3256835}{5 \cdot 7^5} \\
 &= \frac{651367}{16807}.
 \end{aligned}$$

Therefore our final answer is $651367 + 16807 + 28795 = \boxed{696969}$.