Euclidean Geometry in Mathematical Olympiads Solutions

Ryder Pham

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§1 Angle Chasing

Problem (1.51, IMO 1985/1)

A circle has center on the side $\overline{AB}$ of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD+BC = AB$.

Proof. **INCOMPLETE** Call the center of the circle point O. Let the point T be where $AD = AT$. $\square$

§2 Circles

Lemma (2.19)

Prove that the A-exradius has length

$$r_a = \frac{s}{s-a} r,$$

where r is the inradius.

Proof. Drop perpendiculars from I and I_A to AB. Call the feet of these perpendiculars B_1 and B_2 respectively. Notice that $IB_1 = r$ and $I_A B_2 = r_a$ and that $\triangle AB_1 I \sim \triangle AB_2 I_A$. Therefore

$$\frac{r}{r_a} = \frac{AB_1}{AB_2},$$

but by Lemmas 2.15 and 2.17, we know that $AB_1 = s - a$ and $AB_2 = s$, hence

$$r_a = \frac{s}{s-a} r,$$

and we are done. □

Lemma (2.20)

Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to BC at D and X, respectively. Show that $BD = CX$ and $BX = CD$.

Proof. We will first show that $BD = CX$. Let the incircle be tangent to side AB at point F and let to side AC at point E. Let the A-excircle be tangent to the extension of line AC at C_1 and to the extension of line AB at B_1. Then

$$\begin{aligned}
BD &= BF \\
&= AB_1 - AF - BB_1 \\
&= (AC_1 - AE) - BX \\
&= (CC_1 + CE) - (BC - CX) \\
&= CX + (CD - BC) + CX \\
&= 2CX - BD \\
2BD &= 2CX \to BD = CX.
\end{aligned}$$

It follows that $BX = CD$ because

$$\begin{aligned}
BD &= CX \\
BD + DX &= DX + CX \\
BX &= CD.
\end{aligned}$$

□

Lemma (2.24)

Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.

Proof. By the Incenter-Excenter Lemma, we know that AI_A, BI_B, and CI_C coincide at the incenter I. We also know from the Lemma that II_A is the diameter of circle $BICI_A$. Therefore we have that

$$\angle I_CCI_A = \angle ICI_A = 90^\circ \text{ and } \angle I_BBI_A = \angle IBI_A = 90^\circ.$$

This follows similarily for II_B and II_C. Now we know that AI_A, BI_B, CI_C are in fact the altitudes of triangle $I_AI_BI_C$, therefore I is the orthocenter of triangle $I_AI_BI_C$. Note that since A, B, and C are the feet of the altitudes, ABC is the orthic triangle of triangle $I_AI_BI_C$. □

Theorem (2.25, The Pitot Theorem)

Let $ABCD$ be a quadrilateral. If a circle can be inscribed in it, prove that $AB + CD = BC + DA$.

Proof. Call the points where AB, BC, CD, DA are tangent to the circle E, F, G, H, respectively. Let $AE = AH = a, BE = BF = b, CF = CG = c, DG = DH = d$. Now note that our condition can be manipulated as follows:

$$\begin{gathered} AB + CD = BC + DA \\ (AE + BE) + (CG + DG) = (BF + CF) + (AH + DH) \\ a + b + c + d = b + c + a + d. \end{gathered}$$

Hence, we are done. $\square$

Problem (2.26, USAMO 1990/5)

An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

Proof. Let the circle with diameter $\overline{AB}$ be called ω_1 and the circle with diameter $\overline{AC}$ be called ω_2. By Theorem 2.9, it suffices to show that the intersection of $\overline{MN}$ and $\overline{PQ}$ lies on the radical axis of ω_1 and ω_2. Since $\overline{MN}$ and $\overline{PQ}$ are altitudes of $\triangle ABC$, their intersection is the orthocenter of $\triangle ABC$. We will call this point H. Note that $\overline{AH}$ is the third altitude of $\triangle ABC$. Call the foot of this altitude A'. Now note that $\angle AA'B = \angle AA'C = 90°$, and since $\overline{AB}$ and $\overline{AC}$ are diameters of their respective circles, ω_1 and ω_2 must intersect at A'. Hence, $\overline{AA'}$ is the radical axis of ω_1 and ω_2, and since A, A', H are colinear, H lies on this line. $\square$

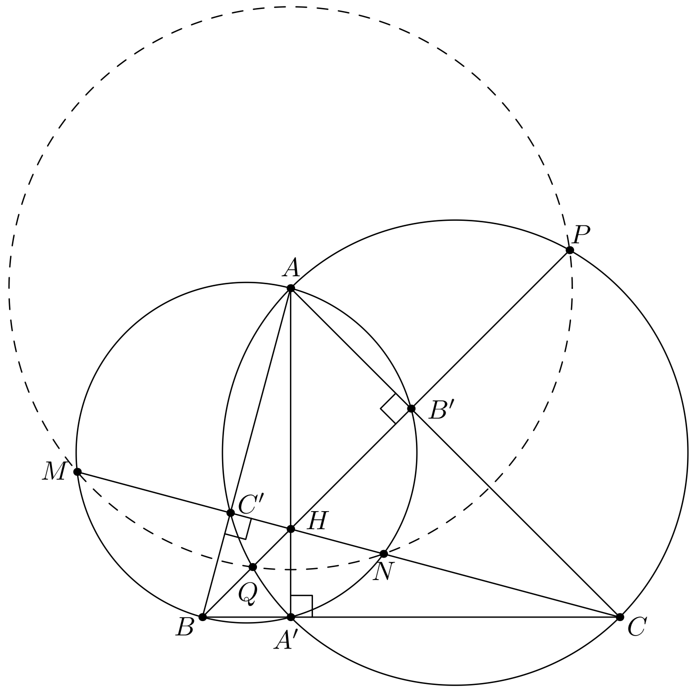

Figure 1: Problem 2.26

Problem (2.27, BAMO 2012/4)

Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.

(a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N.

(b) Prove that no matter where one chooses point M along segment $\overline{AB}$, all lines MN will pass through some fixed point K in the plane.

Proof. We will prove (a) by angle chasing. Notice that since $ACNM$ and $BDNM$ are cyclic, we have that

$$\angle AMC = \angle ANC = \angle ACM = \angle ANM = \angle MDB = \angle MNB = 60^\circ,$$

and since $\angle ANC + \angle ANM + \angle MNB = 60^\circ + 60^\circ + 60^\circ = 180^\circ$, we have

that BC is a straight line passing through N. A very similar argument follows for AD.
We will now prove (b) using radical axes. First, construct an equilateral triangle ABE on the same side as the other two equilateral triangles. Let the circumcircles around triangles AMC, BMD, and ABE be ω_1, ω_2, and ω_3, respectively. Note that MN is the radical axis of circles ω_1 and ω_2, the line tangent to circles ω_1 and ω_3 at point A is the radical axis of circles ω_1 and ω_3, and the line tangent to circles ω_2 and ω_3 at point B is the radical axis of circles ω_2 and ω_3. Since the centers of ω_1, ω_2, and ω_3 are not colinear, their radical axes (one of which is MN) must coincide at the radical center K. Since changing the location of M on AB does not change the tangents at A and B, the point K does not move, hence all possible lines MN must pass through K. $\square$

Problem (2.28, JMO 2012/1)

Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Proof. Since $\angle BPS = \angle PRS$ by the Tangent Criterion, $\overline{AB}$ is tangent to (PRS). Likewise we have that $\overline{AC}$ is tangent to (QRS). Suppose (PRS) and (QRS) are not the same circle. Then since $AP = AQ$ are both tangents to their respective circles, A must lie on the radical axis $\overline{BC}$, but since ABC is a triangle, this is obviously impossible. Hence P, Q, R, S are concyclic. $\square$

Problem (2.29, IMO 2008/1)

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.

Proof. We will first show that B_1, B_2, C_1, C_2 are concyclic. Since $\Gamma_A, \Gamma_B, \Gamma_C$ all intersect at H, H is the radical center. We claim that $\overline{AH}$ is the radical axis of Γ_B and Γ_C. By similar triangles, $M_B M_C$ is parallel to BC, and since $\overline{AH} \perp BC$, $\overline{AH} \perp M_B M_C$. The centers of circles Γ_B and Γ_C are M_B and M_C, respectively, thus $\overline{AH}$ is the radical axis of circles Γ_B and Γ_C. Since $\overline{B_1B_2}$ and $\overline{C_1C_2}$ intersect at A, by Theorem 2.9 we have shown that B_1, B_2, C_1, C_2 are concyclic. Note that the circumcenter of $(B_1B_2C_1C_2)$ is the intersection of the perpendicular bisectors of B_1B_2 and C_1C_2, which is the orthocenter O of triangle ABC. Thus what we have proven is that $OB_1 = OB_2 = OC_1 = OC_2$. A similar argument can be persued for OA_1 and OA_2, hence we are done. □

Problem (2.30, USAMO 1997/2)

Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$ respectively are concurrent.

Proof. Consider the circles with centers D, E, F with chords BC, CA, AB, respectively. Note that the radical axes of these three circles are the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$, and since the centers of these three circles are not colinear, their radical axes must intersect at a point. □

(These centers can be colinear, but we won't talk about that)

Problem (2.31, IMO 1995/1)

Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Proof. Since P lies on the radical axis of these two circles, and $\overline{BN} \cap \overline{CM} = P$, $MNBC$ is cyclic by Theorem 2.9. (Reminder that the symbol $\measuredangle$ denotes the directed angle.) Note that

$$\measuredangle NMC = \measuredangle NBC = \measuredangle NBD = 90^\circ - \measuredangle BDN = 90^\circ - \measuredangle ADN,$$

so

$$\measuredangle NMA = \measuredangle NMC - 90^\circ = (90^\circ - \measuredangle ADN) - 90^\circ = -\measuredangle ADN = \measuredangle NDA,$$

therefore quadrilateral $DAMN$ is cyclic. The radical axes of the circles $(DAMN), (AMC)$, and (BND) are $\overline{AM}, \overline{DN}, \overline{XY}$, and since the centers of these circles are never colinear, they must intersect at the radical center. □

Problem (2.32, USAMO 1998/2)

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

Proof. **INCOMPLETE** Note that $CDEF$ is cyclic (need to prove). M is the center of circle $(CDEF)$. Thus $CM = DM$. □

§3 Lengths and Ratios

Theorem (3.2, Angle Bisector Theorem)
Let ABC be a triangle and D a point on $\overline{BC}$ so that $\overline{AD}$ is the internal angle bisector of $\angle BAC$. Show that

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

Proof. Let $\angle BAD = \alpha = \angle CAD$ and $\angle ADB = \beta$. Note that $\angle ADC = 180^\circ - \beta$. By Law of Sines, we have

$$\frac{DB}{\sin\alpha} = \frac{AB}{\sin\beta} \text{ and } \frac{DC}{\sin\alpha} = \frac{AC}{\sin(180^\circ - \beta)}.$$

Note that $\sin(180^\circ - \beta) = \sin\beta$. Rearranging terms, we have that

$$\frac{\sin\beta}{\sin\alpha} = \frac{AB}{BD} = \frac{AC}{CD}.$$

It follows that $\frac{AB}{AC} = \frac{DB}{DC}$. $\square$

Problem (3.5)
Show the trigonometric form of Ceva holds.

Proof. Recall that the trigonometric from of Ceva's Theorem is as follows: Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be cevians of a triangle ABC. They concur if and only if

$$\frac{\sin\angle BAX \sin\angle CBY \sin\angle ACZ}{\sin\angle XAC \sin\angle YBA \sin\angle ZCB} = 1.$$

By the Law of Sines, we have that

$$\frac{\sin\angle BAX}{BX} = \frac{\sin B}{AX}$$

and

$$\frac{\sin\angle XAC}{XC} = \frac{\sin C}{AX}.$$

Combining these two equations gives us

$$AX = \frac{BX \sin B}{\sin \angle BAX} = \frac{XC \sin C}{\sin \angle XAC} \Rightarrow \frac{\sin \angle BAX}{\sin \angle XAC} = \frac{BX}{XC} \cdot \frac{\sin C}{\sin B}.$$

Similarly, we have that

$$\frac{\sin \angle CBY}{\sin \angle YBA} = \frac{CY}{YA} \cdot \frac{\sin A}{\sin C}$$

and

$$\frac{\sin \angle ACZ}{\sin \angle ZCB} = \frac{AZ}{ZB} \cdot \frac{\sin B}{\sin A}.$$

Plugging these values into the original equation, we have that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,$$

and we know this is true from the original statement of Ceva's Theorem. □

Problem (3.6)

Let $\overline{AM}, \overline{BE}$, and $\overline{CF}$ be concurrent cevians of a triangle ABC. Show that $\overline{EF} \parallel \overline{BC}$ if and only if $BM = MC$.

Proof. Suppose $\overline{EF} \parallel \overline{BC}$. Call the point where $\overline{AM}$ intersects $\overline{EF}$ point Q. Notice that $\triangle BPM \sim \triangle EPQ$ and $\triangle CPM \sim \triangle FPQ$. Thus we have the following relationship:

$$\frac{BM}{EQ} = \frac{MP}{QP} = \frac{CM}{FQ}.$$

Now also notice that $\triangle BAM \sim \triangle FAQ$ and $\triangle CAM \sim \triangle EAQ$. Thus we have the following relationship:

$$\frac{BM}{FQ} = \frac{MA}{QA} = \frac{CM}{EQ}.$$

Putting these two relationships together, it follows that $BM = CM$.

We will now prove the other direction. Suppose $BM = MC$. Then by Ceva's Theorem we have that

$$\frac{CE}{AE} = \frac{BF}{AF}$$
$$\frac{CE}{BF} = \frac{AE}{AF} = \frac{CE+AE}{BF+AF} = \frac{AC}{AB}$$
$$\frac{AE}{AC} = \frac{AF}{AB}.$$

Since $\angle FAE = \angle BAC$, we have that $\triangle FAE \sim \triangle BAC$. Thus $\angle AEF = \angle ACB$, therefore $\overline{EF} \parallel \overline{BC}$. $\square$

Problem (3.12)

Give an alternative proof of Lemma 3.9 by taking a negative homothety.

Proof. Consider a homothety centered at G with $M = h(A), N = h(B), L = h(C)$. Note that $\triangle ACB \sim \triangle NCM$ by midpoints and that $\triangle ALG \sim \triangle Mh(L)G$ by homothety. Also notice that $h(L)$ is the midpoint of NM. Since $AB/NM = 2/1$,

$$\frac{AB}{NM} = \frac{AL}{Mh(L)} = \frac{AG}{MG} = \frac{2}{1}.$$

$\square$

Lemma (3.13, Euler Line)

In triangle ABC, prove that O, G, H (with their usual meanings) are collinear and that G divides $\overline{OH}$ in a $2:1$ ratio.

Proof. We will first show that O, G, H are collinear. Call the point where the perpendicular from O meets $\overline{BC}, \overline{CA}, \overline{AB}$ points A', B', C', respectively. Since $\overline{BC}, \overline{CA}, \overline{AB}$ are chords of the circle (ABC), points A', B', C' are in fact the midpoints of their respective line segments. Thus A' lies on $\overline{AG}$, B' lies on $\overline{BG}$, and C' lies on $\overline{CG}$. Now notice that $\overline{AH} \parallel \overline{OA'}, \overline{BH} \parallel \overline{OB'}, \overline{CH} \parallel \overline{OC'}$ since they are all perpendicular to some side of the triangle ABC. Thus, a homothety h centered at G exists such that $h(A) = A', h(B) = B', h(C) = C'$. Thus, $h(O) = H$, so O, G, H are collinear.

We will now show that G divides $\overline{OH}$ in a $2:1$ ratio. This is equivalent to showing that the homothety h must have a scale factor $k = -2$. From Lemma 3.9 (Centroid Division) we have that $AG/GA' = 2/1$. Since G lies in between A and A', we have that $k = -2$, as desired. (!!!) $\square$

Problem (3.16)

Let ABC be a triangle with contact triangle DEF. Prove that $\overline{AD}, \overline{BE}, \overline{CF}$ concur. The point of concurrency is the Gergonne point of triangle ABC.

Proof. Notice by Lemma 2.15 we have that

$$\begin{aligned} AE &= AF = s-a \\ BD &= BF = s-b \\ CD &= CE = s-c. \end{aligned}$$

Thus, by Ceva's Theorem, we have that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1.$$

$\square$

Lemma (3.17)

In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.

Proof. Reflect X and Y across $\overline{BC}$ and call these points X' and Y' respectively. Notice that X' and Y' lie on $(ABCD)$. Thus $ADX'Y'$ is a cyclic quadrilateral. Then we have that

$$\measuredangle AXY = \measuredangle X'XY = \measuredangle Y'X'X = \measuredangle Y'X'A = \measuredangle Y'DA = \measuredangle YDA.$$

Similarly, we have that $\measuredangle DAX = \measuredangle XYD$. Hence $AXYD$ is a parallelogram. $\square$

Problem (3.18)
Let $\overline{AD}, \overline{BE}, \overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

Proof. By Area Ratios, we can transform each term in our desired equation as follows:

$$\frac{PD}{AD} = \frac{[BPC]}{[BAC]},$$
$$\frac{PE}{BE} = \frac{[CPA]}{[CBA]},$$
$$\frac{PF}{CF} = \frac{[APB]}{[ACB]}.$$

Therefore our desired equation turns into

$$\frac{[BPC]}{[BAC]} + \frac{[CPA]}{[CBA]} + \frac{[APB]}{[ACB]} = 1.$$

Notice that $[BPC] + [CPA] + [APB] = [ABC]$. Hence we are done. $\square$

Problem (3.19, Shortlist 2006/G3)
Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \text{ and } \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$.

Proof. Let B' be intersection of diagonals AC and BD, and let E' be the intersection of diagonals AD and CE. Also let A' be the intersection of ray AP with CD. Notice that the given angle conditions imply that $\triangle ABC \sim \triangle ACD \sim \triangle ADE$. From this it follows that quadrilaterals $ABCD$ and $ACDE$ are similar. Since B' and E' are the intersections of the diagonals of their respective quadrilaterals, we have that $\frac{CB'}{B'A} = \frac{DE'}{E'A}$. By Ceva's on $\triangle ACD$, we have that

$$\frac{AE'}{E'D} \cdot \frac{DA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

Since $\frac{CB'}{B'A} \cdot \frac{AE'}{E'D} = 1$, we have that $DA' = A'C$. □

Problem (3.20, BAMO 2013/3)

Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC.

Proof. Let A', B', C' be the circumcenters of $(BCH), (CAH), (ABH)$, respectively. Note that H is the radical center of $(ABH), (BCH), (CAH)$. Thus $\overline{AH} \perp \overline{B'C'}$. Also notice by properties of circumcenters, A' is on the perpendicular bisector of $\overline{BC}$. Let O be where the perpendicular bisectors of $\triangle ABC$ intersect (namely, the circumcenter of $\triangle ABC$). Since $\overline{A'O} \parallel \overline{AH}$, $\overline{A'O} \perp \overline{B'C'}$. This follows similarly for B' and C', hence O is the orthocenter of $\triangle A'B'C'$. Also notice that, by construction, H is the circumcenter of $\triangle A'B'C'$. Therefore, a homothety of scale factor -1 exists that sends H to O, A to A', B to B', and C to C'. Hence, $\triangle ABC \cong \triangle A'B'C'$. □

Problem (3.21, USAMO 2003/4)

Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Proof. By assuming $MB \cdot MD = MC^2$, we have that $\frac{MB}{MC} = \frac{MC}{MD}$, and since $\angle BMC = \angle CMD$, this implies that $\triangle BMC \sim \triangle CMD$. Since $ABDE$ is a cyclic quadrilateral, $\angle DAE = \angle DBE$. Now we have that

$$\angle CAE = \angle DAE = \angle DBE = \angle MBC = \angle MCD = \angle FCA,$$

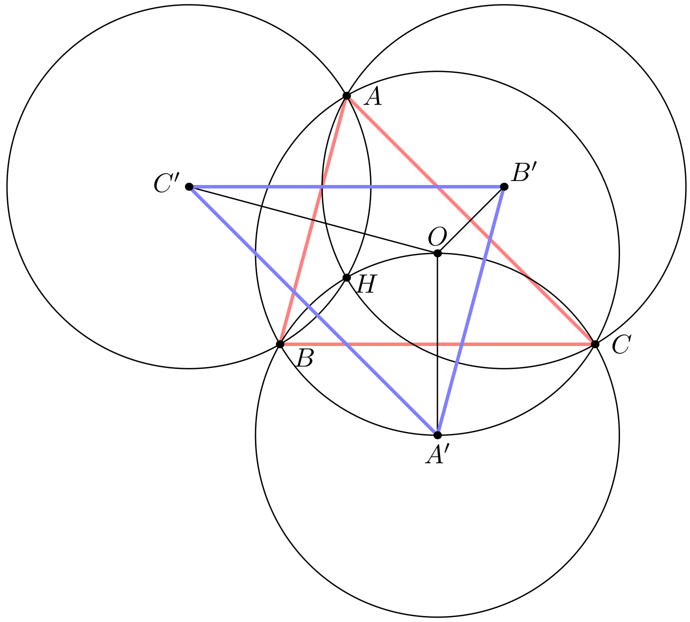

Figure 2: Problem 3.20

hence $\overline{AE} \parallel \overline{CF}$. Therefore $\triangle ABE \sim \triangle FBC$ and $\frac{FB}{AB} = \frac{CB}{EB}$. Then

$$\begin{aligned}
\frac{FB}{AB} &= \frac{CB}{EB} \\
\frac{FA+AB}{AB} &= \frac{CE+EB}{EB} \\
1+\frac{FA}{AB} &= 1+\frac{CE}{EB} \\
\frac{FA}{AB} &= \frac{CE}{EB}.
\end{aligned}$$

By Ceva's on $\triangle BCF$, we have that

$$\frac{FA}{AB} \cdot \frac{BE}{EC} \cdot \frac{CM}{MF} = 1.$$

Since $\frac{FA}{AB} = \frac{CE}{EB}$, we have that $MF = MC$.

We will now go in the reverse direction. We assume $MF = MC$. By Ceva's on $\triangle BCF$,

$$\frac{FA}{AB} \cdot \frac{BE}{EC} \cdot \frac{CM}{MF} = 1.$$

and since $MF = MC$, we have that $\frac{FA}{AB} \cdot \frac{BE}{EC} = 1$. It follows that

$$\begin{aligned} \frac{FA}{AB} &= \frac{CE}{EB} \\ 1 + \frac{FA}{AB} &= 1 + \frac{CE}{EB} \\ \frac{AB}{AB} + \frac{FA}{AB} &= \frac{EB}{EB} + \frac{CE}{EB} \\ \frac{FB}{AB} &= \frac{CB}{EB}. \end{aligned}$$

Thus $\triangle ABE \sim \triangle FBC$. This implies that $\overline{AE} \parallel \overline{CF}$. Since $ABDE$ is a cylic quadrilateral, we have that $\angle FCA = \angle DAE = \angle DBE$, and since $\angle BMC = \angle CMD$, we have that $\triangle BMC \sim \triangle CMD$ by $AA \sim$. Thus $\frac{MB}{MC} = \frac{MC}{MD} \to MB \cdot MD = MC^2$, as desired. $\square$

Theorem (3.22, Monge's Theorem)

Consider disjoint circles $\omega_1, \omega_2, \omega_3$ in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.

Proof. Let the points O_1, O_2, O_3, be the centers of $\omega_1, \omega_2, \omega_3$, respectively. Let the external tangents of ω_1 and ω_2 meet at X, and define Y and Z analogously. Note that X, Y, Z are each on an extension of a side of $\triangle O_1O_2O_3$. Let T_1 and T_2 be points of tangency of ω_1 and ω_2, respectively, where T_1 and T_2 are on the same side of line XO_1O_2. Note that it is impossible for X to be between O_1 and O_2, since X is the intersection of external tangents. Since tangents are always perpendicular to their circles, we have that $\triangle T_1O_1X \sim \triangle T_2O_2X$ by $AA \sim$, thus with directed lengths we have $\frac{O_1X}{XO_2} = -\frac{r_1}{r_2}$, where r_1 and r_2 are the radii of ω_1 and ω_2. Similar arguments can be applied to the other two pairs of circles to give $\frac{O_2Y}{YO_3} = -\frac{r_2}{r_3}$ and $\frac{O_3Z}{ZO_1} = -\frac{r_3}{r_1}$. Thus

$$\frac{O_1X}{XO_2} \cdot \frac{O_2Y}{YO_3} \cdot \frac{O_3Z}{ZO_1} = \left(-\frac{r_1}{r_2}\right)\left(-\frac{r_2}{r_3}\right)\left(-\frac{r_3}{r_1}\right) = -1.$$

By Menelaus's Theorem, this proves that X, Y, Z are collinear. $\square$

Theorem (3.23, Cevian Nest)

Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be concurrent cevians of ABC. Let $\overline{XD}, \overline{YE}, \overline{ZF}$ be concurrent cevians in triangle XYZ. Prove that rays AD, BE, CF concur.

Proof. By the Ratio Lemma on $\triangle ZAY$, we have that

$$\frac{\sin \angle BAD}{\sin \angle CAD} = \frac{\sin \angle ZAD}{\sin \angle YAD} = \frac{AY}{YC} \cdot \frac{ZD}{DY}.$$

Similarly, for $\triangle XBZ$ and $\triangle YCX$ we have that

$$\frac{\sin \angle CBE}{\sin \angle ABE} = \frac{BZ}{XB} \cdot \frac{XE}{EZ}$$

and

$$\frac{\sin \angle ACF}{\sin \angle BCF} = \frac{CX}{YC} \cdot \frac{YF}{FX}.$$

Multiplying these three equations together gives us

$$\begin{aligned}\frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} &= \left(\frac{AY}{YC} \cdot \frac{CX}{XB} \cdot \frac{BZ}{ZA}\right) \cdot \left(\frac{ZD}{DY} \cdot \frac{YF}{FX} \cdot \frac{XE}{EZ}\right) \\ &= 1 \cdot 1 \\ &= 1.\end{aligned}$$

Note that each of the factors in parentheses on the RHS of the first equation are equal to 1 by Ceva's on $\triangle ABC$ and $\triangle XYZ$, respectively. By the trigonometric form of Ceva's, this implies rays AD, BE, CF concur. □

Problem (3.24)

Let ABC be an acute triangle and suppose X is a point on (ABC) with $\overline{AX} \parallel \overline{BC}$ and $X \neq A$. Denote by G the centroid of triangle ABC, and by K the foot of the altitude from A to BC. Prove that K, G, X are collinear.

Proof. Denote by A', B', C' the midpoints of sides $\overline{BC}, \overline{CA}, \overline{AB}$. Note that A', B', C', K are on the nine-point circle of $\triangle ABC$. Also note that each side of $\triangle A'B'C'$ is parallel to a side of $\triangle ABC$. Therefore there exists a homothety h centered at G such that $h(A) = A', h(B) = B', h(C) = C'$. Since $\overline{AX} \parallel \overline{BC} \parallel \overline{A'K}$, h sends K to X. Therefore K, G, X are collinear. □

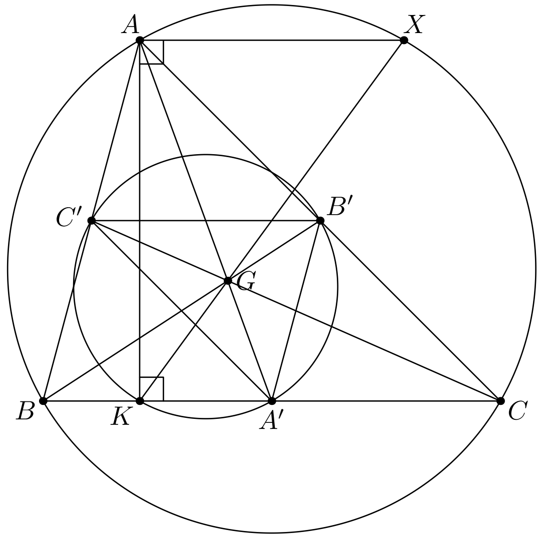

Figure 3: Problem 3.24

Problem (3.25, USAMO 1993/2)

Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are concyclic.

Proof. Denote by P, Q, R, S the projections of E onto $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$, respectively. Note that quadrilaterals $APES, BQEP, CREQ, DSER$ are cyclic. We will show $PQRS$ is cyclic by angle chase.

$$\begin{aligned}
\measuredangle SPQ &= \measuredangle SPE + \measuredangle EPQ \\
&= \measuredangle SAE + \measuredangle EBQ \\
&= 90^\circ - \measuredangle EDS + 90^\circ - \measuredangle QCE \\
&= \measuredangle SRE + \measuredangle ERQ \\
&= \measuredangle SRQ.
\end{aligned}$$

A homothety centered at E with a scale factor of 2 sends $PQRS$ to the desired quadrialteral. Hence we are done. $\square$

Problem (3.26, EGMO 2013/1)

The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that if $AD = BE$ then the triangle ABC is right-angled.

Proof. Let $BC = a, CA = b, AB = c, \angle BAC = \theta$. Thus $CD = a$ and $AE = 2b$. Let F be a point on $\overline{AB}$ such that $\overline{DF} \parallel \overline{AC}$. By similar triangles, $AF = c, DF = 2b$, and $\angle AFD = \theta$. We also have that $\angle EAC = 180^\circ - \theta$. Thus by the Law of Cosines,

$$BE^2 = (2b)^2 + c^2 - 2 \cdot 2b \cdot c \cdot \cos(180^\circ - \theta)$$

and

$$AD^2 = (2b)^2 + c^2 - 2 \cdot 2b \cdot c \cdot \cos(\theta).$$

Since $AD = BE$, we have that $\cos(180^\circ - \theta) = \cos(\theta)$. Therefore $\theta = 90^\circ$, as desired. $\square$

Problem (3.27, APMO 2004/2)

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.

Proof. (NOT ORIGINAL) WLOG let B and C be on the same side of line $\overline{OH}$. Let M be the midpoint of BC. Denote by A', B', C', M' the projections onto OH of A, B, C, M, respectively. Notice that a homothety centered at G with a ratio of -2 sends $\triangle MM'G$ to $\triangle AA'G$, so $AA' = 2MM' = BB' + CC'$, which implies the result. $\square$

Problem (3.28, Shortlist 2001/G1)

Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1 and C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

Proof. (NOT ENTIRELY ORIGINAL) Denote by $A_\square$ the square with center A_1. Denote by P, Q the vertices of $A_\square$ on sides AB and AC, respectively. Consider a homothety centered at A that sends P to B and Q to C. The center A_1', of the new square lies on AA_1. Similar arguments hold for homotheties centered at B and C. Thus we just need to show that AA_1', BB_1', CC_1' are concurrent. By the Law of Sines,

$$\frac{A_1'B}{\sin\angle A_1'AB} = \frac{AA_1'}{\sin(B+45^\circ)} \text{ and } \frac{A_1'C}{\sin\angle A_1'AC} = \frac{AA_1'}{\sin(C+45^\circ)}.$$

Since $A_1'B = A_1'C$, we have that $\dfrac{\sin(B+45^\circ)}{\sin(C+45^\circ)} = \dfrac{\sin\angle A_1'AB}{\sin\angle A_1'AC}$. Doing this for B_1' and C_1' then multiplying all three equations together proves the desired conclusion by the Trigonometric form of Ceva's Theorem. $\square$

§4 Assorted Configurations

Proposition (4.1)

Prove that the Simson line is parallel to $\overline{AK}$ in the notation of Figure 4.1A.

Proof. Notice that $BXPZ$ is cyclic since $\measuredangle BZP = \measuredangle BXP$. Then

$$\measuredangle YXP = \measuredangle ZXP = \measuredangle ZBP = \measuredangle ABP = \measuredangle AKP.$$

$\square$

Problem (4.2)

Let K' be the reflection of K across $\overline{BC}$. Show that K' is the orthocenter of $\triangle PBC$.

Proof. Note that K' is already on the altitude PX. Also note that K is on the circumcircle of $\triangle PBC$. Since K and K' are reflections of each other over BC, by Lemma 1.17 we have that K' is the orthocenter of $\triangle PBC$. $\square$

Problem (4.3)

Show that $LHXP$ is a parallelogram.

Proof. Note that $\overline{LH} \parallel \overline{XP}$ by construction. Thus it suffices to show that $LH = XP$. By Proposition 4.1 we have that $LA = XK$. Also, by the conclusion made after Problem 4.2 we have that $AH = PK'$. Thus

$$LH = LA + AH = XK + PK' = XK' + PK' = XP.$$

$\square$

Problem (4.5)

Check $\angle IAI_B = 90^\circ$ and $\angle IAI_C = 90^\circ$.

Proof. By the Incenter-Excenter Lemma we know that II_B is the diameter of circle $(AICI_B)$. Therefore $\angle IAI_B = 90^\circ$. A similar argument holds to show $\angle IAI_C = 90^\circ$. $\square$

Problem (4.7)

How are Lemma 1.18, Lemma 3.11, and Lemma 4.6 related?

Let L be the midpoint of II_A. Then in Figure 4.2A, $(ABCL)$ is in fact the nine-point circle of $\triangle I_AI_BI_C$ (Lemma 3.11). Moreover, by Lemma 1.18, I, B, C, I_A all lie on a circle centered at L. Finally, by Lemma 4.6, we know I is the orthocenter of $\triangle I_AI_BI_C$, but we also can derive this from Lemma 3.11. This means that any two lemmas can prove the third.

Problem (4.8)

Prove that A, E, and X are collinear and that $\overline{DE}$ is a diameter of the incircle.

Proof. Consider the homothety bringing $\triangle AB'C'$ to $\triangle ABC$. This homothety brings the circle with center I to the circle with center I_A, so it must bring point E to point X. Hence A, E, X are collinear. Now notice that $\overline{BC} \perp \overline{ID}$ and $\overline{B'C'} \perp \overline{IE}$ by tangency and $\overline{B'C'} \parallel \overline{BC}$. This is enough to show I, D, E are collinear, therefore $\overline{DE}$ is a diameter of the incircle. □

Lemma (4.10, Diameter of the Excircle)

In the notation of Lemma 4.9, suppose $\overline{XY}$ is a diameter of the A-excircle. Show that D lies on $\overline{AY}$.

Proof. Consider again the homothety bringing $\triangle AB'C'$ to $\triangle ABC$. Since DE and YX are diameters of the incircle and A-excircle, respectively, the homothety brings D to Y. Hence D lies on $\overline{AY}$. □

Problem (4.11)

If M is the midpoint of $\overline{BC}$, prove that $\overline{AE} \parallel \overline{IM}$.

Proof. Since M is the midpoint of BC, we know that M is also the midpoint of DX since $BD = XC$. We also know that I is the midpoint of DE. Thus we find that $\triangle DEX \sim \triangle DIM$, implying the result. □

Problem (4.12)

Prove that points X, I, M are collinear.

Proof. Note that $AK \parallel ED$, so the homothety centered at X that sends $\triangle AXK$ to $\triangle EXD$ sends M to I because they are midpoints of AK and ED, respectively. This implies that X, I, M are collinear. $\square$

Problem (4.13)

Show that D, I_A, M are collinear.

Proof. Note that $AK \parallel XY$, so the homothety centered at D that sends $\triangle XDY$ to $\triangle KDA$ sends I_A to M since they are midpoints of XY and AK, respectively. This implies that D, I_A, M are collinear. $\square$

Problem (4.15)

Show that I must lie on $(AB'C')$.

Proof. Since $B'C' \parallel BC, \angle IXB' = \angle IXC' = 90°$. Also note that $\angle BFI = 90° = \angle IEC$ by tangency. Thus $\overline{FXE}$ is the Simson line of I with respect to $\triangle AB'C'$. Thus by Lemma 1.48 we are done. $\square$

Problem (4.16)

Prove that $XB' = XC'$.

Proof. Note that since AI bisects $\angle B'AC'$, and by Problem 4.15 I lies on $(AB'C')$, then I must be the midpoint of arc $B'IC'$ by the Incenter-Excenter Lemma. Since X is the foot of the altitude from I to $B'C'$, and $IB' = IC'$, it follows that $XB' = XC'$. $\square$

Problem (4.19)

Show that if two of the angle relations in Lemma 4.18 hold, then so does the third.

Proof. Note by the trigonometric form of Ceva's Theorem we have that

$$\frac{\sin\measuredangle BAP\cdot\sin\measuredangle CBP\cdot\sin\measuredangle ACP}{\sin\measuredangle PAC\cdot\sin\measuredangle PBA\cdot\sin\measuredangle PCB}=1$$

and

$$\frac{\sin\measuredangle P^*AC\cdot\sin\measuredangle P^*BA\cdot\sin\measuredangle P^*CB}{\sin\measuredangle BAP^*\cdot\sin\measuredangle CBP^*\cdot\sin\measuredangle ACP^*}=1.$$

Now WLOG assume $\measuredangle BAP=\measuredangle P^*AC$ and $\measuredangle CBP=\measuredangle P^*BA$. This implies that both $\measuredangle PAC=\measuredangle BAP^*$ and $\measuredangle PBA=\measuredangle CBP^*$. Thus equating the two above equations and simplifying gives us

$$\frac{\sin\measuredangle ACP}{\sin\measuredangle PCB}=\frac{\sin\measuredangle P^*CB}{\sin\measuredangle ACP^*}.$$

Noticing that $\measuredangle ACP+\measuredangle PCB=\measuredangle ACB=\measuredangle ACP^*+\measuredangle P^*CB$, we can easily see that this implies $\measuredangle ACP=\measuredangle P^*CB$. □

Problem (4.20)

Prove that the cevians AX', BY', and CZ' concur as described above.

Proof. By Ceva's we know that

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1.$$

By relfection about the midpoint we know that $BX=X'C$ and $BX'=XC$. Analgous arguments can be applied to the other sides of the triangle. Plugging these into our first equation gives us

$$\frac{X'C}{BX'}\cdot\frac{Y'A}{CY'}\cdot\frac{Z'B}{AZ'}=1,$$

hence we are done. □

Problem (4.21)

Check that if Q is the isogonal conjugate of P, then P is the isogonal conjugate of Q.

Proof. This is trivial by the reflexive property of equality. For instance, if $\measuredangle BAP=\measuredangle P^*AC$, then $\measuredangle P^*AC=\measuredangle BAP$. □

Theorem (4.22, Isogonal Ratios)

Let D and E be points on $\overline{BC}$ so that $\overline{AD}$ and $\overline{AE}$ are isogonal. Then

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2.$$

Proof. By the Ratio Lemma, we have that

$$\frac{BD}{DC} = \frac{AB}{AC} \cdot \frac{\sin \angle BAD}{\sin \angle DAC}$$

and

$$\frac{BE}{EC} = \frac{AB}{AC} \cdot \frac{\sin \angle BAE}{\sin \angle EAC}.$$

Multiplying the two equations together gives us

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2,$$

as desired. (Note that $\angle BAD = \angle EAC$ and $\angle BAE = \angle DAC$ by the definition of being isogonal, so the sin simplify to 1.) $\square$

Problem (4.23)

What is the isogonal conjugate of a triangle's circumcenter?

Proof. We claim this point is the orthocenter, H. Note that

$$\angle BAH = 90^\circ - \angle CBA = 90^\circ - \frac{1}{2}\angle COA = \angle OAC.$$

Analagous arguments can be used for the other vertices of $\triangle ABC$, hence we are done. $\square$

Problem (4.25)

Show that

$$\frac{CM}{MB} = \frac{\sin \angle B \sin \angle BAX}{\sin \angle C \sin \angle CAX} = 1.$$

Proof. By Law of Sines we have

$$\frac{MB}{\sin\angle MAB} = \frac{MB}{\sin\angle CAX} = \frac{AM}{\sin\angle B}$$

and

$$\frac{CM}{\sin\angle CAM} = \frac{CM}{\sin\angle BAX} = \frac{AM}{\sin\angle C}.$$

Solving for AM and combining the two equations gives us

$$\frac{CM}{MB} = \frac{\sin\angle B \sin\angle BAX}{\sin\angle C \sin\angle CAX}.$$

Now note that $\angle ABX = 180° - \angle C$ and $\angle ACX = 180° - \angle B$ by the Tangent Criterion. Also note that $\sin\theta = \sin(180° - \theta)$. Thus by the Law of Sines we have

$$\frac{AX}{\sin\angle ABX} = \frac{AX}{\sin\angle C} = \frac{BX}{\sin\angle BAX}$$

and

$$\frac{AX}{\sin\angle ACX} = \frac{AX}{\sin\angle B} = \frac{CX}{\sin\angle CAX}.$$

Now noting that $BX = CX$ and combining the equations we get

$$\frac{\sin\angle B \sin\angle BAX}{\sin\angle C \sin\angle CAX} = 1,$$

as desired. □

Problem (4.28)

Verify (d) of Lemma 4.26.

Proof. Recall that what we want to show is

$$\frac{AB}{BK} = \frac{AC}{CK}.$$

Note that $MC = MB$ since M is the midpoint of BC. By using property (b) of Lemma 4.26 twice we have

$$\frac{AB}{BK} = \frac{AM}{MC} = \frac{AM}{MB} = \frac{AC}{CK}.$$

Hence, we are done. □

Problem (4.29)

Show that (f) of Lemma 4.26 follows (with some effort) from (d).

Proof. (NOT ENTIRELY ORIGINAL) We will first show that $\overline{BC}$ is the B-symmedian of $\triangle BAK$, and the proof regarding the C-symmedian will follow analagously. Let N be the point that bisects AK. It suffices to show that $\angle KBN = \angle B$, since this would satisfy our isogonality requirement for the symmedian. Note that $\angle NKB = \angle AKB = \angle ACB = \angle C$. Thus, it now suffices to show that $\angle BNK = \angle A$. Note that $\angle BNK = \angle BNX = \angle BCX = \angle A$, where the second equality holds by (e) making B, C, N, X concylic and the third equality holds by $\overline{CX}$ being tangent to (ABC). Hence, we're done. □

Problem (4.31)

Show that this homothety takes K to M, and in particular that T, K, and M are collinear.

Proof. Note that by tangency, $PK \perp AB$. By midpoints, $OM \perp AB$. Thus $PK \parallel OM$. Also note that $TP = PK$ and $TO = OM$, thus $\triangle TPK \sim \triangle TOM$. Thus the homothety centered at T sends P to O and K to M. This also proves that T, K, M are collinear. □

Problem (4.32)

Show that $\triangle TMB \sim \triangle BMK$.

Proof. By angle chasing we have

$$\angle MTB = \angle MAB = \angle MBA = \angle MBK.$$

We also have $\angle TMB = \angle BMK$, thus $\triangle TMB \sim \triangle BMK$ by $AA \sim$. □

Problem (4.34, Curvilinear Incircles)

Prove that the points C, L, I, T are concyclic.

Proof. We want to show that $\measuredangle TCI = \measuredangle TCM = \measuredangle TLK$. This is equivalent to showing that the arc measures of $\widehat{TK}$ and $\widehat{TM}$ are equal. Since there is a homothety centered at T that sends K to M, the circle ω will be sent to the circle Ω, so the two arc measures are equal. Hence, we are done. □

Problem (4.35, Curvilinear Incircles)
Show that $\triangle MKI \sim \triangle MIT$, and that the triangles are oppositely oriented.

Proof. Trivially $\measuredangle KMI = \measuredangle TMI$. We would like to show that $\measuredangle IKM = \measuredangle MIT$ to finish off with $AA \sim$. Note that

$$\begin{aligned}\measuredangle IKM &= \measuredangle IKT \\ &= \measuredangle LKT \\ &= \measuredangle CLT \quad \text{(Tangency Criterion)} \\ &= \measuredangle CIT \quad (C, L, I, T \text{ are concyclic}) \\ &= \measuredangle MIT.\end{aligned}$$

Note that $\measuredangle MIT = -\measuredangle MKI$, so the two triangles are oppositely oriented. □

Problem (4.37, Mixtilinear Incircles)
Using the fact that I lies on $\overline{KL}$, check that I is in fact the midpoint of $\overline{KL}$.

Proof. Note that $\angle KAI = \angle LAI$ because I is the incenter, $\angle AKI = \angle ALI$ by tangency, and $AK = AL$ again by tangency, so $\triangle KAI \cong \triangle LAI$ by $ASA \cong$. This implies that $KI = LI$, and we are done. □

Problem (4.38, Mixtilinear Incircles)
Prove that $\angle ATK = \angle LTI$.

Proof. By angle chase we have

$$\angle LTI = \angle LCI = \angle ACM_C = \angle ATM_C = \angle ATK.$$

□

Problem (4.39, Mixtilinear Incircles)
Prove that S is the midpoint of the arc $\widehat{BC}$ containing A.

Proof. By angle chase we have

$$\begin{aligned} \measuredangle SBC &= \measuredangle STC = \measuredangle ITC = \measuredangle ILC = \measuredangle ILA \\ &= \measuredangle AKI = \measuredangle BKI = \measuredangle BTI = \measuredangle BTS = \measuredangle BCS. \end{aligned}$$

Since we have that $\angle SBC = \angle SCB$, this implies the result. □

Problem (4.41, Hong Kong 1998)
Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90^\circ$ and let H and K be the feet of the altitudes from Q to lines PR and PS. Prove that HK bisects QS.

Proof. Note that $\overline{HK}$ is the Simson Line of Q with respect to $\triangle SPR$. Denote by L the foot of the altitude from Q to $\overline{RS}$. Since $\angle KSL = \angle PSR = 90^\circ$, $\angle QKS = 90^\circ$, and $\angle SLQ = 90^\circ$, quadrilateral $KSLQ$ is a rectangle, so its diagonals bisect each other. Thus, $\overline{KL} = \overline{HK}$ (our Simson Line!) bisects SQ so we are done. □

Problem (4.42, USAMO 1988/4)
Let ABC be a triangle with incenter I and circumcenter O. Show that the circumcircles of $\triangle IBC$, $\triangle ICA$ and $\triangle IAB$ lie on a circle with center O.

Proof. Call the circumcenter of (BIC) point O_A and define O_B and O_C similarly. By the Incenter-Excenter Lemma, O_A is the midpoint of arc $\widehat{BC}$, O_B is the midpoint of arc $\widehat{AC}$, and O_C is the midpoint of arc $\widehat{AB}$. This implies that O_A, O_B, O_C are all on the circumcirle of $\triangle ABC$ so we are done. (trivial) □

Problem (4.43, USAMO 1995/3)

Given a non-isosceles, non-right triangle ABC, let O denote its circumcenter, and let A_1, B_1 and C_1 be the midpoints of its sides. Point A_2 is on ray OA_1 so that $\triangle OAA_1$ is similar to $\triangle OA_2A$. Points B_2 and C_2 are defined similarly. Prove that AA_2, BB_2 and CC_2 are concurrent.

Proof. We begin with the following claim that seems plausible upon inspection.

Claim — $\overline{A_2B}$ and $\overline{A_2C}$ are tangent to (ABC). Equivalently, $\overline{AA_2}$ is the A-symmedian of $\triangle ABC$.

Proof. We leverage the fact that $\triangle OAA_1 \sim \triangle OA_2A$. Note that

$$\frac{OA_1}{OA} = \frac{OA}{OA_2} \Rightarrow OA_1 \cdot OA_2 = OA^2 = R^2 = OC^2$$

where R is the circumradius of $\triangle ABC$. Now consider the power of O with respect to (A_1A_2C). Our equation above clearly implies that OC is tangent to (A_1A_2C), and since $\angle A_2A_1C = 90°$, $\overline{A_2C}$ is a diameter of (A_1A_2C). Thus $\overline{OC} \perp \overline{A_2C}$, and since OC is itself a radius of (ABC), we have shown that $\overline{A_2C}$ is tangent to (ABC). A very similar line of reasoning holds for $\overline{A_2B}$. ■

We can follow two other very similar arguments to prove that $\overline{BB_2}$ and $\overline{CC_2}$ are the B- and C-symmedians of $\triangle ABC$, respectively. Since the three medians coincide at a single point (namely, the centroid), the symmedians (which are the isogonal conjugates of the medians) must also coincide at a single point, namely, the symmedian point. Hence, we are done. □

Problem (4.44, USA TST 2014)

Let ABC be an acute triangle and let X be a variable point on the minor arc BC. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to $\overline{AC}$. Let ℓ be the line through P parallel to $\overline{XR}$. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point.

Proof. We claim this fixed point is the orthocenter H of $\triangle ABC$. Note that $\overline{PQ}$ is the Simson Line of X with respect to $\triangle ABC$. By the Simson Line Bisection Lemma (4.4) we know that $PXRH$ is a parallelogram, so we have that H is always on ℓ. Note that this argument holds for any placement of X along minor arc $\widehat{BC}$, so we are done. $\square$

Problem (4.45, USA TST 2011/1)

In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{CA}$, $\overline{CF} \perp \overline{AB}$. Altitudes $\overline{AB}$, $\overline{BE}$, $\overline{CF}$ meet at the orthocenter H. Points P and Q lie on the segment $\overline{EF}$ such that $\overline{AP} \perp \overline{EF}$ and $\overline{HQ} \perp \overline{EF}$. Lines DP and QH intersect at point R. Compute HQ/HR.

Solution. We claim $HQ/HR = 1$. First, by the Duality of Orthocenters and Excenters (Lemma 4.6) we have that H is the incenter and A is the D-excenter of orthic triangle DEF. Since Q is the foot of the altitude from H to $\overline{EF}$, HQ is a radius of the incircle of $\triangle DEF$.
Now note that since D, R, P are collinear and $\overline{AP} \perp \overline{EF} \perp \overline{QR} \Longrightarrow \overline{AP} \parallel \overline{QR}$, a homothety centered at D that sends A to H will send P to R. Thus, HR is a radius of the incircle of $\triangle DEF$. Hence the ratio of HQ and HR is equal to 1 and we are done. $\square$

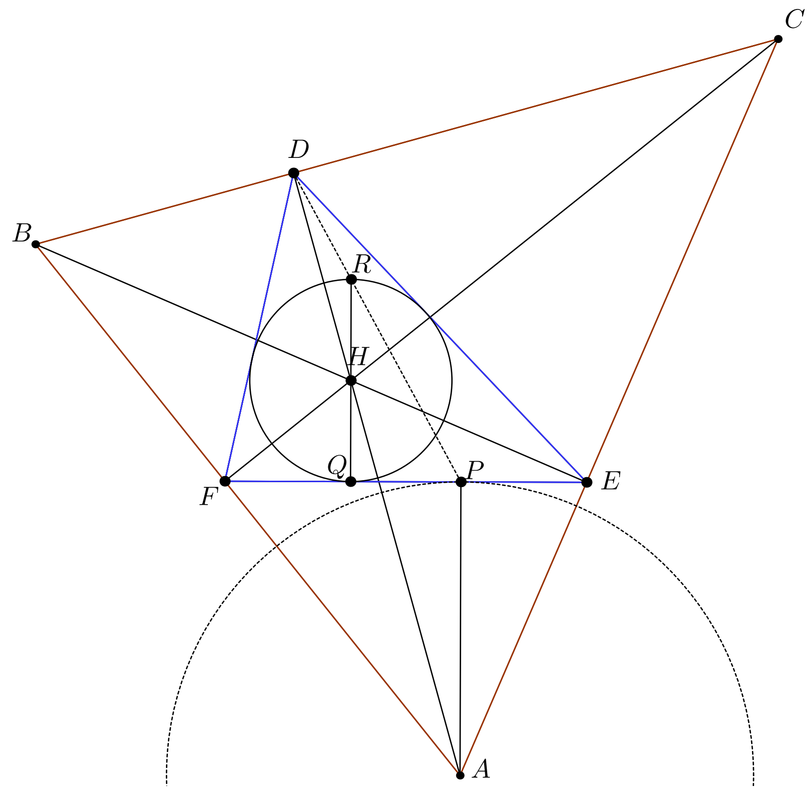

Figure 4: Problem 4.45

Problem (4.46, ELMO Shortlist 2012)

Circles Ω and ω are internally tangent at point C. Chord AB of Ω is tangent to ω at E, where E is the midpoint of $\overline{AB}$. Another circle, ω_1, is tangent to Ω, ω, and $\overline{AB}$ at D, Z, and F respectively. Rays CD and AB meet at P. If $M \neq C$ is the midpoint of major arc AB, show that

$$\tan \angle ZEP = \frac{PE}{CM}.$$

Proof. For ease of notation, denote by O_1, O_2, O_3 the centers of circles Ω, ω, ω_1, respectively. We know that $\angle ZEP = \angle ECZ$ by our tangency condition. Thus it suffices to show that $\frac{EZ}{CZ} = \frac{EP}{CM}$. However, we must take care of a few things before we tackle this claim.

Claim — Point M lies on the radical axis of ω and ω_1.

Proof. It is well known (also by Lemma 4.33) that $\text{Pow}_\omega(M) = \text{Pow}_{\omega_1}(M)$, so we are done with this claim. ■

Claim — Points Z, P lie on $\overline{O_2O_3}$

Proof. Call the second intersection of CD with ω_1 point X. Since $\angle XDF = \angle CDM = 90°$, we have that XF is a diameter of ω_1. So, considering the the homothety centered at P that sends ω to ω_1, we have that O_2, O_3, P are all collinear. Finally note that Z obviously lies on $\overline{O_2O_3}$ since it is the tangency point of the two circles, so we are done with this claim. ■

The equation $\frac{EZ}{CZ} = \frac{EP}{CM}$ is equivalent to the following.

Claim — $\triangle CZM \sim \triangle EZP$

Proof. We already have that $\angle MCZ = \angle ECZ = \angle PEZ$. Note that $\angle CZM = \angle CZE + \angle EZM$ and $\angle EZP = \angle EZM + \angle MZP$. Hence it suffices to show that $\angle CZE = \angle MZP$. We already know that $\angle CZE = 90°$ since CE is a diameter of ω. We also know that the radical axis of two circles is perpendicular to the line connecting their centers, so by our previous claims we have that $\angle MZP = 90°$ and we have proven our claim by $AA \sim$. ■

With this final claim proven, we are done. □

Proof. **INVALID** Denote by P' the foot of the altitude from P to the line passing through M tangent to Ω. Note that because $\angle MEP, \angle EMP'$, and $\angle MP'P$ are all right angles, quadrilateral $MP'PE$ is a rectangle, which implies that $PE = P'M$. Thus it suffices to show that $\angle P'CM = \angle ZEP$ since $\triangle P'CM$ is a right triangle with a right angle at M, implying that $\frac{PE}{CM} = \frac{P'M}{CM} = \tan \angle P'CM$.

Claim — The points C, Z, P' are collinear.

Proof. The homothety centered at Z that sends ω to ω_1 will send C to F (if we say that C is at the "North Pole" of ω, then F is at the "South Pole" of ω_1), so C, Z, F are collinear. A second homothety centered at C **[how do we know this exists?]** that sends ω to Ω will send F to P', so C, F, P' are collinear as well. Hence we have proven our claim. ■

Since $\angle ZEP = \angle ZCE = \angle P'CM$, where the second equality holds by our claim, we are done. □

Problem (4.47, USAMO 2011/5)
Let P be a point inside convex quadrilateral $ABCD$. Points Q_1 and Q_2 are located within $ABCD$ such that

$$\angle Q_1BC = \angle ABP, \quad \angle Q_1CB = \angle DCP,$$
$$\angle Q_2AD = \angle BAP, \quad \angle Q_2DA = \angle CDP.$$

Prove that $\overline{Q_1Q_2} \parallel \overline{AB}$ if and only if $\overline{Q_1Q_2} \parallel \overline{CD}$.

Proof. We begin with the following claim.

Claim — When $\overline{AB}, \overline{CD}$ are not parallel, $\overline{AB}, \overline{CD}, \overline{Q_1Q_2}$ are concurrent at a single point.

Proof. Call the intersection of $\overline{AB}$ and $\overline{CD}$ point X. First consider the triangle $\triangle ADX$. Since $\measuredangle XAP = \measuredangle BAP = \measuredangle Q_2AD$ and $\measuredangle XDP = \measuredangle CDP = \measuredangle Q_2DA$, we have that P and Q_2 are isotomic conjugates with respect to $\triangle ADX$. This implies that

$$\measuredangle AXP = \measuredangle Q_2XD.$$

Similarly, considering triangle $\triangle BCX$, we have that $\measuredangle XBP = \measuredangle ABP = \measuredangle Q_1BC$ and $\measuredangle XCP = \measuredangle DCP = \measuredangle Q_1CB$, so P and Q_1 are isotomic conjugates with respect to $\triangle BCX$. This implies that

$$\measuredangle BXP = \measuredangle Q_1XC.$$

Putting this all together, we have that

$$\measuredangle Q_2XD = \measuredangle AXP = \measuredangle BXP = \measuredangle Q_1XC = \measuredangle Q_1XD,$$

so Q_1, Q_2, X are collinear as desired. ■

Note that the intersection point X is at the point at infinity if $\overline{AB} \parallel \overline{CD}$, so we have that $\overline{Q_1Q_2}$ must meet $\overline{AB}, \overline{CD}$ at the point at infinity, implying that all 3 lines must be parallel. □

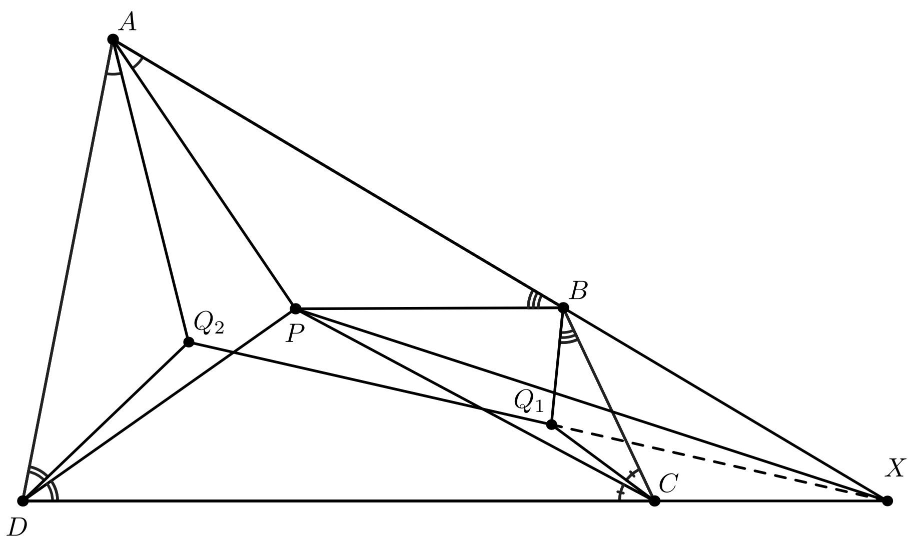

Figure 5: Problem 4.47

Problem (4.48, Japanese Olympiad 2009)

Triangle ABC is inscribed in circle Γ. A circle with center O is drawn, tangent to side BC at a point P, and internally tangent to the arc BC of Γ not containing A at a point Q. Show that if $\angle BAO = \angle CAO$ then $\angle PAO = \angle QAO$.

Proof. Note that $\angle BAO = \angle CAO$ is equivalent to point I, the incenter of $\triangle ABC$, lying on $\overline{AO}$. Denote by M the midpoint of the arc $\widehat{BC}$ on the side containing point A, and denote by N its antipodal point on Γ. It is well known that M lies on $\overline{PQ}$, and we note by Fact 5 that N lies on $\overline{AI}$.

Claim — Points A, P, O, Q are concyclic.

Proof. We first note that $\overline{OP} \parallel \overline{MN}$ since both are perpendicular to $\overline{BC}$. It follows that

$$\angle QPO = \angle QMN = \angle QAN = \angle QAO,$$

hence our claim is proven. ■

Since $\triangle POQ$ is an isosceles triangle, our claim implies that

$$\angle PAO = \angle PQO = \angle QPO = \angle QAO,$$

so we are done. □