Random Problems

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Note: The symbol $\measuredangle$ refers to the directed angle in this text.

Problem (OTIS Excerpts #7)
Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

Proof. We claim that $c = 10$. We know this value works because for $n \geq 1, c^n \in \{10, 100, 1000, \ldots\}$, and since all digits from the 10s and to the left are less than 4, adding 1 to them will not violate our digit condition. We will now check that this is the smallest possible value for c.

- $c = 1$ fails at $n = 1$ since $1 + 4 = 5$.
- $c = 2$ fails at $n = 1$ since $2 + 4 = 6$.
- $c = 3$ fails at $n = 1$ since $3 + 4 = 7$.
- $c = 4$ fails at $n = 1$ since $4 + 4 = 8$.
- $c = 5$ fails at $n = 1$ since $5 + 4 = 9$.
- $c = 6$ fails at $n = 2$ since $36 + 2014 = 2050$.
- $c = 7$ fails at $n = 2$ since $49 + 2014 = 2063$.
- $c = 8$ fails at $n = 2$ since $64 + 2014 = 2078$.
- $c = 9$ fails at $n = 2$ since $81 + 2014 = 2095$.

Since every value of c less than 10 fails, we are done. $\square$

Problem (OTIS Excerpts #77, HMMT February 2013)

Values $a_1, \dots, a_{2013}$ are chosen independently and at random from the set $\{1, \dots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \dots, a_{2013}\}$?

Solution. Let P be the number of distinct values in $a_1, \dots, a_{2013}$, and for each $i = 1, 2, \dots, 2013$ let

$$P_i := \begin{cases} 1 & \text{if } a_i \neq a_j \text{ for all } j < i \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $P = P_1 + \cdots + P_{2013}$. Thus it follows that

$$\begin{aligned} E[P] &= E[P_1] + E[P_2] + \cdots + E[P_{2013}] \\ &= 1 + (1 - 1/2013) + \cdots + (1 - 1/2013)^{2012} \\ &= \frac{1 - (2012/2013)^{2013}}{1 - 2012/2013} \\ &= 2013 \left(1 - \left(\frac{2012}{2013}\right)^{2013}\right). \end{aligned}$$

□

Problem (100 Geometry Problems #8)

Let ABC be a triangle with $\angle CAB$ a right angle. The point L lies on the side BC between B and C. The circle BAL meets the line AC again at M and the circle CAL meets the line AB again at N. Prove that L, M, and N lie on a straight line.

Proof. Since $ANLC$ and $ALBM$ are cyclic quadrilaterals, $\angle CAN = \angle CLN = \angle 90^\circ = \angle BAM = \angle BLM$. Since $\angle CLN + \angle BLN = 180^\circ$, we have $\angle BLN = \angle BLM = 90^\circ$, as desired. □

Problem (100 Geometry Problems #11)

A closed planar shape is said to be equiable if the numerical values of its perimeter and area are the same. For example, a square with side length 4 is equiable since its perimeter and area are both 16. Show that any closed shape in the plane can be dilated to become equiable. (A dilation is an affine transformation in which a shape is stretched or shrunk. In other words, if $\mathcal{A}$ is a dilated version of $\mathcal{B}$ then $\mathcal{A}$ is similar to $\mathcal{B}$.)

Proof. Note that for any scaling of the perimeter by a factor of k, the area increases by a factor of k^2. It is not hard to see that by making the perimeter arbitrarily large, at some point the area must be larger than the perimeter, and by making the perimeter arbitrarily small, the area must be smaller than the perimeter. Thus by the Intermediate Value Theorem there must be a scale factor k such that the perimeter equals the area. □

Problem (100 Geometry Problems #13)

Points A and B are located on circle Γ, and point C is an arbitrary point in the interior of Γ. Extend AC and BC past C so that they hit Γ at M and N respectively. Let X denote the foot of the perpendicular from M to BN, and let Y denote the foot of the perpendicular from N to AM. Prove that $AB \parallel XY$.

Proof. It suffices to show that $\triangle ABC \sim \triangle YXC$, as this would prove that $AB \parallel XY$. Note that $NXYM$ is a cyclic quadrilateral because $\measuredangle NXM = 90^\circ = \measuredangle NYM$. By angle chasing we get

$$\measuredangle ABC = \measuredangle ABN = \measuredangle AMN = \measuredangle YMN = \measuredangle YXN = \measuredangle YXC.$$

We know $\measuredangle ACB = \measuredangle YCX$ by vertical angles, hence $\triangle ABC \sim \triangle YXC$ by $AA \sim$. This completes the proof. □

Problem (100 Geometry Problem #14, AIME 2007)

Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.

Solution. Extend BE and CF to meet at G and extend AE and DF to meet at H. Note by symmetry, $FGHE$ is a square of side length $12+5 = 17$. Thus $EF^2 = 17^2 + 17^2 = \boxed{578}$. □

Problem (100 Geometry Problems #15)

Let Γ be the circumcircle of $\triangle ABC$, and let D, E, F be the midpoints of arcs AB, BC, CA, respectively. Prove that $DF \perp AE$.

Proof. Denote by I the incenter of $\triangle ABC$. By the Incenter-Excenter Lemma, I lies on AE. Also by the Lemma, D and F are the circumcenters of (AIB) and (AIC), respectively. The radical axis of these two circles is AI, thus $AI \perp DF \implies AE \perp DF$. □

Problem (Andrews, NT, Problem 1-1.7)

Denote by F_n the n-th Fibonacci Number. Prove that

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

Proof. We will use induction.

Base Case: $n = 1$. Notice that $1 = F_1 = F_3 - 1 = 2 - 1$.

Induction Hypothesis: Assume our desired equation is true for all $n \leq k$. We will now show that our desired equation holds for $n = k + 1$. Note that

$$(F_1 + \cdots + F_k) + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1,$$

where the first equality holds by our Induction Hypothesis and the second holds by the definition of the Fibonacci Sequence. This concludes the proof. □

Problem (Andrews, NT, Problem 1-1.8)

Prove that
$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

Proof. We will use induction.

Base Case: $n = 1$. $F_1 = F_2 = 1$.

Induction Hypothesis: Assume our desired equation is true for all $n \leq k$. We will now show that it also holds for $n = k + 1$. Note that

$$(F_1 + F_3 + \cdots + F_{2k-1}) + F_{2(k+1)-1} = F_{2k} + F_{2k+1} = F_{2k+2} = F_{2(k+1)}.$$

Hence we are done. □

Problem (Andrews, NT, Problem 1-1.17)

Prove that $n(n^2 - 1)(3n + 2)$ is divisible by 24 for each positive integer n.

Proof. We will use induction.

Base Case: $n = 1$. Note that $(1)((1)^2 - 1)(3(1) + 2) = 0$, which is divisible by 24.

Induction Hypothesis: Assume the problem statement is true for all $n \leq k$. We will now show it is true for $n = k + 1$. Note that

$$\begin{aligned}
&(k+1)((k+1)^2 - 1)(3(k+1) + 2) \\
&= (k+1)(k^2 + 2k)(3k + 5) \\
&= k(k+1)(k+2)(3k+5) \\
&= k(k+1)((k-1) + 3)((3k+2) + 3) \\
&= k(k+1)(k-1)(3k+2) + k(k+1) \cdot [3(3k+2) + 3(k-1) + 3(3)].
\end{aligned}$$

We know the first term of the RHS is divisible by 24 by our Induction Hypothesis. It suffices to show that $k(k+1) \cdot [3(3k+2) + 3(k-1) + 3(3)]$ is divisible by 24. It follows that

$$\begin{aligned}
&k(k+1) \cdot [3(3k+2) + 3(k-1) + 3(3)] \\
&= k(k+1)[9k + 6 + 3k - 3 + 9] \\
&= k(k+1)(12k + 12) \\
&= 12k(k+1)(k+1).
\end{aligned}$$

It is obvious that one of $k, k+1$ is even. Hence we are done. □

Problem (Andrews, NT, Problem 1-2.4)

Prove that each integer may be uniquely represented in the form

$$n = \sum_{j=0}^{s} c_j 3^j,$$

where $c_s \neq 0$, and each c_j is equal to $-1, 0$, or 1.

Proof. It is easy to show that every non-zero integer can be uniquely represented in base 3. To covert from base 3 to the representation described in the problem, replace $2 \cdot 3^k$ with $1 \cdot 3^{k+1} + (-1) \cdot 3^k$ until all coefficients are $-1, 0$, or 1. □

Problem (Andrews, NT, Problem 2-1.4)

Any set of integers J that fulfills the following two conditions is called an *integral ideal:*

(i) if n and m are in J, then $n + m$ and $n - m$ are in J

(ii) if n is in J and r is an integer, then rn is in J.

Let $\mathcal{J}_m$ be the set of all integers that are integral multiples of a particular integer m. Prove that $\mathcal{J}_m$ is an integral ideal.

Proof. Note that criteria (i) is satisfied since for any two multiples of m (call them am and bm), $am + bm = (a + b)m$ and $am - bm = (a - b)m$ are in $\mathcal{J}_m$. Also note that criteria (ii) is satisfied since for any integer r and any element of the set $\mathcal{J}_m$ (call this element cm), $(rc)m$ is in $\mathcal{J}_m$. □

Problem (Andrews, NT, Problem 2-1.5)
Prove that every integral ideal J is identical with $\mathcal{J}_m$ for some m.

Proof. If $J \neq \{0\} = \mathcal{J}_0$, then there exist non-zero integers in J, and with the right choice of r it is not hard to see that there must exist positive integers in J. By the well-ordering principle of the natural numbers, there must be a least positive integer in J, say m. We will now show that $J = \mathcal{J}_m$. It is clear that every multiple of m is also in J by the definition of integral ideals. Also note that m is not the multiple of any positive integer less than it since there are no positive integers in J that are less than m. Finally, no non-multiples of m can be in the set. Say there exists some element k in J such that $m < k$ and k is not a multiple of m. Then by Euclid's Division Lemma we have that $k = qm + r$ for some integers $q, r < m$. However, by the definition of integral ideals, qm is in J, so r must be in J, violating the minimality of m. Hence $J = \mathcal{J}_m$, as desired. $\square$

Problem (Andrews, NT, Problem 2-1.6)
Prove that if a and b are odd integers, then $a^2 - b^2$ is divisible by 8.

Proof. Let $a = 2k + 1$ and $b = 2l + 1$ for some integers k, l. Then

$$a^2 - b^2 = (2k+1)^2 - (2l+1)^2 = 4(k^2 + k - (l^2 + l)) = 4(k(k+1) - l(l+1)).$$

Notice that $k(k+1)$ and $l(l+1)$ are both even. Hence $a^2 - b^2$ is divisible by 8, as desired. $\square$

Problem (Andrews, NT, Problem 2-1.7)
Prove that if a is an odd integer, then $a^2 + (a+2)^2 + (a+4)^2 + 1$ is divisible by 12.

Proof. Let $a = 2k + 1$. Then

$$\begin{aligned} a^2 + (a+2)^2 + (a+4)^2 + 1 &= (2k+1)^2 + (2k+3)^2 + (2k+5)^2 + 1 \\ &= 4k^2 + 4k + 1 + 4k^2 + 12k + 9 + 4k^2 + 20k + 25 + 1 \\ &= 12k^2 + 36k + 36, \end{aligned}$$

which is obviously divisible by 12. $\square$

Problem (Andrews, NT, Problem 2-2.4)
Prove
$$\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}.$$

Proof. Let $a = kd$ and $b = ld$ where $d = \gcd(a,b)$. Then
$$\frac{ab}{\gcd(a,b)} = \frac{kd \cdot ld}{d} = kld.$$
Note that k and l are relatively prime. Also note that kld is a multiple of both a and b. We have not double-counted any divisors since the greatest common divisor of a and b appears only once, hence we are done. □

Problem (100 Geometry Problems #20, Sharygin 2014)
Let ABC be an isosceles triangle with base AB. Line ℓ touches its circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

Proof. Denote by H the orthocenter of $\triangle ABC$. Also, call the intersection of $\overline{CD}$ and $\overline{EH}$ point G.

Lemma 1: G lies on (ABC).

Proof. Note that B, E, G, D are concyclic since $\measuredangle GEB = \measuredangle GDB = 90^\circ$. Thus
$$\measuredangle CGA = \measuredangle CGE = \measuredangle DGE = \measuredangle DBE = \measuredangle DBC = \measuredangle BAC = \measuredangle CBA.$$
Hence A, B, C, G are concyclic, as desired. ■

Lemma 2: $\triangle CEH \sim \triangle CDB$.

Proof. It is well known that the reflection of an orthocenter of a triangle along one of its sides coincides with its circumcircle. Since G lies on $\overline{EH}$ and, by Lemma 1, G lies on (ABC), G is the reflection of H along CB, making $\triangle HCG$ isosceles. Note that $\measuredangle CHE = \measuredangle EGC = \measuredangle CBD$ and $\measuredangle CEH = \measuredangle CDB = 90^\circ$, so $\triangle CEH \sim \triangle CDB$ by $AA \sim$. ■

The previous result shows that there exists a spiral similarity between $\triangle CEH$ and $\triangle CDB$ centered at C. Note that B is on $\overline{HF}$, so it follows that D is on $\overline{EF}$. Hence, we are done. □

Problem (100 Geometry Problems #21, Purple Comet 2013)

Two concentric circles have radii 1 and 4. Six congruent circles form a ring where each of the six circles is tangent to the two circles adjacent to it as shown. The three lightly shaded circles are internally tangent to the circle with radius 4 while the three darkly shaded circles are externally tangent to the circle with radius 1. The radius of the six congruent circles can be written $\frac{k+\sqrt{m}}{n}$, where k, m and n are integers with k and n relatively prime. Find $k+m+n$.

Solution. Number each of the six congruent circles from 1 to 6, with the first circle being lightly colored. We know that the centers of all six congruent circles $(O_1, O_2, \ldots, O_6)$ are evenly spaced around the center of the two concentric circles, O, by symmetry. That is, $\angle O_1OO_2 = 60°$. Denote by r the radius of each of the six circles. Note that $O_1O = 4-r$, $O_2O = r+1$, and $O_1O_2 = 2r$. Then by LoC on $\triangle O_1OO_2$ we have

$$\begin{aligned}
(2r)^2 &= (r+1)^2 + (4-r)^2 - 2(r+1)(4-r)\cos(60°)\\
(2r)^2 &= (r+1)^2 + (4-r)^2 - (r+1)(4-r)\\
4r^2 &= r^2 + 2r + 1 + 16 - 8r + r^2 - (3r + 4 - r^2)\\
0 &= r^2 + 9r - 13.
\end{aligned}$$

Using the quadratic formula gives us

$$r = \frac{-9+\sqrt{133}}{2}.$$

Thus our final answer is $-9+133+2 = \boxed{126}$. □

Problem (100 Geometry Problems #22)

Let A, B, C, D be points in the plane such that $\angle BAC = \angle CBD$. Prove that the circumcircle of $\triangle ABC$ is tangent to BD.

Proof. Let D be on the tangent of B, and let D' be on (ABC) while on the same side of BC as D. Note that $\measuredangle BAC = \measuredangle BD'C$. As D' approaches B, line BD' approaches a tangent to (ABC), but our angle property does not change. Hence, in the limit, $\measuredangle BAC = \measuredangle CBD$. □

Problem (100 Geometry Problems #23, Britain 1995)

Triangle ABC has a right angle at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q respectively. The points M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

Solution. Note that $BCQN$ and $ACPM$ are cyclic quadrilaterals. Now consider $\triangle CMN$. We have

$$\angle CNM = 90^\circ - \angle QNC = 90^\circ - \angle QBC = 90^\circ - \frac{1}{2}\angle B.$$

Similarly, we have

$$\angle CMN = 90^\circ - \angle PMC = 90^\circ - \angle CAP = 90^\circ - \frac{1}{2}\angle A.$$

Considering $\triangle CMN$ as a whole, we have

$$\begin{aligned}\angle MCN &= 180^\circ - \angle CNM + \angle CMN \\ &= 180 - \left(90^\circ - \frac{1}{2}\angle B + 90^\circ - \frac{1}{2}\angle A\right) \\ &= \frac{1}{2}(\angle A + \angle B) \\ &= \frac{1}{2}\angle C \\ &= \boxed{45^\circ}.\end{aligned}$$

□

Problem (100 Geometry Problems #24)

Let $ABCD$ be a parallelogram with $\angle A$ obtuse, and let M and N be the feet of the perpendiculars from A to sides BC and CD. Prove that $\triangle MAN \sim \triangle ABC$.

Proof. Since $\measuredangle AMC = \measuredangle ANC = 90^\circ, AMCN$ is a cyclic quadrilateral. Thus

$$\measuredangle AMN = \measuredangle ACN = \measuredangle CAB$$

where the second equality holds because $AB \parallel CD$. Also note that

$$\measuredangle BCA = \measuredangle MCA = \measuredangle MNA.$$

Therefore $\triangle MAN \sim \triangle ABC$ by $AA \sim$. □

Problem (100 Geometry Problems #25)

For a given triangle $\triangle ABC$, let H denote its orthocenter and O its circumcenter.

(a) Prove that $\angle HAB = \angle OAC$.

(b) Prove that $\angle HAO = |\angle B - \angle C|$.

Proof. We will prove part (a) first. Note that

$$\angle BAH = 90^\circ - \angle CBA = 90^\circ - \frac{1}{2}\angle COA = \angle OAC.$$

(This was Problem 4.23 of Evan Chen's EGMO).
We will now prove part (b). Note that $\angle BAH = 90^\circ - \angle B$ and $\angle CAH = 90^\circ - \angle C$. Finally

$$\angle HAO = |\angle CAH - \angle BAH| = |90^\circ - \angle C - (90^\circ - \angle B)| = |\angle B - \angle C|.$$

The absolute value symbols are required to account for cases where $\angle B < \angle C$. Hence, we are done. $\square$

Problem (100 Geometry Problems #27, AMC 12A 2012)

Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13, OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

Solution. For ease of notation denote by O_1 the center of circle C_1 and by O_2 the center of circle C_2. Also denote by θ the measure of $\angle O_1YZ$. Note that O_1, X, Y, Z all lie on C_2. From this we know that $\angle O_1XZ = 180^\circ - \theta$. Also note that $O_1X = r = O_1Y$, where r is the radius of circle C_1. By LoC on $\triangle O_1YZ$ and $\triangle O_1XZ$ we have

$$\begin{cases} 7^2 + r^2 - 14r\cos\theta = 11^2 \\ 13^2 + r^2 - 26r\cos(180^\circ - \theta) = 11^2. \end{cases}$$

After noticing that $\cos(180^\circ - \theta) = -\cos\theta$, we can solve for $r\cos\theta$ and set the two equations equal to each other, leaving us with

$$\frac{r^2 + 7^2 - 11^2}{14} = \frac{r^2 + 13^2 - 11^2}{-26}.$$

Solving for r gives us a final answer of $\boxed{\sqrt{30}}$. $\square$

Problem (100 Geometry Problems #28)

Let $ABCD$ be a cyclic quadrilateral with no two sides parallel. Lines AD and BC (extended) meet at K, and AB and CD (extended) meet at M. The angle bisector of $\angle DKC$ intersects CD and AB at points E and F, respectively; the angle bisector of $\angle CMB$ intersects BC and AD at points G and H, respectively. Prove that quadrilateral $EGFH$ is a rhombus.

Proof. Let $\alpha = \angle AKF = \angle FKB$ and $\beta = \angle BMG = \angle GMC$. Call the intersection of EF and GH point X. Note that $\angle AFK = 180^\circ - (A+\alpha)$, so $\angle KFB = A + \alpha$, then $\angle MXF = 180^\circ - (A+\alpha+\beta)$. Now note that since $180^\circ - C = A$, $\angle KEC = 180^\circ - (\alpha + A)$, so $\angle XEC = A + \alpha$. Then $\angle MXE = 180^\circ - (A+\alpha+\beta) = \angle MXF$, and since E, X, F are collinear, $\angle MXE = \angle MXF = 90^\circ$. Hence the diagonals of quadrilateral $EGFH$ are perpendicular. Also note that $\triangle MXF \cong \triangle MXE$ by $ASA \cong$, so it follows that $XF = XE$. A very similar process can be done to determine that $XG = XH$, hence $EGFH$ is a rhombus, and we are done. $\square$

Problem (100 Geometry Problems #29)

In $\triangle ABC, AB = 13, AC = 14$, and $BC = 15$. Let M denote the midpoint of $\overline{AC}$. Point P is placed on line segment $\overline{BM}$ such that $\overline{AP} \perp \overline{PC}$. Suppose that p, q, and r are positive integers with p and r relatively prime and q squarefree such that the area of $\triangle APC$ can be written in the form $\frac{p\sqrt{q}}{r}$. What is $p+q+r$?

Solution. Denote by D the foot of the altitude from B to AC. With some simple calculations we find that $AD = 5$, $BD = 12$, and $MD = 2$. Then by the Pythagorean Theorem we find that $BM = 2\sqrt{37}$. Denote by E the foot of the altitude from P to AC. Then we know that $\triangle PME \sim \triangle BMD$ by $AA \sim$. So,

$$\frac{PM}{PE} = \frac{BM}{BD} \implies \frac{7}{PE} = \frac{2\sqrt{37}}{12} \implies PE = \frac{42}{\sqrt{37}}.$$

Note that PE is the height of $\triangle APC$. Thus

$$[APC] = \frac{1}{2} \cdot \frac{42}{\sqrt{37}} \cdot 14 = \frac{294\sqrt{37}}{37},$$

so our final answer is $294 + 37 + 37 = \boxed{368}$. $\square$

Problem (100 Geometry Problems #30)

Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocenter of triangle ADE is the midpoint of BC.

Proof. Denote by M the midpoint of BC. Also denote by D' and E' the intersection of $\overline{MD}$ with AC and $\overline{ME}$ with AB, respectively. Note that $BC \perp MP$. Then since $\measuredangle BDP = 90^\circ = \measuredangle BMP$, $BDPM$ is a cyclic quadrilateral. Then

$$\measuredangle D'DA = \measuredangle MDB = \measuredangle MPB = 90^\circ - \measuredangle PBM = 90^\circ - \measuredangle PBC = 90^\circ - \measuredangle BAC,$$

hence $\measuredangle AD'D = 90^\circ$. Similarly, $\measuredangle EE'A = 90^\circ$, so we are done. □

Problem (100 Geometry Problems #31)

For an acute triangle $\triangle ABC$ with orthocenter H, let H_A be the foot of the altitude from A to BC, and define H_B and H_C similarly. Show that H is the incenter of $\triangle H_A H_B H_C$.

Proof. Note that HH_ABH_C and HH_BCH_A are cyclic quadrilaterals. Then by angle chasing we get

$$\measuredangle HH_AH_C = \measuredangle HBH_C = \measuredangle H_BBA = 90^\circ - \measuredangle BAC$$

and

$$\measuredangle H_BH_AH = \measuredangle H_BCH = \measuredangle ACH_C = 90^\circ - \measuredangle BAC.$$

Thus, $\measuredangle HH_AH_C = \measuredangle H_BH_AH$, so H lies on the angle bisector of $\measuredangle H_BH_AH_C$. Similarly, H lies on the angle bisectors of $\measuredangle H_AH_CH_B$ and $\measuredangle H_CH_BH_A$, so H is the incenter of $\triangle H_AH_BH_C$, as desired. □

Problem (100 Geometry Problems #32, AMC 10A 2013)

In $\triangle ABC, AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

Solution. Let $BX = x$ and $CX = y$. By Power of a Point on C, we have

$$(97 - 86)(97 + 86) = 11 \cdot 183 = 3 \cdot 11 \cdot 61 = y(x + y).$$

Note that $y, x + y$ are both positive integers, and $y < x + y$. If $(y, x + y) = (1, 3 \cdot 11 \cdot 61), (3, 11 \cdot 61)$, or $(11, 3 \cdot 61)$, the triangle inequality on $\triangle ACX$ would be violated. Therefore $BC = x + y = \boxed{61}$, our only remaining possibility. $\square$

Problem (The CALT Induction Handout Exercise 2.3, NICE Spring 2021)

Fifty rooms of a castle are lined in a row. The first room contains 100 knights, while the remaining 49 rooms contain one knight each. These knights wish to escape the castle by breaking the barriers between consecutive rooms, ending with the barrier from room 50 to the outside. At the stroke of midnight, each knight in the i-th room begins breaking the barrier between the i-th and $(i + 1)$-st rooms, where we count the 51st room as the exterior. Each person works at a constant rate and is able to break down a barrier in 1 hour, and once a group of knights breaks down the i-th barrier, they immediately join the knight breaking down the $(i + 1)$-st barrier. The number of hours it takes for the knights to escape the castle is $\frac{m}{n}$, where m and n are positive relatively prime integers. Compute the product mn.

Solution. Denote by "room 0" the first room and every remaining room analogously. Denote by t_n the time it takes to break down the wall in room

n. Note that

$$\begin{aligned}
t_0 &= \frac{1}{100} \\
t_1 &= t_0 + (1 - t_0) \cdot \frac{1}{100+1} = \frac{100}{101} t_0 + \frac{1}{101} = \frac{2}{101} \\
t_2 &= t_1 + (1 - t_1) \cdot \frac{1}{100+2} = \frac{101}{102} t_1 + \frac{1}{102} = \frac{3}{102} \\
t_3 &= t_2 + (1 - t_2) \cdot \frac{1}{100+3} = \frac{102}{103} t_2 + \frac{1}{103} = \frac{4}{103} \\
&\vdots \\
t_i &= \underbrace{t_{i-1}}_{\substack{\text{current wall already} \\ \text{done by } i\text{-th person}}} + \underbrace{(1 - t_{i-1}) \cdot \frac{1}{100+i}}_{\substack{\text{rest of wall with} \\ 100+i \text{ people}}} = \frac{i+1}{100+i}.
\end{aligned}$$

Thus, $t_{49} = \frac{50}{149}$, so our final answer is $50 \cdot 149 = \boxed{7450}$. □

Problem (CALT Induction Exercise 2.4, PUMaC 2018)
If $a_1, a_2, \ldots$ is a sequence of real numbers such that for all n,

$$\sum_{k=1}^{n} a_k \left(\frac{k}{n}\right)^2 = 1,$$

find the smallest n such that $a_n < \frac{1}{2018}$.

Solution. We will first begin by finding the first few values of a_k. By some simple calculations it follows that

$$\begin{aligned}
a_1 \cdot \left(\tfrac{1}{1}\right)^2 = 1 &\implies a_1 = \tfrac{1}{1} \\
a_1 \cdot \left(\tfrac{1}{2}\right)^2 + a_2 \cdot \left(\tfrac{2}{2}\right)^2 = 1 &\implies a_2 = \tfrac{3}{4} \\
a_1 \cdot \left(\tfrac{1}{3}\right)^2 + a_2 \cdot \left(\tfrac{2}{3}\right)^2 + a_3 \cdot \left(\tfrac{3}{3}\right)^2 = 1 &\implies a_3 = \tfrac{5}{9} \\
a_1 \cdot \left(\tfrac{1}{4}\right)^2 + a_2 \cdot \left(\tfrac{2}{4}\right)^2 + a_3 \cdot \left(\tfrac{3}{4}\right)^2 + a_4 \cdot \left(\tfrac{4}{4}\right)^2 = 1 &\implies a_4 = \tfrac{7}{16}.
\end{aligned}$$

We can now conjecture that

$$a_k = \frac{2k-1}{k^2}.$$

Now we need to find an n such that

$$\frac{2n-1}{n^2} < \frac{1}{2018}.$$

Reciprocating both sides and doing polynomial long division on the LHS gives us

$$\frac{n^2}{2n-1} = \frac{1}{2}n + \frac{1}{4} + \frac{\frac{1}{4}}{2n-1} > 2018.$$

Multiplying both sides by 4 gives us

$$2n + 1 + \frac{1}{2n-1} > 4 \cdot 2018.$$

Note that $n = 2 \cdot 2018$ satisfies this inequality while $n = 2 \cdot 2018 - 1$ does not. Hence our answer is $2 \cdot 2018 = \boxed{4036}$. □

Problem (2016 AMC 12B #21)

Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \ldots$, let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i\,?$$

Solution. We will use coordbash. First, let $D = (0,0), A = (0,1), B = (1,1),$ and $C = (1,0)$. Note that $\overline{AQ_1}$ is equivalent to $y = -2x+1$ and $\overline{BD}$ is equivalent to $y = x$ in this setup. Solving this system gives us $P_1 = (\frac{1}{3}, \frac{1}{3})$. Since Q_2 is the foot of the altitude from P_1 to $\overline{CD}$, the x-coordinate of Q_2 is the same as for P_1 while the y-coordinate is just 0. In other words, $Q_2 = (\frac{1}{3}, 0)$. We now see that $\overline{AQ_2} = (y = -3x+1)$, $P_2 = (\frac{1}{4}, \frac{1}{4})$, and $Q_3 = (\frac{1}{4}, 0)$. In general, we have

$$P_i = \left(\frac{1}{i+2}, \frac{1}{i+2}\right)$$
$$Q_i = \left(\frac{1}{i+1}, 0\right).$$

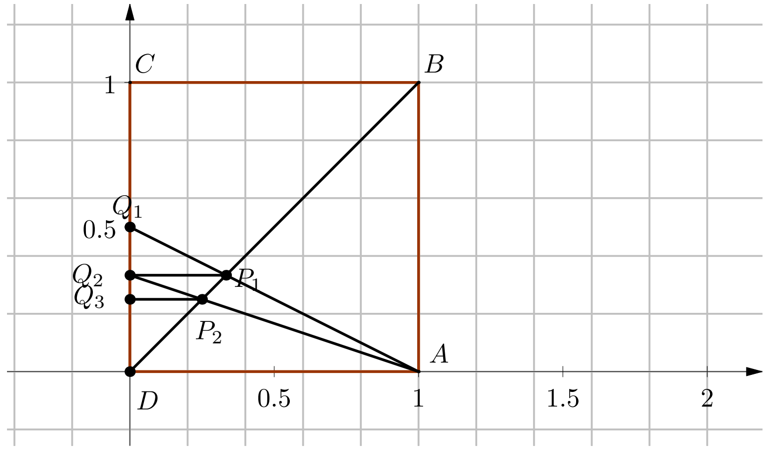

Figure 1: 2016 AMC 12B #21

In general, to find $[DQ_iP_i]$, note that DQ_i is the base of the triangle and P_iQ_{i+1} is the height. Thus

$$[DQ_iP_i] = \frac{1}{2} \cdot \frac{1}{i+1} \cdot \frac{1}{i+2}.$$

What we have left to evaluate is the following sum

$$\frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{i+1} \cdot \frac{1}{i+2}.$$

Using partial fraction decomposition on the summand gets us

$$\frac{1}{2}\left[\sum_{i=1}^{\infty} \frac{1}{i+1} - \sum_{i=1}^{\infty} \frac{1}{i+2}\right].$$

Note that we have a telescoping sum. It is not hard to see by writing out a few terms that our sum is equal to $\boxed{\frac{1}{4}}$. $\square$

Problem (Christmas Math Competitions 12B 2021 #15)
There are n values of x which satisfy

$$\lfloor x \rfloor^2 = \{x\}^2 + \frac{2020}{2021}x^2.$$

What is the remainder when n is divided by 5? (Here, $\lfloor \bullet \rfloor$ is the greatest integer function and $\{\bullet\}$ is the fractional part function.)

Solution. Let $x = A + \varepsilon$ where A is the integer part of x and ε is the fractional part. Substituting this into our given equation gives us

$$A^2 = \varepsilon^2 + \frac{2020}{2021}(A+\varepsilon)^2 = \varepsilon^2 + \frac{2020}{2021}(A^2 + 2A\varepsilon + \varepsilon^2).$$

Distributing $\frac{2020}{2021}$ and moving all the terms to the LHS gives us

$$\begin{aligned} \frac{1}{2021}A^2 - \frac{4040}{2021}A\varepsilon - \frac{4041}{2021}\varepsilon^2 &= 0 \\ A^2 - 4040A\varepsilon - 4041\varepsilon^2 &= 0 \\ (A - 4041\varepsilon)(A+\varepsilon) &= 0. \end{aligned}$$

Thus we have two cases.

- For the case where $A + \varepsilon = 0$, since A is an integer, the only possible value for ε is 0, which then implies that $A = 0$, giving us one value of x in this case.
- For the case where $A - 4041\varepsilon = 0$, for the LHS to be an integer, the denominator of ε must be 4041. Thus $\varepsilon = \frac{1}{4041}, \frac{2}{4041}, \ldots, \frac{4040}{4041}$, giving us 4040 values of x in this case.

Finally, note that $1 + 4040 \equiv \boxed{1} \pmod 5$. $\square$

Problem (AIME 1998 # 6)

Let $ABCD$ be a parallelogram. Extend $\overline{DA}$ through A to a point P, and let $\overline{PC}$ meet $\overline{AB}$ at Q and $\overline{DB}$ at R. Given that $PQ = 735$ and $QR = 112$, find RC.

Solution. Let $RC = x$. First, note that $\triangle PAQ \sim \triangle PDC$. By projecting $\overline{PC}$ onto $\overline{PD}$, we have

$$(Q, P; R, C) \stackrel{B}{=} (A, P; D, P_\infty).$$

Computing cross ratios gives us the following.

$$\begin{aligned}
\frac{QR}{PR} \cdot \frac{PC}{QC} &= \frac{AD}{PD} \\
&= \frac{PD - PA}{PD} \\
&= 1 - \frac{PA}{PD} \\
&= 1 - \frac{PQ}{PC} \quad (\text{since } \triangle PAQ \sim \triangle PDC).
\end{aligned}$$

Substituting the values given in the problem and letting $u = 112 + x$ gives us

$$\begin{aligned}
\frac{112}{735 + 112} \cdot \frac{735 + 112 + x}{112 + x} &= 1 - \frac{735}{735 + 112 + x} \\
\frac{112}{847} \cdot \frac{735 + u}{u} &= \frac{u}{735 + u} \\
112(735 + u)^2 &= 847u^2 \\
735u^2 - 2 \cdot 112 \cdot 735u - 112 \cdot 735^2 &= 0 \\
u^2 - 2 \cdot 112u - 112 \cdot 735 &= 0
\end{aligned}$$

Applying the quadratic formula then gives us

$$u = \frac{2 \cdot 112 \pm \sqrt{2^2 \cdot 112^2 + 4 \cdot 112 \cdot 735}}{2} = 112 \pm \sqrt{112(112 + 735)}.$$

Since $u = x + 112$, we have that

$$x = \sqrt{112 \cdot 847} = \boxed{308}.$$

□

Problem (2015 AIME II #11)

The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC and P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

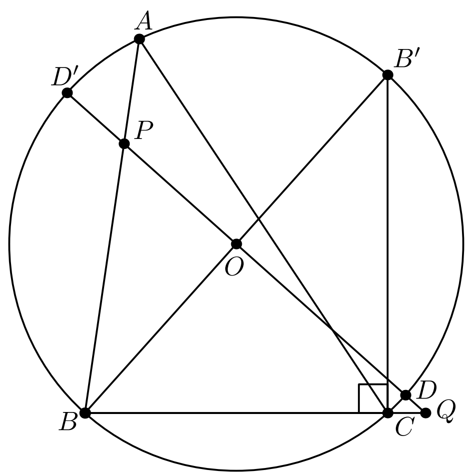

Figure 2: 2015 AIME II #11

Solution. Let ω be the circumcircle in question and D be the intersection point of OQ and ω. Also denote by B', D' the antipodal points of B, D, respectively. Finally denote by R the circumradius.

By Pythagoras on $\triangle BOQ$ we have

$$\begin{aligned} OQ^2 &= BQ^2 - BO^2 \\ &= \left(\frac{9}{2}\right)^2 - R^2. \end{aligned}$$

Also note that

$$\begin{aligned}
\text{Pow}_\omega(Q) = QC \cdot QB &= QD \cdot QD' \\
\frac{1}{2} \cdot \frac{9}{2} &= (OQ - R)(OQ + R) \\
\frac{9}{4} &= OQ^2 - R^2 \\
\frac{9}{4} &= \left(\frac{9}{2}\right)^2 - 2R^2.
\end{aligned}$$

From this we determine that $R = 3$.

Now, since BB' is a diameter of ω, we have that $\angle BCB' = 90^\circ$. Let $\angle PBO = \alpha$. By angle chase we have the following:

$$\alpha = \angle PBO = \angle ABB' = \angle ACB' = 90^\circ - \angle BCA.$$

Rearranging this gives us $\angle BCA = 90^\circ - \alpha$.

Next, let $\angle BAC = \angle BB'C = \theta$. Since $\triangle BB'C$ is a right triangle with a right angle at C, we have

$$\sin(\theta) = \frac{BC}{BB'} = \frac{4}{6} = \frac{2}{3}.$$

Lastly, note that since $\triangle PBO$ is a right triangle with a right angle at O, we have that $\cos(\alpha) = \frac{3}{PB}$. Applying the Law of Sines on $\triangle ABC$ gives us

$$\begin{aligned}
\frac{5}{\sin(90^\circ - \alpha)} &= \frac{4}{\sin(\theta)} \\
\cos(\alpha) &= \frac{5}{4} \cdot \frac{2}{3} \\
\frac{3}{PB} &= \frac{5}{6}.
\end{aligned}$$

From this we determine that $PB = \frac{18}{5}$, so our desired answer is $\boxed{023}$. □

Problem (2018 HMMT Geometry # 5)

In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

Solution. Let $\theta = \angle MAE = \angle RAE$. Since $\angle ARM = 90^\circ$, $\angle RMA = 90^\circ - 2\theta$. Since we know $AM = 2$, then $AR = 2\sin(90^\circ - 2\theta) = 2\cos(2\theta)$.

Next, let F denote the foot of the altitude from E to MR. Since $[RAM] = [REM]$, and MR is a base of both triangles, we have that $AR = EF$. Since AR and EF are also both perpendicular to MR, we know that $AR \parallel EF$.

Note that $\triangle REM$ is isosceles since $\angle MRE = \angle RME = \theta$. Thus the center of ω lies on $\overline{EF}$. If we then consider rotating $\triangle REM$ about the center of ω by 180°, calling $E'F'$ the segment that is the image of EF after this rotation, then we can translate AR so that A coincides with F' and R with F. Clearly $EF + AR + E'F' = EE'$ is a diameter of ω and $EF = AR = E'F'$. Thus,

$$3 \cdot 2\cos(2\theta) = 2.$$

From this we derive that $\cos(2\theta) = \frac{1}{3}$ and $\sin(2\theta) = \frac{2\sqrt{2}}{3}$. Finally, it follows that

$$\begin{aligned}
[MARE] &= 2[RAM] \\
&= \tfrac{1}{2} \cdot 2 \cdot AM \cdot AR \cdot \sin(\angle RAM) \\
&= \tfrac{1}{2} \cdot 2 \cdot 2\cos(2\theta)\sin(2\theta) \\
&= 2 \cdot \tfrac{1}{3} \cdot \tfrac{2\sqrt{2}}{3} \\
&= \boxed{\frac{8\sqrt{2}}{9}}
\end{aligned}$$

□

Problem (2018 PUMaC Geometry A5)

Let $\triangle ABC$ be a triangle with side lengths $AB = 9, BC = 10, CA = 11$. Let O be the circumcenter of $\triangle ABC$. Denote $D = AO \cap BC, E = BO \cap CA, F = CO \cap AB$. If $\frac{1}{AD} + \frac{1}{BE} + \frac{1}{FC}$ can be written in simplest form as $\frac{a\sqrt{b}}{c}$, find $a + b + c$.

Solution. Note that $[ABC] = 30\sqrt{2}$ by Heron's Formula, and using the formula $[ABC] = \frac{abc}{4R}$, we have that the circumradius $R = \frac{33\sqrt{2}}{8}$.

Let M_A denote the midpoint of BC and F_A denote the foot of the altitude from A to BC. Denote M_B, M_C, F_B, F_C similarly. It is clear that

$\triangle AF_AD \sim \triangle OM_AD$. From this we derive that

$$\frac{OM_A}{AF_A} = \frac{OD}{AD} = \frac{AD - R}{AD} = 1 - \frac{R}{AD}.$$

Rearranging this we have that

$$\frac{1}{AD} = \frac{1}{R}\left(1 - \frac{OM_A}{AF_A}\right).$$

Similarly, we have that

$$\frac{1}{BE} = \frac{1}{R}\left(1 - \frac{OM_B}{BF_B}\right)$$

$$\frac{1}{CF} = \frac{1}{R}\left(1 - \frac{OM_C}{CF_C}\right).$$

Adding these three equations together we have that

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{FC} = \frac{1}{R}\left(3 - \frac{OM_A}{AF_A} - \frac{OM_B}{BF_B} - \frac{OM_C}{CF_C}\right).$$

Using the facts that $OM_A = \sqrt{R^2 - 5^2}$ and $AF_A = \frac{2[ABC]}{10}$ (and the analogous facts for B and C), after a big bash, we compute

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{FC} = \frac{8\sqrt{2}}{33},$$

so our desired answer is $\boxed{43}$. $\square$

Problem (IMO 1961/2)
Let a, b, and c be the lengths of a triangle whose area is S. Prove that

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}.$$

In what case does equality hold?

Proof. We begin by proving the following lemma.
Lemma. The following inequality holds for all angles C:

$$2 \geq \sqrt{3}\sin(C) + \cos(C).$$

Proof. Rearranging, we have

$$\begin{aligned} 1 &\geq \frac{\sqrt{3}}{2}\sin(C) + \frac{1}{2}\cos(C) \\ &= \cos\left(\frac{\pi}{6}\right)\sin(C) + \sin\left(\frac{\pi}{6}\right)\cos(C) \\ &= \sin\left(\frac{\pi}{6} + C\right). \end{aligned}$$

This clearly holds for all values of C, and equality is achieved when $C = \frac{\pi}{3} = 60^\circ$. ∎

Now note by AM-GM and our Lemma that

$$\begin{aligned} a^2 + b^2 &\geq 2ab \\ &\geq \left(\sqrt{3}\sin(C) + \cos(C)\right)ab \\ a^2 + b^2 - ab\cos(C) &\geq ab\sin(C)\sqrt{3} \\ a^2 + b^2 + (a^2 + b^2 - 2ab\cos(C)) &\geq 2 \cdot \left(2 \cdot \tfrac{1}{2}ab\sin(C)\right)\sqrt{3} \\ a^2 + b^2 + c^2 &\geq 4S\sqrt{3}, \end{aligned}$$

as desired, where equality holds when both $C = 60^\circ$ and $a = b$, meaning the triangle in question is equilateral. □

Problem (Evan Chen's Intro to Functional Equations Handout 8.1)
Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real x and y,
$$f(f(x+y)) = f(x) + f(y).$$

Proof. The answer is $f(x) = x + b$ and $f(x) = 0$, both of which clearly work for all values of b. We will now show that these are the only possible forms of f.

Plugging in $y = 0$ into our given equation gives us
$$f(f(x)) = f(x) + f(0).$$
Now our original equation simplifies to
$$\begin{aligned} f(x+y) + f(0) &= f(x) + f(y) \\ f(x+y) - f(0) &= f(x) - f(0) + f(y) - f(0). \end{aligned}$$
Now setting $g(x) = f(x) - f(0)$ gives us
$$g(x+y) = g(x) + g(y).$$
By Cauchy we have that $g(x) = ax$, so $f(x) = ax + b$. Plugging this new f into our original equation gives us
$$a^2(x+y) + ab + b = a(x+y) + 2b,$$
so $a^2 = a$ and $ab = b$. Thus for $a = 0$ we have $f(x) = 0$, and when $a = 1$ we have $f(x) = x + b$, as desired. $\square$

Problem (Evan Chen's Intro to Functional Equations Handout 8.2)
Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y$$

for all real numbers x and y.

Proof. We claim the only solutions are $f(x) = 0$ and $f(x) = x^2$, both of which clearly work. We would like $f(x^2 + y)$ and $f(f(x) - y)$ to cancel, so setting $x^2 + y = f(x) - y$ we have

$$4f(x)y = 0 \text{ and } y = \frac{1}{2}\left(f(x) - x^2\right).$$

It follows that

$$4f(x) \cdot \frac{1}{2}(f(x) - x^2) = 0,$$

so $f(x) = 0$ or $f(x) = x^2$ for all values of x.

We must now deal with functions f for which there exist nonzero a, b where $a \neq b$ and $f(a) = 0$ and $f(b) = b^2$ (the so-called "Pointwise Trap"). Plugging in $x = 0$ into our given equation, recalling that $f(0) = 0$, we have that $f(y) = f(-y)$, so f is even. Thus without loss of generality let b be negative. Plugging in $x = a, y = -b$ into our given equation gives us

$$\begin{aligned} f(a^2 - b) &= f(b) \\ (a^2 - b)^2 &= b^2 \\ a^4 - 2a^2b &= 0 \\ a^2(a^2 - 2b) &= 0. \end{aligned}$$

Since a is nonzero, we have that $a^2 = 2b$, but the LHS is positive while the RHS is negative, a contradiction. Hence the only valid functions are $f(x) = 0$ and $f(x) = x^2$, as desired. $\square$

Problem (Evan Chen's Intro to Functional Equations Handout 8.3)
Solve $f(t^2+u) = tf(t) + f(u)$ over $\mathbb{R}$.

Proof. We claim $f(t) = mt$ for any value of m. This clearly works.

First, setting $t = 1$ in our original equation gives us

$$f(u+1) = f(u) + f(1). \quad (\clubsuit)$$

Also, setting $u = 0$ in ($\clubsuit$) gives us $f(0) = 0$.

Second, setting $u = 0$ in our original equation gives us

$$f(t^2) = tf(t). \quad (\spadesuit)$$

Substituting the $tf(t)$ term in our original equation for $f(t^2)$ gives us

$$f(t^2+u) = f(t^2) + f(u).$$

Therefore f is additive over the nonnegative reals (since t^2 is always nonnegative).

Next, substituting $-t$ for t in ($\spadesuit$) gives us

$$-tf(-t) = f((-t)^2) = f(t^2) = tf(t).$$

This implies f is odd. Hence f is additive over all reals now.

We are almost done. Substituting $t+1$ for t in ($\spadesuit$) and using additivity gives us both

$$f((t+1)^2) = f(t^2+2t+1) = f(t^2) + 2f(t) + f(1) = tf(t) + 2f(t) + f(1)$$
$$f((t+1)^2) = (t+1)f(t+1) \overset{\clubsuit}{=} (t+1)(f(t) + f(1)) = tf(t) + f(t) + tf(1) + f(1).$$

Subtracting these two gives us $f(t) = tf(1)$, as desired. $\square$

Problem (OTIS Excerpts Problem #2.2.4.1)
If $a+b+c=1$, then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le 3+2\cdot\frac{\left(a^3+b^3+c^3\right)}{abc}$.

Proof. Note that $(2,1,0) \prec (3,0,0)$, so by Muirhead's inequality we proceed as follows:

$$\begin{aligned}
\sum_{\text{sym}} a^2b &\le \sum_{\text{sym}} a^3 \\
a^2b+ab^2+b^2c+bc^2+c^2a+ca^2 &\le a^3+a^3+b^3+b^3+c^3+c^3 \\
ab(a+b)+bc(b+c)+ca(c+a) &\le 2\left(a^3+b^3+c^3\right).
\end{aligned}$$

Dividing both sides by abc gives us

$$\begin{aligned}
\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} &\le 2\cdot\frac{\left(a^3+b^3+c^3\right)}{abc} \\
1+\frac{a}{c}+\frac{b}{c}+1+\frac{b}{a}+\frac{c}{a}+1+\frac{c}{b}+\frac{a}{b} &\le 3+2\cdot\frac{\left(a^3+b^3+c^3\right)}{abc} \\
(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) &\le 3+2\cdot\frac{\left(a^3+b^3+c^3\right)}{abc}.
\end{aligned}$$

Since $a+b+c=1$, we have the desired result. $\square$

Problem (Evan Chen's Intro to Olympiad Inequalities Problem #2.4.3)
If $a + b + c = 3$ then

$$\sum_{\text{cyc}} \frac{a}{2a^2 + a + 1} \leq \frac{3}{4}.$$

Proof. First note that a, b, c are on the interval $[0, 3]$. Define $f(x) = \frac{x}{2x^2+x+1}$. By taking the derivative twice we have that

$$f''(x) = \frac{8x^3 - 12x - 2}{\left(2x^2 + x + 1\right)^3}.$$

We want to find the inflection points of f, so we define a new function $g(x) = 8x^3 - 12x - 2$ in the hopes of finding where this new function has roots. It is not important to know exactly what the roots of g are but only if they are in the interval $[0, 3]$ or not.

x	$g(x)$
-2	-42
-1	2
0	-2
3	178

By the Intermediate Value Theorem, since g is continuous, if g switches sign on an interval, then g must have a root on that interval. Since g is a degree 3 polynomial, it is evident that only one root of g (and thus only one inflection point of f) falls in the interval $[0, 3]$.

We are now motivated to use the $n-1$ EV principle. In other words, when the quantity $f(a)+f(b)+f(c)$ achieves a maximum, two of the variables are equal, and without loss of generality we let $b = c$. Given that $a+b+c = 3$, we only have to check the maximum value of $f(a)+2f(b) = f(3-2b)+2f(b)$, represented in the following inequality, which is left for us to verify.

$$f(3 - 2b) + 2f(b) = \frac{3 - 2b}{2(3 - 2b)^2 + (3 - 2b) + 1} + \frac{2b}{2b^2 + b + 1} \leq \frac{3}{4}$$
$$\iff (b - 1)^2(8b^2 - 14b + 9) \geq 0.$$

Calculating the discriminant of $8b^2 - 14b + 9$ reveals that this factor is always positive. Thus we have verified this inequality with the equality case at $a, b, c = 1$, so we are done. $\square$

Problem (USAJMO 2015/1)

Given a sequence of real numbers, a move consists of choosing two terms and replacing each with their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence – no matter what move – there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.

Proof. Denote a move between two terms x, y by (x, y). They are replaced by $\frac{x+y}{2}, \frac{x+y}{2}$. Note that the sum of the two terms before and after the move is invariant.

We are motivated by this invariance to consider the sequence

$$\{a_n\} = -1007, \ldots, -2, -1, 0, 1, 2, \ldots, 1007,$$

whence our goal contains 2015 zeroes since the sums of all a_n (as well as all the terms in the final sequence) is 0. We claim that after any initial move done to a_n, there will always be a set of moves that will take the resulting sequence all the way to b_n. We proceed with casework.

Case 1: The first move does not contain the 0 term.

Say the first move was (a, b) for nonzero a, b where $a \neq b$. We can then perform the moves

$$(-a, -b), (\tfrac{a+b}{2}, \tfrac{-a-b}{2}), (\tfrac{a+b}{2}, \tfrac{-a-b}{2})$$

and then $(c, -c)$ for all remaining pairs of nonzero terms $\{c, -c\}$. If $a = -b$, our work is even easier since we can just perform the move $(a, -b)$ and finish off with $(c, -c)$ for all remaining nonzero pairs of terms. In either case, we are left with a sequence of 2015 zeroes, as desired.

Case 2: The first move does contain the 0 term.

Say the first move was $(0, a)$. We can combine all pairs $(c, -c)$ similarly as before leaving a sequence with only the terms $\frac{a}{2}, \frac{a}{2}, -a$, and 2012 zeroes. We can then perform the remaining moves

$$(-a, 0), (\tfrac{a}{2}, \tfrac{-a}{2}), (\tfrac{a}{2}, \tfrac{-a}{2})$$

to end up with a sequence of 2015 zeroes, as desired. $\square$

Problem (Evan Chen's Intro to Olympiad Inequalities Problem #3.3.2)
If $a^2+b^2+c^2=12$, then $a\cdot\sqrt[3]{b^2+c^2}+b\cdot\sqrt[3]{c^2+a^2}+c\cdot\sqrt[3]{a^2+b^2}\le 12$.

Proof. We first make the substitution $a^2=x, b^2=y, c^2=z$ so that $x+y+z=12$. We can rewrite the LHS of our inequality in the following way.

$$\sum_{cyc} a\sqrt[3]{b^2+c^2}=\sum_{cyc} a\sqrt[3]{12-a^2}=\sum_{cyc}\sqrt{x}\sqrt[3]{12-x}.$$

Define $f(x)=\sqrt{x}\sqrt[3]{12-x}$. The interval that we care about is when $x\in(0,12)$, otherwise the value of $f(x)$ is clearly nonpositive or complex. Note that

$$f'(x)=\frac{\sqrt[3]{12-x}}{2\sqrt{x}}-\frac{\sqrt{x}}{3\left(12-x\right)^{2/3}}=\frac{36-5x}{6\sqrt{x}\left(12-x\right)^{2/3}}.$$

Now note that f' is decreasing since f' has a decreasing function in the numerator and an increasing function in the denominator for all x on our interval. This then implies that f is concave on the interval $(0,12)$. We are motivated to use Jensen's Inequality.

Our inequality is finished as follows.

$$\begin{aligned}\sum_{cyc}\sqrt{x}\sqrt[3]{12-x}&=f(x)+f(y)+f(z)\\&\le 3f\left(\frac{x+y+z}{3}\right)\\&=3f(4)\\&=12.\end{aligned}$$

$\square$