# CNCM Problem of the Day Solutions

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# 6 August 2021

Notice that the expected number of dollars Tommy expects to win is equivalent to the following infinite series:

$$\frac{1}{6}\sum_{n=0}^{\infty} n^2 \left(\frac{5}{6}\right)^n.$$

Define

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n$$

where we want to find the value of $f(5/6)$. Then

$$\begin{aligned}
f(x) &= x\sum_{n=0}^{\infty} n^2 x^{n-1} \\
&= x\frac{d}{dx}\left[\sum_{n=0}^{\infty} n x^n\right] \\
&= x\frac{d}{dx}\left[x\sum_{n=0}^{\infty} n x^{n-1}\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\frac{1}{1-x}\right]\right] \\
&= x\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] \\
&= x\left[\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}\right] \\
&= x\left[\frac{(1-x) + 2x}{(1-x)^3}\right] \\
&= x\left[\frac{1+x}{(1-x)^3}\right] \\
f(5/6) &= \frac{5}{6}\left[\frac{1+5/6}{(1-5/6)^3}\right] \\
&= \frac{5}{6}\cdot\frac{11}{6}\cdot\frac{6^3}{1} \\
&= 330.
\end{aligned}$$

Then our final answer is $\frac{1}{6}f(\frac{5}{6}) = \frac{330}{6} = \boxed{55}$.

# 11 August 2021

Here let $R_n$ denote the remaining water after the $n$-th pour.

$$\begin{aligned} R_0 &= 1 \\ R_1 &= \left(1 - \frac{1}{2}\right) R_0 = \frac{1}{2} \\ R_2 &= \left(1 - \frac{1}{3}\right) R_1 = \frac{1}{3} \\ R_3 &= \left(1 - \frac{1}{4}\right) R_2 = \frac{1}{4} \end{aligned}$$

Therefore we can assume by Engineer's Induction that $R_n = \frac{1}{n+1}$. Hence $R_9 = \frac{1}{10}$ for a final answer of $\boxed{9}$.

# 12 August 2021

For a two-game block, there is a probability of $\dfrac{3}{4} \cdot \dfrac{1}{4} + \dfrac{1}{4} \cdot \dfrac{3}{4} = \dfrac{3}{8}$ of the entire match ending then and there. The only other outcome after two games is a tie, since each game must declare a winner, and this happens with probability 5/8. Thus the expected number of games in a match is the following:

$$\begin{aligned} \frac{3}{8} \cdot 2 + \frac{3}{8} \cdot \frac{5}{8} \cdot 4 + \frac{3}{8} \cdot \left(\frac{5}{8}\right)^2 \cdot 6 + \cdots &= \sum_{n=0}^{\infty} 2(n+1) \cdot \frac{3}{8} \cdot \left(\frac{5}{8}\right)^n \\ &= \frac{3}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{5}{8}\right)^n \\ &= \frac{3}{4} \sum_{n=0}^{\infty} n \left(\frac{5}{8}\right)^n + \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n. \end{aligned}$$

Define $f(x) = \sum_{n=0}^{\infty} nx^n$. We would like to find the value of $f(5/8)$. Note that

$$\begin{aligned} f(x) &= x \sum_{n=0}^{\infty} nx^{n-1} \\ &= x \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ &= x \frac{d}{dx} \left[\frac{1}{1-x}\right] \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

Thus $f(5/8) = \dfrac{5/8}{(1-5/8)^2} = \dfrac{5}{8} \cdot \dfrac{8^2}{3^2} = \dfrac{40}{9}$. Then our original sum becomes

$$\begin{aligned}
\frac{3}{4}\sum_{n=0}^{\infty} n\left(\frac{5}{8}\right)^n + \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n &= \frac{3}{4}\cdot\frac{40}{9} + \frac{3}{4}\cdot\frac{1}{1-5/8} \\
&= \frac{10}{3} + \frac{3}{4}\cdot\frac{8}{3} \\
&= \frac{10}{3} + 2 \\
&= \frac{16}{3}.
\end{aligned}$$

Our final answer is $160 + 3 = \boxed{163}$.