Random Problems

RYDER PHAM

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Note: The symbol $\measuredangle$ refers to the directed angle in this text.

Problem (OTIS Excerpts #7)

Determine, with proof, the smallest positive integer c such that for any positive integer n, the decimal representation of the number $c^n + 2014$ has digits all less than 5.

Proof. We claim that $c = 10$. We know this value works because for $n \geq 1, c^n \in \{10, 100, 1000, \dots\}$, and since all digits from the 10s and to the left are less than 4, adding 1 to them will not violate our digit condition. We will now check that this is the smallest possible value for c.

- $c = 1$ fails at $n = 1$ since $1 + 4 = 5$.
- $c = 2$ fails at $n = 1$ since $2 + 4 = 6$.
- $c = 3$ fails at $n = 1$ since $3 + 4 = 7$.
- $c = 4$ fails at $n = 1$ since $4 + 4 = 8$.
- $c = 5$ fails at $n = 1$ since $5 + 4 = 9$.
- $c = 6$ fails at $n = 2$ since $36 + 2014 = 2050$.
- $c = 7$ fails at $n = 2$ since $49 + 2014 = 2063$.
- $c = 8$ fails at $n = 2$ since $64 + 2014 = 2078$.
- $c = 9$ fails at $n = 2$ since $81 + 2014 = 2095$.

Since every value of c less than 10 fails, we are done. $\square$

Problem (OTIS Excerpts #77, HMMT Februrary 2013)

Values $a_1, \ldots, a_{2013}$ are chosen independently and at random from the set $\{1, \ldots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \ldots, a_{2013}\}$?

Solution. Let P be the number of distinct values in $a_1, \ldots, a_{2013}$, and for each $i = 1, 2, \ldots, 2013$ let

$$P_i := \begin{cases} 1 & \text{if } a_i \neq a_j \text{ for all } j < i \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $P = P_1 + \cdots + P_{2013}$. Thus it follows that

$$\begin{aligned} E[P] &= E[P_1] + E[P_2] + \cdots + E[P_{2013}] \\ &= 1 + (1 - 1/2013) + \cdots + (1 - 1/2013)^{2012} \\ &= \frac{1 - (2012/2013)^{2013}}{1 - 2012/2013} \\ &= 2013 \left(1 - \left(\frac{2012}{2013}\right)^{2013}\right). \end{aligned}$$

□

Problem (100 Geometry Problems #8)

Let ABC be a triangle with $\angle CAB$ a right angle. The point L lies on the side BC between B and C. The circle BAL meets the line AC again at M and the circle CAL meets the line AB again at N. Prove that L, M, and N lie on a straight line.

Proof. Since $ANLC$ and $ALBM$ are cyclic quadrilaterals, $\angle CAN = \angle CLN = \angle 90^\circ = \angle BAM = \angle BLM$. Since $\angle CLN + \angle BLN = 180^\circ$, we have $\angle BLN = \angle BLM = 90^\circ$, as desired. □

Problem (100 Geometry Problems #11)

A closed planar shape is said to be equiable if the numerical values of its perimeter and area are the same. For example, a square with side length 4 is equiable since its perimeter and area are both 16. Show that any closed shape in the plane can be dilated to become equiable. (A dilation is an affine transformation in which a shape is stretched or shrunk. In other words, if $\mathcal{A}$ is a dilated version of $\mathcal{B}$ then $\mathcal{A}$ is similar to $\mathcal{B}$.)

Proof. Note that for any scaling of the perimeter by a factor of k, the area increases by a factor of k^2. It is not hard to see that by making the perimeter arbitrarily large, at some point the area must be larger than the perimeter, and by making the perimeter arbitrarily small, the area must be smaller than the perimeter. Thus by the Intermediate Value Theorem there must be a scale factor k such that the perimeter equals the area. $\square$

Problem (100 Geometry Problems #13)

Points A and B are located on circle Γ, and point C is an arbitrary point in the interior of Γ. Extend AC and BC past C so that they hit Γ at M and N respectively. Let X denote the foot of the perpendicular from M to BN, and let Y denote the foot of the perpendicular from N to AM. Prove that $AB \parallel XY$.

Proof. It suffices to show that $\triangle ABC \sim \triangle YXC$, as this would prove that $AB \parallel XY$. Note that $NXYM$ is a cyclic quadrilateral because $\measuredangle NXM = 90^\circ = \measuredangle NYM$. By angle chasing we get

$$\measuredangle ABC = \measuredangle ABN = \measuredangle AMN = \measuredangle YMN = \measuredangle YXN = \measuredangle YXC.$$

We know $\measuredangle ACB = \measuredangle YCX$ by vertical angles, hence $\triangle ABC \sim \triangle YXC$ by $AA\sim$. This completes the proof. $\square$

Problem (100 Geometry Problem #14, AIME 2007)

Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.

Solution. Extend BE and CF to meet at G and extend AE and DF to meet at H. Note by symmetry, $FGHE$ is a square of sidelength $12+5=17$. Thus $EF^2 = 17^2 + 17^2 = \boxed{578}$. □

Problem (100 Geometry Problems #15)

Let Γ be the circumcircle of $\triangle ABC$, and let D, E, F be the midpoints of arcs AB, BC, CA, respectively. Prove that $DF \perp AE$.

Proof. Denote by I the incenter of $\triangle ABC$. By the Incenter-Excenter Lemma, I lies on AE. Also by the Lemma, D and F are the circumcenters of (AIB) and (AIC), respectively. The radical axis of these two circles is AI, thus $AI \perp DF \implies AE \perp DF$. □

Problem (Andrews, NT, Problem 1-1.7)

Denote by F_n the n-th Fibonacci Number. Prove that

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

Proof. We will use induction.

Base Case: $n = 1$. Notice that $1 = F_1 = F_3 - 1 = 2 - 1$.

Induction Hypothesis: Assume our desired equation is true for all $n \leq k$. We will now show that our desired equation holds for $n = k + 1$. Note that

$$(F_1 + \cdots + F_k) + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1,$$

where the first equality holds by our Induction Hypothesis and the second holds by the definition of the Fibonacci Sequence. This concludes the proof. □

Problem (Andrews, NT, Problem 1-1.8)

Prove that
$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

Proof. We will use induction.

Base Case: $n = 1$. $F_1 = F_2 = 1$.

Induction Hypothesis: Assume our desired equation is true for all $n \leq k$. We will now show that it also holds for $n = k + 1$. Note that

$$(F_1 + F_3 + \cdots + F_{2k-1}) + F_{2(k+1)-1} = F_{2k} + F_{2k+1} = F_{2k+2} = F_{2(k+1)}.$$

Hence we are done. $\square$

Problem (Andrews, NT, Problem 1-1.17)

Prove that $n(n^2-1)(3n+2)$ is divisible by 24 for each positive integer n.

Proof. We will use induction.

Base Case: $n = 1$. Note that $(1)((1)^2-1)(3(1)+2) = 0$, which is divisible by 24.

Induction Hypothesis: Assume the problem statement is true for all $n \leq k$. We will now show it is true for $n = k + 1$. Note that

$$\begin{aligned}
&(k+1)((k+1)^2-1)(3(k+1)+2) \\
&= (k+1)(k^2+2k)(3k+5) \\
&= k(k+1)(k+2)(3k+5) \\
&= k(k+1)((k-1)+3)((3k+2)+3) \\
&= k(k+1)(k-1)(3k+2) + k(k+1) \cdot [3(3k+2) + 3(k-1) + 3(3)].
\end{aligned}$$

We know the first term of the RHS is divisible by 24 by our Induction Hypothesis. It suffices to show that $k(k+1) \cdot [3(3k+2) + 3(k-1) + 3(3)]$ is divisible by 24. It follows that

$$\begin{aligned}
&k(k+1) \cdot [3(3k+2) + 3(k-1) + 3(3)] \\
&= k(k+1)[9k + 6 + 3k - 3 + 9] \\
&= k(k+1)(12k+12) \\
&= 12k(k+1)(k+1).
\end{aligned}$$

It is obvious that one of $k, k+1$ is even. Hence we are done. $\square$

Problem (Andrews, NT, Problem 1-2.4)

Prove that each integer may be uniquely represented in the form

$$n = \sum_{j=0}^{s} c_j 3^j,$$

where $c_s \neq 0$, and each c_j is equal to $-1, 0$, or 1.

Proof. It is easy to show that every non-zero integer can be uniquely represented in base 3. To covert from base 3 to the representation described in the problem, replace $2 \cdot 3^k$ with $1 \cdot 3^{k+1} + (-1) \cdot 3^k$ until all coefficients are $-1, 0$, or 1. $\square$

Problem (Andrews, NT, Problem 2-1.4)

Any set of integers J that fulfills the following two conditions is called an *integral ideal:*

(i) if n and m are in J, then $n + m$ and $n - m$ are in J

(ii) if n is in J and r is an integer, then rn is in J.

Let $\mathcal{J}_m$ be the set of all integers that are integral multiples of a particular integer m. Prove that $\mathcal{J}_m$ is an integral ideal.

Proof. Note that criteria (i) is satified since for any two multiples of m (call them am and bm), $am + bm = (a + b)m$ and $am - bm = (a - b)m$ are in $\mathcal{J}_m$. Also note that criteria (ii) is satisfied since for any integer r and any element of the set $\mathcal{J}_m$ (call this element cm), $(rc)m$ is in $\mathcal{J}_m$. $\square$

Problem (Andrews, NT, Problem 2-1.5)

Prove that every integral ideal J is identical with $\mathcal{J}_m$ for some m.

Proof. If $J \neq \{0\} = \mathcal{J}_0$, then there exist non-zero integers in J, and with the right choice of r it is not hard to see that there must exist positive integers in J. By the well-ordering principle of the natural numbers, there must be a least positive integer in J, say m. We will now show that $J = \mathcal{J}_m$. It is clear that every multiple of m is also in J by the definition of integral ideals. Also note that m is not the multiple of any positive integer less than it since there are no positive integers in J that are less than m. Finally, no non-multiples of m can be in the set. Say there exists some element k in J such that $m < k$ and k is not a multiple of m. Then by Euclid's Division Lemma we have that $k = qm + r$ for some integers $q, r < m$. However, by the definition of integral ideals, qm is in J, so r must be in J, violating the minimality of m. Hence $J = \mathcal{J}_m$, as desired. $\square$

Problem (Andrews, NT, Problem 2-1.6)

Prove that if a and b are odd integers, then $a^2 - b^2$ is divisble by 8.

Proof. Let $a = 2k + 1$ and $b = 2l + 1$ for some integers k, l. Then

$$a^2 - b^2 = (2k+1)^2 - (2l+1)^2 = 4(k^2 + k - (l^2 + l)) = 4(k(k+1) - l(l+1)).$$

Notice that $k(k + 1)$ and $l(l + 1)$ are both even. Hence $a^2 - b^2$ is divisible by 8, as desired. $\square$

Problem (Andrews, NT, Problem 2-1.7)

Prove that if a is an odd integer, then $a^2 + (a + 2)^2 + (a + 4)^2 + 1$ is divisible by 12.

Proof. Let $a = 2k + 1$. Then

$$\begin{aligned} a^2 + (a+2)^2 + (a+4)^2 + 1 &= (2k+1)^2 + (2k+3)^2 + (2k+5)^2 + 1 \\ &= 4k^2 + 4k + 1 + 4k^2 + 12k + 9 + 4k^2 + 20k + 25 + 1 \\ &= 12k^2 + 36k + 36, \end{aligned}$$

which is obviously divisible by 12. $\square$

Problem (Andrews, NT, Problem 2-2.4)
Prove
$$\text{lcm}(a,b) = \frac{ab}{\gcd(a,b)}.$$

Proof. Let $a = kd$ and $b = ld$ where $d = \gcd(a,b)$. Then
$$\frac{ab}{\gcd(a,b)} = \frac{kd \cdot ld}{d} = kld.$$
Note that k and l are relatively prime. Also note that kld is a multiple of both a and b. We have not double-counted any divisors since the greatest common divisor of a and b appears only once, hence we are done. □

Problem (100 Geometry Problems #20, Sharygin 2014)
Let ABC be an isosceles triangle with base AB. Line ℓ touches its circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

Proof. Denote by H the orthocenter of $\triangle ABC$. Also, call the intersection of $\overline{CD}$ and $\overline{EH}$ point G.

Lemma 1: G lies on (ABC).

Proof. Note that B, E, G, D are concyclic since $\measuredangle GEB = \measuredangle GDB = 90°$. Thus
$$\measuredangle CGA = \measuredangle CGE = \measuredangle DGE = \measuredangle DBE = \measuredangle DBC = \measuredangle BAC = \measuredangle CBA.$$
Hence A, B, C, G are concylic, as desired. ■

Lemma 2: $\triangle CEH \sim \triangle CDB$.

Proof. It is well known that the reflection of an orthocenter of a triangle along one of its sides coincides with its circumcircle. Since G lies on $\overline{EH}$ and, by Lemma 1, G lies on (ABC), G is the reflection of H along CB, making $\triangle HCG$ isosceles. Note that $\measuredangle CHE = \measuredangle EGC = \measuredangle CBD$ and $\measuredangle CEH = \measuredangle CDB = 90°$, so $\triangle CEH \sim \triangle CDB$ by $AA \sim$. ■

The previous result shows that there exists a spiral similarity between $\triangle CEH$ and $\triangle CDB$ centered at C. Note that B is on $\overline{HF}$, so it follows that D is on $\overline{EF}$. Hence, we are done. □

Problem (100 Geometry Problems #21, Purple Comet 2013)

Two concentric circles have radii 1 and 4. Six congruent circles form a ring where each of the six circles is tangent to the two circles adjacent to it as shown. The three lightly shaded circles are internally tangent to the circle with radius 4 while the three darkly shaded circles are externally tangent to the circle with radius 1. The radius of the six congruent circles can be written $\frac{k+\sqrt{m}}{n}$, where k, m and n are integers with k and n relatively prime. Find $k + m + n$.

Solution. Number each of the six congruent circles from 1 to 6, with the first circle being lightly colored. We know that the centers of all six congruent circles $(O_1, O_2, \ldots, O_6)$ are evenly spaced around the center of the two concentric circles, O, by symmetry. That is, $\angle O_1OO_2 = 60°$. Denote by r the radius of each of the six circles. Note that $O_1O = 4 - r$, $O_2O = r + 1$, and $O_1O_2 = 2r$. Then by LoC on $\triangle O_1OO_2$ we have

$$\begin{aligned}
(2r)^2 &= (r+1)^2 + (4-r)^2 - 2(r+1)(4-r)\cos(60°) \\
(2r)^2 &= (r+1)^2 + (4-r)^2 - (r+1)(4-r) \\
4r^2 &= r^2 + 2r + 1 + 16 - 8r + r^2 - (3r + 4 - r^2) \\
0 &= r^2 + 9r - 13.
\end{aligned}$$

Using the quadratic formula gives us

$$r = \frac{-9 + \sqrt{133}}{2}.$$

Thus our final answer is $-9 + 133 + 2 = \boxed{126}$. □

Problem (100 Geometry Problems #22)

Let A, B, C, D be points in the plane such that $\angle BAC = \angle CBD$. Prove that the circumcircle of $\triangle ABC$ is tangent to BD.

Proof. Let D be on the tangent of B, and let D' be on (ABC) while on the same side of BC as D. Note that $\measuredangle BAC = \measuredangle BD'C$. As D' approaches B, line BD' approaches a tangent to (ABC), but our angle property does not change. Hence, in the limit, $\measuredangle BAC = \measuredangle CBD$. □

Problem (100 Geometry Problems #23, Britain 1995)

Triangle ABC has a right angle at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q respectively. The points M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

Solution. Note that $BCQN$ and $ACPM$ are cyclic quadrilaterals. Now consider $\triangle CMN$. We have

$$\angle CNM = 90^\circ - \angle QNC = 90^\circ - \angle QBC = 90^\circ - \frac{1}{2}\angle B.$$

Similarly, we have

$$\angle CMN = 90^\circ - \angle PMC = 90^\circ - \angle CAP = 90^\circ - \frac{1}{2}\angle A.$$

Considering $\triangle CMN$ as a whole, we have

$$\begin{aligned}\angle MCN &= 180^\circ - \angle CNM + \angle CMN \\ &= 180 - \left(90^\circ - \frac{1}{2}\angle B + 90^\circ - \frac{1}{2}\angle A\right) \\ &= \frac{1}{2}(\angle A + \angle B) \\ &= \frac{1}{2}\angle C \\ &= \boxed{45^\circ}.\end{aligned}$$

□

Problem (100 Geometry Problems #24)

Let $ABCD$ be a parallelogram with $\angle A$ obtuse, and let M and N be the feet of the perpendicualrs from A to sides BC and CD. Prove that $\triangle MAN \sim \triangle ABC$.

Proof. Since $\measuredangle AMC = \measuredangle ANC = 90^\circ, AMCN$ is a cyclic quadrilateral. Thus

$$\measuredangle AMN = \measuredangle ACN = \measuredangle CAB$$

where the second equality holds because $AB \parallel CD$. Also note that

$$\measuredangle BCA = \measuredangle MCA = \measuredangle MNA.$$

Therefore $\triangle MAN \sim \triangle ABC$ by $AA \sim$. □

Problem (100 Geometry Problems #25)

For a given triangle $\triangle ABC$, let H denote its orthocenter and O its circumcenter.

(a) Prove that $\angle HAB = \angle OAC$.

(b) Prove that $\angle HAO = |\angle B - \angle C|$.

Proof. We will prove part (a) first. Note that

$$\angle BAH = 90^\circ - \angle CBA = 90^\circ - \frac{1}{2}\angle COA = \angle OAC.$$

(This was Problem 4.23 of Evan Chen's EGMO).
We will now prove part (b). Note that $\angle BAH = 90^\circ - \angle B$ and $\angle CAH = 90^\circ - \angle C$. Finally

$$\angle HAO = |\angle CAH - \angle BAH| = |90^\circ - \angle C - (90^\circ - \angle B)| = |\angle B - \angle C|.$$

The absolute value symbols are required to account for cases where $\angle B < \angle C$. Hence, we are done. □

Problem (100 Geometry Problems #27, AMC 12A 2012)

Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13, OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

Solution. For ease of notation denote by O_1 the center of circle C_1 and by O_2 the center of circle C_2. Also denote by θ the measure of $\angle O_1YZ$. Note that O_1, X, Y, Z all lie on C_2. From this we know that $\angle O_1XZ = 180^\circ - \theta$. Also note that $O_1X = r = O_1Y$, where r is the radius of circle C_1. By LoC on $\triangle O_1YZ$ and $\triangle O_1XZ$ we have

$$\begin{cases} 7^2 + r^2 - 14r\cos\theta = 11^2 \\ 13^2 + r^2 - 26r\cos(180^\circ - \theta) = 11^2. \end{cases}$$

After noticing that $\cos(180^\circ - \theta) = -\cos\theta$, we can solve for $r\cos\theta$ and set the two equations equal to each other, leaving us with

$$\frac{r^2 + 7^2 - 11^2}{14} = \frac{r^2 + 13^2 - 11^2}{-26}.$$

Solving for r gives us a final answer of $\boxed{\sqrt{30}}$. □

Problem (100 Geometry Problems #28)

Let $ABCD$ be a cyclic quadrilateral with no two sides parallel. Lines AD and BC (extended) meet at K, and AB and CD (extended) meet at M. The angle bisector of $\angle DKC$ intersects CD and AB at points E and F, respectively; the angle bisector of $\angle CMB$ intersects BC and AD at points G and H, respectively. Prove that quadrilateral $EGFH$ is a rhombus.

Proof. Let $\alpha = \angle AKF = \angle FKB$ and $\beta = \angle BMG = \angle GMC$. Call the intersection of EF and GH point X. Note that $\angle AFK = 180° - (A + \alpha)$, so $\angle KFB = A + \alpha$, then $\angle MXF = 180° - (A + \alpha + \beta)$. Now note that since $180° - C = A$, $\angle KEC = 180° - (\alpha + A)$, so $\angle XEC = A + \alpha$. Then $\angle MXE = 180° - (A + \alpha + \beta) = \angle MXF$, and since E, X, F are collinear, $\angle MXE = \angle MXF = 90°$. Hence the diagonals of quadrilateral $EGFH$ are perpendicular. Also note that $\triangle MXF \cong \triangle MXE$ by $ASA \cong$, so it follows that $XF = XE$. A very similar process can be done to determine that $XG = XH$, hence $EGFH$ is a rhombus, and we are done. $\square$