Euclidean Geometry in Mathematical Olympiads
Solutions

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Chapter 1

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Chapter 2

Lemma 2.19. Prove that the A-exradius has length

$$r_a = \frac{s}{s-a}r,$$

where r is the inradius.

Proof. Drop perpendiculars from I and I_A to AB. Call the feet of these perpendiculars B_1 and B_2 respectively. Notice that $IB_1 = r$ and $I_AB_2 = r_a$ and that $\triangle AB_1I \sim \triangle AB_2I_A$. Therefore

$$\frac{r}{r_a} = \frac{AB_1}{AB_2},$$

but by Lemmas 2.15 and 2.17, we know that $AB_1 = s - a$ and $AB_2 = s$, hence

$$r_a = \frac{s}{s-a}r,$$

and we are done. □

Lemma 2.20. Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to BC at D and X, respectively. Show that $BD = CX$ and $BX = CD$.

Proof. We will first show that $BD = CX$. Let the incircle be tangent to side AB at point F and let to side AC at point E. Let the A-excircle be tangent to the extension of line AC at C_1 and to the extension of line AB at B_1. Then

$$\begin{aligned}
BD &= BF \\
&= AB_1 - AF - BB_1 \\
&= (AC_1 - AE) - BX \\
&= (CC_1 + CE) - (BC - CX) \\
&= CX + (CD - BC) + CX \\
&= 2CX - BD \\
2BD &= 2CX \to BD = CX.
\end{aligned}$$

It follows that $BX = CD$ because

$$\begin{aligned} BD &= CX \\ BD + DX &= DX + CX \\ BX &= CD. \end{aligned}$$

$\square$

Lemma 2.24. Let ABC be a triangle with I_A, I_B, and I_C as excenters. Prove that triangle $I_A I_B I_C$ has orthocenter I and that triangle ABC is its orthic triangle.

Proof. By the Incenter-Excenter Lemma, we know that AI_A, BI_B, and CI_C coincide at the incenter I. We also know from the Lemma that II_A is the diameter of circle $BICI_A$. Therefore we have that

$$\angle I_C C I_A = \angle I C I_A = 90^\circ \text{ and } \angle I_B B I_A = \angle I B I_A = 90^\circ.$$

This follows similarly for II_B and II_C. Now we know that AI_A, BI_B, CI_C are in fact the altitudes of triangle $I_A I_B I_C$, therefore I is the orthocenter of triangle $I_A I_B I_C$. Note that since A, B, and C are the feet of the altitudes, ABC is the orthic triangle of triangle $I_A I_B I_C$. $\square$

Theorem 2.25 (The Pitot Theorem). Let $ABCD$ be a quadrilateral. If a circle can be inscribed in it, prove that $AB + CD = BC + DA$.

Proof. Call the points where AB, BC, CD, DA are tangent to the circle E, F, G, H, respectively. Let $AE = AH = a, BE = BF = b, CF = CG = c, DG = DH = d$. Now note that our condition can be manipulated as follows:

$$\begin{aligned} AB + CD &= BC + DA \\ (AE + BE) + (CG + DG) &= (BF + CF) + (AH + DH) \\ a + b + c + d &= b + c + a + d. \end{aligned}$$

Hence, we are done. $\square$

Problem 2.26 (USAMO 1990/5). An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

Proof. Let the circle with diameter $\overline{AB}$ be called ω_1 and the circle with diameter $\overline{AC}$ be called ω_2. By Theorem 2.9, it suffices to show that the intersection of $\overline{MN}$ and $\overline{PQ}$ lies on the radical axis of ω_1 and ω_2. Since $\overline{MN}$ and $\overline{PQ}$ are altitudes of $\triangle ABC$, their intersection is the orthocenter of $\triangle ABC$. We will call this point H. Note that $\overline{AH}$ is the third altitude of $\triangle ABC$. Call the foot of this altitude A'. Now note that $\angle AA'B = \angle AA'C = 90°$, and since $\overline{AB}$ and $\overline{AC}$ are diameters of their respective circles, ω_1 and ω_2 must intersect at A'. Hence, $\overline{AA'}$ is the radical axis of ω_1 and ω_2, and since A, A', H are colinear, H lies on this line. □

Problem 2.27 (BAMO 2012/4). Given a segment $\overline{AB}$ in the plane, choose on it a point M different from A and B. Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment $\overline{AB}$. The circumcircles of the two triangles intersect in point M and another point N.

(a) Prove that $\overline{AD}$ and $\overline{BC}$ pass through point N.

(b) Prove that no matter where one chooses point M along segment $\overline{AB}$, all lines MN will pass through some fixed point K in the plane.

Proof. We will prove (a) by angle chasing. Notice that since $ACNM$ and $BDNM$ are cyclic, we have that

$$\angle AMC = \angle ANC = \angle ACM = \angle ANM = \angle MDB = \angle MNB = 60°,$$

and since $\angle ANC + \angle ANM + \angle MNB = 60° + 60° + 60° = 180°$, we have that BC is a straight line passing through N. A very similar argument follows for AD.

We will now prove (b) using radical axes. First, construct an equilateral triangle ABE on the same side as the other two equilateral triangles. Let the circumcircles around triangles AMC, BMD, and ABE be ω_1, ω_2, and ω_3, respectively. Note that MN is the radical axis of circles ω_1 and ω_2, the line tangent to circles ω_1 and ω_3 at point A is the radical axis of circles ω_1 and ω_3, and the line tangent to circles ω_2 and ω_3 at point B is the radical axis of circles ω_2 and ω_3. Since the centers of ω_1, ω_2, and ω_3 are not colinear, their radical axes (one of which is MN) must coincide at the radical center K. Since changing the location of M on AB does not change the tangents at A and B, the point K does not move, hence all possible lines MN must pass through K. □

Problem 2.28 (JMO 2012/1). Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Proof. Since $\angle BPS = \angle PRS$ by the Tangent Criterion, $\overline{AB}$ is tangent to (PRS). Likewise we have that $\overline{AC}$ is tangent to (QRS). Suppose (PRS) and (QRS) are not the same circle. Then since $AP = AQ$ are both tangents to their respective circles, A must lie on the radical axis $\overline{BC}$, but since ABC is a triangle, this is obviously impossible. Hence P, Q, R, S are concyclic. □

Problem 2.29 (IMO 2008/1). Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of $\overline{BC}$ and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points

B_1, B_2, C_1, and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.

Proof. We will first show that B_1, B_2, C_1, C_2 are concyclic. Since $\Gamma_A, \Gamma_B, \Gamma_C$ all intersect at H, H is the radical center. We claim that $\overline{AH}$ is the radical axis of Γ_B and Γ_C. By similar triangles, $M_B M_C$ is parallel to BC, and since $\overline{AH} \perp BC$, $\overline{AH} \perp M_B M_C$. The centers of circles Γ_B and Γ_C are M_B and M_C, respectively, thus $\overline{AH}$ is the radical axis of circles Γ_B and Γ_C. Since $\overline{B_1 B_2}$ and $\overline{C_1 C_2}$ intersect at A, by Theorem 2.9 we have shown that B_1, B_2, C_1, C_2 are concyclic. Note that the circumcenter of $(B_1 B_2 C_1 C_2)$ is the intersection of the perpendicular bisectors of $B_1 B_2$ and $C_1 C_2$, which is the orthocenter O of triangle ABC. Thus what we have proven is that $OB_1 = OB_2 = OC_1 = OC_2$. A similar argument can be persued for OA_1 and OA_2, hence we are done. □

Problem 2.30 (USAMO 1997/2). Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$ respectively are concurrent.

Proof. Consider the circles with centers D, E, F with chords BC, CA, AB, respectively. Note that the radical axes of these three circles are the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$, and since the centers of these three circles are not colinear, their radical axes must intersect at a point. □

(These centers can be colinear, but we won't talk about that)

Problem 2.31 (IMO 1995/1). Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Proof. Since P lies on the radical axis of these two circles, and $\overline{BN} \cap \overline{CM} = P$, $MNBC$ is cyclic by Theorem 2.9. (Reminder that the symbol $\measuredangle$ denotes the directed angle.) Note that

$$\measuredangle NMC = \measuredangle NBC = \measuredangle NBD = 90^\circ - \measuredangle BDN = 90^\circ - \measuredangle ADN,$$

so

$$\measuredangle NMA = \measuredangle NMC - 90^\circ = (90^\circ - \measuredangle ADN) - 90^\circ = -\measuredangle ADN = \measuredangle NDA,$$

therefore quadrilateral $DAMN$ is cyclic. The radical axes of the circles $(DAMN), (AMC)$, and (BND) are $\overline{AM}, \overline{DN}, \overline{XY}$, and since the centers of these circles are never colinear, they must intersect at the radical center. □

Problem 2.32 (USAMO 1998/2). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent $\overline{AB}$ to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and $\mathcal{C}_1$, and let D be the midpoint of $\overline{AB}$. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

Proof. **INCOMPLETE** Note that $CDEF$ is cyclic (need to prove). M is the center of circle $(CDEF)$. Thus $CM = DM$. $\square$

Chapter 3

Theorem 3.2 (Angle Bisector Theorem). Let ABC be a triangle and D a point on $\overline{BC}$ so that $\overline{AD}$ is the internal angle bisector of $\angle BAC$. Show that

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

Proof. Let $\angle BAD = \alpha = \angle CAD$ and $\angle ADB = \beta$. Note that $\angle ADC = 180^\circ - \beta$. By Law of Sines, we have

$$\frac{DB}{\sin\alpha} = \frac{AB}{\sin\beta} \text{ and } \frac{DC}{\sin\alpha} = \frac{AC}{\sin(180^\circ - \beta)}.$$

Note that $\sin(180^\circ - \beta) = \sin\beta$. Rearranging terms, we have that

$$\frac{\sin\beta}{\sin\alpha} = \frac{AB}{BD} = \frac{AC}{CD}.$$

It follows that $\frac{AB}{AC} = \frac{DB}{DC}$. $\square$

Problem 3.5. Show the trigonometric form of Ceva holds.

Proof. Recall that the trigonometric from of Ceva's Theorem is as follows:

Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be cevians of a triangle ABC. They concur if and only if

$$\frac{\sin\angle BAX \sin\angle CBY \sin\angle ACZ}{\sin\angle XAC \sin\angle YBA \sin\angle ZCB} = 1.$$

By the Law of Sines, we have that

$$\frac{\sin\angle BAX}{BX} = \frac{\sin B}{AX}$$

and

$$\frac{\sin\angle XAC}{XC} = \frac{\sin C}{AX}.$$

Combining these two equations gives us

$$AX = \frac{BX\sin B}{\sin\angle BAX} = \frac{XC\sin C}{\sin\angle XAC} \Rightarrow \frac{\sin\angle BAX}{\sin\angle XAC} = \frac{BX}{XC}\cdot\frac{\sin C}{\sin B}.$$

Similarly, we have that

$$\frac{\sin \angle CBY}{\sin \angle YBA} = \frac{CY}{YA} \cdot \frac{\sin A}{\sin C}$$

and

$$\frac{\sin \angle ACZ}{\sin \angle ZCB} = \frac{AZ}{ZB} \cdot \frac{\sin B}{\sin A}.$$

Plugging these values into the original equation, we have that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,$$

and we know this is true from the original statement of Ceva's Theorem. □

Problem 3.6. Let $\overline{AM}, \overline{BE}$, and $\overline{CF}$ be concurrent cevians of a triangle ABC. Show that $\overline{EF} \parallel \overline{BC}$ if and only if $BM = MC$.

Proof. Suppose $\overline{EF} \parallel \overline{BC}$. Call the point where $\overline{AM}$ intersects $\overline{EF}$ point Q. Notice that $\triangle BPM \sim \triangle EPQ$ and $\triangle CPM \sim \triangle FPQ$. Thus we have the following relationship:

$$\frac{BM}{EQ} = \frac{MP}{QP} = \frac{CM}{FQ}.$$

Now also notice that $\triangle BAM \sim \triangle FAQ$ and $\triangle CAM \sim \triangle EAQ$. Thus we have the following relationship:

$$\frac{BM}{FQ} = \frac{MA}{QA} = \frac{CM}{EQ}.$$

Putting these two relationships together, it follows that $BM = CM$.

We will now prove the other direction. Suppose $BM = MC$. Then by Ceva's Theorem we have that

$$\begin{aligned}
&\frac{CE}{AE} = \frac{BF}{AF} \\
&\frac{CE}{BF} = \frac{AE}{AF} = \frac{CE + AE}{BF + AF} = \frac{AC}{AB} \\
&\frac{AE}{AC} = \frac{AF}{AB}.
\end{aligned}$$

Since $\angle FAE = \angle BAC$, we have that $\triangle FAE \sim \triangle BAC$. Thus $\angle AEF = \angle ACB$, therefore $\overline{EF} \parallel \overline{BC}$. □

Problem 3.12. Give an alternative proof of Lemma 3.9 by taking a negative homothety.

Proof. Consider a homothety centered at G with $M = h(A), N = h(B), L = h(C)$. Note that $\triangle ACB \sim \triangle NCM$ by midpoints and that $\triangle ALG \sim \triangle Mh(L)G$ by homothety. Also notice that $h(L)$ is the midpoint of NM. Since $AB/NM = 2/1$,

$$\frac{AB}{NM} = \frac{AL}{Mh(L)} = \frac{AG}{MG} = \frac{2}{1}.$$

□

Lemma 3.13 (Euler Line). In triangle ABC, prove that O, G, H (with their usual meanings) are collinear and that G divides $\overline{OH}$ in a $2:1$ ratio.

Proof. We will first show that O, G, H are collinear. Call the point where the perpendicular from O meets $\overline{BC}, \overline{CA}, \overline{AB}$ points A', B', C', respectively. Since $\overline{BC}, \overline{CA}, \overline{AB}$ are chords of the circle (ABC), points A', B', C' are in fact the midpoints of their respective line segments. Thus A' lies on $\overline{AG}$, B' lies on $\overline{BG}$, and C' lies on $\overline{CG}$. Now notice that $\overline{AH} \parallel \overline{OA'}, \overline{BH} \parallel \overline{OB'}, \overline{CH} \parallel \overline{OC'}$ since they are all perpendicular to some side of the triangle ABC. Thus, a homothety h centered at G exists such that $h(A) = A', h(B) = B', h(C) = C'$. Thus, $h(O) = H$, so O, G, H are collinear.

We will now show that G divides $\overline{OH}$ in a $2:1$ ratio. This is equivalent to showing that the homothety h must have a scale factor $k = -2$. From Lemma 3.9 (Centroid Division) we have that $AG/GA' = 2/1$. Since G lies in between A and A', we have that $k = -2$, as desired. (!!!) □

Problem 3.16. Let ABC be a triangle with contact triangle DEF. Prove that $\overline{AD}, \overline{BE}, \overline{CF}$ concur. The point of concurrency is the Gergonne point of triangle ABC.

Proof. Notice by Lemma 2.15 we have that

$$\begin{aligned} AE = AF &= s - a \\ BD = BF &= s - b \\ CD = CE &= s - c. \end{aligned}$$

Thus, by Ceva's Theorem, we have that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1.$$

□

Lemma 3.17. In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.

Proof. Reflect X and Y across $\overline{BC}$ and call these points X' and Y' respectively. Notice that X' and Y' lie on $(ABCD)$. Thus $ADX'Y'$ is a cyclic quadrilateral. Then we have that

$$\measuredangle AXY = \measuredangle X'XY = \measuredangle Y'X'X = \measuredangle Y'X'A = \measuredangle Y'DA = \measuredangle YDA.$$

Similarly, we have that $\measuredangle DAX = \measuredangle XYD$. Hence $AXYD$ is a parallelogram. □

Problem 3.18. Let $\overline{AD}, \overline{BE}, \overline{CF}$ be concurrent cevians in a triangle, meeting at P. Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

Proof. By Area Ratios, we can transform each term in our desired equation as follows:

$$\begin{aligned}\frac{PD}{AD} &= \frac{[BPC]}{[BAC]},\\ \frac{PE}{BE} &= \frac{[CPA]}{[CBA]},\\ \frac{PF}{CF} &= \frac{[APB]}{[ACB]}.\end{aligned}$$

Therefore our desired equation turns into

$$\frac{[BPC]}{[BAC]} + \frac{[CPA]}{[CBA]} + \frac{[APB]}{[ACB]} = 1.$$

Notice that $[BPC] + [CPA] + [APB] = [ABC]$. Hence we are done. □

Problem 3.19 (Shortlist 2006/G3). Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \text{ and } \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P. Prove that ray AP bisects $\overline{CD}$.

Proof. Let B' be intersection of diagonals AC and BD, and let E' be the intersection of diagonals AD and CE. Also let A' be the intersection of ray AP with CD. Notice that the given angle conditions imply that $\triangle ABC \sim \triangle ACD \sim \triangle ADE$. From this it follows that quadrilaterals $ABCD$ and $ACDE$ are similar. Since B' and E' are the intersections of the diagonals of their respective quadrilaterals, we have that $\frac{CB'}{B'A} = \frac{DE'}{E'A}$. By Ceva's on $\triangle ACD$, we have that

$$\frac{AE'}{E'D} \cdot \frac{DA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

Since $\frac{CB'}{B'A} \cdot \frac{AE'}{E'D} = 1$, we have that $DA' = A'C$. $\square$

Problem 3.20 (BAMO 2013/3). Let H be the orthocenter of an acute triangle ABC. Consider the circumcenters of triangles ABH, BCH, and CAH. Prove that they are the vertices of a triangle that is congruent to ABC.

Proof. Let A', B', C' be the circumcenters of $(BCH), (CAH), (ABH)$, respectively. Note that H is the radical center of $(ABH), (BCH), (CAH)$. Thus $\overline{AH} \perp \overline{B'C'}$. Also notice by properties of circumcenters, A' is on the perpendicular bisector of $\overline{BC}$. Let O be where the perpendicular bisectors of $\triangle ABC$ intersect (namely, the circumcenter of $\triangle ABC$). Since $\overline{A'O} \parallel \overline{AH}$, $\overline{A'O} \perp \overline{B'C'}$. This follows similarly for B' and C', hence O is the orthocenter of $\triangle A'B'C'$. Also notice that, by construction, H is the circumcenter of $\triangle A'B'C'$. Therefore, a homothety of scale factor -1 exists that sends H to O, A to A', B to B', and C to C'. Hence, $\triangle ABC \cong \triangle A'B'C'$. $\square$