CNCM Problem of the Day Solutions

Ryder Pham

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6 August 2021

Notice that the expected number of dollars Tommy expects to win is equivalent to the following infinite series:

$$\frac{1}{6}\sum_{n=0}^{\infty} n^2 \left(\frac{5}{6}\right)^n .$$

Define

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n$$

where we want to find the value of $f(5/6)$. Then

$$\begin{aligned}
f(x) &= x\sum_{n=0}^{\infty} n^2 x^{n-1} \\
&= x\frac{d}{dx}\left[\sum_{n=0}^{\infty} n x^n\right] \\
&= x\frac{d}{dx}\left[x\sum_{n=0}^{\infty} n x^{n-1}\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\frac{1}{1-x}\right]\right] \\
&= x\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] \\
&= x\left[\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}\right] \\
&= x\left[\frac{(1-x) + 2x}{(1-x)^3}\right] \\
&= x\left[\frac{1+x}{(1-x)^3}\right] \\
f(5/6) &= \frac{5}{6}\left[\frac{1+5/6}{(1-5/6)^3}\right] \\
&= \frac{5}{6}\cdot\frac{11}{6}\cdot\frac{6^3}{1} \\
&= 330.
\end{aligned}$$

Then our final answer is $\frac{1}{6}f(\frac{5}{6}) = \frac{330}{6} = \boxed{55}$.

11 August 2021

Here let R_n denote the remaining water after the n-th pour.

$$\begin{aligned} R_0 &= 1 \\ R_1 &= \left(1 - \frac{1}{2}\right) R_0 = \frac{1}{2} \\ R_2 &= \left(1 - \frac{1}{3}\right) R_1 = \frac{1}{3} \\ R_3 &= \left(1 - \frac{1}{4}\right) R_2 = \frac{1}{4} \end{aligned}$$

Therefore we can assume by Engineer's Induction that $R_n = \frac{1}{n+1}$. Hence $R_9 = \frac{1}{10}$ for a final answer of $\boxed{9}$.

12 August 2021

For a two-game block, there is a probability of $\dfrac{3}{4} \cdot \dfrac{1}{4} + \dfrac{1}{4} \cdot \dfrac{3}{4} = \dfrac{3}{8}$ of the entire match ending then and there. The only other outcome after two games is a tie, since each game must declare a winner, and this happens with probability 5/8. Thus the expected number of games in a match is the following:

$$\begin{aligned} \frac{3}{8} \cdot 2 + \frac{3}{8} \cdot \frac{5}{8} \cdot 4 + \frac{3}{8} \cdot \left(\frac{5}{8}\right)^2 \cdot 6 + \cdots &= \sum_{n=0}^{\infty} 2(n+1) \cdot \frac{3}{8} \cdot \left(\frac{5}{8}\right)^n \\ &= \frac{3}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{5}{8}\right)^n \\ &= \frac{3}{4} \sum_{n=0}^{\infty} n \left(\frac{5}{8}\right)^n + \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n. \end{aligned}$$

Define $f(x) = \sum_{n=0}^{\infty} nx^n$. We would like to find the value of $f(5/8)$. Note that

$$\begin{aligned} f(x) &= x\sum_{n=0}^{\infty} nx^{n-1} \\ &= x\frac{d}{dx}\sum_{n=0}^{\infty} x^n \\ &= x\frac{d}{dx}\left[\frac{1}{1-x}\right] \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

Thus $f(5/8) = \dfrac{5/8}{(1-5/8)^2} = \dfrac{5}{8}\cdot\dfrac{8^2}{3^2} = \dfrac{40}{9}$. Then our original sum becomes

$$\begin{aligned} \frac{3}{4}\sum_{n=0}^{\infty} n\left(\frac{5}{8}\right)^n + \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n &= \frac{3}{4}\cdot\frac{40}{9} + \frac{3}{4}\cdot\frac{1}{1-5/8} \\ &= \frac{10}{3} + \frac{3}{4}\cdot\frac{8}{3} \\ &= \frac{10}{3} + 2 \\ &= \frac{16}{3}. \end{aligned}$$

Our final answer is $160 + 3 = \boxed{163}$.

26 August 2021

Our recurrence relation is

$$7a_n = -a_{n-1} + 8a_{n-2}.$$

By simple calulations we determine that $a_1 = 25$. Note that the recurrence is linear and homogenous. Its characteristic equation is

$$\begin{aligned} 7r^2 + r - 8 &= 0 \\ (7r+8)(r-1) &= 0 \\ r_{1,2} &= -\frac{8}{7}, 1. \end{aligned}$$

So by some theorem (idk) $a_n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2(1)^n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2$ is a solution. To find α_1, α_2 we must solve the following system:

$$\begin{cases} a_0 = \alpha_1 + \alpha_2 = 4 \\ a_1 = -\frac{8}{7}\alpha_1 + \alpha_2 = 25. \end{cases}$$

Solving this gets us $(\alpha_1, \alpha_2) = (-49/5, 69/5)$. Thus what we have left to evaluate is

$$\begin{aligned} a_7 &= -\frac{49}{5}\left(-\frac{8}{7}\right)^7 + \frac{69}{5} \\ &= \frac{49}{5}\left(\frac{8}{7}\right)^7 + \frac{69}{5} \\ &= \frac{1}{5} \cdot \frac{8^7}{7^5} + \frac{69}{5} \\ &= \frac{8^7 + 69 \cdot 7^5}{5 \cdot 7^5} \\ &= \frac{8^7 + 69 \cdot 16807}{5 \cdot 7^5} \\ &= \frac{2097152 + 69 \cdot 16807}{5 \cdot 7^5} \\ &= \frac{2097152 + 1159683}{5 \cdot 7^5} \\ &= \frac{3256835}{5 \cdot 7^5} \\ &= \frac{651367}{16807}. \end{aligned}$$

Therefore our final answer is $651367 + 16807 + 28795 = \boxed{696969}$.

2 September 2021

Let $BD = x$. By the Angle Bisector Theorem

$$\frac{8}{12} = \frac{x}{10 - x}.$$

Solving for x gives us $x = 4$. Thus $BD = 4$ and $CD = 6$. By Stewart's Theorem on $\triangle ABC$ we have

$$\begin{aligned} b^2m + c^2n &= a(d^2 + mn) \\ 8^2 \cdot 6 + 12^2 \cdot 4 &= 10(d^2 + 6 \cdot 4) \\ 384 + 576 &= 10d^2 + 240 \\ d^2 &= 72 \\ AD &= 6\sqrt{2}. \end{aligned}$$

We will now find BD'. Applying Stewart's Theorem again on $\triangle ABD'$ gives us

$$\begin{aligned} 8^2 \cdot 2\sqrt{2} + c^2 \cdot 6\sqrt{2} &= 8\sqrt{2}(4^2 + 24) \\ 128 + 6c^2 &= 128 + 192 \\ c^2 &= 32 \\ BD' &= 4\sqrt{2}. \end{aligned}$$

Similarly to find CD' we apply Stewart's Theorem a third time on $\triangle ACD'$, which gives us

$$\begin{aligned} b^2m + c^2n &= a(d^2 + mn) \\ 12^2 \cdot 2\sqrt{2} + c^2 \cdot 6\sqrt{2} &= 8\sqrt{2}(6^2 + 24) \\ 288 + 6c^2 &= 288 + 192 \\ c^2 &= 32 \\ CD' &= 4\sqrt{2}. \end{aligned}$$

Thus $BD' \cdot CD' = (4\sqrt{2})^2 = \boxed{32}$.