

Euclidean Geometry in Mathematical Olympiads Solutions

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Chapter 1

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Chapter 2

Lemma 2.19. Prove that the A-exradius has length

$$r_a = \frac{s}{s-a}r,$$

where r is the inradius.

Proof. Drop perpendiculars from I and I_A to AB . Call the feet of these perpendiculars B_1 and B_2 respectively. Notice that $IB_1 = r$ and $I_AB_2 = r_a$ and that $\triangle AB_1I \sim \triangle AB_2I_A$. Therefore

$$\frac{r}{r_a} = \frac{AB_1}{AB_2},$$

but by Lemmas 2.15 and 2.17, we know that $AB_1 = s - a$ and $AB_2 = s$, hence

$$r_a = \frac{s}{s-a}r,$$

and we are done. \square

Lemma 2.20. Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to BC at D and X , respectively. Show that $BD = CX$ and $BX = CD$.

Proof. We will first show that $BD = CX$. Let the incircle be tangent to side AB at point F and let to side AC at point E . Let the A-excircle be tangent to the extension of line AC at C_1 and to the extension of line AB at B_1 . Then

$$\begin{aligned} BD &= BF \\ &= AB_1 - AF - BB_1 \\ &= (AC_1 - AE) - BX \\ &= (CC_1 + CE) - (BC - CX) \\ &= CX + (CD - BC) + CX \\ &= 2CX - BD \\ 2BD &= 2CX \rightarrow BD = CX. \end{aligned}$$

It follows that $BX = CD$ because

$$\begin{aligned} BD &= CX \\ BD + DX &= DX + CX \\ BX &= CD. \end{aligned}$$

□

Lemma 2.24. Let ABC be a triangle with I_A, I_B , and I_C as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.

Proof. By the Incenter-Excenter Lemma, we know that AI_A, BI_B , and CI_C coincide at the incenter I . We also know from the Lemma that II_A is the diameter of circle $BICI_A$. Therefore we have that

$$\angle I_CCI_A = \angle ICI_A = 90^\circ \text{ and } \angle I_BBI_A = \angle IBI_A = 90^\circ.$$

This follows similarly for II_B and II_C . Now we know that AI_A, BI_B, CI_C are in fact the altitudes of triangle $I_AI_BI_C$, therefore I is the orthocenter of triangle $I_AI_BI_C$. Note that since A, B , and C are the feet of the altitudes, ABC is the orthic triangle of triangle $I_AI_BI_C$. □

Theorem 2.25 (The Pitot Theorem). Let $ABCD$ be a quadrilateral. If a circle can be inscribed in it, prove that $AB + CD = BC + DA$.

Proof. Call the points where AB, BC, CD, DA are tangent to the circle E, F, G, H , respectively. Let $AE = AH = a, BE = BF = b, CF = CG = c, DG = DH = d$. Now note that our condition can be manipulated as follows:

$$\begin{aligned} AB + CD &= BC + DA \\ (AE + BE) + (CG + DG) &= (BF + CF) + (AH + DH) \\ a + b + c + d &= b + c + a + d. \end{aligned}$$

Hence, we are done. □

Problem 2.26 (USAMO 1990/5). An acute-angled triangle ABC is given in the plane. The circle with diameter \overline{AB} intersects altitude $\overline{CC'}$ and its extension at points M and N , and the circle with diameter \overline{AC} intersects altitude \overline{BB} and its extensions at P and Q . Prove that the points M, N, P, Q lie on a common circle.

Proof. Let the circle with diameter \overline{AB} be called ω_1 and the circle with diameter \overline{AC} be called ω_2 . By Theorem 2.9, it suffices to show that the intersection of \overline{MN} and \overline{PQ} lies on the radical axis of ω_1 and ω_2 . Since \overline{MN} and \overline{PQ} are altitudes of $\triangle ABC$, their intersection is the orthocenter of $\triangle ABC$. We will call

this point H . Note that \overline{AH} is the third altitude of $\triangle ABC$. Call the foot of this altitude A' . Now note that $\angle AA'B = \angle AA'C = 90^\circ$, and since \overline{AB} and \overline{AC} are diameters of their respective circles, ω_1 and ω_2 must intersect at A' . Hence, $\overline{AA'}$ is the radical axis of ω_1 and ω_2 , and since A, A', H are colinear, H lies on this line. \square

Problem 2.27 (BAMO 2012/4). Given a segment \overline{AB} in the plane, choose on it a point M different from A and B . Two equilateral triangles AMC and BMD in the plane are constructed on the same side of segment \overline{AB} . The circumcircles of the two triangles intersect in point M and another point N .

(a) Prove that \overline{AD} and \overline{BC} pass through point N .

(b) Prove that no matter where one chooses point M along segment \overline{AB} , all lines MN will pass through some fixed point K in the plane.

Proof. We will prove (a) by angle chasing. Notice that since $ACNM$ and $BDNM$ are cyclic, we have that

$$\angle AMC = \angle ANC = \angle ACM = \angle ANM = \angle MDB = \angle MNB = 60^\circ,$$

and since $\angle ANC + \angle ANM + \angle MNB = 60^\circ + 60^\circ + 60^\circ = 180^\circ$, we have that BC is a straight line passing through N . A very similar argument follows for AD .

We will now prove (b) using radical axes. First, construct an equilateral triangle ABE on the same side as the other two equilateral triangles. Let the circumcircles around triangles AMC, BMD , and ABE be ω_1, ω_2 , and ω_3 , respectively. Note that MN is the radical axis of circles ω_1 and ω_2 , the line tangent to circles ω_1 and ω_3 at point A is the radical axis of circles ω_1 and ω_3 , and the line tangent to circles ω_2 and ω_3 at point B is the radical axis of circles ω_2 and ω_3 . Since the centers of ω_1, ω_2 , and ω_3 are not colinear, their radical axes (one of which is MN) must coincide at the radical center K . Since changing the location of M on AB does not change the tangents at A and B , the point K does not move, hence all possible lines MN must pass through K . \square

Problem 2.28 (JMO 2012/1). Given a triangle ABC , let P and Q be points on segments \overline{AB} and \overline{AC} , respectively, such that $AP = AQ$. Let S and R be distinct points on segment \overline{BC} such that S lies between B and R , $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Proof. Since $\angle BPS = \angle PRS$ by the Tangent Criterion, \overline{AB} is tangent to (PRS) . Likewise we have that \overline{AC} is tangent to (QRS) . Suppose (PRS) and (QRS) are not the same circle. Then since $AP = AQ$ are both tangents to their respective circles, A must lie on the radical axis \overline{BC} , but since ABC is a triangle, this is obviously impossible. Hence P, Q, R, S are concyclic. \square

Problem 2.29 (IMO 2008/1). Let H be the orthocenter of an acute-angled triangle ABC . The circle Γ_A centered at the midpoint of \overline{BC} and passing through H intersects the sideline BC at points A_1 and A_2 . Similarly, define the points

B_1, B_2, C_1 , and C_2 . Prove that six points A_1, A_2, B_1, B_2, C_1 , and C_2 are concyclic.

Proof. We will first show that B_1, B_2, C_1, C_2 are concyclic. Since $\Gamma_A, \Gamma_B, \Gamma_C$ all intersect at H , H is the radical center. We claim that \overline{AH} is the radical axis of Γ_B and Γ_C . By similar triangles, $M_B M_C$ is parallel to BC , and since $\overline{AH} \perp BC$, $\overline{AH} \perp M_B M_C$. The centers of circles Γ_B and Γ_C are M_B and M_C , respectively, thus \overline{AH} is the radical axis of circles Γ_B and Γ_C . Since $\overline{B_1 B_2}$ and $\overline{C_1 C_2}$ intersect at A , by Theorem 2.9 we have shown that B_1, B_2, C_1, C_2 are concyclic. Note that the circumcenter of $(B_1 B_2 C_1 C_2)$ is the intersection of the perpendicular bisectors of $B_1 B_2$ and $C_1 C_2$, which is the orthocenter O of triangle ABC . Thus what we have proven is that $OB_1 = OB_2 = OC_1 = OC_2$. A similar argument can be pursued for OA_1 and OA_2 , hence we are done. \square

Problem 2.30 (USAMO 1997/2). Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$ respectively are concurrent.

Proof. Consider the circles with centers D, E, F with chords BC, CA, AB , respectively. Note that the radical axes of these three circles are the lines through A, B, C perpendicular to $\overline{EF}, \overline{FD}, \overline{DE}$, and since the centers of these three circles are not colinear, their radical axes must intersect at a point. \square

(These centers can be colinear, but we won't talk about that)

Problem 2.31 (IMO 1995/1). Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters \overline{AC} and \overline{BD} intersect at X and Y . The line XY meets \overline{BC} at Z . Let P be a point on the line XY other than Z . The line CP intersects the circle with diameter AC at C and M , and the line BP intersects the circle with diameter BD at B and N . Prove that the lines AM, DN, XY are concurrent.

Proof. Since P lies on the radical axis of these two circles, and $\overline{BN} \cap \overline{CM} = P$, $MNBC$ is cyclic by Theorem 2.9. (Reminder that the symbol \angle denotes the directed angle.) Note that

$$\angle NMC = \angle NBC = \angle NBD = 90^\circ - \angle BDN = 90^\circ - \angle ADN,$$

so

$$\angle NMA = \angle NMC - 90^\circ = (90^\circ - \angle ADN) - 90^\circ = -\angle ADN = \angle NDA,$$

therefore quadrilateral $DAMN$ is cyclic. The radical axes of the circles $(DAMN)$, (AMC) , and (BND) are $\overline{AM}, \overline{DN}, \overline{XY}$, and since the centers of these circles are never colinear, they must intersect at the radical center. \square

Problem 2.32 (USAMO 1998/2). Let \mathcal{C}_1 and \mathcal{C}_2 be concentric circles, with \mathcal{C}_2 in the interior of \mathcal{C}_1 . From a point A on \mathcal{C}_1 one draws the tangent \overline{AB} to \mathcal{C}_2 ($B \in \mathcal{C}_2$). Let C be the second point of intersection of ray AB and \mathcal{C}_1 , and let D be the midpoint of \overline{AB} . A line passing through A intersects \mathcal{C}_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB . Find, with proof, the ratio AM/MC .

Proof. **INCOMPLETE** Note that $CDEF$ is cyclic (need to prove). M is the center of circle $(CDEF)$. Thus $CM = DM$. \square

Chapter 3

Theorem 3.2 (Angle Bisector Theorem). Let ABC be a triangle and D a point on \overline{BC} so that \overline{AD} is the internal angle bisector of $\angle BAC$. Show that

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

Proof. Let $\angle BAD = \alpha = \angle CAD$ and $\angle ADB = \beta$. Note that $\angle ADC = 180^\circ - \beta$. By Law of Sines, we have

$$\frac{DB}{\sin \alpha} = \frac{AB}{\sin \beta} \text{ and } \frac{DC}{\sin \alpha} = \frac{AC}{\sin(180^\circ - \beta)}.$$

Note that $\sin(180^\circ - \beta) = \sin \beta$. Rearranging terms, we have that

$$\frac{\sin \beta}{\sin \alpha} = \frac{AB}{BD} = \frac{AC}{CD}.$$

It follows that $\frac{AB}{AC} = \frac{DB}{DC}$. □

Problem 3.5. Show the trigonometric form of Ceva holds.

Proof. Recall that the trigonometric form of Ceva's Theorem is as follows:

Let $\overline{AX}, \overline{BY}, \overline{CZ}$ be cevians of a triangle ABC . They concur if and only if

$$\frac{\sin \angle BAX \sin \angle CBY \sin \angle ACZ}{\sin \angle XAC \sin \angle YBA \sin \angle ZCB} = 1.$$

By the Law of Sines, we have that

$$\frac{\sin \angle BAX}{BX} = \frac{\sin B}{AX}$$

and

$$\frac{\sin \angle XAC}{XC} = \frac{\sin C}{AX}.$$

Combining these two equations gives us

$$AX = \frac{BX \sin B}{\sin \angle BAX} = \frac{XC \sin C}{\sin \angle XAC} \Rightarrow \frac{\sin \angle BAX}{\sin \angle XAC} = \frac{BX}{XC} \cdot \frac{\sin C}{\sin B}.$$

Similarly, we have that

$$\frac{\sin \angle CBY}{\sin \angle YBA} = \frac{CY}{YA} \cdot \frac{\sin A}{\sin C}$$

and

$$\frac{\sin \angle ACZ}{\sin \angle ZCB} = \frac{AZ}{ZB} \cdot \frac{\sin B}{\sin A}.$$

Plugging these values into the original equation, we have that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1,$$

and we know this is true from the original statement of Ceva's Theorem. \square

Problem 3.6. Let \overline{AM} , \overline{BE} , and \overline{CF} be concurrent cevians of a triangle ABC . Show that $\overline{EF} \parallel \overline{BC}$ if and only if $BM = MC$.

Proof. Suppose $\overline{EF} \parallel \overline{BC}$. Call the point where \overline{AM} intersects \overline{EF} point Q . Notice that $\triangle BPM \sim \triangle EPQ$ and $\triangle CPM \sim \triangle FPQ$. Thus we have the following relationship:

$$\frac{BM}{EQ} = \frac{MP}{QP} = \frac{CM}{FQ}.$$

Now also notice that $\triangle BAM \sim \triangle FAQ$ and $\triangle CAM \sim \triangle EAQ$. Thus we have the following relationship:

$$\frac{BM}{FQ} = \frac{MA}{QA} = \frac{CM}{EQ}.$$

Putting these two relationships together, it follows that $BM = CM$.

We will now prove the other direction. Suppose $BM = MC$. Then by Ceva's Theorem we have that

$$\begin{aligned} \frac{CE}{AE} &= \frac{BF}{AF} \\ \frac{CE}{BF} &= \frac{AE}{AF} = \frac{CE + AE}{BF + AF} = \frac{AC}{AB} \\ \frac{AE}{AC} &= \frac{AF}{AB}. \end{aligned}$$

Since $\angle FAE = \angle BAC$, we have that $\triangle FAE \sim \triangle BAC$. Thus $\angle AEF = \angle ACB$, therefore $\overline{EF} \parallel \overline{BC}$. \square

Problem 3.12. Give an alternative proof of Lemma 3.9 by taking a negative homothety.

Proof. Consider a homothety centered at G with $M = h(A), N = h(B), L = h(C)$. Note that $\triangle ACB \sim \triangle NCM$ by midpoints and that $\triangle ALG \sim \triangle Mh(L)G$ by homothety. Also notice that $h(L)$ is the midpoint of NM . Since $AB/NM = 2/1$,

$$\frac{AB}{NM} = \frac{AL}{Mh(L)} = \frac{AG}{MG} = \frac{2}{1}.$$

□

Lemma 3.13 (Euler Line). In triangle ABC , prove that O, G, H (with their usual meanings) are collinear and that G divides \overline{OH} in a $2 : 1$ ratio.

Proof. We will first show that O, G, H are collinear. Call the point where the perpendicular from O meets $\overline{BC}, \overline{CA}, \overline{AB}$ points A', B', C' , respectively. Since $\overline{BC}, \overline{CA}, \overline{AB}$ are chords of the circle (ABC) , points A', B', C' are in fact the midpoints of their respective line segments. Thus A' lies on \overline{AG} , B' lies on \overline{BG} , and C' lies on \overline{CG} . Now notice that $\overline{AH} \parallel \overline{OA'}, \overline{BH} \parallel \overline{OB'}, \overline{CH} \parallel \overline{OC'}$ since they are all perpendicular to some side of the triangle ABC . Thus, a homothety h centered at G exists such that $h(A) = A', h(B) = B', h(C) = C'$. Thus, $h(O) = H$, so O, G, H are collinear.

We will now show that G divides \overline{OH} in a $2 : 1$ ratio. This is equivalent to showing that the homothety h must have a scale factor $k = -2$. From Lemma 3.9 (Centroid Division) we have that $AG/GA' = 2/1$. Since G lies in between A and A' , we have that $k = -2$, as desired. (!!!) □

Problem 3.16. Let ABC be a triangle with contact triangle DEF . Prove that $\overline{AD}, \overline{BE}, \overline{CF}$ concur. The point of concurrency is the Gergonne point of triangle ABC .

Proof. Notice by Lemma 2.15 we have that

$$\begin{aligned} AE &= AF = s - a \\ BD &= BF = s - b \\ CD &= CE = s - c. \end{aligned}$$

Thus, by Ceva's Theorem, we have that

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1.$$

□

Lemma 3.17. In cyclic quadrilateral $ABCD$, points X and Y are the orthocenters of $\triangle ABC$ and $\triangle BCD$. Show that $AXYD$ is a parallelogram.

Proof. Reflect X and Y across \overline{BC} and call these points X' and Y' respectively. Notice that X' and Y' lie on $(ABCD)$. Thus $ADX'Y'$ is a cyclic quadrilateral. Then we have that

$$\angle AXY = \angle X'XY = \angle Y'X'X = \angle Y'X'A = \angle Y'DA = \angle YDA.$$

Similarly, we have that $\angle DAX = \angle XYD$. Hence $AXYD$ is a parallelogram. \square

Problem 3.18. Let $\overline{AD}, \overline{BE}, \overline{CF}$ be concurrent cevians in a triangle, meeting at P . Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

Proof. By Area Ratios, we can transform each term in our desired equation as follows:

$$\begin{aligned} \frac{PD}{AD} &= \frac{[BPC]}{[BAC]}, \\ \frac{PE}{BE} &= \frac{[CPA]}{[CBA]}, \\ \frac{PF}{CF} &= \frac{[APB]}{[ACB]}. \end{aligned}$$

Therefore our desired equation turns into

$$\frac{[BPC]}{[BAC]} + \frac{[CPA]}{[CBA]} + \frac{[APB]}{[ACB]} = 1.$$

Notice that $[BPC] + [CPA] + [APB] = [ABC]$. Hence we are done. \square

Problem 3.19 (Shortlist 2006/G3). Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \text{ and } \angle ABC = \angle ACD = \angle ADE.$$

Diagonals BD and CE meet at P . Prove that ray AP bisects \overline{CD} .

Proof. Let B' be intersection of diagonals AC and BD , and let E' be the intersection of diagonals AD and CE . Also let A' be the intersection of ray AP with CD . Notice that the given angle conditions imply that $\triangle ABC \sim \triangle ACD \sim \triangle ADE$. From this it follows that quadrilaterals $ABCD$ and $ACDE$ are similar. Since B' and E' are the intersections of the diagonals of their respective quadrilaterals, we have that $\frac{CB'}{B'A} = \frac{DE'}{E'A}$. By Ceva's on $\triangle ACD$, we have that

$$\frac{AE'}{E'D} \cdot \frac{DA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

Since $\frac{CB'}{B'A} \cdot \frac{AE'}{E'D} = 1$, we have that $DA' = A'C$. \square

Problem 3.20 (BAMO 2013/3). Let H be the orthocenter of an acute triangle ABC . Consider the circumcenters of triangles ABH , BCH , and CAH . Prove that they are the vertices of a triangle that is congruent to ABC .

Proof. Let A', B', C' be the circumcenters of (BCH) , (CAH) , (ABH) , respectively. Note that H is the radical center of (ABH) , (BCH) , (CAH) . Thus $\overline{AH} \perp \overline{B'C'}$. Also notice by properties of circumcenters, A' is on the perpendicular bisector of \overline{BC} . Let O be where the perpendicular bisectors of $\triangle ABC$ intersect (namely, the circumcenter of $\triangle ABC$). Since $\overline{A'O} \parallel \overline{AH}$, $\overline{A'O} \perp \overline{B'C'}$. This follows similarly for B' and C' , hence O is the orthocenter of $\triangle A'B'C'$. Also notice that, by construction, H is the circumcenter of $\triangle A'B'C'$. Therefore, a homothety of scale factor -1 exists that sends H to O , A to A' , B to B' , and C to C' . Hence, $\triangle ABC \cong \triangle A'B'C'$. \square

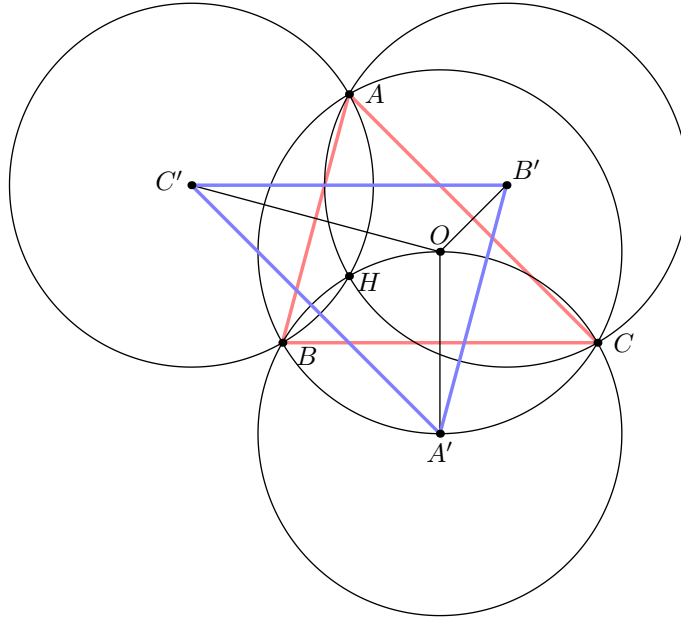


Figure 3.1: Problem 3.20

Problem 3.21 (USAMO 2003/4). Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E , respectively. Lines AB and DE intersect at F , while lines BD and CF intersect at M . Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Proof.

□