# CNCM Problem of the Day Solutions

Ryder Pham

October 26, 2021

## 6 August 2021

Notice that the expected number of dollars Tommy expects to win is equivalent to the following infinite series:

$$\frac{1}{6}\sum_{n=0}^{\infty} n^2 \left(\frac{5}{6}\right)^n.$$

Define

$$f(x) = \sum_{n=0}^{\infty} n^2 x^n$$

where we want to find the value of $f(5/6)$. Then

$$\begin{aligned}
f(x) &= x\sum_{n=0}^{\infty} n^2 x^{n-1} \\
&= x\frac{d}{dx}\left[\sum_{n=0}^{\infty} n x^n\right] \\
&= x\frac{d}{dx}\left[x\sum_{n=0}^{\infty} n x^{n-1}\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]\right] \\
&= x\frac{d}{dx}\left[x\frac{d}{dx}\left[\frac{1}{1-x}\right]\right] \\
&= x\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] \\
&= x\left[\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}\right] \\
&= x\left[\frac{(1-x) + 2x}{(1-x)^3}\right] \\
&= x\left[\frac{1+x}{(1-x)^3}\right] \\
f(5/6) &= \frac{5}{6}\left[\frac{1+5/6}{(1-5/6)^3}\right] \\
&= \frac{5}{6}\cdot\frac{11}{6}\cdot\frac{6^3}{1} \\
&= 330.
\end{aligned}$$

Then our final answer is $\frac{1}{6}f(\frac{5}{6}) = \frac{330}{6} = \boxed{55}$.

# 11 August 2021

Here let $R_n$ denote the remaining water after the $n$-th pour.

$$\begin{aligned}
R_0 &= 1 \\
R_1 &= \left(1 - \frac{1}{2}\right) R_0 = \frac{1}{2} \\
R_2 &= \left(1 - \frac{1}{3}\right) R_1 = \frac{1}{3} \\
R_3 &= \left(1 - \frac{1}{4}\right) R_2 = \frac{1}{4}
\end{aligned}$$

Therefore we can assume by Engineer's Induction that $R_n = \frac{1}{n+1}$. Hence $R_9 = \frac{1}{10}$ for a final answer of $\boxed{9}$.

# 12 August 2021

For a two-game block, there is a probability of $\dfrac{3}{4} \cdot \dfrac{1}{4} + \dfrac{1}{4} \cdot \dfrac{3}{4} = \dfrac{3}{8}$ of the entire match ending then and there. The only other outcome after two games is a tie, since each game must declare a winner, and this happens with probability 5/8. Thus the expected number of games in a match is the following:

$$\begin{aligned}
\frac{3}{8} \cdot 2 + \frac{3}{8} \cdot \frac{5}{8} \cdot 4 + \frac{3}{8} \cdot \left(\frac{5}{8}\right)^2 \cdot 6 + \cdots &= \sum_{n=0}^{\infty} 2(n+1) \cdot \frac{3}{8} \cdot \left(\frac{5}{8}\right)^n \\
&= \frac{3}{4} \sum_{n=0}^{\infty} (n+1) \left(\frac{5}{8}\right)^n \\
&= \frac{3}{4} \sum_{n=0}^{\infty} n \left(\frac{5}{8}\right)^n + \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n.
\end{aligned}$$

Define $f(x) = \sum_{n=0}^{\infty} nx^n$. We would like to find the value of $f(5/8)$. Note that

$$\begin{aligned} f(x) &= x\sum_{n=0}^{\infty} nx^{n-1} \\ &= x\frac{d}{dx}\sum_{n=0}^{\infty} x^n \\ &= x\frac{d}{dx}\left[\frac{1}{1-x}\right] \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

Thus $f(5/8) = \dfrac{5/8}{(1-5/8)^2} = \dfrac{5}{8}\cdot\dfrac{8^2}{3^2} = \dfrac{40}{9}$. Then our original sum becomes

$$\begin{aligned} \frac{3}{4}\sum_{n=0}^{\infty} n\left(\frac{5}{8}\right)^n + \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n &= \frac{3}{4}\cdot\frac{40}{9} + \frac{3}{4}\cdot\frac{1}{1-5/8} \\ &= \frac{10}{3} + \frac{3}{4}\cdot\frac{8}{3} \\ &= \frac{10}{3} + 2 \\ &= \frac{16}{3}. \end{aligned}$$

Our final answer is $160 + 3 = \boxed{163}$.

## 26 August 2021

Our recurrence relation is

$$7a_n = -a_{n-1} + 8a_{n-2}.$$

By simple calulations we determine that $a_1 = 25$. Note that the recurrence is linear and homogenous. Its characteristic equation is

$$\begin{aligned} 7r^2 + r - 8 &= 0 \\ (7r+8)(r-1) &= 0 \\ r_{1,2} &= -\frac{8}{7}, 1. \end{aligned}$$

So by some theorem (idk) $a_n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2(1)^n = \alpha_1 \left(-\frac{8}{7}\right)^n + \alpha_2$ is a solution. To find $\alpha_1, \alpha_2$ we must solve the following system:

$$\begin{cases} a_0 = \alpha_1 + \alpha_2 = 4 \\ a_1 = -\frac{8}{7}\alpha_1 + \alpha_2 = 25. \end{cases}$$

Solving this gets us $(\alpha_1, \alpha_2) = (-49/5, 69/5)$. Thus what we have left to evaluate is

$$\begin{aligned} a_7 &= -\frac{49}{5}\left(-\frac{8}{7}\right)^7 + \frac{69}{5} \\ &= \frac{49}{5}\left(\frac{8}{7}\right)^7 + \frac{69}{5} \\ &= \frac{1}{5} \cdot \frac{8^7}{7^5} + \frac{69}{5} \\ &= \frac{8^7 + 69 \cdot 7^5}{5 \cdot 7^5} \\ &= \frac{8^7 + 69 \cdot 16807}{5 \cdot 7^5} \\ &= \frac{2097152 + 69 \cdot 16807}{5 \cdot 7^5} \\ &= \frac{2097152 + 1159683}{5 \cdot 7^5} \\ &= \frac{3256835}{5 \cdot 7^5} \\ &= \frac{651367}{16807}. \end{aligned}$$

Therefore our final answer is $651367 + 16807 + 28795 = \boxed{696969}$.

## 2 September 2021

Let $BD = x$. By the Angle Bisector Theorem

$$\frac{8}{12} = \frac{x}{10-x}.$$

Solving for $x$ gives us $x = 4$. Thus $BD = 4$ and $CD = 6$. By Stewart's Theorem on $\triangle ABC$ we have

$$\begin{aligned} b^2m + c^2n &= a(d^2 + mn) \\ 8^2 \cdot 6 + 12^2 \cdot 4 &= 10(d^2 + 6 \cdot 4) \\ 384 + 576 &= 10d^2 + 240 \\ d^2 &= 72 \\ AD &= 6\sqrt{2}. \end{aligned}$$

We will now find $BD'$. Applying Stewart's Theorem again on $\triangle ABD'$ gives us

$$\begin{aligned} 8^2 \cdot 2\sqrt{2} + c^2 \cdot 6\sqrt{2} &= 8\sqrt{2}(4^2 + 24) \\ 128 + 6c^2 &= 128 + 192 \\ c^2 &= 32 \\ BD' &= 4\sqrt{2}. \end{aligned}$$

Similarly to find $CD'$ we apply Stewart's Theorem a third time on $\triangle ACD'$, which gives us

$$\begin{aligned} b^2m + c^2n &= a(d^2 + mn) \\ 12^2 \cdot 2\sqrt{2} + c^2 \cdot 6\sqrt{2} &= 8\sqrt{2}(6^2 + 24) \\ 288 + 6c^2 &= 288 + 192 \\ c^2 &= 32 \\ CD' &= 4\sqrt{2}. \end{aligned}$$

Thus $BD' \cdot CD' = (4\sqrt{2})^2 = \boxed{32}$.

# 6 September 2021

Let $PY = x$ and $QY = y$. Note that the area of $\triangle XYZ$ (henceforth denoted $[XYZ]$) is $\frac{1}{2} \cdot 12 \cdot 2004$. Also note that $[XYZ] = [XPY] + [YPZ]$. It follows that

$$\begin{aligned}
[XYZ] &= [XPY] + [YPZ] \\
\frac{1}{2} \cdot 12 \cdot 2004 &= \frac{1}{2} \cdot 12 \cdot \frac{x}{2} + \frac{1}{2} \cdot 2004 \cdot \frac{x\sqrt{3}}{2} \\
12 \cdot 2004 &= 6x + 1002\sqrt{3}x \\
x &= \frac{2 \cdot 2004}{1 + 167\sqrt{3}}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
[XYZ] &= [XQY] + [YQZ] \\
\frac{1}{2} \cdot 12 \cdot 2004 &= \frac{1}{2} \cdot 12 \cdot \frac{y\sqrt{3}}{2} + \frac{1}{2} \cdot 2004 \cdot \frac{y}{2} \\
12 \cdot 2004 &= 6\sqrt{3}y + 1002y \\
y &= \frac{2 \cdot 2004}{167 + \sqrt{3}}
\end{aligned}$$

Thus

$$\begin{aligned}
(PY + YZ)(QY + XY) &= \left(\frac{2 \cdot 12 \cdot 167}{1 + 167\sqrt{3}} + 12 \cdot 167\right)\left(\frac{2 \cdot 12 \cdot 167}{167 + \sqrt{3}} + 12\right) \\
&= 12 \cdot 167 \cdot 12 \left(\frac{2 + 1 + 167\sqrt{3}}{1 + 167\sqrt{3}}\right)\left(\frac{2 \cdot 167 + 167 + \sqrt{3}}{167 + \sqrt{3}}\right) \\
&= 12 \cdot 167 \cdot 12 \left(\frac{3 + 167\sqrt{3}}{1 + 167\sqrt{3}}\right)\left(\frac{3 \cdot 167 + \sqrt{3}}{167 + \sqrt{3}}\right) \\
&= 12 \cdot 167 \cdot 12 \left(\frac{167^2 \cdot 3\sqrt{3} + 12 \cdot 167 + 3\sqrt{3}}{167^2\sqrt{3} + 4 \cdot 167 + \sqrt{3}}\right) \\
&= 12 \cdot 167 \cdot 12 \cdot 3 \left(\frac{167^2\sqrt{3} + 4 \cdot 167 + \sqrt{3}}{167^2\sqrt{3} + 4 \cdot 167 + \sqrt{3}}\right) \\
&= \boxed{72144}.
\end{aligned}$$

## 8 September 2021

We have $f(x) = \log_2(x + \underbrace{\log_2(x + \log_2(x + \cdots))}_{f(x)}) = \log_2(x + f(x))$. Solving for $x$ gives us $x = 2^{f(x)} - f(x)$. Thus

$$f^{-1}(x) = 2^x - x.$$

It follows that

$$\begin{aligned}
\sum_{k=2}^{10} \left[2^k - k\right] &= \sum_{k=2}^{10} 2^k - \sum_{k=2}^{10} k \\
&= \sum_{k=0}^{8} 2^{k+2} - \sum_{k=0}^{8} (k+2) \\
&= 4 \cdot \frac{2^9 - 1}{2 - 1} - \frac{8 \cdot 9}{2} - 2 \cdot 9 \\
&= 4(511) - 36 - 18 \\
&= \boxed{1990}.
\end{aligned}$$

## 17 September 2021

Right off the bat, we can ignore dividing by 1 since every number is divisible by 1.

**Claim** — All $n = 11(2k+1), k \in \mathbb{Z}$ fail.

*Proof.* Note that

$$\begin{aligned}
n &\equiv 22k + 11 \equiv 1 \pmod 2 \\
n &\equiv 0 \pmod{11}.
\end{aligned}$$

Since $1 > 0$, this is a decerasing sequence. $\square$

**Claim** — The remainders of 11, 22, 121, and 242 always follow a non-decreasing sequence except for $n \in [121, 131]$.

*Proof.* We will start with $(11, 22)$. We can represent $n = 22q_1 + r_1$ with $0 \le r_1 < 22$. Then

$$\begin{aligned} n &= 22q_1 + r_1 \\ &= 11(2q_1) + r_1 \\ &= 11(2q_1) + 11k + r_2 \\ &= 11(2q_1 + k) + r_2, \end{aligned}$$

where $0 \le r_2 < 11$. If $r_1 \ge 11$, then $k = 1$ and $r_1 > r_2$. If $r_1 < 11$, then $k = 0$ and $r_1 = r_2$. Thus $n \pmod{22} > n \pmod{11}$. Very similar arguments apply between $(11, 121), (11, 242)$, and $(121, 242)$ since, in each pair, one is a multiple of the other. Additionally, since each pair cannot decrease the remainder of $n$, together they must form a non-decreasing sequence.

However, we are not done, since we must still consider $(22, 121)$ because 121 is not a multiple of 22. Write $n = 121q_1 + r_1$ with $0 \le r_1 < 121$. Note that if $q_1$ is even, we are done since we can follow a very similar line of reasoning as above to show that $r_1 \ge r_2$. We will now deal with the case if $q_1$ is odd. Note that

$$\begin{aligned} n &= 121q_1 + r_1 \\ &= 121(2k + 1) + r_1 \\ &= 2 \cdot 121k + 121 + r_1 \\ &= 22(11k + 5) + 11 + r_1, \end{aligned}$$

where $k \ge 0$. If $r_1 \ge 11$, then $r_1 > r_2$, so we are good here. However, if $r_1 < 11$, then $r_2 = 11 + r_1 > r_1$, so this case fails here. The only possible odd value that $q_1$ can be and still have $n \le 242$ is 1, so $n \in \{121, 122, ..., 131\}$ all fail. $\square$

Now, counting up how many $n$ fail and subtracting from 242 we get $242 - \underbrace{11}_{\text{first claim}} - \underbrace{11}_{\text{second claim}} + \underbrace{1}_{\text{double counting 121}} = \boxed{221}$.

## 20 September 2021

First note that $a_k = \frac{k(k+1)}{2}$. We will now manipulate our product as follows:

$$\begin{aligned}
\prod_{t=2}^{2021} \frac{a_t}{a_t - 1} &= \prod_{t=2}^{2021} \frac{\frac{t(t+1)}{2}}{\frac{t(t+1)}{2} - 1} \\
&= \prod_{t=2}^{2021} \frac{\frac{t(t+1)}{2}}{\frac{t(t+1)-2}{2}} \\
&= \prod_{t=2}^{2021} \frac{t(t+1)}{t(t+1) - 2} \\
&= \prod_{t=2}^{2021} \frac{t(t+1)}{(t+2)(t-1)} \\
&= \frac{2 \cdot 3}{4 \cdot 1} \cdot \frac{3 \cdot 4}{5 \cdot 2} \cdot \frac{4 \cdot 5}{6 \cdot 3} \cdots \frac{2021 \cdot 2022}{2023 \cdot 2020} \\
&= \frac{3}{1} \cdot \frac{2021}{2023} \\
&= \frac{6063}{2023}.
\end{aligned}$$

Hence our final answer is $6063 + 2023 = \boxed{8086}$.

## 6 October 2021 INCOMPLETE SOLUTION

Note that for $x, y = 0$, we have $f(0) = -1$. For $y = 1$ and $x$ free, we have

$$f(x+1) - f(x) = x + 2.$$

Making the substitution $x \mapsto x + 1$, we obtain

$$f(x+2) - f(x+1) = x + 3.$$

Then, for any integer $k$, making the substitution $x \mapsto x + k$ gets us

$$f(x+k) - f(x+k-1) = x + k + 1.$$

Adding up all the equations from $k = 1$ to $k = n$ gives us (this equation does not match what I have on paper)

$$f(x+n) - f(x) = nx + \frac{n(n+1)}{2}.$$

## 25 October 2021

Denote by $D$ the foot of the altitude from $M$ to $OC$, and let $x = MD$. Note that since $AM = 3, AC = 6$, and $\angle AMC = 90°$, we have that $MC = 3\sqrt{3}$ by Pythagoras on $\triangle AMC$. Similarly, by Pythagoras on $\triangle AMO$, we have that $MO = 6\sqrt{2}$. Note that $OD + DC = 9$. Thus, by using Pythagoras on $\triangle MOD$ and $\triangle MCD$, we have the following equation:

$$\begin{aligned}\sqrt{(6\sqrt{2})^2 - x^2} + \sqrt{(3\sqrt{3})^2 - x^2} &= 9 \\ \sqrt{72 - x^2} + \sqrt{27 - x^2} &= 9.\end{aligned}$$

We can make the substitution $u^2 = 27 - x^2$. Note that $u^2 + 45 = 72 - x^2$. Now we have

$$\begin{aligned}\sqrt{u^2 + 45} + \sqrt{u^2} &= 9 \\ u^2 + 45 &= (9 - u)^2 \\ 45 &= 81 - 18u \\ u &= 2.\end{aligned}$$

Hence $x^2 = 27 - u^2 = \boxed{23}$.