Lesson 3

Boundary Value Problems for ODEs

3.2 Shooting Method

3.3 Finite Difference Method

3.4 Variational Methods

- Collocation Method

Akash Anand
MATH, IIT KANPUR

Boundary Value Problems: Variational Methods

Akash Anand
MATH, IIT KANPUR

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that is,

$$\sum_{j=1}^{n} y_j \varphi_j''(t) = f\left(t_i, \sum_{j=1}^{n} y_j \varphi_j(t_i), \sum_{j=1}^{n} y_j \varphi_j'(t_i)\right), \qquad i = 2, \dots, n-1.$$

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In addition, we also enforce the boundary condition at the end-points:

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This yields a system of n *algebraic equation in* n *unknowns. This may be linear or non-linear depending on whether* f *is linear or non-linear.*

In particular, for the approximation space consisting of polynomials of degree $n-1$ *or less (i.e.,* $\wp_{n-1}$*), we have*

$$u(t) \approx v(t,y) = \sum_{i=1}^{n} y_i \ell_i(t) \in \wp_{n-1},$$

where the Lagrange basis, given by

$$\ell_i(t) = \frac{\prod_{k=1,k\neq i}^{n}(t-t_k)}{\prod_{k=1,k\neq i}^{n}(t_i-t_k)} \in \wp_{n-1}$$

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is used. Then the collocation method yields the following system of algebraic equations: $Ay = F(y)$ where

$$Ay = \begin{bmatrix} \ell_1(t_1) & \ell_2(t_1) & \cdots & & \ell_n(t_1) \\ \ell_1''(t_2) & \ell_2''(t_2) & & & \\ \vdots & \vdots & \ddots & & \vdots \\ & & & \ell_{n-1}''(t_{n-1}) & \ell_n''(t_{n-1}) \\ \ell_1(t_n) & \cdots & & \ell_{n-1}(t_n) & \ell_n(t_n) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}, F(y) = \begin{bmatrix} \alpha \\ f\left(t_2, \sum_{i=1}^{n} y_i \ell_i(t_2), \sum_{i=1}^{n} y_i \ell_i'(t_2)\right) \\ \vdots \\ f\left(t_{n-1}, \sum_{i=1}^{n} y_i \ell_i(t_{n-1}), \sum_{i=1}^{n} y_i \ell_i'(t_{n-1})\right) \\ \beta \end{bmatrix}.$$

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The Newton iterations take the form $y^{(m+1)} = y^{(m)} - \left(A - F'\left(y^{(m)}\right)\right)^{-1} \left(Ay^{(m)} - F\left(y^{(m)}\right)\right)$.

Example

Consider the two-point BVP

$$u'' = 6t, \qquad 0 < t < 1,$$
$$u(0) = 0, \qquad u(1) = 1.$$

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We seek the approximate solution in the space of quadratic polynomials of the form

$$v(t, y) = 2\left(t - \frac{1}{2}\right)(t - 1)y_1 - 4t(t - 1)y_2 + 2t\left(t - \frac{1}{2}\right)y_3.$$

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$$v(t, y) = y_1 + y_2 t + y_3 t^2.$$

The system of algebraic equation that we solve is

$$y_1 = 0,$$
$$2y_3 = 6\left(\frac{1}{2}\right),$$
$$y_1 + y_2 + y_3 = 1.$$

Thus, the approximate solution is

$$u(t) \approx -\frac{1}{2}t + \frac{3}{2}t^2.$$