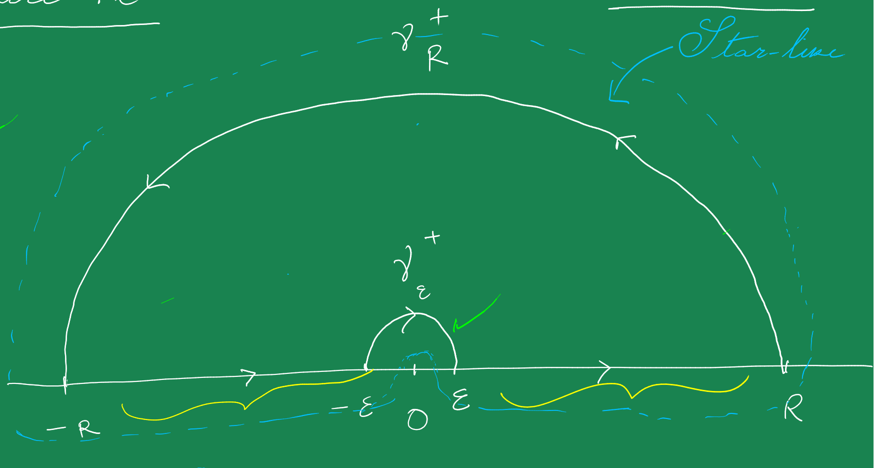
# Lecture 12

28/08/2023

$$\checkmark \int_0^\infty \frac{1-\cos x}{x^2}\,dx = \pi/2$$

Enough to show

$$\boxed{\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi}$$




$$f(z) = \frac{1-e^{iz}}{z^2}, \quad z \in \mathbb{C}\setminus\{0\},$$

$R>0$, $\varepsilon>0$, $\varepsilon<R$, $\varepsilon<1$

as $R\to\infty$

$$\int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,dx + \int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx + \cancel{\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz} = 0$$

(IV)

$z = x+iy$, $y \geq 0$

$$|e^{iz}| = |e^{-y+ix}| = e^{-y} \leq 1 \quad \Rightarrow \quad \frac{|1-e^{iz}|}{|z^2|} \leq \frac{2}{|z|^2}$$

$$\left|\int_{\gamma_R^+} \frac{1-e^{iz}}{z^2}\,dz\right| \leq \frac{2}{R^2}\cdot \pi R = \frac{2\pi}{R} \xrightarrow[R\to\infty]{} 0$$

As $R\to\infty$

$$\int_{|x|\geq\varepsilon} \frac{1-e^{it}}{t^2}\,dt = -\int_{\gamma_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz \quad \cdots \quad (*)$$

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$$

$$\Rightarrow 1-e^{iz} = -iz - \frac{(iz)^2}{2!} - \frac{(iz)^3}{3!} - \cdots = -i - \left[\frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots\right]$$

$$\Rightarrow \frac{1-e^{iz}}{z^2} = -\frac{i}{z} - \left[\frac{i^2}{2!} + \frac{i^3 z}{3!} + \cdots\right].$$

Let $E(z) = \frac{i}{2!} + \frac{i^3 z}{3!} + \cdots$, $\forall z\in\mathbb{C}$

$\forall\, |z| \le 1$, $\quad |E(z)| \le \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \cdots \le 1$

$$< \frac{1}{2^2} + \frac{1}{2^3}$$

*When* $z \neq 0$, $\quad \dfrac{1 - e^{iz}}{z^2} = -\dfrac{i}{z} + E(z)$

$$\Rightarrow \int_{\gamma_\varepsilon^+} \frac{1 - e^{iz}}{z^2}\, dz = -i \int_{\gamma_\varepsilon^+} \frac{dz}{z} + \int_{\gamma_\varepsilon^+} E(z)\, dz \Longrightarrow$$

*As* $\varepsilon \to 0$

$\text{RHS} \to \pi$

$\forall \varepsilon \in (0,1)$

$$\Rightarrow \left| \int_{\gamma_\varepsilon^+} E(z)\, dz \right| \overset{ML}{\le} 1 \cdot \pi \varepsilon \xrightarrow[\varepsilon \to 0]{} 0$$

$$\int_{\gamma_\varepsilon^+} \frac{dz}{z} = \int_0^\pi \frac{1}{\cancel{\varepsilon} \cancel{e^{i(\pi - t)}}} \cancel{\varepsilon} \cdot i\, \cancel{e^{i(\pi-t)}}\, dt \qquad \gamma_\varepsilon^+(t) = \varepsilon e^{i(\pi - t)}, \quad 0 \le t \le \pi$$

$$= \pi i, \qquad \forall \varepsilon > 0$$

*From* $(*)$, *one obtains that*

$$\lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} \frac{1 - e^{it}}{t^2}\, dt = \pi$$

$$\parallel$$

$$\int_{-\infty}^{\infty} \frac{1 - e^{it}}{t^2}\, dt = \pi.$$ *Now taking the real part,* $$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2}\, dx = \pi$$

$$\Rightarrow \int_0^\infty \frac{1 - \cos x}{x^2}\, dx = \pi/2$$

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*Equicontinuity* : $\quad X$ – *metric sp*

- $\mathcal{F}$ – *family of functions,* $f : X \to \mathbb{C}$. *Fix* $x_0 \in X$

*We say* $\mathcal{F}$ *is equicts. at* $x_0 \in X$, *if* $\forall \varepsilon > 0\ \exists\, \delta > 0$ *s.t.* $d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon, \quad \forall f \in \mathcal{F}$

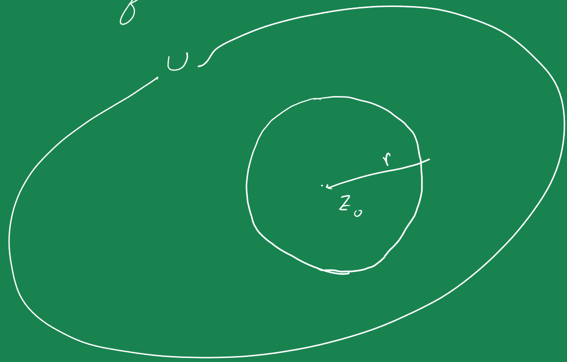
NOTE: *If* $\mathcal{F}$ *is equicts. at* $x_0 \in X$, *then clearly every* $f \in \mathcal{F}$ *is cts. at* $x_0$.

PROP: Let $U \underset{\text{open}}{\subseteq} \mathbb{C}$, $\mathcal{F} \subseteq H(U)$. Assume $\mathcal{F}$ is 'uniformly' bdd on each $K \underset{\text{cpt.}}{\subseteq} U$'

(i.e., $\forall\, K \underset{\text{cpt.}}{\subseteq} U$, $\exists\, M > 0$ s.t. $\forall\, z \in K$, $f \in \mathcal{F}$, $|f(z)| \le M$)

Then $\mathcal{F}$ is equicts.

PROOF :-




$D(z_0; R) \subseteq U$, $0 < r < R$

$\Rightarrow \overline{D(z_0; r)} \subseteq U$

Let $f \in \mathcal{F}$

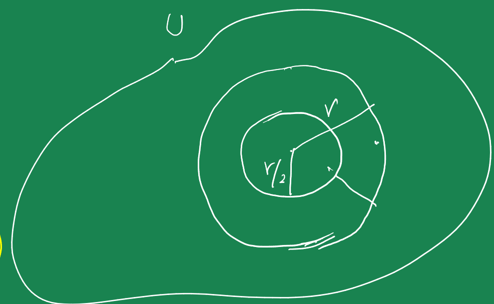
$$f(z) - f(z_0) \overset{\text{CIF}}{=} \frac{1}{2\pi i} \int_{C(z_0; r)} f(w) \left( \frac{1}{w - z} - \frac{1}{w - z_0} \right) dw$$

$$= \frac{1}{2\pi i} \int_{C(z_0; r)} f(w) \frac{z - z_0}{(w - z)(w - z_0)} \, dw$$

Let $M \geq 0$ s.t. $\forall\, f \in \mathcal{F}$ & $|w - z_0| = r$, $|f(w)| \le M$ (from uniformly bdd on cpt. subsets)

$$\Rightarrow |f(z) - f(z_0)| \le \frac{1}{2\pi} M \cdot |z - z_0| \cdots$$

$$\left| \frac{1}{(w - z)(w - z_0)} \right| = \frac{1}{|w - z|\, r}$$




If $z \in D(z_0, r/2)$, Then we get, $\forall\, f \in \mathcal{F}$

$$\boxed{|f(z) - f(z_0)| \le \frac{M}{\cancel{2}\pi} |z - z_0| \cdot \frac{\cancel{2}}{r} \cdot \frac{1}{r} = \frac{M}{\pi r^2} |z - z_0|}$$

Given $\varepsilon > 0$, take $\delta = \min\left\{ \frac{\pi r^2 \varepsilon}{M}, \frac{r}{2} \right\}$. Now $|z - z_0| < \delta \Rightarrow$ (indep. of $f$.)

$$|f(z) - f(z_0)| \le \frac{M}{\pi r^2} |z - z_0| < \frac{M\delta}{\pi r^2} \le \varepsilon, \quad \forall\, f \in \mathcal{F}.$$