Lesson 4

Numerical Solution of PDE

4.1 BVP for 2nd Order Elliptic PDE

- Finite Difference Method

- More Stability Analysis – Fourier Analysis

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Stability Analysis using Fourier Analysis

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First we consider the one dimensional case. With $h = 1/N$, *let* $I_h = \{h, 2h, \ldots, (N-1)h\}$ *and let*

$$L(I_h) = \left\{u: \bar{\mathrm{I}}_h \to \mathbb{R} : u(0) = 0, u(1) = 0\right\}.$$

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The functions $\varphi_m, m = 1,2,\ldots,N-1$ form an orthogonal basis of $L(I_h)$. Consequently, any function $v \in L(I_h)$ can be expanded as $v = \sum_{m=1}^{N-1} a_m \varphi_m$ with

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$$a_m = \frac{\langle v, \varphi_m \rangle_h}{\|\varphi_m\|_h^2}, \qquad \|v\|_h^2 = \sum_{m=1}^{N-1} a_m^2 \|\varphi_m\|_h^2.$$

From this, we obtain the stability result for the one-dimensional Laplacian: if $f = D_h^2 v = -\sum_{m=1}^{N-1} \lambda_m a_m \varphi_m$, then

$$\|f\|_h^2 = \sum_{m=1}^{N-1} \lambda_m^2 a_m^2 \|\varphi_m\|_h^2 \geq 8^2 \|v\|_h^2$$

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The extension to two-dimensional case is straightforward.

Let $L(\Omega_\mathrm{h}) = \{u\colon \overline{\Omega}_h \to \mathbb{R} : u(x) = 0, x \in \Gamma_h\}$ *so that* $L(\Omega_\mathrm{h})$ *is isomorphic to* $\mathbb{R}^M, M = (N-1)^2$.

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We use the basis

$$\varphi_{mn}(x_1, x_2) = \varphi_m(x_1)\varphi_n(x_2), \qquad m, n = 1, \dots, N-1.$$

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It is easy to check (exercise) that these $(N-1)^2$ *functions form an orthogonal basis for* $L(\Omega_\mathrm{h})$ *equipped with the inner product*

$$\langle u, v\rangle_h = h^2 \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} u(mh, nh)v(mh, nh)$$

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$$\Delta_\mathrm{h}\varphi_{mn} = -\lambda_{mn}\varphi_{mn}$$

where $\lambda_{mn} = \lambda_m + \lambda_n \geq 16.$

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Theorem

We have $\|v\|_h \leq \frac{1}{16}\|f\|_h$ *as the stability estimate where* v *solves the discrete problem* $\Delta_h v = f, \text{ on } \Omega_\mathrm{h}, v = 0, \text{on } \Gamma_\mathrm{h}.$