MTH 101-Calculus
Spring-2021
Assignment 3 : Derivatives, Maxima and Minima, Rolle's Theorem

1. Show that the function $f(x) = x \mid x \mid$ is differentiable at 0. More generally, if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0.

2. Let f be defined for all real x, and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all real x and y. Prove that f is a constant function.

3. Show that among all triangles with given base and the corresponding vertex angle, the isosceles triangle has the maximum area.

4. Show that exactly two real values of x satisfy the equation $x^2 = xsinx + cosx$.

5. Let $g : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b) such that $g(a) = g(b) = 0$ and $g(x) \neq 0$ for all $x \in (a, b)$. Show that the function $h : (a, b) \to \mathbb{R}$, defined by $h(x) = \frac{g'(x)}{g(x)}$ is an onto function.

6. Suppose f is continuous on $[a,\ b]$, differentiable on $(a,\ b)$ and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. Then show that the equation $f'(x)f(x) = x$ has at least one root in $(a,\ b)$.

7. Let $f : (-1, 1) \to \mathbb{R}$ be twice differentiable. Then show the following:

   (a) If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$ then $f'(0) = f''(0) = 0$.

   (b) If $f''(0) > 0$ then there exists $n \in \mathbb{N}$ such that $f(\frac{1}{n}) \neq 1$.