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$u_{xx}+u_{yy}=0$ in $[0,a]\times[0,b]$

Aim: To find the representation of solution
'seperation of variable'

- Works well for very specific domains
- Useful to find sol^n in dim 2
- For specific boundary conditions

$$u_{xx}+u_{yy}=0 \quad \text{in} \quad [0,\pi]\times[0,\pi]$$
$$u(0,y)=0=u(\pi,y)$$
$$u(x,0)=0$$
$$u(x,\pi)=g$$

Assume $u(x,y)=X(x)Y(y)$ put u in the eq^n
$$X''(x)Y(y)+X(x)Y''(y)=0$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\lambda$$

Hence $X''(x)=\lambda X(x)$ in $[0,\pi]$
and $X(0)=0=X(\pi)$ — (A)

$?\ Y(\pi)=\frac{g(x)}{X(x)}$

also we get
$Y''(y)=-\lambda Y(y)$ in $[0,\pi]$
and $Y(0)=0$, $Y(\pi)=g(x)$ — (B)

From (A) (for $\lambda>0$)
$$X(x)=a\sinh\sqrt{\lambda}\,x+b\cosh\sqrt{\lambda}\,x$$
putting $X(0)=0=X(\pi)$ $\Rightarrow a=b=0$
we should not take $\lambda>0$!
for $\lambda=0$
$$X(x)=ax+b \qquad \text{Boundary cond}^n \Rightarrow a=b=0$$
Avoid this too!

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for $\lambda < 0$

$$X(x) = a\sin\sqrt{-\lambda}x + b\cos\sqrt{\lambda}x$$

$$X(0) = b = 0$$

$$X(\pi) = a\sin\sqrt{-\lambda}\pi = 0$$

$$\Rightarrow \sqrt{-\lambda} = n$$

$$\Rightarrow \lambda = -n^2 \quad n \in \mathbb{N}$$

$$X(x) = a_n \sin nx$$

Next we solve for $Y(y)$

$$Y''(y) = n^2 Y(y) \quad \text{in } [0,\pi]$$

$$Y(0) = 0, \quad Y(\pi) = g$$

$$Y(y) = c_n \sinh ny + d_n \cosh ny$$

$$Y(0) = 0 \Rightarrow d_n = 0$$

$$\therefore Y(y) = c_n \sinh ny$$

we take

$$\boxed{u(x,y) = \sum A_n \sinh(ny)\sin(nx)}$$

Aim to find A_n !!

Multiply the expression by $\sin mx$ & Integrate over 0 to π

$$g(x) = u(x,\pi) = \sum_{n\in\mathbb{N}} A_n \sinh(n\pi)\sin(nx)$$

$$\int_0^\pi \sin(mx)\, g(x)\,dx = \sum_{n\in\mathbb{N}} A_n \sinh(n\pi)\int_0^\pi \sin nx \sin mx\,dx$$

Note $\int_0^\pi \sin nx \sin mx\,dx = \begin{cases} \pi/2 & m = n \\ 0 & m \neq n \end{cases}$

$$A_n = \frac{2}{\pi\sinh(n\pi)}\int_0^\pi \sin(nx)\,g(x)\,dx$$

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Parabolic Eq^n : Canonical form ($u_t - \Delta u = 0$)
Heat Eq^n

Aim to solve heat eq^n via fourier transform!

$f \in C_c^\infty(\mathbb{R}^n)$

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$$

$$\check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi$$

$f : S \to S$

$S = \{ f \in C^\infty(\mathbb{R}^n) : \| (1+|x|)^N (D^m f) \| < \infty \ \forall N, m \in \mathbb{N} \}$

$C_c^\infty(\mathbb{R}^n) \subseteq S(\mathbb{R}^n)$

$\hat{f} : \mathbb{R}^n \to \mathbb{R}$

$$\widehat{\left(\frac{\partial f}{\partial x_i}\right)}(\xi) = -i\xi_i \hat{f}(\xi)$$

$$\frac{\partial \hat{f}(\xi)}{\partial \xi_i} = (\widehat{-ig})(\xi) \quad \text{where } g(x) = x_i f(x)$$

Heat Eq^n : $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$

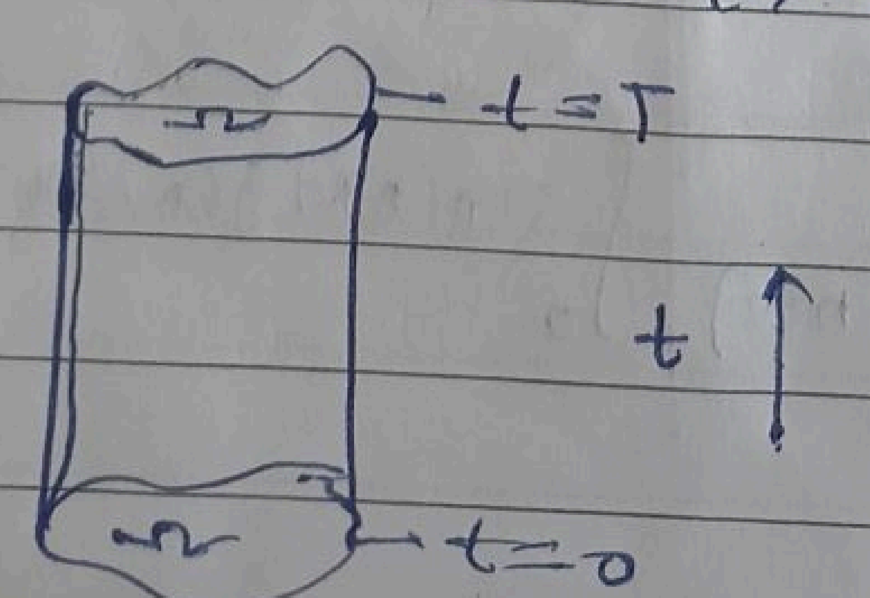
$u(x,0) = g(x) \quad x \in \mathbb{R}^n$

First let's solve in bounded domain

$u_t - \Delta u = 0$ in $\Omega \times (0,T)$

$u(x,0) = g(x)$ on $x \in \Omega$

$u = h$ on $\partial\Omega \times (0,T)$



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Lemma : $f \in S$ $\quad (f \in C_c^\infty(\mathbb{R}^n))$

$$(\hat{f})^{\vee} = f$$

Lemma: $f \in L^1(\mathbb{R}^n)$ (Integrable function) then $\hat{f}$ is a bounded function

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}^n} |e^{-ix\cdot\xi}| |f(x)| \, dx = \int_{\mathbb{R}^n} |f(x)| \, dx < \underline{\underline{\infty}}$$

Lemma: $\dfrac{\partial (\hat{f})(\xi)}{\partial \xi_i} = \hat{g}(\xi)$ where $g(x) = -ix_i f(x)$

Proof: $\dfrac{\partial (\hat{f})(\xi)}{\partial \xi_i} = \dfrac{\partial}{\partial \xi_i}\left[\dfrac{1}{(2\pi)^{n/2}} \displaystyle\int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx\right]$

for $f \in C_c^\infty(\mathbb{R}^n)$, so f is uniformly cont. on c

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_i}\left(e^{-ix\cdot\xi}\right) f(x)\,dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} -ix_i e^{-ix\cdot\xi} f(x)\,dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \left[-ix_i f(x)\right] dx$$

Take, $g(x) = -ix_i f(x)$

$$\frac{\partial}{\partial \xi_i} (\hat{f})(\xi) = \hat{g}(\xi)$$
$$= \widehat{(-ix_i f(x))}(\xi)$$

Lemma : $\dfrac{\partial^k}{\partial \xi_i^k} \hat{f}(\xi) = \widehat{\left((-ix_i)^k f\right)}(\xi)$

Green's formula

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, ds \qquad \frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$$

$$\int_{\Omega} (u\Delta u - v\Delta u)\,dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) ds$$

$$\int_{\Omega} \nabla v \cdot \nabla u \, dx = -\int_{\Omega} u\Delta v \, dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, ds$$

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Lemma: $\widehat{\left(\frac{\partial^k f}{\partial x_i^k}\right)}(\xi) = (i\xi_i)^k \hat{f}(\xi)$

Proof: $$\widehat{\left(\frac{\partial f}{\partial x_i}\right)}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\partial f}{\partial x_i} \, dx$$

$$= \frac{-1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} e^{-ix\cdot\xi} f \, dx \qquad \left[\begin{array}{c} \partial\mathbb{R}^n \text{ ??} \\ \text{term} \\ ! \end{array}\right]$$

maybe u or v become zero at infinite region $\boxed{e^{-x}}$

$$= \frac{i\xi_i}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

$$= i\xi_i \hat{f}(\xi)$$

Note: $$\int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\partial f}{\partial x_i} \, dx = \lim_{R\to\infty} \int_{B(0,R)} e^{-ix\cdot\xi} \frac{\partial f}{\partial x_i} \, dx$$

$$= \lim_{R\to\infty} \left[-\int_{B(0,R)} \frac{\partial (e^{-ix\cdot\xi})}{\partial x_i} \, dx + \underbrace{\int_{\partial B(0,R)} e^{-ix\cdot\xi} \cdot f(x) \nu_i \, dx}_{0}\right.$$

for $R >> 0$ as f has compact support

Heat Eq^n: $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$

$u(x,0) = g(x)$ for $x \in \mathbb{R}^n$

Assume the solution is "smooth". Take fourier transform in x-variable

$$\widehat{\left(\frac{\partial u}{\partial t} - \Delta u\right)} = \hat{0} \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

$$\Rightarrow \widehat{\left(\frac{\partial u}{\partial t}\right)} - \widehat{(\Delta u)} = 0$$

IBP: $\int_{\Omega} u_{x_i} v \, dx = -\int_{\Omega} u v_{x_i} \, dx + \int_{\partial\Omega} u v \nu_i \, dS$ where $\nu_i(x)$ is the ith component of unit normal vector ν of $\partial\Omega$ at $x \in \partial\Omega$.

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$\Rightarrow$ $\frac{\partial (\hat{u})}{\partial t} - \widehat{(\Delta u)} = 0$ in $\mathbb{R}^n \times (0,\infty)$

$$\hat{u}(\xi,0) = \hat{g}(\xi)$$

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$$

$$\check{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi$$

$$\widehat{(\Delta u)}(\xi) = \sum_{i=1}^{n} \widehat{\frac{\partial^2 u}{\partial x_i^2}}(\xi)$$

By lemma 2; $= \sum (i\xi_i)^2 \hat{u}(\xi)$

$= -|\xi|^2 \hat{u}(\xi)$

Thus our heat eq^n looks like,

$$\frac{\partial \hat{u}}{\partial t} + |\xi|^2 \hat{u}(\xi) = 0$$

keeping ξ fixed, we get an ODE in t-variable and solving the ODE we get

$$\hat{u}(\xi,t) = c e^{-|\xi|^2 t}$$

and $\hat{u}(\xi,0) = \hat{g}(\xi)$

Therefore

$$\hat{u}(\xi,t) = \hat{g}(\xi) e^{-|\xi|^2 t}$$

Therefore, using the def^n of inverse fourier tranf

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} \hat{g}(\xi) e^{-|\xi|^2 t} \, d\xi$$

$$= \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{-|\xi|^2 t} g(y)\,dy\,d\xi$$

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$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} g(y) \left[\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{-|\xi|^2 t} d\xi \right] dy$$

Define $K(x,y,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} e^{-|\xi|^2 t} d\xi$

: final aim is to calculate $K(x,y,t)$

via contour integral on C_1 to find $K(x,y,t)$
for $n=1$:-

$$K(x,y,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}}$$

for general dim

$$K(x,y,t) = \prod_{i=1}^{n} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x_i-y_i|^2}{4t}} = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}$$

Hence

$$\boxed{u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy} \quad \text{— (A)}$$

continuous & bounded

Theorem: Assume $g \in C_b(\mathbb{R}^n)$ and define u by (A). Then
(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$
(ii) $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$
(iii) $\lim_{\substack{(x,t)\to(x_0,0) \\ (x,t)\in\mathbb{R}^n\times(0,\infty)}} u(x,t) = g(x_0)$
for any $x_0 \in \mathbb{R}^n$

Fundamental solⁿ of Heat Eqⁿ:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

Properties of ϕ :- take $y = \frac{x}{2\sqrt{t}}$

(i) $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $\int_{\mathbb{R}^n} \phi(x,t) = 1 \Rightarrow \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx = (2\sqrt{t})^n \int_{\mathbb{R}^n} e^{-|y|^2} dy = (2\sqrt{t})^n (\sqrt{\pi})^n$
$= (4\pi t)^{n/2}$

Proof: (i) $u = \phi * g$ as ϕ is smooth we get $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t - \Delta u = \frac{\partial}{\partial t}\left[\int_{\mathbb{R}^n} \phi(x-y,t)\, g(y)\, dy\right] - \Delta \int_{\mathbb{R}^n} \phi(x-y,t)\, g(y)\, dy$

$$= \int_{\mathbb{R}^n} \left[\frac{\partial \phi(x-y,t)}{\partial t} - \Delta\phi(x-y,t)\right] g(y)\, dy$$

As ϕ decays rapidly to zero when $|x| \to \infty$ → (?? why need)

$= 0$ as ϕ is sol^n

(iii) Fix $\epsilon > 0$, $\exists\, \delta << 1$, $|x - x_0| < \delta/2$

Take $|u(x,t) - g(x_0)|$

$= \left| \int_{\mathbb{R}^n} \phi(x-y,t)\,(g(y) - g(x_0))\, dy \right| = \left| \int_{B(x_0,\delta)} \phi(x-y,t)(\;) + \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \right|$

I + J

$$I = \left| \int_{B(x_0,\delta)} \phi(x-y,t)\,(g(y) - g(x_0))\, dy \right|$$

$$\le \epsilon \int_{B(x_0,\delta)} \phi(x-y,t)\, dy \le \epsilon$$

$$J = \left| \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} (g(y) - g(x_0))\, dy \right|$$

$y \in B(x_0,\delta)^c \ni |x_0 - y| > \delta$

$|x_0 - y| \le |x-y| + |x - x_0| \le |x-y| + \delta/2 \le |x-y| + \frac{|x_0 - y|}{2}$

$|x-y| > \frac{|x_0 - y|}{2}$

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$$|J| \le \frac{2\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x-y|^2}{4t}} dy$$

$$\le \frac{2\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|x_0-y|^2}{16t}} dy$$

$$= \frac{2\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi t)^{n/2}} \int_\delta^\infty \int_{\partial B(x_0,r)} e^{-\frac{|x_0-y|^2}{16t}} dr\, dS(y)$$

$$= \frac{2\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi t)^{n/2}} \int_\delta^\infty e^{-\frac{r^2}{16t}} r^{n-1} \omega_n\, dr$$ (using fubini change integrals)

$$\frac{r}{4\sqrt{t}} = s$$

$$= \frac{2\|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi \cancel{t})^{n/2}} \int_{\frac{\delta}{4\sqrt{t}}}^\infty e^{-s^2} (4s\sqrt{t})^{n-1}\, 4\cancel{\sqrt{t}}\, ds$$

$$= \frac{2\, 4^n \|g\|_{L^\infty(\mathbb{R}^n)}}{(4\pi)^{n/2}} \int_{\delta/4\sqrt{t}}^\infty e^{-s^2} s^{n-1}\, ds$$

as $t \to 0$ As $\int_0^\infty s^{n-1} e^{-s^2} ds < \infty$

we get $\lim_{t\to 0} \int_{\delta/4\sqrt{t}}^\infty s^{n-1} e^{-s^2} ds = 0$??

thus $\lim_{\substack{(x,t)\to(x_0,0)\\ (x,t)\in\mathbb{R}^n\times(0,\infty)}} u(x,t) = g(x_0)$

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We have $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$
$u(x,0) = g(x)$ on $\mathbb{R}^n$

solution $u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy$ where $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}}$

For $u_t - \Delta u = f(x,t)$ in $\mathbb{R}^n \times (0,\infty)$ — ①
$u(x,0) = 0$ in $\mathbb{R}^n$

Note $\Phi(x-y,t)$ a fundamental sol^n of $u_t - \Delta u = 0$ for fixed $y \in \mathbb{R}^n$.
$x \mapsto \Phi(x-y, t-s)$ a fundamental sol^n of $u_t - \Delta u = 0$ for any fixed $y \in \mathbb{R}^n$, $s>0$ st. $t>s$. (we just use change of variable)

$$u(x,t,s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy$$

$u(x,t,s)$ solves the following eq^n
$u_t(x,t,s) - \Delta u(x,t,s) = 0$ in $\mathbb{R}^n \times (s,\infty)$
$u(x,s,s) = f(x,s)$ for all $x \in \mathbb{R}^n$

Duhamel's Principle
Define $u(x,t) = \int_0^t u(x,t,s)\, ds$

we check that $u(x,t)$ solves ①

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, f(y,s)\, dy\, ds \quad \text{— ②}$$

Note ϕ has singularity at $t=0$, therefore it is crucial to take $\frac{\partial}{\partial t}$ on f ??

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Theorem: Assume $f \in C^2(\mathbb{R}^n \times [0,\infty))$ and assume f has compact support. Then u defined by ②

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t - \Delta u = f$ in $\mathbb{R}^n \times (0,\infty)$

Do this!! ← (iii) $\lim_{(x,t)\to(x_0,0)} u(x,t) = 0$ for any $x_0 \in \mathbb{R}^n$

(i) By convolution property it's clear that $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u(x,t) = \int_0^t \int_{\mathbb{R}^n} \phi(y,s) f(x-y, t-s)\, ds\, dy$

(Leibnitz rule for diff.) $u_t = \int_0^t \int_{\mathbb{R}^n} \phi(y,s)\, \partial_t f(x-y, t-s)\, ds\, dy + \int_{\mathbb{R}^n} \phi(y,t) f(x-y,0)\, dy$

$\left\{ \frac{d}{dt}\int_0^t f(t,x)\,dx = f(t,t) + \int_0^t f_t(x,t)\,dx \right\}$

$\Delta u = \int_0^t \int_{\mathbb{R}^n} \phi(y,s)\, \Delta f(x-y, t-s)\, ds\, dy$

$$(u_t - \Delta u)(x,t) = \int_0^t \int_{\mathbb{R}^n} \phi(y,s) \left(\frac{\partial}{\partial t} - \Delta\right) f(x-y, t-s)\, ds\, dy + \int_{\mathbb{R}^n} \phi(y,t) f(x-y,0)\, dy$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \phi(y,s) \left(\frac{\partial}{\partial t} - \Delta\right) f(x-y, t-s)\, ds\, dy + \int_0^\varepsilon \int_{\mathbb{R}^n} \phi(y,s) \left(\frac{\partial}{\partial t} - \Delta\right) f(x-y, t-s)\, ds\, dy + \int_{\mathbb{R}^n} \phi(y,t) f(x-y,0)\, dy$$

$= I_\varepsilon + J_\varepsilon + K$

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$$|J_\varepsilon| = \left| \int_0^\varepsilon \int_{\mathbb{R}^n} \phi(y,s) \left(\frac{\partial}{\partial t} - \Delta\right) f(x-y, t-s)\, dy\, ds \right|$$

(2)

$$\leq \|D^2 f\|_{L^\infty(\mathbb{R}^n \times [0,t])} \underbrace{\int_0^\varepsilon \int_{\mathbb{R}^n} \phi(y,s)\, dy\, ds}_{=1}$$

$$\leq \varepsilon \|D^2 f\|_{L^\infty}$$

$$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^n} \phi(y,s) \left(-\frac{\partial}{\partial s} - \Delta\right) f(x-y, t-s)\, dy\, ds$$

(IBP, 1 wrt time 2 wrt space)

$$\int_{\mathbb{R}^n} \left[\int_\varepsilon^t \phi(y,s) \left[-\frac{\partial f(x-y,t-s)}{\partial s} \right] ds \right] dy$$

$$= \int_{\mathbb{R}^n} -\left[\phi(y,s) f(x-y,t-s) \right]_\varepsilon^t dy + \int_{\mathbb{R}^n} \int_\varepsilon^t \frac{\partial \phi(y,s)}{\partial s} f(x-y,t-s)\, ds\, dy$$

and

$$\int_\varepsilon^t - \int_{\mathbb{R}^n} \phi(y,s) \underbrace{\Delta f(x-y,t-s)}\, dy\, ds$$

wrt x or y, same result.

$$= \Sigma \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial y_i} \frac{\partial f(x-y,t-s)}{\partial y_i}\, dy - \lim_{R\to\infty} \int_{\partial B(0,R)} \phi \frac{\partial f}{\partial y_i} \cdot v_i\, dy$$

$$= -\Sigma \left(\int_{\mathbb{R}^n} \frac{\partial^2 \phi}{\partial y_i^2} f(x-y,t-s)\, dy + \lim_{R\to\infty} \int_{\partial B(0,R)} \frac{\partial \phi \cdot v_i}{\partial y_i} f(x-y,t-s)\, dy \right)$$

By green th

$$\int_{\partial B} \left(f \frac{\partial \phi}{\partial v} - \phi \frac{\partial f}{\partial v} \right) dy = \int_{\mathbb{R}^n} (f \Delta \phi - \phi \Delta f)\, dy$$

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$$u_t - \Delta u = \lim_{\epsilon \to 0} \left[I_\epsilon + J_\epsilon + K \right]$$

$$= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \phi(y, s) f(x-y, t-s)\, dy$$

$$= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \phi(x-y, s)\, f(y, t-s)\, dy$$

$$= \lim_{\epsilon \to 0} f(x, t-s)$$

$$= \underline{\underline{f(x,t)}}$$

from last theorem

$$\lim_{x,t \to x_0, 0} u(x,t) = \lim \int \phi(x-y) g(y)\, dy = \underline{\underline{g(x_0)}}$$

Result:

$$u_t - \Delta u = f \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$u(x, 0) = g \quad \text{on } \mathbb{R}^n$$

$$u(x,t) = \int_{\mathbb{R}^n} \phi(x-y, t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \phi(x-y, t-s)\, f(y,s)\, dy\, ds$$