PHI455: Normal Modal Logic

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Agenda

1. Background: Limitations of Classical Logic
2. What is Modal Logic? Why Modal Logic?
3. Origin of Modal Logic: C. I. Lewis on Strict Implication
4. Syntax: Various modal logic Axiomatic systems, Proofs in K, T, D, B, S4, S5.
5. Semantics: What are Possible Worlds?
6. Semantics of Modal Logic: Kripke model theoretic semantics: Examples
7. Validity of Modal Logic formulas in various frames: Semantic tableaux Method.

Clasical Logic

1. Systematic study of principles of valid reasoning, study of logical connectives $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$
2. Law of Identity, Law of excluded middle $(\alpha \vee \neg \alpha)$, Law of Non-contradiction $\neg(\alpha \wedge \neg \alpha)$.
3. Double negation $\neg \neg \alpha \rightarrow \alpha$.
4. Reductio ad absurdum $(\neg \alpha \rightarrow \alpha) \rightarrow \alpha$ If the asumption that α does not hold implies α then α must hold.
5. Pierce's law: $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$

Limitations of Classical Logic

1. The failure to provide a clear and satisfactory explanation of the bizarre but mathematically correct and combinatorially necessary properties of implication is a major weakness of classical logic.
2. Paradoxes of Implication: If liars have integrity, cats can walk on water.
3. It fails to distinguuish possibility, actualitiy, necessity. formalizing statements about what must be true, what could be true, and what cannot be true is essential.
4. Arguments about knowledge, belief, and free will

Limitations of Classical Logic

1. Classical logic studies formulas that are true (especially those that are true in all interpretations, i.e., valid) and how truth is preserved in reasoning such that true premises only have true consequences.
2. We write $\Gamma \models A$ if formula A is a logical consequence of Γ. We just write $\models A$ if formula A holds in all interpretations and say that A is valid.
3. Formulas of the form $A \vee \neg A$ are always trivially valid in classical logic, because each interpretation Γ satisfies either $\Gamma \models A$ or $\Gamma \models \neg A$. Consequently, $\models A \vee \neg A$.
4. Classical Logic is extensional whereas Modal Logics are intensional.

Various other types of Modal Logic:

1. **deontic modal logic**, dealing with statements such as
   It is compulsory that p
   It is forbidden that p
   It is permissible that p, etc
2. **temporal modal logic**, dealing with statements such as
   It is always true that p
   It is sometimes true that p.
3. **ethical modal logic**, dealing with statements such as
   It is good that p
   It is bad that p

Limitations of Classical Logic: Example

First order predicate logic can't deal with modality very well. Take the following arguments:

1. Wittgenstein is a philosopher. Therefore Wittgenstein could have been a philosopher
2. The metal piece Copper is essentially a material object. Therefore Copper piece is a material object

Try formalizing them in predicate logic: Call the former F and the latter G. The form of the first argument would then be: Fa and the second, Ga

Dissatisfaction with Material Implication

1. **Irrelevance/non-causality:** If the Sun is hot, then 2+2=4.
2. **Ex falsum quodlibet**: If 2+2=5 then the Moon is made of cheese.
3. **Monotonicity:** If I put sugar in my tea, then it will taste good. Therefore, If I put sugar and I put petrol in my tea then it will taste good.

Dorothy Edgington's Proof of the Existence of God

there are counter-intuitive results to formulate if...then as $\neg p \vee q$.

Argument

If God does not exist($\neg G$), then it is not the case that if I pray (P), my prayers will be answered(A): $(\neg G \to \neg(P \to A))$.
I dont Pray $(\neg P)$.
Therefore, God Exist (G).

Contd...

Formal Analysis

1. $(\neg G \rightarrow \neg(P \rightarrow A))$
2. $(\neg G \rightarrow (P \wedge \neg A))$
3. $(G \vee (P \wedge \neg A))$
4. I dont pray: $\mathrm{P} = 0$
5. $G \vee (0 \wedge \neg A)$
6. $G \vee 0$
7. **therefore** G (God Exist).

Example

Prosecutor: If Eric is guilty then he had an accomplice.
Defense: I disagree
Judge: I agree with the defense.

Example

Example (Example1)

If a new course is to be offered next year, then submission must be made to the faculty board before April. If submissions are to be made to the faculty member before April, then Departmental meeting must be called. A week's notice must be given if a Departmental meeting is to be called. Since it is not possible to give such notice, it follows that it is not possible to offer a new course next year.

Example: Logical representation

1. N = A new course is to be offered next year.
2. S = Submissions will be made to the Faculty Board before April.
3. D = A departmental meeting is to be called.
4. W = A week's notice is to be given
5. $\{N \rightarrow S, S \rightarrow D, D \rightarrow W, \neg W\} \vdash \neg N$

What is Modal Logic?

1. Modal logic studies reasoning that involves the use of the expressions necessarily and possibly.
2. Ways things must be, and the ways things might have been.

What is Modal Logic?

Modal logic is a type of formal logic primarily developed in the 1960s that extends classical propositional and predicate logic to include operators expressing modality. A modal—a word that expresses a modality—qualifies a statement (Wikipedia).

Modal Propisitions in English language

There are numerous kinds of expression that have modal meanings, the following is just a subset of the variety one finds in English:

1. Modal auxiliaries: Krishna must/should/might/may/could be home.
2. Semimodal Verbs: Krishna has to/ought to/needs to be home.
3. Adverbs: Perhaps, Krishna is home.
4. Nouns: There is a slight possibility that Krishna is home.
5. Adjectives: It is far from necessary that Krishna is home.
6. Conditionals: If the light is on, Krishna is home.

Example

Example

It is Summer in Kanpur.

Think of ways in which we intend its truth or falsity of proposition.

1. Is it necessarily Summer? Is it known that it is summer? It it believed to be summer? Is it summer now or will be summer in future? If I fly to Melbourne, will it be Summer?
2. These are modifications of our initial assertion: It is summer in Kanpur

Sronger commitment of modalities

1. Ravi must(has) to be home by now.
2. Ravi should be home by now.
3. Ravi might be home by now.

statement 1 expresses stronger commitment on the part of the speaker to the truth of the base proposition

Epistemic and Deontic Modalities

1. Epistemic modality indicates possibility and necessity relative to the speaker's knowledge of the situation, i.e., whether the proposition is possibly or necessarily true in light of available evidence.
2. Deontic modality indicates possibility and necessity relative to some authoritative person or code of conduct which is relevant to the current situation.

1. Ravi didn't show up for work. He must be sick (Epistemic)
2. Ravi didn't show up for work. He must be fired (Deontic)

Modal Propositions

1. It is possible that it will rain tomorrow.
2. It is possible for humans to travel to Mars.
3. It is not possible that: every person is mortal, Socrates is a person, and Socrates is not mortal.
4. It is necessary that either it is raining here now or it is not raining here now.
5. A proposition p is not possible if and only if the negation of p is necessary. $\Diamond p \equiv \neg \Box \neg p$.
6. It is necessary that All grandmothers are mothers of some parents.

What is Modal Logic?

1. Modal logic is the study of modal propositions and the logical relationships that they bear to one another.
2. Modal Logics are [the] Logics of Qualified truth.
3. A systematic study of Logic of *Necessity* and *possibility*.

Von Wright on catalogue of Different Modalities

- Truth or False
- Alethic: Necessary, contingent, possibility.
- Epistemic Modalities: Knowledge, Belief (doxastic).
- Temporal: Always, Sometimes
- Boulomatic modalities: desire, wishes etc
- Deontinc modalities: duty, permission, obligation, commitment, juristic norms.
- evaluative modalities: good, bad , ethical norms
- physical, causal modalities
- Modalities of action (somebody does, did, will do, starts doing
- Mellonic modalities: modalities of process, of becoming

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Alethic Modalities

1. Modal Phrases: perhaps, can, might, must
2. Study of Alethic (truth) modalities: Possibilities and Necessity.
3. Traditional terminology: Necessarily P is an apodeictic judgment, Possibly P a problematic judgment,and P, by itself, an assertoric judgment.
4. Modal Operators: Necessary, obligatory, true after action, (running a computer program), known, knowable, believed, provable, from now on, so far, since and until.

Applications of Modal Logic

Researchers from mathematics, philosophy, computer science, linguistics, political science and economics work on variety of modal logics, focusing on numerous different topics with many amazingly different applications.

1. **Mathematics** Provability Logic: If it is provable that if P is provable then P is true, then P is provable ($\Box(\Box P \to P) \to \Box P$). Mathematicians approach it mostly from a model theoretical point of view.
2. **Philosophers**: It is a powerful tool for semantics. Many concepts in philosophy of language can be formalized in modal logic

1. **Computer Scientists**: Modal logic is used to represent the programs. Model checking and temporal logic are very hot research areas in computer science which use modal logics extensively used.
2. **Linguists**: Modal logics are used in formulating appropriate theory of semantics of various linguistic expressions.
3. **Political Theorists and Economists** Modal Logic is important in analyzing fair division algorithms. Especially, welfare theories that have game theoretical motivation with an underlying modal logical intuition.

Example

Example (Example1)

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Intension vs Extension

1. The Morning star is the evening star.
2. The Morning star is the Morning star.

1. The two phrases, “morning star” and “evening star” may designate the same object, but they do not have the same meaning. Meanings, in this sense, are often called intensions, and things designated, extensions.
2. Contexts in which extension is all that matters are, naturally, called extensional, while contexts in which extension is not enough are *intensional*.

Frege

Frege: Predicators (general terms) have

1. Sinn = meaning (sense, intension, comprehension) vs.
2. Bedeutung = reference (denotation, extension, ´etendue)

Example:

The morning star is identical with the evening star.
Reference of a name (nominator) is the named object,
meaning is the mode how the object is given.

Remarks

1. Mathematics is typically extensional throughout. we usually write $3 + 2 = 2 + 3$
2. In classical first order logic intension plays no role. It is extensional by design since primarily it evolved to model the reasoning needed in mathematics.

Extending Classical Logic

Modalities

Modality is any word or phrase that can be applied to a given statement X to create a new statement that makes an assertion about the mode of truth of X:
These Modalities are about when, where or how X is true, or about the circumstances under which S may be true.

Modes of truth

In modal logic we provide extensions to the concept X is true. For example, we define concepts (or modalities) such as:

1. X is believed to be true
2. X is known to be true
3. X ought to be true
4. X is eventually true
5. X is necessarily true

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Types of Modal Logic

Alethic modal logic : AML is dealing with statements such as,

It is necessary that p,*It is possible that p*

Epistemic Modal logic . EML deals with statements such as

I know that p

I believe that p

Necessity and Possibility:

1. $\Box p$**:** It is necessary that p, It must be that p
2. $\Diamond p$**:** It is possible that p, Possibly p, It could be, might be that p, It can be that p, It might be that p. It is possible that p imply also that it might not be possible, in some other situation.

$\Diamond$ p is usually translated as possibly p, or it is possible that p, or p might be true.
$\Box$ p is standardly translated as Necessarily p, or it is necessary that p, or p must, should be true.

Logical Equivalences

1. $\neg \Diamond p \equiv \Box \neg p$
2. $\neg \Box p \equiv \Diamond \neg p$.
3. $\Box p \equiv \neg \Diamond \neg p$.
4. $\Diamond p \equiv \neg \Box \neg p$.

Translation in Propositional Modal Logic

Translation

1. p is contingent translates to $\nabla p \; = \; (\Diamond p \wedge \Diamond \neg p)$
2. p is not contingent translates to $\neg((\Diamond p \wedge \Diamond \neg p))$. It is same as $(\Box \neg p \vee \Box p)$
3. p is contradictory: $\Box \neg p$
4. p is analytic: $(\Box p \vee \Box \neg p)$.
5. p is consistent , compatible with q: $\Diamond (p \wedge q)$
6. p is compatible with q: $(p \circ q) \; \equiv \Diamond (p \wedge q)$
7. p and q are contradictory $\neg \Diamond (p \wedge q)$ and $\neg\Diamond((\neg p \wedge \neg q)$

Some Modal Propositions

1. There are nine planets in the solar system.
2. The Square root of 9 is 3.
3. It is possible that tomorrow it will rain in Kanpur
4. It is possible for humans to travel to Mars and it might have been the case that there is water on Mars.
5. It is necessary that 2+2 is 4
6. It is known to us that Mr. Narendra Damodhar Modi is the current Prime Minister of India.
7. It is obligatory that Doctors needs to address emergency cases.
8. A proposition p is not possible if and only if the negation of p is necessary.

Modal Propositions

Here are numerous kinds of expression that have modal meanings, the following is just a subset of the variety one finds in English:

Modal auxiliaries : Sandy must/should/might/may/could be home.

Semimodal Verbs : Sandy has to/ought to/needs to be home.

Adverbs :Perhaps, Sandy is home.

Nouns : There is a slight possibility that Sandy is home.

Adjectives : It is far from necessary that Sandy is home.

Conditionals : If the light is on, Sandy is home.

These extensions makes sense in the context of possible worlds or alternate universes. An alternate universe is one whose characteristics or history differs from our own.

What could have been

Modal statements are about what could have been and they occur for example in the following examples.

1. Hitler could have won World War II;
2. I could have been a fisherman or farmer.
3. The speed of light could have been twice as fast as it actually is;
4. Swans could have been black;
5. It is impossible for there to be round squares
6. Necessarily, $2+2=4$.

Counterfactuals

Modal statements also include counterfactual statements. Counterfactual statements are conditional statements in which the antecedent is always false.

1. **Scientific:** If the speed of light were faster, atomic explosions would be more deadly;
2. **Ethical:** If you hadn't taken bribes and involved in corruption, he would not have lost his job;
3. **Everyday:** If I hadn't taken antibiotics last night for my toothache,I wouldnt have slept well.

Syntactic Approach

1. It is concerned with systems of modal sentences.
2. A system is a class of modal sentences, say Γ, with the property that of $A \in \Gamma$, $A \to B \in \Gamma$, then $B \in \Gamma$.
3. systems are closed with respect to Modus ponenes.
4. If S is a system, we refer to members of S as theorems of S
5. We write $\vdash_S A$ in place of $A \in S$.
6. System already includes PC.

Axioms of PC

$$
\begin{aligned}
\text{A1} &: \quad A \to (B \to A) \\
\text{A2} &: \quad A \to (B \to C) \to (A \to B) \to (A \to C). \\
\text{A3} &: ((A \to \bot) \to \bot) \to A \\
\text{K} &: \quad \Box(A \to B) \to \Box A \to \Box B
\end{aligned}
$$

MP: From A and $A \to B$, to derive B.

Language of Modal propositional Logic

1. Propositional Logic$+$ $\Box$, $\Diamond$.
2. $\{p, q, r, s\}$ are atomic formulas.

Definition (Backus Naur Form(BNF))

$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid (\phi \wedge \psi) \mid \phi \vee \psi \mid (\phi \to \psi) \mid \phi \leftrightarrow \psi \mid \Box\phi \mid \Diamond\phi$. p is any atomic formula.

Definitions

$$
\begin{aligned}
\text{D1} &: \neg A = (A \to \bot) \\
\text{D2} &: A \vee B = \neg A \to B \\
\text{D3} &: A \wedge B = \neg(\neg A \vee \neg B) \\
\text{D4} &: A \leftrightarrow B = (A \to B) \wedge (B \to A) \\
\text{D5} &: \bot = \neg \top.
\end{aligned}
$$

Modal Definitions

D6 : $\Diamond A = \neg \Box \neg A$

D7 : $\Box^n A = \Box \ldots\ldots \Box A$, $\Box^o A = A$

Example

1. $(p \wedge \Diamond(p \rightarrow \Box\neg r)$
2. $\Box((\Diamond q \wedge \neg r) \rightarrow \Box p)$

Propositional Modal Formulas

The set of formulas is specified by the following rules:

1. Every propositional letter (p, q r) is a formula.
2. If X is a formula, so is $\neg X$.
3. If X and Y are formulas, and $\circ$ is binary connective,, $X \circ Y$ is a formula.
4. If X is a formula, so are $\Box X$ and $\Diamond X$.

Some text books follow L, M for Modal Operators.

Language of Modal Logic

1. Examples of well formed formulas: $\Box\Diamond p$, $q \to \Box\Diamond(p \to q)$.
2. $q\Box\neg \to\ p \lor\ \land q$ is not a well formed formula.
3. Language of Propositional logic: Atomic propositions or propositional variables(p,q,r...), Metavariables (α,β, γ or ϕ, ψ, Logical connectives $\{\neg, \to, \land, \lor, \leftrightarrow\}$.

Language of Modal Logic

1. The symbol $\top$ is used for a constant true formula, equivalent to any tautology, while $\perp$ is a constant (always false) false formula, equivalent to $\neg\top$. It is defined as follows: $\neg p =_{\mathrm{Def}} p \to \perp$.
2. **Dual and Unary Operators** $\Box p$: It is necessary that p, $\Diamond p$: It is possible that p.
3. $\Box p = \neg \Diamond \neg p$; $\Diamond p = \neg \Box \neg p$

Convention

We assume that the unary connectives ($\neg, \Box, \Diamond$), bind most closely, followed by $\wedge$, $\vee$ and then followed by $\rightarrow$, $\leftrightarrow$

Example (Parse trees)

1. $(p \wedge \Diamond(p \rightarrow \Box\neg r)$
2. $((\Box\Diamond q \wedge \neg r \rightarrow \Box p))$.

Normal Modal Logic

Definition (Properties of Normal Modal Logic L)

1. L contains all Tautologies
2. L contains all the instances of distribution axiom (K): $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$.
3. L contains all instances of the formula scheme Dual: $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$.
4. L is closed under Uniform substitution and Modus ponens
5. L is closed under the rule of necessitation.

Non-Normal Modal Logic

Non-normal worlds are those worlds where the truth conditions of modal operators are different.

If the World is non-normal then the following holds:

1. $v_w(\Box A) = 0$
2. $v_w(\Diamond A) = 1$

In a non-normal world, A is false, no matter what A is. So, even $p \vee \neg p$ and $p \rightarrow p$ are false at such worlds. In turn, this means that the rule of necessitation can not be universally applied.

Minimal Modal Logic: K

1. Its axioms include all tautologies of propositional logic(PL) plus the following sentence, which is called K.
   $\Box\,(p \rightarrow\ q)\ \rightarrow\ (\Box p \rightarrow \Box q)$.
2. **Uniform Substitution:** It says that the result of uniformly replacing sentence letters in a theorem with arbitrary sentences is a theorem. Uniformly substituting $\neg p$ for p in **??**results in the formula: $\Box(\neg p \rightarrow q) \rightarrow (\Box\neg p \rightarrow \Box q)$
3. Necessitation: $\vdash \alpha \Rightarrow \Box\alpha$. Necessitation says that if ϕ is a theorem then necessarily ϕ is also a theorem. The intuition behind Necessitation is the following: if ϕ is a theorem, then it is valid; if it is valid, it is true in all worlds in all models
4. **Modus Ponens:** From ϕ and $\phi \rightarrow \psi$ derive ψ.

Definition

Definition ($\vdash_k \alpha$)

$\vdash_K \alpha$ if and only if there is a sequence of formulas of which α is the last such that every formula in the sequence is either an axiom of K or else is derived by means of one of the rules from formulas appearing higher up in the sequence.

Proofs

$p \to q \vdash_K \Box(p \to q)$

1. $p \to q$ Assumption
2. $\Box(p \to q)$ Necessitation Rule on 1
3. $\Box(p \to q) \to \Box p \to \Box q$ Axiom K
4. $\Box p \to \Box q$ MP 2,3 $\Box$

$$(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$$

Proof.

1. $\vdash \phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)$ Tautology
2. $\vdash \Box(\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$ RN
3. $\Box(\phi \rightarrow \psi) \rightarrow \Box\phi \rightarrow \Box\psi$ K-Axiom
4. $\vdash \Box(\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)) \rightarrow (\Box\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)))$ Uniform Substitution, 3.
5. $\vdash (\Box\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)))$ 2,4 Modus ponens
6. $\vdash \Box(\phi \rightarrow (\phi \wedge \psi)) \rightarrow (\Box\phi \rightarrow \Box(\psi))$.
7. $\vdash \Box\phi \rightarrow (\Box\psi \rightarrow \Box(\phi \wedge \psi))$.
8. $\vdash (\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$ $\Box$

$\Box$

Other Modal Logic Systems:

Other axiomatic systems of Modal Logic

1. **D** $\Box\phi \to \Diamond\phi$
2. **T** $\Box\phi \to \phi$.
3. **B** $\phi \to \Box\Diamond\phi$
4. **4** $\Box\phi \to \Box\Box\phi$.
5. **5** $\Diamond\phi \to \Box\Diamond\phi$.
6. **G:** $\Diamond(\Box\phi \to \phi) \to \Box\phi$.
7. **Tr:** $\Box\phi \leftrightarrow \phi$.
8. **W:** $\Box[\Box\phi \to \phi] \to \phi$.

Other Normal Systems

1. System K= PL+Axiom K+ Necessitation (from $\vdash \phi$ infer $\vdash \Box\phi$)
2. System T: K + Axiom D
3. System S4: K + Axioms T and 4
4. System S5: K + Axiom T, 4, 5 System G: K+ Axiom 4, G

Distribution

1. $\Box$ distributes over $\rightarrow$ but not $\vee$- disjunction
2. Tigers exist or Tigers doesnt exist.
3. It is neither necessary that they exist nor not necessary that they dont exist.

Translations in Propositional Logic

1. Not p: $\neg P$
2. Both p and q: $(p \wedge q)$
3. Either p or q: $(p \vee q)$
4. If p then q: $p \rightarrow q$.
5. Possibly p: $\Diamond p$
6. Neccesarily p: $\Box p$
7. p inspite of q: $p \wedge q$.
8. p only if q; p is sufficient for q, q is necessary for p: $p \rightarrow q$.

Examples: Scope of Modal Operator

1. $\Box(p \to p) \wedge (q \to r))$
2. $\Box(p \wedge q) \to (p \to q)$
3. If Socrates is human then he must be mortal.
4. Of necessity, if Socrates is human then he is moral.

Modalities operate over the whole conditional.

Translation:

1. p is necessary:$\Box p$; p is contradictory: $\Box\neg p$; p is possible: $\Diamond p$.
2. p is contingent (p) translates to $(\Diamond p \wedge \Diamond\neg p)$.$p$ is not contingent translates to $\neg(\Diamond p \wedge \Diamond\neg p)$.
3. p is analytic: $(\Box p \vee \Box\neg p)$.
4. p is consistent with q is translated as $(\Diamond(p \wedge q)$; p is compatible with q $(p \circ q)$ as $\Diamond(p \wedge q)$.
5. p is incompatible with q: $\neg\Diamond(p \wedge q)$.

Examples

1. Necessarily, if snow is white, then snow is white or Grass is green.: $\Box[S \to (S \vee G)]$.
2. I will go if I must: $\Box G \to G$.
3. If snow could have been green, then grass could have been white: $\Diamond G \to \Diamond W$.
4. It is impossible for snow to be both white and and not white: $\neg\Diamond(W \wedge \neg W)$.
5. God's being merciful is inconsistent with your imperfection being incompatible with your going to heaven: $\neg\Diamond(M \wedge \neg\Diamond(I \wedge H))$.
6. Nothing is absolutely relative: $\neg\Box(\Diamond p \wedge \Diamond\neg p)$

Origin of Modal Logic:2

1. The problem of future contingents: a logical paradox by Diodorus Cronus, Megarian school of philosophy.
2. what happens was necessarily going to happen
3. what does not happen was necessarily going to not happen
4. Aristotle: statements about the future are neither true nor false

Aristotle De Interpretatione

Translation by Ross

……if a thing is white now, it was true before to say that it would be white, so that of anything that has taken place, it was always true to say it is or it will be. But if it was always true to say that a thing is or will be, it is not possible that it should not be or not come to be, and when a thing cannot not come to be, it is impossible that it should not come to be, and when it is impossible that it should not come to be, it must come to be. All then, that is about to be must of necessity take place. It results from this that nothing is uncertain or fortuitous, for if it were fortuitous it would not be necessary.

Ross, W.D. 1923. Aristotle. London: Methuen and Co.

Origin of Modal Logic:2

1. After a long tradition, starting with the *father of logic*, Aristotle, himself, modal logic had been expelled from logical consideration by Frege.
2. Material implication is chosen to explain the strict implication.
3. C. I Lewis : Entailment to be a much stronger relation between propositions than is expressed by the arrow entails not merely means that it is not the case that p and $\neg p$, but rather that it is impossible for this to be the case

Contd

1. $pq =_{\textbf{Def}} \Box(p \to q) =_{\textbf{Def}} \neg\Diamond(p \wedge \neg q)$.
2. Since then, the study of the notions of *possible* and *necessary* overshadowed the original occupation with strict implication and thus modalities were brought back to the attention of logicians.

C I Lewis on Implication

……the expositors of the algebra of logic have not always taken pains to indicate that there is a difference between the algebraic and ordinary meanings of implication[p.522].[1]

[1] C. I. Lewis. Implication and the algebra of logic. Mind (New Series), 21:522-531, 1912.

C. I Lewis on Material Implication

As indicated in the preceding chapter, we have here the further purpose to develop a calculus based upon a meaning of implies such that p implies q will be synonymous with *q is deducible from p.* The relation of material implication, which figures in most logistic calculus of propositions, does not accord with this usual meaning of *implies*. It leads to those paradoxes such as A false proposition implies every proposition and A true proposition is implied by any which have been set forth in Chapter **IV a** we shall see, it is entirely possible so to develop the calculus of propositions that it accords with the usual meaning of *implies*......[2](p. 122-3)

[2] Lewis and Langford, Symbolic Logic, Chapter VI, The Logistic Calculus of Unanalyzed Propositions

Paradox of Material Implication

The paradoxes of material implication are a group of formulas which are truths of classical logic, but which are intuitively problematic. One of these paradoxes is the paradox of entailment.

1. $(\neg p \land p) \to q$, which is the paradox of entailment.
2. $\neg\alpha \to (\alpha \to \beta)$
3. $\alpha \to (\beta \to \alpha)$.
4. $(\alpha \to \beta) \lor (\beta \to \alpha)$

The first one2 can be said to mean that if a proposition is true any proposition implies it. The sense of the second3 is that anything is implied by a proposition which is false.
If the moon is made of green cheese, then the world is coming to an end is true merely because the moon isn't made of green cheese.

Usage of Material Implication

Valid Inferences in Propositional Logic

1. $(p \rightarrow q) \wedge (r \rightarrow s) \vdash (p \rightarrow s) \vee (r \rightarrow q)$
2. $(p \wedge q) \rightarrow r \vdash (p \rightarrow r) \vee (q \rightarrow r)$

Example

If Ravi is in Kapur (p) then he is in UttarPradesh (q), and if he is in Hyderabad(r) then he is in Telangana(s). Therefore, it is either true that if Ravi is in Kanpur then he is in Telangana, or that if he is in Uttar Pradesh, then he is in Hyderabad.

Either Ravi is in Kanpur or Ravi is not in Kanpur If Ravi is in Kanpur, then Ravi is in Uttar Pradesh.

Examples

Usage of Material Implication

if Ravi is in Hyderabad, then Ravi is in Uttar Pradesh holds because we have prior knowledge that the conclusion is true. If Ravi is not in Kanpur, then the proposition if Ravi is in Kanpur, then Ravi is in Telangana state is true because we have prior knowledge that the premise is false.

Example-2

If both switch A and switch B are closed, then the light is on. Therefore, it is either true that if switch A is closed, the light is on, or if switch B is closed, the light is on.

If the two switches are in series, then the premise is true but the conclusion is false

For Lewis the ordinary meaning of α implies β is that β can be validly inferred from α, or is deducible from α. If we take this interpretation then we can avoid paradoxes.

C.I. Lews 1912

$A\,B =_{\mathsf{Def}} \Box(A \to B)$.

1. **Extensional Disjunction:** This is simply the material (algebraic) implication synonymous with it is false that α is true and β is false.
2. **Intensional Disjunction:** Intensional disjunction is such that at least one of the disjoined propositions is necessarily true.

Lewis S_1 System

Axioms of S_1

1. $(p \wedge q)(q \wedge p)$
2. $(p \wedge q)p$
3. $p(p \wedge p)$
4. $((p \wedge q) \wedge r)(p \wedge (q \wedge r))$
5. $((pq) \wedge (qr))(pr)$.
6. $(p \wedge (pq))q$

S_2 to S_5

1. $S_2 = S_1 + \Diamond(p \wedge q)$.
2. $S_3 = S_1 + (pq)(\neg\Diamond q \neg\Box p)$.
3. $S_4 = S_1 + \Diamond\Diamond pp$ or Equivalently, $\Box\Box pp$.
4. $S_5 = S_1 + \Diamond p \Box\Diamond p$.

Further References

1. Refer [1, 2]

Some Rules of Inference

Lewis formulated five systems of modal logic of increasing strength: S1 to S5. Only the last two are normal systems.

Rules of Inference

1. Uniform substitution of formulas for propositional variables.
2. Substitution of strict equivalents: from $(\alpha = \beta)$ and Γ infer any formula obtained from Γ by substituting β for some occurrence(s) of α.
3. Adjunction: from α and β infer $\alpha \wedge \beta$.
4. Strict detachment: from α and $\alpha\beta$ infer β.

Intension and Possible Worlds

Definition

The intension of an expression A is the function which assigns to every possible world the extension of A in that world.

1. The intension of an expression determines its extensions in all possible worlds, and vice versa. Two expressions have the same intension iff they have the same extension in any possible world.
2. Assumption: For a set of possible worlds I there is one and the same unique domain of individuals U.
3. The intension of a predicator p is the function which assigns for each possible world $i \in I$ the extension of p in i.
4. The intension of a sentence A is the function which assigns for each possible world $i \in I$ the truth value of A in i.

Towards Intensional Semantics

1. Extensional semantics that models the meaning of sentences based on the extensions of linguistic expressions is limited and cannot handle intensional constructions.
2. What is common in these intensional constructions is that they call for a consideration of extensions that an expression may have in circumstances other than the one in which it is evaluated. This is called an INTENSION of a linguistic expression.
3. In order to get at the intensions, we need to consider alternative ways in which the world might have been, alternative sets of circumstances, or Possible Worlds.
4. The framework that models the meaning of sentences based on the intensions of linguistic expressions is called Intensional Semantics or Possible Worlds semantics.

Possible Worlds and Intensions

Possible Worlds

Possible worlds are possible circumstances in which some (or all) events or states are different from what they in fact are in the actual circumstance.
W is a set of all possible worlds. $W = \{w1w2, w3, ...\}$

Intensions

The intension of a sentence S: proposition (e.g., Raj is funny)
The set of possible worlds in which it is true.
Function from possible worlds to truth values

It is a function from possible worlds to sets of individuals

Possible Worlds: Features

1. It has more objects, events, and individuals than the real one, such as a cancer cure, a bridge between the earth and the moon, a third world war, a ten feet tall person, a Mars expedition, or Superman.
2. It has less objects, events, and individuals than the real one, such as no Pyramids, no Gulf war, and no Shakespeare.
3. It differs with respect to properties, for example, the Sydney opera house is red, and I am a multi-millionaire.
4. It differs with respect to relations, for example, the Great Wall of China is located in the Middle East.

Spheres around Actuality and the necessity scale

1. A proposition has the lowest grade of necessity just in case it is true throughout that sphere, which is so just in case the proposition is true.
2. A proposition P has the lowest degree of necessity just in case it is actually true but is not true throughout any sphere larger than the singleton of the actual world
3. For any conceivable departure from actuality, however small, the departure required to make P false is no larger than that.
4. The largest sphere around actuality is the class of all worlds. A true proposition P has the highest degree (and the highest grade) of necessity just in case P is true throughout that sphere, i.e., true at every world whatsoever.
5. A proposition is necessary iff it is true at all possible worlds. A proposition is possible iff it is true at some possible world.

Logical and Physical Possibility: Examples

1. physical constraint: the speed limit of fastest train of India is 115km/h
2. the actual scenario: in some day, I take the train which runs in a speed, say only 200km/h
3. logical possibility: although the speed is now lower than 125km/h, it is logically possible that it is higher than 200km/h
4. logical impossibility: it is logically impossible that the speed is lower than 125km/h and higher than 200km/h simultaneously.
5. physical possibility: although the speed of fasted movable partcle is now lower than the speed of velocity of light, it is logically possible that it is higher than the speed of light.
6. physical impossibility: it is physically impossible that your Bicycle moves with speed higher than 100km/h

Alternative Universes: Possible Worlds

1. Interpretations for possible worlds, ranging from metaphysical worlds to worlds in science-fiction, states of a computer, board positions in chess, or deals in a card game.[6]
2. However, we require that these alternate universes are logically consistent. There may be alternate universes where Narendra Modi is not the PM of India, but there are no alternate universes where $2 + 2 = 5$.
3. The extent to which alternate universes actually exist is a deep metaphysical question with strong connections to theology and physics. However, for our purposes we will assume that anything that can be imagined without contradiction is a valid alternate universe.

Intuitive Meaning:

formula $\Box\phi$ holds in a given world *w*, if ϕ holds in all worlds alternative for *w*.
formula $\Diamond\phi$ holds in a given world *w*, if ϕ holds in some world alternative for *w*.

Different Kinds of Modalities

1. Deontic possibility (what is compatible with the dictates of morality)
2. logical possibility (what is compatible with the laws of logic)
3. bouletic possibility (what is compatible with a person's desires): Krishna has to work hard if he wants to retire at age 50
4. nomological possibility (what is compatible with the laws of nature)
5. epistemic possibility (often thought of as what is compatible with what is known or believed)

Kai van Fintel 2006

1. It has to be raining. [after observing people coming inside with wet umbrellas; epistemic modality]
2. Visitors have to leave by six pm. [hospital regulations; deontic]
3. John has to work hard if he wants to retire at age 50. [to attain desires; bouletic]2
4. I have to sneeze. [given the current state of one's nose; dynamic]3
5. To get home in time, you have to take a taxi. [in order to achieve the stated purpose; teleological]

https://web.mit.edu/fintel/fintel-2006-modality.pdf

Frame

A Frame consists of a non-empty set F, whose members are generally called Possible Worlds, and a binary relation R, on F, generally called Accessibility relation. if u and v are possible worlds, then $(u\,R\,v)$, means v is accessible from u, or u is accessible to v, or v is an alternative world to u.

Some Definitions

Definition (Kripke Frame)

A Kripke frame $\langle W, R\rangle$ consists of a non empty set of Possible Worlds W and a relation $R = W \times W$ on worlds. The elements of W are called possible worlds and R is called accessibility relation.

Definition (Kripke structure)

A Kripke structure $S = \langle W, R, v\rangle$ consists of Kripke frame $\langle W, R\rangle$ and a mapping $\mathbf{v}$: Atoms$\{\top, \bot\}$ that assigns truth values to all the propositional letters in all worlds.

Kripke Model

A semantics for the basic modal language was developed by Saul Kripke[5], Stag Kanger, Jaakko Hinitkka and others in the 1960s and 1970s.
A model in propositional logic is simply a valuation function assigning truth values to the set of atoms.

Definition (Relational Structure)

A Relational Structure (also called a possible worlds model, Kripke model or a modal model) is a triple $M = \langle W, R, V \rangle$ where W is a nonempty set (elements of W are called states), R is a relation on W (formally, $R \subseteq (W \times W)$ and V is a valuation function assigning truth values $V(p, w)$ to atomic propositions p at state w (formally V: $S \times W \rightarrow \{T, F\}$;) where S is the set of sentence letters.

Example

I know that, if the door is open,then either I forgot to close it, or I was burglarized. Secondly, I also know that, if the window is broken,then either there was a football accident, or I was burglarized. Now, as a matter of fact, the window is broken and the door is open. But, it is inconceivable that I both forgot to close the door and there was a football accident. Therefore, I know that I was burglarized.

Rod Girle: Modal Logic for Philosophers PP 21

PP: 21

Truth in a Model

Let (W, R, V) be a model. The relation V is extended to arbitrary formulas as follows. For $\Gamma \in W$:

1. $\Gamma \models \neg X$ iff $\Gamma \not\in X$. $\Gamma \models X$ means X is true in Γ.
2. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.
3. $\Gamma \models \Box X$, iff for every $vinW$, if $u\,R\,v$ then $v \models X$.
4. $\Gamma \models \Diamond X$ iff for some $v \in W$, $(u\,R\,v)$ and $v \models X$.

Example

Example (Example of Relational Structure:)

Often relational structures are drawn instead of formally defined. For example, the following picture represents the relational structure $M = \langle W, R, V \rangle$ where $W = \{w1, w2, w3, w4\}$, $R = \{(w1, w2), (w1, w3), (w1, w4), (w2, w2), (w2, w4), (w3, w4)\}$ and $V(p, w2) = V(p, w3) = V(q, w3) = V(q, w4) = T$ (with all other propositional variables assigned F at the states)

Accessibility Relation

1. Axiom (T) will hold (in all PWS models) just in case the relation R is reflexive. That is, ($T = \Box p \rightarrow p$) will hold just in case $R(w, w)$, for all worlds w. Or, intuitively, if all possible worlds are accessible from themselves.
2. Axiom (S4) will hold iff R is transitive. That is, ($S4$) will hold iff $R(w1, w2)$ and $R(w2, w3)$ implies that $R(w1, w3)$.
3. Axiom (E or S5) will hold iff $R$is euclidean. That is, (E) will hold iff $R(w1, w2)$ and $R(w1, w3)$ implies that $R(w2, w3)$.
4. Axiom (B) will hold iff R is symmetric, That is, (B) will hold iff $R(w1, w2)$ implies $R(w2, w1)$.

Truth in Relational Structure:

Definition (Truth of Modal Formulas)

Truth of a modal formula ϕ at a state w in a relational structure $M = \langle W, RV \rangle$ denoted $M, w \models \phi$ is defined inductively as follows:

1. $M, w \models p$ iff $V(p, w) = T$ (where $p \in S$)
2. $M, w \models \top$ and $(M, w) \not\models \bot$
3. $M, w \models \neg\phi$ iff $M, w \not\models \phi$.
4. $M, w \models \phi \wedge \psi$ iff $M, w \models \phi$ and $M, w \models \psi$.
5. $M, w \models \phi \rightarrow \psi$ iff $M, w \models \neg\phi$ or $M, w \models \psi$
6. $M, w \models \Box\phi$ iff for all $v \in W$, if wRv then $M, v \models \phi$.
7. $M, w \models \Diamond\phi$ iff there is a $v \in W$, such that wRv and $M, v \models \phi$.

$\Box$ and $\Diamond$

Definition ($\Diamond$)

$$v_M(\Diamond\phi, w) = \begin{cases} 1 & \text{if } \exists_u \, w \, R \, u \ \text{ and } \ V_M(\phi, u) = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Definition ($\Box$)

$$v_M(\Box\phi, w) = \begin{cases} 1 & \text{if } \forall_u \, wRu \ \Rightarrow V_M(\phi, u) = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Alternative notation for $v_M(\Box\phi, w)$ is $M, w \models \phi$.

Read: $\Box\phi$ holds at state/point/world w (in M)

or: $\Box\phi$ is satisfied at state/point/world w (in M)

or: $\Box\phi$ is true at state/point/world w (in M)

Frame Properties:

1. **Reflexive:** $\forall_w (wRw)$
2. Serial: $\forall_w \exists_v (wRv)$
3. **Transitive:**$\forall_w \forall_v \forall_u (wRv), (vRu) \rightarrow (wRu)$
4. **Symmetric:**$\forall_w \forall_v (wRv) \rightarrow (vRu)$
5. **Eucledean:**$\forall_w \forall_v \forall_u (wRv) \wedge (wRu) \rightarrow (vRu)$
6. **Dense:**$\forall_w \forall_v (wRv) \rightarrow \exists_u (wRu) \wedge (uRv)$

Frame Properties:

1. **irreflexive:** $\forall_x \neg(xRx)$.
2. **asymmetric** $\forall_x \forall_y (x \not\equiv y \wedge (xRy) \rightarrow \neg(yRx)$
3. antisymmetric $\forall_x \forall_y (xRy \wedge (yRx) \rightarrow x = y)$
4. **weakly ordered** $\forall_x \forall_y (xRy \vee (yRx) \vee x = y)$
5. **partial order:** reflexive, transitive and antisymmetric
6. **equivalence relation:** reflexive, transitive and symmetric
7. **serial** $\forall_x \exists_x (xRy)$
8. **completely disconnected"** $\forall_x \forall_y \neg(xRy)$

Logic and Frame Conditions

Logic

1. K
2. D
3. T
4. B
5. K4
6. S4
7. S5

Frame Conditions

1. No conditions
2. Serial
3. Reflexive
4. Reflexive, symmetric
5. Transitive
6. Reflexive, Transitive
7. Reflexive, Symmetric, Transitive.

Exercises

1. Show that $\Box P \to \Diamond P$ is valid in all serial models.
2. $\vdash_{S4} \Box P \to \Box\Box P$.
3. $\nvdash_{K} \Diamond P \to \Box P$.

Four levels of truth in Kripke semantics

1. Truth at a world: $(M, w) \models A$.
2. Truth in a model: $M \models A$.
3. Validity (truth) in a frame: A is valid in frame $F = \langle W, R \rangle$ if A is true in all interpretations based on F. Notation: $F \models A$
4. Validity (truth) in a class of frames: If θ is a class (set) of frames, we say that A is true in θ if $F \models A$ for all $F \in \theta$.

Kripke Frame

Kripke Frame and Kripke Model

Definition

A Kripke frame is a pair $\{W; R\}$, where W is a non-empty set (of possible worlds) and R is a binary relation on W. A Kripke model is a triple $\{W; R; V\}$, where the valuation V is a mapping from primitive propositions to sets of worlds. Thus, $V(p)$ is the set of worlds at which p is true under the valuation V. We write $wRw^{'}$ iff $\{(w; w^{'}\} \in R$, and we say that world $w^{'}$ is reachable from world w, and that $w^{'}$ is a successor of w.

Validity

Definition (Modal Validity)

A modal formula ϕ is valid, written as $\models \phi$, if $M, w \models \phi$ for all models and worlds.

Example (Some Valid Modal Logic formulas)

1. $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$
2. $\Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$
3. $\Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi)$.

Validity of K, T, B, D, S4, S5

1. All K-valid formulas are T valid (but not vice versa).
2. All T valid formulas are S4 valid.
3. All S4 valid are S5 valid.
4. All the formulas of all systems are S5 valid. It is in this sense called Strong Modal Logic.

Validities

1. $K \rightarrow D \rightarrow \ T \rightarrow B \ \rightarrow S5$
2. $K \rightarrow K4 \ \rightarrow \ S4 \ \rightarrow \ S5$
3. $K \rightarrow D \rightarrow \ T \rightarrow S4 \ \rightarrow S5$

K valid formulas are valid in all Systems

Logic

1. K
2. D
3. T
4. B
5. S4
6. S5

Accesibility Relation

1. No constraint on R
2. Serial
3. Reflexive
4. Reflexive, symmetric
5. Reflecive and transitive
6. Reflecive, transitive, Symmetric.

Charateristic Axiom:

1. $\Box(p \to q) \to \Box p \to \Box q$
2. $\Box\phi \to \Diamond\phi$.
3. $\Box\phi \to \phi$
4. $\phi \to \Box\Diamond\phi$
5. $\Box\phi \to \Box\Box\phi$.
6. $\Diamond\phi \to \Box\Diamond\phi$.

Prefix

A prefix is a finite sequence of positive integers. A prefixed formula is an expression of the form $\sigma\, X$, where σ is a prefix and X is a formula.

Intuitively, the prefix σ, names a possible world in some model, and $\sigma\, X$ tells us that X is true at the world σ names.

Possibility Rules

If the prefix $\sigma\,.\eta$ is new to the branch:

1. $$\frac{\sigma\, \Diamond X}{\sigma\,.\eta\, X}$$
2. $$\frac{\sigma\, \neg\Box X}{\sigma.\eta\, \neg X.}$$

Basic Necessity Rules

Necessity Rules

If the prefix $\sigma.\eta$ already occurs on the branch:

1. $$\frac{\sigma\ \Box X}{\sigma.\eta\ \ X}$$

2. $$\frac{\sigma\ \neg\Diamond X}{\sigma.\eta\ \ \neg X.}$$

Closure of Tree

Closure

A Tableau branch is closed if it contains both $\sigma\ X$ and $\sigma\ \neg X$ for some formula X. A branch that is not closed is open. A tableau is closed if every branch is closed.

Tableau Proof

A closed Tableau $\neg Z$ is a tableau proof of Z, and Z is a theorem if ihas a tableau proof.

Special Necessitation Rules

T

1. $$\frac{\sigma\ \Box X}{\sigma\ X}$$

2. $$\frac{\sigma\ \neg \Diamond X}{\sigma\ \neg X}$$

D

1. $$\frac{\sigma\ \Box X}{\sigma\ \Diamond X}$$

2. $$\frac{\sigma\ \neg \Diamond X}{\sigma\ \neg \Box X}$$

Special Necessitation Rules

B

1. $$\frac{\sigma\ .\eta\ \Box X}{\sigma\ X}$$

2. $$\frac{\sigma\ \eta\ \neg\Diamond\ X}{\sigma\ \neg X}$$

4

1. $$\frac{\sigma\ .\ \Box X}{\sigma\ .\eta\ \ \Box X}$$

2. $$\frac{\sigma\ \ \neg.\Diamond\ X}{\sigma\ .\eta\ \neg\Diamond X}$$

Necessitation Rules

4r

1. $$\frac{\sigma\ .\eta\ \Box X}{\sigma\ \Box\ X}$$

2. $$\frac{\sigma\ .\eta\ \neg \Diamond\ X}{\sigma\ \neg \Diamond\ X}$$

S5 Simplified Modal Rules

S5 Possibility Rule

If the integer τ is new to the branch:

$$\text{1.}\quad \frac{\eta \ \Diamond X}{\tau \ \ X}$$

$$\text{2.}\quad \frac{\eta \ \neg \Diamond \ X}{\tau \ \neg \ X}$$

S5 Necessity Rule

If the integer τ is new to the branch:

$$\text{1.}\quad \frac{\eta \ \Box X}{\tau \ \ X}$$

$$\text{2.}\quad \frac{\eta \ \neg \Diamond \ X}{\tau \ \neg \ X}$$

Logic and Frame Conditions

Logic

1. K
2. D
3. T
4. B
5. K4
6. S4
7. S5

Rules

1. No Rule
2. Serial: D
3. Reflexive: T
4. Reflexive, symmetric: K4, B
5. Transitive: 4
6. Reflexive, Transitive. T, 4
7. Reflexive, Symmetric, Transitive. T, 4, 4r

Example

Example (Modal Argument)

I know that, if the door is open(D), then either I forgot to close it(F), or I was burglarized(B). Secondly, I also know that, if the window is broken(W), then either there was a football accident (A), or I was burglarized (B). Now, as a matter of fact, the window is broken and the door is open. But, it is inconceivable that I both forgot to close the door and there was a football accident. Therefore, I know that I was burglarized.

1. $\Box[D \to (F \vee B)]$.
2. $\Box[W \to (A \vee B)]$
3. $W \wedge D$
4. $\neg \Diamond(F \wedge A)$.
5. therefore $\Box B$.

Aristotle Sea battle Example:

Example (Aristotle's Sea Battle Argument)

A general is contemplating whether or not to give an order to attack. The general reasons as follows:

1. If I give the order to attack, then, necessarily, there will be a sea battle tomorrow
2. If not, then, necessarily, there will not be one.
3. Now, I give the order or I do not.
4. Hence, either it is necessary that there is a sea battle tomorrow or it is necessary that none occurs.
5. **Conclusion:** Either it is inevitable that there is a sea battle tomorrow or it is inevitable that there is no battle

why should the general bother giving the order?

Analysis of Sea Battle Argument:

1. what happens was necessarily going to happen.
2. what does not happen was necessarily going to not happen.
3. Aristotle: statements about the future are neither true nor false

Two Versions:

Version-1

1. $A \to \Box B$
2. $\neg A \to \Box\neg B$.
3. $A \vee \neg A$.
4. Therefore, $\Box B \vee \Box\neg B$.

Valid with Question begging premises.

Version-2

1. $\Box(A \to B)$
2. $\Box(\neg A \to \neg B)$.
3. $A \vee \neg A$.
4. **Therefore**, $\Box B \vee \Box\neg B$.

Invalid argument with Plaussible Premises

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Other Links

1. Link: `http://www.st-andrews.ac.uk/~ac117/teaching/minicourse2.pdf`
2. Modal Logic notes:
   `https://mally.stanford.edu/notes.pdf`
3. Three months of Modal Logics:
   `https://www.youtube.com/watch?v=JHyfy0Chcs4`
4. Internet Encyclopedia of Logic:
   `https://iep.utm.edu/modal-lo/`
5. AIML: `http://www.cs.man.ac.uk/~schmidt/tools/`
6. Gary Hardegree:
   `http://courses.umass.edu/phil511-gmh/text.htm`
7. Philpapers:
   `https://philpapers.org/browse/modal-logic`
8. `https://logic.berkeley.edu/logic@UCB/van_Benthem_logic@UCB.pdf`

Misc

1. More about BNF: https://www.sciencedirect.com/topics/computer-science/backus-naur-form
2. SEP: https://plato.stanford.edu/entries/logic-modal/
3. Philosophy Doumentation on Modal Logic: https://www.pdcnet.org/collection-anonymous/search?q=modal+logic&rows=20

The End