Boundary Value Problems: Finite Difference Method

Akash Anand
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Theorem

There is a unique solution to the discrete BVP

$$D_h^2 u_h(t_i) = f(t_i), \;\; t_i, i = 1, \dots, n,$$
$$u_h(a) = \alpha, \qquad u_h(b) = \beta.$$

Proof:

It is sufficient to show that, if $D_h^2 u_h(t_i) = 0$ *for* $t_i, i = 1, \dots, n,$ *and* $u_h(a) = 0,\; u_h(b) = 0,$ *then* $u_h \equiv 0.$

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$$\max_{1\le i\le n} u_h(t_i) \le \max_{1\le i\le n}(u_h(t_i) + w(t_i)\|f\|_{\infty,h}) \le \max\left\{u_h(a) + w(a)\|f\|_{\infty,h}, (u_h(b) + w(b)\|f\|_{\infty,h}\right\}.$$

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A similar argument applies to $-u_h$ giving the theorem.

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Let u be the solution to

$$\begin{aligned} u'' = f(t), &\qquad a < t < b, \\ u(a) = \alpha, &\qquad u(b) = \beta, \end{aligned}$$

and u_h be the solution to the corresponding discrete problem

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Proof:

The error estimate follows from the previous theorem applied to the discrete problem

$$\begin{aligned} D_h^2(u_h - u)(t_i) = D_h^2 u_h(t_i) - D_h^2 u(t_i) = f(t_i) - D_h^2 u(t_i), &\qquad i = 1, \dots, n, \\ (u_h - u)(a) = 0, &\qquad (u_h - u)(b) = 0. \end{aligned}$$

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Corollary

If $u \in C^2([a,b])$, then

$$\lim_{h \to 0} \|u_h - u\|_{\infty,h} = 0.$$

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If $u \in C^4([a,b])$, then

$$\|u_h - u\|_{\infty,h} \le \frac{h^2 (b-a)^2}{96} \left\| u^{(4)} \right\|_{\infty,[a,b]}.$$

Remark

The quantity $\left\|f - D_h^2 u\right\| = \left\|u'' - D_h^2 u\right\|$ *is the consistency error of the discretization and the statement* $\lim_{h\to 0}\left\|u'' - D_h^2 u\right\| = 0$ *means that the discretization is consistent.*

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An estimate of the form $\|v\| \le C_h\|f\|$ *whenever* $D_h^2 v(t_i) = f(t_i), i = 1, \dots, n,$ *and* $v(a) = 0, v(b) = 0,$ *is a stability estimate and if it holds with* C_h *independent of* h*, we say that the discretization is stable.*

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As a final remark, the finite difference method helped us find the solution values at the mesh points, but the solution at non-mesh points are not readily available from the method. If needed, one can obtain the solution at non-mesh points through interpolation or try other approximation approaches …

Lesson 3

Boundary Value Problems for ODEs

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One way to rectify the non-availability of the solution at non-mesh points is to try to approximate the solution using functions coming from a finite dimensional space.

Boundary Value Problems: Variational Methods

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We again consider a scalar two-point boundary value problem

$$u'' = f(t, u, u'), \qquad a < t < b,$$

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$$u(t) \approx v(t, y) = \sum_{i=1}^{n} y_i \varphi_i(t),$$

where $\varphi_i(t)$ *are basis functions defined on* $[a, b]$ *and* y *is an* n*-vector of parameters to be determined.*

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The residual $r(t, y) \coloneqq v''(t, y) - f(t, v(t, y), v'(t, y))$ measures how well the approximation satisfies the ODE. For an exact approximation, that is, $u(t) = v(t, y)$, we have $r(t, y) = 0$.