MTH 430: Numerical Analysis & Scientific Computing II
Department of Mathematics and Statistics
Indian Institute of Technology - Kanpur

August 12, 2024

Problem Set 1

SUBMISSION INSTRUCTIONS

Submit **before the end of the day**, that is, by 11:59 pm, on August 21, 2022.

EXERCISES

1. Determine Lipschitz constant L given by

$$L = \max_{(t,y)\in D} \left| \frac{\partial f(t,y)}{\partial y} \right|$$

for the following functions:

   (a) $f(t, y) = 2y/t,\ D = \{(t, y) : t \geq 1, y \in \mathbb{R}\}$
   (b) $f(t, y) = \tan^{-1} y,\ D = \mathbb{R}^2$
   (c) $f(t, y) = (t^3 - 2)^{27}/(17t^2 + 4),\ D = \mathbb{R}^2$
   (d) $f(t, y) = t - y^2,\ D = \{(t, y) : t \in \mathbb{R}, |y| \leq 10\}$

2. Write each of the following ODEs with given initial conditions as an equivalent first order system of ODEs:

   (a) $y'' = t + y + y',\ y(0) = 1,\ y'(0) = 1$
   (b) $y''' = ty + y'',\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 1$
   (c) $y''' = y'' - 2y' + y - t + 1,\ y(0) = 1,\ y'(0) = 1,\ y''(0) = 1$
   (d) $y'' = y'(1 - y^2) - y,\ y(0) = 1,\ y'(0) = 0$
   (e) $y''' = -yy'',\ y(0) = 1,\ y'(0) = 0.25,\ y''(0) = 0.5$
   (f) $y_1'' = \alpha y_1/(y_1^2 + y_2^2)^{3/2},\quad y_2'' = \alpha y_2/(y_1^2 + y_2^2)^{3/2},\ y_1(0) = 0.4,\ y_1'(0) = 0,\ y_2(0) = 0,\ y_2'(0) = 2$

3. Let $u(t)$ be the solution, if it exists, to the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0. \tag{1}$$

By integrating, show that u satisfies

$$u(t) = y_0 + \int_{t_0}^{t} f(s, u(s))\, ds.$$

Conversely, show that if this equation has a continuous solution on the $t_0 \leq t \leq T$, then the initial value problem (1) has the same solution.

4. Show that Euler's method fails to approximate the solution $y(t) = (2t/3)^{3/2}, t \geq 0$, of the problem $y' = y^{1/3}, y(0) = 0$. Explain why.

5. Let $A, B, \eta_0, \ldots, \eta_N$ be non-negative numbers satisfying

$$\eta_{n+1} \leq A\eta_n + B, \quad n = 0, \ldots, N-1.$$

Then, show that

$$\eta_n \leq A^n \eta_0 + \left(\sum_{i=0}^{n-1} A^i \right) B, \quad n = 1, \ldots, N.$$

6. Let $A_n > 1$ and $B_n \geq 0$ for $n = 0, 1, \ldots, N-1$ and let $\eta_0, \ldots, \eta_N \geq 0$. Suppose that

$$\eta_{n+1} \leq A_n \eta_n + B_n, \quad n = 0, \ldots, N-1.$$

Then, show that

$$\eta_n \leq \left(\prod_{i=0}^{n-1} A_i \right) \eta_0 + \left(\prod_{i=0}^{n-1} A_i - 1 \right) \sup_{0 \leq i \leq n-1} \frac{B_i}{A_i - 1}, \quad n = 1, \ldots, N.$$

7. Consider the initial value problem

$$\begin{aligned} y' &= f(t, y), \quad (t, y) \in [t_0, t^*] \times [a, b] \\ y(t_0) &= y_0, \end{aligned}$$

with a continuous function f that is Lipschitz continuous in y with Lipschitz constant L. Show that, for every $\epsilon > 0$, there exists $\tilde{h}$ such that for any choice of steps $\{h_n = t_{n+1} - t_n\}_{n=0}^{N-1}$ with $t_N = t^*$ satisfying $\max_{0 \leq n \leq N-1} h_n \leq \tilde{h}$, we have that error $e_n = y_n - y(t_n)$ at $t = t_n$ in the Euler's method $y_{n+1} = y_n + h_n f(t_n, y_n), n \geq 1$, satisfies $\|e_n\| \leq \epsilon$ for all $n = 0, \ldots, N$.

Moreover, if the solution $y \in C^2[t_0, t^*]$, $\max_{0 \leq n \leq N} \|e_n\| \leq C\tilde{h}$ where

$$C = \|y''\|_\infty \frac{e^{L|t^* - t_0|} - 1}{2L}.$$

8. Repeat the previous problem for backward Euler's method.