MTH 101-Calculus

Spring-2021

Assignment 2 : Continuity, Existence of minimum, Intermediate Value Property

1. Let $[x]$ denote the integer part of the real number x. Suppose $f(x) = g(x)h(x)$ where $g(x) = [x^2]$ and $h(x) = \sin 2\pi x$. Discuss the continuity/discontinuity of f, g and h at $x = 2$ and $x = \sqrt{2}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \in \mathbb{Q} \text{ with } x = \frac{p}{q} \text{ for } p, q \in \mathbb{N} \text{ coprime} \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

   Show that f is continuous at every irrational and discontinuous at every rational.

3. Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S converging to a number $x_0 \notin S$. Show that there exists an unbounded continuous function on S.

4. Let $f : [a, b] \to \mathbb{R}$ and for every $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| < \frac{1}{2}|f(x)|$. Find $\inf\{|f(x)| : x \in [a, b]\}$. Show that f is not continuous on $[a, b]$.

5. Using the intermediate value theorem, prove that every odd degree real polynomial has a real root.

6. Show that the function $f(x) = \begin{cases} sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ satisfies the intermediate value property but it is not continuous.

7. Assuming that every continuous function $g : [a, b] \to [a, b]$ has a fixed point, show that every continuous function $f : [a, b] \to \mathbb{R}$ has intermediate value property.

8. Let p be an odd degree polynomial and $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$. Further show that the equation $x^{13} - 3x^{10} + 4x + sinx = \frac{1}{1+x^2} + cos^2x$ has a solution in $\mathbb{R}$.