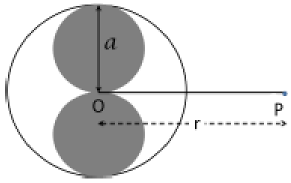
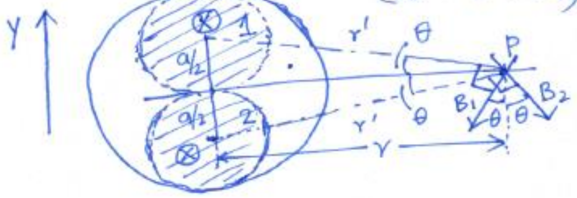
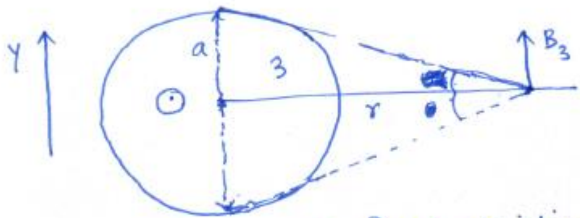
1. An infinitely long solid conducting cylinder of radius a has two hollow cylindrical regions each of diameter a as shown by the shaded regions in the figure below. The conductor carries a current I flowing uniformly through the solid portion (unshaded region). Find the field at point P, a distance r from the center on the axis ($r > a$) as shown in the figure. What is the magnetic field at p when $r >> a$?



Given, Current density $= \dfrac{I}{(\pi a^2 - 2\cdot \pi a^2/4)} = \dfrac{I}{\frac{1}{2}\pi a^2}$ (assume current pointing out of page)



2 current carrying wires (shaded) each of radius $\frac{a}{2}$ with current into the page (I_1, I_2), $I_1 = I_2$



1 (unshaded) wire of radius a with current I_3 out of page

Use principle of superposition → Imagine the system as a superposition of magnetic fields due to 3 current carrying wires: the two shaded wires carry current in opposite direction w.r.t the whole unshaded wire.

$$\vec{B} = (B_3 - B_1\cos\theta - B_2\cos\theta)\,\hat{y}$$

$$= \left(\frac{\mu_0 I_3}{2\pi r} - \frac{\mu_0 I_1}{2\pi r'}\cos\theta - \frac{\mu_0 I_2}{2\pi r'}\cos\theta\right)\hat{y}$$

$$= \left(\frac{\mu_0 I_3}{2\pi r} - \frac{2\mu_0 I_1}{2\pi r'}\cos\theta\right)\hat{y}$$

Now $I_3 = \dfrac{I}{\frac{1}{2}\pi a^2}\cdot \pi a^2$ and $I_1 = I_2 = \dfrac{I}{\frac{1}{2}\pi a^2}\cdot\dfrac{\pi a^2}{4}$

$$\Rightarrow \vec{B} = \left(\frac{\mu_0\cdot 2I}{2\pi r} - \frac{2\cdot\mu_0\cdot\frac{I}{2}}{2\pi\sqrt{r^2+\frac{a^2}{4}}}\cdot\frac{r}{\sqrt{r^2+\frac{a^2}{4}}}\right)\hat{y}$$

$$= \left[\frac{\mu_0\cdot 2I}{2\pi r} - \frac{\mu_0 I}{2\pi}\cdot\frac{r}{\left(r^2+\frac{a^2}{4}\right)}\right]\hat{y}$$

$$= \frac{\mu_0 I}{2\pi r}\left[2 - \frac{1}{\left(1+\frac{a^2}{4r^2}\right)}\right]\hat{y}$$

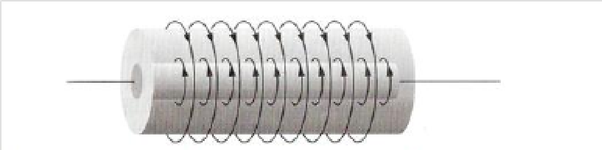
[P.S.: for $a << r$, $\vec{B} = \dfrac{\mu_0 I}{2\pi r}\left(1+\dfrac{a^2}{4r^2}\right)\hat{y}$]

2. Consider two long coaxial solenoids each carrying current I, but in opposite direction as shown in Fig. 3. The inner solenoid of radius a has n_1 turns per unit length, and the outer one of radius $b(> a)$ has n_2. Find the magnetic field in each of the three regions:

(a) inside the inner solenoid;

(b) in between the two solenoids;

(c) outside both the solenoids.

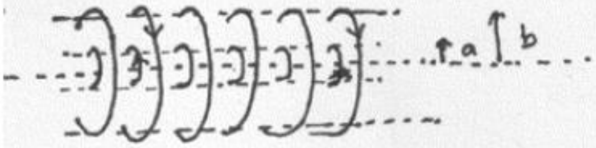


Compare the results of the above situation with the following problem. A long cylindrical wire with inner radius $r = a$ and outer radius $r = b$ carries a uniform current I along the axis. Find the magnetic field

(a) inside the hollow region $(r < a)$;

(b) inside the conducting region $(a < r < b)$;

(c) outside the conductor $(r > a)$.



i) Inside the inner solenoid, both outer solenoid and inner solenoid will contribute to the magnetic field.

outer solenoid $\rightarrow$ $\mu_0 n_2 I$, towards right

Inner solenoid $\rightarrow$ $\mu_0 n_1 I$, towards left

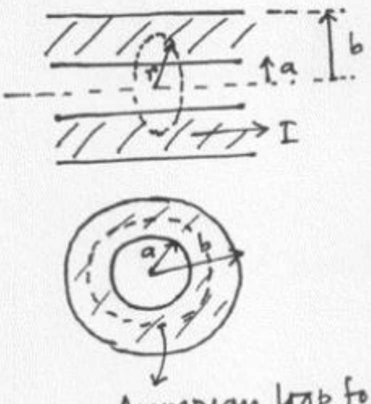
Net magnetic field inside the inner solenoid, $\mu_0 I (n_2 - n_1)$

ii) In between the two solenoids, there is no contribution from the inner solenoid. The magnetic field is $\mu_0 n_2 I$, towards right.

iii) Outside both the solenoids, the magnetic field is zero.

In case of cylindrical wire

The current density $J = \frac{I}{\pi(b^2-a^2)}$



i) Inside the hollow region ($r<a$), the Amperian loop enclosed no current. Hence $B=0$

ii) Inside the conductor ($a<r<b$), the current through the Amperian loop is

$$I(r) = J \cdot \pi(r^2-a^2)$$

iii) outside the conductor ($r>b$), the Amperian loop encloses current I.

Amperian loop for $a<r<b$

i) $r<a$: $\oint \vec{B}\cdot\vec{dl} = 0 \Rightarrow B=0$

ii) $a<r<b$: $\oint \vec{B}\cdot\vec{dl} = \mu_0 J \cdot \pi(r^2-a^2) = \mu_0 \frac{I}{\pi(b^2-a^2)} \cdot \pi(r^2-a^2)$

$$\Rightarrow B \cdot 2\pi r = \mu_0 I \frac{(r^2-a^2)}{(b^2-a^2)}$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}\left(\frac{r^2-a^2}{b^2-a^2}\right)$$, along circumferential ($\hat{\theta}$) direction

iii) $r>b$: $\oint \vec{B}\cdot\vec{dl} = \mu_0 I$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$, again along $\hat{\theta}$ direction

Unlike the case of solenoids, the current here is along the axial direction, not the circumferential direction. Hence, there is no magnetic field along the axis of the wire.

3. A spherical shell with radius R and uniform surface charge density σ spins with angular frequency ω around its diameter. Find the magnetic field at the center of the sphere using Biot-Savart law involving surface current distribution.

Take $\vec{\omega}$ along z axis.

Surface current density at (R, θ, φ)

$$\vec{K} = \sigma \vec{\omega} \times \vec{r}' = \sigma \omega R \; \hat{z} \times \hat{r}' \qquad (\vec{r}' = R\hat{r}')$$

$$= \sigma \omega R \; \hat{z} \times [\sin\theta \sin\varphi \; \hat{x}' + \sin\theta \sin\varphi \; \hat{y}' + \cos\theta \; \hat{z}']$$

$$= \sigma \omega R \; [\sin\theta \cos\varphi \; \hat{y}' - \sin\theta \sin\varphi \; \hat{x}'] \qquad (\hat{z} = \hat{z}')$$

Here the field point is ~~#~~ at the origin

Hence the magnetic field due to surface element da'

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{K} \times (-R\hat{r}')}{R^3} da'$$

$$= \frac{-\mu_0}{4\pi} \frac{\sigma \omega R^2}{R^3} [\sin\theta \cos\varphi \; \hat{y}' - \sin\theta \sin\varphi \; \hat{x}'] \times [\sin\theta \cos\varphi \; \hat{x}' + \sin\theta \sin\varphi \; \hat{y}' + \cos\theta \; \hat{z}'] \, da'$$

$$= \frac{-\mu_0}{4\pi} \frac{\sigma \omega}{R} [-\sin^2\theta \cos^2\varphi \; \hat{z}' - \sin^2\theta \sin^2\varphi \; \hat{z}' + \sin\theta \cos\theta \cos\varphi \; \hat{x}' + \sin\theta \cos\theta \sin\varphi \; \hat{y}'] \, da'$$

$$= \frac{-\mu_0}{4\pi} \frac{\sigma \omega}{R} \left[-\sin^2\theta \; \hat{z}' + \sin\theta \cos\theta \left(\cos\varphi \; \hat{x}' + \sin\varphi \; \hat{y}'\right)\right] da'$$

Magnetic field due to the spherical shell

$$\vec{B} = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} d\vec{B} \; R^2 \sin\theta \, d\theta \, d\varphi$$

Since, $\int_0^{2\pi} \sin\varphi \, d\varphi = \int_0^{2\pi} \cos\varphi \, d\varphi = 0$,

$$\vec{B} = \hat{z}' \frac{\mu_0}{4\pi} \sigma \omega R \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= \hat{z}' \frac{\mu_0}{4\pi} \sigma \omega R \cdot \frac{4}{3} \cdot 2\pi$$

$$\Rightarrow \boxed{\vec{B} = \frac{2}{3} \mu_0 \sigma \omega R \; \hat{z}}$$

4. We have discussed in the lecture that the vector potential for a volume current density $\mathbf{J}$ is given by,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d\tau'.$$

Calculate $\nabla \times \mathbf{B}$ using the above expression assuming a steady current distribution.

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

$$= \frac{\mu_0}{4\pi}\int \left[\vec{\nabla}\left(\vec{\nabla}\cdot \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \nabla^2\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\right] d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \left[\vec{\nabla}\left(-\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\cdot \vec{J}(\vec{r}')\right) + \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\,\vec{\nabla}\cdot\cancel{\vec{J}(\vec{r}')}\right)\right] d\tau'$$

$$- \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\,\nabla^2\frac{1}{|\vec{r}-\vec{r}'|}\, d\tau'$$

switch from unprimed to primed is accompanied by '−' sign

goes to zero as $\vec{J}$ is a function of primed variables.

taken outside the laplacian as $\vec{J}$ is a function of primed variables

$$\Rightarrow \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int\left[\vec{\nabla}\left(-\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\cdot\vec{J}(\vec{r}')\right) + \vec{J}(\vec{r}')\,4\pi\,\delta(\vec{r}-\vec{r}')\right] d\tau'$$

$$= \frac{\mu_0}{4\pi}\,\vec{\nabla}\int\left[-\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\cdot\vec{J}(\vec{r}')\right] d\tau' + \mu_0\vec{J}(\vec{r})$$

Now, $\displaystyle\int -\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}\cdot\vec{J}(\vec{r}')\, d\tau'$

$$= \int\left[-\vec{\nabla}'\cdot\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{1}{|\vec{r}-\vec{r}'|}\,\vec{\nabla}'\cdot\vec{J}(\vec{r}')\right] d\tau$$

0 for steady current distribution

divergence term → can be turned into a surface integral

The surface integral vanishes when the surface is pushed outside the current distribution.

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$