MTH636A (2023-24, EVEN SEMESTER)

PROBLEM SET 2

1. There are n individuals who witness a crime. Everybody would like the police to be called. If this happens, each individual derives satisfaction $v > 0$ from it. Calling the police has a cost of c, where $0 < c < v$. The police will come if at least one person calls. Hence, this is an n-person game in which each player chooses from $\{C, N\}$; C means 'call the police' and N means 'do not call the police'. The payoff to person i is 0 if nobody calls the police, $v - c$ if i (and perhaps others) call the police, and v if the police is called but not by person i.

   (a) What are the Nash equilibria of this game in pure strategies? Does the game have a symmetric Nash equilibrium in pure strategies (a Nash equilibrium is symmetric if every player plays the same strategy)? No symmetric NE
   Nash equilibria-> only one person calls

   (b) Compute the symmetric Nash equilibrium or equilibria in mixed strategies.
   p(N) = (c / v)^(1 / n-1), but I couldn't prove it is NE

   (c) For the Nash equilibrium/equilibria in (b), compute the probability of the crime being reported. What happens to this probability if n becomes large? p(rep) = 1 - (c / v)^(n / n-1)
   n->inf; p(rep)= 1 - c / v

2. There are two cities S and T. A total of 4000 cars need to go from S to T. There are two paths from S to T. Path 1 goes via city A — we will denote this as $S \to A \to T$. Path 2 goes via city B — we will denote this as $S \to B \to T$. Note that roads go in one direction.

   On Path 1, the road from S to A takes $\frac{x}{100}$ units of time, where x is the number of cars traveling on this road. On the other hand, the road from A to T takes 45 units of time, irrespective of the number of cars traveling on this road. On Path 2, the road from S to B takes 45 units of time, irrespective of the number of cars traveling on this road. On the other hand, the road from B to T takes $\frac{y}{100}$ units of time, where y is the number of cars traveling on this road.

Suppose the utility of every car is the negative of the amount of time spent in going from S to T.

(a) Suppose each car (driver) is strategic in choosing his path and all cars choose their paths simultaneously. Describe a pure strategy Nash equilibrium of this game.

2000 cars on each path

The planner decided to build another road to travel from A to B (you cannot travel from B to A on this road). The travel time from A to B is zero (irrespective of the number of cars traveling on this road). So, now there are three paths from S to T: Path 1 is $S \to A \to T$, Path 2 is $S \to B \to T$, and Path 3 is $S \to A \to B \to T$.

(b) Suppose each car (driver) is strategic in choosing his path and all cars choose their paths simultaneously. Does the Nash equilibrium that you found when the road from A to B did not exist still a Nash equilibrium now? Describe a pure strategy Nash equilibrium of this game. No its not more NE. all cars on Path 3

(c) What conclusions can be drawn from this Nash equilibrium about whether the planner should build the road from A to B? No, it increases journey time.

3. For each of the statements below, prove the result if it is true, give a counter example if it is not true. Suppose a game $G = \langle N, (S_i)_{\{i \in N\}}, (u_i)_{\{i \in N\}} \rangle$ has exactly two pure strategy Nash equilibria s and s' such that $u_i(s) \neq u_i(s')$ for all $i \in N$. Then,

   (a) there must be at least two distinct players $k, l \in N$ such that $s_k \neq s'_k$ and $s_l \neq s'_l$ True. easy.

   (b) there is a mixed strategy Nash equilibrium σ of G such that $\sigma \notin \{s, s'\}$.

4. Consider a game $G = \langle N, (S_i)_{\{i \in N\}}, (u_i)_{\{i \in N\}} \rangle$ where S_i is finite for all $i \in N$. Suppose the players are pessimistic and the utility of player i for a mixed strategy profile σ is defined as follows:

$$\hat{u}_i(\sigma) = \min\{u_i(s) \mid s \in \text{Supp}(\sigma)\}$$

where $\text{Supp}(\sigma) = \{(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n \mid \sigma_i(s_i) > 0 \text{ for all } i \in N\}$. Either prove or disprove the following: The game G has a mixed strategy NE when players are pessimistic.

5. Suppose that a mixed strategy σ_i of player i strictly dominates another of his mixed strategies, $\hat{\sigma}_i$. Prove or disprove each of the following claims:

(a) Player i has a pure strategy $s_i \in S_i$ satisfying: (i) $\hat{\sigma}_i(s_i) > 0$ and (ii) strategy s_i is not chosen by player i in any equilibrium.

(b) For each equilibrium $\sigma^* = (\sigma_i^*)_{\{i \in N\}}$, player i has a pure strategy $s_i \in S_i$ satisfying (i) $\hat{\sigma}_i(s_i) > 0$ and (ii) $\sigma_i^*(s_i) = 0$.

Hint: Suppose not.

6. Let $(a_{i,j})_{1 \le i,j \le n}$ be non-negative numbers satisfying $\sum_{j \neq i} a_{i,j} = a_{i,i}$ for all $i \in \{1, \ldots, n\}$. Julie and Sam are playing the following game. Julie writes down a natural number i, $1 \le i \le n$, on a slip of paper. Sam does not see the number that Julie has written. Sam then guesses what number Julie has chosen, and writes his guess, which is a natural number j, $1 \le i \le n$, on a slip of paper. The two players simultaneously show each other the numbers they have written down. If Sam has guessed correctly, Julie pays him $a_{i,i}$ dollars, where i is the number that Julie chose (and that Sam correctly guesses). If Sam was wrong in his guess $(i \neq j)$, Sam pays Julie $a_{i,j}$ dollars. Depict this game as a two-player zero-sum game in strategic form, and prove that the value in mixed strategies of the game is 0.

Hint: get max-min value <= 0, min-max value >= 0