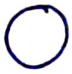
Topics In Topology

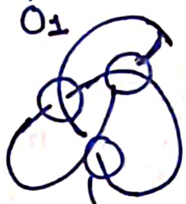
Knot Theory

Definition: Let S : unit circle $x^2+y^2=1$. Then, a knot K is an embedding of S^1 into $\mathbb{R}^3$.

Example: ① Unknot  O_1

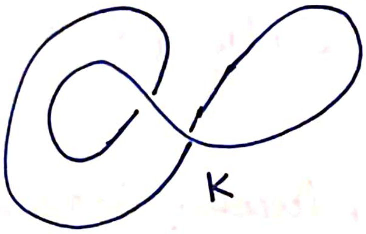
crossing points

② Trefoil knot



3_1 (Right Hand Trefoil)

3 crossing points



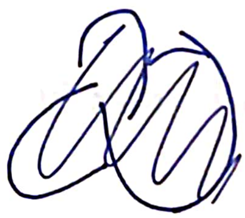
K

→ same as unknot. (convert by twisting)

$K \cong O_1$

$3_1 \ncong O_1$

③

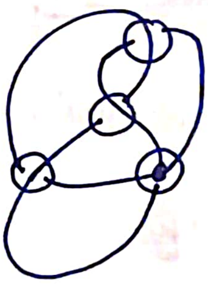


3_1^*

Left Hand Trefoil (not equivalent to the right hand Trefoil)

$3_1 \ncong 3_1^*$

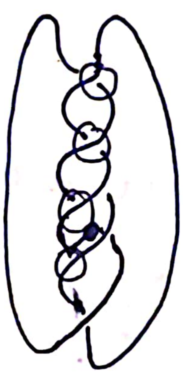
③ Figure 8



4 crossing

4_1

④



5_1

HISTORY OF KNOT THEORY

Began in 1880, chemist William Thomson. (Lord Kelvin) proposed that atomic structures were knots in ether. This persuaded his friend PG Tate, who decided to make a list of knots, He correctly classified up to the first 800 knots.

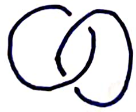
PG Tait – Founder of Knot Theory. made 4 conjectures called Tait conjectures. At the end of his study, it turned out that atomic structures did not need knots. So, chemists / physicists lost interest in Knot theory. At this point, mathematicians began studying knots ⇒

Classical Knot Theory (1930 – 1984)

In 1984, a mathematician in operator theory V. Jones discovered a new knot. 'Jones polynomial'. He got a Fields medal in 1990 for this.

LINKS: A link is a finite ordered collection of knots which do not intersect each other.

Examples: ① Unlink O O O^2

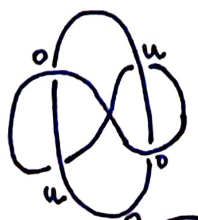
②  Hopf link 2^2_1

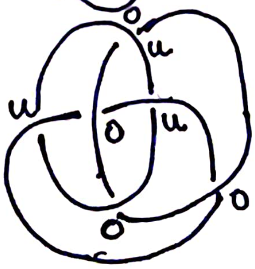
2 → 2 component

1 → first non trivial link

2 → # crossings

③ Whitehead link

 5^2_1

④ 

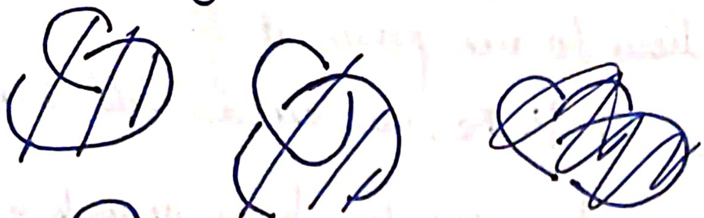
Any 2 components are unlinked

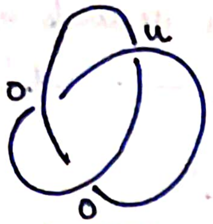
Borromean Ring

alternating

Definition: Alternating knot / link
If atleast one diagram which goes over under alternately

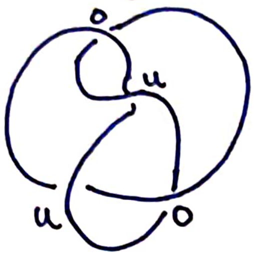
3_1





→ alternating

4_1



$8_{19}, 8_{20}, 8_{21}$ → only non alternating

~~Definition : Let S^1 be the~~
Let S^1 be the unit circle $x^2+y^2=1$. Then,

(a) K is the image of a map δ from $S^1 \longrightarrow \mathbb{R}^3$ which is one-one and continuous.

(b) (Links) : A link is a finite ordered collection of knots which do not intersect each other.

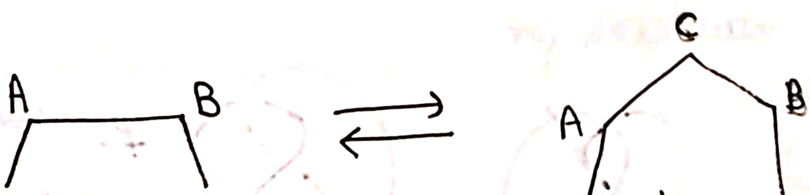
$L = \{K_1, \dots K_n\}$.

FIRST QUESTION: (a) When are two knots K_1 and K_2 equivalent?
(b) When are two links L_1 and L_2 equivalent.

Definition : (~~El~~ Elementary Knot Moves)

(1)

(2)



Def^n : Knots K_1 and K_2 are said to be equivalent if we can go from K_1 to K_2 by finitely many knot moves.

Note: (1) If $K_1 \simeq K_2$, we should prove ∃ finitely many knot moves which take K_1 to K_2 and vice versa.

(2) If $K_1 \not\simeq K_2$, how do we prove it?

If we want to prove $K_1 \not\simeq K_2$, we need knot invariants.

Definition: A knot invariant $u(K)$ (can be a number, polynomial or a property) such that, if $K_1 \simeq K_2 \Rightarrow u(K_1) = u(K_2)$

$\Rightarrow$ If $u(K_1) \neq u(K_2)$, then $K_1 \not\simeq K_2$

Knot Invariant

① $c(K)$: ~~cro~~ Crossing number

= minimum number of crossing points of K over all possible diagrams of K.

$c(K) = 3$

$\simeq$

reducible crossing

② $u(K)$: Unknotting number

switch any one crossing

$\simeq$ unknot

The minimum number of exchanges needed to make K trivial (unknot)

$\boxed{u(3_1) = 1}$

Find unknotting numbers for:

switch

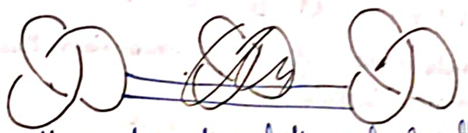
$\not\simeq$ unknot

$u(3_2) = 1$

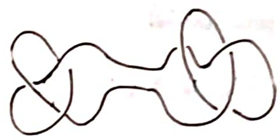
$\simeq$ unknot

$u(6_1) = 1$

Definition: Connected sum of knots K_1 and K_2



cut a small arc from K_1 and K_2 and glue the ends.



→ connected sum of 2 trefoil knots.

Def^n: Prime Knot: A knot K is prime whenever $K = K_1 \# K_2$ either $K_1 \simeq O_1$ or $K_2 \simeq O_2$.

Theorem: (Prime Decomposition theorem).

Any knot K has a unique decomposition $K \simeq K_1 \# K_2 \# \cdots \# K_n$ where each K_i is prime.

Q1 $\boxed{c(K_1 \# K_2) = c(K_1) + c(K_2)}$

Proved to be true for the class of alternating knots, but the question is open for non alternating knots).

Q2 $\boxed{u(K_1 \# K_2) = u(K_1) + u(K_2)}$

Open problem: even for alternating knots.

→ HOMEWORK:

Find unknotting numbers for $K = 5_2, 6_2, 6_3, 7_2, 6_1, 3_2$

In 1928, a topologist Alexander discovered a nontrivial invariant – Alexander Polynomial

$\Delta_K(t)$

→ All polynomials are symmetric.

- 3_1 : $(-1)(1-1+1) = t^{-1}(1 - t + t^2)$

The first 35 knots have different polynomials. However, it is not a complete invariant.

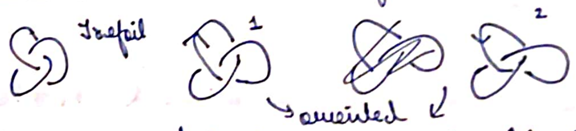
$\exists K_1, K_2$, $K_1 \neq K_2$, $\Delta_{K_1}(t) = \Delta_{K_2}(t)$

#1950 : Alexander-Conway polynomial.

We can obtain $\Delta_K(t)$ from $\nabla_K(z)$ by:

$$z = \sqrt{t} - \frac{1}{\sqrt{t}}.$$

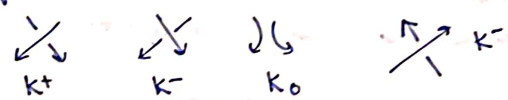
ORIENTED KNOT, (Orientation: choice of direction)



Given an oriented knot / link, we assign AC polynomial $\nabla_K(z)$ by 2 axioms

① If K is the trivial knot, $\nabla_K(z) = 1$

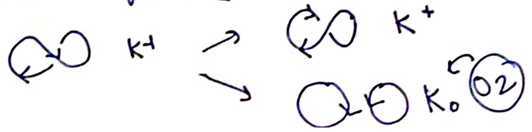
→



If a knot differs at only one crossing point,

$\nabla_{K^+}(z) - \nabla_{K^-}(z) = z\nabla_{K^0}(z)$

$\nabla_K(z) = 1$ for O_1

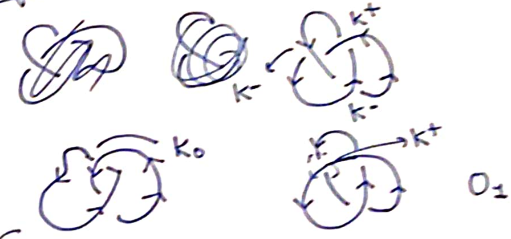


$\nabla_{K^+} - \nabla_{K^+} = z\nabla_{O_2}$

$\nabla O_1 = \nabla O_1 - z O_2$

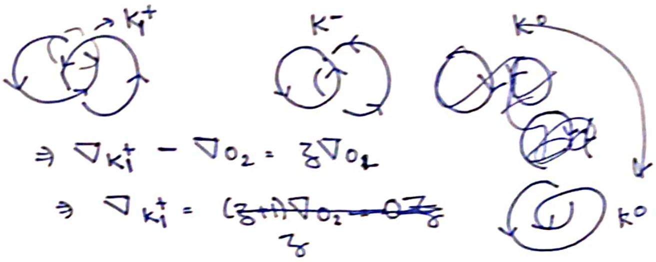
$\Rightarrow \nabla_{O_2} = 0$

Trefoil



$\nabla_{K-} = \nabla_{K+} - z\nabla_{K_0}$

$= 1 - z\nabla_{K_0}$ ——— (*)



$\Rightarrow \nabla_{K_1^+} - \nabla_{O_2} = z\nabla_{O_2}$

$\Rightarrow \nabla_{K_1^+} = \cancel{(z+1)\nabla_{O_2} = 0 z}$ z

$\Rightarrow \cancel{\nabla_{K^+}(z)}$

(*) : $\cancel{\nabla_{K^-} = 1 - z = 1}$ z

$\nabla_{K^-} = 1 = z$ $\boxed{1+z^2}$?

$\Delta = t + \frac{1}{t} - 1$

$$\# \nabla_{5_1} = 1\nabla_{3_1} + z^2\nabla_{3_1} + z^2 V_{O_1} - z V_{O_2}$$

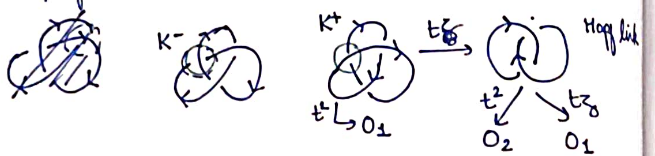
$$\nabla_{3_1} = 1 + z^2$$

$$V_{5_1} = t^{-2}V_{3_1} + t^{-2}z^2 V_{3_1} + t^{-4}z^2 V_{O_1} - t^{-5} z V_{O_2}.$$

THEOREM : Suppose K^* is the mirror image of a knot (or link) K, then $V_{K^*}(t) = V_K(1/t)$.

PROOF: Suppose D is a diagram of K and D^* is its mirror image. If the skein tree diagram of D is R (say), we form the skein tree diagram of D^* (R^*) as follows.

When we perform a skein operation at a crossing point c of D to make R, at the equivalent crossing point of D^*, also perform a skein operation, so forming R^*.



For the mirror image, diagram will unravel the same way. The only difference in R and R^* is in the coefficients assigned.

$$t^2 \longrightarrow t^{-2} \quad | \quad t^{-2} \longrightarrow t^2$$

$$tz = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)t \longrightarrow -\frac{z}{t} = \frac{1}{t}\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right) \quad | \quad -\frac{z}{t} \longrightarrow tz$$

$$= t\sqrt{t} - \sqrt{t} \longrightarrow \frac{1}{t\sqrt{t}} - \frac{1}{\sqrt{t}}$$

This is equivalent to $t \longrightarrow \frac{1}{t}$ everywhere

$$\Rightarrow \boxed{V_{K^*}(t) = V_K\left(\frac{1}{t}\right)}$$

DEFINITION: Homfly polynomial (1985) ~~is defined as follows.~~ $P_K(v,z)$ a 2 variable polynomial for an oriented knot K (or link) is defined as.

AXIOM 1 : If K is O_1 (trivial knot), then $P_K(v,z) = 1$

AXIOM 2 : K^+ K^- K^0.

$$\frac{1}{v}P_{K^+}(v,z) - vP_{K^-}(v,z) = zP_{K^0}(v,z)$$

Corollary 1: If $v = 1$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$, $P_K\left(1, \sqrt{t} - \frac{1}{\sqrt{t}}\right) = \Delta_K(t)$

Corollary 2: If $v = t$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$, $P_K\left(t, \sqrt{t} - \frac{1}{\sqrt{t}}\right) = V_K(t)$

OPEN PROBLEM (Jones' Unknotting Conjecture)

If $K \cong O_1 \Rightarrow V_{O_1} = 1$

If $V_K = 1$, is $K \cong O_1$?

PROPERTIES OF JONES POLYNOMIAL

① If K is an alternating knot (or link) and $V_K(t)$ is its Jones polynomial.

$$V_K(t) = a_{-m}t^{-m} + a_{-m+1}t^{-m+1} + c + a_1 t + \dots + a_n t^n$$

Max degree of $V_K(t) = n$

Min degree of $V_K(t) = -m$

$\boxed{\text{span } V_K(t) = n - (-m) = n + m}$

Then, $\boxed{\text{span } V_K(t) = c(K)}$.

∫ (K Murasugi-(1984) : The Jones Polynomial & classical conjectures in Knot theory.

PROPERTY 3

If $K = K_1 \# K_2$, then $V(K) = V(K_1) \cdot V(K_2)$

Lemma 1 If O_m is the trivial m-component link, $O_m = O \cdot O \cdot O \cdots O$
then $V_{O_m}(t) = (-1)^{m-1}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{m-1}$.

Defn : We denote by $X \sqcup Y$ (the disjoint union of X and Y) the union of 2 sets X and Y with no points in common.

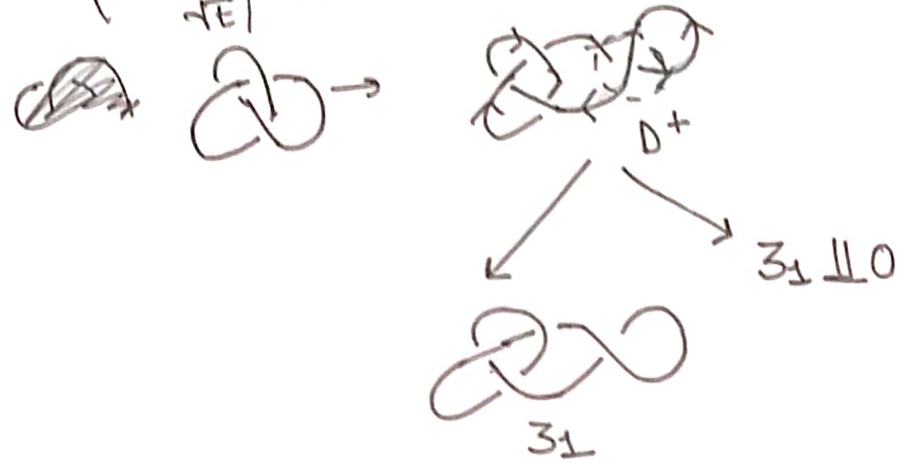
$$O_2 = OO = O_1 \sqcup O_1$$

Lemma 2 $V_{K \sqcup O_m} = (-1)^m \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^m V_K(t)$ —— $(*)$

PROOF By INDUCTION :

$$V_{K \sqcup O} = (-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_K(t)$$

$K = 3_1$



Writing the Jones polynomial,

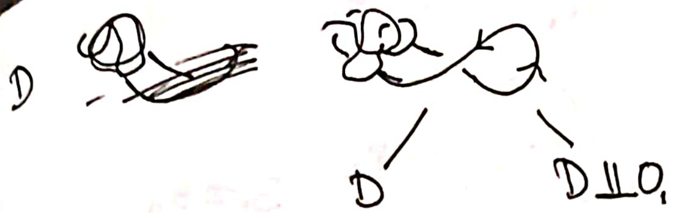
$$\frac{1}{t}V_{3_1} - tV_{3_1} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{3_1 \sqcup O}$$

$$V_3 (-1)\left(t - \frac{1}{t}\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{3_1 \sqcup O}$$

$$\Rightarrow (-1) V_{3_1}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) = V_{3_1 \sqcup O}$$

$$\Rightarrow V_{3_1 \sqcup O} = (-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_{3_1}$$

By induction, we first prove $(*)$ for $m = 1$.
For any knot K, we can create an artificial (reducible) crossing as before.



$\Rightarrow \frac{1}{t}V_K - tV_K = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K \amalg O_1}$

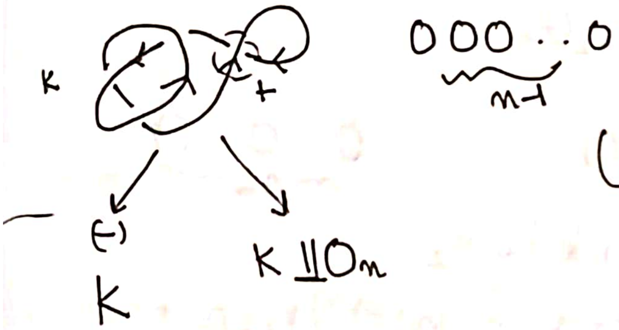
$\Rightarrow -V_K\left(t - \frac{1}{t}\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K \amalg O_1}$

$\Rightarrow -V_K\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) = V_{K \amalg O_1}$

$\Rightarrow V_{K \amalg O_1} = (-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_K$

$\therefore$ We have shown (*) for $m = 1$.

Let us assume by induction that (*) is true for $i \leq m-1$ and we will prove it for $i = m$. $\Rightarrow$ By induction (*) holds for all m...



$(K_+ = K_- = K \amalg O_{m-1})$

$\frac{1}{t}V_{K_+} - tV_{K_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K_0}$

$\Rightarrow \left(\frac{1}{t} - t\right) V_{K \amalg O_{m-1}} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K \amalg O_m}$

$\Rightarrow \left(\frac{1}{t} - t\right) V_{K \amalg O_{m-1}} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K \amalg O_m}$

$\Rightarrow \left(\frac{1}{t} - t\right) (-1)^{m-1} \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{m-1} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K \amalg O_m}$

$\Rightarrow (-1)^m \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{m-1} \cancel{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)} \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) = \cancel{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)} V_{K \amalg O_m}$

$$\Rightarrow \boxed{(-1)^m \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^m = V_{K \amalg O_m}}$$

$\therefore$ By induction, lemma 2 is true.

Theorem: $V_{K_1 \# K_2}(t) = V_{K_1}(t) \circ V_{K_2}(t)$

Proof:

K_2 $\quad 3_1 \# 3_2$

K_1

Just for a while represent K_2 by a single dot. •

Using this representation, (ignoring K_2 for a short while) we create a skein tree diagram for the Jones polynomial of K_1 as follows

$$V_{K_1}(t) = f_1(t) V_{O_1}(t) + f_2(t) V_{O_2}(t) + \dots + f_m(t) V_{O_m}(t)$$

$$V_{K_1 \# K_2}(t) = V_{K_1}(t) \circ V_{K_2}(t)$$

$O_1 \quad OO \quad OOO \cdots \quad \underbrace{O \cdots O \cdots O}_{O_m}$

Now, we replace the dot back by K_2.

$$V_{K_1 \# K_2}(t) = f_1(t) V_{K_2}(t) + f_2(t) V_{K_2 \amalg O_1} + f_3(t) V_{K_2 \amalg O_2} + \dots f_m(t) V_{K_2 \amalg O_{m-1}}$$

By lemma 2,

$$V_{K_2 \amalg O_k}(t) = (-1)^k \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^k V_{K_2}$$

$$V_{K_1 \# K_2}(t) = V_{K_2}(t) \left\{ f_1(t) V_{O_1}(t) + f_2(t) V_{O_2}(t) + \dots + f_m(t) V_{O_m}(t) \right\}$$

$$= V_{K_2}(t) V_{K_1}(t)$$

■

$K_1 = 4_1 \# 4_1$ (Kanenobu 1985)

$K_2 = 8_9$

$$V_{K_1}(t) = V_{K_2}(t)$$

$$V_{4_1}(t) = \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$$

~~$V_{4_1 \# 4_2}$~~ $V_{4_1 \# 4_2}(t) = \left(\frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2\right)^2 = V_{8_9}(t)$

$$\Delta_{K_1}(t) \neq \Delta_{K_2}(t)$$

$$\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t)\,\Delta_{K_2}(t)$$

$$\Delta_{4_1 \# 4_2}(t) = \frac{1}{t}(-1+3t-t^2) \times \frac{1}{t}(-1+3t-t^2)$$

$$\Delta_{8_9} = \frac{1}{t^3}(-1+3t-5t^2+7t^3-5t^4+3t^5-t^6)$$

$$\Delta_{4_1 \# 4_2}(t) = \frac{1}{t^2}(t^2-3t+1)^2$$

$$\Delta_{8_9} = \cancel{-t^3} \quad -t^3+3t^2-5t+7-\frac{5}{t}+\frac{3}{t^2}-\frac{1}{t^3}$$

$$\frac{1}{t^2}(t^4+9t^2+1-6t^3-6t+2t^2)$$

$$= t^2-6t+11-\frac{6}{t}+\frac{1}{t^2} \neq \Delta_{8_9}$$

$$= (-2)(1-6+11-6+1)$$

$$\Delta_{8_9} = (-3)(-1+3-5+7-5+3-1)$$

$$V_{4_1}(t) = \left(t^2-t-\frac{1}{t}+\frac{1}{t^2}+1\right)\left(t^2-t-\frac{1}{t}+\frac{1}{t^2}+1\right)$$

$$\cancel{t^4}-\cancel{t^3}-\cancel{t}+\cancel{1}+\cancel{t^2}-\cancel{t^3}+\cancel{t^2}+\cancel{1}-\cancel{\frac{1}{t}}-\cancel{t}-\cancel{t}+\cancel{1}+\cancel{\frac{1}{t^2}}-\frac{1}{t^3}-\cancel{\frac{1}{t}}$$

$$+\cancel{1}-\cancel{\frac{1}{t}}-\frac{1}{t^3}+\frac{1}{t^4}+\cancel{\frac{1}{t^2}}+\cancel{t^2}-\cancel{t}-\cancel{\frac{1}{t}}+\cancel{\frac{1}{t^2}}+\cancel{1}$$

$$= t^4-2t^3+3t^2-4t+5-\frac{4}{t}+\frac{3}{t^2}-\frac{2}{t^3}$$

$$= (-3)(-2+3-4+5-4+3-2+1)$$

→ BRAID GROUP (B_n)

We defined n braids α, β. Then we defined a 'product' of braids $\alpha \cdot \beta$, and we showed the set of n braids B_n with the product (composition) satisfies the group properties.

① Closure : If α, β are n braids i.e $\alpha, \beta \in B_n$, then $\alpha \cdot \beta \in B_n$

② Associativity: $\alpha, \beta, \gamma \to$ n braids $\Rightarrow \alpha \circ (\beta \bullet \gamma) = (\alpha \circ \beta) \circ \gamma$

③ Existence of identity : $\exists$ an n braid denoted by e_n such that $\alpha \circ e_n = e_n \circ \alpha = \alpha$

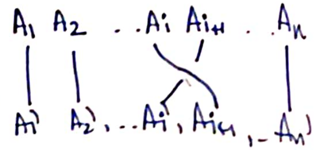
④ Existence of inverse: Given $\alpha \in B_n$, $\exists$ an n-braid α^{-1} in B_n such that $\alpha \cdot \alpha^{-1} = e_n \cdot \alpha^{-1} \cdot \alpha$

$\{B_n, \bullet\}$ is a group (n^{th} Braid Group)

~~Among the~~ Among the n-braids in B_n, $\exists$ some special braids $A_1 \dots A_n \to$ the generators of B_n.

$A_1, A_2 \dots A_i, A_{i+1}, \dots A_n \longleftrightarrow \sigma_i$

$A_1, A_2' \cdot A_i', A_{i+1}', \dots A_n'$ $\quad \sigma_i^{-1}$

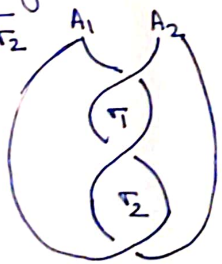


* B_n has $n-1$ generators : $\sigma_1, \sigma_2, \dots \sigma_{n-1}$, and their inverses $\sigma_1^{-1}, \dots \sigma_{n-1}^{-1}$.

Claim: Any n-braid $\alpha \in B_n$ is a finite product of generators and ~~the~~ their inverses.

Given an n-braid α, we form its closure $\bar{\alpha}$ by connecting A_1 to A_1', A_2 to A_2', ... A_n to A_n' by non intersecting arcs outside the braid diagram. $\bar{\alpha}$ (Knot or link).

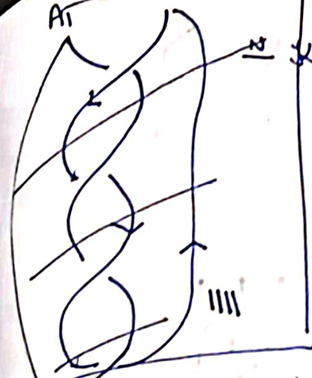
① $\overline{\sigma_1^2} = \overline{\sigma_1 \circ \sigma_2}$



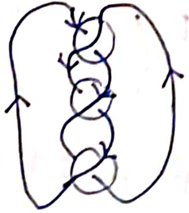
$\cong$ Hopf link

(closure)

② $\overline{\sigma_1^3} = \overline{\sigma_1 \circ \sigma_1 \circ \sigma_1}$ ⟶



$\cong$ knot



3 crossings

$\cong 3_1$ (Trefoil)

③ $n = 2$

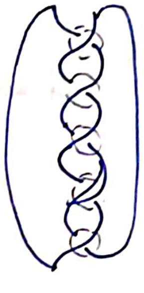
$B_2 \to$ generated by σ_1 & σ_1^{-1}

$\alpha \in B_2 \Rightarrow \alpha = \sigma_1^{m}, m \in \mathbb{Z}$

B_2 is the only abelian braid group.

$B_2 \cong (\mathbb{Z}, +)$ (isomorphic) ($\phi : \mathbb{Z} \longrightarrow B_2$, $1 \longrightarrow \sigma_1$)

$\overline{\sigma_1^5}$



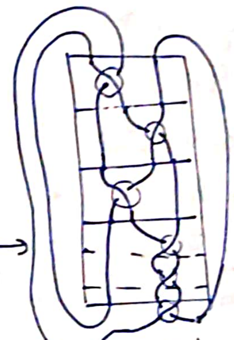
$\cong 5_1$

Similarly, $\overline{\sigma_1^7} = 7_1$

B_3

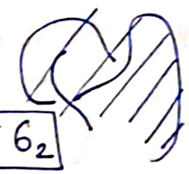
→ $\overline{\sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2} = 4_1$

→ $\overline{\sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{3}}$ ⟶



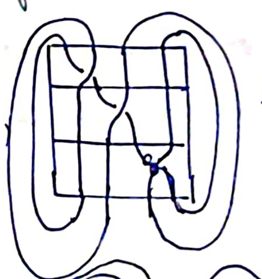
6 crossings

$\cong$ $\boxed{6_2}$



Theorem (Alexander) : Any knot or link can always be written as the closure of some n-braid α.

Eg $\overline{\sigma_1 \sigma_2 \sigma_3}$

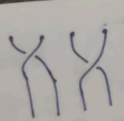


$\cong O_1$ (unknot)

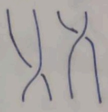
Eg $\overline{\sigma_1 \sigma_2 \sigma_3 \sigma_4}$

$\cong O_1$ (unknot)

$\underline{B_4}$ $\sigma_1\sigma_3$



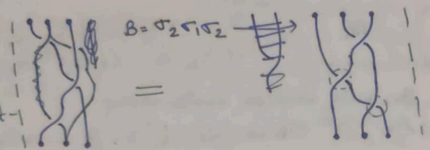
$\cong$ $\sigma_3\sigma_1$



*For B_n, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $|i-j| \geq 2$.

$\underline{B_3}$

$\alpha = \sigma_1\sigma_2\sigma_1$ $B = \sigma_2\sigma_1\sigma_2$



mirror image about axis

$\therefore \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

BRAID GROUP B_n

B_n is a free group on n-generators $\sigma_1, \sigma_2, \dots \sigma_{n-1}$. and 2 fundamental relations

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

- $B_1 \to$ Trivial group $= \{e\}$
- $B_2 = \{\sigma_1^m \mid m \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$ (Abelian)
- $B_3 = \{\underbrace{\sigma_1, \sigma_2}_{\text{generators}} \mid \underbrace{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2}_{\text{fundamental relations}}\}$
- $B_4 = \{\sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1,\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3\}$
- $B_m = \{\sigma_1, \dots \sigma_{m-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\}$

Ex : $\omega_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4$. Show that $\omega_1 \cong \omega_2$

$\omega_2 = \sigma_2\sigma_1\sigma_2^2$

$= \cancel{\sigma_2\sigma_1\sigma_2\sigma}$

$$\begin{aligned}
\omega_1 &= \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\underline{\sigma_2\sigma_4} \\
&= \sigma_1\sigma_2\sigma_4^{-1}\underline{\sigma_1\sigma_4}\sigma_2 \\
&= \sigma_1\sigma_2\underbrace{\sigma_4^{-1}\sigma_4}_{e}\sigma_1\sigma_2 \\
&= \underline{\sigma_1\sigma_2\sigma_1}\sigma_2 = \boxed{\sigma_2\sigma_1\sigma_2^2}
\end{aligned}$$

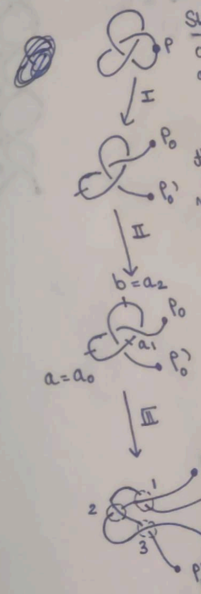
ALEXANDER THEOREM:

*Definition : Given a braid $\alpha \in B_n$, we form its closure by joining A_1 to A_1', A_2 to A_2', ... A_n to A_n' by large non intersecting arcs which do not intersect the braid diagram. $\bar{\alpha}$ is a knot or link.

Alexander Theorem (1930)

Any knot (or link) K is the closure of some braid α, i.e. $\bar{\alpha} = K$.

Proof :



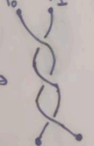
Step 1 : Cut K at a point P (not a crossing point) and pull the loose ends apart, call them P_0 and P_0'.

Step 2 : If there is a local maxima, (b) then there exists a local minima (a). Call $a = a_0$. The arc $\overline{ab}$ - whenever it meets a crossing point, call it $a_1, a_2, \dots$ $a_m = b$

Step 3 : Replace the arc $\overline{a_0a_1}$ by the larger arc $a_0P_1'P_1a_1$.
(cut at a_1)

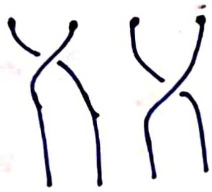
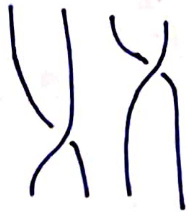
$\cong$

P_0 P_1 σ_1^3



Step 4: Continue in this manner till there is no maxima or minima left. What we have left then is a braid such that $\bar{\alpha} = K$.

Examples : ① 3_1 (done above)
② 5_1.

$\underline{B_4}$ $\sigma_1\sigma_3$  $\sim$ $\sigma_3\sigma_1$ 

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$\underline{B_3}$

$\alpha = \sigma_1\sigma_2\sigma_1$  $B = \sigma_2\sigma_1\sigma_2$  

mirror image about axis

$=$

$\therefore \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

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$\underline{\text{Ex}}$: $w_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4$. Show that $w_1 \sim w_2$

$w_2 = \sigma_2\sigma_1\sigma_2^2$

$= \cancel{\sigma_2\sigma_1\sigma_2\sigma}$

$w_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\underline{\sigma_2\sigma_4}$

$= \sigma_1\sigma_2\sigma_4^{-1}\underline{\sigma_1\sigma_4}\sigma_2$

$= \sigma_1\sigma_2\underbrace{\sigma_4^{-1}\sigma_4}_{e}\sigma_1\sigma_2$

$= \underline{\sigma_1\sigma_2\sigma_1}\sigma_2 = \sigma \ \boxed{\sigma_2\sigma_1\sigma_2^2}$

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Definition : Given a braid $\alpha \in B_n$, we form its closure by joining A_1 to A_1', A_2 to A_2', . . . A_n to A_n' by large non intersecting lines which do not intersect the braid diagram. $\bar{\alpha}$ is a knot or link.

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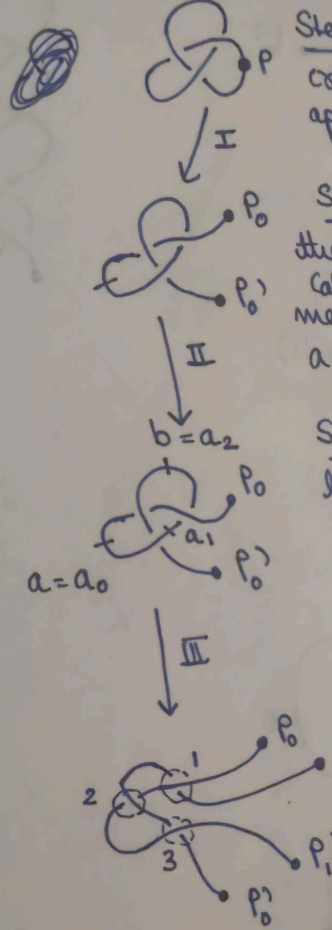

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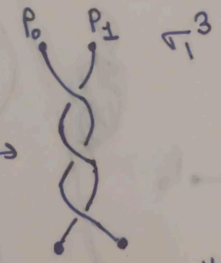

Step 4: Continue in this manner till there is no maxima or minima left. What we have left then is a braid such that $\bar{\alpha} = K$.

Examples : (1) 3_1 (done above)

(2) 5_1.

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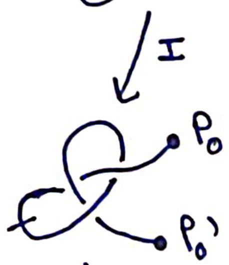

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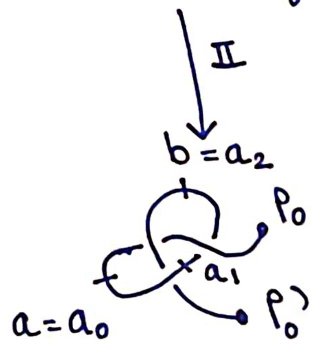

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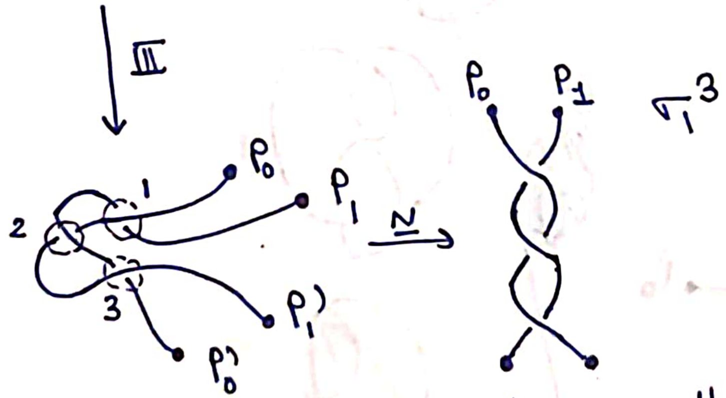


Step 4: Continue in this manner till there is no maxima or minima left. What we have left then is a braid such that $\bar{\alpha} = K$.

Examples : ① 3_1 (done above)
② 5_1.

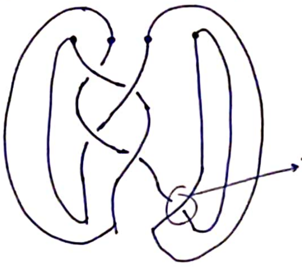
Theorem: $\alpha \in B_n$, $\gamma \in B_n$

M_1 : $\alpha \longrightarrow \gamma\alpha\gamma^{-1}$ then $\overline{\alpha} \simeq \overline{\gamma\alpha\gamma^{-1}}$

M_2 : $\alpha \longrightarrow \alpha\sigma_n$ or $\alpha\sigma_n^{-1}$ then $\overline{\alpha} \simeq \overline{\alpha\sigma_n}$

$\alpha \simeq_M \beta$ if we get α from β by finitely many Markov ~~mv~~
$\overline{\alpha} \simeq \overline{\beta}$.

Eg $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3$



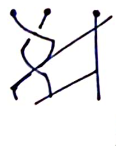
$\simeq$ Figure 8 knot

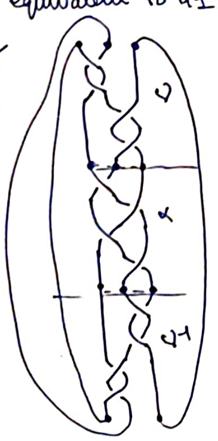
Eg $\sigma_1^{-2}\sigma_2^{2}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_2^{-2}\sigma_1^{2}$
$\simeq$ Figure 8

$\underbrace{\sigma_1^{-2}\sigma_2^{2}}_{\gamma}\underbrace{(\overline{\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2})}_{4_1}\underbrace{(\sigma_2^{-2}\sigma_1^{2})}_{\gamma^{-1}}$

4_1 → figure 8 knot

~~$\gamma \neq \gamma$~~ $\gamma\alpha\gamma^{-1} \Rightarrow$ By first Markov move, equivalent to 4_1.





γ & γ^{-1} when connected, get untwisted. (see)

Eg $\omega_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_4 \in B_5$
$\omega_2 = \sigma_2^{2}\sigma_3\sigma_1\sigma_3^{-1}\sigma_4 \in B_5$

$\omega_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{-1}$ (Remove σ_4 ~~(M_1)~~) (M_2)

$\downarrow$

$\sigma_2(\sigma_1\sigma_2\sigma_1)\sigma_2^{-1}$

$\downarrow$ (M_1)

$\sigma_1\sigma_2\sigma_1$

$\Rightarrow \omega_1 \simeq \sigma_1\sigma_2\sigma_1$

ω_2 : $\sigma_2^{2}\sigma_3\sigma_1\sigma_3^{-1}\sigma_4$

$\downarrow$ (M_2)

$\sigma_2^{2}\sigma_3\sigma_1\sigma_3^{-1}$ $(\sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| \geq 2)$

$\downarrow$

$\sigma_2^{2}\sigma_3\sigma_3^{-1}\sigma_1$

$\downarrow$

$\sigma_2^{2}\sigma_1$

$\Rightarrow \omega_2 \simeq \sigma_2^{2}\sigma_1$

→ BRAID INDEX Thus a given braid knot K can be formed as the closure of infinitely many distinct braids (Markov equivalent). Among them, $\exists$ a braid α with minimum number of strings. The number of strings of α is called the braid index of K.

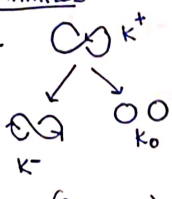
Braid index : ~~B(K) is~~ b(K) is an invariant of K (1930)

→ K is a knot formed as a closure of ~~B_n~~ $\alpha \in B_n$; $P_K(V, B)$ → Homfly polynomial of K.

* V-span $P_K(V,B) \leq 2(n-1)$
* $b(K) \geq \frac{1}{2}[\text{V-span } P_K(V,B)] + 1$
* $b(K) = \frac{1}{2}(\text{V-span } P_K) + 1$

→ EXAMPLES

① P_{O_2}



$(\frac{1}{V}P_{K^+} - VP_{K^-} = zP_{K_0})$
$P_{K^+} = v^2 P_{K^-} \pm vz P_{K_0}$

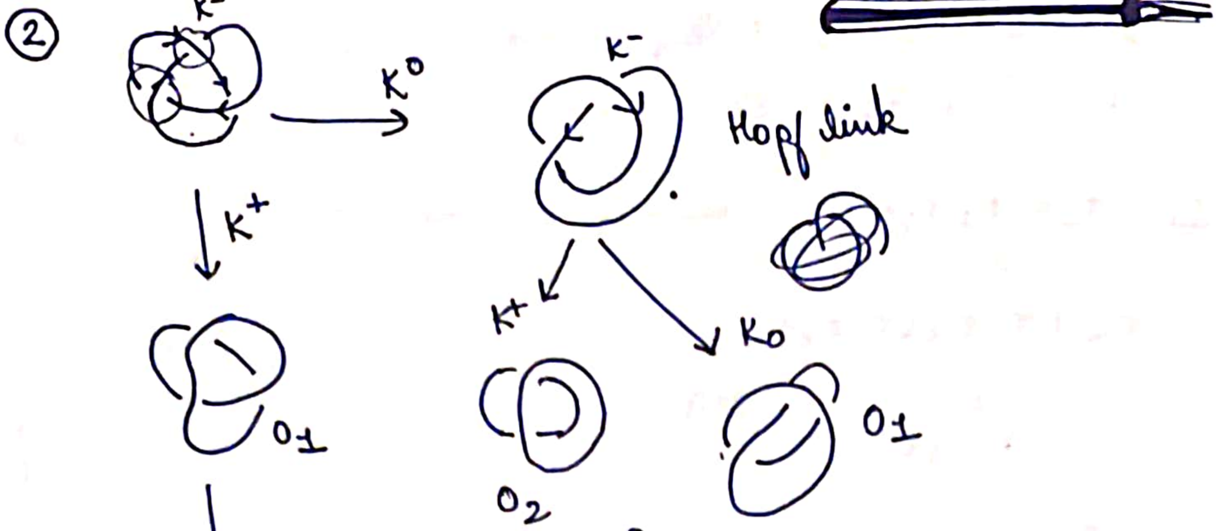
$P_{O_2} \to$ $P_{O_1} = v^2 P_{O_1} + vz P_{O_2}$

~~$v_{O_1} = v^2 v_{O_1}$~~

$(P_{O_1} = 1)$

$\Rightarrow \boxed{P_{O_2} = \frac{1-v^2}{vz}}$

②



$$v^2 P_{Hopf} = P_{O_2} - vz$$
$$= \frac{1-v^2}{vz} - vz$$

$$v^2 P_{3_1} = P_{K^+} - vz\left(\frac{1-v^2}{vz} - vz\right)$$
$$= 1 - (1 - v^2 - v^2 z^2)$$
$$= v^2(1+z^2)$$
$$\cancel{\boxed{P_{3_1} = 1+z^2}}$$

$$\boxed{v^2z^2 + 2v^2\cancel{z} - v^4}$$

max v degree = 4
min v degree = 2
v-span = 2

② Trefoil

K^+ — $K^- \cong O_1$

$K^0 \cong$ Hopf link — O_2 K_-

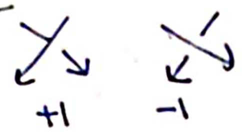
K^+ — K_0 O_1

$$P_H = v^{\cancel{2}}\left(\frac{1-v^2}{\cancel{v}z}\right) + vz$$
$$= \frac{v - v^3 + vz^2}{z}$$

$$P_{3_1} = v^2 + v^2 - v^4 + v^2 z^2$$
$$= 2v^2 + v^2z^2 - v^4$$

$$b(K) = \frac{1}{2}(\text{v-span}) + 1 = \frac{1}{2}(2) + 1 = \boxed{2}$$

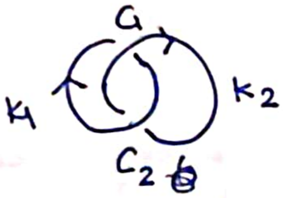
→ Linking Number:

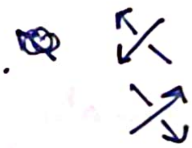


Def^n (linking number)
Let D be a diagram of a 2 component link L. $L = \{K_1, K_2\}$.
suppose crossing points between $K_1 \times K_2$ are $c_1, c_2, \ldots c_m$, then the linking number $(K_1, K_2) = \frac{1}{2}\{\operatorname{sgn} c_1 + \operatorname{sgn} c_2 + \cdots + \operatorname{sgn} c_m\}$
$lk(K_1, K_2)$

Theorem: $lk(K_1, K_2)$ is an invariant of the link $L = \{K_1, K_2\}$.

Eg

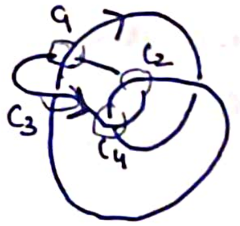


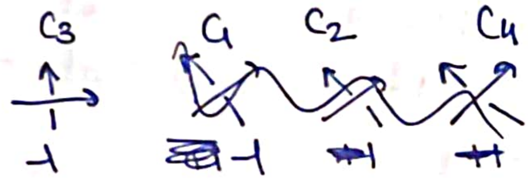


$\operatorname{sgn}(c_2) = -1$ $\quad L(K_1, K_2) =$

$\operatorname{sgn}(c_1) = -1$ $\quad \frac{1}{2}(-1-1) = \boxed{-1}$

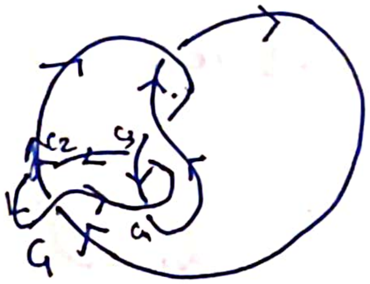
(Only count the crossings where 2 knots intersect Each Other.

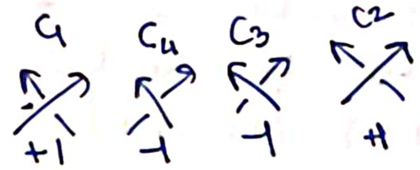




$L(K_1, K_2) = \frac{1}{2}(3-1) = \cancel{0}\ -2$

L_2





$L(K_1, K_2) = 0$

→ $L = (K_1, K_2)$

L^* is got by ~~not~~ exchanging all the crossings keeping orientations the same.

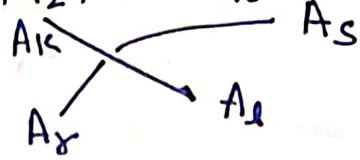
$lk(L) = -lk(L^*)$

If $L \cong L^*$ (achiral) $\Rightarrow lk L = 0$

If $lk(L) \neq 0 \Rightarrow L$ is not achiral

→ TRICOLOURABILITY

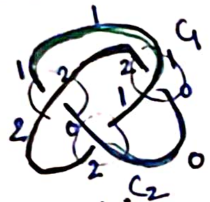
Suppose K is a knot or link with n crossing points $P_1, P_2, \ldots P_n$,
At each crossing point P_i $P_1, P_2, \ldots P_n$
we assign 3 colours:



such that

① A_K & A_ℓ have the same colours
② A_K, A_r, A_S either ~~have~~ all have the same colour OR all have different colours.

Definition If we can do this tricolouration at all crossing points of K (using atleast 2 diff colours), K is said to be tricolourable.



Definition: given a prime p, we assign a number λ_r from the set $\{0, 1, \ldots, p-1\}$. We want that ~~$\lambda_K = \lambda_\ell$~~

① $\lambda_K = \lambda_\ell$

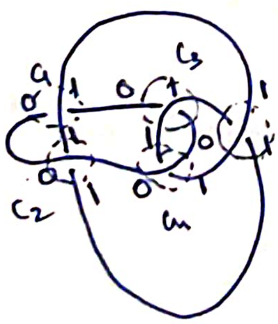
② $\lambda_r + \lambda_S \equiv \lambda_K + \lambda_\ell \pmod p$

Eg - Trefoil done above. p = 3 here

$C_1 \to$ ~~$(0+0) \bmod 3 = 0$~~ $0 + 0 = (1+2) \bmod 3$

$C_2 : (2+0) = (1+1) \bmod 3$

$C_3 : (0+1) = (2+2) \bmod 3$



C_1 : $(1+1) = (0+0) \bmod 2$

C_2 : $(1+1) = (0+0) \bmod 2$

C_3 : $(0+0) = (1+1) \bmod 2$

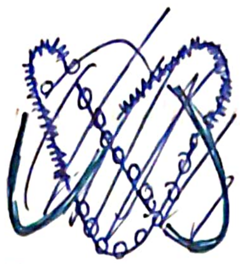
C_4 : $(0+0) = (1+1) \bmod 2$

FIGURE-8 – 5 colourable (congruent mod 5)

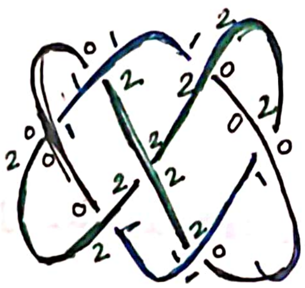
0, 1, 2, 3

CLASSICAL KNOT INVARIANTS – linking number and tricolourability

7_4



7_4.



Eg 2

(similarly number this)

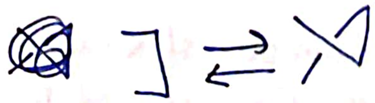
Definition.

…nots K_1 and K_2 are equivalent if we can go from K_1 to K_2 in …tely many knot moves.

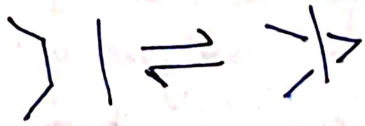
Representing K in 2D is called a knot diagram.

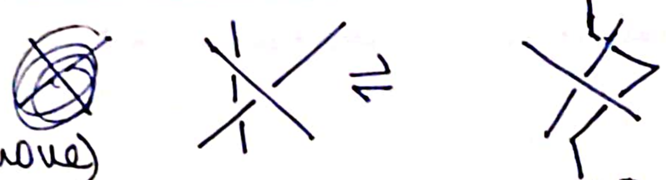
Reidemeister Moves

① Ω_1



② Ω_2



③ Ω_3 

(3rd R-move)

<u>Definition</u> If D, D' are knot diagrams, we can get D from D' by finitely many of 3-R moves, then we say $D \cong D'$.

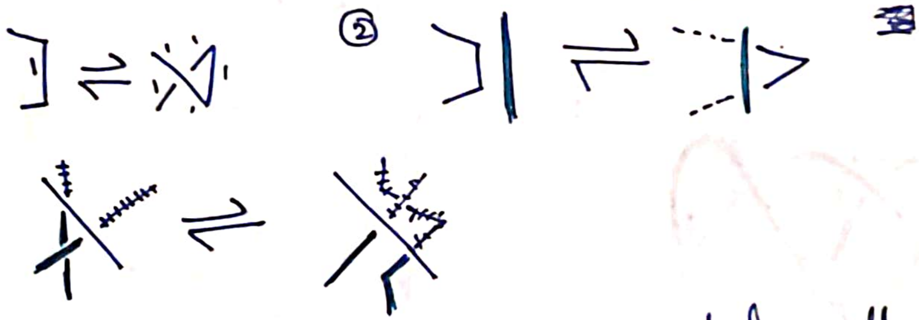
<u>Reidemeister theorem</u>

Suppose K, K' are knots with diagrams D, D'. Then $K \cong K' \Leftrightarrow D \cong D'$

<u>Theorem</u>: Tricolourability is a knot invariant

<u>Proof</u>: It is enough to show that tricolourability is preserved under 3 R-moves.

① ② ③



K ~~is~~, a knot (or link) $K \subset \mathbb{R}^3$, is represented on the 2D plane by a knot diagram D.

 → This is a representation of the 3 dim knot 3_1 on the 2D plane.

<u>Knot diagram D of 3_1</u>

→ <u>Knot Diagrams:</u>

$D_1 \cong D_2$ if we can go from D_1 to D_2 by finitely many R-moves

→ <u>Reidemeister's theorem:</u>

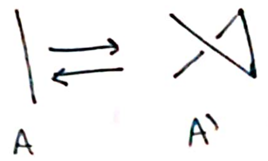
$K_1 \cong K_2 \Longleftrightarrow D_1 \cong D_2$

Let K be a knot (or link). If there is a regular diagram of K which is 3 colourable, then every diagram D' of K is 3 colourable. Such a knot K is 3 colourable.

<u>Proof</u>: Suppose D is a regular diagram of K which is 3 colourable. Let D' be another diagram of K, then we know we can get D' from D by finitely many R-moves & their inverses. To prove 3 colourability ~~it is~~ is a knot

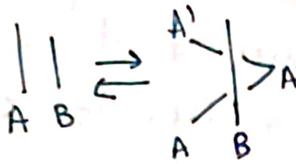
invariant, it is enough to show tricolourability is preserved under 3 R moves.

① Ω_1



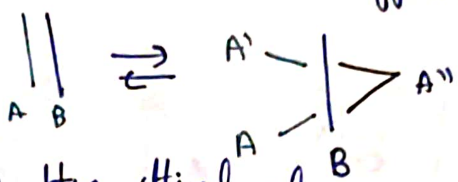
we will colour each segment of A' the same colour as A

② Ω_2



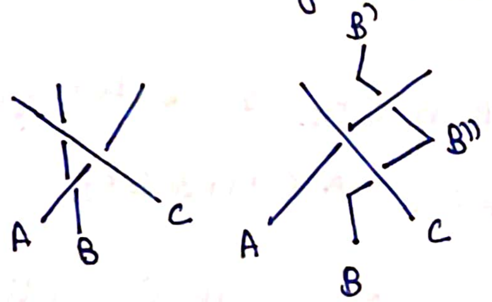
Case 1: If A & B are the same colour, then A, A', A'' $\longrightarrow$ same colour

Case 2: A & B have different colours



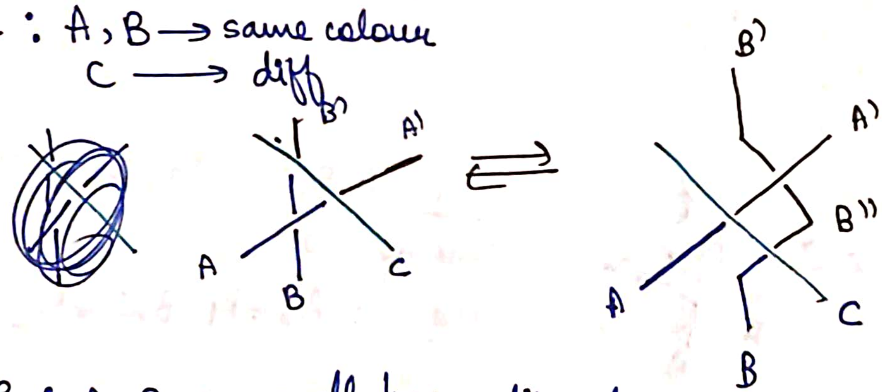
A & B retain their old colours, & A'' is given the third colour

③ Ω_3

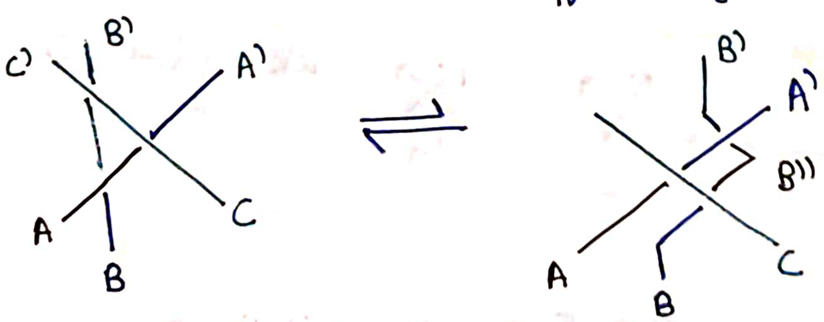


Case 1 : A, B, C $\rightarrow$ same colour, A, C, B, B', B'' all have the same colour

Case 2 : A, B $\rightarrow$ same colour
C $\longrightarrow$ diff



Case 3 : A, B, C $\longrightarrow$ all have diff colours

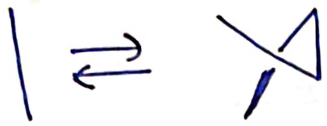


∴ Tricolourability is preserved.

Theorem: Let $L = \{K_1, K_2\}$ be a 2 component link. Then, the linking number $lk(L) = \frac{1}{2}(\text{sgn } c_1 + \dots \text{sgn}(c_n))$ is a link invariant.

Proof: Suppose D is a diagram of L and D' is another diagram of L. We know that we can get D' from D by applying 3 R-moves $\Omega_1, \Omega_2, \Omega_3$ finitely many times. So it is enough to show that L is invariant under Ω_1, Ω_2 & Ω_3.

① Ω_1

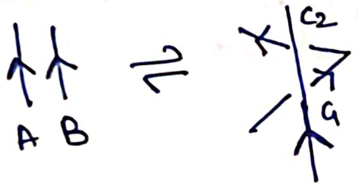


No effect on linking number

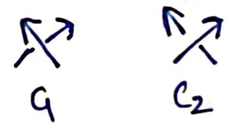
$\boxed{lk = 0}$

② Ω_2 If $A, B \in K_1$ or $A, B \in K_2 \Rightarrow$ no change

Case 2: $A \in K_1, B \in K_2$

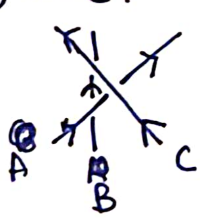


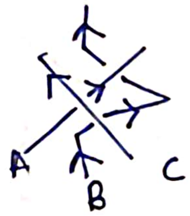
$lk = 0$



$\therefore lk(L) = \frac{1}{2}(1-1) = \boxed{0}$

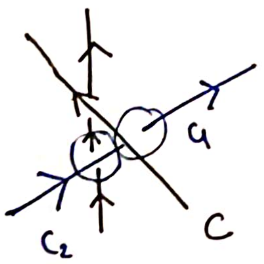
③ Ω_3:

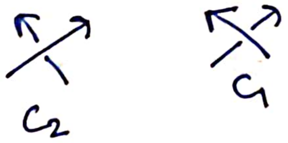




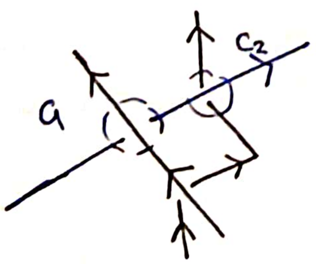
Case 1: $A, B, C \in K_1 / K_2 \Rightarrow$ no change (0)

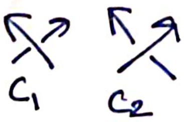
Case 2: $A \in K_1$ $B, C \in K_2$





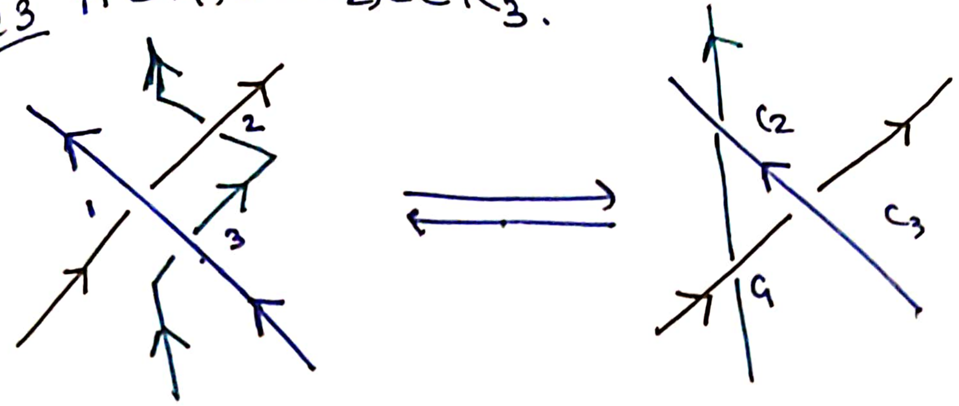
$lk = \frac{1}{2}(1-1) = 0$





$lk = \frac{1}{2}(1-1) = 0.$

Case 3 $A \in K_1, B \in K_2, C \in K_3$.



c_1

c_2

c_3

$$l = \frac{1}{2}(1-1+1) = \boxed{\frac{1}{2}}$$