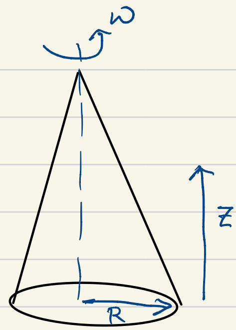
Section 2

2.1.



Key stratogy $\vec{m}$,

then $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$

- Calculate $\vec{m}_{disc}$ at any z:

Total charge q.

$$\vec{m}_{disc} = \hat{z} \int_0^r \left(\frac{q}{\pi r^2}\right) 2\pi s \, ds \cdot \frac{\omega}{2\pi} \left(\pi s^2\right) = \frac{q\omega}{r^2} \frac{r^4}{4}$$

$$= \frac{q\omega r^2}{4} \hat{z}$$

$$\vec{m} \text{ for cone: } \vec{m} = \hat{z} \int_0^h \pi \underbrace{\left(1 - \frac{z}{h}\right)^2}\underbrace{\left(R^2 dz \, \rho\right)} \frac{\omega}{4} \underbrace{\left(1 - \frac{z}{h}\right)^2 R^2}$$

$$= \hat{z} \frac{\pi \rho \omega}{4} R^4 \int_0^h \left(1 - \frac{z}{h}\right)^4 dz = \boxed{\hat{z} \frac{\pi \rho \omega R^4 h}{20}}$$

For $r \gg h$:

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \bigg|_{\theta = \pi/2} = \boxed{\frac{\mu_0 \rho \omega R^4 h}{80 r^2} \hat{\varphi}}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \bigg|_{\theta = \frac{\pi}{2}} = \boxed{- \frac{\mu_0 \rho \omega R^4 h}{80 r^3} \hat{z}}$$

2.2

Magnetic field inside $\vec{B} = \frac{2}{3}\mu_0 \sigma R \vec{\omega}$

Standard Result Quoted in the Question.

$\dot{\vec{\omega}} \Rightarrow$ Induced $\vec{E}$

$$E_{ind.} \cdot 2\pi R \sin\theta = \pi R^2 \sin^2\theta \frac{dB}{dt}$$

$$= \pi R^2 \sin^2\theta \left(\frac{2}{3}\mu_0 \sigma R \dot{\omega}\right)$$

$$\Rightarrow E_{ind} = \boxed{\frac{\mu_0 \sigma R^2}{3} \sin\theta\, \dot{\omega}} \qquad \left(\alpha = \frac{d\omega}{dt}\right)$$

Torque on a ring: $dQ = 2\pi R^2 \underbrace{\sin\theta\, d\theta}\, \sigma$

$$N = \int_0^{\pi} (R\sin\theta)\left(\frac{\mu_0 \sigma R^2 \alpha}{3}\right) \sin\theta\, 2\pi R^2 \sigma \;\; d(\cos\theta)$$

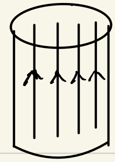
$$= \frac{2\pi}{3} \mu_0 \sigma^2 R^5 \alpha \int_{-1}^{1} (1 - \cos^2\theta)\, d(\cos\theta)$$

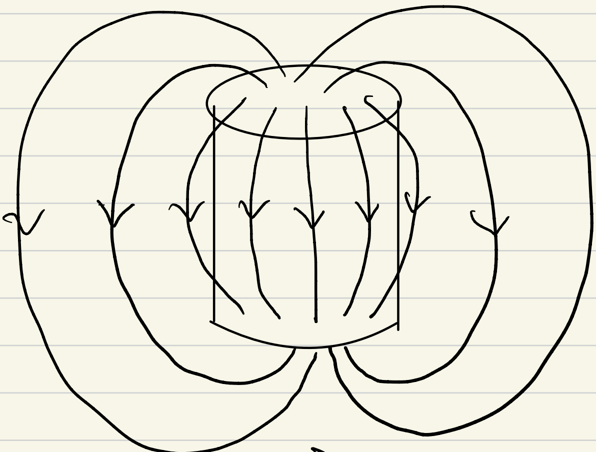
$$= \boxed{\frac{8\pi \mu_0 \sigma^2 R^5 \alpha}{9}}$$

$$\text{Energy} = \int N\, d\theta$$

$$= \frac{8}{9}\mu_0 \sigma^2 R^5 \int \frac{d\omega}{dt}\, \omega\, dt = \boxed{\frac{4\pi}{9}\mu_0 \sigma^2 R^5 \omega_0^2}$$

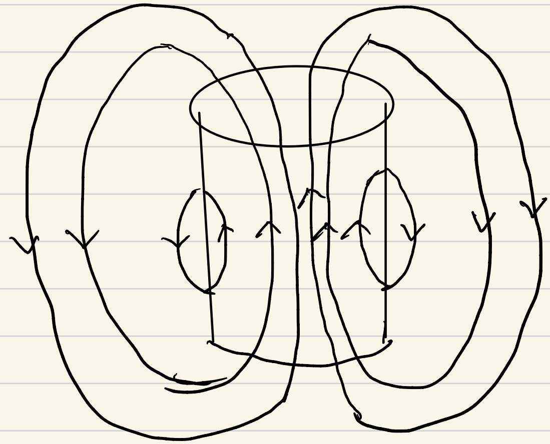
2.3





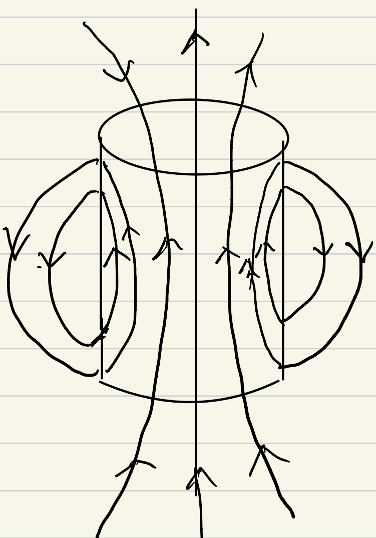
$\vec{E}$

Field of two circular plates

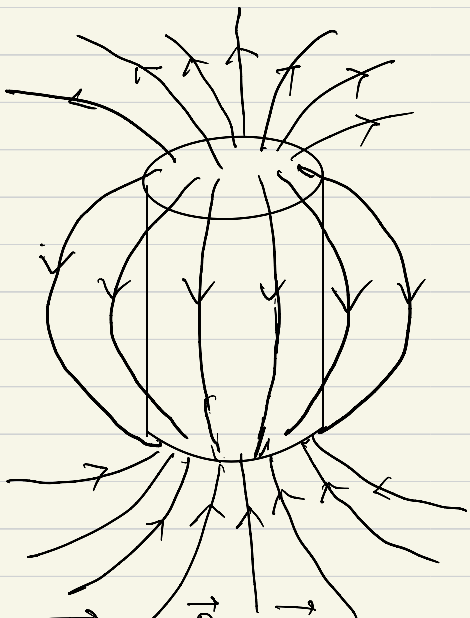


$\vec{D}$

$\vec{\nabla}\cdot\vec{D}=0$, hence lines are continuous, similar to $\vec{E}$ outside.



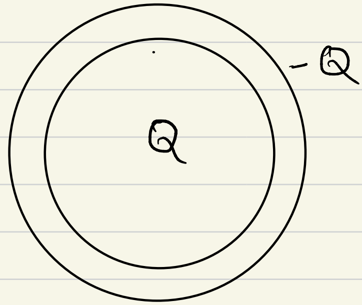
same as field of short solenoid.



$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

Section - 3

3.1.



$$\vec{E}_{in} = \frac{\rho}{3\epsilon_0}\vec{r}$$

$$\vec{E}_{out} = 0$$

$$\vec{B}_{in} = \frac{2}{3}\mu_0 \vec{M} = \frac{2}{3}\mu_0 M \hat{z}$$

Momentum density $= \vec{g} = \frac{\vec{S}}{c^2} = \frac{1}{\mu_0 c^2} \vec{E}\times\vec{B}$

$$\vec{g} = \left[\frac{1}{\mu_0 c^2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)\vec{r}\right] \times \left[\frac{2}{3}\mu_0 M \hat{z}\right]$$

$$= \frac{QM}{6\pi\epsilon_0 c^2 R^3}\left(\vec{r}\times\hat{z}\right) = -\underbrace{\left(\frac{QM}{6\pi\epsilon_0 c^2 R^3}\right)}_{\beta} r\sin\theta\,\hat{\varphi}$$

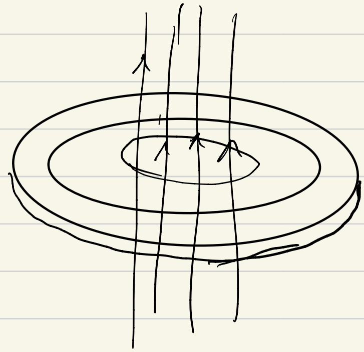
Angular Momentum $= -\beta\int\left(\vec{r}\times\hat{\varphi}\right) r\sin\theta\, d\tau$

$$= -\beta\int\left(r^2\sin\theta\ \hat{\theta}\right) r^2 dr\ d(\cos\theta)\ d\theta$$

$$= -\beta\int_0^R r^4 dr \int \sin^2\theta\ d(\cos\theta)\ 2\pi\ \hat{z}$$

$$= -\beta\cdot\frac{R^5}{5}\cdot\frac{4}{3}\,2\pi\hat{z} = -\frac{4}{45}\frac{QMR^2}{\epsilon_0 c^2}\hat{z}$$

3.2.



$$\int \vec{E}(s)\cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$= \mu_0 n \pi a^2 I_0 \omega \sin(\omega t)$$

$$E = \frac{\mu_0 n a^2 I_0 \omega \sin(\omega t)}{2s}$$

$$J(s) = \sigma E, \qquad dI = J(s)\cdot ds(d)$$

$$= \frac{\sigma \mu_0 n a^2 I_0 \sin(\omega t)}{2s} d \; ds$$

$$\boxed{I_c = \frac{\sigma \mu_0 n a^2 \omega d \sin(\omega t)}{2} \ln\left(\frac{c}{b}\right)}$$

$$J_d = \epsilon_0 \frac{d\vec{E}}{dt} = \frac{\mu_0 \epsilon_0 n a^2 I_0 \omega^2 \cos(\omega t)}{2s}$$

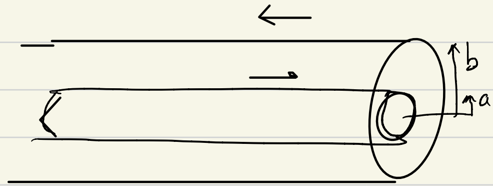
$$I_d = \int \frac{d \cdot \mu_0 \epsilon_0 n a^2 I_0 \omega^2 \cos(\omega t)}{2s}$$

$$= \boxed{\frac{d \mu_0 \epsilon_0 n a^2 I_0 \omega^2}{2} \cos(\omega t) \ln\left(\frac{c}{b}\right)}$$

$$\frac{I_c}{I_d} = \frac{\sigma}{\omega \epsilon_0} = \frac{10^7}{9\pi \times 10^9 \times \frac{1}{36\pi} \times 9 \times 10^{-9}} \approx \boxed{4 \times 10^7}$$

3.3.

(a)



$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

$$\frac{\text{Energy}}{\text{Vol.}} = \frac{1}{2\mu}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}$$

shell of length ℓ, radiu s, and thickness ds

$$\left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right)\ell 2\pi s\, ds = \frac{\mu_0 I^2 \ell}{4\pi}\left(\frac{ds}{s}\right)$$

$$\boxed{\left(\frac{W}{\ell}\right) = \frac{\mu_0 I^2}{4\pi}\ln\left(\frac{b}{a}\right)}$$

(b) It was broadcast in the beginning of the exam that they can take the inner conductor to be carrying λ linear charge density.

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s} \qquad \vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{s}\hat{\varphi}$$

$$\vec{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\hat{z} \Rightarrow P = \int \vec{S}\cdot d\vec{a}$$

$$= \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{1}{s^2} 2\pi s\, ds$$

$$= \boxed{\frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)} = IV$$

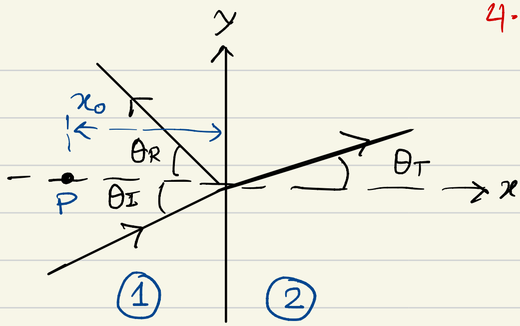
Since $V = \int_a^b \vec{E}\cdot d\vec{\ell} = \frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{1}{s}ds = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right).$

4.1

1/2

4.1

a)



$\theta_I = \theta_B$, μ_0

$$\tan\theta_B = \frac{n_2}{n_1} = \sqrt{3}$$

$$\theta_B = 60^\circ = \theta_I$$

Since the reflected wave vanishes, the electric field vector must lie in x-y plane.

$$\therefore \hat{e} = \left(-\sin 60^\circ\, \hat{i} + \cos 60^\circ\, \hat{j}\right) = \left(-\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$a = -\frac{\sqrt{3}}{2}$

$b = \frac{1}{2}$

$c = 0$

b) $\theta_I = 30^\circ$, $\tilde{E}_{0T} = \frac{2}{\alpha+\beta}\tilde{E}_{0I}$

$$\beta = \frac{n_2}{n_1} = \sqrt{3}, \quad \alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\sqrt{3}/2}$$

$$= \frac{\sqrt{1-\frac{\sin^2\theta_I}{n_2^2}}}{\sqrt{3}/2}$$

$$\therefore \alpha = \frac{\sqrt{1-\frac{1}{4\cdot 3}}}{\sqrt{3}} \times 2 = \frac{\sqrt{11}}{3} = 1.1$$

$$E_{0T} = \frac{2}{\frac{\sqrt{11}}{3}+\sqrt{3}} E_{0I} = \frac{2\times 3}{\sqrt{11}+3\sqrt{3}} E_{0I} = \frac{6}{\sqrt{11}+3\sqrt{3}} E_{0I}$$

$$= 0.7\, E_{0I}$$

$$\cos\theta_T = \sqrt{1-\frac{1}{4\times 3}} = \sqrt{\frac{11}{12}} = 0.96$$

$$\sin\theta_T = \sqrt{\frac{1}{12}} = 0.29$$

$$E_{oT} = E_{oT}\left(-\sin\theta_T\,\hat{i} + \cos\theta_T\,\hat{j}\right)\cos\left(\vec{k}_T\cdot\vec{r} - \omega t\right)$$

$$= E_{oI}\,\frac{6}{\sqrt{11}+3\sqrt{3}}\left(-\sqrt{\frac{1}{12}}\,\hat{i} + \sqrt{\frac{11}{12}}\,\hat{j}\right)\times$$

$$\cos\left[x\sqrt{\frac{11}{12}} + y\sqrt{\frac{1}{12}} - \omega t\right]$$

c) $\vec{E} = \tilde{E}_I + \tilde{E}_R$

$$\vec{k}_R = \left(-\cos 30^\circ\,\hat{i} + \sin 30^\circ\,\hat{j}\right)k_I$$

$$\tilde{E}_{oR} = E_{oR}\left(\sin 30^\circ\,\hat{i} + \cos 30^\circ\,\hat{j}\right)$$

$$\vec{E} = E_{oI}\left(-\sin 30^\circ\,\hat{i} + \cos 30^\circ\,\hat{j}\right)\cos\left(\vec{k}_I\cdot\vec{r} - \omega t\right)$$

$$+ E_{oR}\left(\sin 30^\circ\,\hat{i} + \cos 30^\circ\,\hat{j}\right)\cos\left(\vec{k}_R\cdot\vec{r} - \omega t + \pi\right)$$

$\tilde{E}_R$ is out of phase with $\tilde{E}_I$ by π

$$E_{oR} = \frac{|\alpha-\beta|}{\alpha+\beta}E_{oI} = \frac{\left|\frac{\sqrt{11}}{3} - \sqrt{3}\right|}{\frac{\sqrt{11}}{3} + \sqrt{3}}E_{oI} = 0.22\,E_{oI}$$

$$\boxed{\begin{aligned}\vec{E} &= E_{oI}\left(-\tfrac{1}{2}\hat{i} + \tfrac{\sqrt{3}}{2}\hat{j}\right)\cos\left(-\tfrac{\sqrt{3}}{2}x_0 k_I - \omega t\right) \\ &+ E_{oR}\left(\tfrac{1}{2}\hat{i} + \tfrac{\sqrt{3}}{2}\hat{j}\right)\cos\left(\tfrac{\sqrt{3}}{2}x_0 k_I - \omega t + \pi\right)\end{aligned}}$$

$\frac{4.2}{1}$

4.2

$$\frac{\sigma}{\epsilon\omega} >> 1.$$ 2

Good conductor limit

$$K_2\Big|_{real} = K_2\Big|_{imag.} = \sqrt{\frac{\mu\omega\sigma}{2}}$$

$$= \sqrt{\frac{4\pi\times10^{-7}\times3\times10^{15}\times5.8\times10^{7}}{2}}$$ 2

$$= 4.6\times10^{8}\,m^{-1} \qquad d = \frac{1}{K_2}$$

$$k_1 = \frac{\omega}{c} = \frac{3\times10^{15}}{3\times10^{8}} = 10^{7}\,m^{-1}$$ 2

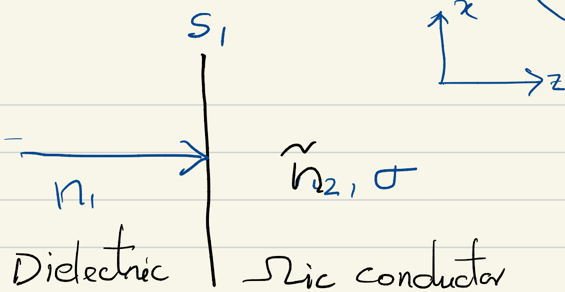
$$R = \frac{(k_1-k_2)^2 + K_2^2}{(k_1+k_2')^2 + K_2^2}$$

$$= \frac{(1-46)^2+46^2}{(1+46)^2+46^2} \simeq 0.96$$ 2

96% of the light will be reflected back.

4.3/1

4.3



a) $\vec{E}_I = \hat{x} E_{0I}\, e^{i(k_1 z - \omega t)}$, $\quad \vec{B}_I = \frac{E_{0I}}{v_1} e^{i(k_1 z - \omega t)}\hat{y}$

$$= \hat{y}\, \frac{E_{0I}\, n_1}{c}\, e^{i(k_1 z - \omega t)}$$

b) $\vec{E}_T = E_{0T}\, e^{i(\tilde{k}_2 z - \omega t)}\, \hat{x} \quad = \frac{E_{0T}\tilde{n}_2}{c} e^{i(\tilde{k}_2 z - \omega t)}$

c) $\left.\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \right\}$ $\quad E_1^{\perp},\ E_2^{\perp} = 0 \Rightarrow \sigma_f = 0$

d) $\vec{K}_f = 0$

e) $E_1^{\parallel} - E_2^{\parallel} = 0 \Rightarrow \boxed{E_{0I} + E_{0R} = E_{0T}}$ — (1)

$$\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = 0 \Rightarrow B_1^{\parallel} - B_2^{\parallel} = 0 \qquad (\mu_1 = \mu_2 = \mu_0)$$

$$\Rightarrow \frac{E_{0I}}{v_1} - \frac{E_{0R}}{v_1} = \frac{E_{0T}}{v_2} \Rightarrow E_{0I} - E_{0R} = \frac{v_1}{v_2} E_{0T} = \frac{n_2}{n_1} E_{0T}.$$

$$\Rightarrow \boxed{E_{0I} - E_{0R} = (1 + ip)\, E_{0T}} \quad —(2)$$

(1) & (2) $\Rightarrow$ $E_{0R} = \frac{-ip}{2+ip} E_{0I} = \frac{-p(p+2i)}{p^2+4} E_{0I}$

$$= \frac{-p}{\sqrt{p^2+4}}\, e^{i\varphi} E_{0I} \Rightarrow \boxed{\tan\varphi = \frac{2}{p}}$$

Section 5

5.1.

Localized Volume $- \mathcal{V}$

IMPORTANT

Since it is about local charge conservation,

$$Q(t) = \int_{\mathcal{V}} \rho(\vec{r}, t)\, d\tau$$

$$\frac{dQ}{dt} = - \oint_S \vec{J} \cdot d\vec{a}$$

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = - \int_S (\vec{\nabla} \cdot \vec{J})\, d\tau$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} = - \vec{\nabla} \cdot \vec{J}}$$

Since it is valid for arb. vol. and its surface.

Given: $\rho(t=0) = \rho_0$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \qquad \vec{J} = \sigma \vec{E}$$

$$\frac{\partial \rho}{\partial t} = - \vec{\nabla} \cdot \vec{J}$$

$$= -\sigma \vec{\nabla} \cdot \vec{E}$$

$$= - \left(\frac{\sigma}{\epsilon}\right) \rho(t)$$

$$\rho(t) = \rho_0 e^{-t/\left(\frac{\epsilon}{\sigma}\right)} = \boxed{\rho_0 e^{-t/\tau}} \qquad \boxed{\tau = \frac{\epsilon}{\sigma}}$$

5.2. $\vec{A} = A_0 \cos(\alpha x - \omega t)\, \hat{y}$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left[A_0 \cos(\alpha x - \omega t)\right] \hat{y}$$

$$\boxed{\vec{E} = -A_0 \omega \sin(\alpha x - \omega t)\, \hat{y}}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{z} \frac{\partial}{\partial x}\left[A_0 \cos(\alpha x - \omega t)\right]$$

$$\boxed{\vec{B} = -A_0 \alpha \sin(\alpha x - \omega t)\, \hat{z}}$$

$$\vec{\nabla} \times \vec{E} = \hat{z} \frac{\partial}{\partial x}\left[A_0 \omega \sin(\alpha x - \omega t)\right]$$

$$\boxed{= \hat{z} A_0 \omega \alpha \cos(\alpha x - \omega t)} \quad \text{or,} \quad \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla} \times \vec{B} = -\hat{y} \frac{\partial}{\partial x}\left[-A_0 \alpha \sin(\alpha x - \omega t)\right]$$

$$= \boxed{A_0 \alpha^2 \cos(\alpha x - \omega t)\, \hat{y}}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Big| \text{ Maxwell-Ampere's Law in vac.}$$

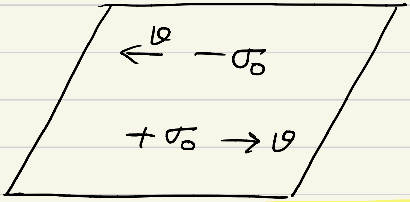
$$\frac{\partial \vec{E}}{\partial t} = A_0 \omega^2 \cos(\alpha x - \omega t)\, \hat{y}$$

$$\Rightarrow \quad A_0 \alpha^2 \cos(\alpha x - \omega t)\, \hat{y} = \mu_0 \epsilon_0 \left[A_0 \omega^2 \cos(\alpha x - \omega t)\right] \hat{y}$$

$$\Rightarrow \quad \alpha^2 = \mu_0 \epsilon_0 \omega^2$$

$$\Rightarrow \quad \boxed{\omega = c\alpha}$$

5.3.



In frame S, $E = 0$.

$$\vec{K} = 2\sigma_0 v \hat{z}$$

$$\vec{B} = \frac{\mu_0}{2} \vec{K} \times \hat{n} = \mu_0 \sigma_0 v \hat{z} \times \hat{k} = \boxed{-\mu_0 \sigma_0 v \hat{j}}$$

In frame S', the charge density for the positive sheet $\sigma_+' = \sigma_0 \sqrt{1 - v^2/c^2}$

In S, $E_x = E_y = E_z = 0$,

$B_x = 0$, $B_y = -\mu_0 \sigma_0 v$, $B_z = 0$

$$E_x' = E_x = 0, \quad E_y' = \frac{E_y - v B_z}{\sqrt{1 - v^2/c^2}} = 0,$$

$$E_z' = \frac{E_z + v B_y}{\sqrt{1 - v^2/c^2}} = -\frac{\mu_0 \sigma_0 v^2}{\sqrt{1 - v^2/c^2}} = \frac{\sigma_0 (v^2/c^2)}{\epsilon_0 \sqrt{1 - v^2/c^2}}$$

$$= \frac{\sigma_0}{\epsilon_0} \left[\frac{v^2/c^2}{\sqrt{1 - v^2/c^2}} \right]$$

$$B_x' = B_x = 0, \quad B_y' = \frac{B_y + \frac{v^2}{c^2} E_z}{\sqrt{1 - v^2/c^2}} = \frac{-\mu_0 \sigma_0 v}{\sqrt{1 - v^2/c^2}}, \quad B_z' = 0$$