# Numerical Analysis & Scientific Computing II

## Lesson 3

# Boundary Value Problems for ODEs

Akash Anand
MATH, IIT KANPUR

# Boundary Value Problems: Well-posedness

Akash Anand
MATH, IIT KANPUR

*In many practical problems involving ODEs, instead of initial values, specifying additional at more that one points may be more relevant. In such case, we say that the problem is a Boundary Value Problem (BVP) for ODE.*

*For example, if you want to throw a projectile from location A and want it to hit location B, you would need to solve the equation of motion together with conditions at A and B.*

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## 3.1 Well-posedness

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*A general first-order two-point BVP for an ODE has the form*

$$y' = f(t, y), \qquad a < t < b,$$

*with boundary conditions*

$$g\big(y(a), y(b)\big) = 0$$

*where* $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$ *and* $g: \mathbb{R}^{2n} \to \mathbb{R}^n$.

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*The boundary condition is said to be linear if they have the form*

$$B_a y(a) + B_b y(b) = c,$$

*where* $B_a, B_b \in \mathbb{R}^{n\times n}$ *and* $c \in \mathbb{R}^n$.

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*If the boundary conditions are both separated and linear, then for each* $i, 1 \le i \le n$*, either the* $i$*th row* $B_a$ *or the* $i$*th row of* $B_b$ *contains only zero entries.*

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*The BVP is said to be linear if both the ODE and the boundary conditions are linear.*

# Boundary Value Problems: Well-posedness

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MATH, IIT KANPUR

## Example

Consider the two-point BVP for the second-order scalar ODE

$$u'' = f(t, u, u'), \qquad a < t < b,$$

with boundary conditions

$$u(a) = \alpha, \qquad u(b) = \beta.$$

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*This problem is equivalent to the first-order system of ODEs*

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ f(t, y_1, y_2) \end{bmatrix}, \qquad a < t < b,$$

*with separated linear boundary conditions*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(a) \\ y_2(a) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(b) \\ y_2(b) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

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MATH, IIT KANPUR

## Existence, Uniqueness and Conditioning

*Unlike the IVPs, with the BVP, there is no single point at which complete state information is given, and hence no point at which local existence of a solution can be established.*

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*Consider the two-point BVP*

$$u'' = -u, \qquad 0 < t < b,$$

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$$u(0) = 0, \qquad u(b) = \beta.$$

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*The general solution of the ODE satisfying* $u(0) = 0$ *is* $u(t) = c \sin t$ *for a constant* $c$. *If* $b = m\pi, m \in \mathbb{Z}$, *then* $c \sin b = 0$ *for any* $c$, *so there are infinitely many solutions of the BVP if* $\beta = 0$, *but there is no solution* $\beta \neq 0$.

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*Let* $y(t; x)$ *denote the solution to the IVP* $y' = f(t, y),\ y(a) = x,\ x \in \mathbb{R}^n$. *This solution is a solution to the BVP if*

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*The solvability of the BVP therefore depends on the existence and uniqueness of solutions to the system of nonlinear algebraic equations $h(x) = 0$, where $h(x) = g(x, y(b; x))$. We have seen (in the first course) that this is not always true, and therefore, can not expect a general theorem for existence and uniqueness of solutions for BVP.*

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*Such results are available only in certain specialized and simplified conditions.*