Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Complex Analysis (MTH 403)
Exercise Sheet 7

1. Zeros of analytic functions

Throughout this section, unless otherwise mentioned, U always stands for a region.

1.1. Show that $H(U)$ is an integral domain with respect to pointwise addition and multiplication.

1.2. Let U be an open connected subset of $\mathbb{C}$ and $f \in H(U)$. Assume that for all $z \in U$ there exists $n \geq 0$ such that $f^{(n)}(z) = 0$. What can you conclude about f?

1.3. Let $f : \mathbb{D} \longrightarrow \mathbb{C}$. Show that, if f^2 and f^3 both are holomorphic, then so if f.

Hint. *Observe that* $f = \frac{f^3}{f^2}$ *at all ponts* $z \in \mathbb{D}$ *such that* $f(z) \neq 0$. *So zeros are needed to be taken care of.*

1.4. (L'Hôpital's rule). Let $U \subseteq_{open} \mathbb{C}$ and $f, g \in H(U)$. Suppose that $z_0 \in U$ is such that on some neighbourhood of z_0 in U, none of f and g vanishes identically, but $\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to z_0} g(z) = 0$. Show that $\frac{f(z)}{g(z)}$ approaches to a finite limit or ∞ as $z \to z_0$, and furthermore, $\dfrac{\lim\limits_{z \to z_0} f(z)}{\lim\limits_{z \to z_0} g(z)} = \dfrac{\lim\limits_{z \to z_0} f'(z)}{\lim\limits_{z \to z_0} g'(z)}$.

1.5.∗ Let U be a region in $\mathbb{C}$. Assume that U is *symmetric* with respect the real axis, i.e., $z \in U \Longrightarrow \bar{z} \in U$. Suppose that $f \in H(U)$ is such that $f(J) \subseteq \mathbb{R}$, for some open subinterval of containied in $U \cap \mathbb{R}$. Show that $f(U \cap \mathbb{R}) \subseteq \mathbb{R}$ and $f(\bar{z}) = \overline{f(z)}$, for all $z \in U$.

1.6. Let f be a nonzero entire function such that $f(0) = 0$ and $f(\mathbb{R}) \subseteq \mathbb{R}$. Show that if the image of the imaginary axis under f is contained in a line, then that line must be either the real axis or the imaginary axis.

Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$. We say that $w \in \mathbb{C}$ is a *period* of φ if $\varphi(w + z) = \varphi(z)$, for all $z \in \mathbb{C}$.

1.7. Let L_1 and L_2 stand for the lines $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \pi$ respectively. Suppose that f is an entire function such that $f(L_j) \subseteq \mathbb{R}$, for $j = 1, 2$. Show that f is 2π-periodic.

1.8. Show that a nonconstant entire function can have at most countably many periods.

2. Maximum modulus principle

2.1. Formulate and prove the 'Minimum modulus principle'. Conclude that, for any region U in $\mathbb{C}$ and nonconstant holomorphic function $f : U \longrightarrow \mathbb{C}$, $|f|$ can attain a local minima only at zeros of f.

2.2. Find the maximum and minimum of $|f|$ in each of the following cases:

(a) $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} \dfrac{z^2}{z+2}$.
(b) $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} z^2 - z$.
(c) $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} e^{z^2}$.
(d) $f : \{z \in \mathbb{C} : |z|^2 \leq 4, \operatorname{Re} z, \operatorname{Im} z \geq 0\} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} ze^z$.

2.3. Let $n \in \mathbb{N}$ and $P(z) \stackrel{\text{def}}{=} z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients.

(a) Choose $r > 1 + 2|a_0| + a_1 + \cdots + |a_{n-1}|$. Show that, for any $t \in [0, 2\pi]$, one has $|P(re^{it})| > |P(0)|$.
(b) Using Maximum modulus principle, show that P must have a zero.
(c) Conclude the Fundamental theorem of algebra.

2.4.$*$ (a) Let $U \subseteq \mathbb{C}$ be a bounded region and $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\overline{U}$ converging uniformly on ∂U. Show that, if each $f_n \in H(U)$, then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\overline{U}$.
(b) Find all functions $f : \partial\mathbb{D} \longrightarrow \mathbb{C}$ such that there is a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ which converges uniformly to f (on $\partial\mathbb{D}$).

Hint. *It follows from 2.4.a that there exists a continuous $g : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ such that $g \in H(\mathbb{D})$ and $g|_{\partial\mathbb{D}} = f$. Conversely, let $g : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous and $g \in H(\mathbb{D})$. For each $n \in \mathbb{N}$, consider $g_n(z) = g\left(\frac{n}{n+1}z\right)$, for all $z \in D\left(0; \frac{n+1}{n}\right)$. Clearly all g_n's are holomorphic. Now for any $n \in \mathbb{N}$, choose a polynomial P_n such that $|P_n(z) - g_n(z)| < \frac{1}{n}$, for all $z \in \overline{\mathbb{D}}$. Using the uniform continuity of g, show that $\{P_n\}_{n=1}^{\infty}$ which converges uniformly to g on $\overline{\mathbb{D}}$.*

2.5. Let $U \subseteq \mathbb{C}$ be as above in 2.4.a and $f : \overline{U} \longrightarrow \mathbb{C}$ be continuous and holomorphic on U. Show the following:

(a) If f is nonconstant and $|f|$ is constant on ∂U, then f must have a zero in U.
(b) if $f \equiv 0$ on ∂U then f must be identically zero everywhere.
(c) If f is real valued on ∂U, then f is constant. What if f assumes purely imaginary values on ∂U?
(d) If $U = \mathbb{D}$, $|f(z)| > 1$ whenever $|z| = 1$, and $f(0) = i$, then f has a zero on $\mathbb{D}$.

2.6. Let $U \subseteq_{open} \mathbb{C}$ and $f \in H(U)$ be nonconstant. Can $\operatorname{Re} f$ and $\operatorname{Im} f$ have local maxima or minima?

2.7. Show that, for any finite subset $\{a_1, \ldots, a_n\}$ of the unit circle, $\max\limits_{|z|=1} |z - a_1| \ldots |z - a_n| \geq 1$.

2.8. (a) Let U be a bounded region in $\mathbb{C}$ and $f \in H(U)$. Suppose that, for every $\{z_n\}_{n=1}^{\infty}$ in U converging to a point of ∂U, $f(z_n) \xrightarrow[n\to\infty]{} 0$. Then show that $f \equiv 0$ on U.
(b)$*$ Let $U \stackrel{\text{def}}{=} \mathbb{D}$ in 2.8.a. Suppose that the hypothesis is weakened as follows: for every $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ converging to a point of an arc $\{e^{it} : \alpha \leq t \leq \beta\}$, where $\alpha < \beta$, $f(z_n) \xrightarrow[n\to\infty]{} 0$. Show that one can arrive at the same conclusion, i.e., $f \equiv 0$ on $\mathbb{D}$.
(c) Conclude that if $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ is continuous and holomorphic on $\mathbb{D}$ and vanishes identically on an arc of the boundary, then $f \equiv 0$.

2.9.$*$ Suppose that $f \in H(\mathbb{D})$ is such that $f(0) = 0$ and $\forall z \in \mathbb{D}$, $|f(z)| \leq 1$. Show that, if f has any other fixed point different from 0 then it must be the identity function.

Hint. *Consider the the following function:*

$$g(z) \stackrel{def}{=} \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Then g is holomorphic. Use Maximum modulus principle to show that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, and from this show that $g \equiv 1$ on $\mathbb{D}$.

2.10. Let $f : \mathbb{D} \longrightarrow \mathbb{C}$ be holomorphic.

(a) Show that there exists $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that $|z_n| \xrightarrow[n\to\infty]{} 1$ and $\{f(z_n)\}_{n=1}^{\infty}$ is convergent.

Hint. *Enough to work with the case f is nonconstant and has finitely many zeros in $\mathbb{D}$ (why?) Then dividing it by a suitable polynomial, we may further assume that f is zero-free. Now use Minimum modulus principle to construct such a sequence.*

(b)$*$ Assume that f is nonconstant. Show that there are sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that $|z_n|, |w_n| \xrightarrow[n\to\infty]{} 1$, both $\{f(z_n)\}_{n=1}^{\infty}$ and $\{f(w_n)\}_{n=1}^{\infty}$ are convergent but limits are not equal.

Hint. *Let $\{z_n\}_{n=1}^{\infty}$ be as obtained in 2.10.a. If necessary, subtracting a constant from f we may assume that $f(z_n) \xrightarrow[n\to\infty]{} 0$. Passing through a subsequence if needed, we may further assume that $\{|z_n|\}_{n=1}^{\infty}$ is strictly increasing. Now, for each $n \in \mathbb{N}$, consider $M_n \overset{def}{=} \max_{|w|=|z_n|} |f(w)|$. What can you say about the sequence $\{M_n\}_{n=1}^{\infty}$? For n sufficiently large, find b_n with $|w_n| = |z_n|$ such that $|f(w_n)| = M_1$.*

2.11. Let $f, g \in H(\mathbb{D})$ be nowhere vanishing. Assume that $\dfrac{f'}{f}\left(\dfrac{1}{n}\right) = \dfrac{g'}{g}\left(\dfrac{1}{n}\right)$, for all $n \in \mathbb{N} \setminus \{1\}$.

2.12. Let $P(z)$ and $Q(z)$ be nonconstant complex polynomials of the same degree. Assume that there exists $r > 0$ such that $|P(z)| = |Q(z)|$, whenever $|z| = r$, and all zeros of $P(z)$ and $Q(z)$ lie in $D(0; r)$. Show that there exists $\lambda \in S^1$ such that $P(z) = \lambda Q(z)$, for all $z \in \mathbb{D}$.

3. Open mapping theorem

3.1. Prove that there cannot exist bijective holomorphic map from $\mathbb{D}$ to $A(1,2) \overset{\text{def}}{=} \{z \in \mathbb{C} : 1 < |z| < 2\}$.

3.2. Let $U \subseteq \mathbb{C}$ be a region and $f \in H(U)$ be nonconstant. Deduce from Open mapping theorem that neither $|f|$ nor $\operatorname{Re} f$ nor $\operatorname{Im} f$ can have a local maxima.

3.3. Let $U, V \subseteq \mathbb{C}$ be open and connected and $f \in H(U)$ be such that $f(U) \subseteq V$. If the inverse image of every compact subset of V under f is compact, then show that $f(U) = V$. Does the above statement remain true if holomorphic is replaced by continuous in the hypothesis?