Name: ____________________

Roll Number: ______________

Practice Midterm Solutions

MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2022/23, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.
2. This exam contains **4 + 1** questions and is worth **40%** of your grade.
3. Answer **ALL** questions.

Question 1. [5 × 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) For every infinite limit ordinal $\alpha < \omega_1$, there is an ordinal β such that $\alpha = \beta + \omega$.

(ii) The set of irrationals numbers has the same cardinality as the set of real numbers.

(iii) For every function $f : \mathbb{R} \to \mathbb{R}$, there are injective functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = g - h$.

(iv) There exists a sequence $\langle X_n : n < \omega \rangle$ such that $|X_{n+1}| < |X_n|$ for every $n < \omega$.

(v) If ϕ and $(\phi \implies (\phi \implies \psi))$ are tautologies, then ψ is also a tautology.

Solution

(i) **False**. Take $\alpha = \omega \cdot \omega$ and note that for every $x \in \alpha$, there exists $y \in \alpha$ such that $x < y$ and the interval $[x, y]$ is infinite. It is clear that this fails for $\beta + \omega$ (Take $x = \beta$, then for every $y \in \beta + \omega$, $[x, y]$ is finite). Hence $\alpha \neq \beta + \omega$ for any ordinal β.

(ii) **True**.

(iii) **True**. By HW problem 19, there are injective functions g, h such that $f = g + h = g - (-h)$. Since h is injective, $-h$ is also injective.

(iv) **False**. Otherwise, $\{|X_n| : n < \omega\}$ is a nonempty set of ordinals with no least member (See Theorem (e) on slide 32).

(v) **True**.

Question 2. [10 Points]

(a) [**4 Points**] Let $\mathcal{F}$ be the set of all functions from ω to ω. Show that $|\mathcal{F}| = \mathfrak{c}$.

(b) [**4 Points**] Let $\mathcal{B}$ be the set of all bijections from ω to ω. Show that $|\mathcal{B}| = \mathfrak{c}$.

(c) [**2 Points**] State the continuum hypothesis.

Solution

(a) Note that $\mathcal{F} = \omega^\omega$. Since $2^\omega \subseteq \omega^\omega$, we get $\mathfrak{c} = |2^\omega| \leq |\omega^\omega|$.

Since every function $f : \omega \to \omega$ is a subset of $\omega \times \omega$, we get $\omega^\omega \subseteq \mathcal{P}(\omega \times \omega)$ which implies that $|\omega^\omega| \leq |\mathcal{P}(\omega \times \omega)|$. Since $|\omega \times \omega| = \omega$, we get $|\mathcal{P}(\omega \times \omega)| = |\mathcal{P}(\omega)| = \mathfrak{c}$. Hence $|\omega^\omega| \leq \mathfrak{c}$.

It follows that $|\mathcal{F}| = \mathfrak{c}$.

(b) Since $\mathcal{B} \subseteq \mathcal{F}$, we get $|\mathcal{B}| \leq |\mathcal{F}| = \mathfrak{c}$. So it suffices to show that $|\mathcal{B}| \geq \mathfrak{c}$. For this, we will construct an injection $H : \mathcal{P}(E) \to \mathcal{B}$ where $E = \{2n : n < \omega\}$ is the set of all even natural numbers.

Observe that for every infinite $X \subseteq \omega$, there is a bijection $f : \omega \to \omega$ such that for every $n < \omega$,

$$f(n) = n \iff n \in (\omega \setminus X)$$

To see why this is true, let $X = \{n_0, n_1, n_2, \cdots\}$ list X in increasing order and define

$$f(n) = \begin{cases} n & \text{if } n \in \omega \setminus X \\ n_{2k+1} & \text{if } n = n_{2k} \\ n_{2k} & \text{if } n = n_{2k+1} \end{cases} \tag{1}$$

For each $Y \subseteq E$, define $H(Y) = f$ where $f : \omega \to \omega$ is a bijection satisfying

$$(\forall n < \omega)(f(n) = n \iff n \in Y)$$

It is easy to see that H is an injection from $\mathcal{P}(E)$ to $\mathcal{B}$.

(c) $\mathfrak{c} = \omega_1$.

Question 3. [10 Points]

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$,

$$f(x+y) = f(x) + f(y) + f(x)f(y) \tag{2}$$

(a) [**2 Points**] Define $h = 1 + f$. Show that $h(x+y) = h(x)h(y)$.

(b) [**5 Points**] Suppose f is continuous and not identically equal to -1. Show that $f(x) = a^x - 1$ for some $a > 0$.

(c) [**3 Points**] Show that there is a discontinuous f satisfying Equation (2).

Solution

(a) $h(x+y) = 1 + f(x+y) = 1 + f(x) + f(y) + f(x)f(y) = (1 + f(x))(1 + f(y)) = h(x)h(y)$.

(b) By HW problem 17, $h(x) = a^x$ for some $a > 0$. So $f(x) = a^x - 1$.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function. Define $f(x) = 2^{g(x)} - 1$.

Question 4. [10 Points]

(a) [**2 Points**] State Zorn's lemma.

(b) [**4 Points**] Use Zorn's lemma to show the following. For any two sets X and Y, **either** there exists an injective function $f : X \to Y$ **or** there exists an injective function $g : Y \to X$.

(c) [**4 Points**] Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying (i)-(iii) below.

(i) For every $x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$.

(ii) For every $x \in \mathbb{R}$, $f(x + 1) = f(x)$.

(iii) f is not identically zero.

Solution

(a) Suppose $(P, \preceq)$ is a partial ordering such that every chain in P has an upper bound in P. Then P has a maximal element.

(b) See Lecture notes Slide 59.

(c) Let H be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ with $1 \in H$. Define $h : H \to \mathbb{R}$ by $h(1) = 0$ and $h(x) = 1$ for every $x \in H \setminus \{1\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the unique additive extension of h.

Bonus Question [5 Points]

Show that there is a subset of plane that meets every line at exactly 10 points.

Solution

Let $\mathcal{L}$ be the family of all lines in plane. Note that $|\mathcal{L}| = |\mathbb{R}^2 \times \mathbb{R}^2| = |\mathbb{R}^2| = \mathfrak{c}$. Let $\langle \ell_\alpha : \alpha < \mathfrak{c} \rangle$ be an injective sequence with range $\mathcal{L}$. Using transfinite recursion, construct a sequence $\langle S_\alpha : \alpha < \mathfrak{c} \rangle$ of subsets of $\mathbb{R}^2$ such that the following hold.

1. $S_0 = 0$ and if γ is limit, then $S_\gamma = \bigcup_{\alpha<\gamma} S_\alpha$.
2. $|S_\alpha| \leq |\alpha + \omega| < \mathfrak{c}$.
3. No 11 points in S_α are collinear.
4. $\beta < \alpha \implies |S_\alpha \cap \ell_\beta| = 10$.

Having constructed S_α, $S_{\alpha+1}$ is obtained as follows. Let $\mathcal{T}$ be the set of all lines that pass through at least 2 points in S_α. Let B be the set of points of intersection of ℓ_α with the lines in $\mathcal{T}$. Note that $|B| \leq |\alpha + \omega| < \mathfrak{c}$. By clause 3, $|S_\alpha \cap \ell_\alpha| \leq 10$ so we can add $10 - |S_\alpha \cap \ell_\alpha|$ points from $\ell_\alpha \setminus B$ to S_α to get $S_{\alpha+1}$. Having completed the construction, put $S = \bigcup_{\alpha<\mathfrak{c}} S_\alpha$. Then S is as required. $\square$

End of Exam