Department of Mathematics

Calculus of Several Variables and Differential Geometry

ASSIGNMENT-I

1. Let $n, k \geq 1$ and $T : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that T is continuous.

2. Let $n \geq 2$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ is differentiable and find the derivative df_x.

3. Use the series expansion of e^x and $e^{x+y} = e^x e^y$ for $x, y \in \mathbb{R}$ to show that the function $f(x) = e^x$ is differentiable. What is the derivative df_x of f?

4. Show that the following maps are differentiable and find the derivative in each case. Let $m, n \geq 2$ and $k \geq 1$.

   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) := (e^x \cos y, e^x \sin y)$.

   (b) Let $f : M(n, \mathbb{R}) \to M(n, \mathbb{R})$ be the map defined by $f(A) = A^k$.

   (c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be the map defined by $f(x) := \langle x, x \rangle$.

   (d) Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x, y) := \langle x, y \rangle$.

   (e) Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function defined by $f(x_1, x_2, \ldots, x_n) := \prod_{i=1}^n x_i$.

   (f) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) := \langle Ax, x \rangle$.

   (g) Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be linear and $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x, y) := \langle Ax, y \rangle$.

   (h) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two differentiable functions and $h : \mathbb{R}^n \to \mathbb{R}$ be defined by $h(x) = f(x)g(x)$.

   (i) Let f and g be as above and $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $h(x, y) = f(x)g(y)$.

5. Let $n \geq 1$ and U be an open subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a differentiable function. Show that f is continuous on U.

6. Let U be as above and $f : U \to \mathbb{R}$ be a differentiable function such that for all x in U, $f(x) \neq 0$. Show that the function $\frac{1}{f} : U \to \mathbb{R}$ defined by $\frac{1}{f(x)}$ is differentiable on U and find its derivative.

7. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function such that, for all $x \in \mathbb{R}$, we have $|f(x)| \leq x^2$. Is the function f differentiable at 0? If it is find the derivative of f at 0.

8. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function such that, for all $x \in \mathbb{R}$, we have $|f(x) - f(y)| \leq |x - y|^2$. Find the pounts where the function f differentiable. If it is find the derivative of f.

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that for all x, y in $\mathbb{R}^n$, $|f(x) - f(y)| \leq \|x - y\|^2$. Is the function f differentiable? If it is, what is its derivative?

10. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a function such that for all x, y in $\mathbb{R}^n$, $\|f(x) - f(y)\| \leq \|x - y\|^2$. Is the function f differentiable? If it is, what is its derivative?

11. Check the continuity, existence of partial derivatives, directional derivatives and the differentiability of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$.

(a) $f(x,y) := \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}.$

(b) $f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}.$

(c) $f(x,y) := \begin{cases} \frac{x^2y}{x^4+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}.$

12. Find the directional derivative of the functions

    (a) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := xy$ at $x = (0,1)$ and $(\cos\alpha, \sin\alpha)$.

    (b) $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x,y,z) := xyz$ at $x = (0,1,0)$ and $v = (\cos\alpha, \sin\alpha\cos\beta, \sin\alpha\sin\beta)$.

13. Find the Jacobian matrix of the differentiable functions

    (a) $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(u,v) := (u^2+v^2, uv, u^2-v^2)$.

    (b) $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(u,v) := (u+v, u-v, u^2-v^2)$.

    (c) $g : \mathbb{R}^3 \to \mathbb{R}$ defined by $g(x,y,z) := x^2+y^2+z^2$.

    (d) $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \langle Ax, x\rangle$ where $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

14. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(u,v) := (u+v, u-v, u^2-v^2)$ and $g : \mathbb{R}^3 \to \mathbb{R}$ defined by $g(x,y,z) := x^2+y^2+z^2$. Show that $g \circ f$ is differentiable, find its derivative and the Jacobian matrix.

15. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sin(xy)$ is differentiable and find its derivative.