Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)

Exercise Sheet 4

1. Integration of complex valued functions

1.1. For each of the following, formulate and prove the analogous statement for complex valued functions:

(a) Linearity property.
(b) First and second fundamental theorems of Calculus.
(c) Integration by parts.
(d) Substitution principle.
(e)$*$ *Triangle inequality*: $\forall$ Riemann integrable $f : [a,b] \longrightarrow \mathbb{C}$, $\left|\int_a^b f\right| \le \int_a^b |f|$.

(**Hint:** We may assume $\int_a^b f \neq 0$. Write $\int_a^b f = re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi)$. Now observe that $|\int_a^b f| = |e^{-i\theta} \int_a^b f| = |\int_a^b e^{-i\theta} f| = |\int_a^b \mathrm{Re}\,(e^{-i\theta} f)| \le \int_a^b |\mathrm{Re}\,(e^{-i\theta} f)|$.)

1.2. Let $\{f_n\}_{n=1}^{\infty}$ be sequence of complex valued functions defined over a closed and bounded interval $[a,b]$. Assume that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f : [a,b] \longrightarrow \mathbb{C}$ on $[a,b]$. Show that f is Riemann integrable and furthermore

$$\int_a^b f_n \xrightarrow[n\to\infty]{} \int_a^b f.$$

1.3.$*$ Let $z_0 \in \mathbb{C}$ and $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with radius of convergence $R \in (0, \infty]$. Define

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \forall z \in B(z_0; R).$$

Pick any $r \in (0, R)$. Suppose that $|f(z)| \le M$, whenever $|z - z_0| = r$.

(a) Show that, for any $n \ge 0$, $\frac{1}{2\pi} \int_0^{2\pi} \left| s_n\left(z_0 + re^{it}\right)\right|^2 dt = \sum_{k=0}^{n} |a_k|^2 r^{2k}$, where s_n denotes the n-th partial sum of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

(b) Deduce from 1.3.a that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(z_0 + re^{it}\right)\right|^2 dt = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

(**Hint:** Recall that, if $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are two uniformly bounded sequences of complex valued functions defined over a set X and converge uniformly (on X) to f and g respectively, then $\{f_n g_n\}_{n=1}^{\infty}$ converges to fg uniformly on X.)

(c) Using 1.3.b show that $\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \le M^2$.

(d) Conclude from 1.3.c that if $R = \infty$ and f is bounded then f must be constant.

(e) Set $M(r) \overset{\text{def}}{=} \sup_{|z - z_0| = r} |f(z)|$. Conclude from 1.3.b that, for all $n \ge 0$, $|f^{(n)}(z_0)| \le n! \frac{M(r)}{r^n}$.

(f) Conclude from 1.3.b that, if $|f(z_0)| = M(r)$, for some $0 < r < R$, then f is constant.

2. Integration over paths

In what follows, for $z_0 \in \mathbb{C}$ and $r > 0$, by *positively oriented circle centered at z_0 with radius r* we mean the following closed path:

$$t \mapsto z_0 + re^{it}, \ \forall t \in [0, 2\pi].$$

We usually denote this by $C(z_0; r)$. When $z_0 = 0$ and $r = 1$, we simply refer to this as the *positively oriented unit circle*.

2.1. Evaluate the following integrals along the specified paths:

(a) z^n along the path $\gamma(t) = e^{ikt}$, $\forall t \in [0, 2\pi]$, where $n, k \in \mathbb{Z}$.
(b) e^z along positively oriented unit circle.
(c) $\log_\pi z$ along the semicircle joining $-i$ to i on the right half place.

2.2. Let $\alpha \in [0, \pi]$ and $n \in \mathbb{N}$. Consider the n-th root function $f(z) = z^{\frac{1}{n}} \overset{\text{def}}{=} e^{\frac{1}{n}\log_\alpha z}$, for all $z \in \mathbb{C} \setminus \overline{R_\alpha}$. Can it be integrated along the path $\gamma(t) \overset{\text{def}}{=} e^{it}$, $\forall t \in [0, \pi]$? If so, then evaluate the integral.

2.3. (a) Evaluate the integral of the function z^{i-1} (using the principal branch) along the path $\gamma(t) \overset{\text{def}}{=} e^{it}$, $\forall t \in [-\pi, \pi]$.
(b) What if the path in 2.3.a would have been changed to the positively oriented unit circle? Do you see any change in the value of the integral with that of 2.3.a? Can you explain this?

2.4. Show that, for any $n \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_0^{2\pi} (2\cos\theta)^{2n}\, d\theta = \frac{(2n)!}{(n!)^2}.$$

(**Hint:** Consider the integral of the function $\frac{1}{z} \cdot (z + \frac{1}{z})^{2n}$ along the positively oriented unit circle.)

3. Applications of ML inequality

3.1. (a) Show that $\left|\int_\gamma \frac{e^z}{z}\, dz\right| \le 2\pi e$, where γ is the positively oriented unit circle.
(b) Let $z_0 \in \mathbb{C}$, $r > 0$ and $f : D(z_0; r) \longrightarrow \mathbb{C}$ be continuous. Find $\displaystyle\lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int_{C(z_0;\varepsilon)} \frac{f(z)}{z - z_0}\, dz$.
(c) Find $\displaystyle\lim_{R \to \infty} \int_{C(0;R)} \frac{z}{z^5 + 9}\, dz$.
(d) Find $\displaystyle\lim_{R \to \infty} \int_{[-R, -R+i]} \frac{z^3 e^z}{z + 3}\, dz$.

3.2. Let γ be a closed path in $\mathbb{C}$ and $\{f_n\}_{n=1}^\infty$ be a sequence of complex valued functions defined on γ^*.

(a) Assume that $\{f_n\}_{n=1}^\infty$ converges uniformly to $f : \gamma^* \longrightarrow \mathbb{C}$ on γ^*. Show that $\displaystyle\int_\gamma f_n \xrightarrow[n\to\infty]{} \int_\gamma f$.
(b) Assume that $\displaystyle\sum_{n=1}^\infty f_n$ is uniformly convergent. Show that $\displaystyle\sum_{n=1}^\infty \int_\gamma f_n = \int_\gamma \sum_{n=1}^\infty f_n$.

3.3. (a) Using series expansion method, show that

$$\frac{1}{2\pi i}\int_{C(z_0;r)} \frac{dw}{w - z} = \begin{cases} 0 & \text{if } z \notin \overline{D(z_0, r)} \\ 1 & \text{if } z \in D(z_0, r) \end{cases}.$$

(b) Let $P(z) \in \mathbb{C}[z]$ and $\alpha \in \mathbb{C} \setminus \gamma_r^*$. Evaluate $\displaystyle\frac{1}{2\pi i}\int_{C(z_0;r)} \frac{P(z)}{z - \alpha}\, dz$.
(c) Let $a \ne b \in \mathbb{C}$. Evaluate the following for every $r \ne |a|, |b|$: $\displaystyle\int_{C(0;r)} \frac{dz}{(z - a)(z - b)}$.

(d) Consider $\gamma_1 \overset{\text{def}}{=} [z_0 - r, z_0 + r]$ and $\gamma_2 : [0, \pi] \longrightarrow \mathbb{C}$, $\gamma_2(t) \overset{\text{def}}{=} z_0 + re^{it}$. Let $z \notin \gamma_1^* \cup \gamma_2^*$. Evaluate $\dfrac{1}{2\pi i} \displaystyle\int_{\gamma_1 * \gamma_2} \frac{dw}{w - z}\, dw$.

3.4. Let $U \subseteq \mathbb{C}$ be open and connected, and $f : U \longrightarrow \mathbb{C}$ be continuous. Show that the following are equivalent:

(a) Whenever two paths γ_1, γ_2 with $\gamma_1^*, \gamma_2^* \subseteq U$, have same initial and end points, $\int_{\gamma_1} f = \int_{\gamma_2} f$.

(b) $\int_\gamma f = 0$, for every closed path γ such that $\gamma^* \subseteq U$.

(c)∗ Show that, f has a primitive. (**Hint:** Fix $z_0 \in U$. For any $z \in U$, there exists a path γ joining $z_0$and z (why?). Define $F(z) = \int_\gamma f$. First check that this does not depend on the choice of γ. Show that, the function F thus obtained is a primitive of f.)

Can you drop the connectedness assumption from the hypothesis?

3.5. Let $U \subseteq \mathbb{C}$ be open and convex and $f : U \longrightarrow \mathbb{C}$. Assume that, there exists $z_0 \in U$ such that, $|f'(z) - f'(z_0)| < |f'(z_0)|$, for all $z \in U$. Show that f is injective. (**Hint:** Suppose that, $f(z_1) = f(z_2)$, for some $z_1 \neq z_2 \in U$. Then observe that $\displaystyle\int_{[z_1,z_2]} f' = 0$. What is $\displaystyle\int_{[z_1,z_2]} (f'(z) - f'(z_0))\, dz$?)

4. Continuity of zeros of polynomials

Fix $d \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $P_n(z) \overset{\text{def}}{=} a_{n,d}z^d + a_{n,d-1}z^{d-1} + \ldots a_{n,1}z + a_{n,0} \in \mathbb{C}[z]$ have degree d. Assume that $P(z) \overset{\text{def}}{=} a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$ is of degree d and P_n *converges to P coefficientwise* as $n \to \infty$, i.e., for every $j = 0, \ldots, d$, $a_{n,j} \xrightarrow[n\to\infty]{} a_j$.

4.1. Show that $P_n \xrightarrow[n\to\infty]{} P$ uniformly on every bounded subset of $\mathbb{C}$.

4.2.∗ Consider a zero $z_0 \in \mathbb{C}$ $P(z)$ with multiplicity m. Suppose that $R > 0$ is such that $P(z)$ does not vanish anywhere on the closed disc $\overline{D(z_0; R)}$ except the center z_0. Then show that there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $P_n(z)$ does not have a zero on the boundary of the open disc $D(z_0; R)$ but has precisely m zeros inside.

Let $\alpha_1, \ldots, \alpha_k$ be all distinct zeros of $P(z)$ with multiplicities $n_1, \ldots, n_k$ respectively. Thus

$$P(z) = a_d \prod_{i=1}^{k} (z - \alpha_i)^{n_i}.$$

Pick any $R > 0$ such that $D(\alpha_i; R) \cap D(\alpha_j; R) = \emptyset$, for all $i \neq j$. For any $i = 1, \ldots, k$ and $n \in \mathbb{N}$, we define

$$\Phi_{i,n,R}(z) = \prod (z - \beta),$$

where β runs over all zeros of $P_n(z)$, counting multiplicities, in $D(\alpha_i; R)$.

4.3. Show that, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\deg \Phi_{i,n,R}(z) = n_i$, for all $i = 1, \ldots, k$.

4.4. Let N be as above in 4.3. Show that, for all $n \geq N$, $P_n(z) = a_{n,d} \displaystyle\prod_{i=1}^{k} \Phi_{i,n,R}(z)$.

4.5. Let $0 < \varepsilon < R$. For any $i = 1, \ldots, k$ and $n \in \mathbb{N}$, we define $\Phi_{i,n,\varepsilon}(z)$ in the similar way as before. Show that, there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, one has $\Phi_{i,n,\varepsilon}(z) = \Phi_{i,n,R}(z)$, for all $i = 1, \ldots, k$.

4.6.∗∗Show that, for each $n \in \mathbb{N}$, one can arrange the zero of $P_n(z)$ in an order, say $\alpha_{n,1}, \ldots, \alpha_{n,d}$, such that $\alpha_{n,j} \xrightarrow[n\to\infty]{} \alpha_j$, for all $j = 1, \ldots, d$.

Use 4.6. to the following:

4.7.**Let $\{A_n\}_{n\geq 1}$ be a sequence of $d \times d$ complex matrices, where $d \in \mathbb{N}$, and $A \in M_d(\mathbb{C})$. Assume that $A_n \xrightarrow[n\to\infty]{} A$. Denote the eigenvalues of A by $\lambda_1, \ldots, \lambda_d$. Prove that for each $n \in \mathbb{N}$, there exists an ordering $\lambda_{n,1}, \ldots, \lambda_{n,d}$ of the eigenvalues of A_n such that $\lambda_{n,j} \xrightarrow[n\to\infty]{} \lambda_j$, for all $j = 1, \ldots, d$.

Note: The above statement 4.7. can also be proved without using Complex analysis. You may see Section 5.2 of Artin's Algebra book (2nd edition).

4.8.$*$ Let $d \geq 2$. Show that the following subset of $\mathrm{GL}_d(\mathbb{R})$ is not dense in $\mathrm{GL}_d(\mathbb{R})$:

$$\{A \in \mathrm{GL}_d(\mathbb{R}) : \exists P \in \mathrm{GL}_d(\mathbb{R}) \text{ s.t. } PAP^{-1} \text{ is doagonal}\}$$

(**Hint:** First try to prove for $d = 2$. Can $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the limit of a sequence of 2×2 matrices with real entries that are diagonalizable over $\mathbb{R}$?)

Note: The analogous statement holds over $\mathbb{C}$, as every matrix is similar to an upper traingular matrix.