Lesson 4

Numerical Solution of PDE

4.1 BVP for 2nd Order Elliptic PDE

- Finite Difference Method

- More Stability Analysis – Energy Estimate

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Stability Analysis using an energy estimate

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Let $v \in L(\Omega_{\mathrm{h}})$ *and define the backward difference operator*

$$\partial_{x_1} v(mh, nh) = \frac{v(mh, nh) - v((m-1)h, nh)}{h}, 1 \leq m \leq N, 0 \leq n \leq N.$$

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$$\leq \left(h\sum_{k=1}^{N}\left|\partial_{x_1} v(kh, nh)\right|\right)^2 \leq \left(h\sum_{k=1}^{N}\left|\partial_{x_1} v(kh, nh)\right|^2\right)\left(h\sum_{k=1}^{N} 1^2\right) = h\sum_{k=1}^{N}\left|\partial_{x_1} v(kh, nh)\right|^2.$$

Therefore,

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Lemma

If $v, w \in L(\Omega_{\mathrm{h}})$*, then*

$$-\langle \Delta_{\mathrm{h}} v, w \rangle_h = \left\langle \partial_{x_1} v, \partial_{x_1} w \right\rangle_h + \left\langle \partial_{x_2} v, \partial_{x_2} w \right\rangle_h.$$

Proof:

For $v_0, v_1, \dots, v_N, w_0, w_1, \dots, w_N \in \mathbb{R}$ *with* $w_0 = w_N = 0$*. Then,*

$$\sum_{k=1}^{N} (v_k - v_{k-1})(w_k - w_{k-1}) = \sum_{k=1}^{N} v_k w_k + \sum_{k=1}^{N} v_{k-1} w_{k-1} - \sum_{k=1}^{N} v_{k-1} w_k - \sum_{k=1}^{N} v_k w_{k-1}$$

$$= 2 \sum_{k=1}^{N-1} v_k w_k - \sum_{k=1}^{N-1} v_{k-1} w_k - \sum_{k=1}^{N-1} v_{k+1} w_k = - \sum_{k=1}^{N-1} (v_{k+1} - 2 v_k + v_{k-1}) w_k.$$

Hence,

$$-h \sum_{k=1}^{N-1} \frac{v\big((k+1)h, nh\big) - 2v(kh, nh) + v\big((k-1)h, nh\big)}{h^2} w(kh, nh) = h \sum_{k=1}^{N} \partial_{x_1} v(kh, nh)\, \partial_{x_1} w(kh, nh)$$

and thus

$$-\left\langle D_{h,x_1}^2 v, w \right\rangle_h = \left\langle \partial_{x_1} v, \partial_{x_1} w \right\rangle_h.$$

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and thus

$$-\left\langle D^2_{h,x_1} v, w\right\rangle_h = \left\langle \partial_{x_1} v, \partial_{x_1} w\right\rangle_h.$$

Similarly

$$-\left\langle D^2_{h,x_2} v, w\right\rangle_h = \left\langle \partial_{x_2} v, \partial_2 w\right\rangle_h.$$

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Combining the two results, we get the following stability result:

If $v \in L(\Omega_{\mathrm{h}})$, then

$$\|v\|_h^2 \leq \left\|\partial_{x_1} v\right\|_h^2 \leq \left\|\partial_{x_1} v\right\|_h^2 + \left\|\partial_{x_2} v\right\|_h^2 = -\langle \Delta_{\mathrm{h}} v, v \rangle_h \leq \|\Delta_{\mathrm{h}} v\|_h \|v\|_h,$$

where $\Delta_{\mathrm{h}} v$ is extended to the boundary grid Γ_{h} as zero while computing $\|\Delta_{\mathrm{h}} v\|_h$.

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If $v, w \in L(\Omega_\mathrm{h})$, *then*

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where $\Delta_\mathrm{h} v$ *is extended to the boundary grid* Γ_h *as zero while computing* $\|\Delta_\mathrm{h} v\|_h$. *Thus,* $\|v\|_h \le \|\Delta_h v\|_h$.

Lesson 4

Numerical Solution of PDE

4.1 BVP for 2nd Order Elliptic PDE

- Finite Difference Method

- General Domains

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Numerical Methods for PDE: 2nd Order Elliptic PDE

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A natural generalization to the two-point BVP

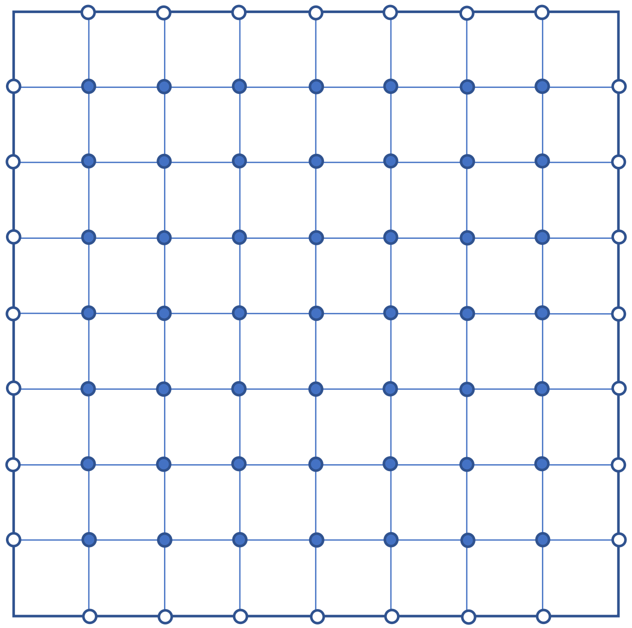
$$u'' = f(t), \qquad a < t < b,$$
$$u(a) = 0, \qquad u(b) = 0,$$

to two dimensions is

$$\Delta u \coloneqq u_{x_1x_1} + u_{x_2x_2} = f, \qquad \text{in } \Omega,$$
$$u = g, \qquad \text{on } \Gamma.$$

For simplicity, we will first consider a very simple domain $\Omega = (0,1) \times (0,1)$.

$N = 8$



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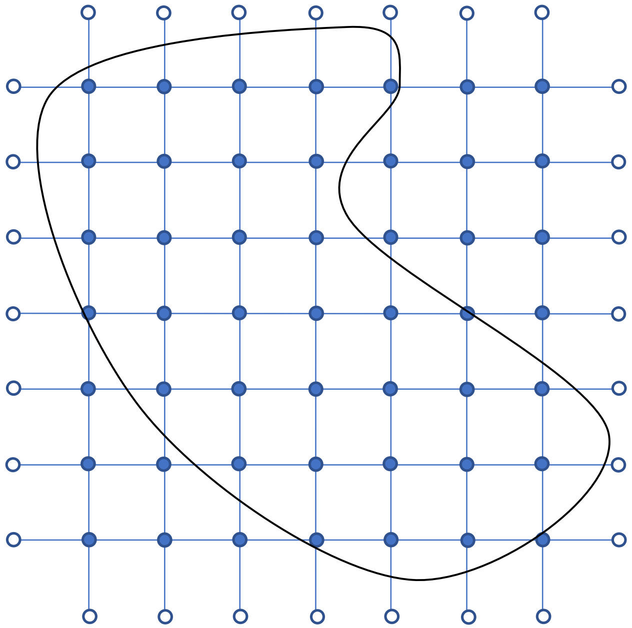

Numerical Methods for PDE: 2nd Order Elliptic PDE

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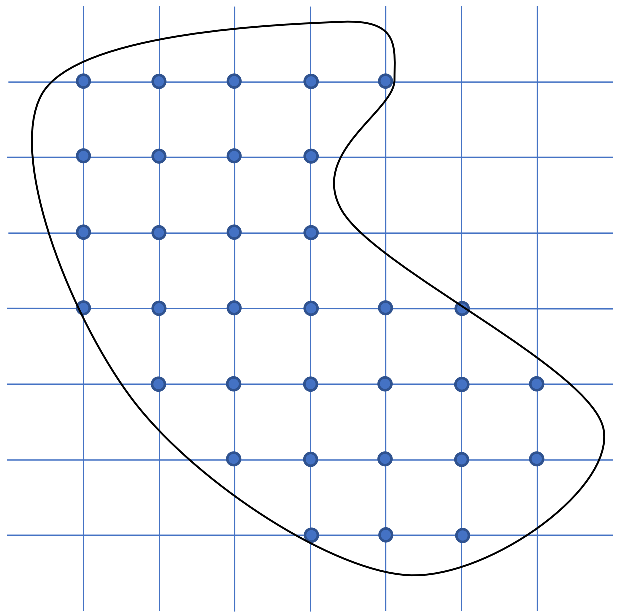

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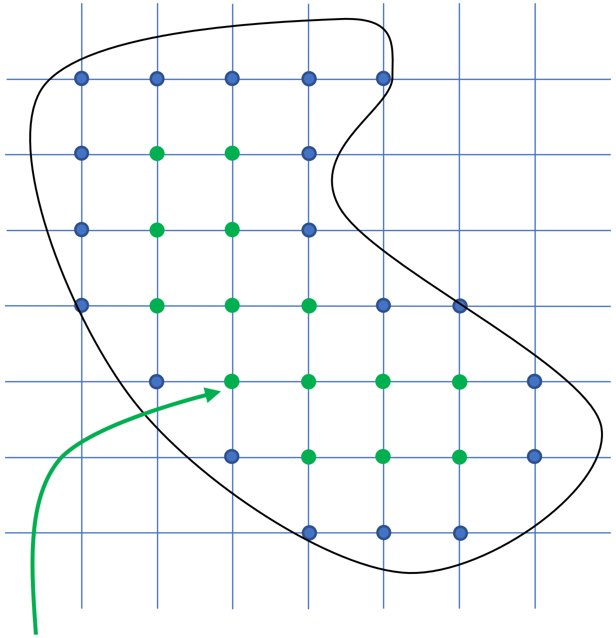

A mesh point with all four nearest neighbors in the interior

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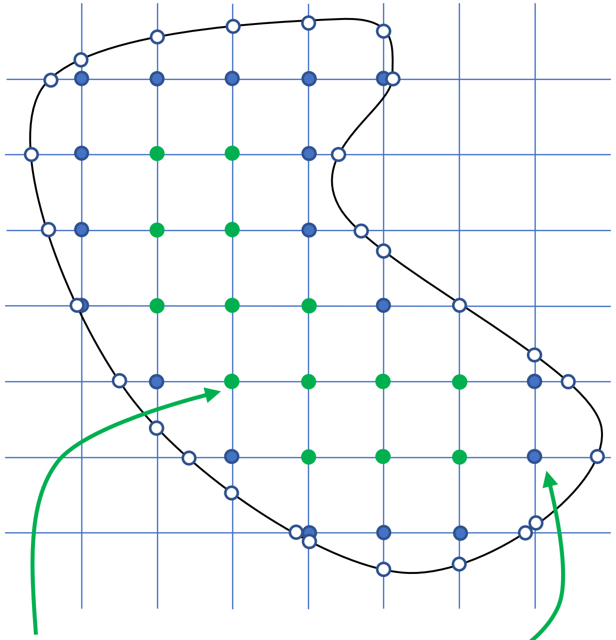

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A mesh point with at least one nearest neighbor on the boundary

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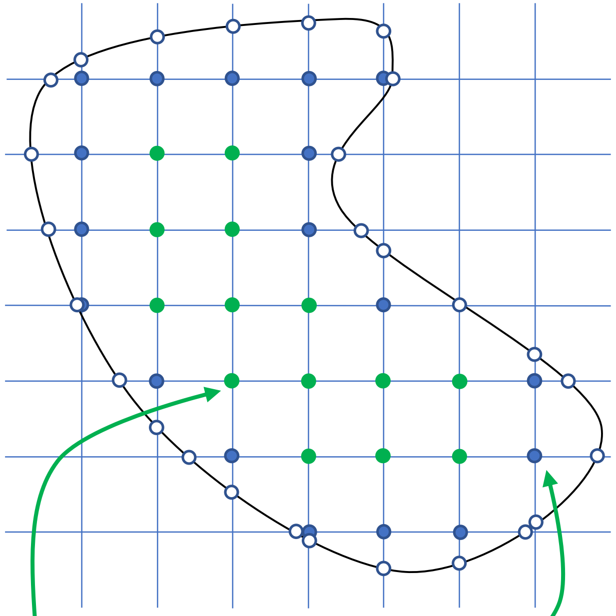

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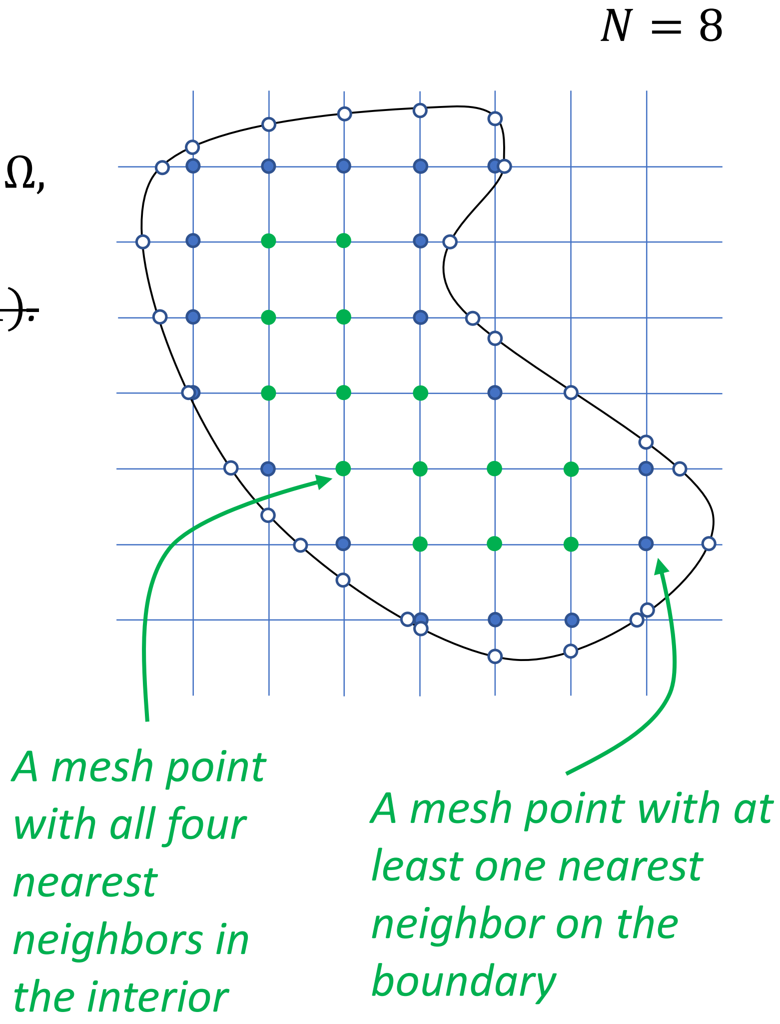
Also, $\Gamma_\mathrm{h} = \Gamma \cap \mathbb{R}_h^2$ *and* $\overline{\Omega}_h = \Omega \cup \Gamma_h$.

On Ω_h°, $\Delta_\mathrm{h} v$ *is defined as the usual 5-point Laplacian.*



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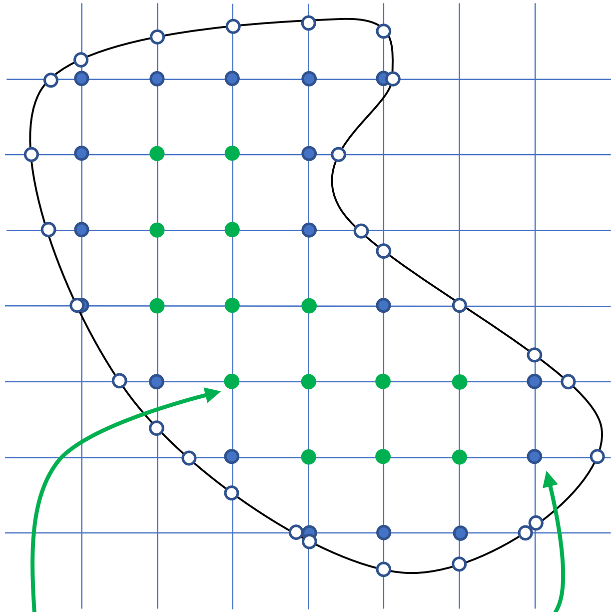
For $(x_1, x_2) \in \Omega_\mathrm{h} \setminus \Omega_h^\circ$, let $(x_1 + h_1, x_2)$, $(x_1, x_2 + h_2)$, $(x_1 - h_1, x_2)$, and $(x_1, x_2 - h_4)$ be the nearest neighbors (with $0 \le h_k \le h$), and let v_1, v_2, v_3 and v_4 denote the values of v at these four points. And finally, let $v_0 = v(x_1, x_2)$.

$N = 8$



A mesh point with all four nearest neighbors in the interior

A mesh point with at least one nearest neighbor on the boundary

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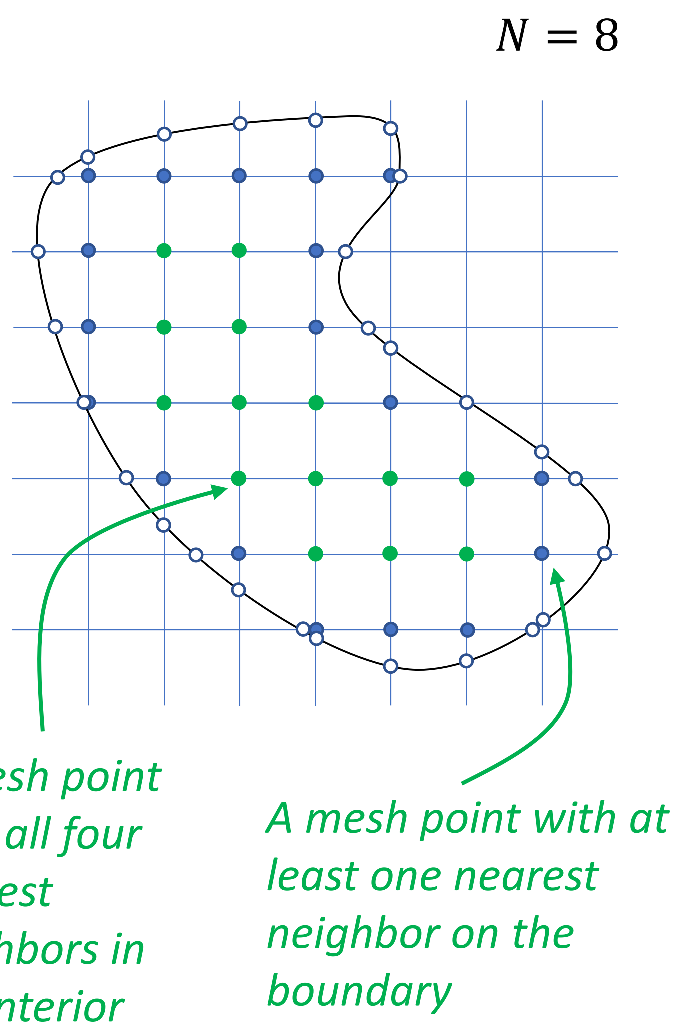

Numerical Methods for PDE: 2nd Order Elliptic PDE

Akash Anand
MATH, IIT KANPUR

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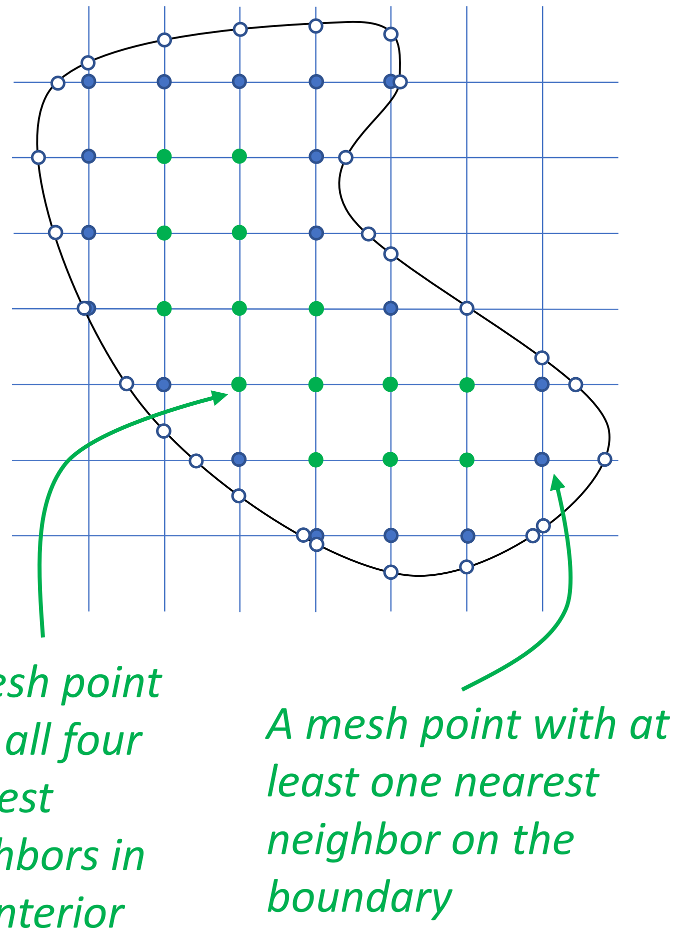

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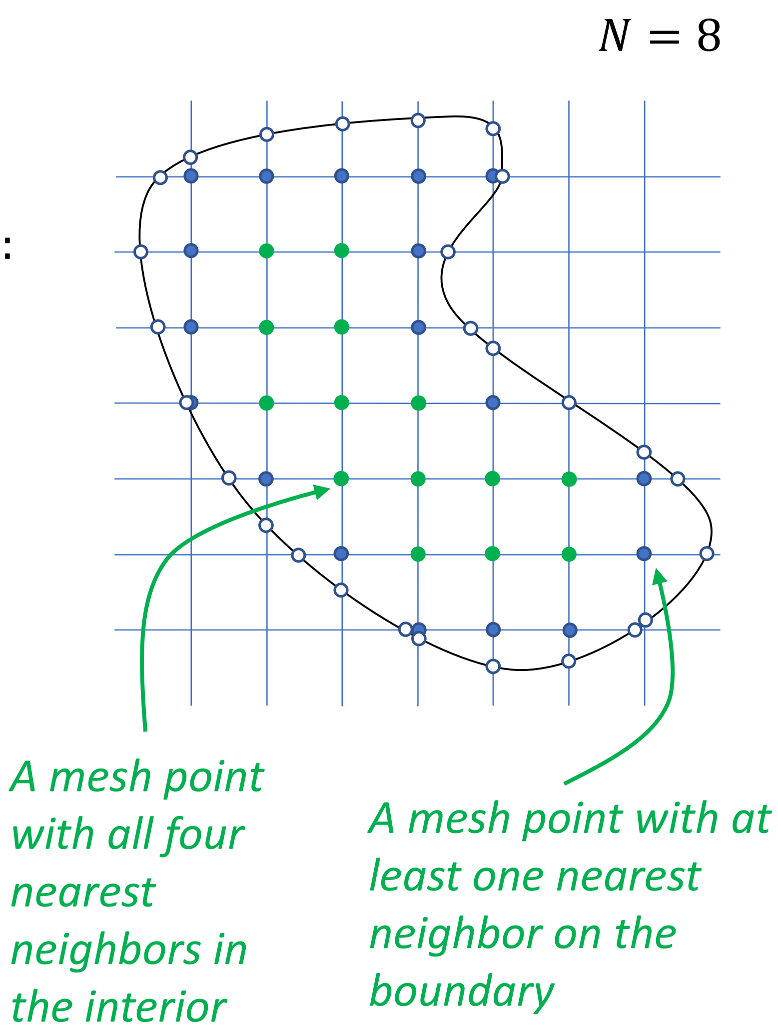

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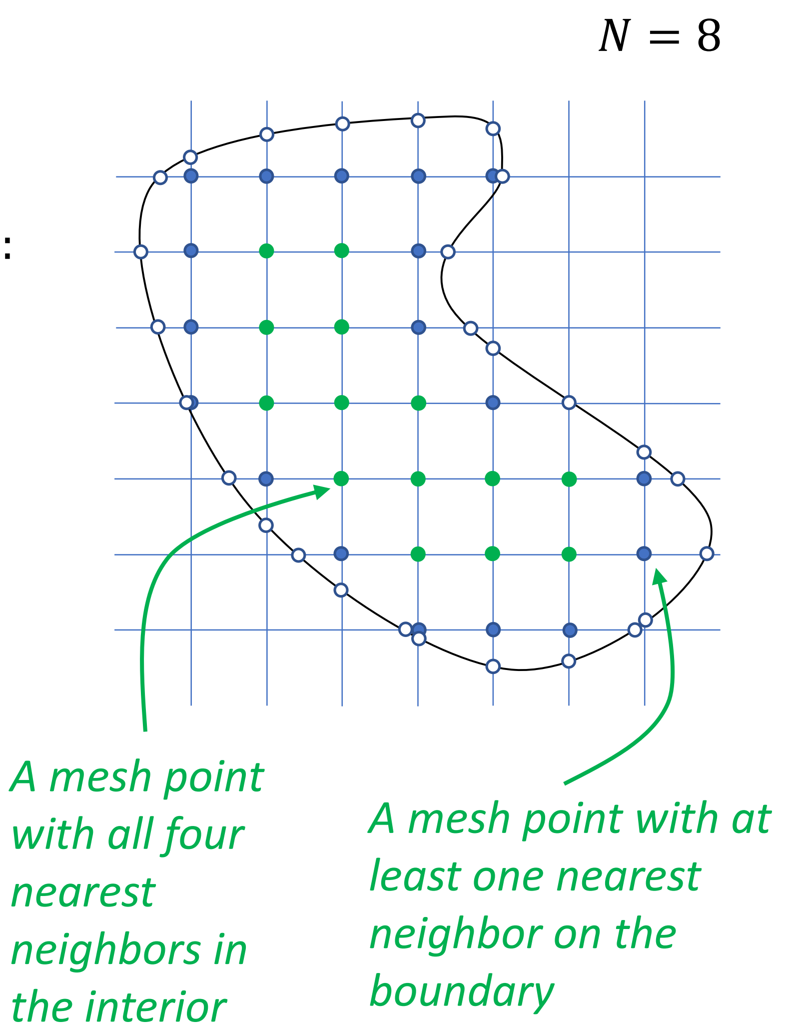

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This calculation leads us to the Shortley-Weller formula for $\Delta_\mathrm{h} v$:

$$\Delta_\mathrm{h} v(x_1, x_2) = \frac{2}{h_1(h_1+h_3)} v_1 + \frac{2}{h_2(h_2+h_4)} v_2 + \frac{2}{h_3(h_1+h_3)} v_3 + \frac{2}{h_4(h_2+h_4)} v_4 - \left(\frac{2}{h_1 h_3} + \frac{2}{h_2 h_4}\right) v_0$$

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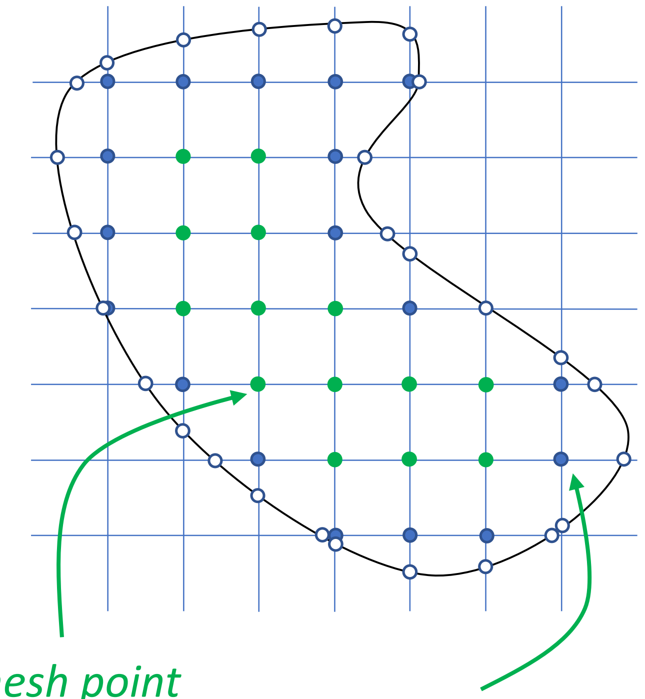

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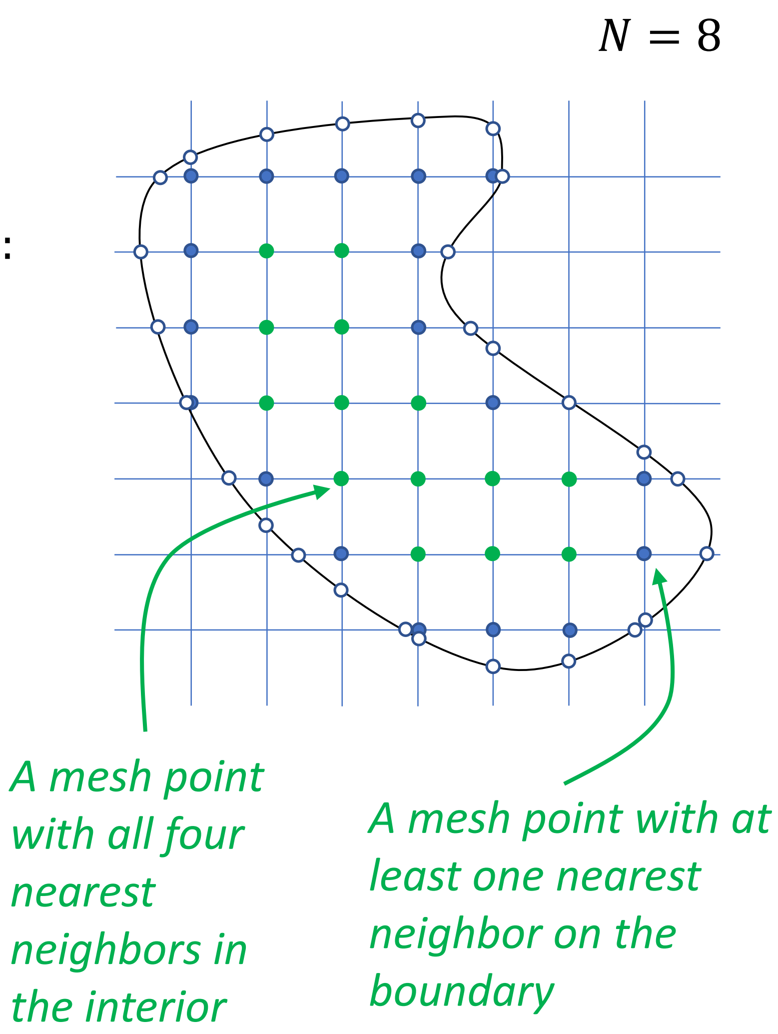
Using Taylor's theorem with remainder, we can easily see that for $v \in C^3(\overline{\Omega})$,

$$\|\Delta v - \Delta_h v\|_{\infty, \Omega_h} = \frac{2M_3}{3} h,$$

where M_3 is the maximum of the L^∞ norms of the third derivative of v.



Numerical Methods for PDE: 2nd Order Elliptic PDE

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We can obtain the discrete maximum/minimum principle with virtually the same proof as for the square domain and then a stability result follows as before (exercise). In this way, we can obtain an $O(h)$ convergence result.

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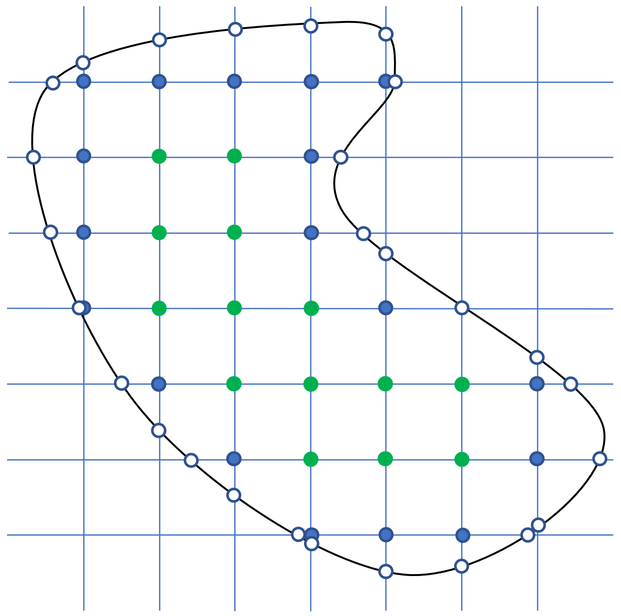

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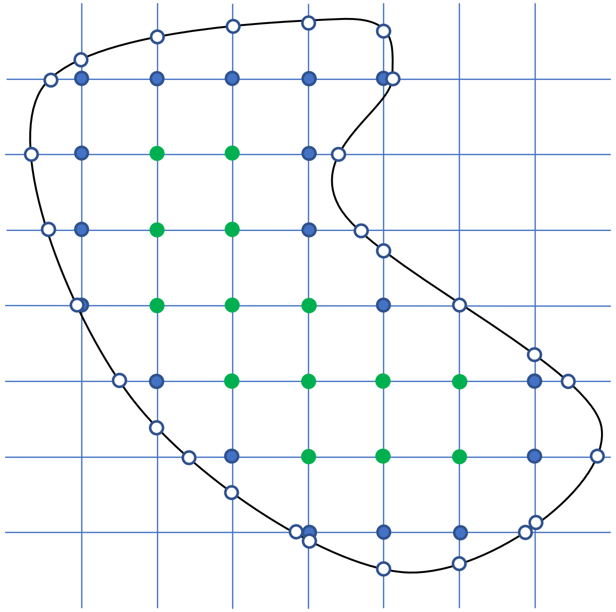

In fact, this result can be improved. Although the truncation error is only $O(h)$, it is $O(h^2)$ all points except those neighboring the boundary and these account for only $O(h^{-1})$ of the total $O(h^{-2})$ points in Ω_h. Moreover, these points are within h of the boundary where the solution is known exactly. For both these reasons, the contribution to the error from these points is smaller than what we observed earlier.

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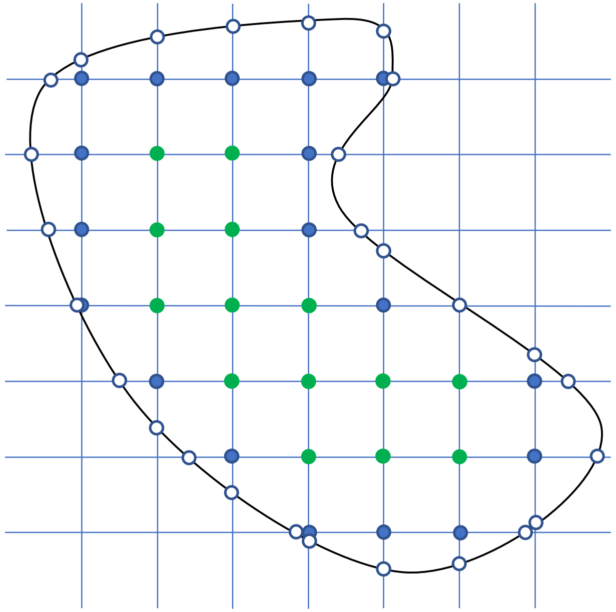

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Theorem

Let u be the solution to

$$\begin{aligned} \Delta u &= f, \qquad \text{in } \Omega, \\ u &= g, \qquad \text{on } \Gamma, \end{aligned}$$

and u_h be the solution to the corresponding discrete problem

$$\begin{aligned} \Delta_h u_h &= f, \qquad \text{on } \Omega_{\mathrm{h}}, \\ u_h &= g, \qquad \text{on } \Gamma_{\mathrm{h}}. \end{aligned}$$

Then,

$$\|u_h - u\|_{\infty,\overline{\Omega}_h} \le \frac{M_4 d^2}{96} h^2 + \frac{2M_3}{3} h^3,$$

where d is the diameter of the smallest disc containing Ω and M_k is the maximum of the L^∞ norms of the kth derivative of v.