1·2: Recall that, if A is a closed subset of a metric sp. X & $x \in X$, $d(x,A) := \inf\{d(x,a) : a \in A\}$.
One has $d(x,A) = 0 \Leftrightarrow x \in A$.

Furthermore, $\forall \underset{\text{compact}}{K \subseteq X}$, $d(K,A) := \inf\left\{d(k,a) : \begin{matrix} k \in K \\ a \in A \end{matrix}\right\}$

It is also easy to see that $d(K,A) = 0 \Leftrightarrow K \cap A \neq \emptyset$

(a) In view of the above, we note that,

$$K_n := \overline{D(0;n)} \cap \left\{z \in U : d(z, \mathbb{C}\setminus U) \geq \tfrac{1}{n}\right\}$$

As, for any closed $A \subseteq X$, $x \mapsto d(x,A)$ is cts.,
we see that $\left\{z \in U : d(z, \mathbb{C}\setminus U) \geq \frac{1}{n}\right\}$ is a closed subset of $\mathbb{C}$. Hence K_n is closed & bdd, so compact.

(b) $$K_n = \overline{D(0;n)} \cap \left\{z \in U : d(z, \mathbb{C}\setminus U) \geq \tfrac{1}{n}\right\}$$
$$\subseteq D(0;n+1) \cap \left\{z \in U : d(z, \mathbb{C}\setminus U) > \tfrac{1}{n+1}\right\}$$
$$\subseteq \overline{D(0;n+1)} \cap \left\{z \in U : d(z, \mathbb{C}\setminus U) \geq \tfrac{1}{n+1}\right\} = K_{n+1}$$

As $D(0;n+1) \cap \left\{z \in U : d(z, \mathbb{C}\setminus U) > \frac{1}{n+1}\right\}$ is open so it is contained in the interior of K_{n+1}.

(c) $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $K \subseteq \overline{D(0;n)}$ $\left(\begin{matrix}\text{as } K \text{ is} \\ \text{bdd}\end{matrix}\right)$

Since $K \cap \mathbb{C}\setminus U = \emptyset$ so, $d(K, \mathbb{C}\setminus U) > 0$.

$\Rightarrow \exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$, $d(K, \mathbb{C}\setminus U) \geq \frac{1}{n}$.

Take $N := \max\{N_1, N_2\}$. Then clearly

$$K \subseteq \overline{D(0;N)} \cap \left\{z \in U : d(z, \mathbb{C}\setminus U) \geq \tfrac{1}{N}\right\} = K_N.$$

1.3. (a) Suppose that $\alpha \in D(0;1)$ is a zero of $P(z)$.

Then $(1-\alpha)P(\alpha) = 0 \Rightarrow \left(a_n\alpha^n + \cdots + a_1\alpha + a_0\right)$

$$- \left(a_n\alpha^{n+1} + \cdots + a_1\alpha^2 + a_0\alpha\right) = 0$$

$$\Rightarrow -a_n\alpha^{n+1} + (a_n - a_{n-1})\alpha^n + \cdots$$

$$+ (a_1 - a_0)\alpha + a_0 = 0$$

$$\Rightarrow a_0 = a_n\alpha^{n+1} + (a_{n-1} - a_n)\alpha^n + \cdots + (a_0 - a_1)\alpha$$

$$\Rightarrow a_0 = \left|a_n\alpha^{n+1} + (a_{n-1} - a_n)\alpha^n + \cdots + (a_0 - a_1)\alpha\right|$$

$$\underset{\text{TI}}{<} \cancel{a_n} + (a_{n-1} - \cancel{a_n}) + \cdots + (a_0 - \cancel{a_1})$$ [Here we use triangle inequality & $|\alpha| < 1$]

$$= a_0,$$

which is a contradiction.

(b) Assume contrary, i.e; $\exists\, \alpha \in \overline{D(0;1)}$ s.t. $P(\alpha) = 0$

From part (a), it is clear that $|\alpha| = 1$.

Proceeding as before,

$$a_0 = \left|a_n\alpha^{n+1} + (a_{n-1} - a_n)\alpha^n + \cdots + (a_0 - a_1)\alpha\right|$$

$$\leq a_n + (a_{n-1} - a_n) + \cdots + (a_0 - a_1) = a_1$$

$\Rightarrow$ Equality must occur in triangle inequality

$\Rightarrow$ all $a_n\alpha^{n+1}, (a_{n-1} - a_n)\alpha^n, \ldots, (a_0 - a_1)\alpha$ must be nonnegative multiple of each other.

$\Rightarrow \alpha \in \mathbb{R} \Rightarrow \alpha = 1$ or -1 as $|\alpha| = 1$.

It is easy to see that $\alpha = -1$ is not possible, so $\alpha = 1$. But $P(1) = a_n + \cdots + a_0 > 0$, which is a contradiction.

2.3. $\displaystyle\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k (B_k - B_{k-1})$ $[B_0 := 0]$

$$= \sum_{k=1}^{n} a_k B_k - \sum_{k=1}^{n} a_k B_{k-1}$$

$$= a_n B_n + \sum_{k=1}^{n-1} a_k B_k - \sum_{k=1}^{n} a_k B_{k-1}$$

$$= a_n B_n + \sum_{k=1}^{n-1} a_k B_k - \sum_{k=1}^{n-1} a_{k+1} B_k$$

$$= a_n B_n + \sum_{k=1}^{n-1} (a_k - a_{k+1}) B_k .$$

2.4. (b) We use 2.3 to show that $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ ~~does not converge anywhere on the boundary~~ ~~$|z|=1$, except at $z=$~~ converges everywhere on the boundary $|z|=1$, except at $z=1$. Observe that $\forall z \neq 1$, one has

$$\sum_{k=1}^{n} \frac{z^k}{k} = \frac{1}{n} \cdot \frac{1-z^{n+1}}{1-z} + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \frac{1-z^{k+1}}{1-z}$$

$$= \frac{1}{n} \cdot \frac{1-z^{n+1}}{1-z} + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \cdot \frac{1-z^{k+1}}{1-z} \quad \cdots (*)$$

If $|z|=1$ & $z \neq 1$, we have

$$\left| \frac{1-z^{k+1}}{1-z} \right| \leq \frac{1+|z|^{k+1}}{|1-z|} \leq \frac{2}{|1-z|}, \quad \forall k \in \mathbb{N}.$$

This shows that $\displaystyle\frac{1}{n} \frac{1-z^{n+1}}{1-z} \xrightarrow[n\to\infty]{} 0$ & also

$$\sum_{k=1}^{\cancel{n-1}\ \infty} \frac{1}{k(k+1)} \left| \frac{1-z^{k+1}}{1-z} \right| \leq \frac{2}{|1-z|} \sum_{k=1}^{\cancel{n-1}\ \infty} \frac{1}{k(k+1)} < \infty, \text{ so}$$

$\displaystyle\sum_{k=1}^{\cancel{n-1}\ \infty} \frac{1}{k(k+1)} \frac{1-z^{k+1}}{1-z}$ converges

2.7. (a) Use the formula mentioned in 2.6.(a).

(b) Since $\sum_{n=0}^{\infty} a_n r^n$ is cgt. so $a_n r^n \xrightarrow[n\to\infty]{} 0$.

$\Rightarrow \exists M \geq 0$ s.t. $\forall n \geq 1$, $a_n r^n \leq M$

Now, $\forall z \in D$, we have $\sum_{n=0}^{\infty} \frac{|a_n|}{n!} |z|^n$ is cgt.

and furthermore

$$|h(z)| \leq \sum_{n=0}^{\infty} \frac{|a_n|}{n!} |z|^n \leq \sum_{n=0}^{\infty} M \cdot \left(\frac{|z|}{\cancel{R} r}\right)^n \cdot \frac{1}{n!} = M \sum_{n=0}^{\infty} \frac{\left(\frac{|z|}{r}\right)^n}{n!}$$

$$= M e^{\frac{|z|}{r}}.$$

2.8. $\forall z \in D(z_0;R)$, $f(z) = S_N(z) + \sum_{n=N+1}^{\infty} a_n (z-z_0)^n$

$\Rightarrow$ " $|f(z) - S_N(z)| \leq \sum_{n=N+1}^{\infty} |a_n||z-z_0|^n$

[Note that we have absolute conv. at every pt. of $D(z_0;R)$]

Observe that, $\forall l \geq 0$, $z \in D(z_0;R)$

$$\sum_{N=0}^{l} |f(z) - S_N(z)| \leq \sum_{n=1}^{\infty} |a_n||z-z_0|^n + \sum_{n=2}^{\infty} |a_n||z-z_0|^n$$

$$+ \cdots + \sum_{n=l+1}^{\infty} |a_n||z-z_0|^n$$

$$\cancel{= |a_1||z-z_0| + 2|a_2||z-z_0|^2 + \cdots + (l+1)|a_{l+1}||z-z_0|^{l+1}}$$

$$= |a_1||z-z_0| + |a_2||z-z_0|^2 + \cdots + |a_l||z-z_0|^l + \sum_{n=l+1}^{\infty} |a_n||z-z_0|^n$$

$$+ \quad |a_2||z-z_0|^2 + \cdots + |a_l||z-z_0|^l + \text{"}$$

$$\vdots$$

$$+ \quad |a_l||z-z_0|^l + \text{"}$$

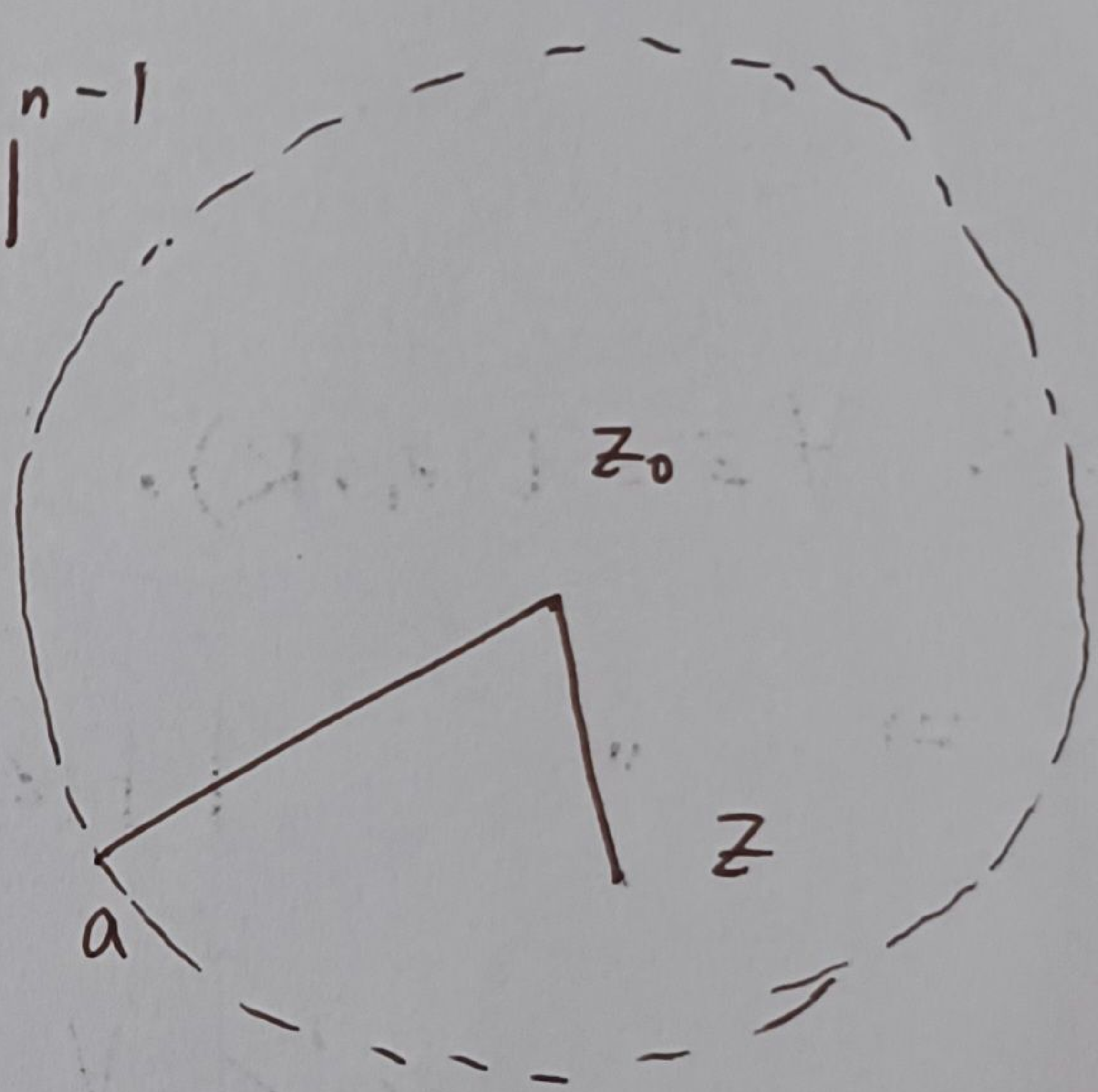
$$= |a_1||z-z_0| + 2|a_2||z-z_0|^2 + \cdots + l|a_l||z-z_0|^l + (l+1) \text{"}$$

$$= |z-z_0|\left(|a_1| + 2|a_2||z-z_0| + \cdots + l|a_l||z-z_0|^{l-1}\right) + (l+1)\sum_{n=l+1}^{\infty} |a_n||z-z_0|^n$$

$$\cancel{\leq |z-z_0| \sum_{n=1}^{\infty} |a_n||z-z_0|^{n-1} + (l+1)\sum_{n=l+1}^{\infty} |a_n||z-z_0|^n}$$

$$\leq |z-z_0|\left(|a_1| + 2|a_2||z-z_0| + \cdots + l|a_l||z-z_0|^{l}\right) + |z-z_0| \sum_{n=l+1}^{\infty} n|a_n||z-z_0|^{n-1}$$

$$= |z-z_0| \sum_{n=1}^{\infty} n|a_n||z-z_0|^{n-1}$$



2.10. *Let* $z \in D(z_0; |z_0 - a|)$

Then $$\frac{1}{z-a} = \frac{1}{(z-z_0)+(z_0-a)}$$

$$= \frac{1}{z_0-a} \cdot \frac{1}{1 - \frac{-(z-z_0)}{z_0-a}}$$

$$= \frac{1}{z_0-a} \cdot \frac{1}{1 - \left(-\frac{z-z_0}{z_0-a}\right)}$$

Recall that, whenever $|w|<1$, $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$

Hence we obtain that $$\frac{1}{1-\left(-\frac{z-z_0}{z_0-a}\right)} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-z_0)^n}{(z_0-a)^n}$$

$$\Rightarrow \frac{1}{z-a} = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{(z_0-a)^{n+1}}}_{=c_n} \cdot (z-z_0)^n.$$

As $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ *conv. for all* $z \in D(z_0; |z_0-a|)$, *its radius of conv. cannot be* $< |z_0-a|$.