Sample Solution of Quiz-1

1. (i) False: Propagated Error in an approximation is affected by the condition number of a function at the point of approximation. Propagated error can be very bad even if the algorithm is very good.

(ii) False: Let δ be such that

$$\frac{\varepsilon_{mach}}{2} < \delta < \varepsilon_{mach}.$$

Then in $\mathbb{F}$,

$$(1+\delta)+\delta = 1+\delta = 1$$

$$1+(\delta+\delta) = 1+2\delta \neq 1$$

So $(1+\delta)+\delta \neq 1+(\delta+\delta)$.

(iii) False: $K(cA) = \|cA\| \, \|(cA)^{-1}\| = |c| \, \|A\| \frac{1}{|c|} \|A^{-1}\|$

$$= \|A\| \, \|A^{-1}\|$$

$$= K(A)$$

$\forall\ c \in \mathbb{R} - \{0\}$.

P-2

(2) Let $x = \beta^{L-1}$ = UFL in $\mathbb{F}$ with $L-1<0$.

$y = \beta^{L-1}$, $z = \beta^{1-L}$

$(x \times y) \times z = \beta^{2(L-1)} \times z = 0 \times z = 0$

($\because \beta^{2(L-1)} = 0$ in $\mathbb{F}$)

$x \times (y \times z) = x \times 1 = x = \beta^{L-1}$

So $(x \times y) \times z \neq x \times (y \times z)$.

(3) $x = 2 = 2.00$, $y = -.600$, $z = .602$

$(x \times y) = -1.20$, $(x \times z) = 1.204$

$= 1.20$ in 3-digit rounding

$(x \times y) + (x \times z) = 0.00$

$x \times (y+z) = 2.00(.002) = .004$.

Hence, $(x \times y) + (x \times z) \neq x \times (y+z)$.

Floating pt. Arithmatic in $\mathbb{F}$ is not distributive.

(4) (i) $x = -0.04518$, $x_A = 0.045113$

$x = (-1)^1 \times .4518 \times 10^{-1}$ $\therefore e = -1$

$s = e-1 = -1-1 = -2.$

$(x - x_A) = -0.000067$

$$\begin{array}{r} .04518 \\ -.045113 \\ \hline .000067 \end{array}$$

$|x - x_A| = .000067$

$= .067 \times 10^{-3}$

$= \frac{1}{2} \times .134 \times 10^{-3}$

$\leq \frac{1}{2} \times 10^{-3}$

Comparing $s+1-r = -3 \Rightarrow -2+1-r = -3$

$\boxed{r = 2.}$

(ii) $x = 23.4604 = (-1)^0 \times .234604 \times 10^2$,

$e = 2$, $s = e-1 = 2-1 = 1.$

$$\begin{array}{r} 23.4604 \\ 23.4213 \\ \hline 00.0391 \end{array}$$

$|x - x_A| = .0391 = .391 \times 10^{-1}$

$= \frac{1}{2} \times .782 \times 10^{-1}$

$\leq \frac{1}{2} \times 10^{-1}$

P-4

Comparing $s+1-r=-1$

$1+1-r=-1$

$\boxed{r=3}$

5 $c=0$, $K_f(0)=|f'(0)|=\ln\pi$

$f(x)=\pi^x$, $\ln f(x)=x\ln\pi$

Diff. $\dfrac{f'(x)}{f(x)}=\ln\pi$

$f'(x)=f(x)\ln\pi=(\ln\pi)\,\pi^x$

$c\neq0$

$$K_f(c)=\frac{|f'(c)||c|}{|f(c)|}=\frac{\pi^c\cdot\ln\pi\cdot|c|}{\pi^c}=|c|\ln\pi.$$

$$K_f(c)=\begin{cases}\ln\pi, & c=0\\ |c|\ln\pi, & c\neq0.\end{cases}$$

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$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 7 \\ 2 & 2 & 4 & 5 \\ 2 & 3 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 - 2R_1 \\ R_4 - 2R_1}]{R_2 - R_1} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -2 & 2 & 5 \\ 0 & 0 & 6 & 1 \\ 0 & 1 & 6 & 2 \end{bmatrix}$$

$$\xrightarrow{R_4 + \frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & -2 & 2 & 5 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 7 & \frac{9}{2} \end{bmatrix}$$

$$\xrightarrow{R_4 - \frac{7}{6}R_3} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -2 & 2 & 5 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & \frac{10}{3} \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & -\frac{1}{2} & \frac{7}{6} & 1 \end{bmatrix}$$

$$\|A\|_1 = \max\{6, 7, 10, 20\} = 20.$$

$$\|A\|_\infty = \max\{5, 10, 13, 15\} = 15$$

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$$B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \xrightarrow{R_2 + \frac{1}{4}R_1} \begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{15}{4} & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 + \frac{4}{15}R_2} \begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{15}{4} & -1 \\ 0 & 0 & \frac{56}{15} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & -\frac{4}{15} & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{15}{4} & -1 \\ 0 & 0 & \frac{56}{15} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & -\frac{4}{15} & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{15}{4} & 0 \\ 0 & 0 & \frac{56}{15} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{4}{15} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{15}}{2} & 0 \\ 0 & -\frac{2}{\sqrt{15}} & \sqrt{\frac{56}{15}} \end{bmatrix} \begin{bmatrix} 2 & -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{15}}{2} & -\frac{2}{\sqrt{15}} \\ 0 & 0 & \sqrt{\frac{56}{15}} \end{bmatrix}.$$

One could also directly find LL^T decomposition by considering

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Then comparing the elements. find l_{ij}.

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(6) The way we have defined $K_f(x)$ for $x=0$ or $f(x)=0$

It is contradicting the assumptions given.

For example if $x=0$, $K_f(0) = |f'(0)|$

$$K_g(0) = |g'(0)|$$

Since f is well conditioned

g is ill "

Therefore $K_f(0) < K_g(0)$

ie $|f'(0)| < |g'(0)|$.

It is contradiction to the fact

$|g'(0)| \leq M \leq |f'(0)|$.

P-8 We will show the conclusion

$$|g(x)| < |f(x)|$$

only for $x \in \mathbb{R} \setminus \left[\{0\} \cup \{x_1 \mid f(x_1) = 0\}\right]$

1st case: $x \neq 0$, $f(x) \neq 0 \neq g(x)$.

Then, $K_f(x) = \dfrac{|f'(x)||x|}{|f(x)|}$

$$K_g(x) = \frac{|g'(x)||x|}{|g(x)|}$$

Since f is well conditioned and g is ill conditioned.

$$K_f(x) < K_g(x)$$

$$\frac{|f'(x)||x|}{|f(x)|} < \frac{|g'(x)||x|}{|g(x)|}$$

$$\Rightarrow \frac{M|x|}{|f(x)|} \leq \frac{|f'(x)||x|}{|f(x)|} < \frac{|g'(x)||x|}{|g(x)|} \leq \frac{M|x|}{|g(x)|}$$

(by assumption (1))

$$\Rightarrow \frac{1}{|f(x)|} < \frac{1}{|g(x)|}$$

$$\Rightarrow |g(x)| < |f(x)|.$$

2nd case

Let $x \neq 0$, $g(x) = 0$, $f(x) \neq 0$

$$K_f(x) = \frac{|f'(x)||x|}{|f(x)|}, \quad K_g(x) = |g'(x)|.$$

Again as f is well conditioned and

g is ill ".

$K_f(x) < K_g(x)$

$$\frac{M|x|}{|f(x)|} \leq \frac{|f'(x)|\,|x|}{|f(x)|} < |g'(x)| \leq M$$

[by assumption (1)]

$$\Rightarrow \frac{|x|}{|f(x)|} < 1 \Rightarrow |x| < |f(x)|$$

$$\Rightarrow |g(x)| = 0 < |x| < |f(x)|.$$

$$\Rightarrow |g(x)| < |f(x)|.$$

Those who will consider at least 1st case, will get full marks.

For $x \in \{0\} \cup \{x_0 \mid f(x_0)\}$, the conclusion is not true.

End