Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Complex Analysis (MTH 403)
Exercise Sheet 5

1. Cauchy's theorem

1.1. Let f be a continuous function defined on a closed disc $\overline{D(z_0;R)}$, where $z_0 \in \mathbb{C}$ and $R > 0$. Assume that f is holomorphic on $D(z_0;R)$.

(a) Is it necessarily true that $\displaystyle\int_{C(z_0;R)} f = 0$?

(b) Prove or disprove the following:

$$f(z) = \frac{1}{2\pi i}\int_{C(z_0;R)} \frac{f(w)}{(w-z)}\,dw,\ \forall z \in D(z_0;R).$$

1.2. Justify the following:

(a) The star-like assumption in the hypothesis of Cauchy's theorem cannot be dropped.

(b) Cauchy's estimate cannot be improved in general.

1.3.∗ Let γ be a closed path in $\mathbb{C}$ and $g : \gamma^* \longrightarrow \mathbb{C}$ be a continuous function. For $n \in \mathbb{N}$, consider

$$\varphi(z) \overset{\text{def}}{=} \int_\gamma \frac{g(w)}{(w-z)^n}\,dw,\ \forall z \in \mathbb{C}\setminus\gamma^*. \tag{1.1}$$

Show that φ, defined as in (1.1), is holomorphic and indeed, for all $z \in \mathbb{C}\setminus\gamma^*$,

$$\varphi'(z) = n\int_\gamma \frac{g(w)}{(w-z)^{n+1}}\,dw.$$

(**Hint:** Consider the case $n = 1$ first, i.e., the function $\Phi(z) \overset{\text{def}}{=} \displaystyle\int_\gamma \frac{g(w)}{(w-z)}\,dw,\ \forall z \in \mathbb{C}\setminus\gamma^*$. Is Φ holomorphic? What is $\Phi^{(n-1)}$? Is it related to φ?)

1.4. Let $z_0 \in U \subseteq_{open} \mathbb{C}$, $D(z_0;R) \subseteq U$ and $f : U \longrightarrow \mathbb{C}$ be holomorphic. Pick any $0 < r < R$. Show that, for any $k \geq 0$, one has the following integral formula for the n-th derivative of f:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C(z_0;r)} \frac{f(w)}{(w-z)^{n+1}}\,dw,\ \forall z \in D(z_0;r).$$

(**Hint:** Prove by induction. Use 1.3.)

In 1.5. and 1.6., we let z_0, U, R, f and r be as in 1.4.. For any $n \geq 0$, denote the n-th Taylor polynomial of f at z_0 and the corresponding remainder term by s_n and R_n respectively.

1.5. Show that, for all $z \in D(z_0;r)$,

$$s_n(z) = \frac{n!}{2\pi i}\int_{C(z_0;r)} \frac{f(w)}{(w-z_0)^{n+1}}\left(\frac{(w-z_0)^{n+1}-(z-z_0)^{n+1}}{z-z_0}\right)dw.$$

1.6. (a) Analogous to the Cauchy integral formula, can you provide integral representation of R_n?

(b) Let $0 < \rho < r$. Show that, for all $z \in D(z_0; \rho)$, one has

$$|R_n(z)| \le \sup_{|w-z_0|=r} |f(w)| \cdot \frac{r}{r-\rho} \cdot \left(\frac{\rho}{r}\right)^{n+1}.$$

1.7. Evaluate the following integrals:

(a) $\displaystyle\int_0^{2\pi} e^{e^{it}}\, dt.$

(b) $\displaystyle\int_0^{2\pi} e^{(e^{it}-t)}\, dt.$

(c) $\displaystyle\int_{C(0;1)} \frac{|dz|}{|z-a|^2}$, where $a \in \mathbb{D}$.

(d) $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|ae^{it}-b|^4\, dt}$, where $0 < a < b$.

(e)$*$ $\displaystyle\int_\gamma \frac{\cos z}{z(z^2+8)}\, dz$, where γ is the positively oriented square whose sides are the lines $x = \pm 2$ and $y = \pm 2$.

1.8. Let $f : \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \longrightarrow \mathbb{C}$ be holomorphic and bounded. Prove that, for any $\alpha > 0$, f is uniformly continuous on the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$.

1.9. Let $D(z_0; R) \subseteq U \subseteq_{open} \mathbb{C}$, and $f \in H(U)$. Show that, for any $0 < r < R$,

$$|f(z_0)|^2 \le \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R \left|f\left(z_0 + re^{it}\right)\right|^2 r\, dr dt.$$

1.10. Let $f(w) \stackrel{\text{def}}{=} \displaystyle\sum_{n=0}^{\infty} \frac{2^n w^n}{3^n}$, for all $|w| < \frac{3}{2}$ and $g(w) \stackrel{\text{def}}{=} \displaystyle\sum_{n=0}^{\infty} \frac{1}{(2w)^n}$, for all $|w| > \frac{1}{2}$. For $|z| \neq 1$, consider

$$h(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{C(0;1)} \left(\frac{f(w)}{w-z} + \frac{z^2 g(w)}{w^2 - wz}\right) dw.$$

Show that

$$h(z) = \begin{cases} \frac{3}{3-2z} & \text{if } |z| < 1 \\ \frac{2z^2}{1-2z} & \text{if } |z| > 1. \end{cases}$$

1.11. Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$ be holomorphic. Consider any closed disc $\overline{D(z_0; r)} \subseteq U$. Denote by γ the image of the circle $C(z_0; r)$, oriented positively, under f. Show that

$$L_\gamma \ge 2\pi r |f'(z_0)|.$$

1.12. Let $f : \mathbb{D} \longrightarrow \mathbb{C}$ be a holomorphic function.

(a) If $|f(z)| \le \dfrac{1}{1-|z|}$, for all $z \in \mathbb{D}$, then show that, for any $n \ge 0$,

$$|f^{(n)}(0)| \le (n+1)!\left(1 + \frac{1}{n}\right)^n < e(n+1)!.$$

(b)$*$ Assume that there exists $C > 0$ such that, $|f(z)| \le \dfrac{C}{1-|z|}$, for all $z \in \mathbb{D}$. Show that, for any $z \in \mathbb{D}$,

$$|f'(z)| \le \frac{4C}{(1-|z|)^2}.$$

(**Hint:** Apply Caushy's integral formula on a disc with center z and suitable radius.)

1.13. Let f be an entire function. Suppose that there exist positive numbers A, B and α such that
$$|f(z)| \leq A + B|z|^\alpha, \ \forall z \in \mathbb{C}.$$
What can you conclude about f from this?

1.14. (a) If f is an entire function such that $\lim_{z\to\infty} \frac{f(z)}{z} = 0$, then f is a constant function.
(b) Using 1.14.a or otherwise, show that if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic and bounded then it must be a constant function.

1.15.$*$ Let f be an entire function. Assume that, for all $z \in \mathbb{C} \setminus (-\infty, 0]$, $|f(z)| \leq |\log_{-\pi} z|$. What can you conclude about f?

1.16.$*$ Let f be an entire function. Assume that, for all $r > 0$, $\int_0^{2\pi} \left|f\left(re^{it}\right)\right| dt \leq r^{\frac{17}{3}}$. Prove that $f \equiv 0$.

In 1.17. and 1.18., let $z_0 \in U \subseteq_{open} \mathbb{C}$ and $f : U \setminus \{z_0\} \longrightarrow \mathbb{C}$ be holomorphic.

1.17. Suppose that there exists $r > 0$ such that $D(z_0; r) \subseteq U$ and f is bounded on $D(z_0; r) \setminus \{z_0\}$. Show that, f can be defined at z_0 so that the extended function is holomorphic on U.

1.18. Assume that there exists $r, \delta > 0$ and $w \in \mathbb{C}$ such that $f(D(z_0; r) \setminus \{z_0\}) \cap D(w, \delta) = \emptyset$. Consider the function
$$g(z) \stackrel{\text{def}}{=} \frac{1}{f(z) - w}, \ \forall z \in D(z_0; r) \setminus \{z_0\}. \tag{1.2}$$

(a) Show that the function g, defined as above in (1.2), can be defined at z_0 so that the extended function is holomorphic on $D(z_0; r)$. (**Hint:** Use 1.17.)
(b) Denote the extended function mentioned above in 1.18.a by g as well. Show that, if $g(z_0) \neq 0$, then f can be defined at z_0 so that the extended function is holomorphic on U.

1.19.$*$ Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be continuous. Assume that f is holomorphic on $\{z \in \mathbb{C} : \operatorname{Im} z \neq 0\}$. Prove that f must be analytic. (**Hint:** Use Morera's theorem.)

2. Poisson integral formula

Let $z \in \mathbb{D}$. Consider the two functions
$$P_z : \mathbb{R} \longrightarrow (0, \infty), \ P_z(t) \stackrel{\text{def}}{=} \frac{1 - |z|^2}{|e^{it} - z|^2}, \tag{2.1}$$
and
$$Q_z : \mathbb{R} \longrightarrow \mathbb{C}, \ Q_z(t) \stackrel{\text{def}}{=} \frac{e^{it} + z}{e^{it} - z}. \tag{2.2}$$

2.1. Show that $\operatorname{Re} Q_z = P_z$.

2.2. Show that, if $z = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$, then for any $t \in \mathbb{R}$,
$$P_z(t) = P_r(t - \theta) = \frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2} = P_r(\theta - t).$$

P_z and Q_z are called the *Poisson kernel* and *Cauchy kernel* respectively.

2.3.$*$ Let $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous and holomorphic on $\mathbb{D}$. For any $z \in \mathbb{D}$, show that

$$f(z) = \frac{1}{2\pi i} \int_{C(0;1)} \left(\frac{1}{w - z} - \frac{1}{w - \frac{1}{\bar{z}}} \right) f(w)\, dw. \tag{2.3}$$

(**Hint:** Use Cauchy's theorem.)

2.4. Show that, for any $t \in [0, 2\pi]$,

$$\left(\frac{1}{e^{it} - z} - \frac{1}{e^{it} - \frac{1}{\bar{z}}} \right) e^{it} = \frac{e^{it}}{e^{it} - z} + \frac{\bar{z}e^{it}}{1 - \bar{z}e^{it}} = \frac{e^{it}}{e^{it} - z} + \frac{\bar{z}}{e^{-it} - \bar{z}} = \frac{1 - |z|^2}{|e^{it} - z|^2} f(e^{it})..$$

2.5. Let f be as above in 2.3.. From 2.3. and 2.4., conclude that,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} P_z(t) f(e^{it})\, dt,\ \forall z \in \mathbb{D}. \tag{2.4}$$

Note that, if $z = 0$, then $P_0 \equiv 1$, so that (2.4) reduces to the Mean value property (MVP) for $\mathbb{D}$:

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt.$$

The formula established as above in (2.4) thus generalizes the MVP to the value of the function at an arbitrary point of $\mathbb{D}$. It is just the weighted average of values of f on the boundary, where the weights are given by the Poisson kernel. We refer to (2.4) as the *Poisson integral formula* for $\mathbb{D}$.

2.6. Can you generalize the Poisson inetgral formula for any arbitrary disc?

3. Proof of Cayley-Hamilton theorem using Cauchy's theorem

This section is aimed to prove the Cayley-Hamilton theorem using Cauchy's theorem. To this end, we need to consider sequence of series of functions that take values in matrices with complex entries. We first make a few natural definitions.

Definition 3.1. *Fix $d \in \mathbb{N}$. For each $n \in \mathbb{N}$, let A_n be an $d \times d$ matrix with complex entries. Denote the (i, j)-th entry of A_n by $a_{ij}^{(n)}$, for all $i, j = 1, \ldots, d$.*

(i) We say that the sequence $\{A_n\}_{n=1}^{\infty}$ converges to $A = [a_{ij}]_{1 \le i,j \le d} \in M_d(\mathbb{C})$ if, for all $i, j = 1, \ldots, d$, one has $a_{ij}^{(n)} \xrightarrow[n\to\infty]{} a_{ij}$. For example,

$$\begin{bmatrix} \frac{1}{n} & 2 + \frac{3}{n} \\ 5 & \frac{i}{2^n} \end{bmatrix} \xrightarrow[n\to\infty]{} \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}.$$

(ii) The convergence of the series $\sum_{n=1}^{\infty} A_n$ can now be defined in the usual way, i.e., if the sequence $\left\{ \sum_{k=1}^{n} A_k \right\}_{n=1}^{\infty}$ converges.

Definition 3.2. *Let $X \neq \emptyset$ and $f_n : X \longrightarrow M_d(\mathbb{C})$, for all $n \in \mathbb{N}$.*

(i) We say that $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise, where $f : X \longrightarrow M_d(\mathbb{C})$, if for every $x \in X$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of matrices converges to $f(x)$.

(ii) For any $n \in \mathbb{N}$, $i, j = 1, \dots, d$ and $x \in X$, denote the (i, j)-th entry of $f_n(x)$ and $f(x)$ by $f_{i,j}^{(n)}(x)$ and $f_{i,j}(x)$ respectively. If, for any $i, j = 1, \dots, d$, the sequence $\left\{f_{i,j}^{(n)}\right\}_{n=1}^{\infty}$ of functions converges uniformly to $f_{i,j}$, we then say that $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly.

Definition 3.3. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a path and $f : \gamma^* \longrightarrow M_d(\mathbb{C})$ be continuous. Then $\int_\gamma f$ is defined to be the matrix whose (i, j)-th entry is $\left[\int_\gamma f_{i,j}\right]_{1 \le i,j \le d}$, for all $i, j = 1, \dots, d$, where $f_{i,j}$ is defined as in Definition 3.2.*

Let $A \in M_d(\mathbb{C})$ be nonzero. Consider the following series:

$$\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} A^n. \tag{3.1}$$

3.1. Show that the series as given in (3.2.) converges uniformly on $\mathbb{C} \setminus D(0; 2\|A\|)$. (**Hint:** Denote the (i, j)-th entry of A^n by $a_{ij}^{(n)}$, for all $i, j = 1, \dots, d$. Then it is easy to see that, for all $n \ge 0$, $|a_{ij}^{(n)}| \le \|A^n\| \le \|A\|^n$. Using this, estimate $\dfrac{|a_{ij}^{(n)}|}{|z|^{n+1}}$.)

3.2. Show that, for all $z \in \mathbb{C}$ with $|z| \ge 2\|A\|$, $(zI - A)$ is invertible, and in fact, the inverse is given by (3.1).

3.3.$*$ Show that, for all $k \ge 0$,

$$A^k = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} z^k (zI - A)^{-1}\, dz.$$

(**Hint:** Do you see $a_{i,j}^{(k)} = \dfrac{1}{2\pi i} \displaystyle\int_{C(0;2\|A\|)} \left(\sum_{n=0}^{\infty} \frac{z^k}{z^{n+1}} a_{i,j}^{(n)}\right) dz$, for all $i, j = 1, \dots, d$? This shows that $A^k = \dfrac{1}{2\pi i} \displaystyle\int_{C(0;2\|A\|)} z^k \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} A^n\, dz$. Now use 3.2.)

3.4. Deduce that, for any polynomial $P(z)$ with complex coefficients, one has

$$P(A) = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} P(z)(zI - A)^{-1}\, dz. \tag{3.2}$$

(This is an analogue of Cauchy's integral formula.)

3.5. Let $\chi(z) = \det(zI - A)$ be the characteristic polynomial of A. Conclude from (3.2) that

$$\chi(A) = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} \det(zI - A)(zI - A)^{-1}\, dz. \tag{3.3}$$

3.6.$*$ Recall that, for any invertible $B \in M_d(\mathbb{C})$, $(\det B)B^{-1} = \operatorname{Adj} B$. Using this, conclude from (3.3) that $\chi(A) = 0$.

4. Basic properties of sine and cosine functions

The sine and cosine functions over $\mathbb{C}$ are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The aim of the exercises 4.1.-4.4. is to realize the sine and cosine functions as the solutions of some IVP's. Further to that, using this, we can prove the basic properties of these functions.

4.1. Consider the following differential equation:

$$f'' + f = 0. \tag{4.1}$$

Find all entire functions f satisfying (4.1) with the initial conditions $f(0) = 0$ and $f'(0) = 1$. What if the initial conditions are changed to $f(0) = f'(0) = 0$?

4.2. Find all entire functions satisfying the following: $f'' - f = 0$, $f(0) = 1$ and $f'(0) = 0$.

4.3. Find all solutions (entire functions) of (4.1). (**Hint:** Assuming f is a solution of (4.1), can you see that $\varphi(z) \overset{\text{def}}{=} f(0)\cos z + f'(0)\sin z$, $\forall z \in \mathbb{C}$, satisfies some IVP?)

4.4. Show the following:

(a) $\forall z \in \mathbb{C}$, $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.

(b) For every $z, w \in \mathbb{C}$, $\sin(z+w) = \sin z \cos w + \cos z \sin w$ and $\cos(z+w) = \cos z \cos w - \sin z \sin w$. (**Hint:** Fix $w \in \mathbb{C}$. Consider the function $f(z) \overset{\text{def}}{=} \cos(z+w)$, $\forall z \in \mathbb{C}$. Do you see that f solves (4.1)?)