MTH302A: *Set Theory and Mathematical Logic*
Midterm Exam Solutions Odd Semester 2022/23

Question 1

For each of the following statements, determine whether it is **true or false**. No justification required. [**5 × 2 Points**]

(a) There exists a surjective function $f : \mathbb{R} \to \mathbb{R}^2$.

(b) If $(\phi \vee \psi)$ is a tautology, then either ϕ is a tautology or ψ is a tautology.

(c) Every countable linear ordering is isomorphic to a subordering of the rationals.

(d) There are injective functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) + g(x) = e^{-x^2} \sin x$.

(e) Every chain in $(\mathcal{P}(\omega), \subseteq)$ is countable.

Solution

(a) True. Since $|\mathbb{R}^2| = |\mathbb{R}|$.

(b) False. Consider $(p \vee \neg p)$.

(c) True. See the solution to HW 11.

(d) True. See HW 19.

(e) False. See HW 10.

Question 2

(a) [**2 Points**] State the Schröder-Bernstein theorem.

(b) [**2 Points**] Show that there is a bijection from $[0, 1]$ to $(0, 2)$.

(c) [**6 Points**] Let $\mathcal{I}$ be the set of all bijections from ω to ω. Show that $|\mathcal{I}| = \mathfrak{c}$.

Solution

(a) For any two sets A and B, if there exist injections from A to B and from B to A, then there exists a bijection from A to B.

(b) $x \mapsto x + 1/2$ is an injection from $[0, 1]$ to $(0, 2)$ and $x \mapsto x/2$ is an injection from $(0, 2)$ to $[0, 1]$. By the Schröder Bernstein theorem, there is a bijection from $[0, 1]$ to $(0, 2)$.

(c) See Practice midterm (2)(b).

Question 3

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies: For every $x, y \in \mathbb{R}$,

$$f(x + y) + f(x)f(y) = f(x) + f(y) \tag{1}$$

(a) [**2 Points**] Show that $f(x) = 1 - 2^x$ satisfies Equation (1).

(b) [**5 Points**] Suppose f is continuous. Show that **either** f is identically 1 **or** $f(x) = 1 - a^x$ for some constant $a > 0$.

(c) [**3 Points**] Show that there is a discontinuous f satisfying Equation (1).

Solution

(a) $f(x + y) + f(x)f(y) = 1 - 2^{x+y} + (1 - 2^x)(1 - 2^y) = 1 - 2^x2^y + 1 - 2^x - 2^y + 2^x2^y = 2 - 2^x - 2^y = (1 - 2^x) + (1 - 2^y) = f(x) + f(y)$.

(b) Assume f is continuous. Put $g(x) = 1 - f(x)$. Then g is continuous and $g(x + y) = 2 - f(x + y) = 2 - f(x) - f(y) + f(x)f(y) = (1 - f(x))(1 - f(y)) = g(x)g(y)$. By HW 17, either g is identically 0 or $g(x) = a^x$ for some $a > 0$. Hence f is either identically 1 or $f(x) = 1 - a^x$ for some $a > 0$.

(c) Let $h : \mathbb{R} \to \mathbb{R}$ be any discontinuous additive function. Define $f(x) = 1 - e^{h(x)}$. It is easy to check that f satisfies Equation (1). Note that $1 - f(x) > 0$ for every x (as $e^y > 0$ for every y). It follows that f cannot be continuous otherwise $h(x) = \ln(1 - f(x))$ (being a composition of two continuous functions) would also be continuous.

Question 4

(a) [**2 Points**] State the Axiom of choice.

(b) [**2 Points**] State Zorn's lemma.

(c) [**6 Points**] Let $(P, \preceq)$ be a partial ordering. Let $\mathcal{F} = \{C \subseteq P : C \text{ is a chain in } (P, \preceq)\}$. Show that $(\mathcal{F}, \subseteq)$ has a maximal member.

Solution

(a) For every family $\mathcal{F}$ of nonempty sets, there exists a function $h : \mathcal{F} \to \bigcup \mathcal{F}$ such that for every $A \in \mathcal{F}$, $h(A) \in A$.

(b) Let $(P, \preceq)$ be a partial ordering such that every chain in $(P, \preceq)$ has an upper bound in P. Then $(P, \preceq)$ has a maximal element.

(c) We will show that every chain in $(\mathcal{F}, \subseteq)$ has an upper bound in $\mathcal{F}$. Zorn's lemma will then imply that $(\mathcal{F}, \subseteq)$ has a maximal member.

Let $\mathcal{C}$ be a chain in $(\mathcal{F}, \subseteq)$. Put $W = \bigcup \mathcal{C}$. We claim that $W \in \mathcal{F}$ or equivalently, W is a chain in $(P, \preceq)$. Clearly, $W \subseteq P$. Suppose $x, y \in W$. Choose $A, B \in \mathcal{C}$ such that $x \in A$ and $y \in B$. Since $\mathcal{C}$ is a $\subseteq$-chain, either $A \subseteq B$ or $B \subseteq A$. WLOG assume $A \subseteq B$. Then both $x, y \in B$. As $B \in \mathcal{F}$, B is a chain in $(P, \preceq)$. So either $x \preceq y$ or $y \preceq x$. It follows that any two members in W are $\preceq$-comparable. Hence $W \in \mathcal{F}$. It is clear that for every $A \in \mathcal{C}$, $A \subseteq W$. Hence W is an upper bound for $\mathcal{C}$ in $(\mathcal{F}, \subseteq)$.

Bonus problem

Show that $\mathbb{R}^3$ can be partitioned into circles of unit radius. [**5 Points**]

Solution

Let $\mathcal{C}$ be the family of all circles of unit radius in $\mathbb{R}^3$. Let $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ be an injective sequence whose range is $\mathbb{R}^3$. Using transfinite recursion, construct $\langle \mathcal{C}_\alpha : \alpha < \mathfrak{c}\rangle$ such that the following hold.
(1) Each $\mathcal{C}_\alpha \subseteq \mathcal{C}$ consists of pairwise disjoint circles of unit radius and $\mathcal{C}_0 = \emptyset$.
(2) If $\alpha < \beta < \mathfrak{c}$, then $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$.
(3) If $\alpha < \mathfrak{c}$ is limit, then $\mathcal{C}_\alpha = \bigcup\{\mathcal{C}_\beta : \beta < \alpha\}$.
(4) For every $\alpha < \mathfrak{c}$, $|\mathcal{C}_\alpha| \leq \max(\{\omega, |\alpha|\})$.
(5) For every $\alpha < \mathfrak{c}$, $x_\alpha \in \bigcup \mathcal{C}_{\alpha+1}$.
At limit stages $\alpha < \mathfrak{c}$, we simply define $\mathcal{C}_\alpha$ by Clause (3) above. Having constructed $\mathcal{C}_\alpha$, we define $\mathcal{C}_{\alpha+1}$ as follows. If $x_\alpha \in \bigcup \mathcal{C}_\alpha$, then we put $\mathcal{C}_{\alpha+1} = \mathcal{C}_\alpha$. Now assume that x_α does not lie on any circle in $\mathcal{C}_\alpha$.

Claim: There is a circle C of radius 1 such that C passes through x_α and for every circle $T \in \mathcal{C}_\alpha$, $T \cap C = \emptyset$.

Proof of Claim: Let $\mathcal{P}_\alpha$ be the family of all planes P such that some circle in $\mathcal{C}_\alpha$ lies completely within P. Then $|\mathcal{P}_\alpha| \leq |\mathcal{C}_\alpha| \leq \max(\{|\alpha|, \omega\}) < \mathfrak{c}$. Choose a plane P such that $x_\alpha \in P$ and $P \notin \mathcal{P}_\alpha$. This can be done because there are continuum many planes passing through x_α. Let B be the set of all points in P which also lie on some circle in $\mathcal{C}_\alpha$. Since each circle in $\mathcal{C}_\alpha$ meets P at ≤ 2 points, we get $|B| < \mathfrak{c}$. Note that $x_\alpha \notin B$ as $x_\alpha \notin \bigcup \mathcal{C}_\alpha$.
Consider the family $\mathcal{E}$ of all circles of unit radius inside the plane P that pass through x_α. Observe that $|\mathcal{E}| = \mathfrak{c}$. Define a function H with domain B by $H(y) = \{S \in \mathcal{E} : y \in S\}$. Note that for every $y \in B$, $H(y)$ contains at most two circles from $\mathcal{E}$. Put $\mathcal{R} = \bigcup \text{range}(H)$. Then $|\mathcal{R}| \leq \max(\{2 \cdot |B|, \omega\}) < \mathfrak{c}$. Hence we can choose $C \in \mathcal{E} \setminus \mathcal{R}$. $\square$

Let C be as in the claim. Define $\mathcal{C}_{\alpha+1} = \mathcal{C}_\alpha \cup \{C\}$ and note that $x_\alpha \in \bigcup \mathcal{C}_{\alpha+1}$. This completes the construction. Let $\mathcal{F} = \bigcup\{\mathcal{C}_\alpha : \alpha < \mathfrak{c}\}$. By Clause (1), it is clear that $\mathcal{F}$ is a disjoint family of circles. Also, by Clause (5), $\bigcup \mathcal{F} = \mathbb{R}^3$. Hence $\mathcal{F}$ is a partition of $\mathbb{R}^3$ into circles of unit radius.