Boundary Value Problems: Shooting Method

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and choose $\gamma > 0$ *to minimize*

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Lesson 3

Boundary Value Problems for ODEs

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Boundary Value Problems: Finite Difference Method

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$$u'' = f(t, u, u'), \qquad a < t < b,$$

with boundary conditions

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$$u'(t_i) \approx \frac{u_{i+1} - u_{i-1}}{2h}, \;\; u''(t_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

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This yields a system of algebraic equations

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), i = 1, \dots, n,$$

to be solved for the unknowns $u_i, i = 1, \dots, n$.

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In the matrix form, we have

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & & & \cdots & 0 \\ 1 & -2 & 1 & & & & \\ \vdots & \vdots & & \ddots & \vdots & & \vdots \\ & & & & 1 & -2 & 1 \\ 0 & \cdots & & & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}=\begin{bmatrix} f\left(t_1,u_1,\frac{u_2-\alpha}{2h}\right) \\ f\left(t_2,u_2,\frac{u_3-u_1}{2h}\right) \\ \vdots \\ f\left(t_n,u_n,\frac{\beta-u_n}{2h}\right) \end{bmatrix}$$

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$$\frac{1}{h^2} Au = F(u) + g.$$

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Thus, the Newton's method for solving the system of algebraic equations is given by

$$u^{(m+1)} = u^{(m)} - \left[\frac{1}{h^2}A - F'\left(u^{(m)}\right)\right]^{-1}\left[\frac{1}{h^2}Au^{(m)} - F\left(u^{(m)}\right) - g\right]$$

where the Jacobian matrix is given by $[F(u)]_{ij} = [\partial f(t_i, u_i, (u_{i+1} - u_{i-1})/(2h))/\partial u_j]$.