MTH 301: Analysis I — *September 22, 2017*
Department of Mathematics and Statistics — Time: 120 minutes
Indian Institute of Technology - Kanpur — Maximum Score: 30

Mid-semester Examination

1. Indicate whether following statements are true or false. Justify your answer (to justify a claim that a statement is true, a proof is required; to justify a claim that a statement is false, a single counterexample is sufficient.) **Note that no credit will be given to a correct guess without any explanation or followed by an incorrect justification. Also make sure that you say true or false before the justification to get credit.**

   (a) The set $([-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]) \cap \mathbb{Q}$ is open in $\mathbb{Q}$. **[2]**

   Solution: *True.* $A = ([-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]) \cap \mathbb{Q} = ((-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})) \cap \mathbb{Q}$ is open in $\mathbb{Q}$

   (b) Let X be an ordered set with order "$<_X$" and $d : X \times X \to [0, \infty)$ be a distance function. If (x_n) is a bounded and monotonically increasing sequence (that is, $x_n \leq_X x_{n+1}$, and $d(x_n, x) \leq_X C$, for some $C > 0$, $x \in X$, for all $n \in \mathbb{N}$), then (x_n) converges. **[2]**

   Solution: *False.* Consider $X = \mathbb{R}^2$ with the $<_X$ as dictionary order. The sequenced $((1 - 1/n, -n))$ is a bounded and monotonically increasing sequence but does not converge.

   (c) For $x \in \ell_1$ and $y \in \ell_\infty$, $\sum_{n=1}^{\infty} |x_n||y_n| \leq \|x\|_1 \|y\|_\infty$. **[2]**

   Solution: *True.* Follows from $|x_n||y_n| \leq |x_n|\|y\|_\infty$.

   (d) Every bounded sequence in a complete metric space X has a subsequence that converges to a point in X. **[2]**

   Solution: *False.* Consider the sequence $(e_n)_{n\in\mathbb{N}} \in \ell_2$ where

   $$e_n = (0, \ldots, \underbrace{1}_{n^{th} \text{ term}}, 0, \ldots),$$

   that is,

   $$e_n(k) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

   Clearly, $(e_n)_{n\in\mathbb{N}}$ is a bounded sequence in ℓ_2 but has no convergent subsequence as $d(e_m, e_n) = \sqrt{2}$ for all $m \neq n$.

(e) The Cantor set $C = \cap_{n=0}^{\infty} I_n$, where $I_0 = [0,1]$ and for $n \in \mathbb{N}$, $I_n = I_{n-1} \setminus J_{n-1}$ with $J_n = \cup_{k=1}^{3^n}((3k-2)3^{-(n+1)}, (3k-1)3^{-(n+1)})$ is compact. **[2]**

Solution: *True.* Clearly, C is a closed and bounded set of $\mathbb{R}$ and hence compact.

(f) A bounded set in $\mathbb{R}^k$, $k \geq 1$, is totally bounded. **[2]**

Solution: *True.* Consider a bounded set A in $\mathbb{R}^k$. Clearly, $\overline{A}$ is a closed and bounded ($\mathrm{diam}(\overline{A}) = \mathrm{diam}(A) < \infty$) in $\mathbb{R}^k$ and hence compact. Thus, $\overline{A}$ is totally bounded, that is, for every $\epsilon > 0$, there exist $x_1, x_2, \dots, x_n \in \mathbb{R}^k$ such that $\overline{A} \subseteq \cup_{i=1}^{n} N_\epsilon(x_i)$. Then, we also have

$$A \subseteq \cup_{i=1}^{n} N_\epsilon(x_i),$$

and, therefore, A is totally bounded.

2. (a) Let (X, d) be a metric space. Given a set $\tilde{X}$ and $f : \tilde{X} \to X$, a one-to-one function, let $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ be given by $\tilde{d}(x,y) = d(f(x), f(y))$. Then, show that $(\tilde{X}, \tilde{d})$ is a metric space. **[2]**

Solution:

$$\begin{aligned}
&\text{If } x \in \tilde{X}, \text{then } \tilde{d}(x,x) = d(f(x), f(x)) = 0\\
&\text{If } x, y \in \tilde{X}, x \neq y, \text{then } \tilde{d}(x,y) = d(f(x), f(y)) > 0 \text{ (as } f(x) \neq f(y))\\
&\text{If } x, y \in \tilde{X}, \text{then } \tilde{d}(x,y) = d(f(x), f(y)) = d(f(y), f(x)) = \tilde{d}(y,x)\\
&\text{If } x, y, z \in \tilde{X}, \text{then}\\
&\tilde{d}(x,y) = d(f(x), f(y)) \leq d(f(x), f(z)) + d(f(z), f(y)) = \tilde{d}(x,z) + \tilde{d}(z,y)
\end{aligned}$$

(b) Explain why is $(\mathbb{R}, d_0)$ with $d_0 : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ given by $d_0(x,y) = |\arctan x - \arctan y|$ a metric space. Show that $(\mathbb{R}, d_0)$ is not a complete metric space. **[1+2]**

Solution: The fact that d_0 is a metric follows from part (a) where $\tilde{X} = X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}, f(x) = \arctan x$ is a one-to-one function and $d(x,y) = |x - y|$.
Now, let, for $n \in \mathbb{N}$,

$$x_n = \tan\left(\frac{\pi}{2} - \frac{1}{n}\right).$$

Clearly,

$$d_0(x_m, x_n) = \left|\left(\frac{\pi}{2} - \frac{1}{m}\right) - \left(\frac{\pi}{2} - \frac{1}{n}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right|$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. Then, for $m, n \geq N$, we have

$$d_0(x_m, x_n) \leq \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{m} + \frac{1}{n} \leq \frac{1}{N} + \frac{1}{N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, (x_n) is a Cauchy sequence but does not converge (in fact, it is unbounded).

3. Let (X,d) be a metric space.

(a) Recall that $C \subseteq X$ is disconnected (not-connected) if there exists a pair $E, F \subseteq X$ such that $E \cup F = C$ and $\overline{E} \cap F = \emptyset$, $E \cap \overline{F} = \emptyset$; otherwise C is connected. Prove that C is connected if and only if a pair of nonempty sets A and B, both open in C, with $A \cap B = \emptyset$, $A \cup B = C$ does not exist. **[2+2]**

Solution: ($\Longrightarrow$) Suppose a pair of nonempty sets A and B, both open in C with $A \cap B = \emptyset$, $A \cup B = C$ exists. Let $E = C \setminus A$ and $F = C \setminus B$. Then E and F are closed in C and, therefore, $\mathrm{cl}_C(E) = E$, $\mathrm{cl}_C(F) = F$. We then have,

$$E \cup F = (C \cap A^c) \cup (C \cap B^c) = C \cap (A^c \cup B^c) = C \cap (A \cap B)^c = C \cap X = C, \tag{1}$$

and

$$\mathrm{cl}_C(E) \cap F = E \cap \mathrm{cl}_C(F) = E \cap F = (C \cap A^c) \cap (C \cap B^c) = C \cap (A \cup B)^c = \emptyset. \tag{2}$$

Note that

$$\overline{E} = \mathrm{cl}_X(E) = (\mathrm{cl}_X(E) \cap C) \cup (\mathrm{cl}_X(E) \cap C^c) = \mathrm{cl}_C(E) \cup (\mathrm{cl}_X(E) \cap C^c).$$

Similarly,

$$\overline{F} = \mathrm{cl}_C(F) \cup (\mathrm{cl}_X(F) \cap C^c).$$

Thus, we have

$$\overline{E} \cap F = (\mathrm{cl}_C(E) \cap F) \cup (\mathrm{cl}_X(E) \cap C^c \cap F) = \emptyset$$

(follows from (2) and the fact that $F \subseteq C$). Similarly,

$$E \cap \overline{F} = (E \cap \mathrm{cl}_C(F)) \cup (\mathrm{cl}_X(F) \cap C^c \cap E) = \emptyset$$

Thus, C is not connected (disconnected).

($\Longleftarrow$) Suppose C is not connected. Then, there exists a pair E, F such that $E \cup F = C$ and $\overline{E} \cap F = \emptyset$, $E \cap \overline{F} = \emptyset$. Consider $U = (\overline{E})^c = X \setminus \overline{E}$ and $V = (\overline{F})^c = X \setminus \overline{F}$. Clearly, U and V are open in X.

$$U \cap V = (\overline{E})^c \cap (\overline{F})^c = (\overline{E} \cup \overline{F})^c = (\overline{C})^c.$$

Also, if $x \in C$, then $x \in E$ or $x \in F$ which implies that $x \notin \overline{F}$ or $x \notin \overline{E}$, that is, $x \in (\overline{F})^c$ or $x \in (\overline{E})^c$. Thus,

$$C \subseteq (\overline{E})^c \cup (\overline{F})^c = U \cup V.$$

Clearly, $A = C \cap U$ and $B = C \cap V$ are open in C with $A \cap B = C \cap U \cap V = C \cap (\overline{C})^c = \emptyset$, $A \cup B = (C \cap U) \cup (C \cap V) = C \cap (U \cup V) = C$.

(b) If C is a connected subset of X and if A and B are disjoint non-empty open sets in X with $C \subseteq A \cup B$, prove that $C \subseteq A$ or $C \subseteq B$. **[2]**

Solution: We can always write $C = (C \cap A) \cup (C \cap B)$. Since A and B are open in X, the sets $C \cap A$ and $C \cap B$ are open in C. If C is connected, then one of those sets has to be empty. But if $C \cap A = \emptyset$, then $C \cap B = C$, that is, $C \subseteq B$. Similarly, if $C \cap B = \emptyset$, then $C \subseteq A$.

(c) If E and F are connected subsets of X with $E \cap F \neq \emptyset$, then show that $E \cup F$ is connected. **[2]**

Solution: Suppose $E \cup F$ is not connected. Then, there exist non-empty open sets A and B such that

$$E \cup F = A \cup B, \tag{3}$$
$$A \cap B = \emptyset. \tag{4}$$

Part (a) and (3) imply that

$$E \subseteq A \text{ or } F \subseteq B, \text{ and}$$
$$F \subseteq A \text{ or } F \subseteq B.$$

But $E \subseteq A$ and $F \subseteq B$ or $E \subseteq B$ and $F \subseteq A$ contradicts $E \cap F \neq \emptyset$ (because of (4)) whereas $E \subseteq A$ and $F \subseteq A$ or $E \subseteq B$ and $F \subseteq B$ contradicts (3).

4. Consider $\mathbb{R}$ with the standard metric. Let $A \subseteq \mathbb{R}$ be a subset which has no limit points.

(a) Construct a map $f : A \to \mathbb{Q}$ such that f is one-to-one and then show that it is indeed one-to-one. **[2+2]**

Solution: Fix $a \in A$. Since a is an isolated point, there exists an $r_a > 0$ such that $N_{r_a}(a) \cap A = \{a\}$. Now, pick an element $q_a \in \mathbb{Q} \cap N_{r_a/2}(a)$. Doing this for each $a \in A$ defines a function

$$f : A \to \mathbb{Q}, \quad f(a) = q_a.$$

Now we show that f is one-to-one. Suppose $f(a) = f(b)$ and let $q = f(a)$. Then, $q = q_a = q_b \in N_{r_a/2}(a) \cap N_{r_b/2}(b)$. Thus,

$$d(a, b) \leq d(a, q) + d(q, b) \leq r_a/2 + r_b/2 \leq \max\{r_a, r_b\},$$

and therefore, either $a \in N_{r_b}(b)$ or $b \in N_{r_a}(a)$ resulting in $a = b$.

(b) Explain why is A at most countable. **[1]**

Solution: From part (a), we see that $A \sim \text{Range}(f) = \{q \in \mathbb{Q} \ : \ \text{there is } a \in A \text{ such that } f(a) = q\} \subseteq \mathbb{Q}$. As subsets of a countable set is countable, we have that A is countable.