Numerical Analysis & Scientific Computing II

Lesson 2
Initial Value Problems

2.2 Stability

2.3 Euler's method

2.4 Implicit method

- Trapezoidal method

Akash Anand
MATH, IIT KANPUR

How do we obtain a higher-order method?

Initial Value Problems: Implicit Methods

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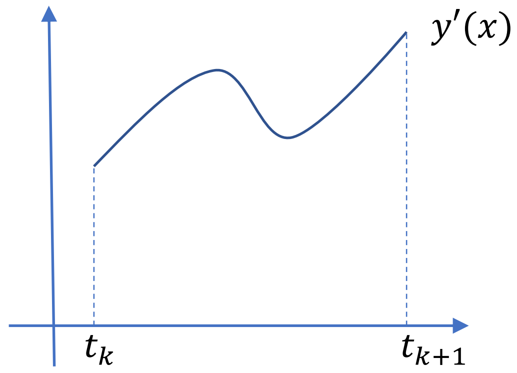

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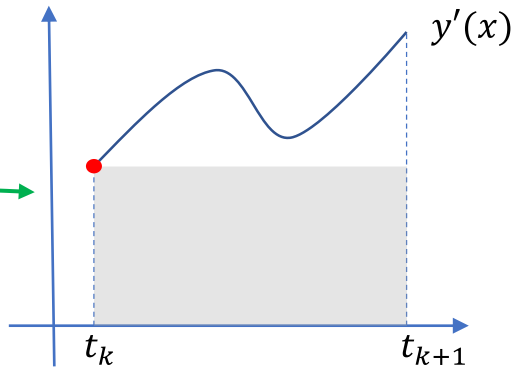

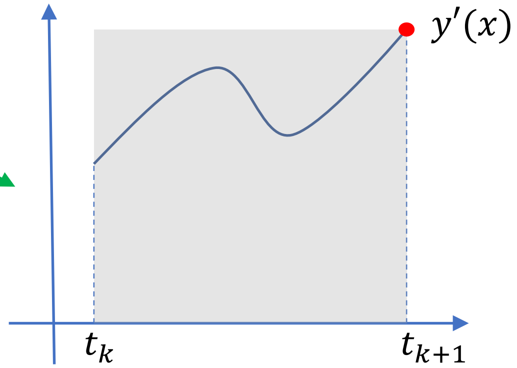

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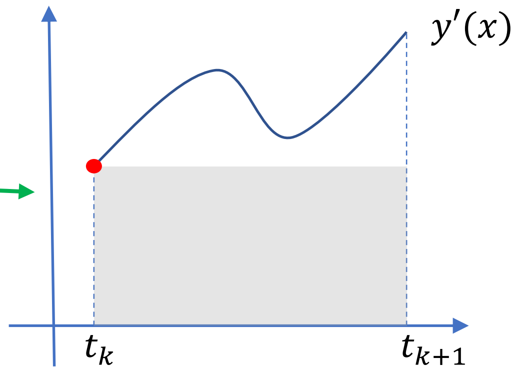

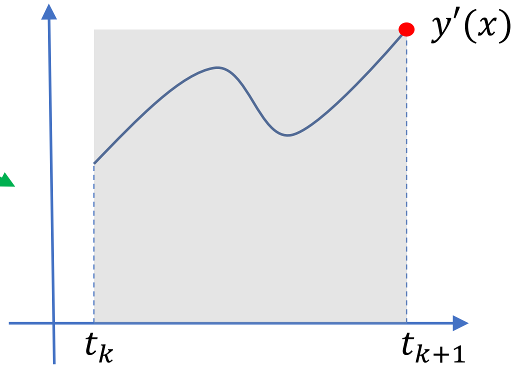

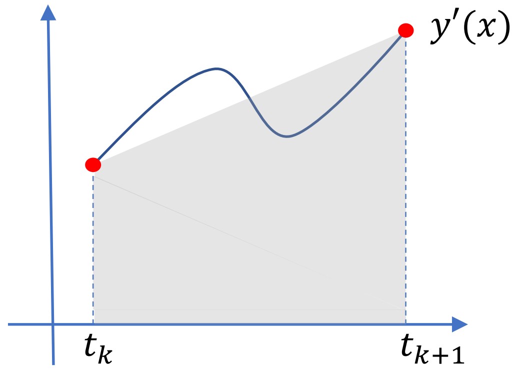

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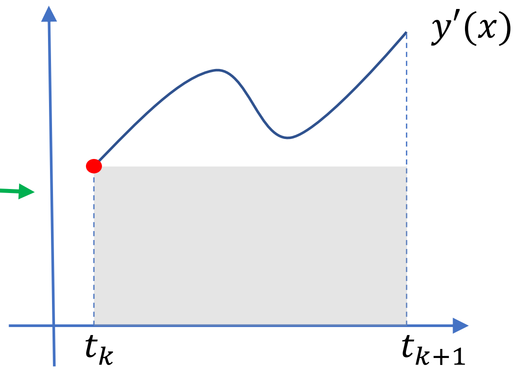

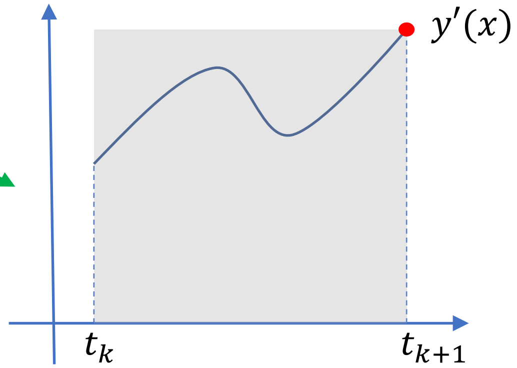

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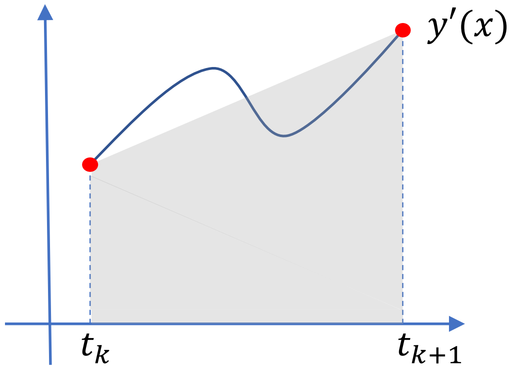

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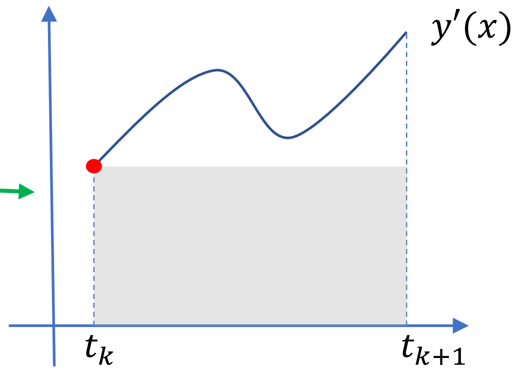

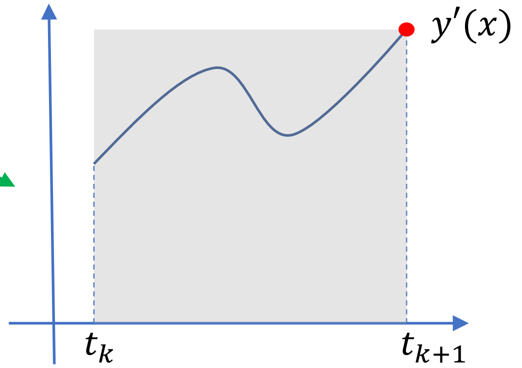

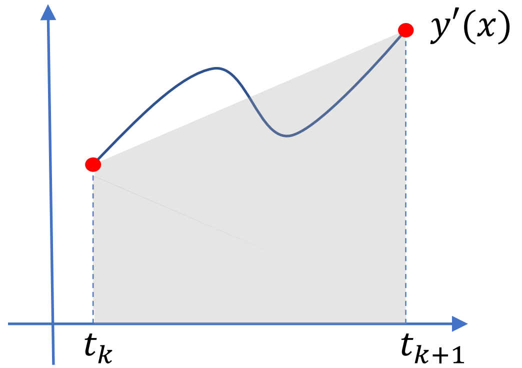

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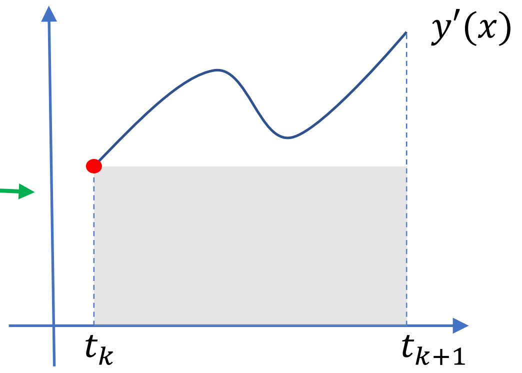

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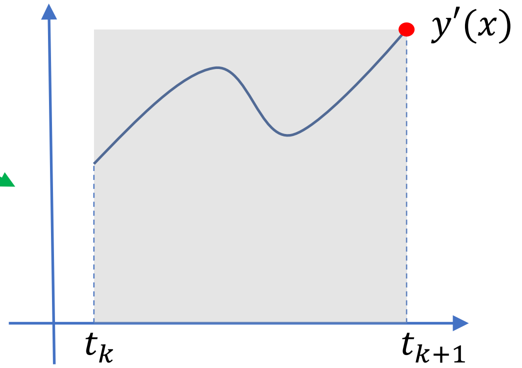

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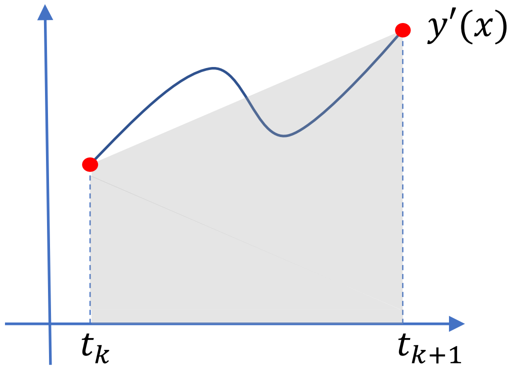

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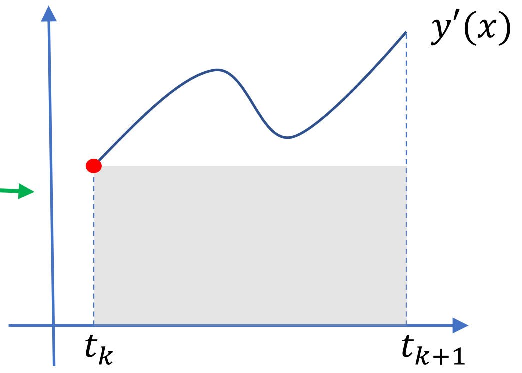

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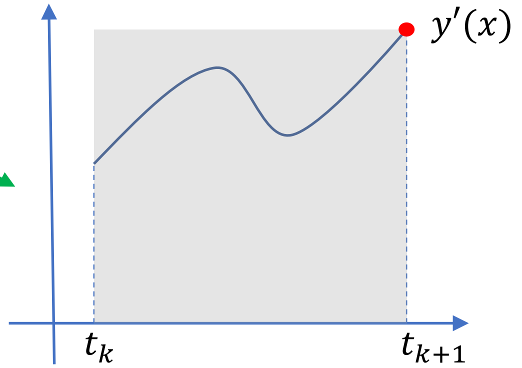

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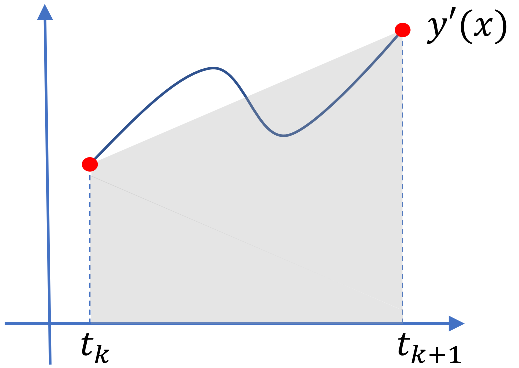

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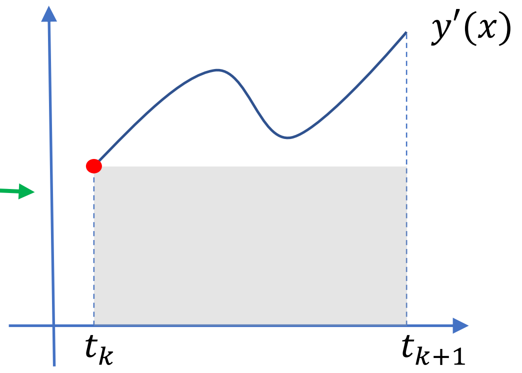

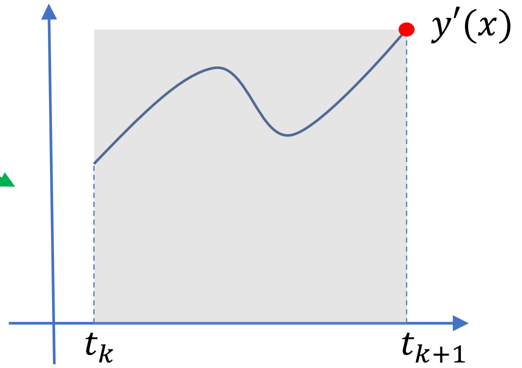

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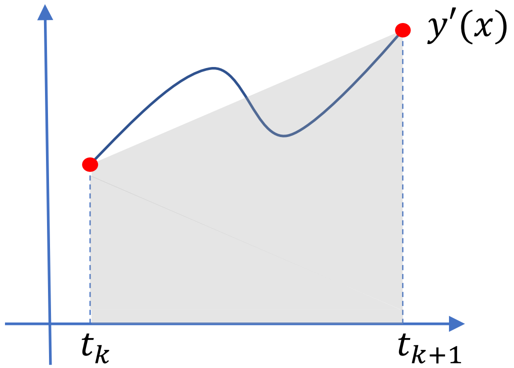

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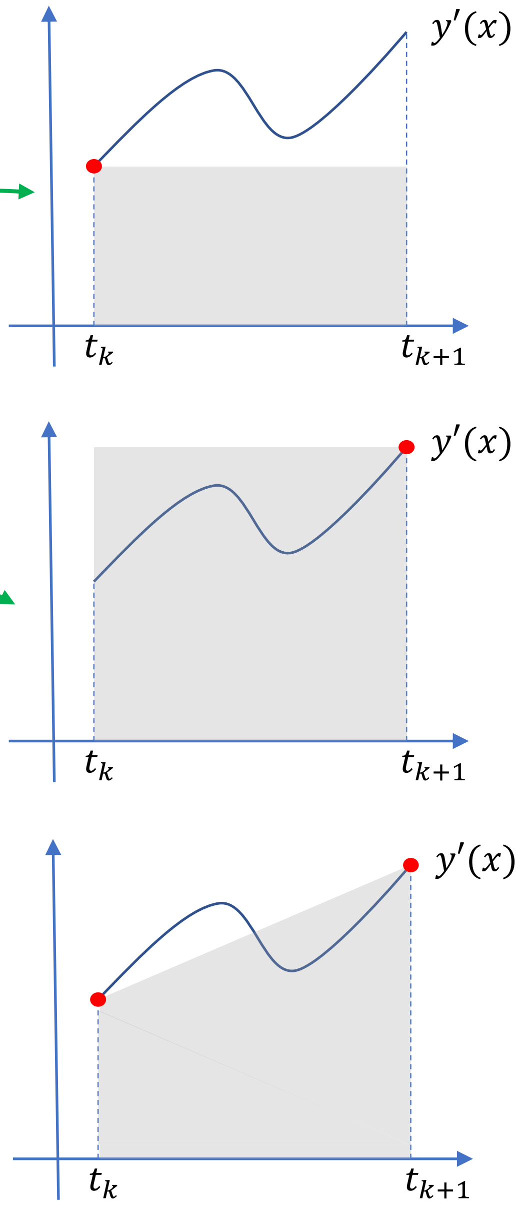

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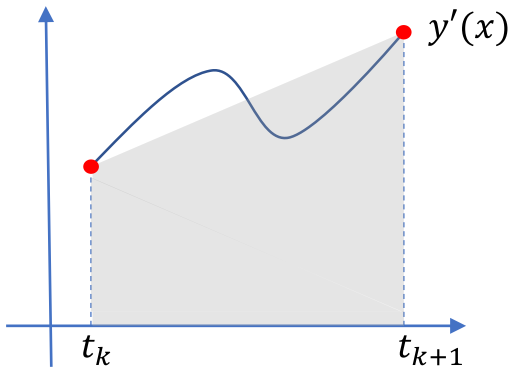

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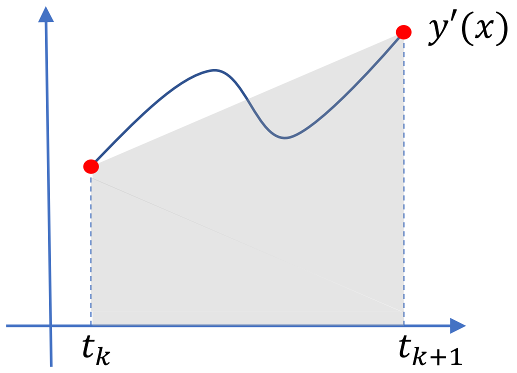

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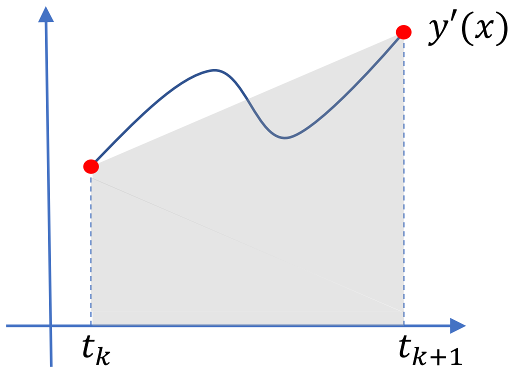

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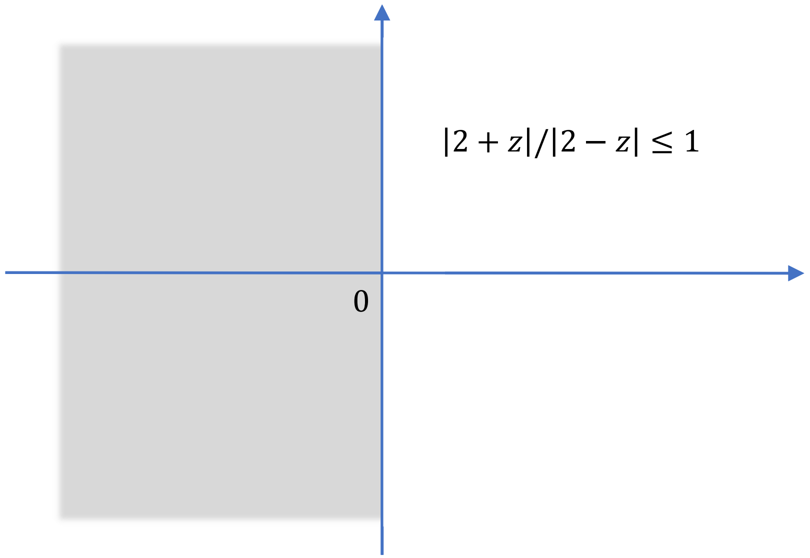

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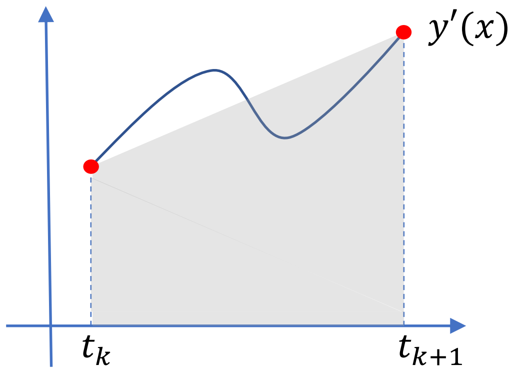

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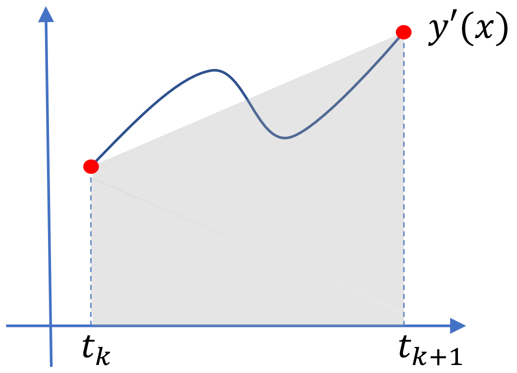

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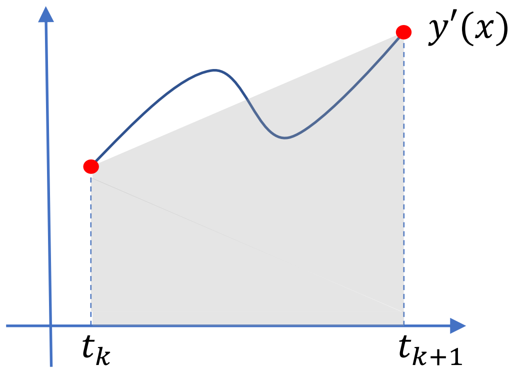

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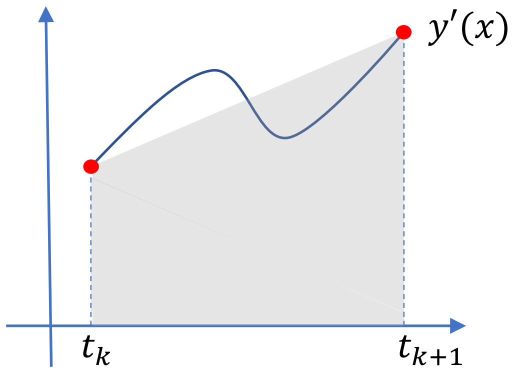

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$$= h_k\big(f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))\big)/2 - (y(t_{k+1}) - y(t_k))/2 + \big(y(t_k) - y(t_{k+1})\big)/2$$

$$= h_k\big(f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))\big)/2 - \left(h_k y'(t_k) + h_k^2 y''(t_k)/2 + O(h_k^3)\right)/2$$

$$+ \left(-h_k y'(t_{k+1}) + h_k^2 y''(t_{k+1})/2 + O(h_k^3)\right)/2 \quad = \ O(h_k^3)$$

Initial Value Problems: Implicit Methods

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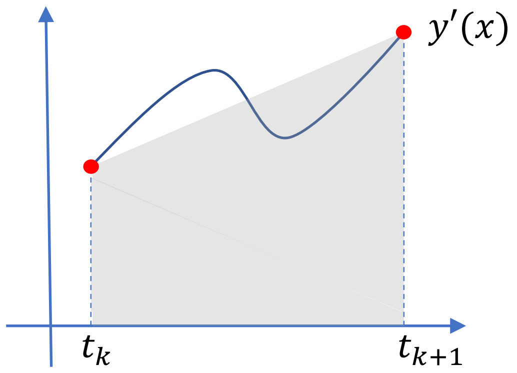
Approximating the integral by trapezoidal rule, we get

$$y(t_{k+1}) = y(t_k) + h_k\big(f(t_k, y_k) + f(t_{k+1}, y_{k+1})\big)/2$$



To study the stability, we apply the method to $y' = \lambda y$:

$$y_{k+1} = y_k + \lambda h_k (y_k + y_{k+1})/2$$

$$(1 - \lambda h_k/2) y_{k+1} = (1 + \lambda h_k/2) y_k$$

$$y_{k+1} = y_k\big((1 + \lambda h_k/2)/(1 - \lambda h_k/2)\big)$$

$$y_k = \big((1 + \lambda h_k/2)/(1 - \lambda h_k/2)\big)^k y_0$$

For stability, we need $|(1 + \lambda h_k/2)/(1 - \lambda h_k/2)| \le 1.$

For order of accuracy, we look at the local error

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$$+ \left(-h_k y'(t_{k+1}) + h_k^2 y''(t_{k+1})/2 + O\left(h_k^3\right)\right)/2 \quad = \quad O\left(h_k^3\right)$$

second-order accurate!

Initial Value Problems: Implicit Methods

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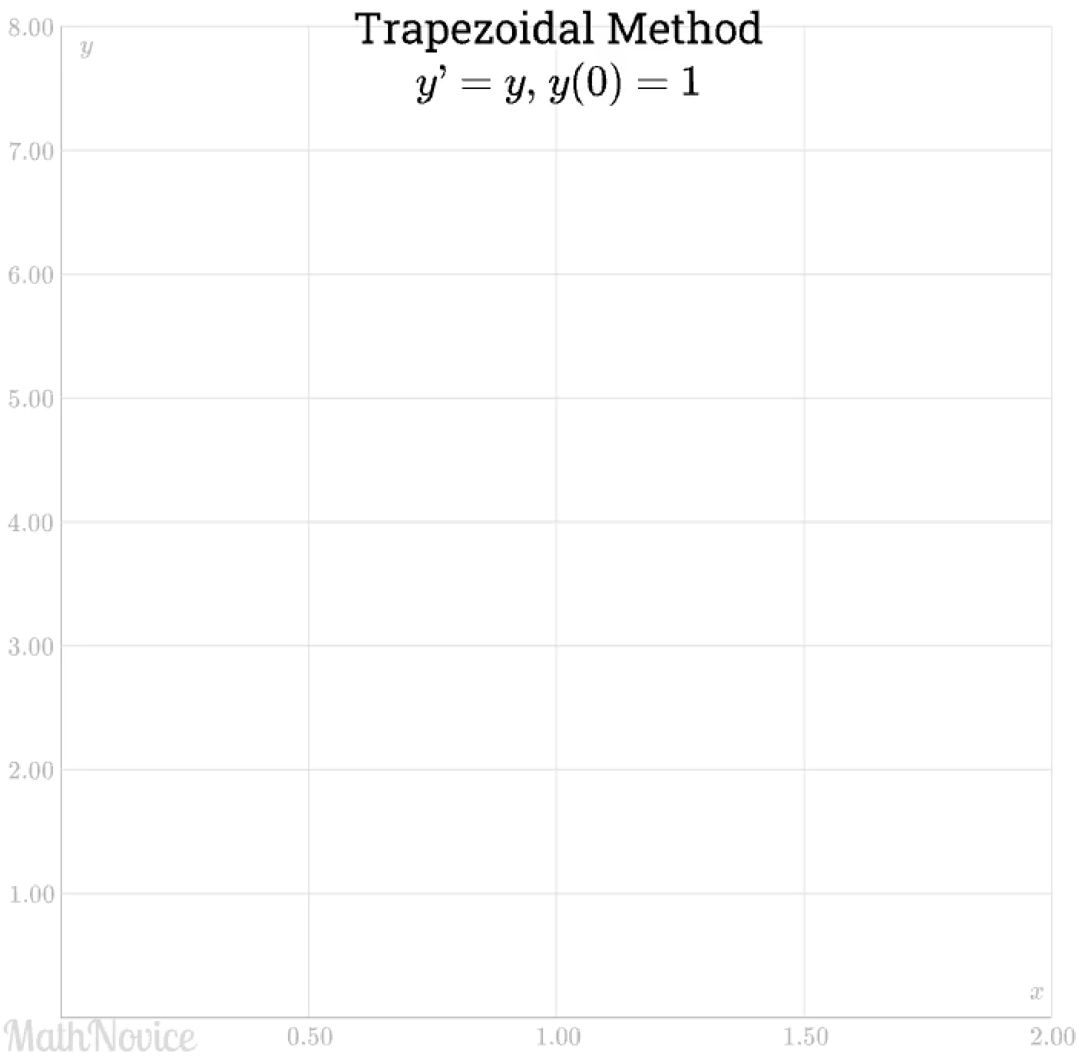

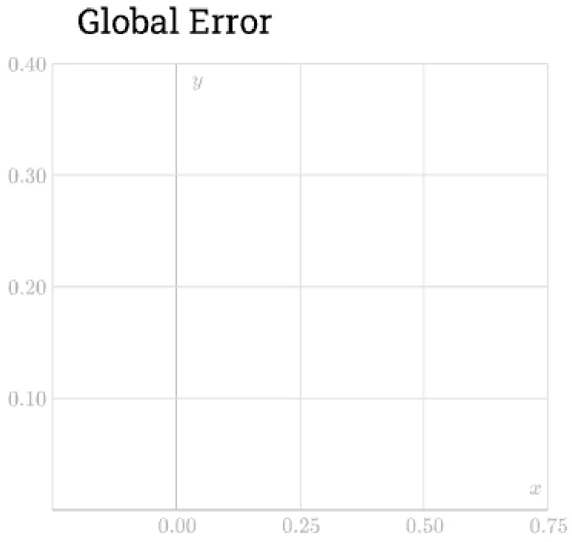

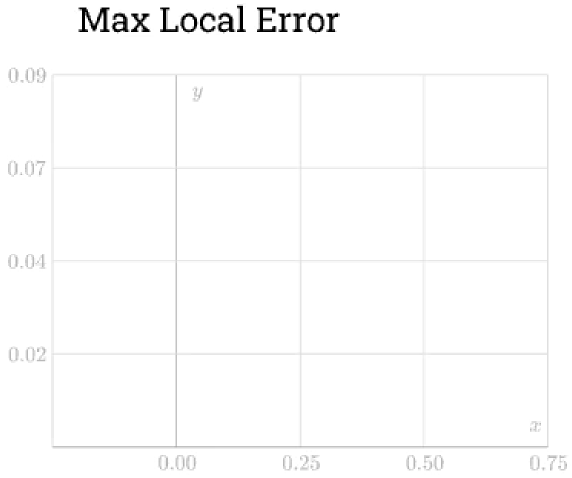

Source: https://www.youtube.com/watch?v=kcgtbXgDNE8

Initial Value Problems: Implicit Methods

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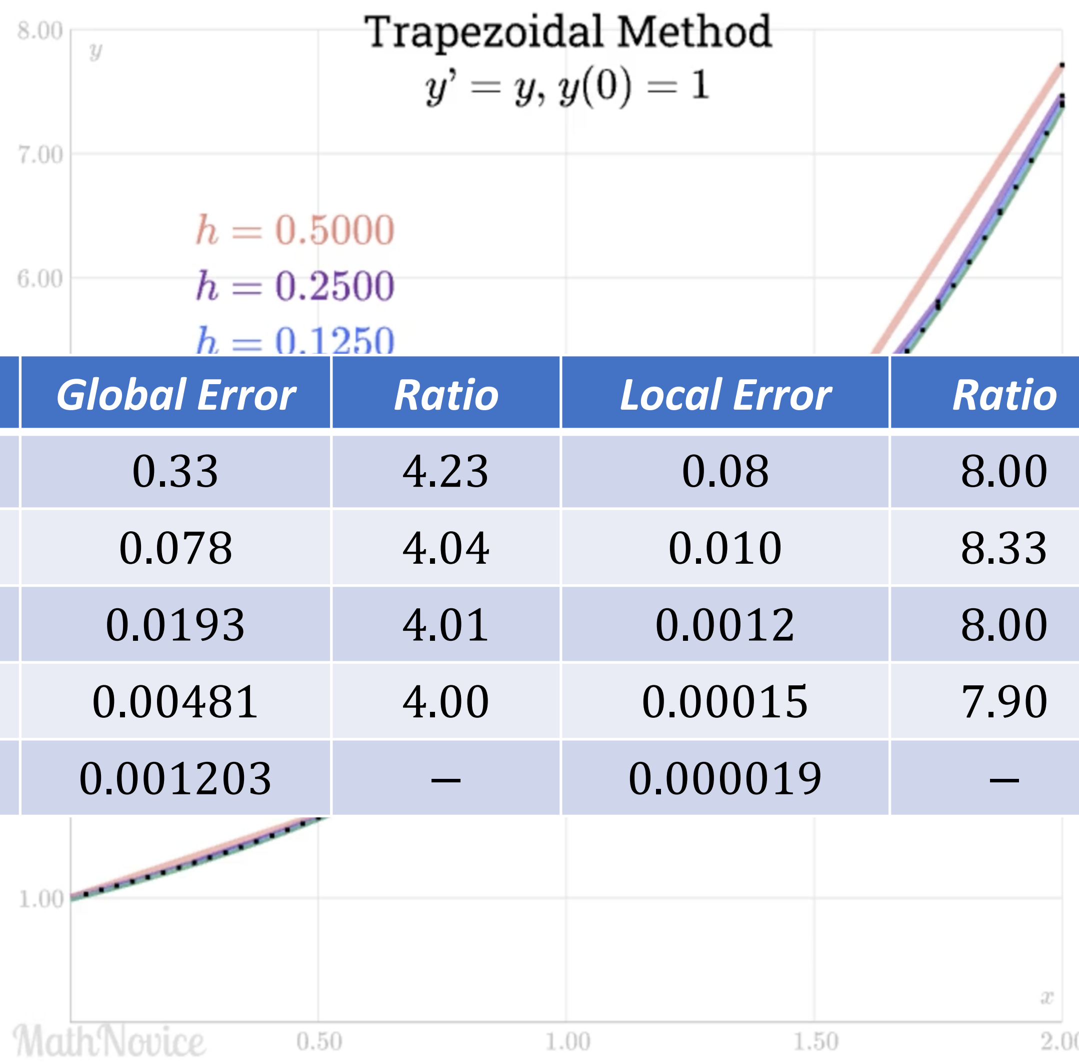
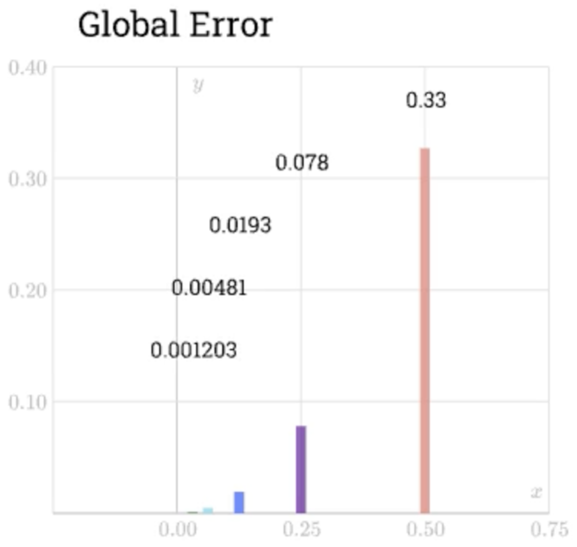


h	Global Error	Ratio	Local Error	Ratio
1/2	0.33	4.23	0.08	8.00
1/4	0.078	4.04	0.010	8.33
1/8	0.0193	4.01	0.0012	8.00
1/16	0.00481	4.00	0.00015	7.90
1/32	0.001203	–	0.000019	–



Module 2

Initial Value Problems

2.3 Euler's method

2.4 Implicit method

2.5 Stiffness

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Initial Value Problems: Stiffness

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Example

Consider the following IVP, $y' = f(t, y),\ y(0) = y_0$, *where*

$$\mathrm{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad f(t, y) = \begin{bmatrix} -1 & 0 \\ 0 & -100 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \textit{and} \quad y_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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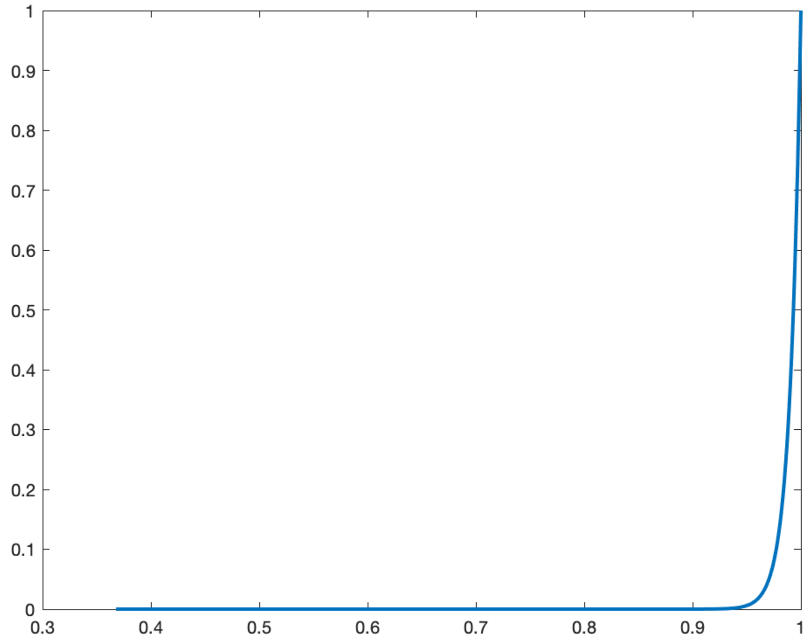
Exact solution



Initial Value Problems: Stiffness

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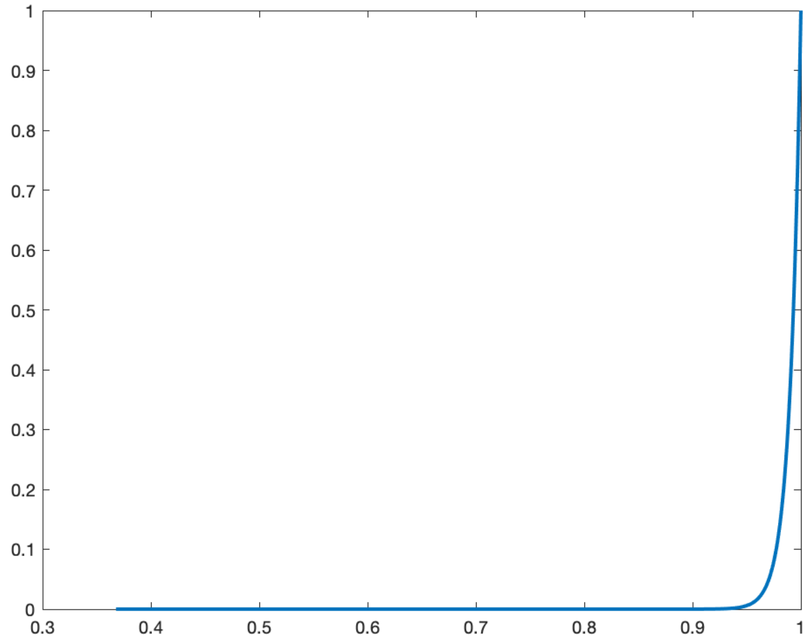

Euler's method with $h = 0.04$

Initial Value Problems: Stiffness

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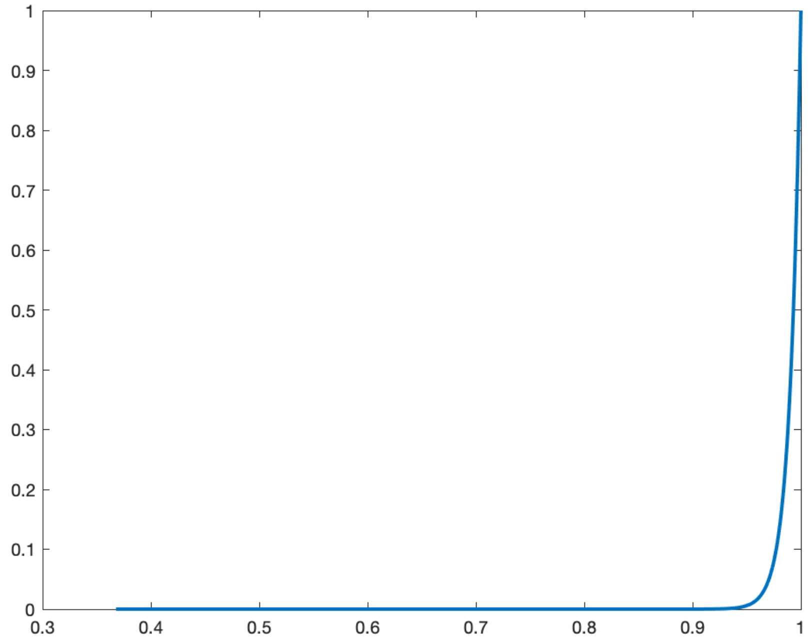

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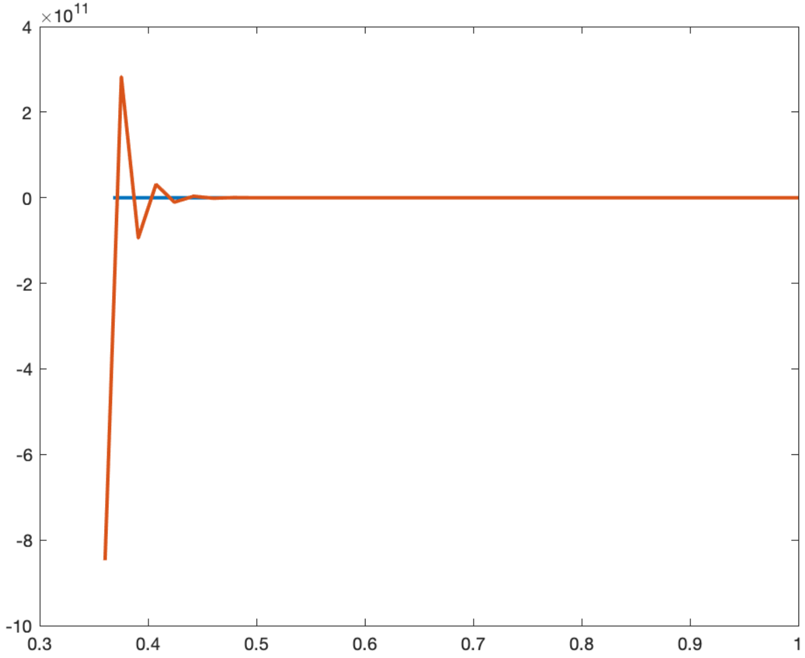

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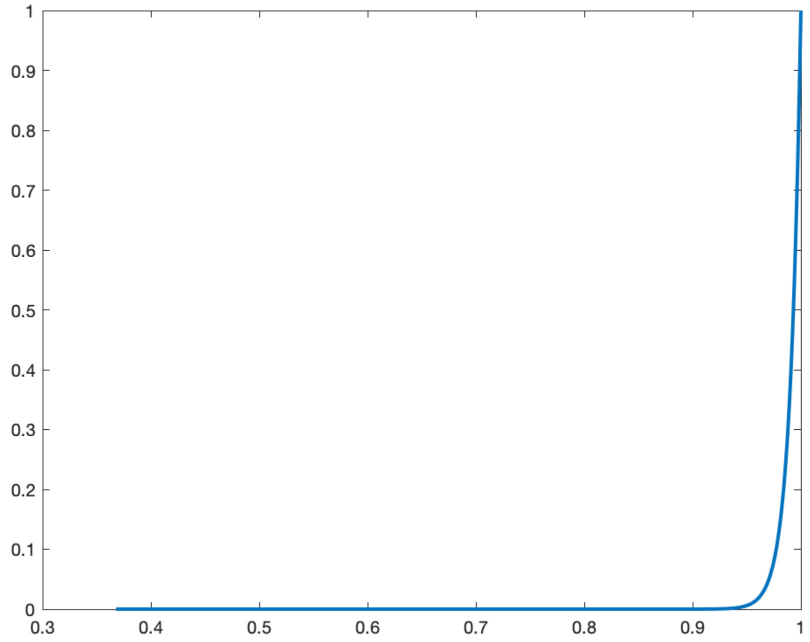
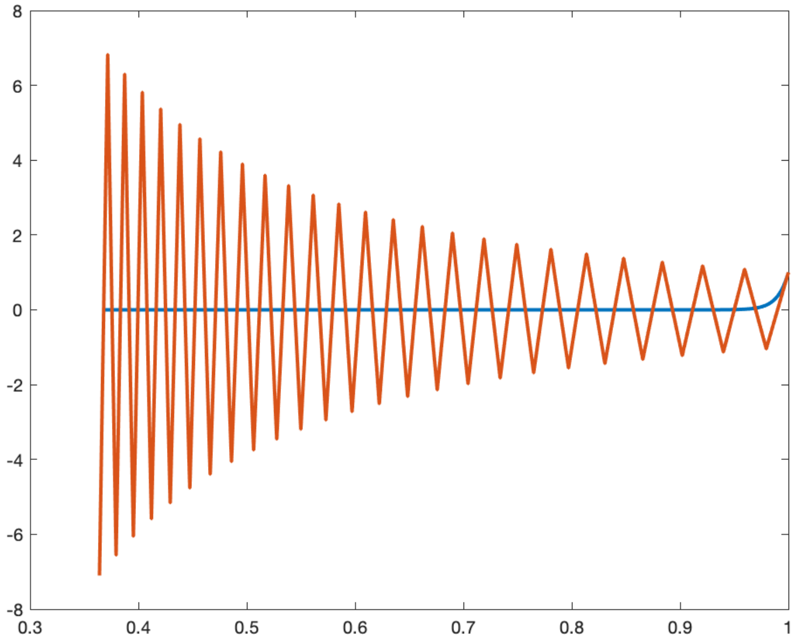


Euler's method with $h = 0.0204$



Initial Value Problems: Stiffness

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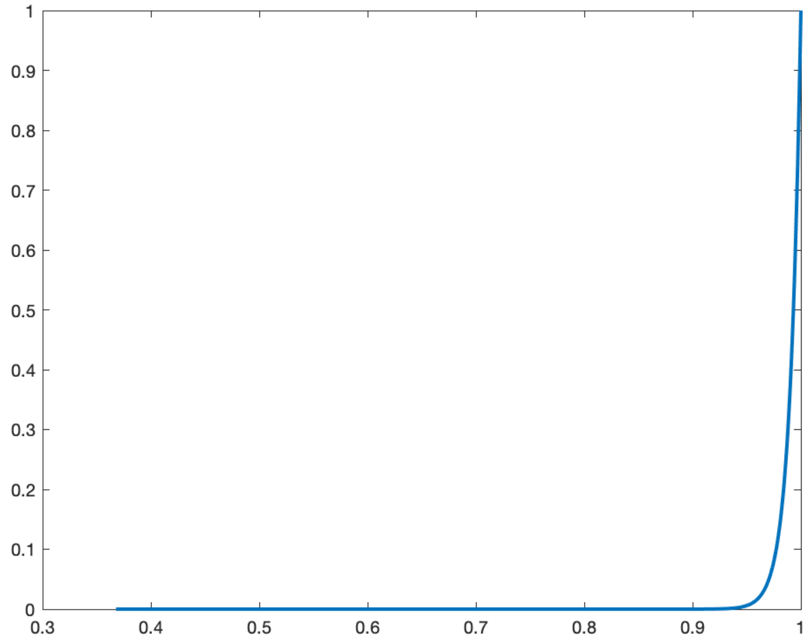
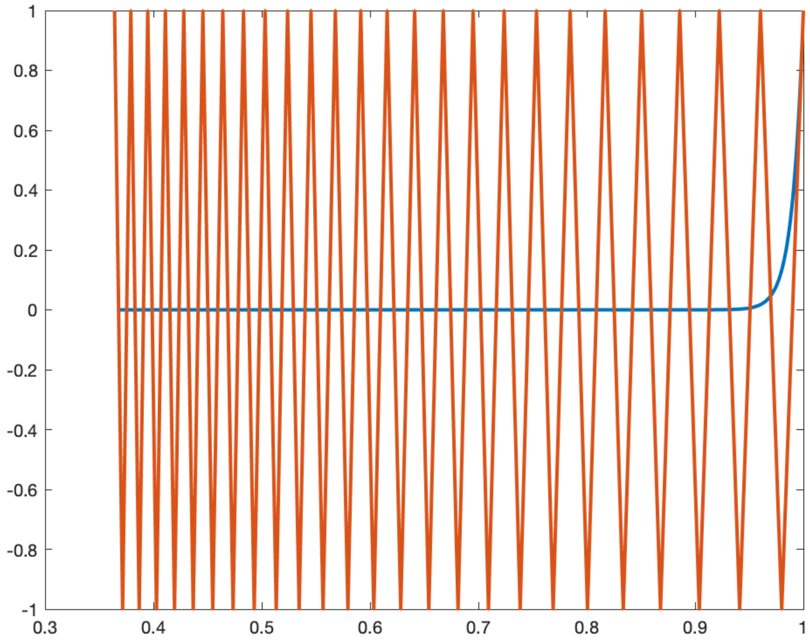


Euler's method with $h = 0.02$



Example

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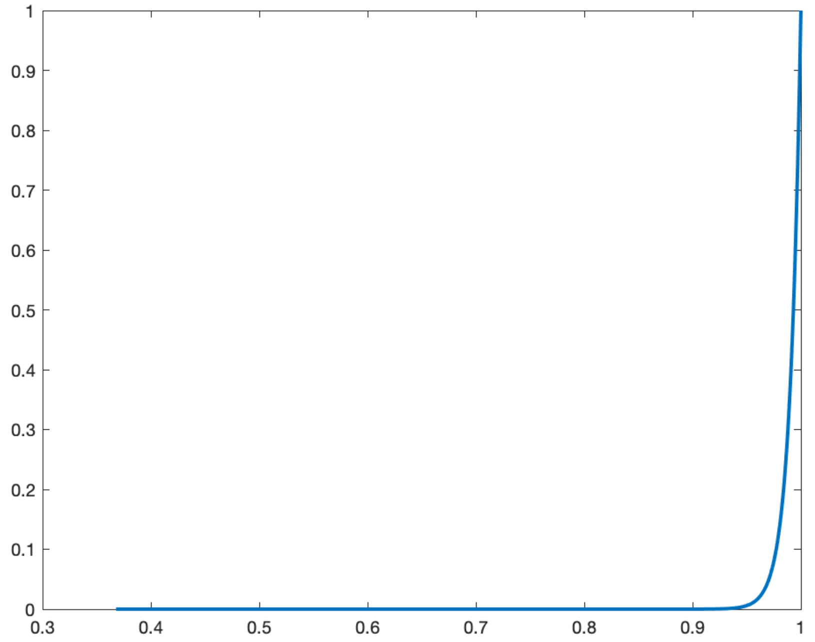
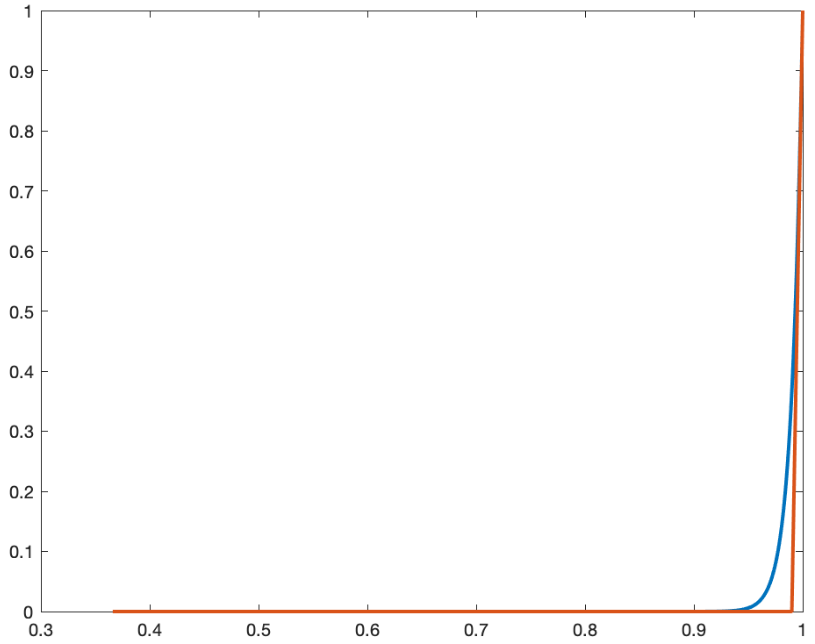


Euler's method with $h = 0.01$



Initial Value Problems: Stiffness

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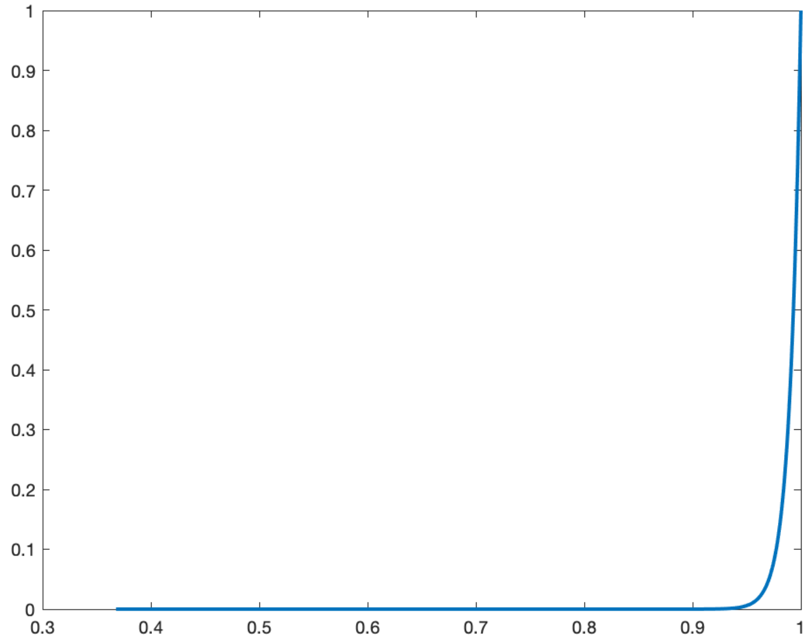
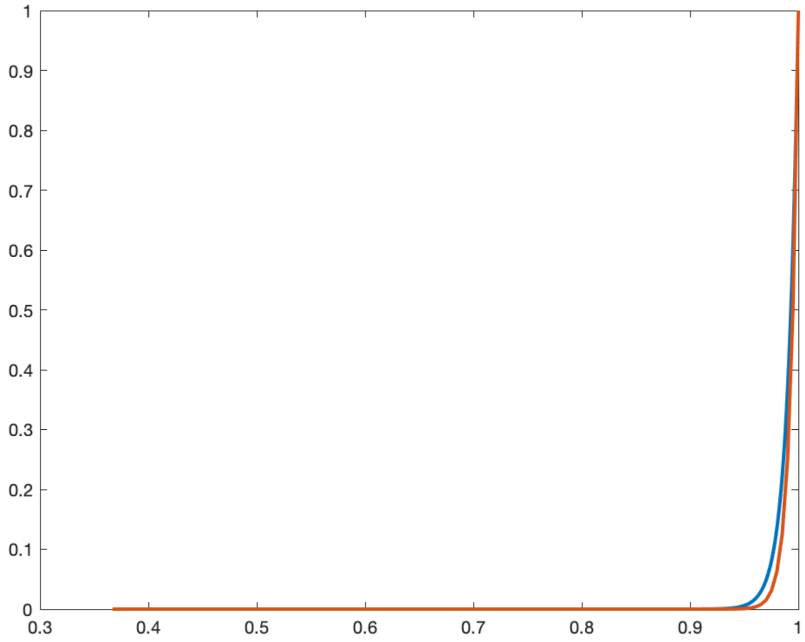


Euler's method with $h = 0.005$



Initial Value Problems: Stiffness

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How do we explain this?

Initial Value Problems: Stiffness

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How do we explain this?

$$f'(t, y) = \begin{bmatrix} -1 & 0 \\ 0 & -100 \end{bmatrix}, \quad I + hf' = \begin{bmatrix} 1 - h & 0 \\ 0 & 1 - 100h \end{bmatrix}$$

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For $-1 \leq 1 - 100h \leq 1$, *we need* $h \leq 1/50 = 0.02.$

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Consider the following IVP, $y' = f(t,y),\ y(0) = y_0$, *where*

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Note that, for solving the first equation $y_1' = -y_1$ *only, the Euler's method is stable for all* $h \le 2$.

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Note that, for solving the first equation $y_1' = -y_1$ *only, the Euler's method is stable for all* $h \le 2.$

The more severe restriction $h \le 0.02$ *is primarily due to the second equation* $y_2' = -100y_2$ *which governs the component that varies much more rapidly than the first component* y_1.

Initial Value Problems: Stiffness

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MATH, IIT KANPUR

Stiffness

A stable ODE $y' = f(t, y)$ is stiff if the Jacobian matrix f' has eigenvalues that differ greatly in magnitude.

There may be eigenvalues with relatively large negative real parts (corresponding to strongly damped components of the solution) or relatively large imaginary parts (corresponding to rapidly oscillating components of the solution).

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Going back to the previous example and using backward Euler, we see that …

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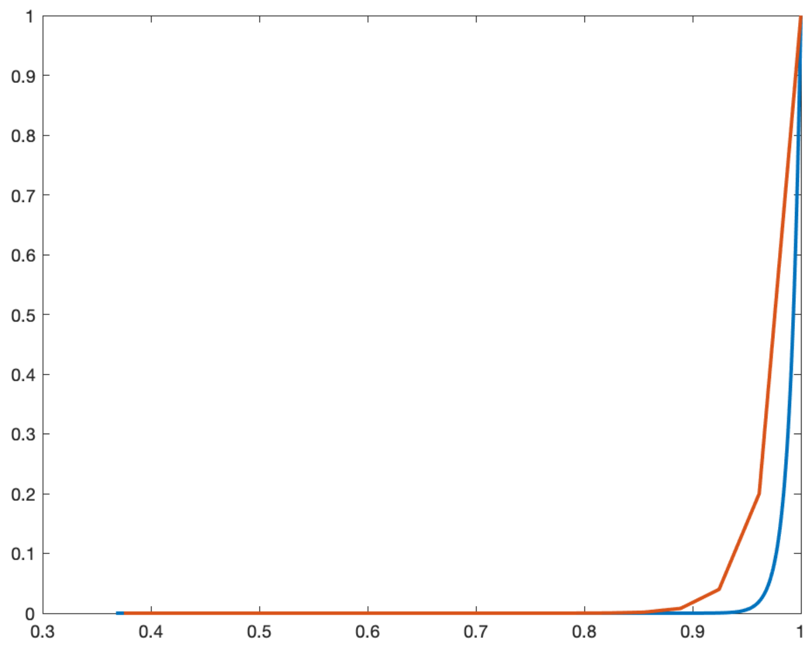
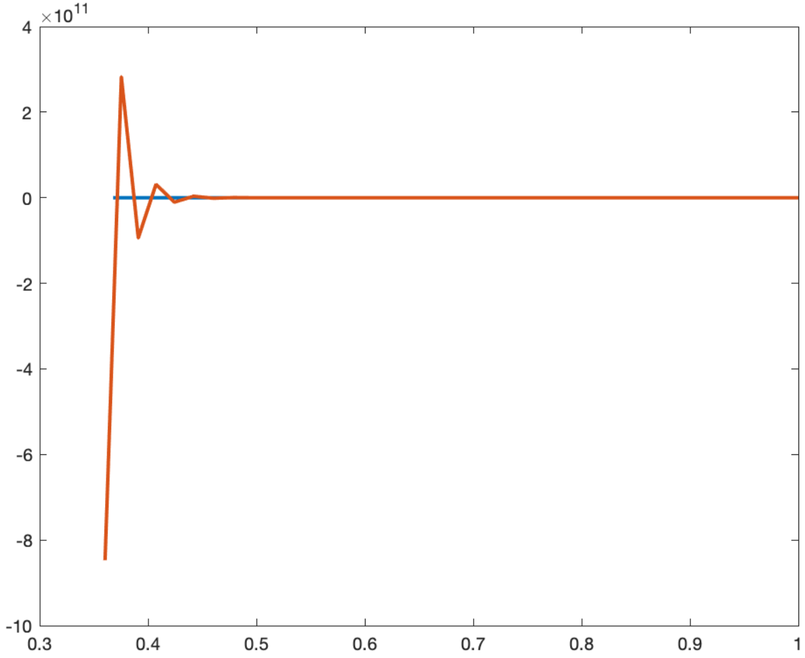
Backward Euler with $h = 0.04$



Euler's method with $h = 0.04$



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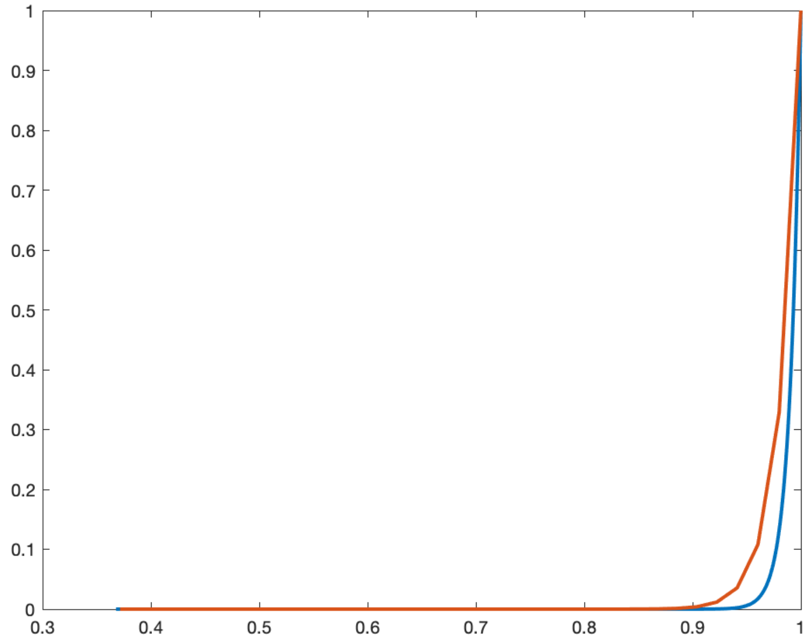
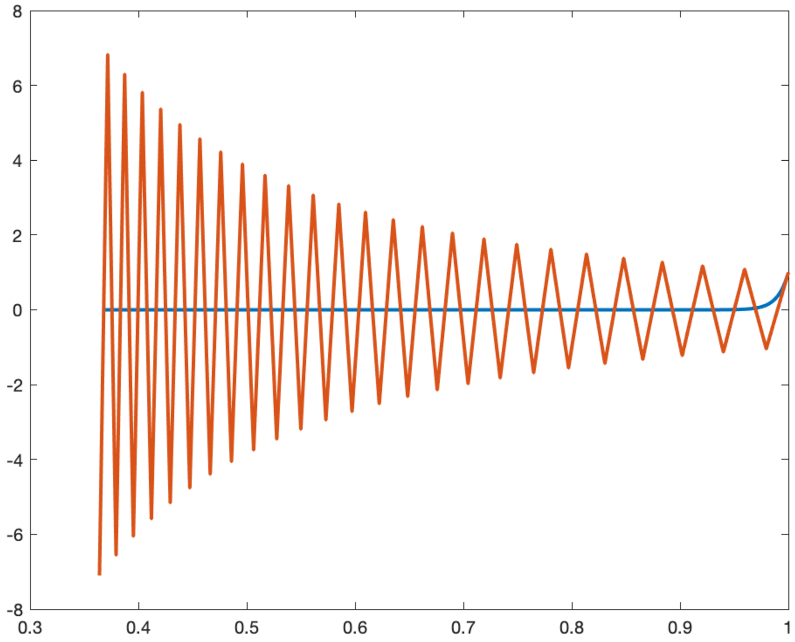


Euler's method with $h = 0.0204$



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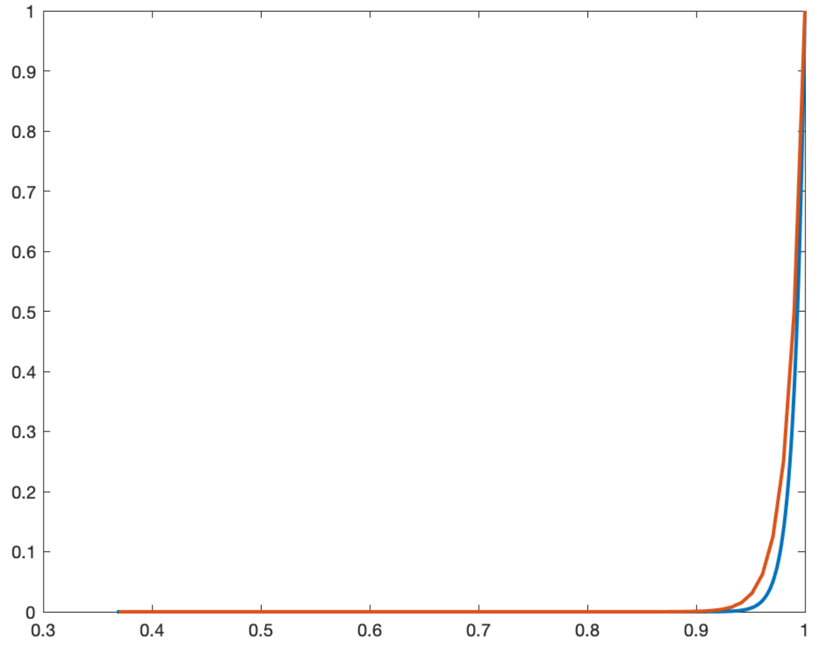
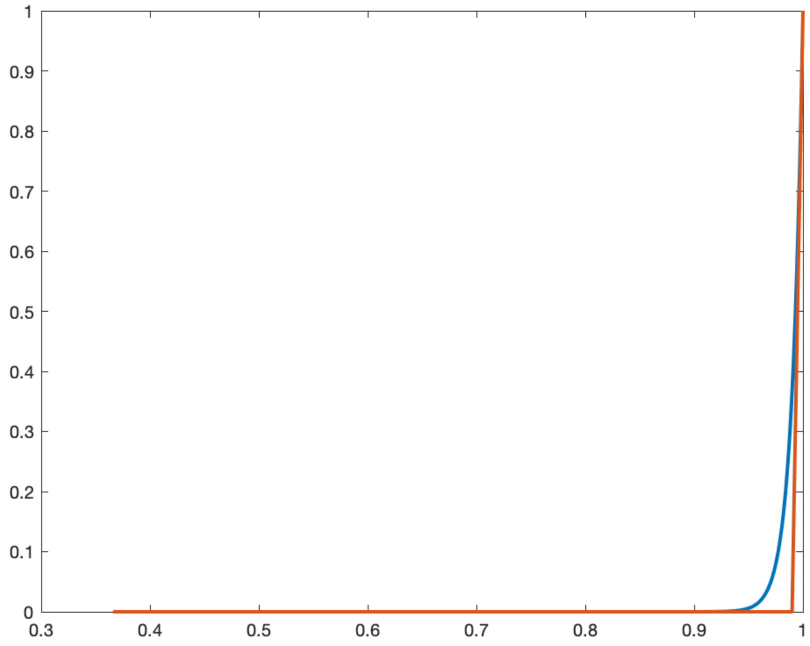
Backward Euler with $h = 0.01$



Euler's method with $h = 0.01$



Example

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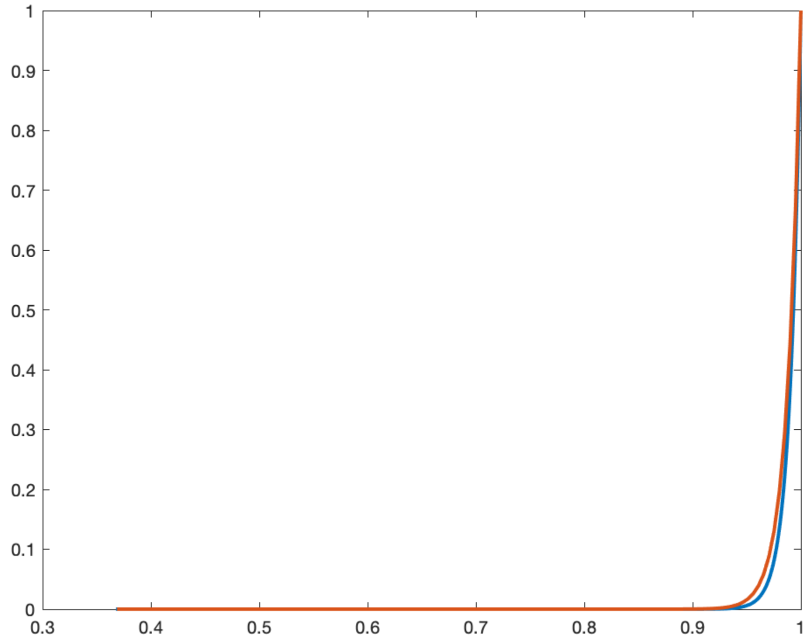
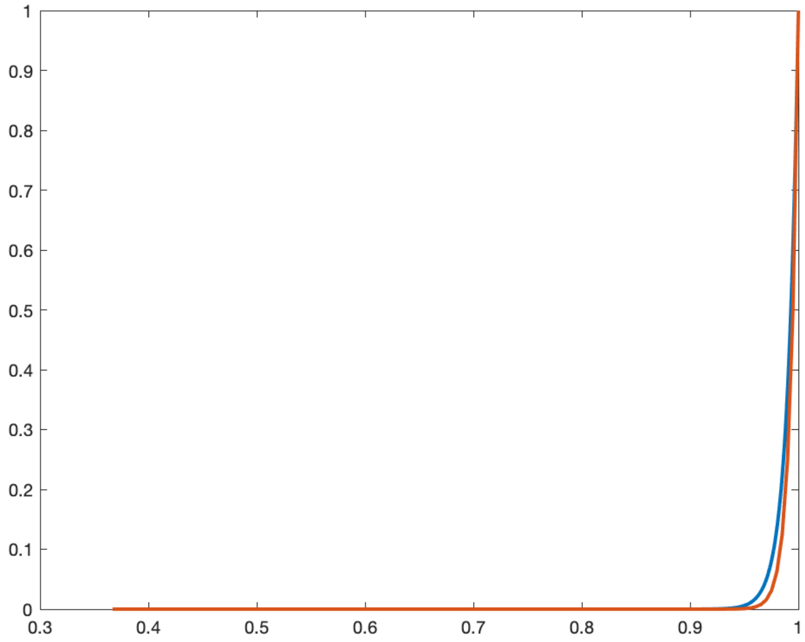
Backward Euler with $h = 0.005$



Euler's method with $h = 0.005$



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… is stable throughout. This is consistent with the unconditional stability of the method.

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Therefore, implicit methods are required for stiff ODEs. Thus, a nonlinear (generally) equation must be solved at each step. For fixed point iterations to converge, step size h must be small which defeats the purpose of using implicit method at the first place. As a result, Newton's method or its variants are used.