Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)
Exercise Sheet 9

1. Conformal equivalence

1.1. Let $\alpha \in [0,1]$ and $\mathbb{D}_\alpha \overset{\text{def}}{=} \mathbb{D} \setminus [\alpha, 1]$.

(a) Show that $\mathbb{D}_\alpha$ is conformally equivalent to $\mathbb{D}_0$.

Hint. *Would some automorphism of $\mathbb{D}$ be of any help?*

(b) Is $\mathbb{D}_\alpha$ conformally equivalent to the upper-half of the unit disc $\mathbb{D}^+ \overset{\text{def}}{=} \mathbb{D} \cap \mathbb{H}$?

Hint. *The map $z \mapsto z^2$ may be useful.*

1.2. Let $f(z) \overset{\text{def}}{=} \exp(2\pi i z)$, for all $z \in \mathbb{H}$.

(a) Show that $f(\mathbb{H}) \subseteq \mathbb{D} \setminus \{0\}$.
(b) For $r > 0$, find the image of $\{z \in \mathbb{H} : \operatorname{Im} z > r\}$.
(c) Is $\{z \in \mathbb{H} : \operatorname{Im} z > r\}$ conformally equivalent to its image under f? If not, what needs to be done so as to obtain a conformal equivalence?

1.3. In each of the following, exhibit a bijective holomorphic map between the given subsets:

(a) The first quadrant $\{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ and $\mathbb{H}$.
(b) The quarter disc $\{z \in \mathbb{D} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ and $\mathbb{D}^+$.
(c) $\mathbb{D}^+$ and the first quadrant $\{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$.
(d) The quarter disc $\{z \in \mathbb{D} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ and $\mathbb{H}$.
(e) $\mathbb{D}^+$ and the half strip $\{z \in \mathbb{C} : \operatorname{Re} z < 0, 0 < \operatorname{Im} z < \pi\}$.
(f) $\mathbb{H}$ and the strip $\{z \in \mathbb{H} : 0 < \operatorname{Im} z < \pi\}$.
(g) $\{z \in \mathbb{D} : \operatorname{Re} z > 0\}$ and $\mathbb{D}$.
(h) $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$ and $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$, where r_1, r_2, R_1 and $R_2 > 0$ and $\frac{r_1}{r_2} = \frac{R_1}{R_2}$.

1.4.∗ (a) Let $\alpha \in [0, \pi]$. Show that $\mathbb{H} \setminus \left\{e^{it} : t \in [0, \alpha]\right\}$ is conformally equivalent to $\mathbb{H} \setminus \left\{it : 0 \leq t \leq \frac{1}{2} \tan \frac{\alpha}{2}\right\}$.
(b) Let $\beta \geq 0$. Show that $\{z \in \mathbb{C} : \operatorname{Re} z > 0\} \setminus [0, \beta]$ is confonrmally equivalent to $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Hint. *What is the image of $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ under the map $z \mapsto z^2$? Can you use an analytic square root function?*

(c) Show that, for any $a > 0$, $\mathbb{H} \setminus \{it : 0 \leq t \leq a\}$ is conformally equivalent to the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

1.5. Show that the following map establishes a conformal equivalence between $\{z \in \mathbb{H} : |z| > 1\}$ and $\mathbb{H}$:

$$f : \{z \in \mathbb{H} : |z| > 1\} \longrightarrow \mathbb{H}, \; f(z) \overset{\text{def}}{=} z + \frac{1}{z}.$$

2. Families of analytic functions

Recall that, for $U \subseteq_{open} \mathbb{C}$, one has $U = \bigcup_{n=1}^{\infty} K_n$, where

$$K_n \overset{\text{def}}{=} \overline{D(0;n)} \cap \left\{ z \in U : |w - z| \geq \frac{1}{n}, \; \forall w \in \mathbb{C} \setminus U \right\}.$$

These compact sets K_n's have the following properties:

(i) For all $n \in \mathbb{N}$, K_n is contained in the interior of K_{n+1}.

(ii) For every compact subset K of U, there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$.

Let $C(U)$ denote the set of all complex valued continuous functions on U. For $f, r \in C(U)$, define

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}} \right), \tag{2.1}$$

where for any $n \in \mathbb{N}$,

$$\|f-g\|_{K_n} \stackrel{\text{def}}{=} \begin{cases} \sup_{z \in K_n} |f(z)-g(z)| & \text{if } K_n \neq \emptyset \\ 0 & \text{if } K_n = \emptyset \end{cases}.$$

2.1. Show that d, defined as above in (2.1), is a metric on $C(U)$.

Hint. *If $a, b, c \geq 0$ with $a \leq b + c$ then $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$. This can be seen considering the function $\frac{x}{1+x}$, for all $x \geq 0$.*

2.2. Show that the metric d on $C(U)$ is bounded.

From now on, unless otherwise mentioned, $C(U)$ is will always be endowed with the metric d.

2.3.∗ Let $\{f_n\}_{n=1}^{\infty}$ in $C(U)$ be a sequence in $C(U)$.

(a) Show that $\{f_n\}_{n=1}^{\infty}$ is convergent (with respect to the metric d) if and only if it is uniformly convergent on each compact subset of U.
(b) Show that $\{f_n\}_{n=1}^{\infty}$ is Cauchy (with respect to the metric d) if and only if it is uniformly Cauchy on each compact subset of U.

2.4. (a) Show that $C(U)$ is a complete metric space.
(b) Show that $H(U)$ is closed in $C(U)$.
(c) Conclude that $H(U)$ is a complete metric space.

2.5. Show that a sequence $\{f_n\}_{n=1}^{\infty}$ in $H(\mathbb{D})$ converges to f if and only if $\displaystyle\int_{C(0;1)} |f_n(z) - f(z)|\,|dz| \xrightarrow[n\to\infty]{} 0$, for all $0 < r < 1$.

2.6. Let U and V are open subsets of $\mathbb{C}$.

(a) Suppose $\varphi : U \longrightarrow V$ is a bijective holomorphic map. Show that, if $\mathscr{F} \subseteq H(V)$ is relatively compact, then so is $\{f \circ \varphi : f \in \mathscr{F}\}$.
(b)∗ Let $\mathscr{F} \subseteq H(U)$ be relatively compact. Assume that $f(U) \subseteq V$, for all $f \in \mathscr{F}$. Show that, for any $g \in H(V)$, $\{g \circ f : f \in \mathscr{F}\}$ is relatively compact.

2.7. Let $\mathscr{F} \stackrel{\text{def}}{=} \{f \in H(\mathbb{D}) : \operatorname{Re} f > 0 \text{ and } |f(0)| \leq 1\}$. Show that $\mathscr{F}$ is relatively compact, but not compact.

2.8. Let $U \subseteq_{open} \mathbb{C}$, $w \in \mathbb{C}$ and $r > 0$. Consider $\mathscr{F} \stackrel{\text{def}}{=} \{f \in H(U) : |f(z) - w| \geq r\}$. Show that for any sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathscr{F}$, one has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which either converges (in $H(U)$) to some $f \in H(U)$ or diverges to ∞ uniformly on every compact subset of U.

2.9. Let $U \subseteq_{open} \mathbb{C}$ and $\mathscr{F} \subseteq H(U)$. Denote $\mathscr{F}' \stackrel{\text{def}}{=} \{f' : f \in \mathscr{F}\}$.

(a) Show that, if $\mathscr{F}$ is relatively compact, then so is $\mathscr{F}'$.
(b) Is the converse of 2.9.a true?
(c)∗ Prove the converse of 2.9.a when U is an open disc under the additional hypothesis that there exists $z_0 \in U$ such that the set $\{f(z_0) : f \in \mathscr{F}\}$ is bounded.

Hint. *Let $U = D(a; R)$. One can choose a convergent subsequence $\{f_{n_k}\}_{n=1}^{\infty}$ in such a way that $\{f_{n_k}(z_0)\}_{n=1}^{\infty}$ also converges. Show that $\{f_{n_k}\}_{n=1}^{\infty}$ is uniformly Cauchy on $\overline{D(a; r)}$, for every $0 < r < R$.*

(d)$*$ Let the additional assumption be as above in 2.9.c. Denote by V the set of all $z \in U$ such that $\{f|_{D(z_0;r)} : f \in \mathscr{F}\}$ is relatively compact in $H(D(z_0; r))$, for some $r > 0$. Show that V is nonempty and both open and closed in U.

(e)$*$ Assume that U is connected. Prove the converse of 2.9.a under the additional assumption mentioned in 2.9.c.

2.10. Show that $\mathscr{F} \subseteq H(\mathbb{D})$ is relatively compact if and only if there exists a sequence $\{M_n\}_{n=0}^{\infty}$ of nonnegative reals such that $\limsup_{n\to\infty} M^{\frac{1}{n}} \leq 1$ and $\left|\dfrac{f^{(n)}(0)}{n!}\right| \leq M_n$, for all $f \in \mathscr{F}$ and $n = 0, 1, 2, \ldots$.

2.11.$*$ Let $U \subseteq_{open} \mathbb{C}$ and $L : H(U) \longrightarrow \mathbb{C}$ is a linear map. Assume that L is *multiplicative*, i.e., $L(fg) = L(f)L(g)$, for all $f, g \in H(U)$. Suppose that L is nonzero.

(a) Show that, if $f \equiv 1$, then $L(f) = 1$.

(b) Denote the identity map on U by I. Show that $L(I) \in U$.

Hint. *Assume $z_0 \overset{def}{=} L(I) \notin U$. Then the function $I - z_0$ is nowhere vanishing, so that you can consider the holomorphic function $\frac{1}{I-z_0}$ on U.*

(c) Show that, for every $f \in H(U)$, $L(f) = f(z_0)$.

Hint. *Consider $g : U \longrightarrow \mathbb{C}$, $g(z) \overset{def}{=} \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0. \end{cases}$ Is g analytic? $g(I - z_0)$ might be useful.*

(d) Find all linear maps from $H(U)$ to $\mathbb{C}$ that are multiplicative.

3. Riemann mapping theorem

3.1. Let U be a nonempty proper simply connected region in $\mathbb{C}$ and $z_0 \in U$. If f is the Riemann map from U to $\mathbb{D}$, i.e., f is bijective, holomorphic, $f(z_0) = 0$ and $f'(z_0) > 0$. Express any arbitrary bijective holomorphic map $g : U \longrightarrow \mathbb{D}$ in terms of f.

3.2. Let U and V be nonempty proper simply connected open subsets of $\mathbb{C}$. Show that, for any $z_1 \in U$ and $z_2 \in V$, there exists a unique bijective holomorphic map $f : U \longrightarrow V$ such that $f(z_1) = z_2$ and $f'(z_1) > 0$.

3.3. Let U, V, z_1 and z_2 be as above in 3.2. Suppose that $g : U \longrightarrow V$ is a bijective holomorphic map with $g(z_1) = z_2$ and $h : U \longrightarrow V$ be any holomorphic map satisfying $h(z_1) = z_2$. Show that $|h'(z_1)| \leq |g'(z_1)|$. What about the equality case?

3.4. Let $U, V \subseteq \mathbb{C}$ be open and connected. Assume further that $V \neq \mathbb{C}$ and it is simply connected. Show that the family $\{f \in H(U) : f(U) \subseteq V\}$ is relatively compact.