MTH 101-Calculus

Spring-2021

Assignment-10-Solns: Directional derivatives, Maxima, Minima, Lagrange Multipliers

1. $|f(x,y) - f(0,0)| \leq |x| + |y|$. Thus f is continuous at $(0,0)$.

   $\lim_{t \to 0} \dfrac{f(tu_1, tu_2) - f(0,0)}{t} = \dfrac{|t|}{2t}\{|\,|u_1| - |u_2|\,| - |u_1| - |u_2|\}$.

   Hence, the directional derivatives of f exist at $(0,0)$ if and only if $|\,|u_1| - |u_2|\,| = |u_1| + |u_2|$, that is, either $u_1 = 0$ or $u_2 = 0$. Since the directional derivatives in all direction do not exist, the function cannot be differentiable at $(0,0)$.

2. Let $(u,v) \in \mathbb{R}^2$ be such that $\|(u,v)\| = 1$. Then $D_{(u,v)}f(0,0) = \lim_{t\to 0} \frac{f(tu,tv)}{t} = u^2v$ but $D_{(0,0)}f(u,v) \neq \nabla f(0,0) \cdot (u,v)$ if u and v are non-zeros. Therefore f is not differentiable.

3. Since f_x and f_y are continuous, f is differentiable. Therefore $D_{(1,2)}f(\frac{3}{5}, \frac{4}{5}) = f_x(1,2) \cdot \frac{3}{5} + f_y(1,2) \cdot \frac{4}{5}$.

4. The normal to the given surface is $N = (1 + yz^2, 2z + xz^2, 2y + 2xyz)$. The normal at a point on the z-axis is $(1, 2t, 0)$. If (x,y,z) is any point on the given surface generated then $\dfrac{x}{1} = \dfrac{y}{2t}$, $z = t$. Hence, the surface generated is $y = 2xz$ (by eliminating t).

5. (i) For $f(x,y) = 4xy - x^4 - y^4, f_x(x_0,y_0) = f_x(x_0,y_0) = 0$ for $(x_0,y_0) = (0,0), (1,1)$ or $(-1,-1)$. These are the critical points. By second derivative test, $(0,0)$ ia a saddle point and $(-1,1)$ and $(1,1)$ are local maxima.

   (ii) $f(x,y) = x^3 - 3xy^2, f_x(x_0,y_0) = f_x(x_0,y_0) = 0$ for $(x_0,y_0) = (0,0)$. So $(0,0)$ is the only critical point . Second derivative fails here. Along $y = 0, f(x,y) = x^3$, hence $(0,0)$ is a saddle point.

6. $f(x,y) = xy \Rightarrow f_x = y, f_y = x$. Clearly, $(0,0)$ is the only critical point. $f(0,0) = 0$.

   Let us use the method of lagrange multipliers on $x^2 + y^2 = 1$. Consider the function $F(x,y,z) = xy - \lambda(x^2 + y^2 - 1)$. Here, $F_x = y - 2\lambda x$, $F_y = x - 2\lambda y$ and $F_\lambda = x^2 + y^2 - 1$. Therefore, $y = 2\lambda x$, $x = 2\lambda y \Rightarrow x = 0 \iff y = 0$. But, $x^2 + y^2 = 1$. Hence, $y = 4\lambda^2 y$.

   $\lambda = \pm\frac{1}{2}$ and $y = \pm x \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ and $x = \pm\frac{1}{\sqrt{2}}$.

   Hence, we need to compute the absolute maximum and minimum at the points $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The absolute maximum is attained at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.