Conditional Logic C_b and its Tableau System

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Abstract. Conditional logic is a kind of modal logic for analyzing the truth conditions and inferences of conditional sentences in natural language. However, it has been pointed out in the literature that empirical problems plague all of the previously proposed conditional logics. Moreover, C_1 and C_2 are defined by imposing certain restrictions on their Kripke frames, and there exist no corresponding proof systems.
In order to solve these problems, we propose a new system of conditional logic, which we call C_b. C_b is an extension of C^+ through the addition of new rules on accessibility, and it has a corresponding tableau system. We show that C_b has empirical advantages over C_1 and C_2 as a model of inference in natural language, and compare it with other proof systems of conditional logic.

1 Introduction

1.1 Conditional Sentences in Natural Language and Classical Logic

The following inferences are valid in classical logic:

Antecedent strengthening: $A \supset B \vdash (A \wedge C) \supset B$
Transitivity: $A \supset B, B \supset C \vdash A \supset C$
Contraposition: $A \supset B \vdash \neg B \supset \neg A$

If we simply assume classical logic to explain the semantics of natural language, the formulae above give rise to the following infelicitous arguments [6].

(1) If it does not rain tomorrow we will go to the cricket. Hence, if it does not rain tomorrow and I am killed in a car accident tonight then we will go to the cricket.
(2) If the other candidates pull out, John will get the job. If John gets the job, the other candidates will be disappointed. Hence, if the other candidates pull out, they will be disappointed.
(3) If we take the car then it won't break down en route. Hence, if the car does break down en route, we didn't take it.

The reason for such infelicity is that obvious premises can be omitted in conditionals. For example, the first sentence in (1) introduces the condition "it does not rain" and as we usually do not think about the possibility that we may be killed in a car accident, we continue to assume that we will not to be killed in a car accident as an obvious premise. However, the second sentence of (1) has the opposite meaning in that the premise that is omitted as obvious itself leads to the invalid conclusion. In case (1), what we actually mean to say is:

(1') If it does not rain tomorrow and I am not killed in a car accident tonight, then we will go to the cricket tomorrow.

Similar comments can be made about the arguments in (2) and (3). In case (2), the actual wording should be "if John gets the job and the other candidates do not pull out, they will be disappointed." In case (3), the premise "we will take the car" is omitted and the sentence actually states the opposite, "we didn't take it", making case (3) incomprehensible. Thus, we see that we cannot correctly address conditionals of natural language with semantics based on classical logic.

Let us discuss case (1) in more detail. The correct sentence for (1) is "if it does not rain tomorrow then, other things being equal, we will go to the cricket". We can call "other things being equal" *ceteris paribus*. Conditional sentences include some notion of *ceteris paribus*, so G_A in the conditional sentence "if A and G_A, then B" can be referred to as a *ceteris paribus clause*, which depends on A. Therefore, in the example sentence, if A is "it does not rain tomorrow", then G_A includes the condition that we are not invaded by Martians. If A is "flying saucers arrive from Mars", it does not.

Thus, the notion of *ceteris paribus* is important in conditional sentences of natural language. Accordingly, some logic systems based on the notion of *ceteris paribus* have been proposed; however, they do not sufficiently represent truth conditions in conditional sentences. This paper, first, discusses a previously developed logic system for analyzing conditional sentences, and second, presents an extension of it as a new logic system.

2 Modal Tableau

Modal Tableau is a tableau system for modal logic. In this section, we briefly describe the version of modal tableau that we adopt from [6]. In modal tableaux, we put a natural number with each formula to designate a possible world in which the formula is assumed to be true.

$$\begin{array}{ccc} & A \vee B, i & \\ \swarrow & & \searrow \\ A, i & & B, i \end{array}$$

As the diagram above indicates, the tableau rules for truth functors are the same as those for classical logic except for the numbers for possible worlds. Four new rules are added for the modal operators.

$$\begin{array}{cccc} \neg\Box A, i & \neg\Diamond A, i & \Box A, i & \Diamond A, i \\ \downarrow & \downarrow & irj & \downarrow \\ \Diamond\neg A, i & \Box\neg A, i & \downarrow & irj \\ & & A, j & A, j \end{array}$$

In the rules above, r in irj represents the binary accessibility relation R between two worlds i, j in a Kripke frame $\langle W, R, \nu \rangle$ of modal logic. W is a non-empty set and ν is a function that assigns a truth value to each formula, such that either $\nu_w(p) = 1$ or $\nu_w(p) = 0$. i and j are natural numbers, but j must be new and must not occur at any node above in the same branch.

The two rules on the right are deduced by the following interpretation in Kripke semantics for $\Box$ and $\Diamond$. For any world $w \in W$:

- $\nu_w(\Box A) = 1$ if, for all $w' \in W$ such that wRw', $\nu_{w'}(A) = 1$;
  $\nu_w(\Box A) = 0$ otherwise.
- $\nu_w(\Diamond A) = 1$ if, for some $w' \in W$ such that wRw', $\nu_{w'}(A) = 1$;
  $\nu_w(\Diamond A) = 0$ otherwise.

The two rules on the left can be explained by following proofs respectively.

In any world, w,

$$\begin{aligned}
\nu_w(\neg\Box A) = 1 \text{ iff } & \nu_w(\Box A) = 0 \\
& \text{iff it is not the case that, for all } w' \text{ such that } wRw',\ \nu_{w'}(A) = 1 \\
& \text{iff for some } w' \text{ such that } wRw',\ \nu_{w'}(A) = 0 \\
& \text{iff for some } w' \text{ such that } wRw',\ \nu_{w'}(\neg A) = 1 \\
& \text{iff } \nu_w(\Diamond\neg A) = 1
\end{aligned}$$

In any world, w,

$$\begin{aligned}
\nu_w(\neg\Diamond A) = 1 \text{ iff } & \nu_w(\Diamond A) = 0 \\
& \text{iff it is not the case that, for some } w' \text{ such that } wRw',\ \nu_{w'}(A) = 1 \\
& \text{iff for all } w' \text{ such that } wRw',\ \nu_{w'}(A) = 0 \\
& \text{iff for all } w' \text{ such that } wRw',\ \nu_{w'}(\neg A) = 1 \\
& \text{iff } \nu_w(\Box\neg A) = 1
\end{aligned}$$

The following is an example of a modal tableau for: $\vdash \Diamond(A \wedge B) \supset (\Diamond A \wedge \Diamond B)$

$$\begin{array}{ccc}
 & \neg(\Diamond(A \wedge B) \supset (\Diamond A \wedge \Diamond B)), 0 & \\
 & \downarrow & \\
 & \Diamond(A \wedge B), 0 & \\
 & \neg(\Diamond A \wedge \Diamond B), 0 & \\
\swarrow & & \searrow \\
\neg\Diamond A, 0 & & \neg\Diamond B, 0 \\
\downarrow & & \downarrow \\
\Box\neg A, 0 & & \Box\neg B, 0 \\
\downarrow & & \downarrow \\
0r1 & & 0r1 \\
A \wedge B, 1 & & A \wedge B, 1 \\
\downarrow & & \downarrow \\
A, 1 & & A, 1 \\
B, 1 & & B, 1
\end{array}$$

$$\begin{array}{ccc} \downarrow & \qquad\qquad & \downarrow \\ \neg A, 1 & & \neg B, 1 \\ \times & & \times \end{array}$$

As a proof of a theorem (rather than of deduction), the initial list has only one formula, $\neg(\Diamond(A \wedge B) \supset (\Diamond A \wedge \Diamond B)), 0$. The natural numbers for possible worlds start at 0. When the rule $\Diamond(A \wedge B), 0$ is applied, the new formula $A \wedge B$ is introduced for both branches and a new world number 1 is assigned to it. It causes no problem that the same world number is used in both branches as they do not interact with each other. When we judge whether a given branch closes or not in modal tableau, we must compare contradictory formulae which are assigned the same world number, as for $A, 1$ and $\neg A, 1$ in the example above.

3 Conditional Logic

In section 1.1, we explained the concept of *ceteris paribus* in the context of conditional sentences in natural language. The logic equipped with this concept is called *conditional logic* [3, 4, 8], which is a kind of modal logic held to be useful as a semantic framework of natural language.

3.1 Syntax of Conditional Logic

Let us write $A > B$ for a conditional with a *ceteris paribus* condition: "if A, then B". The syntax of conditional logic is defined by the following BNF grammar, where p is a propositional parameter:

$$\mathcal{F} ::= p \mid \neg\mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \to \mathcal{F} \mid \mathcal{F} \leftrightarrow \mathcal{F} \mid \Box\mathcal{F} \mid \Diamond\mathcal{F} \mid \mathcal{F} > \mathcal{F}$$

3.2 Semantics of Conditional Logic

The Kripke frame of conditional logic consists of a quadruple: $\langle W, \{R_A : A \in \mathcal{F}\}, R, \nu\rangle$, where W is a non-empty set, ν is a function that assigns a truth value to each pair comprising a world, w, and a propositional parameter, p, the same as for modal logic. "In world w, p is true (or false)" is written as $\nu_w(p) = 1$ (or $\nu_w(p) = 0$). R is a binary relation on W which is reflexive, symmetrical and transitive. Each R_A is a binary relation on W for any formula A. Intuitively, $w_1 R_A w_2$ly means that w_2 is the same as w_1 except that A is true in w_2, which represents the *ceteris paribus* condition.

In the settings of conditional logic, $\Box$ and $\Diamond$ are treated as those of system K_ν [6].

- $\nu_w(\Box A) = 1$ if, for all $w' \in W$ such that wRw', $\nu_{w'}(A) = 1$; and 0 otherwise.
- For any world $w \in W$
  $\nu_w(\Diamond A) = 1$ if, for some $w' \in W$ such that wRw', $\nu_{w'}(A) = 1$; and 0 otherwise.

By means of the notions introduced above, the semantics of $A > B$ is defined as follows.

- $\nu_w(A > B) = 1$ iff for all w' such that wR_Aw', $\nu_{w'}(B) = 1$ and $\nu_{w'}(B) = 0$ otherwise.

Furthermore, in conditional logic, the following conception, $f_A(w)$, $[A]$, is added.

- $f_A(w) = \{x \in W : wR_Ax\}$
- $[A] = \{w : \nu_w(A) = 1\}$

Here, $f_A(w)$ is the set of worlds accessible to w under R_A. Also, R and $f_A(w)$ are interdefinable, since wR_Aw' iff $w' \in f_A(w)$. $[A]$ is a class of worlds where A is true, $\{w : \nu_w(A) = 1\}$.

With this conception, the semantics of $A > B$ can be simply defined as follows:

- $A > B$ is true in $w \Leftrightarrow f_A(w) \subseteq [B]$

4 Previous Study

It would seem that conditional logic has a close connection with the phenomenon of natural language, but there is as yet no logical system that can represent it. In this section, we briefly explain some logical systems that extend the conditional logic set out in section 3.

4.1 C^+

C^+ [1] is logic system which is an extension of conditional logic C by adding the following conditions on its Kripke frame.

1. $f_A(w) \subseteq [A]$
2. If $w \in [A]$, then $w \in f_A(w)$

C^+ has a tableau system that corresponds to the Kripke frame above. The following three rules are added to the tableau rules for C:

$$\begin{array}{c} A > B, i \\ ir_Aj \\ \downarrow \\ B, j \end{array} \quad \begin{array}{c} \neg(A > B), i \\ \downarrow \\ ir_Aj \\ A, j \\ \neg B, j \end{array} \quad \begin{array}{cc} \multicolumn{2}{c}{.} \\ \swarrow & \searrow \\ \neg A, i & A, i \\ & ir_Ai \end{array}$$

The difference from a modal tableau is that each formula has its own relation of accessible worlds like ir_Aj.

As an example for a tableau proof in C^+, we prove $A, A > B \vdash_{C^+} B$.

$$
\begin{array}{ccc}
 & A, 0 & \\
 & A > B, 0 & \\
 & \neg B, 0 & \\
\swarrow & & \searrow \\
\neg A, 0 & & A, 0 \\
\times & & 0 r_A 0 \\
 & & \downarrow \\
 & & B, 0 \\
 & & \times
\end{array}
$$

First, the rightmost rule of C^+ is applied, which yields $0r_A0$. Then, the leftmost rule can be applied to $A > B, 0$. This is closed by contradiction of $\neg B, 0$ and $B, 0$.

The tableau proof that follows is of $p > q \nvdash_{C^+} (p \wedge r) > q$ in C^+. This tableau proves that the inferences that result in the infelicitous semantics described in section 1.1 are not valid when implementing the concept of *ceteris paribus*.

$$
\begin{array}{c}
p > q, 0 \\
\neg((p \wedge r) > q), 0 \\
\downarrow \\
0 r_{p \wedge r} 1 \\
p \wedge r, 1 \\
\neg q, 1 \\
\downarrow \\
p, 1 \\
r, 1 \\
\begin{array}{cc}
\swarrow & \searrow \\
\begin{array}{c}
\neg p, 0 \\
\begin{array}{cc}
\swarrow & \searrow \\
\neg p, 1 & p, 1 \\
\times & 1 r_p 1
\end{array}
\end{array}
&
\begin{array}{c}
p, 0 \\
0 r_p 0 \\
\downarrow \\
q, 0 \\
\begin{array}{cc}
\swarrow & \searrow \\
\neg q, 1 & q, 1 \\
 & 1 r_q 1 \\
 & \times
\end{array}
\end{array}
\end{array}
\end{array}
$$

Not all branches close completely, and for branches which do not do so, namely $p, 1$ and $\neg q, 1$, a similar formula already exists at the upper nodes. Thus, it seems to apply the same rule infinitely, indicating that this formula is indeed invalid.

Showing that the tree does not close does not mean that the formula is invalid, but indicates the possibility that it is invalid. One good way to *prove* the invalidity of the formula is to draw a counter-model. Counter-models can be read off from an open branch of a tableau in a natural way.

The counter-model of the formula above is as follows: $w_1 R_p w_1$, $w_0 R_{p \wedge r} w_1$ and $\nu_{w_0}(\neg p) = 1$ and $\nu_{w_1}(p) = \nu_{w_1}(\neg q) = \nu_{w_1}(r) = 1$. In regard to other formulae for A, the accessibility relation R_A is defined such that $f_A(w) = [A]$ for all w. Thus, the interpretation can be depicted as follows, from which we can check that the accessibility relation of worlds forms an infinite loop. This means that this tableau is never closed and $p > q \nvdash_{C^+} (p \wedge r) > q$ is proved.

$$\begin{array}{ccc} & & R_p \\ & & \circlearrowright \\ w_0 & \xrightarrow{R_{p\wedge r}} & w_1 \\ \neg p & & p, \neg q, r \end{array}$$

Similarly, cases (2) and (3) in the 1.1 are invalid in C^+. Thus, by extending to C^+, we have solved the problem outlined in the 1.1 that classical logic is too weak as semantics in natural language.

4.2 S

In section 3.2, we said that $w_1 R_A w_2$ means that w_2 is the same as w_1 except that A is true in w_2. Thus, we need somehow to consider the notion of similarity between two worlds, in order to reflect the intuition behind the ceteris paribus condition. In C and C^+, however, the relation R_A by no means represents such notion of similarity.

S is an extension of C^+ through the addition of the following three conditions on a Kripke frame[1], which express a certain "similarity" between the two arguments of R_A.

3. If $[A] \neq \phi$, then $f_A(w) \neq \phi$
4. If $f_A(w) \subseteq [B]$ and $f_B(w) \subseteq [A]$, then $f_A(w) = f_B(w)$
5. If $f_A(w) \cap [B] \neq \phi$, then $f_{A\wedge B}(w) \subseteq f_A(w)$

As a result, for example, the inference $p > q, q > p \vdash (p > r) \equiv (q > r)$ is valid in S, but not in C^+.

Proof. Suppose that the premise is true in world w, i.e., $f_p(w) \subseteq [q]$ and $f_q(w) \subseteq [p]$. Then, by applying condition **4**, $f_p(w) = f_q(w)$. Hence, $f_p(w) \subseteq [r]$ iff $f_q(w) \subseteq [r]$, i.e., $(p > r)$is true in w iff $(q > r)$ is true in w. i.e., $(p > r) \equiv (q > r)$ is true in w.

Because S does not have a corresponding tableau system, the proof in S can rely on the semantic notions alone.

4.3 C_1, C_2

Although S is a stronger logic system than C^+, it is still weak as semantics of natural language. This prompted Stalnaker and Lewis to propose extensions[2] of S: C_1 [8] and C_2 [3, 4]. Conditions **1** to **5** of S above are common conceptions of C_1 and C_2 [7].

C_1 and C_2 also make a difference through the addition of the following conditions. C_1 adds the following condition.

[1] The system S is defined as a common part of the conditional logics proposed by Stalnaker [8] and Lewis [3, 4]. Following the convention in [6], we call it S.
[2] The names C_1 and C_2 are taken from [6].

6. If $w \in [A]$ and $w' \in f_A(w)$, then $w = w'$

C_2, on the other hand, adds the following instead of 6.

7. If $x \in f_A(w)$ and $y \in f_A(w)$, then $x = y$

Both condition **6** and condition **7** concern the relationship between two worlds, but the difference between them is that **7** entails **6**, according to **2**. So we can say that C_2 is stronger than C_1. We can sort these systems by increasing strength as follows: $C^+ < S < C_1 < C_2$.

However, C_1 and C_2 are not without problems [2]. For example, $A \wedge B \vdash A > B$ is one of the formulae that is valid in C_1 but not in S.

Suppose you go to a fake fortune-teller, who says that you will come into a large sum of money. And suppose that, purely by accident, you do. The statement "If the fortune-teller says that you will come into a large sum of money, you will" still, however, would appear to be false.

Similarly, $\vdash (A > B) \vee (A > \neg B)$ is an example that it is valid in C_2 but not in S. However, both of the following conditionals would appear to be false: "If it will either rain tomorrow or it won't, then it will rain tomorrow" and "If it will either rain tomorrow or it won't, then it won't rain tomorrow."

Thus, the two conceptions yield empirically wrong predictions, which are good illustrations that each logic system is too strong to be semantics of natural language. Given the above, although C_1 and C_2 are the strongest existing conditional logic, they do not seem to be suitable as a logic system for natural language.

5 Proposal: A New Conditional Logic C_b

In section 4, we pointed out the drawbacks with existing systems of conditional logic. However, viewing them purely from the perspective of the semantics of natural language, C^+ and S are too weak and C_1 and C_2 are too strong.

Moreover, S, C_1 and C_2 have no known tableau systems or any other proof systems. In developing a logic system for natural language, especially in the context of natural language processing, whether it enables us to *compute* the entailment relation between given sentences is an important feature, and is a feature that has not been achieved in previous studies with a few notable exceptions, such as [5].

Against such background, we propose here, as semantics of natural language, a logic system we call C_b that properly extends C^+ and has a corresponding tableau system.

One of the common problems with C^+, S, C_1 and C_2 is when $\wedge$ or $\vee$ appears on the accessibility relations: when such truth functions occur in the antecedents of formulae, we cannot apply any rules, and all inferences including these shall be invalid.

The system C_b is obtained by adding the following new conditions on the Kripke frame. It allows the nature that is inherited from C^+ to be preserved.

8. $\bigcup_{w' \in f_A(w)} f_B(w') \subseteq f_{A \wedge B}(w)$

9. $f_A(w) \subseteq f_{A \vee B}(w)$
$f_B(w) \subseteq f_{A \vee B}(w)$

C_b has a tableau system which is an extension of that of C^+ through the addition of the following three rules:

$$\begin{array}{ccc} ir_A j & ir_A j & ir_A j \\ jr_B k & \downarrow & \downarrow \\ \downarrow & ir_{A \vee B} j & ir_{B \vee A} j \\ ir_{A \wedge B} k & & \end{array}$$

The additional rules refer only to an accessibility relation not formulae. The left rule is for $\wedge$. For every $ir_A j$ and $jr_B k$ on the branch, we can derive $ir_{A \wedge B} k$. The right two rules are for $\vee$. For every $ir_A j$ on the branch, we can derive $ir_{A \vee B} j$ and $ir_{B \vee A} j$ for any B. In section 7 and section 8, we prove that this tableau system is sound and complete with respect to the Kripke semantics introduced above.

Note that any theorem of C^+ is a theorem of C_b. In the next section, we will verify what inferences are valid in C_b.

6 Empirical Verification

The following tableau is an example of a proof in C_b. We note, in passing, that all the following inferences are valid in C_b but not in C^+, S, C_1 or C_2. [3]

Ex.1 $(p \wedge q) > r \vdash p > (q > r)$

$$\begin{array}{c} (p \wedge q) > r, 0 \\ \neg(p > (q > r)), 0 \\ \downarrow \\ 0r_p 1 \\ p, 1 \\ \neg(q > r), 1 \\ \downarrow \\ 1r_q 2 \\ q, 2 \\ \neg r, 2 \\ \downarrow \\ 0r_{p \wedge q} 2 \\ \downarrow \\ r, 2 \\ \times \end{array}$$

[3] As discussed in section 4.1, we need to construct a counter-model for proving the invalidity of a formula, but we omit this here due to the space limitations.

This is an example of applying the left rule proposed in section 5.

$0r_{p\wedge q}2$ is derived by applying the C_b rule to the second line $0r_p1$ and the third line $1r_q2$. This formula creates a sentence like the following.

"If the rain stops and the water temperature is more than $25°C$, then we can swim in the pool."
⇒"If the rain stops, then additionally, if the water temperature is more than $25°C$, we can swim in the pool."

Ex.2 $(p \wedge q) > r, p > q \vdash p > r$

$$
\begin{array}{c}
(p \wedge q) > r, 0 \\
p > q, 0 \\
\neg(p > r), 0 \\
\downarrow \\
0r_p1 \\
p, 1 \\
\neg r, 1 \\
\downarrow \\
q, 1 \\
\begin{array}{cc}
\swarrow & \searrow \\
\neg q, 1 & q, 1 \\
\times & 1r_q1 \\
 & \downarrow \\
 & 0r_{p\wedge q}1 \\
 & \downarrow \\
 & r, 1 \\
 & \times
\end{array}
\end{array}
$$

This inference is little changed from Ex. 1, and similarly to Ex. 1, it applies the left rule as in section 5 above.

This formula creates a sentences like the following:
"If A and B come, C will also come. And if A comes then B comes."
⇒ "If A comes then C comes."

Ex.3 $(p \vee q) > r \vdash (p > r) \wedge (q > r)$

$$
\begin{array}{c}
(p \vee q) > r, 0 \\
\neg((p > r) \wedge (q > r)), 0 \\
\begin{array}{cc}
\swarrow & \searrow \\
\neg(p > r), 0 & \neg(q > r), 0 \\
\downarrow & \downarrow \\
0r_p1 & 0r_q2 \\
p, 1 & q, 2
\end{array}
\end{array}
$$

$$\begin{array}{cc}
\neg r, 1 & \neg r, 2 \\
\downarrow & \downarrow \\
0r_{p\vee q}1 & 0r_{p\vee q}2 \\
\downarrow & \downarrow \\
r, 1 & r, 2 \\
\times & \times
\end{array}$$

This is an example of applying the right rules proposed in section 5 above.

In order to apply the rule to the antecedent of the formula, it is necessary to account for the accessibility relation of $p \vee q$. Then, after $0r_p1$ and $0r_q2$ are produced, they derive $0r_{p\vee q}1$ and $0r_{p\vee q}2$, respectively. Hence, closing becomes possible by applying the rule to the antecedent of the given formula.

This formula creates sentences like the following:
"If it rains or snows, the game will be cancelled."
$\Rightarrow$"If it rains, the game will be cancelled and if it snows, the game will be cancelled."

Ex.4 $p > q, (p \wedge \neg r) > \neg q \vdash p > r$

$$\begin{array}{c}
p > q, 0 \\
(p \wedge \neg r) > \neg q, 0 \\
\neg(p > r), 0 \\
\downarrow \\
0r_p1 \\
p, 1 \\
\neg r, 1 \\
\downarrow \\
q, 1 \\
\begin{array}{cc}
\swarrow & \searrow \\
\neg\neg r, 1 & \neg r, 1 \\
\downarrow & 1r_{\neg r}1 \\
r, 1 & \downarrow \\
\times & 0r_{p\wedge\neg r}1 \\
 & \downarrow \\
 & \neg q, 1 \\
 & \times
\end{array}
\end{array}$$

Although the above inference is valid in C_b, there is a problem in regard to its empirical validity. For example, the conditional corresponding to this formula seems to be false: 'If there are various drinks there, I will go. Even though there are various drinks, if there is no pizza, I won't go."
$\Rightarrow$?∗"If there are various drinks, there will be pizza."

Ex.5 $(p \wedge r) > (r \wedge q), p > r \vdash p > q$

$$(p \wedge r) > (r \wedge q), 0$$

$$
\begin{array}{ccc}
 & p > r, 0 & \\
 & \neg(p > q), 0 & \\
 & \downarrow & \\
 & 0 r_p 1 & \\
 & p, 1 & \\
 & \neg q, 1 & \\
 & \downarrow & \\
 & r, 1 & \\
\swarrow & & \searrow \\
\neg r, 1 & & r, 1 \\
\times & & 1 r_r 1 \\
 & & \downarrow \\
 & & 0 r_{p \wedge r} 1 \\
 & & \downarrow \\
 & & r \wedge q, 1 \\
 & & \downarrow \\
 & & r, 1 \\
 & & q, 1 \\
 & & \times
\end{array}
$$

This inference is also valid in C_b, but we cannot find a conditional to correspond to it. In such instance, the system may be too strong. Therefore, in future research, we need to find a suitable limitation.

7 Soundness

Our proof of the soundness and completeness of C_b is based on the proof of soundness and completeness of C^+ given in [6].

Definition 1 (Faithfulness). *Let $I = \langle W, R, \nu \rangle$ be any Kripke interpretation, and b be any branch of a tableau. Then, I is faithful to b iff there is a map $g : \mathbb{N} \to W$ such that:*

1. *For every node A, i on b, A is true at $g(i)$ in I.*
2. *If $ir_A j$ is on b, $g(i) R_A g(j)$ in I.*

Lemma 2 (Soundness Lemma). *Let b be any branch of a tableau and $I = \langle w, R, \nu \rangle$ be any Kripke interpretation. If I is faithful to b and a tableau rule is applied to it, then it produces at least one extension b' such that I is faithful to b'.*

Proof. Since C^+ is proved to be sound with respect to its semantics given in section 4.1, we merely have to check the case for each rule of C_b.

The argument for the left tableau rule in section 5 is the following. Suppose there are $ir_p j$ and $jr_q k$ on a branch to which I is faithful and we apply the left rule and obtain $ir_{p \wedge q} k$. According to the definition of faithfulness, $g(i) r_p g(j)$ and $g(j) r_q g(k)$ are in I. Hence:

$$\begin{aligned}
g(k) &\in \{x \mid \exists w' \ (g(i) R_p w \wedge w R_q x)\} \\
&\equiv \bigcup_{w' \in f_p(w)} f_q(g(i)) \\
&\subseteq f_{p \wedge q}(g(i)) && \text{(according to Condition } \mathbf{8} \text{ in section 5)} \\
&\equiv \{x \mid g(i) R_{p \wedge q} x\} && \text{(according to the definition of } f_A(w))
\end{aligned}$$

Therefore, $g(i)R_{p\wedge q}g(k)$, which shows that I is faithful to this extension.

For the right rules, suppose that there is ir_pj on a branch to which I is faithful and from applying the rule we obtain $ir_{p\vee q}j$. According to the definition of faithfulness, $g(i)r_pg(j)$ is in I. Hence:

$$\begin{aligned}
g(j) &\in \{x \in W \mid g(i) R_p x\} \\
&\equiv f_p(g(i)) && \text{(according to the definition of } f_A(w)) \\
&\subseteq f_{p \vee q}(g(i)) && \text{(according to Condition } \mathbf{9} \text{ in section 5)} \\
&\equiv \{x \in W \mid g(i) R_{p \vee q} x\}
\end{aligned}$$

Therefore, $g(i)R_{p\vee q}(j)$, which shows that I is faithful to this extension. The case for ir_qj can be proved in the same way.

Theorem 3 (Soundness Theorem). *The tableau system of C_b is sound with respect to its semantics, i.e. for finite Σ, if $\Sigma \vdash A$ then $\Sigma \models A$.*

Proof. Suppose that $\Sigma \not\models A$. Then we have the interpretation $I = \langle W, R, \nu\rangle$ that makes every formula in Σ true and A false, in some world w. Let $h : \mathbb{N} \to W$ be a map such that $h(0) = w$, which makes I faithful to the initial list. When we apply a rule to the list, there is at least one extension to which I is faithful, due to the Soundness Lemma. Thus, if the tableau is closed, there is at least one branch b for which the interpretation I is faithful, and there is a formula B such that both B and $\neg B$ are on b. This is impossible, however, because it means $\nu(B) = 1$ and $\nu(\neg B) = 1$. Therefore, the tableau must be open, i.e. $\Sigma \not\vdash A$.

8 Completeness

Definition 4 (Induced Interpretation). *Let b be an open branch of a tableau. The interpretation $I = \langle W, R, \nu\rangle$ induced by b is defined as follows:*

- $W = \{w_i \mid i$ *occurs in* $b\}$
- *For any formula A:*
  - $w_iR_Aw_j$ *iff* ir_Aj *is on* b, *if* A *occurs as the antecedent of a conditional or negated conditional at a node of* b.
  - $w_iR_Aw_j$ *iff* $\nu_{w_j}(A) = 1$ *otherwise.*
- $\nu_{w_i}(A) = \begin{cases} 1 & \textit{if } A, i \textit{ occurs on } b \\ 0 & \textit{if } \neg A, i \textit{ occurs on } b \\ 1 \textit{ or } 0 \textit{ otherwise} \end{cases}$

Lemma 5 (Completeness Lemma). *Let b be any open complete branch of a tableau. Let $I = \langle W, R, \nu\rangle$ be the interpretation induced by b. Then:*

– *if* A, i *is on* b*, then* $\nu_{w_i}(A) = 1$
– *if* $\neg A, i$ *is on* b*, then* $\nu_{w_i}(A) = 0$

Proof. Since the syntax of C_b is the same as that of C and C^+, the Completeness Lemma can be proved in the same way [6].

Theorem 6 (Completeness Theorem). *The tableau system of* C_b *is complete with respect to its Kripke semantics: for finite* Σ*, if* $\Sigma \models A$ *then* $\Sigma \vdash A$*.*

Proof. Suppose that $\Sigma \nvdash A$. Given an open branch b of the tableau, the interpretation induced by b makes all the formulae in Σ true and A false in w_0, by the Completeness Lemma.

Now we must check that the induced interpretation satisfies conditions 1 and 2 of C^+, whose proof is given in [6], and conditions 8 and 9 of C_b given in section 5.

Suppose that b is a completed open branch. For any formula A, either A occurs on b as an antecedent or not. In the former case, the result holds according to the definition of R_A. In the latter case, let us check the two conditions of C_b in turn.

– For **8**, let w_x be any world. Suppose that there exists a world w_y such that $w_i R_A w_y$ and $w_y R_B w_x$. Then, according to the definition of induced interpretations, $ir_A y$ and $yr_B x$ occur on b. Since b is completed, $ir_{A\wedge B} x$ is also on b. Again, according to the definition of induced interpretations, $w_i R_{A\wedge B} w_x$ holds, as required.
– For **9**, let w_x be any world such that $w_i R_A w_x$. According to the definition of induced interpretations, $ir_A x$ occurs on b. Since b is completed, $ir_{A\vee B} x$ and $ir_{B\vee A} x$ occurs on b for any formula B. So, according to the definition of induced interpretations, $w_i R_{A\vee B} w_x$ and $w_i R_{B\vee A} w_x$ hold, as required.

Hence $\Sigma \not\models A$.

9 Conclusion and Future Work

In this paper, we have explored in detail the problems of conditional sentences in natural language and proposed a new logic system C_b by extending existing conditional logics.

There are two advantages to our logic C_b: first, as a semantic theory of natural language, it is empirically more correct than preceding analyses; second, a tableau proof is available for C_b, which is sound and complete with respect to its Kripke semantics.

The following figure shows a comparison between S and C_b in regard to their valid inferences.

S \ C_b	valid	invalid
valid	$p > (q \wedge r) \vdash p > q$	$p > q, q > r \vdash p > r$
	$p > (p > q) \vdash p > q$	$p > q, \neg(p > \neg r) \vdash (p \wedge r) > q$
invalid	$(p \vee q) > r \vdash (p > r) \wedge (q > r)$	
	$(p \wedge q) > r \vdash p > (q > r)$	$(p \wedge r) > (r \wedge q) \vdash (p > q) \vee (r > q)$
	$(p \wedge q) > r, p > q \vdash p > r$	
	$p > q, (p \wedge \neg r) > \neg q \vdash p > r$	
	$(p \wedge r) > (r \wedge q), p > r \vdash p > q$	

This figure indicates that inferences with conditional formulae that include $\wedge$ or $\vee$ in the antecedent are valid only in C_b, constituting what we believe to be a substantial extension.

For future work, further examination of the empirical validity of C_b is required. Cases such as Ex. 4 and Ex. 5 in section 6 may be problematic for the current version of C_b. Moreover, we should think about how to treat inferences with a conditional formula whose antecedent contains the negation symbol $\neg A$. The availability of an automatic proof and its implementation remain as topics for future work.

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