Angles under holomorphic maps

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1. Orientation in a real inner product space

Let V be an n ($\in \mathbb{N}$) dimensional real inner product space. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ and consider the determinant function with respect to this orthonormal basis. For any two ordered bases $\mathcal{B} \overset{\text{def}}{=} \{v_1, \ldots, v_n\}$ and $\mathcal{B}' \overset{\text{def}}{=} \{w_1, \ldots, w_n\}$ of V, denote by $T_{\mathcal{B},\mathcal{B}'}$ the unique linear operator on V such that $Tv_i = w_i$, for all $i = 1, \ldots, n$. If $\det T_{\mathcal{B},\mathcal{B}'} > 0$ (or < 0) then we say that the two ordered bases *have same orientation* (or *opposite orientation* respectively).

Theorem 1.1. *For two ordered bases $\mathcal{B}$ and $\mathcal{B}'$ of V, define a relation '$\sim$' as follows:*

$$\mathcal{B} \sim \mathcal{B}' \text{ if and only if } \mathcal{B} \text{ and } \mathcal{B}' \text{ have same orientation.} \tag{1.1}$$

Then the relation defined as above in (1.1) *is an equivalence relation, and furthermore it has precisely two equivalence classes.*

Proof. The proof of Theorem 1.1 is straightforward and hence, we leave it to the reader. □

We say that an order basis $\{v_1, \ldots, v_n\}$ of V is *positively oriented* if it has the same orientation with $\{e_1, \ldots, e_n\}$. Otherwise we say it is *negatively oriented*. It is clear that $\{v_1, \ldots, v_n\}$ has positive (or negative) orientation if and only if $\det(v_1, \ldots, v_n)$ is positive (or negative respectively). When $V = \mathbb{R}^n$, where $n \in \mathbb{N}$, the determinant function is always considered with respect to the standard basis unless otherwise mentioned. For $n = 2$, we can geometrically interpret positive and negative orientation as follows. An ordered basis $\{v_1, v_2\}$ has positive (or negative) orientation if v_1 can be brought to the line spanned by v_2 by applying a rotation by an angle less that 180 degrees in the anticlockwise (or clockwise respectively) direction.

Let $T : V \longrightarrow V$ be an invertible linear operator. It is obvious that the following conditions are equivalent:

(O.1) For any ordered basis $\{v_1, \ldots, v_n\}$ of V, it has same orientation with $\{Tv_1, \ldots, Tv_n\}$.

(O.2) There exists an ordered basis $\{v_1, \ldots, v_n\}$ of V which has same orientation with $\{Tv_1, \ldots, Tv_n\}$.

(O.3) $\det T > 0$.

An invertible linear operator T on V satisfying at least one (equivalently, all) the conditions listed above in (O.1)-(O.3) is referred to as *orientation preserving*. It is easy to find the analogues of the equivalent statements listed above in (O.1)-(O.3) when T *reverses* the orientation.

2. Preservation of angles

We let V be as in §1. For $v, w \in V \setminus \{0\}$, the angle between v and w is defined by

$$\cos^{-1}\left(\frac{\langle v, w\rangle}{||v||||w||}\right). \tag{2.1}$$

Geometrically, one considers the plane $\mathrm{Span}\{v, w\}$, provided they are linearly independent. Then (2.1) is the angle, with the smaller magnitude, formed by the vectors v and w in the plane $\mathrm{Span}\{v, w\}$.

Remark 2.1. *Recall that, if $V = \mathbb{R}^2 \equiv \mathbb{C}$ then $\langle v, w\rangle = \mathrm{Re}(v\overline{w})$. So, for any two nonzero complex number v, w, if $\theta \in [0, \pi]$ is the angle between them then $\cos\theta = \dfrac{\mathrm{Re}(v\overline{w})}{|v||w|}$.*

Definition 2.1. *A linear operator $T : V \longrightarrow V$ is said to* preserve angles *if for any $v, w \in V \setminus \{0\}$, the angle between v and w equals to the angle between Tv and Tw, i.e.,*

$$\frac{\langle v, w\rangle}{||v||||w||} = \frac{\langle Tv, Tw\rangle}{||Tv||||Tw||}. \tag{2.2}$$

Note that, in the above definition, it has been tacitly assumed that $Tv \neq 0$ if $v \neq 0$. Since V is finite dimensional, T must be invertible.

It is clear that any orthogonal operator on V is angle preserving. Since any nonzero scalar multiple of an angle preserving operator is angle preserving, any λT, where $\lambda \neq 0$ and $T : V \longrightarrow V$ is orthogonal, preserves angles. The following theorem shows that any angle preserving linear map has to be of this form.

Theorem 2.1. *If $T : V \longrightarrow V$ is angle preserving then there exists $\lambda > 0$ such that λT is orthoginal.*

Proof. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of V. It is easy to see that, for any $i \neq j \in \{1, \ldots, n\}$, $\langle Te_i, Te_j\rangle = 0$ as $\langle e_i, e_j\rangle = 0$. So $\{Te_1, \ldots, Te_n\}$ is orthogonal. It follows that, for any $i, j = 1, \ldots, n$,

$$\langle T^*Te_i, e_j\rangle = \langle Te_i, Te_j\rangle = \begin{cases} ||Te_i||^2 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.3}$$

Let $j \in \{2, \ldots, n\}$. Then, from (2.3), one has

$$\begin{aligned} \frac{\langle e_1, e_1 + e_j \rangle}{\sqrt{2}} &= \frac{\langle Te_1, T(e_1 + e_j) \rangle}{\|Te_1\| \|T(e_1 + e_j)\|} \\ &= \frac{\langle e_1, T^*T(e_1 + e_j) \rangle}{\|Te_1\| \|T(e_1 + e_j)\|} \\ &= \frac{\|Te_1\|^2}{\|Te_1\| \|T(e_1 + e_j)\|} \\ &= \frac{\|Te_1\|}{\|Te_1 + Te_j\|}. \end{aligned} \tag{2.4}$$

As Te_1 and Te_j are orthogonal, we have $\|Te_1 + Te_j\| = \|Te_1\| + \|Te_j\|$. Hence, from (2.4), we obtain that, $\frac{1}{\sqrt{2}} = \frac{\|Te_1\|}{\|Te_1\| + \|Te_j\|}$, which implies $\|Te_1\|^2 = \|Te_j\|^2$. Set $\alpha \overset{\text{def}}{=} \|Te_1\|^2$. In view of (2.3), it is now clear that $T^*T = \alpha I$, which shows that $\frac{T}{\sqrt{\alpha}}$ is orthogonal. □

From the equivalent statements (O.1)-(O.3), one now concludes the following:

Corollary 2.1. *Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be orientation preserving. If T also preserves angles then, the matrix of T with respect to the standard bases is of the following form:*

$$\lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.5}$$

where $\lambda > 0$ and $\theta \in \mathbb{R}$.

Let $\{v_1, v_2\}$ be an ordered basis in $\mathbb{R}^2$. The angle between v_1 and v_2 is said to be *positively (or negatively) oriented* if the ordered basis $\{v_1, v_2\}$ has the positive (or negative respectively) orientation. We say that a linear operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *preserves oriented angles* if it is angle preserving and for every ordered basis $\{v_1, v_2\}$ of $\mathbb{R}^2$, the angles between v_1 and v_2 and Tv_1 and Tv_2 have same orientation. In other words, T preserves angle and orientation both. Corollary 2.1 yields that the matrix representation of any oriented angle preserving linear operator on the plane $\mathbb{R}^2$ with respect to the standard basis is given by (2.5).

3. Angles under differentiable maps

Let $\mathbf{x}_0 \in U \subseteq_{open} \mathbb{R}^n$, where $n \in \mathbb{N}$. Suppose that $\gamma_1 : [a_1, b_1] \longrightarrow U$ and $\gamma_2 : [a_2, b_2] \longrightarrow U$ are two curves *intersecting at* $\mathbf{x}_0$, i.e., say $\gamma_1(t_1) = \gamma_2(t_2) = \mathbf{x}_0$, where $t_1 \in [a_1, b_1]$ and $t_2 \in [a_2, b_2]$. Assume that γ_i is differentiable at t_i and the tangent vector $\gamma_i'(t_i)$ at $\mathbf{x}_0$ is nonzero, for $i = 1, 2$. Then the angle between the curves γ_1 and γ_2 at $\mathbf{x_0}$ is defined to be the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$. Let $f : U \longrightarrow \mathbb{R}^n$ be differentiable at $\mathbf{x}_0$.

Definition 3.1. *We say that f* preserves angles at $\mathbf{x}_0$ *if, the angle between the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(\mathbf{x}_0)$ equals to that of γ_1 and γ_2 at $\mathbf{x}_0$, for any curves γ_1 and γ_2 in U intersecting at $\mathbf{x}_0$.*

In addition, when $n = 2$, if whenever the angle between γ_1 and γ_2 at $\mathbf{x}_0$ has an orientation, the angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(\mathbf{x}_0)$ also has the same orientation, we then call f an oriented angle preserving map

Note:

1. It has been assumed in the above definition that the tangent vectors of γ_1 and γ_2 at $\mathbf{x}_0$ and $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(\mathbf{x}_0)$ are all nonzero.
2. When $n = 2$, we have also assumed that, if tangent vectors of γ_1 and γ_2 at $\mathbf{x}_0$ constitutes an ordered basis of $\mathbb{R}^2$, tangent vectors of $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(\mathbf{x}_0)$ will also form an ordered basis, and the latter has the same orientation with the former.

Proposition 3.1. *Let U and $\mathbf{x}_0$ be as above. Then, for every $\mathbf{v} \in \mathbb{R}^n$, there exists a differentiable curve γ in U defined on an interval $[-\varepsilon, \varepsilon]$, where $\varepsilon > 0$, such that $\gamma(0) = \mathbf{x}_0$ and $\gamma'(0) = \mathbf{v}$.*

Proof. If $\mathbf{v} = \mathbf{0}$, the constant curve $\mathbf{x}_0$ works. So let us assume $\mathbf{v} \neq \mathbf{0}$. Choose $r > 0$ such that $B(\mathbf{x}_0; r) \subseteq U$. Consider the curve

$$\gamma : \left[-\frac{r}{\mathbf{2}\|\mathbf{v}\|}, \frac{r}{2\|\mathbf{v}\|}\right] \longrightarrow \mathbb{R}^n,\ \gamma(t) \overset{\text{def}}{=} \mathbf{x}_0 + t\mathbf{v}.$$

Clearly $\gamma^* \subseteq B(\mathbf{x}_0; r) \subseteq U$ and $\gamma'(0) = \mathbf{v}$. □

Recall that, if $\gamma : [a, b] \longrightarrow U$ is differentiable at $t_0 \in [a, b]$ and $\gamma(t_0) = \mathbf{x}_0$, then the tangent vector to the curve $f \circ \gamma$ at the point $f(\mathbf{x}_0)$ is $f'(\mathbf{x}_0)\gamma'(t_0)$, by the chain rule. Hence in view of Proposition 3.1, it is easy to see that f preserves angles at $\mathbf{x}_0$ if and only if the linear operator $f'(\mathbf{x}_0)$ is angle preserving. Furthermore, in the case $n = 2$, saying that f preserves oriented angles at $\mathbf{x}_0$ is same as saying that $f'(\mathbf{x}_0)$ is both orientation and angle preserving. Consequently, matrix representation of $f'(\mathbf{x}_0)$ with respect to the standard basis will be given by (2.5).

4. Preservation of angles by holomorphic maps

Let $z_0 \overset{\text{def}}{=} x_0 + iy_0 \in U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$. Assume that f is holomorphic at z_0 and $f'(z_0) \neq 0$. Consider two differentiable curves $\gamma_1 : [a_1, b_1] \longrightarrow U$ and $\gamma_2 : [a_2, b_2] \longrightarrow U$ such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, where $t_1 \in [a_1, b_1]$ and $t_2 \in [a_2, b_2]$. Assume that $\gamma'(t_i) \neq 0$, for $i = 1, 2$. It is then easy to see that,

$$\begin{aligned}\frac{\operatorname{Re}\left((f \circ \gamma_1)'(t_1)\overline{(f \circ \gamma_2)'(t_2)}\right)}{|(f \circ \gamma_1)'(t_1)||(f \circ \gamma_2)'(t_2)|} = \frac{\operatorname{Re}\left(f'(z_0)\gamma_1'(t_1)\overline{f'(z_0)\gamma_2'(t_2)}\right)}{|f'(z_0)\gamma_1'(t_1)||f'(z_0)\gamma_2'(t_2)|} &= \frac{\operatorname{Re}\left(|f'(z_0)|^2\gamma_1'(t_1)\overline{\gamma_2'(t_2)}\right)}{|f'(z_0)|^2|\gamma_1'(t_1)||\gamma_2'(t_2)|} \\ &= \frac{\operatorname{Re}\left(\gamma_1'(t_1)\overline{\gamma_2'(t_2)}\right)}{|\gamma_1'(t_1)||\gamma_2'(t_2)|}.\end{aligned}$$

Hence f preserves angles at z_0. Furthermore, since $\det f'(x_0, y_0) = |f'(z_0)|^2 > 0$, one obtains that f preserves oriented angles.

Conversely, if f is differentiable at (x_0, y_0) and it preserves oriented angles, then from the matrix representation of $f'(x_0, y_0)$ with respect to the standard basis, as given in (2.5), it is clear that Cauchy-Riemann equations are satisfied. This implies that f is holomorphic at z_0. Further to that, one has $f'(z_0) \neq 0$ as $\det f'(x_0, y_0) > 0$. Thus we prove the following:

Theorem 4.1. *Let $z_0 \stackrel{def}{=} x_0 + iy_0 \in U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$. Then the following are equivalent:*

(C.1) *f is holomorphic at z_0 and $f'(z_0) \neq 0$.*

(C.2) *f is differentiable at (x_0, y_0) and it preserves oriented angles.*

Recall that, $f : U(\subseteq_{open} \mathbb{C}) \longrightarrow \mathbb{C}$ is said to be *conformal* if it is holomorphic and f' does not vanish anywhere in U. The following necessary and sufficient condition for conformality is an imemdiate consequence of Theorem 4.1:

Corollary 4.1. *Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$ be differentiable. Then f is conformal if and only if it preserves oriented angles at every point of U.*