- What is a partial differential equation (PDE):

It is an equation which relates an unknown function u and its partial derivatives togther with independent variables.

⊛ In PDE, 2 or more "independent variable" should present.

⊛ If we are only dealing with one independent variable, then it becomes an ODE.

General form :- Let $x \in \mathbb{R}^n$;

$$F(x_1, x_2, \dots, x_n, u_{x_1}, u_{x_2}, \dots u_{x_n}, u_{x_1 x_1}, \dots) = 0.$$

Example :-

(i) $u_x + e\, u_y = 1$.

(ii) $u_{xx} + u_{yy} = 0$ or $\Delta u = 0$.

(iii) $u_x + x\, u_y = 0$.

A heuristic modeling of variation of temperature in a room

We define the temperature by u.

- depends on position (x, y, z) and time t.

Let we are in a large room.

Assume we could measure the temperature at time $t=0$; and at the boundary. (known quantity)

- At a fixed time, temperature can vary from place to place, and change is measured by u_x, u_y, u_z.

- On the other hand; at a fixed place the temperature can change with time, and measured by u_t.

- If we assume their is no heat source or sink. and the temperature was initialy constant; therfore $u_x = u_y = u_z = 0$ at $t=0$, then there will be no temporal change in temperature, i.e $u_t = 0$ for all space and all ~~op~~ over the time.
  And we will have no ~~special~~ spatial variation for positive time, i.e $u_x = u_y = u_z = 0$ for $t > 0$.

- If we have a "non-zero spatial gradiant"; heat would flow from hot to cold as time varies.

- Therefor ~~time~~ change in time u_t should have some relation with gradiant. ($\nabla u = (u_x, u_y, u_z)$).

- A modeling with temperature flux, along with divergence theorem, will yield the diffusion Eq. $\boxed{u_t = \alpha \, \mathrm{div}(\nabla u)}$.

Solution of PDE:—

A solution (more precisely, clasical) to a PDE in a domain $\Omega \subseteq \mathbb{R}^n$, is a sufficiently smooth function $u(x)$ which satisfies the defining equation F for all values of the independent ~~var~~ variables in Ω.

Note:—

1. For most of the PDE's, it is impossible to guess a function which satisfies the equation.

2. In this course, although, we will look for solutions of PDEs in various special cases (trivial cases). But, that should not give an impression that PDEs are usually solvable.

3. There are weaker notion of solution aswell. As in most of the cases, "classical solution" may not exists. But that should not restrict us to study PDEs.

- Eikonal Equation:—

  In 1-d, $|u'(x)| = 1$.

  physical interpretation of this kind of equation comes from "continuous shortest path problem".

  → The solution of $|\nabla u(x)| = \frac{1}{f(x)}$, $x \in \Omega \subset \mathbb{R}^n$

  with $u = 0$ on $\partial\Omega$,

  can be interpreted as the lowest time required to travel from x to $\partial\Omega$, where $f : \bar{\Omega} \to (0, +\infty)$ is the speed of travel.

  → By simple application of Role's theorem (for dimension 1) we see that solution of $|u'(x)| = 1$ with $u = 0$ at 0 and 1 can not be differentiable everywhere in $(0,1)$.

→ But, $u(x) = |x| - 1$; is already a solution for it except $x=0$.

→ Goal; we should generalize the notion of solution so that we can include "good candidat

→ For general dimension, this led to the ~~developm~~ discovery of "viscosity solution" theory. Pioneering contribution by P.L. Lions & M.G. Crana Lions got Fields Medal due to that.

Order of PDE:—

The order of the PDE is the order of the heighst derivatives ~~which~~ present in the equation.

Example:- $u_x + c u_y = 0$ → Order 1

$\Delta u = 0$ → order 2

$u_x u_{xy} + u_{xx} + (u_x)^3 = 0$ → order 2.

Linear PDE:—

- First write the PDE is following form:

All the terms containing u and its partial derivativ $=$ all the terms involving "only" the independent variable.

Denote $\mathcal{L}(u) = f(x)$.

Definition:- We say a PDE is linear if $\mathcal{L}$ is linear in u. That is

$$\mathcal{L}(u_1 + u_2) = \mathcal{L}u_1 + \mathcal{L}u_2$$

Nonlinear PDEs :-

Definition :- A PDE is called non-linear, if it is not linear.

- There are different type of nonlinearity, which have profound effect of complexity.

Definition (Semilinear, Quasilinear, fullynonlinear)

A PDE of order K is called

Semilinear : if all the K^{th} order partial derivatives having co-efficient depending only on independent variable.

Quasilinear : if all the K^{th} order partial derivatives having co-efficient depending only on the independent variables, u and its partial derivatives of order strictly less than K.

Fullynonlinear : if it is not quasi linear.

Note :(i) We should focus on ~~heighst deriv~~ the coefficient of height partial derivatives to determine linearity or nonlinearity.

(ii) ~~Linear~~ Linear PDE $\subset$ Semilinear $\subset$ Quasilinear

Examples :-

(i) General form of Linear First Order PDE :

$$a(x,y)\, u_x + b(x,y)\, u_y = c(x,y)\, u + d(x,y).$$

(PDE in dim 2).

(ii) $x u_x + e^y u_y = e^x \rightarrow$ Linear

(iii) $x u_x + e^y u_y = u(e^x) \rightarrow$ ~~Semilinear~~ linear

(iv) $u_{xy} = 0 \rightarrow$ Linear

(v) $x^2 u_{xy} = 0 \rightarrow$ Linear

(vi) $u^2 u_{xy} = x^2 \rightarrow$ quasilinear

(vii) $(u_x)^2 + (u_y)^2 = u^2 \rightarrow$ fully nonlinear.

Homogeneous / nonhomogeneous PDE :-

The general form of PDE can be writen as

$$\mathcal{L}(u) = f(x).$$

Definition :- If $f(x) = 0$, then we say the PDE is homogeneous. Otherwise, we say it is nonhomogeneous.

Principle of superposition for Linear PDEs :-

- If u_1 is a solution of $\mathcal{L}(u_1) = f_1$ and u_2 is a soution of $\mathcal{L}(u_2) = f_2$. Then
$$\mathcal{L}(au_1 + bu_2) = af_1 + bf_2.$$

General Solution, Auxiliary Condition, Well-posedness

- For ODE it has infinitely many solution and the class of infinitely many solution is parametrized by arbitrary constant.

- PDE will also have infinitely many solution and parametrized by arbitrary function.

Example:- in $\mathbb{R}^2$; $u(x,y)$ solves

$$u_x = 0,$$

then $u(x,y) = f(y)$ is general solution.

similarly, in $\mathbb{R}^3$, $u(x,y,z)$ solves

$$u_x = 0.$$

then $u(x,y,z) = f(y,z)$ is the general solution.

Auxiliary Condition :—

- PDE is often be supplemented by an auxiliary condition.
- This is only physically relavant.
- Auxiliary Condition :- We specify (a-priori), in some subset of the domain, the value of the solution u and/or its partial derivatives.
- In general; enforcing an auxiliary condition will yield a "unique solution".

- Generally; if there are n-independent variable; then the auxiliary condition is set on an $(N-1)$-dimensional subset Γ of the domain Ω.

  Example; for $n=2$; the auxiliary condition are specifie on a curve.

(*) There are two natural classes of auxiliary conditi

(i) Initial value problems (IVP)

(ii) Boundary value problems (BVP).

(i) IVP :- When one of the independent variables represent time "t". The solution is given by $u(x,t)$ and a natural condition is to specify value (and/or it value of derivatives) at $t=0$.

Example :- a) IVP of Heat eq$\underline{n}$:-

$$\begin{cases} u_t = \Delta u & \text{in } (x,t) \in \mathbb{R}^n \times (0,\infty). \\ u(x,0) = g(x). & \text{for } x \in \mathbb{R}^n \end{cases}$$

b) IVP of wave Eq. :-

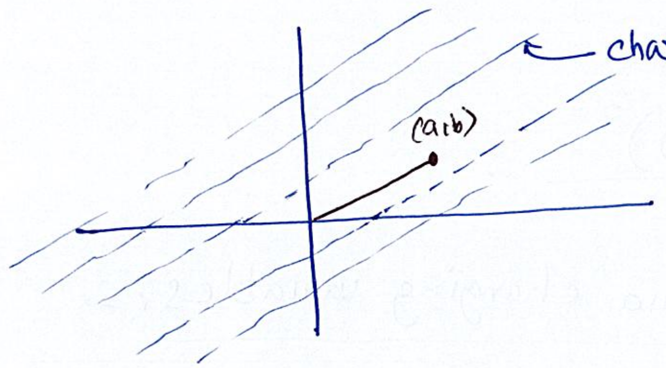
$$\begin{cases} u_{tt} = c\,\Delta u & \text{in } (x,t) \in \mathbb{R}^n \times (0,\infty). \\ u(x,0) = g(x);\ u_t(x,0) = f(x) & \text{for } x \in \mathbb{R}^n. \end{cases}$$

A simple example :- $a u_x + b u_y = 0$.

$$\Leftrightarrow (a,b)\cdot(u_x, u_y) = 0. \Leftrightarrow (a,b)\cdot \nabla u = 0$$

Or equivalently;

$$\frac{\partial u}{\partial p} = 0, \text{ where } p = \frac{(a,b)}{\sqrt{a^2+b^2}}$$



Therefore the directional derivative is 0 along the direction p.

· Therefore, on any line parallel to (a,b)-vector; the solution u of the PDE must be constant.

Any line parallel to (a,b) is given by.

$$bx - ay = c;$$

For different values of 'c' we get different line. Therefore $(x_1,y_1), (x_2,y_2)$ are at same line when

$$bx_1 - ay_1 = bx_2 - ay_2.$$

The values may change from one line to another

Hence, the solution is given by

$$u(x,y) = f(bx-ay)$$

for some arbitrary fn.

particular Example :-

$$2u_x + 3u_y = 0$$

$$u(x,0) = x^3 \quad \leftarrow \text{Auxiliary condition.}$$

The solution is given by

$$u(x,y) = f(3x - 2y)$$

where $u(x,0) = x^3 = f(3x)$

$$\Leftrightarrow f(x) = \frac{x^3}{27}.$$

Therefore, the solution is

$$u(x,y) = \frac{(3x-2y)^3}{27}.$$

Solution of $a u_x + b u_y = 0$ via changing variables :-

Take $\xi = ax + by$, $\eta = bx - ay$.

Replace derivatives of u_x, u_y, by u_ξ, u_η.

$$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = a u_\xi + b u_\eta .$$

$$u_y = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = b u_\xi - a u_\eta$$

Then, $a u_x + b u_y = a^2 u_\xi + ab u_\eta + b^2 u_\xi - ab u_\eta = (a^2 + b^2) u_\xi$

As $a u_x + b u_y = 0 \Rightarrow u_\xi = 0.$

Therefore $u = f(\eta) = f(bx - ay)$ $\square$

Example 2:- $u_x + y u_y = 0.$ ⟶ (i)

Therefore, $(1, y) \cdot \nabla u = 0$.

So, the above information sugests that any solution must not change in certain direction, But now this direction changes point to point. If we can find a curve $y(x)$ which "move" in this direction, then the solution should be constant on these curves.

How do we find these curves?

Ans:- At any point $(x, y(x))$ on such a curve the tangent vector $\left(1, \frac{dy}{dx}\right)$ to be parallel to $(1, y)$. Therefore, we want

$$\frac{dy}{dx} = \frac{y}{1} = y. \longrightarrow \text{(ii)}$$

or $y = Ce^x$, for arbitrary constant C.

Now take the curve $y(x)$ which solves (ii); and take the solution u of (i) on the graph. In other word, take $u(x, y(x))$. We get

$$\frac{d}{dx}[u(x, y(x))] = u_x + u_y \frac{dy}{dx} = u_x + y u_y = 0.$$

Therefore; u is constant on the curve, the curve is given by (x, ce^x).

Now; $u(x, Ce^x) = u(0, Ce^0) = u(0, C) = u(0, e^{-x}y)$

Therefore, $u(x, y) = f(e^{-x}y)$ [Auxiliary condn. Take, $u(0, y) = f(y)$]

Example 3 :— $au_x + bu_y + c\,u = 0$. (directional derivatives are not zero now!).

By change of variable

$$\xi = ax + by, \quad \eta = bx - ay.$$

$$(a^2+b^2)\, u_\xi(\xi,\eta) = -u(\xi,\eta).$$

Solving an ODE we get

$$u(\xi,\eta) = f(\eta)\, e^{-\frac{\xi}{a^2+b^2}}$$

In terms of original variable

$$u(x,y) = f(bx - ay)\, e^{-\frac{ax+by}{a^2+b^2}}$$

Example 4 :- $u_x + y u_y + u = 0$?

(*) The charectaristics are not line;
and solution is also not constant on charectaristic curves.

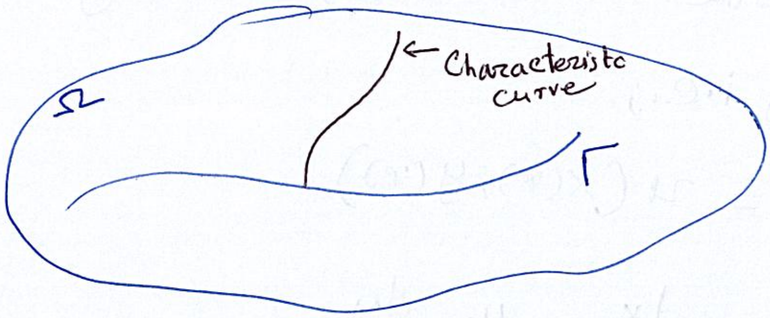
Note :- ① So far; in previous examples, we proceed by finding general solution, then chose aux. condn.

② For the method of charactaristic based upon a direct link with auxiliary condn.

Method of Characteristics (Linear Eq.)

We consider this method for 2-independent variable.

Suppose; the PDE to be solved in $\Omega \subset \mathbb{R}^2$ and the auxiliary condition given on $\Gamma \subset \bar{\Omega}$.



The most general linear PDE is given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y) \quad \rightarrow (1)$$

Let Γ_0 be the curve where data is prescribed. (Initial curve).

Assume Γ_0 is given in a parametric form;

$$\Gamma_0 = \{ (x_0(s), y_0(s)) : s \in I \subset \mathbb{R} \}.$$

The given data:— $u(x_0(s), y_0(s)) = u_0(s)$.

Note that; left hand side of (1) is the directional derivative of u in the direction of $(a(x,y), b(x,y))$;

Thus, if we consider any curve $(x(r), y(r))$, parametrized by r variable; such that the tangent of it at each point on this curve is $(a(x,y), b(x,y))$; then the left hand side of (1) is the total derivative ~~along~~ of u, along curve $(x(r), y(r))$.

characteristic curve

In parametric form

$$\frac{dx}{dr} = a(x(r), y(r)) \quad ; \quad \frac{dy}{dr} = b(x(r), y(r)) \quad \rightarrow (2)$$

The curve $(x(r), y(r)) \longrightarrow$ characteristic curve.

Let $z(r)$ denote the solution u along the characteristics ; i.e.,;

$$z(r) := u(x(r), y(r)).$$

Then;
$$\frac{dz}{dr} = u_x \frac{dx}{dr} + u_y \frac{dy}{dr}$$
$$= a(x(r), y(r))\, u_x + b(x(r), y(r))\, u_y.$$

Now, by (1) we get

$$\frac{dz}{dr} = c_0(x(r), y(r))\, z(r) + d(x(r), y(r)).$$

We solve (2) with different initial conditions $(x(0), y(0))$, to get different characteristic curve.

Now, we choose ~~charact~~ auxiliary condition (Γ_0 such a way that the characteristic curve interset it

For <u>fixed</u> s; we consider

$$x(0) = x_0(s) \quad ; \quad y(0) = y_0(s).$$

(*) Under the appropriate assumptions on a and b, for example, taking a, b are C^1-functions, the "existen and unqiueness result from ODE", there exists a unique solution (characteristics curve) in a nbd. of $r = ($

Again, the curve depends on the initial values, hence (4)
on s (we fixed it, while solving the ODEs). To emphasise the dependence, we denote the curve by

$$\left(x(r,s),\, y(r,s)\right).$$

More precisely; for fixed s, the points $(x(r,s), y(r,s))$ moves along the charateristic curve for ~~~~ r small around $r=0$. As s varies; we get different characteristic curves.

Now; for fixed s; we get an equation of u (or in terms of z) as

$$\frac{d}{dr} z(r,s) = c\left(x(r,s), y(r,s)\right) z + d\left(x(r,s), y(r,s)\right).$$

with the initial value

$$u\left(x(0,s), y(0,s)\right) = u_0(s).$$

or, $$z(0,s) = u_0(s).$$

We obtain following system of ODEs, which is called <u>Characteristic Equation</u>

$$\begin{cases} \cancel{\frac{dx}{dr} = a(x(r), y(r))} \quad \frac{dx}{dr} = a\left(x(r,s), y(r,s)\right). \\ \qquad\qquad\qquad\qquad \frac{dy}{dr} = b\left(x(r,s), y(r,s)\right) \\ \frac{dz}{dr} = c\left(x(r,s), y(r,s)\right) z + d\left(x(r,s), y(r,s)\right). \\ x(0,s) = x_0(s) \quad ; \quad y(0,s) = y_0(s) \\ z(0,s) = u_0(s) \end{cases}$$

Notice; as the PDE is Linear; we can solve first two equation together without solving the third.

Terminology :-

The curve, $r \longmapsto (x(r,s), y(r,s), z(r,s))$, for fixed s is called the "Characteristic Curve", while the curve $(x(r,s), y(r,s))$ is called the Projected ~~Chad~~ Characteristics.

Let us verify this method through some example.

Example 1 :- $\begin{cases} a u_x + b u_y = 0 \\ u(x,0) = x^3 \end{cases}$

The Γ_0 curve is given by $(x_0(s), y_0(s)) = (s, 0)$.

and $u_0(s) = s^3$.

Now the Characteristic eqns are

$$\begin{cases} \dfrac{dx}{dr} = a \ , \ \dfrac{dy}{dr} = b \ , \ x(0,s) = s, \ y(0,s) = 0. \\ \dfrac{dz}{dr} = 0 . \text{ and } z(0,s) = s^3. \end{cases}$$

The characteristic curve (Projected Char.) is given by

$x = ar + c_1$, $y = br + c_2$, now as $x(0,s) = s$ and $y(0,s) = 0$

$\Rightarrow c_1 = s$ and $c_2 = 0$.

$\Rightarrow \boxed{x = ar + s, \quad y = br.}$

Again, $z(r,s) = c_3$ and as $z(0,s) = s^3$. $\Rightarrow c_3 = s^3$.

$\Rightarrow z(r,s) = s^3$ $\quad (u(x(r,s), y(r,s)) = z(r,s))$

Now, we write, r and s in terms of x and y.

From Projected Char.

$$r = \frac{y}{b} \quad \text{and} \quad x = a \cdot \frac{y}{b} + s.$$

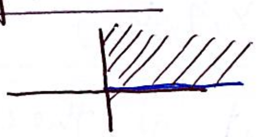
$$s = \frac{bx - ay}{b}.$$

(To find; for point (x,y), what characteristic curve it corresponds to, and this Char curve interset Γ_0, i.e, x-axis at what point)

Therefore the solution.

$$\boxed{u(x,y) = \frac{(bx-ay)^3}{b^3}.}$$

Example 2 :—



$$\begin{cases} -y u_x + x u_y = u \\ u(x,0) = g(x). \quad \text{for } x > 0. \end{cases}$$

The auxiliary condn. is given on $\Gamma_0 = (x_0(s), y_0(s)) = (s, 0)$ $\forall s \in \mathbb{R}$

If r is the parameter for • PC (Projected Characteristic) then

$$\frac{dx}{dr} = y, \quad \frac{dy}{dr} = x \quad \text{and} \quad \frac{dz}{dr} = z.$$

initial data: $x(0,s) = s$, $y(0,s) = 0$ and $z(0,s) = g(s)$.

1st two eqn. corresponds to system of ODE, and the solution is given by

$$\left.\begin{aligned} x(r,s) &= c_2 \cos r - c_1 \sin r \\ y(r,s) &= c_1 \cos r + c_2 \sin r \end{aligned}\right\}$$ where, c_1, c_2 are constants to be determined from initial data.

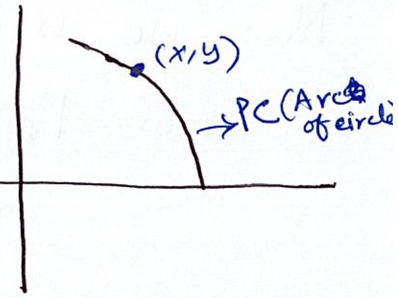
We get from initial data that

$$c_2 = s, \quad c_1 = 0.$$

Then, $x = s\cos r$, $y = s\sin r$.

On the other hand, $z(r,s) = g(s)\, e^{r}$.

Note that the PC are the arcs of circles.



Now; $s^2 = x^2 + y^2$ and $r = \arctan\frac{y}{x}$;

then plugging into $Z(r,s)$ we get

$$\boxed{u(x,y) = g\left(\sqrt{x^2+y^2}\right) e^{\arctan\frac{y}{x}}}$$

Method of Characteristics for Semilinear PDE:-

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y,u)$$

Although this is a nonlinear PDE; but as there is no nonlinear term involving derivatives of u; the PDE behaves more like linear PDE.

Our previous method of characteristic will work exacty same way with one difference: Once we determine $x(r,s)$, $y(r,s)$, we have to solve a nonlinear ODE:

$$\frac{dZ}{dr} = c\big(x(r,s), y(r,s), Z(r,s)\big).$$

Example 1:-

$$\begin{cases} u_x + 2u_y = u^2 \\ u(x,0) = g(x). \end{cases}$$

Again as previous solving PC, we get

$$x(r,s) = r + s, \quad y(r,s) = 2r.$$

We now need to solve $\frac{dZ}{dr}(r,s) = Z^2(r,s)$.

the general solution is given by

$$z(r,s) = \frac{1}{-c-r},$$

Now as $z(0,s) = g(s)$, then

$$c = \frac{-1}{g(s)} \Rightarrow z(r,s) = \frac{g(s)}{1 - r\,g(s)}$$

Again as; ~~...~~

$$r = \frac{y}{2} \quad ; \quad s = x - \frac{y}{2}$$

Finally, the solution of the PDE is given by.

$$\boxed{u(x,y) = \frac{g\left(x - \frac{y}{2}\right)}{1 - \frac{y}{2}\, g(x - y/2)}.}$$

$$\cancel{\lim_{h \to 0} \frac{u(t_0+h)-u(t_0)}{h}}$$

Theorem :– If a, b are assumed to be C function...

Then "PC" of any semilinear PDE

0!

$$a(x,y)\,u_x + b(x,y)\,u_t = c(x,y,u) \quad \text{in } \Omega.$$
$$u(x,0) = g(x).$$

can never intersect.

proof: The proof is by contradiction. ~~which start~~

let $(x_1(r,s_1), y_1(r,s_1))$, $(x_2(r,s_2), y_2(r,s_2))$

intersect each other at (x,y).

$$x_1(r_1,s_1),\ y_1(r_1,s_1) = \big(x_2(r_2,s_2),\ y_2(r_2,s_2)\big) = (x,y).$$

Define a new curve

$$r \longmapsto (x_3(r,s),\ y_3(r,s)) \text{ by.}$$

$$x_3(r) = x_2(r - r_1 + r_2),\quad y_3(r) = y_2(r - r_1 + r_2)$$

(Reparametrization of ~~curve~~ (x_2, y_2) curve, so that it reache (x,y), at r_1).

Therfore;
$$x_3(r_1) = x_2(r_2) = x_1(r_1) = x,$$
$$y_3(r_1) = y_2(r_2) = y_1(r_1) = y.$$

Now,
$$x_3'(r) = x_2' = (r - r_1 + r_2) = a\big(x_2(r - r_1 + r_2),\ y_2(r - r_1 + r_2)\big)$$
$$= a(x_3(r), y_3(r)).$$

slly; $y_3'(r) = b(x_3(r), y_3(r))$.

with the initial condition $x_3(r_1) = x$, $y_3(r_1) = y$.

$$\begin{cases} x_3'(r) = a(x_3'(r), y_3'(r)). \\ y_3'(r) = b(x_3'(r), y_3'(r)) \\ x_3(r_1) = x,\ y_3(r_1) = y. \end{cases}$$

But; (x_1, y_1) also solve the same equation. That is a contradiction to the uniqueness of ODE. $\square$.

Example (Quasilinear).

Consider; $\begin{cases} u_t + u u_x = 0 \\ u(x,0) = g(x). \end{cases}$

By solving the Characteristic equation we get

$$x(r,s) = g(s)\, r + s \;;\quad t(r,s) = r.$$

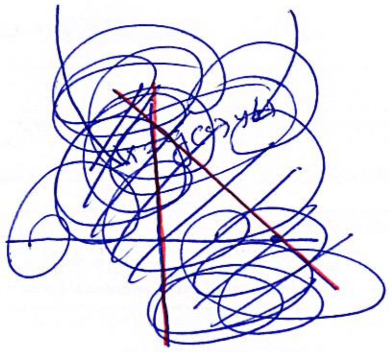
If we try to draw the characteristic curve

For fixed s; $\boxed{x = g(s)\, t + s}$

For example; if we consider; $g(x) = x^2$

then; taking $s=0$; we get the chachacteristiclin

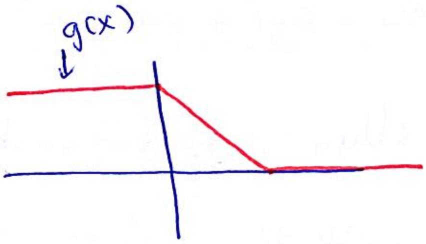
$x = 0$. On the otherhand taking $s = 1$; $x = t + 1$



particular Example :-

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = g(x). \end{cases}$$

where, $g(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 1-x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$



Then $z(r,s) = g(s)$, $x(r,s) = g(s)\, r + s$, $t(r,s) = r$.

Now, we explicitly solve x, y in terms of $t, s)$, in general.

Solution $u(x,t) = \begin{cases} 1 & x \le t \\ \frac{1-x}{1-t} & t < x < 1 \\ 0 & x > 1 \end{cases}$

Tangent Vectors:-

$v \in \mathbb{R}^3$ is called a tangent vector to S $\subseteq \mathbb{R}^3$ at a point A, if there exists a curve $t \mapsto \gamma(t) \subseteq S$ such that

$$\gamma(0) = A, \quad \gamma'(0) = v.$$

Tangent Plane/space:-

The set of all tangent vectors at a point A forms the tangent space at A of S.

It is a vector space of dimension 2.

Normal Space:-

The set of vectors in $\mathbb{R}^3$ that are perpendicular to all the vectors in tangent ~~pa~~ plane.

Dimension of normal space is 1.

⊙ Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Then the graph of the function is given by the set

$$S := \{(x,y,z) : z = u(x,y)\}.$$

Then S is a 2-dimensional surface in $\mathbb{R}^3$.

Lemma:- $(u_x, u_y, -1)$ is normal to the surface S.

proof:- ~~We have to show; <~~ Let, $N = (u_x, u_y, -1)$.
We have to show $\langle N, v \rangle = 0$ $\forall v$ in tangent space of S at A.
v being a tangent vector at A, $\exists \gamma(t) = (x(t), y(t), z(t))$ curve in S such that $\gamma'(0) = v$ and $\gamma(0) = A$.

By definition of surface

$$z(t) = u(x(t), y(t)).$$

Then $\dot{z}(t) = u_x x'(t) + u_y y'(t)$.

Therefore.

$$u_x(x(0), y(0))\, x'(0) + u_y(x(0), y(0))\, y'(0) - z'(0) = 0.$$

$$\Rightarrow (u_x, u_y, -1) \cdot (x'(0), y'(0), z'(0)) = 0.$$

$$\Rightarrow \langle N, \gamma'(0) \rangle = 0 \Rightarrow \langle N, v \rangle = 0.$$

Integral surface :-

Let $V: \mathbb{R}^3 \to \mathbb{R}^3$ (vector field). A 2-dimensional surface S is said to be integral surface of V if $V(x)$ is a tangent vector of S at x for all x.

Example :- Take $V = (-y, -x, 0)$.

Then. $S: x^2+y^2+z^2 = k \rightarrow$ integral surface.

S can be writen as $z = \pm\sqrt{k - x^2 - y^2}$

Then, $(u_x, u_y, -1)$ is normal to S at (x, y, z).

We can prove $(u_x, u_y, -1) \cdot (-y, -x, 0) = 0$.

Geometric Interpretation for the solution

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u).$$

$$\Leftrightarrow (a, b, c) \cdot (u_x, u_y, -1) = 0$$

This says that our solution u has to be such that at it is an integral surface of the vector field (a, b, c).

Example :-

$$y u_x - x u_y = 0.$$

Then, $(a, b, c) = (y, -x, 0)$

As the graph of our solution is the intgral surface to the vector field $(y, -x, 0)$, which implies.

$(x^2 + y^2 + z^2) = k$. $\rightarrow$ This is the solution of the PDE.

Example of Quasilinear PDE :-

$$\begin{cases} x u_x + u u_y = xy. \\ u(x, y) = x \quad \text{on } xy = 1, \ y > 0. \end{cases}$$

Step 1 :- parametrize the initial curve

$\Gamma_0 = (x_0(s), y_0(s)) = (s, \frac{1}{s})$. for $s > 0$.

Step 2 :- The Char. Eq.

$$\frac{dx}{dr} = x \quad , \ x(0, s) = s$$

$$\frac{dy}{dr} = y \quad , \ y(0, s) = \frac{1}{s}.$$

$$\frac{dz}{dr} = xy \quad , \ z(0, s) = u(s, \tfrac{1}{s}) = s.$$

Then, Step 3 :- Solving the ODEs.

$x(r, s) = s e^r$, $y(r, s) = \frac{e^r}{s}$; then $\frac{dz}{dr} = e^{2r}$.

Then $z(r, s) = \frac{e^{2r}}{2} + C$; now $s = z(0, s) = \frac{1}{2} + C$.

then $z(r, s) = e^{2r}/2 + s - 1/2$.

Step 4 :- Write r, s in terms of x and y.

$$x = se^{r}, \quad y = \frac{e^{r}}{s} \quad \Rightarrow \quad xy = e^{2r}$$

Again, $s = \dfrac{\sqrt{xy}}{y} = \dfrac{\sqrt{x}}{\sqrt{y}}$.

Finally, the solution is given by

$$u(x,y) = \frac{xy}{2} + \sqrt{\frac{x}{y}} - 2. \qquad \text{for } \begin{matrix} x>0 \\ y>0. \end{matrix}$$

Step 5 :- tranversality cond$^{\underline{n}}$.

$$\boxed{\det \begin{pmatrix} a & x_0'(s) \\ b & y_0'(s) \end{pmatrix}}$$

0!

$$= a y_0'(s) - b x_0'(s)$$

$$= s\left(-\frac{1}{s^2}\right) - \frac{1}{s}\cdot 1 = \frac{-2}{s} \neq 0, \quad s>0.$$

Example 2 :-

$$\begin{cases} (x+u)\,u_x + y u_y = u + y^2 \\ u(x,1) = x \end{cases}$$

And $\Omega = \{(x,y) : y \geq 1\}$.

~~Sr~~ proof :- $y(r,s) = e^{r}, \quad z(r,s) = e^{2r} + (s-1)e^{r}$

$$x(r,s) = e^{2r} + (s-1)e^{r}(r+1).$$

$$r = \log y, \quad s = 1 + y\,\frac{x - y^2}{\log y + 1}$$

$$u(x,y) = y^2 + \frac{(x-y^2)}{\log y + 1}$$

$$u u_x + z u_y + u_z = y, \quad u(x,y,0) = x$$

$$u u_x + t u_y + u_t = y, \quad u(x,y,0) = x$$

Existence & Uniqueness

Theorem:- Consider the following PDE;

$$\begin{cases} a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u) \text{ in } \Omega. \\ u(\Gamma_0) = g \end{cases}$$

Let, a, b, c are C^1 function from $\mathbb{R}^3$ to $\mathbb{R}$ and $g \in C^1(\mathbb{R})$

Let us assume (transvarsality condition).

$$x_0'(s_0)\, b(x_0(s_0), y_0(s_0), g(s_0)) - y_0'(s_0)\, a(x_0(s_0), y_0(s_0), g(s_0)) \neq 0.$$

Then there exists a neighbourhood Ω around $(x_0(s_0), y_0(s_0))$ where the above quasilinear PDE has a "unique" solution.

proof:- The Characteristic ODE is given by

$$(\text{ODE}) \begin{cases} \dfrac{dx}{dr} = a(x(r,s), y(r,s), z(r,s)), & x(0,s) = x_0(s) \\ \dfrac{dy}{dr} = b(x(r,s), y(r,s), z(r,s)), & y(0,s) = y_0(s) \\ \dfrac{dz}{dr} = c(x(r,s), y(r,s), z(r,s)), & z(0,s) = g(s) \end{cases}$$

Then by the existence and uniqueness of ODE, there exists $\varepsilon, \delta > 0$ such that (ODE) has unique solution $(x(r,s), y(r,s), z(r,s))$ for all $(r,s) \in (-\delta, \delta) \times (s_0-\varepsilon, s_0+\varepsilon)$, and

$$A(s,t) := (x(r,s), y(r,s), z(r,s)) \in C^1.$$

Define the map;

$$B(r,s) = (x(r,s), y(r,s)) : (-\delta,\delta) \times (s_0-\varepsilon, s_0+\varepsilon) \longrightarrow \mathbb{R}^2,$$

which is a C^1 function. Now we calculate it Jacobian at $(x(0,s_0), y(0,s_0))$

$$\det\left(\mathrm{Jac}(B)\left(x_0(s_0), y_0(s_0)\right)\right)$$

$$= \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{pmatrix}\Bigg|_{(0,s_0)}.$$

$$= \det\begin{pmatrix} a(x,y,z)\big|_{(0,s_0)} & x_0'(s_0) \\ b(x,y,z)\big|_{(0,s_0)} & y_0'(s_0) \end{pmatrix}$$

$$= y_0'(s_0)\, a\left(x_0(s_0), y_0(s_0), g(s_0)\right) - x_0'(s_0)\, b\left(x_0(s_0), y_0(s_0), g(s_0)\right)$$

$$\neq 0.$$

Then by Inverse function Theorem $\exists$ a nbd Ω around $(x_0(s_0), y_0(s_0))$ and U around $(0, s_0)$ such that

$B : U \to \Omega$ is one-one and onto and inverse exists which is C^1.

If $B^{-1} = \left(R^{-1}(x,y), S^{-1}(x,y)\right) : \Omega \to U$

Method of Characteristics: General 1 order PDE

- We derive the method in general dimension.

$$F(\underset{z}{x,\ u},\ \underset{p}{\nabla u(x)}) = 0 \longrightarrow (1)$$

for some $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, C^1 function,

where $x = (x_1, \dots, x_n)$, $\nabla u = (u_{x_1}, \dots, u_{x_n})$.

- We denote the third argument of the function by p which is a vector in $\mathbb{R}^n$.

<u>Example</u>:- Let; $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and the PDE is given by

$$u u_{x_1} + u_{x_1} u_{x_2} + x_1 u_{x_3} + u^3 = x_2 x_3.$$

Then, by notation $F(x, z, p)$,

$$F(x, z, p) = z p_1 + p_1 p_2 + x_1 p_3 + z^3 = x_2 x_3.$$

- We plan to solve eq. (1) for all x in its domain with $u(x) = g(x)$ in Γ_0.

<u>The Characteristic Equation</u>:-

As previous; we parametrize the characteristic curve with r. and denote

$$x(r) = (x_1(r), x_2(r), \dots, x_n(r))$$

(*) Let us assume we have a C^2-solution of (1).

Along the characteristic; denote

$$z(s) = u(x(s)) \quad \text{and} \quad p(s) = \nabla u(x(s))$$
$$:= (p_1(s), \dots, p_n(s)).$$

<u>Question</u>:- Why do we introduce $p(s)$ here?

Let us first focus on $p(s)$; gradiant of u along $x(s)$. Suppose we wish to compute how it should evolve along the characteristic

$$\dot{p}_i(s) = \frac{d p_i(s)}{d s} \quad ; \ i = 1, \dots, n.$$

By the chain rule:

$$\dot{p}_i(s) = \frac{d}{ds} u_{x_i}(x(s)) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \, \dot{x}_j(s) \qquad \text{——— (2)}$$

BUT, R.H.S of (2) consist 2nd derivative of u. which we do not keep in track along the characteristic.

<u>Remedy</u>: Choose $x(s)$ such a way that the PDE allows us to compute $\dot{p}(s)$ only from the information of $x(s)$, $z(s)$, $p(s)$.

"Linearize" → Differentiate (1) w.r.to x_i to find

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(x, u, \nabla u) \, u_{x_i x_j} + \frac{\partial F}{\partial z}(x, u, \nabla u) \, u_{x_i} + \frac{\partial F}{\partial x_i}(x, u, \nabla u) = 0.$$

Observe; by taking

$$\dot{x}_j(s) = \frac{\partial F}{\partial p_j}(x(s), z(s), p(s)). \qquad \text{——— (3)}$$

$$\dot{p}_i(r) = -\frac{\partial F}{\partial z}(x(s), z(s), p(s))\, p_i(s) - \frac{\partial F}{\partial x_i}(x(s), z(s), p(s)) \quad \text{--- (4)}$$

On the other hand;

$$\dot{z}(s) = \frac{d}{dr} u(x(s)) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x(r))\, \dot{x}_j(r) = \sum_{j=1}^{n} p_j(x(r))$$

$$= \sum_{j=1}^{n} p_j(x(r)) \frac{\partial F}{\partial p_j}(x(r), z(r), p(r)). \quad \text{--- (5)}$$

Then (3), (4), (5) ~~cons~~ consists of new characteristic Eqn.

<u>For quasilinear Eq.</u>

$$F(x, u, \nabla u) = a(x,u) \cdot \nabla u + b(x,u) = 0.$$

$$F(x, z, p) = a(x,z) \cdot p + b(x,z).$$

Then by char Eq.

$$\begin{cases} \dot{x}_j(r) = a_j(x,z). \\ \dot{z}(r) = -b(x,z). \end{cases}$$

And we do not need p eqn as the are seperate Eq. independent of p.

Compact form:- $\dot{x}(r) = \nabla_p F(x(r), z(r), p(r))$

$$\dot{z}(r) = \nabla_p F(x(r), z(r), p(r)) \cdot p(s)$$

$$\dot{p}(r) = -\nabla_x F(x(r), z(r), p(r)) - \frac{\partial F}{\partial z}(x(r), z(r), p(r))\, p(s)$$