Boundary Value Problems: Shooting Method

Akash Anand
MATH, IIT KANPUR

Example

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$$s_m \to s_*^h \approx s_* = \frac{e - e^{-1}}{(e + e^{-1})^2},$$

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$n = 2/h$	$s_* - s_*^h$	Ratio	$E^h = \max_{0\le i\le n} \lvert u(t_i) - y^h(t_i; s_*^h)\rvert$	Ratio
4	4.01×10^{-3}	–	2.83×10^{-2}	–
8	1.52×10^{-3}	2.64	7.30×10^{-3}	3.88
16	4.64×10^{-4}	3.28	1.82×10^{-3}	4.01
32	1.27×10^{-4}	3.64	4.54×10^{-4}	4.01
64	3.34×10^{-5}	3.82	1.14×10^{-4}	4.00

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$$u'' = p(t)u' + q(t)u + r(t), \qquad a < t < b,$$
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If $u_1(t)$ solves the IVP

$$u'' = p(t)u' + q(t)u + r(t), \qquad u(a) = -\alpha c_1, \qquad u'(a) = -\alpha c_0,$$

and $u_2(t)$ solves the IVP

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$$u(t) = u_1(t) + s\, u_2(t)$$

satisfies

$$a_0u(a) - a_1u'(a) = a_0(-\alpha c_1 + sa_1) - a_1(-\alpha c_0 + sa_0) = \alpha(a_1c_0 - a_0c_1) = \alpha.$$

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that is (why?)

$$s = \frac{\beta - \big(b_0 u_1(b) + b_1 u_1'(b)\big)}{\big(b_0 u_2(b) + b_1 u_2'(b)\big)}.$$

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For approximate solution of the BVP, we form

$$u^h(t_j) = u_1^h(t_j) + s^h\, u_2^h(t_j), \qquad t_j = a + jh, \qquad j = 0,1,\dots,J, \qquad h = (b-a)/J.$$

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However (exercise),

$$s^h - s = -\frac{\left(b_0 e_1(t_J) + b_1 \varepsilon_1(t_J)\right) + s\left(b_0 e_2(t_J) + b_1 \varepsilon_2(t_J)\right)}{\left(b_0 u_2^h(t_J) + b_1 v_2^h(t_J)\right)}.$$

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If $|e_i(t_j)| = O(h^p)$ *and* $|\varepsilon_i(t_j)| = O(h^p)$, *then (why?)*

$$|e(t_j)| = O(h^p).$$