Global Cauchy theorem

Arijit Ganguly
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur

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Theorem 1. *Let $U \subseteq_{open} \mathbb{C}$ and γ be a closed path such that $\gamma^* \subseteq U$. Assume that*

$$\mathrm{Ind}_\gamma(z) = 0,\ \forall z \in \mathbb{C} \setminus U. \tag{$*1$}$$

Then at least one (thence both) of the following equivalent statements holds:

(C.1) $\int_\gamma f = 0$*, for every holomorphic* $f : U \longrightarrow \mathbb{C}$*.*

(C.2) For every holomorphic $f : U \longrightarrow \mathbb{C}$*, one has*

$$\mathrm{Ind}_\gamma(z) f(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{f(w)}{w - z}\, dw,\ \forall z \in U \setminus \gamma^*. \tag{$*2$}$$

We first see the equivalence of (C.1) and (C.2). Assume (C.1). Let $z \in U \setminus \gamma^*$. Consider the following function:

$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z. \end{cases} \tag{$*3$}$$

It is obvious that g is holomorphic on $U \setminus \{z\}$. Choose $r > 0$ such that $D(z; r) \subseteq U$. Since $D(z; r)$ is open and convex and the function g is continuous on $D(z; r)$ and holomorphic on $D(z; r) \setminus \{z\}$, it follows from Morera's theorem that g is holomorphic at z. Now ($*2$) is immediate since $\int_\gamma g = 0$.

To see the converse, fix $z \in U \setminus \gamma^*$. Define $F : U \longrightarrow \mathbb{C}$, $w \mapsto (w - z) f(w)$, $\forall w \in U$. Clearly F is holomorphic. So in view of (C.1), one obtains that

$$\int\limits_\gamma f = \int\limits_\gamma \frac{F(w)}{w - z}\, dw = 2\pi i\ \mathrm{Ind}_\gamma(z)\, F(z) = 0.$$

Remark 1. *If U is star-like and $z \in \mathbb{C} \setminus U$, then the function $w \mapsto \dfrac{1}{w-z}$ is holomorphic on U, and hence it admits a primitive. As a consequence of this, for every closed path γ with $\gamma^* \subseteq U$, we get that*

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z} = 0.$$

In other words, the condition $(*1)$ of Theorem 1 is satisfied by all closed path γ such that $\gamma^* \subseteq U$. Then (C.1) yields that f has a primitive, while the Cauchy's integral formula for a star-like open set is obtained from (C.2). Thus one notes that Theorem 1 generalizes Cauchy's theorem and the integral formula for a star-like region.

Theorem 1 is due to Emil Artin. Artin's proof makes use of greater topological considerations. In fact, in this approach, Theorem 1 reduces to a statement which only involves the index, not the holomorphic function f at all. The proof is highly geometric and surprisingly beautiful, an excellent reference of which is [2, IV, §3]. However, we follow Dixon's approach ([1]) for being succinct and more analytic.

Dixon's proof of Theorem 1. We consider the function $g : U \times U \longrightarrow \mathbb{C}$ defined as follows:

$$g(w,z) = \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases} \tag{$*4$}$$

It is enough to show that $\int_\gamma g(w,z)\,dw = 0$, whenever $z \in U \setminus \gamma^*$.

Step 1. *We show that the function $z \mapsto \int_\gamma g(w,z)\,dw$, $\forall z \in U$, is holomorphic.*

Claim 1. *g is continuous.*

Let $(w_0, z_0) \in U \times U$. The continuity of g at (w_0, z_0) is clear if $w_0 \neq z_0$. So we now assume $w_0 = z_0$. Let $\varepsilon > 0$. Since f' is continuous at z_0, we obtain $\delta > 0$ such that, for all $u \in D(z_0; r)$, $|f'(u) - f'(z_0)| < \varepsilon$. Pick any $z, w \in D(z_0; \delta)$. Then $z + t(w-z) \in D(z_0; \delta)$, for all $t \in [0,1]$. Observe that, if $w \neq z$,

$$g(w,z) = \frac{f(w)-f(z)}{w-z} = \frac{1}{w-z}\int_{[z,w]} f' = \int_0^1 f'(z+t(w-z))\,dt. \tag{$*5$}$$

For $w = z$, $g(w,z) = \int_0^1 f'(z+t(w-z))\,dt$ holds trivially. From this, it follows that,

$$\begin{aligned} |g(w,z) - g(w_0,z_0)| &= \left|\int_0^1 f'(z+t(w-z))\,dt - \int_0^1 f'(z_0)\,dt\right| \\ &\leq \int_0^1 |f'(z+t(w-z)) - f'(z_0)|\,dt \\ &\leq \varepsilon. \end{aligned}$$

This settles Claim 1.

Claim 2. *For any $z \in U$, the one variable function $w \mapsto g(w,z)$ is holomorphic on U. Similarly, for any $w \in U$, the function $z \mapsto g(w,z)$ is holomorphic on U.*

This is apparent the moment we proceed along the line of proof of $(C.1) \Longrightarrow (C.2)$.

Define $\phi : [a,b] \times U \longrightarrow \mathbb{C}$ by $\phi(t,z) = g(\gamma(t),z)\gamma'(t)$, $\forall (t,z) \in [a,b] \times U$. From Claim 1 and Claim 2, it is clear that ϕ is continuous, and for any $t \in [a,b]$, the function $z \mapsto \phi(t,z)$ is holomorphic on U. It now follows that the function $z \mapsto \int_a^b \phi(t,z)\,dt = \int_\gamma g(w,z)\,dw$ is holomorphic on U.

Step 2. *We extend the holomorphic the function $z \mapsto \int_\gamma g(w,z)\,dw$, $\forall z \in U$, to an entire function.*

Let $V \overset{\text{def}}{=} \{z \in \mathbb{C} \setminus \gamma^* : \text{Ind}_\gamma(z) = 0\}$. Since the function Ind_γ is locally constant, V is open. Also, from the hypothesis, we have $\mathbb{C} \setminus U \subseteq V$ so that $U \cup V = \mathbb{C}$. Observe that, whenever $z \in U \cap V$,

$$\int_\gamma g(w,z)\,dw = \int_\gamma \frac{f(w)}{w-z}\,dw - \text{Ind}_\gamma(z) f(z) = \int_\gamma \frac{f(w)}{w-z}\,dw,$$

as $\text{Ind}_\gamma(z) = 0$. This allows us to define the function $h : \mathbb{C} \to \mathbb{C}$ given by,

$$h(z) = \begin{cases} \displaystyle\int_\gamma g(w,z)\,dw & \text{if } z \in U \\ \displaystyle\int_\gamma \frac{f(w)}{w-z}\,dw & \text{if } z \in V. \end{cases}$$

Since the function $z \mapsto \int_\gamma \frac{f(w)}{w-z}\,dw$, is holomorphic on $\mathbb{C} \setminus \gamma^*$, it is obvious that h is holomorphic everywhere on $\mathbb{C}$.

Step 3. *We now show, using Liouville's theorem, that h is constant.*

Choose $R > 0$ such that $\gamma^* \subseteq D(0;R)$. The connected subset $\mathbb{C} \setminus D(0;R)$ must be contained in one of the connected components, say $\mathfrak{C}$, of $\mathbb{C} \setminus \gamma^*$. Hence $\mathfrak{C}$ is unbounded. As Ind_γ is identically 0 on $\mathfrak{C}$, one has $\mathbb{C} \setminus D(0;R) \subseteq \mathfrak{C} \subseteq V$. This implies that $\mathbb{C} \setminus V \subseteq D(0;R)$. From this we see that h is bounded on the complement of V, as it is bounded on $\overline{\mathbb{C} \setminus V}$. Therefore, it suffices to show that h is bounded on V.

Consider any $M > 0$. Let us split V into two parts as follows:

$$V'_M \overset{\text{def}}{=} \left\{z \in V : \min_{a \le t \le b} |\gamma(t) - z| > M\right\}, \tag{$*6$}$$

and

$$V''_M \overset{\text{def}}{=} \left\{z \in V : \min_{a \le t \le b} |\gamma(t) - z| \le M\right\}. \tag{$*7$}$$

Claim 3. *For any $z \in V'_M$, $|h(z)| \le L_\gamma \cdot \sup_{w \in \gamma^*} |f(w)| \cdot \frac{1}{M}$*

This follows directly from ML-inequality.

Claim 4. *V''_M is bounded, whence h is bounded on V''_M.*

Let $z \in V''_M$. Then $\exists t_0 \in [a, b]$ such that $|\gamma(t_0) - z| \leq M$. This implies that $|z| \leq |\gamma(t_0) - z| + |\gamma(t_0)| \leq M + R$.

The boundedness of h is now immediate from Claim 3 and Claim 4. From Liouville's theorem, we conclude that $\exists c \in \mathbb{C}$ such that $h(z) = c$, for every $z \in \mathbb{C}$.

Step 4. *The final stage is to show that $c = 0$.*

As V is not bounded, it is easy to see that, $\forall M > 0$, $V'_M \neq \emptyset$. It thus follows at once from Claim 3 that

$$|c| \leq \frac{L_\gamma \cdot \sup_{w \in \gamma^*} |f(w)|}{M}, \forall M > 0. \tag{$*8$}$$

The estimate given by ($*8$) shows that $c = 0$. This completes Step 4 and hereby the proof of Theorem 1.

$\square$

References

[1] Dixon, John D.; *A brief proof of Cauchy's integral theorem*, Proc. Amer. Math. Soc **29** (1971), 625-626

[2] Lang, Serge; *Complex Analysis*, Springer Verlag, ISBN 978-1-4757-3083-8