1.a. $\int_{-4}^{5} c(x+4)\,dx = 1$

$t = x+4$

$\Rightarrow c \int_{0}^{9} t\,dt = 1$

$\Rightarrow c \cdot \frac{t^2}{2}\Big|_0^9 = c \cdot \frac{81}{2} = 1$

$\Rightarrow c = \frac{2}{81}.$ 1 mark

$y = x|x| = \begin{cases} -x^2, & x<0 \\ x^2, & x<0 \end{cases}$ $\uparrow$ on $\mathbb{R}$

$S_Y = (-16, 25) = (-16, 0) \cup (0, 25)$

- For $-16 < y < 0$, we have

$P(X|X| \leq y) = P(X<0, X^2 \geq -y) + P(X>0, X^2 \leq y)$

$= P(X \leq -\sqrt{-y}) + 0$

$= \frac{2}{81} \int_{-4}^{-\sqrt{-y}} (x+4)\,dx$

$= \frac{(4-\sqrt{-y})^2}{81}.$ 1 mark

- For $0 < y < 25$, we have

$$P(X|X| \leq y) = P(X<0, X^2 \geq -y) + P(X>0, X^2 \leq y)$$

$$= P(X<0) + P(0 < X \leq \sqrt{y})$$

$$= P(X \leq \sqrt{y})$$

$$= \frac{2}{81} \int_{-4}^{\sqrt{y}} (x+4)\, dx$$

$$= \frac{(4+\sqrt{y})^2}{81}.$$

1 mark

So,

$$F_Y(y) = \begin{cases} 0 & , y < -16 \\ \frac{(4-\sqrt{-y})^2}{81} & , -16 \leq y < 0, \\ \frac{(4+\sqrt{y})^2}{81} & , 0 \leq y < 25, \\ 1 & , y \geq 25. \end{cases}$$

1 mark

F_Y is differentiable (except at a finite number of points).

So, the pdf is

$$f_Y(y) = F'_Y(y)$$

$$= \begin{cases} \dfrac{(4-\sqrt{-y})}{81 \cdot \sqrt{-y}}, & -16 < y < 0 \\ \dfrac{(4+\sqrt{y})}{81 \cdot \sqrt{y}}, & 0 < y < 25 \\ 0, & \text{otherwise.} \end{cases}$$

1 mark

1. b. (i) Let $x > 0$.

$$F(x) = \int_0^x 2\beta t e^{-\beta t^2} dt$$

$$z = \beta t^2$$

$$= \int_0^{\beta x^2} e^{-z} dz$$

$$= 1 - e^{-\beta x^2}$$

$$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-\beta x^2} & , x \geq 0. \end{cases}$$

$\boxed{\text{1 mark}}$

Let the median be m_0.

$$F(m_0) = \frac{1}{2}$$

$$\Rightarrow 1 - e^{-\beta m_0^2} = \frac{1}{2}$$

$$\Rightarrow e^{-\beta m_0^2} = \frac{1}{2}$$

$$\Rightarrow \beta m_0^2 = \log_e 2$$

$$\Rightarrow m_0 = \sqrt{\frac{\log_e 2}{\beta}}.$$

$\boxed{\text{1 mark}}$

Now,

$$\log_e f(t) = \log_e 2\beta + \log_e t - \beta t^2$$

$$\Rightarrow f'(t) = f(t)\left[\frac{1}{t} - 2\beta t\right] = 0$$

$$\Rightarrow t_0 = \frac{1}{\sqrt{2\beta}}$$ $\boxed{1 \text{ mark}}$

$$f''(t) = f(t)\left[-\frac{1}{t^2} - 2\beta\right] + f(t)\left[\frac{1}{t} - 2\beta t\right]^2$$

$$= f(t)\left[-4\beta + 4\beta^2 t^2\right]$$

$$f''(t_0) = f(t_0)\cdot\left[-4\beta + 2\beta\right]$$

$$= -2\beta f(t_0) < 0 ..$$

$[\because \beta > 0 \;\&\; f > 0]$ $\boxed{1 \text{ mark}}$

1\. b. (ii) From $M_X(t)$, it is clear that

$$p(x) = \begin{cases} \frac{1}{8} & x = -1, \\ \frac{1}{4} & x = 0, \\ \frac{5}{8} & x = 2. \end{cases}$$

1 mark

$$E(X) = -\frac{1}{8} + \frac{10}{8} = \frac{9}{8}$$

$$E(X^2) = \frac{1}{8} + \frac{20}{8} = \frac{21}{8}$$

$$Var(X) = \frac{21}{8} - \frac{81}{64} = \frac{87}{64}$$

1 mark

2.a.i) $X_1, \dots, X_n \overset{iid}{\sim} N(0,1)$

$\Rightarrow \bar{X}_n$ is independent of $S_n^2 \ \forall\, n \geq 1.$

[Student's theorem]

$\Rightarrow \bar{X}_n$ is independent of $\frac{1}{S_n} \ \forall\, n \geq 1$

Fix $n=5$.

$$E\left[\frac{\bar{X}_5}{S_5}\right] = E\left[\bar{X}_5\right] \cdot E\left[\frac{1}{S_5}\right]$$

$$= 0$$

$\boxed{1 \text{ mark}}$

$$Var\left[\frac{\bar{X}_5}{S_5}\right] = E\left[\frac{\bar{X}_5^{\,2}}{S_5^2}\right] - 0$$

$$= E\left[\bar{X}_5^2\right] \cdot E\left[\frac{1}{S_5^2}\right]$$

$$E\left[\bar{X}_5^2\right] = \frac{1}{5}$$

$\boxed{1 \text{ mark}}$

$$4 S_5^2 \sim \chi_4^2$$

$$= \frac{4}{5} \cdot E\left[\frac{1}{4S_5^2}\right]$$

$$= \frac{4}{5} \cdot \frac{2^{2/2} \cdot \Gamma(2/2)}{2^{4/2} \cdot \Gamma(4/2)}$$

$$= \frac{4}{5} \cdot \frac{2 \cdot 1}{4 \cdot 1}$$

So, $\quad \mathrm{Var}\left[\frac{\bar{X}_5}{S_5}\right] = \frac{2}{5}$.

$\boxed{\text{1 mark}}$

2.a.ii) $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Unif}(0,1)$

<u>Claim</u>: $X_{(1)} \xrightarrow{P} 0$ as $n \to \infty$

$X_{(n)} \xrightarrow{P} 1$ as $n \to \infty$.

<u>Proof</u>: $$E[X_{(1)}^2] = n\int_0^1 y^2(1-y)^{n-1}dy$$

$$= n \cdot B(3,n)$$

$$= n \cdot \frac{2!\,(n-1)!}{(n+2)!}$$

$$= \frac{2}{(n+2)(n+1)} \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow X_{(1)} \xrightarrow{L_2} 0 \text{ as } n \to \infty.$$

$\boxed{\text{1 mark}}$

Similarly,

$$E[1 - X_{(n)}]^2 = n\int_0^1 y^{n-1}(1-y)^2 dy$$

$$= n \cdot B(n,3)$$

$$\to 0 \text{ as } n \to \infty.$$

$$\Rightarrow \quad X_{(n)} \xrightarrow{L_2} 1 \quad \text{as } n \to \infty.$$

$\boxed{\text{1 mark}}$

Combining these two statements, we have

$$X_{(1)} X_{(n)} \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

$\boxed{\text{1 mark}}$

2.b. $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}\right)$

i) $$\max\{Z_1, Z_2\} = \frac{Z_1 + Z_2 + |Z_1 - Z_2|}{2}$$

$$E[\max\{Z_1, Z_2\}] = \frac{0 + E|Z_1 - Z_2|}{2}$$

1 mark

$$Z_1 - Z_2 \sim N(0,1).$$

So, $E|Z_1 - Z_2| = \sqrt{\frac{2}{\pi}}$

1 mark

$$\Rightarrow E[\max\{Z_1, Z_2\}] = \frac{1}{\sqrt{2\pi}}$$

ii) Recall $Z_2 | Z_1 \sim N\left(\frac{Z_1}{2}, \frac{3}{4}\right)$.

1 mark

$$E[e^{tZ_1Z_2}]$$

$$= E_{Z_1}\left\{E[e^{tZ_1Z_2} | Z_1]\right\}$$

$$= E_{Z_1}\left\{e^{\frac{tZ_1^2}{2} + \frac{3t^2Z_1^2}{8}}\right\}$$

$$= E_{Z_1}\left(e^{sZ_1^2}\right), \quad s = \frac{t}{2} + \frac{3t^2}{8}$$

$\boxed{\text{1 mark}}$

$$= \frac{1}{\sqrt{1-2s}} \quad \text{for} \quad s < \frac{1}{2}$$

$$[\because Z_1^2 \sim \chi_1^2]$$

$$= \frac{1}{\sqrt{1-\left(t+\frac{3t^2}{4}\right)}} \quad \text{for} \quad -2 < t < \frac{2}{3}.$$

$\boxed{\text{1 mark}}$

3\. a. $X_1, \dots, X_n \overset{iid}{\sim} \text{unif}(\theta - a, \theta + a)$

i). $L(\theta, \alpha) = \prod_{i=1}^{n} f(X_i)$

$$= \frac{1}{(2a)^n}, \quad \theta - a \leq X_{(1)} < X_{(n)} \leq \theta + a.$$

free of θ!

Now,

$$X_{(n)} - a \leq \theta \leq X_{(1)} + a.$$

given a, L is maximized if

$$\theta \in [X_{(n)} - a, X_{(1)} + a].$$

In particular, let us consider

$$\hat{\theta}_{MLE} = \frac{X_{(n)} - a + X_{(1)} + a}{2}$$

$$= \frac{X_{(1)} + X_{(n)}}{2}. \quad \boxed{1 \text{ mark}}$$

Note that L is $\downarrow$ in a.
So, L is maximized if a is minimized. Note that

$$\underline{\theta - X_{(1)} \leq a \leq X_{(n)} - \theta}$$

$$\Rightarrow \hat{a}_{MLE} = \hat{\theta}_{MLE} - X_{(1)}$$

$$= \frac{X_{(n)} - X_{(1)}}{2}.$$ $\boxed{\text{1 mark}}$

ii) $$E[X_{(n)}] = n\int_{\theta-a}^{\theta+a} x \cdot \frac{1}{2a} \cdot \left[1 - \frac{x-\theta+a}{2a}\right]^{n-1} dx$$

$$z = \frac{x-\theta+a}{2a}, \quad dz = \frac{dx}{2a}$$

$$= n\int_0^1 (2az+\theta-a)\cdot(1-z)^{n-1}\,dz$$

$$= 2an\int_0^1 z(1-z)^{n-1}\,dz +$$

$$(\theta-a)n\int_0^1 (1-z)^{n-1}\,dz$$

$$= 2an\,\mathrm{Beta}(2,n) + \frac{(\theta-a)}{\cancel{n}}\cdot\cancel{n}$$

$$= 2an\cdot\frac{1!\,(n-1)!}{(n+1)!} + (\theta-a)$$

$\boxed{\text{1 mark}}$

$$= \frac{2a}{(n+1)} + (\theta-a).$$

$$E[X_{(n)}] = n\int_{\theta-a}^{\theta+a} x\cdot\frac{1}{2a}\cdot\left(\frac{x-\theta+a}{2a}\right)^{n-1} dx$$

$$z = \frac{x-\theta+a}{2a}, \quad dz = \frac{dx}{2a}$$

$$= n\int_0^1 (2az+\theta-a)\cdot z^{n-1}\,dz$$

$$= 2an\int_0^1 z^n\,dz + (\theta-a)n\int_0^1 z^{n-1}\,dz$$

$$= \frac{2an}{(n+1)} + (\theta - a)$$

$$= 2a - \frac{2a}{(n+1)} + (\theta - a)$$

1 mark

$$= (\theta + a) - \frac{2a}{n+1}$$

Now,

$$E[\hat{\theta}_{MLE}] = E\left[\frac{X_{(1)} + X_{(n)}}{2}\right]$$

$$= \frac{1}{2}\left[\cancel{\frac{2a}{n+1}} + \theta - \cancel{a} + \theta + \cancel{a} - \cancel{\frac{2a}{n+1}}\right]$$

$$= \theta \text{ (Yes)}$$

1 mark

$$E[\hat{a}_{MLE}] = E\left[\frac{X_{(n)} - X_{(1)}}{2}\right]$$

$$= \frac{1}{2}\left[(\cancel{\theta} + a) - \frac{2a}{n+1} - \frac{2a}{n+1} - \cancel{\theta} + a\right]$$

$$= \frac{1}{2}\left[2a - \frac{4a}{n+1}\right] = a - \frac{2a}{n+1}$$

$$= \frac{(n-1)}{(n+1)} \cdot a$$

$$\neq a \quad (\text{No})$$

$\boxed{\text{1 mark}}$

3. b. Using the NP lemma, we get

if $\prod_{i=1}^{n} \frac{2X_i}{1} > c$, then we reject H_0.

Critical region: $\prod_{i=1}^{n} X_i > c/2^n$.

$\boxed{1 \text{ mark}}$

$$P_{H_0}\left(\prod_{i=1}^{n} X_i > c_1\right)$$

$$= P_{H_0}\left(-2\sum_{i=1}^{n} \log X_i < -2\log c_1\right)$$

Under H_0, $X \sim U(0,1)$ $\boxed{1 \text{ mark}}$

$$-2\log X \sim \chi_2^2$$

$\Rightarrow -2\sum_{i=1}^{n} \log X_i \sim \chi_{2n}^2$ (Additivity of χ^2)

Now, $P(\chi_{2n}^2 < K) = \alpha$

$\Rightarrow K = \chi_{2n}^2(1-\alpha)$ $\boxed{1 \text{ mark}}$

For $\alpha = 0.10$ and $n = 10$, we get $k = \chi^2_{20}(0.90) = 12.443$.

$\boxed{\text{1 mark}}$