Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)
Exercise Sheet 11

1. Isolated singularities

1.1.$*$ Let $z_0 \in U \subseteq_{open} \mathbb{C}$, $f \in H(U \setminus \{z_0\})$, $\overline{D(z_0; r_2)} \subseteq U$, where $r_2 > 0$, and $0 < r_1 < r_2$. Suppose that $\{a_n\}_{n=-\infty}^{\infty}$ is a binifinite sequence such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

and the convergence is uniform on every compact subset of $A(z_0; r_1, r_2)$. Show that, for any $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{f(w)}{(w - z_0)^{n+1}}\, dw,$$

for all $r \in (r_1, r_2)$.

1.2. In each of the following cases find the Laurent series expansion:

(a) $\dfrac{\exp(z)}{z}$ about 0.

(b) $\dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$, $c \neq 0$ and $ad - bc \neq 0$, about $-\frac{d}{c}$.

(c) $\dfrac{z+1}{z-1}$ about 0 in the following regions:

(i) $|z| < 1$ (ii) $|z| > 1$.

(d) $\dfrac{1}{(z-a)(z-b)}$, where $0 < |a| < |b|$, about 0 in the following regions:

(i) $0 < |z| < a$ (ii) $a < |z| < b$ (iii) $|z| > b$.

(e) $\dfrac{1}{z(z-a)(z-b)}$, where $0 < |a| < |b|$, about 0 in the following regions:

(i) $0 < |z| < a$ (ii) $a < |z| < b$ (iii) $|z| > b$.

(f) $\dfrac{1}{z^2(1-z)}$ about 0 and 1.

(g) $\dfrac{1}{(z-1)^2(z+1)^2}$ on the annulus $1 < |z| < 2$.

(h)$*$ $\exp\left(\frac{1}{z}\right)$ about 0. Use this to evaluate

$$\frac{1}{\pi} \int_0^{\infty} e^{\cos t} \cos(\sin t - nt)\, dt.$$

(i)$*$ $\exp\left(z + \dfrac{1}{z}\right)$ about 0. Further use this to prove that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{2\cos t} \cos nt\, dt = \sum_{k=0}^{\infty} \frac{1}{(n+k)!k!}.$$

1.3.$*$ Let $a \neq b \in \mathbb{C}$ and ℓ be the line segment in $\mathbb{C}$ joining the points a and b. Consider the function $f(z) \overset{\text{def}}{=} \dfrac{(z-a)}{(z-b)}$, for all $\mathbb{C} \setminus \{a, b\}$.

(a) Show that f has an analytic logarithm on $\mathbb{C} \setminus \ell$.

Hint. *Consider any closed path γ in $\mathbb{C} \setminus \ell$. Then a and b must lie in the same connected component of $\mathbb{C} \setminus \gamma^*$.*

(b) Let g be an analytic logarithm of f on $\mathbb{C} \setminus \{a, b\}$. Find the Laurent series expansion of g about 0.

Hint. *Observe that, for any $z \in \mathbb{C} \setminus \ell$, $g'(z) = \frac{f'(z)}{f(z)} = \frac{1}{z-a} - \frac{1}{z-b}$. Now write laurent series expansions of the latter. From that get the required Laurent series expansions of g.*

1.4. Let $R > 0$ and $f \in H(\{z \in \mathbb{C} : |z| > R\})$. Assume that f is bounded. Denote the n-th coefficient of the Laurent series of f about 0 by c_n, for all $n \in \mathbb{Z}$. Show that $c_n = 0$, for all $n \in \mathbb{N}$.

1.5.* Let $r_0 > 0$ and $f : D(0, r_0) \setminus \{z_0\} \longrightarrow \mathbb{C}$ be holomorphic. Denote the n-th coefficient of the Laurent series of f about 0 by c_n, for all $n \in \mathbb{Z}$. Assume that there exists $M, \alpha > 0$ such that

$$r^\alpha \int_0^{2\pi} \left| f\left(re^{it}\right)\right|^2 dt \leq M,$$

for all $0 < r < r_0$. Show that the singularity of f at 0 cannot be essential.

Hint. *You may need Cauchy-Schwartz inequality.*

1.6.* Let $f : [a, b] \longrightarrow \mathbb{C}$ be continuous. Consider the function

$$F(z) \overset{\text{def}}{=} \int_a^b \frac{f(t)}{t - z}\, dt, \ \forall z \in \mathbb{C} \setminus [a, b]. \tag{1.1}$$

Show that the function F, defined as above in (1.1), determines f uniquely.

1.7. Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function, where $P(z), Q(z) \in \mathbb{C}[z]$. Suppose that $z_1, \ldots z_k$ are precisely all distinct zeros of Q with multiplicities $n_1, \ldots, n_k$ respectively. Show that there exists $S(z) \in \mathbb{C}[z]$ and complex numbers $a_{j,r}$, for $j = 1, \ldots, k$ and $r = 0, \ldots n_j - 1$ such that

$$f(z) = S(z) + \sum_{j=1}^{k} \sum_{r=0}^{n_j - 1} \frac{a_{j,r}}{(z - z_j)^{n_j - r}}.$$

Note: This proves the existence of partial fraction expansion for any rational function.

1.8. Find the radius of convergence of the Taylor series of the function $\dfrac{1}{1 + z^2 + z^4 + z^6 + z^8 + z^{10}}$ about the point 1.

1.9.* Does there exist $f \in H(\mathbb{C} \setminus \{0\})$ such that $|f(z)| \geq \dfrac{1}{\sqrt{|z|}}$, for all $z \neq 0$? Justify your answer.

Hint. *Consider $g \overset{def}{=} \frac{1}{f}$. Show that the singularity of g at 0 is removable. Now estimate $g'(z)$, for all $z \in \mathbb{C}$.*

1.10. Let $U \subseteq_{open} \mathbb{C}$, $z_0 \in U$ and $f \in H(U \setminus \{z_0\})$.

(a) If z_0 is a zero of order m then show that the function $\frac{f(z)}{(z-z_0)^m}$ has a removable singularity at z_0.
(b) If z_0 is a pole of f then show that there exist unique $m \in \mathbb{N}$ and $g \in H(U)$ with $g(z_0) \neq 0$ such that $f(z) = \frac{g(z)}{(z-z_0)^m}$, for all $z \in U \setminus \{z_0\}$.
(c) Show that z_0 is a pole of f of order $m \in \mathbb{N}$ if and only if z_0 is a zero of $\frac{1}{f}$ of order m.

(d) Show that z_0 is a pole of f of order $m \in \mathbb{N}$ if and only if there exist positive numbers r, C_1 and C_2 such that
$$\frac{C_1}{|z-z_0|^m} \leq |f(z)| \leq \frac{C_2}{|z-z_0|^m}, \forall z \in D(z_0; r) \setminus \{z_0\}.$$
(e) Show that z_0 is a pole of f of order $m \in \mathbb{N}$ if and only if
$$\lim_{z \to z_0} (z-z_0)^m f(z) \neq 0 \text{ but } \lim_{z \to z_0} (z-z_0)^{m+1} f(z) = 0.$$
(f) Suppose that U is connected, f is nonconstant and the singularity of at z_0 is removable. Then show that $\frac{1}{f}$ has either a removable singularity or a pole at z_0.
(g) Suppose that U is connected and f is not identically zero. Show that if z_0 is a limit point of $Z(f)$ then the singularity at z_0 must be essential.

1.11. Let f be an entire function.

(a) Show that, if f is not a polynomial then, for every $r > 0$, $f(\mathbb{C} \setminus D(0; r))$ is dense in $\mathbb{C}$.
(b) Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$ if f is nonconstant.

1.12.* Let $R > 0$ and $f \in H(\{z \in \mathbb{C} : |z| > R\})$. Consider $F(z) \stackrel{\text{def}}{=} f\left(\frac{1}{z}\right)$, for all $z \neq 0$. We say that f has a *removable singularity*, a *pole* or an *essential singularity at* ∞ if F has a removable singularity, a pole or an essential singularity at 0 respectively. In the case of pole, the order of the pole of F at 0 is called the *order of the pole at* ∞ of f and denote by $\mathrm{Ord}_\infty(f)$.

(a) Find all entire functions having a removable singularity at ∞.
(b) Find all entire functions having a pole at ∞ or order $m \in \mathbb{N}$.
(c) Find all meromorphic functions having a pole at ∞. What will be the order of the pole at ∞ for such a function?

Hint. *First show that f can have only finitely many poles. Suppose that $z_1, \ldots, z_n$ are the poles of f with order $m_1, \ldots, m_n$ respectively. Consider the function $g(z) \stackrel{def}{=} (z-z_1)^{m_1} \ldots (z-z_n)^{m_n} f(z)$.*

(d) Can you characterize all rational functions having a removable singularity at ∞?
(e) Can you characterize all rational functions having a pole of order $m \in \mathbb{N}$ at ∞?

1.13. Let U be a region in $\mathbb{C}$ and f be a nonconstant meromorphic function on U. Denote the set of all poles of f by $P(f)$. For every $z \in P(f)$, $f(z) \stackrel{\text{def}}{=} \infty$. Show that $f : U \longrightarrow \hat{\mathbb{C}}$ is a continuous open map.

Note: 1.13. generalizes the Open mapping theorem for meromorphic functions.

1.14.* Let $f \in H(\mathbb{H})$ be periodic with period 1.

(a) Show that, for any $z \in \mathbb{H}$, one has $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$, where $a_n \stackrel{\text{def}}{=} \displaystyle\int_0^1 f(x+iy) e^{-2\pi i n(x+iy)}\, dx$, for all $y > 0$.

Hint. *Show that there exists $g \in H(\mathbb{D} \setminus \{0\})$ such that $g\left(e^{2\pi i z}\right) = f(z)$, for all $z \in \mathbb{H}$. Find the Laurent series expansion of g about 0.*

(b) Suppose that there exists $R > 0$ such that f is bounded on $\{z \in \mathbb{H} : \operatorname{Im} z \geq R\}$. Show that $a_n = 0$, for all $n < 0$.
(c) Is the converse of 1.14.b true?

2. Residue

2.1. Let $R > 1$ and $f \in H(D(0; R) \setminus \{1\})$. Show that if f has a simple pole at 1 then $\left\{\dfrac{f^{(n)}(0)}{n!}\right\}_{n=0}^{\infty}$ is convergent.

2.2. Let $U \subseteq_{open} \mathbb{C}$, $z_0 \in U$ and $f \in H(U \setminus \{z_0\})$.

(a) Show that there exists unique $\alpha \in \mathbb{C}$ such that $f(z) - \frac{\alpha}{(z-z_0)}$, for all $z \in U \setminus \{z_0\}$, has a primitive on $U \setminus \{z_0\}$.

(b) Let f have a pole at z_0 of order m. Consider $g(z) \stackrel{\text{def}}{=} (z-z_0)^m f(z)$, for all $z \in U \setminus \{z_0\}$. Express all the coefficients of the principal part of f at z_0 in terms of the derivatives of g. In particular, $\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} g^{(m)}(z)$.

(c) In class we have proved the following: if $g \in H(U)$ and f has a simple pole at z_0, then $\operatorname{Res}(gf, z_0) = g(z_0) \operatorname{Res}(f, z_0)$. Does the same conclusion hold if the pole at z_0 is not simple?

2.3. Let U be a region in $\mathbb{C}$. Suppose that $A \subseteq U$ does not have a limit point in U. Show that $U \setminus A$ is open and connected.

2.4. Prove the following generalization of the Residue theorem: let $U \subseteq_{open} \mathbb{C}$ and $A \subseteq U$ have no limit point in U. Suppose that γ be a cycle in $U \setminus A$ such that $\operatorname{Ind}_\gamma(z) = 0$, for all $z \in \mathbb{C} \setminus U$. Show that, for all $f \in H(U \setminus A)$ one has,

$$\frac{1}{2\pi i} \int_\gamma f = \sum_{a \in A} \operatorname{Res}(f, a) \operatorname{Ind}_\gamma(a).$$

2.5. Let $U \subseteq \mathbb{C}$ be a region in $\mathbb{C}$ and $f \in H(U)$, $\overline{D(z_0; r)} \subseteq U$. Suppose that f has no zeros on the circle $|z - z_0| = r$ and $z_1, \ldots, z_n$ are precisely all zeros of f in $D(z_0; r)$. Show that, for any $g \in H(U)$,

$$\frac{1}{2\pi i} \int_{C(z_0;R)} g \cdot \frac{f'}{f} = \sum_{j=1}^{\infty} \operatorname{Ord}_{z_j}(f) g(z_j).$$

2.6.∗ Let f be a meromorphic function on $\mathbb{C}$ and $P(f)$be its set of poles. Suppose that, for every closed path γ in $\mathbb{C} \setminus P(f)$ and $p(z) \in \mathbb{C}[z]$, one has

$$\int_\gamma p(z)^2 f(z)\, dz = 0.$$

Show that f is entire.

Hint. *First show that* $\int_\gamma p(z) f(z)\, dz = 0$, *for every polynomial* $p(z)$ *and closed path* γ *in* $\mathbb{C} \setminus P(f)$. *Suppose now that* f *has a pole of order* m *at* z_0. *Then the the residue of function* $(z - z_0)^{m-1} f(z)$ *at* z_0 *is nonzero.*

3. Analytic autmorphisms of $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$

3.1.∗ Let f be an $1-1$ meromorphic function on $\mathbb{C}$.

(a) Show that f can have at most one pole in $\mathbb{C}$.

(b) Show that the singularity of f at ∞ cannot be essential.

(c) Show that f has exactly one pole in $\hat{\mathbb{C}}$.

(d) Let z_0 be the pole obtained in 3.1.c. Show that the pole is simple.

Hint. *Consider two cases. If* $z_0 = \infty$, *then show that* f *must be a polynomial with degree* 1. *If* $z_0 \in \mathbb{C}$, *consider*

$$g(z) \stackrel{def}{=} \begin{cases} \frac{1}{f(z)} & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}.$$

Now show that g *is analytic and* $g'(z_0) \neq 0$.

(e) Show that $f(z) - \frac{\operatorname{Res}(f, z_0)}{(z-z_0)}$ has only removable singularities on $\hat{\mathbb{C}}$. Hence it is constant.

(f) Conclude that f must be a Möbius trnasformation.

3.2.$*$ Find all analytic autmorphisms of $\mathbb{C}$ and $\mathbb{C} \setminus \{0\}$.