MTH101A, Mid-Sem. Exam, IIT Kanpur

Date: 20.09.2017 **Time: 13:00-15:00 hrs** **Total Marks:** 70

Instructions:

1. *Please write down the page numbers in the answer book.*
2. *Make a tabular column on the top cover of your answer book and indicate the page number in which the respective question has been answered.*
3. *Answer all parts of a question together at one place.*

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1. (a) Let b_k denotes the number of prime numbers less or equal to k. (For example, $b_4 = 2$ since the prime numbers less than or equal to 4 are 2 and 3.)
Let $a_1 = 2$, $a_2 = 3$ and for $n \geq 3$; $a_n = \sum_{k=3}^{n} \frac{1}{b_k}$. Determine the convergence or divergence for the sequence (a_n) with proper justification for your answer.

[5]

(b) Check convergence or divergence of the series $\sum_{n=1}^{\infty}(1 - n\ \sin\frac{1}{n})$.

[4]

(c) Determine all the values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^2 2^n}$ is convergent.

[4]

2. (a) Let $f : [0, 2] \to \mathbb{R}$ be a twice differentiable function such that given a $\delta > 0$ there is a point $x \in (1 - \delta, 1 + \delta)$ with $f'(x) - 3 = 0$. Prove that $f'(1) = 3$.

[5]

(b) Using Cauchy Mean Value Theorem (CMVT) show that for $x \in (0, \infty)$

$$x - \frac{x^3}{3!} < \sin x.$$

[6]

3. (a) Suppose (a_n) is a decreasing sequence of real numbers converging to 0. Define $b_k = \sum_{i=1}^{k}(-1)^{i+1}a_i$ for $k \in \mathbb{N}$. Prove that both the sequences (b_{2n}) and (b_{2n+1}) converge to the same limit. [6]

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable function which has a local minimum at $x = 0$. Prove that $f''(0) \geq 0$.

[4]

4. (a) Let $f(x) = \frac{2x^2}{1-x^2}$. Find the

(i) asymptotes of f (ii) locate the intervals of decreasing, increasing

(iii) locate the point of maximum, minimum (iv) locate the intervals of concavity, convexity for f.

[3+2+1+2 = 8]

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that $|f'(x)| < \frac{1}{3}$ for all $x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}$. Show that (a_n) is a convergent sequence.

[4]

5. (a) Write down the Taylor series for the function $f(x) = \cos x$ around $x = 0$. Prove that the series converges to $f(x)$ for all $x \in \mathbb{R}$.

[2+4= 6]

(b) Consider the function $f : [0, 2] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number;} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

Show that the function f is not a Riemann integrable function on $[0, 2]$.

[6]

6. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(x + 2) = f(x)$ for all $x \in \mathbb{R}$. Show that the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = \displaystyle\int_x^{x+2} f(t)dt$ is a constant function.

[6]

(b) Compute the following limit

$$\lim_{x \to 1} \frac{1}{(x-1)^3} \int_1^x \frac{(t-1)^2}{1+t^4} dt.$$

[6]

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Marking Scheme

1. (a) Let b_k denotes the number of prime numbers less or equal to k. (For example, $b_4 = 2$ since the prime numbers less than or equal to 4 are 2 and 3.)

Let $a_1 = 2$, $a_2 = 3$ and for $n \geq 3$; $a_n = \sum_{k=3}^{n} \frac{1}{b_k}$. Determine the convergence or divergence for the sequence (a_n) with proper justification for your answer.

[5]

Answer: By definition, $b_k < k$ for all $k \in \mathbb{N}$. [2]

Now, for $k \geq 3$, $\frac{1}{k} < \frac{1}{b_k}$. As $\sum_{k=3}^{\infty} \frac{1}{k}$ is a divergent series, by comparison test $\sum_{k=3}^{\infty} \frac{1}{b_k}$ is a divergent series. [2]

So, the sequence of partial sums $a_n = \sum_{k=3}^{n} \frac{1}{b_k}$ is a divergent sequence [1]

(b) Check convergence or divergence of the series $\sum_{n=1}^{\infty}(1 - n \, \sin \frac{1}{n})$.

[4]

Answer: Let $a_n = (1 - n \, \sin \frac{1}{n})$ and $b_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. [1]

Then $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1}{6} > 0$. [1]

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Limit Comparison Test (or LCT) the series $\sum_{n=1}^{\infty}(1 - n \, \sin \frac{1}{n})$ converges. [1+1]

(c) Determine all the values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^2 2^n}$ is convergent.

[4]

Answer: Let $a_n = \frac{(x-1)^{2n}}{n^2 2^n}$. Then $a_n^{\frac{1}{n}} = \frac{(x-1)^2}{2} \times \frac{1}{n^{\frac{2}{n}}} \to \frac{(x-1)^2}{2}$ as $n \to \infty$.

By Root test, if $x \in \mathbb{R}$ such that $\frac{(x-1)^2}{2} < 1$ then the series

$$\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^2 2^n}$$

is converges. [2]

This implies the given power series converges for $|x-1| < \sqrt{2}$ or $x \in (1-\sqrt{2}, 1+\sqrt{2})$.

[1]

At $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$, the series $\sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. [1]

2. (a) Let $f : [0, 2] \to \mathbb{R}$ be a twice differentiable function such that given a $\delta > 0$ there is a point $x \in (1-\delta, 1+\delta)$ with $f'(x) - 3 = 0$. Prove that $f'(1) = 3$.

[5]

Answer:

For $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$ and $x_n \in (1 - \frac{1}{n}, 1 + \frac{1}{n})$ with $f'(x_n) - 3 = 0$. [2]

So, we get a sequence (x_n) converging to 1 and $(f'(x_n))$ is the constant sequence $3, 3, \cdots$ [1]

As f is twice differentiable, f' is a continuous function. Thus, for a sequence$x_n \to 1$ implies $f'(x_n) \to f'(1)$. Therefore, $f'(1) = 3$. [2]

Alternative solution:

Let $\epsilon > 0$. Then by continuity of f' at $x = 1$, we get $\delta > 0$ such that $|x - 1| < \delta$ implies $|f'(x) - f'(1)| < \epsilon$. Now, for this δ there exists a point $x_\delta \in (1-\delta, 1+\delta)$ with $f'(x_\delta) = 3$. [3]

Therefore, for any $\epsilon > 0$, we get $x_\delta \in (1-\delta, 1+\delta)$ such that $|f'(x_\delta) - f'(1)| = |3 - f'(1)| < \epsilon$. [1]

Since ϵ is arbitrarily chosen, this implies that $f'(1) = 3$. [1]

(b) Using Cauchy Mean Value Theorem (CMVT) show that for $x \in (0,$

$$x - \frac{x^3}{3!} < \sin x.$$

[6]

Answer:

Let $x \in (0, \infty)$. Consider $f(t) = t - \sin t$ and $g(t) = \frac{t^3}{3!}$ for all $t \in [0, x]$

By Cauchy Mean Value Theorem (CMVT) there exists $c \in (0, x)$ such that

$$\frac{f(x) - f(0}{g(x) - g(0)} = \frac{x - \sin x}{\frac{x^3}{3!}} = \frac{1 - \cos c}{\frac{c^2}{2}}$$

[3]

Also, by applying CMVT on $f(t) = 1 - \cos t$ and $g(t) = \frac{t^2}{2}$, we get $1 - \frac{x^2}{2!} < \cos x$ for $x \in (0, \infty)$. [2]

Therefore,

$$\frac{x - \sin x}{\frac{x^3}{3!}} = \frac{1 - \cos c}{\frac{c^2}{2}} < 1$$

This implies $x - \frac{x^3}{3!} < \sin x$. [1]

3. (a) Suppose (a_n) is a decreasing sequence of real numbers converging to 0. Define $b_k = \sum_{i=1}^{k} (-1)^{i+1} a_i$ for $k \in \mathbb{N}$. Prove that both the sequences (b_{2n}) and (b_{2n+1}) converge to the same limit.

[6]

Answer:

For $n \in \mathbb{N}$ we have

$$b_{2n+2} - b_{2n} = (a_{2n+1} - a_{2n+2}) \geq 0$$

and

$$a_1 - b_{2n} = (a_2 - a_3) + \cdots + (a_{2n-2} - a_{2n-1}) + a_{2n} \geq 0.$$

This implies that (b_{2n}) is a motonically increasing sequence bounded above by a_1. [1+1]

Also for $n \in \mathbb{N}$ we have

$$b_{2n+3} - b_{2n+1} = (a_{2n+3} - a_{2n+2}) \leq 0$$

and

$$b_{2n+1} = (a_1 - a_2) + (a_3 - a_4) \cdots + (a_{2n-1} - a_{2n}) + a_{2n+1} \geq (a_1 - a_2) = b_2.$$

So, (b_{2n}) is a motonically decreasing sequence bounded below by b_2. [1]

Therefore, both the sequence (b_{2n}) and (b_{2n+1}) is a convergent sequence. Let (b_{2n}) converges to b and (b_{2n+1}) converges to b'. [1]

Since, $b_{2n+1} - b_{2n} = (-1)^{2n+2} a_{2n+1} = a_{2n+1} \to 0$ as $n \to \infty$. [1]

By limit theorem, we get $b' - b = 0$. Consequently, $b = b'$. [1]

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable function which has a local minimum at $x = 0$. Prove that $f''(0) \geq 0$.

[4]

Answer:

Since 0 is a point of local minimum for f and f is a differentiable function, it follows that $f'(0) = 0$. [1]

Also, there exists a $\delta > 0$ such that $f(x) \geq f(0)$ for all $x \in (-\delta, \delta)$. [1]

Now for any $x \in (-\delta, \delta)$ by applying MVT on the function f on $[x, 0]$ or $[0, x]$ we get that there exists $h \in (x, 0)$ or $h \in (0, x)$ such that $f'(h)(x - 0) = [f(x) - f(0)] \geq 0$. Hence x and $f'(h)$ have the same sign. By taking $x = 1/n$ we get a sequence h_n lying between 0 and $\frac{1}{n}$ such that $f'(h_n)$ and h_n are of same sign.

This implies, $f''(0) = \lim\limits_{h \to 0} \dfrac{f'(h) - f'(0)}{h} = \lim\limits_{h \to 0} \dfrac{f'(h)}{h} = \lim\limits_{h_n \to 0} \dfrac{f'(h_n)}{h_n} \geq 0.$ [2]

MANY HAS WRITTEN AS GIVEN BELOW, BUT THIS IS AN INCORRECT SOLUTION. ONLY 2 MARKS ARE GIVEN FOR THIS.

Since 0 is a point of local minimum for f and f is a differentiable function, it follows that $f'(0) = 0$.

Then there exists a $\delta > 0$ such that f is decreasing in $(-\delta, 0)$ and f is increasing in $(0, \delta)$. **

Hence $f'(h) \leq 0$ for $h \in (-\delta, 0)$ and $f'(h) \geq 0$ for $h \in (0, \delta)$.

Therefore, $f''(0) = \lim\limits_{h \to 0} \dfrac{f'(h) - f'(0)}{h} = \lim\limits_{h \to 0} \dfrac{f'(h)}{h} \geq 0.$

**But this argument is false. For example, consider the function

$$f(x) = \begin{cases} x^4 \sin^2(\frac{1}{x}) & \text{for } x \neq 0; \\ 0 & \text{for } \ x = 0. \end{cases}$$

It is a twice differentiable function with a local minimum at $x = 0$, but there is no such $\delta > 0$ as above.

4. (a) Let $f(x) = \frac{2x^2}{1-x^2}$. Find the
(i) asymptotes of f (ii) locate the intervals of decreasing, increasing
(iii) locate the point of maximum, minimum (iv) locate the intervals of concavity, convexity for f.

[3+2+1+2 = 8]

Answer: $f(x) = \frac{2x^2}{1-x^2}$

(i) **Asymptotes :** $x = +1$, $x = -1$ and $y = -2$ [1+1+1=3]

(ii) **locate the intervals of decreasing, increasing:** $f'(x) = \frac{4x}{(1-x^2)^2} \geq 0$ on $[0, 1) \cup (1, \infty)$. So, f is increasing on $[0, 1) \cup (1, \infty)$. [1]

and $f'(x) = \frac{4x}{(1-x^2)^2} \leq 0$ on $(-\infty, -1) \cup (-1, 0]$. So, f is decreasing on $(-\infty, -1) \cup (-1, 0]$. [1]

(iii) **locate the point of maximum, minimum** $x = 0$ is a local minimum [1]

(iv) locate the intervals of concavity, convexity for f. Then $f''(x) = \frac{4(3x^2+1)}{(1-x^2)^3} > 0$ on $(-1, 1)$ and f is convex. [1]

$f'' = \frac{4(3x^2+1)}{(1-x^2)^3} < 0$ on $(-\infty, -1) \cup (1, \infty)$ and f is concave. [1]

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function such that $|f'(x)| < \frac{1}{3}$ for all $x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}$. Show that (a_n) is a convergent sequence.

[4]

Answer: Let $a_1 \in \mathbb{R}$ and (a_n) be the given sequence where $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}$. By MVT, $f(a_{n+1}) - f(a_n) = f'(c)(a_{n+1} - a_n)$ for some c between a_{n+1} and a_n. [1]

For $n \in \mathbb{N}$,

$|a_{n+2} - a_{n+1}| = |f(a_{n+1}) - f(a_n)| = |f'(c)||(a_{n+1} - a_n)| < \frac{1}{3}|a_{n+1} - a_n|$ [2]

This shows that (a_n) satisfies contractive condition, so it is a Cauchy sequence and hence this is a convergent sequence. [1]

5. (a) Write down the Taylor series for the function $f(x) = \cos x$ around $x = 0$. Prove that the series converges to $f(x)$ for all $x \in \mathbb{R}$.

[2+4= 6]

Answer:

Let us compute the derivative $f^n(0)$ for all $n = 1, 2, \cdots$.

The Taylor series is given by $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ [2]

By Taylor's theorem we can express the error term as

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}$$

for some c between x and 0, where $P_n(x)$ is the Taylor polynomial of degree n with respect to f and the point $x = 0$. [1]

Here, $|f^{(n)}(x)| \leq 1$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. [1]

So $|E_n(x)| \leq \frac{x^{n+1}}{(n+1)!}$.By applying ratio test $\frac{x^{n+1}}{(n+1)!} \to 0$ and so the error term $E_n(x) \longrightarrow 0$. [1]

As $n \longrightarrow \infty$, $P_n(x) \longrightarrow$ the Taylor series of f around 0. Therefore, by applying the limit theorem we get that the Taylor series converges to $f(x)$. [1]

(b) Consider the function $f : [0, 2] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number;} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

Show that the function f is not a Riemann integrable function on $[0, 2]$.

[6]

Answer:

Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{2n-1}{n}, 2\}$ be a partition of $[0, 2]$. [1]

Then for $i = 1, 2, \cdots, 2n$ the length of the i- th subinterval $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$; and the minimum value of the function $m_i = 0$ and the maximum value of the function $M_i = 1$. [2] So,

$$L(P_n, f) = \sum_{i=1}^{2n} m_i \Delta x_i = 0.$$

[1]

$$U(P_n, f) = \sum_{i=1}^{2n} M_i \Delta x_i = 2.$$

[1]

This implies that the lower Riemann integral and the upper Riemann integral does not coincide. Hence, the function f is not a Riemann integrable function on $[0, 2]$. [1]

6. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. Show that the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = \displaystyle\int_x^{x+2} f(t)dt$ is a constant function.

[6]

Answer: Let $F(x) = \displaystyle\int_0^x f(t)dt$ for all $x \in \mathbb{R}$. [1]

Then $g(x) = \displaystyle\int_x^{x+2} f(t)dt = F(x+2) - F(x)$ for all $x \in \mathbb{R}$. [2]

By First Fundamental Theorem of Calculus (First F.T.C) we get, $F'(x) = f(x)$ for all $x \in \mathbb{R}$. [1]

This implies that for all $x \in \mathbb{R}$,

$$g'(x) = F'(x+2) - F'(x) = f(x+2) - f(x) = 0.$$

[2]

Therefore g is a constant function.

(b) Compute the following limit

$$\lim_{x \to 1} \frac{1}{(x-1)^3} \int_1^x \frac{(t-1)^2}{1+t^4} dt.$$

[6]

Answer: Let $F(x) = \displaystyle\int_1^x \frac{(t-1)^2}{1+t^4} dt$ for all $x \in \mathbb{R}$. [1]

By First Fundamental Theorem of Calculus (First F.T.C) we get, $F'(x) = \frac{(x-1)^2}{1+x^4}$ for all $x \in \mathbb{R}$. [1]

Now by L'Hospital rule,

$$\begin{aligned}
\lim_{x \to 1} \frac{1}{(x-1)^3} \int_1^x \frac{(t-1)^2}{1+t^4} dt &= \lim_{x \to 1} \frac{F'(x)}{3(x-1)^2} \\
&= \lim_{x \to 1} \frac{(x-1)^2}{1+x^4} \times \frac{1}{3(x-1)^2} \\
&= \frac{1}{2} \times \frac{1}{3} \\
&= \frac{1}{6}
\end{aligned}$$

[4]