# Initial Value Problems: Linear Multistep Methods

Akash Anand
MATH, IIT KANPUR

## Theorem

*A linear multistep is consistent if and only if*

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

*The method is of order $p$ if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

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*This system of linear equation has a unique solution*

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*This scheme is known as Milne-Simpson method and it is the unique fourth order 2-step method.*

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*Does this method converge?*

## Lesson 2

# Initial Value Problems

2.4 Implicit method

2.5 Stiffness

**2.6 Linear Multistep Methods**

**- Convergence**

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### Convergence

*A linear multistep method is* convergent *if whenever the initial values* $y_n$ *are chosen such that* $\max_{0\le n\le k} |e_n| \to 0$, *as* $h \to 0$, *then* $\max_{0\le n\le N} |e_n| \to 0$.

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to obtain the equation $v_{n+1} = Av_n$, where

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Using, $v_n = A^n v_0$ and $y_0 = 0$, we get

$$y_n = (1 - (-5)^n)y_1/6$$

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*Note that if we take exact starting values $y_0 = y_1 = 0$, then $y_n = 0$ for all $n$.*

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*Apply the method to the IVP* $y' = 0,\ y(0) = 0$. *We have*

$$y_{n+1} = -4y_n + 5y_{n-1}.$$

*Solving the difference equation yields*

$$y_n = (1 - (-5)^n)y_1/6.$$

*If* $h = 1/N$, *then*

$$e_N = y_N - y(1) = (1 - (-5)^N)y_1/6 = \left(1 - (-5)^{1/h}\right)y_1/6$$

*For* $y_1 = h$, *we have* $e_1 = h$. *We see that even as* $e_1 \to 0$ *as* $h \to 0$, $e_N \nrightarrow 0$. *Thus, the method does not converge.*

*Note that if we take exact starting values* $y_0 = y_1 = 0$, *then* $y_n = 0$ *for all* $n$. *Thus, a perturbation of size* $\varepsilon$ *in the starting values leads to a difference of size roughly* $5^{1/h}\varepsilon$ *in the discrete solution.*

## Convergence

*A linear multistep method is convergent if whenever the initial values $y_n$ are chosen such that $\max_{0\le n\le k}|e_n| \to 0$, as $h \to 0$, then $\max_{0\le n\le N}|e_n| \to 0$.*

## Example

*Does*

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*Note that if we take exact starting values $y_0 = y_1 = 0$, then $y_n = 0$ for all $n$. Thus, a perturbation of size $\varepsilon$ in the starting values leads to a difference of size roughly $5^{1/h}\varepsilon$ in the discrete solution. The method is, therefore, not stable.*

Module 2

# Initial Value Problems

2.4 Implicit method

2.5 Stiffness

**2.6 Linear Multistep Methods**

**- Stability**

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## Stability

*A linear $k+1$ step method is stable if for any initial value problem with Lipschitz continuous $f$ and of $\varepsilon > 0$, there exists $\delta, h_0 > 0$ such that if $h \le h_0$ and two choices of starting values $y_j$ and $\hat{y}_j$ are chosen satisfying*

$$\max_{0 \le j \le k} \left| y_j - \hat{y}_j \right| \le \delta,$$

*then the corresponding approximate solutions satisfy*

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# Initial Value Problems: Linear Multistep Methods

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Thus, for $\rho(t) = \prod_{j=1}^{J}(t-\lambda_j)^{M_j}$ where $\sum_{j=1}^{J} M_j = k+1$, the general solution is $y_n = \sum_{j=1}^{J}\sum_{m=0}^{M_j-1} c_{jm} n^m \lambda_j^n$.