1. (a) Calculate the magnitude of $\vec{E}$ and $\vec{B}$ fields associated with a monochromatic light beam with $2mW$ power ($\lambda = 632.8$nm) propagating in i) vacuum and ii) glass of refractive index 1.5. The beam cross-section is 0.5mm^2. Comment on the relative strengths of $\vec{E}$ and $\vec{B}$ fields and the way light propagates in a non-conducting medium.

(b) Calculate the radiation pressure exerted by the light beam on a perfectly absorbing medium and also a perfectly reflecting medium.

a) Intensity $I = \dfrac{\text{power}}{\text{Area}} = 4 \times 10^3 \, W/m^2$

In vacuum, $I = \frac{1}{2} \epsilon_0 c E_0^2 \Rightarrow E_0 = \sqrt{\dfrac{2I}{\epsilon_0 c}} = 1.75 \times 10^3 \, V/m$

$B_0 = \dfrac{E_0}{c} = 5.8 \times 10^{-6} \, T$

$E_0^{glass} = \sqrt{\dfrac{2I}{\epsilon v}} = \sqrt{\dfrac{2In}{\epsilon_0 \epsilon_r c}} = \sqrt{\dfrac{2I}{\epsilon_0 c}} \dfrac{1}{\sqrt{n}} = \dfrac{E_0}{\sqrt{n}}$

$\left(v = \dfrac{c}{n} , \; n = \sqrt{\epsilon_r \mu_r} \simeq \sqrt{\epsilon_r} \right)$

$B_0^{glass} = \dfrac{E_0^{glass}}{v} = \dfrac{E_0^{glass}}{c} n$

Again $E_0^{glass} = \dfrac{E_0}{\sqrt{n}} \Rightarrow B_0^{glass} = \dfrac{E_0}{\sqrt{n}} \cdot \dfrac{n}{c} = B_0 \sqrt{n}$

$= 7.1 \times 10^{-6} \, T$

$S_{vac} = \dfrac{1}{\mu_0} E_0 B_0 = \dfrac{E_0^2}{c}$, $S_{glass} = \dfrac{1}{\mu_0} E_0^{glass} B_0^{glass}$

$= \dfrac{1}{\mu_0} \dfrac{E_0}{\sqrt{n}} B_0 \sqrt{n} = \dfrac{1}{\mu_0} E_0 B_0$

Energy flow is the same in vacuum and in glass.

In a dielectric medium $\vec{B}$ gets stronger at the cost of $\vec{E}$ fields.

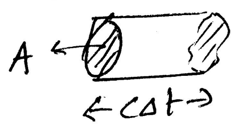
b) Normal incidence, $p = \dfrac{1}{A} \dfrac{dP}{dt}$ → change in momentum

pressure

$\vec{g} = \mu_0 \epsilon_0 \vec{S} = \dfrac{\vec{S}}{c^2}$

For perfectly absorbing surface,

$\Rightarrow \Delta \vec{p} = \dfrac{\vec{S}}{c^2} A . c . \Delta t$



$\Rightarrow p = \dfrac{1}{A} \dfrac{\Delta P}{\Delta t} = \dfrac{\vec{S}}{c}$

Avg pressure $\langle \vec{p} \rangle = \dfrac{\langle \vec{S} \rangle}{c} = \dfrac{I}{c} = \dfrac{4}{3} \times 10^{-5} \, N/m^2$

For perfectly reflecting surface (perfect conductor) change in momentum is twice that of perfectly absorbing surface

Hence corresponding pressure $\dfrac{8}{3} \times 10^{-5} \, N/m^2$

2. A plane electromagnetic wave traveling in air ($\mu_r = 1$; $\epsilon_r = 1$) has $\mathbf{E} = \hat{y} 10\, e^{i(4x-3z-\omega t)}$ $\mathrm{Vm^{-1}}$. The wave falls on a dielectric medium with $\mu_r = 1$ and $\epsilon_r = 1.44$ at $z = 0$ (the surface of the medium is in x-y plane).

(a) Find the expression for the electric field of the reflected wave.

(b) Find the expression for the electric and the magnetic fields of the transmitted wave.

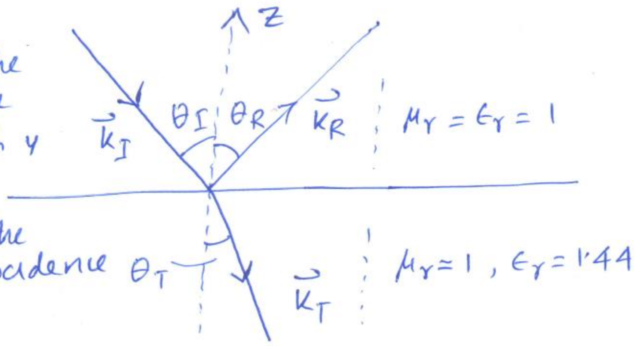
a) $\vec{E}_I = 10\, e^{i(4x-3z-\omega t)}\,\hat{y}$ → incident E field

$\vec{k}_I \cdot \vec{r} = 4x - 3z \Rightarrow \vec{k}_I = 4\hat{x} - 3\hat{z} \Rightarrow |\vec{k}_I| = 5\ \mathrm{m^{-1}}$

[wavelength of incident wave → $\lambda = \frac{2\pi}{k} = \frac{2\pi}{5}$ m

angular freq of incident wave → $\omega = ck = 15 \times 10^8$ rad/s]

$\hat{k}$ is in the xz plane and $\vec{E}$ is in y direction $\Rightarrow \vec{E} \perp$ to the plane of incidence



For reflected wave, $(k_I)_x = (k_R)_x$

$(k_I)_z = -(k_R)_z$

$\Rightarrow \vec{k}_R = 4\hat{x} + 3\hat{z}$

$\Rightarrow \vec{E}_R = E_{0R}\, e^{i(4x+3z-\omega t)}\,\hat{y}$

$$\cos\theta_I = -\frac{\vec{k}_I \cdot \hat{z}}{|\vec{k}_I|} = \frac{-(4\hat{x} - 3\hat{z})\cdot\hat{z}}{5} = \frac{3}{5}$$

$\Rightarrow \sin\theta_I = \frac{4}{5}$

Again, $\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \approx n_2 \quad (n_1 = 1)$

and $n_2 = \sqrt{\mu_r \epsilon_r} = \sqrt{1.44} = 1.2$

$\Rightarrow \sin\theta_T = \frac{4}{5 \times 1.2} = \frac{2}{3} \Rightarrow \cos\theta_T = \frac{\sqrt{5}}{3}$

Put $\frac{\cos\theta_T}{\cos\theta_I} = \alpha$ and $\frac{n_2}{n_1} = \beta \Rightarrow \frac{E_{0R}}{E_{0I}} = \frac{1-\alpha\beta}{1+\alpha\beta}$

and $\frac{E_{0T}}{E_{0I}} = \frac{2}{1+\alpha\beta}$

Here $\alpha = \dfrac{\cos\theta_T}{\cos\theta_I} \simeq 1.24$, $\beta = \dfrac{n_2}{n_1} = 1.2$

$$\Rightarrow \frac{E_{0R}}{E_{0I}} = \frac{1-\alpha\beta}{1+\alpha\beta} = -0.197$$

$$\Rightarrow E_{0R} = -0.197\, E_{0I} = -0.197 \times 10 \ V/m$$

$$\Rightarrow \boxed{\vec{E}_R = -1.97\left(\frac{V}{m}\right) e^{i(4x+3z-\omega t)}\,\hat{y}}$$

π phase difference due to reflection from a denser medium

b) For the transmitted wave, $\hat{k}_T = -\cos\theta_T\,\hat{z} + \sin\theta_T\,\hat{x}$

$$= -\frac{\sqrt{5}}{3}\hat{z} + \frac{2}{3}\hat{x} = \frac{2\hat{x}-\sqrt{5}\,\hat{z}}{3}$$

$$v = \frac{c}{n} = \frac{3\times10^8}{1.2} \ m/s$$

$$|\vec{k}_T| = \frac{\omega}{v} = \frac{5\times 3\times10^8}{3\times10^8} \times 1.2 = 6 \ m^{-1}$$

$$\Rightarrow \vec{k}_T = 6\left(\frac{2\hat{x}-\sqrt{5}\,\hat{z}}{3}\right) = 4\hat{x} - 2\sqrt{5}\,\hat{z}$$

Again, $\dfrac{E_{0T}}{E_{0I}} = \dfrac{2}{1+\alpha\beta} = 0.803 \Rightarrow E_{0T} = 8.03 \ V/m$

$$\Rightarrow \boxed{\vec{E}_T = 8.03\left(\frac{V}{m}\right) e^{i(4\hat{x}-2\sqrt{5}\,\hat{z}-\omega t)}\,\hat{y}}$$

$$\Rightarrow \vec{B}_T = \frac{\hat{k}_T\times\vec{E}_T}{v} = \frac{\left(\frac{\sqrt{5}}{3}\hat{x} + \frac{2}{3}\hat{z}\right)\times 8.03\times e^{i(4x-2\sqrt{5}z-\omega t)}\times1.2}{3\times10^8} \ (T)$$

$$\Rightarrow \boxed{B_T = 3.2\times10^{-8}\left(\frac{\sqrt{5}}{3}\hat{x} + \frac{2}{3}\hat{z}\right)(T)\, e^{i(4x-2\sqrt{5}z-\omega t)}}$$

3. A light wave is incident from air on crown glass ($n = 1.52$) at an angle $\theta = \frac{\pi}{6}$. The beam is linearly polarized in the plane of incidence. Assume that the magnetic permeabilities are same across the boundary between the two media.

(a) Determine the amplitude reflection and transmission coefficients, i.e., $\frac{E_{0R}}{E_{0I}}$ and $\frac{E_{0T}}{E_{0I}}$, respectively.

(b) Find the angle at which the reflected wave would be completely extinguished.

a) $\vec{E}$ $\parallel$ to the plane of incidence

$\Rightarrow$ putting $\dfrac{\cos\theta_T}{\cos\theta_I} = \alpha$ and $\dfrac{n_2}{n_1} = \beta$,

$$\frac{E_{0R}}{E_{0I}} = \frac{\alpha - \beta}{\alpha + \beta}, \quad \frac{E_{0T}}{E_{0I}} = \frac{2}{\alpha + \beta}$$

Here $\theta_I = \pi/6 \Rightarrow \sin\theta_T = \sin\pi/6 \, \dfrac{n_1}{n_2} \simeq 0.33$

$\Rightarrow \cos\theta_T \simeq 0.94$

$$\frac{E_{0R}}{E_{0I}} \simeq -0.17, \quad \frac{E_{0T}}{E_{0I}} \simeq 0.77$$

b) For reflected wave to be completely extinguished,

$\dfrac{E_{0R}}{E_{0I}} = 0 \Rightarrow \alpha = \beta \Rightarrow$ corresponds to the Brewster's angle $\theta_I = \theta_B$

At $\theta_I = \theta_B$, $\quad \theta_B + \theta_T = \pi/2$

Again, $\alpha = \beta \Rightarrow n_1 \cos\theta_T = n_2 \cos\theta_B$

$\Rightarrow n_1 \cos(\pi/2 - \theta_B) = n_2 \cos\theta_B$

$\Rightarrow \tan\theta_B = \dfrac{n_2}{n_1} = 1.52$

$\Rightarrow \theta_B = 56.66^\circ$ or 0.99 rad

4. Calculate the time averaged energy density of an electromagnetic plane wave in a conductor. Comment on the contributions due to the magnetic field and electric field in a conducting medium.

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{B^2}{\mu}\right)$$

$$= \frac{1}{2} e^{-2\beta z}\left[\epsilon \tilde{E}_0^{\,2} \cos^2(\quad) + \frac{\tilde{B}_0^{\,2}}{\mu}\cos^2(\quad)\right]$$

$$\langle \cos^2 \rangle = \frac{1}{2} \Rightarrow \langle u \rangle = \frac{1}{2} e^{-2\beta z}\left[\frac{1}{2}\epsilon \tilde{E}_0^{\,2} + \frac{1}{2\mu}\tilde{B}_0^{\,2}\right]$$

$$\frac{\tilde{B}_0}{\tilde{E}_0} = \frac{1}{V_2} = \frac{\tilde{k}_2}{\omega}, \quad \tilde{k}_2 = \alpha + i\beta$$

$$\langle u \rangle = \frac{1}{2} e^{-2\beta z}\left[\frac{\epsilon}{2}\tilde{E}_0^{\,2} + \frac{1}{2\mu}\frac{\tilde{k}_2 \cdot \tilde{k}_2^*}{\omega^2}\tilde{E}_0^{\,2}\right]$$

$$= \frac{1}{4} e^{-2\beta z}\tilde{E}_0^{\,2}\left[\epsilon + \frac{1}{\mu}\frac{\alpha^2+\beta^2}{\omega^2}\right]$$

Good conductor $\Rightarrow \alpha \simeq \beta \simeq \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \sigma \gg \epsilon\omega$

$$\langle u \rangle = \frac{1}{4} e^{-2\beta z}\tilde{E}_0^{\,2}\left[\epsilon + \frac{1}{\mu}\frac{\omega\mu\sigma}{\omega^2}\right]$$

$$= \frac{1}{4} e^{-2\beta z}\epsilon\tilde{E}_0^{\,2}\left[1 + \frac{\sigma}{\epsilon\omega}\right]$$

Magnetic contribution is dominant

$$\simeq \frac{\sigma}{4\omega} e^{-2\beta z}\tilde{E}_0^{\,2}$$

Also $$\frac{\langle u_m \rangle}{\langle u_e \rangle} = \frac{\tilde{B}_0^{\,2}/\mu}{\tilde{E}_0^{\,2}\epsilon} = \frac{1}{\mu\epsilon}\frac{\tilde{B}_0^{\,2}}{\tilde{E}_0^{\,2}} = \frac{1}{\mu\epsilon}\frac{\alpha^2+\beta^2}{\omega^2}$$

$$= \frac{1}{\mu\epsilon}\frac{\omega\mu\sigma}{\omega^2} = \frac{\sigma}{\epsilon\omega}$$

good conductor $\Rightarrow \sigma \gg \epsilon\omega$

$\Rightarrow \langle u_m \rangle \gg \langle u_e \rangle$