Complex power functions

Arijit Ganguly
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur

August 20, 2023

To begin with, let us recall how power functions are defined over $\mathbb{R}$. For $x > 0$ and $a \in \mathbb{R}$,

$$x^a \stackrel{\text{def}}{=} e^{a \log x}. \tag{$*1$}$$

It follows from the definition, given as above in $(*1)$, that for any fixed $a \in \mathbb{R}$, $x \mapsto x^a$ is a C^∞ function from $(0, \infty)$ to $(0, \infty)$. We now adapt $(*1)$ over $\mathbb{C}$ to define complex power functions.

Let $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$. Unlike the real scenario, here we have too many (branches of) logarithm, and moreover, there is no canonical choice among them. Hence To make the definition of z^w, we first need to make a choice of $\alpha \in \mathbb{R}$, and then we proceed as before:

$$z^w \stackrel{\text{def}}{=} \exp(w \log_\alpha z). \tag{$*2$}$$

Thus we can see that, for given $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$, z^w does depend upon the choice of the branch $\log_\alpha$. Indeed, varying α, one obtains all possible values of z^w. It is now immediate in view of the definition $(*2)$ that, given $\alpha \in \mathbb{R}$,

$$\mathbb{C} \setminus \overline{R_\alpha} \longrightarrow \mathbb{C} \setminus \{0\},\ z \mapsto z^w, \tag{$*3$}$$

is a holomorphic function. From now on, once we have made the choice of α, we shall assume that the domain of the power function $z \mapsto z^w$ is $\mathbb{C} \setminus \overline{R_\alpha}$, unless otherwise mentioned. It is clear that the derivative of $(*3)$ at any point $z \in \mathbb{C} \setminus \overline{R_\alpha}$ is wz^{w-1}.

As mentioned above, unlike the scenario over $\mathbb{R}$ where power of a positive number is a unique number, here z^w has more than one values, until and unless we make a choice of α. In fact, for given $x > 0$ and $a \in \mathbb{R}$, the value of x^a obtained from $(*1)$ is included in the set of values that $(*2)$ yields by varying α. To see this, take $\alpha = -\pi$. Then

$$x^a = \exp(a \log_{-\pi} x) = \exp(a \log x) = e^{a \log x},$$

as exp and $\log_{-\pi}$ extends the exponential and logarithm functions over $\mathbb{R}$ respectively. In what follows, when $a = \frac{1}{n}$, where $n \in \mathbb{N}$, we shall see how all n-th roots of a, including the complex ones, can be obtained by making different choices of α.

Let us first consider the simple case where $a = \frac{1}{2}$. As seen above, taking $\alpha = -\pi$, we get the $\sqrt{x}$, i.e., the positive square root of x. Now observe that

$$\exp\left(\frac{1}{2}\log_{\pi} x\right) = \exp\left(\frac{1}{2}\log_{\pi}(\exp(\log x + 2\pi i))\right) = \exp\left(\frac{1}{2}(\log x + 2\pi i)\right) = \sqrt{x}\exp(i\pi) = -\sqrt{x}.$$

Thus $\alpha = \pi$ gives us the negative square root. As, for any $\alpha \in \mathbb{R}$, $\left(\exp\left(\frac{1}{2}\log_{\pi} x\right)\right)^2 = \exp(\log_{\alpha} x) = x$, so $\sqrt{x}$ and $-\sqrt{x}$ are precisely all the values that we obtain from $(*2)$ by varying α. Geometrically the curve $y^2 = x$ has two branches. It is customary to choose the branch in which $y > 0$ and denote y by $\sqrt{x}$. We obtain this branch from that of the complex square root function corresponding to $\alpha = -\pi$, while $\alpha = \pi$ yields us the other branch, i.e., in which $y < 0$.

Let $n \in \mathbb{N}$. Choose $k \in \mathbb{Z}$ such that $2k\pi \leq \alpha < 2(k+1)\pi$. If $\alpha = 2k\pi$, we see that

$$\begin{aligned}
\exp\left(\frac{1}{n}\log_{\alpha} x\right) &= \exp\left(\frac{1}{n}\log_{\alpha}\left(\exp(\log x + 2k\pi i)\right)\right) \\
&= \exp\left(\frac{1}{n}(\log x + 2k\pi i)\right) \\
&= \sqrt[n]{x}\cdot\exp\left(\frac{2k\pi i}{n}\right).
\end{aligned}$$

Similarly one obtains that $\exp\left(\frac{1}{n}\log_{\alpha} x\right) = \sqrt[n]{x}\cdot\exp\left(\frac{2(k+1)\pi i}{n}\right)$, when $\alpha > 2k\pi$, as then one has $2(k+1)\pi$ lies in $\in [\alpha, \alpha + 2\pi)$. Thus $\exp\left(\frac{1}{n}\log_{\alpha} x\right)$ is always of the form $\sqrt[n]{x}\cdot\exp\left(\frac{2k\pi i}{n}\right)$, where $k \in \mathbb{Z}$. Write $k = nq + r$, where $q \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, n-1\}$. Then it is easy to see that $\exp\left(\frac{2k\pi i}{n}\right) = \exp\left(\frac{2r\pi i}{n}\right)$. This shows that, the values of $x^{\frac{1}{n}}$ that we obtain from $(*2)$ by varying α are precisely the following:

$$\sqrt[n]{x},\ \sqrt[n]{x}\cdot\exp\left(\frac{2\pi i}{n}\right),\ \sqrt[n]{x}\cdot\exp\left(\frac{4\pi i}{n}\right), \ldots, \sqrt[n]{x}\cdot\exp\left(\frac{2(n-1)\pi i}{n}\right). \qquad (*4)$$

Note that, the complex numbers as listed above in $(*4)$, are precisely all roots of the equation $z^n = x$.