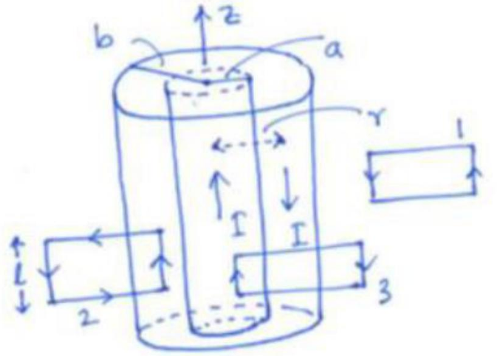
1. Two long coaxial cylindrical shells of radii a and b ($b > a$) are placed with their axis along the z-direction. A current I goes through the inner shell in the z-direction and returns through the outer shell, the current being uniformly distributed on the surface. If $I = I_0 \exp -t/\tau$, find the induced electric field everywhere.



$\vec{B} = 0$, for $r<a$ inside both shell

$= \dfrac{\mu_0 I}{2\pi r}\hat{\varphi}$, $a<r<b$, in between the shells

$= 0$, $r>b$, outside both shell

$r>b$, $\boxed{\vec{E}=0}$ (no flux change)

$a<r<b$, Apply $\oint \vec{E}\cdot\vec{dl} = \int \left(\dfrac{-\partial \vec{B}}{\partial t}\right)\cdot\vec{da}$ over loop 2

$$\Rightarrow \quad El = \int \frac{-\partial}{\partial t}\left(\frac{\mu_0 I_0 e^{-t/\tau}}{2\pi r}\right)\hat{\varphi}\cdot\vec{da}$$

$$= \frac{\mu_0 I_0}{2\pi\tau}e^{-t/\tau}\int_r^b \frac{1}{r}\hat{\varphi}\cdot l\,dr\,\hat{\varphi}$$

$$= \frac{\mu_0 I_0 l}{2\pi\tau}e^{-t/\tau}\ln\frac{b}{r}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\mu_0 I_0}{2\pi\tau}e^{-t/\tau}\ln\frac{b}{r}\,\hat{z}}$$

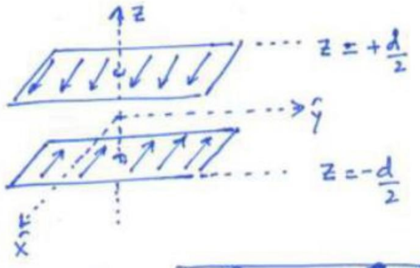
$r<a$, Apply $\oint \vec{E}\cdot\vec{dl} = \int\left(\dfrac{-\partial \vec{B}}{\partial t}\right)\cdot\vec{da}$ over loop 3

$$\Rightarrow El = \int_a^b \frac{-\partial}{\partial t}\left(\frac{\mu_0 I_0 e^{-t/\tau}}{2\pi r}\right)\hat{\varphi}\cdot\vec{da}$$

$$= \frac{\mu_0 I_0}{2\pi\tau}e^{-t/\tau}\int_a^b \frac{1}{r}\hat{\varphi}\cdot l\,dr\,\hat{\varphi} = \frac{\mu_0 I_0 l}{2\pi\tau}e^{-t/\tau}\ln\frac{b}{a}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\mu_0 I_0}{2\pi\tau}e^{-t/\tau}\ln\frac{b}{a}\,\hat{z}}$$

2. Two large plates at $z = \pm\frac{d}{2}$ carry slow time-varying surface currents $K(t)\hat{x}$ and $-K(t)\hat{x}$, respectively. Find the magnetic field everywhere using quasi-static approximation. Find the induced electric field everywhere.



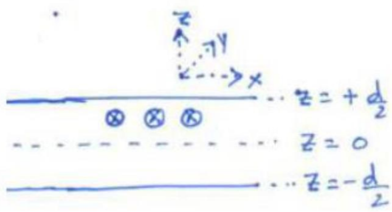
For $-\frac{d}{2} < z < \frac{d}{2}$, (using quasi-static approximation)

$$\vec{B} = 2 \cdot \frac{\mu_0}{2} k \hat{y} = \mu_0 k \hat{y}$$

$\vec{B} = 0$ elsewhere due to equal and opposite contribution of the two plates

Hence $\boxed{-\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{dk}{dt} \hat{y}}$, for $-\frac{d}{2} < z < \frac{d}{2}$ and zero elsewhere

compare $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\Rightarrow$ Map the present problem onto a problem of a slab with current density $\left(-\frac{dk}{dt}\right)$ in the $\hat{y}$ direction

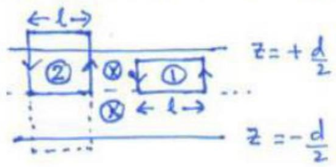


The imaginary slab is symmetric under $180°$ rotation about Y axis

It is also symmetric under translations in the x-direction $\Rightarrow$ field must be independent of x. Similarly the field is independent of y

x component of the field must be an odd function of z, otherwise the field would not look the same after a rotation by $180°$ about Y axis.

$\Rightarrow$ $\vec{E} = 0$ at $z = 0$, with $\vec{E}$ changing sign on either side of $z = 0$



For $z > \frac{d}{2}$ use loop ②

$$E l = \mu_0 \frac{dk}{dt} \cdot l \cdot \frac{d}{2}$$

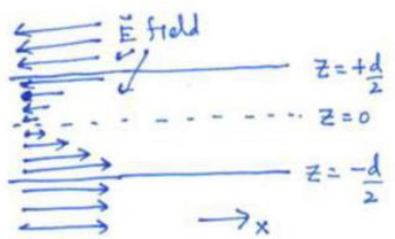
$$\Rightarrow \boxed{\vec{E} = -\frac{1}{2} \mu_0 d \frac{dk}{dt} \hat{x}}$$

Similarly for $z < -\frac{d}{2}$ $\boxed{\vec{E} = +\frac{1}{2} \mu_0 d \frac{dk}{dt} \hat{x}}$

For $-\frac{d}{2} < z < \frac{d}{2}$, use loop ①,

$$E l = \mu_0 \frac{dk}{dt} \cdot l \cdot z$$

$$\Rightarrow \boxed{\vec{E} = -\mu_0 z \frac{dk}{dt} \hat{x}}$$



3. A parallel plate capacitor is made of two circular sheets of radius R with a separation $d \ll R$. The capacitor is getting charged at a very slow rate with charge $Q(t) = Q_0\{1 - \exp(-t/t_0)\}$

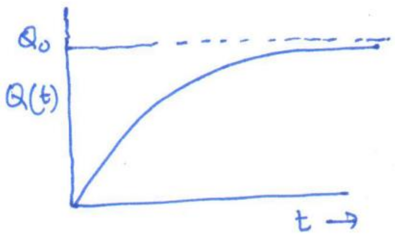
(a) Plot the charge as a function of time.

(b) Determine and plot the displacement current as a function of time

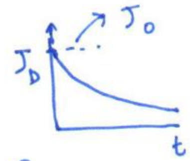
(c) Determine the magnetic field in between the plates.

(d) What is the origin of the magnetic field? When and where would the above analysis represent the true fields in between the plates?

a)



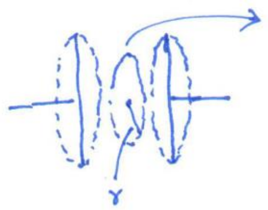
b)



$$E(t) = \frac{Q(t)}{\epsilon_0 A} = \frac{Q_0}{\epsilon_0 A}\left(1 - e^{-t/t_0}\right) \quad , \; A = \pi R^2$$

$$\Rightarrow J_D = \epsilon_0 \frac{\partial E}{\partial t} = \frac{Q_0}{\pi R^2 t_0} e^{-t/t_0} = J_0 e^{-t/t_0} \quad , \; J_0 = \frac{Q_0}{\pi R^2 t_0}$$

c)



$$B . 2\pi r = \mu_0 \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} . d\vec{a}$$

$$= \mu_0 J_0 e^{-t/t_0} . \pi r^2$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 Q_0}{2\pi t_0} \frac{r}{R^2} e^{-t/t_0} \hat{\varphi}}$$

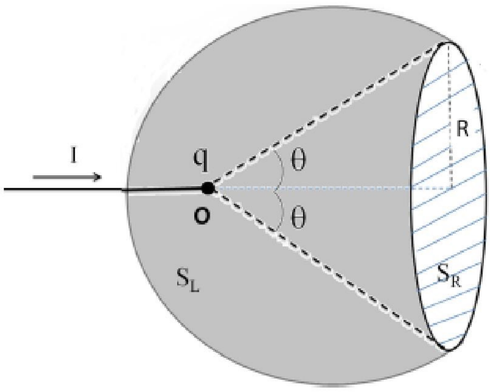
d) Source of $\vec{B}$ field is displacement current J_D

For calculation of $\vec{B}$ field we have used Ampere's law of magnetostatics in the quasi-static approximation.

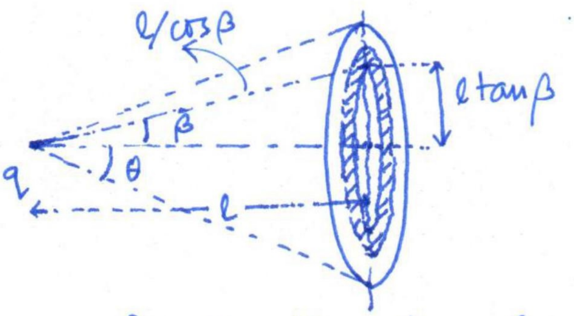
Quasistatic approximation is valid when t_0 is large

The estimation of $\vec{B}$ field is reasonably accurate in the region between the plates far from the edges.

4. A half-infinite straight wire carries current I from negative infinity to the origin O as shown by the solid straight line in the figure below. The termination of the wire leads to a build-up (increase) of charge q at the origin with time (so that $\frac{dq}{dt} = I$). Consider the circle shown in the figure below which has radius R and subtends an angle 2θ with respect to the charge. Compute the integral $\oint \vec{B}\cdot\vec{dl}$ around the circle for the two surfaces S_R and S_L as shown in the figure.



1) $$\int_{S_R} \vec{E}\cdot d\vec{a} = \int_{S_R} E\cos\beta\, da = \int_0^\theta \frac{q}{4\pi\epsilon_0}\frac{1}{(\ell/\cos\beta)^2}\cdot\cos\beta\cdot 2\pi\cdot\ell\tan\beta\cdot d(\ell\tan\beta)$$



$$= \frac{q}{2\epsilon_0}\int_0^\theta \frac{\cos^3\beta}{\ell^2}\cdot\ell^2\tan\beta\cdot\frac{1}{\cos^2\beta}\,d\beta$$

$$= \frac{q}{2\epsilon_0}\int_0^\theta \sin\beta\,d\beta$$

$$= \frac{q}{2\epsilon_0}(1-\cos\theta)$$

2) Flux through surface S_L + flux through surface $S_R = \frac{q}{\epsilon_0}$

$$\Rightarrow \int_{S_L}\vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0} - \frac{q}{2\epsilon_0}(1-\cos\theta) = \frac{q}{2\epsilon_0}(1+\cos\theta)$$

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc} + \mu_0\epsilon_0 \int_S \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot \vec{da}$$

1) For S_R, $I_{enc} = 0$

$$\oint \vec{B}\cdot\vec{dl} = \mu_0\epsilon_0 \int_{S_R} \frac{\partial \vec{E}}{\partial t}\cdot \vec{da}$$

$$= \mu_0\epsilon_0 \frac{d}{dt}\int_{S_R} \vec{E}\cdot\vec{da}$$

$$= \frac{\mu_0}{2}(1-\cos\theta)\frac{dq}{dt} = \frac{\mu_0 I}{2}(1-\cos\theta)$$

2) For S_L, $I_{enc} = I$

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I + \mu_0\epsilon_0 \int_{S_L} \frac{\partial \vec{E}}{\partial t}\cdot\vec{da}$$

$$= \mu_0 I - \mu_0\epsilon_0 \frac{1}{2\epsilon_0}(1+\cos\theta)\frac{dq}{dt}$$

(−ve sign since $\vec{da}$ is in a direction opposite to current)

$$= \mu_0 I\left[1 - \frac{1}{2}(1+\cos\theta)\right]$$

$$= \frac{\mu_0 I}{2}(1-\cos\theta)$$

$\Rightarrow$ $\oint \vec{B}\cdot\vec{dl}$ is surface independent, as expected.