Analytic functions

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Theorem 1. *Let* $\sum_{n=0}^{\infty} a_n(z-a)^n$ *be a power series in* $\mathbb{C}$ *with radius of convergence* $R \in (0, \infty]$. *Define*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, \ \forall z \in D(a; R). \tag{$*1$}$$

Then f *is holomorphic everywhere in* $D(a; R)$*, and furthermore,*

$$\forall z \in D(a; R), \ f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}. \tag{$*2$}$$

We first need the following simple lemma that will be used in the proof of Theorem 1.

Lemma 1. *For all* $\alpha, \beta \in \mathbb{C}$ *and* $n \in \mathbb{N}$*,*

$$|(\alpha+\beta)^n - \alpha^n| \le n|\beta|(|\alpha| + |\beta|)^{n-1}. \tag{$*3$}$$

Proof. Observe that $(\alpha+\beta)^n - \alpha^n = \beta\left(\sum_{k=0}^{n-1}(\alpha+\beta)^k \alpha^{n-1-k}\right)$. Triangle inequality yields that, for all $k = 0, 1, \ldots, n-1$, $|\alpha+\beta|^k \le (|\alpha| + |\beta|)^k$. Hence

$$\begin{aligned}
|(\alpha+\beta)^n - \alpha^n| &\le |\beta| \left|\sum_{k=0}^{n-1}(\alpha+\beta)^k \alpha^{n-1-k}\right| \\
&\le |\beta| \sum_{k=0}^{n-1} |\alpha+\beta|^k |\alpha|^{n-1-k} \\
&\le |\beta| \sum_{k=0}^{n-1}(|\alpha| + |\beta|)^k (|\alpha| + |\beta|)^{n-1-k} \\
&= n|\beta|(|\alpha| + |\beta|)^{n-1}.
\end{aligned}$$

From this $(*3)$ is immediate. □

Proof of Theorem 1. Let $z_0 \in D(a;R)$ and $\varepsilon > 0$. Fix $r > 0$ such that $|z_0 - a| < r < R$. In view of Lemma 1, whenever $0 < |z - z_0| < r - |z_0 - a|$, we have the following for all $n \in \mathbb{N}$:

$$|a_n| \left| \frac{(z-a)^n - (z_0 - a)^n}{z - z_0} \right| \leq n|a_n|(|z - z_0| + |z_0 - a|)^{n-1} \leq n|a_n|r^{n-1}. \tag{$*4$}$$

Since $\sum_{n=1}^{\infty} na_n(z-a)^{n-1}$ also has the radius of convergence R, one has $\sum_{n=1}^{\infty} n|a_n|r^{n-1}$ converges. Choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < \frac{\varepsilon}{4}$. From ($*4$), using the 'comparison test', one obtains that the series $\sum_{n=N+1}^{\infty} \left(a_n \frac{(z-a)^n - (z_0 - a)^n}{z - z_0} \right)$ converges absolutely, and subsequently,

$$\begin{aligned} \left| \sum_{n=N+1}^{\infty} \left(a_n \frac{(z-a)^n - (z_0 - a)^n}{z - z_0} \right) \right| &\leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z-a)^n - (z_0 - a)^n}{z - z_0} \right| \\ &\leq \sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < \frac{\varepsilon}{4}, \ \forall z \in D(z_0; r - |z_0 - a|). \end{aligned} \tag{$*5$}$$

We also have the following for the same reason:

$$\left| \sum_{n=N+1}^{\infty} na_n(z_0 - a)^{n-1} \right| \leq \sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < \frac{\varepsilon}{4}, \ \forall z \in D(z_0; r - |z_0 - a|). \tag{$*6$}$$

Now consider the polynomial function $\sum_{n=0}^{N} a_n(z-a)^n$. Since it is holomorphic with derivative $\sum_{n=1}^{N} na_n(z-a)^{n-1}$, there exists $\delta_1 > 0$ such that one has

$$\left| \frac{\sum_{n=0}^{N} a_n(z-a)^n - \sum_{n=0}^{N} a_n(z_0-a)^n}{z - z_0} - \sum_{n=1}^{N} na_n(z_0 - a)^{n-1} \right| < \frac{\varepsilon}{2}, \text{ whenever } 0 < |z - z_0| < \delta_1. \tag{$*7$}$$

Set $\delta \overset{\text{def}}{=} \min\{\delta_1, r - |z_0 - a|\}$. It follows at once from ($*5$), ($*6$) and ($*7$) that, for all $0 < |z - z_0| < \delta$,

$$\begin{aligned} \left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} na_n(z_0 - a)^{n-1} \right| &\leq \left| \frac{\sum_{n=0}^{N} a_n(z-a)^n - \sum_{n=0}^{N} a_n(z_0-a)^n}{z - z_0} - \sum_{n=1}^{N} na_n(z_0 - a)^{n-1} \right| \\ &\quad + \left| \sum_{n=N+1}^{\infty} \left(a_n \frac{(z-a)^n - (z_0 - a)^n}{z - z_0} \right) \right| + \left| \sum_{n=N+1}^{\infty} na_n(z_0 - a)^{n-1} \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This completes the proof of Theorem 1. □

Definition 1. *Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$. We say that f is analytic if for every $a \in U$ there exists $r > 0$ such that $D(a; r) \subseteq U$ and f is 'represented by a power series centered at a on $D(a; r)$', i.e., there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \ \forall z \in D(a; r).$$

From Theorem 1, we conclude that any analytic function is holomorphic everywhere in its domain.

Theorem 2. *Let $f : [a, b] \longrightarrow \mathbb{C}$ be Riemann integrable and $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a continuous curve. Denote the image of γ by γ^*. Define*

$$F(z) = \int_a^b \frac{f(t)}{\gamma(t) - z}\, dt, \ \forall z \notin \gamma^*.$$

Then F is analytic.

Note: For any fixed $z \notin \gamma^*$, the map $t \mapsto (\gamma(t) - z)$ is continuous and nowhere vanishing on $[a, b]$. Hence, $t \mapsto \dfrac{1}{(\gamma(t) - z)}$ is also continuous on $[a, b]$, and consequently Riemann integrable. From this, it follows that the function $[a, b] \longrightarrow \mathbb{C}$, $t \mapsto \dfrac{f(t)}{\gamma(t) - z}$, is Riemann integrable. Thus the integral that appears in the definition of F makes sense.

Proof. Let $z_0 \notin \gamma^*$. Since γ^* is compact, it is closed, and hence there exists $r > 0$ such that $D(z_0; r) \cap \gamma^* = \emptyset$. Now, we observe that, for all $z \in D(z_0; r)$ and $t \in [a, b]$,

$$\begin{aligned}
\frac{f(t)}{\gamma(t) - z} = \frac{f(t)}{(\gamma(t) - z_0) - (z - z_0)} &= \frac{f(t)}{(\gamma(t) - z_0)} \cdot \frac{1}{1 - \left(\dfrac{z - z_0}{\gamma(t) - z_0}\right)} \\
&= \frac{f(t)}{(\gamma(t) - z_0)} \cdot \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\gamma(t) - z_0}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{f(t)}{(\gamma(t) - z_0)^{n+1}} \cdot (z - z_0)^n,
\end{aligned}$$

since $\left|\dfrac{z - z_0}{\gamma(t) - z_0}\right| \le \dfrac{|z - z_0|}{r} < 1$. Fix $z \in D(z_0; r)$. Since f is Riemann integrable, it is bounded. Then one has

$$\left|\frac{f(t)}{(\gamma(t) - z_0)^{n+1}} \cdot (z - z_0)^n\right| \le \frac{\sup\limits_{t \in [a,b]} |f(t)|}{r} \cdot \left(\frac{|z - z_0|}{r}\right)^n,$$

for all $n \in \mathbb{N}$ and $t \in [a, b]$. From Weirstrass M-test, it now follows that $\displaystyle\sum_{n=0}^{\infty} \frac{f(t)(z - z_0)^n}{(\gamma(t) - z_0)^{n+1}}$ converges uniformly on $[a, b]$. This yields that

$$F(z) = \int_a^b \frac{f(t)}{\gamma(t) - z}\, dt = \sum_{n=0}^{\infty} \left(\int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{n+1}}\, dt\right)(z - z_0)^n, \ \forall z \in D(z_0; r). \tag{$*8$}$$

It now follows from $(*8)$ that F is holomorphic on $D(z_0; r)$. □

Corollary 1. *Let f and γ be as above in the hypothesis of Theorem 2. For any $n \in \mathbb{N}$, define the function $F_n : \mathbb{C} \setminus \gamma^* \longrightarrow \mathbb{C}$ as follows:*

$$F_n(z) = \int_a^b \frac{f(t)}{(\gamma(t) - z)^n}\, dt,\ \forall z \notin \gamma^*.$$

Then F_n is holomorphic and

$$F_n'(z) = n \int_a^b \frac{f(t)}{(\gamma(t) - z)^{n+1}}\, dt,\ \forall z \notin \gamma^*.$$

Proof. From $(*8)$, it is clear that,

$$\frac{F^{(n)}(z_0)}{n!} = \int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{n+1}}\, dt,\ \forall n \geq 0. \tag{$*9$}$$

Since z_0 is arbitrary, we have, for any $n \in \mathbb{N}$, $F_n = \dfrac{F^{(n-1)}}{(n-1)!}$. Hence, for all $n \in \mathbb{N}$, F_n is differentiable and, for all $z \in \mathbb{C} \setminus \gamma^*$,

$$F_n'(z) = \frac{F^{(n)}(z)}{(n-1)!} = n \int_a^b \frac{f(t)}{(\gamma(t) - z)^{n+1}}\, dt.$$

□

In view of $(*9)$, we now see that $(*8)$ is the Taylor series expansion of F at the point z_0. For any $n \geq 0$, we denote the n-th remainder term in the above-mentioned expansion by R_n. The following Corollary provides an integral representation of R_n.

Corollary 2. *Let r be as in the proof of Theorem 2. Then, for every $z \in D(z_0; r)$,*

$$R_n(z) = (z - z_0)^{n+1} \int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{n+1}(\gamma(t) - z)}\, dt. \tag{$*10$}$$

Proof. Let $z \in D(z_0; r)$. Then

$$\begin{aligned}
R_n(z) &= \sum_{k=n+1}^{\infty} \left(\int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{k+1}}\, dt \right) (z - z_0)^k \\
&= (z - z_0)^{n+1} \sum_{k=n+1}^{\infty} \left(\int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{k+1}}\, dt \right) (z - z_0)^{k-(n+1)} \\
&= (z - z_0)^{n+1} \sum_{k=n+1}^{\infty} \left(\int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{n+1}} \cdot \frac{1}{(\gamma(t) - z_0)} \cdot \left(\frac{z - z_0}{\gamma(t) - z_0} \right)^{k-(n+1)} \right) dt \\
&= (z - z_0)^{n+1} \sum_{k=0}^{\infty} \left(\int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{n+1}} \cdot \frac{1}{(\gamma(t) - z_0)} \cdot \left(\frac{z - z_0}{\gamma(t) - z_0} \right)^{k} \right) dt
\end{aligned} \tag{$*11$}$$

Similar to what has been done in the proof of Theorem 2, using Weirstrass M-test, we get that

$$\sum_{k=0}^{\infty}\left(\frac{f(t)}{(\gamma(t)-z_0)^{n+1}}\cdot\frac{1}{(\gamma(t)-z_0)}\cdot\left(\frac{z-z_0}{\gamma(t)-z_0}\right)^k\right)$$

converges uniformly on $[a,b]$. Hence, from $(*11)$, one obtains that

$$\begin{aligned}
R_n(z) &= (z-z_0)^{n+1}\int_a^b\sum_{k=0}^{\infty}\left(\frac{f(t)}{(\gamma(t)-z_0)^{n+1}}\cdot\frac{1}{(\gamma(t)-z_0)}\cdot\left(\frac{z-z_0}{\gamma(t)-z_0}\right)^k\right)dt\\
&= (z-z_0)^{n+1}\int_a^b\left(\frac{f(t)}{(\gamma(t)-z_0)^{n+1}}\cdot\frac{1}{(\gamma(t)-z_0)}\cdot\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\gamma(t)-z_0}\right)^k\right)dt\\
&= (z-z_0)^{n+1}\int_a^b\left(\frac{f(t)}{(\gamma(t)-z_0)^{n+1}}\cdot\frac{1}{(\gamma(t)-z_0)}\cdot\frac{1}{1-\left(\frac{z-z_0}{\gamma(t)-z_0}\right)}\right)dt\\
&= (z-z_0)^{n+1}\int_a^b\frac{f(t)}{(\gamma(t)-z_0)^{n+1}(\gamma(t)-z)}\,dt.
\end{aligned}$$

□