Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Complex Analysis (MTH 403)
Hints for Exercise Sheet 6

1. Integrals along real line

1.1. Let $b > 0$. Show the following:

$$\int_0^\infty e^{-x^2} \cos 2bx \, dx = e^{-b^2} \frac{\sqrt{\pi}}{2}$$

and

$$\int_0^\infty e^{-x^2} \sin 2bx \, dx = e^{-b^2} \int_0^b e^{t^2} \, dt.$$

You may assume that $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$.

Hint. *For any $R > 0$, the integral of e^{-z^2} along the positively boundary of the rectangle with vertices $0, R, R + ib$ and ib is 0.*

1.2. Show that

$$\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Hint. *For any $R > 0$, consider the sector of $C(0; R)$ made of the line segment joining 0 and R and having the central angle $\frac{\pi}{4}$ in the upper half plane. The integral of e^{iz^2} along the boundary of this sector is 0 by Cauchy's theorem. You may proceed as follows:*

(a) First, using Calculus or otherwise, show that $\dfrac{\sin t}{t} \geq \dfrac{2}{\pi}$, for all $t \in (0, \frac{\pi}{2}]$.
(b) Show that, as $R \longrightarrow \infty$, the integral along the arc goes to 0.
(c) Show that, as $R \longrightarrow \infty$, the integral along the line segment joining 0 and $Re^{i\frac{\pi}{4}}$ converges to $e^{i\frac{\pi}{4}} \cdot \frac{\sqrt{\pi}}{2}$

1.3. Evaluate $\int_0^\infty e^{-x^2} \cos x^2 \, dx$.

Hint. *The above integral is the real part of the following:*

$$\int_0^\infty e^{-(1+i)x^2} \, dx = \int_0^\infty e^{-\sqrt{2}e^{\frac{i\pi}{4}} x^2} \, dx$$
$$= \int_0^\infty e^{-\sqrt{2}\left(e^{\frac{i\pi}{8}} x\right)^2} \, dx.$$

Now proceed along the lines exactly similar to 1.2., but the central angle is $\frac{\pi}{8}$ in this case. The final answer is $\dfrac{\cos \frac{\pi}{8}}{2^{\frac{5}{4}}} \sqrt{\pi}$.

2. Liouville's theorem

2.1. Find all entire functions f with the property that, for all $z \in \mathbb{C}$, $f(z + 1) = f(z) = f(z + i)$.

Sketch of the solution. *It is easy to see that, for any $m, n \in \mathbb{Z}$ and $z \in \mathbb{C}$ one has $f(z+m+in) = f(z)$. It follows that $f(\mathbb{C}) = f(\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z, \operatorname{Im} z \leq 1\})$. Since f is continuous and $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z, \operatorname{Im} z \leq 1\}$ is compact, there exists $M > 0$ such that $|f(z)| \leq M$ whenever $0 \leq \operatorname{Re} z, \operatorname{Im} z \leq 1$. This imples that f is bounded. Hence, from Liouville's theorem, f is constant.*

2.2. Let f and g be two entire functions. Assume that there exists $c \in \mathbb{R}$ such that

$$\operatorname{Re} f(z) \leq c \operatorname{Re} g(z), \ \forall z \in \mathbb{C}.$$

Show that $f = ag + b$, for some $a, b \in \mathbb{C}$. In particular, if $\operatorname{Re} f$ is bounded above then f must be a constant function.

Sketch of the solution. *Let* $h \stackrel{def}{=} f - cg$. *Then* $|e^h| = \left|e^{\operatorname{Re}(f-cg)}\right|$ *which is bounded from above by* 1. *Hence, from Liouville's theorem,* $f - cg$ *is constant.*

2.3. (a) Let f be an entire function such that $\operatorname{Re} f > 0$. Show that f is constant.

Hint. *Consider* $\frac{1}{1+f}$.

Sketch of the solution. *Observe that* $|1+f| \geq |\operatorname{Re}(1+f)| = ||1+\operatorname{Re} f| = 1 + \operatorname{Re} f \geq 1$. *So* $\left|\frac{1}{1+f}\right| \leq 1$. *This makes* $\frac{1}{1+f}$ *constant, due to Liouville's theorem, which implies that* f *is constant.*

(b) Suppose that f is an entire function such that $\operatorname{Re} f$ or $\operatorname{Im} f$ has no zeros. What can you conclude about f?

Sketch of the solution. *Suppose that* $\operatorname{Re} f$ *has no zero. From intermediate value theorem, it follows that either* $\operatorname{Re} f > 0$ *or* $\operatorname{Re} f < 0$ *(how?). If* $\operatorname{Re} f > 0$ *then from 2.3.a, we get that* f *is constant. If* $\operatorname{Re} f < 0$ *then* $\operatorname{Re}(-f) > 0$, *so that* $-f$ *is constant, due to 2.3.a. Hence in either case* f *is constant. If* $\operatorname{Im} f$ *does not have a zero then consider the function* if *and use the previous case.*

(c) Prove the analogue of 2.3.a for 2×2 valued functions: If $F : \mathbb{C} \longrightarrow M_2(\mathbb{C})$ is such that F_{jk} is entire for each $j, k = 1, 2$, and $\forall z \in \mathbb{C}$, $F(z) + F(z)^*$ is positive definite. Then show that F is constant.

Hint. *Diagonal entries of a positive definite matrix are positive, and the determinant of a positive definite matrix is positive. So from 2.3.a, it follows that* f_{11} *and* f_{22} *are constant. This implies that* $f_{12} + \overline{f_{21}}$ *is bounded. Thus* $f_{12} + f_{21}$ *is constant. Similarly, one obtains* $f_{12} - f_{21}$ *is constant.*

Sketch of the solution. *Follow the hints given above. Observe that, if* $f_{12} + \overline{f_{21}}$ *is bounded, then so is* $\operatorname{Re}(f_{12} + f_{21})$. *From 2.2., it follows that* $f_{12} + f_{21}$ *is constant. Observe that* $\operatorname{Re}(f_{21} - f_{12}) = \operatorname{Re}(i(f_{12} + \overline{f_{21}}))$. *The latter is bounded as* $f_{12} + \overline{f_{21}}$ *is bounded. Hence again from 2.2., it follows that* $f_{12} - f_{21}$ *is constant. The rest is immediate.*

2.4.$*$ Let f be an entire function satisfying the following:

$$|f(z)| \leq \frac{1}{\sqrt{|\operatorname{Re} z|}}, \text{ whenever } \operatorname{Re} z \neq 0. \tag{2.1}$$

(a) Suppose that $|\operatorname{Re} z_0| < \frac{1}{2}$. Write $z_0 = x_0 + iy_0$. Consider the square S with the following vertices:

$$-1 + i(y_0 - 1), 1 + i(y_0 - 1), 1 + i(y_0 + 1) \text{ and } -1 + i(y_0 + 1).$$

Assume that S is equipped with the counterclockwise orientation. Show that

$$f(z_0) = \frac{1}{2\pi i} \int_S \frac{f(z)}{z - z_0}\, dz. \tag{2.2}$$

(b) Using (2.1), show that in the integral appearing in (2.2), the contribution from each horizontal edge of S is at most 8 in absolute value.

(c) Show that, in the integral appearing in (2.2), the contribution from each vertical edge of S is at most 4 in absolute value.

(d) Deduce from 2.4.b and 2.4.c that $|f(z_0)| \leq \frac{12}{\pi}$.

(e) Conclude that f must be a constant function.

3. Homology and homotopy version of Cauchy's theorem

3.1. Is the condition $\operatorname{Ind}_\gamma(z) = 0$, for all $z \in \mathbb{C} \setminus U$, necessary in the Global Cauchy theorem?

Hint. *Let $z \in \mathbb{C} \setminus U$. Then the function $w \mapsto \frac{1}{w-z}$ is holomorphic on U.*

3.2. Let $U \subseteq \mathbb{C}$ be open, γ be a cycle in U which is homologous to 0 in U. Consider finitely many distinct points, say $z_1, z_2, \cdots, z_n$, of U. For each $j \in \{1, \cdots, n\}$, choose a closed disc $D_j \subseteq U$ centered at z_j such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Denote the boundary of D_j oriented anticlockwise by γ_j and let $k_j \stackrel{\text{def}}{=} \operatorname{Ind}_\gamma(z_j)$ for all $j = 1, \cdots, n$.

(a) Show that γ is homologous to $\displaystyle\sum_{j=1}^{n} k_j\gamma_j$ in $U \setminus \{z_1, z_2, \cdots, z_n\}$.

(b) Show that, for every $f \in H(U \setminus \{z_1, z_2, \cdots, z_n\})$,

$$\int_\gamma f = \sum_{j=1}^{n} k_j \int_{\gamma_j} f.$$

3.3. Let U, γ and f be as in Global Cauchy theorem. Show that, for all $n \geq 0$,

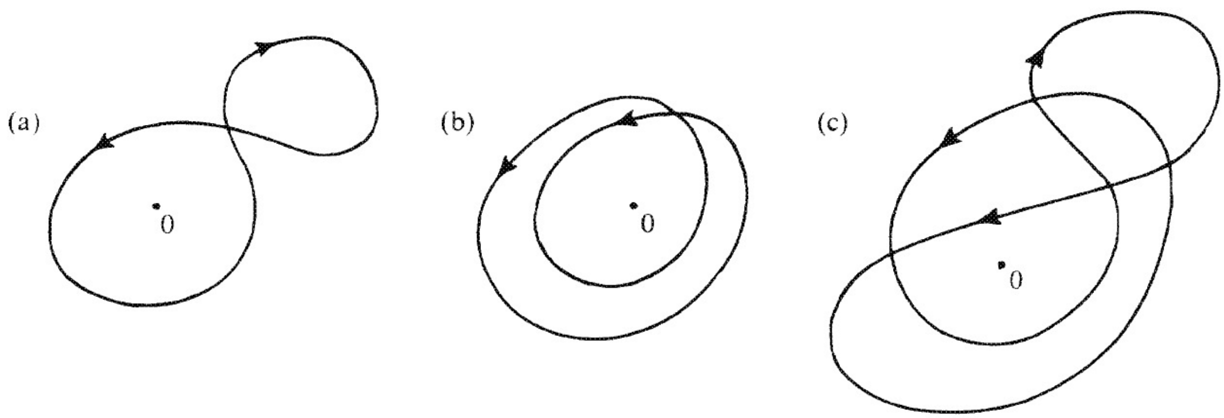
$$\operatorname{Ind}_\gamma(z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw, \ \forall z \in U \setminus \gamma^*.$$

Sketch of the solution. *Let V be a connected component of $U \setminus \gamma^*$. being the connected component of an open set, clearly V is open. Note that the function $\operatorname{Ind}_\gamma$ is constant on V, say α. From Global Cauchy theorem, we obtain that,*

$$\alpha f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)}\, dw, \ \forall z \in U \cap V.$$

Now use Exercise 1.3 of Exercise Sheet 5.

3.4. Along each of the following paths, evaluate the integral of $\dfrac{e^z - e^{-z}}{z^4}$:



Answer. *(a) $\frac{2\pi i}{3}$ (b), (c) $\frac{4\pi i}{3}$ (use Global Cauchy theorem).*

3.5. Let $a, b > 0$. Show that $\displaystyle\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}$.

Hint. *Consider the path $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ given by $\gamma(t) \stackrel{def}{=} a\cos t + ib\sin t$. What is $\displaystyle\int_\gamma \frac{dz}{z}$? Can yo use that here?*

3.6. Let $P(z) \in \mathbb{C}[z]$ whose all distinct zeros are $a_1, \ldots, a_k$. Suppose γ is a closed curve in $\mathbb{C}$ such that $a_j \notin \gamma^*$, for all $j = 1, \ldots, n$. Find $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{P'}{P}$.

Answer. *Let $m_1, \ldots, m_k$ be the multiplicities of $a_1, \ldots, a_k$ respectively. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{P'}{P} = m_1 \operatorname{Ind}_\gamma(a_1) + \cdots + m_k \operatorname{Ind}_\gamma(a_k).$$

3.7. Let $U \subseteq_{open} \mathbb{C}$, $h \in H(U)$ be zero-free, $z_0 \in U$, $m \in \mathbb{N}$ and $f(z) \stackrel{\text{def}}{=} (z-z_0)^m h(z)$, for all $z \in U$. Suppose that γ is a closed path in U such that it is homologous to 0 in U. Assume that $z_0 \notin \gamma^*$. Prove that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = m \operatorname{Ind}_\gamma(z_0).$$

Solution. *Observe that, for all $z \in U$, one has*

$$\frac{f'(z)}{f(z)} = \frac{m}{(z-z_0)} + \frac{h'(z)}{h(z)}.$$

Since $\frac{h'}{h} \in H(U)$, it follows from Global Cauchy theorem that $\int_\gamma \frac{h'}{h} = 0$. The rest is obvious.

3.8. Let $U \subseteq \mathbb{C}$ be open such that every closed path in U is homologous to 0 in U. Consider finitely many distinct points, say $z_1, z_2, \cdots, z_n$, of U. For each $j \in \{1, \cdots, n\}$, choose a closed disc $D_j \subseteq U$ centered at z_j such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Denote the boundary of D_j oriented anticlockwise by γ_j. Suppose that $f \in H(U \setminus \{z_1, \ldots, z_n\})$ and $a_k \stackrel{\text{def}}{=} \frac{1}{2\pi i}\int_{\gamma_k} f$ for all $k = 1, \cdots, n$. Define

$$h(z) = f(z) - \sum_{k=1}^{n} \frac{a_k}{z - z_k}, \ \forall z \in U \setminus \{z_1, \ldots, z_n\}. \tag{3.1}$$

Show that the function h, defined as above in (3.1), has a primitive.

Sketch of the solution. *Let γ be a closed path in $U \setminus \{z_1, \ldots, z_n\}$. Let $\sigma \stackrel{def}{=} \sum_{k=1}^{n} \operatorname{Ind}_\gamma(z_k)\gamma_k$. Verify that γ is homologous to σ in U. From Golbal Cauchy theorem, it follows that,*

$$\begin{aligned}
\int_\gamma f &= \int_\sigma f \\
&= \sum_{k=1}^{n} \operatorname{Ind}_\gamma(z_k) \int_{\gamma_k} f \\
&= \sum_{k=1}^{n} 2\pi i \cdot \operatorname{Ind}_\gamma(z_k) \cdot a_k \\
&= \sum_{k=1}^{n} \int_\gamma \frac{a_k}{w - z_k}\, dw \\
&= \int_\gamma \left(\sum_{k=1}^{n} \frac{a_k}{w - z_k} \right) dw.
\end{aligned}$$

Thus $\int_\gamma h\, dw = 0$. Hence h has a primitive.

4. Analytic logarithms and n-th roots of functions

Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C} \setminus \{0\}$ be an analytic function.

4.1. Show that f has an analytic logarithm if and only if its *logarithmic derivative*, i.e., $\frac{f'}{f}$ has a primitive.

Sketch of the solution. *This has been done in the class.*

4.2. Show that the following are equivalent:

(a) f has an analytic logarithm on U.

(b) For every $n \in \mathbb{N}$, f has an *analytic n-th root on* U, i.e., there exists an analytic $g : U \longrightarrow \mathbb{C}$ such that $g^n = f$.

(c)∗ For infinitely may $n \in \mathbb{N}$, f has an analytic n-th root on U.

Hint. *Fix a closed path γ in U.*

(i) If g_n is an analytic n-th root of f, then observe that $\frac{1}{2\pi i}\int_\gamma \frac{g_n'}{g_n} = \frac{1}{2\pi i n}\int_\gamma \frac{f'}{f}$.

(ii) Observe that $\mathrm{Ind}_{g_n\circ\gamma}(0) = \frac{1}{2\pi i}\int_\gamma \frac{g_n'}{g_n}$.

(iii) Conclude from 4.2.(c)i and 4.2.(c)ii that $\int_\gamma \frac{f'}{f} = 0$.

Note: It is not enough to have an analytic n-th root for a particular n in order to conclude that f has an analytic logarithm. The next exercise provides an example.

4.3. Consider the function $f(z) \stackrel{\text{def}}{=} z^2$, for all $z \in \mathbb{D} \setminus \{0\}$. Show that f does not have an analytic logarithm despite having an analytic square root.

Sketch of the solution. *Use 4.1.*

4.4.∗ Let $U \subseteq_{open} \mathbb{C}$. Show that the following are equivalent:

(a) For every cycle γ in U and $z \in \mathbb{C} \setminus U$, $\mathrm{Ind}_\gamma(z) = 0$.

(b) For every cycle γ in U and $f \in H(U)$, $\int_\gamma f = 0$.

(c) Every analytic function on U has a primitive.

(d) Every zero-free analytic function on U has an analytic logarithm.

(e) Every zero-free analytic function on U has an analytic n-th root for all $n \in \mathbb{N}$.