Indian Institute of Technology Kanpur
Department of Mathematics and Statistics
Complex Analysis (MTH 403)
Exercise Sheet 1

1. Basic properties of $\mathbb{C}$

1.1. Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$. Show that
$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|. \tag{1.1}$$
Find a necessary and sufficient condition so that equality holds in (1.1).

1.2.∗ Let $U \subseteq_{open} \mathbb{C}$. For each $n \in \mathbb{N}$, define
$$K_n = \overline{D(0; n)} \cap \left\{ z \in U : |w - z| \geq \frac{1}{n},\ \forall w \in \mathbb{C} \setminus U \right\}. \tag{1.2}$$

(a) Show that K_n is compact, for all $n \in \mathbb{N}$.
(b) Show that, for all $n \in \mathbb{N}$, K_n is contained in the interior of K_{n+1}.
(c) Show that, for every compact subset K of U, there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$. In particular, conclude that $U = \bigcup_{n=1}^{\infty} K_n$.

1.3.∗ Let $P(z) \stackrel{\text{def}}{=} a_n z^n + \cdots + a_1 z + a_0$, where $n \in \mathbb{N}$, be a polynomial with complex coefficients. Assume that $0 < a_n \leq \cdots \leq a_1 \leq a_0$.

(a) Show that no zero of $P(z)$ can lie in $D(0; 1)$. (**Hint:** Consider $(1 - z)P(z)$.)
(b) Find all zeros of $P(z)$ inside the closed unit disk $\overline{D(0; 1)}$, provided $a_{j+1} < a_j$, for all $j = 0, 1, \ldots, n - 1$.

2. Power series

Throughout this section, we always assume $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a complex power series, i.e., $z_0 \in \mathbb{C}$ and $a_n \in \mathbb{C}$, for all $n \in \mathbb{N}$.

2.1. Suppose that $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges for a complex number $z_1 \neq z_0$. Then show that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ whenever $|z - z_0| < |z_1 - z_0|$.

(**Hint:** Note that $a_n (z_1 - z_0)^n \xrightarrow[n \to \infty]{} 0$, hence one obtains $M > 0$ such that $|a_n (z_1 - z_0)^n| \leq M$, for all $n \geq 0$. Now observe that, for any n, $|a_n||z - z_0|^n = |a_n||z_1 - z_0|^n \left(\frac{|z - z_0|}{|z_1 - z_0|} \right)^n \leq M \left(\frac{|z - z_0|}{|z_1 - z_0|} \right)^n$.)

2.2. Let
$$R \stackrel{\text{def}}{=} \sup \left\{ |z - z_0| : \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges} \right\}. \tag{2.1}$$
Show the following:

(a) $|z - z_0| < R \Longrightarrow \sum_{n=0}^{\infty} |a_n||z - z_0|^n < \infty$.

(b) $|z-z_0|>R \Longrightarrow \sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges.

Conclude from (2.2.a) and (2.2.b) that R, defined as above in (2.1), is the only number in $[0,\infty]$ for which (2.2.a) and (2.2.b) hold together. We call R the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. The disc $D(z_0;R)$ is called the *disc of convergence* of the power series.

2.3. Let $n \geq 2$ and $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ are complex numbers. Show the following:

$$\sum_{k=1}^{n} a_k b_k = a_n B_n + \sum_{k=1}^{n-1}(a_k - a_{k+1})B_k,$$

where $B_k \stackrel{\text{def}}{=} b_1+\cdots+b_k$, for all $k=1,\ldots,n$. (**Hint:** Write $\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k(B_k - B_{k-1})$, $B_0 \stackrel{\text{def}}{=} 0$.)

2.4. Find all points of convergence for each of the following power series:

(a) $\sum_{n=0}^{\infty} z^n$ (b) $\sum_{n=1}^{\infty} \frac{z^n}{n}$ (c) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

For 2.5. and 2.6., we let R be as above in 2.2.

2.5. (a) Show that the convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is uniform on every compact subset of $D(z_0;R)$.

(b) Give an example where $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ does not converge uniformly on $D(z_0;R)$.

2.6. (a) Show that $R = \dfrac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$.

(b) If $a_n \neq 0$, for all $n \in \mathbb{N}$, then show that

$$\liminf_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \leq R \leq \limsup_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|. \tag{2.2}$$

(c) Conclude from (2.2) that, if $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists in $[0,\infty]$ then it must be equal to R.

(d) Conclude from 2.6.a that the radius of convergence of the power series $\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ is R.

2.7. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series in $\mathbb{C}$ with radius of convergence $R>0$.

(a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$.

(b) Let h be the function represented by the power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ on its disc of convergence, say D. Show that, for any $0<r<R$, there exists $M>0$ such that $|h(z)| \leq M e^{\frac{|z|}{r}}$, for all $z \in D$.

2.8.* Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series in $\mathbb{C}$ with radius of convergence $R > 0$ and f be the function it represents on $D(z_0; R)$. For $N \geq 0$, denote by S_N the N-th partial sum of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Show that

$$\sum_{N=0}^{\infty} |f(z) - S_N(z)| < \infty,$$

for each $z \in D(z_0; R)$.

2.9.* Suppose that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for every $z \in D(z_0; r)$, where $r > 0$. Let $z_1 \in D(z_0; r)$. Show that there exists a sequence $\{c_n\}_{n \geq 0}$ in $\mathbb{C}$ such that

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} c_n(z-z_1)^n \text{ whenever } |z-z_1| < r - |z_1 - z_0|.$$

2.10.* Fix $a \in \mathbb{C}$. Show that, for every $z_0 \in \mathbb{C} \setminus \{a\}$, there exists a power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ in $\mathbb{C}$ having radius of convergence at least $|z_0 - a|$ such that

$$\frac{1}{z-a} = \sum_{n=0}^{\infty} c_n(z-z_0)^n \text{ whenever } |z-z_0| < |z_0 - a|.$$