Module 2

Initial Value Problems

2.2 Stability

2.3 Euler's method

2.4 Implicit method

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Initial Value Problems: Implicit Methods

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- Backward Euler method

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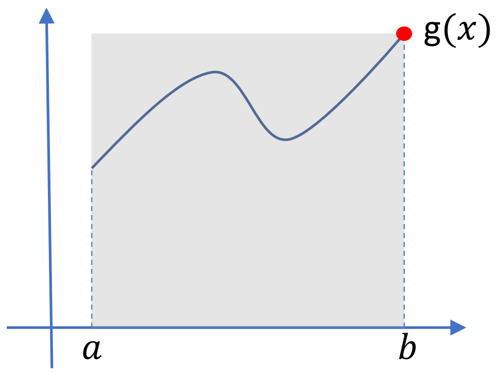

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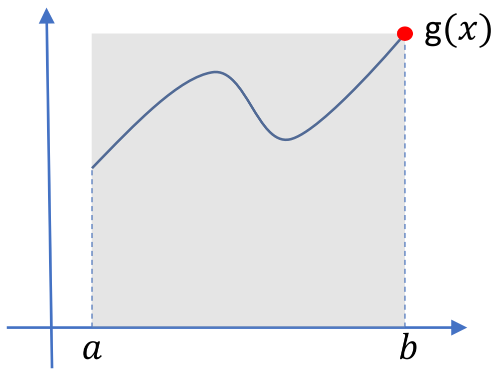

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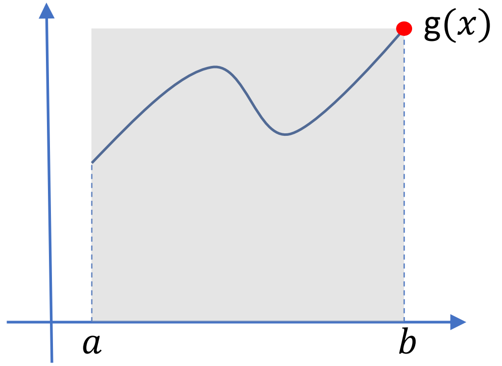

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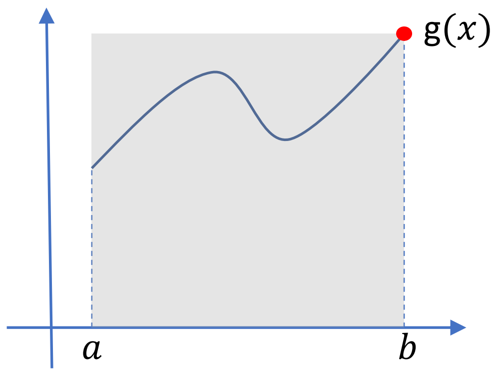

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Background Euler method

When y_{k+1} *appears on the right-hand side of the method*

$$y_{k+1} = \Phi(f, y_{k+1}, y_k, h_k)$$

then the method is called an implicit method.

Example

Let us solve $y' = -y^3,\ y(0) = 1$ *using the backward Euler method taking the uniform step size* $h = h_k = t_{k+1} - t_k = 0.5.$

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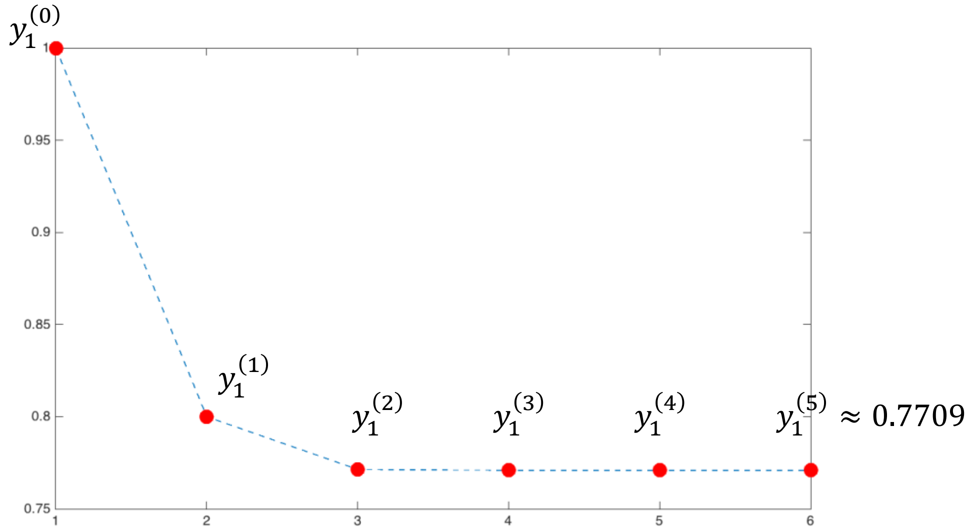

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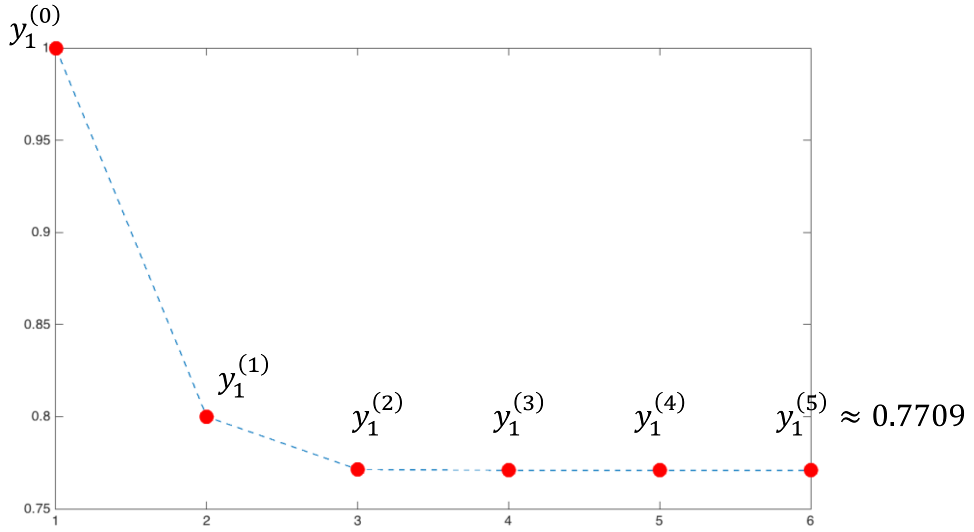

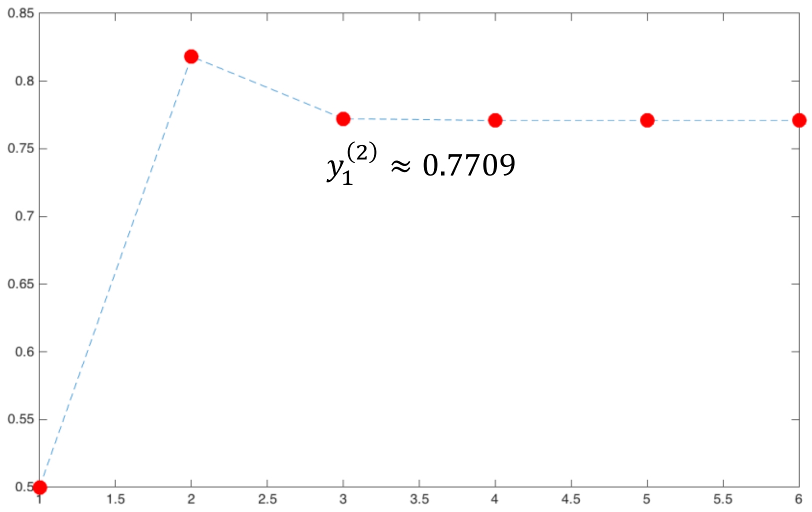

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Remark:

In the previous example, we could have used the fixed point iteration $y_{k+1}^{(n+1)} = y_k + hf\left(t_{k+1}, y_{k+1}^{(n)}\right)$ *in place of Newton's iterations*

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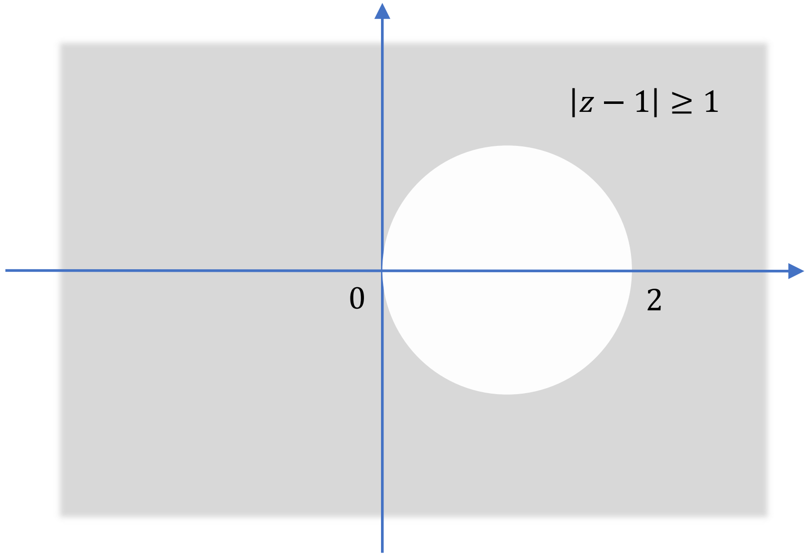

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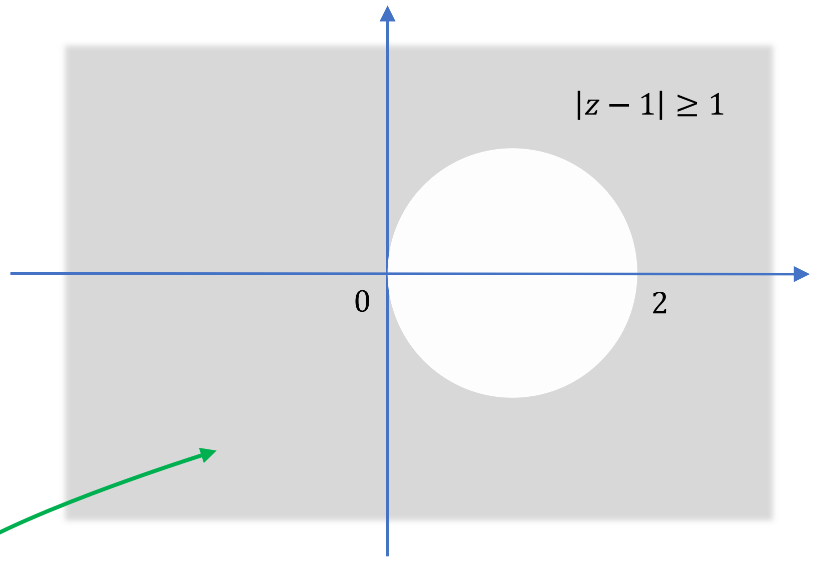
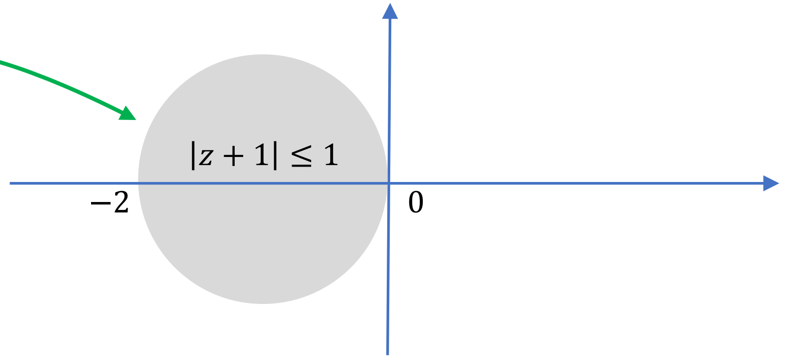


Compare with Euler's method



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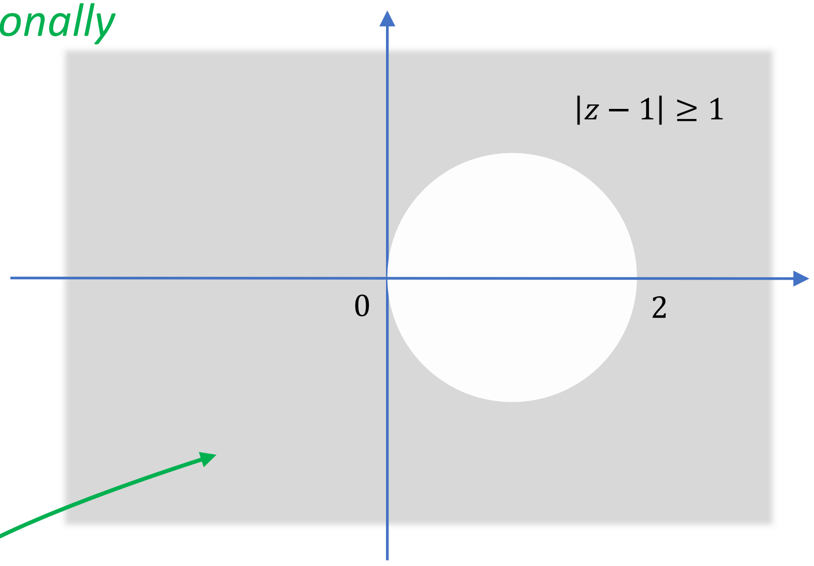

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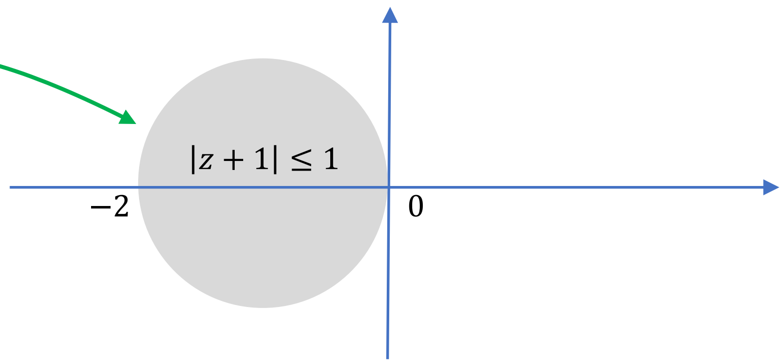

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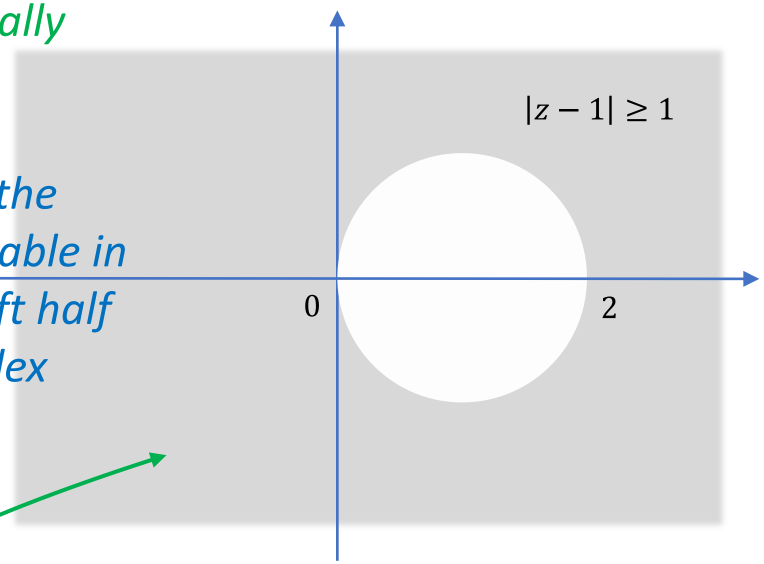

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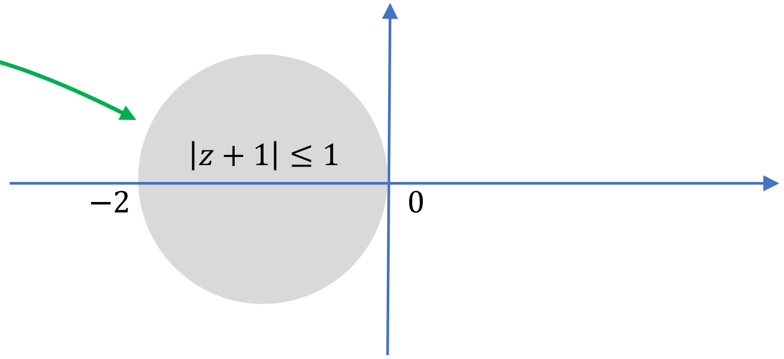

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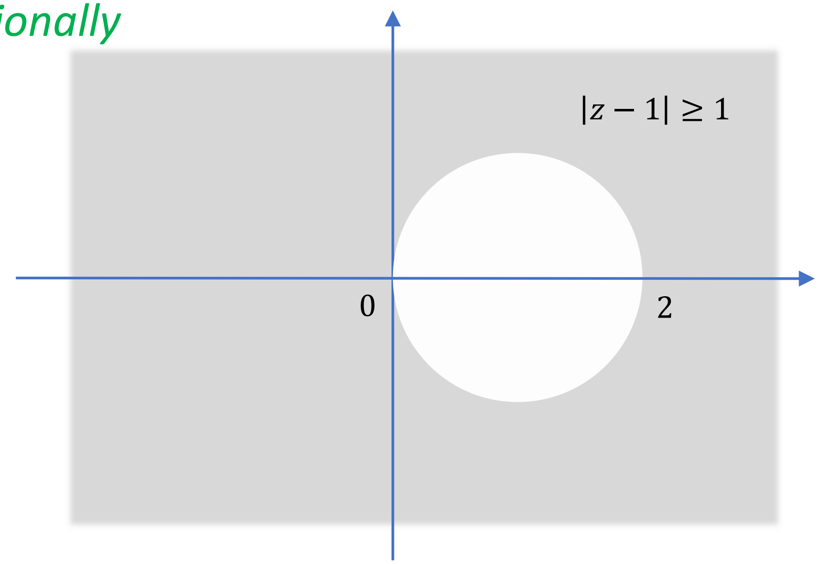

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$$\left|\frac{1}{1-h\lambda}\right| \leq 1 \quad \text{or} \quad |1 - \lambda h| \geq 1.$$

Note that this condition holds for all $h > 0$ where $Re(\lambda) < 0$.

For the general ODE $y' = f(t, y)$, we have

Initial Value Problems: Implicit Methods

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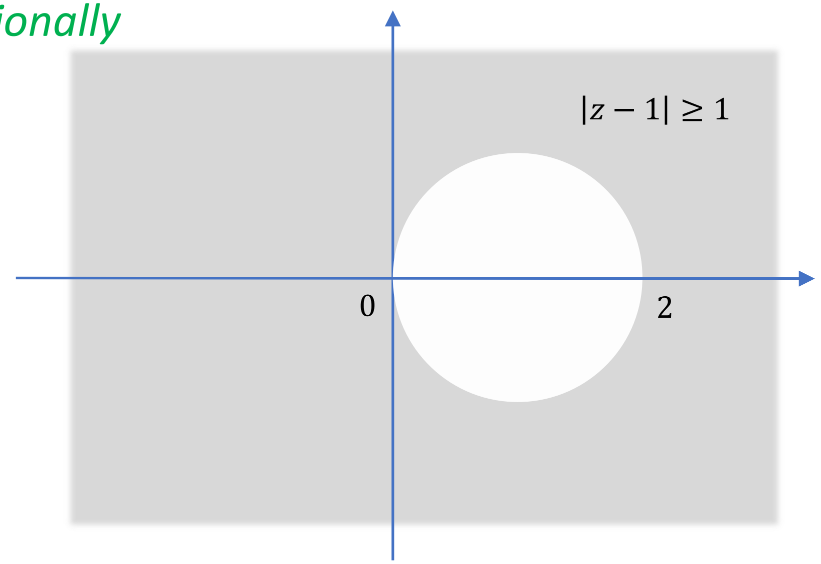
Unconditionally stable!



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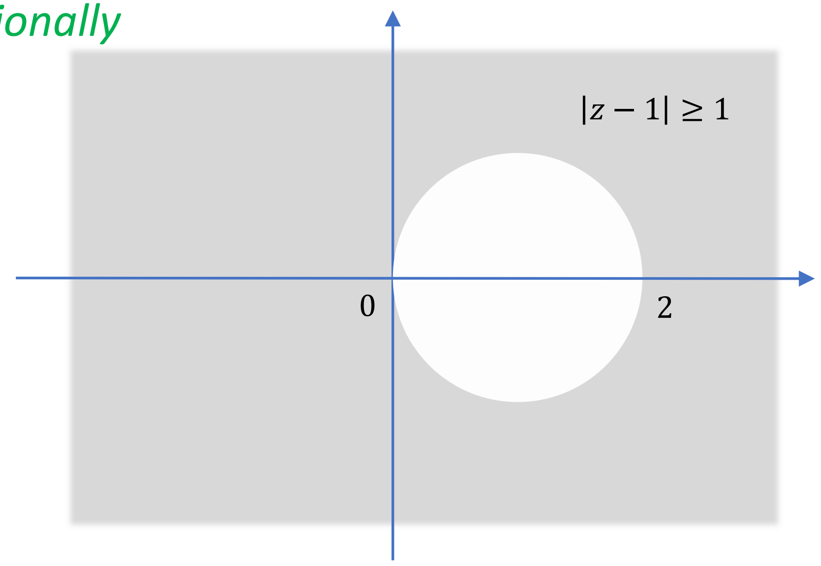

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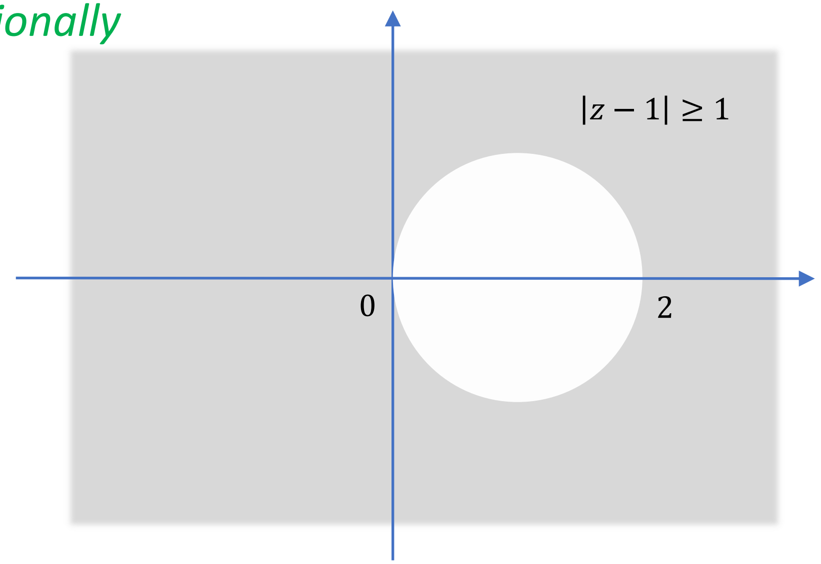

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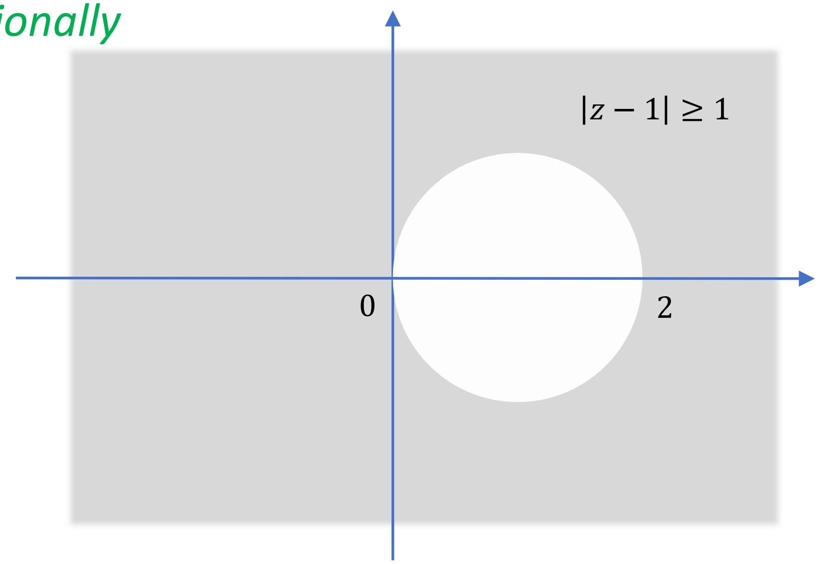

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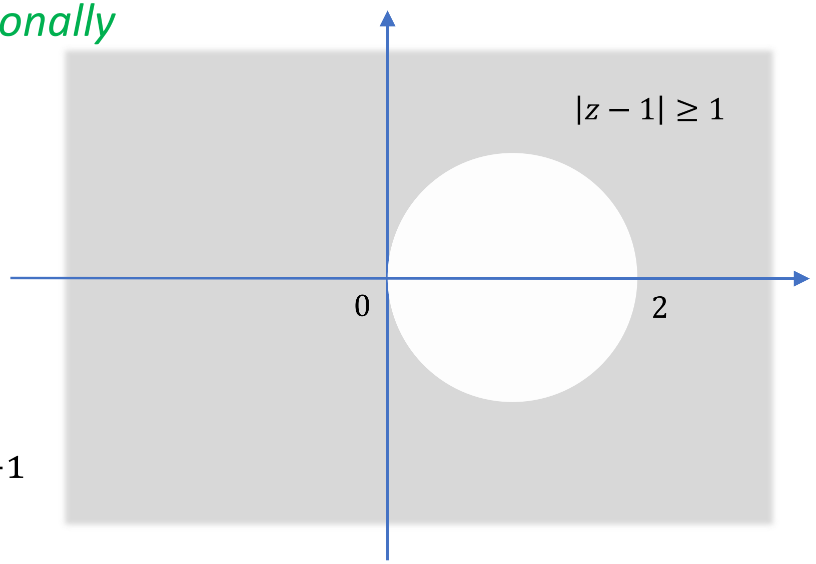

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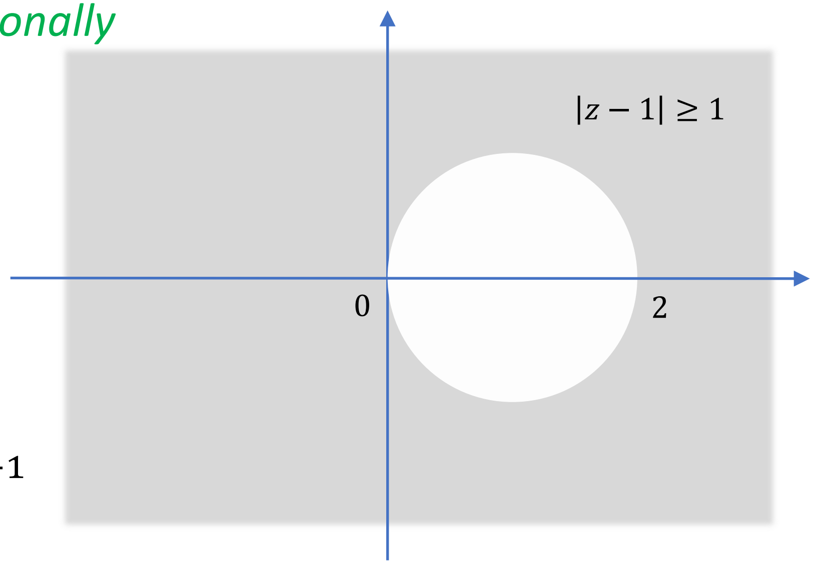

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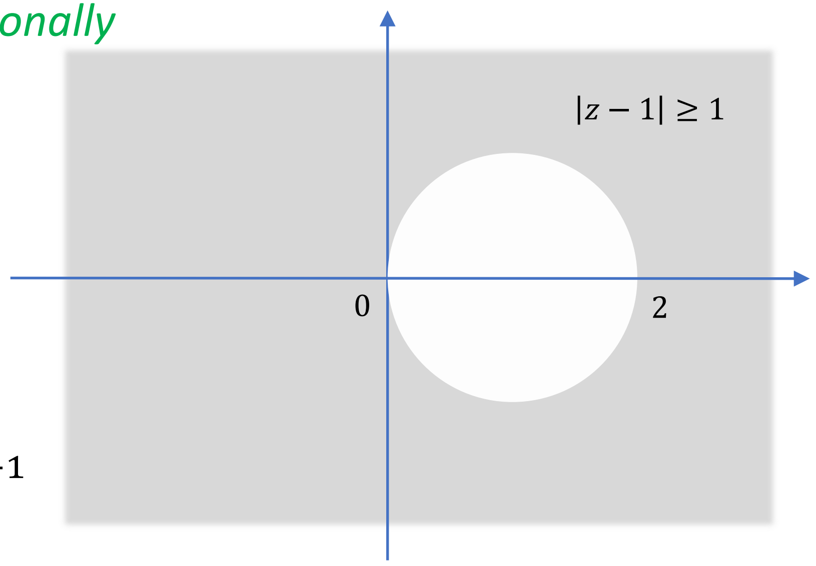

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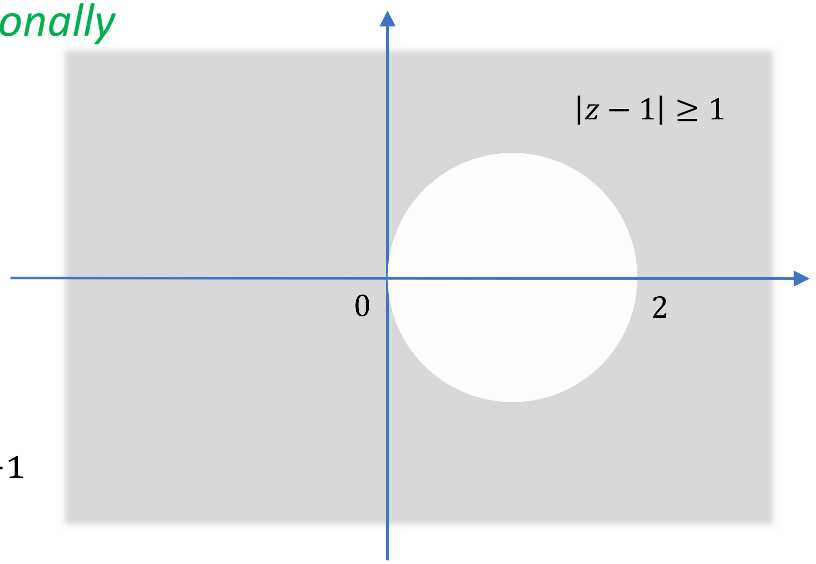

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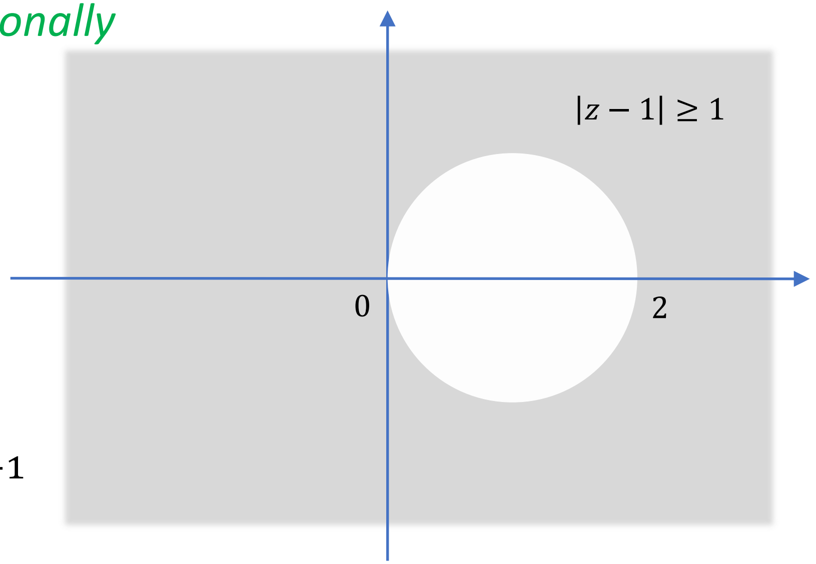

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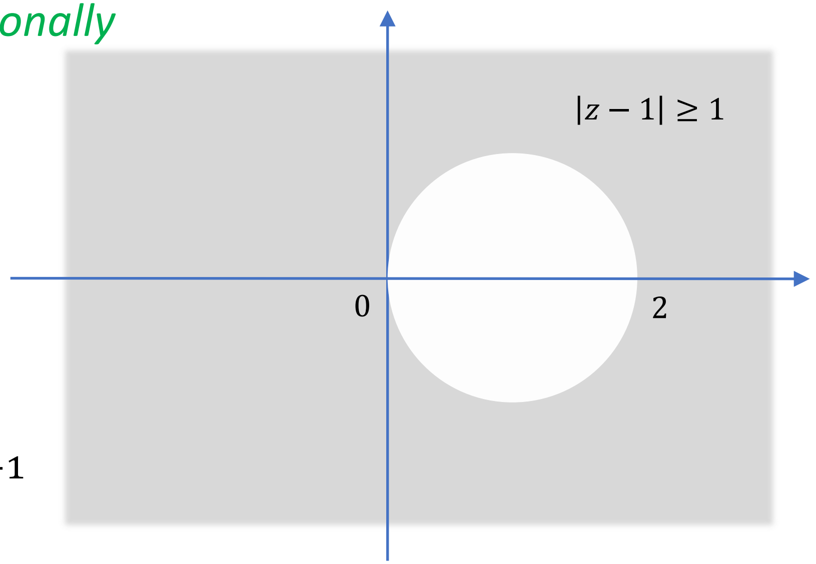

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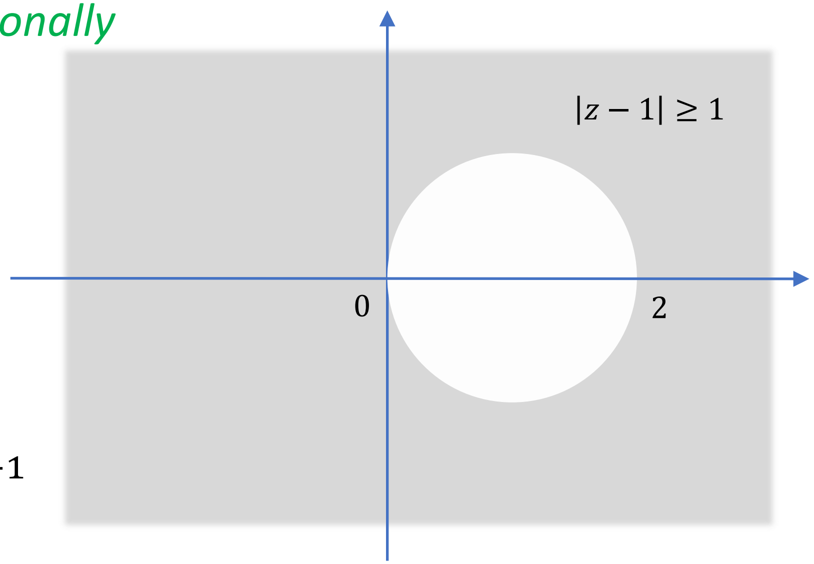

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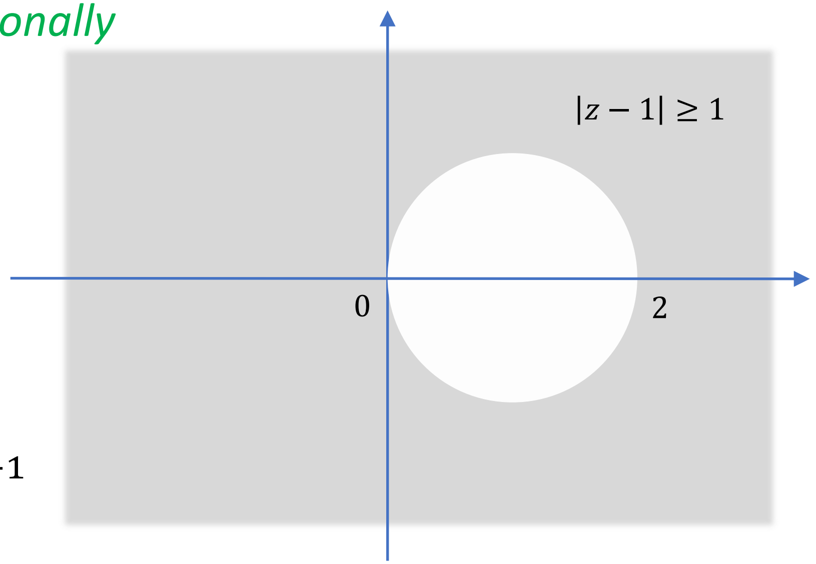

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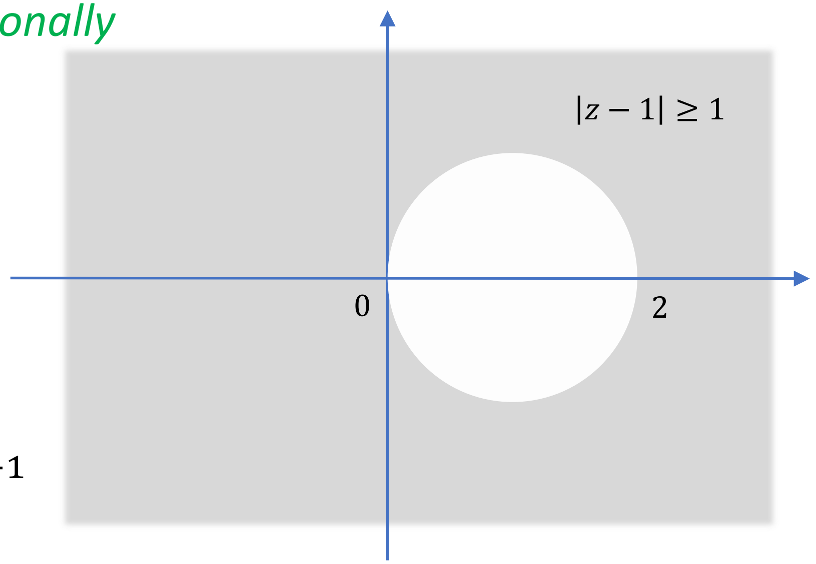

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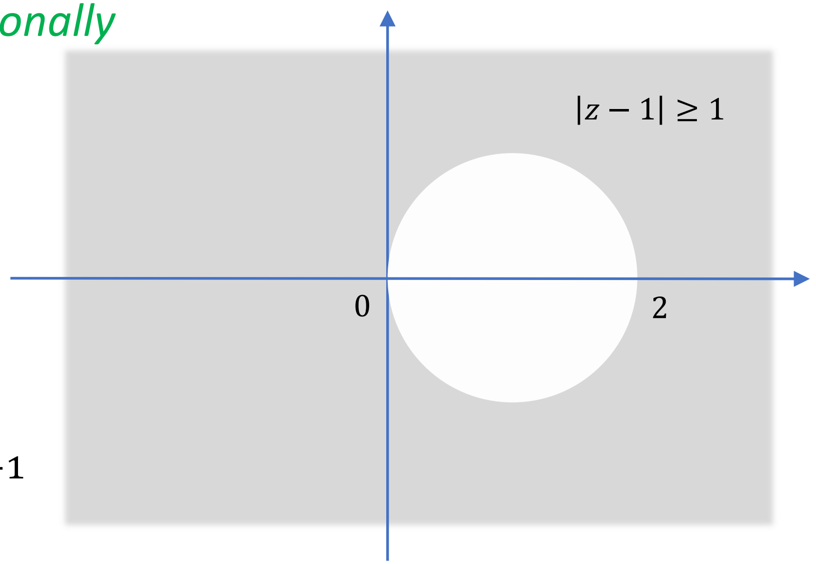

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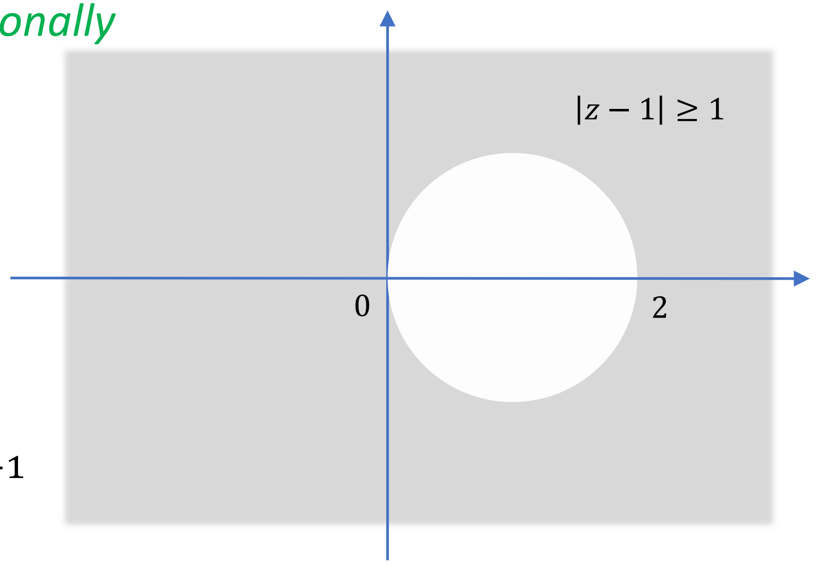

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first-order accurate!