Module 2
Initial Value Problems

2.5 Stiffness

2.6 Linear Multistep Method

2.7 Non-Linear Methods

- Runge-Kutta methods

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Initial Value Problems: Non-Linear Methods

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The Tayler series methods require evaluation of partial derivatives of f; *can we develop high-order one step methods that depends only on evaluation of* f *and not its derivatives?*

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For motivation, let's look at the predictor-corrector Heun's method

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in terms of the relative increment function Ψ, then for the Heun's method, we have

$$\Psi(f; t_n, y_n, h) = \left(f(t_n, y_n) + f\left(t_{n+1}, y_n + h_n f(t_n, y_n)\right)\right)/2.$$

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$$\Psi = \big(f + f + hf_t + hff_y + O(h^2)\big)/2 = f + \frac{h}{2}\big(f_t + ff_y\big) + O(h^2).$$

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Backtracking this calculation, we see that

$$\Psi = \frac{1}{2}k_1 + \frac{1}{2}k_2$$

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$$k_1 = f(t_n, y_n), \qquad k_2 = f(t_n + h, y_n + hk_1).$$

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More generally,

$$\Psi = b_1 k_1 + b_2 k_2 + \cdots + b_q k_q$$

where $k_i = f(t_n + c_i h, p_i)$ *and*

$$\begin{gathered} p_1 = y_n \\ p_2 = y_n + h(a_{21}k_1) \\ p_3 = y_n + h(a_{31}k_1 + a_{32}k_2) \\ \vdots \\ p_q = y_n + h\big(a_{q1}k_1 + a_{q2}k_2 + \cdots + a_{q,q-1}k_{q-1}\big) \end{gathered}$$

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To specify a particular method of this form we must specify the coefficients b_i, c_i*,* $1 \le i \le q$*, and* a_{ij}*,* $1 \le i \le q$*,* $1 \le j \le i$*. The* b_i *are called weights, the* c_i *(or the points* $t_n + c_i h$*) the nodes, and* p_i *or, sometimes, the* k_i*, are called the stages.*

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A Runge-Kutta method is often recorded in a tableau of the form

$$\begin{array}{c|c} \boldsymbol{c} & A \\ \hline & \boldsymbol{b}^T \end{array}$$

For example, the tableau for Heun's method is

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

where we have omitted the zeros in the upper triangle of A. The other well known RK methods are given below:

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

Modified Euler method (order 2)

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Heun's 3-stage method (order 3)

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Runge-Kutta-Simpson 4-stage method (the RK method) (order 4)

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- Consistency and Convergence of one step methods

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Remark

1. We assume that Ψ is defined for $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, and is continuous there. Moreover, we assume the uniform Lipschitz condition

$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$

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2. *Note that a single step method* $y_{n+1} = y_n + h\Psi(f; t_n, y_n, h)$ *is consistent if*

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3. In view of the continuity, consistency condition is equivalent to $\Psi(f; t_n, y, 0) = f(t_n, y)$.

4. The method has order p if
$$\left|\Psi(f; t, y_n, h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right| \leq Ch^p$$
for all $h \leq h_0$ for some constants $C, h_0 > 0$, for all $y \in C^{p+1}$.