Classification Theorems

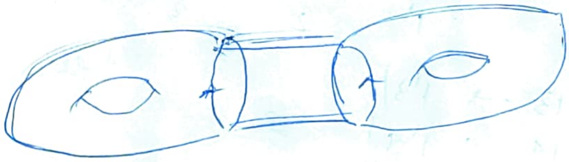
Classification Thm for 2D surfaces which are oriented, compact and without boundary

1) Sphere : $x^2+y^2+z^2=1$

2) Torus ; $S^1 \times S^1 = T_1$

Connected sum : $S_1 \# S_2$

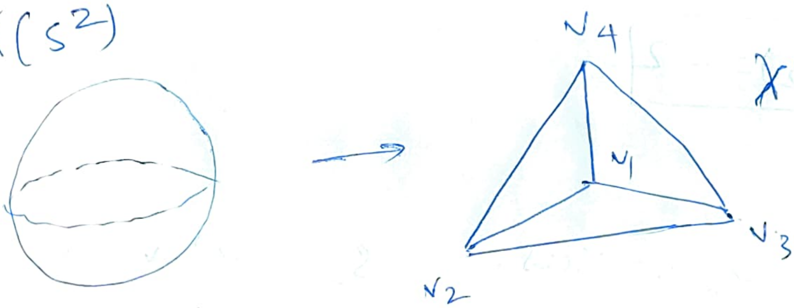
ex : $T \# T$ — T_2 or double torus



ex : $T \# S^1 = T$!

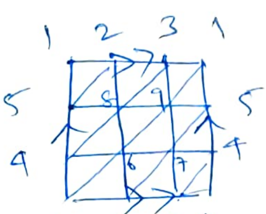
$\chi = V - E + F$ is a topological invariant.

1) $\chi(S^2)$



$\chi(S^2) = 4 - 6 + 4$
$= 2$.

2) $\chi(S_1 \times S_1)$

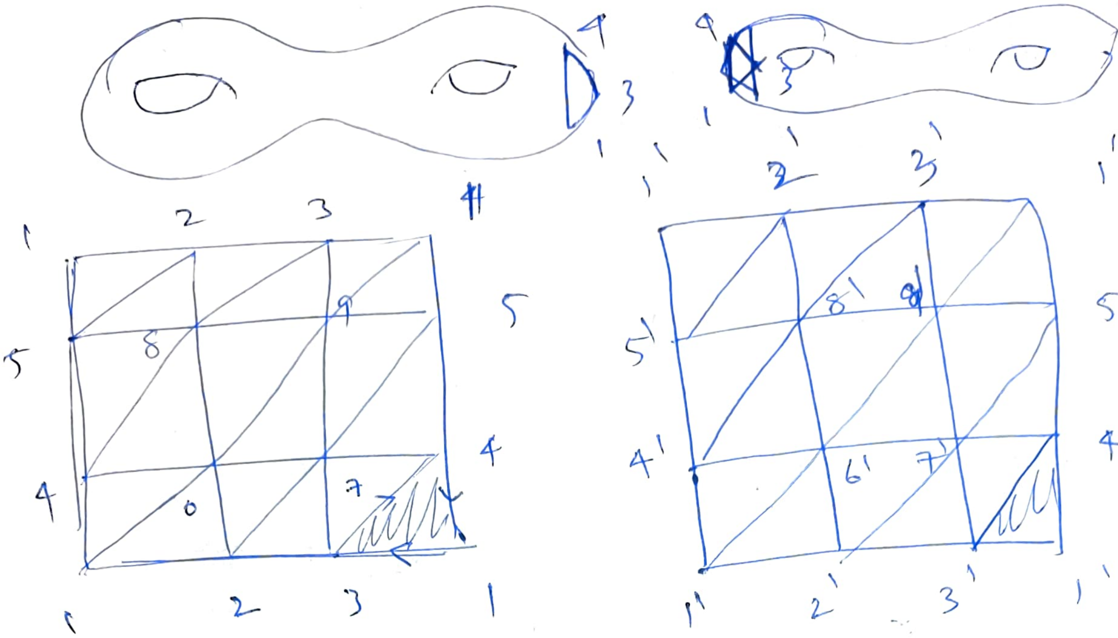


- fold into a cylinder and extend to a torus.

$\chi(S_1 \times S_1) = 0$

$\alpha_0 = 9$
$\alpha_1 = 27$
$\alpha_2 = 18$

(3) Double Torus T_2



Join $13 \to 1'3'$
$14 \to 1'4'$
$34 \to 3'4'$

$\alpha_0 = 9 \times 2 - 3$

$\alpha_1 = 27 \times 2 - 3$

$\alpha_2 = 18 \times 2 - 2$

$$\boxed{\chi(T_2) = -2}$$

In general,

S_1 — α_0 vertices, α_1 edges, α_2 faces

S_2 — α_0' vertices, α_1' edges, α_2' faces.

$S_1 \# S_2$ — $\alpha_0'' = \alpha_0 + \alpha_0' - 3$

$\alpha_1'' = \alpha_1 + \alpha_1' - 3$

$\alpha_2'' = \alpha_2 + \alpha_2' - 2$.

$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$

using induction,

$T_g = T_{g-1} \# T$

$\chi(T_g) = \chi(T_{g-1}) + \chi(T) - 2.$

$= -2(g-2) - 2 = -2(g-1)$

Thm: Classification Thm for 2D surfaces which are oriented and compact without boundary. is either

1. The sphere S^2
2. The Torus T
3. The connected sum of g tori T_g.

Simplicial Homology

Defn: Let K^n be an n-dimensional simplicial complex, dim = n means simplex σ_i of highest dim in K^n is σ_n.
We will associate with K^n, a set of $(n+1)$ abelian groups, denoted by $H_0(K^n)$, $H_1(K^n)$ —— $H_n(K^n)$ called the simplicial homology groups of K^n; and each of them is a topological invariant of K^n.

Defn (1): Let K & L be simp complexes. K is isomorphic to L if there is a map ϕ from the vertices of K to the vertices of L which is 1-1, onto; s.t if $v_1, v_2 \ldots v_s$ spans a simplex if and only if $\phi(v_1), \phi(v_2) \ldots \phi(v_s)$ spans a simplex of L.

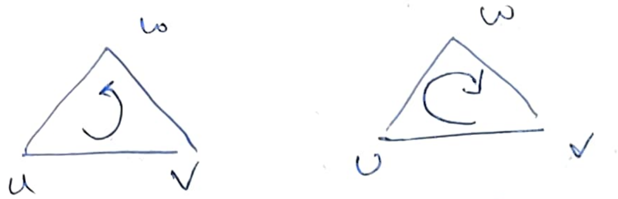
Defn (2): orientation of a simplex.

0 - simplex $\rightarrow$ 1 orientation

1 - simplex vw $\rightarrow$ 2 orientations

$\overrightarrow{(w,v)} = -\overrightarrow{(v,w)}$

2 - simplex



uvw

anticlockwise

vuw

clockwise

for an i - simplex

$\sigma_i = (v_0 v_1 \text{——} v_i)$ with a given orientation.

$-\sigma_i = (v_0 v_1 v_2 \text{——} v_i)$ with opposite orientation.

Let θ be a permution of $(1, 2, 3 \text{——} i)$

$\operatorname{sgn}\theta = \pm 1$.

$\left(v_{\theta(1)}, v_{\theta(2)} \text{——} v_{\theta(i)}\right) = \operatorname{sgn}\theta = (v_1 v_2 \text{——} v_i)$.

Defn ③ : Boundary of a simplex.

for oriented 1 simplex

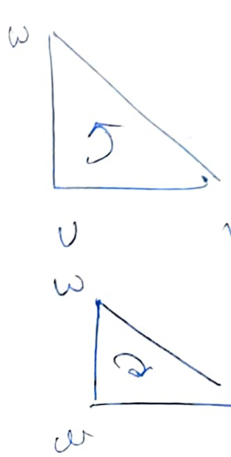
vw

$\partial(\vec{vw}) = w - v$

wv

$\partial(\vec{wv}) = v - w$

$= -\partial\,\vec{vw}$



$\partial(\overrightarrow{uvw}) = \vec{uv} + \vec{vw} + \vec{wu}$

$\partial(\overrightarrow{vuw}) = \vec{vu} + \vec{uw} + \vec{wv}$

$\partial(\overrightarrow{uvw}) = -\partial(\overrightarrow{vuw})$.

for an i-simplex

$$\sigma_i = (\overrightarrow{v_0 v_1 \text{---} v_i}).$$

$$\partial \sigma_i = \sum_{j=0}^{i} (-1)^j v_0 v_1 \text{---} \hat{v_j} \text{---} v_i$$

where $\hat{v_j}$ means that v_j is omitted

Defn ④: $C_q(K^n)$ — The q^{th} chain group of K^n as follows. Suppose K^n has finitely many q simplexes σ_1^q — σ_p^q

$$C_q(K^n) = \{\lambda_1 \sigma_1^q + \lambda_2 \sigma_2^q \text{———} \lambda_p \sigma_p^q \mid \lambda_i \in \mathbb{Z}\}$$

$C_q(K^n)$ is an abelian group. The group action is component wise addition.

$$\lambda_1 \sigma_1^q \text{———} \lambda_p \sigma_p^q$$
$$+ \lambda_1' \sigma_1^q \text{———} \lambda_p' \sigma_p^q$$
$$= (\lambda_1 + \lambda_1') \sigma_1^q \text{———} (\lambda_p + \lambda_p') \sigma_p^q$$

① identity: λ_1 — $\lambda_p = 0$

② inverse: $-\lambda_1 \sigma_1^q$ — $-\lambda_p \sigma_p^q$

$C_q(K)$ is an abelian group $\simeq \underbrace{\mathbb{Z} \times \text{—} \mathbb{Z}}_{p}$

$$C_n(K^n) \xrightarrow{\partial_n} C_{n-1}(K^n) \xrightarrow{\partial_{n-1}} C_{n-2} — — —$$
$$\longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_0$$

$$\boxed{\partial_{i-1} \circ \partial_i = 0}$$

$$C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1}$$

$$\partial_i \circ \partial_{i+1} = 0$$

$\ker \partial_i = Z_i \Rightarrow$ subgroup of i-cycles.

$\operatorname{img} \partial_{i+1} = B_i \Rightarrow$ i - boundaries.

$$B_i \subset Z_i$$

finally H_i, i^{th} hom gp $= \underline{Z_i / B_i}$.

Boundary homeomorphism for every $0 \le i \le n$ as $\sigma_i = v_0 v_1 — v_i$ $\Rightarrow$ $\partial(\sigma_i) = \underline{\Sigma(-1)^i v_0 v_1 \hat{v}_j — v_i}$

$$\partial : C_i \longrightarrow C_{i-1}$$

$$c_i = \lambda \sigma_1^i + \lambda_2 \sigma_2^i \quad — \quad \lambda_{\alpha(i)} \sigma^i_{\alpha(i)}$$

$$\underline{\partial(c_i) = \lambda_1 \partial \sigma_1^i + \lambda_2 \partial \sigma_2^i — \lambda_{\alpha(i)} \partial \sigma^i_{\alpha(i)}}$$

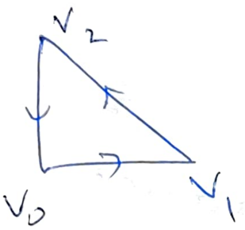
chain complex.

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

$\Rightarrow \partial \circ \partial = 0$ i.e. $\partial_{i-1} \partial_i = 0$.

ex:

$\sigma_2 = v_0 v_1 v_2$



$$\begin{aligned}
&\partial_1(\partial_2(\sigma_2)) \\
&= \partial_1(v_0v_1 + v_1v_2 + v_2v_0) \\
&= \partial_1(v_0v_1) + \partial_1(v_1v_2) + \partial_1(v_2v_0) \\
&= v_1 - v_0 + v_2 - v_1 + v_0 - v_2 \\
&= 0.
\end{aligned}$$

Lemma:

For k^4!

$$C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \longrightarrow \cdots \xrightarrow{\partial_1} 0$$

$$\partial_{q+1} \circ \partial_q = 0 \qquad \forall q$$

proof:

$\sigma_{q+1} = v_0 v_1 \cdots v_{q+1}$

$$\partial_{q+1}(v_0 \cdots v_{q+1}) = \sum_i (-1)^i v_0 v_1 \cdots \hat{v_i} \cdots v_{q+1}$$

$$\partial_q \circ \partial_{q+1}(v_0 \cdots v_{q+1}) = \partial_q \sum_i (-1)^i (v_0 \cdots \hat{v_i} \cdots v_{q+1})$$

$$= \sum_{i=0}^{q+1} (-1)^i \sum_{j=i+1}^{q+1} (-1)^{j-1} v_0 - \hat{v}_i - \hat{v}_j - v_{q+1}$$

$$+ \sum_{i=0}^{q+1} (-1)^i \sum_{j=0}^{i-1} (-1)^j v_0 - \hat{v}_j - \hat{v}_i - v_{q+1}$$

Note: All the terms will cancel pairwise since each oriented $(q-1)$ simplex

$(v_0 - \hat{v}_i - \hat{v}_j - v_{q+1})$ appears twice with coefficients $(-1)^{i+j-1}$ & $(-1)^{i+j}$.

$$C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1}.$$

$\ker \partial_q = Z_q \Rightarrow$ q-cycles

$\operatorname{Im} \partial_{q+1} = B_q \Rightarrow$ q-boundaries.

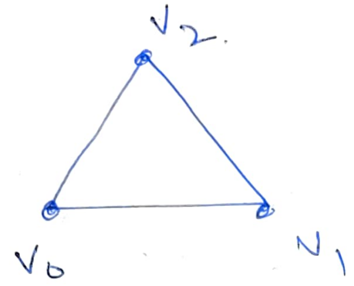
$B_q \subset Z_q$.

$\forall \quad H_q = Z_q / B_q \Rightarrow q^{th}$ homology.

Thm :-

If K^n is connected, path connected
$H_0(K^n) \cong \mathbb{Z}$.

Ext S^1



K^1

$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$H_1 = Z_1 / B_1 = \ker \partial_1 / \operatorname{img} \partial_2$

$B_1 = \phi \Rightarrow H_1 = Z_1$

$= V_1 V_0 + V_0 V_2 + V_2 V_1$

$\partial_1(Z_1) = 0$.

Z_1 is 1 cycle of the circle

$Z_1 = \alpha(V_1 V_0 + V_0 V_2 + V_2 V_1) \qquad \alpha \in \mathbb{Z}$

$Z_1 \cong \mathbb{Z}$.

hence $H_0(S^1) \cong \mathbb{Z}$.

Calculate H_i for

1) figure 8

2) g order at a point

3) Sphere

4) Torus.

Examples.

Thm: If K^n is a simplicial complex which is connected and path connected then $H_0(X) = \mathbb{Z}$.

① Circle S^1

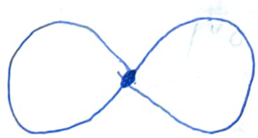
$C_0 = \{v_0, v_1, v_2\} \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

$C_1 = \{v_0v_1, v_1v_2, v_2v_3\} \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

$Z_1 = \ker \partial_1 \simeq (\mathbb{Z}, +)$ $\quad H_1(S^1) \simeq Z_1 / B_1 \simeq Z_1 \simeq (\mathbb{Z}, +)$

$B_1 = \operatorname{Im} \partial_2 \simeq \phi$ $\quad H_0(S^1) \simeq (\mathbb{Z}, +)$.

② Figure 8



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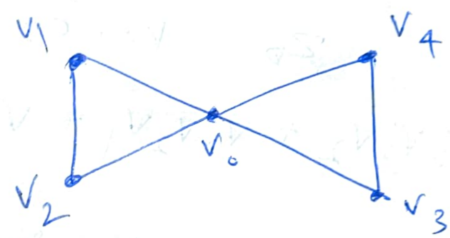


$\mathcal{C}: \; C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$.

$B_1 = \operatorname{Im} \partial_2 \simeq \phi$

$Z_1 = \ker \partial_1$

$\cancel{B_1 = v_1v_0 + v_0v_3 + v_3v_4 + v_4v_0 + v_0v_2 + v_2v_1}$

$z_1 = v_0v_1 + v_1v_2 + v_2v_0$

$z_2 = v_0v_4 + v_4v_3 + v_3v_0$

$Z_1 = \lambda_1(v_0v_1 + v_1v_2 + v_2v_0) + \lambda_2(v_0v_4 + v_4v_3 + v_3v_0)$.

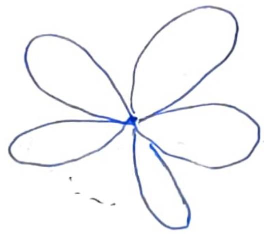
$Z_1 \simeq \mathbb{Z} \times \mathbb{Z}$ $\quad$, $H_1(X) = \mathbb{Z} \times \mathbb{Z}$

$H_0(X) = \mathbb{Z}$.

③ Similary g circles at a point.

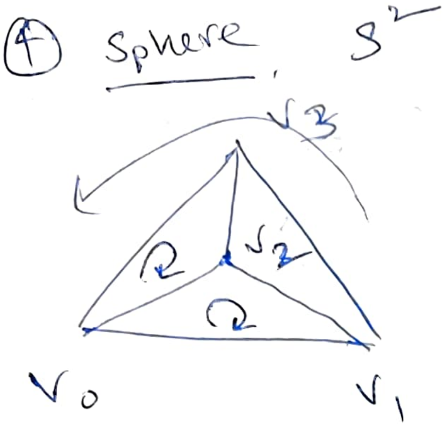
$H_1(X) = \underbrace{\mathbb{Z}\times\mathbb{Z} - \mathbb{Z}}_{r}$

$H_0(X) = \mathbb{Z}$



r circles

④ Sphere S^2



$\mathcal{C}: C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$H_2 = Z_2 / B_2$

$B_2 = \operatorname{im} \partial_3 = \phi$.

$Z_2 = \ker \partial_2$.

$\xi = v_0v_3v_2 + v_0v_2v_1 + v_2v_3v_1 + v_3v_0v_1$

$Z_2 = \partial(v_0v_3v_2 + v_0v_2v_1 + v_2v_3v_1 + v_3v_0v_1) \simeq \mathbb{Z}$

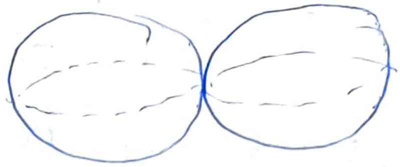
$\Rightarrow \quad H_2 = Z_2 / B_2 = \mathbb{Z}$.

$H_1 = Z_1 / B_1 = \phi$

since Z_1 and B_1 are the same groups.

$H_0 = \mathbb{Z}$.

Two spheres touching ($S^2 \vee S^2$)



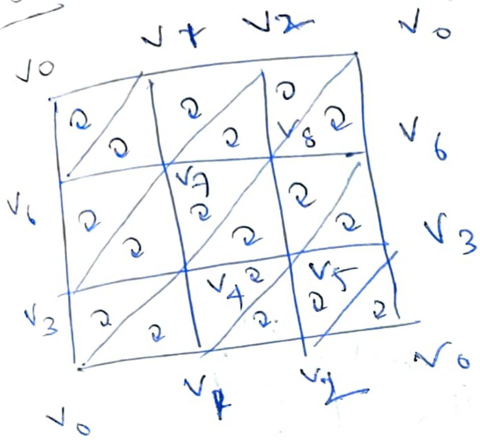
$H_2 = Z_2/B_2$, $Z_2 \simeq \mathbb{Z} \times \mathbb{Z}$

$H_2 = \mathbb{Z} \times \mathbb{Z}$.

$H_1 = \phi$

$H_0 = \mathbb{Z}$.

Torus $\simeq S^1 \times S^1$

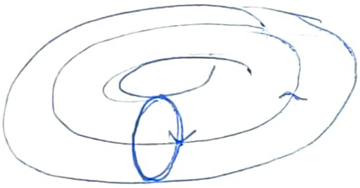


$\mathcal{C} : C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

$H_2 = Z_2/B_2$; $B_2 = \phi$

$H_2 = Z_2 = \mathbb{Z}$

$\boxed{H_1 = \mathbb{Z} \times \mathbb{Z}}$?

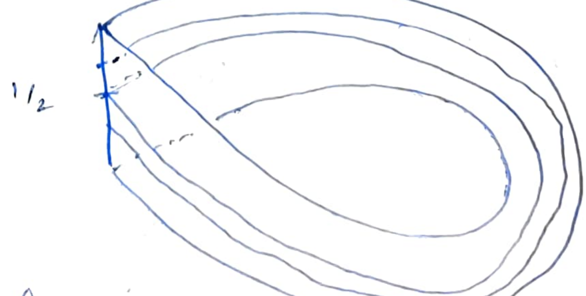


Thm: K^n is a simplicial complex of dim n, A^m is a sub complex of K^n and A^m is a deformation of K^n. Then $H_i(K^n) \underset{iso}{\cong} H_i(A^m)$. $\forall i$

Example: Moebius strip M

$H_i(M)$

compress M to the central circle.

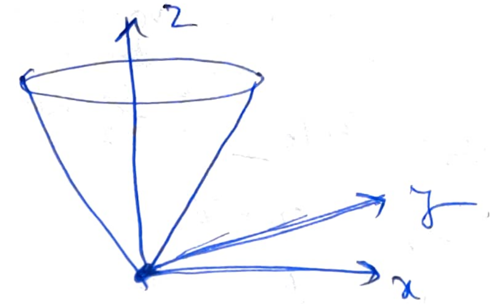


$A^m \cong S^1$

M deforms to central circle.

$H_0(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}$.

Example: C $x^2+y^2=z^2$
$0 \le z \le 1$



retract to the point at origin

$H: C \times I \longrightarrow C$

$H(x,y,z,t) = (1-t)(x,y,z).$

$H(x,y,z,1) = 0$

$H_0(C) = \mathbb{Z}$

$H_i(C) = \phi$

Ex: $S^n: \{(x_1, x_2 - x_{n+1}) \in \mathbb{R}^{n+1} / \sum_{i=1}^{n+1} x_i^2 = 1\}$

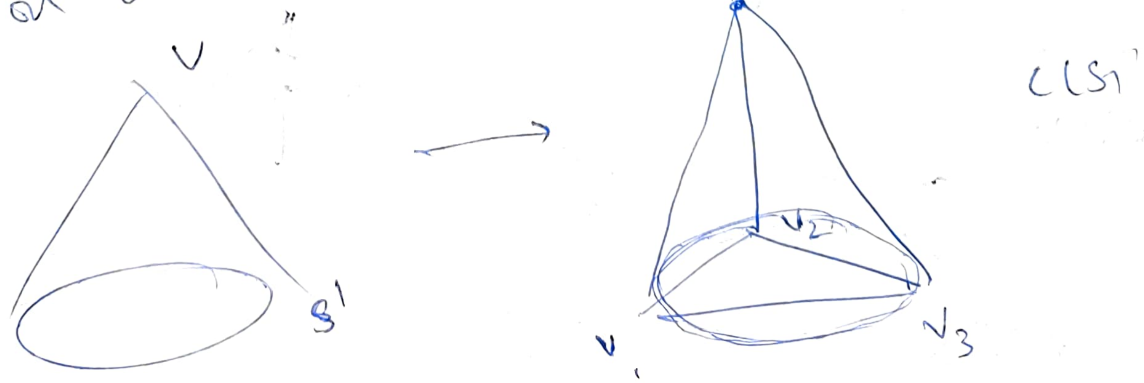
$H_i(S^n) \cong 0$ if $i = 1, 2$ —

$H_n(S^n) \cong (\mathbb{Z}, +)$

Proof:

Via the calculation of a homology group of cone complex $C K^n$ of and simplicial complex K^n.

cone on a circle



$c(s^1) = (v_1, v_2, v_3, v)$.

c_1 : $(v_1v_2, v_2v_3, v_3v_1$
$vv_1, vv_2, vv_3)$

c_2 : $\{vv_1v_3, vv_3v_2, vv_1v_2\}$.

Join of v,

let L be a simplicial complex in $\mathbb{R}^n$
let $v = (\underbrace{0, \ldots, 0, 1}_{n+1})$. Construct the cone complex CL called the cone on L as follows.

If A is a k-simplex of L, with vertices $(v_0, v_1 \ldots v_k)$. Then the points $v - v_0, v - v_1 \ldots v - v_k$ are

Thus $vv_0v_1 \ldots v_k$ is a $(k+1)$ simplex in $\mathbb{R}^{n+1}$ called the join of v with A.

The cone complex CL consists of

$C_0(CL)$: 1) vertices of L + vertex v.
$C_1(CL)$ = 2) Edges of L + vv_i $\forall v_i \in K$

Q :

$C_i(CL)$ = i) i simplexes of L + v ($i-1$ simplexes of L).

Armstrong Chapter 8 pg 181-182 Ex 7, 5.

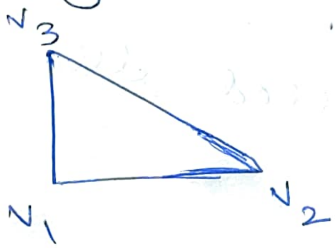
Thm ①:- If CL is any cone complex, then

$$H_0(CL) = \mathbb{Z}$$
$$H_i(CL) = 0 \quad \forall i \neq 0.$$

We will use theorem ~~one~~ ① to calculate Homology groups of S^n for any n.

step ①

Δ_2 2-simplex



$$\partial(\Delta_2) = S^1$$

Δ_3 3-simplex $\quad \partial(\Delta_3) = S^2$

$\vdots$

$$\boxed{\partial(\Delta_{n+1}) = S^n}$$

step 2

0-simplex Δ_0 • v_0

$$C\Delta_0 = \Delta_1$$
$$C\Delta_1 = \Delta_2$$
$$\vdots$$
$$C\Delta_n = \Delta_{n+1}$$

$$\boxed{C\Delta_n = \Delta_{n+1}}$$

$S^n = \partial(\Delta^{n+1})$.

i - simplexes of S^n are the same as of The same as i simplexes of Δ^{n+1} for $0 \leq i \leq n$.

Only diff is one extra $(n+1)$ simplex in Δ_{n+1}.

hence $H_i(S^n) = H_i(\Delta^{n+1})$.

$1 \leq i \leq n-1$

$\Rightarrow H_i(S^n) = 0 \quad \forall\, 1 \leq i \leq n-1$

$H_0(S^n) = \mathbb{Z}$

$H_n(S^n) = \dfrac{Z_n(S^n)}{B_n(S^n)}$. $\qquad Z_n(S^n) = Z_n(\Delta^{n+1})$.

$H_n(\Delta_{n+1}) = \dfrac{Z_n(\Delta_{n+1})}{B_n(\Delta_{n+1})} = 0$

$\Rightarrow Z_n(\Delta_{n+1}) \cong B_n(\Delta_{n+1})$.

but $B_n(\Delta_{n+1}) = \mathbb{Z}$.

$\Rightarrow Z_n(\Delta_{n+1}) = \mathbb{Z}$

$H_n(S^n) = Z_n(S^n) = \mathbb{Z}$. $\qquad$ since $B_n(S^n) = 0$