Question: Let $(a_{i,j})_{1\le i,j\le n}$ be non-negative numbers satisfying $\sum_{j\neq i} a_{i,j} = a_{i,i}$ for all $i \in \{1,\dots,n\}$. Julie and Sam are playing the following game. Julie writes down a natural number i, $1 \le i \le n$, on a slip of paper. Sam does not see the number that Julie has written. Sam then guesses what number Julie has chosen, and writes his guess, which is a natural number j, $1 \le i \le n$, on a slip of paper. The two players simultaneously show each other the numbers they have written down. If Sam has guessed correctly, Julie pays him $a_{i,i}$ dollars, where i is the number that Julie chose (and that Sam correctly guesses). If Sam was wrong in his guess $(i \neq j)$, Sam pays Julie $a_{i,j}$ dollars. Depict this game as a two-player zero-sum game in strategic form, and prove that the value in mixed strategies of the game is 0.

Proof. Let Julie be the first player and Sam the second player. Then the matrix of the game becomes the following:

$$\begin{bmatrix} -a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & -a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & -a_{n,n} \end{bmatrix}$$

Note that $Ay = 0$ where $y^T = (\frac{1}{n},\dots,\frac{1}{n})$. Thus, $w^T Ay = 0$ for all $w \in \Delta\{1,\dots,n\}$. Hence, $v(A) \ge 0$. Suppose $v(A) > 0$. This means there exists $x \in \Delta\{1,\dots,n\}$ such that $x^T Aw > 0$ for all $w \in \Delta\{1,\dots,n\}$ implying $x^T A > 0$. But this contradicts the following lemma: (we did this in the class)

Lemma 0.1 (Theorem of the Alternative for Matrices)**.** *Let A be an $m \times n$ matrix. Exactly one of the following two statements is true.*

(a) There are $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ with $(y,z) \ge 0$, $(y,z) \neq 0$ and $Ay + z = 0$.

(b) There is an $x \in \mathbb{R}^m$ with $x > 0$ and $x^T A > 0$.

■