# MSO201A/ESO 209: PROBABILITY & STATISTICS
# MID SEMESTER EXAMINATION
# FULL MARKS: 60

**[1]** **(a)** Let $\Omega = [0,1]$ and $\mathcal{F} = \{\phi, [0,1], [0,1/2), [1/2,1], \{1\}\}$ be a class of subsets of $\Omega$. Verify whether $\mathcal{F}$ is a $\sigma-$field of subsets of $\Omega$. If not, then adding as few sets as possible, extend the class $\mathcal{F}$ to obtain a $\sigma-$field.

**(b)** Let $\Omega = \{1,2,3,4\}$ and $\mathcal{F}^{0} = \{\phi, \{1,2,3,4\}, \{1\}, \{1,2\}, \{2,3,4\}, \{3,4\}, \{1,3,4\}, \{2\}\}$ be a $\sigma-$field of subsets of $\Omega$. Verify whether the real valued function $X$ defined on $\Omega$ by $X(\omega) = \omega^2 + 2, \omega \in \Omega$; is a random variable with respect to $\mathcal{F}^{0}$. 8 (4+4) Marks

**[2]** A slip of paper is given to $A$, who marks it with either a $+$ or a $-$ sign, with a probability $1/3$ of writing a $+$ sign. $A$ passes the slip to $B$, who may either leave it unchanged or change the sign before passing it to $C$. $C$ in turn passes the slip to $D$ after perhaps changing the sign; finally $D$ passes it to a referee after perhaps changing the sign. It is further known that $B, C$ and $D$ each change the sign with probability $2/3$. Find the probability that $A$ originally wrote a $+$ given that the referee sees a $+$ sign on the slip. 5 Marks

**[3]** The cumulative distribution function of a random variable $X$ is given by

$$F(x) = \begin{cases} 0, & \text{if } x < -1, \\ \frac{1}{8}, & \text{if } -1 \le x < 0, \\ \frac{17}{24} - \frac{e^{-x}}{4}\left(1+e^{-x}\right), & \text{if } 0 \le x < 1, \\ \frac{23}{24} - \frac{e^{-x}}{4}\left(1+e^{-x}\right), & \text{if } 1 \le x < 2, \\ 1 - \frac{e^{-x}}{4}\left(1+e^{-x}\right), & \text{if } x \ge 2. \end{cases}$$

**(a)** Find $F_d(x)$ and $F_c(x)$ such that $F(x) = \alpha\, F_d(x) + (1-\alpha)\, F_c(x)$, where, $F_d(x)$ is cumulative distribution function of a discrete random variable and $F_c(x)$ is cumulative distribution function of an absolutely continuous random variable, $0 < \alpha < 1$.

**(b)** Find $P(1 \le X_d < 2 \mid X_d > 0)$ and the m.g.f. of $X_d$, where $X_d$ is the discrete random variable having $F_d(x)$ as its cumulative distribution function. 11 (8+3) Marks

**[4]** $X$ and $Y$ are independent random variables each having $N(0,1)$ distribution. Find $E\left(|X|\, e^{|Y|}\right)$. 7 Marks

**[5]** Let $T$ be a continuous random variable with probability density function (p.d.f.)

$$f_T(t) = \begin{cases} \dfrac{1}{4}, & \text{if } |t| \le 1 \\ \dfrac{1}{4t^2}, & \text{if } |t| > 1. \end{cases}$$

Consider the transformed random variable $W = 1/T$. Find the cumulative distribution function of $W$ and use it to fin d its p.d.f.

8 Marks

**[6]** The joint probability mass function (p.m.f.) of $X$ and $Y$ is given by

$$P(X = x, Y = y) = \begin{cases} \dfrac{(x+y+\alpha-1)!}{x!\ y!\ (\alpha-1)!}\left(\dfrac{1}{3}\right)^{x+y+\alpha}, & \text{if } x = 0,1,\ldots;\ y = 0,1,\ldots, \\ 0, & \text{otherwise} \end{cases};$$

here, $\alpha$ is a positive integer.

**(i)** Find the conditional p.m.f. of $Y$ given $X = x$.

**(ii)** Find $E(Y \mid X = x)$.

8 (6+2) Marks

**[7]** The joint probability density function (p.d.f.) of $X$ and $Y$ is given by

$$f_{X,Y}(x,y) = \begin{cases} 1, & \text{if } \{x < y < x+1, -1/2 < x < 0\} \quad \text{or} \quad \{-x < y < 1-x, 0 < x < 1/2\} \\ 0, & \text{otherwise.} \end{cases}$$

**(a)** Find the marginal p.d.f. of $Y$.

**(b)** Verify whether $X$ and $Y$ are independent random variables.

**(c)** Find correlation between $X$ and $Y$.

13 (7+3+3) Marks

**Useful data:**

- If $X \sim N(\mu, \sigma^2); \mu \in \mathbb{R}, \sigma > 0$ then p.d.f. of $X$ is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}; -\infty < x < \infty$$

- If $X \sim NB(r, p), r > 0$ is a positive integer, $0 < p < 1$; then p.m.f. of $X$ is

$$P(X = x) = \begin{cases} \dbinom{x+r-1}{x}(1-p)^x p^r, & x = 0,1,\ldots \\ 0, & \text{otherwise.} \end{cases}$$

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(a)

$$\Omega = [0, 1]$$

$$\mathcal{F} = \{ \phi, \Omega, [0, \tfrac{1}{2}), [\tfrac{1}{2}, 1], \{1\} \}$$

$\{1\}^c = [0, 1) \notin \mathcal{F} \Rightarrow \mathcal{F}$ is not a $\sigma$-field — (1)

$$\mathcal{F}^* = \{ \phi, \Omega, [0, \tfrac{1}{2}), [\tfrac{1}{2}, 1], \{1\}, [0, 1), [0, \tfrac{1}{2}) \cup \{1\}, [\tfrac{1}{2}, 1) \}$$

— (3)

$\mathcal{F}^*$ is closed under complements and countable (finite here) unions

$\Rightarrow \mathcal{F}^*$ is the desired $\sigma$-field.

(b) $\Omega = \{1, 2, 3, 4\}$

$$\mathcal{F}^0 = \{ \phi, \Omega, \{1\}, \{1,2\}, \{2,3,4\}, \{3,4\}, \{1,3,4\}, \{2\} \}.$$

$\mathcal{F}^0$ – $\sigma$-field of subsets of $\Omega$

$X(\omega) = \omega^2 + 2; \quad \omega \in \Omega \qquad \mathcal{X} = \{3, 6, 11, 18\}$

$$X^{-1}(-\infty, x] = \{\omega : X(\omega) \le x\} = \begin{cases} \phi \in \mathcal{F}^0, & x < 3 \\ \{1\} \in \mathcal{F}^0 & 3 \le x < 6 \\ \{1,2\} \in \mathcal{F}^0 & 6 \le x < 11 \\ \{1,2,3\} \notin \mathcal{F}^0 & 11 \le x < 18 \\ \Omega \in \mathcal{F}^0 & x \ge 18 \end{cases}$$

— (4)

$\Rightarrow X^{-1}(-\infty, x] \notin \mathcal{F}^0 \ \forall x \in \mathbb{R}$

$\Rightarrow X$ is not a r.v. w.r.t $\mathcal{F}^0$.

(2) Define events

$A^+$ – A passes with '+' sign

$D^+$ – D passes with '+' sign

$$\text{reqd prob} = P(A^+ \mid D^+) = \frac{P(A^+)\,P(D^+ \mid A^+)}{P(D^+)}.$$

$$= \frac{P(A^+)\,P(D^+ \mid A^+)}{P(A^+)\,P(D^+ \mid A^+) + P(A^{+c})\,P(D^+ \mid A^{+c})} \quad - (1)$$

$$P(A^+) = \frac{1}{3}$$

$$P(D^+ \mid A^+) = \left(\frac{1}{3}\right)^3 + \binom{3}{2}\left(\frac{2}{3}\right)^2 \frac{1}{3} = \frac{13}{27} \quad - (1)$$

$$P(D^+ \mid A^{+c}) = P(\text{D passes with '+'} \mid \text{A passes with '−'})$$

$$= \binom{3}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 + \binom{3}{3}\left(\frac{2}{3}\right)^3 = \frac{14}{27}$$

$$P(D^+) = \frac{1}{3} \times \frac{13}{27} + \frac{2}{3} \times \frac{14}{27} = \frac{41}{81} \quad - (2)$$

$$\text{reqd prob} = P(A^+ \mid D^+) = \frac{\frac{1}{3} \times \frac{13}{27}}{\frac{41}{81}} = \frac{13}{41}. \quad - (1)$$

(3) $$F(x) = \begin{cases} 0, & x < -1 \\ 1/8, & -1 \le x < 0 \\ \frac{17}{24} - \frac{e^{-x}}{4}(1+e^{-x}), & 0 \le x < 1 \\ \frac{23}{24} - \frac{e^{-x}}{4}(1+e^{-x}), & 1 \le x < 2 \\ 1 - \frac{e^{-x}}{4}(1+e^{-x}), & x \geq 2 \end{cases}$$

points of jumps of $F(x)$

$$\left.\begin{aligned} x = -1 &\rightarrow \tfrac{1}{8} = F(-1) - F((-1)-) \\ x = 0 &\rightarrow \tfrac{1}{12} = F(0) - F(0-) \\ x = 1 &\rightarrow \tfrac{1}{4} = F(1) - F(1-) \end{aligned}\right] \quad - (1)$$

$$x = 2 \rightarrow \tfrac{1}{24} = F(2) - F(2-)$$

$$\Rightarrow \alpha F_d(x) = \begin{cases} 0, & x < -1 \\ 1/8, & -1 \le x < 0 \\ 5/24, & 0 \le x < 1 \\ 11/24, & 1 \le x < 2 \\ 12/24, & x \geq 2 \end{cases} \quad - (1)$$

$$\Rightarrow \alpha = \tfrac{1}{2} \ \& \ F_d(x) = \begin{cases} 0, & x < -1 \\ 1/4, & -1 \le x < 0 \\ 5/12, & 0 \le x < 1 \\ 11/12, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases} \quad - (2).$$

$$\Rightarrow (1-\alpha) F_c(x) = F(x) - \alpha F_d(x)$$

$$= \begin{cases} 0, & x < 0 \\ \frac{1}{2} - \frac{e^{-x}}{4}(1+e^{-x}), & x \geq 0. \end{cases} \quad \text{— (2)}$$

$$\Rightarrow F_c(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{e^{-x}}{2}(1+e^{-x}), & x \geq 0. \end{cases} \quad \text{— (2)}$$

(b) $X_d$ discrete r.v. with d.f. $F_d(x)$

p.m.f. of $X_d$

| $x$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|
| $P(X_d = x)$ | $1/4$ | $1/6$ | $1/2$ | $1/12$ |

$$P(1 \leq X_d < 2 \mid X_d > 0) = \frac{P(1 \leq X_d < 2)}{1 - P(X_d \leq 0)}$$

$$= \frac{F_d(2-) - F_d(1-)}{1 - F_d(0)} = \frac{1/2}{1 - (1/4 + 1/6)} = \frac{1/2}{\frac{7}{12}} = \frac{6}{7}$$

— $(\cancel{2})(1\frac{1}{2})$

m.g.f. of $X_d$

$$M_{X_d}(t) = E(e^{tX_d}) = e^{-t} \cdot \frac{1}{4} + e^{0} \cdot \frac{1}{6} + e^{t} \cdot \frac{1}{2} + e^{2t} \cdot \frac{1}{12}$$

$$= \frac{1}{12}(3e^{-t} + 2 + 6e^{t} + e^{2t}).$$

— $(1\frac{1}{2})$

(4) $X$ & $Y$ indep $N(0,1)$

$$E(|X|\, e^{|Y|}) = E(|X|)\, E(e^{|Y|}) \quad \text{— (1)}$$

$$E(|X|) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|\, e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[ -\int_{-\infty}^{0} x\, e^{-x^2/2}\, dx + \int_{0}^{\infty} x\, e^{-x^2/2}\, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x\, e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-y/2}\, dy$$

$$= \sqrt{\frac{2}{\pi}}. \quad \text{— (3)}$$

$$E(e^{|Y|}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{|y|}\, e^{-y^2/2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{0} e^{-y}\, e^{-y^2/2}\, dy + \int_{0}^{\infty} e^{y}\, e^{-y^2/2}\, dy \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{y}\, e^{-y^2/2}\, dy$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{1}{2}(y^2 - 2y + 1 - 1)}\, dy$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{e} \int_0^{\infty} e^{-\frac{1}{2}(y-1)^2}\, dy = \sqrt{e} \sqrt{\frac{2}{\pi}}\, P(Y > 0)\, \sqrt{2\pi}$$

$Y \sim N(1,1)$

$$= 2\sqrt{e}\; P(Y - 1 > -1)$$

$$= 2\sqrt{e}\, (1 - \Phi(-1)) = 2\sqrt{e}\; \Phi(1) \quad \text{— (3)}$$

(5). $f_T(t) = \begin{cases} 1/4, & |t| \le 1 \\ \frac{1}{4t^2}, & |t| > 1. \end{cases}$

<u>dist$^n$ f$^n$ of T</u>

$-\infty < t < -1 \quad P(T \le t) = \int_{-\infty}^{t} \frac{1}{4x^2}\,dx = -\frac{1}{4t}$

$-1 \le t \le 1 \quad P(T \le t) = \int_{-\infty}^{-1} \frac{1}{4x^2}\,dx + \int_{-1}^{t} \frac{1}{4}\,dx$

$= \frac{1}{2} + \frac{t}{4}$

$t > 1 \quad P(T \le t) = \int_{-\infty}^{-1} \frac{1}{4x^2}\,dx + \int_{-1}^{1} \frac{1}{4}\,dx + \int_{1}^{t} \frac{1}{4x^2}\,dx$

$= 1 - \frac{1}{4t}.$

$$F_T(t) = \begin{cases} -\frac{1}{4t}, & -\infty < t \le -1 \\ \frac{1}{2} + \frac{t}{4}, & -1 < t \le 1 \\ 1 - \frac{1}{4t}, & t > 1. \end{cases}$$

dist$^n$ f$^n$ of $W = \frac{1}{T}$

$F_W(w) = P(W \le w) = P\left(\frac{1}{T} \le w\right). \quad — \quad (*)$

If $-\infty < w < -1$, $\Rightarrow \underline{-1 < \frac{1}{w} < 0}$

$(*) = P\left(T \in \left(\frac{1}{w}, 0\right)\right) = P\left(\frac{1}{w} < T < 0\right)$

$= F_T(0) - F_T\left(\frac{1}{w}\right) = \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{4w}\right) = -\frac{1}{4w} \quad — \left(1\frac{1}{2}\right)$

If $-1 < w < 0$, $\underline{-\infty < \frac{1}{w} < -1}$

$$(*) = P\left(T \in \left(\tfrac{1}{w}, 0\right)\right)$$
$$= P\left(\tfrac{1}{w} < T < 0\right) = F_T(0) - F_T\left(\tfrac{1}{w}\right).$$
$$= \tfrac{1}{2} - \left(-\tfrac{w}{4}\right) = \tfrac{1}{2} + \tfrac{w}{4} \quad - \left(1\tfrac{1}{2}\right)$$

If $0 < w < 1$, $\underline{1 < \frac{1}{w} < \infty}$

$$(*) = P\left(T^{-1} \in (-\infty, w)\right) = P\left(T^{-1} \in (-\infty, 0)\right) + P\left(T^{-1} \in (0, w)\right)$$
$$= P\left(T \in (-\infty, 0)\right) + P\left(T \in \left(\tfrac{1}{w}, \infty\right)\right)$$
$$= P(-\infty < T < 0) + P\left(\tfrac{1}{w} < T < \infty\right)$$
$$= \tfrac{1}{2} + \left(1 - F_T\left(\tfrac{1}{w}\right)\right)$$
$$= \tfrac{1}{2} + \left(1 - \left(1 - \tfrac{w}{4}\right)\right) = \tfrac{1}{2} + \tfrac{w}{4}. \quad - \left(1\tfrac{1}{2}\right).$$

Lastly if $1 < w < \infty$, $\underline{0 < \frac{1}{w} < 1}$

$$(*) = P(-\infty < T < 0) + P\left(\tfrac{1}{w} < T < \infty\right)$$
$$= \tfrac{1}{2} + \left(1 - F_T\left(\tfrac{1}{w}\right)\right)$$
$$= \tfrac{1}{2} + \left(1 - \left(\tfrac{1}{2} + \tfrac{1}{4w}\right)\right) = 1 - \tfrac{1}{4w}. \quad - \left(1\tfrac{1}{2}\right)$$

$$\Rightarrow F_W(w) = \begin{cases} -\frac{1}{4w}, & -\infty < w < -1 \\ \frac{1}{2} + \frac{w}{4}, & -1 \le w \le 1 \\ 1 - \frac{1}{4w}, & w > 1. \end{cases}$$

p.d.f. of W

$$f_W(w) = \begin{cases} \frac{1}{4w^2}, & -\infty < w < -1 \\ \frac{1}{4}, & -1 \le w \le 1 \\ \frac{1}{4w^2}, & 1 < w < \infty \end{cases} \quad -(2)$$

i.e. $$f_W(w) = \begin{cases} \frac{1}{4}, & |w| \le 1 \\ \frac{1}{4w^2}, & |w| > 1 \end{cases}$$

(6) Marginal p.m.f. of $X$

$$P(X=x) = \sum_{y=0}^{\infty} P(X=x, Y=y)$$

$$= \frac{1}{x!\,(\alpha-1)!} \left(\frac{1}{3}\right)^{x+\alpha} \sum_{y=0}^{\infty} \frac{(x+\alpha+y-1)!}{y!} \left(\frac{1}{3}\right)^{y} \quad -(1)$$

$$= \frac{(x+\alpha-1)!}{x!\,(\alpha-1)!} \left(\frac{1}{3}\right)^{x+\alpha} \sum_{y=0}^{\infty} \frac{(x+\alpha+y-1)!}{y!\,(x+\alpha-1)!} \left(\frac{1}{3}\right)^{y}$$

$$= \binom{x+\alpha-1}{x} \left(\frac{1}{3}\right)^{x+\alpha} \sum_{y=0}^{\infty} \binom{x+\alpha+y-1}{y} \left(\frac{1}{3}\right)^{y}$$

$$= \binom{x+\alpha-1}{x} \left(\frac{1}{3}\right)^{x+\alpha} \left(1-\frac{1}{3}\right)^{-(x+\alpha)}$$

$$= \binom{x+\alpha-1}{x} \left(\frac{1}{\cancel{3}}\right)^{x+\alpha} \left(\frac{\cancel{3}}{2}\right)^{x+\alpha}$$

i.e. $P(X=x) = \binom{x+\alpha-1}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{\alpha} \qquad x = 0, 1, \ldots \infty$ $\quad (3)$

i.e. $X \sim NB(\alpha, 1/2)$

Conditional p.m.f. of $Y$ given $X=x$

$$P(Y=y \mid X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$$

$$= \frac{\dfrac{(x+y+\alpha-1)!}{x!\,y!\,(\alpha-1)!} \left(\frac{1}{3}\right)^{x+y+\alpha}}{\dfrac{(x+\alpha-1)!}{x!\,(\alpha-1)!} \left(\frac{1}{2}\right)^{x+\alpha}}$$

$$= \binom{x+y+\alpha-1}{y} \left(\frac{2}{3}\right)^{x+\alpha} \left(\frac{1}{3}\right)^{y} \qquad y = 0, 1, \ldots \infty. \quad (2)$$

i.e. $P(Y=y|X=x) = \binom{x+y+\alpha-1}{y}\left(\frac{1}{3}\right)^{y}\left(\frac{2}{3}\right)^{x+\alpha}$ ; $y=0,1..\infty$.

$Y|X \sim NB\left(x+\alpha, \frac{2}{3}\right)$.

$$E(Y|X) = \sum_{y=0}^{\infty} y \binom{x+y+\alpha-1}{y}\left(\frac{1}{3}\right)^{y}\left(\frac{2}{3}\right)^{x+\alpha}$$

$$= \left(\frac{2}{3}\right)^{x+\alpha} \sum_{y=1}^{\infty} y \binom{x+y+\alpha-1}{y}\left(\frac{1}{3}\right)^{y}$$

$$= \left(\frac{2}{3}\right)^{x+\alpha}(-(x+\alpha))\left[\sum_{y=1}^{\infty}\binom{-(x+\alpha)-1}{y-1}\left(-\frac{1}{3}\right)^{y}\right]$$

$$= \left(\frac{2}{3}\right)^{x+\alpha}(-(x+\alpha))\left(-\frac{1}{3}\right)\sum_{y=0}^{\infty}\binom{-(x+\alpha)-1}{y}\left(-\frac{1}{3}\right)^{y}$$

$$= \left(\frac{2}{3}\right)^{x+\alpha} \frac{1}{3}(x+\alpha).\left(1-\frac{1}{3}\right)^{-(x+\alpha)-1} \quad \text{— (2)}$$

$$= \frac{(x+\alpha).\,1/3}{2/3} = \frac{1}{2}(x+\alpha).$$

(7) $f_{X,Y}(x,y) = \begin{cases} 1, & \{x<y<x+1,\ -\tfrac{1}{2}<x<0\} \text{ or } \{-x<y<1-x,\ 0<x<\tfrac{1}{2}\}. \\ 0, & \text{o/w.} \end{cases}$

$$f_Y(y) = \int_{R_1} f_{X,Y}\,dx + \int_{R_2} f_{XY}\,dx$$

$R_1$ : $\underbrace{\begin{matrix} -\tfrac{1}{2}<x<0 \\ x<y<x+1 \end{matrix}}_{\text{Depends on } y}$ $\qquad$ $R_2$: $\underbrace{\begin{matrix} 0<x<\tfrac{1}{2} \\ -x<y<1-\cancel{x} \end{matrix}}_{\text{Depends on } y}$

$R_1$ : $x<y<x+1$ , $-\tfrac{1}{2}<x<0$

$\Rightarrow y-1<x<y$ & $-\tfrac{1}{2}<x<0$

$\Rightarrow \max(-\tfrac{1}{2}, y-1)<x<\min(0,y)$

$R_2$ : $-x<y<1-x$ & $0<x<\tfrac{1}{2}$

$\Rightarrow -y<x<1-y$ & $0<x<\tfrac{1}{2}$

$\Rightarrow \max(0,-y)<x<\min(\tfrac{1}{2}, 1-y)$.

Thus $$f_Y(y) = \int_{\max(-1/2,\,y-1)}^{\min(0,y)} dx + \int_{\max(0,-y)}^{\min(1/2,\,1-y)} dx \quad — (1).$$

If $-\tfrac{1}{2}<y<0$

$$f_Y(y) = \int_{-1/2}^{y} dx + \int_{-y}^{1/2} dx = \underline{2(y+\tfrac{1}{2}) = (2y+1)}.$$

(2)

If $0 < y < \frac{1}{2}$

$$f_Y(y) = \int_{-1/2}^{0} dx + \int_{0}^{1/2} dx = 1$$

(2)

If $\frac{1}{2} < y < 1$

$$f_Y(y) = \int_{y-1}^{0} dx + \int_{0}^{1-y} dx = 2(1-y)$$

(2).

$$\Rightarrow f_Y(y) = \begin{cases} (2y+1), & -\frac{1}{2} < y < 0 \\ 1, & 0 < y \leq \frac{1}{2} \\ 2(1-y), & \frac{1}{2} < y < 1. \\ 0, & o/w \end{cases}$$

p.d.f. of Y.

Parts of p.d.f. marked earlier along with respective ranges of Y.

(b).

$$f_X(x) = \begin{cases} \int_{x}^{x+1} dy = 1, & -\frac{1}{2} < x < 0 \\ \int_{-x}^{1-x} dy = 1, & 0 < x < \frac{1}{2} \end{cases}$$

$$\text{i.e. } f_X(x) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2} \\ 0, & o/w. \end{cases}$$

— (1)

$$f_{X,Y}(x, y) \neq f_X(x)\, f_Y(y)$$ — (2)

$\Rightarrow$ X & Y are not indep. (provided the marginals $f_X$ & $f_Y$ are correct)

(c) $X \sim U(-\tfrac{1}{2}, \tfrac{1}{2})$ $\quad E(X) = 0$ — (1)

$$Cov(X, Y) = E(XY)$$

$$= \int_{-1/2}^{0} x \int_{x}^{x+1} y\, dy\, dx + \int_{0}^{1/2} x \int_{-x}^{1-x} y\, dy\, dx$$ — (1)

$$= \int_{-1/2}^{0} x \left[\frac{y^2}{2}\right]_{x}^{x+1} dx + \int_{0}^{1/2} x \left[\frac{y^2}{2}\right]_{-x}^{1-x} dx$$

$$= \frac{1}{2}\int_{-1/2}^{0} (2x^2+x)\, dx + \frac{1}{2}\int_{0}^{1/2} (x-2x^2)\, dx.$$

$$= \frac{1}{2}\left[\int_{-1/2}^{0} (2x^2+x)\, dx + \int_{0}^{1/2} (x-2x^2)\, dx\right]$$

$$= 0$$ — (1)

$\Rightarrow$ Corrl$^n$ $(X, Y) = 0$

$\Rightarrow$ X and Y are uncorrelated.