Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

MTH 636M: Game Theory

Quiz 1, Date: February 06, 2024, Tuesday

Timing: 10:30 AM to 11:50 AM

- Answer any four questions from the following. The exam is for 20 marks.
- Try not to use any result not done in the class. However, if you use any such result, clearly state and prove it.
- Write your name, roll no., program name, and seat number clearly on the top of your answer sheet.
- One A4 sheet with ONLY necessary definitions and results are allowed during the exam. Use of a calculator, mobile, and smart watch is strictly prohibited.

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1. Consider the following matrix game:

$$\begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 1 \end{bmatrix}$$

(i) Determine all the max-min (pure) strategies of both the players. What can you conclude about the value of the game in mixed strategies? (**2 marks**)

(ii) The value of the game is $\frac{12}{7}$. Use this to give an argument why player 2 will put zero probability on column 2 in any maxmin strategy. (**3 marks**)

Answer: (i) The max-min pure strategies for Player 1 and Player 2 are row-2, and column-1 and column-2, respectively. Also, we can conclude that (we did the proof in the class)

$$\underline{v}^P \leq v \leq \bar{v}^P$$

where v is the value of the game in mixed strategies, and $\underline{v}^P$ and $\bar{v}^P$ are the max-min and min-max value in pure strategies, respectively. Thus, we have $1 \leq v \leq 3$ for the above game.

(ii) Let σ_2^* be a max-min strategy for Player 2 where $\sigma_2^*(\text{column-2}) > 0$. Consider a max-min strategy σ_1^* of Player 1. Recall that, this means (σ_1^*, σ_2^*) is an NE. Further,

$u(\sigma_1^*, \sigma_2^*) = \frac{12}{7}$. This together with Theorem $*$ (written on the board during the Quiz) and the fact that $\sigma_2^*(\text{column-2}) > 0$, we have

$$u(\sigma_1^*, \text{column-2}) = \frac{12}{7}.$$

But this does not hold as minimum in column-2 is 2.

2. Let A and B be two finite-dimensional matrices with positive entries. Show that the game

A	0
0	B

has no value in pure strategies. (Each 0 here represents a matrix of the proper dimensions, such that all of its entries are 0.) (**5 marks**)

Answer: Note that $\min_{s_2^j} u(s_1^i, s_2^j) = 0$ for all s_1^i as all the entries of A and B are positive and each row has at least one zero element. Thus, $\underline{v} = \max_{s_1^i} \min_{s_2^j} u(s_1^i, s_2^j) = 0$. Further, $\max_{s_1^i} u(s_1^i, s_2^j) > 0$ for all s_2^j as all the entries of A and B are positive and each column all the other elements are zero. Hence, $\bar{v} = \min_{s_2^j} \max_{s_1^i} u(s_1^i, s_2^j) > 0$ as there are finitely many columns. Therefore, $\underline{v} < \bar{v}$, and the game does not have a value in pure strategies.

3. Rohit (Player 1), and Virat (Player 2), are business partners. Each of the partners has to determine the amount of effort he or she will put into the business, which is denoted by e_i , $i = 1, 2$, and may be any non-negative real number. The cost of effort e_i for Player i is ce_i , where $c > 0$ is equal for both players. The success of the business depends on the amount of effort put in by the players; the business's profit is denoted by $r(e_1, e_2) = e_1^{\alpha_1} e_2^{\alpha_2}$, where $\alpha_1, \alpha_2 \in (0, 1)$ are fixed constants known by Rohit and Virat, and the profit is shared equally between the two partners. Each player's utility is given by the difference between the share of the profit received by that player and the cost of the effort he or she put into the business. Answer the following questions:

   (i) Describe this situation as a strategic-form game. (**2 marks**)

   (ii) Find all the Nash equilibria (in pure strategies) of the game. (**3 marks**)

Answer: (i) The situation can be represented as a strategic form game $\langle N, S_1, S_2, u_1, u_2\rangle$ where $N = \{1 \equiv \text{Rohit}, 2 \equiv \text{Virat}\}$, $S_i = [0, \infty)$ for all $i \in \{1, 2\}$, $u_1(x, y) = \frac{x^{\alpha_1} y^{\alpha_2}}{2} - cx$, and $u_2(x, y) = \frac{x^{\alpha_1} y^{\alpha_2}}{2} - cy$.

(ii) First note that $(0, 0)$ is an NE of the game. This is because $u_1(x, 0) = -cx$ is uniquely maximized at $x = 0$, and $u_2(0, y) = -cy$ is uniquely maximized at $y = 0$. For other NEs, suppose (x^*, y^*) is an NE. This means $u_1(x, y^*)$ is maximized at $x = x^*$ and $u_2(x^*, y)$ is maximized at $y = y^*$. As $x^* > 0$ and $y^* > 0$, using partial differentiation of the functions (note that both u_is are partially differentiable), we get

$$x^* = \left[\frac{2c}{\alpha_2^{\alpha_2}\alpha_1^{1-\alpha_2}}\right]^{\frac{1}{\alpha_1+\alpha_2-1}} \text{ and } y^* = \left[\frac{2c}{\alpha_1^{\alpha_1}\alpha_2^{1-\alpha_1}}\right]^{\frac{1}{\alpha_1+\alpha_2-1}}.$$

Thus, the game has two NEs, $(0,0)$ and (x^*, y^*).

4. Two competing coffee house chains, Pete's Coffee and Caribou Coffee, are seeking locations for new branch stores in Cambridge. The town is comprised of only one street, along which all the residents live. Each of the two chains therefore needs to choose a single point within the interval $[0,1]$, which represents the exact location of the branch store along the road. It is assumed that each resident will go to the coffee house that is nearest to his place of residence. If the two chains choose the exact same location, they will each attract an equal number of customers. Each chain, of course, seeks to maximize its number of customers.

   To simplify the analysis required here, suppose that each point along the interval $[0,1]$ represents a town resident, and that the fraction of residents who frequent each coffee house is the fraction of points closer to one store than to the other.

   (i) Describe this situation as a two-player strategic-form game. (**1.5 marks**)

   (ii) Prove that the only equilibrium in this game is that given by both chains selecting the location $x = \frac{1}{2}$. (**1.5 marks**)

   (iii) Prove that if three chains were to compete for a location in Cambridge, the resulting game would have no equilibrium. (Under this scenario, if two or three of the chains choose the same location, they will split the points closest to them equally between them.) (**2 marks**)

   Answer: (i) The situation can be represented as a strategic form game $\langle N, S_1, S_2, u_1, u_2\rangle$ where $N = \{1 \equiv \text{Pete's coffee}, 2 \equiv \text{Caribou coffee}\}$, $S_i = [0,1]$ for all $i \in \{1,2\}$, and

$$u_1(x,y) \begin{cases} \frac{x+y}{2} & \text{if } x < y \\ 1 - \frac{x+y}{2} & \text{if } x > y \\ \frac{1}{2} & \text{if } x = y \end{cases}$$

   and

$$u_2(x,y) \begin{cases} 1 - \frac{x+y}{2} & \text{if } x < y \\ \frac{x+y}{2} & \text{if } x > y \\ \frac{1}{2} & \text{if } x = y. \end{cases}$$

   (ii) Let (x^*, y^*) be an NE. First assume that $x^* \leq y^*$. This means if $x^* + y^* < 1$, we have $u_1(x^*, y^*) < \frac{1}{2} \leq u_1(\frac{1}{2}, y^*)$, and if $x^*+y^* > 1$, we have $u_2(x^*, y^*) < \frac{1}{2} \leq u_2(x^*, \frac{1}{2})$ implying (x^*, y^*) cannot be an equilibrium. Thus, $x^*+y^* = 1$. If $x^* \neq \frac{1}{2}$ then Player 1 can move to $\frac{1}{2}$ and get better off as $u_1(x^*, y^*) = \frac{1}{2}$ and $u_1(\frac{1}{2}, y^*) = \frac{\frac{1}{2}+y^*}{2} > \frac{1}{2}$. Hence, it must be that $x^* = \frac{1}{2}$ and $y^* = 1 - x^* = \frac{1}{2}$. Similarly, we can show that for $y^* \leq x^*$, the only equilibrium is $(\frac{1}{2}, \frac{1}{2})$.

(iii) Let (x^*, y^*, z^*) be an NE. Without loss of generality assume that $x^* \leq y^* \leq z^*$. If $x^* < y^*$ (or $y^* < z^*$) then Player 1 (or Player 3) has an incentive to move closer to y^* to be better off. Therefore, it has to be $x^* = y^* = z^*$. Hence, $u_1(x^*, y^*, z^*) = \frac{1}{3}$. If $x^* \leq \frac{1}{2}$ (or $x^* \geq \frac{1}{2}$) then Player 1 can move slightly towards right (or left) and getting a utility close to $\frac{1}{2}$ implying (x^*, y^*, z^*) cannot be an equilibrium.

5. Consider the following two-player zero-sum game:

	L	M	R
A	2	3	$\frac{7}{3}$
B	3	1	$\frac{7}{3}$

   (i) Find the value of the game and a max-min strategy for both the players. (**3 marks**)

   (ii) Describe all the max-min strategies for Player 2 (Column player). (**2 marks**)

Answer: (i) Do it as we did in the class.

(ii) Let $\sigma_2^* = (\alpha\ (L), \beta\ (M), 1 - \alpha - \beta\ (R))$ be a max-min strategy for Player 2. Recall that as we discussed in the class, it must be that $u(x, \sigma_2^*)$ coincide with the line $y = \frac{7}{3}$ for all $x \in [0, 1]$ where x denotes the mixed strategy $(x\ (A), 1 - x\ (B))$. This means

$$\begin{aligned}\frac{7}{3} &= \alpha u(0, L) + \beta u(0, M) + (1 - \alpha - \beta) u(0, R) \\ &\implies \alpha = 2\beta.\end{aligned}$$

Therefore, the set of all max-min strategies are $\{(\alpha\ (L), \frac{\alpha}{2}\ (M), 1 - \frac{3\alpha}{2}\ (R)) \mid \alpha \in [0, \frac{2}{3}]\}$.