Integral Equations: Numerical Methods

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Thus,

$$\|A_n - A\|_\infty = \max_{x\in\Omega} \int_\Omega |K_n(x,y) - K(x,y)| dy \leq \frac{1}{8} h^2 (b-a) \left\|\frac{\partial^2 K}{\partial x^2}\right\|_\infty.$$

- Example: Equidistant piecewise linear interpolation

Consider the following integral equation

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)\,dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}, \qquad 0 \le x \le 1.$$

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Moreover,

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Table showing the error between the approximate and the exact solution ➡

n	x = 0	x = 0.25	x = 0.5	x = 0.75	x = 1
4	0.004808	0.005430	0.006178	0.007128	0.008331
8	0.001199	0.001354	0.001541	0.001778	0.002078
16	0.000300	0.000385	0.000385	0.000444	0.000519
32	0.000075	0.000085	0.000096	0.000111	0.000130

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  - For example, for the approximation of an integral operator A with a 2π-periodic and m times continuously differentiable kernel K by degenerate kernels via trigonometric interpolation, we have
    $$\|A - A_n\|_\infty \leq O\left(\frac{1}{n^{m-\epsilon}}\right)$$
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for all $\epsilon > 0$.
If the kernel is analytic, then we have
$$\|A - A_n\|_\infty \leq O(e^{-ns})$$
for some $s > 0$.

- Example: Trigonometric interpolation

Let $t_j = j\pi/n, j = 0, \dots, 2n-1$, be an equidistant subdivision of the interval $[0,2\pi]$ with an even number of grid points.

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$$u(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1} [\alpha_k \cos kt + \beta_k \sin kt] + \frac{\alpha_n}{2} \cos nt$$

with the interpolation property $u(t_j) = g_j, j = 0, \dots, 2n-1$. Its coefficients are given by

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To check the interpolation property, note that

$$u(t) =$$
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$$L_j(t) = \frac{1}{2n}\left(1 + 2\sum_{k=1}^{n-1}\cos k(t - t_j) + \cos n(t - t_j)\right)$$

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Exercise

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Lagrange Basis

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Theorem: Let $g\colon \mathbb{R} \to \mathbb{R}$ be analytic and 2π periodic. Then there exists a strip $D = \mathbb{R} \times (-s, s) \subset \mathbb{C}$ with $s > 0$ such that g can be extended to a holomorphic and 2π periodic bounded function $g\colon D \to \mathbb{C}$. The error for the trigonometric interpolation can be extended by

$$\|P_n g - g\|_\infty < M \frac{\coth s/2}{\sinh ns},$$

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Thus, we have the estimate

$$\|P_n g - g\|_\infty \leq C e^{-ns}$$

for the trigonometric interpolation of periodic analytic functions, where C and s are some positive constants depending on g.

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Consider the following integral equation ($a \geq b > 0,\ c = (a-b)/(a+b)$)

$$\varphi(t) + \frac{ab}{\pi}\int_0^{2\pi} \frac{\varphi(\tau)d\tau}{a^2+b^2-(a^2-b^2)\cos(t+\tau)} dy = e^{\cos t}\cos(\sin t) + e^{c\cos t}\cos(c\sin t), \qquad 0 \leq t \leq 2\pi.$$

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Table showing the error between the approximate and the exact solution

	$2n$	$t=0$	$t=\pi/2$	$t=\pi$
$a=1$, $b=0.2$	4	-0.56984945	-0.18357135	0.06022598
	8	-0.14414257	-0.00368787	-0.00571394
	16	-0.00602543	-0.00035953	-0.00045408
	32	-0.00000919	-0.00000055	-0.00000069

Lesson 5

Integral Equations

5.2 An Introduction

5.3 Numerical Methods

- ***Degenerate Kernel Method***
  - ***via orthogonal expansion***

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If $\langle\cdot,\cdot\rangle$ *denotes an inner product on* $C(\Omega)$ *and* $\{u_1, u_2, \dots\}$ *is a complete orthonormal system, then a given continuous kernel* K *is expanded with respect to* x *for each* y*, that is,* $K(x, y)$ *is approximated by the partial sum*

$$K_n(x, y) = \sum_{j=1}^{n} u_j(x)\langle K(\cdot, y), u_j\rangle$$

of the Fourier series.

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For this degenerate kernel, the linear system reads

$$\gamma_j - \sum_{k=1}^{n} \gamma_k \int_{\Omega} u_k(y)\langle K(\cdot,y), u_k \rangle dy = \int_{\Omega} f(y)\langle K(\cdot,y), u_k \rangle dy\,, j = 1,2, \dots, n.$$

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Note that the setting up the linear system requires a double integration for each coefficient and for each right-hand side.

Example (Expansion with respect to the Chebyshev system)

Let $\Omega = (-1,1)$.

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Easy to check that

$$T_0(z) = 1, \qquad T_1(z) = z, \qquad T_{n+1}(z) + T_{n-1}(z) = 2zT_n(z), \qquad n \geq 1.$$

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Check that (exercise)

$$\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx$$

defines an inner product $\langle T_m, T_n \rangle = \frac{\pi}{2}(\delta_{mn} + \delta_{n0})$, *that is,* T_n *form an orthogonal system with respect to* $\langle \cdot,\cdot \rangle$.

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The approximate solution u_n to $(I-A)u = f$ is obtained as

$$u_n(x) = f(x) + \sum_{j=1}^{n} \gamma_j T_j(x)$$

where the coefficients γ_j solve the linear system

$$\gamma_j - \sum_{k=1}^{n} \gamma_k \int_{-1}^{1} T_k(y)\langle K(\cdot,y), T_k\rangle dy = \int_{-1}^{1} f(y)\langle K(\cdot,y), T_k\rangle dy\,, \qquad j = 1,2,\dots,n.$$

Theorem

Let $g: [-1,1] \to \mathbb{R}$ be analytic. Then, there exists an ellipse E with foci at -1 and 1 such that g can be extended to a holomorphic and bounded function $g: D \to \mathbb{C}$ where D denotes the open interior of E. The orthonormal expansion with respect to the Chebyshev polynomials

$$g = \frac{a_0}{2} T_0 + \sum_{n=1}^{\infty} a_n T_n, \qquad a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{g(x) T_n(x)}{\sqrt{1-x^2}} dx = \frac{2}{\pi} \langle g, T_n \rangle$$

is uniformly convergent with the estimate

$$\left\| g - \frac{a_0}{2} T_0 - \sum_{k=1}^{n} a_k T_k \right\|_{\infty} \le \frac{2M}{R-1} R^{-n}.$$

Here R is given through the semi-axis a and b of E by $R = a + b$ and M is a bound on g in D.

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Using this result from the "approximation theory", we see that

$$\|A_n - A\|_\infty = \max_{x \in [-1,1]} \int_{-1}^{1} \left| \frac{1}{\pi} T_0(x) \langle K(\cdot, y), T_0 \rangle - \frac{2}{\pi} \sum_{k=1}^{n} a_k T_k(x) \langle K_n(\cdot, y), T_k \rangle - K(x,y) \right| dy \leq O(R^{-n}).$$

This leads to accurate approximations if the kernel is sufficiently smooth.