Lesson 4

Numerical Solution of PDE

4.3 Hyperbolic PDE

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For the first example, we take the advection equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where c is a constant. We consider an initial value problem so that the function $u = u(x,t)$ is given when $t = 0$ and is to be found for $t > 0$.

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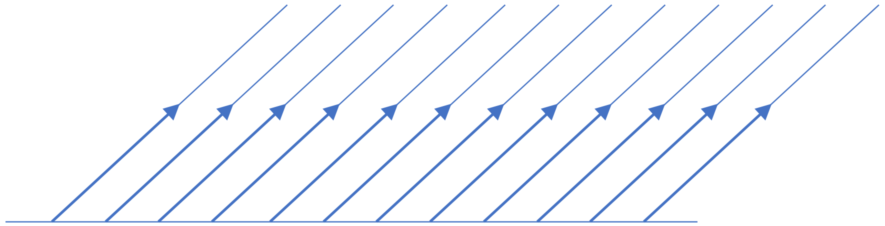

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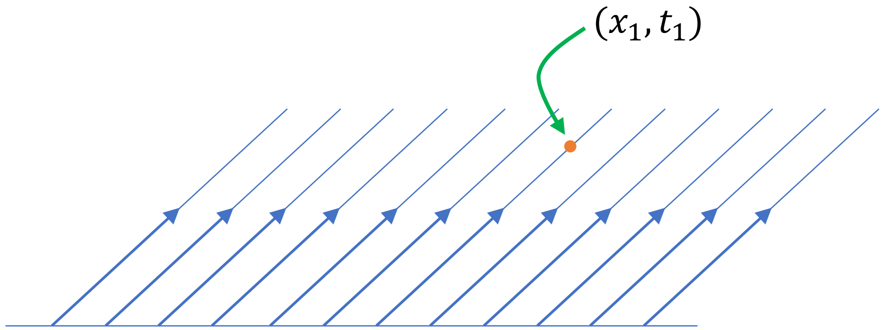

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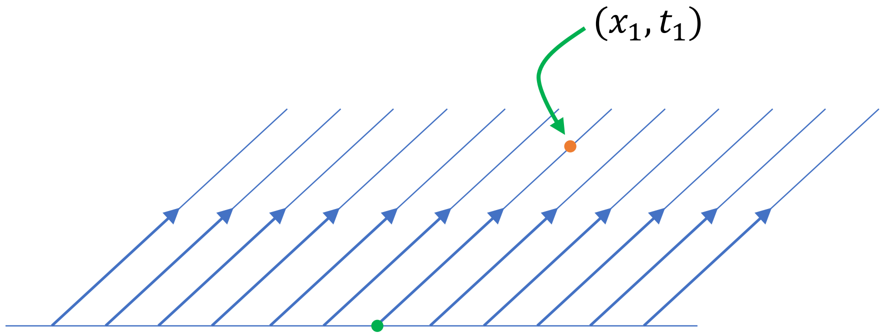

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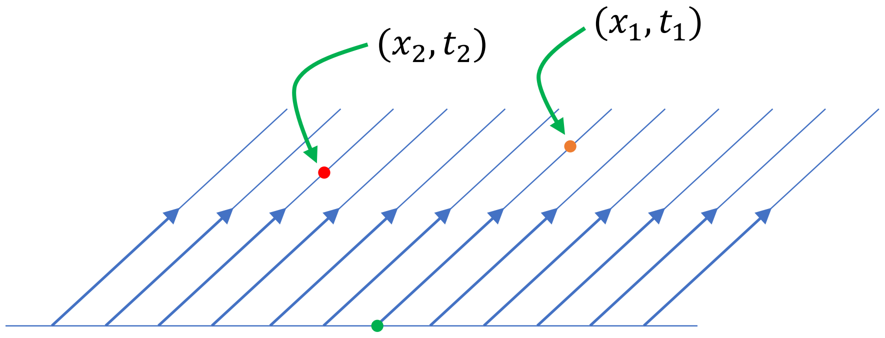

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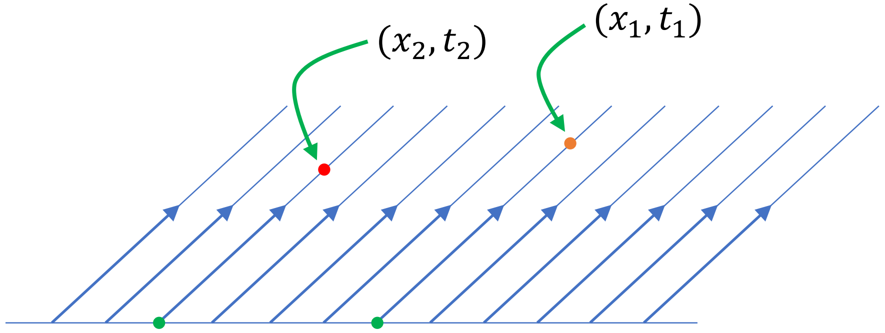

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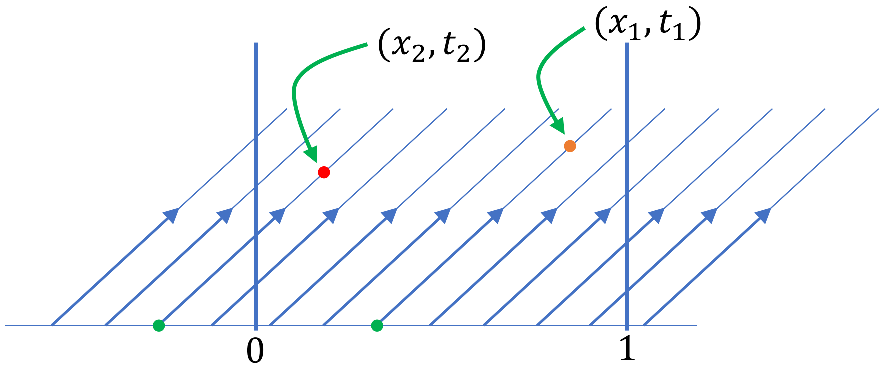

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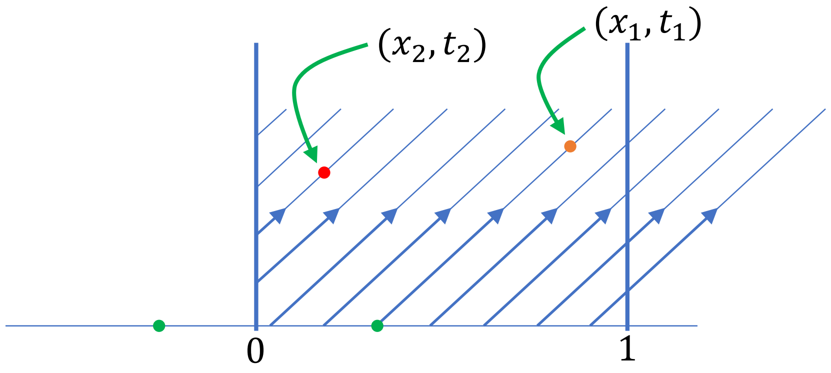

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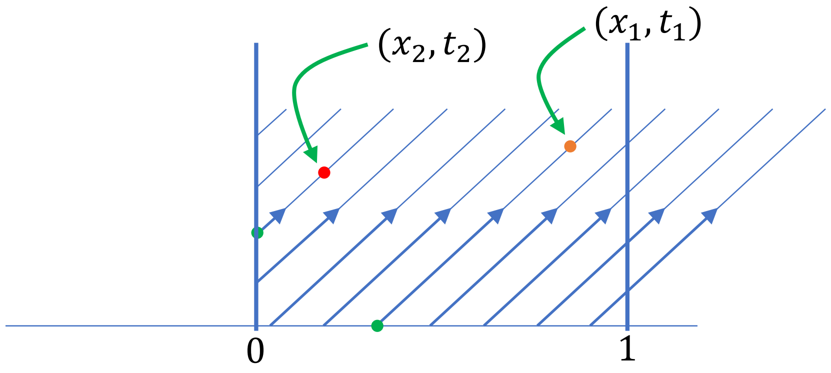

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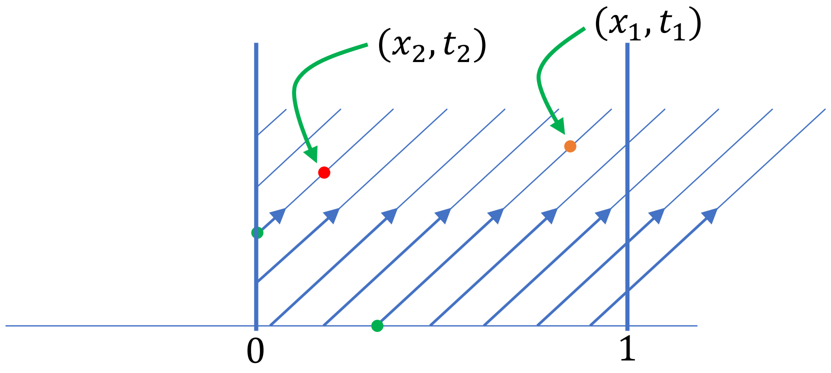
For determining solution at (x,t) with $x - ct < 0$, we need boundary condition at $x = 0$. This can be specified for all $t > 0$, and reads $u(0,t) = g(t)$ for a given g.



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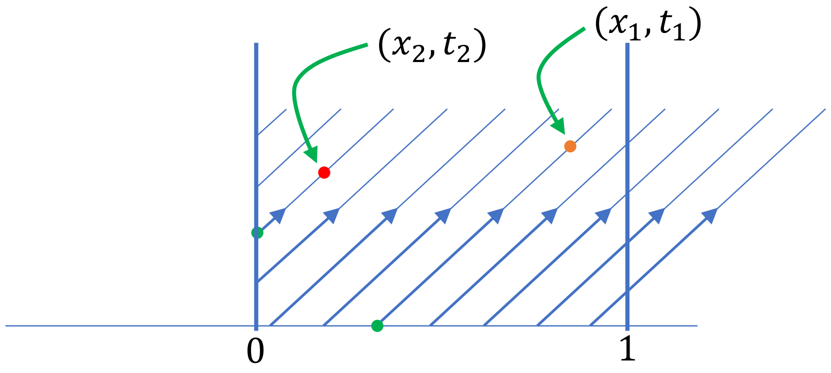

Thus, to obtain a well-posed problem, we need

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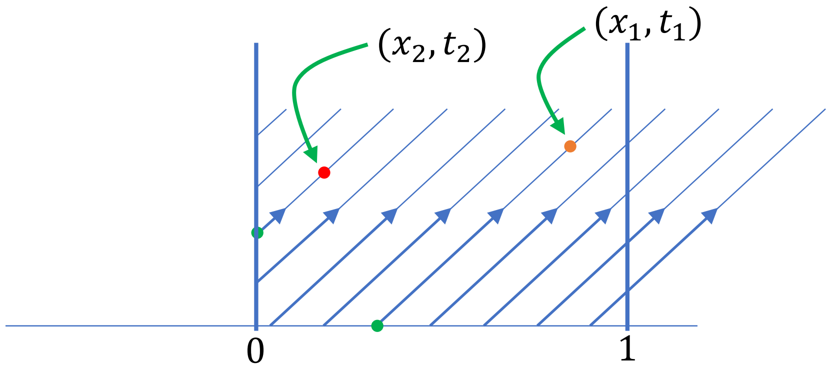

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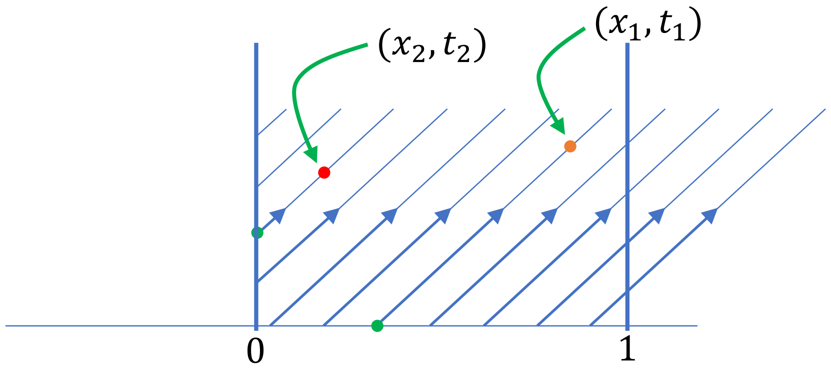

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$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0,$$

where C is an $n \times n$ symmetric matrix (that is, has real eigenvalues). Suppose $C = S^{-1}DS$ with S invertible and D a diagonal matrix.

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Example:

Consider the wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}.$$

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Finally, we find the solution (u_1, u_2) *from* $u_1 + u_2$ *which is a wave moving from right to left and from* $u_1 - u_2$, *a wave moving to the right.*