Initial Value Problems: Linear Multistep Methods

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Definition

A linear multistep method satisfies the root condition if

(1) all roots of the first characteristic polynomial has modulus less that or equal to 1, and

(2) all roots of modulus 1 are simple.

Theorem

The linear multistep method

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j})$$

is stable only if it satisfies the root condition. If the method satisfies the root condition (and f is Lipschitz continuous), then it is stable.

Proof.

Note that

$$\begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix} \begin{bmatrix} y_{n-k} \\ y_{n-k+1} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1} f_{n+1} + b_0 f_n + \cdots + b_k f_{n-k}) \end{bmatrix}$$

Proof.

Note that

$$\begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix} \begin{bmatrix} y_{n-k} \\ y_{n-k+1} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1} f_{n+1} + b_0 f_n + \cdots + b_k f_{n-k}) \end{bmatrix}$$

For solutions y_j and $\hat{y}_j$, let

$$E_n = \begin{bmatrix} y_{n-k} - \hat{y}_{n-k} \\ y_{n-k+1} - \hat{y}_{n-k+1} \\ \vdots \\ y_{n-1} - \hat{y}_{n-1} \\ y_n - \hat{y}_n \end{bmatrix}$$

Proof.

Note that

$$\begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix} \begin{bmatrix} y_{n-k} \\ y_{n-k+1} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1} f_{n+1} + b_0 f_n + \cdots + b_k f_{n-k}) \end{bmatrix}$$

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and

$$Q_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\big(b_{-1}(f(t_{n+1}, y_{n+1}) - f(t_{n+1}, \hat{y}_{n+1})) + \cdots + b_k(f(t_{n-k}, y_{n-k}) - f(t_{n-k}, \hat{y}_{n-k}))\big) \end{bmatrix}$$

to get

$$E_{n+1} = A E_n + Q_n$$

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Proof.

For solutions y_j and $\hat{y}_j$, let

$$E_n = \begin{bmatrix} y_{n-k} - \hat{y}_{n-k} \\ y_{n-k+1} - \hat{y}_{n-k+1} \\ \vdots \\ y_{n-1} - \hat{y}_{n-1} \\ y_n - \hat{y}_n \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix}$$

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to get

$$E_{n+1} = AE_n + Q_n.$$

Note that $|\lambda I - A| = \rho(\lambda)$. (Why?)

Proof.

For solutions y_j and $\hat{y}_j$, let

$$E_n = \begin{bmatrix} y_{n-k} - \hat{y}_{n-k} \\ y_{n-k+1} - \hat{y}_{n-k+1} \\ \vdots \\ y_{n-1} - \hat{y}_{n-1} \\ y_n - \hat{y}_n \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix}$$

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to get

$$E_{n+1} = AE_n + Q_n.$$

Thus, we have

$$E_{k+n} = A^n E_k + \sum_{j=0}^{n-1} A^{n-1-j} Q_{k+j}.$$

Proof.

For solutions y_j and $\hat{y}_j$, let

$$E_n = \begin{bmatrix} y_{n-k} - \hat{y}_{n-k} \\ y_{n-k+1} - \hat{y}_{n-k+1} \\ \vdots \\ y_{n-1} - \hat{y}_{n-1} \\ y_n - \hat{y}_n \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix}$$

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$$E_{k+n} = A^n E_k + \sum_{j=0}^{n-1} A^{n-1-j} Q_{k+j} \,.$$

Thus, using ℓ_∞ norm for vectors and the fact that there is a constant C so that $\|A^m\| \le C$, for all m, we have

$$\|E_{k+n}\| \le C\|E_k\| + C \sum_{j=0}^{n-1} \|Q_{k+j}\|$$

Proof.

For solutions y_j and $\hat{y}_j$, let

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Thus, using ℓ_∞ norm for vectors and the fact that there is a constant C so that $\|A^m\| \le C$, for all m, we have

$$\|E_{k+n}\| \le C\|E_k\| + C\sum_{j=0}^{n-1} \|Q_{k+j}\| \le C\|E_k\| + hC\|b\|_1 L \sum_{j=0}^{n-1} \left(\|E_{k+j}\| + \|E_{k+j+1}\|\right)$$

Proof.

For solutions y_j and $\hat{y}_j$, let

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$$\|E_{k+n}\| \leq C\|E_k\| + 2hC\|b\|_1 L\sum_{j=1}^{n-1}\|E_{k+j}\| + hC\|b\|_1 L\|E_k\| + hC\|b\|_1 L\|E_{k+n}\|$$

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$$(1 - hC\|b\|_1 L)\|E_{k+n}\| \le C\|E_k\| + 2hC\|b\|_1 L\sum_{j=0}^{n-1}\|E_{k+j}\|$$

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For solutions y_j and $\hat{y}_j$, let

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to get

$$(1 - hC\|b\|_1 L)\|E_{k+n}\| \leq C\|E_k\| + 2hC\|b\|_1 L \sum_{j=0}^{n-1} \|E_{k+j}\|.$$

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Thus, for $h \le (2C\|b\|_1 L)^{-1}$, we have

$$\|E_{k+n}\| \le \frac{C\|E_k\|}{(1 - hC\|b\|_1 L)} + \frac{2hC\|b\|_1 L\left(\sum_{j=0}^{n-1}\|E_{k+j}\|\right)}{(1 - hC\|b\|_1 L)}$$

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For solutions y_j and $\hat{y}_j$, let

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$$\|E_{k+n}\| \le \frac{C\|E_k\|}{(1 - hC\|b\|_1 L)} + \frac{2hC\|b\|_1 L\left(\sum_{j=0}^{n-1}\|E_{k+j}\|\right)}{(1 - hC\|b\|_1 L)} \le 2C\|E_k\| + 4hC\|b\|_1 L\left(\sum_{j=0}^{n-1}\|E_{k+j}\|\right)$$

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Proof.

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$$\|E_{k+n}\| \le 2C\|E_k\| + 4hC\|b\|_1 L\left(\sum_{j=0}^{n-1} \|E_{k+j}\|\right).$$

$$\|E_{k+n}\| \le 2C\|E_k\|(1 + 4hC\|b\|_1 L)^n$$

Proof.

For solutions y_j and $\hat{y}_j$, let

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$$\|E_{k+n}\| \leq 2C\|E_k\|(1 + 4hC\|b\|_1 L)^n \leq 2C\|E_k\|(1 + 4hC\|b\|_1 L)^N$$

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$$\|E_{k+n}\| \leq 2C\|E_k\|(1 + 4hC\|b\|_1 L)^n \leq 2C\|E_k\|(1 + 4hC\|b\|_1 L)^N \leq 2C\|E_k\| e^{4NhC\|b\|_1 L}$$

Proof.

For solutions y_j and $\hat{y}_j$, let

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Thus, for $h \leq (2C\|b\|_1 L)^{-1}$, we have

$$\|E_{k+n}\| \leq 2C\|E_k\| + 4hC\|b\|_1 L\left(\sum_{j=0}^{n-1} \|E_{k+j}\|\right).$$

$$\|E_{k+n}\| \leq 2C\|E_k\|(1 + 4hC\|b\|_1 L)^n \leq 2C\|E_k\|(1 + 4hC\|b\|_1 L)^N \leq 2C\|E_k\| e^{4NhC\|b\|_1 L} \leq C e^{4(T-t_0)C\|b\|_1 L}\|E_k\|.$$

Proof.

For solutions y_j and $\hat{y}_j$, let

$$E_n = \begin{bmatrix} y_{n-k} - \hat{y}_{n-k} \\ y_{n-k+1} - \hat{y}_{n-k+1} \\ \vdots \\ y_{n-1} - \hat{y}_{n-1} \\ y_n - \hat{y}_n \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix}$$

and

$$Q_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\big(b_{-1}(f(t_{n+1}, y_{n+1}) - f(t_{n+1}, \hat{y}_{n+1})) + \cdots + b_k(f(t_{n-k}, y_{n-k}) - f(t_{n-k}, \hat{y}_{n-k}))\big) \end{bmatrix}$$

Thus, for $h \le (2C\|b\|_1 L)^{-1}$, we have

$$\|E_{k+n}\| \le 2C\|E_k\| + 4hC\|b\|_1 L\left(\sum_{j=0}^{n-1} \|E_{k+j}\|\right).$$

$$\|E_{k+n}\| \le 2C\|E_k\|(1 + 4hC\|b\|_1 L)^n \le 2C\|E_k\|(1 + 4hC\|b\|_1 L)^N \le 2C\|E_k\| e^{4NhC\|b\|_1 L} \le Ce^{4(T-t_0)C\|b\|_1 L}\|E_k\|.$$

So stability follows.

Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Consistency, stability and convergence

Akash Anand
MATH, IIT KANPUR

Theorem

The linear multistep method is convergent if and only if it is consistent and stable.

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Initial Value Problems: Linear Multistep Methods

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Proof. (sketch)

Convergence $\Rightarrow$ Consistency

Convergence $\Rightarrow$ Stability

Consistency and Stability $\Rightarrow$ Convergence

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Proof. (sketch)

Convergence ⇒ Consistency

Apply the method to $y' = 0, y(0) = 1$ *and* $y' = 1, y(0) = 0$ *for verifying satisfiability of the consistency conditions.*

Convergence ⇒ Stability

Consistency and Stability ⇒ Convergence

Initial Value Problems: Linear Multistep Methods

Theorem

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Remark

This theorem states that it can be determined whether or not a linear multistep method is convergent simply by checking some algebraic conditions concerning its characteristic polynomials.

Proof. (sketch)

Convergence $\Rightarrow$ Consistency

Apply the method to $y' = 0, y(0) = 1$ and $y' = 1, y(0) = 0$ for verifying satisfiability of the consistency conditions.

Convergence $\Rightarrow$ Stability

Apply the method to $y' = 0, y(0) = 0$ for verifying satisfiability of the root condition.

Consistency and Stability $\Rightarrow$ Convergence