Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)
Hints for Exercise Sheet 5

1. Cauchy's theorem

1.1. Let f be a continuous function defined on a closed disc $\overline{D(z_0;R)}$, where $z_0 \in \mathbb{C}$ and $R > 0$. Assume that f is holomorphic on $D(z_0;R)$.

(a) Is it necessarily true that $\displaystyle\int_{C(z_0;R)} f = 0$?

Sketch of the solution. *Let $\{r_n\}_{n=1}^{\infty}$ be a sequence in $(0,r)$ converging to R and $\varepsilon > 0$. Observe that $f\left(z_0 + r_n e^{it}\right) r_n i e^{it} \xrightarrow[n\to\infty]{} f\left(z_0 + Re^{it}\right) Rie^{it}$ uniformly in $t \in [0, 2\pi]$.*

(b) Prove or disprove the following:

$$f(z) = \frac{1}{2\pi i}\int_{C(z_0;R)} \frac{f(w)}{w-z}\,dw,\ \forall z \in D(z_0;R).$$

Sketch of the solution. *Let $z \in D(z_0;R)$. Consider the function $g : \overline{D(z_0;R)} \longrightarrow \mathbb{C}$ defined by*

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z. \end{cases}$$

Then g is continuous on $\overline{D(z_0;R)}$ and holomorphic on $D(z_0;R)$. Now use (1.1.a).

1.2. Justify the following:

(a) The star-like assumption in the hypothesis of Cauchy's theorem cannot be dropped.

Sketch of the solution. *Take $U = \mathbb{C} \setminus \{0\}$ and $f(z) \overset{def}{=} \frac{1}{z}$. The integral of f along the positively oriented unit circle is nonzero.*

(b) Cauchy's estimate cannot be improved in general.

Sketch of the solution. *Take $f(z) \overset{def}{=} z$. Then equality occurs in Cauchy's estimate for $f'(0)$.*

1.3.$*$ Let γ be a closed path in $\mathbb{C}$ and $g : \gamma^* \longrightarrow \mathbb{C}$ be a continuous function. For $n \in \mathbb{N}$, consider

$$\varphi(z) \overset{\text{def}}{=} \int_\gamma \frac{g(w)}{(w-z)^n}\,dw,\ \forall z \in \mathbb{C}\setminus\gamma^*. \tag{1.1}$$

Show that φ, defined as in (1.1), is holomorphic and indeed, for all $z \in \mathbb{C}\setminus\gamma^*$,

$$\varphi'(z) = n\int_\gamma \frac{g(w)}{(w-z)^{n+1}}\,dw.$$

Hint. *Consider the case $n = 1$ first, i.e., the function $\Phi(z) \overset{def}{=} \displaystyle\int_\gamma \frac{g(w)}{(w-z)}\,dw$, $\forall z \in \mathbb{C}\setminus\gamma^*$. Is Φ holomorphic? What is $\Phi^{(n-1)}$? Is it related to φ?*

Sketch of the solution. *Use Corollary 1 of the class notes titled "Analytic functions".*

1.4. Let $z_0 \in U \subseteq_{open} \mathbb{C}$, $D(z_0;R) \subseteq U$ and $f : U \longrightarrow \mathbb{C}$ be holomorphic. Pick any $0 < r < R$. Show that, for any $k \geq 0$, one has the following integral formula for the n-th derivative of f:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C(z_0;r)} \frac{f(w)}{(w-z)^{n+1}}\, dw, \ \forall z \in D(z_0;r).$$

Hint. *Prove by induction. Use 1.3.*

In 1.5. and 1.6., we let z_0, U, R, f and r be as in 1.4.. For any $n \geq 0$, denote the n-th Taylor polynomial of f at z_0 and the corresponding remainder term by s_n and R_n respectively.

1.5. Show that, for all $z \in D(z_0;r)$,

$$s_n(z) = \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{f(w)}{(w-z_0)^{n+1}} \left(\frac{(w-z_0)^{n+1} - (z-z_0)^{n+1}}{w-z} \right) dw.$$

Sketch of the solution. *Simplify the RHS using 1.4.*

1.6. (a) Analogous to the Cauchy integral formula, can you provide integral representation of R_n?

Sketch of the solution. *Use Corollary 2 of the class notes titled "Analytic functions" to show that, for all $z \in D(z_0;r)$,*

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \int_{C(z_0;r)} \frac{f(w)}{(w-z_0)^{n+1}(w-z)}\, dw. \tag{1.2}$$

(b) Let $0 < \rho < r$. Show that, for all $z \in D(z_0;\rho)$, one has

$$|R_n(z)| \leq \sup_{|w-z_0|=r} |f(w)| \cdot \frac{r}{r-\rho} \cdot \left(\frac{\rho}{r}\right)^{n+1}.$$

Sketch of the solution. *Use (1.2).*

1.7. Evaluate the following integrals:

(a) $\displaystyle\int_0^{2\pi} e^{e^{it}}\, dt.$

Hint. *Apply CIF to the function* $\exp(z)$.

(b) $\displaystyle\int_0^{2\pi} e^{(e^{it}-it)}\, dt.$

Hint. *Apply 1.4. to the function* $\exp(z)$.

(c) $\displaystyle\int_{C(0;1)} \frac{|dz|}{|z-a|^2}$, where $a \in \mathbb{D}$.

Solution.

$$\begin{aligned}
\int_{C(0;1)} \frac{|dz|}{|z-a|^2} &= \int_0^{2\pi} \frac{dt}{|e^{it}-a|^2} \\
&= \int_0^{2\pi} \frac{dt}{(e^{it}-a)(e^{-it}-\bar{a})} \\
&= \int_0^{2\pi} \frac{e^{it}}{(e^{it}-a)(1-\bar{a}e^{it})}\, dt \\
&= \frac{1}{i} \int_{C(0;1)} \frac{dz}{(z-a)(1-\bar{a}z)} \\
&= -\frac{1}{i} \int_{C(0;1)} \frac{dz}{(z-a)\left(z-\frac{1}{\bar{a}}\right)}
\end{aligned}$$

Now use Exercise 3.3. (c) of Exercise Sheet 4.

(d) $\displaystyle \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|ae^{it}-b|^4\,dt}$, where $0<a<b$.

Sketch of the solution. *It is easy to see that, for all $t\in[0,2\pi]$,*

$$\frac{1}{|ae^{it}-b|^4}=\frac{1}{(ae^{it}-b)^2(ae^{-it}-b)^2}=\frac{e^{2\pi it}}{(ae^{it}-b)^2(a-be^{it})^2}.$$

From this, it follows that,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|ae^{it}-b|^4\,dt}&=\frac{1}{2\pi i}\int_{C(0;1)}\frac{z}{(az-b)^2(a-bz)^2}\,dz\\&=\frac{1}{2\pi ib^2}\int_{C(0;1)}\frac{z}{(az-b)^2(z-\frac{a}{b})^2}\,dz\end{aligned}$$

Now apply 1.4. to the function $\frac{1}{b^2}\cdot\frac{z}{(az-b)^2}$.

(e)∗ $\displaystyle\int_\gamma\frac{\cos z}{z(z^2+8)}\,dz$, where γ is the positively oriented square whose sides are the lines $x=\pm2$ and $y=\pm2$.

1.8. Let $f:\{z\in\mathbb{C}:\operatorname{Re}z>0\}\longrightarrow\mathbb{C}$ be holomorphic and bounded. Prove that, for any $\alpha>0$, f is uniformly continuous on the half-plane $\{z\in\mathbb{C}:\operatorname{Re}z>\alpha\}$.

1.9. Let $D(z_0;R)\subseteq U\subseteq_{open}\mathbb{C}$, and $f\in H(U)$. Show that, for any $0<r<R$,

$$|f(z_0)|^2\le\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}\left|f\left(z_0+re^{it}\right)\right|^2 r\,drdt.$$

Sketch of the solution. *Use MVP for f^2 so as to obtain*

$$|f(z_0)|^2\le\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(z_0+re^{it}\right)\right|^2dt.$$

Now multiply both sides by r and integrate from 0 to R.

1.10. Let $f(w)\stackrel{\text{def}}{=}\displaystyle\sum_{n=0}^{\infty}\frac{2^nw^n}{3^n}$, for all $|w|<\frac{3}{2}$ and $g(w)\stackrel{\text{def}}{=}\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2w)^n}$, for all $|w|>\frac{1}{2}$. For $|z|\neq1$, consider

$$h(z)\stackrel{\text{def}}{=}\frac{1}{2\pi i}\int_{C(0;1)}\left(\frac{f(w)}{w-z}+\frac{z^2g(w)}{w^2-wz}\right)dw.$$

Show that

$$h(z)=\begin{cases}\frac{3}{3-2z} & \text{if } |z|<1\\ \frac{2z^2}{1-2z} & \text{if } |z|>1.\end{cases}$$

Note to the student. *Actually this exercise can be solved without using Cauchy theory. Just Exercise 3.3. (c) of Exercise Sheet 4 will be enough.*

Sketch of the solution. *It is east to see that $f(w)=\frac{3}{3-2w}$, for $|w|<\frac{3}{2}$, and $g(w)=\frac{2w}{2w-1}$, for $|w|>\frac{1}{2}$. Then one has, for $|z|\neq1$,*

$$\frac{1}{2\pi i}\int_{C(0;1)}\frac{f(w)}{w-z}\,dw=-\frac{3}{2}\cdot\frac{1}{2\pi i}\int_{C(0;1)}\frac{1}{(w-z)\left(w-\frac{3}{2}\right)}\,dw,$$

and

$$\frac{1}{2\pi i}\int_{C(0;1)}\frac{z^2 g(w)}{w^2-wz}\,dw = z^2\cdot\frac{1}{2\pi i}\int_{C(0;1)}\frac{dw}{\left(w-\frac{1}{2}\right)(w-z)}.$$

Now use Exercise 3.3. (c) of Exercise Sheet 4. Note that, if $|z|>1$, *then the function* $w\mapsto\frac{1}{(w-z)\left(w-\frac{3}{2}\right)}$ *is holomorphic on the open convex set* $D(0;r)$ *where* $1<r<|z|$.

1.11. Let $U\subseteq_{open}\mathbb{C}$ and $f:U\longrightarrow\mathbb{C}$ be holomorphic. Consider any closed disc $\overline{D(z_0;r)}\subseteq U$. Denote by γ the image of the circle $C(z_0;r)$, oriented positively, under f. Show that

$$L_\gamma\geq 2\pi r|f'(z_0)|.$$

1.12. Let $f:\mathbb{D}\longrightarrow\mathbb{C}$ be a holomorphic function.

(a) If $|f(z)|\leq\dfrac{1}{1-|z|}$, for all $z\in\mathbb{D}$, then show that, for any $n\geq 0$,

$$|f^{(n)}(0)|\leq(n+1)!\left(1+\frac{1}{n}\right)^n<e(n+1)!.$$

(b)$*$ Assume that there exists $C>0$ such that, $|f(z)|\leq\dfrac{C}{1-|z|}$, for all $z\in\mathbb{D}$. Show that, for any $z\in\mathbb{D}$,

$$|f'(z)|\leq\frac{4C}{(1-|z|)^2}.$$

Hint. *Apply Cauchy's integral formula on a disc with center z and suitable radius.*

1.13. Let f be an entire function. Suppose that there exist positive numbers R,A,B and α such that

$$|f(z)|\leq A+B|z|^\alpha,\ \text{ whenever } |z|\geq R.$$

What can you conclude about f from this?

1.14. (a) If f is an entire function such that $\lim\limits_{z\to\infty}\dfrac{f(z)}{z}=0$, then f is a constant function.

Hint. *Use 1.13.*

(b) Using 1.14.a or otherwise, show that if $f:\mathbb{C}\longrightarrow\mathbb{C}$ is holomorphic and bounded then it must be a constant function.

Hint. *Use 1.14.a*

1.15.$*$ Let f be an entire function. Assume that, for all $z\in\mathbb{C}\setminus(-\infty,0]$, $|f(z)|\leq|\log_{-\pi}z|$. What can you conclude about f?

1.16.$*$ Let f be an entire function. Assume that, for all $r>0$, $\displaystyle\int_0^{2\pi}\left|f\left(re^{it}\right)\right|dt\leq r^{\frac{17}{3}}$. Prove that $f\equiv 0$.

In 1.17. and 1.18., let $z_0\in U\subseteq_{open}\mathbb{C}$ and $f:U\setminus\{z_0\}\longrightarrow\mathbb{C}$ be holomorphic.

1.17. Suppose that there exists $r>0$ such that $D(z_0;r)\subseteq U$ and f is bounded on $D(z_0;r)\setminus\{z_0\}$. Show that, f can be defined at z_0 so that the extended function is holomorphic on U.

Solution. *Let $h:U\longrightarrow\mathbb{C}$ be defined as follows:*

$$h(z)\overset{def}{=}\begin{cases}(z-z_0)^2f(z) & \text{if } z\neq z_0\\ 0 & \text{if } z=z_0.\end{cases}$$

It follows that $h \in H(U)$. Then we have, for all $z \in D(z_0;r)$, $h(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. The boundedness of f implies that $h'(z_0) = 0$. hence $a_0 = a_1 = 0$. This shows that, for aall $z \in D(z_0;r) \setminus \{z_0\}$,

$$f(z) = \frac{h(z)}{(z-z_0)^2} = a_2 + a_3(z-z_0) + a_4(z-z_0)^2 + \ldots$$

It is now clear from this that if $f(z_0) \stackrel{def}{=} a_2$ then the extended function will be holomorphic on U.

1.18. Assume that there exists $r, \delta > 0$ and $w \in \mathbb{C}$ such that $f(D(z_0;r) \setminus \{z_0\}) \cap D(w,\delta) = \emptyset$. Consider the function

$$g(z) \stackrel{\text{def}}{=} \frac{1}{f(z)-w}, \ \forall z \in D(z_0;r) \setminus \{z_0\}. \tag{1.3}$$

(a) Show that the function g, defined as above in (1.3), can be defined at z_0 so that the extended function is holomorphic on $D(z_0;r)$.

Hint. *Use 1.17.*

(b) Denote the extended function mentioned above in 1.18.a by g as well. Show that, if $g(z_0) \neq 0$, then f can be defined at z_0 so that the extended function is holomorphic on U.

Sketch of the solution. *There exists $\rho > 0$ such that, for all $z \in D(z_0;\rho) \setminus \{z_0\}$, $|g(z)| \geq \frac{|g(z_0)|}{2}$. This swhos that $f - w$ is bounded in $D(z_0;\rho) \setminus \{z_0\}$. Use 1.17.*

1.19.∗ Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be continuous. Assume that f is holomorphic on $\{z \in \mathbb{C} : \operatorname{Im} z \neq 0\}$. Prove that f must be analytic.

Hint. *Use Morera's theorem.*

2. Poisson integral formula

Let $z \in \mathbb{D}$. Consider the two functions

$$P_z : \mathbb{R} \longrightarrow (0,\infty), \ P_z(t) \stackrel{\text{def}}{=} \frac{1-|z|^2}{|e^{it}-z|^2}, \tag{2.1}$$

and

$$Q_z : \mathbb{R} \longrightarrow \mathbb{C}, \ Q_z(t) \stackrel{\text{def}}{=} \frac{e^{it}+z}{e^{it}-z}. \tag{2.2}$$

2.1. Show that $\operatorname{Re} Q_z = P_z$.

Sketch of the solution. *Straightforward calculation.*

2.2. Show that, if $z = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$, then for any $t \in \mathbb{R}$,

$$P_z(t) = P_r(t-\theta) = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} = P_r(\theta - t).$$

Sketch of the solution. *Straightforward calculation.*

P_z and Q_z are called the *Poisson kernel* and *Cauchy kernel* respectively.

2.3.∗ Let $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous and holomorphic on $\mathbb{D}$. For any $z \in \mathbb{D}$, show that

$$f(z) = \frac{1}{2\pi i}\int_{C(0;1)} \left(\frac{1}{w-z} - \frac{1}{w - \frac{1}{\bar{z}}} \right) f(w)\, dw. \tag{2.3}$$

Hint. *Use Cauchy's theorem and 1.1. Note that* $\left|\frac{1}{\bar{z}}\right| > 1$ *so that* $w \mapsto \dfrac{f(w)}{w - \frac{1}{\bar{z}}}$ *is a holomorphic function on* $\mathbb{D}$.

2.4. Show that, for any $t \in [0, 2\pi]$,

$$\left(\frac{1}{e^{it} - z} - \frac{1}{e^{it} - \frac{1}{\bar{z}}}\right) e^{it} = \frac{e^{it}}{e^{it} - z} + \frac{\bar{z} e^{it}}{1 - \bar{z} e^{it}} = \frac{e^{it}}{e^{it} - z} + \frac{\bar{z}}{e^{-it} - \bar{z}} = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

2.5. Let f be as above in 2.3.. From 2.3. and 2.4., conclude that,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} P_z(t) f(e^{it})\, dt, \ \forall z \in \mathbb{D}. \tag{2.4}$$

Note that, if $z = 0$, then $P_0 \equiv 1$, so that (2.4) reduces to the Mean value property (MVP) for $\mathbb{D}$:

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt.$$

The formula established as above in (2.4) thus generalizes the MVP to the value of the function at an arbitrary point of $\mathbb{D}$. It is just the weighted average of values of f on the boundary, where the weights are given by the Poisson kernel. We refer to (2.4) as the *Poisson integral formula* for $\mathbb{D}$.

2.6. Can you generalize the Poisson integral formula for any arbitrary disc?

Sketch of the solution. *Let* $z_0 \in \mathbb{C}$ *and* $R > 0$. *Suppose* $f : \overline{D(z_0; R)} \longrightarrow \mathbb{C}$ *is continuous and holomorphic on* $D(z_0; R)$. *Then for any* $z \in D(z_0; R)$, *one has*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} P_{\frac{z - z_0}{R}}(t) f(e^{it})\, dt.$$

To prove this, consider the function $g(w) = f(z_0 + Rw)$, *for all* $w \in \mathbb{D}$. *Now apply* (2.4) *to* g.

3. Proof of Cayley-Hamilton theorem using Cauchy's theorem

This section is aimed to prove the Cayley-Hamilton theorem using Cauchy's theorem. To this end, we need to consider sequence of series of functions that take values in matrices with complex entries. We first make a few natural definitions.

Definition 3.1. *Fix* $d \in \mathbb{N}$. *For each* $n \in \mathbb{N}$, *let* A_n *be an* $d \times d$ *matrix with complex entries. Denote the* (i, j)*-th entry of* A_n *by* $a_{ij}^{(n)}$, *for all* $i, j = 1, \ldots, d$.

(i) *We say that the sequence* $\{A_n\}_{n=1}^{\infty}$ *converges to* $A = [a_{ij}]_{1 \le i,j \le d} \in M_d(\mathbb{C})$ *if, for all* $i, j = 1, \ldots, d$, *one has* $a_{ij}^{(n)} \xrightarrow[n \to \infty]{} a_{ij}$. *For example,*

$$\begin{bmatrix} \frac{1}{n} & 2 + \frac{3}{n} \\ 5 & \frac{i}{2^n} \end{bmatrix} \xrightarrow[n \to \infty]{} \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}.$$

(ii) *The convergence of the series* $\displaystyle\sum_{n=1}^{\infty} A_n$ *can now be defined in the usual way, i.e., if the sequence* $\left\{\displaystyle\sum_{k=1}^{n} A_k\right\}_{n=1}^{\infty}$ *converges.*

Definition 3.2. *Let $X \neq \emptyset$ and $f_n : X \longrightarrow M_d(\mathbb{C})$, for all $n \in \mathbb{N}$.*

(i) We say that $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise, where $f : X \longrightarrow M_d(\mathbb{C})$, if for every $x \in X$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of matrices converges to $f(x)$.

(ii) For any $n \in \mathbb{N}$, $i, j = 1, \ldots, d$ and $x \in X$, denote the (i, j)-th entry of $f_n(x)$ and $f(x)$ by $f_{i,j}^{(n)}(x)$ and $f_{i,j}(x)$ respectively. If, for any $i, j = 1, \ldots, d$, the sequence $\left\{f_{i,j}^{(n)}\right\}_{n=1}^{\infty}$ of functions converges uniformly to $f_{i,j}$, we then say that $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly.

Definition 3.3. *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a path and $f : \gamma^* \longrightarrow M_d(\mathbb{C})$ be continuous. Then $\int_\gamma f$ is defined to be the matrix whose (i, j)-th entry is $\left[\int_\gamma f_{i,j}\right]_{1\leq i,j\leq d}$, for all $i, j = 1, \ldots, d$, where $f_{i,j}$ is defined as in Definition 3.2.*

Let $A \in M_d(\mathbb{C})$ be nonzero. Consider the following series:

$$\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} A^n. \tag{3.1}$$

3.1. Show that the series as given in (3.1) converges uniformly on $\mathbb{C} \setminus D(0; 2\|A\|)$.

Hint. *Denote the (i, j)-th entry of A^n by $a_{ij}^{(n)}$, for all $i, j = 1, \ldots, d$. Then it is easy to see that, for all $n \geq 0$, $|a_{ij}^{(n)}| \leq \|A^n\| \leq \|A\|^n$. Using this, estimate $\dfrac{|a_{ij}^{(n)}|}{|z|^{n+1}}$.*

Solution. *Let $|z| \geq 2\|A\|$. Then for any $n \geq 0$, $|z|^{n+1} \geq 2^{n+1}\|A\|^{n+1}$, so that $\dfrac{|a_{ij}^{(n)}|}{|z|^{n+1}} \leq \dfrac{\|A\|^n}{|z|^{n+1}} \leq \dfrac{1}{\|A\|} \cdot \dfrac{1}{2^{n+1}}$. Now the uniform convergence follows from Weierestrass M-test.*

3.2. Show that, for all $z \in \mathbb{C}$ with $|z| \geq 2\|A\|$, $(zI - A)$ is invertible, and in fact, the inverse is given by (3.1).

Sketch of the solution. *Calculate $(zI - A)\displaystyle\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} A^n$.*

3.3.∗ Show that, for all $k \geq 0$,

$$A^k = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} z^k (zI - A)^{-1}\, dz.$$

Hint. *Do you see $a_{i,j}^{(k)} = \dfrac{1}{2\pi i} \displaystyle\int_{C(0;2\|A\|)} \left(\sum_{n=0}^{\infty} \frac{z^k}{z^{n+1}} a_{i,j}^{(n)}\right) dz$, for all $i, j = 1, \ldots, d$? This shows that $A^k = \dfrac{1}{2\pi i} \displaystyle\int_{C(0;2\|A\|)} z^k \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} A^n\, dz$. Now use 3.2.*

Solution. *Since the series $\displaystyle\sum_{n=0}^{\infty} \frac{a_{i,j}^{(n)}}{z^{n+1}}$ converges uniformly on the circle $|z| = 2\|A\|$, so does $\displaystyle\sum_{n=0}^{\infty} \frac{z^k}{z^{n+1}} a_{i,j}^{(n)}$. Hence*

$$\begin{aligned}
\frac{1}{2\pi i} \int_{C(0;2\|A\|)} \left(\sum_{n=0}^{\infty} \frac{z^k}{z^{n+1}} a_{i,j}^{(n)}\right) dz &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C(0;2\|A\|)} \frac{z^k}{z^{n+1}} a_{i,j}^{(n)}\, dz \\
&= \frac{1}{2\pi i} \int_{C(0;2\|A\|)} \frac{a_{i,j}^{(k)}}{z}\, dz \\
&= a_{i,j}^{(k)}.
\end{aligned}$$

3.4. Deduce that, for any polynomial $P(z)$ with complex coefficients, one has

$$P(A) = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} P(z)(zI - A)^{-1}\, dz. \tag{3.2}$$

(This is an analogue of Cauchy's integral formula.)

Solution. *Let* $P(z) \overset{def}{=} a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$*, where* $a_k \in \mathbb{C}$ *for* $k = 0, \ldots, d$*, and* $a_d \neq 0$*. For any* $k = 0, \ldots, d$*, from 3.3., we have*

$$a_k A^k = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} a_k z^k (zI - A)^{-1}\, dz.$$

Now take sum over k from 0 *to* d*.*

3.5. Let $\chi(z) = \det(zI - A)$ be the characteristic polynomial of A. Conclude from (3.2) that

$$\chi(A) = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} \det(zI - A)(zI - A)^{-1}\, dz. \tag{3.3}$$

3.6.$*$ Recall that, for any invertible $B \in M_d(\mathbb{C})$, $(\det B)B^{-1} = \operatorname{Adj} B$. Using this, conclude from (3.3) that $\chi(A) = 0$.

Solution. *All entries of Adj* $(zI - A)$ *are polynomial in* z*, hence admit primitives. It follows from Fundamental theorem for path integrals that,*

$$\chi(A) = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} \det(zI - A)(zI - A)^{-1}\, dz = \frac{1}{2\pi i} \int_{C(0;2\|A\|)} \mathit{Adj}\,(zI - A)\, dz = 0.$$

4. Basic properties of sine and cosine functions

The sine and cosine functions over $\mathbb{C}$ are defined as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The aim of the exercises 4.1.-4.4. is to realize the sine and cosine functions as the solutions of some IVP's. Further to that, using this, we can prove the basic properties of these functions.

4.1. Consider the following differential equation:

$$f'' + f = 0. \tag{4.1}$$

Find all entire functions f satisfying (4.1) with the initial conditions $f(0) = 0$ and $f'(0) = 1$. What if the initial conditions are changed to $f(0) = f'(0) = 0$?

Sketch of the solution. *Recall that, if* f *is entire than it is represented by a power series* $\sum_{n=0}^{\infty} a_n z^n$ *on* $\mathbb{C}$*. So if* f *is a solution of (4.1) then one must have* $a_n + (n+1)(n+2)a_{n+2} = 0$*, for all* $n \geq 0$*. Now* $a_0 = 0$ *and* $a_1 = 1$ *yield that* $a_1 = 1, a_3 = -\frac{1}{3!}, a_5 = \frac{1}{5!}$ *and so on. Thus we get* $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sin z$*, for all* $z \in \mathbb{C}$*. If* $a_0 = a_1 = 0$ *then clearly* $f \equiv 0$*.*

4.2. Find all entire functions satisfying the following: $f'' + f = 0, f(0) = 1$ and $f'(0) = 0$.

Sketch of the solution. *Proceed as 4.1. In this case the unique solution is* $\cos z \overset{def}{=} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$*.*

4.3. Find all solutions (entire functions) of (4.1).

Hint. *Assuming* f *is a solution of (4.1), can you see that* $\varphi(z) \overset{def}{=} f(0)\cos z + f'(0)\sin z$, $\forall z \in \mathbb{C}$, *satisfies some IVP?*

Solution. *Observe that $\varphi'' - \varphi$. Since both f and φ satisfy (4.1), so does $\psi \overset{def}{=} f - \varphi$. Since $\psi(0) = \psi'(0) = 0$, one must have $\psi \equiv 0$. This shows that the solutions of (4.1) are precisely given by $\alpha \cos z + \beta \sin z$, for $\alpha, \beta \in \mathbb{C}$.*

4.4. Show the following:

(a) $\forall z \in \mathbb{C}$, $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.

Solution. *It is clear from the definitions of $\cos z$ and $\sin z$.*

(b) For every $z, w \in \mathbb{C}$, $\sin(z+w) = \sin z \cos w + \cos z \sin w$ and $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Hint. *Fix $w \in \mathbb{C}$. Consider the function $f(z) \overset{def}{=} \cos(z+w)$, $\forall z \in \mathbb{C}$. Do you see that f solves (4.1)?*

Solution. *It is clear that $f'' + f = 0$. Hence there exists $\alpha, \beta \in \mathbb{C}$ such that $\cos(z+w) = \alpha \cos z + \beta \sin z$, for all $z \in \mathbb{C}$. It follows that $\alpha = \cos w$ and $\beta = \sin w$. The other identity can be proved similarly.*