5. Consider a plane wave of angular frequency $\omega$ traveling in a conducting medium of conductivity $\sigma$. The electric field is given by $\mathbf{E} = E_0 e^{i(kx-\omega t)}\hat{y}$, where $k^2 = i\mu_0\sigma\omega$.

(a) Find $\mathbf{B}$.

(b) Fine the phase difference between $\mathbf{E}$ and $\mathbf{B}$.

(c) Find the contribution of $\mathbf{E}$ and $\mathbf{B}$ to the energy density.

a) $\vec{B} = \dfrac{\vec{\tilde{k}} \times \vec{\tilde{E}}}{\omega}$

since, $\vec{\tilde{E}} = \tilde{E}_0 \, e^{i(kx-\omega t)}\,\hat{y}$

$\vec{\tilde{B}} = \dfrac{\tilde{E}_0 \tilde{k}}{\omega} \, e^{i(kx-\omega t)}\,\hat{z}$

b) $\tilde{k} = \alpha + i\beta$

$k^2 = i\mu_0\sigma\omega$

$\Rightarrow \alpha^2 - \beta^2 + 2i\alpha\beta = i\mu_0\sigma\omega$

$\Rightarrow \alpha^2 = \beta^2, \quad \alpha\beta = \frac{1}{2}\mu_0\sigma\omega$

$\Rightarrow \alpha = \beta = \sqrt{\dfrac{\mu_0\sigma\omega}{2}}$

$\Rightarrow \tilde{k} = \sqrt{\dfrac{\mu_0\sigma\omega}{2}}\,(1+i) = \sqrt{\mu_0\sigma\omega}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right)$

$= \sqrt{\mu_0\sigma\omega}\, e^{i\pi/4}$

$\Rightarrow$ phase difference between $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$ is $\pi/4$

c) $\vec{\tilde{E}} = E_0\, e^{-\sqrt{\frac{\mu_0\sigma\omega}{2}}\,x}\, e^{i\left(\sqrt{\frac{\mu_0\sigma\omega}{2}}\,x - \omega t\right)}\,\hat{y}$

$\langle \frac{1}{2}\epsilon E^2 \rangle = \frac{1}{2}\epsilon E_0^2 e^{-2\sqrt{\frac{\mu_0\sigma\omega}{2}}\,x} \, \frac{1}{2}$ (Taking real part of E)

$\langle \frac{1}{2\mu_0} B^2 \rangle = \frac{1}{2}\frac{1}{\mu_0} B_0^2 e^{-2\sqrt{\frac{\mu_0\sigma\omega}{2}}\,x} \times \frac{1}{2}$ (Taking real part of B)

$= \frac{1}{2}\frac{1}{2\mu_0}\dfrac{E_0^2}{\omega^2}(\mu_0\sigma\omega)\, e^{-2\sqrt{\frac{\mu_0\sigma\omega}{2}}\,x}$

$k^2 = \alpha^2 + \beta^2 = \mu_0\sigma\omega$

$\Rightarrow \dfrac{\langle B^2/2\mu_0 \rangle}{\langle \epsilon E^2/2 \rangle} = \dfrac{1}{\mu_0\epsilon}\dfrac{(\mu_0\sigma\omega)}{\omega^2} = \dfrac{\sigma}{\epsilon\omega}$

For a good conductor $\dfrac{\sigma}{\epsilon\omega} \gg 1 \Rightarrow$ Magnetic contribution dominates over electric contribution.

6. Calculate the reflection coefficient ($R$) for light beam having angular frequency $\omega = 4 \times 10^{15}$ rad/s at an air-to-silver interface. [Given, $\mu_{air} = \mu_{Ag} = \mu_0$; $\epsilon_{Ag} \approx \epsilon_0$; $\sigma = 6 \times 10^7 (\Omega m)^{-1}$].

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$$r = \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} = \frac{1 - \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2}{1 + \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2}$$

given $\left.\begin{array}{l}\mu_1 = \mu_{air} = \mu_0 \\ \mu_2 = \mu_{Ag} = \mu_0 \\ \epsilon_{Ag} \approx \epsilon_0\end{array}\right\} \Rightarrow \frac{\tilde{E}_{0R}}{E_{0I}} = \frac{1 - \frac{v_1}{\omega}\tilde{k}_2}{1 + \frac{v_1}{\omega}\tilde{k}_2}$

$\tilde{k}_2 = \alpha + i\beta$, where $\alpha, \beta = \omega\sqrt{\frac{\mu_0\epsilon_0}{2}}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon_0^2}} \pm 1\right]^{1/2}$

Also given $\sigma = 6 \times 10^7 \,(\text{ohm m})^{-1}$, $\omega = 4 \times 10^{15}$ rad/s

$$\Rightarrow \frac{\sigma}{\omega\epsilon_{Ag}} = \frac{6 \times 10^7}{8.85 \times 10^{-12} \times 10^{15} \times 4} \gg 1$$

$\Rightarrow$ good conductor

$$\Rightarrow \alpha = \beta = \sqrt{\frac{\omega\mu_0\sigma}{2}}$$

$$\Rightarrow \frac{\tilde{E}_{0R}}{E_{0I}} = \frac{1 - \frac{v_1}{\omega}\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)}{1 + \frac{v_1}{\omega}\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)}$$

Let us put $\tilde{\gamma} = \frac{v_1}{\omega}\sqrt{\frac{\omega\mu_0\sigma}{2}}(1+i)$

$$= v_1\sqrt{\frac{\mu_0\sigma}{2\omega}}(1+i) = \gamma_0(1+i)$$

$$R = \left|\frac{\tilde{E}_{0R}}{E_{0I}}\right|^2 = \left|\frac{1-\tilde{\gamma}}{1+\tilde{\gamma}}\right|^2 = \frac{(1-\tilde{\gamma})(1-\tilde{\gamma}^*)}{(1+\tilde{\gamma})(1+\tilde{\gamma}^*)}$$

$$= \frac{(1-\gamma_0-i\gamma_0)(1-\gamma_0+i\gamma_0)}{(1+\gamma_0+i\gamma_0)(1+\gamma_0-i\gamma_0)}$$

$$= \frac{(1-\gamma_0)^2+\gamma_0^2}{(1+\gamma_0)^2+\gamma_0^2}$$

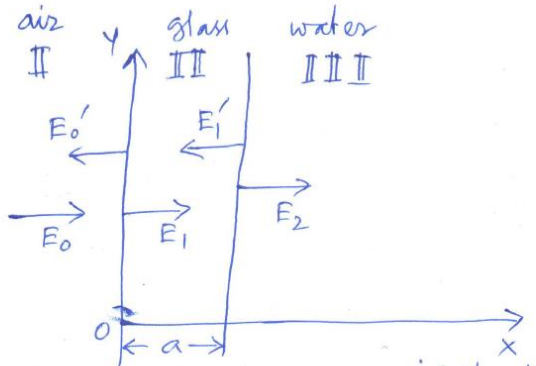
Now $\gamma_0 = v_1\sqrt{\frac{\mu_0\sigma}{2\omega}} = \frac{c}{n}\sqrt{\frac{\mu_0\sigma}{2\omega}}$, where $n = \sqrt{\epsilon_r} = 1$

$$= 3 \times 10^8\sqrt{\frac{4\pi \times 10^{-7} \times 6 \times 10^7}{2 \times 4 \times 10^{15}}} = 3\sqrt{\frac{4\pi \times 6 \times 10}{2 \times 4}} = 3\sqrt{30\pi}$$

$\Rightarrow \gamma_0 = 29.1 \Rightarrow R = \frac{(28.1)^2+(29.1)^2}{(30.1)^2+(29.1)^2} = \frac{789.6+846.8}{906.0+846.8}$

$\Rightarrow R = \frac{1636.4}{1752.8} = 0.93 \Rightarrow$ 93% of incident light will be reflected

7. Consider light traveling in air ($n = 1$) which is incident normally on the wall of a glass plate ($n_1 = 1.5$) of thickness $a$ and eventually passes into water. Find the overall transmission coefficient $T$ (from air to water) and plot it as a function of $k_1 a$ where $k_1$ is the wave-number of the light in glass. The refractive index of water is $n_2 = 1.3$.




I $\rightarrow$ Incident : $\vec{E}_0 = E_0 \, e^{-i(\omega t - kx)} \hat{y}$

Reflected : $\vec{E}_0' = E_0' \, e^{-i(\omega t + kx)} \hat{y}$

II $\rightarrow$ Incident (transmitted) : $\vec{E}_1 = E_1 \, e^{-i(\omega t - k_1 x)} \hat{y}$

Reflected $\vec{E}_1' = E_1' \, e^{-i(\omega t + k_1 x)} \hat{y}$

III $\rightarrow$ $\vec{E}_2 = E_2 \, e^{-i(\omega t - k_2 x)} \hat{y}$

B.C at $x = 0$

$$\boxed{E_0 + E_0' = E_1 + E_1'} \quad \text{---- (1)}$$

$$\frac{1}{\mu_0}\left(\frac{E_0}{c} - \frac{E_0'}{c}\right) = \frac{1}{\mu_{glass}}\left(\frac{E_1}{v_1} - \frac{E_1'}{v_1}\right)$$

$$\Rightarrow \boxed{E_0 - E_0' = n_1 (E_1 - E_1')} \text{--(2)} \left[\because \mu_0 = \mu_{glass} \text{ and } v_1 = \frac{c}{n_1}\right]$$

B.C at $x = a$

$$\boxed{E_1 e^{ik_1 a} + E_1' e^{-ik_1 a} = E_2 e^{ik_2 a}} \quad \text{---- (3)}$$

$$\frac{1}{\mu_{glass}}\left(\frac{E_1 e^{ik_1 a}}{v_1} - \frac{E_1' e^{-ik_1 a}}{v_1}\right) = \frac{1}{\mu_{water}} \frac{E_2 e^{+ik_2 a}}{v_2}$$

$$\Rightarrow \boxed{n_1\left(E_1 e^{ik_1 a} - E_1' e^{-ik_1 a}\right) = n_2 E_2 e^{ik_2 a}} \quad \text{-- (4)}$$

(1) & (2) $\Rightarrow$ $2E_0 = (1+n_1) E_1 + (1-n_1) E_1'$

$$\Rightarrow E_1' = \frac{2E_0 - (1+n_1)E_1}{1-n_1}$$

Eqn (3) $\Rightarrow$ $E_1 e^{ik_1 a} + \left[\frac{2E_0 - (1+n_1)E_1}{1-n_1}\right] e^{-ik_1 a} = E_2 e^{ik_2 a}$

$$\Rightarrow \boxed{E_1\left[(1-n_1) e^{2ik_1 a} - (1+n_1) e^{-2ik_1 a}\right] = (1-n_1) E_2 e^{2k_2 a} - 2E_0 e^{-ik_1 a}} \quad \text{-- (5)}$$

Similarly, from (4) $\Rightarrow$

$$n_1\left[E_1 e^{2k_1a} - \left\{\frac{2E_0 - (1+n_1)E_1}{1-n_1}\right\} e^{-ik_1a}\right] = n_2 E_2 e^{2k_2a}$$

$$\Rightarrow n_1 E_1\left[(1-n_1)e^{2k_1a} + (1+n_1)e^{-2k_1a}\right]$$
$$- \frac{n_1}{\cancel{1-n_1}} 2E_0 e^{-ik_1a} = n_2(1-n_1)E_2 e^{ik_2a}$$

$$\Rightarrow \boxed{E_1\left[(1-n_1)e^{2k_1a} + (1+n_1)e^{-2k_1a}\right] = (1-n_1)\frac{n_2}{n_1}E_2 e^{2k_2a} + 2E_0 e^{-2k_1a}} \quad \text{(6)}$$

Eliminating $E_1$ from (5) & (6) $\Rightarrow$

$$\frac{(1-n_1)E_2 e^{ik_2a} - 2E_0 e^{-2k_1a}}{(1-n_1)e^{2k_1a} - (1+n_1)e^{-2k_1a}} = \frac{(1-n_1)\frac{n_2}{n_1}E_2 e^{2k_2a} + 2E_0 e^{-2k_1a}}{(1-n_1)e^{2k_1a} + (1+n_1)e^{-2k_1a}}$$

put $(1-n_1)e^{2k_1a} - (1+n_1)e^{-2k_1a} = \eta$

& $(1-n_1)e^{2k_1a} + (1+n_1)e^{-2k_1a} = \xi$

$$\Rightarrow \frac{1}{\eta}\left[(1-n_1)E_2 e^{2k_2a} - 2E_0 e^{-2k_1a}\right] = \frac{1}{\xi}\left[(1-n_1)\frac{n_2}{n_1}E_2 e^{ik_2a} + 2E_0 e^{-2k_1a}\right]$$

$$\Rightarrow \left[\xi(1-n_1)e^{2k_2a} - \eta(1-n_1)\frac{n_2}{n_1}e^{2k_2a}\right]E_2 = 2(\eta+\xi)E_0 e^{-2k_1a}$$

$$\Rightarrow (1-n_1)e^{2k_2a}\left[\xi - \eta\frac{n_2}{n_1}\right]E_2 = 2(\eta+\xi)E_0 e^{-ik_1a}$$

$$\Rightarrow \frac{(1-n_1)}{n_1}(n_1\xi - n_2\eta)E_2 e^{ik_2a} = 2(\eta+\xi)E_0 e^{-2k_1a} \quad \text{...(7)}$$

[ Again $\eta + \xi = 2(1-n_1)e^{ik_1a}$

and $n_1\xi - n_2\eta = n_1(1-n_1)e^{ik_1a} + n_1(1+n_1)e^{-ik_1a} - n_2(1-n_1)e^{ik_1a} + n_2(1+n_1)e^{-ik_1a}$

$= 2n_1(1+n_2)\cos k_1a - 2i(n_1^2+n_2)\sin k_1a$

Plugging the values in (7),

$$\frac{(1-n_1)}{n_1}\left[2n_1(1+n_2)\cos k_1 a - 2i(n_1^2+n_2)\sin k_1 a\right]E_2 e^{ik_2 a}$$

$$= 2\cdot 2(1-n_1)\,\cancel{e^{ik_1 a}}\cdot E_0\,\cancel{e^{-ik_1 a}}$$

$$\frac{E_2}{E_0} = \frac{4(1-\cancel{n_1})\,n_1 \cdot e^{-ik_2 a}}{(\cancel{1-n_1})\left[2n_1(1+n_2)\cos k_1 a - 2i(n_1^2+n_2)\sin k_1 a\right]}$$

$$= \frac{2n_1 \quad e^{-ik_2 a}}{\cancel{2}n_1(1+n_2)\cos k_1 a - \cancel{2}i(n_1^2+n_2)\sin k_1 a}$$

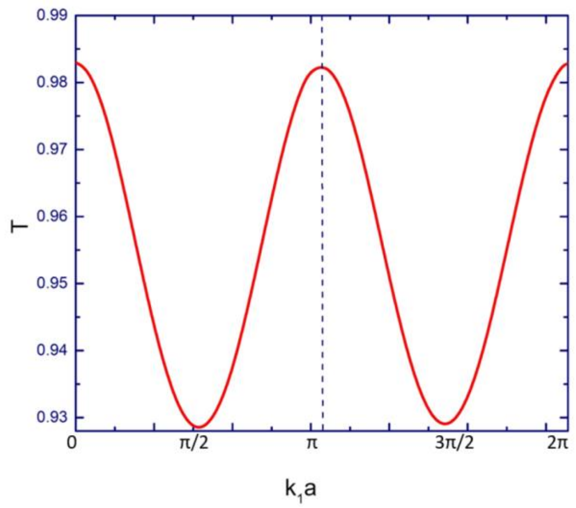
Transmission coefficient $T = \dfrac{\frac{1}{2}\epsilon_2 v_2 E_2^2}{\frac{1}{2}\epsilon_0 v_0 E_0^2}$, $\quad v_0 = c$, $v_2 = \dfrac{c}{n_2}$

$$= n_2\left(\frac{E_2}{E_0}\right)^2$$

$$\therefore\ T = \frac{4n_1^2 n_2}{\left[n_1^2(1+n_2)^2\cos^2 k_1 a + (n_1^2+n_2)^2\sin^2 k_1 a\right]}$$

$$\boxed{T = \frac{4n_1^2 n_2}{\left[n_1^2(1+n_2)^2 + \{(n_1^2+n_2)^2 - n_1^2(1+n_2)^2\}\sin^2 k_1 a\right]}}$$

$n_1=1.5$, $n_2=1.3$, plot $T$ vs $k_1a$ for $k_1a$ $0 \rightarrow 2\pi$



8. Consider a plane polarized electromagnetic wave traveling along $z$ direction in a dielectric of refractive index $n_1$ and incident normally on a ohmic conductor of conductivity $\sigma$ and refractive index $n_2 = n_1(1+i\beta)$, where $\beta$ is a dimensionless real number. The dielectric-conductor interface $S_1$ lies in the $XY$ plane. The incident electromagnetic wave is linearly polarized in the $x$ direction and the corresponding electric field is represented as $\vec{E}_I = E_{0I}e^{-i(\omega t - k_1 z)}\hat{x}$. Assume $\mu_1 \approx \mu_2 \approx \mu_0$ (the free space permeability). The amplitudes of reflected and transmitted electric fields are $E_{0R}$ and $E_{0T}$, respectively.

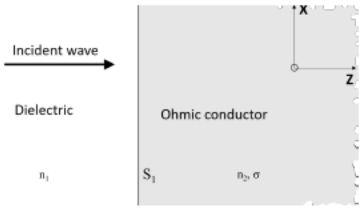
(a) Write down the expression for the incident magnetic field.

(b) Write down the expressions for the electric field and magnetic field corresponding to the transmitted wave.

(c) Find out the free charge density at $S_1$ using appropriate boundary conditions.

(d) What is the free surface current density at $S_1$?

(e) Write down the boundary conditions at the dielectric-conductor interface $S_1$ for the components of $\vec{E}$ and $\vec{B}$ fields parallel to the interface to find out the phase change undergone by the electric field vector of the reflected wave.




a) $\vec{E}_I = E_{0I}\, e^{-i(\omega t - k_1 z)}\,\hat{x}$ , $\vec{B}_I = \frac{E_{0I}}{v_1}\, e^{-i(\omega t - k_1 z)}\,\hat{y} = \frac{E_{0I}\, n_1}{c}\, e^{-i(\omega t - k_1 z)}\,\hat{y}$

b) $\vec{E}_T = E_{0T}\, e^{-i(\omega t - k_2 z)}\,\hat{x}$ , $\vec{B}_T = \frac{E_{0T}}{v_2}\, e^{-i(\omega t - k_2 z)}\,\hat{y} = \frac{E_{0T}\, n_2}{c}\, e^{-i(\omega t - k_2 z)}\,\hat{y}$

c) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$

$E_1^{\perp}$: normal component of Electric field in 1

$E_2^{\perp}$: normal component of Electric field in 2

$E_1^{\perp}, E_2^{\perp} = 0$

$\Rightarrow \sigma_f = 0$

d) $\vec{K}_f = 0$

e) $E_1^{\parallel} - E_2^{\parallel} = 0 \Rightarrow \boxed{E_{0I} + E_{0R} = E_{0T}}$ ---- (i)

$\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = 0 \Rightarrow B_1^{\parallel} - B_2^{\parallel} = 0 \quad (\mu_1 \approx \mu_2)$

$\Rightarrow \frac{E_{0I}}{v_1} - \frac{E_{0R}}{v_1} = \frac{E_{0T}}{v_2}$

$\Rightarrow E_{0I} - E_{0R} = \frac{v_1}{v_2} E_{0T} = \frac{n_2}{n_1} E_{0T}$

$\Rightarrow \boxed{E_{0I} - E_{0R} = (1+i\beta) E_{0T}}$ (ii)

Solve (i) & (ii) $\Rightarrow$ $E_{0R} = \frac{-i\beta}{(2+i\beta)} E_{0I}$

$= \frac{-\beta(\beta+2i)}{\beta^2+4} E_{0I} = \frac{-\beta}{\sqrt{\beta^2+4}} e^{i\varphi} E_{0I}$

phase change $\boxed{\varphi = \tan^{-1}\frac{2}{\beta}}$