3. Is the followi... or both are... urable. Show...

29 January 2024 | MTH 628 Topics in Topology | Name – Manan Kabra

Max Marks = 25 | Quiz 1 | Roll No – 210578

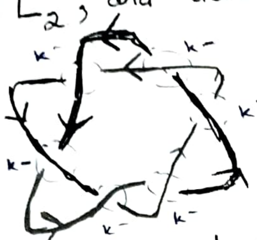
25/25

Excellent Work

There are 9 questions carrying 5 marks each. **Do any 5 out of 9**

1) Determine the linking number of the following 2-component links L_1 and L_2, and determine whether L_1 is equivalent to L_2 or not?

L_1



$L(L_1) = \frac{1}{2}(-6) = -3$

L_2

$L(L_2) = \frac{1}{2}(+4) = +2$

$L_1 \not\cong L_2$

2) Determine the Unknotting Number of the given prime knot 6_1 giving complete reasons for your answer.

$u(6_1) = 1$

6_1

3) Prove that if a knot K is achiral i.e. $K \simeq K^*$, then its Jones polynomial is palindromic i.e. $V_K(t) = V_K(1/t)$.

4) Is the prime knot 7_1 (given here) achiral or not? Justify your answer.

No

7_1

5) State the axioms defining the 2-variable HOMFLY polynomial $P_K(v, z)$. Show that the Jones polynomial and the Alexander polynomial can be obtained as special cases of the HOMFLY polynomial

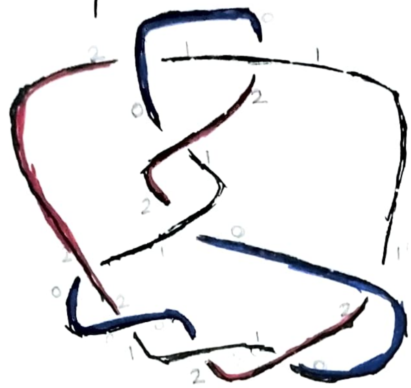
6) Calculate the Jones polynomial for the figure eight knot 4_1. Is the figure eight knot 4_1 achiral or not? Justify your answer.

4_1

7) Can the polynomial $t^2 - t + 2 - \frac{1}{t} + \frac{1}{t^2}$ be the Alexander polynomial of a knot or not? Justify?

8) Determine whether the following knot (of 9 crossings) is tricolorable by giving a tricoloration for it, or show that it is not possible

Yes, it is possible.



9) Suppose K is an alternating knot whose Jones polynomial is given as $t^2+2t^4-2t^5+t^6-2t^7+t^8$. K is the connected sum of two familiar knots. Identify them. What is the crossing number of K?

Ans. 9

$$V_K(t) = t^2+2t^4-2t^5+t^6-2t^7+t^8$$

$$\cancel{= t^2(1+2t^2-2t^3+t^4-2t^5+t^6)}$$

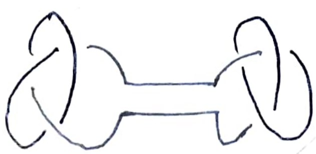
$$= (t+t^3-t^4)^2$$

$$= \underbrace{(t+t^3-t^4)}_{V_{K_1}(t)} \times \underbrace{(t+t^3-t^4)}_{V_{K_2}(t)}$$

where $K_1, K_2 \to$ trefoil

$\therefore K = K_1 \# K_2$, $K_1, K_2 \to$ trefoil (3_1)

$c(K) = c(K_1) + c(K_2) = 3+3 = 6$

$K \cong$ 

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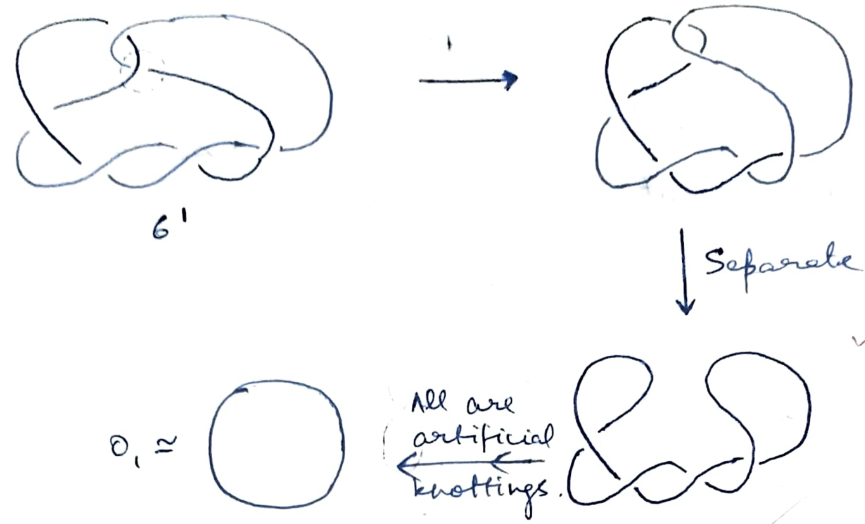


Very Good

$\therefore \mu(6_1) = 1$

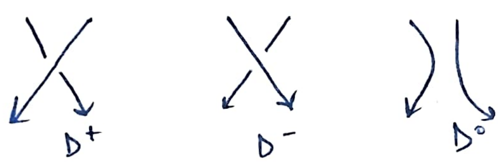
Ans. 4 We note that 7_1 knot is alternating knot.
$\therefore$ By the theorem proved in class, any alternating achiral knot has even crossing number.
But 7 is not even $\Rightarrow$ 7_1 is ~~achiral~~ chiral knot.

Very Good

Ans. 5 The 2 axioms are defined as follows –
(1) The HOMFLY polynomial of trivial knot is equal to 1, i.e

$$K \simeq O_1 \Rightarrow P_K(v, z) = 1$$

(2)



$$\frac{1}{v} P_{K^+}(v, z) - v\, P_{K^-}(v, z) = z\, P_{D^0}(v, z)$$

Alexander Polynomial $\rightarrow \Delta_K(t) = P_K\left(1, \sqrt{t} - \frac{1}{\sqrt{t}}\right)$

$$\left(\Delta_{K^+}(t) - \Delta_{K^-}(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) \Delta_{K^0}(t) \right)$$

5
Excellent

Jones Polynomial $\rightarrow V_K(t) = P_K\left(t, \sqrt{t} - \frac{1}{\sqrt{t}}\right)$

$$\left(\frac{1}{t} V_{K^+}(t) - t\, V_{K^-}(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{K^0}(t) \right)$$

$\therefore$ Alexander & Jones polynomial are the special cases of the HOMFLY poly

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Ans 7 $\Delta_K(t) = t^2 - t + 2 - \frac{1}{t} + \frac{1}{t^2}$

We calculate $\Delta_K(1) = 1 - 1 + 2 - 1 + 1 = 2 \neq 1$

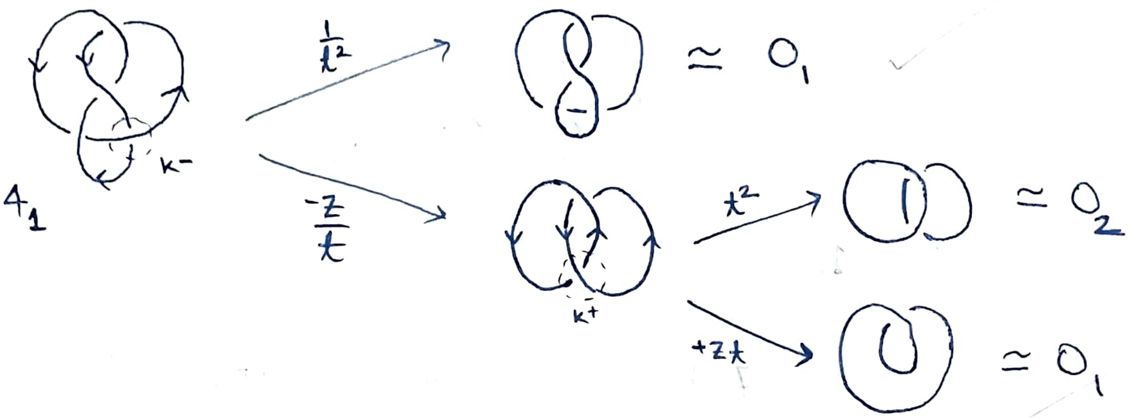
5

Good

But we know that $\Delta_K(1) = 1$ for all knots (Proved in class)

$\Rightarrow$ There exists no such knot.

Ans. 6



$$V_{4_1}(t) = \frac{1}{t^2} V_{O_1} - \frac{Z}{t}\left[t^2 V_{O_2} + Zt\, V_{O_1}\right]$$

$$= \frac{1}{t^2}\cdot 1 - \frac{Z}{t}\left[t^2 (-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) + Zt\right]$$

$$= \frac{1}{t^2} + t\, Z\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) - Z^2$$

$$= \frac{1}{t^2} + t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) - \left(t + \frac{1}{t} - 2\right)$$

$$= \frac{1}{t^2} + t\left(t - \frac{1}{t}\right) - \left(t + \frac{1}{t} - 2\right)$$

$$= \frac{1}{t^2} + (t^2 - 1) - \left(t + \frac{1}{t} - 2\right)$$

5

$$V_{4_1}(t) = \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$$

Excellent

$$= [-2][1 \; -1 \; +1 \; -1 \; +1]$$

Since $V_{4_1}(t) = V_{4_1}\left(\frac{1}{t}\right) \Rightarrow 4_1$ is achiral.

...plete reasons for your answer and

19. February. 2024.

Time 13.00 to 15.00

Department of Mathematics & Statistics
MTH-628, Topics in Topology
Mid-Semester Examination

210578

Max Marks: 60

There are **15** questions. Each question carries 5 marks. Do any 12.

1. Calculate the linking number of the following two-links, and determine if they are equivalent or not

1 0

2. Define when a knot K is said to be tricolourable. Show that if K_1 and K_2 are equivalent knots, then both are tricolourable or both are not tricolourable.

3. Is the following knot tricolourable or not. Justify your answer.

4. State the Alexander's Theorem for knots.

   Use the Alexander's Theorem to find a braid representation for the knot 6_2.

6_2

5. Draw the knot 6_3 obtained as the closure of the braid $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_2\sigma_1^{-1}$.

   What is the unknotting number of 6_3? Justify your answer.

   2

6_3

6. Show that the two words w_1 and w_2 represent the same five string braid up to equivalence

$$w_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_4$$

and $w_2 = \sigma_2^2\sigma_3\sigma_1\sigma_3^{-1}\sigma_4$

$\sigma_2^2\sigma_1\cancel{\sigma_2\sigma_2^{-1}}\sigma_4 = \sigma_2^2\sigma_3\sigma_1\sigma_3^{-1}\sigma$

Please answer clearly ANY

question carries 5 Marks. Maximum Mar

answer and

7. Calculate the HOMFLY polynomial for the left hand trefoil knot 3_1

3_1

8. Using question 7 or otherwise calculate the Jones polynomial for the knot 3_1

9. Calculate the Alexander polynomial for the knot 5_1

5_1

10. State the Markov Theorem for closed n −braids.

11. Use the Alexander's Theorem to find a braid representation for the knot 7_1

7_1

12. Is K amphicheiral or not ? Justify your answer.

K

13. Determine the Jones polynomial for a connected sum of two knots K_1 and K_2
Justify your answer. (Assume the Lemma to be true)

14. Determine the following closed braid Justify your answer.

a.) $\sigma_1^{-1}\sigma_2^3$ 4_1

b.) $\sigma_1^{-3}\ \sigma_2^3$ 6_1

c.) $\sigma_1^{-2}\sigma_2$ Hopf Link

15. Sketch the rational knot T[4,2] and T[2, 1 ,1 ,2], and determine the rational number corresponding to it.

and determine if they are equivalent or not Justify your answer.

$4+\frac{1}{2} = \frac{9}{2}$

$2+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}$

$t^2\left[-t^2+\frac{1}{t}+t-z+2\right]$

$-t^4+t^3+t$

$1+\frac{2}{3}$

$2+\frac{3}{5}$

$=\frac{13}{5}$

25 April 2024

8-11 A.M.

INDIAN INSTITUTE OF TECHNOLOGY KANPUR
DEPARTMENT OF MATHEMTICS & STATISTICS
MTH-628 (TOPICS IN TOPOLOGY)
END-SEMESTER EXAMINATION

Max Marks=75

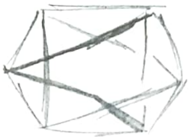
Please answer clearly ANY 15 QUESTIONS ONLY out of the given 18 questions. Each question carries 5 Marks. Maximum Marks are 15 x 5 = 75 Marks.

1. Compute the fundamental group of the circle S^1 giving complete reasons for your answer and full statements of theorems used.

2. (a) Let A be a subspace of X. Define what it means for A to be a strong deformation retract of X. Give an example.

(b) Compute the fundamental group of the following spaces, giving brief reasons for your answer.

Cone on a circle ; $z^2 = x^2 + y^2$



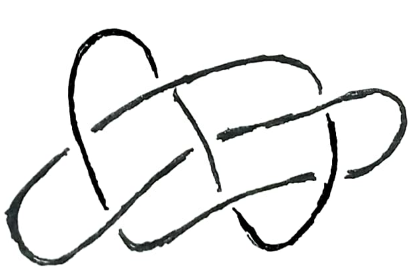
3. Define a covering space E of B. Let $p: E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$; then $p^{-1}(b)$ has k elements for every $b \in B$ (a k-fold cover).

4. Define the fundamental group of a topological space X and show that it is a topological invariant. However, show that the Converse is False by giving an example of topological spaces X and Y such that $\pi_1(X) = \pi_1(Y)$ but X is not homeomorphic to Y.

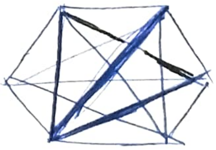
5. State the classification Theorem for 2-dimensional surfaces that are compact and without boundary.

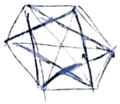
6. What is the Euler characteristic of the connected sum of two surfaces, S_1 and S_2 ? Using this or otherwise, calculate the Euler Characteristic of T_g (The connected sum g – torus)

7. Determine whether the following knot 7_4 is tricolourable or not.



7_4





8. Construct a spatial graph for the complete graph on 6 vertices K_6 , and show that it contains a pair of linked triangles (i.e. the Hopf link).

9. Prove that if a knot K is amphicheiral then its Jones polynomial is palindromic i.e. $V_k(t) = V_k\left(\frac{1}{t}\right)$.

$t^{3/2} - t^{1/2}$

$t^{-3/2} - t^{-1/2}$

10. Determine the linking numbers of the following 2-component links, and determine whether they are equivalent or not.

L_1

L_2

11. State and prove the BROUWER'S. Fixed Point Theorem for the disc D.

12, 13. For the following spaces (i) construct a triangulation (ii) write down the simplicial homology groups with brief reasons (iii) calculate the Euler Characteristic and (iv) show that

$$\chi(K) = \sum_{i=0}^{n} (-1)^i \beta_i .$$

[1] Four circles meeting at a point.
[2] [redacted] TORUS $S^1 \times S^1$ [10.]
[3] Double Torus
[4] One point union of two spheres
[5] One point union of a sphere S^2 and a circle S^1

14. Prove that R^n is homeomorphic to R^m if and only if $m = n$.

15. (a) Draw the rational knot $K_1 = [5,1,4]$ $[5,1,4,1]$ and determine the rational number corresponding to it

(b) Draw the knot 6_3 obtained as the closure of the braid $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_2\sigma_1^{-1}$.

16. Determine the fundamental group of the following spaces, giving brief reasons for your answer in each case, and statements of theorems used

(a) Let X be the sphere $x^2 + y^2 + z^2 = 1$ (b) Trefoil Knot 

K Z

17. A compact topological space X is said to have the fixed point property if every continuous map $f : X \to X$ has at least one fixed point. Determine whether the following topological spaces have the fixed point property or not. Justify your answer in each case, with reasons

(i) Sphere S^2
(ii) Torus T
(iii) Finite cone $z^2 = x^2 + y^2$, $0 \le z \le 1$ - C

18. State and prove the Brouwer's Fixed Point Theorem for the n-disc D^n.

$S^1 \xrightarrow{i} D \xrightarrow{f} S^1$