Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

MTH 636A: Game Theory

Mid-Semester examination, Date: February 24, 2024, Saturday

Timing: 1:00 PM to 3:00 PM

- This paper has five questions. The exam is for 30 marks, and the maximum you can get is 34.
- **Answer the questions ONLY in the spaces provided after the questions. Answers written anywhere else will not be graded.** You may take additional sheets for rough work.
- Try not to use any result not done in the class. However, if you use any such result, clearly state and prove it.
- Write your name, roll no., program name, and seat number clearly in the appropriate place.
- **For prove or disprove type questions, clearly state whether it's a prove or a disprove and then provide the arguments.**
- One A4 sheet with ONLY necessary definitions and results is allowed during the exam. Use of a calculator, mobile, and smart watch is strictly prohibited.

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Name and Program: ______________________________

Roll number and Seat number: ______________________________

1. There are n individuals who witness a crime. Everybody would like the police to be called. If this happens, each individual derives satisfaction $v > 0$ from it. Calling the police has a cost of c, where $0 < c < v$. The police will come if at least one person calls. Hence, this is an n-person game in which each player chooses from $\{C, N\}$; C means 'call the police' and N means 'do not call the police'. The payoff to person i is 0 if nobody calls the police, $v - c$ if i (and perhaps others) calls the police, and v if the police are called but not by person i.

   (a) What are the Nash equilibria of this game in pure strategies? Does the game have a symmetric Nash equilibrium in pure strategies (a Nash equilibrium is symmetric if every player plays the same strategy)? (**2 marks**)

   Answer: It is easy to see that the game has only n many pure strategy Nash Equilibria. Those are one player playing C and the other players playing N. There is no pure strategy Nash Equilibrium.

(b) Compute the symmetric Nash equilibrium or equilibria in mixed strategies. (**4 marks**)

Answer Let's assume there is a symmetric Nash Equilibrium in mixed strategies where each player is playing the strategy $[p\ (C), 1-p\ (N)]$. From (a), it must hold that $0 < p < 1$. Therefore, by the indifference principle, we have Player 1 is indifferent between playing C and N, provided others are playing the strategy $[p\ (C), 1-p\ (N)]$. This means

$$(v-c) = v(1-(1-p)^{n-1})$$
$$\Longrightarrow p = 1-(\frac{c}{v})^{\frac{1}{n-1}}.$$

(c) For the Nash equilibrium/equilibria in (b), compute the probability of the crime being reported. What happens to this probability if n becomes large? (**2 marks**)

Answer As we can see in (b), the probability of the crime being reported is $(1-(1-p)^n) = 1-(\frac{c}{v})^{\frac{n}{n-1}}$. The probability tends to $1-\frac{c}{v}$ as n becomes large.

2. Let $G = \langle V, E \rangle$ be a directed graph, where V is a set of vertices, and E is a set of edges. A directed edge from vertex x to vertex y is denoted by (x, y). Suppose that the graph is complete, i.e., for every pair of edges $x, y \in V$, either $(x, y) \in E$ or $(y, x) \in E$, but not both. In particular, $(x, x) \in E$ for all $x \in E$.

   (a) Define a two-player zero-sum game in which the set of pure strategies of the two players is V, and the payoff function is defined as follows:

$$u(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \text{ and } (x,y) \in E \\ -1 & \text{if } x \neq y \text{ and } (x,y) \notin E \end{cases}$$

   Prove that the value of the game is 0 in mixed strategies. (**4 marks**)
   (Hint: For an antisymmetric matrix $A_{n\times n}$, $x^T Ay = -y^T Ax$ for all $x, y \in \mathbb{R}^n$.)
   Answer: Let A denote the matrix of the two-person zero-sum game. As per the form of the function u, the matrix A is antisymmetric ($A^T = -A$). Suppose the value of the game is v, and p is a max-min strategy for Player 1. This means

$$p^T Aq \geq v \text{ for all } q \in \Delta V. \tag{1}$$

   As A is antisymmetric, (1) implies $p^T Ap = -p^T Ap \geq v$. Hence, $v = 0$.

(b) Show that every max-min strategy $q \in \Delta V$ of Player 2 in this game satisfies the following equation

$$\sum_{\{y \in V \mid (y,x) \in E\}} q(y) \geq \frac{1}{2}.$$

(4 marks)

Answer: Let q be a max-min strategy for Player 2. For $x \in V$, we have $e^x A q \leq 0$ where $e^x \in \Delta V$ is the degenerate probability distribution at x. Thus,

$$\begin{aligned}
& e^x A q \leq 0 \\
\Longrightarrow\ & \sum_{\{y \in V \mid (x,y) \in E\}} q(y) + \sum_{\{y \in V \mid (x,y) \notin E\}} (-q(y)) \leq 0 \\
\Longrightarrow\ & \sum_{\{y \in V \mid (x,y) \in E\}} q(y) \leq \sum_{\{y \in V \mid (y,x) \in E\}} q(y) \quad (\text{as } (x,y) \notin E \implies (y,x) \in E)
\end{aligned} \tag{2}$$

(2) and the fact that q is a probability distribution together imply

$$\sum_{\{y \in V \mid (y,x) \in E\}} q(y) \geq \frac{1}{2}.$$

3. Find all the Nash equilibria/equilibrium of the following game

(5 marks)

Player 2

Player 1

	a	b
A	$(0, 0)$	$(6, -3)$
B	$(-3, 6)$	$(5, 5)$

Answer: For Player 1, the strategy B is strictly dominated, therefore, to calculate the Nash Equilibria, we may eliminate the strategy B. Thus, the game reduces to

Player 2

Player 1

	a	b
A	$(0, 0)$	$(6, -3)$

Now given Player 1 is playing A, the best reply for Player 2 is a. Thus, the only Nash Equilibrium of the game is (A, a).

4. Two agents want to split one unit of a divisible good. Each agent $i = 1, 2$ announces a non-negative real number x_i. Both agents make their announcements simultaneously. Each agent i pays an amount of the good equal to his announcement, i.e., x_i. If $x_i > x_j$ ($i, j \in \{1, 2\}$) then agent i gets the entire 1 unit of the good and agent j receives nothing. If $x_1 = x_2$, then each agent receives $\frac{1}{2}$ a unit. The net utility of each agent is the amount of the good they receive minus the amount they announce. A pure strategy for an agent in this game is a non-negative real number. Does this game have a pure strategy Nash equilibrium? Explain your answer. (**4 marks**)

Answer: It is easy to see that there cannot be any equilibrium where $x_1 \neq x_2$. Suppose there is an equilibrium where $x_1 = x_2$. This means each agent receives $\frac{1}{2}$ a unit, and $u_1(x_1, x_2) = u_2(x_1, x_2) = \frac{1}{2} - x_1$. As this is an NE, we have

$$\begin{aligned}
&u_1(x_1, x_2) \geq u_1(x_1 + \epsilon, x_2) \text{ for all } \epsilon > 0 \\
\implies &\frac{1}{2} - x_1 \geq 1 - (x_1 + \epsilon) \quad (\text{as } x_1 + \epsilon > x_2, \text{ Player 1 will get the good}) \\
\implies &\epsilon \geq \frac{1}{2}.
\end{aligned}$$

Therefore, there is no equilibrium (x_1, x_2) where $x_1 = x_2$.

5. Prove or disprove the following statements:

(a) Let $\langle \{1, 2\}, S_1, S_2, u\rangle$ be a two-player zero-sum game with value v in mixed strategies. Suppose there is a strategy profile (mixed) (σ_1, σ_2) such that $u(\sigma_1, \sigma_2) = v$. Then (σ_1, σ_2) is a max-min strategy profile. **(3 marks)**

Answer: *(Disprove)* Consider the following game:

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Note that the game has a saddle point $(1, 1)$. Therefore, the game has value $u(1, 1) = 2$. Also, for the strategy-profile $(2, 2)$, we have $u(2, 2) = 2$ but neither row-2 nor column-2 is a max-min strategy for the corresponding players.

(b) Suppose that a mixed strategy σ_i of player i strictly dominates another of his mixed strategies, $\hat{\sigma}_i$. Player i has a pure strategy $s_i \in S_i$ satisfying: (i) $\hat{\sigma}_i(s_i) > 0$ and (ii) strategy s_i is not chosen by player i in any equilibrium. (**3 marks**)

Answer: *(Disprove)* Consider the following game:

	a	b
A	$(6,0)$	$(6,0)$
B	$(10,0)$	$(0,0)$
C	$(0,0)$	$(10,0)$

Note that the strategy $\sigma_1 = A$ strictly dominates the strategy $\hat{\sigma}_1 = [\frac{1}{2}\ (B), \frac{1}{2}\ (C)]$. The two strategies that get positive probabilities at $\hat{\sigma}_1$ are B and C. For the above statement to be true, there must exist $s_i \in S_i$ satisfying: (i) $\hat{\sigma}_i(s_i) > 0$ and (ii) strategy s_i is not chosen by player i in any equilibrium. As (B, a) and (C, b) are both Nash Equilibria, the statement is false.

(c) Suppose a game $G = \langle \{1, 2\}, S_1, S_2, u_1, u_2\}\rangle$ has exactly two pure strategy Nash equilibria s and s' such that $u_i(s) \neq u_i(s')$ for all $i \in \{1, 2\}$. Then there is a mixed strategy Nash equilibrium σ of G such that $\sigma \notin \{s, s'\}$. **(3 marks)**

Answer: *(Disprove)* Consider the following game:

	a	b
A	$(1, 1)$	$(0, 0)$
B	$(0, 0)$	$(0, 0)$

The game satisfies the claims in the question, but it does not have any other mixed strategy NE. **Can you think of any other restriction on $u_i(s)$ and $u_i(s')$ such that the above statement holds true?**