MTH-424 PARTIAL DIFFERENTIAL EQ^n

Lecture 1

* **Wellposedness of PDE**
  - → existence of sol^n
  - → uniqueness of sol^n
  - → cont^n dependance [closer the eq^n closer the sol^n]

* **PDEs**

PDE is a combination of independant variables (x, y), dependant variable $u : \mathbb{R}^2 \to \mathbb{R}$ and all their partial derivatives

$\therefore F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy} \ldots) = 0$

and if 1 independant variable == ODE.

* **Notion of Solutions**

A "classical sol^n" of PDE is a func^n, $u : \mathbb{R}^2 \to \mathbb{R}$ such that it solves the given eq^n pointwise.

Example →

a) $u_{xx} + u_{yy} = 0$ (laplace eq^n), $u : \mathbb{R}^2 \to \mathbb{R}$, $u \in C^2(\mathbb{R}^2)$

b) $u_x + c u_y = 0$, $u \in C^1(\mathbb{R}^2)$

this condition depends on the PDE itself.

→ Remark : Not always $C^1/C^2 \ldots$ work for a PDE. such that

Eikonal Eq^n :- $|u'(x)| = f(x)$ $\quad u : \mathbb{R} \to \mathbb{R}$

let's take $f(x) = 1$

$$① \begin{cases} \therefore |u'(x)| = 1 & x \in (-1, 1) \quad \therefore u : [-1, 1] \to \mathbb{R}^2 \\ u(-1) = u(1) = 0. \end{cases}$$

if $u \in C^1([-1, 1])$ then by rolle's theorem. $\exists\, t \in (-1, 1)$

$$\frac{u(-1) - u(1)}{2} = u'(t) = 0 \quad (\Rightarrow\Leftarrow).$$

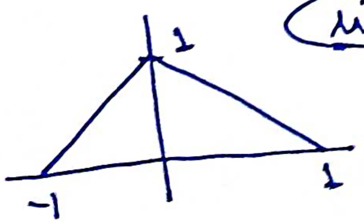
$\therefore$ THINK OUT OF THE BOX !!

It doesn't have a 'classical sol^n'.

If we take $v(x) = 1 - |x| \;\forall\, x \in [-1, 1]$

lipschitz cont^n for $x \in [-1, 1]$. (not C^1).

and it solves the above ①



* Radamacher Theorem

A Lipchitz contn funcn is almost everywhere differential

Heance, a "weak solution"

The eqn satisfies "almost everywhere", now the eqn has ∞ly many soln.

Lecture 2

Order of PDE $\quad (u : \mathbb{R}^2 \to \mathbb{R})$.

A PDE is said to be of 'order k' if the highest partial derivative present in the eqn is of order k.

Example

a) $u_x + u_y = 0$ (Linear)
↳ PDE of order 1

b) $\Delta u = u_{xx} + u_{yy} = 0$ (Linear) → LAPLACE EQn
↳ PDE of order 2

c) $u_x + x u_{xy} + u_{xx} \cdot u_x = 0$ → PDE of order 2 (Non-linear)

d) $u_x + c(u_y)^2 = 0$ → order 1. (Non-linear)

* Linear PDE $\quad (u : \mathbb{R}^n \to \mathbb{R})$. A $(X = (x_1, \dots, x_n))$.

Rewrite $F(u, x_1, x_2, \dots, x_n, u_{x_1}, \dots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2} \dots) =$ in terms of $L(u) = f(X)$.

consider all such elements consisting u and its partial derivatives.

Terms consisting "only" of independent variable

A PDE is called linear if $L(u)$ is linear i.e.
$$L(au_1 + bu_2) = aL(u_1) + bL(u_2).$$

A PDE is called non-linear if it is not linear.

Semilinear PDE

A PDE of order k, is said to be semilinear, if all its k^{th} ~~derivatives~~ partial derivatives coefficients consists only of independent variables.

check the highest degree (k).

Quasilinear PDE

A PDE of order k, is said to be quasilinear, if the coeff of k^{th} order partial derivatives consists independent variables, dependent variable (u) and ~~all the~~ only the partial derivatives of order upto $(k-1)$.

Fully Non Linear

A PDE, is said to be fully linear are all PDEs which are not Quasilinear non semilinear.

... or not ?

Example

(i) $u_x + c(u_y)^2 = 0 \rightarrow$ No

$\underbrace{c u_y}_{\text{coeff.}} \cdot u_y$

(ii) $u_x + c(u_y)^2 + \underline{u_{xx}} = 0 \rightarrow$ yes   only this 2 order

b) Quasilinear or not?

(i) $u_x + x^2 \underline{u_{xy}} + \underline{u_x u_{xx}} = 0 \rightarrow$ yes.

(ii) $u_x + c(u_y)^2 = 0 \rightarrow$ No.

c) Fully Non-linear

(i) $u_x + c(u_y)^2 = 0 \rightarrow$ yes.

(ii) $u_x + \sin(u_y) = 0 \rightarrow$ yes.

* **Homogeneous PDE**

It is of the form $L(u) = 0$ $[f(\underline{x}) = 0]$.

A non-homogeneous PDE $\rightarrow$ if $f(\underline{x}) \neq 0$.

* **General solⁿ**

For ODE (without assuming initial solⁿ), we have ∞ly many solⁿ. And it is parametrized by constant

- PDE will also have ∞ly many solⁿ but it will be parametrized by funcⁿ of independent variable.

Example

(i) $u_x = 0$ $u: \mathbb{R}^2 \rightarrow \mathbb{R}$

$u(x, y) = f(y)$. $\rightarrow$ arbitrariness comes from funcⁿ.

Lecture 3

Auxiliary condition $\rightarrow$

A PDE is usually supplemented with an a-priori condition. This is called auxiliary condition.

- PDE only with auxiliary condition are meaningful/ physically useful.
- We may get uniqueness of solution with auxiliary condition.

Generally, if we have n-independent variables in a given domain Ω; $(u: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R})$, then the auxiliary condition is set in a $\underline{(n-1)}$ dimⁿ subdomain Γ of Ω.

↳ b/c of best balance (not understrain nor overstrain).

$\exists$ 2 most imp. auxiliary condⁿ.

a) IVP ⎤ Initial main hai (n-1)

b) BVP ⎦ मतलब है and boundary में भी.

IVP → whenever a PDE is based on time as independent variable, t, then we often give condition at initial time.

Example (i) Heat Eqn $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, (t, x)

$$u_t - \Delta u = 0$$

$u(0, x) = g(x)$ → auxiliary condn.

(ii) Wave Eqn

$$u_{tt} - \Delta u = 0$$

$$\left.\begin{aligned} u(0,x) &= g(x) \\ u_t(0,x) &= h(x) \end{aligned}\right\} \text{auxiliary cond}^n$$

BVP → not only t is the independent variable.

Example (i) Laplace Eqn

$$\Delta u = f(x) \text{ in } \Omega \subseteq \mathbb{R}^n.$$

auxiliary condn ← $u(x) = g(x)$ on $x \in \partial\Omega$

* FIRST ORDER PDE

Example (i) $u_x = 0$; $u : \mathbb{R}^2 \to \mathbb{R}$

general soln → $u(x, y) = f(y)$

assume $u(0, y) = y^2$ ∴ $f(y) = y^2$.

Transport eqn (ii) $a u_x + b u_y = 0$ $\left[\nabla u = (u_x, u_y) \text{ ↳ gradient}\right]$

$$\underline{(a, b) \cdot (u_x, u_y)} = 0$$

$\dfrac{\partial u}{\partial p} = 0$ where $p = \dfrac{(a,b)}{\sqrt{a^2 + b^2}}$

↳ basically u is not incrementing the direction of p.

← not changing value on these parallel lines. (i.e. $u(x,y)$ remains constant on these parallel lines.)

$$bx - ay = c.$$

$$\boxed{u(x, y) = f(bx - ay)}$$

(iii) $3u_x + 2u_y = 0$

$u(x, 0) = x^3$.   ∴ $f(t) = \dfrac{t^3}{8}$

$u(x, y) = f(2x - 3y)$.

$u(x, 0) = f(2x) = x^3 = \dfrac{(2x)^3}{8}$

$\therefore \quad u(x,y) = \dfrac{(2x-3y)^3}{8}$ * (GOAL: FIND THE CURVE)

* Another way to solve

$$a u_x + b u_y = 0.$$

Change in variable: $\xi = ax + by \quad \eta = bx - ay$

$$u_\xi = 0 \Rightarrow u(\xi,\eta) = f(\eta) = f(bx - ay).$$

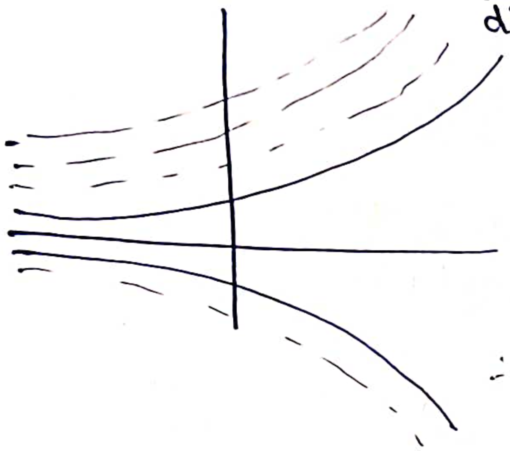
(iv) $u_x + y \cdot u_y = 0$ — the sol^n $u(x,y)$ will remain constant through a curve where the tangent is $(1,y)$ (parallel to this).

$\nabla u \cdot (1,y) = 0$

(directional derivative)

Let curve be $(x, y(x))$ then tangent is $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1,y)$.

$\therefore \frac{dy}{dx} = y \qquad \ln y = x + c \qquad \therefore y = ce^x$.



$\therefore u(x,y)$ remain constant through (x, ce^x).

Now, $u(x, ce^x) = u(0, ce^0)$
$= u(0, c)$
$= u(0, ye^{-x})$.

$\therefore u(x,y) = f(ye^{-x})$

Lecture 4

1^{st} order linear PDEs

- General form →

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y).$$

- Method of characteristics →

$$(1)\ \begin{cases} a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y) \text{ in } \Omega. \\ u(x,y) = u_0 \quad \text{for } (x,y) \in \Gamma_0. \end{cases}$$

we write $\Gamma_0 = \{(x_0(s), y_0(s)) : s \in [0,1]\}$. where s is the parametrization of Γ_0.

Then, the auxiliary condition can be written as.

$$u(x_0(s), y_0(s)) = u_0(s).$$

The left hand side of the PDE gives us the directional derivative along "some curves" whose tangent is $(a(x,y), b(x,y))$ at the point (x,y).

→ find those curves

Let us choose a curve $(x(r), y(r))$ [parametrized by r] on Ω whose tangent at $(x(r), y(r))$ is $(a(x,y), b(x,y))$

Then we get.

$$\Rightarrow \frac{dx}{dr} = a(x(r), y(r)), \quad \frac{dy}{dr} = b(x(r), y(r)).$$

∴ The LHS is then the total derivative of u along the curve $(x(r), y(r))$*

$$\frac{d}{dr}(u(x(r), y(r))) = \cancel{\frac{d(x)}{d}}\, u_x \cdot \frac{dx(r)}{dr} + u_y \frac{dy(r)}{dr}$$

$$= a(x(r), y(r)) \cdot u_x + b(x(r), y(r)) \cdot u_y.$$

By PDE,

$$= c(x(r), y(r)) \cdot u + d(x(r), y(r)).$$

Now, we impose that $x(0) = x_0(s)$, $y(0) = y_0(s)$ and solve the following ODE.

[for different values of s, we will different yet similar curves. (parallels) which pass through the Γ_0.]

$$\frac{dx}{dr} = a(x(r), y(r)) \; ; \quad \frac{dy}{dr} = b(x(r), y(r))$$

$$x(0) = x_0(s) \qquad y(0) = y_0(s)$$

Existence of solution of above ODES can be guaranteed by assuming 'some' ~~certain~~ regularity of a & b.

∴ We call $(x(r), y(r))_s$ → CHARACTERISTIC CURVE.

Along the characteristic curve, we solve the PDE.

$$\boxed{\begin{gathered}\frac{du(x(r), y(r))}{dr} = c(x(r), y(r)) \cdot u + d(x(r), y(r)) \\ u(x(0), y(0)) = u_0(s).\end{gathered}}$$

whenever s changes, the curves starting pt. changes and hence the curve changes.

we define the characteristic curve as.

$$(x(r,s), y(r,s)).$$

In order to solve (1) we write the ODES to be →

$$\left.\begin{aligned} \frac{dx(r,s)}{dr} &= a(x(r,s), y(r,s)) \\ x(0,s) &= x_0(s) \end{aligned}\right\} \text{(i)}$$

$$\left.\begin{aligned} \frac{dy(r,s)}{dr} &= b(x(r,s), y(r,s)) \\ y(0,s) &= y_0(s) \end{aligned}\right\} \text{(ii)}$$

$$\left.\begin{aligned} \frac{d\overbrace{u(x(r,s), y(r,s))}^{z(r,s)}}{dr} &= c(x(r,s), y(r,s)).z(r,s) \\ &\quad + d(x(r,s), y(r,s)) \\ z(0,s) &= u_0(s) \end{aligned}\right\} \text{(iii)}.$$

Lecture 5

Step 1: Check whether there exists a characteristic curve $(x(r,s), y(r,s))$ passing through (x, y).

(Remark: Two distinct characteristic curve do not intersect.)

Step 2: check characteristic curves do not intersect with each other.

$\boxed{\text{Theorem 1}}$ If a, b are C^1 funcn (not necessary but sufficient) semilinear PDE, then characteristic curves of any

$$a(x,y)u_x + b(x,y)u_y = c(x, y, u). \text{ in } \Omega$$

↳ May be non-linear in terms of u.

can never intersect

Proof:- Prove by contradiction,

Assume there exists 2 characteristic curves. $(x(r,s_1), y(r,s_1))$, $(x(r,s_2), y(r,s_2))$ which intersect at the point (x_0, y_0) for the first time such that

$$x(r_1, s_1) = x(r_2, s_2) = x_0$$
$$y(r_1, s_1) = y(r_2, s_2) = y_0$$

We drop s_1 & s_2 and rewrite

$$(x(r, s_1), y(r, s_1)) = (x_1(r), y_1(r))$$
$$(x(r, s_2), y(r, s_2)) = (x_2(r), y_2(r))$$

Now, consider a curve

$$\left.\begin{aligned} r &\mapsto (x_3(r), y_3(r)) \\ x_3(r) &= x_2(r - r_1 + r_2) \\ y_3(r) &= y_2(r - r_1 + r_2) \end{aligned}\right\} \text{change of variables}$$

We note that

$$\left.\begin{aligned} x_3(r_1) &= x_2(r_2) = x_0. \\ y_3(r_1) &= y_2(r_2) = y_0. \end{aligned}\right\} \text{①}$$

We know that

$$\frac{dx_2}{dr} = a(x_2(r), y_2(r))$$

$$\frac{dy_2}{dr} = b(x_2(r), y_2(r)).$$

$$\therefore \frac{dx_3}{dr} = \frac{dx_2}{dr}(r - r_1 + r_2). = a(x_2(r - r_1 + r_2), y_2(r - r_1 + r_2))$$
$$= a(x_3(r), y_3(r)). \quad \text{—— ②}$$

Similarly,

$$\frac{dy_3}{dr} = b(x_3(r), y_3(r)). \quad \text{—— ③}$$

By ①, ②, ③, we get that (x_3, y_3) is solving the same ODE that (x_1, y_1) ~~and~~ is ~~(x_1, y_1) are~~ solving.

which is

$$\left.\begin{aligned} \frac{dx}{dr} &= a(x(r), y(r)). \\ \frac{dy}{dr} &= b(x(r), y(r)). \\ x(r_1) &= x_0 \ \& \ y(r_1) = y_0. \end{aligned}\right\}$$

$\therefore$ By uniqueness of ODE in nbd of (x_0, y_0) (as a, b as C^1), $\exists$ only one sol^n. $\therefore$ $(x_1, y_1) = (x_3, y_3)$ $\forall$ r.

There is a contradiction to the fact that (x_1, y_1) and (x_2, y_2) intersect first time as (x_0, y_0).

■

Example

(i) $a u_x + b u_y = 0$

(ii) $-y u_x + x u_y = u$.

$u(x,0) = g(x)$ for $x > 0$

a) step 1 → $T_0 := \{ (s,0) : s \in \mathbb{R}^+ \}$.

$x_0(s) = s \quad y_0(s) = 0 \quad u(x_0(s), y_0(s)) = g(s)$.

b) step 2 → $\frac{dx}{dr} = -y \quad \frac{dy}{dr} = x$.

$x(0,s) = s \quad y(0,s) = 0$.

As a system of ODE,

$x = c_1 \cos r - c_2 \sin r$

$y = c_1 \sin r + c_2 \cos r$.

$s = c_1. \quad c_2 = 0. \quad \therefore \; x = s\cos r$
$y = s \sin r$.

$\therefore (x(r), y(r)) = (s\cos r, s \sin r)$.

c) Step 3 → $\frac{dz}{dr} = z$ where $z = u(x(r,s), y(r,s))$
$z(0,s) = g(s)$

$\therefore z(r,s) = g(s) e^{r}$.

d) Step 4 → write s & r in terms of x & y, inverse func^n th.,

$x^2 + y^2 = s^2$

$\frac{y}{x} = \operatorname{Tan} r$.

$\therefore u(x,y) = g(\sqrt{x^2+y^2}) \cdot e^{\arctan\left(\frac{y}{x}\right) + 2n\pi}$.

M. C. McOwen,

Lecture 6

1st order linear / semi linear PDEs

$\frac{dX}{dr} = a(X(r), Y(r))$.

$\frac{dY}{dr} = b(X(r), Y(r))$

$\frac{dZ}{dr} = c(X(r), Y(r), Z(r))$

$X(0,s) = X_0(s)$

$Y(0,s) = Y_0(s)$.

$Z(0,s) = u_0(s)$

Example (i) $u_x + 2u_y = u^2$

$u(x,0) = g(x)$.

Step 1: $\Gamma_0 = \{(s,0), s \in \mathbb{R}\}$.

$x_0(s) = s$, $y_0(s) = 0$, $\underbrace{u(x(s), y(s))}_{z(0,s)} = g(s)$

Step 2: $a(x(r), y(r)) = 1$

$b(x(r), y(r)) = 2$

$\therefore \frac{dx(r,s)}{dr} = 1$ $\qquad \frac{dy(r,s)}{dr} = 2$

$x_0(0,s) = s$ $\qquad y(0,s) = 0$.

$cx + d$

$x(r,s) = r + s$ $\qquad y(r,s) = 2r$.

$\therefore (x(r,s), y(r,s)) = (r+s, 2r)$.

Step 3:

$\frac{dz}{dr} = z^2$ $\quad \therefore \quad -\frac{1}{z} = r + \alpha$.

$= -\frac{1}{z} + c$

$z = \frac{-1}{r + \alpha}$

and $z(0,s) = u(s,0) = g(s)$.

$\therefore$

$\therefore -\frac{1}{\alpha} = g(s)$ $\quad \therefore \quad \alpha = \frac{-1}{g(s)}$.

$z = \frac{1}{\frac{1}{g(s)} - r}$ $\qquad 2r = y$

Step 4: $\qquad r = \frac{y}{2}$

$\therefore u(x,y) = \frac{1}{\frac{1}{g(x - \frac{y}{2})} - \frac{y}{2}}$ $\qquad \therefore s = x - \frac{y}{2}$

we want $u(x,y)$, a C^1 funcⁿ for it to be a classical solution.

Q. Can we solve a semi-linear PDE everytime? NO!

data / auxiliary condⁿ $\rightarrow$ ques.

We cannot just make $1 - \frac{y}{2} g(x - \frac{y}{2}) \neq 0$ as ~~it~~ it is not a sufficient cond^n for the $u(x,y)$ to be C^1.

$\therefore$ we need $1 - \frac{y}{2} g(x - \frac{y}{2}) > 0$. (or < 0)

QUASILINEAR PDE

$$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u).$$

- Method of characteristics

$$\left.\begin{aligned} \frac{dX}{dr} &= a(X(r,s), Y(r,s), Z(r,s)) \\ \frac{dY}{dr} &= b(X(r,s), Y(r,s), Z(r,s)) \\ \frac{dZ}{dr} &= c(X(r,s), Y(r,s), Z(r,s)) \end{aligned}\right\}$$

the ~~two~~ characteristic ODEs are not independent that is why we need take each eq^n into account.

Example (a) Burger Eq^n.

$$\begin{cases} u u_x + u_y = 0. \\ u(x,0) = g(x) \end{cases}$$

step 1: $\Gamma_0 = \{ (s,0),\ s \in \mathbb{R} \}$.

step 2: $\frac{dx}{dr} = Z$ $\qquad \frac{dy}{dr} = 1$ $\qquad \frac{dZ}{dr} = 0.$

this tells that $Z(r,s)$ will be const^n along characteristic curve.

$x_0(s) = s$ $\qquad y_0(s) = 0$ $\qquad z_0(s) = g(s).$

$\underline{Y(r,s) = r}$ $\qquad \therefore \underline{Z(r,s) = g(s)}.$

$\therefore \frac{dX}{dr} = g(s)$

$X = g(s) r + c.$

$g(s) \cdot 0 + c = s$

$\therefore c = s$

$\therefore \underline{X(r,s)} = r \cdot g(s) + s.$

$\therefore y = r$

$\boxed{x = y \cdot g(s) + s.}$

$\hookrightarrow$ characteristic curve.

choose $g(s) = s^2$.

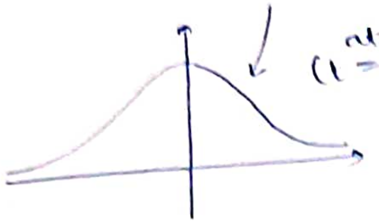
$x = ys^2 + s.$

☆ Note: for $s = 0$, $x = 0$; $s = 1$, $x = y + 1.$ $\}$ they are not parallel, and also intersect.

● any characteristic curves can intersect for quasi-linear eq^n.

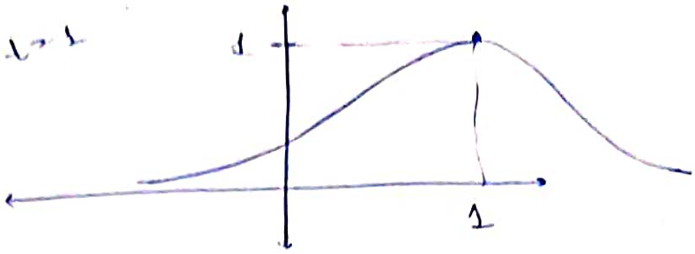
(b) $u_x + u_t = 0$ $(t == y)$. (Transport Eqⁿ)

$u(x,0) = \frac{1}{1+x^2}$



$(t=0)$

$t=1$



Lecture 7

$u u_x + u_y = 0$

$u(x,0) = g(x) = \frac{1}{1+x^2}$

[graphs for $u(x,0)$ & $u(x,0.5)$ .....]

[Shock wave in Burger Eqⁿ when the deformation is such that ∃ more than 1 solⁿ for 1 pt. for a char. curve.]

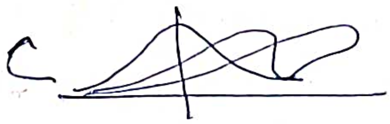
$\frac{dx}{dr} = z$ $\frac{dy}{dr} = 1$ $\frac{dz}{dr} = 0$. $z(r,s) = \frac{1}{1+s^2}$

$x_0(s) = s$ $y_0(s) = 0$ $z_0(s) = g(s) = \frac{1}{1+s^2}$ $x(r,s) = \frac{r}{1+s^2} + s$

$y = r$ $x = \frac{r}{1+s^2}$ $1+s^2 = \frac{y}{x} - 1$ $y(r,s) = r + \text{const.}$ → change 0.5.

$s = \pm\sqrt{\frac{y}{x} - 1}$.

$\therefore z(r,s) = u(x,y) = \frac{1}{y/x} = \frac{x}{y}$.



Burger Eqⁿ

Tangent vector (v)

$v \in \mathbb{R}^3$ is called a tangent vector to S at a point $a \in S$. If ∃ a curve $\gamma : t \mapsto \gamma(t) \in S$ such that $\gamma(0) = a$. $\gamma'(t) = v$.

Tangent space (T_a).

The set of all tangent vectors at a point a to the surface S is called the tangent space. It is a vector space of dim ≤ 2. It is a subset of $\mathbb{R}^3$.

Normal space (N_a)

The set of all vectors that are ⊥ar to tangent space T_a is called normal space.

for a $w \in N_a$, $\langle w, v \rangle = 0$ $\forall$ $v \in T_a$.

Dim of the normal space is 1.

$\Rightarrow S := \{ (x, y, z) : z = u(x,y) \}$ → surface

S is 2-dim space in $\mathbb{R}^3$

2 vectors in its basis.

Lemma

$(u_x, u_y, -1)$ is normal to the surface S.

proof:

$z = u(x,y)$

we know that

$a(x,y,u)\cdot u_x + b(x,y,u)\cdot u_y = c(x,y,u)$

$\therefore \underbrace{(a(x,y,u), b(x,y,u), c(x,y,u))}\cdot(u_x, u_y, -1) = 0$

$\frac{dx}{dr} = a \quad \frac{dy}{dr} = b \quad \frac{dz}{dr} = c$ and this (x,y,z) forms the surface solⁿ of the PDE. $\therefore (u_x, u_y, -1)$ is normal to the surface S.

Integral surface

Let $V: \mathbb{R}^3 \to \mathbb{R}^3$ (vector field). A 2-dim surface S is set to be an integral surface of V if $V(x)$ is a tangent vector of S at x $\forall$ x.

Example

~~$(x, y, -1)$~~

$S := x^2+y^2+z^2 = k^2$

$V = (-y, +x, 0)$.
that is
$V(x,y,z) = (-y, +x, 0)$.

check S is integral surface to V.

$S := \{(x, y, \sqrt{k^2-x^2-y^2})\}$ $\quad u(x,y) = \sqrt{k^2-x^2-y^2}$

Then define $(u_x, u_y, -1)$

$$\left(\frac{-2x}{\sqrt{k^2-x^2-y^2}}, \frac{-2y}{\sqrt{k^2-x^2-y^2}}, -1\right)$$

checking now that $(-y,+x,0)\cdot(u_x, u_y, -1)$.

$$(-y, +x, 0)\cdot\left(\underset{(u_x)}{\frac{-2x}{\sqrt{k^2-x^2-y^2}}}, \underset{(u_y)}{\frac{-2y}{\sqrt{k^2-x^2-y^2}}}, -1\right)$$

$\Rightarrow 0$ ✗

GEometrical interpretation of the solⁿ (Quasilinear PDE).

$a(x,y,u)\,u_x + b(x,y,u)\,u_y - c(x,y,u) = 0$

$(a(x,y,u), b(x,y,u), c(x,y,u))\cdot(u_x, u_y, -1) = 0$

If u solves the PDE, then the surface is given by

$$S := \{ (x, y, u(x,y)) \}$$

Now take $V = (a(x,y,u), b(x,y,u), c(x,y,u))$.
then S is an integral surface of the vector field V.

Example.

$$-y u_x + x u_y = 0.$$

~~$(-y, x) \cdot (u_x$~~ $\qquad (-y, x, 0) \cdot (u_x, u_y, -1) = 0$

Here the integral surface of $(-y, x, 0)$ is
$x^2 + y^2 + z^2 = K^2$ (from above example).
Then the solⁿ of this PDE is

$$z = u(x,y) = \pm\sqrt{K^2 - x^2 - y^2}$$

So, अभी तक 2 methods है,

① Method of characteristics
② Integral surface

} to find the solⁿ of a PDE.

Theorem

Consider the following PDE:

$$\left\{ \begin{array}{l} a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u) \text{ in } \Omega \\ u(\Gamma_0) = g \end{array} \right\}$$

Let a, b, c are C^1 funcⁿ from $\mathbb{R}^3 \to \mathbb{R}$ and $g \in C^1(\mathbb{R})$.
Let us assume, Transversality condition i.e.

$$0 \neq x_0'(s_0)\, b(x_0(s_0), y_0(s_0), g(s_0)) - y_0'(s_0)\, a(x_0(s_0), y_0(s_0), g(s_0))$$

Then $\exists$ a nbd Ω around $(x_0(s), y_0(s))$ where the above quasi linear PDE has a unique solⁿ.

[basically det(Jacobian) $\neq 0$].

proof: $\quad \Gamma_0 := \{ (x_0(s), y_0(s)) : s \in I \subseteq \mathbb{R} \}$.

system of PDE.

$$\left\{ \begin{array}{ll} \dfrac{dx}{dr} = a(x(r,s), y(r,s), z(r,s)), & x(0,s) = x_0(s) \\ \dfrac{dy}{dr} = b(x(r,s), y(r,s), z(r,s)), & y(0,s) = y_0(s) \\ \dfrac{dz}{dr} = c(x(r,s), y(r,s), z(r,s)), & z(0,s) = g(s) \end{array} \right.$$

By existence & uniqueness of ODE, $\exists$ a ~~nbd~~ of that $(x(r,s), y(r,s), z(r,s))$ is the unique sol^n of the above ODE in the nbd of $(-\delta, \delta) \times (s-\varepsilon, s+\varepsilon)$

and $A(r,s) = \left((x(r,s), y(r,s), z(r,s))\right) \in C'$ ↳ [this is due to cont^n dependence of s and we want $g \in C'(\mathbb{R})$ even if it does not play any part in getting a sol^n of ODE.]

Define a map.

$$B(r,s) = (x(r,s), y(r,s)).$$

$$B: (\delta,\delta) \times (s-\varepsilon, s+\varepsilon) \longrightarrow \mathbb{R}^2 \text{ is } C'.$$

~~now using inverse~~ Now, Jacobian of B

$$\det(J_B) = \det \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{ds} \\ \frac{dy}{dr} & \frac{dy}{ds} \end{pmatrix} (0,s) \neq 0.$$

|||

Transversality cond^n

$\therefore$ By using inverse func^n theorem, $\exists$ a nbd $\tilde{\Omega}$ around $(x_0(s_0), y_0(s_0))$ and v around $(0, s_0)$ s.t. $B: V \longrightarrow \tilde{\Omega}$ is one-one, onto and inverse exists and C^1.

$$B^{-1} = (r^{-1}(x,y), s^{-1}(x,y)).$$

* So, basically, linear PDEs $\Rightarrow$ characteristic lines are parallel otherwise non-linear $\Rightarrow$, they ~~are~~ can be anything. → 1st ORDER.

Initial pt → gives curve ✓
diff initial pt → diff orbit → never insects
∴ it should be autonomous ODE

Lecture 8

For transport eq^n, the characteristic eq^n lines are always parallel. But for burger's eq^n the lines have different slopes ∴ the burger's eq^n graph gets deformed by changing the auxiliary cond^n. as the small y are not changed and high value of y gets deformed easily. On the other hand, transport eq^n graph transports across y-axis.

General 1st order PDE

Fully Non-Linear PDE ⟹

now, it cannot be segragated.

$$F(x, y, u, u_x, u_y) = 0$$

Let us consider

$$u: \mathbb{R}^n \to \mathbb{R} \text{ and } X = (x_1, \ldots, x_n)$$

$\therefore\ F(X, u, \nabla u) = 0$

$F: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}.$

we denote the 'second' variable as z, and 'third' variable as P.

~~we done~~ $\therefore\ u = z$

$\nabla u = P$

$\therefore\ F(X, Z, P) = 0.$

Example

$F(X, u, \nabla u) = 0$ in Ω

$u(X) = g$ in $\Gamma_0 \subseteq \Omega$

(i) $u_{x_1} u_{x_2} + 3x_1^2 (u_{x_1})^2 = 0.$

$F(X, Z, P) = P_1 P_2 + 3x_1^2 P_1^2 = 0.$

* characteristic eq^n:-

Assume $u \in C^2$ and take ∂x_i derivative to eq^n.

by chain rule, $F_{x_i}(X, z, P) + F_z(X, z, P) \cdot u_{x_i} + \sum_{j=1}^{n} \frac{\partial F}{\partial P_j}(X, z, P) P_{j x_i} = 0.$ (1)

$P = (P_1, P_2 \ldots P_n)$; $P(r, s) = (P_1(r, s), P_2(r, s) \ldots P_n(r, s))$

$$\frac{\partial}{\partial r} P_i(r, s) = \frac{\partial}{\partial r} u_{x_i}(r, s)$$

$$\boxed{\frac{\partial P_i(r, s)}{\partial r} = \sum_{j=1}^{n} u_{x_j x_i}(r, s) \cdot \frac{\partial x_j}{\partial r}}$$

take, $\boxed{\frac{\partial x_j}{\partial r} = -\frac{\partial F}{\partial P_j}(X, z, P) - \frac{\partial F}{\partial x_j}(X, z, P)}$ (from (1))

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(u(X(r))).$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(X(r)) \cdot \frac{\partial x_j}{\partial r}$$

$$= \sum_{j=1}^{n} P_j(X(r)) \cdot \frac{\partial F}{\partial P_j}(X, z, P)$$

Compact form of characteristic ODE :-

$$\frac{\partial X}{\partial r} = \nabla_p F(X(r), Z(r); P(r)).$$

$$\frac{\partial Z}{\partial r} = \nabla_p F(X(r), Z(r), P(r)) \cdot P(r).$$

$$\frac{\partial P}{\partial r} = -\nabla_x F(X(r), Z(r), P(r)) - \frac{\partial F}{\partial Z}(X(r), Z(r), P(r)) \cdot P.$$

Remark: we need to solve $(2n+1)$ ODE, but via auxiliary condn, we can have $\underline{n+1 \text{ initial data}}$ for ODEs. We need to $\underline{(n) \text{ more initial data}}$ to solve the $\underline{2n+1 \text{ ODES}}$ completely. We need to predict these P_is at initial time for $r=0$ by using the given eqn & auxiliary data at $r=0$. ∴ the soln to fully non-linear PDES are $\underline{\text{not}}$ $\underline{\text{unique}}$ always. [(n+1) initial condn → parametrize Γ_0 & write auxiliary condn]

Example (i)
$$\begin{cases} u_x u_y = u & \text{in } \Omega := \{(x,y) \mid x>0\}. \\ u(0,y) = y^2 \end{cases}$$

$F(x, z, p) = p_1 p_2 - z$. $\quad P = (P_1, P_2)$

Parametrise $\Gamma_0 = \{(0, s) : s \in \mathbb{R}\}$.

$Z(0,s) = s^2$.

$$\left(\frac{dx}{dr}, \frac{dy}{dr}\right) = \frac{dX}{dr} = \nabla_p(F(X,Z,P)) = (P_2, P_1). \quad \left.\begin{matrix} x(0,s) = 0 \\ y(0,s) = s \\ z(0,s) = s^2 \end{matrix}\right] \text{①}$$

$$\frac{dZ}{dr} = \nabla_p F(X,Z,P) \cdot P \Rightarrow (P_2, P_1) \cdot (P_1, P_2) = 2P_1P_2.$$

$$\left(\frac{dP_1}{dr}, \frac{dP_2}{dr}\right) = \frac{dP}{dr} = -\underbrace{\nabla_x F(X,Z,P)}_{0} - \frac{\partial F}{\partial Z}(X,Z,P) \cdot P$$
$$-(-1)(P_1, P_2) \Rightarrow$$

$$\left(\frac{dP_1}{dr}, \frac{dP_2}{dr}\right) = +(P_1, P_2)$$

continue

From auxiliary condn $\underline{u_y(0,y)} = 2y$.

$u_x(0,s)\, u_y(0,s) = u(0,s)$ $\quad P_2(0,s) = 2s.$ ← with ①

$2u_x(0,s)\, s = s^2$ $\quad$ ∴ $P_1(0,s) = s/2$

Tut 1

(1) (i) Semilinear
(ii) ~~Semilinear~~ linear
(iii) linear
(iv) Quasilinear
(v) fully non-linear
(vi) $u_t + \underbrace{\sup_\alpha \left\{ \frac{1}{2}|\alpha|^2 - \cancel{\alpha u_x} \alpha u_x \right\}}_{\text{how?}} \to \frac{1}{2}(u_x)^2$ → Hamilton Jacobi Bellmann eq^n.
fully ✓ non-linear
(vii) Fully non-linear

$\alpha \to$ control parameter.

(2) (i) $u_y = 0$ in $\mathbb{R}^2$

$\boxed{x_0(s) = s}$

$u(x, 0) = x^2$

$u_y = c$ and $u(x,0) = c = x^2$

$\therefore u(x,y) = x^2$.

(ii) $u_y = 0$ in $\mathbb{R}^2$

$u(x,y) = c_{,...}$, $u(0,y) = \cancel{c}$ const. $\to y^2$

$u(0,y) = y^2 = c$. which is contradictory as.

$u(0,y) = y^2$, $x = 0$

here, auxiliary cond^n is not matching with the PDE given. only at $(x=0, y=0)$ this matches.

(Transversality cond^n is not being followed here).

* Auxiliary data is not fitting for any line.
* even if it does, we cannot use this data to get the answer we need. $f(x) + c = y^2$ Cos

---

(iii) $au_x + bu_y = 0$ in $\mathbb{R}$ $a, b \neq 0$.

parallel to $(0,0)$ to (a,b) lines.

$bx - ay = $ constant.

↳ initial data on this then we get same problem as in (ii).

* $\Rightarrow$ [characteristic line should not be equal to auxiliary line]

(3)

$$\left.\begin{aligned} xu_x + u_y &= 0 \\ u(x,0) &= \frac{1}{x^2+1} \end{aligned}\right\}$$

$\frac{du}{dx} = a(x,y)$ and $\frac{du}{dy} = b(x,y) = 1$

$\frac{du}{dx} = x$

$u = \frac{x^2}{2} + d.$ $\qquad u = y + c$

$\therefore u(x,y) = (\frac{x^2}{2} + y) + c.$

$u(x,0) = \frac{x^2}{2} + c = \frac{1}{x^2+1} - \frac{x^2}{2}.$

$u(x,y) = \frac{x^2}{2} + \frac{1}{x^2+1} - \frac{x^2}{2} + y$

Do it in copy!

$= \frac{1}{x^2+1} + y.$

☆ Visualize for the origin (sink).

(4) $\Gamma_0 = \{x^2+y^2 = 1\}$ $\qquad (\cos\theta, \sin\theta)$ $\qquad x = \cos\theta,\ y = \sin\theta$ $= 1$

$\therefore g(\theta) = u(\cos\theta, \sin\theta) = 1.$ $\qquad u = 1$

$\frac{dX}{dr} = x$ $\qquad \frac{dY}{dr} = y$ $\qquad \frac{dZ}{dr} = 2Z$

$X_0(\theta) = \cos\theta$ $\qquad Y_0(\theta) = \sin\theta$ $\qquad Z_0(s) = 1$

$x = ce^r$ $\qquad y = \sin\theta\, e^r$ $\qquad Z(r,s) = e^{2r}$

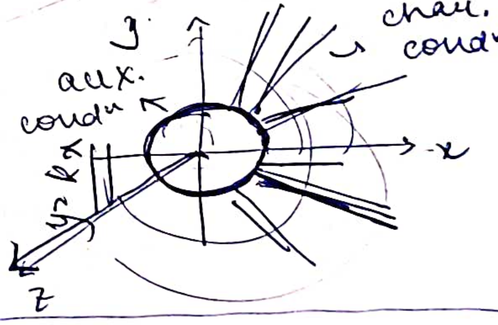
$x = \cos\theta\, e^r$ $\qquad 0 < \theta < 2\pi$

$x^2 + y^2 = e^{2r}$

$Z(r,s) = u(x,y) = x^2 + y^2$

characteristic line → $x^2 + y^2 > 1$ → solved on this.

☆ PDE $x^2+y^2>1$



continued here →

Initial condn

$x(0,s) = 0$ $\qquad p_1(0,s) = \frac{s}{2}$

$y(0,s) = s$ $\qquad p_2(0,s) = 2s$

$z(0,s) = s^2$

$\frac{dx}{dr} = p_2$ $\qquad \frac{dy}{dr} = p_1$ $\qquad \frac{dz}{dr} = 2p_1p_2$

$u(x,y) = \frac{(x+4y)^2}{16}$

$p_1 = ce^r$

$p_1 = \frac{s}{2}e^r$ $\qquad p_2 = 2se^r$

$Z = s^2 e^{2r}.$

$\frac{dp_1}{dr} = p_1$ $\qquad \frac{dp_2}{dr} = p_2$

Lecture 9

* Conservation of mass : Transport Eq^n

General form of transport eq^n. ($\mathbb{R}^n$)

$$\left.\begin{array}{l} u_t + a(x)\nabla u = 0 \\ u(x,0) = g(x). \end{array}\right| \begin{array}{l} x \in \mathbb{R}^n, \quad t > 0 \\ \therefore u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \end{array}$$

condition : (i) $a \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and bounded

(ii) $\operatorname{div}(a(x)) = 0. \quad \forall\ x \in \mathbb{R}^n$

$$\Downarrow$$

$$\frac{\partial a_i(x)}{\partial x_i} \ ; \ a(x) = (a_1, \dots a_n).$$

$\boxed{\text{Theorem}}$:

If $u(x,t)$ is a solution of the above PDE and g is ~~completely~~ compactly supported, then,

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} g(x)\,dx. \quad \forall\ t > 0. \quad \text{—(1).}$$

proof:

To prove: $\dfrac{d}{dt}\left(\displaystyle\int_{\mathbb{R}^n} u(x,t)\,dx\right) = 0.$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} u(t,x)\,dt\right) = \int_{\mathbb{R}^n} \frac{\partial u(t,x)}{\partial t}\,dx$$

why? b/oz $a(x)$ is C^1 and g is compactly supported

$u(x,t)$ is also compactly supported for any fixed time $t > 0$.

$g(cx - t)$ $\quad [c \in \mathbb{R}, \ n = 1$

we know that g is compactly supported; $\Rightarrow g \equiv 0$ on $|x| > R_0$.

solution travels along the char. curve.

By (1), a is bounded, consider $\sup |a(x)| < A$.

for a fixed time $\exists$ a compact domain $B(0, R_0 + tA]$ s.t $u(x,t) \equiv 0$ on $B(0, R_0 + tA)^c$ ★ [cone of interference]

$$\int_{\mathbb{R}^n} \frac{d}{dt} u(x, t)\, dX = \int_{B(0, R_0+tA)} \frac{d}{dt} u(X, t)\, dX.$$

$$= -\int_{B(0, R_0+tA)} a(X) \cdot \nabla u(X, t)\, dX.$$

(we know) $\downarrow$ integration by parts.

$$\int_{B(0, R_0+tA)} \cancel{\operatorname{div}(a(x))}^{\,0} \cdot u(x)\, dx - \int_{\partial B(0, R_0+tA)} a(x) \cdot \nabla \cancel{u} \cdot \nu\, d\sigma$$

ν → outward normal vector

on boundary $u(x, y) = 0$.

$\therefore$ $(= 0.)$

$$\therefore \int_{\mathbb{R}^n} u(x, t)\, dX = \int_{\mathbb{R}^n} g(x)\, dX$$

(constant w.r.to t.)

Lecture 10

2^{nd} order PDE

* Linear 2^{nd} order PDE

⇒ General 2^{nd} order linear PDE →

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u(x) = f(x)$$

$u : \mathbb{R}^n \to \mathbb{R}$; a_{ij}, b, c, f are "smooth" func^n

⇒ Classification:

1. Elliptical 2^{nd} order PDE ($\Delta u = f$ [laplace])
2. Parabolic 2^{nd} order PDE ([Heat Eq]).
3. Hyperbolic 2^{nd} order PDE. ([Wave Eq.])

Restrict to $n=2$. $u(x,y)$.

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}+fu+g=0$$

consider the highest order tnt in the eqn and we write $L(\xi,\eta)$ as,

$$① - a\xi^2+b\xi\eta+c\eta^2 \rightarrow$$ observe which conic section it represents.

Eq ① $\rightarrow$ elliptic $b^2-4ac<0$

$\rightarrow$ parabolic $b^2-4ac=0$

$\rightarrow$ hyperbolic $b^2-4ac>0$.

Remark The names has nothing to do with behaviour of the soln.

Example

① wave Eqn. $\rightarrow$ hyperbolic

$$u_{tt}-\Delta u=0 \quad ; \quad \overset{(n=2)}{u_{tt}-u_{xx}=0}.$$

$$b^2-4ac \rightarrow 0-4(-1)\cdot 1 \rightarrow 4>0.$$

② laplace eqn $\rightarrow$ elliptic

$$u_{xx}+u_{yy}=0$$
$$b^2-4ac \rightarrow -4<0$$

③ heat eqn $\rightarrow$ parabolic.

$$u_t-u_{xx}=0$$
$$b^2-4ac \Rightarrow 0-0 \Rightarrow 0.$$

- wave, heat, laplace eqn are called canonical forms for 2nd order linear PDEs
- For general dimn n, we call laplace eqn, heat & wave eqn as elliptic, parabolic and hyperbolic resp.

Use a transformation → choose 2 new variables ξ & η:

$\xi := \xi(x,y) \quad \eta = \eta(x,y)$

$w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$

also, $u(x,y) = w(\xi(x,y), \eta(x,y))$.

$$u_x = w_\xi \frac{\partial \xi}{\partial x} + w_\eta \frac{\partial \eta}{\partial x} = w_\xi \xi_x + w_\eta \eta_x$$

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{xx} = w_{\xi\xi}(\xi_x)^2 + w_\xi \xi_{xx} + w_{\xi\eta} \xi_x \eta_x + w_{\xi\eta} \xi_x \eta_x + w_{\eta\eta}(\eta_x)^2 + w_\eta \eta_{xx}.$$

$$u_{yy} = w_{\xi\xi}(\xi_y)^2 + 2 w_{\eta\xi} \xi_y \eta_y + w_\xi \xi_{yy} + w_{\eta\eta}(\eta_y)^2 + w_\eta \eta_{yy}.$$

$$u_{xy} = w_{\xi\xi} \xi_x \xi_y + w_{\xi\eta}(\xi_x \eta_y + \xi_y \eta_x) + w_{\eta\eta} \eta_x \eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}$$

$$u_{yx} = w_{\xi\xi} \xi_x \xi_y + w_{\xi\eta}[\xi_x \eta_y + \xi_y \eta_x] + w_{\eta\eta} \eta_x \eta_y + w_\xi \xi_{yx} + w_\eta \eta_{yx}.$$

and put them in eq^n.

$$A w_{\xi\xi} + B w_{\eta\eta} + C w_{\eta\xi} = f(w, \xi, \eta, w_\xi, w_\eta).$$

then, $A = a \xi_x^2 + b \xi_x \xi_y + c \xi_y^2$

$C = 2 \xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + 2 \xi_y \eta_y$

$B = a \eta_x^2 + b \eta_x \eta_y + c \eta_y^2$

• Hyperbolic $(b^2 - 4ac > 0)$

transform to canonical form.

$(u_{tt} - u_{xx} = 0)$. / $u_{tx} = 0$.

transform (particular)

* find η, ξ ; →

we need

$$A = a\xi_x^2 + 2b\,\xi_x \xi_y + c\,\xi_y^2 = 0$$

$$C = a\eta_y^2 + 2b\,\eta_x \eta_y + c\,\eta_y^2 = 0.$$

$$B = 2\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + 2\xi_y \eta_y$$

fully non-linear 1st order PDE. $= 1$

we can write it as,

$$0 = \left(2a\xi_x + (b - \sqrt{b^2 - 4ac})\,\xi_y\right) \cdot \left(2a\xi_x - (b - \sqrt{b^2 - 4ac})\,\xi_y\right)$$

Lecture 11

• Parabolic $(b^2 - 4ac = 0)$.

transform to canonical form.

It deals to ↓

$$(2a\xi_x + b\xi_y)^2 = 0.$$

we don't have a separate eq^n to find η. we find ξ by solving $2a\xi_x + b\xi_y = 0$. or by finding the char. eq^n

$$\frac{dy}{dx} = \frac{b}{2a}$$ → find the family of curves $f(x,y) = c$.

choose $\xi(x,y) = f(x,y)$

To find $\eta(x,y) = x$ → find J (check it is non singular ?).

- Elliptical $(b^2-4ac<0)$.

canonical form. $u_{xx}+u_{tt}=0$

$B=0$ and $A=C$.

$\Rightarrow a\xi_x^2+2b\xi_x\xi_y+c\xi_y^2=a\eta_x^2+2b\eta_x\eta_y+c\eta_y^2$ — ①

$\Rightarrow 2a\xi_x\eta_x+b(\xi_x\eta_y+\xi_y\eta_x)+2c\xi_y\eta_y=0$ — ②

consider ① $+i$②. → solve this complex system of eqⁿ.

$\Rightarrow a(\xi_x^2-2i\xi_x\eta_x-\eta_x^2)+c(\xi_y^2-2i\xi_y\eta_y-\eta_y^2)$.

$+b[2\xi_x\xi_y-i\xi_x\eta_y-i\xi_y\eta_x-2\eta_x\eta_y]=0$.

choose $\phi(x,y)=\xi(x,y)+i\eta(x,y)$

then,

$\Rightarrow a\phi_x^2+c\phi_y^2+2b\phi_x\phi_y=0$.

For elliptical case, we consider, a,b,c to be real analytic functions,

the we again write,

$\Rightarrow (2a\phi_x+(b+\sqrt{b^2-4ac})\phi_y)(2a\phi_x+(b-\sqrt{b^2-4ac})\phi_y)=0$

then using similar arguement as hyperbolic case, we get

$$\frac{dy}{dx}=\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

Remark

the solⁿ for $2a\phi_x+(b+\sqrt{b^2-4ac})\phi_y=0$ is ϕ and solⁿ to $2a\phi_x+(b-\sqrt{b^2-4ac})\phi_y=0$ is complex conjugate to ϕ.

$\therefore$ we consider $\xi=\mathrm{Re}(\phi)$ and $\eta=\mathrm{Im}(\phi)$.

Then, this leads to the canonical eqⁿ.

Lecture 12

Laplace Equation

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \qquad \left(\begin{array}{l} n=2 \\ \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \end{array} \right)$$

classification of linear 2nd order PDE

a) hyperbolic $(u_{tt} - \Delta u = f)$

b) parabolic $(u_t - \Delta u = f)$

c) elliptical $(-\Delta u = f)$.

- Regularization property → u is smoother than f
- Maximum Principle
- Mean Value Property

elliptical eq^n

$n=1$ → $\mathbb{R}$

$u_{xx} = f(x) \geq 0 \quad (I \subseteq \mathbb{R})$ → this says that u is convex.

↓

$\boxed{\Delta u \geq 0}$ — these type of func^n has maximum only at boundary

MVP → $\boxed{\Delta u = 0}$ iff

$$u(x) = \int_{B(x,r)} u(y)\, dy \qquad \forall \boxed{x \in \Omega} \text{ and } r > 0.$$

$B(x,r)$ ↓ Ball centred at x with radius r

this leads to regular-ization prop.

Now, for laplace eq^n $\Delta u = 0$

BVP → Boundary value problem are of 2 types:

a) Dirichlet Boundary cond^n.

Poisson Eq^n
$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega \end{cases}$$
$\partial\Omega$ ↳ boundary

b) Neumann Boundary cond^n
$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g & \text{in } \partial\Omega \end{cases}$$

where, $\frac{\partial u}{\partial \nu} \to$ dir^n derivative towards outer nor...

⇒ Harmonic function

A func^n ~~[illegible]~~ $u \in C^2(\Omega)$ is said to be harmonic in Ω if
$$\Delta u = 0 \text{ in } \Omega$$

example
(i) u = constant
(ii) $u(x) = ax + b$. → we get trouble when we assign a boundary cond^n ∴ not good sol^n

∴ it doesn't have no max. or min. property (values).

[surface of harmonic func^n has 0 convexity]
↓
★★ Read online (of $\Delta u = 0$)

⇒ A specific harmonic function
(Fundamental solution)

⇒ सह तो पता है ऐसा ही dir^n में चलेगा, तो 2nd order derivative का ...

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n$$

Laplace eq^n are invariant under rotation and translation ...

any orthonormal matrices (in $\mathbb{R}^2$) can be written as $O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

[Rotation ⇒ mult. with orthogonal/orthonormal matrices with vector]

Define $x' = Ox$ s.t. $\boxed{OO^T = I}$.
Define $w(x') = u(Ox)$.
then $\boxed{\Delta u(x) = \Delta_{x'} w}$

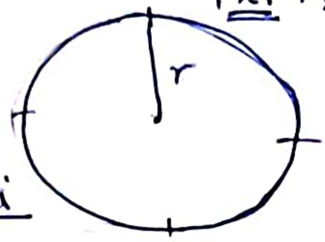
Exercise*.

$w(x') = u(Ox)$

proof:

This suggests to look for sol^n of the form

$|x|$ invaria



$$v(x) = u(|x|) = u(r)$$

take $r = |x| = \sqrt{x_1^2 + x_2^2 - - x_n^2}$

$$\frac{\partial v}{\partial x_i} = u'(r)\frac{\partial r}{\partial x_i} \to u'(r)\cdot\frac{\cancel{2}x_i}{\cancel{2}\sqrt{x_1^2+x_2^2--x_n^2}} \to \frac{u'(r)\,x_i}{r}$$

$$\frac{\partial^2 v}{\partial x_i^2} = u''(r)\frac{x_i^2}{r^2} + \frac{u'(r)}{r} + u'(r)\,x_i\left(\frac{-1}{r^2}\right)\cdot\frac{x_i}{r}.$$

$$= u''(r)\frac{x_i^2}{r^2} + \frac{u'(r)}{r} - \frac{u'(r)\,x_i^2}{r^3}$$

$$\frac{\partial^2 v}{\partial x_i^2} = u''(r)\frac{x_i^2}{r^2} + u'(r)\left[\frac{1}{r} - \frac{x_i^2}{r^3}\right]$$

$$\Delta v = \sum_{i=1}^{n}\frac{\partial^2 v}{\partial x_i^2} = \frac{u''(r)\,\cancel{[x_1^2+--x_n^2]}\ \cancel{r^2}}{\cancel{r^2}} + u'(r)\left[\frac{n}{r} - \frac{\overset{r^2}{\cancel{[x_1^2+x_2^2--x_n^2]}}}{r^3}\right]$$

$$\boxed{\Delta v = u''(r) + \frac{u'(r)\,[n-1]}{r}}$$

where $r = \sqrt{x_1^2+x_2^2+ ---x_n^2}$.

$$\therefore\ \Delta v = 0 \iff \underbrace{u''(r) + \frac{u'(r)\cdot(n-1)}{r}}_{} = 0.$$

Assumption 1:- assume $u'(r) \neq 0$.

$$u''(r) + \frac{u'(r)\,[n-1]}{r} = 0$$

$$\frac{u''(r)}{u'(r)} = \frac{-(n-1)}{r}$$

(ofc) assume $(r>0)$

$$[\log|u'|]' = \frac{(1-n)}{r}$$

$$\log|u'| = (1-n)\log r + c$$

$$\log|u'| = \log c_1 r^{(1-n)}.$$

$\therefore |u'| = c_1 r^{(1-n)}$

integrate again.

$$u(r) = \begin{cases} A \log r & n=2 \\ B \dfrac{1}{r^{n-2}} & n \geq 3 \end{cases}$$

For laplace eq^n $(n \geq 2)$ always, 1 के लिए हम देखेंगे नहीं

$n \geq 2$

$(n=1)$

~~$u(x) = u(r)$ and $\Delta u = 0$~~

we know that $v(x) = u(r)$ and $\Delta v = 0$ in $\mathbb{R}^n \setminus \{0\}$.

Fundamental solution : $\rightarrow$

Define $u(x) = \begin{cases} \dfrac{1}{2\pi} \log|x| & n=2 \\ \dfrac{1}{(n-2)\omega_n} \dfrac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$

Basically जो मैंने $v(x)$ लिखा है वो ।.

where $\omega_n \rightarrow$ surface area of unit sphere in $\mathbb{R}^n$.

$\rightarrow$ we get harmonic sol^n not defined on whole domain but on some of it.

Lecture 13

$$\Phi(x) = \begin{cases} \dfrac{1}{2\pi} \log|x| & n=2 \\ \dfrac{1}{(n-2)\,\omega_n} \dfrac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

$\Delta \Phi(x) = 0$ in $\mathbb{R}^n \setminus \{0\}$.

$\Delta \Phi(x-y) = 0$ in $\mathbb{R}^n \setminus \{y\}$

Poisson Equation

$-\Delta u = f$ in $\mathbb{R}^n$ just for now (without auxiliary cond^n)

regular is just for smooth func^n [same prop passes on]

find a candidate of sol^n?

$\hookrightarrow$ guess? — do it :

$$\rightarrow u(x) := \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

convolution operator $\rightarrow$ $\Phi * f(x)$.

1

[If f is bound & g is integrable then $f*g$ can make sense.

• If one of the func^n is "regular", $f*g$ is also regular]

• Differentiation & convolutions commutes.

$g \in C^1 \rightarrow \partial_i(f * g)(x) = f * \partial_i g(x)$

$f \in C^1 \rightarrow \partial_i(f * g)(x) = \partial_i f(x) * g.$

regular / smooth / C^∞ / nice

[vision: If f is defined as C^2 then u हमारा C_0^2 आएगा]

basically $\Phi = \delta_0$ (Dirac distribution)

$$\delta_0 \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Theorem:— Assume $f \in C_c^2(\mathbb{R}^n)$ (diff upto order 2 with compact set on $\mathbb{R}^n$) and consider as in ①.

then
(i) $u \in C^2(\mathbb{R}^n)$
(ii) $-\Delta u = f$ in $\mathbb{R}^n$.

$\boxed{-\Delta u = f}$

proof:

$$u(x) = \int_{\mathbb{R}^n} f(x-y)\,\Phi(y)\,dy$$

$$= \int_{\mathbb{R}^n} \Phi(x-y)\,f(y)\,dy$$

$$\Delta u = \int_{\mathbb{R}^n} \Delta\Phi(x-y)\,f(y)\,dy.$$

$\mathbb{R}^n = \mathbb{R}^n \setminus \{x\}$ → singelton चीज़ निकालने से कुछ नहीं हुआ.

$$\Delta u = 0$$

Δ दोनों अंदर ले गया इसलिए wrong है। (interchange of integration & diff doesn't make sense here).

for (i) To prove: $\frac{\partial^2 u}{\partial x_i \partial x_j} \in C(\mathbb{R}^n)$ (contn.)

proof existence of partial derivatives of u

To prove: $\lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h}$ exists

$$\frac{1}{h}\left[\int_{\mathbb{R}^n} \Phi(y)\left[f(x-y+he_i) - f(x-y)\right]dy\right.$$

$$\Rightarrow \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x-y+he_i) - f(x-y)}{h} \right] dy$$

$$\therefore \lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h} = \lim_{h\to 0} \int_{\mathbb{R}^n} \Phi(y) \left[\frac{f(x-y+he_i) - f(x-y)}{h} \right] dy$$

To show: Lim and $\int$ are interchangeable

f has compact support,

$\rightarrow$ f is bounded.

$\Rightarrow$ Df and $D^2 f$ are also bounded.

(As $\partial_i f$ and $\partial_i \partial_j f$ also have same compact support).

we show that $\frac{f(x+he_i-y) - f(x-y)}{h} \xrightarrow{h\to 0} \partial_i f(x-y)$.

$$\left| \frac{f(x+he_i-y) - f(x-y)}{h} - \partial_i f(x-y) \right|$$

$$= \cancel{\tfrac{1}{h}} \left| \frac{1}{h} \int_0^1 \frac{d}{dt} f(x-y+the_i)\,dt - \partial_i f(x-y) \right|$$

$$= \left| \frac{1}{\cancel{h}} \int_0^1 \nabla f(x-y+the_i) \cdot \cancel{h} e_i \,dt - \partial_i f(x-y) \right|$$

$$= \left| \int_0^1 \partial x_i f(x-y+the_i)\,dt - \partial x_i f(x-y) \right|$$

$$= \left| \int_0^1 [\partial x_i f(x-y+the_i) - \partial x_i f(x-y)]\,dt \right|$$

by mean-value prop,

$$\leq \int_0^1 \left| \partial x_i x_i f(x-y+\xi_{t,h,i}) \right| th\,dt$$

$$\leq \sup_{x\in\mathbb{R}^n} \left| D^2 f(x) \right| h \int_0^1 t\,dt$$

$$\leq h\, C_f$$

$\hookrightarrow$ independent of $x, y \in \mathbb{R}^n$, we find.

$$\frac{f(x+he_i-y) - f(x,y)}{h} \longrightarrow \partial x_i f(x-y)$$

(uniformly on $\mathbb{R}^n$).

$\therefore$ lim & $\int$ are interchangle

As we have uniform convergence, then

$$= \int_{\mathbb{R}^n} \Phi(y)\, \partial x_i f(x-y)\,dy.$$

Note that, $\int_{\mathbb{R}^n} \Phi(y)\, \partial_{x_i} f(x-y)\, dy$

$$= \int_K \Phi(y)\, \partial_{x_i} f(x-y)\, dy.$$

And we note that Φ is locally integrable func^n near 0.

$$\int_{B(0,1)} \Phi(x)\, dx \to \text{exists and } < \infty.$$

change to polar coordinates and use $\int_0^{\varepsilon} \frac{dx}{x^{\alpha}} < \infty$ if $\alpha < 1$.

hence, $\partial_{x_i} u$ is well-defined

similarly, we can also prove that $u \in C^2(\mathbb{R}^n)$.

Lecture 14.

PDE quiz Next Tuesday
check 6/28 quiz ↳ class notes (wed)

for (ii).

$$\Delta u(x) = \int_{\mathbb{R}^n} \Phi(y)\, \Delta_x f(x-y)\, dy$$

$$= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy + \int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy$$

Let $I_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy$. and $J_\varepsilon = \int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy$

↓

It's easy to show. $\Delta_x f(x-y) = \Delta_y f(x-y)$.

$\int AB$

$\therefore I_\varepsilon \Rightarrow \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \Delta_y f(x-y)\, dy$..

Integration by parts →

$$\Rightarrow \sum_{i=1}^{n} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\, \frac{\partial^2 f(x-y)}{\partial y_i^2}\, dy.$$

$$\Rightarrow \sum_{i=1}^{n} \left[- \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial y_i}\, \frac{\partial f(x-y)}{\partial y_i}\, dy \right.$$

$$\left. + \int_{\partial B(0,\varepsilon)} \Phi(y)\, \frac{\partial f(x-y)}{\partial y_i}\, \nu_i\, dS(y) \right]$$

Formula Integration by parts: $\int_\Omega f \frac{\partial g}{\partial x_i} dx = -\int_\Omega \frac{\partial f}{\partial x_i} g \, dx + \int_{\partial\Omega} f g \, \nu_i \, dS(x)$ ** New

ν_i // outward unit normal vector.

$$\Rightarrow \sum_{i=1}^{n} \left[-\int_{\mathbb{R}^n / B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial y_i} \cdot \frac{\partial f(x-y)}{\partial y_i} dy \right] + \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu} dS(y).$$

integration by parts →

dir^n der^v.

$$\Rightarrow \sum_{i=1}^{n} \left[\int_{\mathbb{R}^n / B(0,\varepsilon)} \frac{\partial^2 \Phi(y)}{\partial y_i^2} f(x-y) dy - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial y_i} \cdot f(x-y) \nu_i \, dS(y) \right]$$

$$+ \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu} dS(y).$$

$$\Rightarrow \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y \Phi(y) f(x-y) dy - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) \cdot f(x-y) \, dS(y)$$

harmonic func^n ∴ $(=0)$
as $\Phi(y)$ is smooth away from the ball

$$+ \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) \, dS(y)$$

$$\Rightarrow K_\varepsilon + L_\varepsilon$$

let $K_\varepsilon = -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) \cdot f(x-y) dS(y)$

$L_\varepsilon = \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) \, dS(y).$

$\therefore |L_\varepsilon| \Rightarrow \left| \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) dS(y). \right|$

as $f \in C^2_c(\mathbb{R}^n)$ ∴ 1st & 2nd derivative is bounded

$\Rightarrow \sup_{x \in \mathbb{R}^n} |Df(x)| \leq K$

$$\therefore |L_\varepsilon| \leq \overbrace{\sup_{x \in \mathbb{R}^n} |Df(x)|}^{\|Df\|_{L^\infty}} \int_{\partial B(0,\varepsilon)} \Phi(y) dS(y)$$

Again on $\partial B(0,\varepsilon) \rightarrow$

Lendi saala

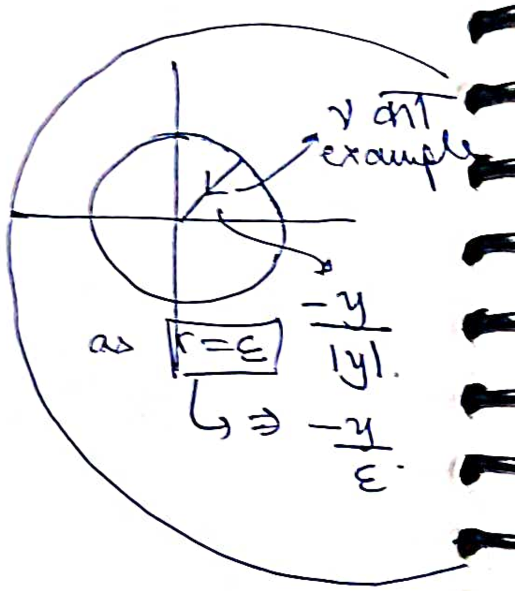
$$\Phi(y) \begin{cases} \frac{1}{2\pi}\log|\varepsilon| , & n=2 \\ \frac{1}{(n-2)w_n}\frac{1}{\varepsilon^{n-2}} , & n\geq 3 \end{cases}$$

$$|L_\varepsilon| \leq \|Df\|_{L^\infty(\mathbb{R}^n)} . \begin{cases} \frac{1}{2\pi}\log|\varepsilon| & n=2 \\ \frac{1}{(n-2)w_n}\frac{1}{\varepsilon^{n-2}} & n\geq 3 \end{cases} \int_{\partial B(0,\varepsilon)} dS(y)$$

int. of surface of radius ε. $\rightarrow w_n \varepsilon^{n-1}$.

$$|L_\varepsilon| \leq \|Df\|_{L^\infty(\mathbb{R}^n)} \begin{cases} \varepsilon\log|\varepsilon| & n=2 \\ \frac{1}{(n-2)}\cdot\varepsilon & n\geq 3. \end{cases}$$

as $\underline{\varepsilon \rightarrow 0}$, $\underline{|L_\varepsilon| \rightarrow 0}$



$$K_\varepsilon \Rightarrow \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y)\cdot f(x-y)\, dS(y).$$

for $n\geq 3$, $\nabla\Phi(y)\cdot\nu(y)$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial\Phi(y)}{\partial y_i}\cdot\left(\frac{-y_i}{\varepsilon}\right)$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial\left(\frac{1}{(n-2)w_n}\cdot\frac{1}{|y|^{n-2}}\right)}{\partial y_i}\left(\frac{-y_i}{\varepsilon}\right)$$

$$\Rightarrow \sum_{i=1}^{n} \frac{-\cancel{(n-2)}}{\cancel{(n-2)}w_n}\cdot\frac{y_i}{|y|^{n}}\cdot\left(\frac{-y_i}{\varepsilon}\right).$$

$$\Rightarrow \sum_{i=1}^{n} \frac{y_i^2}{w_n\cdot\varepsilon^{n+1}}$$

$$\Rightarrow \frac{1}{w_n}\cdot\frac{1}{\varepsilon^{n-1}}.$$

$$|K_\varepsilon| = \left|\frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\, dS(y)\right|.$$

as $\varepsilon \to 0$. $\longrightarrow f(x)$

$$\Rightarrow |K_\varepsilon| = \left|\frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y)\, dS(y) - \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x)\, dS(y)\right|$$

$$= \left|\frac{1}{\omega_n \varepsilon^{n-1}} \left[\int_{\partial B(0,\varepsilon)} (f(x-y) - f(x))\, dS(y)\right]\right|.$$

by contn of f. $\rightarrow$ use modulus of contn of $f \rightarrow \omega_f$.

$$\leq \frac{1}{\omega_n \varepsilon^{n-1}} \left|\int_{\partial B(0,\varepsilon)} \omega_f(\varepsilon) . \underbrace{dS(y)}_{\omega_n \varepsilon^{n-1}}\right|.$$

$$\leq \omega_f(\varepsilon) \quad \text{as as } \varepsilon \to 0$$

$\therefore$ $\boxed{K_\varepsilon \longrightarrow f(x)}$ ✗.

and as $K_\varepsilon = -f(x)$

देख लेना मिल जाएगा

$\boxed{L_\varepsilon \longrightarrow 0}$ as $\varepsilon \longrightarrow 0$.

$\therefore$ $I_\varepsilon = L_\varepsilon + K_\varepsilon$

as $\varepsilon \to 0$ then $\boxed{I_\varepsilon \longrightarrow -f(x)}$.

Now for J_ε,

$$|J_\varepsilon| = \int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy.$$

using similar estimate in L_ε, we can show that

$$|J_\varepsilon| \leq C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \begin{cases} \varepsilon^2 \log \varepsilon & n=2 \\ \varepsilon^2 & n \geq 3. \end{cases}$$

$\hookrightarrow$ change in variable into polar coordinate

$\therefore$ as $\varepsilon \to 0$

then $|J_\varepsilon| \to 0$

$\therefore$ $\boxed{\Delta u(x) = -f(x)}$

Lecture 15

Remark → The condition $f \in C^2_c(\mathbb{R}^n)$ is a stronger condition. we can define the sol^n u for "more general" f:

Example $f \in C^\alpha(\mathbb{R}^n)$.
$\downarrow$
Holder space of exponent α.

def^n: $|\Omega| =$ Vol. of Ω

$$\fint_\Omega f(y)\,dy = \frac{1}{|\Omega|} \cdot \int_\Omega f(y)\,dy$$

PROPERTY OF HARMONIC FUNCTION

$\boxed{\text{Theorem}}$ (Mean Value Property).

Let $u \in C^2(\Omega)$ be harmonic then for any $x \in \Omega$

$$u(x) = \underbrace{\fint_{B(x,r)} u(y)\,dy}_{①} = \underbrace{\fint_{\partial B(x,r)} u(y)\,dy}_{②}.$$

for any $B(x,r) \subset \Omega$.

proof: Define $\psi(r) = \fint_{\partial B(x,y)} u(y)\,dS(y)$.

To show: $\psi'(r) = 0$

change of variable,

$$\psi(r) = \frac{1}{\omega_n \cancel{r^{n-1}}} \int_{\partial B(0,1)} u(x+zr)\,\cancel{r^{n-1}}\,dS(z).$$

[Take $y = x+zr$ variable change]

Differentiate w.r.to r.

$$\psi'(r) = \frac{1}{\omega_n} \int_{\partial B(0,1)} \frac{d\,u(x+zr)}{dr}\,.\,dS(z).$$

{int. & diff can interchange as u is $C^2(\Omega)$ and $B(0,1) \subset \Omega$.}

$$\psi'(r) = \frac{1}{\omega_n} \int_{\partial B(0,1)} (\nabla u(x+zr)\cdot z)\,dS(z).$$

change of variable, [let $x + rz = y$].

$$\psi'(r) = \frac{1}{\omega_n} \int_{\partial B(x,r)} \nabla u(x) \cdot \left(\frac{y-x}{r}\right) \boxed{\times \frac{1}{r^{n-1}}}\,dS(y).$$

coming from Jacobian

$$\psi'(r) = \underbrace{\frac{1}{\omega_n r^{n-1}}}_{|\partial B(x,r)|} \int_{\partial B(x,r)} \nabla u(x) \cdot \underbrace{\frac{(y-x)}{r}}_{\text{outward unit normal of } \partial B(x,r)}\,dS(y)$$

$$\therefore \psi'(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\, dS(y).$$

Integration by parts, (Green's formula can also be done)

$$\psi'(r) = \frac{-1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \Delta u(y) \cdot dy.$$

0 cas u is harmonic in Ω and $B(x,r) \subset \Omega$.

$\therefore \psi'(r) = 0.$

then ~~...~~ $\psi(r) = \psi(t)$ $\forall\ t \le r$ [to be ऐड़ाई particular].

$$\therefore \psi(r) = \lim_{t\to 0} \psi(t)$$

$$= \lim_{t\to 0} \frac{1}{\omega_n t^{n-1}} \int_{\partial B(x,t)} u(y)\, dS(y).$$

$$= u(x)$$ [see K_ε. (prev. page)].

(2) done //

let $\displaystyle\int_{B(x,r)} u(y)\,dy.$ → see, $B(x,r)$ integration करना is same as $\partial B(x,l)$ $0 \le l \le r$. and सब l पे ऊसको वापिस integrate कर देना

integrate, then vary l from 0 to r.

$$\Rightarrow \frac{1}{\alpha_n r^n} \int_{B(x,r)} u(y)\,dy$$

$$\Rightarrow \frac{1}{\alpha_n r^n} \int_0^r \left[\int_{\partial B(x,t)} u(y)\, dS(y) \right] dt.$$

[ω_n → surface, α_n → volume]

$\left(\frac{\alpha_2}{\omega_2} = \frac{1}{2} \text{ if } r=1 \quad \therefore \omega_n = n\alpha_n \ (r=1) \right)$

$$\Rightarrow \frac{1}{\alpha_n r^n} \int_0^r \frac{n\alpha_n}{\omega_n} t^{n-1}\, u(x)\, dt$$

[from prev. calculation]

$$\Rightarrow \frac{n\, u(x)}{r^n} \int_0^r t^{n-1}\, dt \Rightarrow \frac{\cancel{n}\, u(x)}{\cancel{r^n}} \cdot \frac{\cancel{r^n}}{\cancel{n}}$$

$\Rightarrow u(x).$ (1) done //

Reverse of MVP

If for a funcⁿ $u \in C^2(\Omega)$ and ① and ② holds, then u is harmonic

proof: By contradiction.
Assume $\exists\, x_0 \in \Omega$ s.t. $\Delta u(x_0) \neq 0$.

Then $\exists\, B(x_0, r_0)$ s.t. $\forall\, y \in B(x_0, r_0)$.
$\Delta u(y)$ ~~its~~ have same sign.

wlog, we assume $\Delta u(y) > 0$. $\forall\, y \in B(x_0, r_0)$.

$\psi'(r_0) < 0$ [had a minus sign]

(from previous calc.

[but ① holds and we know that $\psi'(r) = 0$ for any r as $\psi(r) =$ constant $= u(x_0)$.]

$\therefore\ \psi'(r) = 0 \quad \forall\, r \leq r_0 + \varepsilon$.

$\therefore$ contradiction.

$\therefore\ \Delta u(x) = 0 \quad \forall\, x \in \Omega$.

$\boxed{\text{Theorem}}$ (Maximum Principle)

Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and harmonic, then

a) $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. [weak maximum principle]

b) In addition, if Ω is connected and if $u(x_0) = \max_{\bar{\Omega}} u(x)$ for some $x_0 \in \Omega$, then u is constant. [strong maximum principle].

If instead of u, we replace statement by $-u$
↳ minimum principle.