Lesson 3

Boundary Value Problems for ODEs

3.2 Shooting Method

3.3 Finite Difference Method

3.4 Variational Methods

- **Least Squares Method**

Akash Anand
MATH, IIT KANPUR

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where

$$\langle u, v\rangle = \int_a^b u(t)v(t)dt\,, \qquad r(t,y) = \sum_{j=1}^n y_j\varphi_j''(t) - f\left(t, \sum_{j=1}^n y_j\varphi_j(t), \sum_{j=1}^n y_j\varphi_j'(t)\right).$$

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$$= \int_a^b r(t,y)\left(\varphi_i''(t) - f_2\left(t, \sum_{j=1}^{n} y_j\varphi_j(t), \sum_{j=1}^{n} y_j\varphi_j'(t)\right)\varphi_i(t) - f_3\left(t, \sum_{j=1}^{n} y_j\varphi_j(t), \sum_{j=1}^{n} y_j\varphi_j'(t)\right)\varphi_i'(t)\right) dt.$$

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This yields a system of algebraic equations

$$\sum_{j=1}^n y_j\langle\varphi_j'', \varphi_i''\rangle = \langle f, \varphi_i''\rangle +$$
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In particular, for the linear problem with $f(t,u,v) = f(t)$, we have

$$\sum_{j=1}^n y_j\langle\varphi_j'', \varphi_i''\rangle = \langle f, \varphi_i''\rangle,$$

a symmetric system.

Lesson 3

Boundary Value Problems for ODEs

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MATH, IIT KANPUR

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$$-\sum_{j=1}^{n} y_j \langle \varphi_j', \varphi_i' \rangle = \langle f, \varphi_i \rangle, \qquad i = 1, \dots, n.$$

Example

Consider the two-point BVP

$$\begin{aligned} u'' &= 6t, \qquad 0 < t < 1, \\ u(0) &= 0, \qquad u(1) = 1. \end{aligned}$$

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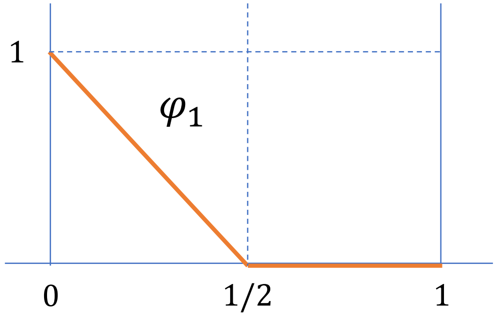

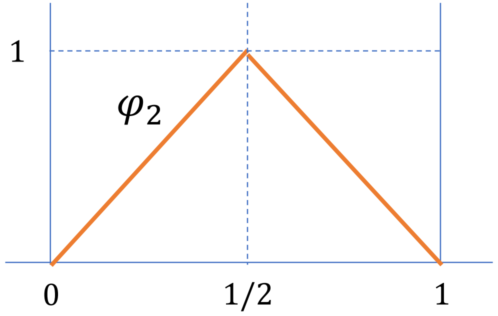

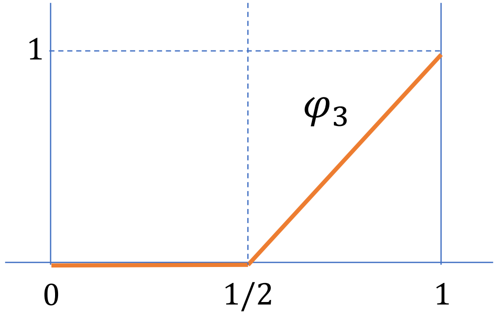

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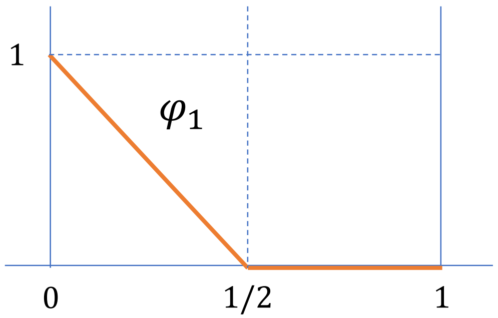

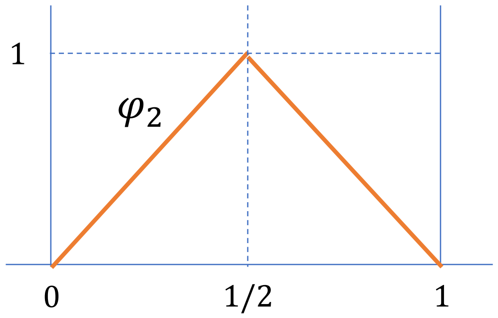

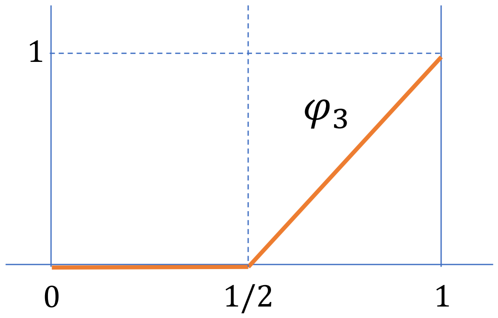

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where y_2 can be found as solution to the equation

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that is, $-4y_2 = 3/2$.

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