1. (a) $u(x,0) = 4x(1-x)$.

By Maximum principle.

$$\min_{\overline{(0,1)\times(0,T)}} u = \min_{\Gamma_{(0,1)\times(0,T)}} u = 0.$$

and $$\max_{\overline{(0,1)\times(0,T)}} u = \max_{\Gamma_{(0,1)\times(0,T)}} u = \max_{\overline{(0,1)}} \left[4x(1-x)\right] = 1$$

~~The~~ As $T$ is ~~ab~~ arbitrary.

$$0 \leq u \leq 1.$$

Again by Strong maximum principle, if $u=1$ for some $(x_0,t_0) \in (0,1)\times(0,T)$; then $u \equiv 1$ on $(0,1)\times(0,T)$.

But $u(x,0)$ is not identically zero.

Therefore $u < 1$ on $(0,1)\times(0,T)$ and again as $T$ is arbitrary, we get $u<1$ on $(0,1)\times(0,\infty)$,

Using ~~weak~~ strong minimum principle, we can show similarly that $u > 0$ on $(0,1)\times(0,\infty)$.

(b) Note that

for $v(x,t) = u(1-x,t)$,

$v$ solve the same eqn. $\begin{cases} v_t = v_{xx} & \text{in } (0,1)\times(0,\infty). \\ v(x,0) = 4x(1-x). \\ v(0,t) = v(1,t) = 0 & \forall t \end{cases}$

Then by using uniqueness on bounded domain we get

$$u(x,t) = v(x,t) \quad \forall (x,t) \in (0,1)\times(0,T).$$

Again as T is arbitrary we see

$$u(x,t) = u(1-x,t) \qquad \forall x \in (0,1) \text{ and } t \in (0,\infty).$$

2. Take $w(x,t) = u(x,t) - v(x,t)$.

Then $\quad w_t - \Delta w \leq 0 \quad \forall \cancel{(x,t)}\ (0,l)\times(0,T) = \Omega_T$

for any $T > 0$.

Then by maximum principle.

$$\max_{\overline{\Omega}_T} w = \max_{\Gamma_T} w.$$

By hypothesis. $u \leq v$ on $\Gamma_T$ hence $\max_{\Gamma_T} w \leq 0$.

Then

$$u(x,t) \leq v(x,t) \quad \forall (x,t) \in (0,l)\times(0,T).$$

As T is arbitrary, the result follows.

1.(c) Taking $e(t) = \int_0^1 u^2(x,t)\,dx$ and differentiating w.r.to $t$, we get

$$e'(t) = -2\int_0^1 (u_x)^2\,dx \leq 0$$

Hence $e$ is dicreasing. Now, if $e'(t)=0$ for some $t_0 \in (0,\infty)$, then.

$$\int_0^1 (u_x)^2(x,t_0)\,dx = 0 \Rightarrow u_x(x,t_0) = 0 \quad \forall x \in (0,1)$$

$$\Rightarrow u(x,t_0) = \text{constant} \quad \forall x \in (0,1).$$

As $u(0,t_0) = u(1,t_0) = 1$, by continuity we get

$$u(x,t_0) = 0 \quad \forall x \in (0,1).$$

But in part (a) we saw $u > 0 \;\forall x \in (0,1)$

Therefore $e'(t) < 0 \quad \forall t \in (0,\infty)$.

and $\int_0^1 u^2(x,t)\,dx$ is a strictly dicreasing function of $t$.

3. Let $f$ be bounded continuous function. Show that if $|v(x,t)| \le A e^{b|x|^2}$ satisfies

$$\begin{cases} v_t - k v_{xx} + f(x,t) & \text{in } \mathbb{R}\times(0,T). \\ v(x,0) = 0 & \text{on } \mathbb{R} \end{cases}$$

then

$$v(x,t) \le \sup_{\mathbb{R}\times(0,T)} |f|.$$

proof:- ~~By~~ We know that

$$u(x,t) = \int_0^t \int_{\mathbb{R}} \Phi^k(x-y, t-s)\, f(y,s)\, dy\, ds$$

is a solution of the equation, where

$$\Phi^k(x, t) = \frac{1}{(4\pi k t)^{1/2}} e^{-\frac{|x|^2}{4kt}}.$$

Clearly, we know $\displaystyle\int_{\mathbb{R}^n} \Phi^k(x,t)\, dx = 1 \quad \forall\, t>0$

Now,

$$|u(x,t)| \le \int_0^t \int_{\mathbb{R}} \Phi^k(x-y, t-s)\, |f(y,s)|\, dy\, ds.$$

$$\le \sup_{\mathbb{R}\times(0,T)} |f| \int_0^t \left[\int_{\mathbb{R}} \Phi^k(x-y, t-s)\, dy\right] ds.$$

$$= \sup_{\mathbb{R}\times(0,T)} |f| \int_0^t \underbrace{\left[\int_{\mathbb{R}} \Phi(y, t-s)\, dy\right]}_{=1} ds.$$

$$= t \sup_{\mathbb{R}\times(0,T)} |f|.$$

$$\le T \sup_{\mathbb{R}\times(0,T)} |f| \quad , \quad \forall\, t \le T.$$