## Module 2

# Initial Value Problems

2.4 Implicit method

2.5 Stiffness

**2.6 Linear Multistep Methods**

**- Consistency and Order**

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# Initial Value Problems: Linear Multistep Methods

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## Consistency and Order

*For the linear multistep method*

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j})$$

*define the local error as*

$$\ell_{n+1}(y,h) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh)$$

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*for any* $y \in C^1$, *and* $h > 0$.

*The linear multistep method is consistent if*

$$\lim_{h \to 0} \max_{k \le n < N} \left| \frac{\ell_{n+1}(y,h)}{h} \right| = 0$$

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The method has order $p$ is for all $y \in C^{p+1}$ there exists constants $C, h_0 > 0$ such that

$$\max_{k \le n < N} \left| \frac{\ell_{n+1}(y,h)}{h} \right| \le C h^p$$

whenever $h < h_0$.

## Note

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## Theorem

*A linear multistep is consistent if and only if*

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

*The method is of order $p$ if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\dots,p.$$

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$$= -y(t_n)C_0 + hy'(t_n)C_1 + R,$$

*where*

$$C_0 = \sum_{j=-1}^{k} a_j,\ C_1 = \sum_{j=-1}^{k} ja_j + \sum_{j=-1}^{k} b_j,$$

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**Proof.** ...

*Similarly, if* $y \in C^{p+1}$ *we have*

$$y(t_n - jh) = \sum_{m=0}^{p} \frac{(-j)^m}{m!} h^m y^{(m)}(t_n) + \frac{(-j)^{p+1}}{(p+1)!} h^{p+1} y^{(p+1)}(\xi_j)$$

$$y'(t_n - jh) = \sum_{m=1}^{p} \frac{(-j)^{m-1}}{(m-1)!} h^{m-1} y^{(m)}(t_n) + \frac{(-j)^p}{p!} h^p y^{(p+1)}(\zeta_j)$$

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for some $\xi_j, \zeta_j \in (t_n - kh, t_n + h), j = -1, \ldots, k$. This yields

$$\ell_{n+1}(y,h) = \sum_{m=0}^{p} h^m y^{(m)}(t_n) C_m + R$$

where

$$C_m = \frac{1}{m!}\left[ m \sum_{j=-1}^{k} (-j)^{m-1} b_j - \sum_{j=-1}^{k} (-j)^m a_j \right], \qquad R = h^{p+1} \sum_{j=-1}^{k} \left[ b_j \frac{(-j)^p}{p!} y^{(p+1)}(\zeta_j) - a_j \frac{(-j)^{p+1}}{(p+1)!} y^{(p+1)}(\xi_j) \right].$$

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*Since* $R = O(h^{p+1})$, $\ell_{n+1}(y,h)/h = O(h^p)$ *if and only if all the* $C_m$ *vanish.*

# Initial Value Problems: Linear Multistep Methods

## Theorem

*A linear multistep is consistent if and only if*

$$\sum_{j=-1}^{k} a_j = 0, \qquad \sum_{j=-1}^{k} j a_j + \sum_{j=-1}^{k} b_j = 0\,.$$

*The method is of order* $p$ *if and only if*

$$\sum_{j=-1}^{k} (-j)^m a_j - m \sum_{j=-1}^{k} (-j)^{m-1} b_j = 0\,, \qquad m = 0,1,\ldots,p.$$

## Remarks

1. *This theorem is an example of how a complicated analytic condition may sometimes reduce to a simple algebraic criterion.*

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## Remarks

1. *This theorem is an example of how a complicated analytic condition may sometimes reduce to a simple algebraic criterion.*
2. *Such algebraic criteria for multistep methods can be expressed in terms of characteristic polynomials of the method:*

$$\rho(z) = \sum_{j=-1}^{k} a_j z^{k-j}\,, \qquad \sigma(z) = \sum_{j=-1}^{k} b_j z^{k-j}\,.$$

*For example, the consistency conditions are* $\rho(1) = 0$ *and* $\rho'(1) = \sigma(1)$.