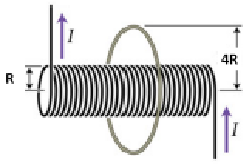
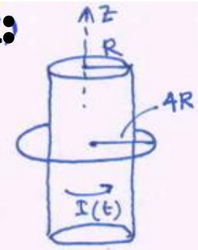
1. A long solenoid of radius R carries a weakly time dependent current $I(t)$. The solenoid is encircled by a symmetrically placed conducting loop of radius $4R$. Calculate the energy inflow to the external loop which replenishes the energy dissipated by joule heating.



Soln:



Magnetic field due to the solenoid $|\vec{B}(t)| = \mu_0 n I(t)$ along $\hat{z}$

changing $\vec{B}$ field induces $\vec{E}$ field in the external loop

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$\Rightarrow E.\,2\pi.4R = \pi R^2 \frac{dB}{dt} \Rightarrow \vec{E} = \frac{\dot{B}(t)}{8\pi R}.\pi R^2\,\hat{\varphi} = \frac{\dot{B}(t) R}{8}\,\hat{\varphi}$$

In the external loop, where the magnetic field due to the solenoid is zero, a magnetic field $\vec{B}_{ext}$ is generated by current flow in the loop.

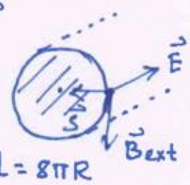
$$\oint \vec{B}_{ext}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\Rightarrow B_{ext}.2\pi r = \mu_0 J A \Rightarrow B_{ext} = \frac{\mu_0 J A}{2\pi r}$$

Again $\vec{J} = \sigma\vec{E}$, σ = conductivity of the wire loop

$$= \sigma.\frac{\dot{B}(t)R}{8}\,\hat{\varphi} \quad , \text{ Hence } I = JA = \frac{\sigma\dot{B}(t)R}{8}.A$$

Hence $B_{ext} = \dfrac{\mu_0 J A}{2\pi r} = \dfrac{\mu_0\,\sigma\dot{B}(t)R}{8}.\dfrac{A}{2\pi r}$, $\vec{E}\perp\vec{B}_{ext}$



Total surface area of the wire loop $2\pi r L$, where $L = 8\pi R$

$$|\vec{S}| = \frac{1}{\mu_0}|\vec{E}\times\vec{B}| = \frac{1}{\mu_0}.\frac{\dot{B}(t)R}{8}.\frac{\mu_0\,\sigma\dot{B}(t)R}{8}.\frac{A}{2\pi r}$$

, radially inward from the surface of the wire

Total power transferred,

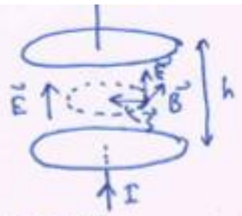
$$S.\,2\pi r L = \frac{1}{\mu_0}\frac{\dot{B}(t)R}{8}.\frac{\mu_0\,\sigma\dot{B}(t)R}{8}.\frac{A}{2\pi r}.2\pi r L$$

$$= \underbrace{\left(\frac{\sigma\dot{B}(t)RA}{8}\right)^2}_{I} . \underbrace{\frac{L}{\sigma A}}_{R} = \underbrace{I^2 R}_{\text{Joule heating}}$$

, R = resistance of the external loop

2. A capacitor with two circular plates of radius R is being charged by a constant current. Calculate the Poynting vector at radius r inside the capacitor, and verify that its flux equals the rate of change of the energy stored in the region bounded by radius r.

Soln:



Far from the edges, inside the capacitor $\vec{E}$ is uniform. constant current $\Rightarrow$ constant $\frac{d\sigma}{dt}$ ($\pm\sigma$ charges on the two plates) $\Rightarrow$ constant $\frac{\partial E}{\partial t}$ $\Rightarrow$ constant $\vec{B}$

$$\oint \vec{B}\cdot d\vec{l} = \epsilon_0\mu_0 \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a} \Rightarrow B.\,2\pi r = \epsilon_0\mu_0 \frac{\partial E}{\partial t}.\,\pi r^2$$

$$\Rightarrow B = \frac{\epsilon_0\mu_0 r}{2}\cdot\frac{\partial E}{\partial t}$$, pointing tangentially around circle of radius r.

Since $\vec{E}$ increases upward, $\vec{B}$ is directed counter-clockwise, as seen from above. $\Rightarrow$ $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$ points radially inward everywhere on the circle of radius r

$$|\vec{S}| = \frac{1}{\mu_0}\cdot E.\,\frac{\epsilon_0\mu_0 r}{2}\frac{\partial E}{\partial t} = \frac{\epsilon_0 r}{2}.\,E\frac{\partial E}{\partial t}$$

curved surface area of the cylindrical region inside the capacitor with radius r is $2\pi r h$ (h = distance between the two plates)

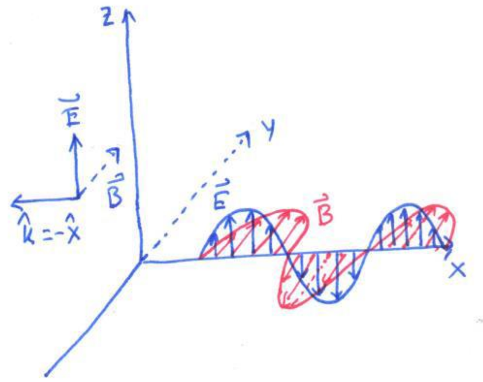
Total power inflow to the cylinder

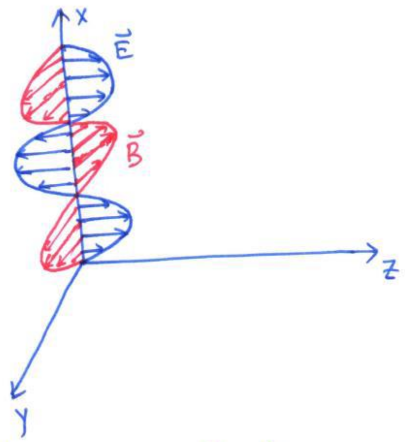
$$P = \frac{\epsilon_0 r}{2}.\,E\frac{\partial E}{\partial t}.\,2\pi r h = \pi r^2 h\,\epsilon_0 E\frac{\partial E}{\partial t} = \pi r^2 h \frac{d}{dt}\left(\epsilon_0 \frac{E^2}{2}\right)$$
$$= \frac{dU_{em}}{dt}$$

where U_{em} = total electromagnetic energy stored in the region. The magnetic energy density is constant and does not affect $\frac{dU_{em}}{dt}$.

3. Write down the real electric and magnetic field for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle (δ) zero that is (i) traveling in the negative x direction and polarized in the z direction; (ii) traveling along (1, 1, 1) direction, with polarization parallel to the xz plane. In each case, sketch the wave, and give the explicit Cartesian components of $\vec{k}$ and $\hat{n}$.

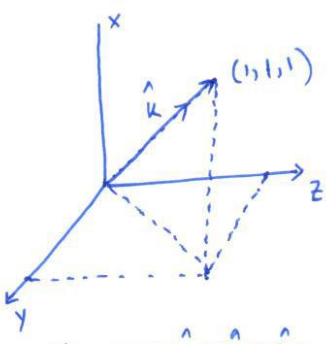
Soln: i) $\vec{E}$ in $\hat{z}$ direction, $\hat{k} = -\hat{x}$ $\Rightarrow$ $\vec{B}$ in $\hat{y}$ direction, $\hat{n} = \hat{z}$

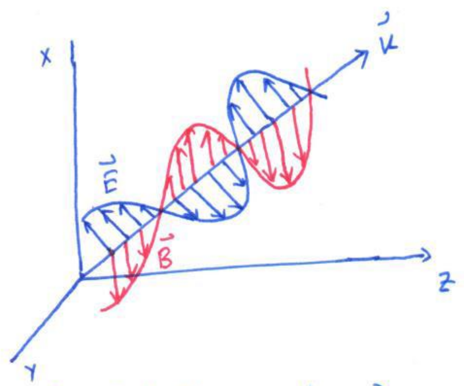




$$\left.\begin{aligned} \vec{E} &= E_0 \cos(\omega t + kx)\, \hat{z} \\ \vec{B} &= \frac{E_0}{c} \cos(\omega t + kx)\, \hat{y} \end{aligned}\right\} \quad \vec{k} = -\frac{\omega}{c}\hat{x}, \quad \hat{n} = \hat{z}$$

ii)





$$\vec{k} = \frac{\omega}{c}\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right), \quad \hat{n} = \alpha\hat{x} + \beta\hat{z} \quad (\text{in the xz plane})$$

Again $\vec{k}\cdot\hat{n} = 0 \Rightarrow \alpha + \beta = 0 \Rightarrow \alpha = -\beta \Rightarrow \hat{n} = \dfrac{\hat{x}-\hat{z}}{\sqrt{2}}$

$$\vec{k}\cdot\vec{r} = \frac{\omega}{c}\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right)\cdot(x\hat{x}+y\hat{y}+z\hat{z}) = \frac{\omega}{\sqrt{3}c}(x+y+z)$$

$$\hat{k}\times\hat{n} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{6}}(-\hat{x}+2\hat{y}-\hat{z})$$

$$\vec{E}(\vec{r},t) = \vec{E}(x,y,z,t) = E_0 \cos\left\{\omega t - \frac{\omega}{\sqrt{3}c}(x+y+z)\right\}\hat{n}, \quad \hat{n} = \frac{\hat{x}-\hat{z}}{\sqrt{2}}$$

$$\vec{B}(\vec{r},t) = \vec{B}(x,y,z,t) = \frac{E_0}{c}\cos\left\{\omega t - \frac{\omega}{\sqrt{3}c}(x+y+z)\right\}\hat{k}\times\hat{n}, \quad \hat{k}\times\hat{n} = \frac{1}{\sqrt{6}}(-\hat{x}+2\hat{y}-\hat{z})$$

4. Two electromagnetic waves traveling along $+z$ and $-z$ direction, respectively, are given by, $E_{x1} = E_0 \sin(\omega t - kz)\hat{x}; B_{y1} = \frac{E_0}{c}\sin(\omega t - kz)\hat{y}$ and $E_{x2} = E_0 \sin(\omega t + kz)\hat{x}; B_{y2} = -\frac{E_0}{c}\sin(\omega t + kz)\hat{y}$. Find out the nature of the wave resulting from superposition of the two traveling waves. Plot the electromagnetic energy density $u_{em}(z,t)$ and the z component of the Poynting vector $S_z(z,t)$ at ωt values of 0, $\pi/4$, $\pi/2$, and $3\pi/4$ and π. Interpret the result.

Soln: $\vec{E}_x = \vec{E}_{x1} + \vec{E}_{x2} = E_0\left[\sin(\omega t - kz) + \sin(\omega t + kz)\right]\hat{x} = 2E_0 \sin\omega t \cos kz\,\hat{x}$

$\vec{B}_y = \vec{B}_{y1} + \vec{B}_{y2} = \frac{E_0}{c}\left[\sin(\omega t - kz) - \sin(\omega t + kz)\right]\hat{y} = -\frac{2E_0}{c}\sin kz \cos\omega t\,\hat{y}$

The electromagnetic field described by $\vec{E}_x, \vec{B}_y$ is a standing wave!

The energy density $u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$

$$\boxed{u_{em} = 2\epsilon_0 E^2\left[\sin^2\omega t \cos^2 kz + \sin^2 kz \cos^2\omega t\right]}$$

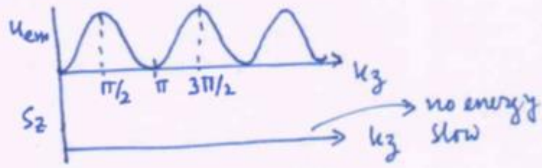
The Poynting vector $\vec{S}$ in $\hat{z}$ direction

$$S_z = -\frac{1}{\mu_0}\cdot 2E_0 \sin\omega t \cos kz \cdot \frac{2E_0}{c}\sin kz \cos\omega t$$

$$\boxed{S_z = -E_0^2\,\epsilon_0 c \sin 2\omega t \sin 2kz} \quad \leftarrow \left(\text{use } c^2 = \frac{1}{\mu_0\epsilon_0}\right)$$

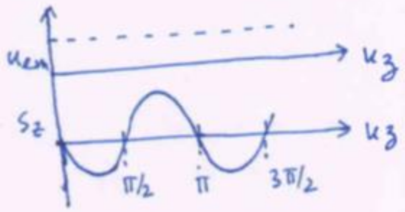
$\boxed{\omega t = 0}$ $u_{em} = 2\epsilon_0 E^2 \sin^2 kz$

$S_z = 0$



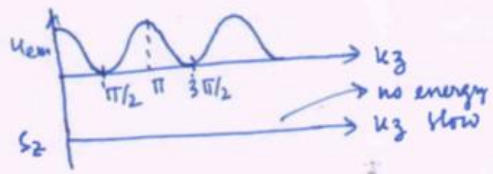
$\boxed{\omega t = \pi/4}$ $u_{em} = \epsilon_0 E^2$

$S_z = -\epsilon_0 c E_0^2 \sin 2kz$



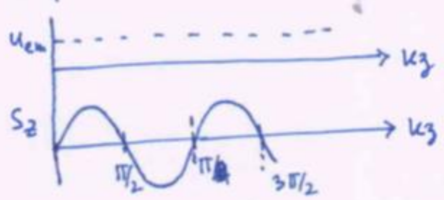
$\boxed{\omega t = \pi/2}$ $u_{em} = 2\epsilon_0 E^2 \cos^2 kz$

$S_z = 0$



$\boxed{\omega t = 3\pi/4}$ $u_{em} = \epsilon_0 E^2$

$S_z = +\epsilon_0 c E_0^2 \sin 2kz$



$\boxed{\omega t = \pi}$ $\left.\begin{array}{l} u_{em} = 2\epsilon_0 E^2 \sin^2 kz \\ S_z = 0 \end{array}\right\}$ same as, $\omega t = 0$

S_z changes sign with time and in space $\rightarrow$ back and forth energy flow

For $\omega t = 0, \pi/2, \pi$ $\rightarrow$ there is no energy flow