MTH101A, End-Sem. Exam Marking, IIT Kanpur

Date: 20.11.2017 **Time: 9:00-12:00 hrs** **Total Marks:** 100

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1. (a) Let $x_1 = 1$ and $x_{n+1} = \dfrac{4+3x_n}{3+2x_n}$ for all $n \geq 1$. Show that the sequence (x_n) is bounded and monotone and find its limit. [4]

ANSWER:

- Notice that $x_2 > x_1$ and $x_{n+1} - x_n = \dfrac{x_n - x_{n-1}}{(3+2x_n)(3+2x_{n-1})}$. [1]

  By Induction principle, the sequence (x_n) is an increasing sequence. [1]

- Also,

$$x_{n+1} = 1 + \frac{1+x_n}{3+2x_n} \leq 2$$

  for all $n \geq 1$. Thus the sequence is also bounded. [1]

- Let l be the limit of the sequence (x_n) then $l = \frac{4+3l}{3+2l}$. This gives $l = \sqrt{2}$. [1]

  Remark: *If anyone uses contractive condition where $\alpha \in (0,1)$ does not depends on the $n \in \mathbb{N}$, to show that the sequence is Cauchy and so bounded then 1-mark is given.*

(b) Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)\log n}{n}$. [4]

ANSWER:

- For $n \in \mathbb{N}$ we have

$$\cos(n\pi) = (-1)^n$$

  and let for $n > 2$

$$a_n = (\frac{\log n}{n}).$$

[1]

  Then a_n is a decreasing sequence. [1 for checking]

  Also, a_n converging to 0. [1 for checking]

- By Leibniz test, the series $\displaystyle\sum_{n=3}^{\infty} \frac{\cos(n\pi)\log n}{n}$ is convergent and hence the given series converges. [1]

(c) Let $f : [0,2] \to \mathbb{R}$ be a twice differentiable function. If $f(0) = 0$, $f(1) = 2$, and $f(2) = 4$ then show that there exists $x_0 \in (0,2)$ such that $f''(x_0) = 0$.

[4]

ANSWER:

- By applying Mean Value Theorem on $f : [0, 1] \longrightarrow \mathbb{R}$ we get

$$f'(c_1)(1 - 0) = f(1) - f(0) = 2.$$

[1]

- Also by applying Mean Value Theorem on $f : [1, 2] \longrightarrow \mathbb{R}$ we get

$$f'(c_2)(2 - 1) = f(2) - f(1) = 2.$$

[1]

- Then by Rolle's theorem on $f : [c_1, c_2] \longrightarrow \mathbb{R}$ we get that $f''(x_0) = 0$ for some $x_0 \in (c_1, c_2)$. [2]

(d) Does there exist a vector field $F(x, y, z)$ such that $curl(F) = (x \sin y, \cos y, z - xy)$? Justify your answer. [2]

ANSWER:

No.

We know that $div(curl(F)) = 0$. [1]

But, $div(x \sin y, \cos y, z - xy) = 1$. [1]

2. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous at $x = 0$ and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then show that f is a continuous function.
Assume that $f(1) = 1$ and find the values of $f(5)$, $f(\frac{5}{3})$ and $f(\sqrt{5})$.

[4+1+2+3=10]

ANSWER:

- Using the condition
$$f(x + y) = f(x) + f(y)$$
for all $x, y \in \mathbb{R}$, we get $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. [1]
- For any $c \in \mathbb{R}$ we have $f(x - c) = f(x) - f(c)$ for all $x \in \mathbb{R}$. Thus, $\lim_{x \to c} f(x - c) = \lim_{x \to c}[f(x) - f(c)]$. [1]
- Since f is continuous at $x = 0$, we have
$$\lim_{x \to c} f(x - c) = \lim_{(x-c) \to 0} f(x - c) = f(0) = 0.$$
Therefore,
$$\lim_{x \to c} f(x) = f(c).$$
[2]
- $f(5) = f(1) + f(1) + f(1) + f(1) + f(1) = 5$ [1]
- Now, $f(5) = f(\frac{5}{3}) + f(\frac{5}{3}) + f(\frac{5}{3}) = 3f(\frac{5}{3})$. So,
$$f(\frac{5}{3}) = \frac{1}{3}f(5) = \frac{5}{3}.$$
[2]
- Similarly, we can show that $f(\frac{p}{q}) = \frac{p}{q}$ for any rational number $\frac{p}{q} \in \mathbb{R}$. For any irrational number $c \in \mathbb{R}$, let (x_n) be a sequence of rational number converging to c. Then by continuity of f we get,
$$f(c) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = c.$$
Therefore $f(\sqrt{5}) = \sqrt{5}$. [3]

Remark: *No mark is given if it is assumed that the function is differentiable and an expression of $f(x)$ is derived by differentiating the condition*
$$f(x + y) = f(x) + f(y).$$
Also, to compute $f(\sqrt{5})$ we need to consider sequences with rational entries converging to $\sqrt{5}$. Otherwise, just by mentioning that $f(\frac{p}{q}) = \frac{p}{q}$ for any rational number $\frac{p}{q} \in \mathbb{R}$. So, $f(\sqrt{5}) = \sqrt{5}$ one will not get marks.

(b) Let $f, g : [0,1] \longrightarrow \mathbb{R}$ be continuous functions such that

$$\inf\{f(x) : x \in [0,1]\} = \inf\{g(x) : x \in [0,1]\}.$$

Show that there exists a point $c \in [0,1]$ such that $f(c) = g(c)$. [6]

ANSWER:

- As f, g are continuous functions on $[0,1]$ there exists $x_1, x_2 \in [0,1]$ such that

$$f(x_1) = \inf\{f(x) : x \in [0,1]\}$$

and

$$g(x_2) = \inf\{g(x) : x \in [0,1]\}$$

[2]

- Note that from definition of infimum we have,

$$f(x_1) \leq g(x_1) \text{ and } f(x_2) \geq g(x_2).$$

[2]

- Let $\phi(x) = f(x) - g(x)$. Then $\phi(x_1) \leq 0$ and $\phi(x_2) \geq 0$. Applying Intermediate Value Property to the function ϕ we get a point $c \in [0,1]$ such that $f(c) = g(c)$. [2]

(c) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^2$ if x is a rational number and $f(x) = 0$ if x is an irrational number. Prove that $f'(0) = 0$. [2]

ANSWER: Note that for all h we have

$$0 \leq |\frac{f(h)}{h}| \leq |h|.$$

So, $f'(0) = \lim\limits_{h \to 0} \frac{f(h)-f(0)}{h} = 0$.

Remark: *If anyone uses sequences (h_n) with only rational entries and only irrational entries and say that $f'(0) = \lim\limits_{h_n \to 0} \frac{f(h_n)-f(0)}{h_n} = 0$, then only 1 marks is given. Because, there might be sequence with both the rational and irrational entries, and we need to give a proof for that case as well.*

3. (a) A rectangular box without a lid is to be made of 12 square meters of cardboard. Assuming that there exists a box of maximum volume, find its dimensions. [6]

Marking Scheme

- Let x, y, z be the dimensions of the box (in meters). We would like to maximize the volume of the box $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = xy + 2xz + 2yz = 12$. [2]
- Lagrange equations: $\nabla f = \lambda \nabla g$ gives $yz = \lambda(y + 2z)$; $xz = \lambda(x + 2z)$; $xy = \lambda(2x + 2y)$. [2]
- Observe that $\lambda \neq 0$ otherwise $x = y = z = 0$ which does not satisfy the constraint equation $g(x, y, z) = xy+2xz+2yz = 12$. Similarly $x, y, z \neq 0$. Solving $x = y = 2$ and $z = 1$. [2]
  [Multiplying the equations appropriately: $xyz = x\lambda(y + 2z) = y\lambda(x + 2z) = z\lambda(2x + 2y)$. From the second equality conclude $x = y$ and from third equality $y = 2z$. Putting in the constraint equation, we get the values.]

(If someone by mistake takes $g(x, y, z) = 2xy + 2xz + 2yz = 12$ and does rest of the calculations correctly then he/she should get the answer as $x = y = z = \sqrt{2}$. In this case 4 marks are to be awarded.)

(b) Find the unit tangent vector, unit normal vector and the curvature at any point of the curve $R(t) = (a \cos t, a \sin t, bt)$, $t \in \mathbb{R}$, $a, b > 0$ [2+2+2]

Marking Scheme

- $R'(t) = (-a \sin t, a \cos t, b)$ and $|R'(t)| = \sqrt{a^2 + b^2}$. The unit tangent vector $T(t) = \frac{R'(t)}{|R'(t)|} = \frac{1}{\sqrt{a^2+b^2}}(-a \sin t, a \cos t, b)$ [2]
- $T'(t) = \frac{1}{\sqrt{a^2+b^2}}(-a \cos t, -a \sin t, 0)$ and $|T'(t)| = \frac{a}{\sqrt{a^2+b^2}}$.
  The unit normal vector $N(t) = \frac{T'(t)}{|T'(t)|} = (-\cos t, -\sin t, 0)$ [2]
- Curvature $\kappa(t) = \left|\frac{dT}{ds}\right| = |\frac{dT}{dt}|/|\frac{ds}{dt}| = |T'(t)|/|R'(t)| = a/(a^2 + b^2)$. [2]

  ALTERNATIVE:

  $$\kappa(t) = \frac{|R'(t) \times R''(t)|}{|R'(t)|^3}.$$
  Now $R''(t) = (-a \cos t, -a \sin t, 0)$ and so
  $R'(t) \times R''(t) = (ab \sin t, -ab \cos t, a^2)$
  $|R'(t) \times R''(t)| = a\sqrt{a^2 + b^2}$. Hence $\kappa(t) = a/(a^2 + b^2)$. [2]

(c) Suppose that at a given instant a rectangular block has dimensions $x = 3m$, $y = 2m$ and $z = 1m$ (m= meter). Now assume that x and y are increasing at 1 cm/min and 2 cm/min respectively, while z is decreasing at 2 cm/min. Determine the rates at which the block's volume and surface area are increasing or decreasing at the given instant. [6]

Marking Scheme

- Let V and S denote the volume and surface area of the rectangular box respectively. Then $V = xyz$ and $S = 2(xy + yz + zx)$, and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 2, \frac{dz}{dt} = -2.$ [2]
- Differentiating by chain rule,
  $$\frac{dV}{dt} = yz\frac{dx}{dt} + zx\frac{dy}{dt} + xy\frac{dz}{dt}$$
  [1]

  $= (200)(100)(1) + (300)(100)(2) + (300)(200)(-2) = -40000.$
  Thus volume is decreasing at the rate 40000 cm^3/min.

  [1]
- Differentiating by chain rule,
  $$\frac{dS}{dt} = 2(y+z)\frac{dx}{dt} + 2(z+x)\frac{dy}{dt} + 2(x+y)\frac{dz}{dt}$$
  [1]

  $= 2(300)(1) + 2(400)(2) + 2(500)(-2) = 200.$
  Thus surface area is increasing at the rate 200 cm^2/min. [1]

4. (a) Let (x_0, y_0) be the centroid of the parametric curve $x(t) = \cos^3 t,\ y(t) = \sin^3 t,\ 0 \le t \le \pi/2$. Using Pappus theorem find y_0. [8]

Marking Scheme

[Pappus Theorem: Let C be a plane curve. Suppose C is revolved about a line which does not cut C, then the area of the surface generated is $S = 2\pi\rho L$, where ρ is the distance of the centroid from the axis of revolution and L is the arc length of the plane curve C.]

- The length of the arc $x(t) = \cos^3 t,\ y(t) = \sin^3 t,\ 0 \le t \le \pi/2$:

$$L = \int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2}dt \qquad [1]$$

$$= \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}dt$$

$$= \int_0^{\pi/2} 3\sin t \cos t dt$$

$$= 3/2. \qquad [2]$$

- The surface area generated by revolving the arc $x(t) = \cos^3 t,\ y(t) = \sin^3 t,\ 0 \le t \le \pi/2$ about x-axis:

$$S = \int_0^{\pi/2} 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}dt \qquad [1]$$

$$= \int_0^{\pi/2} 2\pi \sin^3 t 3 \sin t \cos t dt$$

$$= 6\pi \int_0^{\pi/2} \sin^4 t \cos t dt$$

$$= \frac{6\pi}{5} \text{ (using } \sin t = u \text{ substitution)} \qquad [2]$$

- Therefore, by Pappus theorem $2\pi y_0 L = S,\quad 2\pi y_0 \frac{3}{2} = 6\pi/5$ and so $y_0 = 2/5$. [2]

(b) Find the point(s) on the surface $x^2 - y^2 + 2z^2 = 1$ where the tangent plane is perpendicular to the line joining $(3, -1, 0)$ and $(5, 3, 6)$. Find the equation(s) of the tangent planes there. [6+2]

Marking Scheme

- The normal vector at any point (x_0, y_0, z_0) of the surface is given by $(2x_0, -2y_0, 4z_0)$. [2]
- The direction of line joining $(3-1, 0)$ and $(5, 3, 6)$ is $(2, 4, 6)$ (or $(-2, -4, -6)$). [1]
- By the given condition $2x_0 = 2k, \ -2y_0 = 4k, \ 4z_0 = 6k$ for some $k \in \mathbb{R}$ [1]
- Putting in the given equation of the surface we get $k = \pm\sqrt{6}/3$. [1]
- The require points are $x_0 = k, \ y_0 = -2k, \ z_0 = 3/2k$ for $k = \pm\sqrt{6}/3$. [1]

SECOND PART:

- Tangent planes are given by $2(x - x_0) + 4(y - y_0) + 6(z - z_0) = 0$. Now $2x + 4y + 6z = 2x_0 + 4y_0 + 6z_0 = 3k = \pm\sqrt{6}$. Hence tangent planes are $2x + 4y + 6z = \pm\sqrt{6}$.

  [2]

5. (a) Determine whether the limit exists. Justify your answer.

(i) $\lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{x^4+y^2}$ (ii) $\lim_{(x,y)\to(0,0)} \frac{y^2+(1-\cos(2x))^2}{x^4+y^2}$

[4+4]

Marking Scheme

(i):

- Notice that $0 \le \frac{y^2}{x^4+y^2} \le 1$.
  Since $(1-\cos 2x)$ is a non-negative number, we can multiply all sides of the inequality by it without changing the order of inequality. Thus we get,
  $$0 \le \frac{y^2(1-\cos 2x)}{x^4+y^2} \le (1-\cos 2x).$$
  [3]
- Both left and right hand approach 0 as $x \to 0$. Hence
  $$\lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(2x))}{x^4+y^2} = 0.$$
  [1]

ALTERNATIVE:

- $\frac{y^2(1-\cos 2x)}{x^4+y^2} = \frac{y^2 2\sin^2 x}{x^4+y^2} \le \frac{2y^2x^2}{x^4+y^2}$ (using $\sin x < x$) [1]
  $\le |y|\frac{x^4+y^2}{x^4+y^2}$ (Using A.M $\ge$ G.M) [2]
  $= |y| \to 0$ as $y \to 0$. [1]

Remark: In $\mathbb{R}^2$, to prove a limit exists it is not enough to check for $y = mx$ or $y = mx^2$. We need to prove it from definition. But to show that a limit does not exist, it is enough to show that along two different curves, we get two different limits

(ii):

- We choose paths of the form $y = mx^2$ to show that the limit does not exist. [1]
- $\frac{y^2+(1-\cos(2x))^2}{x^4+y^2} = \frac{m^2x^4+(2\sin^2 x)^2}{x^4+m^2x^4} = \frac{m^2+4\frac{\sin^4 x}{x^4}}{1+m^2} \to \frac{m^2+4}{1+m^2}$ as $x \to 0$. [2]

- Since the limit along $y = mx^2$ depends on m, the given double limit does not exist. [1]

(b) Let $f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}$ for $y \neq 0$ and $f(x,y) = 0$ if $y = 0$. Prove the following:

(i) Directional derivatives of f exist in all direction at the origin.

(ii) The function f is not differentiable at the origin. [2+4]

Marking Scheme:

(i)

- Let $U = (u,v)$ with $u^2+v^2 = 1$. The directional derivative in the direction U is $D_{(0,0)}f(u,v) = \lim_{t\to 0}\frac{f(tu,tv)}{t} = \lim_{t\to 0}\frac{0}{t} = 0$ if $v = 0$.
- Also if $v \neq 0$, then

$$\lim_{t\to 0}\frac{f(tu,tv)}{t} = \lim_{t\to 0}\frac{1}{t}\frac{tv}{|tv|}\sqrt{t^2u^2+t^2v^2} = \frac{v}{|v|}.$$

[2]

(ii)

- We get $f_x(0,0) = 0$ and $f_y(0,0) = 1$. [1]
- Then

$$\epsilon(h,k) = \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = \frac{f(h,k) - k}{\sqrt{h^2+k^2}}$$

[1]

- Letting $(h,k) \to (0,0)$ along y-axis (or x-axis), we see that $\epsilon(h,k) \to 0$. But along $h = k$ line, $\epsilon(h,k) = (\sqrt{2}-1)\frac{k}{|k|}$ whose limit does exist as $k \to 0$. Hence $\lim_{(h,k)\to(0,0)} \epsilon(h,k)$ does not exist, in particular it doest not tend to 0. So f is not differentiable at the origin. [2]

ALTERNATIVE for b (ii):

- We get $f_x(0,0) = 0$ and $f_y(0,0) = 1$. [1]
- If f is differentiable at the origin, then $f'(0,0) = (f_x(0,0), f_y(0,0)) = (0,1)$. Then, for in any direction $U = (u,v)$, we must have $D_{(0,0)}f(u,v) = f'(0,0).U = v$. This does not match with the calculations of part (i). So f is not differentiable at the origin. [3]

(c) Is the following statement true? If yes then give a proof, if no then give a counter example.

Let $f : \mathbb{R} \to \mathbb{R}^2$ be a differentiable function. For any $a, b \in \mathbb{R}$ with $a < b$, there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. [4]

Marking Scheme

- Not true. $f(t) = (\cos t, \sin t)$ and $a = 0, b = 2\pi$. [2]
- Then $f(b) - f(a) = (0, 0)$ but $f'(c) = (-\sin c, \cos c) \neq (0, 0)$ for any $c \in \mathbb{R}$ [2]

Remark: Another example is $f(t) = (t^2, t^3), \quad a = 0, b = 1$.

6. (a) Evaluate the double integral: $\iint\limits_R \left(\frac{x-y}{x+y+2}\right)^2 dxdy$, where R is the region bounded by the lines $x \pm y = \pm 1$. [6]

Marking Scheme

- Make a change of coordinate $u = x + y, \quad v = x - y$. So that the given region mapped onto $R' = \{(u, v): \quad -1 \leq u, v \leq 1\}$ in the u-v plane. [2]

- The Jacobian $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = -2$. So $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$. [1]

- $\iint\limits_R \left(\frac{x-y}{x+y+2}\right)^2 dxdy = \iint\limits_{R'} \left(\frac{v}{u+2}\right)^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

$$= \int\limits_{v=-1}^{1} \int\limits_{u=-1}^{1} \left(\frac{v}{u+2}\right)^2 \frac{1}{2} dudv = \frac{1}{2}\left[v^3/3\right]_{v=-1}^{1} \left[-\frac{1}{2+u}\right]_{u=-1}^{1} = 2/9$$

[3]

(b) Let S be the surface $x^2 + y^2 + z^2 = 4$. Compute the surface integral

$$\iint\limits_S (2x^2 - y^2 + 2z^2 + 3e^{z^2}x - e^{x^2}y + z\cos^2 y)d\sigma.$$

[6]

Marking Scheme

- The unit outward normal to S is given by $n = \frac{1}{2}(x, y, z)$ [1]
- Observe that $2x^2 - y^2 + 2z^2 + 3e^{z^2}x - e^{x^2}y + z\cos^2 y = F.n$ where $F(x, y, z) = (4x + 6e^{z^2}, \ -2y - 2e^{x^2}, \ 4z + 2\cos^2 y)$ [2]
- $div(F) = 6$ [1]
- So by Divergence Theorem,

$$\iint\limits_S (2x^2-y^2+2z^2+3e^{z^2}x-e^{x^2}y+z\cos^2 y)d\sigma = \iint\limits_S F.nd\sigma = \iiint\limits_V div(F)dv$$

$$= 6\iiint\limits_V dv = 6 \times 4\pi/3 \times 2^3 = 64\pi.$$ [2]

(If someone misses the factor 1/2 in first step, but does rest calculations correctly then the answer should be 32π. In this case 4 marks are to to awarded.)

(c) Does there exist a function $\phi(x, y)$ such that $\nabla\phi(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ for all $(x, y) \neq (0, 0)$? Justify your answer. [4]

Marking Scheme:

- By Fundamental theorem of Line integral of $\oint_C \nabla\phi.dR = 0$ for ANY closed curve C (NEED NOT be a SIMPLE closed curve). [2]
- But the line integral $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 2\pi$ where C is the unit circle with anticlockwise orientation.
  To see that, consider $x = \cos t, y = \sin t, \quad 0 \leq t \leq 2\pi$. Then
  $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt = 2\pi.$ [2]
- **Explanation why the choice $\phi(x, y) = \tan^{-1}(y/x)$ does not work:**
  Fist of all, for a given $(x, y) \neq (0, 0)$, $\tan^{-1}(y/x)$ can have many values. Even if we fix a range length of 2π, for example $[0, 2\pi)$, then $\tan^{-1}(y/x)$ becomes a well defined function on $\mathbb{R}^2 - (0, 0)$. But still it does not satisfy the required condition. We can see it as follows:
  Observe that any ϕ which satisfies $\nabla\phi(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ for all $(x, y) \neq (0, 0)$ has to be differentiable on $\mathbb{R}^2 - (0, 0)$ (since the partial derivatives are continuous.) But our choice of $\tan^{-1}(y/x)$ is discontinuous along positive x-axis (since it takes value small positive values just above positive x-axis and takes value near 2π just below the positive x-axis.)