MTH 101-Calculus

Spring-2021

Assignment 7 : Improper Integrals, Applications of Integration, Pappus Theorem

1. Test the convergence/divergence of the following improper integrals:

   $(a)\ \int_0^1 \frac{dx}{\log(1+\sqrt{x})}$ $\quad (b)\ \int_0^1 \frac{dx}{x-\log(1+x)}$ $\quad (c)\ \int_0^1 \frac{\log x}{\sqrt{x}}$ $\quad (d)\ \int_0^1 \sin(1/x)dx.$

   $(e)\ \int_1^\infty \frac{\sin(1/x)}{x}dx$ $\quad (f)\ \int_0^\infty e^{-x^2}dx$ $\quad (g)\ \int_0^\infty \sin x^2 dx,$ $\quad (h)\ \int_0^{\pi/2} \cot x dx,$ $\quad (i)\ \int_0^\infty \frac{x\log x}{(1+x^2)^2}dx.$

2. (a) Determine all those values of p for which the improper integral $\int_0^\infty \frac{1-e^{-x}}{x^p}dx$ converges.

   (b) Show that the integrals $\int_0^\infty \frac{\sin^2 x}{x^2}dx$ and $\int_0^\infty \frac{\sin x}{x}dx$ converge. Further, prove that $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx.$

3. Prove the following statements.

   (a) Let f be an increasing function on (0,1) and the improper integral $\int_0^1 f(x)$ exist. Then

      i. $\int_0^{1-\frac{1}{n}} f(x)dx \le \frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-1}{n})}{n} \le \int_{\frac{1}{n}}^1 f(x)dx.$

      ii. $\lim_{n\to\infty} \frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-1}{n})}{n} = \int_0^1 f(x)dx.$

   (b) $\lim_{n\to\infty} \frac{ln\frac{1}{n}+ln\frac{2}{n}+\cdots+ln\frac{n-1}{n}}{n} = -1.$

   (c) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$

4. Sketch the graphs of $r = \cos(2\theta)$ and $r = \sin(2\theta)$. Also, find their points of intersection.

5. A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

6. Let C denote the circular disc of radius b centered at $(a, 0)$ where $0 < b < a$. Find the volume of the torus that is generated by revolving C around the y-axis using

   (a) the Washer Method

   (b) the Shell Method.

7. Consider the curve C defined by $x(t) = \cos^3(t),\ y(t) = \sin^3 t,\ 0 \le t \le \frac{\pi}{2}.$

   (a) Find the length of the curve.

   (b) Find the area of the surface generated by revolving C about the x-axis.

   (c) If $(\overline{x}, \overline{y})$ is the centroid of C then find $\overline{y}$.

8. A square is rotated about an axis lying in the plane of the square, which intersects the square only at one of its vertices. For what position of the axis, is the volume of the resulting solid of revolution the largest?

9. Find the centroid of the semicircular arc $(x - r)^2 + y^2 = r^2,\ r > 0$ described in the first quadrant. If this arc is rotated about the line $y + mx = 0,\ m > 0$, determine the generated surface area A and show that A is maximum when $m = \pi/2$.