# MTH 424 - PARTIAL DIFFERENTIAL EQUSTION

**IIT KANPUR**

**Instructor: Indranil Chowdhury**

**Odd Semester, 2024-25**

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**Assignment** 3

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1. Prove that Laplace equation, ($\triangle u = 0$) is rotation invariant. That is, if $O$ is an orthogonal $n \times n$ matrix and $v(x) := u(Ox)$, then $\triangle v = 0$.

2. Consider the following function
$$\phi(x) = \frac{1}{|x - x_0|^{n-2}}.$$
   (a) By direct calculation show that $\phi$ is harmonic in $\mathbb{R}^n \setminus \{x_0\}$.
   (b) For $n = 2$ and $\phi(x) = \frac{1}{|x|}$, show that $\triangle \phi(x) \neq 0$ for $x \neq 0$.

3. For $n = 2$, show that the Laplace equation in polar coordinates for function $v(r\theta)$ be given by
$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex function. Assume $u$ is harmonic. Show that $v := \phi(u)$ is *subharmonic* (i.e. $-\triangle v \leq 0$).

5. Assume $u \in C(\Omega)$. Let $B(x_0, r) \subset \Omega$, show that
$$⨍_{B(x_0,\epsilon)} u(y)dy \xrightarrow[\epsilon \to 0]{} u(x) \quad \text{and} \quad ⨍_{\partial B(x_0,\epsilon)} u(y)dy \xrightarrow[\epsilon \to 0]{} u(x).$$

6. Suppose $\Omega$ is a bounded domain and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy
$$-\triangle u \leq 0 \quad \text{and} \quad -\triangle v \leq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u \leq v \quad \text{in} \quad \partial\Omega.$$
   Prove that $u \leq v \quad \text{in} \quad \Omega$.

7. Let $f \in C_c^2(\mathbb{R}^n)$ for $n \geq 3$ and $\Phi$ be the fundamental solution of the Laplace operator. Define
$$I_\epsilon := \int_{B(0,\epsilon)} \Phi(y) \triangle f(x - y) dy$$
   for $0 < \epsilon << 1$. Show that there exists a constant $C > 0$ such that
$$|I_\epsilon| \leq C\epsilon^2.$$

8. let $\Omega$ be a domain in $\mathbb{R}^2$ symmetric about the $x$-axis and let $\Omega^+ = \{(x, y) \in \Omega : y > 0\}$ be the upper part of $\Omega$. Assume $u \in C^2(\overline{\Omega^+})$ is harmonic in $\Omega^+$ with $u = 0$ on $\partial\Omega^+ \cap \{y = 0\}$. Define for $(x, y) \in \Omega$
$$v(x, y) = \begin{cases} u(x, y) & \text{if} \quad y \geq 0, \\ -u(x, -y) & \text{if} \quad y < 0. \end{cases}$$
   Show that $v$ is harmonic.