MTH 101-Calculus

Spring-2021

Assignment-12: Line and Surface Integrals, Green's /Stokes' /Gauss' Theorems

1. Let $\mathcal{C}$ denotes a differentiable parametric curve, non intersecting curve parametrised by some vector valued function $\gamma : [a, b] \to \mathbb{R}^3$. Also let $\tilde{\mathcal{C}}$ denotes the same curve $\mathcal{C}$ but oriented in reverse direction. Then prove that
$$\int_{\mathcal{C}} f_1 dx + f_2 dy + f_3 dz = -\left(\int_{\tilde{\mathcal{C}}} f_1 dx + f_2 dy + f_3 dz\right)$$
whenever the integral exists.

2. What is the integral of the function $x^2 z$ taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2 + y^2 = a^2$. What is the integral of the given function taken throughout the volume of the cylinder.

3. Find the line integral of the vector field $F(x, y, z) = y\vec{i} - x\vec{j} + \vec{k}$ along the path $\mathbf{c}(t) = (\cos t, \sin t, \frac{t}{2\pi}), \quad 0 \leq t \leq 2\pi$ joining $(1, 0, 0)$ to $(1, 0, 1)$.

4. Evaluate $\int\limits_C T \cdot dR$, where C is the circle $x^2 + y^2 = 1$ and T is the unit tangent vector.

5. Show that the integral $\int\limits_C yzdx + (xz + 1)dy + xydz$ is independent of the path C joining $(1, 0, 0)$ and $(2, 1, 4)$.

6. Use Green's Theorem to compute $\int\limits_C (2x^2 - y^2)\, dx + (x^2 + y^2)\, dy$ where C is the boundary of the region $\{(x, y) : x, y \geq 0 \ \& \ x^2 + y^2 \leq 1\}$.

7. Use Stokes' Theorem to evaluate the line integral $\int\limits_C -y^3\, dx + x^3\, dy - z^3\, dz$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

8. Let $\overrightarrow{F} = \frac{\overrightarrow{r}}{|\overrightarrow{r}|^3}$ where $\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$ and let S be any surface that surrounds the origin. Prove that $\iint\limits_S \overrightarrow{F}.n\, d\sigma = 4\pi$.

9. Let D be the domain inside the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$. If $\vec{F} = (x^2 + ye^z, y^2 + ze^x, z + xe^y)$, use the divergence theorem to evaluate $\iint\limits_{\partial D} F \cdot \mathbf{n}\, d\sigma$.