MTH 101-Calculus
Spring-2021
Assignment-12-Solns: Line and Surface Integrals, Green's /Stokes' /Gauss' Theorems

1. Let $f = (f_1, f_2, f_3)$. Then
$$\int_{\mathcal{C}} f_1 dx + f_2 dy + f_3 dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt.$$
Now consider $\Gamma : [0,1] \to \mathcal{C}$, defined as
$$\Gamma(t) = \gamma(b + (a-b)t).$$
Clearly Γ is a parametrisation of $\tilde{\mathcal{C}}$. Now,
$$\int_{\tilde{\mathcal{C}}} f_1 dx + f_2 dy + f_3 dz = \int_0^1 f(\Gamma(s)) \cdot \Gamma'(s) ds = (a-b) \int_0^1 f(\gamma(b+(a-b)s) \cdot \gamma'(b+(a-b)s) ds.$$
Now substituting $b + (a-b)s = m$ the result follows easily.

2. In the cylinder there are three surfaces S_1, S_2 and S_3 where
   (a) S_1 : The base of the cylinder, i.e., $z = 0$,
   (b) S_2 : The top of the cylinder i.e., $z = h$,
   (c) S_3 : The curved surface of the cylinder.

   (a) On S_1, the integral is zero.
   (b) The surface integral over $S_2 = \iint\limits_{S_2} x^2 z d\sigma = \int\limits_0^a \int\limits_0^{2\pi} (r\cos\theta)^2 hr d\theta dr = \dfrac{ha^4\pi}{4}$.
   (c) A parametric representation of S_3 is
   $$r(u,v) = (a\cos u, a\sin u, v), 0 \le u \le 2\pi, 0 \le v \le h.$$
   The surface integral over $S_3 = \iint\limits_{S_3} x^2 z d\sigma = \int\limits_0^h \int\limits_0^{2\pi} x^2 z \parallel r_u \times r_v \parallel dudv$
   $= \int\limits_0^h \int\limits_0^{2\pi} (a\cos u)^2 v\sqrt{EG - F^2} dudv$, where $E = r_u \cdot r_u$, $G = r_v \cdot r_v$ and $F = r_u \cdot r_v$.
   Note that $\sqrt{EG - F^2} = a$. Therefore, $\iint\limits_{S_3} x^2 z d\sigma = \frac{a^3h^2\pi}{2}$.

   Hence, the required integral is $\frac{ha^4\pi}{4} + \frac{a^3h^2\pi}{2}$.
   Over the entire volume, the integral is
   $$V = \int\limits_0^h \int\limits_0^{2\pi} \int\limits_0^a (r\cos\theta)^2 zr dr d\theta dz = \frac{h^2\pi a^4}{8}.$$

3. $\int\limits_C (y, -x, 1) \cdot dR = \int\limits_0^{2\pi} ((\sin t)(-\sin t) dt - \cos t \cos t + \frac{1}{2\pi}) dt.$

4. Take $C = R(t) = (cost, sint),\ 0 \le t \le 2\pi$. Then
$$\int\limits_C T \cdot dR = \int\limits_0^{2\pi} T(t) \cdot R'(t) dt = \int\limits_0^{2\pi} \frac{R'(t)}{\parallel R'(t) \parallel} \cdot R'(t) dt = 2\pi$$

5. If $F = yzi + (xz+1)j + xyk$, then $F = \nabla\varphi$, where $\varphi(x,y,z) = xyz + y$. Hence, by the 2nd fundamental theorem of calculus for line integrals, the problem follows.

6. $M = 2x^2 - y^2$ and $N = x^2 + y^2$. By Green's Theorem

$\int\limits_C (2x^2 - y^2)dx + (x^2+y^2)dy = \int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} (N_x - M_y)dy\,dx$

$= \int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} 2(x+y)dy\,dx = \frac{4}{3}.$

7. Let $F = -y^3\vec{i} + x^3\vec{j} - z^3\vec{k}$. By Stoke's Theorem, $\int\limits_{\partial S} F.dr = \int\int\limits_S (curl\, F).\vec{n}d\sigma$.

Note that $\nabla \times F = 3(x^2+y^2)\vec{k}$. Hence, $\int\limits_{\partial S} F.dr = \iint\limits_D 3(x^2+y^2)dxdy = \frac{3\pi}{2}$.

8. Note that $div\ F = 0$. By divergence theorem

$$\iint\limits_S F\cdot n d\sigma = \iint\limits_{S_\rho} F\cdot n d\sigma$$

where S_ρ is a sphere of (small) radius ρ with center at origin. On S_ρ, $n = \frac{1}{\rho}(xi+yj+zk)$ and hence $F\cdot n = \frac{1}{\rho^2}$. Therefore,

$$\iint\limits_{S_\rho} F\cdot n d\sigma = \frac{1}{\rho^2}\iint\limits_{S_\rho} d\sigma = \frac{1}{\rho^2}4\pi\rho^2 = 4\pi.$$

9. $div\, F = 2x + 2y + 2z$. By the divergence theorem,

$$\int\int\limits_{\partial D} F.\vec{n}d\sigma = \int\int\int\limits_D 2(x+y+z)dV = 2\int\int\limits_{x^2+y^2\le 1} (\int_0^{x+2} (x+y+z)dz)dxdy = \frac{19\pi}{4}$$

10. Discuss the differentiability of the function $f(x,y) = \sin(x)\sqrt{|xy|}$ at all the on the x-axis.