MTH 424 - PARTIAL DIFFERENTIAL EQUSTION

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Assignment 4

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1. Show that the Fundamental solution Φ is locally integrable for any dimension $n \geq 2$. Check integrability of its partial derivatives.

2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Show that the following problem

$$\begin{cases} \triangle u = u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

has only solution $u \equiv 0$. [HINT: Multiply both side of the equation by u and integrate over Ω.]

3. (a) Define a C^2 *sub-harmonic* function and prove that for any bounded domain Ω

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

(b) Construct a C^2 function such that

$$\begin{cases} \Delta u = 2n \text{ in } B_1(0) \in \mathbb{R}^n \\ u = 1 \text{ on } \partial B_1(0) \end{cases}$$

(c) Assume, $u \in C^2(\overline{B(0,1)})$ which solves the following equation

$$\begin{cases} -\Delta u = f \text{ in } B_1(0) \\ u = 0 \text{ on } \partial B_1(0). \end{cases}$$

Show that

$$\max_{B_1(0)} |u(x)| \leq \max_{B_1(0)} |f(x)|.$$

4. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$Lu = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \geq 0,$$

where $a_{ij} = a_{ji}, b_i,\ c$ are real valued smooth functions on Ω and $c < 0$ and $(a_{i,j})_{i,j}$ is a non-negative definite matrix. Then show that u can not take interior positive maximum in Ω.

[HINT: If $x_0 \in \Omega$ is point of maximum, then prove first that $\displaystyle\sum_{i,j=1}^{n} a_{i,j}(x_0) \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \leq 0$]

5. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of

$$\triangle u + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) = 0 \quad \text{in} \quad \Omega$$

with $c(x) < 0$ in Ω, $u = 0$ on $\partial\Omega$. Show that $u \equiv 0$. [HINT: Use previous result with u and $-u$]

6* Let $K(x, y)$ be the Poisson Kernel of $-\triangle$ in $B(0, 1)$. Show that

(a) $\int_{\partial B(0,1)} K(x, y)dy = 1$,

(b) If $u(x) = \int_{\partial B(0,1)} K(x, y)g(y)dy$, for some $g \in C(\partial B(0, 1))$ and $x \in B(0, 1)$, then $\triangle u = 0$ in $B(0, 1)$.

(c) If u is as above, then $\lim_{x \to x_0} u(x) = g(x_0)$ for all $x_0 \in \partial B(0, 1)$.

7* Use Poisson formula for unit ball to show that

$$\frac{1 - |x|^2}{(1 + |x|)^n} u(0) \le u(x) \le \frac{1 - |x|^2}{(1 - |x|)^n} u(0).$$