CS648A : Randomized Algorithms

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An interesting problem based on partition theorem

Sketch of the solution is given on the last page. Use it only after making sincere attempt(s).

Recall the partition theorem, which states that if events $\mathcal{E}_1, ..., \mathcal{E}_\ell$ form a partition of a sample space Ω, and A is any event, then

$$\mathbf{P}[A] = \sum_{j=1}^{\ell} \mathbf{P}[A|\mathcal{E}_j] \cdot \mathbf{P}[\mathcal{E}_j]$$

This theorem can sometimes be used very effectively to calculate probability of event A when any direct method of calculating $\mathbf{P}[A]$ appears difficult. But applying this theorem effectively requires some creative skills and a better insight into the problem. In general, it works when the partition formed is such that calculating $\mathbf{P}[A|\mathcal{E}_j]$ is very easy for each $\mathcal{E}_j$ (in fact usually it turns out to be independent of j). We have already seen an application of partition theorem in the lectures.

As a warm up, try to solve the following problem.

There are n sticks each of different heights. There are n vacant slots arranged along a line and numbered from 1 to n as we move from left to right. The sticks are placed into the slots according to a uniformly random permutation. A stick placed at ith slot is said to be a dominating stick if its height is largest among all sticks placed in slots 1 to $i-1$. Let Let A be the event that the ith slot contains a dominating stick. Find the probability of A.

The first, and perhaps the most natural, approach to solve this problem would be to use the partition defined by the rank of the stick occupying ith slot. Conditioned on the event that jth smallest stick occupied ith slot ($j \geq i$), the probability of event A would be the following.

$$\frac{\binom{j-1}{i-1}(i-1)!(n-i)!}{n!}$$

In order to calculate unconditional probability $\mathbf{P}(A)$, you need to sum the above expression for all $i \leq j \leq n$. Convince yourself that this approach, and hence this partition scheme, does not work. Now think of a better partition.

With the above problem as a warm-up, now solve the following main problem. Elgoog, a very reputed company, is going to visit IITK to hire the best qualified (based on knowledge of the fundamentals of computer science, analytical & creative skills) student in his/her final year. There are n applicants and n is obviously really huge since Elgoog is offering a huge package. They will select a person based totally on his/her qualification which can be revealed only through interview. Assume that there is a total order among all n applicants as far as their qualifications are concerned. Since n is huge, it is not possible to interview every applicant. Furthermore, the placement office requires that each applicant should be informed about his/her selection or rejection immediately after the interview. Therefore, the following strategy is followed by Elgoog. They fix a number $k < n$. They interview and reject first k applicants. After that they continue taking interviews and stop as soon as they find an applicant better than the first k applicants. If they don't find any applicant better than the first k applicants, they return without hiring any one.

1. Assuming the applicants appear in a uniformly random order (all permutations are equally likely), what is the probability in terms of k and n that Elgoog will be successful in selecting the best qualified applicant ?
2. For what value of k, is the probability of selecting the best qualified applicant maximum ?

Sketch:
Step 1. Observe that the probability is going to depend upon the location of the best candidate. Suppose the best candidate occupies location i, where $i > k$. Conditioned on this event, formulate a necessary and sufficient condition to select the best candidate.

Go to the next page only after spending sufficient time on Step 1 given above.

Step 2. The condition: The best candidate among the 1st $(i-1)$ candidates occupies a position $\leq k$.

Step 3. Partition the sample space *suitably* to find the probability of the condition mentioned above.

Go to the next page only after spending sufficient time on the steps given above.

The probability is:

$$\frac{k}{i-1}$$

Step 4. Use the above expression to calculate the required probability of selecting the best candidate.

Step 5. *Differentiate* it *suitably* with respect to k to get the maximum probability.