DMS625: Practice Assignment

November 16, 2024

Renewal theory

1. The production phase in a factory follows an exponential distribution with a mean of 5 hours. After the production phase the factory undergoes maintenance, which is uniformly distributed between 1 and 3 hours. Assume that the durations of production and maintenance phases are independent of each other. In the long run, what fraction of time is the factory in production?

2. Passengers arrive at a city's central bus station following a Poisson process with rate of 1.5 per minute to catch a shuttle to the airport. The shuttle departs when n passengers have boarded. The shuttle company pays the driver \$25 for each trip to the airport. Additionally, the company estimates it loses \$0.08 in goodwill for each minute a passenger waits for the shuttle. What shuttle group size n minimizes the average cost per passenger?

3. A scientist has a machine for measuring ozone in the atmosphere that is located in the mountains just north of Los Angeles. At times of a Poisson process with rate 1, storms or animals disturb the equipment so that it can no longer collect data. The scientist comes every L units of time to check the equipment. If the equipment has been disturbed then she can usually fix it quickly so we will assume the repairs take 0 time.

   (a) What is the limiting fraction of time the machine is working?

   (b) Suppose that the data that is being collected is worth k dollars per unit time, while each inspection costs $c < k$. Find the best value of the inspection time L.

4. Let $N_1(t)$ and $N_2(t)$ be two independent renewal processes. Let $N(t) = N_1(t) + N_2(t)$. Then is $N(t)$ necessarily a renewal process?

5. In a football match between Barcelona and Real Madrid, the teams alternate possession of the ball. On average, Barcelona maintains possession for 4 minutes each time they get the ball, while Real Madrid maintains possession for 3 minutes.

   (a) In the long run, what fraction of time does Barcelona have possession of the ball?

   (b) Suppose that on each possession, Barcelona has a 20% chance of scoring a goal and Real Madrid has a 15% chance of scoring a goal. On average, how many goals will each team score per hour?

Hints

1. Think of this as a renewal reward process, where the length of each interval is production + maintenance time and the reward is the expected amount of time the factory is in production. Therefore, the answer is,
$$\frac{\mathbb{E}[R]}{\mathbb{E}[X]} = \frac{5}{5+2}$$

2. S_n is the waiting time for n arrivals, since the arrivals happen via a Poisson process. Therefore, $\mathbb{E}[S_n] = \frac{n}{1.5}$

   Suppose now n passengers are in the system, the first passenger that came would have to wait on average $\frac{n-1}{1.5}$ time for the n-th passenger to arrive. The second passenger would have to wait on average $\frac{n-2}{1.5}$ time for the n-th passenger to arrive and so on. The $n-1$-th passenger would have to wait $\frac{1}{1.5}$ time on average for the n-th passenger to arrive. Therefore, the collective waiting time incurred by all the n passengers is,
$$\frac{n-1}{1.5} + \frac{n-2}{1.5} + \ldots + \frac{1}{1.5} = \frac{n(n-1)}{3}$$

   Let W denote the average waiting time *per-passenger* then,
$$W = \frac{n(n-1)}{3n} = \frac{n-1}{3}$$

   Then average per-passenger goodwill cost is,
$$G = W \times 0.08$$

   Let D denote the average per-passenger shuttle cost,
$$D = \frac{25}{n}$$

   Now minimize $G + D$ to obtain an optimal value of n.

3. (a) See it as a renewal reward process, same as Q1.

   (b) Similar to Q2.

4. No. Find out a counterexample.

5. (a) The possession of the ball goes from Barcelona to Real Madrid to Barcelona to Real Madrid and so on. Define the length of an interval to be possession by Barcelona and by Real Madrid, i.e., average length is 7 mins. The reward is the amount of time Barcelona has possession in this interval.

   (b) In the above defined interval, Barcelona will have possession once and Real Madrid will have possession once. Count the average number of intervals in an hour, therefore you will get the average number of times Barcelona and Real Madrid will have possession in an hour. Then obtain the average goals each team will score in an hour using the probabilities.