Initial Value Problems: Accuracy and Stability

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Global error

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where y_k is the computed solution at $t = t_k$ and $y(t)$ is the exact/true solution of the ODE passing through the initial point (t_0, y_0).

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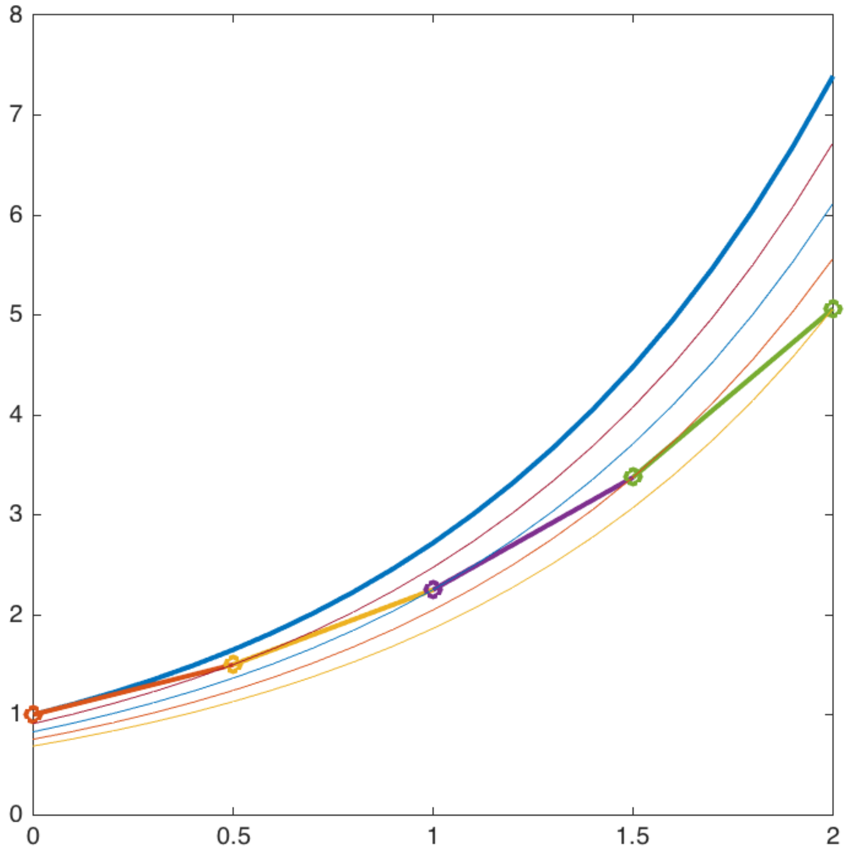

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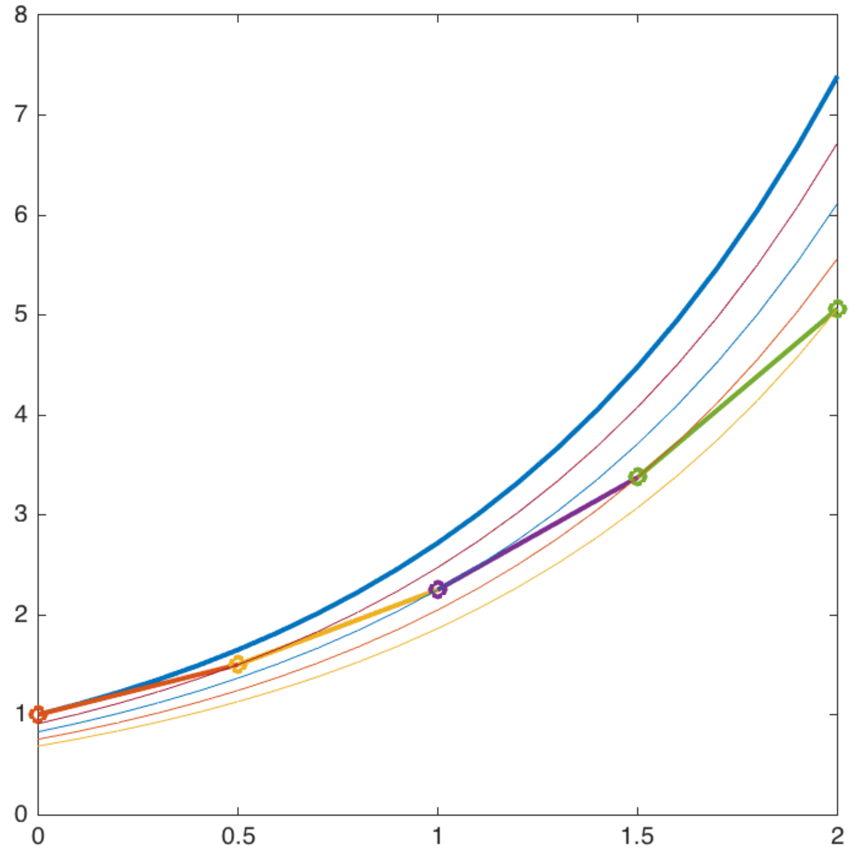

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We see that the local error at a given time step is simply the amount by which the solution of the ODE fails to satisfy the method.

More generally, for a one step method

$$y_{k+1} = y_k + h_k \phi(t_k, y_k, h_k)$$

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In order to assess the effectiveness of a numerical method, we need to characterize both

a) *its local error (accuracy), and*
b) *the compounding effects over multiple steps (stability).*

Module 2
Initial Value Problems

2.1 Well-posedness

2.2 Stability

2.3 Euler's method

- Accuracy and Stability

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The accuracy of a numerical method is said to be of order p if

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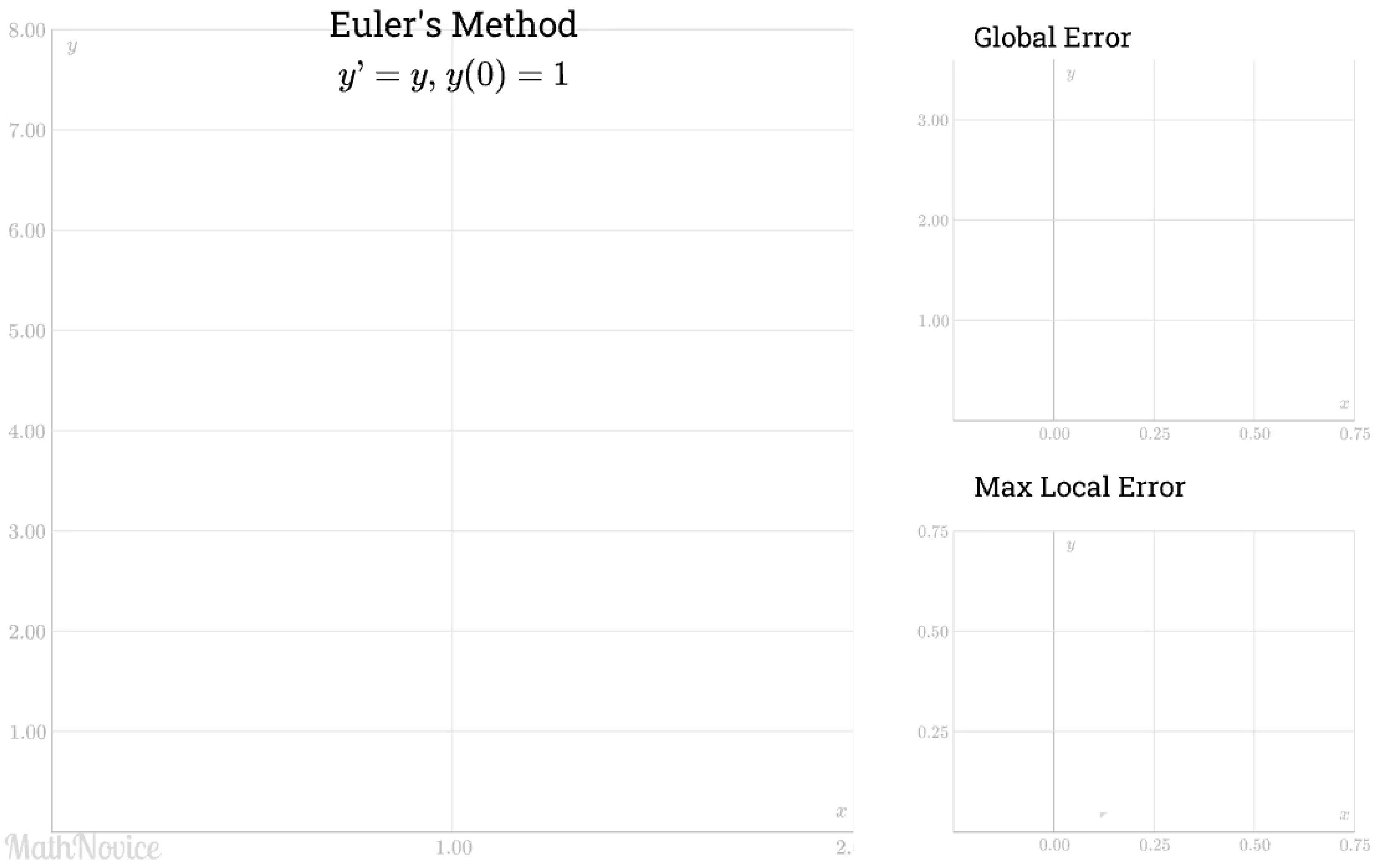

Source: https://www.youtube.com/watch?v=ivyNR1w9-zk

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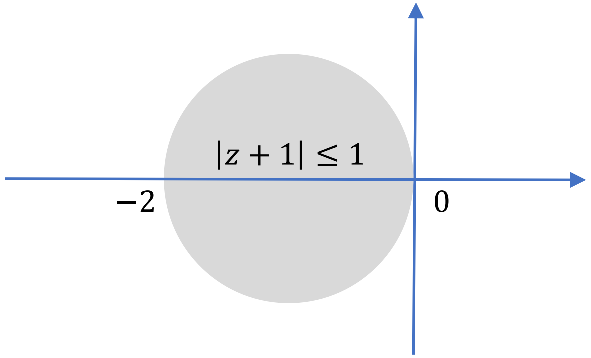

Initial Value Problems: Accuracy and Stability

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which is satisfied if all eigenvalues of $h_k f'$ lie inside the circle in the complex plane of radius 1 and centered at -1.