Lesson 4

Numerical Solution of PDE

4.3 Hyperbolic PDE

- Finite Difference Methods

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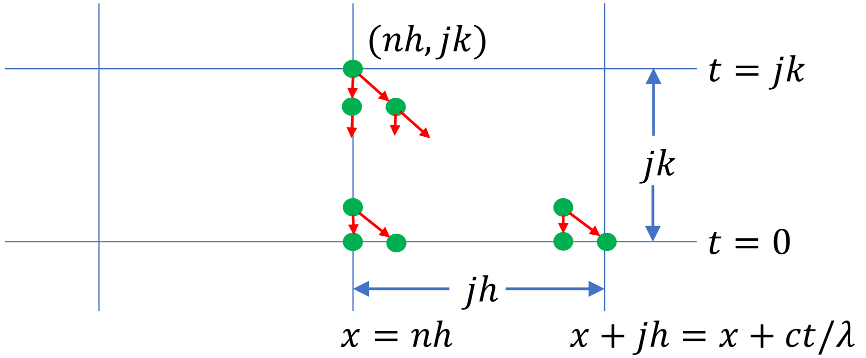

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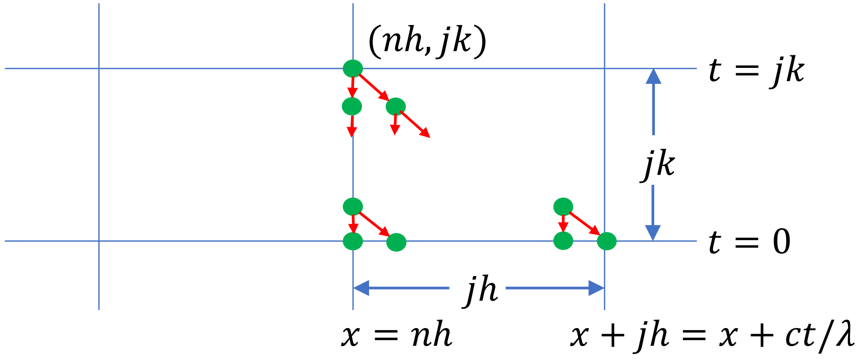

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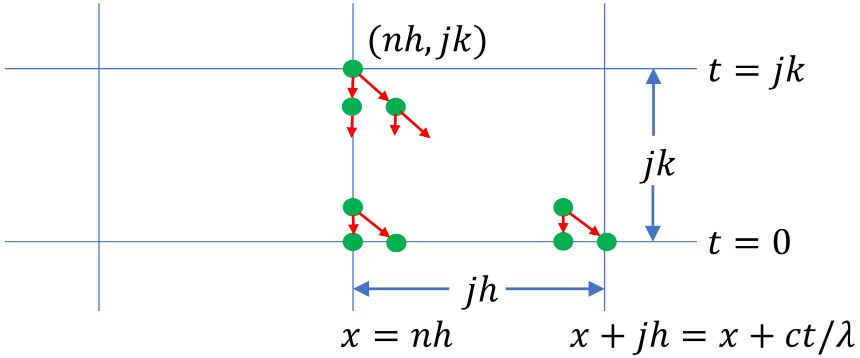

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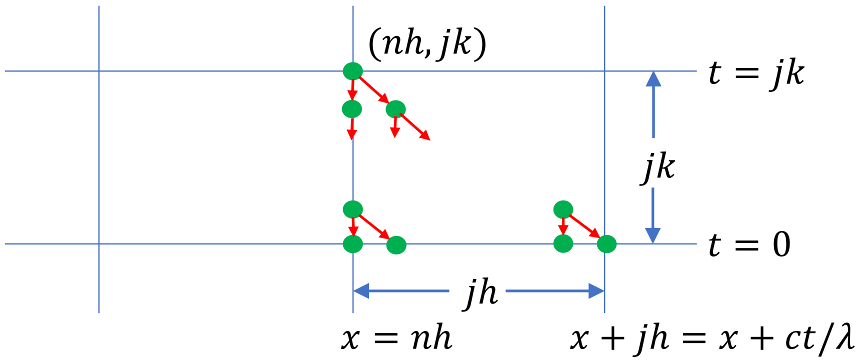

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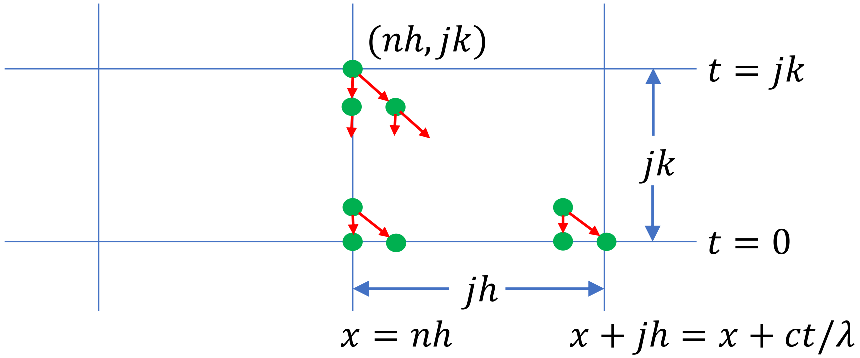

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This necessary condition, which fails for forward-forward difference method, is called the Courant-Friedrichs-Levy condition, or CFL condition.

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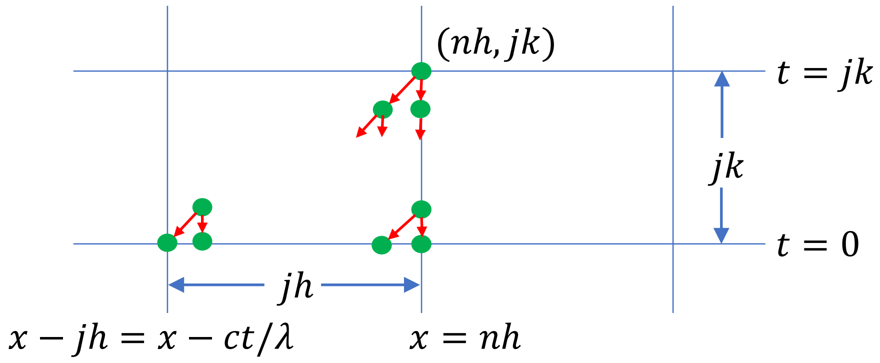

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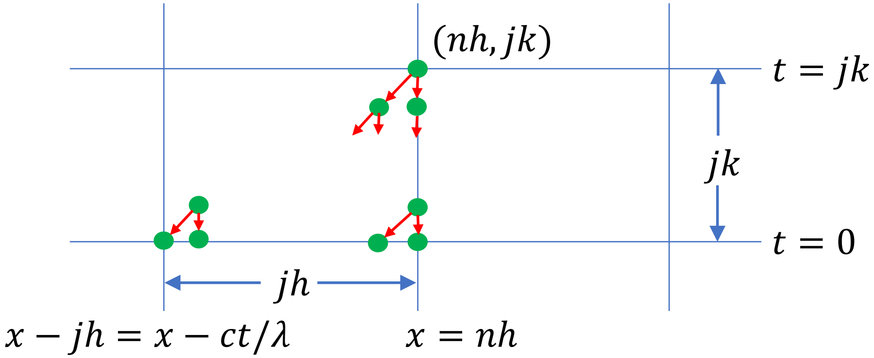

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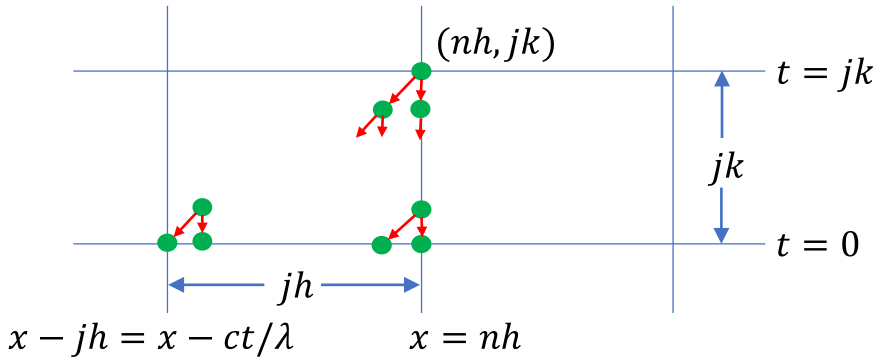

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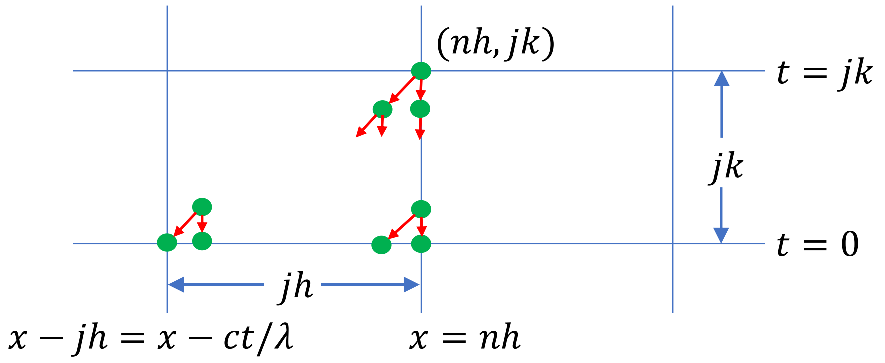

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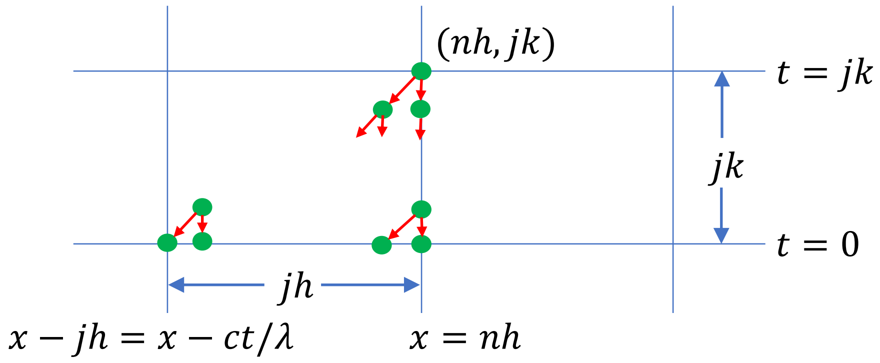

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In general, however, CFL is not sufficient for convergence. It turns out that, for forward-central scheme, while the CFL condition is $|\lambda| \leq 1$, the method is unconditionally unstable.

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- *Finite Difference Methods*
- **Stability Analysis**

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Let’s use the Fourier analysis (von Neumann stability analysis) to study the stability of these methods.

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For simplicity, let's consider a 1-periodic problem rather that a boundary value problem:

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\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0, \qquad x \in \mathbb{R}, t > 0,\\
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For fixed $N > 0$, let $h = 1/N$, we take the spatial mesh $h\mathbb{Z} = \{nh: n \in \mathbb{Z}\}$ and we seek a solution $u_n^j \approx u(nh, jk)$ which satisfies the periodicity condition

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The ψ_m are orthogonal with respect to the inner product (exercise)

$$\langle \phi, \psi \rangle_h = h \sum_{n=0}^{N-1} \phi(nh)\, \overline{\psi(nh)}.$$

Note that ψ_m is an eigenvector for the forward difference operator D_h^+, the backward difference operator D_h^- and the centered difference operator D_h. For example,

$$D_h^- \psi_m(x) = \frac{\psi_m(x) - \psi_m(x-h)}{h} = \frac{1 - e^{-2\pi i m h}}{h} \psi_m(x).$$

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$$D_h^-\psi_m(x) = \frac{\psi_m(x) - \psi_m(x-h)}{h} = \frac{1 - e^{-2\pi imh}}{h}\psi_m(x).$$

It is also an eigenvector for the forward and backward shift operators S_h^+ and S_h^- defined as

$$S_h^+v(x) = v(x+h), \qquad S_h^-v(x) = v(x-h),$$

where

$$S_h^+\psi_m(x) = e^{2\pi im(x+h)} = e^{2\pi imh}\psi_m(x),$$
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Now, consider forward-backward difference method in the operator form:

$$u^{j+1} = (1-\lambda)u^j + \lambda S_h^- u^j = \left((1-\lambda)I + \lambda S_h^-\right)u^j.$$

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Thus, for stability of the method, we need that the eigenvalues are less than or equal to one in magnitude, that is

$$\left|1 - \lambda + \lambda e^{-2\pi imh}\right| \leq 1.$$

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As $1 - \lambda + \lambda e^{-2\pi i m h}$ describes are circle centered at $1 - \lambda$ of radius $|\lambda|$, we see that the method is stable if and only if $0 \leq \lambda \leq 1$.