Solution of Quiz-2

(1) $f(1)f(3) < 0$ and f be continuous (assume it).

$\exists\ r \in (1,3)$ such that $f(r) = 0$.

Let $\{x_k\}_{k=1}^{\infty}$ be bisection sequence

such that $\lim_{k\to\infty} x_k = r$.

We know by error estimate (don't need to prove it in exam hall)

that $\forall\ k \geq$

$$|x_k - r| \leq \frac{1}{2^k}(b_0 - a_0), \quad \forall\ k \in \mathbb{N}.$$

To determine the number of iteration k

so that

$$\frac{1}{2^k}(b_0 - a_0) \leq \varepsilon.$$

$$\Rightarrow \ln(b_0 - a_0) - \ln 2^k \leq \ln \varepsilon.$$

$$\Rightarrow \ln(b_0 - a_0) - k\ln 2 \leq \ln \varepsilon.$$

$$\Rightarrow \ln(b_0 - a_0) - \ln \varepsilon \leq k \ln 2$$

$$\Rightarrow \frac{\ln(b_0 - a_0) - \ln \varepsilon}{\ln 2} \leq k.$$

Taking $a_0 = 1,\ b_0 = 3 \Rightarrow k \geq \frac{\ln 2 - \ln \varepsilon}{\ln 2} = \left(1 - \frac{\ln \varepsilon}{\ln 2}\right)$

P-2

(2) $n \in \mathbb{N}$, $n \geq 2$, $y > 0$ given.

To find $y^{1/n}$, we form the equation

$$x^n - y = 0.$$

Let $f(x) = x^n - y$, $x \geq 0$.

Then, $f'(x) = n x^{n-1}$, $x \geq 0$.

Newton-Raphson Scheme is given by, $\forall$ $k \geq 0$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{(x_k^n - y)}{n x_k^{n-1}}$$

$$= x_k - \frac{1}{n} x_k + \frac{y}{n} \cdot \frac{1}{x_k^{n-1}}$$

$$\boxed{x_{k+1} = \left(1 - \frac{1}{n}\right) x_k + \frac{y}{n} \cdot \frac{1}{x_k^{n-1}}}$$

$\forall$ $k \geq 0$.

(3)

Advantage: Secant method does not require the evaluation of $f'(x_k)$ for any $k \in \mathbb{N} \cup \{0\}$, whereas Newton-Raphson method requires ~~to~~ the evaluation $f'(x_k)$ at every $k \in \mathbb{N} \cup \{0\}$.

Dis-advantage: (i) Secant method requires two initial approximations say x_0, x_1, whereas N-R rerequires only one initial approx.

(ii) Convergence rate of Secant method is 1.68, whereas convergence rate of N-R method is 2.

i.e. Secant method is slower than N-R.

$\boxed{P-4}$

(4) To ~~solve~~ find root r of the eqn. $f(x) = 0$, using fixed pt. iteration

we need to choose g so that

~~an interval $[a,b] \subset \mathbb{R}$ so that~~

~~$r \in [a,b]$ and~~

~~$g : [a,b] \to [a,b]$ continuous.~~

~~This gives~~ (i) $g(r) = r$ and

(ii) $\exists$ $L \in (0,1)$ and $\delta > 0$ such that

g' is contn. and $|g'(x)| \leq L$ $\forall$ $x \in [r-\delta, r+\delta]$

Then, the sequence

$\{x_k\}_{k \geq 0}$ defined by

$$x_k = g(x_{k-1}) \quad \forall\ k \geq 1,$$

and $x_0 \in [r-\delta, r+\delta]$.

is well-defined and converges to r.

(ii) Rate of convergence: $\forall\ k \geq 1$

$$|x_{k+1} - r| = |g(x_k) - g(r)|$$

$$= |g'(\xi_k)||x_k - r| \quad \text{by Lagrange M.V.T.}$$

for some $\xi_k \in (x_k, r)$

or $\xi_k \in (r, x_k)$

$$\frac{|x_{k+1} - r|}{|x_k - r|} = |g'(\xi_k)|$$

Assume $x_k \neq r$ $\forall\ k \geq 1$

As $\lim_{k \to \infty} x_k = r \Rightarrow \lim_{k \to \infty} \xi_k = r$

$$\Rightarrow \lim_{k \to \infty} g'(\xi_k) = g'(r).$$

$$\lim_{k \to \infty} \frac{|x_{k+1} - r|}{|x_k - r|} = \lim_{k \to \infty} |g'(\xi_k)| = |g'(r)|.$$

If $g'(r) \neq 0$, Rate of convergence is 1.

(iii) If $g'(r) = 0 = g''(r)$, $g'''(r) \neq 0$.

and g is C^3. By Taylor's theorem,

$$x_{k+1} - r = g(x_k) - g(r)$$

$$= g'(r)(x_k - r) + \frac{g''(r)}{\lfloor 2}(x_k - r)^2 + \frac{g'''(\xi_k)}{\lfloor 3}(x_k - r)^3$$

P-6

for some $\xi_K \in (x_K, r)$ or (r, x_K).

$$x_{K+1} - r = \frac{g'''(\xi_K)\,(x_K - r)^3}{\underline{|3}}$$

$$\lim_{K\to\infty} \frac{|x_{K+1} - r|}{|x_K - r|^3} = \lim_{K\to\infty} \frac{1}{6}\,|g'''(\xi_K)|$$

$$= \frac{1}{6}\,|g'''(r)| \neq 0. \text{ as. } g \text{ is } C^3.$$

Therefore, assumption for cubic convergence.

$$g'(r) = 0 = g''(r)$$

$$g'''(r) \neq 0$$

$$g \text{ is } C^3.$$

5

Over	Run	1st	2nd	3rd.	
1	3				
		$\frac{7-3}{2-1} = 4$			
2	7		$\frac{7-4}{4-1} = 1$		
		$\frac{21-7}{4-2} = 7$		$\frac{1-1}{8-1} = 0$	
			$\frac{13-7}{8-2} = 1$		
4	21				
		$\frac{73-21}{8-4} = 13$			
8	73				

Solution of Quiz-2

Interpolating polynomial of degree ≤ 3 is given by

$$P_3(x) = 3 + 4(x-1) + 1\cdot(x-1)(x-2)$$
$$+ 0\cdot(x-1)(x-2)(x-4)$$

$$P_3(x) = 3 + 4(x-1) + (x-1)(x-2)$$

$$P_3(10) = 3 + 4\cdot(10-1) + (10-1)(10-2)$$
$$= 3 + 36 + 72 = 111$$ ← Run at over 10

$$P_3(3) = 3 + 4(3-1) + (3-1)(3-2)$$
$$= 3 + 8 + 2 = 13$$ ← Run at over 3.

⑥

$F(x)$	x		
-1	3		
		$\frac{4-3}{2+1} = \frac{1}{3}$	
2	4		$\frac{\frac{3}{4}-\frac{1}{3}}{-2+1} = -\frac{(9-4)}{12} = -\frac{5}{12}$
		$\frac{1-4}{-2-2} = \frac{3}{4}$	
-2	1		

To approximate f^{-1} we construct the following table.

P-8

To ~~#~~ approximate $f^{-1}(y)$, from the above ~~following~~ data, ~~we~~ corresponding interpolating polynomial of degree ≤ 2 is

$$P_3(y) = 3 + \frac{1}{3}(y+1) - \frac{5}{12}(y+1)(y-2)$$

To find $f^{-1}(0)$, we calculate an approx.

$$P_3(0) = 3 + \frac{1}{3}(0+1) - \frac{5}{12}(0+1)(0-2)$$

$$= 3 + \frac{1}{3} + \frac{5}{6}$$

$$= \frac{18+2+5}{6} = 25/6.$$

(f) To construct the cubic spline

$$S(x) = \begin{cases} S_0(x), & x \in [1,2] \\ S_1(x), & x \in [2,3]. \end{cases}$$

~~$S_i(x) = a_i + b_i($~~

$$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3. \quad x \in [1,2]$$

$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3, \quad x \in [2,3].$$

Solution of Quiz - 2

$$S_0'(x) = b_0 + 2c_0(x-1) + 3d_0(x-1)^2$$

$$S_1'(x) = b_1 + 2c_1(x-2) + 3d_1(x-2)^2$$

$$S_0''(x) = 2c_0 + 6d_0(x-1)$$

$$S_1''(x) = 2c_1 + 6d_1(x-2).$$

$$\left.\begin{aligned}
&S_0(1) = a_0 = 2 \\
&S_0(2) = S_1(2) = a_1 \\
&a_1 = a_0 + b_0 + c_0 + d_0 \\
&S_1(3) = a_2 = 5 \\
&a_2 = a_1 + b_1 + c_1 + d_1
\end{aligned}\right\}$$

$$S_0'(2) = S_1'(2)$$

$$\left.\begin{aligned}
\Rightarrow\ & b_1 = b_0 + 2c_0 + 3d_0 \\
& b_2 := S_1'(3) = b_1 + 2c_1 + 3d_1
\end{aligned}\right\}$$

$$S_0''(2) = S_1''(2)$$

$$\left.\begin{aligned}
\Rightarrow\ & 2c_1 = 2c_0 + 6d_0 \Rightarrow c_1 = c_0 + 3d_0 \\
& c_2 := \tfrac{1}{2} S_1''(3) \Rightarrow c_2 = c_1 + 3d_1
\end{aligned}\right\}$$

$$\Rightarrow d_i = \frac{1}{3}(c_{i+1} - c_i), \quad i = 0, 1$$

$\boxed{P-10}$ For natural cubic spline

$$S_0''(1) = 0 = S_1''(3)$$

$$S_0''(1) = 0 \Rightarrow \quad 2c_0 + 6d_0(1-1) = 0 \Rightarrow c_0 = 0$$

$$S_1''(3) = 0 \Rightarrow \quad c_2 = \tfrac{1}{2} \cdot S_1''(3) = 0 \Rightarrow c_2 = 0.$$

$$\Rightarrow \quad b_i = b_{i-1} + 2c_{i-1} + 3d_{i-1} \qquad i = 1,2$$

$$= b_{i-1} + 2c_{i-1} + (c_i - c_{i-1})$$

$$b_i = b_{i-1} + c_{i-1} + c_i \quad , i = 1,2$$

$$\Rightarrow \begin{cases} b_1 = b_0 + c_0 + c_1 \\ b_2 = b_1 + c_1 + c_2 \end{cases}$$

Now, Again $b_i = a_{i+1} - a_i - c_i - d_i$

$$= a_{i+1} - a_i - c_i - \tfrac{1}{3}(c_{i+1} - c_i)$$

$$= (a_{i+1} - a_i) - \tfrac{1}{3}(c_{i+1} + 2c_i)$$

$$i = 0, 1.$$

$$\Rightarrow (a_2 - a_1) - \tfrac{1}{3}(c_2 + 2c_1) = (a_1 - a_0) - \tfrac{1}{3}(c_1 + 2c_0) + c_0 + c_1$$

$$\Rightarrow (a_2 - a_1) = (a_1 - a_0) + c_0(1 - \tfrac{2}{3}) + c_1(1 - \tfrac{1}{3} + \tfrac{2}{3}) + c_2(\tfrac{1}{3})$$

Solution of Quiz-2

$$\tfrac{1}{3}c_0 + \tfrac{4}{3}c_1 + \tfrac{1}{3}c_2 = (a_2 - a_1) - (a_1 - a_0)$$

$$c_0 + 4c_1 + c_2 = 3(a_2 - a_1) - 3(a_1 - a_0). \longrightarrow \circledast_1$$

One can write the following system directly without doing the previous calculation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3(a_2-a_1) - 3(a_1-a_0) \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad c_0 = 0, \quad c_2 = 0$$

$$\Rightarrow \quad 4c_1 = 3(5-3) - 3(3-2)$$

$$\Rightarrow \quad 4c_1 = 3\times 2 - 3\times 1 = 6 - 3 = 3$$

$$c_1 = 3/4.$$

$$b_1 = (5-3) - \tfrac{1}{3}(2\times \tfrac{3}{4} + 0) = 2 - \tfrac{1}{2} = \tfrac{3}{2}$$

$$b_0 = (3-2) - \tfrac{1}{3}(2\times 0 + \tfrac{3}{4}) = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

$$d_1 = \tfrac{1}{3}(c_2 - c_1) = -\tfrac{1}{3}\times\tfrac{3}{4} = -\tfrac{1}{4}$$

$$d_0 = \tfrac{1}{3}(c_1 - c_0) = \tfrac{1}{3}\times\tfrac{3}{4} = \tfrac{1}{4}.$$

$\boxed{P-12}$

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, & 1 \le x \le 2 \\ 3 + 3/2\,(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, & 2 \le x \le 3. \end{cases}$$

(8) Error for trapezoidal formula.

$$E_T(f, h) = -\frac{h^2}{12}(b-a) f''(\eta) \quad \text{for some } \eta \in [a,b]$$

$$|E_T(f,h)| \le \frac{h^2(b-a)}{12} \sup_{[a,b]} |f''(x)|.$$

Trapezoidal rule gives ~~accu~~ accurate result

if $\sup_{[a,b]} |f''(x)| = 0$

$\Rightarrow f''(x) = 0 \quad \forall x \in [a,b]$

$\Rightarrow f'(x) = c_1$ for some $c_1 \in \mathbb{R}$.

$\Rightarrow f(x) = c_1 x + d_1$

(1-degree polynomial)

Error for Simpson's 1/3 rule.

$$E_S(f,h) = -\frac{h^4(b-a)}{180} f^{(iv)}(\eta)$$

$$|E_S(f,h)| \le \frac{h^4(b-a)}{180} \sup_{[a,b]} |f^{(iv)}(x)|.$$

Simpson's 1/3 $\Rightarrow$ accurate result. $f^{(iv)}(x) = 0 \quad \forall x \in [a,b]$

$\Rightarrow f'''(x) = c_2 \Rightarrow f''(x) = c_2 x + d_2 \Rightarrow f(x) = \frac{c_2}{6}x^3 + \frac{d_2 x^2}{2} + e_2 x + a_2$

(cubic poly.)