Lesson 5

Integral Equations

5.2 An Introduction

5.3 Numerical Methods

- Degenerate Kernel Method
- **Projection Method**

Akash Anand
MATH, IIT KANPUR

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

Finally, we will look as some other numerical solution ideas that can be employed for numerical solution of the Fredholm integral equations without going into the detailed convergence analysis.

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

Finally, we will look as some other numerical solution ideas that can be employed for numerical solution of the Fredholm integral equations without going into the detailed convergence analysis.

We begin by noting that the collocation method and the Galerkin method can also be employed to numerically solve these equations.

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

Finally, we will look as some other numerical solution ideas that can be employed for numerical solution of the Fredholm integral equations without going into the detailed convergence analysis.

We begin by noting that the collocation method and the Galerkin method can also be employed to numerically solve these equations.

For example, if we let the approximation space be given by

$$X_n = \mathrm{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

so that we seek the approximate solution in the form

$$u_n = \sum_{j=1}^{n} \gamma_j \varphi_j \,.$$

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

Finally, we will look as some other numerical solution ideas that can be employed for numerical solution of the Fredholm integral equations without going into the detailed convergence analysis.

We begin by noting that the collocation method and the Galerkin method can also be employed to numerically solve these equations.

For example, if we let the approximation space be given by

$$X_n = \mathrm{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

so that we seek the approximate solution in the form

$$u_n = \sum_{j=1}^{n} \gamma_j \varphi_j \,.$$

Recall that, in the collocation method, we enforce that the residual

$$r_n(x) = f(x) - \big(u_n(x) - (Au_n)(x)\big)$$

is zero at a finite number of points in Ω*, say* $x_1, x_2, \dots, x_n \in \Omega$*.*

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

Finally, we will look as some other numerical solution ideas that can be employed for numerical solution of the Fredholm integral equations without going into the detailed convergence analysis.

We begin by noting that the collocation method and the Galerkin method can also be employed to numerically solve these equations.

For example, if we let the approximation space be given by

$$X_n = \mathrm{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

so that we seek the approximate solution in the form

$$u_n = \sum_{j=1}^{n} \gamma_j \varphi_j \,.$$

Recall that, in the collocation method, we enforce that the residual

$$r_n(x) = f(x) - \big(u_n(x) - (Au_n)(x)\big)$$

is zero at a finite number of points in Ω, say $x_1, x_2, \dots, x_n \in \Omega$. Therefore, we have

$$u_n(x_i) - (Au_n)(x_i) = f(x_i), i = 1, \dots, n,$$

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

Finally, we will look as some other numerical solution ideas that can be employed for numerical solution of the Fredholm integral equations without going into the detailed convergence analysis.

We begin by noting that the collocation method and the Galerkin method can also be employed to numerically solve these equations.

For example, if we let the approximation space be given by

$$X_n = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

so that we seek the approximate solution in the form

$$u_n = \sum_{j=1}^{n} \gamma_j \varphi_j \,.$$

Recall that, in the collocation method, we enforce that the residual

$$r_n(x) = f(x) - \big(u_n(x) - (Au_n)(x)\big)$$

is zero at a finite number of points in Ω*, say* $x_1, x_2, \dots, x_n \in \Omega$*. Therefore, we have*

$$u_n(x_i) - (Au_n)(x_i) = f(x_i), i = 1, \dots, n,$$

that is,

$$\sum_{j=1}^{n} \gamma_j \varphi_j\,(x_i) - \sum_{j=1}^{n} \gamma_j \int_{\Omega} K(x_i, y)\varphi_j(y)dy = f(x_i), \qquad i = 1,2, \dots, n.$$

Finally, we will look as some other numerical solution ideas that can be employed for numerical solution of the Fredholm integral equations without going into the detailed convergence analysis.

We begin by noting that the collocation method and the Galerkin method can also be employed to numerically solve these equations.

For example, if we let the approximation space be given by

$$X_n = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

so that we seek the approximate solution in the form

$$u_n = \sum_{j=1}^{n} \gamma_j \varphi_j \,.$$

Recall that, in the collocation method, we enforce that the residual

$$r_n(x) = f(x) - \big(u_n(x) - (Au_n)(x)\big)$$

is zero at a finite number of points in Ω, say $x_1, x_2, \dots, x_n \in \Omega$. Therefore, we have

$$u_n(x_i) - (Au_n)(x_i) = f(x_i), i = 1, \dots, n,$$

that is,

$$\sum_{j=1}^{n} \gamma_j \varphi_j\,(x_i) - \sum_{j=1}^{n} \gamma_j \int_{\Omega} K(x_i, y)\varphi_j(y)dy = f(x_i), \qquad i = 1,2, \dots, n.$$

The solution of this linear system thus yields the numerical solution to the integral equation.

Similarly, the Galerkin method leads to

$$\langle r_n, \varphi_i \rangle = 0, \qquad i = 1, \dots, n,$$

or

$$\langle u_n, \varphi_i \rangle - \langle A u_n, \varphi_i \rangle = \langle f, \varphi_i \rangle, \qquad i = 1, \dots, n.$$

Similarly, the Galerkin method leads to

$$\langle r_n, \varphi_i \rangle = 0, \qquad i = 1, \dots, n,$$

or

$$\langle u_n, \varphi_i \rangle - \langle A u_n, \varphi_i \rangle = \langle f, \varphi_i \rangle, \qquad i = 1, \dots, n.$$

Thus, the resulting linear system that needs to be solved to obtain the coefficient γ_j reads

$$\sum_{j=1}^{n} \gamma_j \langle \varphi_j, \varphi_i \rangle - \sum_{j=1}^{n} \gamma_j \int_{\Omega} \langle K(\cdot, y), \varphi_i \rangle \varphi_j(y) dy = \langle f, \varphi_i \rangle, \qquad i = 1, \dots, n.$$

Lesson 5
Integral Equations

5.2 An Introduction

5.3 Numerical Methods

- Degenerate Kernel Method
- Projection Method
- **Quadrature/Nyström Method**

Akash Anand
MATH, IIT KANPUR

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

Similarly, the Galerkin method leads to

$$\langle r_n, \varphi_i \rangle = 0, \qquad i = 1, \ldots, n,$$

or

$$\langle u_n, \varphi_i \rangle - \langle A u_n, \varphi_i \rangle = \langle f, \varphi_i \rangle, \qquad i = 1, \ldots, n.$$

Thus, the resulting linear system that needs to be solved to obtain the coefficient γ_j *reads*

$$\sum_{j=1}^{n} \gamma_j \langle \varphi_j, \varphi_i \rangle - \sum_{j=1}^{n} \gamma_j \int_{\Omega} \langle K(\cdot, y), \varphi_i \rangle \varphi_j(y) dy = \langle f, \varphi_i \rangle, \qquad i = 1, \ldots, n.$$

One last numerical solution idea, known as the quadrature method or Nyström method, utilizes quadrature for numerical approximation of the integral operator.

Similarly, the Galerkin method leads to

$$\langle r_n, \varphi_i \rangle = 0, \qquad i = 1, \dots, n,$$

or

$$\langle u_n, \varphi_i \rangle - \langle A u_n, \varphi_i \rangle = \langle f, \varphi_i \rangle, \qquad i = 1, \dots, n.$$

Thus, the resulting linear system that needs to be solved to obtain the coefficient γ_j reads

$$\sum_{j=1}^{n} \gamma_j \langle \varphi_j, \varphi_i \rangle - \sum_{j=1}^{n} \gamma_j \int_{\Omega} \langle K(\cdot, y), \varphi_i \rangle \varphi_j(y) dy = \langle f, \varphi_i \rangle, \qquad i = 1, \dots, n.$$

One last numerical solution idea, known as the quadrature method or Nyström method, utilizes quadrature for numerical approximation of the integral operator. For example, if Q_n is a quadrature with weights w_j and quadrature points x_j so that

$$\int_a^b g(x) dx \approx Q_n(g) = \sum_{j=1}^{n} w_j g(x_j),$$

then the integral in equation is approximated through the quadrature and collocated at the quadrature points to yield

$$u_n(x_i) - Q_n(K(x_i, \cdot) u_n) = f(x_i), \qquad i = 1, \dots, n.$$

Integral Equations: Numerical Methods

Akash Anand
MATH, IIT KANPUR

One last numerical solution idea, known as the quadrature method or Nyström method, utilizes quadrature for numerical approximation of the integral operator. For example, if Q_n is a quadrature with weights w_j and quadrature points x_j so that

$$\int_a^b g(x)dx \approx Q_n(g) = \sum_{j=1}^{n} w_j g(x_j),$$

then the integral operator A is approximated as

$$(A_n v)(x) = \sum_{j=1}^{n} w_j K(x, x_j) v(x_j)$$

and the integral in equation is approximated through the quadrature and collocated at the quadrature points to yield

$$u_n(x_i) - (A_n u_n)(x_i) = f(x_i), \qquad i = 1, \dots, n.$$

Thus, the linear system to solve in this case is

$$u_n(x_i) - \sum_{j=1}^{n} w_j K(x_i, x_j) u_n(x_j) = f(x_i), \qquad i = 1, \dots, n.$$

Once we solve for $u_n(x_i)$ at the quadrature points, then the approximate solution is obtained using the Nystrom interpolation formula given by

$$u_n(x) = f(x) + \sum_{j=1}^{n} w_j K(x, x_j) u_n(x_j).$$

While we will not go through all the details in the analysis of the Nyström method, the convergence of this method depends on the accuracy of approximation of the integral operator by the quadrature. We finish this discussion by stating the theorem that establishes the connection between the rate of convergences of the operator approximation and the numerical solution.

Theorem

Let $A: C(\Omega) \to C(\Omega)$ *be an integral operator with weakly singular kernel* K*, that is,*

$$(Av)(x) = \int_{\Omega} K(x,y)v(y)dy$$

such that $(I-A)^{-1}$ *exists. Let* $A_n: C(\Omega) \to C(\Omega)$ *be given by* $(A_n v)(x) = Q_n(K(x,\cdot)v)$

where the quadrature formulas Q_n *are convergent. Then, for sufficiently large* n*, more precisely for all* n *with*

$$\|(I-A)^{-1}(A_n - A)A_n\| < 1,$$

the inverse operators $(I-A_n)^{-1}: C(\Omega) \to C(\Omega)$ *exist and are bounded by*

$$\|(I-A_n)^{-1}\| \leq \frac{1 + \|(I-A)^{-1}A_n\|}{1 - \|(I-A)^{-1}(A_n - A)A_n\|}.$$

For the solution of the equation $u - Au = f$ *and* $u_n - A_n u_n = f_n$*, we have the error estimate*

$$\|u_n - u\| \leq \frac{1 + \|(I-A)^{-1}A_n\|}{1 - \|(I-A)^{-1}(A_n - A)A_n\|}\{\|(A_n - A)u\| + \|f_n - f\|\}.$$