MTH 424.

FB-579 [Requires MTH421 oWo]

Well Posedness of PDE

→ Existence

→ Uniqueness

→ cts dependence.

Def^{n} analogously $u: \mathbb{R}^n \to \mathbb{R}$

$u: \mathbb{R}^2 \to \mathbb{R}$. Let x, y be the independent variables. A PDE is then any combination of u, x, y and any partial derivatives.

$$F(x, y, u, u_x, u_y, u_{xy}, \ldots) = 0 \quad - (*)$$

Def^{n}: (Solution of PDE)

A "classical" solution of a PDE is a function u such that it solves $(*)$ pointwise.

Eg For $u_{xx} + u_{yy} = 0$, if $u: \mathbb{R}^2 \to \mathbb{R}$ is a classical sol^n it needs to be C^2.

For classical solutions, we need to know the regularity.

→ LVG FML

$$\begin{cases} x - 1, & x > 0 \\ 1 + x & x < 0 \end{cases}$$

Detour: Classical-ness

Eikonal Equation

$u: \mathbb{R} \to \mathbb{R}$

$|u'(x)| = f(x)$

$|u'(x)| = 1$ on $(-1, 1)$

$u(\pm 1) = 0$ → No C^1 solution by Rolle's theorem.

Radamaker's theorem

A Lipschitz continuous function is differentiable a.e.

Def^N (Weak Solution)

Equation is satisfied almost everywhere.

Order of PDE

A PDE is said to be of order k if the highest partial deriv-ative(s) present in the equation is of order k.

Eg: $u_x + u_y = 0$ ← PDE of order 1.

$u_{xx} + u_{yy} = 0$ ← order 2

↳ Laplace's Equation.

$u_x + x u_{xy} + u_x u_{xx} = 0$ ← order 2.

Linear PDE

$u: R^n \to R \quad x = (x_1 \ldots x_n)$

$L(u) = f(x)$ ⇝ consisting of indep-variables.

operator on u [u, its partial derivatives]

$$\underbrace{u_x + x u_{xy} + u_x u_{xx}}_{L(u)} + \underbrace{x^2 \sin y}_{f(x,y)} = 0$$

A PDE is called linear if $L(u)$ is linear.

$L(\alpha u_1 + \beta u_2) = \alpha L u_1 + \beta L u_2 \qquad \alpha, \beta \in \mathbb{R}$

$u_1, u_2 \in R^n \to R$.

Otherwise it is called non-linear.

Non-Linear PDEs

→ Three categories

a) Semi-linear PDEs: A PDE of order k is semi-linear if all k-order terms' coeffs are just independent variables

b) Quasi-linear PDEs: coeffs of k^{th} order derivatives consist of independent variables, dependent variables, and partial derivs of orders upto k-1.

c) Fully non-linear PDEs:

If n-l PDE is not quasilinear.

Homogeneous PDE

$\mathcal{L}(u) = 0$

Auxilliary Condition:

→ A PDE is usually supplemented with an a-priori condition.

→ This is called an auxilliary condition to a given PDE.

→ PDE only with auxiliary conditions are meaningful / physically relevant.

→ we may get "unique" solution with auxiliary condition.

→ n-independent variables in a given domain Ω then the auxiliary condition is set on a $(N-1)$ - dimensional subdomain Γ on Ω

Two prominent boundary conditions

→ Initial Value Problem (IVP)

→ Boundary Value Problem (BVP)

FIRST ORDER PDEs

1. $u_x = 0$ $\quad u: R^2 \to R$

$u(x,y) = g(y) \qquad g: R \to R$

now need auxilliary conditions.

2. $au_x + bu_y = 0$ → This class of eq^{ns}: Transport Eq

$$u(x,y) = f(x)\, g(y)$$

$$a\, g(y)\, f'(y) + b\, g'(y)\, f(y) = 0$$

$$a \frac{f'(x)}{f(x)} = -b \frac{g'(y)}{g(y)}$$

$$\frac{f'(x)}{f(x)} = \frac{k}{a}$$

$$f(x) = c e^{kx/a}$$

$$g(x) = d e^{-ky/b}$$

$$u(x,y) = \alpha e^{\frac{kx}{a} - \frac{ky}{b}}$$

one of sols.

$$(a, b) \cdot (u_x, u_y) = 0 \qquad (b, -a)$$

$$\therefore \quad \frac{\partial u}{\partial p} = 0 \qquad p = \frac{(a, b)}{\sqrt{a^2 + b^2}}.$$

$$\therefore \quad u = f(p^{\perp}) = f(bx - ay)$$

Eg : $3u_x + 2u_y = 0$

$u(x, 0) = x^3$

$u(x,y) = f(2x - 3y)$

$f(2x) = x^3$

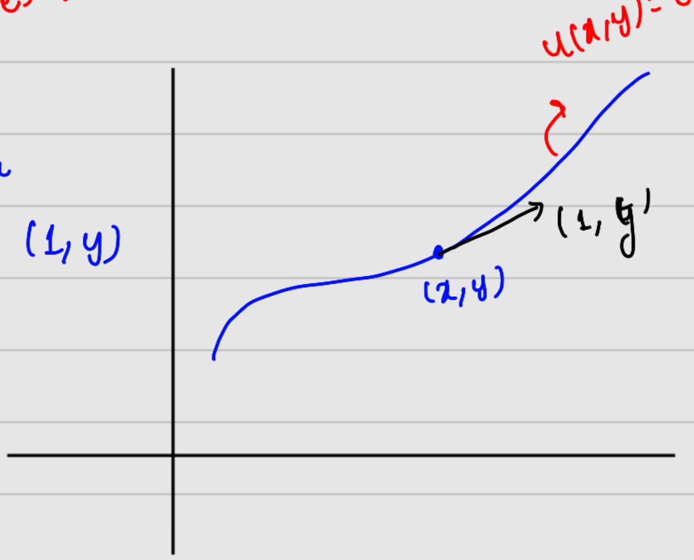
$f(x) = \dfrac{x^3}{8}$

$u(x,y) = \dfrac{(2x-3y)^3}{8}$ — ans

$u_x + y \cdot u_y = 0$ → constant on curves with tgt $(1, y)$

$\nabla u \cdot (1, y) = 0$

$u(x,y) = c$ if $(x, y(x))$ is a curve with tgt at (x,y) $(1,y)$



$(1, y'(x)) = (1, y)$

$\therefore \quad \dfrac{dy}{dx} = y$

$y = ce^x \qquad (x, ce^x)$

$u(x,y) = f(c) = f(ye^{-x})$

1^{st} Order Linear PDE

General form:

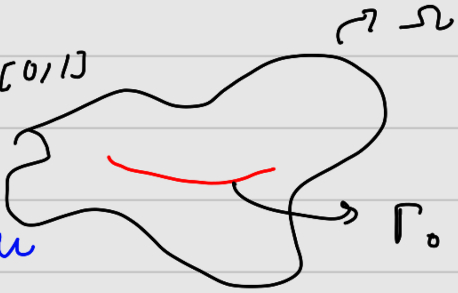
$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)$$

Method of Characteristics

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y) \quad \text{in } \Omega$$

$$u(x,y) = u_0 \quad \text{for} \quad (x,y) \in \Gamma_0$$

we write $\Gamma_0 \equiv (x(s), y(s))\ s \in [0,1]$



The auxiliary condition can be written as

$$u(x(s), y(s)) = u_0(s) \quad s \in [0,1]$$

LHS becomes

$$\begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} \cdot \nabla u = c(x,y)\,u + d(x,y)$$

Let $(x(r), y(r))$ be a curve on Ω parametrized by r whose tgt is $(a(x,y) \quad b(x,y))$.

$$\frac{dx}{dr} = a(x(r), y(r)) \qquad \frac{dy}{dr} = b(x(r), y(r))$$

The LHS is therefore the TOTAL DERIVATIVE of u along curve $(x(r), y(r))$

$$\frac{d}{dr} u(x(r), y(r)) = c(x(r), y(r))\, u(x(r), y(r)) + d(x(r), y(r))$$

$$x(0) = x_0(s) \qquad y(0) = y_0(s)$$

$$\frac{dx}{dr} = a(x(r), y(r)) \qquad \frac{dy}{dr} = b(x(r), y(r))$$

$$x(0) = x_0(s) \qquad y(0) = y_0(s)$$

→ Existence of solution to above ODEs can be guaranteed by assuming some regularity conditions on a, b.

$(x(r), y(r)) \rightarrow$ characteristic curve.

Along the characteristic curve we solve the pde

$$\frac{d\,u(x_r, y_r)}{dr} = c(x_r, y_r)\, u + d(x_r, y_r)$$

$$u(x(0), y(0)) = u_0(s)$$

We redefine the characteristic curve as $(x(r,s), y(r,s))$. Then in order to solve the given PDE, we solve the following.

$$u(x(r,s), y(r,s)) = z(r,s)$$

$$\left.\begin{array}{l} \frac{d}{dr} x(r,s) = a(r,s) \qquad \frac{d}{dr} y(r,s) = b(r,s) \\ x(0,s) = x_0(s) \qquad y(0,s) = y_0(s) \\ \frac{d}{dr} z(r,s) = c(r,s)\, z(r,s) + d(r,s) \\ z(0,s) = u_0(s) \end{array}\right\}$$

Step 1: Check whether there exists a char-curve $(x(r,s), y(r,s))$ passing through (x, y)

Step 2: Check that char. curves do not intersect each other.

Theorem:

If a, b are C^1 functions then the characteristic curves of any semilinear PDE

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

can never intersect.

Pf: suppose they do. suppose through (x_0, y_0) there are two characteristic curves. $(x(r,s_1), y(r,s_1))$ $(x(r,s_2), y(r,s_2))$

$$\left.\begin{aligned} x(r_1,s_1) &= x(r_2,s_2) = x_0 \\ y(r_1,s_1) &= y(r_2,s_2) = y_0 \end{aligned}\right\}$$ does not hold for $r < r_1$ $r < r_2$.

Rewrite for convenience

$$x_1(r_1) = x_2(r_2) = x_0$$
$$y_1(r_1) = y_2(r_2) = y_0$$

consider

$$r \to (x_3(r), y_3(r)) \qquad x_3(r) = x_2(r - r_1 + r_2)$$
$$y_3(r) = y_2(r - r_1 + r_2)$$

NOTE

$x_3 y_3$ now reaches (x_0, y_0) at r_1 now.

we know that $\dfrac{dx_1}{dr} = a(x_1(r), y_1(r))$

$$\frac{dy_1}{dr} = b(x_1(r), y_1(r))$$

$$\frac{dx_3}{dr} = a(x_2(r - r_1 + r_2), y_2(r - r_1 + r_2)) \qquad \frac{dy_3}{dr} = b(x_2(r - r_1 + r_2), y_2(r - r_1 + r_2))$$

$$x_3(r_1) = x_0 \qquad y_3(r_1) = y_0$$

x_3, x_1 solve the same IVP.

$\underline{x_1 \equiv x_3}$ (a, b are C^1 $\Rightarrow$ uniqueness) in a nbd of (x_0, y_0)

Example

Solve $au_x + bu_y = 0$ solve via method of char.

Solve

$$-y u_x + x u_y = u$$

$$\frac{dx(r,s)}{dr} = -y(r,s) \qquad \frac{dy(r,s)}{dr} = x(r,s)$$

$x_0(s) = s$

$$\frac{d^2}{dr^2} y = -y(r,s)$$

$y_0(s) = 0$

$x = c_1 \cos r - c_2 \sin r$

$y = c_1 \sin r + c_2 \cos r$

$$\begin{Bmatrix} x = s\cos r \\ y = s\sin r \end{Bmatrix}$$

$x(0,s) = s \Rightarrow c_1 = s$

$y(0,s) = 0$

$(x(r), y(r)) = (s\cos r, s\sin r)$

$$\frac{du}{dr} = u \qquad z(0,s) = g(s)$$

$z(r,s) = g(s)e^{r}$

$u(s\cos r, s\sin r) = g(s)e^{r}$

$x^2 + y^2 = s^2$

$\tan r = y/x$

$$u(x,y) = g(\sqrt{x^2+y^2})\, e^{\tan^{-1}(y/x)}$$

Reference- M.C Mac Smith

Example

$$u_x + 2u_y = u^2$$

$$u(x,0) = g(x)$$

$$\frac{dx}{dr} = 1 \qquad x(0,s) = s \qquad \Bigg| \qquad \frac{dy}{dr} = 2 \qquad y(0,s)$$

$$x = r + c \qquad \Big| \qquad y = 2r + c$$

$$x = r + s \qquad \qquad y = 2r$$

$$\frac{d}{dr} u(x(r,s), y(r,s)) = u^2$$

$$u(0,s) = g(r+s)$$

$$\frac{du}{u^2} = dr$$

$$\frac{-1}{u(x(r,s), y(r,s)} + \frac{1}{u(x(0,s), y(0,s))} = r$$

$$\frac{1}{u(s,0)} - \frac{1}{u(r+s, 2r)} = r$$

$$x = r + s$$

$$y = 2r$$

$$s = x - \frac{y}{2}$$

$$\frac{1}{g(s)} - r = \frac{1}{u(r+s, 2r)}$$

$$\frac{g(s)}{1 - rg(s)} = u(r+s, 2r)$$

$$\frac{g(x - y/2)}{\underbrace{1 - \frac{y}{2} g(x - \frac{y}{2})}_{>0 \text{ for } C_1}} = u(x,y)$$

$$y\, g(x - \tfrac{y}{2}) \neq 2$$

Quasi-linear PDE

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

Method of Characteristics

$$\frac{dx}{dr} = a(x(r,s), y(r,s), z(r,s))$$

$$\frac{dy}{dr} = b(x(r,s), y(r,s), z(r,s))$$

$$\frac{dz}{dr} = c(x(r,s), y(r,s), z(r,s))$$

Burger's Equation

$$u\,u_x + u_y = 0$$
$$u(x,0) = g(x)$$

$\frac{dx}{dr} = z$ $\quad x(0) = s$ $\qquad \frac{dy}{dr} = 1$ $\quad y(0) = 0$

$y = r$

$\frac{dz}{dr} = 0$ $\quad z(0,s) = g(s)$ $\qquad \Rightarrow \quad z = g(s)$

$z(r,s) = g(s)$

$\frac{dx}{dr} = g(s)$ $\quad x(0) = s$

$g(s)\,r + s = x(r)$

$$\boxed{g(s) = \frac{x-s}{y}} \quad ??$$

$z(r,s) = g(s)$

$x = g(s)y + s$ $\qquad x = s^2 y + s$

$g(x) = x^2$ $\qquad s(0) = \quad x = 0$

$S(1) = \quad x = y+1 \quad \int$ Not u

For quasilinear equations, characteristic curves can intersect.

$u_t + u_x = 0 \qquad u(x,0) = \dfrac{1}{1+x^2}$

$u(x,t) = g(t-x) \qquad g(1-x)$

$u u_x + u_y = 0$

$u(x,0) = g(x)$

$\dfrac{d}{dr} z = 0 \qquad z(0,s) = g(s)$

$\dfrac{dx}{dr} = z \qquad \dfrac{dy}{dr} = 1 \qquad y(0,s) = 0$

$x(0,s) = s \qquad y = r$

$\dfrac{dx}{dr} = g(s)$

$x = r g(s) + s$

$u(x(r), y(r)) = g(s)$

Def^N (Tangent Vector)

$v \in \mathbb{R}^3$ is called a tangent vector to a surface S at $a \in S$, if $\exists$ a curve $r(t) \subseteq S$ such that

$r(0) = a \quad , \quad r'(0) = v$

Def^N (Tangent space) to the surface S

Set of all tangent vectors at point $a \in S$.

$T_a = \{ v \in \mathbb{R}^3 \mid \exists\, r \text{ st } r(0) = a \quad r'(0) = v \}$

$\hookrightarrow$ Subspace of dimension 2.

Def^N (Normal Space)

Set of all vectors that are $\perp$ to tangent space T.

$\langle w, v \rangle = 0 \qquad \forall\, v \in T_A$.

Dimension $\Rightarrow$ 1

$\rightarrow$ 2-D space in $\mathbb{R}^3$.

$S = \{ (x, y, z) : z = u(x, y) \}$

Lemma: $(u_x, u_y, -1)$ is a normal to S. at (x, y)

Def^N (Integral Surface)

Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ (called a vector field). A 2-D surface S is said to be an integral surface of v if $v(x)$ is tangent vector of S at $x \in S$.

Example:

$v : \rightarrow (-y, x, 0)$

$S = x^2 + y^2 + z^2 = \mu^2$ is an integral surface.

$u(x, y) = \pm (k^2 - x^2 - y^2)^{1/2}$

$u_x = \pm \frac{1}{2} (k^2 - x^2 - y^2) (-2x) = \pm x (k^2 - x^2 - y^2)$

$u_y = \qquad\qquad \pm y (k^2 - x^2 - y^2)$

$(-y, x, 0) \cdot (x(k^2 - x^2 - y^2), y(k^2 - x^2 - y^2), 0)$

$-xy (k^2 - x^2 - y^2) + xy (\quad) = 0$

$\hookrightarrow$ NOPE ??

Geometrical Interpretation of Solution

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \quad — (*)$$

$$(a(x,y,u),\ b(x,y,u),\ c(x,y,u))\,(u_x,\ u_y,\ -1) = 0$$

If $V := (a, b, c)$

If u solves $(*)$, the surface is given by
$S = \{(x, y, u(x,y))\}$

$\therefore$ S is the integral surface of V.

Example:

$$-y u_x + x u_y = 0$$

$$(-y, x, 0) \cdot (u_x, u_y, -1) = 0$$

$$\overset{/\!/}{V}$$

$$u = \pm (K^2 - x^2 - y^2)^{1/2}$$

Existence + Uniqueness for Quasilinear PDEs

Thm: consider the following PDE $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$ in Ω

$$u(\Gamma_0) = g.$$

Let a, b, c be C^1 functions from $\mathbb{R}^3$ to $\mathbb{R}$ and $g \in C^1(\mathbb{R})$.

Assumption (Transversality condition)

$*$ $x_0'(s)\, b(x_0(s), y_0(s), g(s)) - y_0'(s)\, a(x_0(s), y_0(s), g(s))$

Then $\exists$ a nbd around $(x_0(s), y_0(s))$ where $\neq 0$ the quasilinear PDE has a unique soln.

Pf

$\Gamma_0 = \{ (x_0(s), y_0(s)) : s \in I \subseteq \mathbb{R} \}$

$$\begin{cases} \dfrac{dx}{dr} = a(x(r,s), y(r,s), z(r,s)) & x(0,s) = x_0(s) \\[2ex] \dfrac{dy}{dr} = b(x(r,s), y(r,s), z(r,s)) & y(0,s) = y_0(s) \end{cases}$$

$$\frac{dz}{dr} = c(x(r,s), y(r,s), z(r,s)) \qquad z(0,s) = g(s)$$

By existence & uniqueness of ODE $\exists$ $\delta, \varepsilon > 0$

such that $(x, y, z)(r,s)$ is the unique soln in the nbd $(-\delta, \delta) \times (s-\varepsilon, s+\varepsilon)$

$A(r,s) = \{ (x(rs), y(rs), z(rs)) \} \in C^1$

Define a map $B(r,s) : \mathbb{R}^2 \to \mathbb{R}^2$

$(r,s) \to (x(r,s), y(r,s))$

$(-\delta, \delta) \times (s-\varepsilon, s+\varepsilon) \to \mathbb{R}^2$

B is C^1. The jacobian

$$\begin{bmatrix} x_r & y_r \\ x_s & y_s \end{bmatrix} = \begin{vmatrix} a & b \\ x' & y' \end{vmatrix} \neq 0.$$

can get x, y through inverse function in a nbd of $(0, s_0)$

By using IFT, $\exists$ a nbd $\tilde{\Omega}$ around $(x_0 s_0, y_0 s_0)$ and $\tilde{U}$ around $(0, s_0)$ that

$$B : \tilde{U} \to \tilde{\Omega} \quad 1,1, \text{ onto}.$$

$$B^{-1} = (r^{-1}(x,y), s^{-1}(x,y)) : \tilde{\Omega} \to \tilde{U}.$$

General First Order PDE

↳ fully non-linear

Cannot write the LHS as a directional derivative.

$\therefore$ NO GENERIC METHOD

$f(x, y, u, u_x, u_y) = 0$

$$u : \mathbb{R}^n \to \mathbb{R}, \quad x = (x_1, x_2, \dots, x_n)$$

$$\mathbb{F}(x, u, \nabla u) = 0; \ \mathbb{F} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

General form.

We denote the "second" variable as z, and "third" variable as p.

$$\mathbb{F}(x, z, p) = 0$$

Eqn:
$$\mathbb{F}(x, u, \nabla u) = 0 \quad \text{on } \Omega$$
$$u(x) = g \quad \Gamma_0 \subseteq \Omega$$

Eg.
$$u_{x_1} u_{x_2} + 3x_1^2 (u_{x_1})^2 = 0$$

$$F(x, z, p) = p_1 p_2 + 3x_1^2 p_1^2 = 0$$

The characteristic Equation

Assume u is C^2, and take partial derivative.

$$F_{x_i}(x, z, p) + F_z(x, z, p)\, u_{x_i} + \sum \frac{\partial F}{\partial p_j}(x, z, p)\, p_{x_i} = 0$$

↳ chain rule ??

Q: No summation here because $f(x_1 \dots x_k)$ $\frac{dx_i}{dx_j} = 0$?!!

A: That is the definition of partial derivative you cump at.

$p = (p_1 \dots p_n)$ $\qquad$ $p(r,s) = (p_1(r,s), \dots, p_n(r,s))$

$$\frac{d}{dr} p_i(r,s) = \frac{\partial}{\partial r} u_{x_i}(r,s) = \sum u_{x_j x_i}(r,s) \frac{dx_i}{dr}$$

$$\boxed{\frac{\partial x_j}{dr} = \frac{\partial F}{\partial p_j}(x, z, p)}$$

$$\frac{\partial}{\partial r} p_i = -\frac{\partial F}{\partial z}(x, z, p)\, p_i - \frac{\partial F}{\partial x_i}(x, z, p)$$

↳ rearranging - a

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r} u(x(r)) = \sum \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial r} = \sum p_j \frac{\partial F}{\partial p_j}(x, z, p)$$

Compact Form of Char. ODE

$$\frac{dx}{dr} = \nabla_p F(x(r), z(r), p(r))$$

$$\frac{dz}{dr} = \nabla_p F(x(r), z(r), p(r)) \cdot p(r)$$

$$\frac{\partial p}{\partial r} = -\nabla_x F(x(r), z(r), p(r)) - \frac{\partial F}{\partial z}(x(r), z(r), p(r))$$

Brush.

* Need to solve 2n+1 ODEs. (n for x, 1 for u(x)).
* Via auxilliary condition $\Rightarrow$ n constraints.
* need n+1 constraints for complete solution.
  additional

Need to predict initial values of p_i by using the PDE & auxilliary data.

The solution, therefore to fully non-linear PDE, are not always unique.

Example

$\Rightarrow u(x,y) = u$ in Ω $\{(x,y): x>0\}$

$u(0,y) = y^2$

$F(x, u, \nabla u) = 0$

$2n+1$ characteristic ODE if $x \in \mathbb{R}^n$.

$$\frac{dx}{d\tau} = \nabla_p F(x(\tau), z(\tau), p(\tau))$$

$$\frac{dz}{d\tau} = \nabla_p F(x(\tau), z(\tau), p(\tau)) \cdot p(\tau)$$

$$\frac{dp}{d\tau} = -\nabla_x F(x(\tau), z(\tau), p(\tau)) - \frac{dF}{dz}(x(\tau), z(\tau), p(\tau)) \cdot p$$

Parametrize Γ_0, this yields $n+1$ initial conditions.
We sacrifice uniqueness, need to prescribe n more constraints.

Example: $u_{x_1} u_{x_2} = u$ on $\Omega = \{(x_1, x_2) : x_1 > 0\}$

sol $F = u_{x_1} u_{x_2} - u = 0$ $\quad u(0, x_2) = x_2^2$

$F(x, z, p) = p_1 p_2 - z = 0$

Parametrize $\Gamma_0 = \{(0, s) : s \in \mathbb{R}\}$ $\quad z(0, s) = s^2$

$$\left(\frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}\right) = (p_2, p_1) \qquad (0, s)$$

$$\frac{dz}{d\tau} = (p_2, p_1) \cdot (p_1, p_2)$$
$$= 2 p_1 p_2$$

$p_2(0, s) = 2s$

$$\frac{dp}{d\tau} = -1 \, (p_1, p_2)$$

$$\frac{dp_1}{d\tau} = -p_1 \qquad \frac{dp_2}{d\tau} = -p_2$$

$$p_1(\tau) = c_1 e^{-\tau} \qquad p_2(\tau) = c_2 e^{-\tau}$$

$P_1 = \frac{s}{2} e^{-x}$ $\qquad 2se^{-x}$

$\frac{dz}{dx} = 2s^2 e^{-2x}$ $\qquad (P_1)(2s) = s^2$

$P_1 = \frac{s}{2}$

$2e^{-2x}$

$u(x_1, x_2)$

$= \left(\frac{x_1}{4} + x_2\right)^2 \Bigg| \quad -\frac{2}{2} e^{-2x} \Big]_0^x \qquad -e^{-2x} + 1$

Thm (Conservation of Mass for transport Equation)

General Form

$u_t + a(x) \cdot \nabla u = 0 \qquad x \in \mathbb{R}^n \quad t \geq 0$

$u(x, 0) = g(x) \qquad u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}.$

(i) $a \in C^1(\mathbb{R}^n)$ $\quad$ a is bounded.

(ii) divergence $(a(x)) = 0 \quad \forall x \in \mathbb{R}^n$

$\sum \frac{\partial}{\partial x_i} a(x_i) \cdot \qquad a \equiv (a_1, a_2 - -)(x)$

If $u(x,t)$ is a solution of the above PDE and g is compactly supported then

$$\int_{\mathbb{R}^n} u(x,t)\, dx = \underbrace{\int_{\mathbb{R}^n} g(x)\, dx}_{\text{"mass"}} \quad \forall t \geq 0$$

Pf. We want to show

$$\frac{d}{dt}\left(\int_{\mathbb{R}^n} u(x,t)\, dx\right) = 0$$

$$\frac{d}{dt} \int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} u(x,t)\, dx$$

possible if u is C^1 and bounded

$$= \int_{\mathbb{R}^n} -a(x) \cdot \nabla u \, dx$$

$\{ a \in \mathbb{R}, n = 1$

$g(ax - t)$

$u(x,t)$ is also compactly supported because g is compactly supported, and u at some time t is just g along some curve.

$\equiv \quad g \equiv 0$ on $|x| > R_0$

By assumption, a is bounded consider

$$\sup_x \| a(x) \| < A$$

↳ bound on slope of char curve.

For a fixed time $\exists$ a compact domain $B(0, R_0 + tA)$ that supports $u(x,t)$

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t} u(x,t)\, dx = \int_{B(0, R_0 + tA)} \frac{\partial}{\partial t} u(x,t)\, dx$$

$$= - \int_{B(0, R_0 + tA)} a(x) \cdot \nabla u \, dx$$

Use integration by parts

$$= - \int_{B(0, R_0 + ta)} \operatorname{div}(a(x)) \cdot u(x)\, dx$$

$$- \int_{\partial B} a(x) \cdot \nabla u \cdot \nu \, d\sigma$$

∂

Second Order Partial Differential Equations

* Only concerned with linear second order PDEs.

General linear Second order PDE

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(u)u(x) = f(x) \quad — (*)$$

$u: \mathbb{R}^n \to R$, $\quad a_{ij}$, b_i, c, f are "smooth".

Classification of linear 2nd order

→ Elliptic (Eg: $\Delta u = 0$ (laplace))

→ Parabolic (Heat Equation)

→ Hyperbolic (wave Equation)

Restricting (*) to

$$\underbrace{a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2}} + d\frac{\partial v}{\partial x} + e\frac{\partial v}{\partial y} + fu + g = 0 \quad — (*)(*)$$

Restricting our attention to, we write

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2}$$

$$L(\xi,\eta) = a\xi^2 + b\xi\eta + c\eta^2$$

Eqn (*)(*) is

→ Elliptic $\quad b^2 - 4ac < 0$

→ Parabolic $\quad b^2 - 4ac = 0$

→ Hyperbolic $\quad b^2 - 4ac > 0$

Remark: The names have nothing to do with the behaviour of the solution.

No connection to the eponymous conic section..

Verification

wave eqn:

$u_{tt} - u_{xx} = 0$

$b = 0 \quad a = 1 \quad c = -1 \qquad b^2 - 4ac = 4 > 0.$

$\therefore$ Is hyperbolic.

Laplace

$u_{xx} + u_{yy} = 0 \qquad b^2 - 4ac = -4 < 0$

$\therefore$ Is elliptic

Heat Eqn:

$u_t - u_{xx} = 0 \qquad b^2 - 4ac = 0$

$\therefore$ Is parabolic.

Remark Laplace, Heat & Wave Equations are called canonical forms for second order linear equations.

Claim: Any second order linear PDE can be transformed to its corresponding canonical form.

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu + g = 0$$

$\xi = \xi(x,y) \quad \eta = \eta(x,y) \leftarrow$ transformation

$$\left.\begin{aligned} w(\xi,\eta) &:= u(x(\xi,\eta), y(\xi,\eta)) \\ u(x,y) &= w(\xi(x,y), \eta(x,y)) \end{aligned}\right\} \text{assumption: IFT 99 times.}$$

$$u_x = w_\xi \frac{\partial \xi}{\partial x} + w_\eta \frac{\partial \eta}{\partial x}$$

$$u_x = w_\xi \xi_x + w_\eta \eta_x$$
$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{xx} = w_{\xi\xi} \xi_x^2 + w_\xi \xi_{xx} + 2 w_{\eta\xi} \xi_x \eta_x + w_\eta \eta_{xx}$$
$$+ w_{\eta\eta} \eta_x^2$$

$$u_{xy} = w_\xi \xi_{xy} + w_{\xi\xi} \xi_x \xi_y + w_{\xi\eta} \eta_y \xi_x$$
$$w_\eta \eta_{xy} + w_{\eta\xi} \xi_y \eta_x + w_{\eta\eta} \eta_x \eta_y$$

$$u_{yy} = w_{\xi\xi} \xi_y^2 + w_\xi \xi_{yy} + 2 w_{\eta\xi} \xi_y \eta_y + w_\eta \eta_{yy}$$
$$+ w_{\eta\eta} \eta_y^2$$

Substitute Back in Eqn 1

$$A w_{\xi\xi} + C w_{\eta\eta} + B w_{\eta\xi} = \phi(\xi, \eta, w_\xi, w_\eta, w) \quad — ***$$

$$A = a \xi_x^2 + 2b \xi_x \xi_y + c \xi_y^2$$
$$B = 2a \xi_x \eta_x + (\xi_x \eta_y + \xi_y \eta_x) + 2c \xi_y \eta_y$$
$$C = a \eta_x^2 + 2b \eta_x \eta_y + c \eta_y^2$$

Case I: $b^2 > 4ac$ ** is hyperbolic

we want to transform it $(u_{tt} - u_{xx} = 0 \; / \; u_{tx} = 0)$

↳ $b^2 - 4ac = 1$

Need $A, C = 0$ $\quad B = 1$

$$A = a \xi_x^2 + 2b \xi_x \xi_y + c \xi_y^2$$
$$C = a \eta_x^2 + 2b \eta_x \eta_y + c \eta_y^2$$

↳ 1st order fully non-linear.

we try to write the equation as a quadratic form.

$$(2a\xi_x + (b-\sqrt{b^2-4ac})\xi_y)\cdot(2a\xi_x - (b-\sqrt{b^2-4ac})\xi_y)$$
$$= 0$$

$$4a^2\xi_x^2 - (b-\sqrt{b^2-4ac})\xi_y^2$$

$4a^2$

$$ax^2 + 2bxy + cy^2$$
$$a(x/y) + 2b(x/y) + c = 0$$

$$\frac{x}{y} = \frac{-2b \pm \sqrt{4b^2-4ac}}{2a}$$

$$\frac{x}{y} = \frac{-b \pm \sqrt{b^2-ac}}{a}$$

$$\frac{x}{y} = \frac{-b - \sqrt{b^2-ac}}{a}$$

$$ax + (b+\sqrt{b^2-ac})y = 0$$
$$ax + (b-\sqrt{b^2-ac})y = 0$$

$$[ax + (b+\sqrt{b^2-ac})][ax + (b-\sqrt{b^2-ac})]y$$

Laplace Equation

$\Delta u = f$

Regularization Property $\to$ solution should be smoother than f.

Maximum Principle $\to$ Maximum value occurs on the boundary (or the function is const)

Mean-value Property $\to$

$\Delta u = 0 \quad \leftrightarrow \quad u(x) = \int_{B(x,r)} u(y)\,dy$. $\leadsto$ lol big phy103 moment

For elliptic equations, auxilliary conditions are prescribed at boundary.

$\to$ Dirichlet Boundary.

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$ $\leadsto$ boundary of Ω

$\to$ Neumann Boundary

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

$\dfrac{\partial u}{\partial \nu}$ is the directional derivative towards outer normal ν.

Def^n: $u \in C^2(\Omega)$ is called a harmonic function in Ω if $\Delta u = 0$.

A specific Harmonic function

$\Delta u = 0$ in $\mathbb{R}^2$.

Laplace equations are invariant under rotation & translation.

Rotation Matrices in $\mathbb{R}^2$ are orthonormal.

Rotation corresponds to multiplying vector with ortho. n.

orthonormal

Define $\tilde{x} = Qx$ $\quad Q^T Q = I$

$w\, x = u(Qx)$

$\Delta u(x) = \Delta_{x'} w$

we therefore look for solutions of the form $v(x) = V(|x|)$

$= v(r)$

$$\frac{\partial u}{\partial x_i} = v'(r)\frac{1}{2r}(2x_i)$$

$$= v'(r)\frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{v'(r)}{r} + v''(r)\frac{x_i^2}{r^2} + x_i\, v'\left[\frac{-1}{r^2}\frac{\partial r}{\partial x_i}\right]$$

$$= \frac{v'(r)}{r} - \frac{v'}{r^3}x_i^2 + \frac{v''(r)\, x_i^2}{r^2}$$

$$\Delta u = \sum v''(r)\frac{x_i^2}{r^2} + \frac{v'(r)}{r} - \frac{v'}{r^3}x_i^2$$

$$= v''(r) + \frac{n\, v'(r)}{r} - \frac{v'}{r}$$

$$= v''(r) + \frac{(n-1)}{r}v'(r) = 0$$

$$v''(r) = \frac{1-n}{r}v'(r)$$

Assumption

$v'(r) \neq 0$ $\quad r > 0$

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \Rightarrow \log|v'(r)| = C \quad \log r^{1-n}$$

$$|v'(r)| = C r^{1-n}$$

$$v(r) = \begin{cases} A \log r & n = 2 \\ \dfrac{B}{r^{n-2}} & n \geq 3 \end{cases}$$

$\Delta u = 0$ in $R^n \setminus \{0\}$

$$u(x) = \begin{cases} \dfrac{1}{2\pi} \log|x| & n = 2 \\ \dfrac{1}{(n-2)\omega_n} \dfrac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

$\omega_n \Rightarrow$ surface area of unit sphere in $\mathbb{R}^n$.

$$\Phi(y) = \begin{cases} \dfrac{1}{2\pi} \log|y| & n = 2 \\ \dfrac{1}{(n-2)\omega_n} \dfrac{1}{|y|^{n-2}} & n \geq 3 \end{cases}$$

$\Phi(y)$ is harmonic on $R^n \setminus \{0\}$.

$\Phi(y-x)$ is harmonic on $R^n \setminus \{x\}$

Poisson Equation

$$-\Delta u = f \text{ on } \mathbb{R}^n$$

Q. How to find a candidate solution?

Guess lol! ← fairly motivated guess

$$u(x) := \int_{R^n} \Phi(y) f(x-y)\, dy$$

↳ convolution operator

$$\Phi * f(x) = \int_{R^n} \Phi(y) f(x-y)\, dy$$

Detour: convolutions

$$f * g(x) = \int_{R^n} f(x-y) g(y)\, dy = \int_{R^n} f(y) g(x-y)\, dy$$

$\Rightarrow$ $f * g$ makes sense if f is bdd, g is integrable.

$\Rightarrow$ If one of the functions is regular, $f * g$ is also regular.

$\Rightarrow$ Differential and conv operators commute.

$$\partial_i (f * g) = f * \partial_i g \quad \text{if } g \text{ is } C^1$$
$$= \partial_i f * g \quad \text{if } f \text{ is } C^1$$

$\Phi * f$ is C^2 if f is a C^2 function.

Theorem

compact support

Assume $f \in C_c^2(\mathbb{R}^n)$ and consider $u = \Phi * f(x)$ then

(a) $u \in C^2(\mathbb{R}^n)$

(b) $-\Delta u = f$ on $\mathbb{R}^n$.

Proof:

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$ wrong

$$\Delta u = \int_{\mathbb{R}^n \setminus \{x\}} \Delta \Phi(x-y) f(y)\, dy = 0$$ wrong

Lap - $\int$ commute /

(a) suffices to $\dfrac{\partial^2 u}{\partial x_i x_j} \in C(\mathbb{R})$

we first prove that partial derivatives of u exist.

To show

$$\lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h} = \frac{1}{h}\left[\int_{\mathbb{R}^n} \Phi(y) f(x+he_i - y) - \int_{\mathbb{R}^n} \Phi(y) f(x-y)\right]$$

$$= \frac{1}{h}\left[\int \Phi(y)\left[f(x+he_i - y)\, dy - f(x-y)\right]\right]$$

True modulo the need to show lt and $\int$ can be interchanged.

$$\leftarrow = \int \Phi(y)\ \partial_i f(x-y)\,dy$$

∂f is C^2

f is compactly supported $\Rightarrow$ f is bounded

Df $D^2 f$ both are bounded

$$\frac{f(x-y+he_i)-f(x-y)}{h} \xrightarrow{h\to 0} \partial_i f(x-y)$$

$$\left|\frac{f(x-y+he_i)-f(x-y)}{h} - \partial_i f(x-y)\right|$$

uniformly cts.

$$|\partial_i f(c) - \partial_i f(x-y)|$$

Need to show $-\Delta u = f$ in $\mathbb{R}^n$

Δu is defined.

$$\Delta u^{(x)} = \int_{\mathbb{R}^n} \Phi(y)\ \Delta_x f(x-y)\,dy$$

$$= \underbrace{\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\ \Delta_x f(x-y)\,dy}_{I_\varepsilon} + \underbrace{\int_{B(0,\varepsilon)} \Phi(y)\ \Delta_x f^{(x-y)}\,dy}_{J_\varepsilon}$$

$$I_\varepsilon := \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\ \Delta_x f(x-y)\,dy$$

It is easy to show $\Delta_x f(x-y) = \Delta_y f(x-y)$

$$\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\ \Delta_y f(x-y)\,dy = \sum \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y)\ \frac{\partial^2}{\partial y_i^2} f(x-y)\,dy$$

$$= \sum -\int \frac{\partial}{\partial y_i} \Phi(y) \frac{\partial}{\partial y_i} f(x-y)\, dy + \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial y_i} \nu_i \, dy$$

Integration By Parts

$$\int_{\Omega} f \frac{\partial g}{\partial x_i}\, dx = -\int_{\Omega} \frac{\partial f}{\partial x_i} g \, dx + \int_{\partial\Omega} f \cdot g \, \nu_i \, dS(x)$$

$$I_\varepsilon = \sum -\int_{R^n \setminus B(0,\varepsilon)} \frac{\partial \Phi}{\partial y_i}(y) \frac{\partial f}{\partial y_i}(x-y)\, dy + \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu} \partial S_y$$

$$= \sum \int_{R^n \setminus B(0,\varepsilon)} \frac{\partial^2 \Phi}{\partial y_i^2} f \, dy - \int \frac{\partial \Phi}{\partial y_i} f(x-y) \nu_i \, \partial S(y)$$

$$+$$

$$= \int_{R^n \setminus B(0,\varepsilon)} \Delta_y \Phi(y) \cancel{f}(x-y)\, dy - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu} f(x-y)\, dS(y)$$

$$0$$

$$+ \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y)$$

$$:= K_\varepsilon + L_\varepsilon$$

$$L_\varepsilon = \int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y)$$

As $f \in C_c^2$, $\sup |DF(x)| \leq K$

$$|L_\varepsilon| \leq \sup |DF(x)| \int_{\partial B(0,\varepsilon)} \Phi(y)\, dS(y)$$

$$\Phi(y) = \begin{cases} \frac{1}{2\pi} \log \varepsilon & \text{for } n=2 \\ \frac{1}{(n-2)\omega_n} \frac{1}{\varepsilon^{n-2}} \end{cases}$$

$$|L_\varepsilon| \leq \|DF(x)\|_\infty \int \Phi(y) \, dS(y)$$

$$\int dS(y) = \omega_n \varepsilon^{n-1}$$

$$\leq \|DF(x)\|_\infty \; \varepsilon \log \varepsilon \qquad n=2$$

$$\|DF(x)\|_\infty \frac{\varepsilon}{(n-2)} \longrightarrow 0 \text{ as } \varepsilon \to 0$$

$$K_\varepsilon := \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(y) \, f(x-y) \, dS(y)$$

$$\frac{\partial \Phi}{\partial \nu}(y) = \nabla \phi(y) \cdot \nu(y)$$

$\underline{n \geq 3}$

$$\sum \frac{\partial \Phi}{\partial y_i}(y) \cdot \frac{-y_i}{\varepsilon}$$

$$= \sum \frac{1}{\omega_n} \frac{y_i^2}{|y|^n} \frac{1}{\varepsilon}$$

$$= \frac{1}{\omega_n \varepsilon} \frac{|y|^2}{|y|^n} = \frac{1}{\omega_n \varepsilon^{n-1}}$$

$$-K_\varepsilon = \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} f(x-y) \, dS(y) \xrightarrow[\varepsilon \to 0]{} f(x).$$

To show $J_\varepsilon \to 0$

$$|J_\varepsilon| = \left| \int_{B(0,\varepsilon)} \Phi(y) \, \Delta_x f(x-y) \, dy \right|$$

Using similar estimate as I_ε,

$$|J_\varepsilon| \le C \|D^2 f\|_\infty \begin{cases} \varepsilon^2 \log \varepsilon & n=2 \\ \varepsilon^2 & n \geq 3 \end{cases}$$

$\to 0$ as $\varepsilon \to 0$

solution of poisson equation $-\Delta u = f$ is $\Phi * f(x)$.
The condition $f = C_c^2(\mathbb{R}^n)$ is a stronger condition.
we can define $\Phi * f(x)$ for more "general" f.
Eg: $f \in C^\alpha(\mathbb{R}^n)$
$\subset$ Hölder cts α.

Properties of Harmonic function

Thm: Mean Value Property.
Let $u \in C^2(\Omega)$ be harmonic. Then for any $x \in \Omega$,

$$u(x) = ⨍_{B(x,r)} u(y) \, dy = ⨍_{\partial B(x,r)} u(y) \, dS(y)$$

for any $B(x,r) \subset \Omega$

Pf suffices to show that $\frac{\partial}{\partial r} \varphi(r) = 0$

$$\varphi(r) := ⨍_{\partial B(x,r)} u(y) \, dS(y)$$

Using change of variable

$$\varphi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(0,1)} u(x+zr) \, r^{n-1} \, dS(z)$$

↳ unit SA

$$\varphi(r) = \frac{1}{w_n} \int_{\partial B(0,1)} u(x+zr)\, dS(z)$$

$$\varphi'(r) = \frac{1}{w_n} \int_{\partial B(0,1)} \frac{\partial}{\partial r} u(x+zr)\, dS(z)$$

if function is smooth enough and we are integrating over a compact set, can switch.

$$\varphi'(r) = \frac{1}{w_n} \int_{\partial B(0,1)} \nabla u(rz+x) \cdot z \, dS(z)$$

PDE by Evans Appendix

change of variables.

$x + rz = y$

$$\varphi'(r) = \frac{1}{w_n r^{n-1}} \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{(y-x)}{r}\, dS(y)$$

surface area

outward unit radial vector

$$= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(u)\, dS(y)$$

By Green's formula,

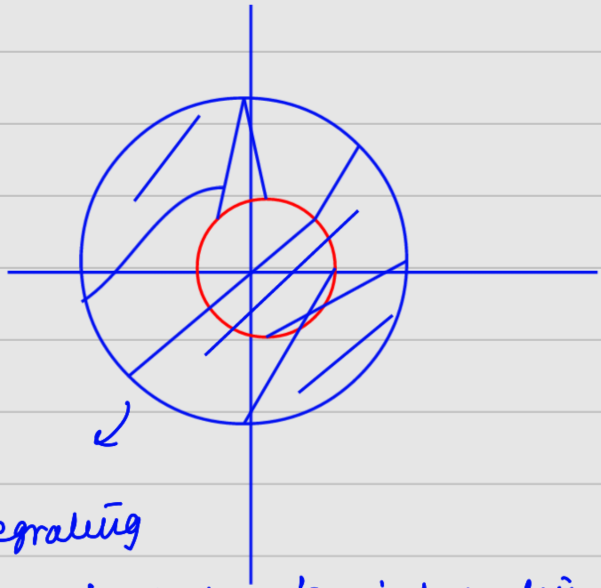
$$\varphi'(r) = \frac{1}{|\partial B(x,r)|} - \int_{B(x,r)} \underbrace{\Delta u(y)}_{\to 0}\, dy = \varphi'(r) = 0$$

$\varphi(r) = \varphi(t) \quad \forall\, t \leq r$

$$\varphi(r) = \lim_{t \to 0} \varphi(t) = \lim_{t \to 0} \frac{1}{w_n t^{n-1}} \int_{B(x,t)} u(y)\, dS(y)$$

$$= u(x)$$

$$\fint_{B(x,r)} u(y)\,dy$$



Integrating over disc is equiv. to integrating over balls.

$$\fint u(y)\,dy = \frac{1}{\alpha_n r^n}\int_0^r \int_{\partial B(x,t)} u(y)\, d\,S(y)\, dy$$

$$= \frac{1}{\alpha_n r^n}\int_0^r \omega_n t^{n-1}\, u(x)\, d\,t$$

$$\omega_n = n\,\alpha_n$$

$$= \frac{n}{r^n}\int_0^r u(x)\, t^{n-1}\, dt$$

$$= \frac{n}{\cancel{r^n}} \cdot \frac{\cancel{r^n}}{\cancel{n}} \cdot u(x) \quad \text{QED}$$

converse of MVT

If $u \in C^2(\Omega)$, MVT holds, $\Delta u = 0$ on Ω.

Proof: Towards a contradiction, suppose not,

$\exists\, x_0 \in \Omega$ s.t $\Delta u(x_0) \neq 0$. As $u \in C^2(\Omega)$ $\exists\, r$ st

$\forall\, y \in B(x_0, y_0)$, $\Delta u(y) > 0$ WLOG

From previous proof,

$$\varphi'(r_0) = \frac{-1}{\partial B(x_0, r_0)} \int_{B(x_0, r_0)} \Delta u(y)\, dy < 0$$

$\rightarrow\leftarrow$ Because MVT $\rightarrow$ $\varphi'(r) = 0$ $\forall\, r$.

Thm: Maximum Principle (for like the 10th time)

Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $\Delta u = 0$. Ω bounded domain

(i) Then $\max_{\bar{\Omega}} u = \max_{\partial\bar{\Omega}} u$ (weak Maximum)

(ii) If Ω is connected and if $u(x_0) = \max_{\bar{\Omega}} u(x)$, $x_0 \in \Omega$ then u is constant.

(strong Maximum)

Pf: we only prove part (ii). Assume $u(x_0) = \max_{\bar{\Omega}} u = M$. $x_0 \in \Omega$.

Now consider $S = \{x \in \Omega : u(x) = M\}$

To show $S = \Omega$. Suffices to show S is open & closed.

$\Rightarrow$ S is closed. $x_0 \in S \Rightarrow S$ is non-empty.

$\Rightarrow$ To show S is open. Take $z \in S$.

$M = u(z) = \fint_{B(z,r)} u(y)\, dy$ $\quad B(z,r) \subseteq \Omega$.

$$\fint_{B(z,r)} u(y)\, dy \leq M \fint_{B(z,r)} dy = M.$$

we claim this identity holds if $u(y) = M \ \forall y \in B(z,r)$

Towards a contradiction $u(y_0) < M$ $y_0 \in B(z,r)$

Then $\exists$ r_0 $y \in B(y_0, r_0) \Rightarrow u(y) < M$.

Take mvp, get contradiction.

S is open & closed non-empty.

Theorem (Uniqueness)

There exists atmost one solution to

$$-\Delta u = f \quad \text{on } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

Ω is connected & bounded

Towards a contradiction..

Pf: u_1, u_2 are two solutions.

Define $w = u_1 - u_2$

w solves

$$-\Delta w = 0 \quad \text{on } \Omega$$
$$w = 0 \quad \text{on } \partial\Omega$$

constant on boundary $\Rightarrow$ $w = 0$ on $\overline{\Omega}$

$\therefore$ $u_1 = u_2$

Theorem: (Smoothness)

Let $u \in C(\Omega)$ satisfying the mean value property for any $B(x,r) \subset \Omega$. Then $u \in C^{\infty}(\Omega)$

Pf: consider a standard mollifier η given by

$$\eta(x) = \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{|x|}{\varepsilon}\right)$$

$\eta \in C_c^\infty(\mathbb{R}) \qquad \int_{\mathbb{R}} \eta(x)\,dx = 1$

$\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = 1$

$supp(\eta_\varepsilon) \subseteq B(0,\varepsilon)$

Now consider $\Omega_\varepsilon = \{ x \in \Omega : d(x, \delta\Omega) > \varepsilon \}$

$u_\varepsilon(x) = u * \eta_\varepsilon(x) \qquad x \in \Omega_\varepsilon$

By the property of convolution,

$u_\varepsilon \in C^\infty(\Omega_\varepsilon)$

we show $u = u_\varepsilon$ in Ω_ε.

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\, u(y)\, dy = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, u(y)\, dy$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \int_{\partial B(x,r)} \eta\left(\frac{r}{\varepsilon}\right) u(y)\, dS(y)\, dr$$

$\omega_n r^{n-1}$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta(r/\varepsilon) \quad u(x)\, dr$$

$$= u(x) \qquad x \in \Omega_\varepsilon$$

$$= \frac{u(x)}{\varepsilon^n} \int_0^\varepsilon \int_{\partial B(x,r)} \eta\left(\frac{|x-y|}{\varepsilon}\right) dS(y)\, dr$$

$$= \frac{u(x)}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right) dy = u(x)$$