Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)
Hints for Exercise Sheet 3

1. Exponential and logarithm functions

1.1. Show that, $\forall r > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N$, the polynomial $1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}$ has all zeros outside the closed disc $\overline{D(0;r)}$.

Solution. *As* $\exp$ *is nowhere zero on* $\overline{D(0;r)}$, $\varepsilon \overset{def}{=} \min\{|\exp(z)| : z \in \overline{D(0;r)}\} > 0$. *From uniform convergence of* $1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}$ *on* $\overline{D(0;r)}$, *one obtains that there exists* $N \in \mathbb{N}$ *such that, for all* $n \geq N$ *and* $z \in \overline{D(0;r)}$,

$$\left|\left|1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}\right| - \left|\exp(z)\right|\right| \leq \left|1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}-\exp(z)\right| < \frac{\varepsilon}{2}.$$

This implies that, for all $n \geq N$ *and* $z \in \overline{D(0;r)}$,

$$\left|1+z+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}\right| > |\exp(z)| - \frac{\varepsilon}{2} > 0.$$

1.2.$*$ (a) Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonnegative reals such that $a_n \underset{n\to\infty}{\longrightarrow} 0$. Show that, for any $x \in \mathbb{R}$, $\sum_{n=1}^{\infty} a_n \sin nx$ converges.

Solution. *Observe that,* $\forall x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $\sum_{k=1}^{n} e^{ikx} = e^{ix}\cdot\frac{1-e^{inx}}{1-e^{ix}} = e^{i\frac{(n+1)x}{2}}\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}$. *Taking imaginary part, we get that* $\left|\sum_{k=1}^{n}\sin kx\right| \leq \frac{1}{\left|\sin\frac{x}{2}\right|}$, *for all* $x \in \mathbb{R}\setminus 2\pi\mathbb{Z}$. *Since the sequence* $\left\{\sum_{k=1}^{n}\sin kx\right\}_{n=1}^{\infty}$ *is bounded and* $\{a_n\}_{n=1}^{\infty}$ *decreases to* 0, $\sum_{n=1}^{\infty} a_n \sin nx$ *converges.*

(b) Show that the convergence in 1.2.a is uniform on each $[2k\pi+\delta, 2(k+1)\pi-\delta]$, where $\delta > 0$ and $k \in \mathbb{Z}$.

Sketch of the solution. *For any* $k \in \mathbb{N}$, *let* $b_k(x) \overset{def}{=} \sin kx$, *for all* $x \in [2k\pi+\delta, 2(k+1)\pi-\delta]$. *Since* $\sum_{k=1}^{n} a_k b_k(x) = a_n B_n(x) + \sum_{k=1}^{n-1}(a_k - a_{k+1})B_k(x)$. *Clearly there exists* $M \geq 0$ *such that, for all* $x \in [2k\pi+\delta, 2(k+1)\pi-\delta]$,

$$|B_n(x)| \leq \frac{1}{\left|\sin\frac{x}{2}\right|} \leq M.$$

From this it follows that $\sum_{k=1}^{\infty}(a_k - a_{k+1})B_k(x)$ *converges uniformly on* $[2k\pi+\delta, 2(k+1)\pi-\delta]$, *and* $a_n B_n(x) \xrightarrow[n\to\infty]{uniformly} 0$.

(c) For $z \in \mathbb{C}$, $\sin z \overset{\text{def}}{=} \frac{e^{iz}-e^{-iz}}{2i} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$. Show that, $\sum_{n=1}^{\infty}\frac{\sin nz}{n}$ cannot converge unless z is real.

Sketch of the solution. *For $z = x + iy$ and $n \in \mathbb{N}$, observe that*

$$\sin nz = \frac{\frac{e^{inx}}{e^{ny}} - \frac{e^{ny}}{e^{inx}}}{2i}.$$

If $y \neq 0$, can you see that $\left|\frac{\sin nz}{n}\right| \xrightarrow[n\to\infty]{} \infty$?

1.3. Let $\alpha \in \mathbb{R}$. Consider the *ray* $R_\alpha \overset{\text{def}}{=} \{e^x(\cos\alpha + i\sin\alpha) : x \in \mathbb{R}\}$. Pick any $z_0 \in \overline{R_\alpha}$. Show that, $\lim\limits_{z\to z_0} \log_\alpha z$ and $\lim\limits_{z\to z_0} \arg_\alpha z$ do not exist. Conclude from this that, the functions $\log_\alpha$ and $\arg_\alpha$ are not continuous at any point R_α.

Sketch of the solution. *Let $z_0 = \exp(x_0 + i\alpha)$. Consider the following sequences*

$$z_n \overset{def}{=} \exp\left(x_0 + i\left(\alpha + \frac{1}{n}\right)\right)$$

and

$$w_n \overset{def}{=} \exp\left(x_0 + i\left(\alpha + 2\pi - \frac{1}{n}\right)\right),$$

for all $n \in \mathbb{N}$. Clearly, for any $n \in \mathbb{N}$, $z_n, w_n \neq z_0$ and $z_n, w_n \xrightarrow[n\to\infty]{} z_0$. What can you say about $\lim_{n\to\infty} \log_\alpha(z_n)$ and $\lim_{n\to\infty} \log_\alpha(w_n)$?

1.4.* Let $r > 0$. Show that there cannot exist any continuous $\theta : C(0; r) \longrightarrow \mathbb{R}$ with the property

$$z = re^{i\theta(z)}, \text{ whenever } |z| = r.$$

In words, it is not possible to make a continuous choice of arguments on $C(0; r)$.

(**Hint:** First show that, if such a θ exists then $\theta - \arg_{-\pi}$ must be constant on $C(0; r) \setminus \{-r\}$. Let $z_n \overset{\text{def}}{=} \log r + i\left(-\pi + \frac{1}{n}\right)$, for all $n \in \mathbb{N}$. Where does the sequence $\{e^{z_n}\}_{n=1}^\infty$ converge to? Use this to find $\theta(-r) - \arg_{-\pi}(-r)$.

Alternatively, first show that, if such a θ exists then it must be injective. Next consider the function $f : C(0; r) \longrightarrow \{\pm 1\}$ defined by $f(z) = \dfrac{\theta(z) - \theta(-z)}{|\theta(z) - \theta(-z)|}$, for all $z \in C(0; r)$. Does connectedness of $C(0; r)$ help now?)

1.5.* Give an example of an open connected $U \subseteq \mathbb{C}$ such that $\forall \alpha \in \mathbb{R}$, $U \cap R_\alpha = \emptyset$ but the function $f(z) = z$ has a continuous argument, and also a holomorphic logarithm on U.

1.6. Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C} \setminus \{0\}$ be holomorphic. Show that every continuous logarithm of f is holomorphic.

Solution. *Suppose that $g : U \longrightarrow \mathbb{C}$ is a continuous logarithm of f. Let $z_0 \in U$. Since f is continuous at z_0 and $f(z_0) \neq 0$, there exists $r > 0$ such that $\forall z \in D(z_0; r)$, $f(z) \in D\left(f(z_0); \frac{|f(z_0)|}{2}\right)$. As $0 \notin D\left(f(z_0); \frac{|f(z_0)|}{2}\right)$, there exists a holomorphic logarithm of f, say h on $D(z_0; r)$. From connectedness of $D(z_0; r)$, one obtains that $g - h$ is constant, which imples that g is holomorphic on $D(z_0; r)$.*

Note to the student. *Thus for a zero-free holomorphic funcstion, continuous and holomorphic arguments are same.*

1.7. Let X be a metric space with the property that every bounded sequence in X admits a convergent subsequence. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence with exactly one subseqeuntial limit ℓ, i.e., any convergent subsequence of $\{x_n\}_{n=1}^\infty$ converges to ℓ. Show that $x_n \xrightarrow[n\to\infty]{} \ell$.

Sketch of the solution. *Prove by contradiction. If* $\{x_n\}_{n=1}^{\infty}$ *does not converge to* ℓ*, then there exists* $\varepsilon > 0$ *such that for all* $N \in \mathbb{N}$*,* $d(x_n, \ell) \geq \varepsilon$*. Using this, get a subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *with the property that* $d(x_{n_k}, \ell) \geq \varepsilon$*, for all* $k \geq 1$*. Since* $\{x_n\}_{n=1}^{\infty}$ *is bounded, so is* $\{x_{n_k}\}_{k=1}^{\infty}$*. It follows that* $\{x_{n_k}\}_{k=1}^{\infty}$ *has a convergent subsequence, and that has to converge to* ℓ*. But this is not possible, as* $d(x_{n_k}, \ell) \geq \varepsilon$*, for all* $k \geq 1$*.*

2. Power function

Let $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$. Choose $\alpha \in \mathbb{R}$ in such a way that $z \notin R_\alpha$. We define

$$z^w = e^{w \log_\alpha z}. \tag{2.1}$$

2.1. Show that, for any $\alpha \in \mathbb{R}$, the function $\mathbb{C} \setminus \overline{R_\alpha} \longrightarrow \mathbb{C}$, $z \mapsto z^w$, where z^w is defined above in (2.1), is holomorphic. What is the derivative?

Sketch of the solution. *The function is holomorphic as it is the composition of two holomorphic functions. Next use Chain rule to find the derivative.*

2.2. Let $x > 0$ and $n \in \mathbb{N}$. Find all possible values of $x^{\frac{1}{n}}$ for $\alpha \neq 0$.

3. Basic properties of index

3.1. Let γ, γ_1 and $\gamma_2 : [a, b] \longrightarrow \mathbb{C}$ be closed curves. Prove the following:

(a) $z \notin \gamma^* \implies \mathrm{Ind}_\gamma(z) = \mathrm{Ind}_{\gamma - z}(0)$.

(b) $0 \notin \gamma_1^* \cup \gamma_2^* \implies \mathrm{Ind}_{\gamma_1\gamma_2}(0) = \mathrm{Ind}_{\gamma_1}(0) + \mathrm{Ind}_{\gamma_2}(0)$ and $\mathrm{Ind}_{\gamma_1/\gamma_2}(0) = \mathrm{Ind}_{\gamma_1}(0) - \mathrm{Ind}_{\gamma_2}(0)$.

(c) If $\gamma^* \subseteq D(z_0; r)$ then $\mathrm{Ind}_\gamma(z) = 0$, for all $z \notin D(z_0; r)$.

Solution. *Let* $|z - z_0| \geq r$*. It is clear that the image of* $\eta \overset{def}{=} \gamma - z$ *is contained in* $D(z_0 - z; r)$*. As* $0 \notin D(z_0 - z; r)$*, there exists* $\alpha \in \mathbb{R}$ *such that* $D(z_0 - z; r) \cap \overline{R_\alpha} = \emptyset$*. Then* $\arg_\alpha \circ \eta$ *is a continuous argument of* η*. So* $\mathrm{Ind}_\gamma(z) = \mathrm{Ind}_\eta(0) = \dfrac{\arg_\alpha(\eta(b)) - \arg_\alpha(\eta(a))}{2\pi} = 0.$

3.2.∗ If $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$, for all $t \in [a, b]$, then $0 \notin \gamma_1^* \cup \gamma_2^*$ and $\mathrm{Ind}_{\gamma_1}(0) = \mathrm{Ind}_{\gamma_2}(0)$. (**Hint:** Consider $\gamma \overset{\text{def}}{=} \frac{\gamma_2}{\gamma_1}$. Now use 3.1.b and 3.1.c.)

Solution. *Clearly* $\gamma_1(t) \neq 0$ *for any* $t \in [a, b]$*. Consider* $\gamma \overset{def}{=} \frac{\gamma_2}{\gamma_1}$*. Then we have, for all* $t \in [a, b]$*,* $|1 - \gamma(t)| < 1$*, i.e.,* $\gamma^* \subseteq D(1; 1)$*. From 3.1.b and 3.1.c, we obtain that* $\mathrm{Ind}_{\gamma_1}(0) - \mathrm{Ind}_{\gamma_2}(0) = \mathrm{Ind}_\gamma(0) = 0$.

Note: 3.2. has a nice interpretation. Suppose that a man is walking with his pet dog on a leash with variable length. A tree is located in the origin. At time t, the position of the man and his dog are $\gamma_1(t)$ and $\gamma_2(t)$ respectively. Then 3.2. shows that, if the length of the leash is always less than the distance between the man and the tree, then the man and his dog must walk around the tree exactly same number of times. That is why sometimes 3.2. is also referred to as the *Dog-on-a-Leash lemma* or *Dog-walking lemma*.