Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

Akash Anand
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Can we make the method higher order?

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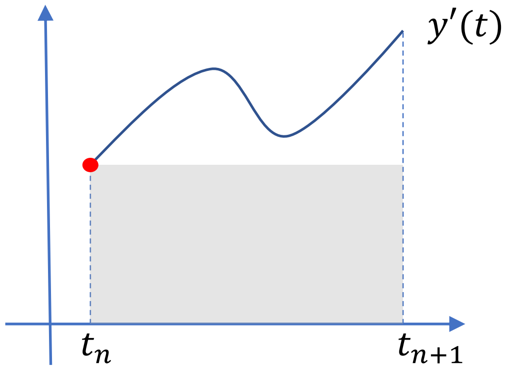

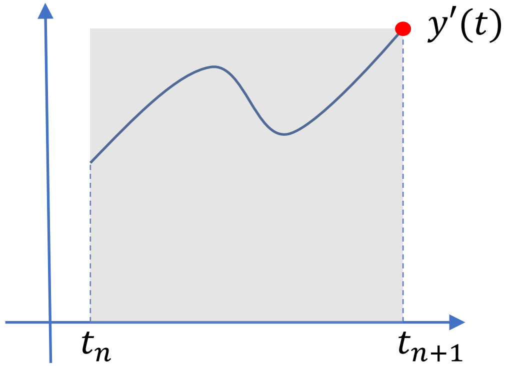

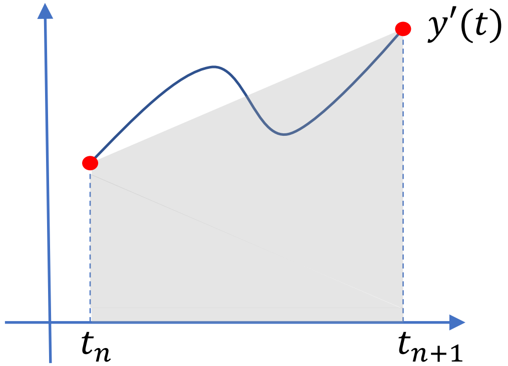

Initial Value Problems: Linear Multistep Methods

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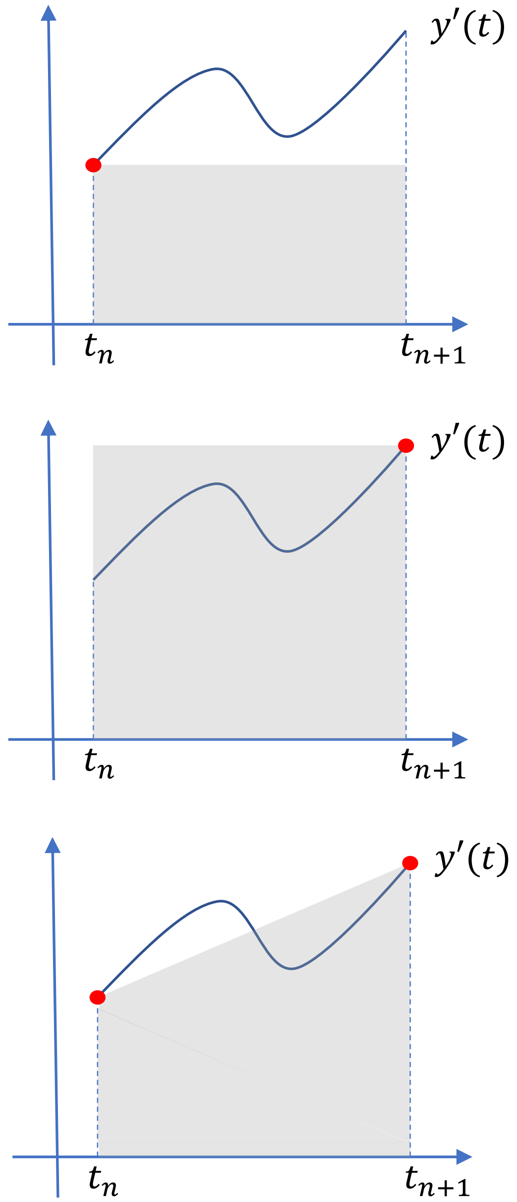
Following the idea that we used to derive trapezoidal method, we could use higher order polynomial approximations.



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Following the idea that we used to derive trapezoidal method, we could use higher order polynomial approximations.

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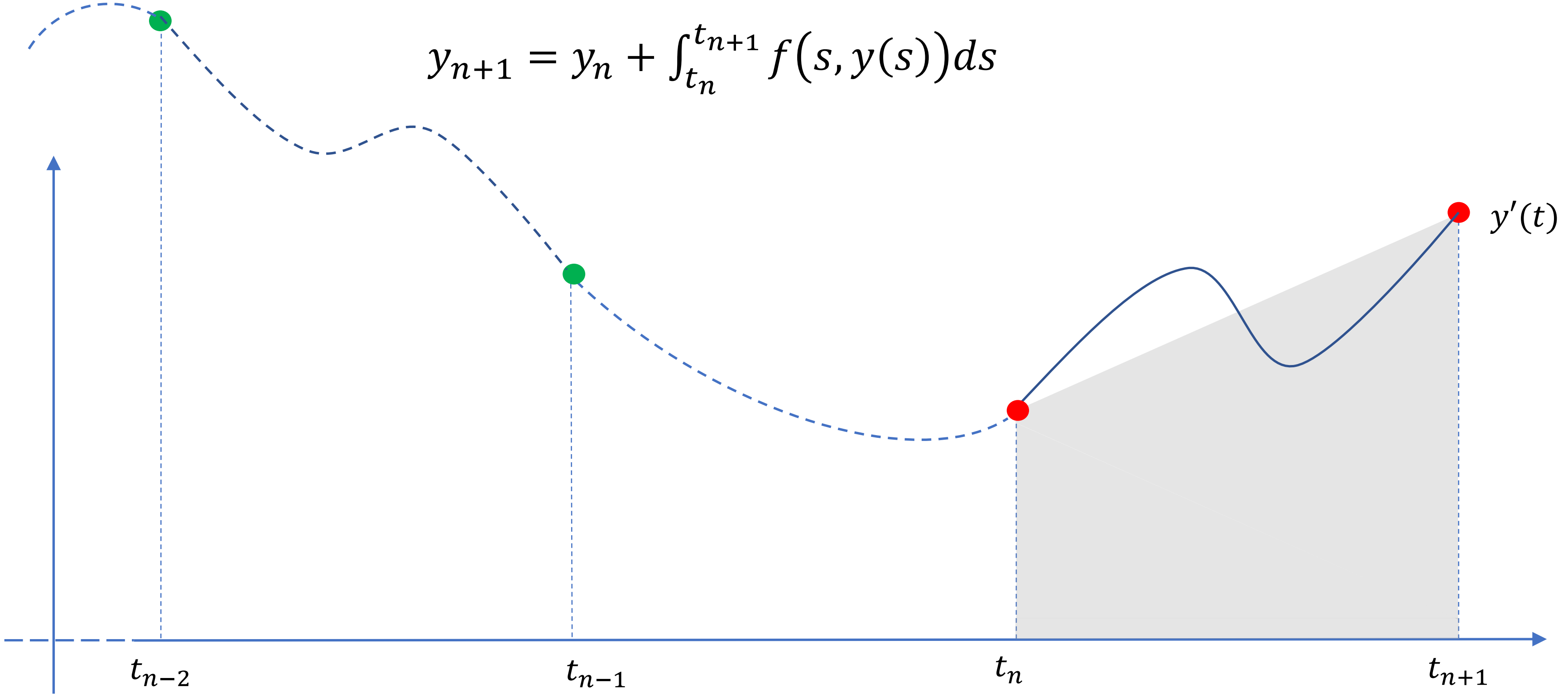

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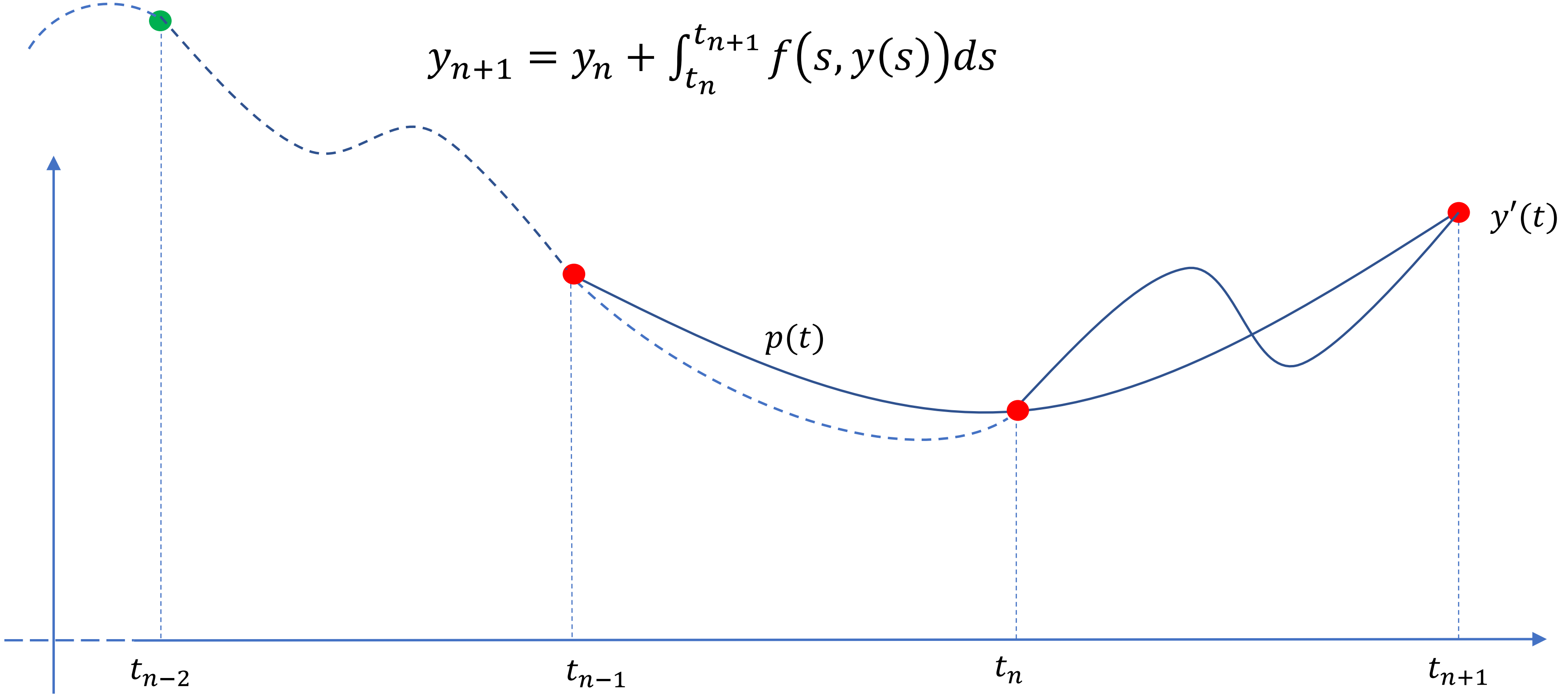

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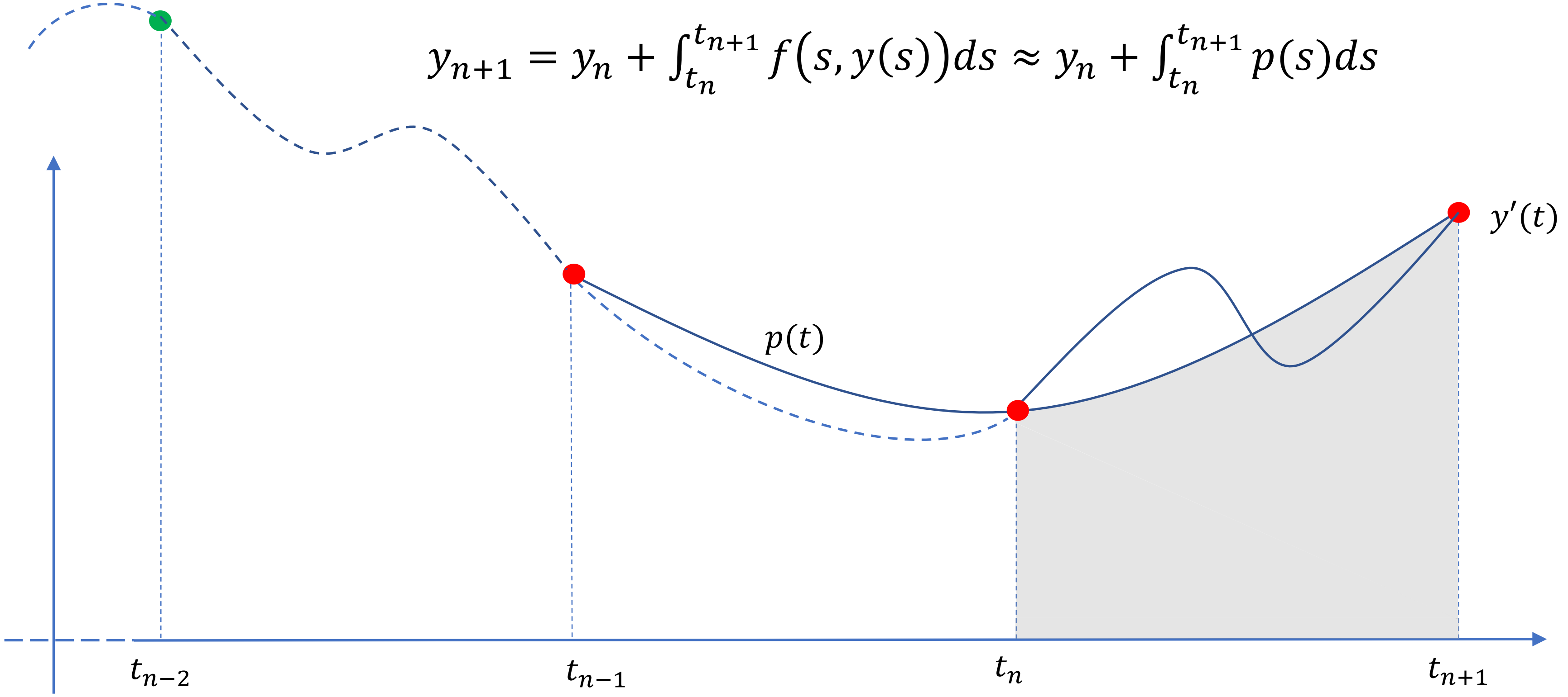

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&= y_n + b_{-1} f(t_{n+1}, y_{n+1}) + b_0 f(t_n, y_n) + b_1 f(t_{n-1}, y_{n-1})
\end{aligned}
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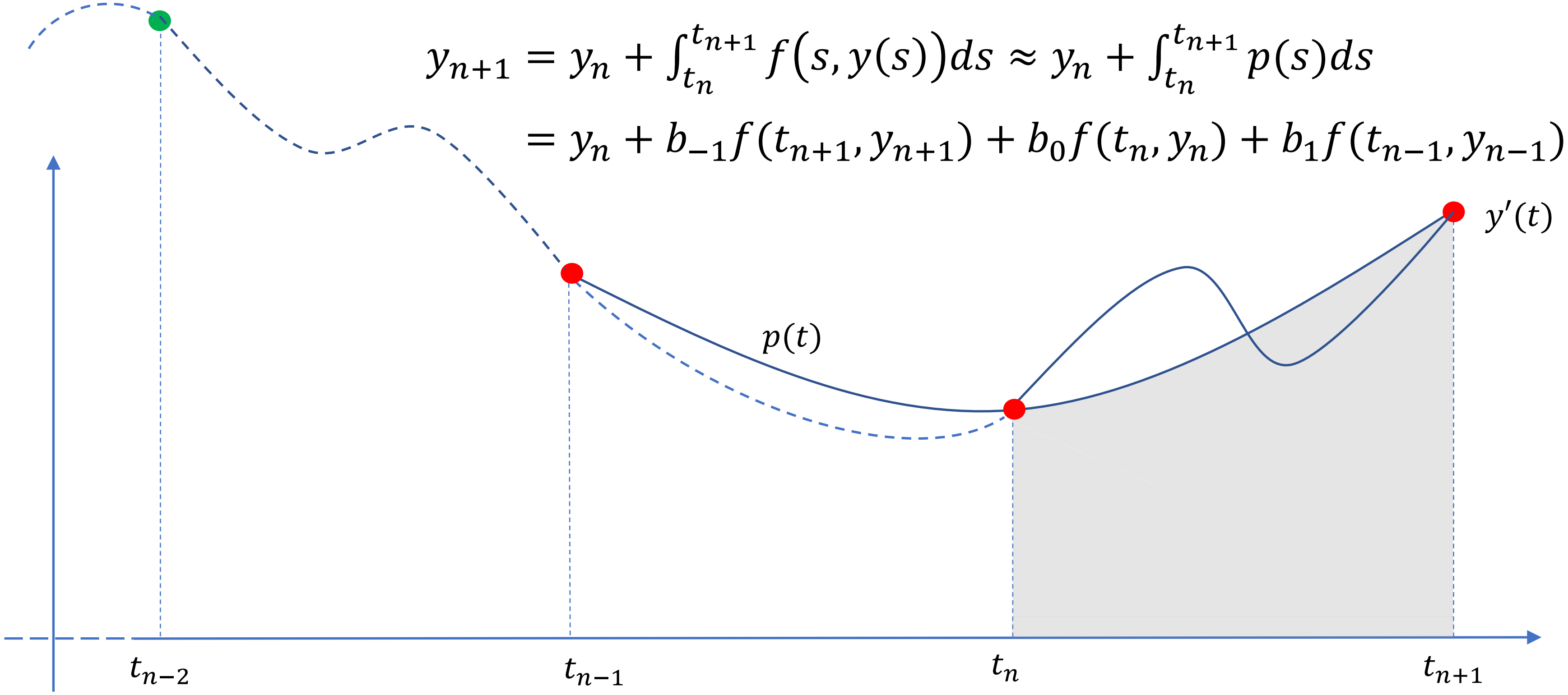

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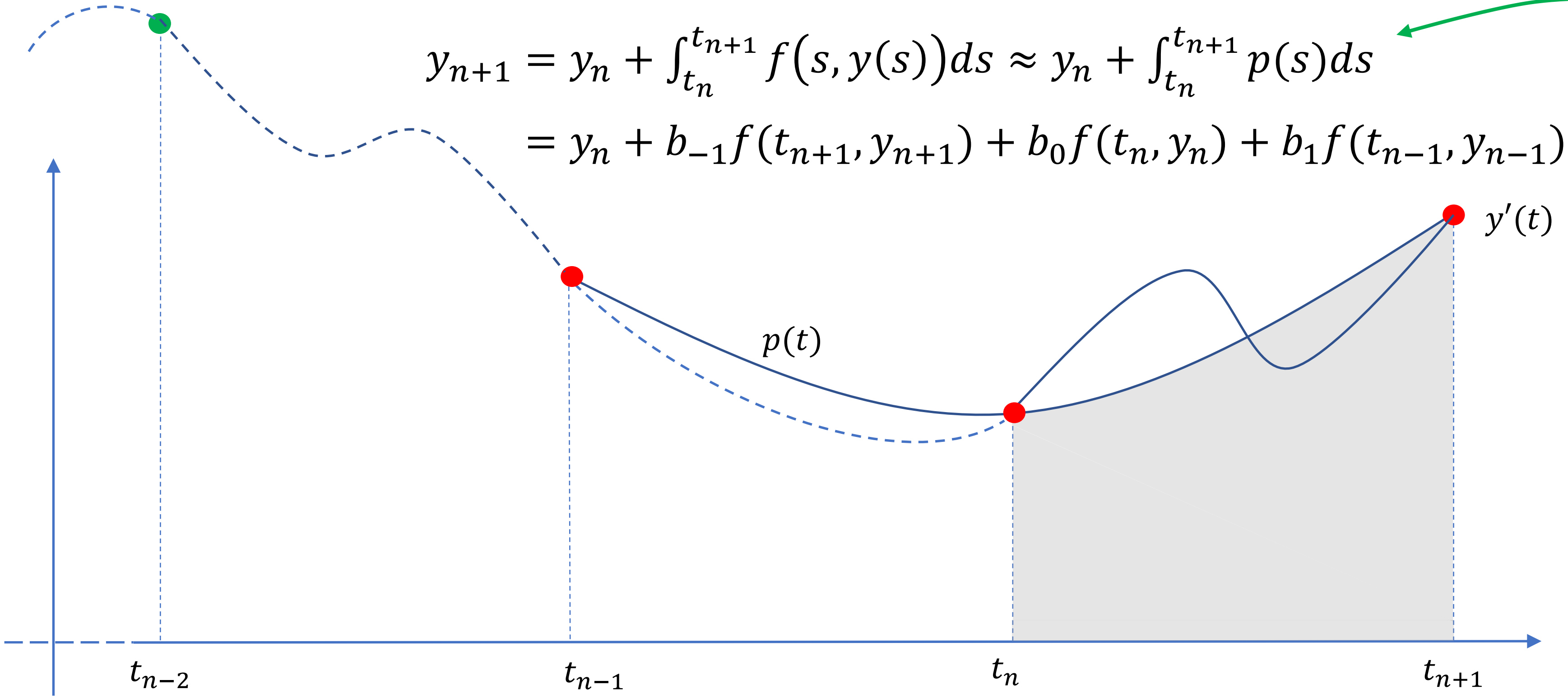
Adams—Moulton method



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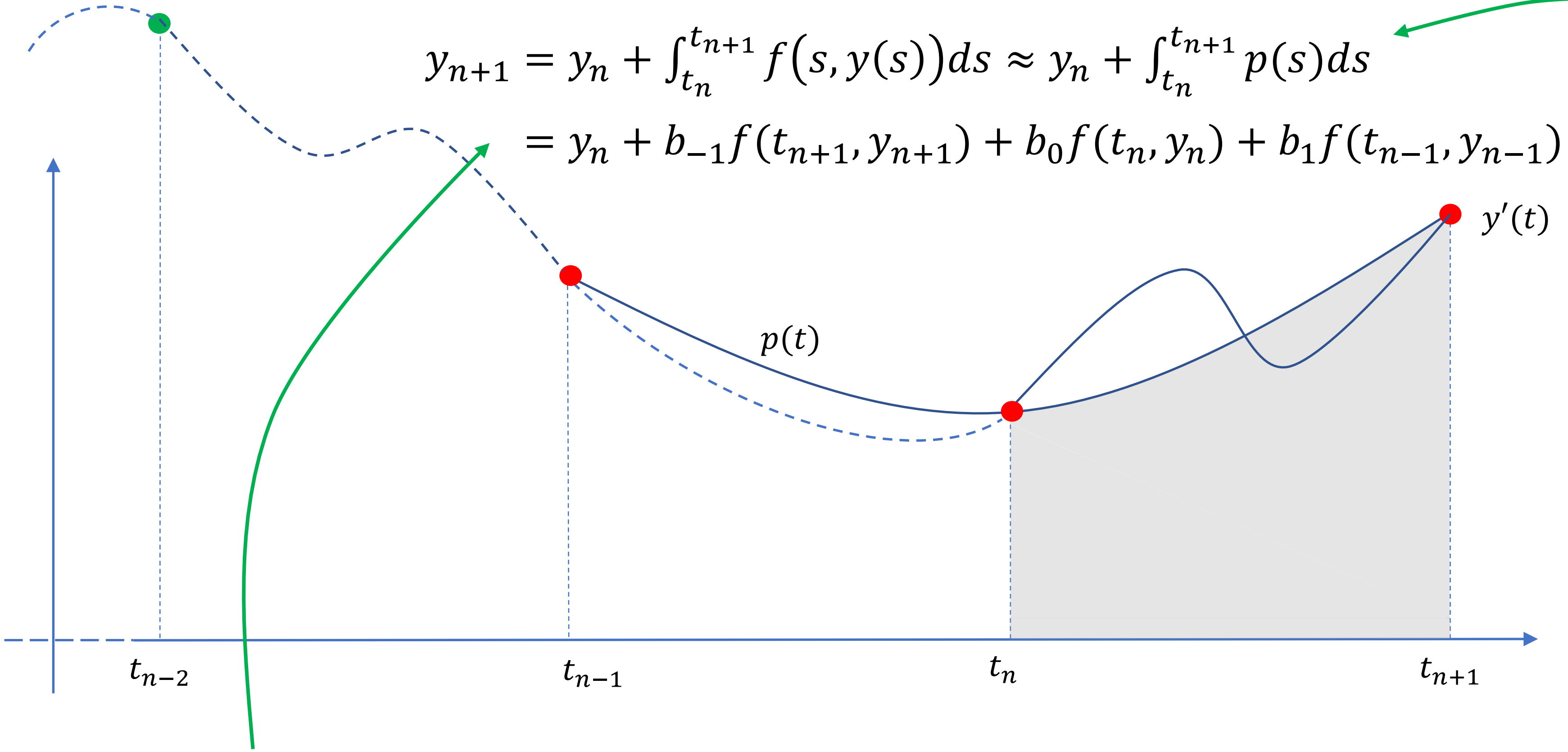

An implicit method

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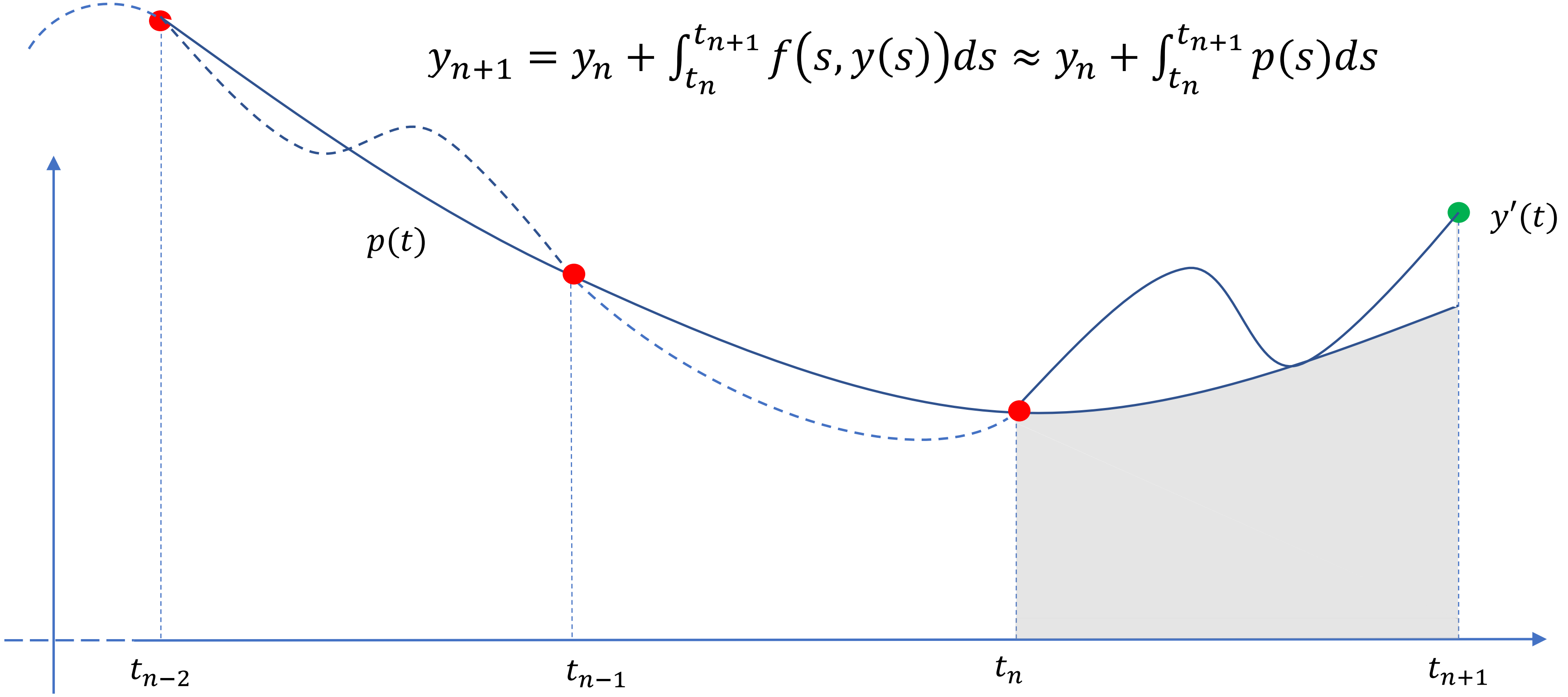

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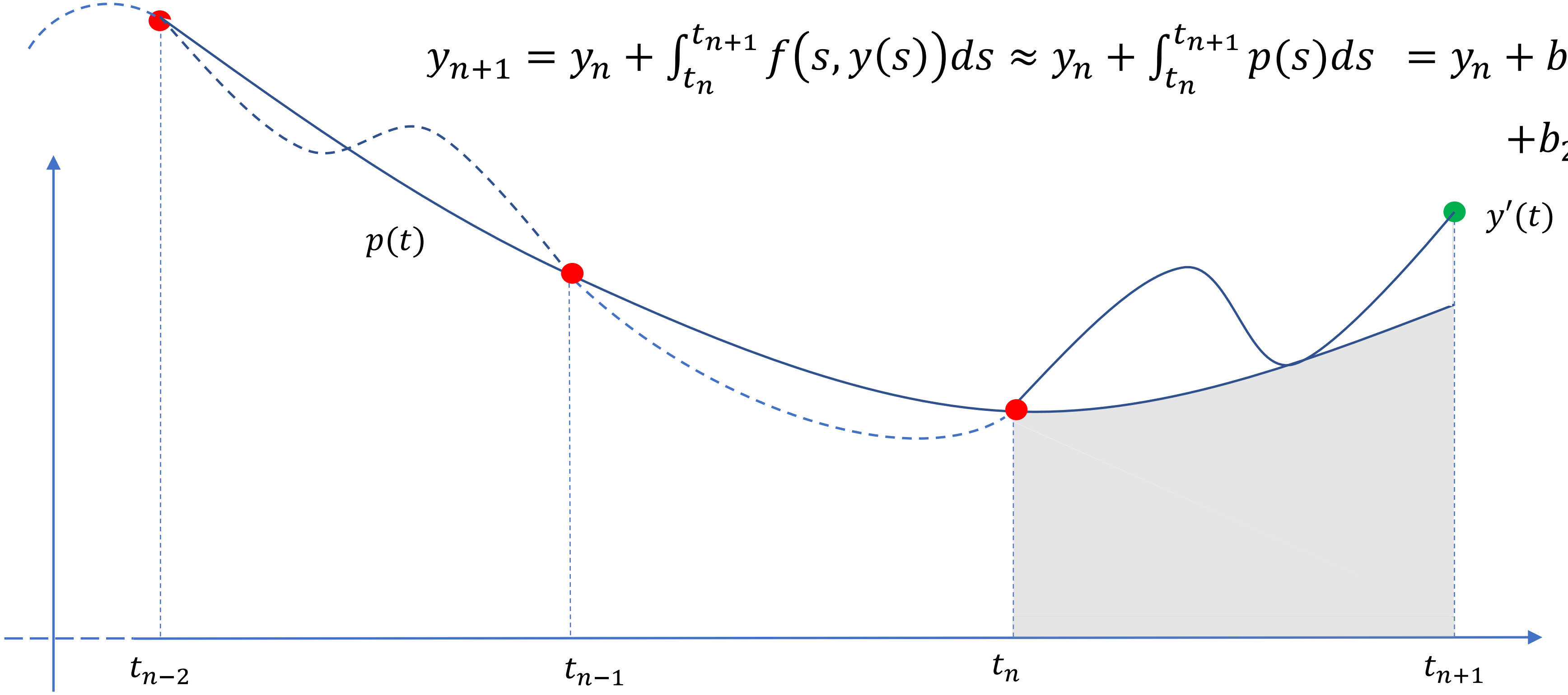

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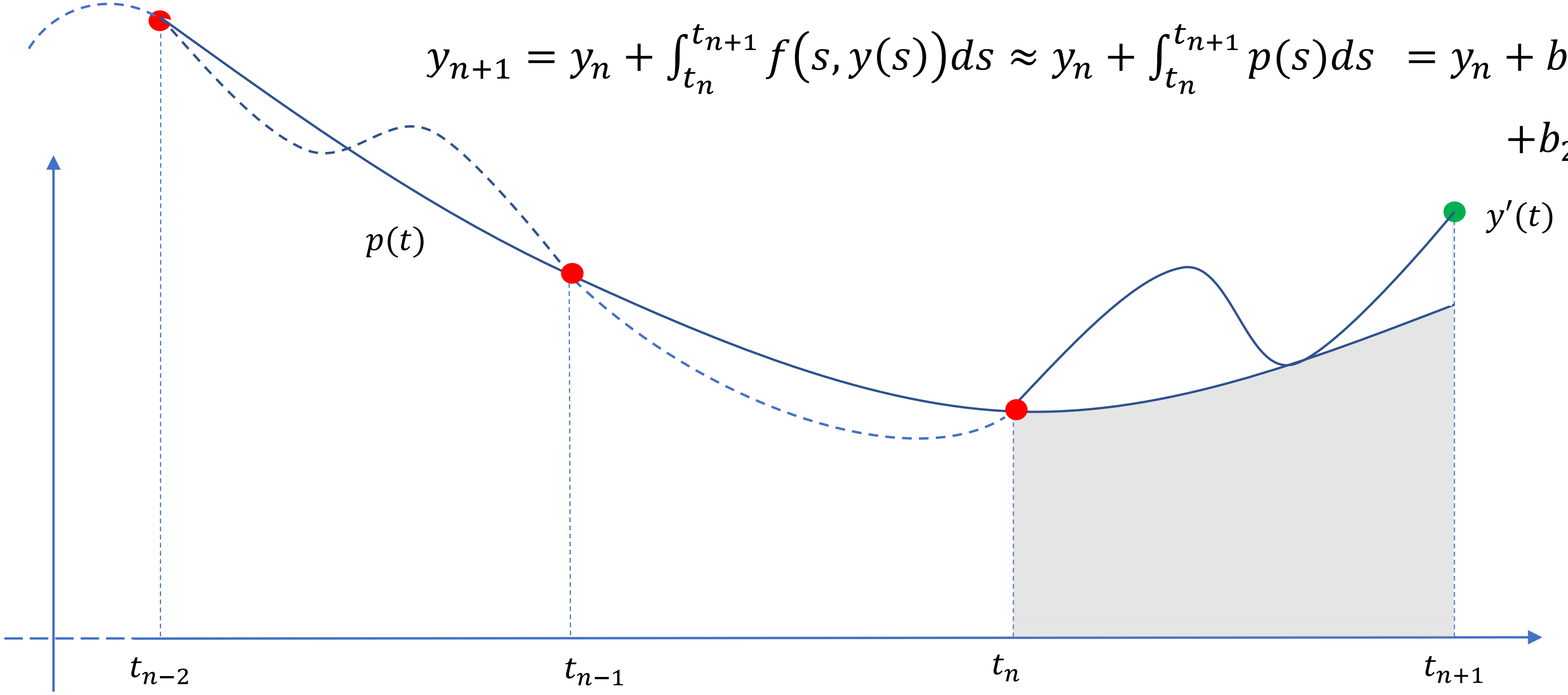

Adams-Bashford method

Initial Value Problems: Linear Multistep Methods

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We consider methods that take constant step size h and determine y_{n+1} using the values from several preceding steps:

$$y_{n+1} = \Phi(f, t_n, y_{n+1}, y_n, y_{n-1}, \dots, y_{n-k}, h).$$

Here y_{n+1} depends on $k+1$ previous values, so this is called a $(k+1)$-step method.

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For $k \geq 1$, that is, for a 2 or more step method, how do we start the time marching? Note that we need to know $y_0, \dots, y_k$, to compute y_{k+1} and we typically only know y_0!

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Examples:

1. $y_{n+1} = y_n + hf(t_n, y_n)$ *… an explicit one step method*
2. $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$ *… an implicit one step method*
3. $y_{n+1} = y_n + h\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)/2$ *… an implicit one step method*

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Improved Euler Method

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Φ Is linear in y_n, $f(t_n, y_n)$, $f(t_{n+1}, y_{n+1})$, etc.

non-linear Φ

Initial Value Problems: Linear Multistep Methods

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We consider linear multistep methods with constant step size, which by definition, are methods of the form

$$y_{n+1} = -\mathrm{a}_0\mathrm{y}_\mathrm{n} - \mathrm{a}_1\mathrm{y}_{\mathrm{n}-1} - \cdots - a_k y_{n-k} + h[b_{-1}f_{n+1} + b_0 f_n + \cdots + b_k f_{n-k}]$$

where f_n *denotes* $f(t_n, y_n)$ *(for brevity) and* a_j*,* b_j *are constants which must be given and determine the specific method.*

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It is also convenient to define $a_{-1} = 1$*, so that the method can we written more concisely as*

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f_{n-j} \, .$$

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Theorem

Let $h_0 = 1/(|b_{-1}|L)$ where L is the Lipschitz constant for f. Then for any $h < h_0$ and any $y_n, y_{n-1}, \dots, y_{n-k}$, there is a unique y_{n+1} such that

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$$F(z) = -\sum_{j=0}^{k} a_j y_{n-j} + h \sum_{j=0}^{k} b_j f(t_{n-j}, y_{n-j}) \; + h b_{-1} f(t_{n+1}, z).$$

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Now, $F(z)$ is Lipschitz with Lipschitz constant less than or equal to $h|b_{-1}|L$ (why?).

Initial Value Problems: Linear Multistep Methods

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Let $h_0 = 1/(|b_{-1}|L)$ *where* L *is the Lipschitz constant for* f*. Then for any* $h < h_0$ *and any* $y_n, y_{n-1}, \dots, y_{n-k}$*, there is a unique* y_{n+1} *such that*

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Proof.

Define

$$F(z) = -\sum_{j=0}^{k} a_j y_{n-j} + h \sum_{j=0}^{k} b_j f(t_{n-j}, y_{n-j}) + h b_{-1} f(t_{n+1}, z).$$

Now, $F(z)$ *is Lipschitz with Lipschitz constant less than or equal to* $h|b_{-1}|L$ *(why?). By hypothesis* $(h < h_0)$*, the Lipschitz constant is strictly less than 1, that is,* $F(z)$ *is a contraction. The contraction mapping theorem then guarantees a unique fixed point, say* y_{n+1}*. Thus, we have* $y_{n+1} = F(y_{n+1})$*, as desired.*

Theorem

Let $h_0 = 1/(|b_{-1}|L)$ where L is the Lipschitz constant for f. Then for any $h < h_0$ and any $y_n, y_{n-1}, \dots, y_{n-k}$, there is a unique y_{n+1} such that

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j}).$$

Proof.

Define

$$F(z) = -\sum_{j=0}^{k} a_j y_{n-j} + h \sum_{j=0}^{k} b_j f(t_{n-j}, y_{n-j}) + h b_{-1} f(t_{n+1}, z).$$

Now, $F(z)$ is Lipschitz with Lipschitz constant less than or equal to $h|b_{-1}|L$ (why?). By hypothesis ($h < h_0$), the Lipschitz constant is strictly less than 1, that is, $F(z)$ is a contraction. The contraction mapping theorem then guarantees a unique fixed point, say y_{n+1}. Thus, we have $y_{n+1} = F(y_{n+1})$, as desired.

Remark

The contraction mapping theorem also implies that the solution can be computed by fixed point iteration as is often done in practice. Moreover, only a fixed (small) number of iterations are made (introducing an additional error).

Module 2

Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Adams methods

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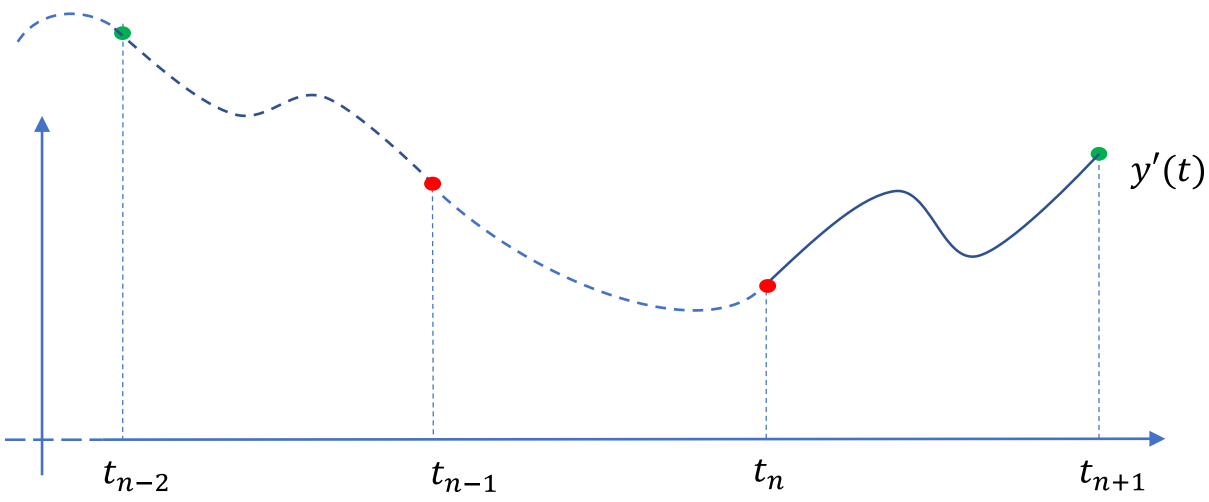
Examples

Adams Bashford methods -

Examples

Adams Bashford methods -

-- 2-step method

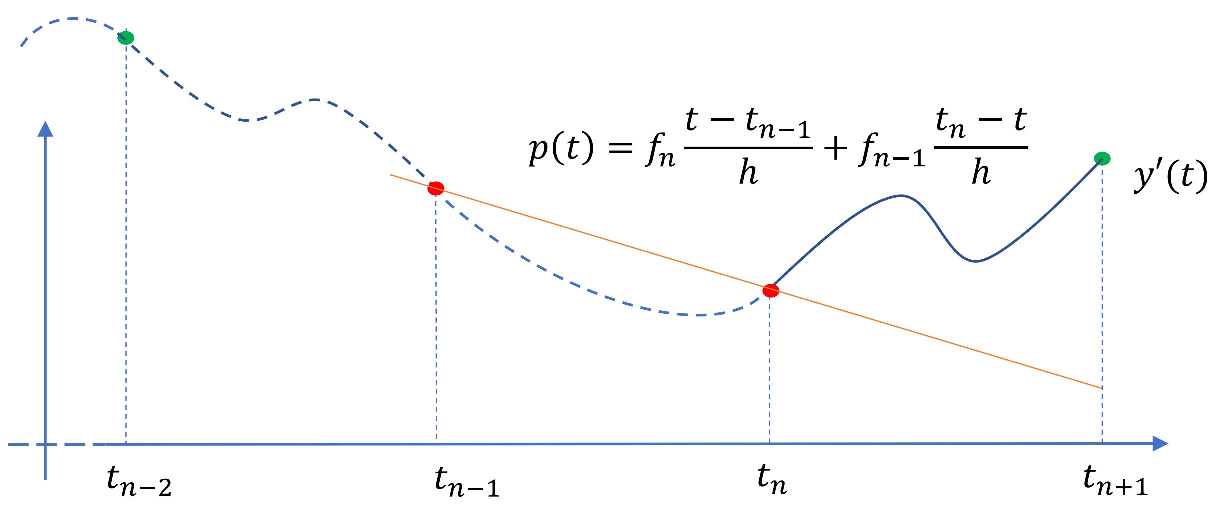


Initial Value Problems: Linear Multistep Methods

Examples

Adams Bashford methods -

-- 2-step method



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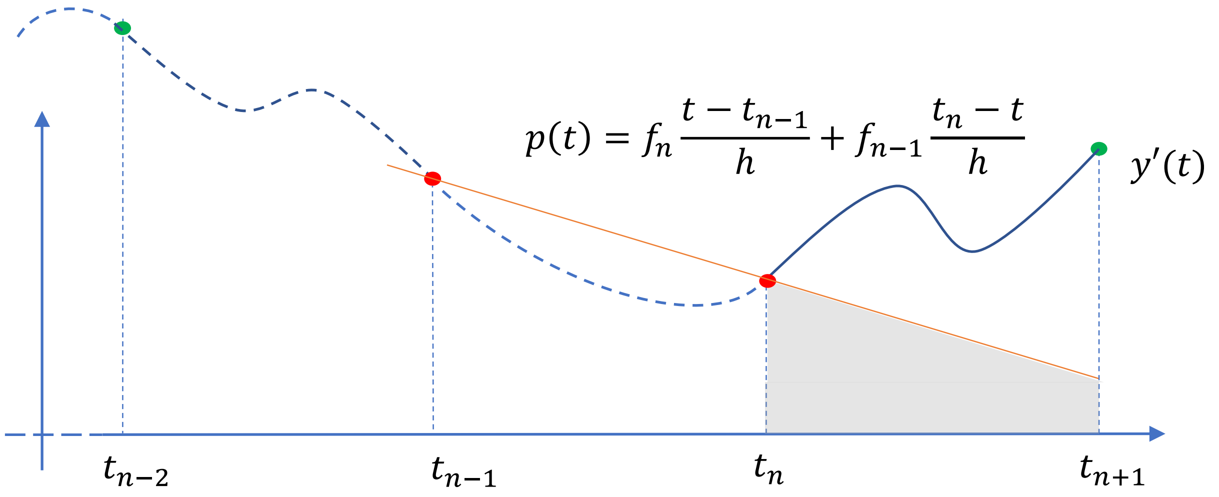

Initial Value Problems: Linear Multistep Methods

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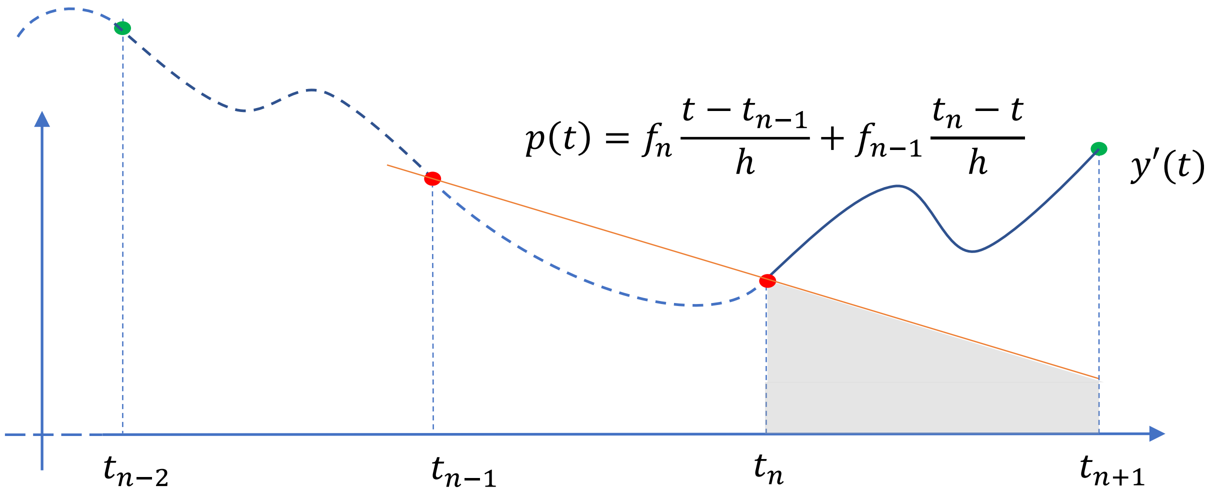
Examples

Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt$$



Initial Value Problems: Linear Multistep Methods

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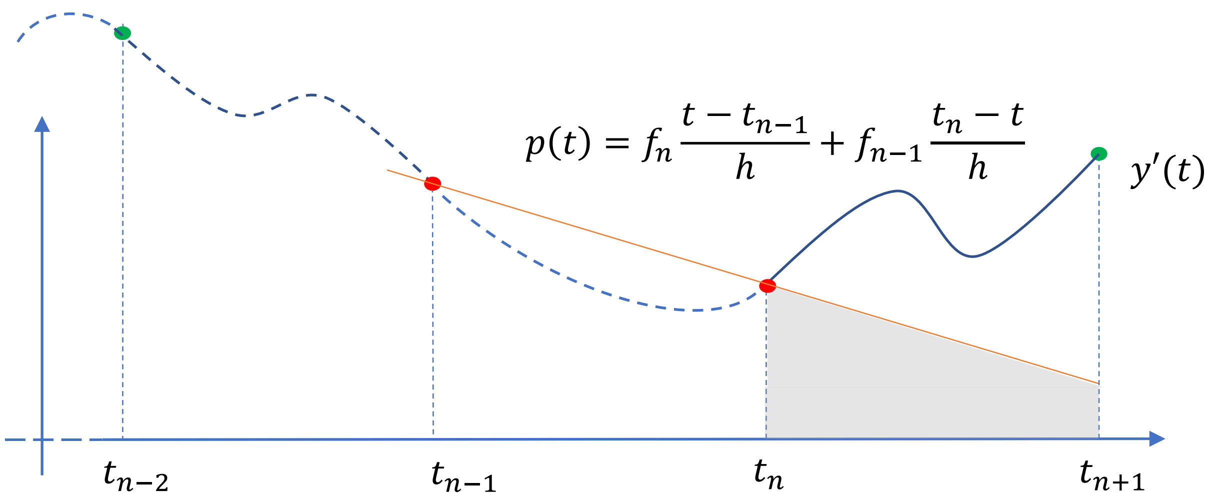
Examples

Adams Bashford methods -

-- 2-step method

$$\int_{t_n}^{t_{n+1}} p(t)dt =$$

$$\int_{t_n}^{t_{n+1}} \left(f_n \frac{t-t_{n-1}}{h} + f_{n-1} \frac{t_n - t}{h}\right) dt$$



Initial Value Problems: Linear Multistep Methods

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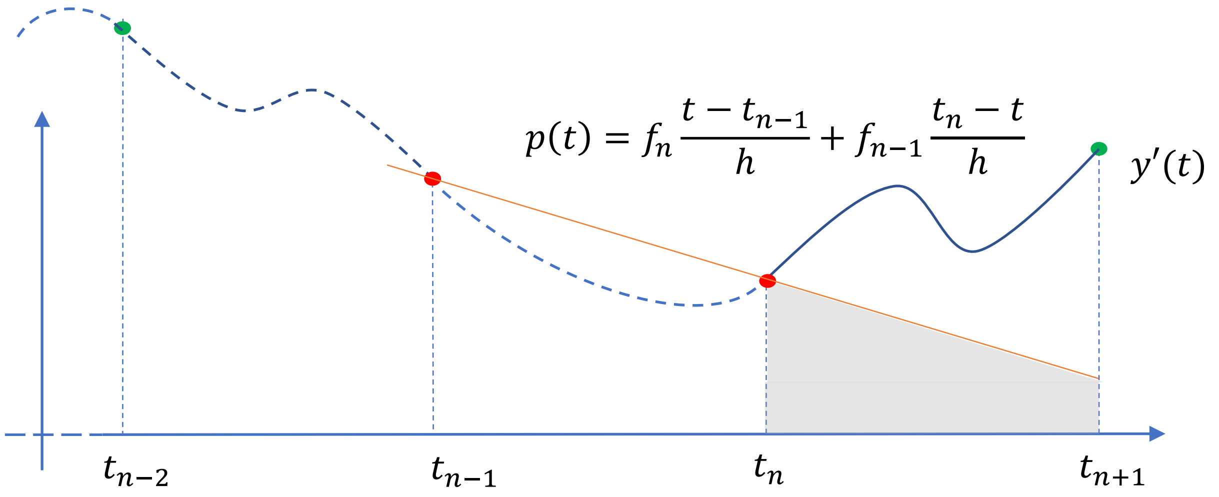

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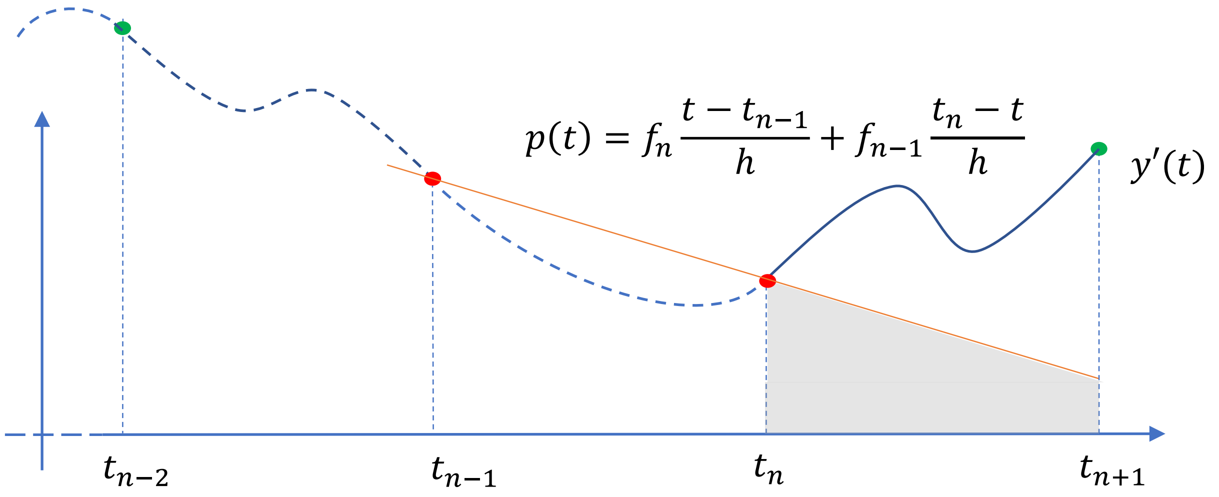


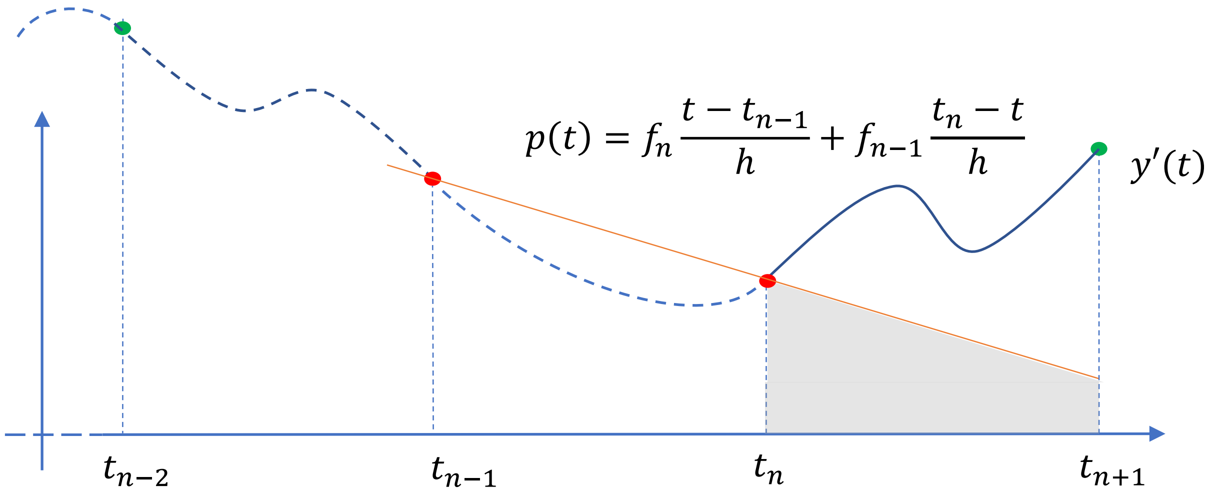
Thus,

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

Examples

Adams Bashford methods -

-- 2-step method



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-- General (k+1) step method

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Adams Bashford methods -

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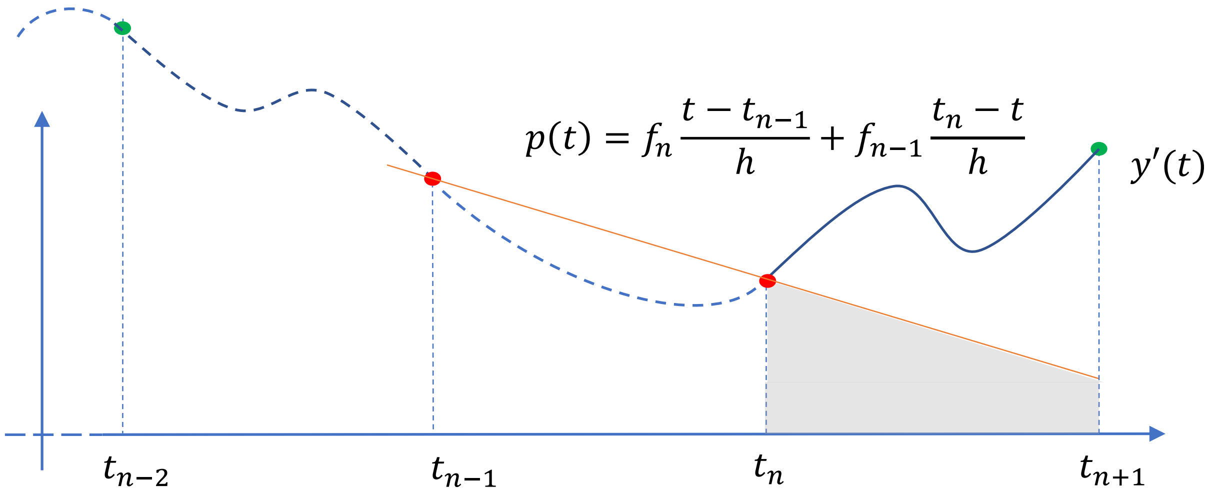

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-- General (k+1) step method

$$p(t) = \sum_{j=0}^{k} l_j^{(k)}(t) f_{n-j}, \quad \text{where} \quad l_j^{(k)}(t) = \prod_{i=0, i\neq j}^{k} (t - t_{n-i})/(t_{n-j} - t_{n-i})$$

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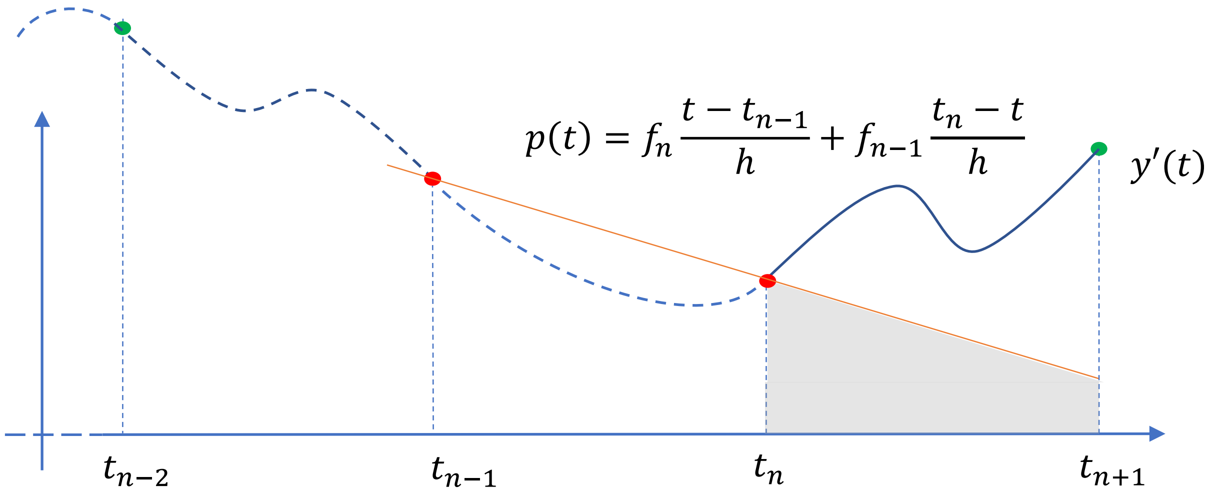

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Thus,

$$y_{n+1} = y_n + \sum_{j=0}^{k} b_j f_{n-j}, \qquad \text{with} \qquad b_j = \int_{t_n}^{t_{n+1}} l_j^{(k)}(t)dt.$$

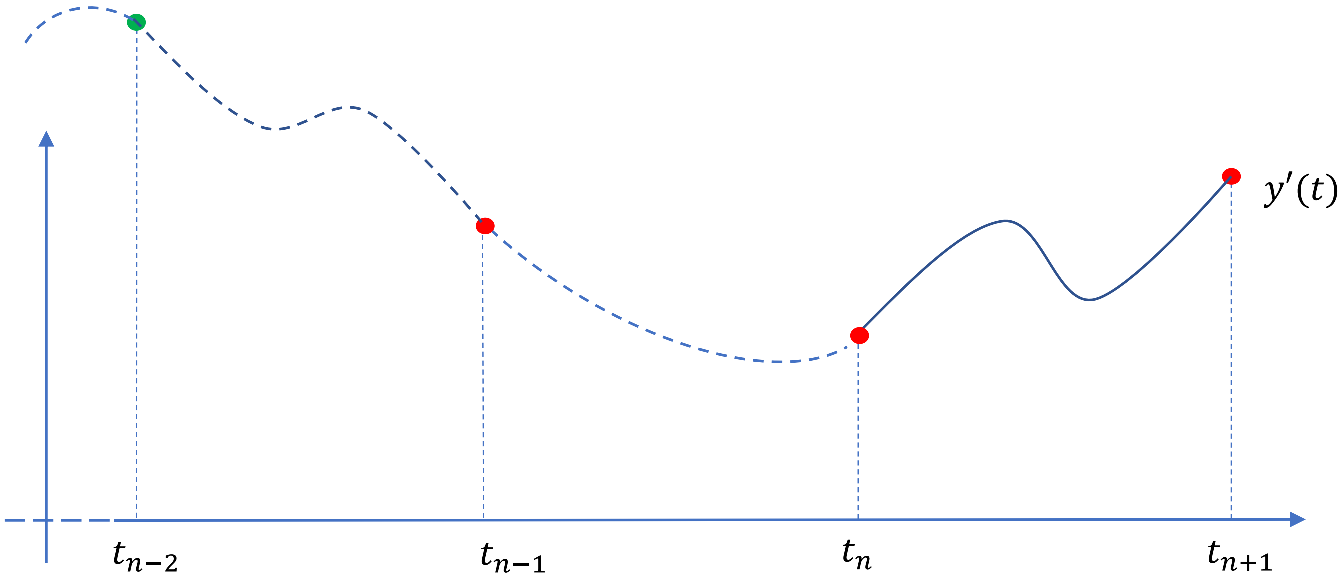
Examples

Adams Moulton methods -

Examples

Adams Moulton methods -

-- 2-step method



Examples

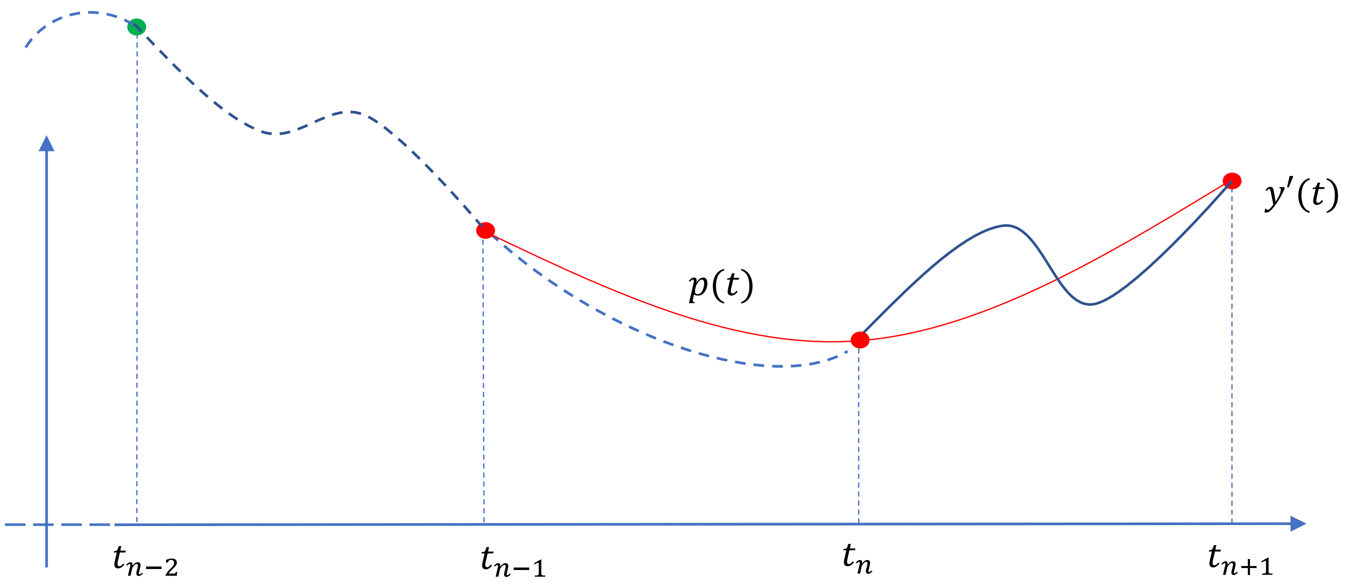
Adams Moulton methods -

-- 2-step method

$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2}$$

$$- f_n\frac{(t-t_{n+1})(t-t_{n-1})}{h^2}$$

$$+ f_{n-1}\frac{(t-t_{n+1})(t-t_n)}{2h^2}$$



Initial Value Problems: Linear Multistep Methods

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Examples

Adams Moulton methods -

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$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2}$$

$$-\, f_n\frac{(t-t_{n+1})(t-t_{n-1})}{h^2}$$

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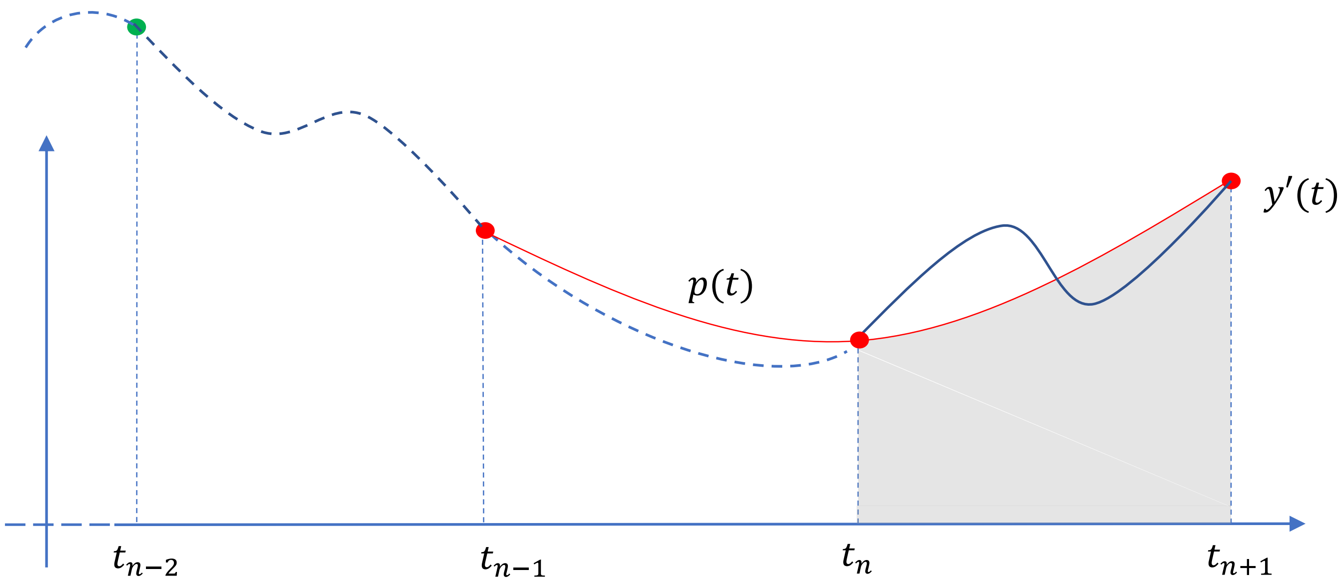


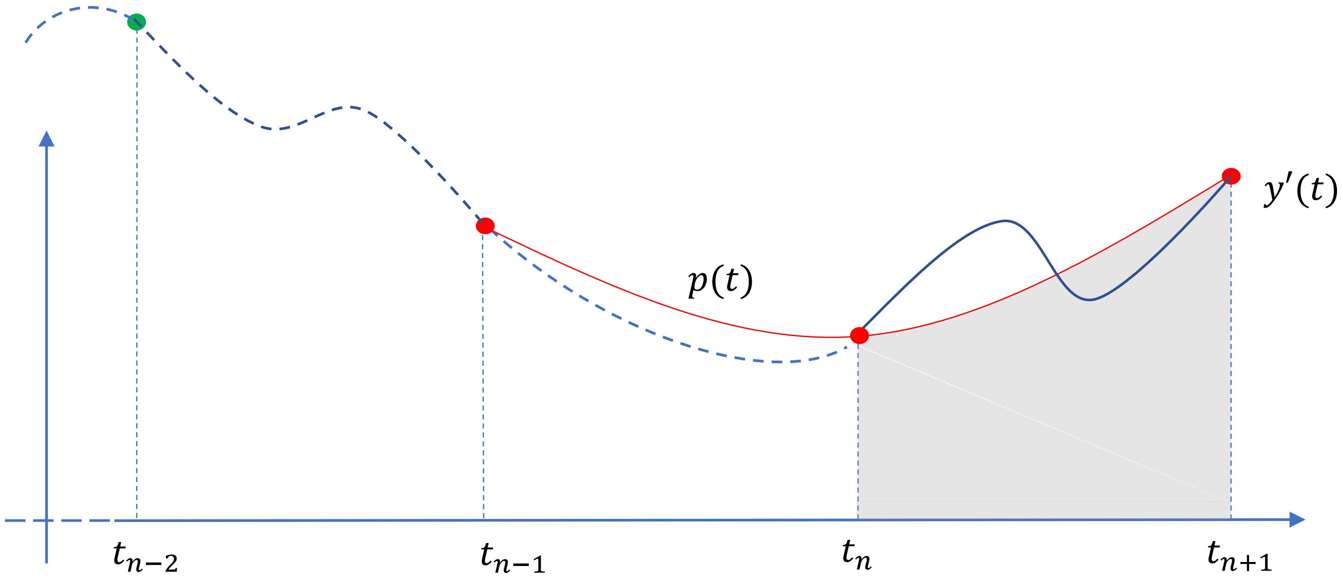
$$\int_{t_n}^{t_{n+1}} p(t)dt$$

Examples

Adams Moulton methods -

-- 2-step method

$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2} - f_n\frac{(t-t_{n+1})(t-t_{n-1})}{h^2} + f_{n-1}\frac{(t-t_{n+1})(t-t_n)}{2h^2}$$



$$\int_{t_n}^{t_{n+1}} p(t)dt = f_{n+1}\left(\frac{5h}{12}\right) - f_n\left(-\frac{2h}{3}\right) + f_{n-1}\left(-\frac{h}{12}\right)$$

Examples

Adams Moulton methods -

-- 2-step method

$$p(t) = f_{n+1}\frac{(t-t_n)(t-t_{n-1})}{2h^2}$$

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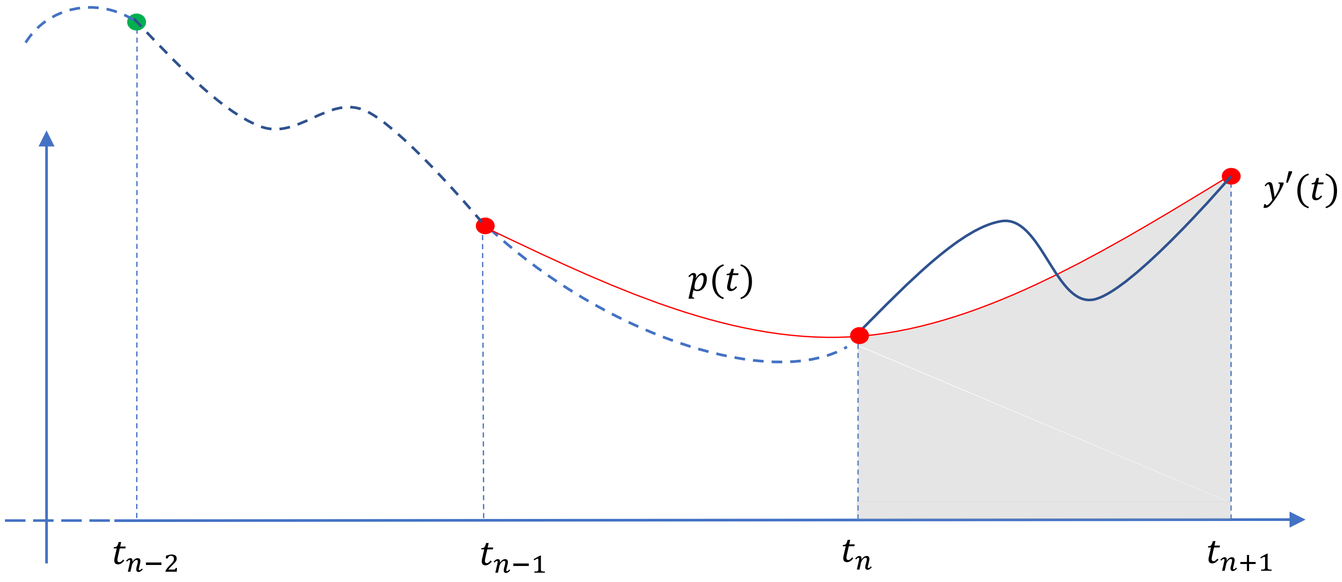

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Thus,

$$y_{n+1} = y_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$$

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Adams Moulton methods -

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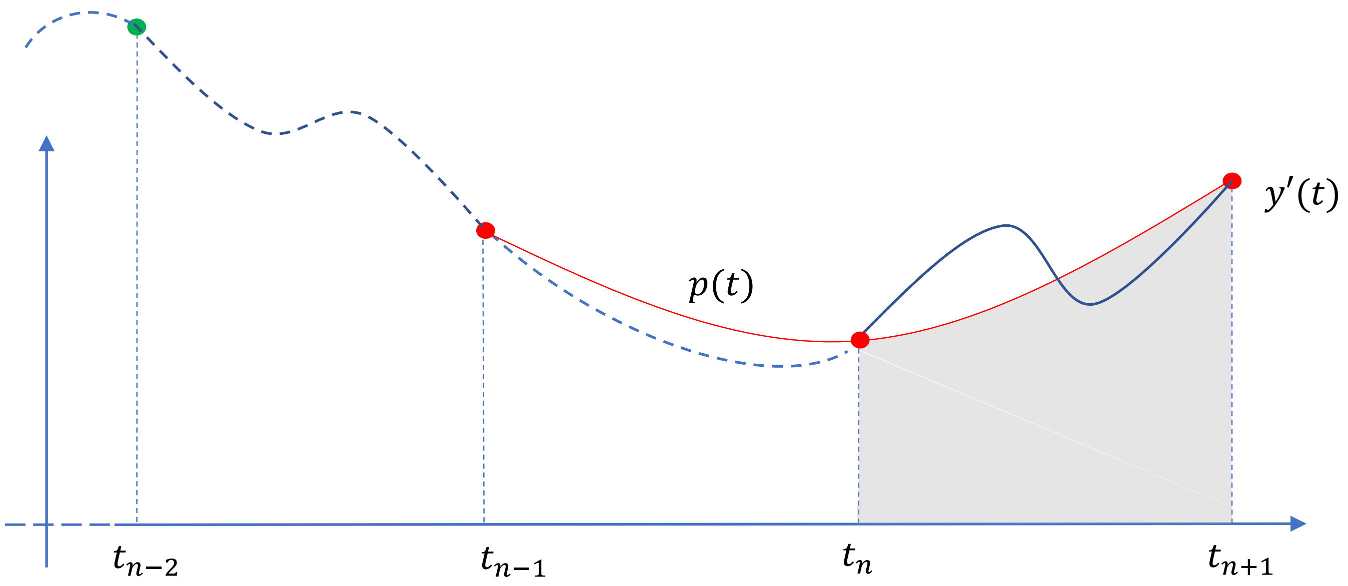

-- General (k+1)-step method

$$p(t) = \sum_{j=-1}^{k} l_j^{(k)}(t)\, f_{n-j}, \qquad \textit{where} \qquad l_j^{(k)}(t) = \prod_{i=-1, i\neq j}^{k} (t - t_{n-i})/(t_{n-j} - t_{n-i})$$

Thus,

$$y_{n+1} = y_n + \sum_{j=-1}^{k} b_j f_{n-j}, \quad \textit{with} \qquad b_j = \int_{t_n}^{t_{n+1}} l_j^{(k)}(t) dt.$$

Module 2
Initial Value Problems

2.4 Implicit method

2.5 Stiffness

2.6 Linear Multistep Methods

- Consistency and Order

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Consistency and Order

For the linear multistep method

$$\sum_{j=-1}^{k} a_j y_{n-j} = h \sum_{j=-1}^{k} b_j f(t_{n-j}, y_{n-j})$$

define the local error as

$$\ell_{n+1}(y,h) = h \sum_{j=-1}^{k} b_j y'(t_n - jh) - \sum_{j=-1}^{k} a_j y(t_n - jh)$$

for any $y \in C^1$, *and* $h > 0$.

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for any $y \in C^1$, and $h > 0$.

The linear multistep method is consistent if

$$\lim_{h\to 0} \max_{k \le n < N} \left| \frac{\ell_{n+1}(y,h)}{h} \right| = 0$$

for all $y \in C^1$.