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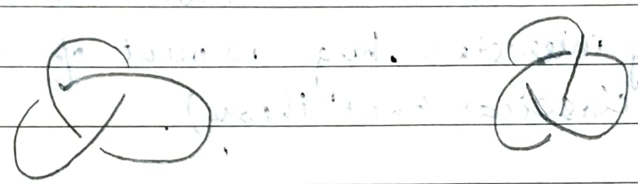
Knot Theory - An Introduction

Def^n Let S^1 be the unit circle $x^2+y^2=1$. Then a knot K is an embedding of the circle S^1 into $\mathbb{R}^3 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ i.e. a knot is the image of a map from $S^1 \to \mathbb{R}^3$ which is 1-1, cont.

Examples (1) S^1 itself - the trivial knot ○.

(2) Trefoil knot

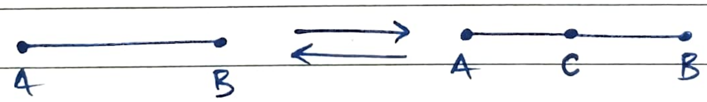


Right Hand Trefoil Knot. Left Hand Trefoil Knot.

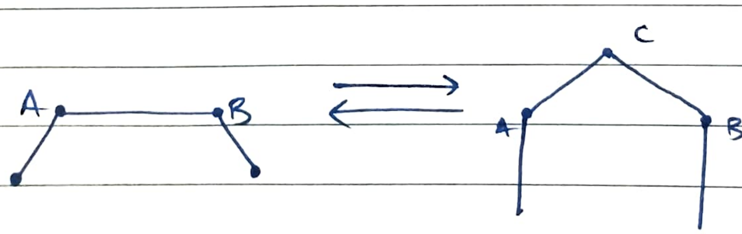
Q. When are 2 knots K_1, K_2 equivalent?

Def^n (Knot Moves)

(1)



(2)



$K_1 \underset{\text{equi.}}{\simeq} K_2$ iff and only if K_1 can be transformed to K_2 by finitely many knot moves.

(Right hand trefoil knot $\not\simeq$ Left Hand trefoil knot)

History – Begins in 1880 when chemist William Thomson (Lord Kelvin) proposed atomic structures were knots in ether.

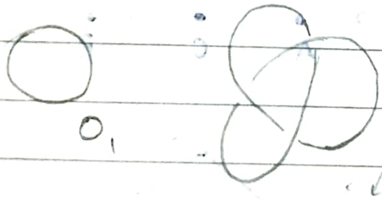
This suggestion led to his friend P.G. Tait to make a list of knots — Founder of knot Theory

He correctly made a list of 800 distinct knots.

When this had no connection with atoms, mathematicians got very interested, huge amount of research was done (classical knot theory).

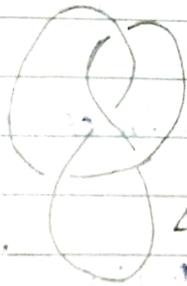
In 1984, by chance in his work on Stat. mechanics/operator theory V. Jones discovered a knot invariant, Jones polynomial settled all Tait conjectures. (Fields medal in 1990).

Currently knot Theory is an active area of research



0_1 3_1

no. of crossings no. of knots



4 ← Fig 8

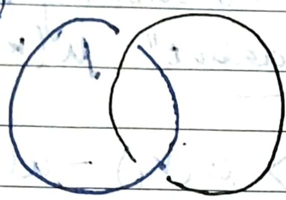
Similarly 5_1

(1998) The first 1.6 million knots Math Intelligencer

Links

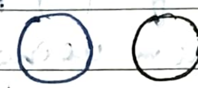
2 components ~~links~~
links

(1) Half Link (Hopf)



2_1^2

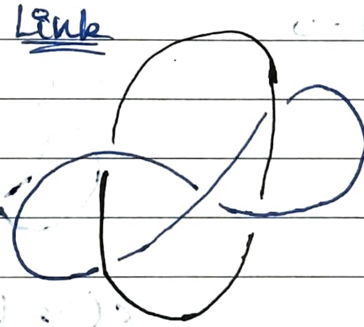
(2) Trivial link



O_2

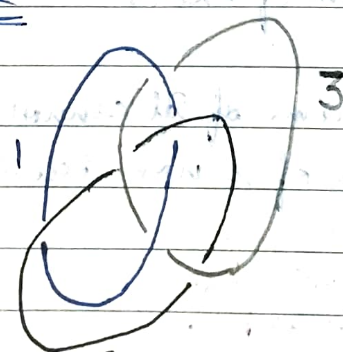
Def^n (Links) A link is a finite ordered collection of knots which do not intersect each other. #

Whitehead Link



5_1^2

Borromean Link



Q. When are 2 knots K_1, K_2 equivalent?:

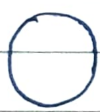
Ans. (a) If $K_1 \simeq K_2$ we have to show it by a collection of knot moves.

(b) If $K_1 \not\simeq K_2$, we need to distinguish between them using "Knot Invariant" $\mu(K)$

i.e. if $K_1 \simeq K_2 \implies \mu(K_1) = \mu(K_2)$

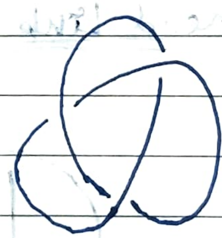
⊙ Inverse is not true.

Crossing Number $c(K)$ — minimum number of crossings of K (over all knot diagrams)



$c(O_1) = 0$

O_1



$c(\text{Trefoil}) = 3$

Def^n Connected Sum of knots

→ A knot is prime if it cannot be written as a connected sum of 2 non-trivial knots.

will get clear as we go on.
(fdl. will come)
in Jones poly.

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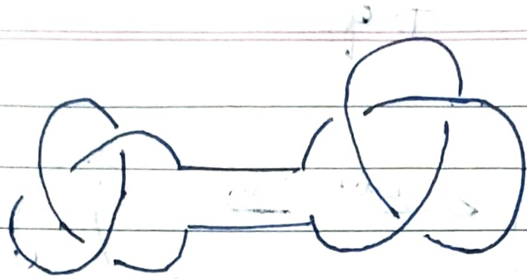

$3_1 \# 3_1$

Open Problem → Given non-trivial knots K_1, K_2

Is $c(K_1 \# K_2) = c(K_1) + c(K_2)$??

Def^n — Alternating knot - A knot where the crossings go over and under alternately.

Alternating knots have excellent properties & well-understood.

→ True proved for → if K_1, K_2 are alternating.

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n- Component Links

2 links $L = \{K_1, K_2, \ldots, K_m\}$ and $L' = \{K_1', K_2', \ldots, K_n'\}$ are equivalent if →

(1) $m = n$

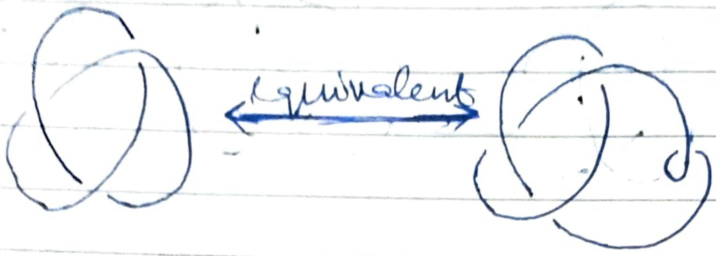
(2) We can change L to L' by finite number of knot moves.

Def^n → K is an alternating knot if it has atleast one alternating diagram.

⟹

Ques Q. Can we better the definition of alternating knots?

Trefoil



Alternating diagram — Non-alternating diagram

Def^n Nonalternating knot is a knot K which has NO alternating diagram.

- A link $L = \{K_1, K_2, \ldots, K_n\}$ is said to be alternating if it has atleast one alternating diagram.

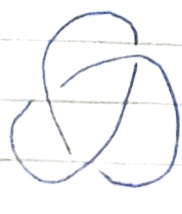
- A link L is nonalternating if it has NO alternating diagram.

Knot Invariants –

(1) Crossing number $c(K)$ = min. no. of crossings of K over all knot diagrams.

(2) Unknotting number $u(K)$ = min. no. of exchanges required to unknot K.

Ex – Trefoil

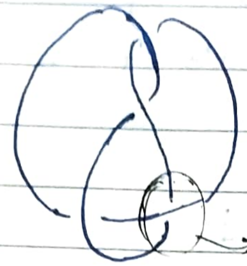


Unknotting number = 1

(100 yrs) Open Problem – Is $u(K)$ additive?

$$u(K_1 \# K_2) = u(K_1) + u(K_2)$$

Ex – Fig8



$u(4_1) = 1$

exchange

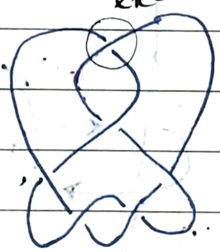
Ex → 5_2

exchange

$u(5_2) = 1$

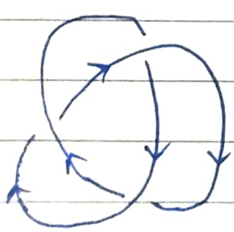
Ex → 6_1

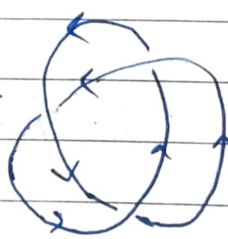
exchange



$u(6_1) = 1$

→ Oriented Knots – An oriented knot K is a knot with a preferred choice of orientations.



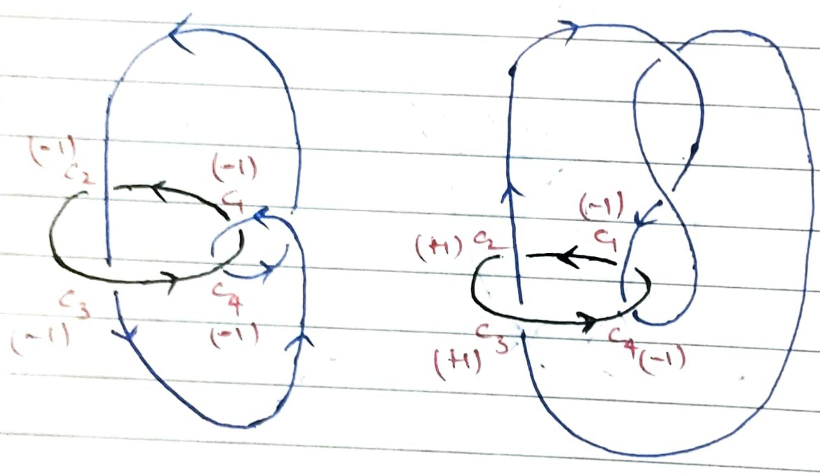


$\Longrightarrow$

(3) Linking number for 2 ~~crossing~~ component links $L = \{K_1, K_2\}$ oriented, crossing points $c_1, c_2, \ldots, c_m$ where K_1 & K_2 intersect.

$$lk(K_1, K_2) = \frac{1}{2}\{\text{sign } c_1 + \ldots + \text{sign } c_m\}.$$

This is a link invariant:



L_1 $\qquad\qquad$ L_2

$lk = \frac{1}{2}(-4) = -2$ $\qquad\qquad$ $lk = \frac{1}{2}(0) = 0$

$\therefore L_1 \neq L_2$

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4) Alexander Polynomial - Discovered by topologist Alexander in 1928.

Used very successfully for ~50 yrs to distinguish b/w knots.

Knot	Alexander Polynomial
3_1	$(-1)[1-1+1] \longrightarrow t^{-1}[1-t+t^2]$ $\Delta_{3_1} = \frac{1}{t} - 1 + t$
4_1	$(-1)[-1+3-1] \longrightarrow t^{-1}[-1+3t-t^2]$ $\Delta_{4_1} = \frac{-1}{t} + 3 - t$
	$3_1 \neq 4_1$
6_1	$(-1)[-2+5-2] \rightarrow \frac{-2}{t} + 5 - 2t$
6_2	$(-2)[-1+3-3+3-1] \rightarrow \frac{-1}{t^2} + 3t - 3 + 3t - t^2$
6_3	$(-2)[1-3+5-3+1] \rightarrow +\frac{1}{t^2} - 3t + 5 - 3t + t^2$
	Hence $6_1 \neq 6_2 \neq 6_3$

→ All knots up to $c(K)=8$ have distinct Alexander polynomial. However $\exists$ an 11-crossing knot.
Kinoshita Terasaki knot T.K. knot K_{11} $\Delta_{K_{11}} = 1$

→ $\therefore$ it is not complete invariant.

Alex Conway Poly

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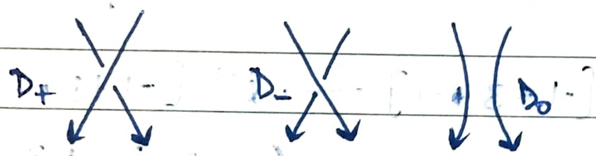
$\nabla_K(z)$

- In 1960's, Conway found an easy way to calculate $\Delta_K(t)$ in terms of another polynomial →
  Alex-Conway poly: $\nabla_K(z)$

Def^n Given an oriented knot or (link) K, we assign a polynomial $\nabla_K(z)$, the alexander-Conway poly-nomial by 2 axioms.

Axiom 1 – If K is trivial $K \simeq O_1$, $\nabla_{O_1}(z) = 1$.

Axiom 2 –



If D_+, D_-, D_0 are diagrams of knots K_+, K_-, K_0 which differs at only one crossing pt.

$$\boxed{\nabla_{K_+}(z) - \nabla_{K_-}(z) = z \cdot \nabla_{K_0}(z)}$$

$$\boxed{\nabla_{K_+}(z) - \nabla_{K_-}(z) = z \cdot \nabla_{K_0}(z)}$$

Def^n → Alexander poly:

$$\Delta_K(t) = \nabla_K\left(z = \sqrt{t} - \frac{1}{\sqrt{t}}\right)$$

Alexander poly: $\Delta_K(t) = \nabla_K\left(z = \sqrt{t} - \frac{1}{\sqrt{t}}\right)$

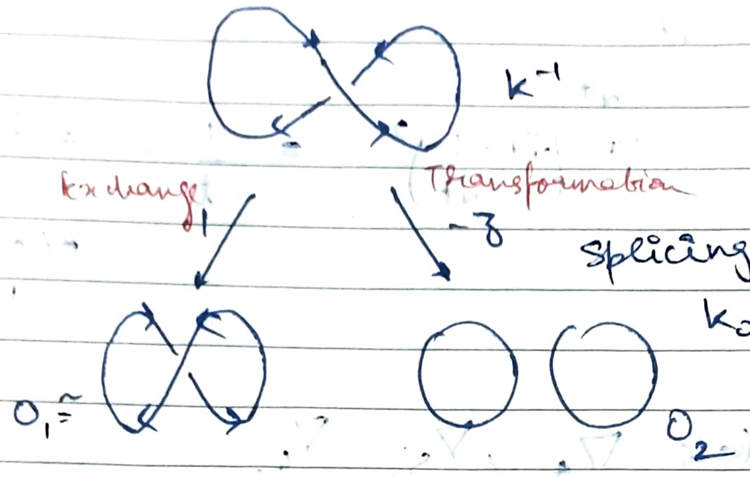
Unknot → $\nabla_{O_1} = 1$

$\nabla_{O_2} \to$? We use a smart trick

$O_2 \equiv$ 

$\Longrightarrow$

$\boxed{\nabla_{K_-} = \nabla_{K_+} - z\nabla_{K_0}} \rightarrow (2*)$ $\quad \boxed{\nabla_{K_+}(z) = \nabla_{K_-} + z\nabla_{K_0}} \rightarrow (1*)$

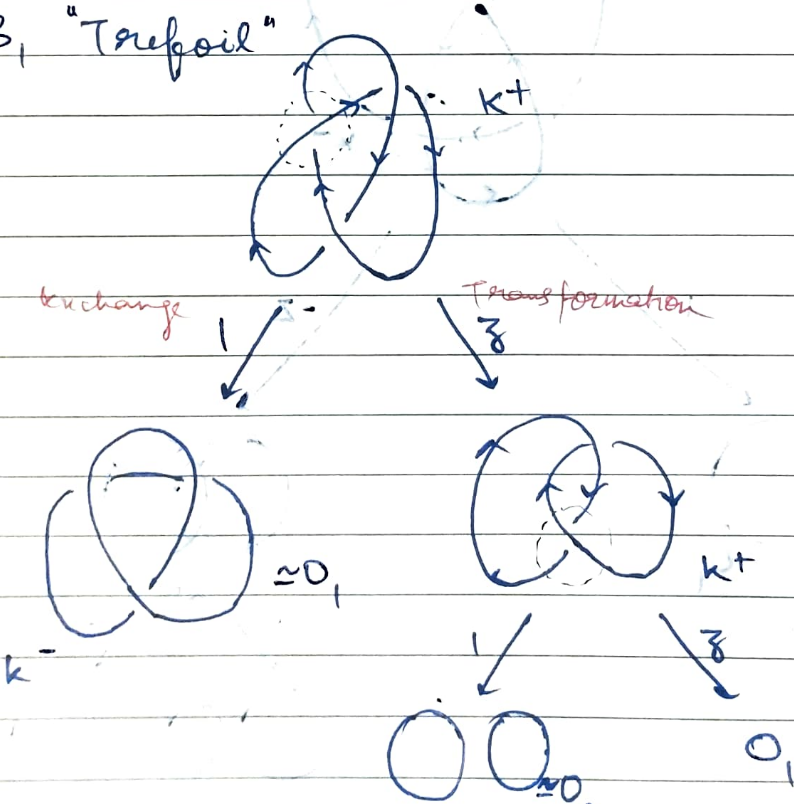


$$\nabla O_1 = \nabla O_1 - z\nabla O_2$$

$$1 = 1 - z\nabla O_2 \Rightarrow \nabla O_2 = 0$$

Ex→ Find ∇O_3, ∇O_4, etc.

Ex→ 3_1 "Trefoil"



$$\nabla_{3_1} = 1\cdot\nabla_{O_1} + z^2\nabla_{O_1} + \cancel{z\nabla_{O_2}} \rightarrow 0$$
$$= 1 + z^2 \Rightarrow$$

$$z = \sqrt{t} - \frac{1}{\sqrt{t}}$$

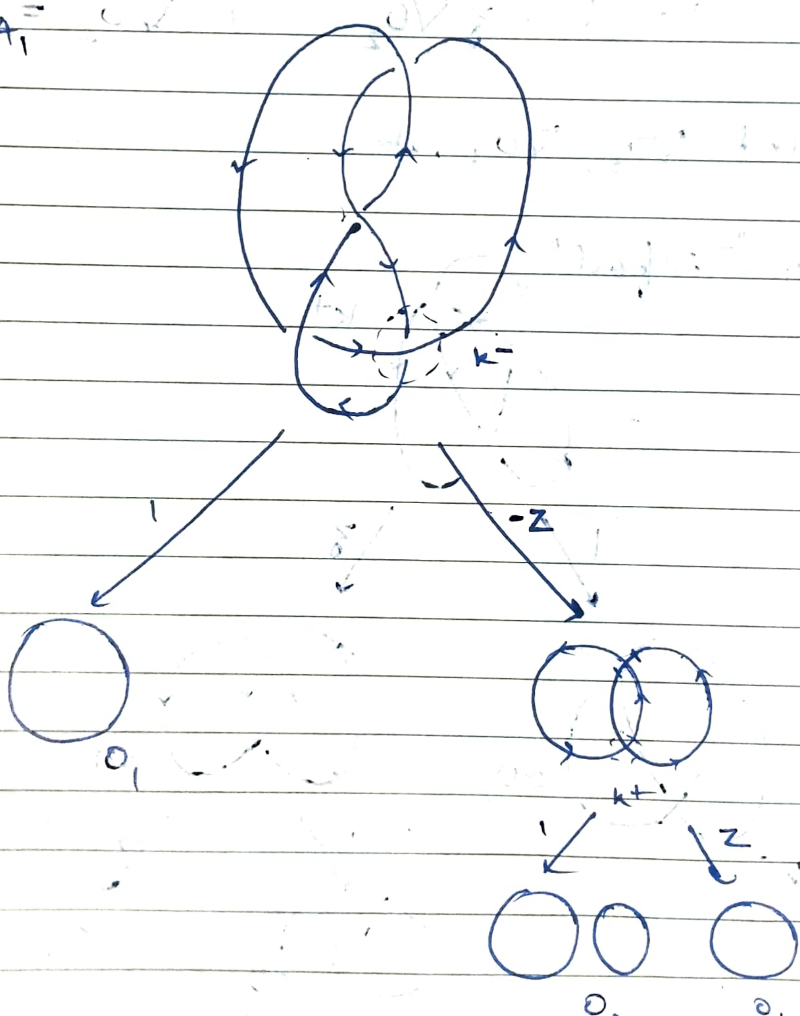
$$1 + \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 = \frac{1}{t} - 1 + t \quad \checkmark$$

Alexander Polynomial

HW Calculate $\nabla_{4_1}, \nabla_{5_1}, \nabla_{6_3}$

$\Delta_{3_1}, \Delta_{4_1}, \Delta_{6_3}$

$\rightarrow \nabla_{4_1} =$



$$\nabla_{4_1} = 1 \cdot \nabla_{O_1} - z\left[1 \cdot \nabla_{O_2} + z\nabla_{O_1}\right]$$
$$= 1 - z^2$$

$$\Delta_{4_1} = 1 - \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2$$

$$= 1 - \left(t + \frac{1}{t} - 2\right)$$

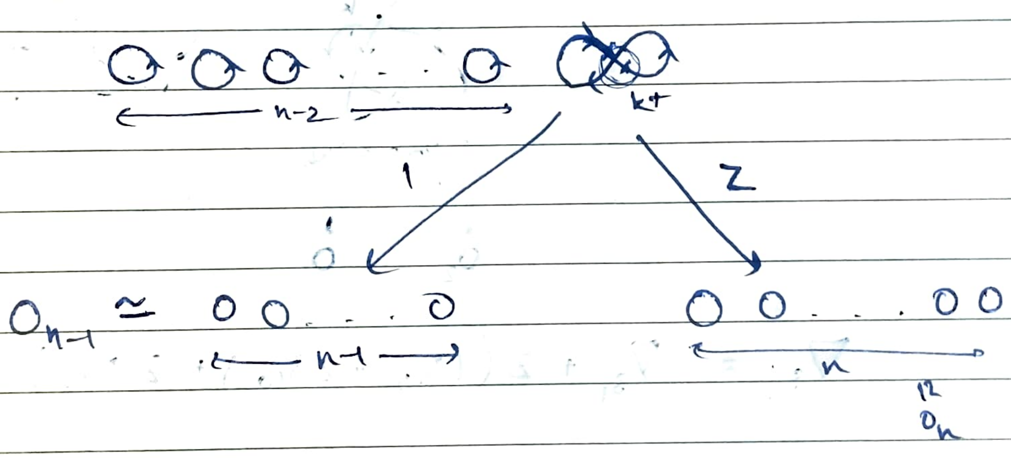
$$= -\frac{1}{t} + 3 - t$$

$$= \frac{1}{t}\left[-1 + 3t - t^2\right]$$

$$\Delta_{4_1} = (-1)\left[-1 + 3 - 1\right]$$ ✓

Ex → ∇_{O_n} $\quad n \geq 2 \quad (=0)$ Prove.

Proof



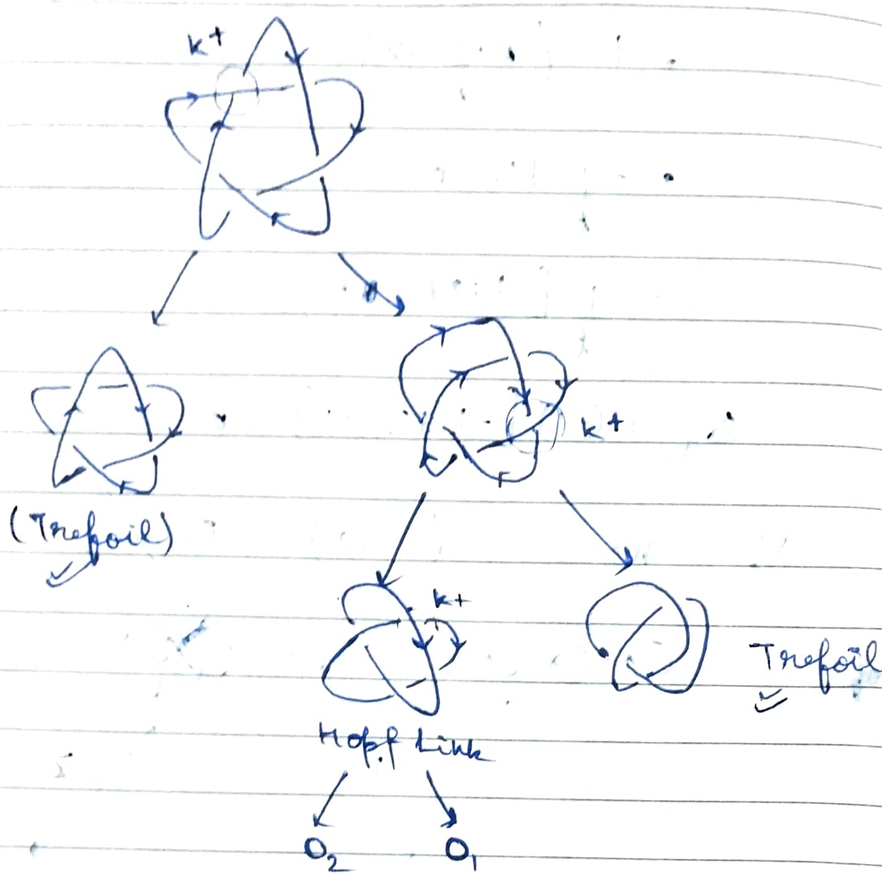
$$\cancel{\nabla_{O_{n-1}}} = \cancel{\nabla_{O_{n-1}}} + z\nabla_{O_n}$$

$$\Rightarrow \nabla_{O_n} = 0$$ QED

~~We cannot~~

Calculate again

Ex → ∇_{5_1} $\quad (= 1+3z^2+z^4)$



$$\nabla_{5_1} = \nabla_{3_1} + z\left(\left(\nabla_{O_2} + z\nabla_{O_1}\right) + z\nabla_{3_1}\right)$$

$$= \nabla_{3_1} + z^2\left(\nabla_{O_1} + \nabla_{3_1}\right)$$

$$= (1+z^2) + z^2(1+1+z^2)$$

$$= z^4 + 3z^2 + 1$$

$$\Delta_{5_1} = \nabla_{5_1}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) = (z^2+1)^2 + z^2$$

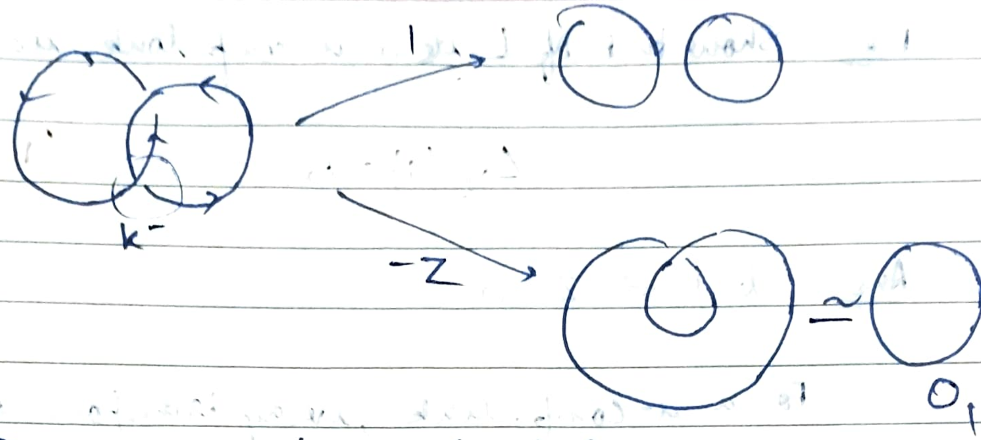
$$= \left(\frac{1}{t} - 1 + t\right)^2 + \left(t + \frac{1}{t} - 2\right)$$

$\Delta_{5_1} = (-2)\ [1-1+1-1+1]$

$$= t^2 + \frac{1}{t^2} + 1 - t - \frac{1}{t}$$

$$= \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$$

Ex →



$$\nabla_{Hopf} = 1(0) + (-Z)(1)$$

$$= -Z$$

$$\Delta_{Hopf} = -\sqrt{t} + \frac{1}{\sqrt{t}}$$

Thm → If k is a knot, then $\Delta_k(1) = 1$ (For knots, not for links.)

Proof: $t = 1$, then $z = \sqrt{t} - \frac{1}{\sqrt{t}} \Rightarrow z = 0$

~~If we are making~~

$$\nabla_{k\pm} = \nabla_{k\mp} \pm z\, \nabla_{k_0}$$

$z = 0$,

$$\nabla_{k+} = \nabla_{k-}$$

Given k, we can unknot it by finitely many exchanges, this gives $\nabla_k = \nabla_{O_1} = 1$.

$$\nabla_k(z=0) = \nabla_{O_1} = 1$$

$$\Delta_k(t=1) = 1$$

For calculating Alexander Poly.
First calculate Alex Conway poly ($\nabla_K(z)$)
Then $z \to \sqrt{t} - \frac{1}{\sqrt{t}}$

Ex → Show that if L is a μ-comp link, $\mu \geq 2$, then

$$\Delta_L(1) = 0 \qquad \text{(For links)}$$

Ans $t=1 \Rightarrow z=0$

For a μ-comp. link, every transformation (involving z) contribute nothing to the final poly. ∵ (as $z=0$)

Only exchanges will give → O_μ & $\nabla_{O_\mu} = 0$ $\forall \mu \geq 2$

$$\text{So, } \nabla_L(z=0) = \nabla_{O_\mu} = 0$$

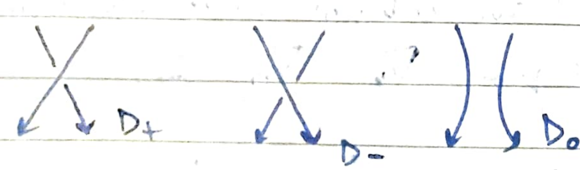
$$\Rightarrow \Delta_L(t=1) = \nabla_{O_\mu} = 0$$

→ In 1984, V. Jones discovered a new knot polynomial which has amazing properties.

Defn → (Jones Polynomial) Suppose K is an oriented knot (or link) and D a diagram for K. Then the Jones polynomial $V_K(t)$ is uniquely defined from 2 axioms.

Axiom 1 → If $K = O_1$, $V_{O_1}(t) = 1$.

Axiom 2 →



$\Rightarrow$

We denote $\left(z = \sqrt{t} - \frac{1}{\sqrt{t}}\right)$

$$\boxed{\begin{aligned} V_{D^+} &= t^2 V_{D^-} + tz\, V_{D_0} \to (1) \\ V_{D^-} &= \frac{1}{t^2} V_{D^+} - \frac{z}{t} V_{D_0} \to (2) \end{aligned}}$$

$$\frac{1}{t} V_{D^+} - t\, V_{D^-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{D_0}$$

$$\frac{1}{t} V_{D^+} - t\, V_{D^-} = Z\, V_{D_0}$$

H.W. Calculate Jones polynomial for $(V_k(t))$

(1) OO O_2

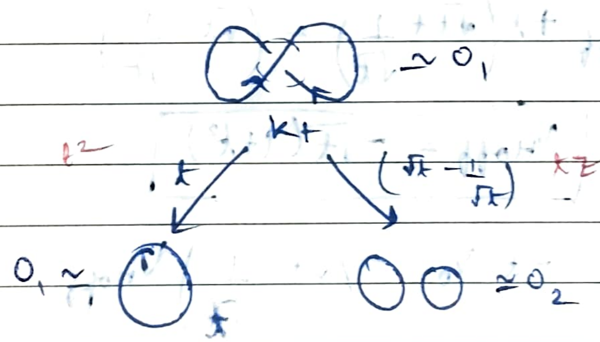
(2) O_n

(3) 3_1

(4) 3_1^* (mirror image)

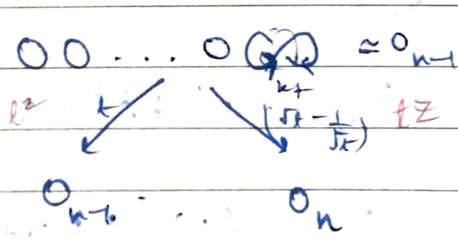
(5) 4_1

→ O_2



$$\frac{1}{t} V_{O_1} - t V_{O_1} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{O_2} \Rightarrow \boxed{V_{O_2} = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)}$$

→ O_n

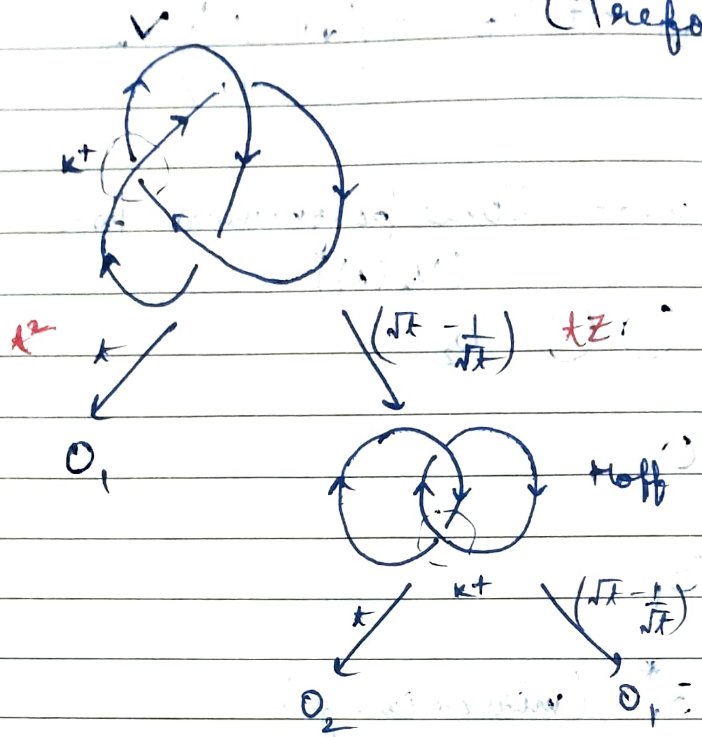


$$\frac{1}{t} V_{O_{n-1}} - t V_{O_{n-1}} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{O_n} \Rightarrow V_{O_n} = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_{O_{n-1}}$$

$$\Rightarrow \boxed{V_{O_n} = \left(-\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)\right)^{n-1}}$$

Left Hand
(Trefoil)

→ 3_1



$$\frac{1}{t}V_{Hopf} - tV_{O_2} = \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)V_{O_1}$$

$$\Rightarrow \frac{1}{t}V_{Hopf} + t\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right) = \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)$$

$$\Rightarrow \boxed{V_{Hopf} = -\sqrt{t}\,(1+t^2)}$$

$$\Rightarrow \frac{1}{t}V_{3_1} - tV_{O_1} = \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)V_{Hopf}$$

$$\Rightarrow \frac{1}{t}V_{3_1} - t = \frac{(t-1)}{\cancel{\sqrt{t}}}(-\cancel{\sqrt{t}})(1+t^2)$$

$$\Rightarrow V_{3_1} = t^2 - t(t^3 - t^2 + t - 1)$$

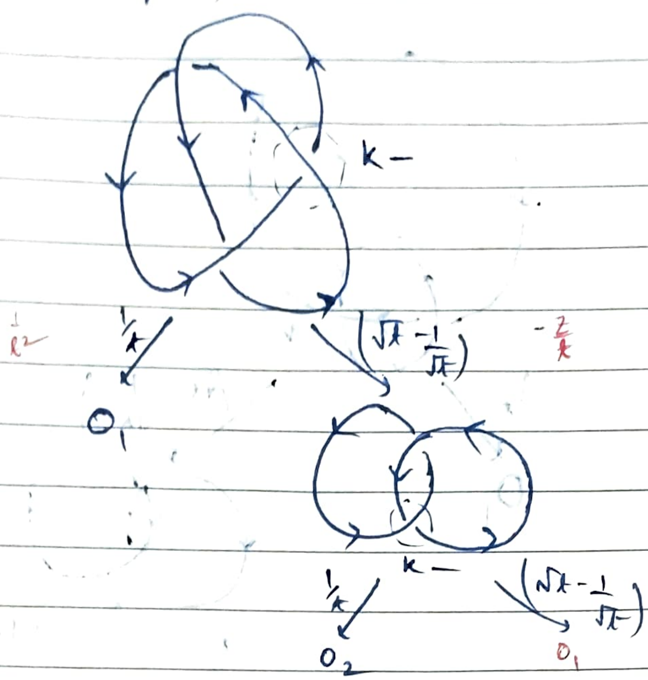
$$\Rightarrow V_{3_1} = -t^4 + t^3 + t$$

$$V_{3_1} = (1)[1+0+1-1]$$

K^* → mirror image of K is got by exchanging all crossings

(Right Hand Trefoil)

→ 3_1^*



$$\frac{1}{t}V_{O_2} - tV_{Hopf} = \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)V_{O_1}$$

$$\Rightarrow \left(\frac{1}{t}\right)\left(-\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\right) - \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right) = tV_{Hopf}$$

$$\Rightarrow \cancel{-\frac{1}{\sqrt{t}}} - \frac{1}{t\sqrt{t}} - \sqrt{t} + \cancel{\frac{1}{\sqrt{t}}} = tV_{Hopf}$$

$$\Rightarrow \boxed{V_{Hopf} = \frac{-(t^2+1)}{t^2\sqrt{t}}}$$

$$\Rightarrow \frac{1}{t}V_{O_1} - tV_{3_1^*} = \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\left(-\frac{(t^2+1)}{t^2\sqrt{t}}\right)$$

$$\Rightarrow V_{3_1^*} = \frac{1}{t^2} + \frac{(t^2+1)(t-1)}{t^4}$$

$$\Rightarrow V_{3_1^*} = \frac{1}{t^2} + \frac{(t^3-t^2+t-1)}{t^4}$$

$$\Rightarrow V_{3_1^*} = \frac{1}{t} + \frac{1}{t^3} - \frac{1}{t^4}$$

$$V_{3_1^*} = (-4)[-1+1+0+1]$$

$\boxed{V_{3_1} \neq V_{3_1^*}}$ ⇒ 3_1 & 3_1^* are not equivalent

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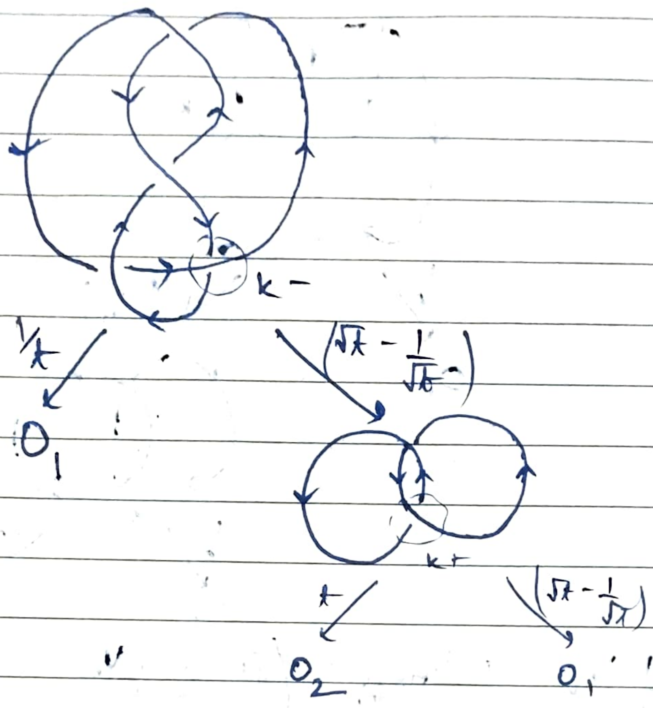
$\rightarrow 4_1$



$$\frac{1}{t} V_{Hopf} - t \cdot V_{O_2} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{O_1}$$

$$\rightarrow V_{Hopf} = (-\sqrt{t})(1+t^2)$$

$$\frac{1}{t} V_{O_1} - t V_{4_1} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left((-\sqrt{t})(1+t^2)\right)$$

$$\rightarrow V_{4_1} = \frac{1}{t^2} + \frac{(t^2+1)(t-1)}{t}$$

$$V_{4_1} = \frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$$

$$V_{4_1} = (-2)\left[1 - 1 + 1 - 1 + 1\right]$$

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Note → $V_{3_1^*}(t) = V_{3_1}\left(\frac{1}{t}\right)$ "Mirror Image"

Thm → $\boxed{V_{k^*}(t) = V_k\left(\frac{1}{t}\right)}$ $k^* \to$ mirror image of k.

Proof → Suppose D is a diagram of K and D^* is a diagram for K^*.

If the skein tree diagram of D is R, we form the skein tree diagram of K^* in exactly the same way as K.

The difference b/w R and R^* is as follows –

At a crossing pt. c of R if it was k^+, then in R^* it will be k^-, and vice versa.

For k_+, coeffs. were t^2 and tz.

In R^*, —— " —— $\frac{1}{t^2}$ and $-\frac{z}{t}$

$$\left.\begin{array}{l} t^2 \to 1/t^2 \\ 1/t^2 \to t^2 \end{array}\right\} t \to \frac{1}{t}$$

$t^{-3/2} - t^{-1/2} \to$ (2)

$t^{3/2} - t^{1/2}$ (1) $\leftarrow$ $$\left.\begin{array}{l} tz \to -\frac{z}{t} \\ -\frac{z}{t} \to tz \end{array}\right\} t \to \frac{1}{t}$$

(2) comes from (1) by replacing $t \to \frac{1}{t}$.

$\therefore$ $V_{k_*}(t) = V_k\left(\frac{1}{t}\right)$.

- If $K \simeq K^*$

$$\boxed{V_K(t) = V_K\left(\frac{1}{t}\right)}$$ Palindromic Symmetry.

(Achiral)

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Jones poly. & its properties

Example 1 If K is trivial 2 component $O_2 = \{O\ O\}$ then

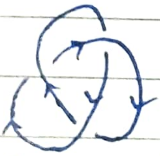
$$V_{O_2}(t) = (-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)$$

Note – $V_{O_2}(1/t) = V_{O_2}(t)$ (as expected)

Example 2

$$V_{3_1} = t + t^3 - t^4$$

Left Hand Trefoil



$$V_{3_1^*} = \frac{1}{t} + \frac{1}{t^3} - \frac{1}{t^4}$$

Right Hand Trefoil

Note → $V_{3_1^*}(t) = V_{3_1}(1/t)$

Since $V_{3_1} \neq V_{3_1^*} \Rightarrow 3_1 \not\simeq 3_1^*$

Thm → If K^* is the mirror image of K, then $V_{K^*}(t) = V_K\left(\frac{1}{t}\right)$

Thm → If O_n is the trivial n-component link →

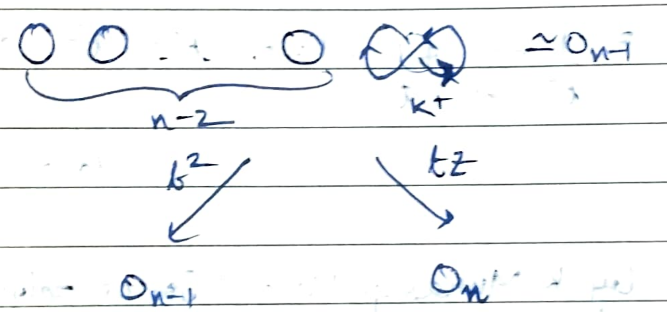
$\underbrace{O\,O \cdots O}_{n}$ then $V_{O_n} = (-1)^{n-1}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^{n-1}$

Proof → By Induction; if $n=1$ $V_{O_1}=1$

$n=2$: $V_{O_2} = (-1)\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)$

Let it holds for $(n-1)$ ⇒

$$V_{O_{n-1}} = (-1)^{n-2}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^{n-2}$$



$$\frac{1}{t}V_{O_{n-1}} - tV_{O_{n-1}} = zV_{O_n}$$

$$\left(\frac{1}{t}-t\right)V_{O_{n-1}} = V_{O_n}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)$$

$$\Rightarrow \left(\frac{1}{t}-t\right)(-1)^{n-2}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^{n-2} = \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)V_{O_n}$$

$$\Rightarrow (-1)^{n-1}\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^{n-1} = V_{O_n}$$

Note → $V_{O_n}\left(\frac{1}{t}\right) = V_{O_n}(t)$

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- If $k = k^* \Rightarrow V_k(t) = V_k\left(\frac{1}{t}\right)$

Q. Is converse true?

→ If $V_k(t) = V_k\left(\frac{1}{t}\right) \Rightarrow k = k^*$?

(True till crossing no. 8, we find a counterexample for crossing no. 9)

9_{42} ← Counterexample

$$V_{9_{42}} = (-3)\{1 -1 +1 -1 +1 -1 +1\}$$

$$= t^{-3}\{1 - t + t^2 - t^3 + t^4 - t^5 + t^6\}$$

$$= \frac{1}{t^3} - \frac{1}{t^2} + \frac{1}{t} - 1 + t - t^2 + t^3 \text{ is palindrome}$$

but $9_{42} \neq 9_{42^*}$

Paper by K. Murasugi (1987) - Jones polynomial & classical conjectures in knot Theory; Topology (1987) ; no. 26

Thm → Let k be an "alternating" knot s.t. $c(k) = n$. Let $V_k(t)$ be its Jones polynomial. Let

"Alternating knot"

$$p = \max \deg V_k(t)$$

$$q = \min \deg V_k(t)$$

Then $c(k) = n = p - q$.

For alternating knots

- Alternating & Achiral $\Rightarrow$ even crossing number.
- Achiral $\not\Rightarrow$ even crossing number.

Ex $\rightarrow$ If k is "alternating" knot, and k is achiral ($k \simeq k^*$) then $c(k) = 2n$ (even)

$$V_k = a_n t^{-n} + a_{n-1} t^{-n+1} + \dots + a_1 t^{-1} + a_0 + a_1' t + \dots + a_n' t^n$$

$p = n$, $q = -n$. $\Rightarrow c(k) = 2n$.

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Thm $\rightarrow$ (b) If k is non-alternating (i.e. k has no alternating diagram) then,

$$\text{span } V_k(t) = p - q < c(k)$$

Tait Conjecture - (100 yrs ago) If k is an achiral knot, i.e. if $k \cong k^*$, then $c(k) = 2n$ (even).

For non-alternating knots $\rightarrow$ Open problem

- For the first 1.6M knots (upto $c(k) = 16$) using computer, just by chance they discovered one 15 crossing non alternating knot k which is achiral $k \cong k^*$

$\Downarrow$

Tait's conjecture is false.

HOMFLY POLYNOMIAL $\rightarrow$

In 1986, all of 6 different mathematician working in groups of 2, all of them discovered a 2-variable polynomial of an oriented knot (or link), $P_k(v, z)$ as follows $\rightarrow$

$\Rightarrow$