Supplementary Notes

(for single non-linear equation)

Multiple zeros

Definition Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $m \in \mathbb{N}$.

A zero r of f (i.e., $f(r)=0$) is called a zero of multiplicity m, if f can be written as

$$f(x) = (x-r)^m q(x), \quad \text{for } x \in N(r)$$

($N(r)$ = neigborhood ~~or~~ of r)

in some neighborhood of r, $N(r)$ and for some continuous function $q: N(r) \longrightarrow \mathbb{R}$ such that $q(r) \neq 0$.

Remark $q(x)$ represents that portion of $f(x)$ which does not contribute to the zero of f.

P-2

Theorem-1

Let $f:[a,b] \to \mathbb{R}$ be C^1.

Then, $r \in (a,b)$ is a simple zero (ie. $m=1$) of f in (a,b) iff $f(r)=0 \neq f'(r)$.

proof: $\Rightarrow$ f has simple zero r. Then,

$f(r) = 0$ and

$$f(x) = (x-r)\,g(x), \quad x \in N(r),$$

where $N(r) \subset [a,b]$ is a nbd of r

and $g: N(r) \to \mathbb{R}$ continuous with

$$g(r) \neq 0.$$

Therefore, $f(x) - f(r) = (x-r)\,g(x)$

$\forall\, x \in N(r)$

$\Rightarrow$ $\forall\, x \in N(r),\, x \neq r$

$$\frac{f(x) - f(r)}{x-r} = g(x)$$

As f is C^1 and $g(r) \neq 0$, $\lim_{x \to r} \dfrac{f(x)-f(r)}{x-r}$ exists and $\quad f'(r) = q(r) \neq 0.$

$\Leftarrow$ (conversely): Let $f(r) = 0 \neq f'(r)$.

Let $x \in [a,b]$. Let us define

$$\gamma : [0,1] \to [a,b] \quad \text{by}$$

$$\gamma(t) = tx + (1-t)r, \quad t \in [0,1].$$

we note that γ is diffble and

$$\gamma(0) = r, \quad \gamma(1) = x.$$

$$\gamma'(t) = (x-r), \quad t \in [0,1].$$

Hence, $\quad f \circ \gamma : [0,1] \to \mathbb{R}$ is diffble,

$$f \circ \gamma(t) = f(\gamma(t)) = f(r + t(x-r)),$$
$$t \in [0,1].$$

and $\quad (f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$

$$= f'(r + t(x-r))\,(x-r).$$

$$\forall \; t \in [0,1].$$

P-4 By Fundamental theorem of Calculus

$$\int_0^1 (f\circ\gamma)'(t)\,dt = (x-r)\int_0^1 f'(r+t(x-r))\,dt$$

$$\Rightarrow f\circ\gamma(1) - f\circ\gamma(0) = (x-r)\,g(x),$$

where, $g:[a,b]\to\mathbb{R}$ be given by

$$g(x) = \int_0^1 f'(r+t(x-r))\,dt,$$

$x\in[a,b]$.

$$\Rightarrow f(x) - f(r) = (x-r)\,g(x)$$

$$\Rightarrow f(x) = (x-r)\,g(x), \quad x\in[a,b].$$

$(\because f(r)=0)$

As f is C^1, g is continuous.

To show $\lim_{x\to r} g(x) = f'(r) \neq 0$.

$$g(x) - f'(r) = \int_0^1 f'(r+t(x-r))\,dt - f'(r)$$

$$= \int_0^1 [f'(r+t(x-r)) - f'(r)]\,dt$$

$\boxed{\text{Supplementary}}$

As f is C^1, so f' is continuous.

Let $\varepsilon > 0$, $\exists$ $\delta > 0$ such that

$$\forall \quad z \in [a,b], \quad |z-r| < \delta \Rightarrow |f'(z) - f'(r)| < \varepsilon/2$$

Note that for, $t \in [0,1]$ and

$$|x-r| < \delta \Rightarrow |r+t(x-r) - r| = |t(x-r)|$$
$$= |t||x-r|$$
$$\leq |x-r| < \delta$$

$$\Rightarrow |f'(r+t(x-r)) - f'(r)| < \varepsilon/2$$

Hence, for $|x-r| < \delta$,

$$|q(x) - f'(r)| \leq \int_0^1 |f'(r+t(x-r)) - f'(r)|\,dt$$
$$\leq \varepsilon/2 \int_0^1 dt = \varepsilon/2 < \varepsilon.$$

Hence q is continuous and

$$\lim_{x \to r} q(x) = f'(r) \neq 0.$$

P-6 More generally

Theorem-2 Let $f \in C^m([a,b])$ and f has a zero r of multiplicity $m \in \mathbb{N}$ in (a,b) if and only if

$$f(r) = f'(r) = \cdots = f^{m-1}(r) = 0 \neq f^m(r).$$

Modified Newton method

One way to handling the problem of multiple root of an ~~#~~ equation $f(x)=0$, is to define

$\mu : N(r) \longrightarrow \mathbb{R}$ by

$$\mu(x) = (x-r)\frac{q(x)}{m\,q(x) + (x-r)\,q'(x)},$$

$x \in N(r)$.

Let f has a zero r of multiplicity $m \in \mathbb{N}$.

Then, $f(x) = (x-r)^m q(x)$, $x \in N(r)$,

$q : N(r) \to \mathbb{R}$ continuous, $q(r) \neq 0$.

Assume further q is C^1 in $N(r)$.

Supplementary

For $x \neq r$, $x \in N(r)$, consider

$$\mu(x) = \frac{f(x)}{f'(x)}$$

$$= \frac{(x-r)^m q(x)}{m(x-r)^{m-1} q(x) + (x-r)^m q'(x)}$$

$$= \frac{(x-r)\, q(x)}{m\, q(x) + (x-r)\, q'(x)}$$

$$= (x-r)\, \tilde{q}(x),$$

where $\tilde{q}(x) = \dfrac{q(x)}{m q(x) + (x-r) q'(x)}$, $x \in N(r)$.

Also, $\lim_{x \to r} \tilde{q}(x) = \dfrac{q(r)}{m\, q(r)} = \dfrac{1}{m}$ $(\because q(r) \neq 0)$.
$\neq 0$.

Hence, r is a zero of μ of multiplicity 1.
we can apply Newton-Raphson method to μ.
For μ, the fixed pt. formulation is

$$x = g(x) = x - \frac{\mu(x)}{\mu'(x)}.$$

$\boxed{P-8}$

$$g(x) = x - \frac{f(x)/f'(x)}{\left(f(x)/f'(x)\right)'}$$

$$= x - \frac{f(x)/f'(x)}{\frac{(f'(x))^2 - f(x)\,f''(x)}{(f'(x))^2}}$$

$$= x - \frac{f(x)\,f'(x)}{(f'(x))^2 - f(x)\,f''(x)}.$$

Remark

Theoretically the above method has only two drawback

(i) additional calculation of $f''(x)$

(ii) lengthy procedure of calculating the iterates.

Practical drawback:

The denominator consists of the difference of two numbers that are both close to 0. It can cause serious round-off error.

Supplementary

Other methods.

Accelerating convergence: Aitken's Δ^2 method

The method called Aitken's Δ^2 method that can be used to accelerate the convergence of a sequence that is linearly convergent, regardles of it origin.

Let $\{x_k\}_{k\geq 0}$ such that $\lim_{k\to\infty} x_k = r$ linearly.

Aitken's Δ^2 method is ~~based on~~ defined by

$$\hat{x}_k := x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k}$$

$$:= x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k} \rightarrow (1)$$

Then, $\{\hat{x}_k\}_{k\geq 0}$ converges faster to r in the sense that

$$\lim_{k\to\infty} \frac{\hat{x}_k - r}{x_k - r} = 0.$$

Ref: Burden & Faires, P-88,
Atkinson, P-83.

P-10

Steffensen's method.

By applying a modification of Aitken's Δ^2 method to a linearly convergent sequence obtained from fixed-point iteration, we can accelerate the convergence to quadratic. This method is known as Steffensen's method:

x_0, $x_1 = g(x_0)$, $x_2 = g(x_1)$, $\hat{x}_0 = \{\Delta^2\}(x_0)$

$x_3 = g(x_2)$, $\hat{x}_1 = \{\Delta^2\}(x_1)$

$x_4 = g(x_3)$, $\hat{x}_2 = \{\Delta^2\}(x_2)$.

where $\{\Delta^2\}$ indicates (1) above.

Steffen's method construct the same 1st four terms $x_0, x_1, x_2, \hat{x}_0$. At this step we assume that $\hat{x}_0$ is a better approximation to r than is x_2 and apply fixed point iteration to $\hat{x}_0$ instead of x_2.

$\boxed{\text{Supplementary}}$

$x_0^{(0)}$; $x_1^{(0)} = g(x_0^{(0)})$ $x_2^{(0)} = g(x_1^{(0)})$

$x_0^{(1)} = \{\Delta^2\}(x_0^{(0)})$ $x_1^{(1)} = g(x_0^{(1)})$, ...

Every 3rd term of the steffenesen's seqn. is generated by ①, the others use fixed-point iteration on the previous term.

x

$\boxed{\text{Horner's method}}$ (synthetic division)

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$a_n \neq 0$

(polynomial of degree n)

Define $b_n = a_n$, $b_k = a_k + b_{k+1} x_0$ for

$k = n-1, n-2, \ldots, 1, 0$.

Then, $b_0 = P(x_0)$.

Moreover, if $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$

then, $P(x) = (x - x_0) Q(x) + b_0$.

proof: See Burden & Faire's, P-92-93.,

Direct computation and comparison.

P-12

Eg: $P(x) = x^4 - 2x^3 - 12x^2 + 16x - 40$.

we want to write

$$P(x) = (x+2)\,Q(x) + b_0.$$

Take $x_0 = -2$. Make the table

	co-eff of x^4	co-eff of x^3	x^2	x	const.
-2	1	-2	-12	16	-40
		-2	8	8	-48
	$1 = b_4$	$-4 = b_3$	$-4 = b_2$	$24 = b_1$	$-88 = b_0 = P(x_0)$
-2		-2	12	-16	
	$1 = c_4$	$-6 = c_3$	$8 = c_2$	$8 = q = Q(x_0)$	

$$P(x) = (x+2)\,Q(x) - 88, \quad P(x_0) = P(-2) = -88$$

$$\cancel{Q(x) = x^3 - 6x^2 + 8x + 8}$$

$$\cancel{Q(x) = x^4 - 4x^3 - 4x^2 + 24x}$$

$$Q(x) = x^3 - 4x^2 - 4x + 24.$$

$$Q(x) = (x+2)(x^2 - 6x + 8) + 8.$$

$$Q(x_0) = Q(-2) = 8.$$

To use Newton's method to locate zeros of polynomial $P(x)$, we need to evaluate $P(x)$ and $P'(x)$ at every iteration.

$$x^k = x^{k-1} - \frac{P(x^{k-1})}{P'(x^{k-1})}$$

Horner's method calculate both $P(x^{k-1}), P'(x^{k-1})$ more efficiently than ~~by~~ direct computation.

At $x = x_0$. We want to know

$$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)}.$$

By synthetic division write $P(x)$ as

$$P(x) = (x - x_0)\, Q(x) + b_0.$$

So $P(x_0) = b_0$.

Again $P'(x) = Q(x) + (x - x_0)\, Q'(x)$

$P'(x_0) = Q(x_0) = c_1$, where.

$Q(x) = (x - x_0) R(x) + c_1$

P-14

Complex zeros: Müller's method.

Ref: Burden & Faires, P-(95-100).

Order of convergence / rate of convergence.

Let $\{x_k\}_{k \geq 0} \subset \mathbb{R}$ be a sequence such that $\lim_{k \to \infty} x_k = r$ and $x_k \neq r$ $\forall\, k \geq 0$.

Defn. If $\exists$ constant $\lambda, \alpha > 0$ such that

$$\lim_{k \to \infty} \frac{|x_{k+1} - r|}{|x_k - r|^{\alpha}} = \lambda,$$

Then, we say $\{x_k\}_{k \geq 0}$ converges to r of order α (or rate α) with asymptotic constant λ.

(i) If $\alpha = 1$, $0 < \lambda < 1$, then, the seqn. is linearly converging to r.

(ii) If $\alpha = 2$, the seqn. is quadratically converging to r.

$\boxed{\text{Supplementary}}$

Eg: 1. In Newton-Raphson method, we have seen

$$\lim_{k \to \infty} \frac{|x_{k+1} - r|}{|x_k - r|^2} = \frac{|f''(r)|}{2|f'(r)|}$$

Hence,

~~Ord~~ Newton-Raphson method quadratically convergent with asymptotic error constant $\lambda = \dfrac{|f''(r)|}{2|f'(r)|}$

provided $f''(r) \neq 0 \neq f'(r)$.

2. In Secant method, we have seen

$$\lim_{k \to \infty} \frac{|x_{k+1} - r|}{|x_k - r|^{\alpha}} = \left(\frac{|f''(r)|}{2|f'(r)|} \right)^{1/\alpha},$$

$$\alpha = \frac{1+\sqrt{5}}{2}.$$

Hence, Secant method has order of convergence $\alpha = \dfrac{1+\sqrt{5}}{2} \approx 1{\cdot}68$ with asymptotic error constant $\lambda = \left(\dfrac{|f''(r)|}{2|f'(r)|} \right)^{1/\alpha}$

provided $f''(r) \neq 0 \neq f'(r)$.

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3. For fixed point iteration, if

$$g'(r) = 0 = \cdots = g^{m-1}(r)$$

$$g^m(r) \neq 0. \quad \text{for } m \in \mathbb{N}$$

Then, $\displaystyle\lim_{k\to\infty} \frac{|x_{k+1} - r|}{|x_k - r|^m} = \frac{1}{\lfloor m} |g^m(r)|$.

Hence, order of convergence is m, with asymptotic error constant

$$\lambda = \frac{1}{\lfloor m} |g^m(r)|.$$

4. If the fn. $f: [a, b] \to \mathbb{R}$ is convex in the interval with $f(a) f(b) < 0$, we choose $x_0 = a$ and $x_1 = b$.

Then, in ~~Regut~~ Regula-Falsi iteration one of the point x_0 or x_1 is always fixed and other pts. ~~vertices~~ varies with k. If x_0 is fixed, then the error equation for Regula-falsi method becomes $(x_{k+1} - r) = c_k (x_0 - r)(x_k - r)$.

If $x_k \neq r \ \forall k \geq 0$, then

$$\lim_{k\to\infty} \frac{|x_{k+1} - r|}{|x_k - r|} = \lim_{k\to\infty} |c_k| |x_0 - r| = |x_0 - r| \cdot \frac{|f''(r)|}{2|f'(r)|}.$$

Hence, in this case ordere of convergence is 1 with asymptotic error constant

$$\lambda = |x_0 - r| \frac{|f''(r)|}{2|f'(r)|}$$ provided

$f''(r) \neq 0 \neq f'(r)$.

The convergence is lineare if $0 < \lambda < 1$.

5. For Biseetion method it is said that the convergence is lineare, but I did not find any proof.

Only we know

$$\lim_{k\to\infty} |x_k - r| \leq \lim_{k\to\infty} \frac{1}{2^{k+1}}(b-a) = 0.$$

6. ~~Conver~~ Ordere of convergence for Müller's method is nearly 1.84. (out of scope).

7. Ordere of convergence of Chebychev's method is 3 (out of scope).

Comparisons of different method

1. Closed domain method.

(i) We need to find initially an interval $[a, b]$ such that $f(a) f(b) < 0$

(ii) These methods surely converge.

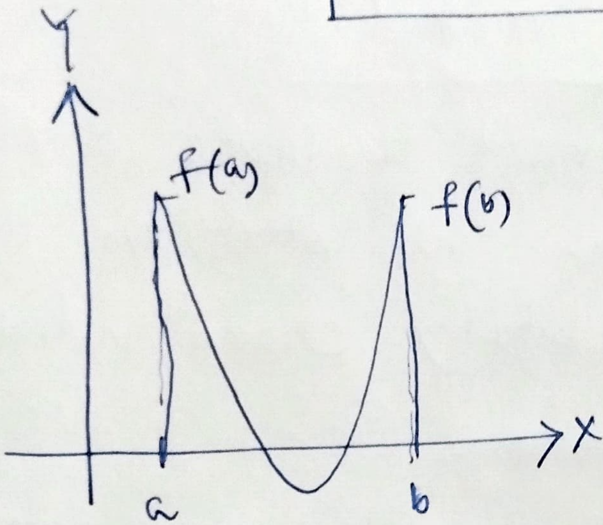
(iii) Bisection method converges slower than Regula-Falsi method

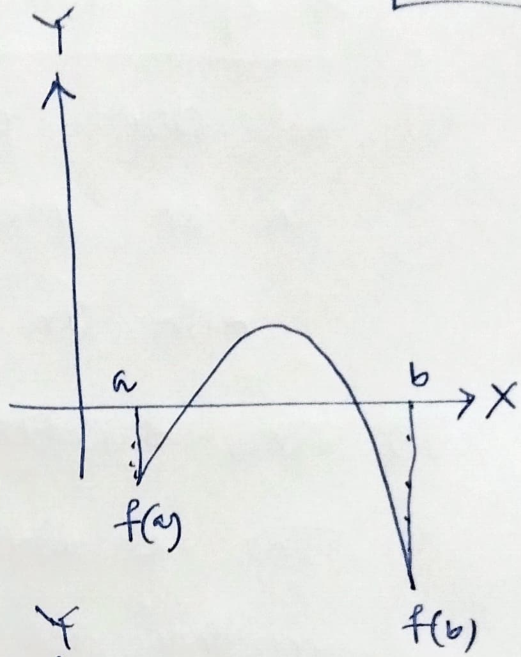
(iv) There might be more than one solution of $f(x) = 0$ in $[a, b]$, but these methods find only one.

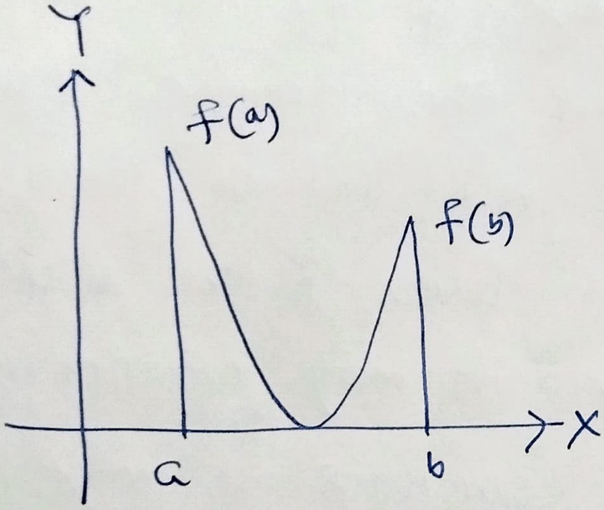
(v) It may also happen that, incase of more than one solution, ~~Bisection~~ the solution found by Bisection method be different from the solution found by Regula-Falsi method.

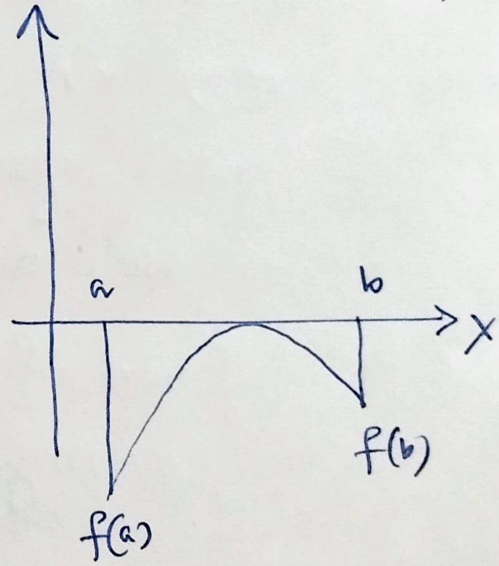
(vi) If $f(a) f(b) < 0$ does not satisfy, then these method cannot locate solution in $[a, b]$, even if there is really a solution of $f(x) = 0$.

Supplementary









In all of the above cases

$f(a)f(b) > 0$. and $\exists\, r \in (a,b)$

such that $f(r) = 0$.

2. Open domain method

(i) We don't need to locate a root in an interval. We start the iteration with an arbitrary chosen initial choice.

(ii) The disadvantage of the open domain method is that the iterative sequence may not be well-defined for all initial guesses. Even if the sequence is well-defined it may not converge. Even if it converge, it may not converge to a specific root.

(iii) In situations where both open and closed domain methods converge, Open domain method are generally faster compared to closed domain methods.

(iv) These method may stuck in a cycle.

Eg: $f(x) = x^3 - x - 3$, $f(x) = 0$. (eqn.)

Take $x_0 = 0$,

(v) If $f(x)=0$ has no real root, there is no indication by these methods and the iterative sequence may simply oscillate.

~~For~~ Eg: $f(x)=0$ with $f(x)=x^2-4x+5$. N-R iteration oscillates.

Secant method	Newton-Raphson method
(i) Need two initial guesses x_0, x_1 to start the iteration.	(i) Need only one initial guess x_0 to start the iteration.
(ii) No need to compute $f'(x_{k-1})$.	(ii) Need to compute $f'(x_{k-1})$ at each iteration.
(iii) Order of convergent is ~ 1.68 i.e, converges ~~faster than~~ slower than N-R method	(iii) Order of convergent is 2 i.e, converges faster than Secant method.

$\boxed{\text{P-22}}$ $\boxed{\text{Exercises (1)}}$ (Conte & de Boor, P-104, Thm-3.2)

Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$ be C^2.

~~f satisfy~~ Let f satisfy the following:

(i) Either $f''(x) \geq 0 \quad \forall\, x \in [a,b]$

or $f''(x) \leq 0 \quad \forall\, x \in [a,b]$

(ii) $f'(x) \neq 0 \quad \forall\, x \in [a,b]$.

(iii) $f(a)\,f(b) < 0$

(iv) $\max\left\{ \dfrac{|f(a)|}{|f'(a)|}, \dfrac{|f(b)|}{|f'(b)|} \right\} < b - a$.

Then, N-R converges to the unique root of $f(x) = 0$ in $[a,b]$ for any choice of initial guess $x_0 \in [a,b]$.

$\boxed{\text{Exercise (2)}}$ (Atkinson, P-77, Lemma-2.5)

Let $a, b \in \mathbb{R}$, $a < b$ and $g : [a,b] \to [a,b]$ be continuous.

Let g satisfy the following:

$\exists\, L \in (0,1)$ such that

$|g(x) - g(y)| \leq L|x - y|, \quad \forall\, x, y \in [a,b]$.

Then, (i) g has a unique fixed pt. $r \in [a,b]$.

(ii) for any $x_0 \in [a,b]$, the seqn.

$x_k = g(x_{k-1}), \quad k \geq 0$

satisfies $\lim_{k \to \infty} x_k = r$, and $|x_k - r| \leq \dfrac{L^k}{1-L} |x_1 - x_0| \quad \forall\, k \geq 1$.

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