Name: ____________________

Roll Number: ______________

Practice Final Solutions

MTH302A - Set Theory and Mathematical Logic

(Odd Semester 2022/23, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.

2. This exam contains **6 + 1** questions and is worth **60%** of your grade.

3. Answer **ALL** questions.

Question 1. **[5 $\times$ 2 Points]**

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) If $(L, \prec)$ is a linear ordering and L is infinite, then there exists an infinite $X \subseteq L$ such that X is well-ordered by $\prec$.

(ii) There exists a bijection $f : \mathbb{R} \to \mathbb{R}^2$ satisfying: For every x, y in $\mathbb{R}$, $f(x - y) = f(x) - f(y)$.

(iii) If $f : \omega \to \omega$ is a strictly increasing computable function, then range(f) is computable.

(iv) There exists a finite $F \subseteq TA$ such that $PA \cup F$ is a complete $\mathcal{L}_{PA}$-theory.

(v) The theory DLO (dense linear orderings without end points) is decidable.

Solution.

(i) False. Take L to be the set of negative integers under the usual ordering.

(ii) True.

(iii) True.

(iv) False. TA is not computably axiomatizable.

(v) True. DLO is both finitely (and hence computably) axiomatizable and complete.

Question 2. [10 Points]

Using transfinite recursion, show that there exists $S \subseteq \mathbb{R}^2$ such that for every line $\ell \subseteq \mathbb{R}^2$,

$$|\ell \cap S| = |\ell \cap (\mathbb{R}^2 \setminus S)| = \mathfrak{c}.$$

Solution 1. Let $\mathcal{L}$ be the family of all lines in plane. Note that $|\mathcal{L}| = |\mathbb{R}^2 \times \mathbb{R}^2| = |\mathbb{R}^2| = \mathfrak{c}$. Fix a sequence $\langle \ell_\alpha : \alpha < \mathfrak{c} \rangle$ such that for every $\ell \in \mathcal{L}$, $\{\alpha < \mathfrak{c} : \ell = \ell_\alpha\} = \mathfrak{c}$ (Why is there such a sequence? Use the fact that $|\mathfrak{c} \times \mathfrak{c}| = \mathfrak{c}$). Using transfinite recursion, construct a sequence $\langle (A_\alpha, B_\alpha) : \alpha < \mathfrak{c} \rangle$ of pairs of **disjoint** subsets of $\mathbb{R}^2$ such that the following hold.

1. $A_0 = B_0 = 0$ and if $\alpha < \beta < \mathfrak{c}$, then $A_\alpha \subseteq A_\beta$ and $B_\alpha \subseteq B_\beta$.
2. If $\gamma < \mathfrak{c}$ is a limit ordinal, then $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ and $B_\gamma = \bigcup_{\alpha<\gamma} B_\alpha$.
3. $|A_\alpha \cup B_\alpha| \leq \max(|\alpha|, \omega) < \mathfrak{c}$.
4. For every $\alpha < \mathfrak{c}$, $|(A_{\alpha+1} \setminus A_\alpha) \cap \ell_\alpha| = |(B_{\alpha+1} \setminus B_\alpha) \cap \ell_\alpha| = 1$.

Start by defining $A_0 = B_0 = 0$. At every limit stage $\gamma < \mathfrak{c}$, Clause 2 forces us to define $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ and $B_\gamma = \bigcup_{\alpha<\gamma} B_\alpha$. So we just need to describe $A_{\alpha+1}, B_{\alpha+1}$ for every $\alpha < \mathfrak{c}$. Clause 4 says we must choose two new points x_α, y_α from $\ell_\alpha \setminus (A_\alpha \cup B_\alpha)$ and define $A_{\alpha+1} = A_\alpha \cup \{x_\alpha\}$ and $B_{\alpha+1} = B_\alpha \cup \{y_\alpha\}$. Note that $|\ell_\alpha| = \mathfrak{c}$ and $|A_\alpha \cup B_\alpha| < \mathfrak{c}$ so we can choose such points.

Having completed the construction, define $A = \bigcup_{\alpha<\mathfrak{c}} A_\alpha$ and $B = \bigcup_{\alpha<\mathfrak{c}} B_\alpha$. It should be clear that A, B are disjoint subsets of $\mathbb{R}^2$. So the only thing to check is $|\ell \cap A| = |\ell \cap B| = \mathfrak{c}$ for every $\ell \in \mathcal{L}$. But this follows from Clause 4 and the fact that each $\ell \in \mathcal{L}$ occurs $\mathfrak{c}$-times in the sequence $\langle \ell_\alpha : \alpha < \mathfrak{c} \rangle$. ☕

Solution 2. For each $n < \omega$, let $C_n = \{x \in \mathbb{R}^2 : n < ||x|| < n+1\}$ ($||x||$ is the distance of x from the origin). Define $A_n = \bigcup_{n<\omega} C_{2n}$ and $B_n = \bigcup_{n<\omega} C_{2n+1}$. Note that for any line ℓ, $\{||x|| : x \in \ell\} = [d, \infty)$ where d is the distance of ℓ from the origin. It follows that both $\ell \cap A$ and $\ell \cap B$ contain line segments of positive length and hence have size continuum. ☕

Remark. Although Solution 2 is simpler, the method of Solution 1 will also work for families other than lines. For example, circles, squares, rectangles, line segments etc.

Question 3. [10 Points]

Let $\mathcal{L} = \{\prec\}$ be the first order language with a binary relation symbol $\prec$.

(a) [**2 Points**] Write down the axioms of the $\mathcal{L}$-theory DLO (dense linear ordering without end points).

(b) [**4 Points**] Show that the $\mathcal{L}$-structures $(\mathbb{R}, <)$ and $(\mathbb{R} \setminus \{0\}, <)$ are not isomorphic. Here $<$ is the usual ordering of real numbers.

(c) [**4 Points**] Show that DLO is not $\mathfrak{c}$-categorical.

Solution.

(a) See Lecture Slide 138. ☕

(b) Suppose not and let $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be an order isomorphism. Put $L = \{h(x) : x < 0\}$ and $R = \{h(x) : x > 0\}$. As $L \subseteq \mathbb{R}$ is bounded from above, it has a least upper bound (supremum) say b. Fix $a \in \mathbb{R} \setminus \{0\}$ such that $h(a) = b$. Note that $a < 0$ or $a > 0$.

If $a < 0$, then $a < a/2 < 0$. So $b = h(a) < h(a/2)$ and $h(a/2) \in L$. So b is not an upper bound of L which is a contradiction. If $a > 0$, then $0 < a/2 < a$. So $h(a/2) < h(a) = b$ and $h(a/2) \in R$. So b is not the **least** upper bound of L which is also a contradiction. ☕

(c) First check that both $(\mathbb{R}, <)$and $(\mathbb{R} \setminus \{0\}, <)$ are models of DLO. By part (b), they are not isomorphic. So DLO is not $\mathfrak{c}$-categorical. ☕

Question 4. [10 Points]

Let $\mathcal{L}$ be the empty language. For each $n \geq 2$, recall that $\exists_{\geq n}$ denotes the following $\mathcal{L}$-sentence:

$$(\exists x_1)(\exists x_2)\ldots(\exists x_n)\left(\bigwedge_{i<j\leq n} \neg(x_i = x_j)\right)$$

Define $T = \{\exists_{\geq n} : n \geq 2\}$.

(a) [**4 Points**] Show that T is a complete $\mathcal{L}$-theory.

(b) [**6 Points**] Show that T is not finitely axiomatizable.

(a) Note that $\mathcal{M}$ is a model of T iff its domain M is infinite. Furthermore, if $\mathcal{M}, \mathcal{N}$ are two models of T with $|M| = |N|$, then $\mathcal{M} \cong \mathcal{N}$ (every bijection $h : M \to N$ is an isomorphism as $\mathcal{L}$ is empty). So T is κ-categorical for every $\kappa \geq \omega$. Hence T is a complete $\mathcal{L}$-theory (Lecture slide 137). ☕

(b) Suppose not and towards a contradiction, fix a finite set $F = \{\psi_k : 1 \leq k \leq n\}$ of $\mathcal{L}$-sentence such that for every $\mathcal{L}$-sentence ϕ, $F \vdash \phi$ iff $T \vdash \phi$. Note that for each k, $T \vdash \psi_k$ so we can fix a finite $S_k \subseteq T$ such that $S_k \vdash \psi_k$. Then $S = \bigcup_{1\leq k\leq n} S_k$ is finite.

Let $S = \{\exists_{n_k} : 1 \leq k \leq m\}$. Define $n = \max(\{n_k : 1 \leq k \leq m\}) + 5$. As $T \vdash \exists_{\geq n}$, we also have $F \vdash \exists_{\geq n}$. As each sentence in F is a theorem of S, it follows that $S \vdash \exists_{\geq n}$ and therefore $S \models \exists_{\geq n}$. But this is impossible since if $\mathcal{M}$ is an $\mathcal{L}$-structure with domain M of size $|M| = \max(\{n_k : 1 \leq k \leq m\})$, then $\mathcal{M} \models S$ and $\mathcal{M} \not\models \exists_{\geq n}$. ☕

Question 5. [10 Points]

(a) [**5 Points**] Suppose $A \subseteq \omega$ and $B \subseteq \omega$ are both computable. Show that $\{x + y : x \in A \text{ and } y \in B\}$ is computable.

(b) [**5 Points**] Let $E \subseteq \omega$ be computable. Show that $\{|x - y| : x, y \in E\}$ is c.e.

Solution.

(a) Put $C = \{x + y : x \in A \text{ and } y \in B\}$. Fix programs P and Q such that P computes 1_A and Q computes 1_B. Consider a program R that on input n does the following.

For each $0 \leq k \leq n$, run P with input k and Q with input $n - k$. If both P and Q output 1 for some $0 \leq k \leq n$, then R outputs 1. Otherwise R outputs 0.

It is clear that R computes 1_C. So C is computable. ☕

(b) Put $W = \{|x - y| : x, y \in E\}$. Fix a program P that computes 1_E. Consider a program R that on input n does the following: Search for the least $k < \omega$ such that P outputs 1 on each of the inputs k and $n + k$. R halts as soon as such a k is found.

It is clear that R halts on input n iff $n \in W$. So W is c.e. ☕

Question 6. [10 Points]

Let $\mathcal{N} = (\omega, 0, S, +, \cdot)$ be the standard model of PA.

(a) [**6 Points**] Define $False_{\mathcal{N}} = \{\ulcorner\psi\urcorner : \mathcal{N} \models \neg\psi\}$. Show that $False_{\mathcal{N}}$ is not definable in $\mathcal{N}$.

(b) [**4 Points**] Show that there are $\mathcal{L}_{PA}$-sentences ϕ and ψ such that PA does not prove either one of the following four sentences.

(i) ϕ.

(ii) $\neg\phi$.

(iii) $\phi \implies \psi$.

(iv) $\phi \implies (\neg\psi)$.

Solution.

(a) Towards a contradiction, assume $False_{\mathcal{N}}$ is definable in $\mathcal{N}$ via the formula $\phi(x)$. So for every $n < \omega$, $n \in False_{\mathcal{N}}$ iff $\mathcal{N} \models \phi(n)$. Define $f : \omega \to \omega$ as follows. If $n = \ulcorner\psi\urcorner$ for some $\mathcal{L}_{PA}$-sentence ψ, then $f(n) = \ulcorner(\neg\psi)\urcorner$. Otherwise, $f(n) = 0$. It is clear that f is computable and hence definable in $\mathcal{N}$ say via the $\mathcal{L}_{PA}$-formula $\eta(y, x)$.

Now consider the formula $\theta(x) \equiv (\exists y)(\eta(y, x) \wedge \phi(y))$. It is easy to check that $n \in True_{\mathcal{N}}$ iff $\mathcal{N} \models \theta(n)$. So $True_{\mathcal{N}}$ is definable in $\mathcal{N}$. A contradiction. ☕

(b) Choose $\phi \in TA$ such that $PA \nvdash \phi$. Note that $PA \nvdash \neg\phi$ since if $PA \vdash \neg\phi$, then as $PA \subseteq TA$, both ϕ and $(\neg\phi)$ are in TA which is impossible.

Since $PA \cup \{\phi\}$ is a computable subset of TA, it cannot axiomatize TA. So we can choose $\psi \in TA$ such that $PA \cup \{\phi\} \nvdash \psi$. By deduction theorem, $PA \nvdash (\phi \implies \psi)$. Finally, to see that PA does not prove $\phi \implies (\neg\psi)$, use the fact that $PA \cup \{\phi\} \nvdash (\neg\psi)$ (otherwise TA would prove both ψ and $\neg\psi$). ☕

Bonus Question [5 Points]

Show that

$$\left|\left\{x > 0 : \lim_{n\to\infty} \mathsf{frac}((n!)x) = 0\right\}\right| = \mathfrak{c}.$$

Here $\mathsf{frac}(x)$ denotes the fractional part of x. For example, $\mathsf{frac}(\pi) = \pi - 3$ and $\mathsf{frac}(7) = 0$.

Solution sketch. Let $G = \{x > 0 : \lim_{n\to\infty} \mathsf{frac}((n!)x) = 0\}$. For each $f \in 2^\omega$, define

$$x_f = \sum_{n\geq 1} \frac{f(n)}{n!}$$

Put $W = \{x_f : f \in 2^\omega\}$. Check that $W \subseteq G$ and $|W| = |2^\omega| = \mathfrak{c}$.

End of Exam