# LINEAR ALGEBRA

The solution of systems of simultaneous linear equations and the calculation of the eigenvalues and eigenvectors of a matrix are two very important problems that arise in a wide variety of contexts. As a preliminary to the discussion of these problems in the following chapters, we present some results from linear algebra. The first section contains a review of material on vector spaces, matrices, and linear systems, which is taught in most undergraduate linear algebra courses. These results are summarized only, and no derivations are included. The remaining sections discuss eigenvalues, canonical forms for matrices, vector and matrix norms, and perturbation theorems for matrix inverses. If necessary, this chapter can be skipped, and the results can be referred back to as they are needed in Chapters 8 and 9. For notation, Section 7.1 and the norm notation of Section 7.3 should be skimmed.

## 7.1 Vector Spaces, Matrices, and Linear Systems

Roughly speaking a *vector space* $V$ is a set of objects, called *vectors*, for which operations of *vector addition* and *scalar multiplication* have been defined. A vector space $V$ has a set of scalars associated with it, and in this text, this set can be either the real numbers **R** or complex numbers **C**. The vector operations must satisfy certain standard associative, commutative, and distributive rules, which we will not list. A subset $W$ of a vector space $V$ is called a *subspace* of $V$ if $W$ is a vector space using the vector operations inherited from $V$. For a complete development of the theory of vector spaces, see any undergraduate text on linear algebra [for example, Anton (1984), chap. 3; Halmos (1958), chap. 1; Noble (1969), chaps. 4 and 14; Strang (1980), chap. 2].

***Example* 1.** $V = \mathbf{R}^n$, the set of all $n$-tuples $(x_1, \dots, x_n)$ with real entries $x_i$, and **R** is the associated set of scalars.

**2.** $V = \mathbf{C}^n$, the set of all $n$-tuples with complex entries, and **C** is the set of scalars.

**3.** $V =$ the set of all polynomials of degree $\le n$, for some given $n$, is a vector space. The scalars can be **R** or **C**, as desired for the application.

**4.** $V = C[a, b]$, the set of all continuous real valued [or complex valued] functions on the interval $[a, b]$, is a vector space with scalar set equal to **R** [or **C**]. The example in (3) is a subspace of $C[a, b]$.

***Definition*** Let $V$ be a vector space and let $v_1, v_2, \ldots, v_m \in V$.

1. We say that $v_1, \ldots, v_m$ are *linearly dependent* if there is a set of scalars $\alpha_1, \ldots, \alpha_m$, with at least one nonzero scalar, for which

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$$

Since at least one scalar is nonzero, say $\alpha_i \neq 0$, we can solve for

$$v_i = -\frac{\alpha_1}{\alpha_i} v_1 \cdots - \frac{\alpha_{i-1}}{\alpha_i} v_{i-1} - \frac{\alpha_{i+1}}{\alpha_i} v_{i+1} - \cdots - \frac{\alpha_m}{\alpha_i} v_m$$

We say that $v_i$ is a *linear combination* of the vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m$. For a set of vectors to be linearly dependent, one of them must be a linear combination of the remaining ones.

2. We say $v_1, \ldots, v_m$ are *linearly independent* if they are not dependent. Equivalently, the only choice of scalars $\alpha_1, \ldots, \alpha_m$ for which

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$$

is the trivial choice $\alpha_1 = \cdots = \alpha_m = 0$. No $v_i$ can be written as a combination of the remaining ones.

3. $\{v_1, \ldots, v_m\}$ is a *basis* for $V$ if for every $v \in V$, there is a unique choice of scalars $\alpha_1, \ldots, \alpha_m$ for which

$$v = \alpha_1 v_1 + \cdots + \alpha_m v_m$$

Note that this implies $v_1, \ldots, v_m$ are independent. If such a finite basis exists, we say $V$ is *finite dimensional.* Otherwise, it is called *infinite dimensional.*

***Theorem 7.1*** If $V$ is a vector space with a basis $\{v_1, \ldots, v_m\}$, then every basis for $V$ will contain exactly $m$ vectors. The number $m$ is called the dimension of $V$.

***Example*** **1.** $\{1, x, x^2, \ldots, x^n\}$ is a basis for the space $V$ of polynomials of degree $\leq n$. Thus dimension $V = n + 1$.

**2.** $\mathbf{R}^n$ and $\mathbf{C}^n$ have the basis $\{e_1, \ldots, e_n\}$, in which

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \tag{7.1.1}$$

with the 1 in position $i$. Dimension $\mathbf{R}^n, \mathbf{C}^n = n$. This is called the standard basis for $\mathbf{R}^n$ and $\mathbf{C}^n$, and the vectors in it are called *unit vectors*.

3. $C[a, b]$ is infinite dimensional.

**Matrices and linear systems** Matrices are rectangular arrays of real or complex numbers, and the general matrix of order $m \times n$ has the form

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix} \tag{7.1.2}$$

A matrix of order $n$ is shorthand for a square matrix of order $n \times n$. Matrices will be denoted by capital letters, and their entries will normally be denoted by lowercase letters, usually corresponding to the name of the matrix, as just given. The following definitions give the common operations on matrices.

***Definition*** **1.** Let $A$ and $B$ have order $m \times n$. The sum of $A$ and $B$ is the matrix $C = A + B$, of order $m \times n$, given by

$$c_{ij} = a_{ij} + b_{ij}$$

**2.** Let $A$ have order $m \times n$, and let $\alpha$ be a scalar. Then the scalar multiple $C = \alpha A$ is of order $m \times n$ and is given by

$$c_{ij} = \alpha a_{ij}$$

**3.** Let $A$ have order $m \times n$ and $B$ have order $n \times p$. Then the product $C = AB$ is of order $m \times p$, and it is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

**4.** Let $A$ have order $m \times n$. The *transpose* $C = A^T$ has order $n \times m$, and is given by

$$c_{ij} = a_{ji}$$

The *conjugate transpose* $C = A^*$ also has order $n \times m$, and

$$c_{ij} = \bar{a}_{ji}$$

The notation $\bar{z}$ denotes the complex conjugate of the complex number $z$, and $z$ is real if and only if $\bar{z} = z$. The conjugate transpose $A^*$ is also called the *adjoint* of $A$.

The following arithmetic properties of matrices can be shown without much difficulty, and they are left to the reader.

$$
\begin{array}{llll}
\text{(a)} & A + B = B + A & \text{(b)} & (A + B) + C = A + (B + C) \\
\text{(c)} & A(B + C) = AB + AC & \text{(d)} & A(BC) = (AB)C \\
\text{(e)} & (A + B)^T = A^T + B^T & \text{(f)} & (AB)^T = B^T A^T
\end{array}
\tag{7.1.3}
$$

It is important for many applications to note that the matrices need not be square for the preceding properties to hold.

The vector spaces $\mathbf{R}^n$ and $\mathbf{C}^n$ will usually be identified with the set of column vectors of order $n \times 1$, with real and complex entries, respectively. The linear system

$$
\begin{gathered}
a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n = b_m
\end{gathered}
\tag{7.1.4}
$$

can be written as $Ax = b$, with $A$ as in (7.1.2), and

$$
x = [x_1, \ldots, x_n]^T \qquad b = [b_1, \ldots, b_m]^T
$$

The vector $b$ is a given vector in $\mathbf{R}^m$, and the solution $x$ is an unknown vector in $\mathbf{R}^n$. The use of matrix multiplication reduces the linear system (7.1.4) to the simpler and more intuitive form $Ax = b$.

We now introduce a few additional definitions for matrices, including some special matrices.

***Definition*** **1.** The *zero matrix* of order $m \times n$ has all entries equal to zero. It is denoted by $0_{m \times n}$, or more simply, by 0. For any matrix $A$ of order $m \times n$,

$$
A + 0 = 0 + A = A
$$

**2.** The *identity matrix* of order $n$ is defined by $I = [\delta_{ij}]$,

$$
\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\tag{7.1.5}
$$

for all $1 \leq i, j \leq n$. For all matrices $A$ of order $m \times n$ and $B$ of order $n \times p$,

$$
AI = A \qquad IB = B
$$

The notation $\delta_{ij}$ denotes the *Kronecker delta function*.

**3.** Let $A$ be a square matrix of order $n$. If there is a square matrix $B$ of order $n$ for which $AB = BA = I$, then we say $A$ is invertible, with *inverse* $B$. The matrix $B$ can be shown to be unique, and we denote the inverse of $A$ by $A^{-1}$.

4. A matrix $A$ is called *symmetric* if $A^T = A$, and it is called *Hermitian* if $A^* = A$. The term symmetric is generally used only with real matrices. The matrix $A$ is *skew-symmetric* if $A^T = -A$. Of necessity, all matrices that are symmetric, Hermitian, or skew-symmetric must also be square.

5. Let $A$ be an $m \times n$ matrix. The row rank of $A$ is the number of linearly independent rows in $A$, regarded as elements of $\mathbf{R}^n$ or $\mathbf{C}^n$, and the column rank is the number of linearly independent columns. It can be shown (Problem 4) that these two numbers are always equal, and this is called the *rank* of $A$.

For the definition and properties of the determinant of a square matrix $A$, see any linear algebra text [for example, Anton (1984), chap. 2; Noble (1969), chap. 7; and Strang (1980), chap. 4]. We summarize many of the results on matrix inverses and the solvability of linear systems in the following theorem.

***Theorem** 7.2* Let $A$ be a square matrix with elements from $\mathbf{R}$ (or $\mathbf{C}$), and let the vector space be $V = \mathbf{R}^n$ (or $\mathbf{C}^n$). Then the following are equivalent statements.

1. $Ax = b$ has a unique solution $x \in V$ for every $b \in V$.
2. $Ax = b$ has a solution $x \in V$ for every $b \in V$.
3. $Ax = 0$ implies $x = 0$.
4. $A^{-1}$ exists.
5. Determinant $(A) \neq 0$.
6. Rank $(A) = n$.

Although no proof is given here, it is an excellent exercise to prove the equivalence of some of these statements. Use the concepts of linear independence and basis, along with Theorem 7.1. Also, use the decomposition

$$Ax = x_1 A_{*1} + \cdots + x_n A_{*n} \qquad x \in \mathbf{R}^n \quad \text{or} \quad \mathbf{C}^n \tag{7.1.6}$$

with $A_{*j}$ denoting column $j$ in $A$. This says that the space of all vectors of the form $Ax$ is spanned by the columns of $A$, although they may be linearly dependent.

**Inner product vector spaces** One of the important reasons for reformulating problems as equivalent linear algebra problems is to introduce some geometric insight. Important to this process are the concepts of inner product and orthogonality.

***Definition* 1.** The *inner product* of two vectors $x, y \in \mathbf{R}^n$ is defined by

$$(x, y) = \sum_{i=1}^{n} x_i y_i = x^T y = y^T x$$

and for vectors $x, y \in \mathbf{C}^n$, define the inner product by

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i = y^* x$$

**2.** The *Euclidean norm* of $x$ in $\mathbf{C}^n$ or $\mathbf{R}^n$ is defined by

$$\|x\|_2 = \sqrt{(x, x)} = \sqrt{|x_1|^2 + \cdots + |x_n|^2} \tag{7.1.7}$$

The following results are fairly straightforward to prove, and they are left to the reader. Let $V$ denote $\mathbf{C}^n$ or $\mathbf{R}^n$.

1. For all $x, y, z \in V$,

$$(x, y + z) = (x, y) + (x, z), \qquad (x + y, z) = (x, z) + (y, z)$$

2. For all $x, y \in V$,

$$(\alpha x, y) = \alpha(x, y)$$

and for $V = \mathbf{C}^n$, $\alpha \in \mathbf{C}$,

$$(x, \alpha y) = \bar{\alpha}(x, y)$$

3. In $\mathbf{C}^n$, $(x, y) = \overline{(y, x)}$; and in $\mathbf{R}^n$, $(x, y) = (y, x)$.
4. For all $x \in V$,

$$(x, x) \geq 0$$

and $(x, x) = 0$ if and only if $x = 0$.

5. For all $x, y \in V$,

$$|(x, y)|^2 \leq (x, x)(y, y) \tag{7.1.8}$$

This is called the *Cauchy–Schwartz inequality*, and it is proved in exactly the same manner as (4.4.3) in Chapter 4. Using the Euclidean norm, we can write it as

$$|(x, y)| \leq \|x\|_2 \|y\|_2 \tag{7.1.9}$$

6. For all $x, y \in V$,

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2 \tag{7.1.10}$$

This is the *triangle inequality*. For a geometric interpretation, see the earlier comments in Section 4.1 of Chapter 4 for the norm $\|f\|_\infty$ on $C[a, b]$. For a proof of (7.1.10), see the derivation of (4.4.4) in Chapter 4.

7. For any square matrix $A$ of order $n$, and for any $x, y \in \mathbf{C}^n$,

$$(Ax, y) = (x, A^*y) \tag{7.1.11}$$

The inner product was used to introduce the Euclidean length, but it is also used to define a sense of angle, at least in spaces in which the scalar set is **R**.

***Definition* 1.** For $x, y$ in $\mathbf{R}^n$, the *angle between x and y* is defined by

$$\mathscr{A}(x, y) = \cos^{-1}\left[\frac{(x, y)}{\|x\|_2\|y\|_2}\right]$$

Note that the argument is between $-1$ and 1, due to the Cauchy–Schwartz inequality (7.1.9). The preceding definition can be written implicitly as

$$(x, y) = \|x\|_2\|y\|_2 \cos(\mathscr{A}) \tag{7.1.12}$$

a familiar formula from the use of the dot product in $\mathbf{R}^2$ and $\mathbf{R}^3$.

2. Two vectors $x$ and $y$ are *orthogonal* if and only if $(x, y) = 0$. This is motivated by (7.1.12). If $\{x^{(1)}, \dots, x^{(n)}\}$ is a basis for $\mathbf{C}^n$ or $\mathbf{R}^n$, and if $(x^{(i)}, x^{(j)}) = 0$ for all $i \neq j$, $1 \leq i, j \leq n$, then we say $\{x^{(1)}, \dots, x^{(n)}\}$ is an *orthogonal basis*. If all basis vectors have Euclidean length 1, the basis is called *orthonormal*.

3. A square matrix $U$ is called *unitary* if

$$U^*U = UU^* = I$$

If the matrix $U$ is real, it is usually called *orthogonal*, rather than unitary. The rows [or columns] of an order $n$ unitary matrix form an orthonormal basis for $\mathbf{C}^n$, and similarly for orthogonal matrices and $\mathbf{R}^n$.

***Example* 1.** The angle between the vectors

$$x \doteq (1, 2, 3) \qquad y = (3, 2, 1)$$

is given by

$$\mathscr{A} = \cos^{-1}\left[\frac{10}{14}\right] \doteq .775 \text{ radians}$$

2. The matrices

$$U_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad U_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

are unitary, with the first being orthogonal.

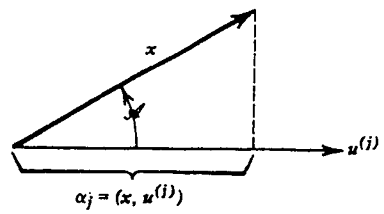


**Figure 7.1** Illustration of (7.1.15).

An orthonormal basis for a vector space $V = \mathbf{R}^n$ or $\mathbf{C}^n$ is desirable, since it is then easy to decompose an arbitrary vector into its components in the direction of the basis vectors. More precisely, let $\{u^{(1)}, \ldots, u^{(n)}\}$ be an orthonormal basis for $V$, and let $x \in V$. Using the basis,

$$x = \alpha_1 u^{(1)} + \cdots + \alpha_n u^{(n)}$$

for some unique choice of coefficients $\alpha_1, \ldots, \alpha_n$. To find $\alpha_j$, form the inner product of $x$ with $u^{(j)}$, and then

$$\begin{aligned}(x, u^{(j)}) &= \alpha_1(u^{(1)}, u^{(j)}) + \cdots + \alpha_n(u^{(n)}, u^{(j)}) \\ &= \alpha_j \end{aligned} \tag{7.1.13}$$

using the orthonormality properties of the basis. Thus

$$x = \sum_{j=1}^{n} (x, u^{(j)}) u^{(j)} \tag{7.1.14}$$

This can be given a geometric interpretation, which is shown in Figure 7.1. Using (7.1.13)

$$\begin{aligned}\alpha_j = (x, u^{(j)}) &= \|x\|_2 \|u^{(j)}\|_2 \cos\left(\mathscr{A}(x, u^{(j)})\right) \\ &= \|x\|_2 \cos\left(\mathscr{A}(x, u^{(j)})\right)\end{aligned} \tag{7.1.15}$$

Thus the coefficient $\alpha_j$ is just the length of the orthogonal projection of $x$ onto the axis determined by $u^{(j)}$. The formula (7.1.14) is a generalization of the decomposition of a vector $x$ using the standard basis $\{e^{(1)}, \ldots, e^{(n)}\}$, defined earlier.

***Example*** Let $V = \mathbf{R}^2$, and consider the orthonormal basis

$$u^{(1)} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad u^{(2)} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

Then for a given vector $x = (x_1, x_2)$, it can be written as

$$x = \alpha_1 u^{(1)} + \alpha_2 u^{(2)}$$

$$\alpha_1 = \left(x, u^{(1)}\right) = \frac{x_1 + x_2\sqrt{3}}{2} \qquad \alpha_2 = \left(x, u^{(2)}\right) = \frac{x_2 - x_1\sqrt{3}}{2}$$

For example,

$$(1,0) = \frac{1}{2}u^{(1)} - \frac{\sqrt{3}}{2}u^{(2)}$$

## 7.2 Eigenvalues and Canonical Forms for Matrices

The number $\lambda$, complex or real, is an *eigenvalue* of the square matrix $A$ if there is a vector $x \in \mathbf{C}^n$, $x \neq 0$, such that

$$Ax = \lambda x \tag{7.2.1}$$

The vector $x$ is called an *eigenvector* corresponding to the eigenvalue $\lambda$. From Theorem 7.2, statements (3) and (5), $\lambda$ is an eigenvalue of $A$ if and only if

$$\det(A - \lambda I) = 0 \tag{7.2.2}$$

This is called the *characteristic equation* for $A$, and to analyze it we introduce the function

$$f_A(\lambda) \equiv \det(A - \lambda I)$$

If $A$ has order $n$, then $f_A(\lambda)$ will be a polynomial of degree exactly $n$, called the *characteristic polynomial* of $A$. To prove it is a polynomial, expand the determinant by minors repeatedly to get

$$\begin{aligned} f_A(\lambda) &= \det(A - \lambda I) \\ &= \det\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & \cdots & a_{nn} - \lambda \end{bmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) \\ &\quad + \text{terms of degree} \leq n - 2 \end{aligned}$$

$$\begin{aligned} f_A(\lambda) &= (-1)^n \lambda^n + (-1)^{n-1}(a_{11} + \cdots + a_{nn})\lambda^{n-1} \\ &\quad + \text{terms of degree} \leq n - 2 \end{aligned} \tag{7.2.3}$$

Also note that the constant term is

$$f_A(0) = \det(A) \tag{7.2.4}$$

From the coefficient of $\lambda^{n-1}$, define

$$\operatorname{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn} \tag{7.2.5}$$

which is often a quantity of interest in the study of $A$.

Since $f_A(\lambda)$ is of degree $n$, there are exactly $n$ eigenvalues for $A$, if we count multiple roots according to their multiplicity. Every matrix has at least one eigenvalue–eigenvector pair, and the $n \times n$ matrix $A$ has at most $n$ distinct eigenvalues.

***Example* 1.** The characteristic polynomial for

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

is

$$f_A(\lambda) = -\lambda^3 + 7\lambda^2 - 14\lambda + 8$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$, and the corresponding eigenvectors are

$$u^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad u^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad u^{(3)} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Note that these eigenvectors are orthogonal to each other, and therefore they are linearly independent. Since the dimension of $\mathbf{R}^3$ (and $\mathbf{C}^3$) is three, these eigenvectors form an orthogonal basis for $\mathbf{R}^3$ (and $\mathbf{C}^3$). This illustrates Theorem 7.4, which is presented later in the section.

**2.** For the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad f_A(\lambda) = (1 - \lambda)^3$$

and there are three linearly independent eigenvectors for the eigenvalue $\lambda = 1$, for example,

$$[1, 0, 0]^T \qquad [0, 1, 0]^T \qquad [0, 0, 1]^T$$

All other eigenvectors are linear combinations of these three vectors.

3. For the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad f_A(\lambda) = (1 - \lambda)^3$$

The matrix $A$ has only one linearly independent eigenvector for the eigenvalue $\lambda = 1$, namely

$$x = [1, 0, 0]^T$$

and multiples of it.

The *algebraic multiplicity* of an eigenvalue of a matrix $A$ is its multiplicity as a root of $f_A(\lambda)$, and its *geometric multiplicity* is the maximum number of linearly independent eigenvectors associated with the eigenvalue. The sum of the algebraic multiplicities of the eigenvalues of an $n \times n$ matrix $A$ is constant with respect to small perturbations in $A$, namely $n$. But the sum of the geometric multiplicities can vary greatly with small perturbations, and this causes the numerical calculation of eigenvectors to often be a very difficult problem. Also, the algebraic and geometric multiplicities need not be equal, as the preceding examples show.

***Definition*** Let $A$ and $B$ be square matrices of the same order. Then $A$ is *similar* to $B$ if there is a nonsingular matrix $P$ for which

$$B = P^{-1}AP \tag{7.2.6}$$

Note that this is a symmetric relation since

$$A = Q^{-1}BQ \qquad Q = P^{-1}$$

The relation (7.2.6) can be interpreted to say that $A$ and $B$ are matrix representations of the same linear transformation $T$ from $V$ to $V$ [$V = \mathbf{R}^n$ or $\mathbf{C}^n$], but with respect to different bases for $V$. The matrix $P$ is called the *change of basis matrix*, and it relates the two representations of a vector $x \in V$ with respect to the two bases being used [see Anton (1984), sec. 5.5 or Noble (1969), sec. 14.5 for greater detail].

We now present a few simple properties about similar matrices and their eigenvalues.

1. If $A$ and $B$ are similar, then $f_A(\lambda) = f_B(\lambda)$. To prove this, use (7.2.6) to show

$$f_B(\lambda) = \det(B - \lambda I) = \det\left[P^{-1}(A - \lambda I)P\right]$$
$$= \det(P^{-1})\det(A - \lambda I)\det(P) = f_A(\lambda)$$

since

$$\det(P)\det(P^{-1}) = \det(PP^{-1}) = \det(I) = 1$$

2. The eigenvalues of similar matrices $A$ and $B$ are exactly the same, and there is a one-to-one correspondence of the eigenvectors. If $Ax = \lambda x$, then using

$$P^{-1}AP(P^{-1}x) = \lambda P^{-1}x$$

$$Bz = \lambda z \qquad z = P^{-1}x \tag{7.2.7}$$

Trivially, $z \neq 0$, since otherwise $x$ would be zero. Also, given any eigenvector $z$ of $B$, this argument can be reversed to produce a corresponding eigenvector $x = Pz$ for $A$.

3. Since $f_A(\lambda)$ is invariant under similarity transformations of $A$, the coefficients of $f_A(\lambda)$ are also invariant under such similarity transformations. In particular, for $A$ similar to $B$,

$$\operatorname{trace}(A) = \operatorname{trace}(B) \qquad \det(A) = \det(B) \tag{7.2.8}$$

**Canonical forms** We now present several important canonical forms for matrices. These forms relate the structure of a matrix to its eigenvalues and eigenvectors, and they are used in a variety of applications in other areas of mathematics and science.

***Theorem** 7.3* (Schur Normal Form) Let $A$ have order $n$ with elements from $\mathbf{C}$. Then there exists a unitary matrix $U$ such that

$$T \equiv U^*AU \tag{7.2.9}$$

is upper triangular.

Since $T$ is triangular, and since $U^* = U^{-1}$,

$$f_A(\lambda) = f_T(\lambda) = (\lambda - t_{11}) \cdots (\lambda - t_{nn}) \tag{7.2.10}$$

and thus the eigenvalues of $A$ are the diagonal elements of $T$.

***Proof*** The proof is by induction on the order $n$ of $A$. The result is trivially true for $n = 1$, using $U = [1]$. We assume the result is true for all matrices of order $n \leq k - 1$, and we will then prove it has to be true for all matrices of order $n = k$.

Let $\lambda_1$ be an eigenvalue of $A$, and let $u^{(1)}$ be an associated eigenvector with $\|u^{(1)}\|_2 = 1$. Beginning with $u^{(1)}$, pick an orthonormal basis for $\mathbf{C}^k$, calling it $\{u^{(1)}, \ldots, u^{(k)}\}$. Define the matrix $P_1$ by

$$P_1 = \left[u^{(1)}, u^{(2)}, \ldots, u^{(k)}\right]$$

which is written in partitioned form, with columns $u^{(1)}, \ldots, u^{(k)}$ that are orthogonal. Then $P_1^*P_1 = I$, and thus $P_1^{-1} = P_1^*$. Define

$$B_1 = P_1^*AP_1$$

Claim:

$$B_1 = \begin{bmatrix} \lambda_1 & \alpha_2 & \cdots & \alpha_k \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{bmatrix}$$

with $A_2$ of order $k-1$ and $\alpha_2, \ldots, \alpha_k$ some numbers. To prove this, multiply using partitioned matrices:

$$AP_1 = A\left[u^{(1)}, \ldots, u^{(k)}\right] = \left[Au^{(1)}, \ldots, Au^{(k)}\right]$$

$$= \left[\lambda_1 u^{(1)}, v^{(2)}, \ldots, v^{(k)}\right] \qquad v^{(j)} = Au^{(j)}$$

$$B_1 = P_1^* A P_1 = \left[\lambda_1 P_1^* u^{(1)}, P_1^* v^{(2)}, \ldots, P_1^* v^{(k)}\right]$$

Since $P_1^* P_1 = I$, it follows that $P_1^* u^{(1)} = e^{(1)} = [1, 0, \ldots, 0]^T$. Thus

$$B_1 = \left[\lambda_1 e^{(1)}, w^{(2)}, \ldots, w^{(k)}\right] \qquad w^{(j)} = P_1^* v^{(j)}$$

which has the desired form.

By the induction hypothesis, there exists a unitary matrix $\hat{P}_2$ of order $k-1$ for which

$$\hat{T} = \hat{P}_2^* A_2 \hat{P}_2$$

is an upper triangular matrix of order $k-1$. Define

$$P_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{P}_2 & \\ 0 & & & \end{bmatrix}$$

Then $P_2$ is unitary, and

$$P_2^* B_1 P_2 = \begin{bmatrix} \lambda_1 & \gamma_2 & & \gamma_k \\ 0 & & & \\ \vdots & & \hat{P}_2^* A_2 \hat{P}_2 & \\ 0 & & & \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \gamma_2 & \cdots & \gamma_k \\ 0 & & & \\ \vdots & & \hat{T} & \\ 0 & & & \end{bmatrix} \equiv T$$

an upper triangular matrix. Thus

$$T = P_2^* B_1 P_2 = P_2^* P_1^* A P_1 P_2 = (P_1 P_2)^* A (P_1 P_2)$$
$$T = U^* A U \qquad U = P_1 P_2$$

and $U$ is easily unitary. This completes the induction and the proof. ■

***Example*** For the matrix

$$A = \begin{bmatrix} .2 & .6 & 0 \\ 1.6 & -.2 & 0 \\ -1.6 & 1.2 & 3.0 \end{bmatrix}$$

the matrices of the theorem and (7.2.9) are

$$T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix} \qquad U = \begin{bmatrix} .6 & 0 & -.8 \\ .8 & 0 & .6 \\ 0 & 1.0 & 0 \end{bmatrix}$$

This is not the usual way in which eigenvalues are calculated, but should be considered only as an illustration of the theorem. The theorem is used generally as a theoretical tool, rather than as a computational tool.

Using (7.2.8) and (7.2.9),

$$\operatorname{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n \qquad \det(A) = \lambda_1 \lambda_2 \ldots \lambda_n \qquad (7.2.11)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$, which must form the diagonal elements of $T$. As a much more important application, we have the following well-known theorem.

***Theorem 7.4*** (Principal Axes Theorem) Let $A$ be a Hermitian matrix of order $n$, that is, $A^* = A$. Then $A$ has $n$ real eigenvalues $\lambda_1, \ldots, \lambda_n$, not necessarily distinct, and $n$ corresponding eigenvectors $u^{(1)}, \ldots, u^{(n)}$ that form an orthonormal basis for $\mathbf{C}^n$. If $A$ is real, the eigenvectors $u^{(1)}, \ldots, u^{(n)}$ can be taken as real, and they form an orthonormal basis of $\mathbf{R}^n$. Finally there is a unitary matrix $U$ for which

$$U^* A U = D \equiv \operatorname{diag}[\lambda_1, \ldots, \lambda_n] \qquad (7.2.12)$$

is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$. If $A$ is also real, then $U$ can be taken as orthogonal.

***Proof*** From Theorem 7.3, there is a unitary matrix $U$ with

$$U^* A U = T$$

with $T$ upper triangular. Form the conjugate transpose of both sides to obtain

$$T^* = (U^*AU)^* = U^*A^*(U^*)^* = U^*AU = T$$

Since $T^*$ is lower triangular, we must have

$$T = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$$

Also, $T^* = T$ involves complex conjugation of all elements of $T$, and thus all diagonal elements of $T$ must be real.

Write $U$ as

$$U = \left[u^{(1)}, \ldots, u^{(n)}\right]$$

Then $T = U^*AU$ implies $AU = UT$,

$$A\left[u^{(1)}, \ldots, u^{(n)}\right] = \left[u^{(1)}, \ldots, u^{(n)}\right]\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\left[Au^{(1)}, \ldots, Au^{(n)}\right] = \left[\lambda_1 u^{(1)}, \ldots, \lambda_n u^{(n)}\right]$$

and

$$Au^{(j)} = \lambda_j u^{(j)} \qquad j = 1, \ldots, n \tag{7.2.13}$$

Since the columns of $U$ are orthonormal, and since the dimension of $\mathbf{C}^n$ is $n$, these must form an orthonormal basis for $\mathbf{C}^n$. We omit the proof of the results that follow from $A$ being real. This completes the proof. ■

***Example*** From an earlier example in this section, the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$ and corresponding orthonormal eigenvectors

$$u^{(1)} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad u^{(2)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad u^{(3)} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

These form an orthonormal basis for $\mathbf{R}^3$ or $\mathbf{C}^3$.

There is a second canonical form that has recently become more important for problems in numerical linear algebra, especially for solving overdetermined systems of linear equations. These systems arise from the fitting of empirical data using the linear least squares procedures [see Golub and Van Loan (1983), chap. 6, and Lawson and Hanson (1974)].

***Theorem** 7.5* (Singular Value Decomposition) Let $A$ be order $n \times m$. Then there are unitary matrices $U$ and $V$, of orders $m$ and $n$, respectively, such that

$$V^*AU = F \tag{7.2.14}$$

is a "diagonal" rectangular matrix of order $n \times m$,

$$F = \begin{bmatrix} \mu_1 & & & 0 & & \\ & \mu_2 & & & & \\ & & \ddots & & & \\ & 0 & & \mu_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \tag{7.2.15}$$

The numbers $\mu_1, \ldots, \mu_r$ are called the *singular values* of $A$. They are all real and positive, and they can be arranged so that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > 0 \tag{7.2.16}$$

where $r$ is the rank of the matrix $A$.

***Proof*** Consider the square matrix $A^*A$ of order $m$. It is a Hermitian matrix, and consequently Theorem 7.4 can be applied to it. The eigenvalues of $A^*A$ are all real; moreover, they are all nonnegative. To see this, assume

$$A^*Ax = \lambda x \qquad x \neq 0.$$

Then

$$(x, A^*Ax) = (x, \lambda x) = \lambda \|x\|_2^2$$

$$(x, A^*Ax) = (Ax, Ax) = \|Ax\|_2^2$$

$$\lambda = \left( \frac{\|Ax\|_2}{\|x\|_2} \right)^2 \geq 0$$

This result also proves that

$$Ax = 0 \quad \text{if and only if} \quad A^*Ax = 0 \qquad x \in \mathbf{C}^n \tag{7.2.17}$$

From Theorem 7.3, there is an $m \times m$ unitary matrix $U$ such that

$$U^*A^*AU = \mathrm{diag}\,[\lambda_1, \ldots, \lambda_r, 0, \ldots, 0] \tag{7.2.18}$$

where all $\lambda_i \neq 0$, $1 \leq i \leq r$, and all are positive. Because $A^*A$ has order

$m$, the index $r \le m$. Introduce the *singular values*

$$\mu_i = \sqrt{\lambda_i} \qquad i = 1, \ldots, r \tag{7.2.19}$$

The $U$ can be chosen so that the ordering (7.2.16) is obtained. Using the diagonal matrix

$$D = \operatorname{diag}[\mu_1, \ldots, \mu_r, 0, \ldots, 0]$$

of order $m$, we can write (7.2.18) as

$$(AU)^*(AU) = D^2 \tag{7.2.20}$$

Let $W = AU$. Then (7.2.20) says $W^*W = D^2$. Writing $W$ as

$$W = \left[W^{(1)}, \ldots, W^{(m)}\right] \qquad W^{(j)} \in \mathbf{C}^n$$

we have

$$\left(W^{(j)}, W^{(j)}\right) = \begin{cases} \mu_j^2 & 1 \le j \le r \\ 0 & j > r \end{cases} \tag{7.2.21}$$

and

$$\left(W^{(i)}, W^{(j)}\right) = 0 \qquad \text{if } i \ne j \tag{7.2.22}$$

From (7.2.21), $W^{(j)} = 0$ if $j > r$. And from (7.2.22), the first $r$ columns of $W$ are orthogonal elements in $\mathbf{C}^n$. Thus the first $r$ columns are linearly independent, and this implies $r \le n$.

Define

$$V^{(j)} = \frac{1}{\mu_j} W^{(j)} \qquad j = 1, \ldots, r \tag{7.2.23}$$

This is an orthonormal set in $\mathbf{C}^n$. If $r < n$, then choose $V^{(r+1)}, \ldots, V^{(n)}$ so that $\{V^{(1)}, \ldots, V^{(n)}\}$ is an orthonormal basis for $\mathbf{C}^n$. Define

$$V = \left[V^{(1)}, \ldots, V^{(n)}\right] \tag{7.2.24}$$

Easily $V$ is an $n \times n$ unitary matrix, and it can be verified directly that $VF = W$, with $F$ as in (7.2.15). Thus

$$VF = AU$$

which proves (7.2.14). The proof that $r = \operatorname{rank}(A)$ and the derivation of other properties of the singular value decomposition are left to Problem 19. The singular value decomposition is used in Chapter 9, in the least squares solution of overdetermined linear systems. ■

To give the most basic canonical form, introduce the following notation. Define the $n \times n$ matrix

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda \end{bmatrix} \qquad n \geq 1 \tag{7.2.25}$$

where $J_n(\lambda)$ has the single eigenvalue $\lambda$, of algebraic multiplicity $n$ and geometric multiplicity 1. It is called a *Jordan block*.

***Theorem 7.6*** (Jordan Canonical Form) Let $A$ have order $n$. Then there is a nonsingular matrix $P$ for which

$$P^{-1}AP = \begin{bmatrix} J_{n_1}(\lambda_1) & & & 0 \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{n_r}(\lambda_r) \end{bmatrix} \tag{7.2.26}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ need not be distinct. For $A$ Hermitian, Theorem 7.4 implies we must have $n_1 = n_2 = \cdots = n_r = 1$, for in that case the sum of the geometric multiplicities must be $n$, the order of the matrix $A$.

It is often convenient to write (7.2.26) as

$$P^{-1}AP = D + N$$
$$D = \operatorname{diag}[\lambda_1, \ldots, \lambda_r] \tag{7.2.27}$$

with each $\lambda_i$ appearing $n_i$ times on the diagonal of $D$. The matrix $N$ has all zero entries, except for possible 1s on the superdiagonal. It is a *nilpotent* matrix, and more precisely, it satisfies

$$N^n = 0 \tag{7.2.28}$$

The Jordan form is not an easy theorem to prove, and the reader is referred to any of the large number of linear algebra texts for a development of this rich topic [e.g., see Franklin (1968), chap. 5; Halmos (1958), sec. 58; or Noble (1969), chap. 11].

## 7.3 Vector and Matrix Norms

The Euclidean norm $\|x\|_2$ has already been introduced, and it is the way in which most people are used to measuring the size of a vector. But there are many situations in which it is more convenient to measure the size of a vector in other ways. Thus we introduce a general concept of the *norm* of a vector.

***Definition*** Let $V$ be a vector space, and let $N(x)$ be a real valued function defined on $V$. Then $N(x)$ is a *norm* if:

**(N1)** $N(x) \geq 0$ for all $x \in V$, and $N(x) = 0$ if and only if $x = 0$.

**(N2)** $N(\alpha x) = |\alpha| N(x)$, for all $x \in V$ and all scalars $\alpha$.

**(N3)** $N(x + y) \leq N(x) + N(y)$, for all $x, y \in V$.

The usual notation is $\|x\| = N(x)$. The notation $N(x)$ is used to emphasize that the norm is a function, with domain $V$ and range the nonnegative real numbers. Define the distance from $x$ to $y$ as $\|x - y\|$. Simple consequences are the *triangular inequality* in its alternative form

$$\|x - z\| \leq \|x - y\| + \|y - z\|$$

and the *reverse triangle inequality*,

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \qquad x, y \in V \tag{7.3.1}$$

***Example* 1.** For $1 \leq p < \infty$, define the *p-norm*,

$$\|x\|_p = \left[\sum_1^n |x_j|^p\right]^{1/p} \qquad x \in \mathbf{C}^n \tag{7.3.2}$$

**2.** The *maximum norm* is

$$\|x\|_\infty = \operatorname*{Max}_{1 \leq j \leq n} |x_j| \qquad x \in \mathbf{C}^n \tag{7.3.3}$$

The use of the subscript $\infty$ on the norm is motivated by the result in Problem 23.

**3.** For the vector space $V = C[a, b]$, the function norms $\|f\|_2$ and $\|f\|_\infty$ were introduced in Chapters 4 and 1, respectively.

***Example*** Consider the vector $x = (1, 0, -1, 2)$. Then

$$\|x\|_1 = 4 \qquad \|x\|_2 = \sqrt{6} \qquad \|x\|_\infty = 2$$

To show that $\|\cdot\|_p$ is a norm for a general $p$ is nontrivial. The cases $p = 1$ and $\infty$ are straightforward, and $\|\cdot\|_2$ has been treated in Section 4.1. But for $1 < p < \infty$, $p \neq 2$, it is difficult to show that $\|\cdot\|_p$ satisfies the triangle inequality. This is not a significant problem for us since the main cases of interest are $p = 1, 2, \infty$. To give some geometrical intuition for these norms, the *unit circles*

$$S_p = \left\{x \in \mathbf{R}^2 \mid \|x\|_p = 1\right\} \qquad p = 1, 2, \infty \tag{7.3.4}$$

are sketched in Figure 7.2.

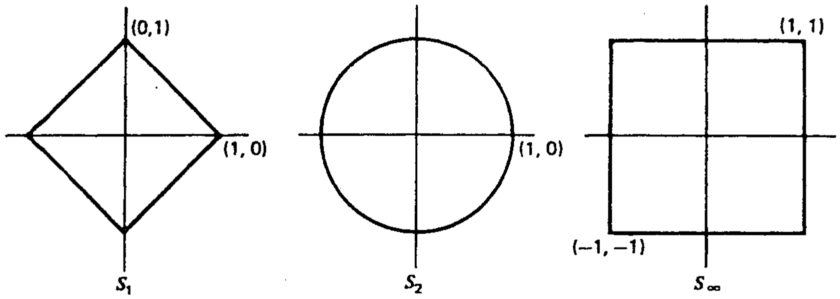


**Figure 7.2** The unit sphere $S_p$ using vector norm $\|\cdot\|_p$.

We now prove some results relating different norms. We begin with the following result on the continuity of $N(x) \equiv \|x\|$, as a function of $x$.

***Lemma*** Let $N(x)$ be a norm on $\mathbf{C}^n$ (or $\mathbf{R}^n$). Then $N(x)$ is a continuous function of the components $x_1, x_2, \ldots, x_n$ of $x$.

***Proof*** We want to show that

$$x_i \doteq y_i \qquad i = 1, 2, \ldots, n$$

implies

$$N(x) \doteq N(y)$$

Using the reverse triangle inequality (7.3.1),

$$|N(x) - N(y)| \le N(x - y) \qquad x, y \in \mathbf{C}^n$$

Recall from (7.1.1) the definition of the standard basis $\{e^{(1)}, \ldots, e^{(n)}\}$ for $\mathbf{C}^n$. Then

$$x - y = \sum_{j=1}^{n} (x_j - y_j) e^{(j)}$$

$$N(x - y) \le \sum_{j=1}^{n} |x_j - y_j| N(e^{(j)}) \le \|x - y\|_\infty \sum_{j=1}^{n} N(e^{(j)})$$

$$|N(x) - N(y)| \le c\|x - y\|_\infty \qquad c = \sum_{j=1}^{n} N(e^{(j)}) \tag{7.3.5}$$

This completes the proof. ■

Note that it also proves that for every vector norm $N$ on $\mathbf{C}^n$, there is a $c > 0$ with

$$N(x) \le c\|x\|_\infty \qquad \text{all } x \in \mathbf{C}^n \tag{7.3.6}$$

Just let $y = 0$ in (7.3.5). The following theorem proves the converse of this result.

***Theorem*** *7.7* (Equivalence of Norms) Let $N$ and $M$ be norms on $V = \mathbf{C}^n$ or $\mathbf{R}^n$. Then there are constants $c_1, c_2 > 0$ for which

$$c_1 M(x) \le N(x) \le c_2 M(x) \qquad \text{all } x \in V \tag{7.3.7}$$

***Proof*** It is sufficient to consider the case in which $N$ is arbitrary and $M(x) = \|x\|_\infty$. Combining two such statements then leads to the general result. Thus we wish to show there are constants $c_1, c_2$ for which

$$c_1 \|x\|_\infty \le N(x) \le c_2 \|x\|_\infty \tag{7.3.8}$$

or equivalently,

$$c_1 \le N(z) \le c_2 \qquad \text{all } z \in S \tag{7.3.9}$$

in which $S$ is the set of all points $z$ in $\mathbf{C}^n$ for which $\|z\|_\infty = 1$. The upper inequality of (7.3.9) follows immediately from (7.3.6).

Note that $S$ is a closed and bounded set in $\mathbf{C}^n$, and $N$ is a continuous function on $S$. It is then a standard result of advanced calculus that $N$ attains its maximum and minimum on $S$ at points of $S$, that is, there are constants $c_1, c_2$ and points $z_1, z_2$ in $S$ for which

$$c_1 = N(z_1) \le N(z) \le N(z_2) = c_2 \quad \text{all } z \in S$$

Clearly, $c_1, c_2 \ge 0$. And if $c_1 = 0$, then $N(z_1) = 0$. But then $z_1 = 0$, contrary to the construction of $S$ that requires $\|z_1\|_\infty = 1$. This proves (7.3.9), completing the proof of the theorem. *Note:* This theorem does not generalize to infinite dimensional spaces. ■

Many numerical methods for problems involving linear systems produce a sequence of vectors $\{x^{(m)} | m \ge 0\}$, and we want to speak of convergence of this sequence to a vector $x$.

***Definition*** A sequence of vectors $\{x^{(1)}, x^{(2)}, \ldots, x^{(m)}, \cdots\}$, in $\mathbf{C}^n$ or $\mathbf{R}^n$ is said to *converge* to a vector $x$ if and only if

$$\|x - x^{(m)}\| \to 0 \qquad \text{as} \quad m \to \infty$$

Note that the choice of norm is left unspecified. For finite dimensional spaces, it doesn't matter which norm is used. Let $M$ and $N$ be two norms on $\mathbf{C}^n$. Then from (7.3.7),

$$c_1 M(x - x^{(m)}) \le N(x - x^{(m)}) \le c_2 M(x - x^{(m)}) \qquad m \ge 0$$

and $M(x - x^{(m)})$ converges to zero if and only if $N(x - x^{(m)})$ does the same. Thus $x^{(m)} \to x$ with the $M$ norm if and only if it converges with the $N$ norm. This is an important result, and it is not true for infinite dimensional spaces.

**Matrix norms** The set of all $n \times n$ matrices with complex entries can be considered as equivalent to the vector space $\mathbf{C}^{n^2}$, with a special multiplicative operation added onto the vector space. Thus a matrix norm should satisfy the usual three requirements N1–N3 of a vector norm. In addition, we also require two other conditions.

***Definition*** A *matrix norm* satisfies N1–N3 and the following:

**(N4)** $\|AB\| \le \|A\|\,\|B\|$.

**(N5)** Usually the vector space we will be working with, $V = \mathbf{C}^n$ or $\mathbf{R}^n$, will have some vector norm, call it $\|x\|_v$, $x \in V$. We require that the matrix and vector norms be *compatible*:

$$\|Ax\|_v \le \|A\|\,\|x\|_v \qquad \text{all } x \in V \qquad \text{all } A$$

***Example*** Let $A$ be $n \times n$, $\|\cdot\|_v = \|\cdot\|_2$. Then for $x \in \mathbf{C}^n$,

$$\|Ax\|_2 = \left[\sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}x_j\right|^2\right]^{1/2}$$

$$\le \left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}|a_{ij}|^2\right\}\left\{\sum_{j=1}^{n}|x_j|^2\right\}\right]^{1/2}$$

by using the Cauchy–Schwartz inequality (7.1.8). Then

$$\|Ax\|_2 \le F(A)\|x\|_2 \qquad F(A) = \left[\sum_{i,j=1}^{n}|a_{ij}|^2\right]^{1/2} \tag{7.3.10}$$

$F(A)$ is called the *Frobenius norm* of $A$. Property N5 is shown using (7.3.10) directly. Properties N1–N3 are satisfied since $F(A)$ is just the Euclidean norm on $\mathbf{C}^{n^2}$. It remains to show N4. Using the Cauchy–Schwartz inequality,

$$F(AB) = \left[\sum_{i,j=1}^{n}\left|\sum_{k=1}^{n} a_{ik}b_{kj}\right|^2\right]^{1/2}$$

$$\le \left[\sum_{i,j=1}^{n}\left\{\sum_{k=1}^{n}|a_{ik}|^2\right\}\left\{\sum_{k=1}^{n}|b_{kj}|^2\right\}\right]^{1/2}$$

$$= F(A)F(B)$$

Thus $F(A)$ is a matrix norm, compatible with the Euclidean norm.

Usually when given a vector space with a norm $\|\cdot\|_v$, an associated matrix norm is defined by

$$\|A\| = \underset{x \ne 0}{\text{Supremum}}\ \frac{\|Ax\|_v}{\|x\|_v} \tag{7.3.11}$$

**Table 7.1 Vector norms and associated operator matrix norms**

| Vector Norm | Matrix Norm |
|---|---|
| $\|x\|_1 = \sum_{i=1}^{n} \|x_i\|$ | $\|A\|_1 = \underset{1 \le j \le n}{\text{Max}} \sum_{i=1}^{n} \|a_{ij}\|$ |
| $\|x\|_2 = \left[ \sum_{j=1}^{n} \|x_j\|^2 \right]^{1/2}$ | $\|A\|_2 = \sqrt{r_\sigma(A^*A)}$ |
| $\|x\|_\infty = \underset{1 \le i \le n}{\text{Max}} \|x_i\|$ | $\|A\|_\infty = \underset{1 \le i \le n}{\text{Max}} \sum_{j=1}^{n} \|a_{ij}\|$ |

It is often called the *operator norm*. By its definition, it satisfies N5:

$$\|Ax\|_v \le \|A\| \, \|x\|_v \qquad x \in \mathbf{C}^n \tag{7.3.12}$$

For a matrix $A$, the operator norm induced by the vector norm $\|x\|_p$ will be denoted by $\|A\|_p$. The most important cases are given in Table 7.1, and the derivations are given later. We need the following definition in order to define $\|A\|_2$.

***Definition*** Let $A$ be an arbitrary matrix. The *spectrum* of $A$ is the set of all eigenvalues of $A$, and it is denoted by $\sigma(A)$. The *spectral radius* is the maximum size of these eigenvalues, and it is denoted by

$$r_\sigma(A) = \underset{\lambda \in \sigma(A)}{\text{Max}} |\lambda| \tag{7.3.13}$$

To show (7.3.11) is a norm in general, we begin by showing it is finite. Recall from Theorem 7.7 that there are constants $c_1, c_2 > 0$ with

$$c_1\|x\|_2 \le \|x\|_v \le c_2\|x\|_2 \qquad x \in \mathbf{C}^n$$

Thus,

$$\frac{\|Ax\|_v}{\|x\|_v} \le \frac{c_2\|Ax\|_2}{c_1\|x\|_2} \le \frac{c_2}{c_1} F(A)$$

which proves $\|A\|$ is finite.

At this point it is interesting to note the geometric significance of $\|A\|$:

$$\|A\| = \underset{x \ne 0}{\text{Supremum}} \frac{\|Ax\|_v}{\|x\|_v} = \underset{x \ne 0}{\text{Supremum}} \left\| A\left(\frac{x}{\|x\|_v}\right) \right\|_v = \underset{\|z\|_v = 1}{\text{Supremum}} \|Az\|_v$$

By noting that the supremum doesn't change if we let $\|z\|_v \le 1$,

$$\|A\| = \underset{\|z\|_v \le 1}{\text{Supremum}} \|Az\| \tag{7.3.14}$$

Let

$$B = \{z \in \mathbf{C}^n \mid \|z\|_v \le 1\}$$

the unit ball with respect to $\|\cdot\|_v$. Then

$$\|A\| = \underset{z \in B}{\text{Supremum}} \|Az\|_v = \underset{w \in A(B)}{\text{Supremum}} \|w\|_v$$

with $A(B)$ the image of $B$ when $A$ is applied to it. Thus $\|A\|$ measures the effect of $A$ on the unit ball, and if $\|A\| > 1$, then $\|A\|$ denotes the maximum stretching of the ball $B$ under the action of $A$.

***Proof*** Following is a proof that the operator norm $\|A\|$ is a matrix norm.

1. Clearly $\|A\| \ge 0$, and if $A = 0$, then $\|A\| = 0$. Conversely, if $\|A\| = 0$, then $\|Ax\|_v = 0$ for all $x$. Thus $Ax = 0$ for all $x$, and this implies $A = 0$.

2. Let $\alpha$ be any scalar. Then

$$\|\alpha A\| = \underset{\|x\|_v \le 1}{\text{Supremum}} \|\alpha A x\|_v = \underset{\|x\|_v \le 1}{\text{Supremum}} |\alpha| \|Ax\|_v = |\alpha| \underset{\|x\|_v \le 1}{\text{Supremum}} \|Ax\|_v$$

$$= |\alpha| \, \|A\|$$

3. For any $x \in \mathbf{C}^n$,

$$\|(A + B)x\|_v = \|Ax + Bx\|_v \le \|Ax\|_v + \|Bx\|_v$$

since $\|\cdot\|_v$ is a norm. Using the property (7.3.12),

$$\|(A + B)x\|_v \le \|A\| \, \|x\|_v + \|B\| \, \|x\|_v$$

$$\frac{\|(A + B)x\|_v}{\|x\|_v} \le \|A\| + \|B\|$$

This implies

$$\|A + B\| \le \|A\| + \|B\|$$

4. For any $x \in \mathbf{C}^n$, use (7.3.12) to get

$$\|(AB)x\|_v = \|A(Bx)\|_v \le \|A\| \, \|Bx\|_v \le \|A\| \, \|B\| \, \|x\|_v$$

$$\frac{\|ABx\|_v}{\|x\|_v} \le \|A\| \, \|B\|$$

This implies

$$\|AB\| \le \|A\| \, \|B\| \qquad \blacksquare$$

We now comment more extensively on the results given in Table 7.1.

***Example* 1.** Use the vector norm

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \qquad x \in \mathbf{C}^n$$

Then

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \le \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|\,|x_j|$$

Changing the order of summation, we can separate the summands,

$$\|Ax\|_1 \le \sum_{j=1}^{n} |x_j| \sum_{i=1}^{n} |a_{ij}|$$

Let

$$c = \underset{1 \le j \le n}{\text{Max}} \sum_{i=1}^{n} |a_{ij}| \tag{7.3.15}$$

Then

$$\|Ax\|_1 \le c\|x\|_1$$

and thus

$$\|A\|_1 \le c$$

To show this is an equality, we demonstrate an $x$ for which

$$\frac{\|Ax\|_1}{\|x\|_1} = c$$

Let $k$ be the column index for which the maximum in (7.3.15) is attained. Let $x = e^{(k)}$, the $k$th unit vector. Then $\|x\|_1 = 1$ and

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| = \sum_{i=1}^{n} |a_{ik}| = c$$

This proves that for the vector norm $\|\cdot\|_1$, the operator norm is

$$\|A\|_1 = \underset{1 \le j \le n}{\text{Max}} \sum_{i=1}^{n} |a_{ij}| \tag{7.3.16}$$

This is often called the *column norm*.

**2.** For $\mathbf{C}^n$ with the norm $\|x\|_\infty$, the operator norm is

$$\|A\|_\infty = \underset{1 \le i \le n}{\text{Max}} \sum_{j=1}^{n} |a_{ij}| \tag{7.3.17}$$

This is called the *row norm* of $A$. The proof of the formula is left as Problem 25, although it is similar to that for $\|A\|_1$.

3. Use the norm $\|x\|_2$ on $\mathbf{C}^n$. From (7.3.10), we conclude that

$$\|A\|_2 \leq F(A) \tag{7.3.18}$$

In general, these are not equal. For example, with $A = I$, the identity matrix, use (7.3.10) and (7.3.11) to obtain

$$F(I) = \sqrt{n} \qquad \|I\|_2 = 1$$

We prove

$$\|A\|_2 = \sqrt{r_\sigma(A^*A)} \tag{7.3.19}$$

as stated earlier in Table 7.1. The matrix $A^*A$ is Hermitian and all of its eigenvalues are nonnegative, as shown in the proof of Theorem 7.5. Let it have the eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

counted according to their multiplicity, and let $u^{(1)}, \ldots, u^{(n)}$ be the corresponding eigenvectors, arranged as an orthonormal basis for $\mathbf{C}^n$.

For a general $x \in \mathbf{C}^n$,

$$\|Ax\|_2^2 = (Ax, Ax) = (x, A^*Ax)$$

Write $x$ as

$$x = \sum_{j=1}^{n} \alpha_j u^{(j)} \qquad \alpha_j \equiv (x, u^{(j)}) \tag{7.3.20}$$

Then

$$A^*Ax = \sum_{j=1}^{n} \alpha_j A^*Au^{(j)} = \sum_{j=1}^{n} \alpha_j \lambda_j u^{(j)}$$

and

$$\|Ax\|_2^2 = \left( \sum_{i=1}^{n} \alpha_i u^{(i)}, \sum_{j=1}^{n} \alpha_j \lambda_j u^{(j)} \right) = \sum_{j=1}^{n} \lambda_j |\alpha_j|^2$$

$$\leq \lambda_1 \sum_{j=1}^{n} |\alpha_j|^2 = \lambda_1 \|x\|_2^2$$

using (7.3.20) to calculate $\|x\|_2$. Thus

$$\|A\|_2^2 \leq \lambda_1$$

Equality follows by noting that if $x = u^{(1)}$, then $\|x\|_2 = 1$ and

$$\|Ax\|_2^2 = (x, A^*Ax) = \left(u^{(1)}, \lambda_1 u^{(1)}\right) = \lambda_1$$

This proves (7.3.19), since $\lambda_1 = r_\sigma(A^*A)$. It can be shown that $AA^*$ and $A^*A$ have the same nonzero eigenvalues (see Problem 19); thus, $r_\sigma(AA^*) = r_\sigma(A^*A)$, an alternative formula for (7.3.19). It also proves

$$\|A\|_2 = \|A^*\|_2 \tag{7.3.21}$$

This is not true for the previous matrix norms.

It can be shown fairly easily that if $A$ is Hermitian, then

$$\|A\|_2 = r_\sigma(A) \tag{7.3.22}$$

This is left as Problem 27.

***Example*** Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

Then.

$$\|A\|_1 = 6 \qquad \|A\|_2 = \sqrt{15 + \sqrt{221}} \doteq 5.46 \qquad \|A\|_\infty = 7$$

As an illustration of the inequality (7.3.23) of the following theorem,

$$r_\sigma(A) = \frac{5 + \sqrt{33}}{2} \doteq 5.37 < \|A\|_2$$

***Theorem 7.8*** Let $A$ be an arbitrary square matrix. Then for any operator matrix norm,

$$r_\sigma(A) \le \|A\| \tag{7.3.23}$$

Moreover, if $\epsilon > 0$ is given, then there is an operator matrix norm, denoted here by $\|\cdot\|_\epsilon$, for which

$$\|A\|_\epsilon \le r_\sigma(A) + \epsilon \tag{7.3.24}$$

***Proof*** To prove (7.3.23), let $\|\cdot\|$ be any matrix norm with an associated compatible vector norm $\|\cdot\|_v$. Let $\lambda$ be the eigenvalue in $\sigma(A)$ for which

$$|\lambda| = r_\sigma(A)$$

and let $x$ be an associated eigenvector, $\|x\|_v = 1$. Then

$$r_\sigma(A) = |\lambda| = \|\lambda x\|_v \le \|Ax\|_v \le \|A\| \, \|x\|_v = \|A\|$$

which proves (7.3.23).

The proof of (7.3.24) is a nontrivial construction, and a proof is given in Isaacson and Keller (1966, p. 12). ■

The following corollary is an easy, but important, consequence of Theorem 7.8.

***Corollary*** For a square matrix $A$, $r_\sigma(A) < 1$ if and only if $\|A\| < 1$ for some operator matrix norm. ■

This result can be used to prove Theorem 7.9 in the next section, but we prefer to use the Jordan canonical form, given in Theorem 7.6. The results (7.3.22) and Theorem 7.8 show that $r_\sigma(A)$ is almost a matrix norm, and this result is used in analyzing the rates of convergence for some of the iteration methods given in Chapter 8 for solving linear systems of equations.

## 7.4 Convergence and Perturbation Theorems

The following results are the theoretical framework from which we later construct error analyses for numerical methods for linear systems of equations.

***Theorem 7.9*** Let $A$ be a square matrix of order $n$. Then $A^m$ converges to the zero matrix as $m \to \infty$ if and only if $r_\sigma(A) < 1$.

***Proof*** We use Theorem 7.6 as a fundamental tool. Let $J$ be the Jordan canonical form for $A$,

$$P^{-1}AP = J$$

Then

$$A^m = (PJP^{-1})^m = PJ^mP^{-1} \tag{7.4.1}$$

and $A^m \to 0$ if and only if $J^m \to 0$. Recall from (7.2.27) and (7.2.28) that $J$ can be written as

$$J = D + N$$

in which

$$D = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$$

contains the eigenvalues of $J$ (and $A$), and $N$ is a matrix for which

$$N^n = 0$$

By examining the structure of $D$ and $N$, we have $DN = ND$. Then

$$J^m = (D+N)^m = \sum_{j=0}^{m} \binom{m}{j} D^{m-j}N^j$$

and using $N^j = 0$ for $j \geq n$,

$$J^m = \sum_{j=0}^{n} \binom{m}{j} D^{m-j} N^j \tag{7.4.2}$$

Notice that the powers of $D$ satisfy

$$m - j \geq m - n \to \infty \qquad \text{as} \quad m \to \infty \tag{7.4.3}$$

We need the following limits: For any positive $c < 1$ and any $r \geq 0$,

$$\underset{m \to \infty}{\text{Limit}}\, m^r c^m = 0 \tag{7.4.4}$$

This can be proved using L'Hospital's rule from elementary calculus.

In (7.4.2), there are a fixed number of terms, $n + 1$, regardless of the size of $m$, and we can consider the convergence of $J^m$ by considering each of the individual terms. Assuming $r_\sigma(A) < 1$, we know that all $|\lambda_i| < 1$, $i = 1, \ldots, n$. And for any matrix norm

$$\left\| \binom{m}{j} D^{m-j} N^j \right\| \leq \frac{m^j}{j!} \|N\|^j \|D^{m-j}\|$$

Using the row norm, we have that the preceding is bounded by

$$\frac{1}{j!} \|N\|_\infty^j m^j \left[r_\sigma(A)\right]^{m-j}$$

which converges to zero as $m \to \infty$, using (7.4.3) and (7.4.4), for $0 \leq j \leq n$. This proves half of the theorem, namely that if $r_\sigma(A) < 1$, then $J^m$ and $A^m$, from (7.4.1), converge to zero as $m \to \infty$.

Suppose that $r_\sigma(A) \geq 1$. Then let $\lambda$ be an eigenvalue of $A$ for which $|\lambda| \geq 1$, and let $x$ be an associated eigenvector, $x \neq 0$. Then

$$A^m x = \lambda^m x$$

and clearly this does not converge to zero as $m \to \infty$. Thus it is not possible that $A^m \to 0$, as that would imply $A^m x \to 0$. This completes the proof. ∎

***Theorem 7.10*** (Geometric Series) Let $A$ be a square matrix. If $r_\sigma(A) < 1$, then $(I - A)^{-1}$ exists, and it can be expressed as a convergent series,

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^m + \cdots \tag{7.4.5}$$

Conversely, if the series in (7.4.5) is convergent, then $r_\sigma(A) < 1$.

***Proof*** Assume $r_\sigma(A) < 1$. We show the existence of $(I - A)^{-1}$ by proving the equivalent statement (3) of Theorem 7.2. Assume

$$(I - A)x = 0$$

Then $Ax = x$, and this implies that 1 is an eigenvalue of $A$ if $x \neq 0$. But we assumed $r_\sigma(A) < 1$, and thus we must have $x = 0$, concluding the proof of the existence of $(I - A)^{-1}$.

We need the following identity:

$$(I - A)(I + A + A^2 + \cdots + A^m) = I - A^{m+1} \tag{7.4.6}$$

which is true for any matrix $A$. Multiplying by $(I - A)^{-1}$,

$$I + A + A^2 + \cdots + A^m = (I - A)^{-1}(I - A^{m+1})$$

The left-hand side has a limit if the right-hand side does. By Theorem 7.9, $r_\sigma(A) < 1$ implies that $A^{m+1} \to 0$ as $m \to \infty$. Thus we have the result (7.4.5).

Conversely, assume the series converges and denote it by

$$B = I + A + A^2 + \cdots + A^m + \cdots$$

Then $B - AB = B - BA = I$, and thus $I - A$ has an inverse, namely $B$. Taking limits on both sides of (7.4.6), the left-hand side has the limit $(I - A)B = I$, and thus the same must be true of the right-hand limit. But that implies

$$A^{m+1} \to 0 \qquad \text{as} \quad m \to \infty$$

By Theorem 7.9, we must have $r_\sigma(A) < 1$. ■

***Theorem 7.11*** Let $A$ be a square matrix. If for some operator matrix norm, $\|A\| < 1$, then $(I - A)^{-1}$ exists and has the geometric series expansion (7.4.5). Moreover,

$$\left\|(I - A)^{-1}\right\| \le \frac{1}{1 - \|A\|} \tag{7.4.7}$$

***Proof*** Since $\|A\| < 1$, it follows from (7.3.23) of Theorem 7.8 that $r_\sigma(A) < 1$. Except for (7.4.7), the other conclusions follow from Theorem 7.10. For (7.4.7), let

$$B_m = I + A + \cdots + A^m$$

From (7.4.6),

$$B_m = (I - A)^{-1}(I - A^{m+1})$$

$$(I - A)^{-1} - B_m = (I - A)^{-1}[I - (I - A^{m+1})]$$

$$= (I - A)^{-1}A^{m+1} \tag{7.4.8}$$

Using the reverse triangle inequality,

$$\left| \|(I-A)^{-1}\| - \|B_m\| \right| \le \|(I-A)^{-1} - B_m\|$$
$$\le \|(I-A)^{-1}\| \, \|A\|^{m+1}$$

Since this converges to zero as $m \to \infty$, we have

$$\|B_m\| \to \|(I-A)^{-1}\| \quad \text{as} \quad m \to \infty \tag{7.4.9}$$

From the definition of $B_m$ and the properties of a matrix norm,

$$\|B_m\| \le \|I\| + \|A\| + \|A\|^2 + \cdots + \|A\|^m$$
$$= \frac{1 - \|A\|^{m+1}}{1 - \|A\|} \le \frac{1}{1 - \|A\|}$$

Combined with (7.4.9), this concludes the proof of (7.4.7). ■

***Theorem 7.12*** Let $A$ and $B$ be square matrices of the same order. Assume $A$ is nonsingular and suppose that

$$\|A - B\| < \frac{1}{\|A^{-1}\|} \tag{7.4.10}$$

Then $B$ is also nonsingular,

$$\|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \, \|A - B\|} \tag{7.4.11}$$

and

$$\|A^{-1} - B^{-1}\| \le \frac{\|A^{-1}\|^2 \|A - B\|}{1 - \|A^{-1}\| \, \|A - B\|} \tag{7.4.12}$$

***Proof*** Note the identity

$$B = A - (A - B) = A\left[I - A^{-1}(A - B)\right] \tag{7.4.13}$$

The matrix $[I - A^{-1}(A - B)]$ is nonsingular using Theorem 7.11, based on the inequality (7.4.10), which implies

$$\left\| A^{-1}(A - B) \right\| \le \|A^{-1}\| \, \|A - B\| < 1$$

Since $B$ is the product of nonsingular matrices, it too is nonsingular,

$$B^{-1} = \left[I - A^{-1}(A - B)\right]^{-1} A^{-1}$$

The bound (7.4.11) follows by taking norms and applying Theorem 7.11. To prove (7.4.12), use

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

Take norms again and apply (7.4.11). ■

This theorem is important in a number of ways. But for the moment, it says that all sufficiently close perturbations of a nonsingular matrix are nonsingular.

***Example*** We illustrate Theorem 7.11 by considering the invertibility of the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & & \\ 0 & 1 & 4 & 1 & & \vdots \\ \vdots & & & \ddots & & \\ & & & 1 & 4 & 1 \\ 0 & \cdots & & 0 & 1 & 4 \end{bmatrix}$$

Rewrite $A$ as

$$A = 4(I + B),$$

$$B = \begin{bmatrix} 0 & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & & 0 \\ 0 & & & \ddots & & \vdots \\ \vdots & & & & & \frac{1}{4} \\ 0 & \cdots & & & \frac{1}{4} & 0 \end{bmatrix}$$

Using the row norm (7.3.17), $\|B\|_\infty = \frac{1}{2}$. Thus $(I + B)^{-1}$ exists from Theorem 7.11, and from (7.4.7),

$$\left\|(I + B)^{-1}\right\|_\infty \le \frac{1}{1 - \frac{1}{2}} = 2$$

Thus $A^{-1}$ exists, $A^{-1} = \frac{1}{4}(I + B)^{-1}$, and

$$\|A^{-1}\|_\infty \le \tfrac{1}{2}$$

By use of the row norm and inequality (7.3.23),

$$r_\sigma(A) \le 6 \qquad r_\sigma(A^{-1}) \le \tfrac{1}{2}$$

Since the eigenvalues of $A^{-1}$ are the reciprocals of those of $A$ (see problem 27), and since all eigenvalues of $A$ are real because $A$ is Hermitian, we have the bound

$$2 \le |\lambda| \le 6 \qquad \text{all } \lambda \in \sigma(A)$$

For better bounds in this case, see the Gerschgorin Circle Theorem of Chapter 9.

## Discussion of the Literature

The subject of this chapter is linear algebra, especially selected for use in deriving and analyzing methods of numerical linear algebra. The books by Anton (1984) and Strang (1980) are introductory-level texts for undergraduate linear algebra. Franklin's (1968) is a higher level introduction to matrix theory, and Halmos's (1958) is a well-known text on abstract linear algebra. Noble's (1969) is a wide-ranging applied linear algebra text. Introductions to the foundations are also contained in Fadeeva (1959), Golub and Van Loan (1982), Parlett (1980), Stewart (1973), and Wilkinson (1965), all of which are devoted entirely to numerical linear algebra. For additional theory at a more detailed and higher level, see the classical accounts of Gantmacher (1960) and Householder (1965). Additional references are given in the bibliographies of Chapters 8 and 9.

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## Problems

**1.** Determine whether the following sets of vectors are dependent or independent.

(a) $(1, 2, -1, 3), (3, -1, 1, 1), (1, 9, -5, 11)$

(b) $(1, 1, 0), (0, 1, 1), (1, 0, 1)$

**2.** Let $A$, $B$, and $C$ be matrices of order $m \times n$, $n \times p$, and $p \times q$, respectively.

(a) Prove the associative law $(AB)C = A(BC)$.

(b) Prove $(AB)^T = B^T A^T$.

**3.** (a) Produce square matrices $A$ and $B$ for which $AB \neq BA$.

(b) Produce square matrices $A$ and $B$, with no zero entries, for which $AB = 0$, $BA \neq 0$.

**4.** Let $A$ be a matrix of order $m \times n$, and let $r$ and $c$ denote the row and column rank of $A$, respectively. Prove that $r = c$. *Hint:* For convenience, assume that the first $r$ rows of $A$ are independent, with the remaining rows dependent on these first $r$ rows, and assume the same for the first $c$ columns of $A$. Let $\hat{A}$ denote the $r \times n$ matrix obtained by deleting the last $m - r$ rows of $A$, and let $\hat{r}$ and $\hat{c}$ denote the row and column rank of $\hat{A}$, respectively. Clearly $\hat{r} = r$. Also, the columns of $\hat{A}$ are elements of $\mathbf{C}^r$, which has dimension $r$, and thus we must have $\hat{c} \leq r$. Show that $\hat{c} = c$, thus proving that $c \leq r$. The reverse inequality will follow by applying the same argument to $A^T$, and taken together, these two inequalities imply $r = c$.

**5.** Prove the equivalence of statements (1)–(4) and (6) of Theorem 7.2. *Hint:* Use Theorem 7.1, the result in Problem 4, and the decomposition (7.1.6).

**6.** Let

$$f_n(x) = \det \begin{bmatrix} x & 1 & 0 & \cdots & & 0 \\ 1 & x & 1 & 0 & & 0 \\ 0 & 1 & x & 1 & & \vdots \\ \vdots & & & \ddots & & 0 \\ & & & & & 1 \\ 0 & \cdots & & 0 & 1 & x \end{bmatrix}$$

with the matrix of order $n$. Also define $f_0(x) \equiv 1$.

**(a)** Show

$$f_{n+1}(x) = xf_n(x) - f_{n-1}(x) \qquad n \geq 1$$

**(b)** Show

$$f_n(x) = S_n\left(\frac{x}{2}\right) \qquad n \geq 0$$

with $S_n(x)$ the Chebyshev polynomial of the second kind of degree $n$ (see Problem 24 in Chapter 4).

**7.** Let $A$ be a square matrix of order $n$ with real entries. The function

$$q(x_1, \ldots, x_n) = (Ax, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \qquad x \in \mathbf{R}^n,$$

is called the *quadratic form* determined by $A$. It is a quadratic polynomial in the $n$ variables $x_1, \ldots, x_n$, and it occurs when considering the maximization or minimization of a function of $n$ variables.

**(a)** Prove that if $A$ is skew-symmetric, then $q(x) \equiv 0$.

**(b)** For a general square matrix $A$, define $A_1 = \frac{1}{2}(A + A^T)$, $A_2 = \frac{1}{2}(A - A^T)$. Then $A = A_1 + A_2$. Show that $A_1$ is symmetric, and $(Ax, x) = (A_1 x, x)$, all $x \in \mathbf{R}^n$. This shows that the coefficient matrix $A$ for a quadratic form can always be assumed to be symmetric, without any loss of generality.

**8.** Given the orthogonal vectors

$$u^{(1)} = (1, 2, -1) \qquad u^{(1)} = (1, 1, 3)$$

produce a third vector $u^{(3)}$ such that $\{u^{(1)}, u^{(2)}, u^{(3)}\}$ is an orthogonal basis for $\mathbf{R}^3$. Normalize these vectors to obtain an orthonormal basis.

**9.** For the column vector $w \in \mathbf{C}^n$ with $\|w\|_2 = \sqrt{w^*w} = 1$, define the $n \times n$ matrix

$$A = I - 2ww^*$$

**(a)** For the special case $w = [\frac{1}{3}, \frac{2}{3}, \frac{2}{3}]^T$, produce the matrix $A$. Verify that it is symmetric and orthogonal.

**(b)** Show that, in general, all such matrices $A$ are Hermitian and unitary.

**10.** Let $W$ be a subspace of $\mathbf{R}^n$. For $x \in \mathbf{R}^n$, define

$$\rho(x) = \underset{y \in W}{\text{Infimum}} \|x - y\|_2$$

Let $\{u_1, \ldots, u_m\}$ be an orthonormal basis of $W$, where $m$ is the dimension of $W$. Extend this to an orthonormal basis $\{u_1, \ldots, u_m, \ldots, u_n\}$ of all of $\mathbf{R}^n$.

**(a)** Show that

$$\rho(x) = \left[ \sum_{j=m+1}^{n} |(x, u_j)|^2 \right]^{1/2}$$

and that it is uniquely attained at

$$y = Px \qquad P = \sum_{j=1}^{m} u_j u_j^T$$

This is called the *orthogonal projection* of $x$ onto $W$.

**(b)** Show $P^2 = P$. Such a matrix is called a *projection matrix.*

**(c)** Show $P^T = P$.

**(d)** Show $(Px, z - Pz) = 0$ for all $x, z \in \mathbf{R}^n$.

**(e)** Show $\|x\|_2^2 = \|Px\|_2^2 + \|x - Px\|_2^2$, for all $x \in \mathbf{R}^n$. This is a version of the *theorem of Pythagoras.*

**11.** Calculate the eigenvalues and eigenvectors of the following matrices.

**(a)** $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$  **(b)** $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

**12.** Let $y \neq 0$ in $\mathbf{R}^n$, and define $A = yy^T$, an $n \times n$ matrix. Show that $\lambda = 0$ is an eigenvalue of multiplicity exactly $n - 1$. What is the single nonzero eigenvalue?

**13.** Let $U$ be an $n \times n$ unitary matrix.

(a) Show $\|Ux\|_2 = \|x\|_2$, all $x \in \mathbf{C}^n$. Use this to prove that the distance between points $x$ and $y$ is the same as the distance between $Ux$ and $Uy$, showing that unitary transformations of $\mathbf{C}^n$ preserve distances between all points.

(b) Let $U$ be orthogonal, and show that

$$(Ux, Uy) = (x, y) \qquad x, y \in \mathbf{R}^n$$

This shows that orthogonal transformations of $\mathbf{R}^n$ also preserve angles between lines, as defined in (7.1.12).

(c) Show that all eigenvalues of a unitary matrix have magnitude one.

**14.** Let $A$ be a Hermitian matrix of order $n$. It is called *positive definite* if and only if $(Ax, x) > 0$ for all $x \neq 0$ in $\mathbf{C}^n$. Show that $A$ is positive definite if and only if all of its eigenvalues are real and positive. *Hint:* Use Theorem 7.4, and expand $(Ax, x)$ by using an eigenvector basis to express an arbitrary $x \in \mathbf{C}^n$.

**15.** Let $A$ be real and symmetric, and denote its eigenvalues by $\lambda_1, \ldots, \lambda_n$, repeated according to their multiplicity. Using a basis of orthonormal eigenvectors, show that the quadratic form of Problem 7, $q(x) = (Ax, x)$, $x \in \mathbf{R}^n$, can be reduced to the simpler form

$$q(x) = \sum_{j=1}^{n} \alpha_j^2 \lambda_j$$

with the $\{\alpha_j\}$ determined from $x$. Using this, explore the possible graphs for

$$(Ax, x) = \text{constant}$$

when $A$ is of order 3.

**16.** Assume $A$ is real, symmetric, positive definite, and of order $n$. Define

$$f(x) = \frac{1}{2} x^T A x - b^T x \qquad x, b \in \mathbf{R}^n$$

Show that the unique minimum of $f(x)$ is given by solving $Ax = b$ for $x = A^{-1} b$.

**17.** Let $f(x)$ be a real valued function of $x \in \mathbf{R}^n$, and assume $f(x)$ is three times continuously differentiable with respect to the components of $x$.

Apply Taylor's theorem 1.5 of Chapter 1, generalized to $n$ variables, to obtain

$$f(x) = f(\alpha) + (x - \alpha)^T \nabla f(\alpha)$$
$$+ \frac{1}{2}(x - \alpha)^T H(\alpha)(x - \alpha) + O(\|x - \alpha\|^3)$$

Here

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right]^T$$

is the *gradient* of $f$, and

$$H(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right] \qquad 1 \le i, j \le n$$

is the *Hessian matrix* for $f(x)$. The final term indicates that the remaining terms are smaller than some multiple of $\|x - \alpha\|^3$ for $x$ close to $\alpha$.

If $\alpha$ is to be a local maximum or minimum, then a necessary condition is that $\nabla f(x) = 0$. Assuming $\nabla f(\alpha) = 0$, show that $\alpha$ is a strict (or unique) local minimum of $f(x)$ if and only if $H(\alpha)$ is positive definite. [Note that $H(x)$ is always symmetric.]

**18.** Demonstrate the relation (7.2.28).

**19.** Recall the notation used in Theorem 7.5 on the singular value decomposition of a matrix $A$.

**(a)** Show that $\mu_1^2, \dots, \mu_r^2$ are the nonzero eigenvalues of $A^*A$ and $AA^*$, with corresponding eigenvectors $U^{(1)}, \dots, U^{(r)}$ and $V^{(1)}, \dots, V^{(r)}$, respectively. The vector $U^{(j)}$ denotes column $j$ of $U$, and similarly for $V^{(j)}$ and $V$.

**(b)** Show that $AU^{(j)} = \mu_j V^{(j)}$, $A^*V^{(j)} = \mu_j U^{(j)}$, $1 \le j \le r$.

**(c)** Prove $r = \text{rank}(A)$.

**20.** For any polynomial $p(x) = b_0 + b_1 x + \cdots + b_m x^m$, and for $A$ any square matrix, define

$$p(A) = b_0 I + b_1 A + \cdots + b_m A^m$$

Let $A$ be a matrix for which the Jordan canonical form is a diagonal

32. Consider the matrix

$$A = \begin{bmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \\ 0 & 1 & 1 \\ \vdots & & \\ 0 & & \cdots \end{bmatrix}$$

Show $A$ is nonsingular. Fi

33. In producing cubic interpo
necessary to solve the line

$$A = \begin{bmatrix} \frac{h_1}{3} & \\ \frac{h_1}{6} & \frac{h_1}{} \\ \vdots & \\ 0 & \end{bmatrix}$$

All $h_i > 0$, $i = 1, \ldots, m$.
show that $A$ is nonsingula

$$\frac{1}{6} \operatorname{Min} (h_i$$

for the eigenvalues of $A$.

34. Let $A$ be a square matrix,
is nonsingular. Such a mat

matrix,

$$P^{-1}AP = D = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$$

For the characteristic polynomial $f_A(\lambda)$ of $A$, prove $f_A(A) = 0$. (This result is the *Cayley–Hamilton theorem*. It is true for any square matrix, not just those that have a diagonal Jordan canonical form.) *Hint:* Use the result $A = PDP^{-1}$ to simplify $f_A(A)$.

21. Prove the following: for $x \in \mathbf{C}^n$

(a) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$

(b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$

(c) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$

22. Let $A$ be a real nonsingular matrix of order $n$, and let $\|\cdot\|_v$, denote a vector norm on $\mathbf{R}^n$. Define

$$\|x\|_* = \|Ax\|_v \qquad x \in \mathbf{R}^n$$

Show that $\|\cdot\|_*$ is a vector norm on $\mathbf{R}^n$.

23. Show

$$\operatorname*{Limit}_{p\to\infty} \left[ \sum_{j=1}^{n} |x_j|^p \right]^{1/p} = \operatorname*{Max}_{1\le i\le n} |x_i| \qquad x \in \mathbf{C}^n$$

This justifies the use of the notation $\|x\|_\infty$ for the right side.

24. For any matrix norm, show that (a) $\|I\| \ge 1$, and (b) $\|A^{-1}\| \ge (1/\|A\|)$. For an operator norm, it is immediate from (7.3.11) that $\|I\| = 1$.

25. Derive formula (7.3.17) for the operator matrix norm $\|A\|_\infty$.

26. Define a vector norm on $\mathbf{R}^n$ by

$$\|x\| = \frac{1}{n} \sum_{j=1}^{n} |x_j| \qquad x \in \mathbf{R}^n$$

What is the operator matrix norm associated with this vector norm?

27. Let $A$ be a square matrix of order $n \times n$.

(a) Given the eigenvalues and eigenvectors of $A$, determine those of (1) $A^m$ for $m \ge 2$, (2) $A^{-1}$, assuming $A$ is nonsingular, and (3) $A + cI$, $c$ = constant.

(b) Prove $\|A\|_2 = r_\sigma(A)$ when $A$ is Hermitian.

(c) For $A$ arbitrary and $U$ unitary of the same order, show $\|AU\|_2 = \|UA\|_2 = \|A\|_2$.

**28.** Let $A$ be square of order $n \times n$.

(a) Show that $F(AU) = F(UA) = F(A)$, for any unitary matrix $U$.

(b) If $A$ is Hermitian, then show that

$$F(A) = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$, repeated according to their multiplicity. Furthermore,

$$\frac{1}{\sqrt{n}} F(A) \leq \|A\|_2 \leq F(A)$$

**29.** Recalling the notation of Theorem 7.5, show

$$\|A\|_2 = \mu_1 \qquad F(A) = \sqrt{\mu_1^2 + \cdots + \mu_r^2}$$

**30.** Let $A$ be of order $n \times n$. Show

$$|\operatorname{trace}(A)| \leq n r_\sigma(A)$$

If $A$ is symmetric and positive definite, show

$$\operatorname{trace}(A) \geq r_\sigma(A)$$

**31.** Show that the infinite series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$$

converges for any square matrix $A$, and denote the sum of the series by $e^A$.

(a) If $A = P^{-1}BP$, show that $e^A = P^{-1}e^B P$.

(b) Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $A$, repeated according to their multiplicity, and show that the eigenvalues of $e^A$ are $e^{\lambda_1}, \ldots, e^{\lambda_n}$.

**32.** Consider the matrix

$$A = \begin{bmatrix} 6 & 1 & 1 & 0 & & \cdots & & & & 0 \\ 1 & 6 & 1 & 1 & 0 & & & & & \vdots \\ 1 & 1 & 6 & 1 & 1 & 0 & & & & \\ 0 & 1 & 1 & 6 & 1 & 1 & 0 & & & 0 \\ \vdots & & & & & & & & & 1 \\ & & & & & & 1 & 1 & 6 & 1 \\ 0 & & \cdots & & & & 0 & 1 & 1 & 6 \end{bmatrix}$$

Show $A$ is nonsingular. Find a bound for $\|A^{-1}\|_\infty$ and $\|A^{-1}\|_2$.

**33.** In producing cubic interpolating splines in Section 3.7 of Chapter 3, it was necessary to solve the linear system $AM = D$ of (3.7.21) with

$$A = \begin{bmatrix} \frac{h_1}{3} & \frac{h_1}{6} & 0 & \cdots & 0 \\ \frac{h_1}{6} & \frac{h_1 + h_2}{3} & \frac{h_2}{6} & & \vdots \\ \vdots & & \ddots & & \\ & & \frac{h_{m-1}}{6} & \frac{h_{m-1} + h_m}{3} & \frac{h_m}{6} \\ 0 & \cdots & 0 & \frac{h_m}{6} & \frac{h_m}{3} \end{bmatrix}$$

All $h_i > 0$, $i = 1, \ldots, m$. Using one or more of the results of Section 7.4, show that $A$ is nonsingular. In addition, derive the bounds

$$\frac{1}{6} \operatorname{Min}(h_i) \le |\lambda| \le \operatorname{Max} h_i \qquad \lambda \in \sigma(A)$$

for the eigenvalues of $A$.

**34.** Let $A$ be a square matrix, with $A^m = 0$ for some $m \ge 2$. Show that $I - A$ is nonsingular. Such a matrix $A$ is called *nilpotent*.