DMS625: Practice Assignment

November 16, 2024

Continuous-time Markov chain

1. *Queues with Balking.* Customers arrive to join a queue at iid exponentially distributed times with rate λ. The clerk serves the customers with exponentially distributed service times with rate μ. The customer upon arriving, joins the queue with probability p. Find the limiting probability of the length of this queue.

2. A shop has two clerks each serving with an exponential rate of 2 customers per hour. Suppose customer arrive at an exponential rate of 4 per hour and the capacity of the shop is that of at most 4 customers.

   (a) In the long run, what fraction of potential customers are able to enter the shop?

   (b) If there was a single clerk who could serve at the rate of 4 customers per hour, then in the long-run fraction of potential customers are able to enter the shop?

   (c) Analyze the difference in the answers to a) and b).

3. Consider a taxi station where taxis and customers arrive at exponential rates of one and two per minute respectively. A taxi will wait no matter how many other taxis are present. However, an arriving customer that doesn't find a taxi waiting leaves. Find

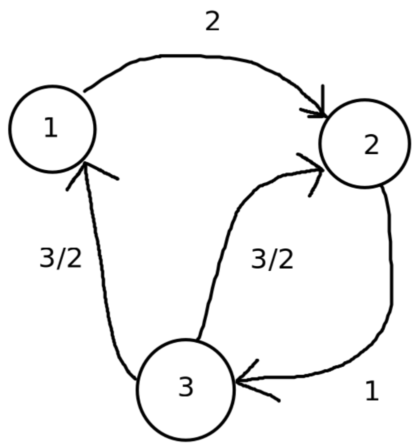
   (a) the average number of taxis waiting in long-run

   (b) the proportion of arriving customers that get taxis in long-run

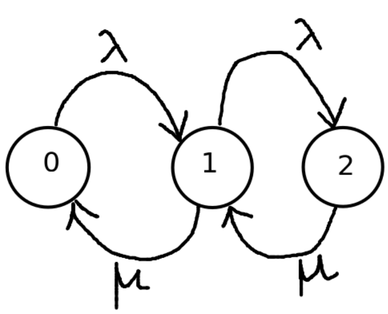
4. Customers arrive at a single-server queue with exponential rate λ. However, an arrival that finds n customers already in the system will only join the system with probability $\frac{1}{n+1}$. The service distribution is exponential with rate μ. Show that the limiting distribution of the number of customers in the system is Poisson with mean $\frac{\lambda}{\mu}$.

5. *Pure death process.* In a Birth and Death process, consider the case where $\lambda_n = 0, \forall n$ and $\mu_n = \mu, \forall n$. Find $P_{i,j}(t)$ for this process.

6. Given below is a CTMC transition diagram for the states $\mathcal{S} = \{1, 2, 3\}$. Above the arrows denotes the exponential rates for transitions in and out of the states. Find the long-run probabilities of this CTMC.



7. Given below is a CTMC transition diagram for states $\mathcal{S} = \{0, 1, 2\}$.



(a) Write the Kolmogorov Backward equation for $P'_{1,0}(t)$.

(b) Write the Kolmogorov Forward equation for $P'_{1,2}(t)$.

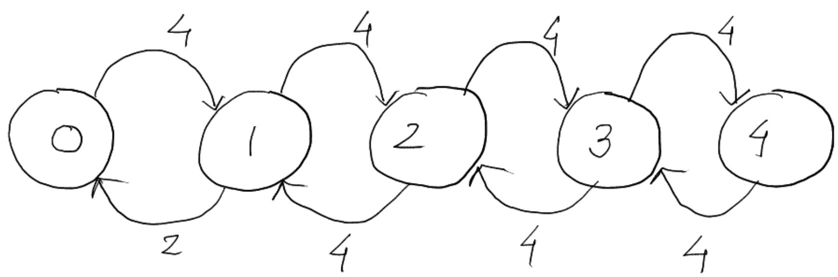
(c) Write the expression for $P_{0,0}(t)$ using uniformization.

(d) Find the limiting probabilities of this system.

Hints

1. This is a Birth and Death process with the rate of birth being λp and death being μ. Why is the rate of birth λp? Hint: think of superposition of Poisson processes.

2. (a) You have to write the equations for limiting probabilities by either writing the Kolmogorov-Forward equation and taking the limit or the inflow-outflow balance heuristic that was discussed in the class. The CTMC diagram for the system is given below.



The system of equations you will obtain are as follows,

$$
\begin{aligned}
4P_0 &= 2P_1 \\
(4+2)P_1 &= 4P_0 + 4P_2 \\
(4+4)P_2 &= 4P_1 + 4P_3 \\
(4+4)P_3 &= 4P_2 + 4P_4 \\
4P_4 &= 4P_3
\end{aligned}
$$

and we are interested in $1 - P_4$.

(b) This is to be solved similarly with appropriate modifications to the death rates.

(c) Why are the limiting probabilities in part a) and b) different?

3. This is just a Birth and Death process.

(a) Expectation of the limiting probabilities.

(b) $1 - P_0$

4. Same idea as Q1 of the problem set. The birth rate at state n is $\frac{\lambda}{n+1}$. Now arrive at the limiting probabilities of this CTMC similarly to the way the limiting probability of the Birth and Death process was derived.

5. Notice that the Pure Birth process with all birth rates being equal to λ is the Poisson process with rate λ. The pure death process (all death rates are identical here) is a pure birth process "backwards", so that should give you a hint to what should be $P_{i,j}(t)$.

This is formally approached by solving the Forward equation. The forward equations are given by,

$$P'_{i,i}(t) = -\mu P_{i,i}(t) \tag{1}$$
$$P'_{i,j}(t) = \mu P_{i,j+1}(t) - \mu P_{i,j}(t) \tag{2}$$

Upon solving (1) we obtain,

$$P_{i,i}(t) = e^{-\mu t}$$

Now in (2) set $j = i - 1$ and we get,

$$P'_{i,i-1}(t) = \mu P_{i,i}(t) - \mu P_{i,i-1}(t)$$

Solving we get $P_{i,i-1}(t) = \mu t e^{-\mu t}$ and similarly we will next set $j = i - 2$ and obtain $P_{i,i-2}$. We will proceed like this recursively.

We can therefore conclude $P_{i,j}(t) = e^{-\mu t} \frac{(\mu t)^{i-j}}{(i-j)!}$ if $i \geq j$ else $P_{i,j}(t) = 0$, which is what we guessed at the beginning.

6. Setup the equations for limiting probabilities.