# Indian Institute of Technology Kanpur
## Department of Mathematics and Statistics

Complex Analysis (MTH 403)

Semester 2023-24-I

Summary of the discussions held till 09 August 2023

### 1. Power series

1.1. **Radius of convergence**: Given a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ over $\mathbb{C}$, there exists a unique $R \in [0, \infty]$ with the following two properties:

(i) $|z - z_0| < R \Longrightarrow \sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely.

(ii) $|z - z_0| > R \Longrightarrow \sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges.

We call $R$ the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. The disc $D(z_0; R)$ is called the *disc of convergence* of the power series.

1.2. **Formula for radius of convergence**:

(a) $R = \dfrac{1}{\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}}}$. This follows from Cauchy's root test.

(b) If $\lim\limits_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right|$ exists in $[0, \infty]$ then it must be equal to $R$. This follows from D'Alembert's ratio test.

**Note:** Since it is easier to compute ratios than roots, 1.2.b is easier to apply despite limited scope. Hence, in the calculation of radius of convergence of a power series, first one may choose to see whether or not 1.2.b applies to that case. If that does not help, then using 1.2.a or other means may be used.

1.3. **Uniform convergence of a power series**:

(a) $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on every compact subset of $D(z_0; R)$.

(b) The radius of convergence of $\sum_{n=0}^{\infty} z^n$ is equal to 1. $\sum_{n=0}^{\infty} z^n$ does not converge uniformly on $\mathbb{D}$.

1.4. **Behaviour on the boundary**: At a boundary point, i.e., $|z - a| = R$, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ may or may not converge. We have seen the following examples:

(a) $\sum_{n=1}^{\infty} z^n$ does not converge at any boundary point.

(b) $\sum_{n=1}^{\infty} \dfrac{z^n}{n}$ converges everywhere on the boundary except at $z = 1$.

(c) $\sum_{n=1}^{\infty} \dfrac{z^n}{n^2}$ converges everywhere on the boundary.

1.5. The radii of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ are same.

## 2. Holomorphic functions

2.1. Definition.

2.2. Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series in $\mathbb{C}$ with radius of convergence $R \in (0, \infty]$. Define

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n,\ \forall z \in D(a; R).$$

Then $f$ is holomorphic everywhere in $D(a; R)$, and furthermore,

$$\forall z \in D(a; R),\ f'(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}.$$

In fact, for any $k \geq 0$, one has

$$\forall z \in D(a; R),\ f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-a)^{n-k}.$$

In particular, $a_k = \dfrac{f^{(k)}(z_0)}{k!}$, for all $k \geq 0$.

2.3. Analytic function. Any analytic function is holomorphic.

2.4. Let $f : [a, b] \longrightarrow \mathbb{C}$ be Riemann integrable and $\gamma : [a, b] \longrightarrow \mathbb{C}$ be continuous. Denote the image of $\gamma$ by $\gamma^*$. Define

$$F(z) = \int_a^b \frac{f(t)}{\gamma(t) - z}\, dt,\ \forall z \notin \gamma^*.$$

Then $F$ is analytic.

## 3. Some more examples

3.1. $F : \mathbb{H} \longrightarrow \mathbb{C}$, $F(z) \stackrel{\text{def}}{=} \dfrac{i-z}{i+z}$ and $G : \mathbb{D} \longrightarrow \mathbb{C}$, $G(w) \stackrel{\text{def}}{=} i\dfrac{1-w}{1+w}$. Both $F$ and $G$ are holomorphic, and they are inverse to each other.

3.2. For $w \in \mathbb{D}$, the function $\varphi_w : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$, $\varphi_w(z) \stackrel{\text{def}}{=} \dfrac{w-z}{1-\bar{w}z}$ is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$. It is self inverse.

3.3. For $g \stackrel{\text{def}}{=} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $gz \stackrel{\text{def}}{=} \dfrac{az+b}{cz+d}$ is a holomorphic map from $\mathbb{H}$ to $\mathbb{H}$, whose inverse is given by the matrix $g^{-1}$, and hence holomorphic.

## 4. Lines and circles

4.1. (a) **Equation of a line**: $\mathrm{Re}(az) = b$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{R}$.

(b) **Equation of a circle**: If the circle is centered at $z_0 \in \mathbb{C}$ and radius is $r > 0$ then the equation is $|z - z_0| = r$. Squaring both sides, we get $|z|^2 - \bar{z}_0 z - \bar{z} z_0 + (|z_0|^2 - r^2) = 0$.

(c) Consider the equation

$$\alpha|z|^2 + \bar{\beta}z + \beta\bar{z} + \gamma = 0, \tag{$*1$}$$

where $\alpha \geq 0, \beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. Then ($*1$) represents:

(i) a line if $\alpha = 0$ and $\beta \neq 0$.
(ii) a circle if $\alpha > 0$ and $|\beta|^2 > \alpha\gamma$.

4.2. Let $g$ be $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in 3.3.. The the image of a line or a circle under the holomorphic map $z \mapsto gz$, defined as above in 3.3., is again a line or a circle.

## 5. Cauchy- Riemann equations

5.1. Being holomorphic at a point is tronger than being differentiable at that point.

5.2. $\bar{z}$ is differentiable at the origin but not holomorphic.

5.3. **Cauchy-Riemann equations**: $\boxed{u_x = v_y \text{ and } v_x = -u_y.}$

5.4. Let $U$ be an open subset of $\mathbb{C}$, $f : U \longrightarrow \mathbb{C}$ and $z_0 = x_0 + iy_0 \in U$. Then the following are equivalent:

(H.1) $f$ is holomorphic at $z_0$.
(H.2) $f$ is differentiable at $(x_0, y_0)$ and the Cauchy-Riemann equations hold at $(x_0, y_0)$.

5.5. A function might satisfy Cauchy-Riemann equations at a given point without being holomorphic at that point. Consider the example

$$f(x + iy) \stackrel{\text{def}}{=} \sqrt{|x||y|},\ \forall (x, y) \in \mathbb{R}^2.$$

The function $f$ defined above satisfies the Cauchy-Riemann equations at the origin, yet it is not holomorphic at 0.

5.6. **A few applications**:

(a) $\bar{z}, |z|, |z|^2$ etc.
(b) $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$, and $\det Df(x_0, y_0) = |f'(z_0)|^2$. In particular, if $f'(z_0) \neq 0$, then $Df(x_0, y_0)$ is invertible.
(c) Let $U \subseteq_{open} \mathbb{C}$ be connected. Then $f$ is constant if any of $\operatorname{Re} f, \operatorname{Im} f$ and $|f|$ is constant.

5.7. **Cauchy-Riemann equations in polar coordinates**: $\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \text{ and } r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.}$

## 6. Exponential and logarithm functions

6.1. Exponential function over $\mathbb{C}$.

6.2. The range of exp is $\mathbb{C} \setminus \{0\}$.

6.3. exp is not injective. In fact, $\exp(z_1) = \exp(z_2)$ if and only if $z_1 - z_2 \in 2\pi i\mathbb{Z}$.

6.4. Let $\alpha \in \mathbb{R}$ and $B_\alpha \stackrel{\text{def}}{=} \mathbb{R} \times [\alpha, \alpha + 2\pi)$. Then $\exp|_{B_\alpha}$ is bijective.

6.5. $\log_\alpha \stackrel{\text{def}}{=} (\exp|_{B_\alpha})^{-1}$ and $\arg_\alpha$ is defined to be the imaginary part of $\log_\alpha$. They have precisely same points of continuity.

6.6. $\log_\alpha$ is not continuous at any point of $\overline{R_\alpha}$. Same for $\arg_\alpha$.

6.7. $\log_\alpha$ and $\arg_\alpha$ are continuous at every point of $\mathbb{C} \setminus \overline{R_\alpha}$.

6.8. $\log_\alpha$ is holomorphic everywhere on $\mathbb{C} \setminus \overline{R_\alpha}$.

6.9. Continuous logarithm and argument of a continuous function $f : X \longrightarrow \mathbb{C} \setminus \{0\}$, where $X$ is a metric space. In fact, any continuous argument must be the imaginary part of a continuous logarithm.

6.10. $f$ has continuous logarithm if and only if it has a continuous argument.

6.11. If $X$ is connected, then any two continuous logarithms will differ by a constant, which is an integral multiple of $2\pi i$. Similarly any two continuous arguments differ constantly by an integral multiple of $2\pi$.

6.12. Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$ be holomorphic. Suppose $\alpha \in \mathbb{R}$ is such that $f(U) \cap \overline{R_\alpha} = \emptyset$. Then $\log_\alpha \circ f$ is a holomorphic (and hence continuous) logarithm of $f$. In particular, if $f(U)$ is contained in an open disc not containing 0 then $f$ has a holomorphic (and hence continuous) logarithm.

6.13. Let $\gamma : [a, b] \longrightarrow \mathbb{C} \setminus \{0\}$ be a curve. Then $\gamma$ has a continuous argument and hence a continuous logarithm. In fact, we will be proving the following generalized version:

Let $f : [a, b] \times [c, d] \longrightarrow \mathbb{C} \setminus \{0\}$ be continuous. Then $f$ has a continuous argument.

6.14. Index of a point with respect to a closed curve.