Lesson 3

Boundary Value Problems for ODEs

3.1 Well-posedness

- Linear two point BVP

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Boundary Value Problems: Well-posedness

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Theorem

Consider the linear two-point BVP

$$y' = A(t)y + r(t), \qquad a < t < b,$$

where $A(t)$ and $b(t)$ are continuous, with boundary conditions

$$B_a y(a) + B_b y(b) = c.$$

The BVP has a unique solution if and only if the matrix

$$Q = B_a Y(a) + B_b Y(b)$$

is non-singular where Y is the fundamental solution matrix for the ODE whose ith column $y_i(t)$ is the solution to the homogeneous ODE $y' = A(t)y$ with initial condition $y(a) = e_i$, where e_i is the ith column of the identity matrix; these columns are called solution modes.

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Proof.

Assume that the matrix Q is invertible.

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Proof.

Assume that the matrix Q is invertible.

Uniqueness of solution follows from the fact that, if $y_1(t)$ and $y_2(t)$ are two solutions to the BVP, then $y(t) = y_1(t) - y_2(t)$ satisfies

$$y'(t) = A(t)y(t),\ B_a y(a) + B_b y(b) = 0,$$

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and hence $y(t)$ must have the form $y(t) = Y(t)d$ for some $d \in \mathbb{R}^n$ satisying, $Qd = 0$.

Proof.

Uniqueness of solution follows from the fact that, if $y_1(t)$ *and* $y_2(t)$ *are two solutions to the BVP, then* $y(t) = y_1(t) - y_2(t)$ *satisfies*

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and hence $y(t)$ *must have the form* $y(t) = Y(t)d$ *for some* $d \in \mathbb{R}^n$ *satisying,* $Qd = 0$.

Now, one can see that the unique solution to the BVP is given by

$$y(t) = Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + Y(t)\int_a^t Y^{-1}(s)r(s)ds$$

by directly verifying that it satisfies the ODE and the boundary condition.

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Proof.

Uniqueness of solution follows from the fact that, if $y_1(t)$ and $y_2(t)$ are two solutions to the BVP, then $y(t) = y_1(t) - y_2(t)$ satisfies

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by directly verifying that it satisfies the ODE and the boundary condition. Indeed,

$$y'(t) = A(t)Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + A(t)Y(t)\int_a^t Y^{-1}(s)r(s)ds + Y(t)Y^{-1}(t)r(t)$$

Proof.

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$$y(a) = Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right),$$

$$y(b) = Y(b)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + Y(b)\int_a^b Y^{-1}(s)r(s)ds\,.$$

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$$y'^{(t)} = A(t)Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + A(t)Y(t)\int_a^t Y^{-1}(s)r(s)ds + Y(t)Y^{-1}(t)r(t),$$

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Therefore,

$$B_a y(a) + B_b y(b)$$

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Therefore,

$$\begin{aligned}
&B_ay(a) + B_by(b)\\
&= B_aY(a)Q^{-1}\left(c - B_bY(b)\int_a^b Y^{-1}(s)r(s)ds\right) + B_bY(b)Q^{-1}\left(c - B_bY(b)\int_a^b Y^{-1}(s)r(s)ds\right)\\
&+ B_bY(b)\int_a^b Y^{-1}(s)r(s)ds
\end{aligned}$$

Proof.

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Therefore,

$$\begin{aligned}
&B_a y(a) + B_b y(b) \\
&= B_a Y(a)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + B_b Y(b)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) \\
&+ B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds \\
&= (B_a Y(a)Q^{-1} + B_b Y(b)Q^{-1})c - (B_a Y(a)Q^{-1} + B_b Y(b)Q^{-1} - I)B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds
\end{aligned}$$

Proof.

Now, one can see that the unique solution to the BVP is given by

$$y(t) = Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + Y(t)\int_a^t Y^{-1}(s)r(s)ds$$

by directly verifying that it satisfies the ODE and the boundary condition. Indeed,

$$y'^{(t)} = A(t)Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + A(t)Y(t)\int_a^t Y^{-1}(s)r(s)ds + Y(t)Y^{-1}(t)r(t),$$

$$y(a) = Y(a)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right),$$

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Therefore,

$$\begin{aligned}
&B_a y(a) + B_b y(b)\\
&= B_a Y(a)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right) + B_b Y(b)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\right)\\
&+ B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds\\
&= (B_a Y(a)Q^{-1} + B_b Y(b)Q^{-1})c - (B_a Y(a)Q^{-1} + B_b Y(b)Q^{-1} - I)B_b Y(b)\int_a^b Y^{-1}(s)r(s)ds = c.
\end{aligned}$$

Proof.

Therefore,

$$
\begin{aligned}
& B_a y(a) + B_b y(b) \\
& = B_a Y(a) Q^{-1} \left(c - B_b Y(b) \int_a^b Y^{-1}(s) r(s) ds \right) + B_b Y(b) Q^{-1} \left(c - B_b Y(b) \int_a^b Y^{-1}(s) r(s) ds \right) \\
& + B_b Y(b) \int_a^b Y^{-1}(s) r(s) ds \\
& = (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1}) c - (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1} - I) B_b Y(b) \int_a^b Y^{-1}(s) r(s) ds = c\,.
\end{aligned}
$$

Finally, for the converse, assume that the BVP has a unique solution, say $y_0(t)$, but Q is not invertible.

Proof.
Therefore,

$$
\begin{aligned}
&B_a y(a) + B_b y(b) \\
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&= (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1})c - (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1} - I) B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds = c\,.
\end{aligned}
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Finally, for the converse, assume that the BVP has a unique solution, say $y_0(t)$, but Q is not invertible.

Then there is $0 \neq d \in \mathbb{R}^n$ such that $Qd = 0$, and therefore, $y(t) = y_0(t) + Yd$ is another solution to the BVP

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Therefore,

$$\begin{aligned}
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&+ B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds \\
&= (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1}) c - (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1} - I) B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds = c\,.
\end{aligned}$$

Finally, for the converse, assume that the BVP has a unique solution, say $y_0(t)$*, but* Q *is not invertible.*

Then there is $0 \neq d \in \mathbb{R}^n$ *such that* $Qd = 0$*, and therefore,* $y(t) = y_0(t) + Yd$ *is another solution to the BVP as*

$$y'(t) = y_0'(t) + A(t)Yd$$

Proof.
Therefore,

$$
\begin{aligned}
&B_a y(a) + B_b y(b) \\
&= B_a Y(a) Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds\right) + B_b Y(b) Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds\right) \\
&+ B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds \\
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y'(t) = y_0'(t) + A(t)Yd = \big(A(t)y_0 + r(t)\big) + A(t)Yd
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$$
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&= (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1}) c - (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1} - I) B_b Y(b) \int_a^b Y^{-1}(s) r(s) ds = c\,.
\end{aligned}
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y'(t) = y_0'(t) + A(t) Y d = \big(A(t) y_0 + r(t) \big) + A(t) Y d = A(t) y + r(t)
$$

Proof.

Therefore,

$$\begin{aligned}&B_a y(a) + B_b y(b)\\&= B_a Y(a) Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds\right) + B_b Y(b) Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds\right)\\&+ B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds\\&= (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1})c - (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1} - I) B_b Y(b)\int_a^b Y^{-1}(s) r(s) ds = c\,.\end{aligned}$$

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and

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&+ B_b Y(b) \int_a^b Y^{-1}(s) r(s) ds \\
&= (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1}) c - (B_a Y(a) Q^{-1} + B_b Y(b) Q^{-1} - I) B_b Y(b) \int_a^b Y^{-1}(s) r(s) ds = c\,.
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and

$$B_a y(a) + B_b y(b) = B_a y_0(a) + B_b y_0(b) + B_a Y d + B_b Y d$$

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and

$$B_a y(a) + B_b y(b) = B_a y_0(a) + B_b y_0(b) + B_a Yd + B_b Yd = c + Qd = c.$$

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$$y'(t) = y_0'(t) + A(t)Yd = \big(A(t)y_0 + r(t)\big) + A(t)Yd = A(t)y + r(t)$$

and

$$B_a y(a) + B_b y(b) = B_a y_0(a) + B_b y_0(b) + B_a Yd + B_b Yd = c + Qd = c.$$

A contradiction.

Boundary Value Problems: Well-posedness

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If we define $\Phi(\mathrm{t}) = \mathrm{Y}(\mathrm{t})\mathrm{Q}^{-1}$ *and the Green's function*

$$G(t,s) = \begin{cases} \Phi(t)B_a\Phi(a)\Phi^{-1}(\mathrm{s}), & a \le s \le t, \\ -\Phi(t)B_b\Phi(b)\Phi^{-1}(s), & t < s \le b. \end{cases}$$

Then the solution $y(t)$ *can be expressed compactly as*

$$y(t) = \Phi(t)c + \int_a^b G(t,s)r(s)ds\,.$$

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Consider the perturbed problem

$$\hat{y}' = A(t)\hat{y} + \hat{r}(t), \qquad a < t < b,$$

with boundary conditions

$$B_a\hat{y}(a) + B_b\hat{y}(b) = \hat{c}.$$

Boundary Value Problems: Well-posedness

If we define $\Phi(\mathrm{t}) = \mathrm{Y}(\mathrm{t})\mathrm{Q}^{-1}$ *and the Green's function*

$$G(t,s) = \begin{cases} \Phi(t)B_a\Phi(a)\Phi^{-1}(\mathrm{s}), & a \le s \le t, \\ -\Phi(t)B_b\Phi(b)\Phi^{-1}(s), & t < s \le b. \end{cases}$$

Then the solution $y(t)$ *can be expressed compactly as*

$$y(t) = \Phi(t)c + \int_a^b G(t,s)r(s)ds\,.$$

Consider the perturbed problem

$$\hat{y}' = A(t)\hat{y} + \hat{r}(t), \qquad a < t < b,$$

with boundary conditions

$$B_a\hat{y}(a) + B_b\hat{y}(b) = \hat{c}.$$

Let $z(t) = \hat{y}(t) - y(t)$, $\Delta r(t) = \hat{r}(t) - r(t)$, *and* $\Delta c(t) = \hat{c}(t) - c(t)$. *Then,* $z(t)$ *satisfies the BVP*

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Absolute Condition Number