Module 2

Initial Value Problems

Akash Anand
MATH, IIT KANPUR

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Initial Value Problems: First order system of ODE

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MATH, IIT KANPUR

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Recall that a kth order ODE is said to be explicit if it can be written in the form

$$u^{(k)} = f\left(t, u, u', u'', \dots, u^{(k-1)}\right)$$

where $f : \mathbb{R}^{kn+1} \to \mathbb{R}^n$.

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Introduce new variables

$$y_1(t) = u(t), y_2(t) = u'(t), \ldots, y_k(t) = u^{(k-1)}(t)$$

so that the original kth order system becomes a system of kn first order equations

$$y' = \begin{bmatrix} y_1' \\ \vdots \\ y_{k-1}' \\ y_k' \end{bmatrix} = \begin{bmatrix} y_2 \\ \vdots \\ y_k \\ f\left(t, y_1, y_2, \cdots, y_k\right) \end{bmatrix}$$

Module 2

Initial Value Problems

2.0 First-order system of ODE

2.1 Well-posedness

2.2 Stability

2.3 Euler's method

Initial Value Problems: Well-posedness

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We want to study the methods for numerical solution of a first order system of ordinary differential equations with initial conditions,

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

where $y: [a, b] \to \mathbb{R}^n$ *and* $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$.

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Before we study the numerical solution, we need to investigate the well-posedness of the problem, that is, we study if the problem has following three properties:

*(i) existence of a solution (**existence**),*

*(ii) uniqueness of the solution (**uniqueness**), and*

*(iii) continuous dependence of the solution of the data (**conditioning**).*

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This investigation, however, is the main subject matter of the course on Ordinary Differential Equations (ODE) and we, in this course, will only recall the relevant discussions.

Initial Value Problems: Well-posedness

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Let $D = [a, b] \times \Omega \subseteq \mathbb{R}^{n+1}$ *be a closed and bounded set.*

Suppose that $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$ *is a Lipschitz continuous function in* y *on* D, *that is, there is a constant* L *such that for any* $t \in [a, b]$ *and for any* y *and* $\hat{y} \in \Omega$,

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Recall, from your ODE course, that for such functions, the following Initial Value Problem (IVP)

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

has a unique solution in a subinterval of $[a, b]$ *containg* y_0*.*

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Example: *If* f *is differentiable, then* f *is Lipschitz continuous with*

$$L = \max_{(t,y)\in D} \|f'(t,y)\|,$$

where f' *is the* $n \times n$ *Jacobian matrix of* f *with respect to* y*,* $[f'(t,y)]_{ij} = \frac{\partial f_i(t,y)}{\partial y_j}$.

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$$\|\hat{y}(t) - y(t)\| \leq e^{L(t-t_0)} \|\hat{y}_0 - y_0\| + \frac{e^{L(t-t_0)} - 1}{L} \|\hat{f} - f\|,$$

where

$$\|\hat{f} - f\| = \max_{(t,y)\in D} \|\hat{f}(t, y) - f(t, y)\|.$$

These perturbation bounds show that the unique solution to the IVP is a continuous function of the problem data, and hence the problem is well-posed.

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Initial Value Problems: Stability

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We will see that the stability of the solution plays a significant role in the propagation of error, introduced by a numerical method, over time; they can either get amplified or diminished depending on the stability.

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Also recall that a solution of the ODE $y' = f(t, y)$ *is said to be stable if for every* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that if* $\hat{y}(t)$ *satisfied the ODE and* $\|\hat{y}(t_0) - y(t_0)\| \leq \delta$, *then* $\|\hat{y}(t) - y(t)\| \leq \epsilon$ *for all* $t \geq t_0$.

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Thus, for a stable solution, if the initial value is perturbed, then the perturbed solution remains close to the original solution, which rules out the exponential divergence of perturbed solution allowed by the perturbation bound

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$$\|\hat{y}(t) - y(t)\| \leq e^{L(t-t_0)} \|\hat{y}_0 - y_0\|.$$

A stable solution is said to be asymptotically stable if $\|\hat{y}(t) - y(t)\| \to 0$ *as* $t \to \infty$. *This stronger form means that the original and perturbed solution not only remain close to each other, but they converge toward each other over time.*

Examples:

(i) Solutions of $y' = b$, for a given b, are …

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(i) *Solutions of $y' = b$, for a given b, are stable but not asymptotically stable.*

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(i) Solutions of $y' = b$, for a given b, are stable but not asymptotically stable.

(ii) Consider the IVP $y' = \lambda y$, for a given constant λ, $y(0) = y_0$. The solution is $y(t) = y_0 e^{\lambda t}$.

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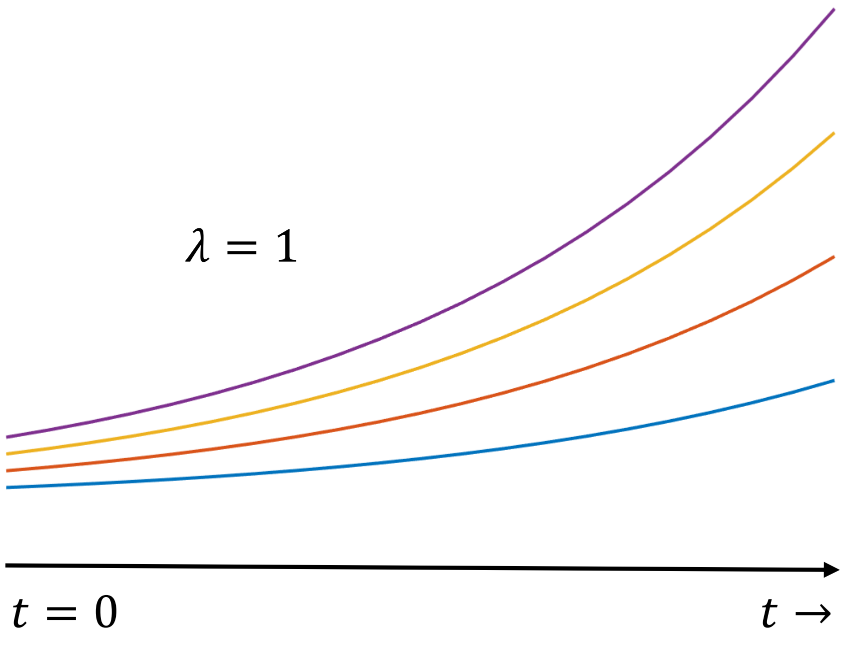

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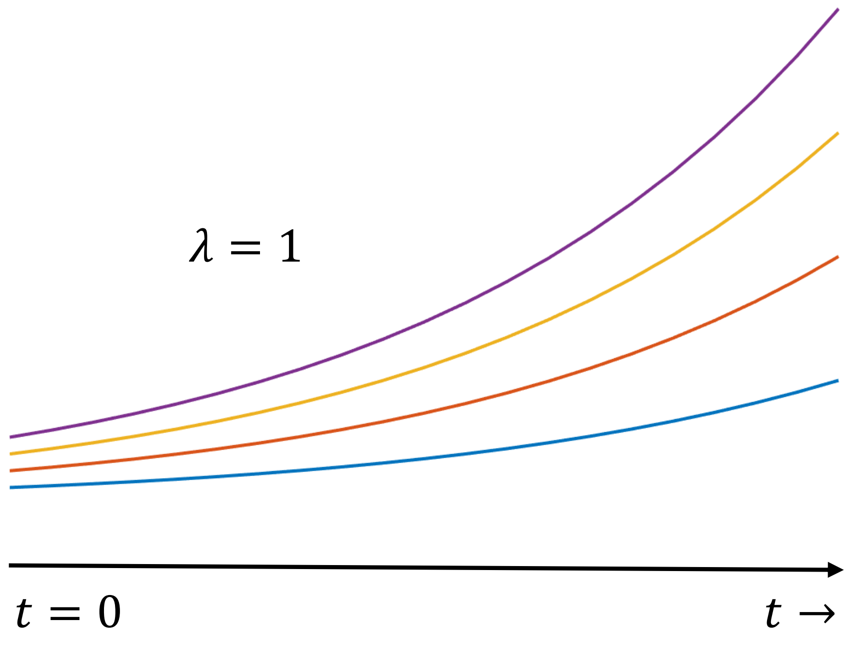
If $Re(\lambda) > 0$, then all non-zero solutions grow exponentially so that any two solutions diverge away, and hence every solution is unstable.



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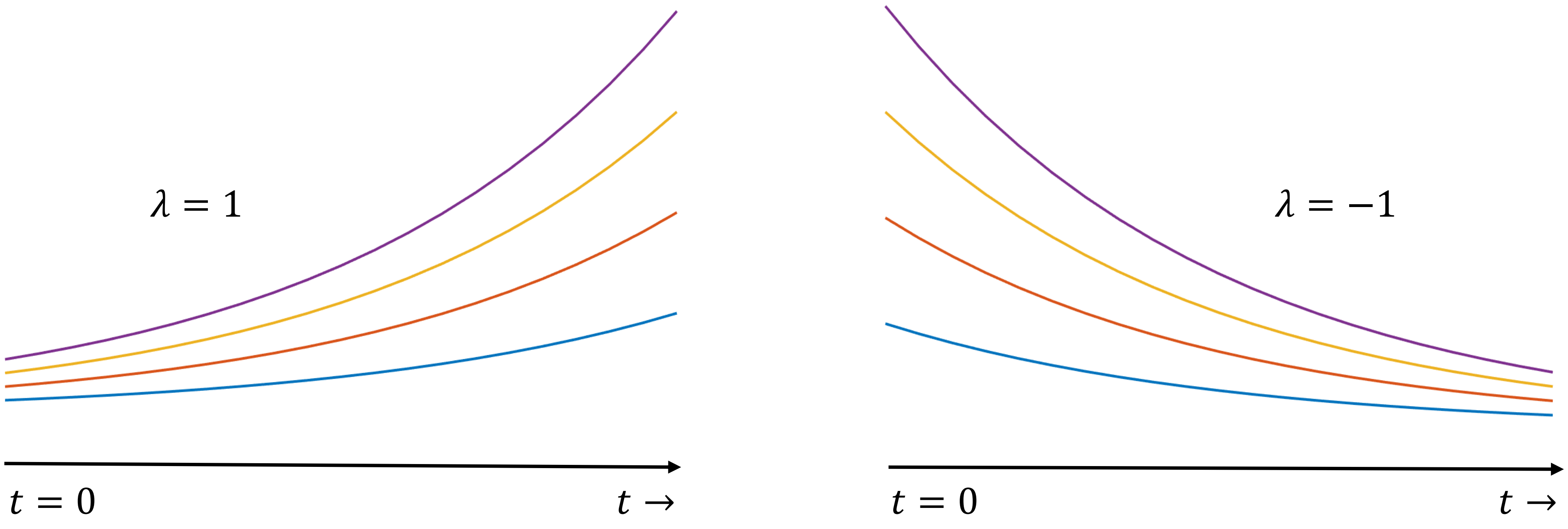

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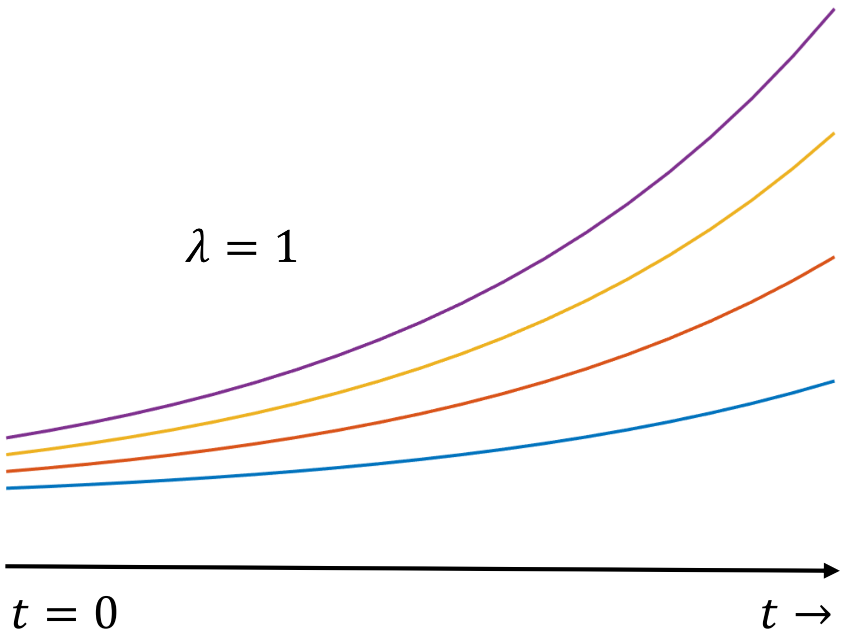

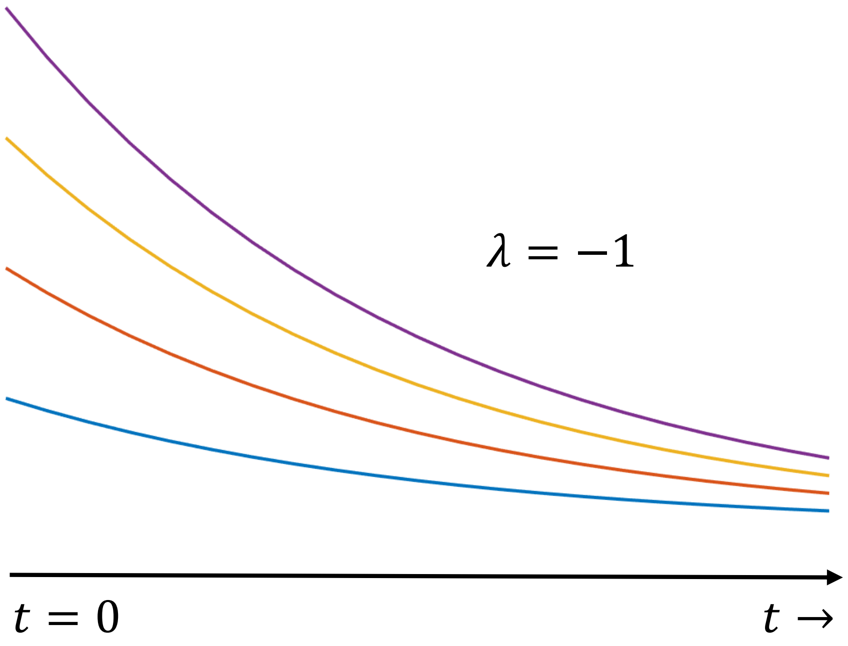

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(iii) A linear homogeneous system of ODEs with constant coefficients has the form

$$y' = Ay,$$

where A is an $n \times n$ matrix. Suppose we have the initial condition $y(0) = y_0$. Discuss the stability of the solutions if

(a) A is diagonalizable, and
(b) A is not diagonalizable.

*(**Exercise**)*

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2.0 First-order system of ODE

2.1 Well-posedness

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Initial Value Problems: Euler's method

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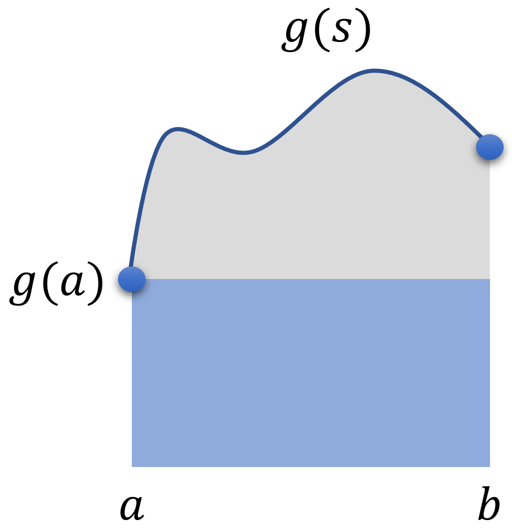

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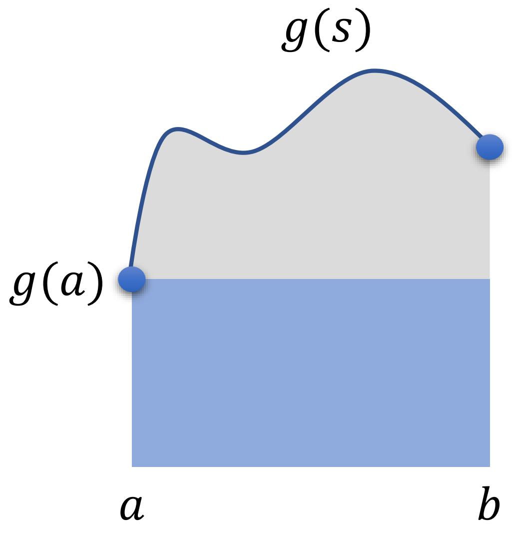

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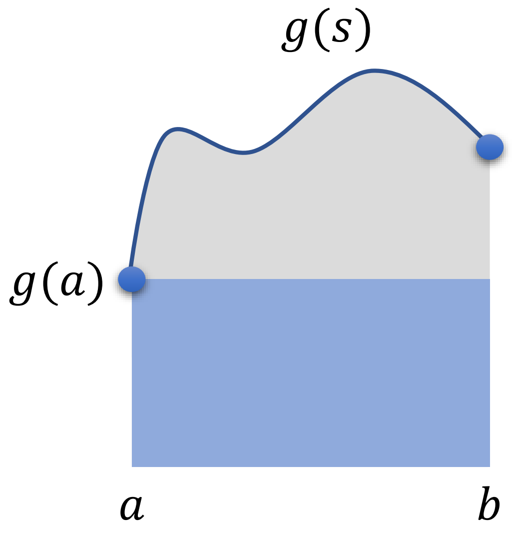

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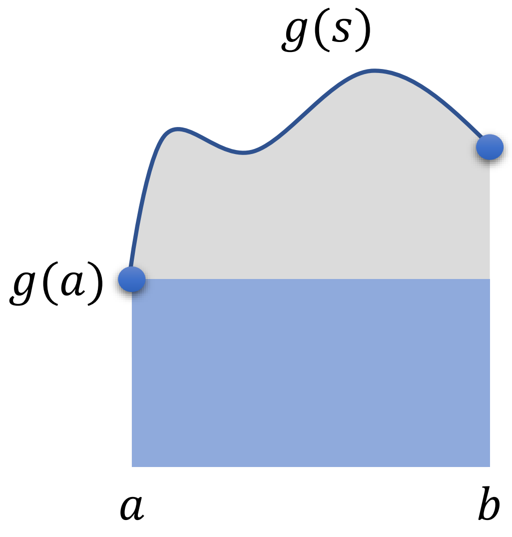

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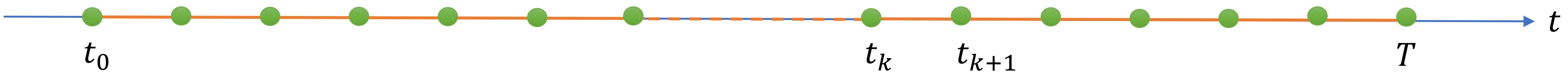

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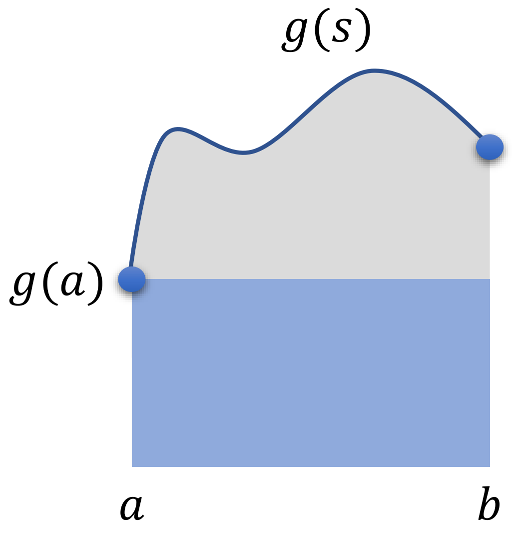

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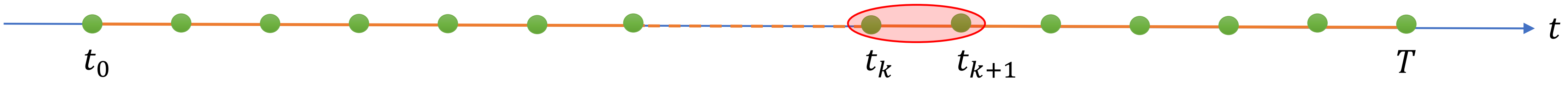

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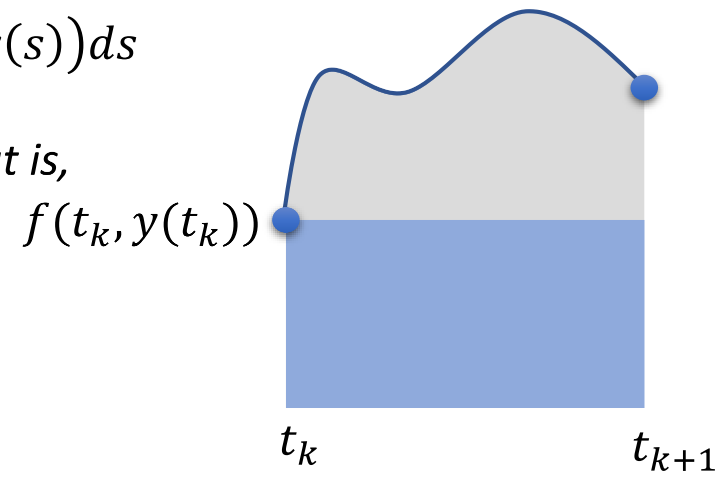

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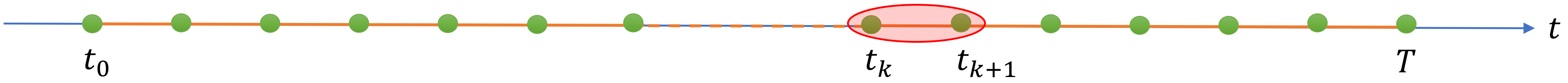


Initial Value Problems: Euler's method

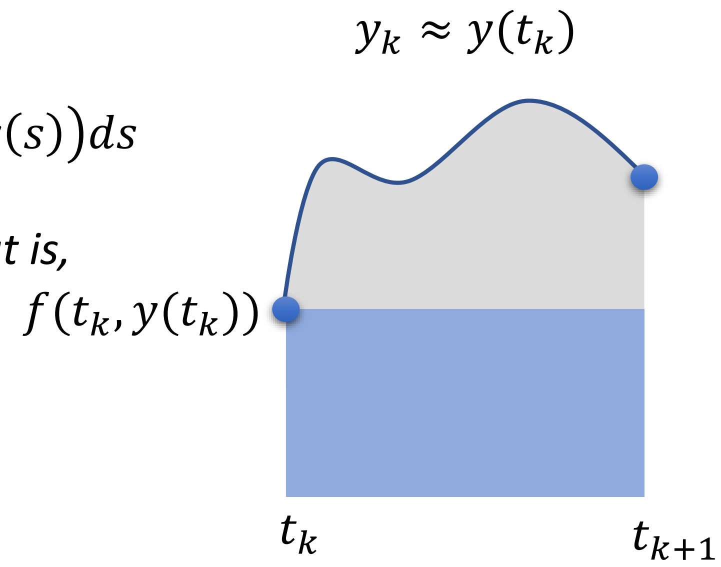
Akash Anand
MATH, IIT KANPUR

Starting at time t_0 with the given initial value y_0, we would like to track the solution tragectory governed by the ODE

$$y' = f(t, y)$$

The fundamental theorem of calculus tells us that

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} y'(s)ds = y(t_k) + \int_{t_k}^{t_{k+1}} f\big(s, y(s)\big)ds$$



The simplest approximation for the integral is by using the left end-point rule, that is,

$$\int_a^b g(s)ds \approx (b-a)g(a)$$

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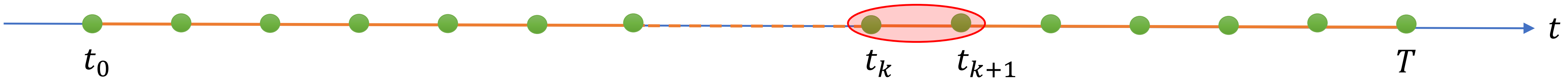

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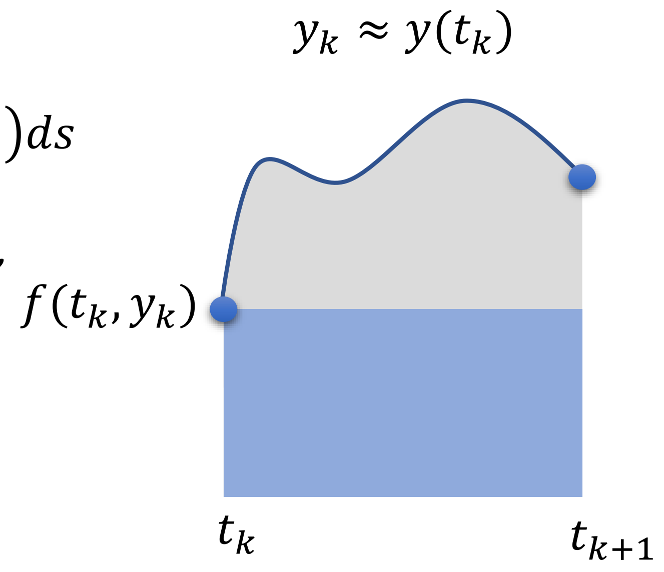

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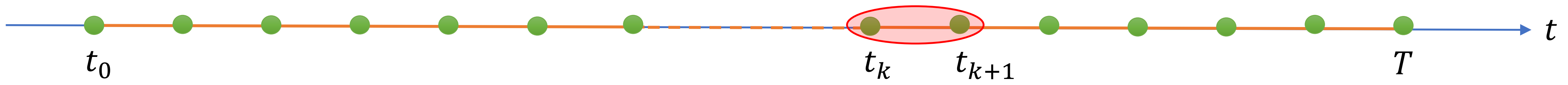

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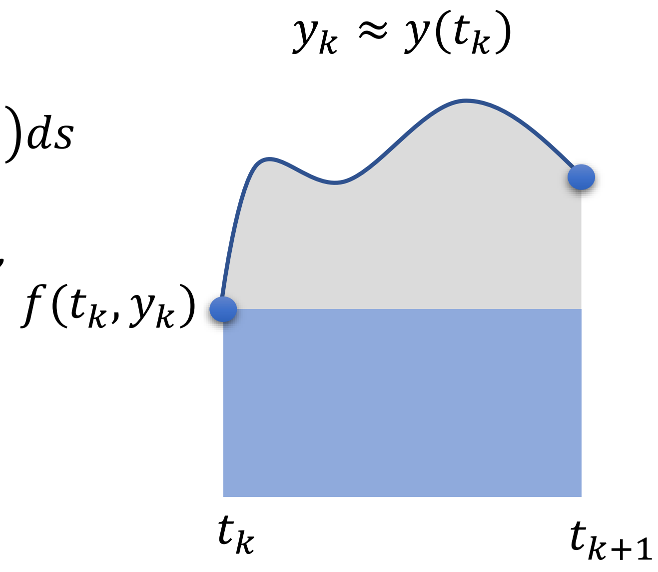

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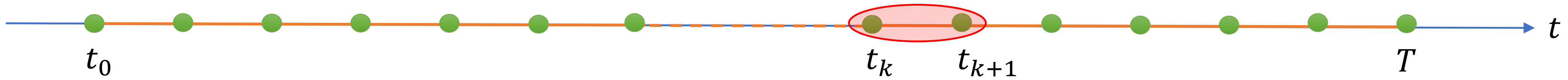

$$y_{k+1} = y_k + (t_{k+1} - t_k)f(t_k, y_k)$$

Initial Value Problems: Euler's method

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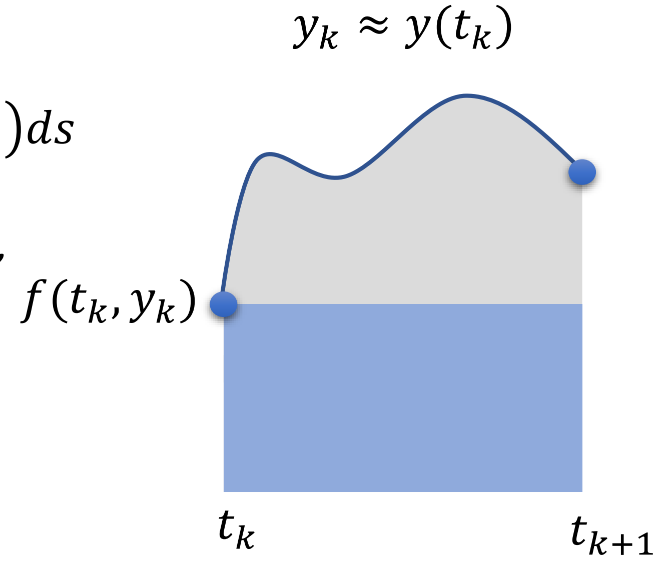

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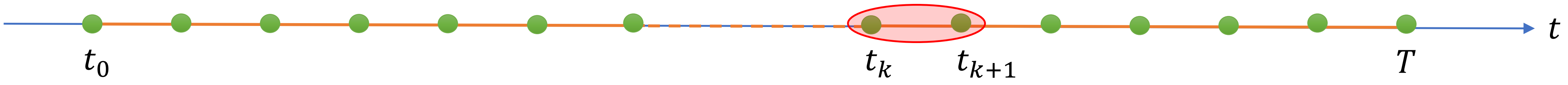

Euler's method

$$\boxed{y_{k+1} = y_k + (t_{k+1} - t_k)f(t_k, y_k)}$$