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Returning to the homogeneous linear difference equation

$$y_{n+1} + \sum_{j=0}^{k} a_j y_{n-j} = 0, \qquad n = k, k+1, \ldots$$

with general solution

$$y_n = \sum_{j=1}^{J} \sum_{m=0}^{M_j - 1} c_{jm} n^m \lambda_j^n$$

is bounded provided …

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For solutions y_j and $\hat{y}_j$, let

$$E_n = \begin{bmatrix} y_{n-k} - \hat{y}_{n-k} \\ y_{n-k+1} - \hat{y}_{n-k+1} \\ \vdots \\ y_{n-1} - \hat{y}_{n-1} \\ y_n - \hat{y}_n \end{bmatrix}$$

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and
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to get
$$E_{n+1} = AE_n + Q_n$$

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Note that $|\lambda I - A| = \rho(\lambda)$. (Why?)

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$$E_{n+1} = AE_n + Q_n.$$

Thus, we have

$$E_{k+n} = A^n E_k + \sum_{j=0}^{n-1} A^{n-1-j} Q_{k+j}.$$

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$$E_{k+n} = A^n E_k + \sum_{j=0}^{n-1} A^{n-1-j} Q_{k+j}\,.$$

Thus, using ℓ_∞ norm for vectors and the fact that there is a constant C so that $\|A^m\| \le C$, for all m, we have

$$\|E_{k+n}\| \le C\|E_k\| + C\sum_{j=0}^{n-1} \|Q_{k+j}\|$$