W7C1 : Boundary Value Problem :-

Consider the DE $p_0(x)y'' + p_1(x)\,y' + p_2(x)\,y = 0$ in $I = [\alpha, \beta]$ subject to the boundary conditions — (1)

$$\left.\begin{aligned} l_1(y) &= a_0\,y(\alpha) + a_1 y'(\alpha) + b_0\,y(\beta) + b_1 y'(\beta) = A \\ \text{and, } l_2(y) &= c_0\,y(\alpha) + c_1 y'(\alpha) + d_0\,y(\beta) + d_1 y'(\beta) = B \end{aligned}\right] \text{— (2)}$$

where a_i, b_i, c_i and d_i's are constant for $i = 1, 2$ along with A and B.

If $l_1(y) = l_2(y) = 0$ then (1) is called a homogeneous B.V.P.

Types of Boundary :

(a) DIRICHLET :- $y(\alpha) = A$ and $y(\beta) = B$.

(b) MIXED :- $y(\alpha) = A$ and $y'(\beta) = B$ or, $y'(\alpha) = A$ and $y(\beta) = B$.

(c) SEPERATED :- $\begin{cases} a_0\,y(\alpha) + a_1\,y'(\alpha) = A \\ d_0\,y(\beta) + d_1\,y'(\beta) = B \end{cases}$ where $a_0^2 + a_1^2 \neq 0$ and, $d_0^2 + d_1^2 \neq 0$.

(iv) NEUMANN BOUNDARY :- $y'(\alpha) = A$ and $y'(\beta) = B$

(v) PERIODIC BOUNDARY :- $y(\alpha) = y(\beta)$ and $y'(\alpha) = y'(\beta)$.

(1) + (2) is called Regular if both α, β are finite and $p_0(x) \neq 0 \ \forall x \in I$.
otherwise we call it singular.

Question we like to pose is if (1) + (2) admits an unique solution.

Some Observation:

(a) $y'' + y = 0$; $y(\alpha) = 0$ and $y(\beta) = \epsilon$ has an unique solution for $\alpha - \beta \neq n\pi$; $n \in \mathbb{Z}$.

[$\because$ $y(x) = A\cos x + B\sin x$

$$\left.\begin{aligned} 0 = y(\alpha) &= A\cos\alpha + B\sin\alpha \\ \text{and, } \epsilon = y(\beta) &= A\cos\beta + B\sin\beta \end{aligned}\right\} \text{— (iii)}$$

(iii) admits unique soln if $\sin(\alpha - \beta) \neq 0$
$\Rightarrow \alpha - \beta \neq n\pi$; $n \in \mathbb{Z}$]

i.e., If $\alpha = 0$, $\beta < \pi$ for (a) to have unique soln.

If $\alpha = 0, \beta = \pi$ and $y(\alpha) = 0$ and $y(\beta) = \varepsilon$ then

$$0 = y(0) = A + B \cdot 0$$
$$\varepsilon = y(\pi) = -A + B \cdot 0$$

Hence, no solution.

Again, if $y(0) = y(\pi) = 0$ then $y(x) = c\sin x$ is a solution for any $c \in \mathbb{R}$

Thus infinitely many solution.

Th: Let y_1 and y_2 be any two linearly independent solution of (I). Then (I)+(II) has only trivial solution iff $\Delta := \begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix} \neq 0.$

Proof: Any solution of (I) is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

This solves (I)+(II) if

$$l_1[c_1y_1 + c_2y_2] = c_1\, l_1(y_1) + c_2\, l_1(y_2) = 0$$

$$l_2[c_1y_1 + c_2y_2] = c_1\, l_2(y_1) + c_2\, l_2(y_2) = 0$$

The above system has only trivial solution iff $\begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix} \neq 0$.

3 options:
no soln
unique soln
infinite soln

Corollary: ① + ② admits infinitely many solutions iff $\Delta = 0$

Ex: $y'' + 2y' + 5y = 0$

$l_1(y) = y(0) = 0$; $l_2(y) = y(\pi/2) = 0$

$\therefore\ y(x) = Ae^{-x}\cos 2x + Be^{-x}\sin 2x$ is the general solution.

$$\Delta = \begin{vmatrix} 1 & 0 \\ -e^{-\pi/2} & 0 \end{vmatrix} = 0 \Rightarrow \text{Infinitely many solutions.}$$

Theorem :- The equation $p_0 y'' + p_1 y' + p_2 y = p_3(x)$ subject to $l_1(y) = A$ and $l_2(y) = B$ — (IV)

has a unique solution iff the homogeneous B.V.P (I) + (II) has only a trivial solution.

Proof :- The general solution of (IV) is given by $\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x) + y_3(x)}$ where (can be easily seen)

y_1 and y_2 are linearly independent solution of (I) and $y_3(x)$ is a particular soln of (II).

$\because$ y solves (IV) + (II) we have,

$$l_1(c_1 y_1 + c_2 y_2 + y_3) = c_1 l_1(y_1) + c_2 l_1(y_2) + l_1(y_3) = A$$

and, $l_2(c_1 y_1 + c_2 y_2 + y_3) = c_1 l_2(y_1) + c_2 l_2(y_2) + l_2(y_3) = B$.

The above equation has an unique solution iff $\Delta \neq 0$

which implies that the homogeneous problem admits only trivial solution.

Ex:- $xy'' - y' - 4x^3y = 1 + 4x^4$. — (a)

$l_1(y) = y(1) = 0$ and $l_2(y) = y(2) = 0$ — (b)

Consider, $xy'' - y' - 4x^3y = 0$ — (c)

Note, $y_1(x) = \cosh(x^2-1)$ and $y_2(x) = \frac{1}{2}\sinh(x^2-1)$ are L.I solutions of (c).

and $\Delta = \begin{vmatrix} 1 & 0 \\ \cosh 3 & \frac{1}{2}\sinh 3 \end{vmatrix} \neq 0$

$\therefore$ (b) + (c) admits an unique solution given by $y(x) \equiv 0$.

Sturm-Liouville Theory :-

Consider the B.V.P

$$(p(x)y')' + q(x)y + \lambda r(x)y = L(y) + \lambda r(x)y = 0 \quad \text{in } [\alpha,\beta] = I \;—\; (1)$$

$$(ii) \;—\; \begin{cases} a_0\, y(\alpha) + a_1\, y'(\alpha) = 0 \;;\; a_0^2 + a_1^2 \neq 0 \\ b_0\, y(\beta) + b_1\, y'(\beta) = 0 \;;\; b_0^2 + b_1^2 \neq 0 . \end{cases}$$

Regular Sturm-Liouvelle Problem

(separated and homogeneous)

(i) + (ii) are called regular Sturm-Liouville Problem.

Assumption :- $\lambda \in \mathbb{C}$ is a parameter, q and r are continuous in I, $p \in C^1(I)$ and $p, r > 0$ in I

Note :- If $\exists\, x_0$ s.t either $p(x_0)$ or $r(x_0)$ is zero, we call such problem as Singular SL BVP.

Question :- Problem is to find λ (eigenvalues) s.t (i) + (ii) admits a non-trivial solution $y_\lambda(x)$ (eigenfunction).

$\boxed{\text{\# Spectrum of (i)+(ii) is the set of all eigenvalues.}}$

Ex: Consider the problem

$$y'' + \lambda y = 0 \; ; \; y(0) = y(\pi) = 0 .$$

Case 1: $\lambda = 0$, we have $y'' = 0 \Rightarrow y(x) = Ax + B$

$$0 = y(0) = A \cdot 0 + B . \Rightarrow B = 0$$

$$0 = y(\pi) = A \cdot \pi + B . \Rightarrow A = 0$$

$\therefore$ 0 is the only solution corresponding to $\lambda = 0$.

Case 2: $\lambda < 0$, $\lambda = -\mu^2$; $\mu \in \mathbb{R}$

$$\therefore y'' - \mu^2 y = 0 \Rightarrow y(x) = Ae^{\mu x} + Be^{-\mu x} .$$

$$0 = y(0) = A + B \Rightarrow A = -B .$$

$$0 = y(\pi) = Ae^{\mu\pi} + Be^{-\mu\pi} \Rightarrow A\left(e^{\mu\pi} - e^{-\mu\pi}\right) = 0 \Rightarrow A = 0$$

$$\Downarrow$$

$$B = 0 .$$

Hence, 0 is the only solution.

Case 3 :- when $\lambda > 0$, $\lambda = \mu^2$; $\mu \in \mathbb{R}$.

$\therefore y'' + \mu^2 y = 0 \Rightarrow y(x) = A \sin \mu x + B \cos \mu x$

$0 = y(0) = A \cdot 0 + B \cdot 1 \Rightarrow B = 0$

and, $0 = y(\pi) = A \sin \mu x \Rightarrow \sin \mu x = \sin n\pi$; $n \in \mathbb{Z}$

$\therefore \mu = n$; $n \in \mathbb{Z}$

$\therefore$ Eigenvalues are $\{n^2 ; n \in \mathbb{Z}\}$ and eigenfunction are $\{\sin nx ; n \in \mathbb{Z}\}$

Question :- Why don't we look for complex λ ?

(1)

Theorem 1: Eigenvalues of a RSLBVP are always real.

Define: $Ly := (py')' + qy$ and let $\lambda = a + ib$ be the complex eigenvalue and

$y_\lambda(x) = y_1(x) + i\,y_2(x)$ be the corresponding eigenfunction.

$$\therefore \left[p(y_1 + i y_2)'\right]' + q(y_1 + i y_2) + (a+ib)\,r(x)\,(y_1 + i y_2) = 0$$

$$\begin{cases} (py_1')' + qy_1 + (ay_1 - by_2)\,r(x) = 0 \Rightarrow Ly_1 + (ay_1 - by_2)\,r(x) = 0 \\ (py_2')' + qy_2 + (by_1 + ay_2)\,r(x) = 0 \Rightarrow Ly_2 + (by_1 + ay_2)\,r(x) = 0. \end{cases}$$

and, $a_0 y_i(\alpha) + a_1 y_i'(\alpha) = 0$

$b_0 y_i(\beta) + b_1 y_i'(\beta) = 0$ for $i = 1, 2$

with (a_o,a_1) not simultaneously zero
and (b_o,b_1) not simultaneously zero

$$\therefore \int_\alpha^\beta \left[y_2 L(y_1) - y_1 L(y_2)\right] = \int_\alpha^\beta \left[-(ay_1 - by_2)\,r y_2 + (by_1 + ay_2)\,r y_1\right] dx = b\int_\alpha^\beta (y_2^2 + y_1^2)\,r(x)\,dx \quad \leftarrow (a)$$

Recall for self-adjoint equation :- (Can be seen through Lagrange / Green's Identity)

$$\int_{\alpha}^{\beta}\left[y_2 L(y_1) - y_1 L(y_2)\right] = \int_{\alpha}^{\beta}\left[p(x)\left(y_2 y_1' - y_1 y_2'\right)\right]'$$

$$= p(x)\, W(y_1, y_2)\Big|_{\beta} - p(x)\, W(y_1, y_2)\Big|_{\alpha}.$$

Again from the B.Values we have,

$$W(y_1, y_2)(\alpha) = 0 \quad \text{and} \quad W(y_1, y_2)(\beta) = 0$$

$$\therefore \int_{\alpha}^{\beta}\left[y_2 L(y_1) - y_1 L(y_2)\right] = 0 \longrightarrow (b)$$

From (a) and (b) we have, $b = 0$ is the only possibility ($\because y_x$ is non-trivial).

Hence, all eigenvalues are real.

Orthogonal Functions :- Let $\{\phi_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a sequence of piecewise continuous fxn on I.

$\{\phi_n(x) : n = 0,1,2,\dots\}$ is said to be orthogonal w.r.t $r(x) \geq 0$ $\forall x \in I$ if

$$(\phi_m, \phi_n) = \int_\alpha^\beta r(x)\,\phi_m(x)\,\phi_n(x)\,dx = 0 \text{ for all } m \neq n$$

and, $\displaystyle\int_\alpha^\beta r(x)\,\phi_n^2(x)\,dx \neq 0 \quad \forall n \in \mathbb{N}$

= ||phi_n(x)||^2

$r(x)$ is called a weight function. We assu that $r(x)$ is also piecewise continuous and

$\displaystyle\int_\alpha^\beta r(x)\,\phi_n^2(x)\,dx$, $n \in \mathbb{N}\cup\{0\}$ exists.

The orthogonal set $\{\phi_n\}_{n=0}^{\infty}$ is said to be orthonormal if $\displaystyle\int_\alpha^\beta r(x)\,\phi_n^2(x)\,dx = 1$ $\forall n \in \mathbb{N}$.

Ex :- Take $r(x) \equiv 1$, then $\{\sin nx\}_{n=0}^{\infty}$ and $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal in $[0,\pi]$.

$$\left[\int_0^\pi \sin mx \sin nx\,dx = \int_0^\pi \cos mx \cos nx\,dx = 0 \;\forall m \neq n \text{ and } \int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \frac{\pi}{2} \;\forall m \in \mathbb{N}\cup\{0\}\right].$$

(2)

Theorem 2 :- Eigenvalues of RSLBVP are simple i.e, if $(\lambda_1, \phi_1(x))$ and $(\lambda_1, \phi_2(x))$ be two distinct eigenpair then $\phi_1(x)$ and $\phi_2(x)$ are linearly dependent.

Proof :- $\because$ ϕ_1 and ϕ_2 are both solutions of (I) + (II).

From Green's Identity,

$$\int_\alpha^\beta \left[\phi_2 L(\phi_1) - \phi_1 L(\phi_2)\right] dx = p_0(x)\left(\phi_2 \phi_1' - \phi_1 \phi_2'\right)\Big|_\alpha^\beta \quad \text{---} (*)$$

Note, $a_0 \phi_1(\alpha) + a_1 \phi_1'(\alpha) = 0$ | $b_0 \phi_1(\beta) + b_1 \phi_1'(\beta) = 0$

$a_0 \phi_2(\alpha) + a_1 \phi_2'(\alpha) = 0$. | $b_0 \phi_2(\beta) + b_1 \phi_2'(\beta) = 0$.

$\because a_0^2 + a_1^2 \neq 0 \Rightarrow W(\phi_1, \phi_2)(\alpha) = 0$

$\because b_0^2 + b_1^2 \neq 0 \Rightarrow W(\phi_1, \phi_2)(\beta) = 0$

from (*) $\underbrace{p_0(x)\, W(\phi_1, \phi_2)(x) = \text{constant}}_{\text{use phi1, phi2 are eigenfuncns}} \Rightarrow W(\phi_1, \phi_2)(x) = 0 \quad \forall x \in I$

(3)

Theorem 3: Let $\{\lambda_n\}_{n=0}^{\infty}$ be the eigenvalues of a RSLBVP and $\{\phi_n\}_{n=0}^{\infty}$ be the corresponding eigenfunctions. Then the set $\{\phi_n(x) : n = 0,1,2,\ldots\}$ is orthogonal in $[\alpha,\beta]$ w.r.t the weight $r(x)$.

Proof: Let $(\lambda_l, \phi_l(x))$ and $(\lambda_K, \phi_K(x))$ $(K \neq l)$ are two distinct eigenpair.

Hence, $L(\phi_K) + \lambda_K r(x) \phi_K(x) = 0$

and, $L(\phi_l) + \lambda_l r(x) \phi_l(x) = 0$

From Green's Identity,

$$(\lambda_K - \lambda_l) \int_{\alpha}^{\beta} r(x) \phi_K(x) \phi_l(x)\, dx = p(x)\, W(\phi_K, \phi_l)(x) \Big|_{\alpha}^{\beta}$$

From the boundary conditions,

$$W(\phi_K, \phi_l)(\beta) = W(\phi_K, \phi_l)(\alpha) = 0$$

Hence, $\int_{\alpha}^{\beta} r(x) \phi_K(x) \phi_l(x)\, dx = 0$ for all $l \neq K$ $(\because \lambda_K \neq \lambda_l)$.

Periodic SLBVP :-

$(py')' + qy + \lambda ry = 0$ for $x \in [\alpha, \beta]$

Define, $L(y) := (py')' + qy$ and hence one has $Ly + \lambda ry = 0$; $\lambda \in \mathbb{C}$ is a parameter.

subject to :- $p(\alpha) = p(\beta)$ and $y(\alpha) = y(\beta)$; $y'(\alpha) = y'(\beta)$

Ex :- $y'' + \lambda y = 0$; $y(0) = y(\pi)$ and $y'(0) = y'(\pi)$

Case 1 :- $(\lambda = 0)$, $y'' = 0 \Rightarrow y(x) = Ax + B$.

Now, $A \cdot 0 + B = A \cdot \pi + B \Rightarrow A = 0$ and $y'(0) = y'(\pi)$ doesnot yield anything.

$\therefore y(x) = B$.

$\therefore (\lambda, y_\lambda) = (\underset{\mathbb{R}}{0}, \underset{\text{function}}{1})$ is an eigenpair

Case 2 :- $(\lambda = \mu^2)$ $(\mu \in \mathbb{R})$.

$\therefore y'' + \mu^2 y = 0 \Rightarrow y(x) = A\cos\mu x + B\sin\mu x.$

$y(0) = y(\pi) \Rightarrow A\cdot 1 + B\cdot 0 = A\cos\mu\pi + B\sin\mu\pi.$

$\Rightarrow A(1-\cos\mu\pi) - B\sin\mu\pi = 0. \quad \text{--- (1)}$

and, $y'(0) = y'(\pi) \Rightarrow -A\mu\cdot 0 + B\mu = -A\mu\sin\mu\pi + B\mu\cos\mu\pi$

$\Rightarrow A\sin\mu\pi + B(1-\cos\mu\pi) = 0 \quad \text{--- (11)}$

For non-trivial solution,

$$\begin{vmatrix} 1-\cos\mu\pi & -\sin\mu\pi \\ \sin\mu\pi & 1-\cos\mu\pi \end{vmatrix} = 0 \Rightarrow (1-\cos\mu\pi)^2 + \sin^2\mu\pi = 0 \Rightarrow 2 - 2\cos\mu\pi = 0$$

$\Rightarrow \cos\mu\pi = 1 = \cos 2n\pi$

$\Rightarrow \mu = 2n \,;\, n \in \mathbb{Z}.$

$\therefore$ Eigenpair $\to \{(4n^2, \cos 2nx), (4n^2, \sin 2nx) \,;\, n \in \mathbb{Z}\}$

(Note: just before we saw that for same eigenvalue in RSLBVP have linearly dep. eigenfuncns but here we're seeing that for Periodic SLBVP it (Thm 2) doesn't holds)

Case 3: $\lambda < 0$, $\boxed{\lambda = -\mu^2}$ $(\mu \neq 0)$

$\therefore\ y'' - \mu^2 y = 0 \Rightarrow y(x) = Ae^{\mu x} + Be^{-\mu x}$; $A, B \in \mathbb{R}$.

$y(0) = y(\pi) \Rightarrow A + B = Ae^{+\mu\pi} + Be^{-\mu\pi} \Rightarrow A(1 - e^{\mu\pi}) + B(1 - e^{-\mu\pi}) = 0$

$y'(0) = y'(\pi) \Rightarrow A - B = Ae^{\mu\pi} - Be^{-\mu\pi} \Rightarrow A(1 - e^{\mu\pi}) + B(-1 + e^{-\mu\pi}) = 0$

$$\left[\therefore\ \underbrace{(1 - e^{\mu\pi})(-1 + e^{-\mu\pi}) - (1 - e^{-\mu\pi})^2}_{\neq 0} \cancel{= (1 - e^{\mu\pi})\left[1 + e^{\mu\pi} - 1 + e^{\mu\pi}\right] \neq 0}\right]$$

Hence, $A = B = 0$ is the only solution.

$\therefore$ There doesnot exists any negative eigenvalue.

Theorem: Eigenvalues of a PSLBVP are real.

Let, $\lambda = a + ib$ is an eigenvalue and $y_1 + iy_2 = y_\lambda$ is corresponding eigenfunction.

$$L(y_1 + iy_2) + (a + ib)r(y_1 + iy_2) = 0 \Rightarrow \begin{cases} Ly_1 + (ay_1 - by_2)r = 0 & \text{— (iii)} \\ Ly_2 + (by_1 + ay_2)r = 0 & \text{— (iv)} \end{cases}$$

$$\int_{\alpha}^{\beta} \left[p(x)\, W(y_1, y_2)(x)\right]' = b \int_{\alpha}^{\beta} (y_1^2 + y_2^2)\, dx$$

Now, $y(\alpha) = y(\beta)$; $y'(\alpha) = y'(\beta)$

Note, $p(x)\, W(y_1, y_2)(x) \Big|_{\alpha}^{\beta} = p(\beta)\left[y_1 y_2' - y_2 y_1'\right]\Big|_{\beta} - p(\alpha)\left[y_1 y_2' - y_2 y_1'\right]\Big|_{\alpha}$

$$= p(\beta)\left[y_1(\beta) y_2'(\beta) - y_2(\beta) y_1'(\beta)\right] - p(\alpha)\left[y_1(\alpha) y_2'(\alpha) - y_2(\alpha) y_1'(\alpha)\right]$$

$$= p(\beta)\left[y_1(\beta) y_2'(\beta) - y_2(\beta) y_1'(\beta) - y_1(\alpha) y_2'(\alpha) + y_2(\alpha) y_1'(\alpha)\right] = 0$$

$\therefore\ b = 0$ and hence all eigenvalues are real.

Question : Is theorem 2 and 3 valid for periodic SLBVP.

thm2,not holds
see note on pg16

thm3 will hold, after green's identity expand RHS and use periodiccondns
(Fun Fact:) To let thm3 hold we'd added p(alpha)=p(beta) in periodiccondns

Before we proceed with the other properties of RSLBVP and PSLBVP let us look at the following problem

(0_0) $y'' + \lambda y = 0$

$y(0) = 0$ and $|y(x)| \leq M < \infty \ \forall x \in (0, \infty)$

When, $\lambda = \mu^2$ $(\mu \neq 0)$

$\therefore y(x) = A\cos\mu x + B\sin\mu x.$

$0 = y(0) = A + B \cdot 0 \Rightarrow A = 0$

$\therefore y(x) = B\sin\mu x.$

$\therefore$ Eigenpair $\{(\mu ; \sin\mu x) ; \mu \in \mathbb{R}\}$.

$\therefore$ The spectrum is no longer discrete for singular BVP.

We next show that the spectrum of RSLBVP and PSLBVP are discrete.

?

<u>Application</u> :-

$$u_{xx} - u_{tt} = 0$$

$$u(x,0) = f(x).$$

$$u_t(x,0) = g(x).$$

$$u(0,t) = u(l,t) = 0.$$

Use seperation of variable to solve it.

i.e., $u(x,t) = X(x)\,T(t)$

Fourier Expansion :-

Note, $x \in \mathbb{R}^n$ may be written as $x = \sum_{i=1}^{n} x_i e_i$.

(x_i are called co-ordinates of x)

Now since $\langle e_i, e_j \rangle = e_i \cdot e_j = 0$ for $i \neq j$

Hence, $\langle x, e_m \rangle = \sum_{i=1}^{n} x_i \langle e_i, e_m \rangle = x_m \|e_m\|^2 = x_m$ ($\because$ e_m is an unit vector)

$\therefore$ $x = \sum_{m=1}^{n} \langle x, e_m \rangle e_m$.

We want to generalize this idea to infinite dimensions.

Question :- Given $\{\phi_n\}_{n=1}^{\infty}$ be an orthogonal set of functions in $I := [\alpha, \beta]$, then can one write an arbitrary function $f(x) \simeq \sum_{n=0}^{\infty} c_n \phi_n(x)$

Note :- Due to the infinite nature of the above sum, one needs to ask the mode of convergence.

Without asking the question of convergence,

$$\int_\alpha^\beta r(x)\phi_m(x) f(x)\,dx = \int_\alpha^\beta \sum_{n=0}^{\infty} c_n r(x)\phi_n(x)\phi_m(x)\,dx$$

Assuming integration and sum be interchanged,

$$\int_\alpha^\beta r(x)\phi_m(x) f(x)\,dx = \sum_{n=0}^{\infty} \int_\alpha^\beta r(x)\phi_m(x)\phi_n(x)\,dx$$

$$= c_m \int_\alpha^\beta r(x)\phi_m^2(x)\,dx.$$

$$= c_m \|\phi_m\|_X^2$$ where X is the ambient vector space.

$$\therefore \boxed{c_m = \frac{\int_\alpha^\beta r(x)\phi_m(x) f(x)\,dx}{\|\phi_m\|_X^2}} \left(= \int_\alpha^\beta r(x)\phi_m(x) f(x)\,dx \text{, if } \{\phi_m\} \text{ are } \cancel{\text{orthogonal}} \text{ orthonormal} \right)$$

c_m's are called Fourier coefficients. [Note: If $\sum_{n=0}^{\infty} c_n\phi_n(x)$ converges uniformly to $f(x)$ in $[\alpha,\beta]$, then all the above procedure].

We shall write, $\boxed{f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x)}$

Example :- $\{1, \cos\frac{n\pi x}{\ell}, \sin\frac{n\pi x}{\ell} ; L>0 \text{ and } n \geq 1\}$ is orthogonal w.r.t $r(x) \equiv 1$ in $[-L, L]$.

Note, $\int_{-\ell}^{\ell} \cos^2\frac{n\pi x}{\ell}\, dx = \begin{cases} 2\ell, & n=0 \\ \ell, & n=1. \end{cases}$

and, $\int_{-\ell}^{\ell} \sin^2\frac{n\pi x}{\ell}\, dx = \ell, \quad n \geq 1$

$\therefore$ Given $f \in \{\text{Nice Enough}\}$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

where, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{\ell}\, dx$; $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{\ell}\, dx$; $n \geq 1$.

Convergence of Fourier Series to $f(x)$:

Assuming, $\{\phi_n(x)\}_{n=0}^{\infty}$ and $f(x)$ are piecewise continuous in I.

Denote, $S_N(x) = \sum_{n=0}^{N} c_n \phi_n(x)$ and consider $|S_N(x) - f(x)|$ for various N and $x \in I$

If for $\epsilon > 0$, $\exists\, N(\epsilon) > 0$ s.t $|S_N(x) - f(x)| < \epsilon$ then the Fourier series converges uniformly to $f(x)$ for all $x \in I$.

On the other hand if $N(x, \epsilon)$, the the convergence is pointwise.

Definition :- (Convergence in mean)

We say $\{\phi_n(x)\}$ converges in mean to $\psi(x)$ (w.r.t to the weight function $r(x)$) if

$$\boxed{\lim_{n\to\infty} \|\psi_n - \psi\|^2 = \lim_{n\to\infty} \int_{\alpha}^{\beta} r(x) \left| \psi_n(x) - \psi(x) \right|^2 dx = 0}$$

Thus the Fourier Series converges to $f(x)$ in the mean provided,

$$\lim_{n\to\infty} \int_{\alpha}^{\beta} r(x) \, |S_n(x) - f(x)|^2 \, dx = 0 .$$

Before we prove the convergence of Fourier Series, consider

$$f(x) = \sum_{n=0}^{\infty} d_n \phi_n(x)$$ where d_n are not necessarily the Fourier Coefficients.

Let, $$T_n(x; d_0, d_1, \ldots, d_N) = \sum_{n=0}^{N} d_n \phi_n(x)$$

and let $e_n = \|T_n - f\|$, then

$$e_N^2 = \|T_N - f\|^2 = \int_{\alpha}^{\beta} r(x) \left(\sum_{n=0}^{N} d_n \phi_n(x) - f(x) \right)^2 dx$$

$$= \sum_{n=0}^{N} d_n^2 \int_{\alpha}^{\beta} r(x) \phi_n^2(x) \, dx - 2 \sum_{n=0}^{N} d_n \int_{\alpha}^{\beta} r(x) \phi_n(x) f(x) \, dx + \int_{\alpha}^{\beta} r(x) \cdot f^2(x) \, dx .$$

$$= \sum_{n=0}^{N} d_n^2 \|\phi_n\|^2 - 2\sum_{n=0}^{N} d_n c_n \|\phi_n\|^2 + \|f\|^2$$

$$= \sum_{n=0}^{N} \|\phi_n\|^2 (d_n - c_n)^2 - \sum_{n=0}^{N} \|\phi_n\|^2 c_n^2 + \|f\|^2 \longleftarrow (*)$$

$\therefore e_N$ is least when $d_n = c_n$ for $n = 0, 1, 2, \dots, N$.

Hence one has,

Theorem :- For any $N \in \mathbb{N}$, the best approximation in mean to f by an expression of the form $\sum_{n=0}^{\infty} d_n \phi_n(x)$ is obtained when d_n are the Fourier coefficients of $f(x)$.

Again from $(*)$, if $d_n = c_n$; $n = 0, 1, 2, \dots, N$ we obtain

$$\|S_n - f\|^2 = \|f\|^2 - \sum_{n=0}^{N} \|\phi_n\|^2 c_n^2 .$$

$$\therefore \|T_n - f\|^2 = \sum_{0}^{N} \|\phi_n\|^2 (d_n - c_n)^2 + \|S_n - f\|^2 \longleftarrow (**)$$

$\Rightarrow \|T_n - f\| \geq \|S_n - f\|$ and since $\sum_{n=0}^{\infty} d_n \phi_n$ converges in mean to $f(x)$; $\|T_n - f\|^2 \to 0$ as $n \to \infty$

$\Rightarrow \|s_n - f\| \to 0$ as $n \to \infty$.

But then (**) implies $\lim_{n\to\infty} \sum_{n=0}^{N} \|\varphi_n\|^2 (d_n - c_n)^2 = 0$

$$\Downarrow$$

$d_n = c_n$; $n = 0, 1, 2 \ldots$

Theorem :- If a series $\sum_{n=0}^{\infty} d_n \varphi_n(x)$ converges in mean to $f(x)$, then the coefficients d_n must be the Fourier series of $f(x)$.

Remark :- $\|S_N - f\|^2 = \|f\|^2 - \sum_{n=0}^{N} \|\varphi_n\|^2 c_n^2$

$$\geq \|f\|^2 - \sum_{n=0}^{N+1} \|\varphi_n\|^2 c_n^2$$

$$= \|S_{N+1} - f\|^2 \geq 0$$

$\therefore \{\|S_n - f\|, n = 0, 1 \ldots\}$ is nonincreasing and bounded below by zero.

If the sequence converges to zero then the Fourier Series of $f(x)$ converges in mean to $f(x)$.

Further, $\|f\|^2 \geq \sum_{n=0}^{N} \|\phi_n\|^2 c_n^2$ for all $N \in \mathbb{N}$.

$$\Rightarrow \|f\|^2 \geq \sum_{n=0}^{\infty} \|\phi_n\|^2 c_n^2$$

If $\{\varphi_n\}$ are orthonormal we have $\boxed{\sum_{n=0}^{\infty} c_n^2 \leq \|f\|^2 \quad \text{(Bessel's Inequality)}}$.

Hence we proved the following :-

Fourier Convergence Theorem :-

Let $\{\phi_n\}$ be a set of orthonormal set and c_n are the Fourier coefficients of 'f' given by
$c_n = \int_\alpha^\beta r(x)\phi_n(x)f(x)\,dx$ then the following holds :-

(a) $\sum_0^\infty c_n^2$ converges and $\lim c_n = 0$

(b) The Fourier series of f converges in the mean to $f(x)$ iff $\|f\|^2 = \sum_{n=0}^\infty c_n^2$. (Parseval Identity).

(c) Bessel Inequality holds.

Denote $C_p[\alpha,\beta]$ be the space of piecewise continuous function in $[\alpha,\beta]$.

The orthonormal set $\{\phi_n\}$ is said to be complete in $C_p[\alpha,\beta]$ if for every $f \in C_p[\alpha,\beta]$ its Fourier series converges in the mean to 'f'.

Property of orthogonal set :-

If an orthogonal set $\{\phi_n(x) ; n=0,1\ldots\}$ is complete in $C_p[\alpha,\beta]$, then any $f \in C_p[\alpha,\beta]$ that is orthogonal to every $\phi_n(x)$ must be zero except possibly at a finite no of points in $[\alpha,\beta]$.

Proof :- Let $\{\phi_n\}$ be orthonormal and f is orthogonal to every $\phi_n(x)$, then

$$c_n := \langle f, \phi_n \rangle = 0 \quad (\text{Fourier coefficient of } f)$$

$\therefore$ By Parseval,

$$\|f\|^2 = \sum_{n=0}^{\infty} c_n^2 = 0$$

$\Rightarrow f$ must be zero only at finitely many zero.

$f_n : [0,1] \to \mathbb{R}$ given by

$$f_n(x) = x^n \longrightarrow f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

$$\int_0^1 |f_n - f|^2 = \int_0^1 |x^n - 0|^2 dx = \int_0^1 x^{2n} = \frac{x^{2n+1}}{2n+1}\Big|_0^1 = \frac{1}{2n+1} \to 0 \text{ as } n \to \infty.$$

Uniform convergence $\Leftarrow\!\!\!\not\Rightarrow$ L^2 convergence

But if $f_n \xrightarrow{u} f$ then given $\varepsilon > 0$, $\exists N(\varepsilon) > 0$ s.t

$|f_n(x) - f(x)| < \varepsilon$. $\forall n \geq N$ and $x \in I$

and, $\displaystyle\int_0^1 |f_n(x) - f(x)|^2 dx < \varepsilon^2 \int_0^1 dx = \varepsilon^2 \quad \forall n \geq N.$

$\Downarrow$

f_n converges to f in mean.

Maximum Principle :-

Theorem 1 :- Let $y \in C^2[\alpha, \beta]$, $y''(x) \geq 0$ in (α, β) and y(x) attains its maximum at the interior pt of $[\alpha, \beta]$, then y(x) must be identically constant in $[\alpha, \beta]$.

Proof :- Let $y''(x) > 0$ in (α, β) ; if y(x) attains its maximum at $x_0 \in (\alpha, \beta)$ then

$y'(x_0) = 0$ and $y''(x_0) \leq 0$.

— a contradiction.

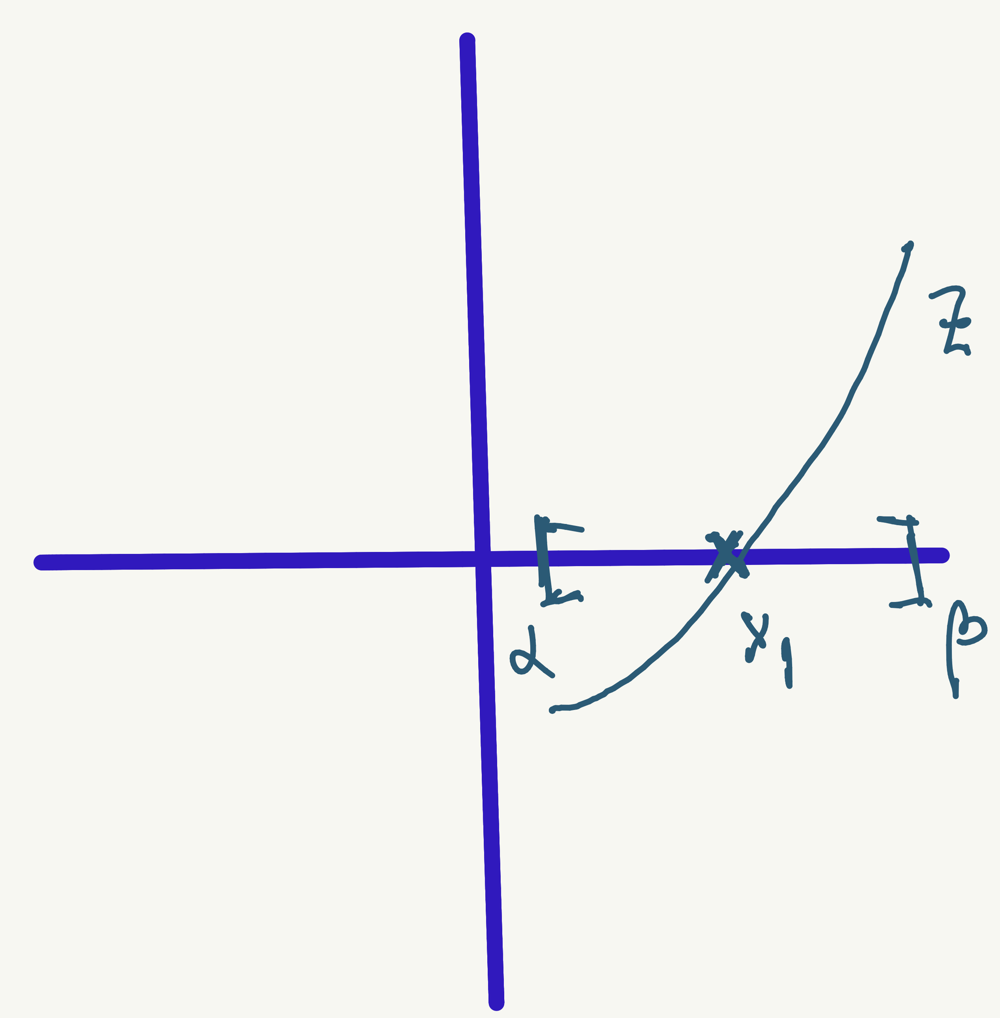
Now suppose $y''(x) \geq 0$ in (α, β) and that $y(x)$ attains its maximum at $x_0 \in (\alpha, \beta)$ say x_1.

If $y(x_1) = M$ then $y(x) \leq M$ in $[\alpha, \beta]$.

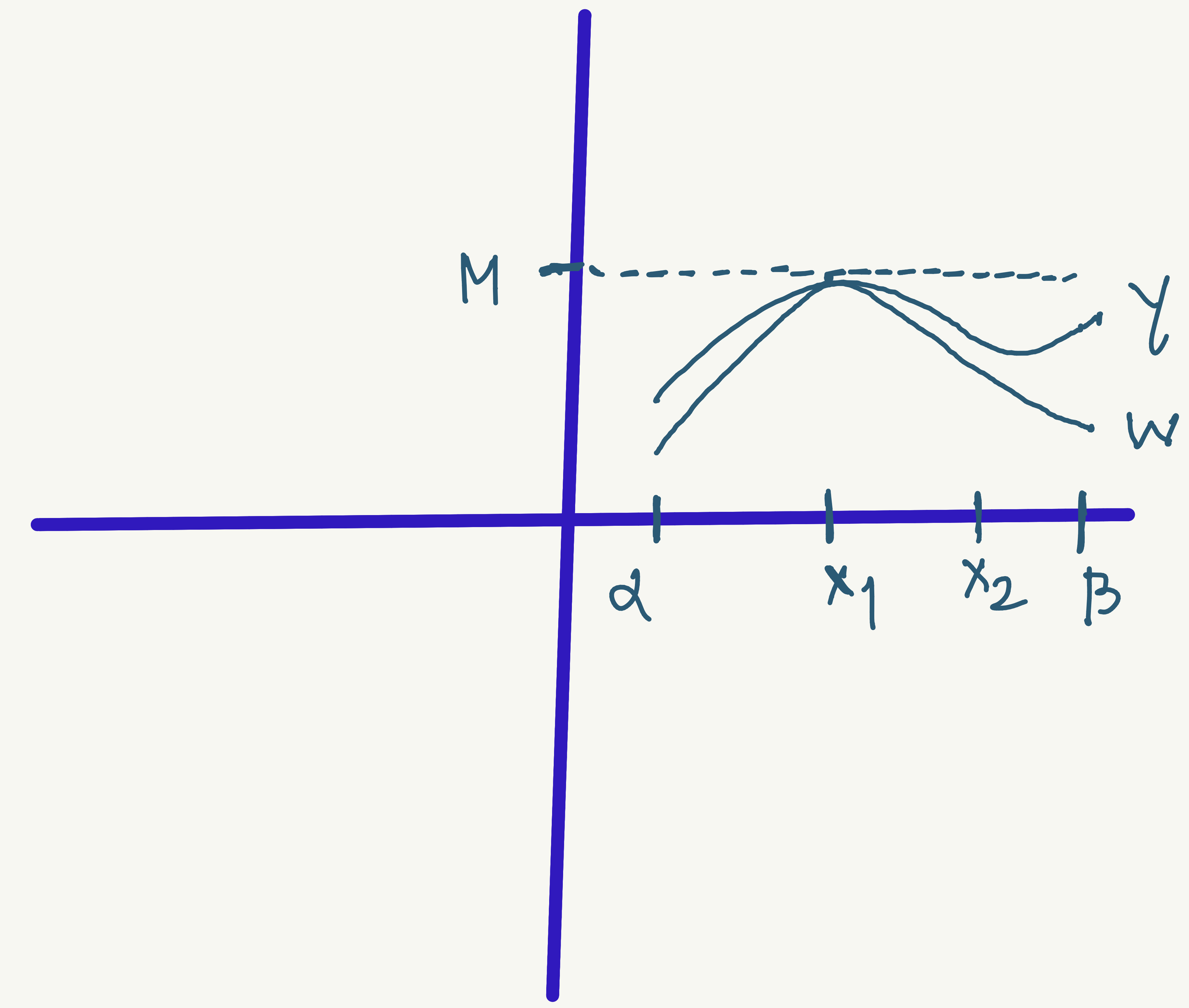
~~But~~ $\exists\, x_2 \in (\alpha, \beta)$ s.t $y(x_2) < M$. If $x_2 > x_1$ then we set $z(x) = \exp[\gamma(x - x_1)] - 1$; $\gamma > 0$.

$\therefore z(x) < 0$ for $x \in [\alpha, x_1)$; $z(x_1) = 0$ and $z(x) > 0$ for $x \in (x_1, \beta]$

and, $z''(x) = \gamma^2 \exp(\gamma(x - x_1)) > 0$ for $x \in [\alpha, \beta]$.



We define, $w(x) = y(x) + \varepsilon z(x)$ where $0 < \varepsilon < \dfrac{M - y(x_2)}{z(x_2)}$



$\because$ $y(x_2) < M$ and $z(x_2) > 0$ such an $\varepsilon > 0$ always exists.

$\therefore$ $w(x) < y(x) \leq M$, $x \in (\alpha, x_1)$

and, $w(x_2) = y(x_2) + \varepsilon z(x_2) < M$ and $w(x_1) = M$.

$\because$ $w''(x) = y''(x) + \varepsilon z''(x) > 0$ in $[\alpha, x_2]$.

$\Rightarrow$ w cannot attain a maximum in the interior of $[\alpha, x_2]$

$\Rightarrow$ w must attain its maximum which is greater than or equal to M at an interior of (α, x_2)

coz w(x1)=M and w(alpha),w(x2) < M

– which is a contradiction.

$\therefore$ $\nexists$ $x_2 \in (\alpha, \beta)$; $x_2 > x_1$ s.t $y(x_2) < M$.

If $x_2 < x_1$ then choose $z(x) = \exp(-\gamma(x - x_1)) - 1$ and use same argument.

$\boxtimes$

We want to extend this idea to $y''+p_0(x)y'+p_1(x)y \geq 0$.

Remark :- $y''+y=0$ has a solution $y(x)=\sin x$ which attains its maximum at $x=\pi/2$, where the equation is defined on $[0,\pi]$.

Theorem 2 :- Let y satisfies $y''+p_0(x)y'+p_1(x)y \geq 0$; $x\in(\alpha,\beta)$ in which $p_0(x)$ and $p_1(x)$ are bounded. Assuming $p_1(x)\leq 0$ in every closed subinterval of (α,β). If $y(x)$ assumes a non-negative maximum in the interior then $y(x)\equiv M$.

Proof :- Let $y''+p_0(x)y'+p_1(x)y>0$ with $p_1\leq 0$.

If $x_0\in(\alpha,\beta)$ s.t $y(x_0)=\max\limits_{[\alpha,\beta]} y$ then $y'(x_0)=0$ and $y''(x_0)\leq 0$

$\therefore$ $y''(x_0)+p_0(x_0)y'(x_0)+p_1(x_0)y(x_0)\leq 0$ – a contradiction.

The rest of the proof follows along the same line as Theorem 1.

Application :-

(a) $y'' = f$; $y(0) = A$ and $y(1) = B$.

Given f is continuous and bounded in $[0,1]$. How many solutions does it have?

Let y_1 and y_2 be any two solution s.t $y_1(x_0) \neq y_2(x_0)$ for some $x_0 \in [0,1]$. WLOG, let $y_1(x_0) > y_2(x_0)$

Define, $\varphi := y_1 - y_2$

$\therefore \varphi'' = y_1'' - y_2'' = 0$

and, $\varphi(0) = \varphi(1) = 0$; $\varphi(x_0) > 0$

By Maximum Principle, If ϕ attains its maximum in the interior then φ is constant.

$\therefore \varphi(x) = \text{constant} = \varphi(0) = 0$

$\therefore y_1(x) = y_2(x)$

$\blacksquare$

W9C1 :- BACK TO SQUARE ONE

PICARD'S FUNDAMENTAL EXISTENCE & UNIQUENESS THEOREM :-

Consider the system
$$\left.\begin{aligned} X' &= F(t,X) \\ X(0) &= X_0 \end{aligned}\right\} \text{— } (1)$$

Let $\Omega (\subseteq \mathbb{R}^n)$ be an open set containing X_0. If F is continuous w.r.t 't' and locally Lipchitz w.r.t X on Ω, then $\exists$ an interval $(-\eta,\eta)$ and a C^1- function $x : (-\eta,\eta) \to \mathbb{R}^n$ such that (1) is satisfied.

(Maximum Interval of Existence)

Consider the system of ODE $x' = F(x)$; $x(0) = x_0$. (F is locally Lipchitz). — (1)

Q:- Find the maximum interval of existence.

Motivation :-

Consider the following problem

$$y' = y^2 \; ; \; y(0) = 1 \quad \text{—— (II)}$$

Clearly, $y(x) = \frac{1}{1-x}$ solves the equation and the maximum interval of existence is given as $(-\infty, 1)$. (By Direct Calculation).

Note by Picard's theorem (II) has a unique solution for $|x| < \min(a, b/M)$ where $M = \max_{S} f(x,y)$ and $S := \{(x,y) \in \mathbb{R}^2 : |x| \le a, |y-1| \le b\}$. the solution exists

f(x,y) = y^2 here ∵ $\max_{S} f(x,y) = (1+b)^2$ and hence for $|x| < \min\left(a, \frac{b}{(1+b)^2}\right)$

Note :- $(1-b)^2 \geq 0 \Rightarrow 1 + 2b + b^2 - 4b \geq 0 \Rightarrow \frac{(1+b)^2}{b} \geq 4 \Rightarrow \frac{1}{4} \geq \frac{b}{(1+b)^2}$

Hence, one may conclude the existence of an unique solution $y_1(x)$ for $|x| \le 1/4$.

Consider the equation :-

$$y' = y^2 \; ; \; y(1/4) = 4/3. \quad \text{—} \; (1.I)$$

Define $S := \{ |x - 1/4| \le a \; ; \; |y - 4/3| \le b \}$ and $\max_S f(x,y) = (b + 4/3)^2$.

$\therefore \exists \; y_2$ solving (1.I) in $|x - 1/4| < \min\left(a, \frac{b}{(b+4/3)^2}\right)$.

$$\because \frac{b}{(b+4/3)^2} \le 3/16$$

One can conclude y_2 to exist in $|x - 1/4| \le 3/16$.

$$\therefore y(x) = \begin{cases} y_1(x) \; ; & -1/4 \le x \le 1/4 \\ y_2(x) \; ; & 1/4 \le x \le 7/16. \end{cases}$$

One may continue the process; but how far can it be continued.

Some Preliminaries :-

Theorem :- Let $F : \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is locally Lipschitz. Let x_1, x_2 be two solutions of the I.V.P

$$x' = F(x) \; ; \; x(0) = x_0$$

defined for $\alpha_j < t < \beta_j$, $j = 1, 2$. Then for all t s.t $\max\{\alpha_1, \alpha_2\} < t < \min\{\beta_1, \beta_2\}$

where both solutions are defined, $x_1(t) = x_2(t)$.

Proof :- Define, $g(t) := |x_1(t) - x_2(t)| \; ; \; 0 \le t < \min\{\beta_1, \beta_2\}$

$$\le \int_0^t |F(x_1(s)) - F(x_2(s))| \, ds$$

For $t \in [0, T]$ where $T < \min\{\beta_1, \beta_2\}$.

$\because [0, T]$ is compact, $K = x_1[0, T] \cup x_2[0, T]$ is compact

Define L as the Lipchitz constant for F on K.

$\therefore |F(x_1(s)) - F(x_2(s))| \le L |x_1(s) - x_2(s)| = L\, g(s)$ for $0 \le s \le T \Rightarrow g(t) \le L \int_0^t g(s) \, ds$

From Gronwall we have $g(t) \le 0$ for $0 \le t \le T$

But since g is non-negative, $g \equiv 0 \Rightarrow x_1(t) = x_2(t)$ for $0 \le t \le T$.

$\because$ T may be arbitrarily close to $\min\{\beta_1, \beta_2\}$ we have our conclusion.

Our 1^{st} result in this direction is as follows :-

Th :- Let $F : \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is locally Lipschitz in Ω. Given $x_0 \in \mathbb{R}^n$, $\exists\, x^* : (-\alpha_*, \beta_*) \to \Omega$ of the I.V.P $x' = F(x)$; $x(0) = x_0$ — ① that is maximal in the following sense :

If another function x solves ① for $t \in I$ (open) then

ⓐ $I \subset (-\alpha_*, \beta_*)$ and ⓑ $x(t) = x^*(t)$ for $t \in I$.

Remark :- It may happen that either α_* or β_* or both equals infinity. The maximum interval of existence is always open, even if α_* or β_* are finite.

Proof :- We only focus on β_* and $t \geq 0$.

Let $\beta_* = \sup\{\beta : (1) \text{ is solvable for } 0 \leq t < \beta\}$

By Picard's theorem $\beta_* > 0$.

For $n = 1, 2, \ldots$ choose solutions x_n of (1) that exists for $t \in [0, \beta_n)$, where

$\beta_n \to \beta_*$ (finite or infinite)

Define, x^* on $[0, \beta_*)$ choose n s.t $\beta_n > t$ and let

$$x^*(t) := x_n(t)$$

The above definition does not depends on the choice of n (uniqueness) and x^* solves (1).

Theorem:- Let the maximal solution $x^*:(-\alpha_*, \beta_*) \to \mathbb{R}^n$ be such that $\beta_* < \infty$. Then for any compact set $K (\subset \Omega)$, $\exists\, \varepsilon > 0$ s.t $x^*(t) \notin K$ for $\beta_* - \varepsilon < t < \beta_*$.

Proof :- Assignment.

Sufficient condition for Global Existence :- (locally lipschitz + bdd. by some linear function)

Th:- If $F:\mathbb{R}^n \to \mathbb{R}^n$ is locally Lipchitz and if $\exists\, B, K (\geq 0)$ s.t $|F(x)| \leq K|x| + B$, $x \in \mathbb{R}^n$ then the solution $x(t)$ of (1) exists for all time $-\infty < t < \infty$ and moreover

$$|x(t)| \leq |b|\, e^{K|t|} + \frac{B}{K}\left(e^{K|t|} - 1\right) \quad ; -\infty < t < \infty .$$

Proof:- We will do it for $t > 0$.

Let (1) has a solution for $t \in [0, \beta)$, then $x(t) = x_0 + \int_0^t F(x(s))\, ds$.

Define, $z(t) := |x(t)|$

$\therefore\ g(t) \le |x_0| + \int_0^t [K g(s) + B]\, ds \ ; \ 0 \le t < \beta.$

Recall Gronwall Lemma :

$g : [0,T] \to \mathbb{R}$ is continuous and $C, K \geq 0$ be s.t. $g(t) \le C + K\int_0^t g(s)\, ds \ ; \ 0 \le t \le T$

then $g(t) \le C e^{Kt} \ ; \ 0 \le t \le T.$

Generalized Gronwall :- If $g(t) \le C + Bt + K\int_0^t g(s)\, ds \Rightarrow g(t) \le Ce^{Kt} + \frac{B}{K}(e^{Kt} - 1)$

$\therefore\ |x(t)| \le |x_0| e^{Kt} + \frac{B}{K}(e^{Kt} - 1)$ for $0 \le t < \beta.$

Hence the solution exists for $0 \le t < \beta.$

Let x^* be the maximal solution of (1) defined on $[0, \beta_*)$ and let $\beta_* < \infty.$

Clearly, $x^*(t) \in K := \left\{ z \in \mathbb{R}^n \,/\, |z| \le |x_0| e^{Kt} + \frac{B}{K}(e^{Kt} - 1) \right\}$ for all $t \in [0, \beta_*].$

— a contradiction.

Stability of solution :-

Consider the problem $y' = y$; $y(0) = y_0$ — ①

① has an unique solution given by $y(x, 0, y_0) = y_0 e^x$.

It follows that, $|\Delta y| = |y(x; 0, y_0) - y(x; 0, y_0 + \Delta y_0)| = |y_0 e^x - (y_0 + \Delta y_0) e^x| = |\Delta y_0 e^x| = |\Delta y_0| e^x$

Change in y

If $x \in [-\varepsilon, \varepsilon]$ then Δy can be made sufficiently small if Δy_0 is small, but if $x \to \infty$ then $|\Delta y|$ may not be bounded ; even for small values of $|\Delta y_0|$.

Lyapunov Stability :- Consider the system $X' = F(X)$; $X(0) = X_0$ —— (*)

Defn 1 :- A solution $X(t) = X(t; 0, X_0)$ of (*) is said to be stable if for each $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon, 0) > 0$ such that $\|\Delta X^0\| < \delta \Rightarrow \|X(t; 0, X_0 + \Delta X_0) - X(t; 0, X_0)\| < \varepsilon$.

initial condn ke respect m cont.

Defn 2:- A solution $x(t) = x(t;0,x_0)$ of ① is said to be unstable if it is not stable.

Defn 3:- A solution $x(t) = x(t;0,x_0)$ of ① is asymptotically stable if it is stable and $\exists\ \delta_0 > 0$ s.t if $\|\Delta x_0\| < \delta_0 \Rightarrow \|x(t;0,x_0+\Delta x_0) - x(t;0,x_0)\| \to 0$ as $t \to \infty$.

Ex:- ⓐ Every solution of $y' = x$ is of the form $y(x) = y(0) - x_0^2/2 + x^2/2$, hence it is stable but not bounded.

ⓑ Every solution of $y' = 0$ is of the form $y(x) = y(0)$ and hence stable but not asymptotically stable.

ⓒ Every solution of $y' = p(x)y$ is of the form $y(x) = y(0)\exp\left(\int_0^x p(s)\,ds\right)$ and hence its trivial solution $y(x) \equiv 0$ is asymptotically stable iff $\int_0^x p(s)\,ds \to -\infty$ as $x \to \infty$.

Theorem :- All solution of $x' = A(t)x$ are stable iff they are bounded.

Proof : Let $\psi(t)$ be the Fundamental matrix of $x' = A(t)x$.

If all solutions are bounded then $\|\psi(t)\| < M$ for all $t \geq 0$

$\therefore$ Given $\epsilon > 0$, choose $\|\Delta x_0\| < \epsilon / M\|\psi^{-1}(0)\| = \delta(\epsilon) > 0$ so that

$$\|x(t;0,x_0) - x(t;0,x_0+\Delta x_0)\| = \|\psi(t)\psi^{-1}(0)\Delta x_0\|$$

$$\leq M\|\psi^{-1}(0)\|\,\|\Delta x_0\|$$

$$< \epsilon,$$

($x_0 \in \mathbb{R}^n$)

Conversely, If all solutions are stable then in particular the trivial solution $x(t,0,0) \equiv 0$ is stable.

Initial pt in $\mathbb{R}$.

$\therefore$ Given $\epsilon > 0$, $\exists\ \delta = \delta(\epsilon) > 0$ s.t $\|\Delta x_0\| < \delta \Rightarrow \|x(t;0,\Delta x^0)\| < \epsilon\ \ \forall\ t \geq 0$.

But, $x(t;0,\Delta x_0) = \psi(t)\psi^{-1}(0)\Delta x_0$ gives $\|\psi(t)\psi^{-1}(0)\Delta x_0\| < \epsilon$

Let, $\Delta x^0 = \frac{\delta}{2} e_i$ (i^{th} unit vector) then, $\|\psi(t)\psi^{-1}(0)\Delta x_0\| = \frac{\delta}{2}\|\psi_i(t)\| < \epsilon$. ($\psi_i$ is the i^{th} column of $\psi(t)\psi^{-1}(0)$)

$\therefore \|\psi(t)\psi^{-1}(0)\| = \max_{1\le i\le n} \|\psi_i(x)\| \le 2\epsilon/\delta$.

Hence, for any Solution $x(t;0,x_0)$ of $X' = A(t)X$,

$$\|x(t;0,x_0)\| = \|\psi(t)\psi^{-1}(0)x_0\| < 2\epsilon/\delta \|x_0\|.$$

$\therefore$ All solutions are bounded.

<u>Theorem</u> :- Let $\psi(t)$ be the Fundamental matrix of the system $X' = A(t)X$. Then all solutions of $X' = A(t)X$ are asymptotically stable iff $\|\psi(t)\| \to 0$ as $t \to \infty$.

Proof :- Every solution of $X' = A(t)X$ may be expressed as $X(t,0,x_0) = \psi(t)\psi^{-1}(0)X_0$.

$\because$ ψ is continuous and $\|\psi(t)\| \to 0$ as $t \to \infty$, $\exists M \in \mathbb{R}^+$ such that $\|\psi(t)\| < M \ \forall t \ge 0$.

Thus, $\|X(t;0,x_0)\| = \|\psi(t)\psi^{-1}(0)x_0\| \le \|\psi(t)\|\|\psi^{-1}(0)\|\|x_0\| < M\|\psi^{-1}(0)\|\|x_0\|$ — Hence all solutions are bdd $\Rightarrow$ all solutions are stable.

$$\|x(t;0,x_0+\Delta x_0) - x(t;0,x_0)\| = \|\psi(t)\psi^{-1}(0)\Delta x_0\|$$

$$\le \|\psi(t)\|\,\|\psi^{-1}(0)\Delta x_0\| \longrightarrow 0 \text{ as } t\to\infty$$

$\therefore$ Every solution is asymptotically stable.

<u>Conversely</u>, if all solutions of $x' = A(t)x$ are asymptotically stable then in particular the trivial solution $x(t;\overset{(\mathbb{R})}{0},\underset{(\mathbb{R}^n)}{0}) \equiv 0$ is A·S (Asymptotically stable)

Hence, $\|x(t;0,\Delta x_0)\| \to 0$ as $t\to\infty$.

$\Rightarrow \|\psi(t)\| \to 0$ as $t\to\infty$

W10 C1 :-

Let Ω be an open subset of $\mathbb{R}^n$ containing x_0 and $f \in C^1(\Omega)$. Let $(-\alpha_*, \beta_*)$ be the maximal interval of existence of the solution $\tilde{x}(t)$ of the I.V.P $x' = f(x)$; $x(0) = x_0$.

Assume $\beta_* < \infty$, then given any compact set $K \subset \Omega$, $\exists$ $t \in (\alpha, \beta)$ s.t $x(t) \notin K$.

Proof :- $f \in C^1(\Omega) \Rightarrow f \in C(K)$ and define $M = \max_K |f|$.

On the contrary let $\tilde{x}(t) \in K$ $\forall$ $t \in (-\alpha_*, \beta_*)$.

Note :- $\lim_{t \to \beta_*^-} \tilde{x}(t)$ exists. ($\because$ If $-\alpha_* < t_1 < t_2 < \beta_*$ then $|\tilde{x}(t_1) - \tilde{x}(t_2)| \leq \int_{t_1}^{t_2} |f(\tilde{x}(s))| ds \leq M(t_2 - t_1)$).

Define $x_1 = \lim_{t \to \beta_*^-} \tilde{x}(t)$. Then $x_1 \in K$

Define, $u : [\alpha_*, \beta_*] \xrightarrow{C^1} \mathbb{R}^n$ by

$$u(t) = \begin{cases} \tilde{x}(t) & , t \in [\alpha_*, \beta_*) \\ x_1 & , t = \beta_* \end{cases}$$

$\therefore\ u(t) = u(0) + \int_0^t f(u(s))\,ds.$

$\Downarrow$

$u'(\beta_*) = f(u(\beta_*)).$

i.e., $u(t)$ is the solution of ① on $[\alpha_*, \beta_*]$.

$\because\ x_1 \in K \subset \Omega$, the problem $x' = f(x)$; $x(\beta_*) = x_1$ has an unique solution $x_1(t)$ on $(\beta_* - \varepsilon, \beta_* + \varepsilon)$; $\varepsilon > 0$

Hence, $x_1(t) = \begin{cases} u(t) & \text{on } (\beta_* - \varepsilon, \beta_*) \\ u(\beta) = x_1 & \text{on } t = \beta_*. \end{cases}$

Define, $v(t) = \begin{cases} u(t), & t \in (-\alpha_*, \beta_*] \\ x_1(t), & t \in [\beta, \beta + \varepsilon) \end{cases}$

Then $v(t)$ is a sol^n of ① in $(-\alpha_*, \beta+\varepsilon)$.

— Contradiction.

$\boxed{N}$

Recall :-

(a) A solution $x(t) := x(t, 0, x_0)$ of the I.V.P $x' = F(t,x)$; $x(0) = x_0$ is said to be stable if for each $\varepsilon > 0$, $\exists\, \delta(\varepsilon, x_0) > 0$ s.t $\|\Delta x_0\| < \delta \Rightarrow \|x(t,0,x_0+\Delta x_0) - x(t,0,x_0)\| < \varepsilon$.

Ex :- $y' = 0$ has all stable solutions.

(b) If a solution is not stable then its called unstable.

Ex :- $y' = y$ has a solution given by $y(t) = e^t$; hence this solution is unstable.

(c) A solution $x(t) := x(t,0,x_0)$ of $\boxed{N}$ is said to be asymptotically stable if it is stable and $\exists\, \delta_0 > 0$ s.t $\|\Delta x_0\| < \delta \Rightarrow \|x(t,0,x_0+\Delta x_0) - x(t,0,x_0)\| \to 0$ as $t \to \infty$.

Ex :- $y' = p(x)y$ has solutions of the form $y(x) = y(x_0) \exp\left(\int_{x_0}^{x} p(s)\,ds\right)$ and hence its trivial solution $y(x) \equiv 0$ is A.S iff $\int_0^x p(s)\,ds \to -\infty$ as $x \to \infty$.

For the linear system, $\boxed{x' = A(t)x}$ L_H with $\Psi(t)$ being its fundamental solution, we have the following result :-

(i) All solutions of (L_H) are stable iff they are bounded.

(ii) All solutions of (L_H) are A.S iff $\|\Psi(t)\| \to 0$ as $t \to \infty$.

COROLLARY :- (iii) If the real parts of the multiple eigenvalues of A are negative and the real part of the simple eigenvalues of A is non-positive, then all solutions of $x' = Ax$ are stable (A = constant Matrix)

(iv) If the real parts of the eigenvalues of a matrix A are negative, then all solutions of $x' = Ax$ are A.S.

Remark :- Note that any solution $x(t,0,x_0)$ of (L_H) is stable $\Rightarrow$

Given $\varepsilon > 0$, $\exists \delta(\varepsilon, x_0)$ s.t $\|\Delta x_0\| < \delta \Rightarrow \|x(t,0,x_0+\Delta x_0) - x(t,0,x_0)\| < \varepsilon$

$\Downarrow$

For, $\|\Delta x_0\| < \delta \Rightarrow \|\Psi(t)\Psi^{-1}(0)\Delta x_0\| < \varepsilon$

$\Uparrow$

Same as saying that $x(t) \equiv 0$ is stable.

UNIFORMLY STABLE :-

A solution $x(t) := x(t, 0, x_0)$ of $\boxed{N}$ is said to be uniformly stable if for each $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t. for any solution $x_1(t) := x_1(t, 0, \tilde{x}_0)$ of the problem $x' = F(t, x)$; $x(0) = \tilde{x}_0$ the inequalities $t_1 \geq 0$ and $\|x_1(t_1) - x(t_1)\| < \delta \Rightarrow \|x_1(t) - x(t)\| < \varepsilon$ $\forall\ t \geq t_1$.

Ex :- Every solution of $y' = p(x)y$ is of the form $y(x) = y(x_0) \exp\left(\int_0^x p(s)\,ds\right)$ and hence its trivial solution $y(x) \equiv 0$ is uniformly stable iff $\int_{x_1}^{x} p(s)\,ds$ is bounded above for all $x \geq x_1 \geq 0$.

Ex :- Every solution of $y' = 0$ is of the form $y(x) = y(0)$ and hence uniformly stable but not asymptotically stable.

Hence, uniform stability does not imply asymptotic stability.

Theorem :- Let $\Psi(t)$ be the Fundamental matrix of the equation $X' = A(t)X$. then all solutions are U.S. iff

$$\|\Psi(t)\Psi^{-1}(t_1)\| \leq c, \quad 0 \leq t_1 \leq t < \infty.$$

where c is a positive constant.

Proof :- Let $X(t) := X(t; 0, x_0)$ solves $X' = A(t)X$. then for $t_1 \geq 0$ we have

$$X(t) = \Psi(t)\Psi^{-1}(t_1)X(t_1)$$

If $X_1(t) = \Psi(t)\Psi^{-1}(t_1)X_1(t_1)$ is any other solution and $\|\Psi(t)\Psi^{-1}(t_1)\| < c$ then

$$\|X_1(t) - X(t)\| \leq \|\Psi(t)\Psi^{-1}(t_1)\| \, \|X(t_1) - X_1(t_1)\| \quad \text{for all } 0 \leq t_1 \leq t < \infty$$

Thus if $\varepsilon > 0$ then $t_1 \geq 0$ and $\|X(t_1) - X_1(t_1)\| < \varepsilon/c = \delta(\varepsilon) > 0 \Rightarrow \|X_1(t) - X(t)\| < \varepsilon$.

Hence the solution is uniformly stable.

Conversely, If all solutions are U.S then the trivial solution is U.S.

$\therefore$ Given $\varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ s.t for $t_1 \geq 0$ and $\|x^1(t_1)\| < \delta \Rightarrow \|x^1(t)\| < \varepsilon \;\; \forall\, t \geq t_1$

thus, $\|\psi(t)\,\psi^{-1}(t_1)\,x^1(t_1)\| < \varepsilon \;\; \forall\, t \geq t_1$

$$\Downarrow$$

$$\|\psi(t)\,\psi^{-1}(t_1)\| < C.$$

L_I

Question :- Can we have similar stability result for $\boxed{x' = A(t)x + B(t)}$?

If yes, under what condition on B are they true?

THEOREM :- Suppose every solution of L_H is bounded in $[0,\infty)$. then every solution of L_I is bounded provided at least one solution is bounded.

- Trivial.

THEOREM :- Suppose every solution of L_H is bounded in $[0,\infty)$ and either

(a) $\lim_{x\to\infty} \int^x \operatorname{Tr} A(t) > -\infty$

(b) $\operatorname{Tr} A(t) = 0$

where we used this ??

Then every solution of L_I is bounded provided, $\int_0^\infty \|B(t)\|\,dt < \infty$

Proof :- $\because$ Every solution of L_H is bounded $\Rightarrow \|\psi(t)\| < M \;\; \forall\, t \geq 0$ and M independent of t.

Also, $\det \psi(t) = \det \psi(0) \exp\left(\int_0^t \operatorname{tr} A(t)\,dt\right)$

$\therefore \; \psi^{-1}(t) = \dfrac{\operatorname{adj} \psi(t)}{\det \psi(t)} = \dfrac{\operatorname{adj} \psi(t)}{\det \psi(0) \exp\left(\int_0^t \operatorname{tr} A(t)\,dt\right)}$

From (a) one has, $\|\psi^{-1}(t)\|$ is bounded.

?

Define $C = \max \left\{ \sup_{t \geq 0} \|\varphi(t)\| , \sup_{t \geq 0} \|\varphi^{-1}(t)\| \right\}$

Hence any solution of L_I with $x(0) = x_0$ gives

$$\|x(t)\| \leq C \|\varphi^{-1}(0)\, x(0)\| + C^2 \int_0^t \|B(s)\|\, ds.$$

$\therefore \|x(t)\|$ is bounded for all t. $\blacksquare$

$\boxed{\text{THEOREM :-}}$ (I) If all solutions of L_I are bounded in $[0, \infty)$, then they are stable.

(II) If all solutions are stable and one is bounded, then all solutions are bounded in $[0, \infty)$.

Quasi-Linear System :-

We now consider the system $\boxed{x' = A(t)x + B(t,x)}$ — (Q) under the assumptions :-

$\|B(t,x)\| = O(\|x\|)$ uniformly in t as $\|x\| \to 0$

i.e., for $\|x\|$ sufficiently close to zero, $\dfrac{\|B(t,x)\|}{\|x\|}$ — (*) can be made arbitrarily small.

Aim :- Can one generalize the results of Linear Homogeneous System for Quasilinear case?

Asymptotic stability of the trivial solution of the unperturbed system $x' = A(t)x$ — (L_H) with (*) does not imply the A.S of trivial solution of the perturbed system (Q)

Ex :-
$$\left.\begin{aligned} x_1' &= -a x_1 \\ x_2' &= (\sin 2x + 2x\cos 2x - 2a)\,x_2 \quad ; \quad 1 < 2a < 3/2 \end{aligned}\right\} \text{— (1)}$$

G.S is $x(t) = \left(c_1 e^{-at},\ c_2 \exp((\sin 2x - 2a)x)\right)$

$\because a > 1/2$, every solution of (1) tends to zero as $t \to \infty$ and hence trivial soln is A.S.

Now, $x_1' = -a x_1$

$$\left.\begin{aligned} x_1' &= -a x_1 \\ x_2' &= (\sin 2x + 2x\cos 2x - 2a)x_2 + x_1^2 \quad ; \quad 1 < 2a < 3/2 \end{aligned}\right\} \leftarrow (2)$$

Here, $B(t,x) = B(t, x_1, x_2) = x_1^2$

General solution of (2) is

$$x_1(t) = c_1 e^{-at}$$

$$x_2(t) = \left(c_2 + c_1^2 \int_0^x e^{-t\sin 2t}\, dt\right) \exp\left((\sin 2t - 2a)t\right)$$

Choose a sequence, $t_n = (n + 1/4)\pi$; $n = 1, 2, 3, \ldots$

then, $-\sin 2s \geq \frac{1}{2}$ for all $t_n + \frac{\pi}{3} \leq t \leq t_n + \frac{\pi}{2}$.

$$\therefore \int_0^{t_n+1} e^{-t\sin 2t}\, dt > \int_{t_n+\pi/3}^{t_n+\pi/2} e^{-t\sin 2t}\, dt > \int_{t_n+\pi/3}^{t_n+\pi/2} e^{t/2}\, dt > 0.4\exp\left(\frac{1}{2}t_n + \pi/4\right).$$

$$\therefore x_2(t_{n+1}) > c_2 e^{(1-2a)x_n} + 0.4 c_1^2 \exp\left[\pi/4 + \left(3/2 - 2a\right)t_n\right] \longrightarrow \infty \quad \text{as } 2a < 3/2,\ c_1 \neq 0.$$

THEOREM :- Suppose the real parts of the eigenvalues of A are negative and suppose $F(t,x)$ satisfies (*). Then the trivial solution of $\boxed{Q}$ are A.S., provided $A(t)$ is independent of 't'.

Proof :- Any solution of $x' = Ax + B(t,x)$ with $x(0) = x_0$ satisfies the integral equation

$$x(t) = e^{At}x_0 + \int_0^t e^{A(-s+t)} B(s, x(s))\, ds.$$

$\because$ Each part of eigenvalues of A are negative, $\exists\, c$ and $\eta = -\delta$ $(\delta > 0)$ s.t

$$\|e^{At}\| \le c\, e^{-\delta t} \quad \forall\, t \ge 0.$$

?

(*) $\Rightarrow$ Given $M > 0$, $\exists\, K > 0$ s.t $\|B(t,x)\| \le M\|x\|$ $\forall\, t \ge 0$ and $\|x\| \le K$. — (iii)

Assuming $\|x_0\| < K$, $\exists\, t_1$ s.t $\|x(t)\| < K$ for all $t \in [0, t_1)$

Using (iii) we obtain,

$$\|x(t)\| e^{\delta t} \le c\|x_0\| + cM \int_0^t \|x(s)\| e^{\delta s}\, ds, \quad t \in [0, t_1)$$

read again...

Gronwall implies :-

$$\|x(t)\| \le C\|x_0\| \exp\left((CM-K)\,t\right) \quad ; t \in [0,t_1)$$

$\because x_0$ and M are on us we choose, $CM < K$ and $x(t_0) = x_0$ so that $\|x_0\| < K/C$

$\Rightarrow \|x(t)\| < K$ for all $t \in [0, t_1)$.

$\because B(t,x)$ is continuous in $[0,\infty) \times \mathbb{R}^n$, we extend $x(t)$ interval by interval preserving K.

$\because K$ can be made arbitrarily small, the trivial soln is stable and $CM - K < 0 \Rightarrow$ A.S. ∎

Q :- what happens when real part of all the eigenvalues of A are non-positive and at least one is zero.

In this case B will influence the stability of trivial solution.

Ex :- Trivial solution of $y' = ay^3$ is asymptotically stable if $a < 0$, stable if $a = 0$ and unstable if $a > 0$.

Week - 11 :-

Consider the system $x' = F(x)$; $F \in C^1(\mathbb{R}^n)$. — (1)

Definition :- A point $b_* \in \mathbb{R}^n$ is said to be equilibrium for an ODE (1) if $F(b_*) = 0$.

Ex 1 :- Consider $x' = Ax$ with A being a (2×2) invertible matrix.

$\therefore F(x) = Ax = 0 \iff x \equiv 0$.

Ex 2 :- $\left.\begin{aligned} x' &= y \\ y' &= x - x^3 - \beta y \end{aligned}\right\}$ Duffing's Equation.

$F(x,y) = (y, x - x^3 - \beta y) = (0,0)$

$\Rightarrow y = 0$ and $x = 0, \pm 1$

$\therefore$ Equilibrium points $(0,0)$ and $(\pm 1, 0)$.

We know that for the linear system $X' = AX$, if the eigenvalues satisfy $\text{Real}(\lambda_j(A)) < 0$, then every solution $x(t)$ of the equation decays to zero as $t \to \infty$.

The following theorem shows similar stability behaviour for solutions of non-linear equation $X' = F(X)$ near an equilibrium point b_*, provided the eigenvalues of $DF(b_*)$ satisfy $\text{Real}(\lambda_j(DF(b_*))) < 0$ for $i = 1, 2, \ldots, n$.

Theorem :- Let b_* be the equilibrium point for $X' = F(X)$ where $F \in C^1(\Omega)$, $\Omega (\subseteq \mathbb{R}^n)$ is open and assume that

$$\text{Real}(\lambda_j(DF(b_*))) < 0 \quad \text{for } i = 1, 2, \ldots, n$$

Then there exists a neighbourhood V of b_* in $\mathbb{R}^n$ s.t for any initial data $X_0 \in V$, the I.V.P $X' = F(X)$; $X(0) = X_0$ has a solution for all $t \geq 0$ and $\lim_{t \to \infty} X(t) = b_*$.

see this proof again...

Remarks:-

(a) $F(x) = Ax$, then at origin, $DF(0) = A$ ($\because DF(x) = A \ \forall x \in \mathbb{R}^n$) and if $R(\lambda_j(A)) < 0$ for $j = 1, 2, \ldots, n$ then one has solutions decaying to zero as $t \to +\infty$.

Proof :- WLOG we assume $b_x = 0$; hence we have $F(0) = 0$.

Write, $F(x) = F(0) + DF(0)x + r(x)$

$= Ax + r(x)$ where,

(a) $F(0) = 0$ is used, and (b) $A := DF(0)$ and (c) $\exists \delta > 0$ s.t if $|x| \leq \delta$, then $\|r(x)\| \leq \eta \|x\|$

Again from Problem (2) of Assignment 10 one has,

$\|e^{At}\| \leq K e^{-\varepsilon t}$; $t \geq 0$ and K, ε are +ive constant. ($K \geq 1$, $\varepsilon > 0$)

Choose $\eta > 0$ s.t $\eta < \varepsilon / K$ then $\exists \delta > 0$ s.t if $\|x\| < \delta$, $\|r(x)\| \leq \eta \|x\|$

Let $V := \{x_0 \in \mathbb{R}^n : \|x_0\| < \delta/K\} \subset B(0,\delta)$

Claim: If $x' = F(x)$; $x(0) = x_0$ is not solvable for $t \in [0,\infty)$, then the solution must leave the ball $B(0,\delta)$.

We seek to find a contradiction by assuming $\exists\, t_* > 0$ s.t $|x(t)| < \delta$ for $t < t_*$.

while $\|x(t_*)\| = \delta$.

Define $g(t) := e^{\varepsilon t}|x(t)|$

and write, $x' - Ax = r(x)$; $x(0) = x_0$.

$$\therefore\ x(t) = e^{At}x_0 + \int_0^t e^{(t-s)A}\, r(x(s))\, ds.$$

$$\therefore\ g(t) \le e^{\varepsilon t}\|e^{At}x_0\| + e^{\varepsilon t}\int_0^t \|e^{(t-s)A}\, r(x(s))\|\, ds.$$

$$\because\ \|e^{At}\| \le K e^{-\varepsilon t} ; t \ge 0 \Rightarrow\ g(t) \le K\|x_0\| + K\int_0^t e^{\varepsilon s}\|r(x(s))\|\, ds. \quad \text{(1)}$$

For $t \le t_*$, $K \int_0^t e^{\varepsilon s} \| r(x(s)) \| \, ds \le K\eta \int_0^t e^{\varepsilon s} \| x(s) \| \, ds = K\eta \int_0^t g(s) \, ds.$

By Gronwall, $g(t) \le K \|x_0\| e^{K\eta t}$ for $0 \le t \le t_*$.

Hence, $\| x(t) \| = e^{-\varepsilon t} g(t) \le K \|x_0\| e^{(K\eta - \varepsilon)t}$ $\longrightarrow$ (*)

$\because$ the exponential is decaying one has, $\| x(t_*) \| \le K \| x_0 \| < \delta$

— a contradiction.

$\therefore$ The solution never leaves the ball of radius 'δ'.

Thus the solution exists for all time $t \ge 0$ and from (*) tends to zero as $t \to \infty$.

Recall Duffing equation :-

$$\left.\begin{aligned} x' &= y \\ y' &= x - x^3 - \beta y \; ; \; \beta > 0. \end{aligned}\right\} - (*)$$

$$F(x,y) = (y, x - x^3 - \beta y)$$

has equilibrium at $(0,0)$, $(\pm 1, 0)$

At $(\pm 1, 0)$, $DF(\pm 1, 0) = \left.\begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & -\beta \end{pmatrix}\right|_{(\pm 1, 0)} = \begin{pmatrix} 0 & 1 \\ -2 & -\beta \end{pmatrix}$

It is easy to check that both eigenvalues of $DF(\pm 1, 0)$ are negative

$\therefore$ Near $(\pm 1, 0)$ solutions of $(*)$ converges asymptotically to the equilibrium.

i.e $\lim_{t \to \infty} X(t) = (\pm 1, 0)$

Lypanov stability ::: for every nbhd V of eqm point b*, there exists V1 s.t. any pt. b in V1 if taken as initial data, its corresponding soln will be contained in V
Attracting ::: for some nbhd V* of b*, for every point in V*, solution exists for all t>=0 and convg. to b*

Another way of looking at Lyapunov Stability :-

An equilibrium b_* of a system $x' = F(x)$ is called L.S if for every neighbourhood V of $b_* \in \mathbb{R}^n$, $\exists\, V_1 (\subset V) \ni b$ and $\tilde{x}(t) \in V \;\forall t \geq 0$ s.t $\tilde{x}' = F(\tilde{x})$ and $\tilde{x}(0) = b$.

The equilibrium b_* is called attracting if there is some nbd $V^* (\ni b_*)$ s.t for all initial data in V^*, the solution exists for all $t \geq 0$ and converges to b_*.

b_* is called Asymptotically stable if it is L.S and attracting.

Note :- If an equilibrium is attracting does not imply it is Lyapunov Stable ?

Consider $\begin{cases} r' = r - r^3 \\ \theta' = 1 - \cos\theta . \end{cases}$

Show!!

All non-zero solution of this system converge to $(r,\theta) = (1,0)$.

A trajectory that starts at $(1,\varepsilon)$; $\varepsilon > 0$, leaves the ball of unit radius around $(1,0)$ and then converge to $(1,0)$.

Hyperbolic Equilibrium :- An equilibrium b_* of an ODE is called hyperbolic if

$$\text{Real}(\lambda_j(DF(b_*))) \neq 0 \;;\; j = 1, 2, \ldots n.$$

Near a hyperbolic equilibrium b_*, locally the flow of the full system resembles the flow of the linearized system $w' = Aw$ where $A = DF(b_*)$.

ACTIVATOR-INHIBITOR (A.I) SYSTEM :-

Determine the equilibrium and stability of

$$x' = \frac{\sigma x^2}{1+y} - x \;;\; \sigma > 0 \quad \text{--- (a)}$$

$$y' = \rho(x^2 - y) \;;\; \rho > 0. \quad \text{--- (b)}$$

Case 1 :-

Clearly $(0,0)$ is one equilibrium point. To determine stability

$$DF(x,y) = \begin{pmatrix} \frac{2\sigma x}{1+y} - 1 & -\frac{\sigma x^2}{(1+y)^2} \\ 2\rho x & -\rho \end{pmatrix}$$

$$\therefore DF(x,y)\Big|_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -\rho \end{pmatrix}$$

$\therefore$ By our previous theorem, this equilibrium is Asymptotically Stable.

Case 2 :- If $\sigma > 2$, the eqn admits two non-trivial equilibria say $P_\pm = (x_\pm, y_\pm)$

where $x_\pm$ satisfy $x^2 - \sigma x + 1 = 0$ and $y_\pm = x_\pm^2$. —— ©

Note, ⓐ vanishes when $x=0$ or $y = \sigma x - 1$.

and ⓑ vanishes when $y = x^2$

$\therefore$ Equilibrium is given as ©.

If $\sigma > 2$, the roots of $x^2 - \sigma x + 1 = 0$ are real. [$\therefore x_\pm = \frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} - 1}$].

The Jacobian $DF(x_\pm, y_\pm) = \begin{pmatrix} 1 & -1/\sigma \\ 2\rho x_\pm & -\rho \end{pmatrix}$

$\therefore$ Trace $(DF(x_+, y_+)) = 1 - \rho \, (= \tilde{A})$

If $\rho > 1$ then $\tilde{A} < 0$ so, P_+ is a sink

If $\rho < 1$ then $\tilde{A} > 0$ so, P_+ is a source.

Can you find what happens at P_- ?? (Saddle !!).

Imp

$\boxed{\text{[Here we used the fact that for a } (2\times 2) \text{ system, } \operatorname{tr} A < 0 \text{ and } \det A > 0 \Leftrightarrow \operatorname{Real}(\lambda_j) < 0 \text{ for } j = 1, 2.\text{]}}$

W12 :- (Lyapunov Function)

Aim :- Analyzing the stability of an equilibrium.

Advantage :- (a) In some cases A.S of equilibrium can be proved in cases where one or more eigenvalues of the Jacobian has zero real part.

(b) Sometimes a Lyapunov function may be used to prove that an equilibrium is globally attracting. !

(c) Stability may be discussed without prior knowledge of solutions.

Lyapunov function:

Let b_* is an equilibrium of $x' = F(x)$ where $F : \Omega \to \mathbb{R}^n$. Let $L : \Omega_1 \to \mathbb{R}$ be a real valued function defined on $\Omega_1 (\subset \Omega) (\ni b_*)$ s.t F is continuous on Ω_1 and C^1 on $\Omega_1 \setminus \{b_*\}$.

We shall call L a Lyapunov function for the system near b_* if it satisfies the following:-

(a) $\forall x \in \Omega_1 \setminus \{b_*\}$, $\langle \nabla L(x), F(x) \rangle \leq 0$ and

(b) $\forall x \in \Omega_1$, $L(x) \geq L(b_*)$ equality holds only if $x = b_*$.

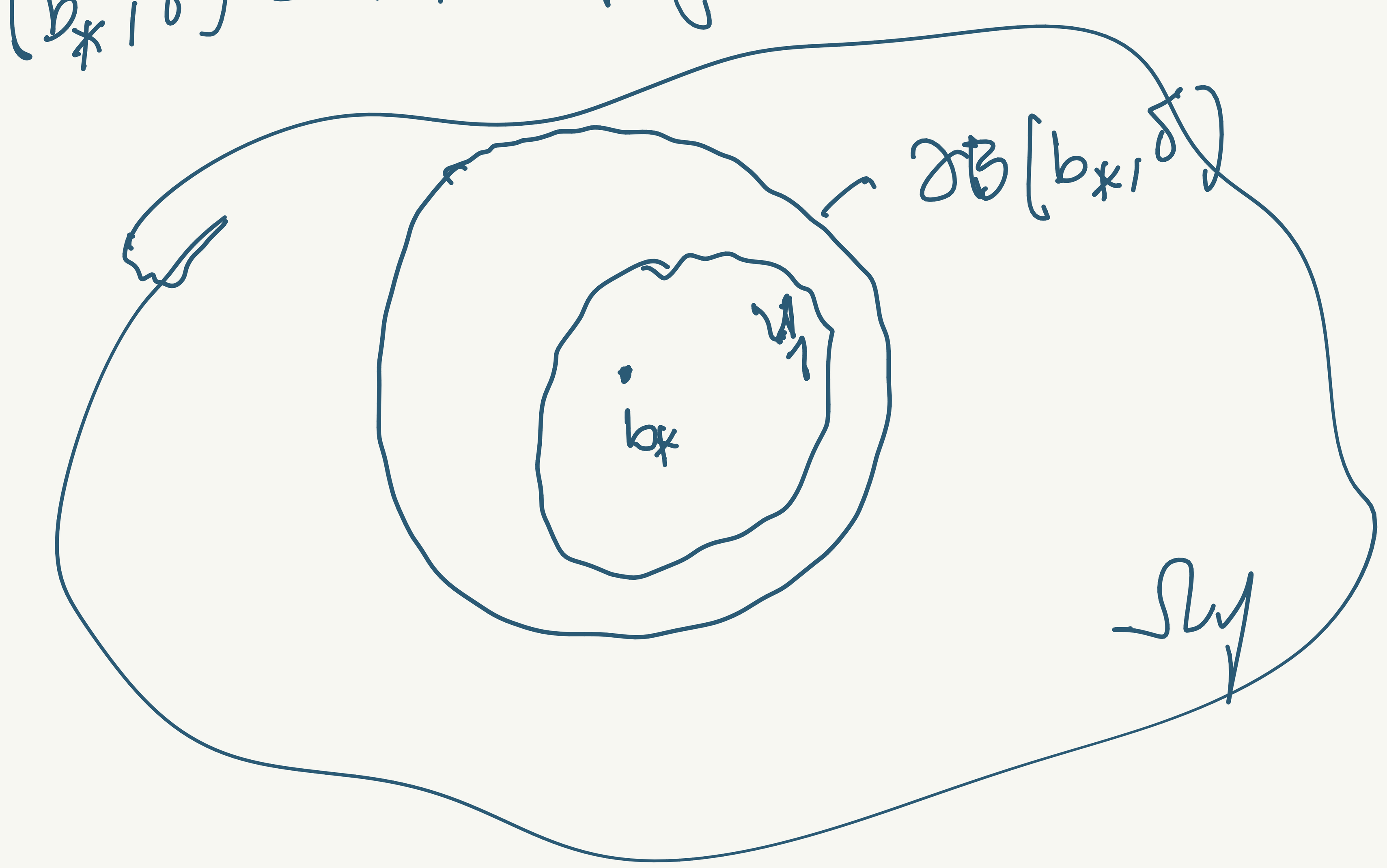
Condition (b) $\Rightarrow$ b_* is the strict minimum of L over Ω_1.

Condition (a) $\Rightarrow$ $L(x)$ is non-increasing along any trajectory of $x' = F(x)$.

$$\boxed{\because \frac{d}{dt} L(x(t)) = \nabla L(x(t)) \cdot x'(t) = \nabla L(x(t)) \cdot F(x(t)) \leq 0}.$$

Theorem 1:- If the eqn $x' = F(x)$ admits a Lyapunov function near b_*, then the equilibrium is Lyapunov stable.

Proof: Let V be a nbd of b_* is given. Choose $\delta > 0$ so that $\overline{B(b_*, \delta)} \subset V \cap \Omega_1$ (Remember L is defined on $b \in \Omega_1$)



$$\text{Let } \alpha = \min_{|x - b_*| = \delta} L(x). \quad \text{--- } (*)$$

By compactness, $\alpha > L(b_*)$.

Define, $V_1 = \{ x \in B(b_*, \delta) : L(x) < \alpha \}$

If $b \in V_1$ and let $x(t) = \varphi(t, b)$ be the solution of $x' = F(x)$; $x(0) = b$.

$\because$ $L(x)$ decreases along the trajectories

$$L(x(t)) \leq L(x(0)) = L(b) < \alpha \quad \text{as long as } x(t) \in \Omega_1$$

$\because$ by $\circledast$, $x(t)$ can't cross $\partial B(b_*, \delta)$ $\Rightarrow$ The trajectory is confined on $\overline{B(b_*, \delta)}$.

Hence, The solution exists for all time $t \geq 0$ and $x(t) \in \overline{B(b_*, \delta)} \subset V$

$\Downarrow$

b_* is Lyapunov Stable.

---

b_* must be strict minimum of $L(x)$ to get a useful concept :-

$\begin{cases} x' = -x \\ y' = y \end{cases}$ $\Rightarrow$ $(0,0)$ is an equilibrium, $L(x,y) = y^2$ satisfy (a) and has minimum at 0, but the equilibrium is unstable. ?

If in the definition one considers $\nabla L(x) \cdot F(x) < 0$ then L is called strict Lyapunov function.

$\Rightarrow$ Along every trajectories $L(x)$ is strictly decreasing.

Theorem 2:- If the eqn $x' = F(x)$ admits a strict Lyapunov function near b_*, then the equilibrium is asymptotically stable.

Proof:- If $x(t)$ starts near b_*, then it exists for all $t \geq 0$ and stays within a compact nbd of b_*.

Let such a trajectory does not converge to b_*. Upto a subsequence, $x(t_n) \to b$ with $b \neq b_*$. $\because$ L is continuous one has

$$\lim_{n \to \infty} L(x(t_n)) = \lim_{t \to \infty} L(x(t)) = L(b) \quad (\because L(x(t)) \text{ is decreasing})$$

Consider, $x' = F(x)$; $x(0) = b$ and denote the solution as $\varphi(s; b)$

For any $s > 0$, $L(\varphi(s, b)) < L(\varphi(0, b)) = L(b)$ $\longrightarrow$ (I)

Again, $\varphi(s, b) = \lim_{n \to \infty} \varphi(s, x(t_n))$. and $\varphi(s, x(t_n)) = x(t_n + s)$ so,

$\varphi(s, b) = \lim_{n \to \infty} x(t_n + s)$ hence,

$L(\varphi(s, b)) = \lim_{n \to \infty} L(x(t_n + s)) = L(b)$. $\longrightarrow$ (II)

Contradicting (I)

Hence, b_* is asymptotically stable.

Lasalle's Invariance Principle :-

Consider Duffing's equation given by

$$x' = y$$
$$y' = x - x^3 - \beta y$$

Let's apply Lyapunov function to show $(\pm 1, 0)$ are A.S

Define, $L(x,y) = y^2/2 - x^2/2 + x^4/4$.

Don't question how we got this.

Clearly, $(\pm 1, 0)$ are strict minima of L.

Also, $\langle \nabla L(x,y) \cdot (x', y') \rangle = \langle (-x + x^3, y) \cdot (y, x - x^3 - \beta y) \rangle$

$\parallel$

$\frac{dL}{dt}$

$$= -xy + x^3 y + xy - x^3 y - \beta y^2 = -\beta y^2 \le 0$$

L is not a strict Lyapunov function since $\frac{dL}{dt}$ vanishes along x-axis.

$\Rightarrow$ $(\pm 1, 0)$ are Lyapunov Stable. in Duffling's eqn

$\because$ L is not strictly decreasing along the trajectories

$\Rightarrow$ One cannot conclude that $(\pm 1, 0)$ is A.S

But from Stable Manifold theorem :-

$$DF(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & -\beta \end{pmatrix}$$ and hence the Real $\left[\lambda_i \left(DF(\pm 1, 0)\right)\right] < 0$.

Hence, $(\pm 1, 0)$ is Asymptotically Stable.

$\therefore$ To Rectify this issue we need information about the set where Lyapunov function fails to be strict.

Define, $S := \{x \in \Omega_1 \setminus \{b_*\} ; \langle \nabla L(x), F(x) \rangle = 0\}$ think about it ??

and let no trajectory that starts in S remains in S. for all positive time 't'. $\rightarrow$ (*)

Lasalle's Invariance Principle:

Theorem:- If near the equilibrium b_*, $x' = F(x)$ has a Lyapunov function L that satisfies (*), then b_* is A.S.

Construction of Lyapunov Function :-

1. Gradient System :-

Consider the system $x_j' = -\dfrac{\partial V}{\partial x_j}$; $j = 1, 2, \ldots, n$ where $V : \Omega \to \mathbb{R}$ is a smooth function.

i.e., $X' = -\nabla V(x) \; (:= F(x))$

The obvious candidate for Lyapunov function in such case is

$$L(x) = V(x)$$

Since ⓐ b_* is an equilibrium iff $\nabla V(b_*) = 0$.

ⓑ $\langle \nabla L(x), F(x) \rangle = \langle \nabla V, -\nabla V \rangle = -|\nabla V|^2 \leq 0$.

Thus, V is a Lyapunov function for an equilibrium of the gradient system provided V has a strict local minima.

Ex:- $x' = -x$

$y' = -y$

$$\therefore \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Clearly, both eigenvalues of A are negative, hence we have A.S (S.M.T) (stable manifold thm)

Alternatively, $-\nabla V(x,y) = (x,y) \Rightarrow V(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$. (Note:- V is a strict Lyapunov function and hence origin is A.S).

#2: Hamiltonian System:

A system of the form $\boxed{x_j' = \dfrac{\partial H}{\partial y_j}, \quad y_j' = -\dfrac{\partial H}{\partial x_j}}$ is called Hamiltonian, where $H(x_1, \dots, x_n, y_1, \dots, y_n)$ is a smooth function in $\mathbb{R}^{2n}$.

Ex: A particle moving in n-dim potential $V(x)$ without friction,

$$x_j' = y_j, \quad y_j' = -\partial V/\partial x_j (x) \quad - \otimes$$

is Hamiltonian with $H(x, y) = |y|^2/2 + V(x)$.

- If $V(x) = -1/\|x\|$ and $(x, y) \in \mathbb{R}^3 \setminus \{0\} \times \mathbb{R}^3$; $\otimes$ describes the two body problem for Gravitational attraction.

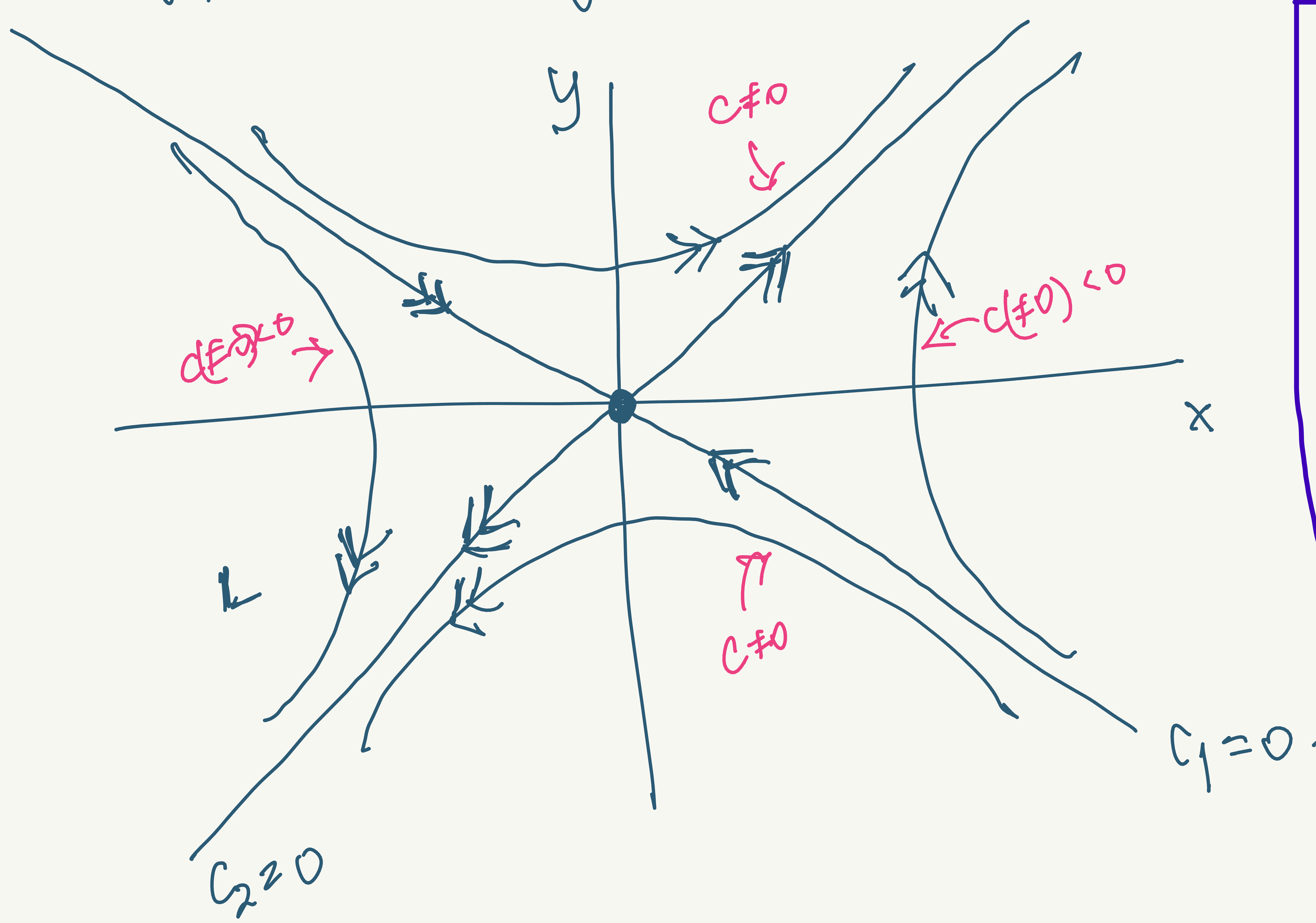
Stable and Unstable Manifolds :-

#1: Consider the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

E.V of the matrix are ± 1 and the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ The trajectories (above) is contained in $y^2 - x^2 = C$; where $C = -4c_1c_2$



$\Rightarrow$ Stable and unstable manifold (curves) relate to the asymptotic behaviour of solution as $t \to \pm\infty$.

$\Rightarrow$ If $(x(0), y(0))$ lies on one of the hyperbola with $C \neq 0$, then the solution tends to infinity as $t \to \pm\infty$.

$\Rightarrow$ In contrast if one starts on the line $x + y = 0$, then the solution tends to equilibrium as $t \to \infty$ and the converse is true.

⇒ We call the line $y=-x$ as stable manifold of the equilibrium denoted by M_s.

⇒ Here the stable manifold is the union of three orbits, the equilibrium and the two rays either side of origin.

⇒ Analogously, $\{y=x\}$ is called the unstable manifold, M_u

In other words, the stable manifold is given by the linear span of all generalized eigenvectors of A, associated with eigenvalues λ such that $\mathrm{Real}(\lambda)<0$

STABLE MANIFOLD THEOREM :-

Consider $X'=F(X)$; $x(0)=X_0$ has an hyperbolic equilibrium at $x=b_*$. ⊛

Suppose n_s eigenvalues of $DF(b_*)$ has negative real part and that remaining $n-n_s$ eigenvalues has positive real part. Let E_s denote the span of all generalized eigenvectors of $DF(b_*)$ associated with $\mathcal{R}(\lambda)<0$, a subspace of dimension n_s.

(Real part)

Statement :- Given a hyperbolic equilibrium b_* of (*), $\exists$ bounded nbd V of b_* in $\mathbb{R}^n$ and a differentiable manifold $M_s \subset V$ of dim n_s tangent to E_s at b_* such that

(i) If $b \in M_s$, then (*) has a solution $\varphi(t, x_0)$ for all time $t \geq 0$ and $\lim_{t \to \infty} \varphi(t, x_0) = b_*$.

(ii) If $b \in V \setminus M_s$, then $\varphi(t, x_0)$ leaves V at some $t > 0$.

LIMIT CYCLE AND POINCARE-BENDIXSON THEOREM :-

Consider the eqn

$$\left.\begin{aligned} x_1' &= -x_2 + x_1(1 - x_1^2 - x_2^2) \\ x_2' &= x_1 + x_2(1 - x_1^2 - x_2^2). \end{aligned}\right\} \longrightarrow (1)$$

Let, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ to obtain

$$\boxed{\frac{d}{dt}(r^2) = 2r\frac{dr}{dt} = 2x_1\frac{dx_1}{dt} + 2x_2\frac{dx_2}{dt}}$$

$$= 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2$$

$$= 2r^2(1 - r^2)$$

$\therefore$ $dr/dt = 2r^2(1-r^2)$

$||^{ly}$, $\dfrac{d\theta}{dt} = \dfrac{d}{dt}\left(\tan^{-1}\dfrac{x_2}{x_1}\right) = 1$ (Please Check).

$\therefore$ (1) is equivalent to $\begin{cases} r'(t) = r(1-r^2) \\ \theta'(t) = 1 \end{cases}$

whose solutions are

$$r(t) = \frac{r(0)}{\left[r(0)^2 + (1-r(0)^2)e^{-2t}\right]^{1/2}} := \alpha(t, r(0))$$

and, $\theta(t) = t + \theta(0)$

Hence the general solutions of (1) are

$$\left.\begin{aligned} x_1(t) &= \alpha(t, r(0)) \cos(t + \theta(0)) \\ \text{and, } x_2(t) &= \alpha(t, r(0)) \sin(t + \theta(0)). \end{aligned}\right\} \text{—— (II)}$$

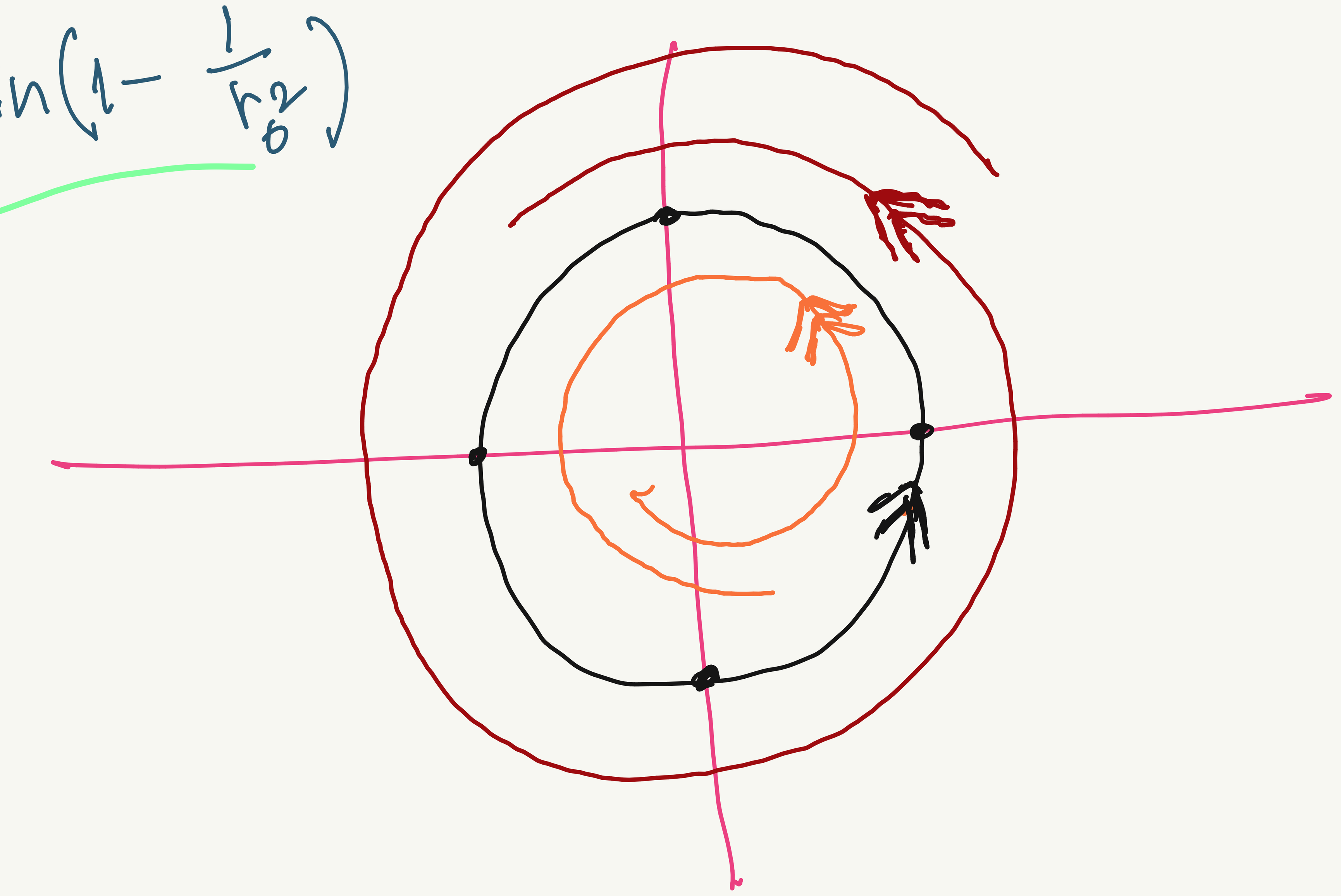
(I) If $r(0)=1$, (II) reduces to $x_1(t)=\cos(t+\theta(0))$ and $x_2(t)=\sin(t+\theta(0))$ (III)

This is a 2π-periodic solution.

(II) If $r(0)\neq 1$, the trajectories given by (II) are not closed but are spiral.

- If $r(0)<1$, the trajectories are spirals lying inside the circle (III) and approach this circle as $t\to\infty$ and approach the only critical point of (I) as $t\to-\infty$.

- If $r(0)>1$, the trajectories are spiral lying outside the circle. These outer trajectories also approach (III) as $t\to\infty$, while as $t\to\frac{1}{2}\ln\left(1-\frac{1}{r_0^2}\right)$ both x_1 and x_2 becomes infinite



Note:- We showed trajectories of a nonlinear system may spiral into simple closed curve.

LIMIT CYCLE (POINCARE) :- A closed trajectory of $X' = F(X)$ which is approached spirally from either the inside or the outside by a non-closed trajectory of $X' = F(X)$ either as $t \to \pm\infty$ is called the Limit Cycle of $X' = F(X)$

SUFFICIENT CONDITION FOR EXISTENCE OF LIMIT CYCLE :-

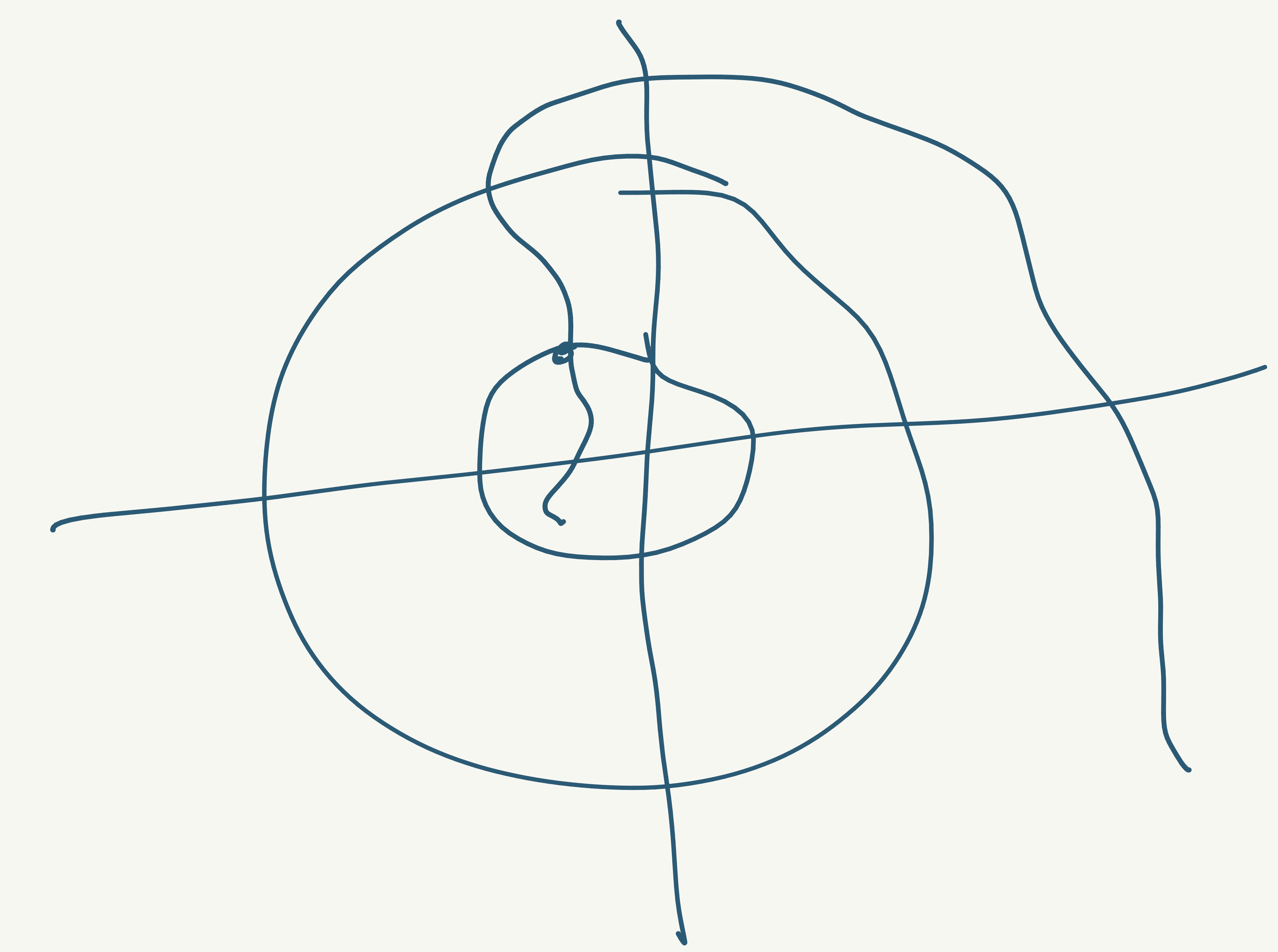
Suppose that a solution $X(t) = (x(t), y(t))$ of $X' = F(X)$ remains in the bounded region in the xy-plane which contains no critical point of $X' = F(X)$. Then its trajectory must spiral into a simple closed curve, which itself is the trajectory corresponding to a periodic solution of $X' = F(X)$.

Example :- $y'' + (2y^2 + 3y'^2 - 1)\, y' + y = 0$ —— (III)

which is equivalent to (*)

$$\begin{cases} x' = y \\ y' = -x + (1 - 2x^2 - 3y^2)\, y \end{cases}$$

For any given solution $X(t) = (x(t), y(t))$ of $(*)$ we note



$$\frac{d}{dt}\left(x^2(t) + y^2(t)\right) = 2x(t)\,x'(t) + 2y(t)\,y'(t)$$

$$= 2\left(1 - 2x^2(t) - 3y^2(t)\right)y^2(t)$$

$\therefore$ $1 - 2x^2 - 3y^2 > 0$ for $x^2 + y^2 < 1/3$

And, $1 - 2x^2 - 3y^2 < 0$ for $x^2 + y^2 > 1/2$.

$\therefore$ $x^2 + y^2$ is increasing when $x^2 + y^2 < 1/3$ and decreasing for $x^2 + y^2 > \frac{1}{2}$.

Thus if $X(t) = (x(t), y(t))$ starts in the annulus $1/3 < x^2(t) + y^2(t) < \frac{1}{2}$ at $t = 0$, then it will stay in the annulus for all $t \geq 0$.

And since this annulus does not contain any critical point of (iii), P-B Theorem says that the trajectory must spiral into a simple closed curve, which itself is a non-trivial periodic solution of (iii).

Sufficient condition for non-existence of closed trajectories :-

Bendixson's Theorem :- Consider the system

$$\left.\begin{aligned} x' &= F(x,y) \\ y' &= G(x,y). \end{aligned}\right\} \quad \text{--- (4)}$$

If $\dfrac{\partial F}{\partial x}(x,y) + \dfrac{\partial G}{\partial y}(x,y)$ has the same sign throughout the domain D, then (4) has no closed trajectory in D.

Proof :- Let S be the region in D which is bounded by a closed curve $\mathcal{C}$.

By Green's theorem

$$\int_C (F\,dy - G\,dx) = \iint_S [F_x + G_y]\,dx\,dy$$

Let $x(t) = (x(t), y(t))$ be the solution of (4) whose trajectory is a closed curve $\mathcal{C}$ in D and is ω-periodic.

$$\int_C [F\,dy - G\,dx] = \int_0^{\omega} \left[F(x(t),y(t))\, y'(t) - G(x(t),y(t))\, x'(t) \right] dt = 0$$

$$\therefore \iint_S [F_x + G_y]\, dx\, dy = 0$$

But this can only be zero if $(F_x + G_y)$ changes sign.

Ex: Consider, $\begin{cases} x' = x(x^2+y^2-2x-3) - y \\ y' = y(x^2+y^2-2x-3) + x \end{cases}$

$$\therefore F(x,y) = x(x^2+y^2-2x-3) - y \Rightarrow F_x = x[2x-2] + (x^2+y^2-2x-3)$$

and, $G(x,y) = y(x^2+y^2-2x-3) + x \Rightarrow G_y = (x^2+y^2-2x-3) + y(2y)$

$$\therefore F_x + G_y = 4(x^2+y^2) - 6x - 6 = 4\left[\left(x - \tfrac{3}{4}\right)^2 + y^2 - \tfrac{33}{16} \right] < 0 \text{ in } D := \left\{ \left(x-\tfrac{3}{4}\right)^2 + y^2 < \tfrac{33}{16} \right\}$$

$\therefore$ Bendixson theorem implies the system does not admit a closed trajectory.