Boundary Value Problems: Finite Difference Method

This yields a system of algebraic equations

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}=f\left(t_i,u_i,\frac{u_{i+1}-u_{i-1}}{2h}\right), i=1,\dots,n,$$

to be solved for the unknowns $u_i, i=1,\dots,n$.

In the matrix form, we have

$$\frac{1}{h^2}\begin{bmatrix}-2 & 1 & 0 & & & \cdots & 0\\ 1 & -2 & 1 & & & & \\ \vdots & \vdots & & \ddots & \vdots & & \vdots\\ & & & & 1 & -2 & 1\\ 0 & \cdots & & & 0 & 1 & -2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ \vdots\\ u_n\end{bmatrix}=\begin{bmatrix}f\left(t_1,u_1,\frac{u_2-\alpha}{2h}\right)\\ f\left(t_2,u_2,\frac{u_3-u_1}{2h}\right)\\ \vdots\\ f\left(t_n,u_n,\frac{\beta-u_n}{2h}\right)\end{bmatrix}-\begin{bmatrix}\frac{\alpha}{h^2}\\ \\ \vdots\\ \frac{\beta}{h^2}\end{bmatrix}$$

which is denoted as

$$\frac{1}{h^2}Au=F(u)+g.$$

Thus, the Newton's method for solving the system of algebraic equations is given by

$$u^{(m+1)}=u^{(m)}-\left[\frac{1}{h^2}A-F'\left(u^{(m)}\right)\right]^{-1}\left[\frac{1}{h^2}Au^{(m)}-F\left(u^{(m)}\right)-g\right]$$

where the Jacobian matrix is given by $[F(u)]_{ij}=[\partial f(t_i,u_i,(u_{i+1}-u_{i-1})/(2h))/\partial u_j]$.

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where the Jacobian matrix is given by $[F'(u)]_{ij} = [\partial f(t_i, u_i, (u_{i+1} - u_{i-1})/(2h))/\partial u_j]$.

In particular,

$$[F'(u)]_{ii} = f_2\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), \qquad 1 \le i \le n,$$
$$[F'(u)]_{i,i-1} = -\frac{1}{2h} f_3\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), \qquad 2 \le i \le n,$$
$$[F'(u)]_{i,i+1} = \frac{1}{2h} f_3\left(t_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right), \qquad 1 \le i \le n-1,$$

where all other entries are $F'(u)$ *are* 0 *and* $f_2(t, u, v)$, $f_3(t, u, v)$ *denote the partial derivatives of* f *with respect to* u *and* v *respectively.*

Example

Consider the two-point BVP

$$u'' = -u + \frac{2(u')^2}{u}, \qquad -1 < t < 1,$$
$$u(-1) = u(1) = (e + e^{-1})^{-1}.$$

The iterative solution via Newton's method satisfies

$$u^{(m+1)} = u^{(m)} - \left[\frac{1}{h^2}A - F'\left(u^{(m)}\right)\right]^{-1}\left[\frac{1}{h^2}Au^{(m)} - F\left(u^{(m)}\right) - g\right],$$

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$$[F'(u)]_{ii} = \qquad\qquad [F'(u)]_{i,i-1} =$$

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$$[F'(u)]_{ii} = -1 - 2\left(\frac{u_{i+1} - u_{i-1}}{2hu_i}\right)^2, [F'(u)]_{i,i-1} =$$

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Boundary Value Problems: Shooting Method

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$$[F'(u)]_{ii} = -1 - 2\left(\frac{u_{i+1} - u_{i-1}}{2hu_i}\right)^2, [F'(u)]_{i,i-1} = -\frac{1}{2h}\left(4\frac{u_{i+1} - u_{i-1}}{2hu_i}\right),$$

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and

$$g = \frac{1}{h^2}\begin{bmatrix}(e + e^{-1})^{-1} \\ 0 \\ \vdots \\ 0 \\ (e + e^{-1})^{-1}\end{bmatrix}.$$

Lesson 3

Boundary Value Problems for ODEs

3.1 Well-posedness

3.2 Shooting Method

3.3 Finite Difference Method

- Error Analysis

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The error analysis in full generality is too complicated to be discussed in this course. We will, therefore, look at simpler situation where the right hand side is independent of u *and* u'*, that is,* $f(t, u, u') = f(t)$.

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$$\begin{aligned} u'' &= f(t), \qquad a < t < b, \\ u(a) &= \alpha, \qquad u(b) = \beta. \end{aligned}$$

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We define

$$D_h^2 v(t) = \frac{v(t+h) - 2v(t) + v(t-h)}{h^2}.$$

Then, the local (truncation) error

$$\ell(t_i, u) = D_h^2 u(t_i) - f(t_i)$$

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Now, if $u \in C^4([a,b])$, then

$$u(t_i + h) = u(t_i) + hu'(t_i) + \frac{h^2}{2}u''(t_i) + \frac{h^3}{6}u'''(t_i) + \frac{h^4}{24}u^{(4)}(\xi_1), \quad \xi_1 \in (t_i, t_i + h),$$

$$u(t_i - h) = u(t_i) - hu'(t_i) + \frac{h^2}{2}u''(t_i) - \frac{h^3}{6}u'''(t_i) + \frac{h^4}{24}u^{(4)}(\xi_2), \quad \xi_2 \in (t_i - h, t_i).$$

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Thus,

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Thus,

$$\ell(t_i, u) = \frac{h^2}{24}\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right) = \frac{h^2}{12}u^{(4)}(\xi), \qquad \xi \in (t_i-h, t_i+h).$$

Boundary Value Problems: Finite Difference Method

Theorem

If $v \in C^2([a,b])$, *then*

$$\lim_{h\to 0}\left\|D_h^2 v - v''\right\|_{\infty,h} = 0.$$

If $v \in C^4([a,b])$, *then*

$$\left\|D_h^2 v - v''\right\|_{\infty,h} \le \frac{h^2}{12}\left\|v^{(4)}\right\|_{\infty,[a,b]}.$$

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$$\left\|D_h^2 v - v''\right\|_{\infty,h} \leq \frac{h^2}{12}\left\|v^{(4)}\right\|_{\infty,[a,b]}.$$

Proof:

We have already seen the proof for the second part. For the first part, we use

$$v(t_i + h) = v(t_i) + hv'(t_i) + \frac{h^2}{2}v''(\xi_1), \quad \xi_1 \in (t_i, t_i + h),$$

$$v(t_i - h) = v(t_i) - hv'(t_i) + \frac{h^2}{2}v''(\xi_2), \quad \xi_2 \in (t_i - h, t_i),$$

yielding

$$D_h^2 v(t_i) - v''(t_i) = \frac{v''(\xi_1) + v''(\xi_2)}{2} - v''(t_i) = v''(\xi) - v''(t_i), \qquad \xi \in (t_i - h, t_i + h).$$

The result follows!

Theorem (Discrete Maximum Principle)

Let v be a function on $[a, b]$ satisfying $D_h^2 v \geq 0$ on $t_i, i = 1, \dots, n$. Then $\max_{1 \leq i \leq n} v(t_i) \leq \max\{v(t_0), v(t_{n+1})\}$. Equality holds if and only if v is constant.

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Suppose $\max\limits_{1\leq i\leq n} v(t_i) \geq \max\{v(t_0), v(t_{n+1})\}$ *and the max is achieved at* t_{i_0}. *Then,*

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since $v(t_i) \leq v(t_{i_0})$. Thus, equality holds throughout and v achieves its maximum at t_{i_0-1} and t_{i_0+1} as well.

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Remark

The analogous discrete minimum principle, obtained by reversing the inequalities and replacing max by min holds.

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Remark

The analogous discrete minimum principle, obtained by reversing the inequalities and replacing max by min holds.

Theorem

There is a unique solution to the discrete BVP

$$\begin{gathered} D_h^2 u_h(t_i) = f(t_i), \;\; t_i, i = 1, \ldots, n, \\ u_h(a) = \alpha, \qquad u_h(b) = \beta. \end{gathered}$$