Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

Akash Anand
MATH, IIT KANPUR

Numerical Methods for PDE: Parabolic PDE

Akash Anand
MATH, IIT KANPUR

As a model problem, consider the heat equation on a spatial domain Ω for a time interval $[0, T]$. The solution u satisfies

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0, T], \qquad c > 0.$$

As a model problem, consider the heat equation on a spatial domain Ω for a time interval $[0, T]$. The solution u satisfies

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0, T], \qquad c > 0.$$

To obtain a well-posed problem, we need to give boundary conditions

$$u = 0, \qquad \mathrm{x} \in \Gamma, \mathrm{t} \in [0, T]$$

and an initial condition

$$u = u_0, \qquad \mathrm{x} \in \Omega, \mathrm{t} = 0.$$

Numerical Methods for PDE: Parabolic PDE

Akash Anand
MATH, IIT KANPUR

As a model problem, consider the heat equation on a spatial domain Ω *for a time interval* $[0, T]$. *The solution* u *satisfies*

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0, T], \qquad c > 0.$$

To obtain a well-posed problem, we need to give boundary conditions

$$u = 0, \qquad \mathrm{x} \in \Gamma, \mathrm{t} \in [0, T]$$

and an initial condition

$$u = u_0, \qquad \mathrm{x} \in \Omega, \mathrm{t} = 0.$$

Let's assume, for simplicity, that $\Omega = (0,1) \times (0,1)$.

Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- Semi-discretization

Akash Anand
MATH, IIT KANPUR

As a model problem, consider the heat equation on a spatial domain Ω for a time interval $[0,T]$. The solution u satisfies

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0,T], \qquad c > 0.$$

To obtain a well-posed problem, we need to give boundary conditions

$$u = 0, \qquad \mathrm{x} \in \Gamma, \mathrm{t} \in [0,T]$$

and an initial condition

$$u = u_0, \qquad \mathrm{x} \in \Omega, \mathrm{t} = 0.$$

Let's assume, for simplicity, that $\Omega = (0,1) \times (0,1)$. Then, we have the 5-point Laplacian Δ_h as the discretization of the Laplacian Δ. Thus, we the solution u_h satisfying

$$\begin{aligned} \frac{\partial u_h}{\partial t} &= \mathrm{c}\Delta_h u_h + f, \qquad && \mathrm{x} \in \Omega_\mathrm{h}, \mathrm{t} \in [0,T] \\ u_h &= 0, && \mathrm{x} \in \Gamma_\mathrm{h}, \mathrm{t} \in [0,T] \\ u_h &= u_0, && \mathrm{x} \in \Omega_\mathrm{h}, \mathrm{t} = 0. \end{aligned}$$

Numerical Methods for PDE: Parabolic PDE

As a model problem, consider the heat equation on a spatial domain Ω for a time interval $[0, T]$. The solution u satisfies

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0, T], \qquad c > 0.$$

To obtain a well-posed problem, we need to give boundary conditions

$$u = 0, \qquad \mathrm{x} \in \Gamma, \mathrm{t} \in [0, T]$$

and an initial condition

$$u = u_0, \qquad \mathrm{x} \in \Omega, \mathrm{t} = 0.$$

Let's assume, for simplicity, that $\Omega = (0,1) \times (0,1)$. Then, we have the 5-point Laplacian Δ_{h} as the discretization of the Laplacian Δ. Thus, we the solution u_h satisfying

$$\begin{aligned} \frac{\partial u_h}{\partial t} &= \mathrm{c}\Delta_h u_h + f, && \mathrm{x} \in \Omega_{\mathrm{h}}, \mathrm{t} \in [0, T] \\ u_h &= 0, && \mathrm{x} \in \Gamma_{\mathrm{h}}, \mathrm{t} \in [0, T] \\ u_h &= u_0, && \mathrm{x} \in \Omega_{\mathrm{h}}, \mathrm{t} = 0. \end{aligned}$$

We may view this as an initial value problem for a system of $(N-1)^2$ ODEs.

Numerical Methods for PDE: Parabolic PDE

Akash Anand
MATH, IIT KANPUR

As a model problem, consider the heat equation on a spatial domain Ω for a time interval $[0, T]$. The solution u satisfies

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0, T], \qquad c > 0.$$

To obtain a well-posed problem, we need to give boundary conditions

$$u = 0, \qquad \mathrm{x} \in \Gamma, \mathrm{t} \in [0, T]$$

and an initial condition

$$u = u_0, \qquad \mathrm{x} \in \Omega, \mathrm{t} = 0.$$

Let's assume, for simplicity, that $\Omega = (0,1) \times (0,1)$. Then, we have the 5-point Laplacian Δ_{h} as the discretization of the Laplacian Δ. Thus, we the solution u_h satisfying

$$\begin{aligned} \frac{\partial u_h}{\partial t} &= \mathrm{c}\Delta_h u_h + f, \qquad && \mathrm{x} \in \Omega_{\mathrm{h}}, \mathrm{t} \in [0, T] \\ u_h &= 0, && \mathrm{x} \in \Gamma_{\mathrm{h}}, \mathrm{t} \in [0, T] \\ u_h &= u_0, && \mathrm{x} \in \Omega_{\mathrm{h}}, \mathrm{t} = 0. \end{aligned}$$

We may view this as an initial value problem for a system of $(N-1)^2$ ODEs.

The process of reducing the evolutionary PDE to a system of ODEs by using a finite difference approximation of the spatial operator is called semi-discretization or the method of lines.

Numerical Methods for PDE: Parabolic PDE

As a model problem, consider the heat equation on a spatial domain Ω for a time interval $[0,T]$. The solution u satisfies

$$\frac{\partial u}{\partial t} = \mathrm{c}\Delta u + f, \qquad \mathrm{x} \in \Omega, \mathrm{t} \in [0,T], \qquad c > 0.$$

To obtain a well-posed problem, we need to give boundary conditions

$$u = 0, \qquad \mathrm{x} \in \Gamma, \mathrm{t} \in [0,T]$$

and an initial condition

$$u = u_0, \qquad \mathrm{x} \in \Omega, \mathrm{t} = 0.$$

Let's assume, for simplicity, that $\Omega = (0,1) \times (0,1)$. Then, we have the 5-point Laplacian Δ_{h} as the discretization of the Laplacian Δ. Thus, we the solution u_h satisfying

$$\begin{aligned} \frac{\partial u_h}{\partial t} &= \mathrm{c}\Delta_h u_h + f, \qquad && \mathrm{x} \in \Omega_{\mathrm{h}}, \mathrm{t} \in [0,T] \\ u_h &= 0, && \mathrm{x} \in \Gamma_{\mathrm{h}}, \mathrm{t} \in [0,T] \\ u_h &= u_0, && \mathrm{x} \in \Omega_{\mathrm{h}}, \mathrm{t} = 0. \end{aligned}$$

We may view this as an initial value problem for a system of $(N-1)^2$ ODEs.

The process of reducing the evolutionary PDE to a system of ODEs by using a finite difference approximation of the spatial operator is called semi-discretization or the method of lines.

This is not a full discretization as we still have to choose a numerical method to solve the ODEs.

Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- Semi-discretization

- **Finite difference discretization**

Akash Anand
MATH, IIT KANPUR

To further simplify the presentation, let's drop down to one space dimension, that is, take $\Omega = (0,1)$.

Numerical Methods for PDE: Parabolic PDE

Akash Anand
MATH, IIT KANPUR

To further simplify the presentation, let's drop down to one space dimension, that is, take $\Omega = (0,1)$.

To solve the ODEs, we first employ the Euler method.

Numerical Methods for PDE: Parabolic PDE

Akash Anand
MATH, IIT KANPUR

To further simplify the presentation, let's drop down to one space dimension, that is, take $\Omega = (0,1)$.

To solve the ODEs, we first employ the Euler method.

Writing $u_n^j = u_h(nh, jk)$, *the fully discrete system reads*

$$\frac{u_n^{j+1} - u_n^j}{k} = c\,\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + f_n^j, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$
$$u_0^j = u_N^j = 0, \qquad j = 0,1,\dots,M-1,$$
$$u_n^0 = u_0(nh), \qquad 0 < n < N.$$

To further simplify the presentation, let's drop down to one space dimension, that is, take $\Omega = (0,1)$.

To solve the ODEs, we first employ the Euler method.

Writing $u_n^j = u_h(nh, jk)$, *the fully discrete system reads*

$$\frac{u_n^{j+1} - u_n^j}{k} = c\,\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + f_n^j, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$

$$u_0^j = u_N^j = 0, \qquad j = 0,1,\dots,M-1,$$

$$u_n^0 = u_0(nh), \qquad 0 < n < N.$$

We call this the forward-centered difference method for the heat equation. Since the Euler's method is explicit, we don't need to solve a linear system:

$$u_n^{j+1} = (1-2\lambda)u_n^j + \lambda u_{n+1}^j + \lambda u_{n-1}^j + k f_n^j, \qquad j = 0,1,\dots,M-1,$$

where $\lambda = ck/h^2$.

To further simplify the presentation, let's drop down to one space dimension, that is, take $\Omega = (0,1)$.

To solve the ODEs, we first employ the Euler method.

Writing $u_n^j = u_h(nh, jk)$, *the fully discrete system reads*

$$\frac{u_n^{j+1} - u_n^j}{k} = c\,\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + f_n^j, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$
$$u_0^j = u_N^j = 0, \qquad j = 0,1,\dots,M-1,$$
$$u_n^0 = u_0(nh), \qquad 0 < n < N.$$

We call this the forward-centered difference method for the heat equation. Since the Euler's method is explicit, we don't need to solve a linear system:

$$u_n^{j+1} = (1-2\lambda)u_n^j + \lambda u_{n+1}^j + \lambda u_{n-1}^j + k f_n^j, \qquad j = 0,1,\dots,M-1,$$

where $\lambda = ck/h^2$.

To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

To further simplify the presentation, let's drop down to one space dimension, that is, take $\Omega = (0,1)$.

To solve the ODEs, we first employ the Euler method.

Writing $u_n^j = u_h(nh, jk)$, the fully discrete system reads

$$\frac{u_n^{j+1} - u_n^j}{k} = c\,\frac{u_{n+1}^j - 2u_n^j + u_{n-1}^j}{h^2} + f_n^j, \qquad 0 < n < N, j = 0,1,\dots,M-1,$$
$$u_0^j = u_N^j = 0, \qquad j = 0,1,\dots,M-1,$$
$$u_n^0 = u_0(nh), \qquad 0 < n < N.$$

We call this the forward-centered difference method for the heat equation. Since the Euler's method is explicit, we don't need to solve a linear system:

$$u_n^{j+1} = (1-2\lambda)u_n^j + \lambda u_{n+1}^j + \lambda u_{n-1}^j + k f_n^j, \qquad j = 0,1,\dots,M-1,$$

where $\lambda = ck/h^2$.

To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

By Taylor's theorem,

$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

where derivatives are evaluated at appropriate points.

To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

By Taylor's theorem,

$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

where derivatives are evaluated at appropriate points.

Let $e_n^j = u_n^j - u(nh,jk)$. Then,

$$e_n^{j+1} = (1-2\lambda)e_n^j + \lambda e_{n+1}^j + \lambda e_{n-1}^j - k\ell_n^j.$$

Numerical Methods for PDE: Parabolic PDE

Akash Anand
MATH, IIT KANPUR

To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

By Taylor's theorem,

$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

where derivatives are evaluated at appropriate points.

Let $e_n^j = u_n^j - u(nh,jk)$. *Then,*

$$e_n^{j+1} = (1-2\lambda)e_n^j + \lambda e_{n+1}^j + \lambda e_{n-1}^j - k\ell_n^j.$$

If we let $E^j = \max_n |e_n^j|$ *denote the maximum norm of the error at the* j*th time step, we get*

$$E^{j+1} \le |1-2\lambda|E^j + \lambda E^j + \lambda E^j + k\max_n\left|\ell_n^j\right|.$$

To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

By Taylor's theorem,

$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

where derivatives are evaluated at appropriate points.

Let $e_n^j = u_n^j - u(nh,jk)$*. Then,*

$$e_n^{j+1} = (1-2\lambda)e_n^j + \lambda e_{n+1}^j + \lambda e_{n-1}^j - k\ell_n^j.$$

If we let $E^j = \max_n |e_n^j|$ *denote the maximum norm of the error at the* j*th time step, we get*

$$E^{j+1} \le |1-2\lambda| E^j + \lambda E^j + \lambda E^j + k \max_n \left|\ell_n^j\right|.$$

If $\lambda \le 1/2$*, and* $\ell = \max_{n,j}\left|\ell_n^j\right|$*, we get*

$$E^{j+1} \le E^j + k\ell.$$

To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

By Taylor's theorem,

$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

where derivatives are evaluated at appropriate points.

Let $e_n^j = u_n^j - u(nh,jk)$. Then,

$$e_n^{j+1} = (1-2\lambda)e_n^j + \lambda e_{n+1}^j + \lambda e_{n-1}^j - k\ell_n^j.$$

If we let $E^j = \max_n |e_n^j|$ denote the maximum norm of the error at the jth time step, we get

$$E^{j+1} \le |1-2\lambda|E^j + \lambda E^j + \lambda E^j + k\max_n\left|\ell_n^j\right|.$$

If $\lambda \le 1/2$, and $\ell = \max_{n,j}\left|\ell_n^j\right|$, we get

$$E^{j+1} \le E^j + k\ell.$$

If follows that

$$E^j \le E^0 + jk\ell$$

for all j.

Numerical Methods for PDE: Parabolic PDE

Akash Anand
MATH, IIT KANPUR

To analyze the method, consider the local (truncation) error

$$\ell_n^j = \frac{u(nh,(j+1)k) - u(nh,jk)}{k} - c\,\frac{u\big((n+1)h,jk\big) - 2u(nh,jk) + u\big((n-1)h,jk\big)}{h^2} - f_n^j.$$

By Taylor's theorem,

$$\ell_n^j = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - c\,\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}$$

where derivatives are evaluated at appropriate points.

Let $e_n^j = u_n^j - u(nh,jk)$. Then,

$$e_n^{j+1} = (1-2\lambda)e_n^j + \lambda e_{n+1}^j + \lambda e_{n-1}^j - k\ell_n^j.$$

If we let $E^j = \max_n |e_n^j|$ denote the maximum norm of the error at the jth time step, we get

$$E^{j+1} \le |1-2\lambda|E^j + \lambda E^j + \lambda E^j + k\max_n \left|\ell_n^j\right|.$$

If $\lambda \le 1/2$, and $\ell = \max_{n,j}\left|\ell_n^j\right|$, we get

$$E^{j+1} \le E^j + k\ell.$$

If follows that

$$E^j \le E^0 + jk\ell$$

for all j. As $jk \le T$ and $k \le h^2/(2c)$, we have

$$\max_{n,j}\left|e_n^j\right| \le T\ell \le C(k+h^2) \le C(1+1/(2c))h^2.$$

Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- *Semi-discretization*
- *Full finite difference discretization*
- ***Fourier Analysis***

Akash Anand
MATH, IIT KANPUR

Another useful way to analyze is to use Fourier analysis.

Another useful way to analyze is to use Fourier analysis.

Recall that, on $L(\mathrm{I_h})$, *we define the inner product*

$$\langle u, v\rangle_h = h\sum_{k=1}^{N-1} u(kh)v(kh)$$

with the corresponding norm $\|v\|_h$ *and* $\varphi_m(x) = \sin \pi m x, m = 1, \dots, N-1,$ *form an orthogonal basis.*

Also,

$$D_h^2\varphi_m = -\lambda_m\varphi_m, \qquad \lambda_m = \frac{2}{h^2}(\cos \pi m h - 1) = \frac{4}{h^2}\sin^2\frac{\pi m h}{2},$$

where the eigenvalues satisfy

$$8 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1} < \frac{4}{h^2}.$$

Another useful way to analyze is to use Fourier analysis.

Recall that, on $L(\mathrm{I_h})$*, we define the inner product*

$$\langle u, v\rangle_h = h\sum_{k=1}^{N-1} u(kh)v(kh)$$

with the corresponding norm $\|v\|_h$ *and* $\varphi_m(x) = \sin \pi m x, m = 1, \dots, N-1,$ *form an orthogonal basis.*

Also,

$$D_h^2\varphi_m = -\lambda_m\varphi_m, \qquad \lambda_m = \frac{2}{h^2}(\cos \pi m h - 1) = \frac{4}{h^2}\sin^2\frac{\pi m h}{2},$$

where the eigenvalues satisfy

$$8 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1} < \frac{4}{h^2}.$$

We write the semi-discrete solution $u_h(t,\cdot) = \sum_{m=1}^{N-1} a_m^h\varphi_m$*. Then,*

$$\frac{\partial u_h}{\partial t} = \sum_{m=1}^{N-1}\frac{da_m^h}{dt}\varphi_m, \qquad D_h^2 u_h = -\sum_{m=1}^{N-1} a_m^h\lambda_m\varphi_m\,.$$

Another useful way to analyze is to use Fourier analysis.

Recall that, on $L(\mathrm{I_h})$, we define the inner product

$$\langle u, v\rangle_h = h\sum_{k=1}^{N-1} u(kh)v(kh)$$

with the corresponding norm $\|v\|_h$ and $\varphi_m(x) = \sin \pi m x, m = 1, \dots, N-1,$ form an orthogonal basis.

Also,

$$D_h^2\varphi_m = -\lambda_m\varphi_m, \qquad \lambda_m = \frac{2}{h^2}(\cos \pi m h - 1) = \frac{4}{h^2}\sin^2\frac{\pi m h}{2},$$

where the eigenvalues satisfy

$$8 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1} < \frac{4}{h^2}.$$

We write the semi-discrete solution $u_h(t,\cdot) = \sum_{m=1}^{N-1} a_m^h\varphi_m$. Then,

$$\frac{\partial u_h}{\partial t} = \sum_{m=1}^{N-1}\frac{da_m^h}{dt}\varphi_m, \qquad D_h^2 u_h = -\sum_{m=1}^{N-1} a_m^h\lambda_m\varphi_m\,.$$

Thus,

$$\frac{da_m^h}{dt} = -ca_m^h\lambda_m \qquad \text{so} \qquad a_m^h(t) = a_m^h(0)e^{-c\lambda_m t}.$$

We write the semi-discrete solution $u_h(t,\cdot) = \sum_{m=1}^{N-1} a_m^h \varphi_m$. *Then,*

$$\frac{\partial u_h}{\partial t} = \sum_{m=1}^{N-1} \frac{d a_m^h}{dt} \varphi_m, \qquad D_h^2 u_h = -\sum_{m=1}^{N-1} a_m^h \lambda_m \varphi_m \,.$$

Thus,

$$\frac{d a_m^h}{dt} = -c a_m^h \lambda_m \qquad so \qquad a_m^h(t) = a_m^h(0) e^{-c\lambda_m t}.$$

The numbers $a_m^h(0)$ *are the coefficients in the discrete Fourier sine transform of the initial data:*

$$u_0(x) = \sum_{m=1}^{N-1} a_m^h(0) \varphi_m(x), \;\; x \in \bar{I}_h.$$

We write the semi-discrete solution $u_h(t,\cdot) = \sum_{m=1}^{N-1} a_m^h \varphi_m$. Then,

$$\frac{\partial u_h}{\partial t} = \sum_{m=1}^{N-1} \frac{da_m^h}{dt}\varphi_m, \qquad D_h^2 u_h = -\sum_{m=1}^{N-1} a_m^h \lambda_m \varphi_m .$$

Thus,

$$\frac{da_m^h}{dt} = -c a_m^h \lambda_m \qquad so \qquad a_m^h(t) = a_m^h(0) e^{-c\lambda_m t}.$$

The numbers $a_m^h(0)$ are the coefficients in the discrete Fourier sine transform of the initial data:

$$u_0(x) = \sum_{m=1}^{N-1} a_m^h(0)\varphi_m(x), \ \ x \in \bar{I}_h.$$

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0) e^{-c\lambda_m t}\varphi_m(x), \quad x \in \bar{I}_h.$$

We write the semi-discrete solution $u_h(t,\cdot) = \sum_{m=1}^{N-1} a_m^h \varphi_m$*. Then,*

$$\frac{\partial u_h}{\partial t} = \sum_{m=1}^{N-1} \frac{da_m^h}{dt}\varphi_m, \qquad D_h^2 u_h = -\sum_{m=1}^{N-1} a_m^h \lambda_m \varphi_m\,.$$

Thus,

$$\frac{da_m^h}{dt} = -c a_m^h \lambda_m \qquad so \qquad a_m^h(t) = a_m^h(0)e^{-c\lambda_m t}.$$

The numbers $a_m^h(0)$ *are the coefficients in the discrete Fourier sine transform of the initial data:*

$$u_0(x) = \sum_{m=1}^{N-1} a_m^h(0)\varphi_m(x), \;\; x \in \bar{I}_h.$$

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0)e^{-c\lambda_m t}\varphi_m(x), \;\;\; x \in \bar{I}_h.$$

For the fully discrete forward-centered scheme, we write the solution as

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0) e^{-c\lambda_m t} \varphi_m(x), \quad x \in \bar{I}_h.$$

For the fully discrete forward-centered scheme, we write the solution at time $t = jk$ as

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

The difference equation then gives

$$\sum_{m=1}^{N-1} A_m^{j+1} \varphi_m(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x) - ck \sum_{m=1}^{N-1} A_m^j \lambda_m \varphi_m(x)$$

yielding

$$A_m^{j+1} = (1 - ck\lambda_m) A_m^j.$$

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0) e^{-c\lambda_m t} \varphi_m(x), \quad x \in \bar{I}_h.$$

For the fully discrete forward-centered scheme, we write the solution at time $t = jk$ *as*

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

The difference equation then gives

$$\sum_{m=1}^{N-1} A_m^{j+1} \varphi_m(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x) - ck \sum_{m=1}^{N-1} A_m^j \lambda_m \varphi_m(x)$$

yielding

$$A_m^{j+1} = (1 - ck\lambda_m) A_m^j.$$

It follows that

$$A_m^j = (1 - ck\lambda_m)^j a_m^h(0)$$

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0) e^{-c\lambda_m t} \varphi_m(x), \quad x \in \bar{I}_h.$$

For the fully discrete forward-centered scheme, we write the solution at time $t = jk$ as

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

The difference equation then gives

$$\sum_{m=1}^{N-1} A_m^{j+1} \varphi_m(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x) - ck \sum_{m=1}^{N-1} A_m^j \lambda_m \varphi_m(x)$$

yielding

$$A_m^{j+1} = (1 - ck\lambda_m) A_m^j.$$

It follows that

$$A_m^j = (1 - ck\lambda_m)^j a_m^h(0), \qquad u^j = \sum_{m=1}^{N-1} (1 - ck\lambda_m)^j a_m^h(0) \varphi_m .$$

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0) e^{-c\lambda_m t} \varphi_m(x), \quad x \in \bar{I}_h.$$

For the fully discrete forward-centered scheme, we write the solution at time $t = jk$ *as*

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

The difference equation then gives

$$\sum_{m=1}^{N-1} A_m^{j+1} \varphi_m(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x) - ck \sum_{m=1}^{N-1} A_m^j \lambda_m \varphi_m(x)$$

yielding

$$A_m^{j+1} = (1 - ck\lambda_m) A_m^j.$$

It follows that

$$A_m^j = (1 - ck\lambda_m)^j a_m^h(0), \qquad u^j = \sum_{m=1}^{N-1} (1 - ck\lambda_m)^j a_m^h(0) \varphi_m .$$

If we assume $ck/h^2 \le 1/2$, *then* $ck\lambda_m \le ck(4/h^2) \le 2$ *and hence* $|1 - ck\lambda_m| \le 1$ *for all* m *and the solution remains bounded.*

Thus, the solution of the semi-discrete system may be written as

$$u(x,t) = \sum_{m=1}^{N-1} a_m^h(0)e^{-c\lambda_m t}\varphi_m(x), \quad x \in \bar{I}_h.$$

For the fully discrete forward-centered scheme, we write the solution at time $t = jk$ as

$$u^j(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x).$$

The difference equation then gives

$$\sum_{m=1}^{N-1} A_m^{j+1}\varphi_m(x) = \sum_{m=1}^{N-1} A_m^j \varphi_m(x) - ck\sum_{m=1}^{N-1} A_m^j \lambda_m \varphi_m(x)$$

yielding

$$A_m^{j+1} = (1 - ck\lambda_m)A_m^j.$$

It follows that

$$A_m^j = (1 - ck\lambda_m)^j a_m^h(0), \qquad u^j = \sum_{m=1}^{N-1} (1 - ck\lambda_m)^j a_m^h(0)\varphi_m\,.$$

If we assume $ck/h^2 \leq 1/2$, then $ck\lambda_m \leq ck(4/h^2) \leq 2$ and hence $|1 - ck\lambda_m| \leq 1$ for all m and the solution remains bounded. On the other hand, if $|1 - ck\lambda_m| > 1$ for some m, the initial data will increase exponentially.