MTH 421A (ODE)

1. Calculate the norm of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Super!! Do it again!

2. Let $F : A (\subseteq \mathbb{R}^n, \text{open}) \to \mathbb{R}^n$ is locally Lipchitz and $B (\subset A)$ is compact, then $F|_B$ is Lipchitz.

see Again!

3. Let Ω $(\subseteq \mathbb{R} \times \mathbb{R}^n)$ be an open set containing $(0, x_0)$ and $F, G : \Omega \to \mathbb{R}^n$ are C^1 function w.r.t X and continuous w.r.t t such that $\| F(t, x_n) - G(t, x_n) \| < \epsilon$. $\forall (t, x) \in \Omega$ and K is the Lipchitz constant in x for $F(t,x)$. If $X(t)$ and $Y(t)$ solves $X'(t) = F(t, X)$; $X(0) = X_0$ and $Y'(t) = G(t, Y)$; $Y(0) = Y_0$ resp. Show, $|X(t) - Y(t)| \leq \epsilon/K \exp(K|t| - 1)$ $\forall t \in J$.

4. Let A be a $(n \times n)$ matrix. Use Picard's iterate to solve the problem $X' = AX$; $X(0) = X_0$.

Picards iterate likho, see pg (8-11) slides, also understand agar A function of t hota to integrate karna padega usko bhi and agar vo commute nhi karega A ke saath to hum common lekar series nhi bna payenge

5. Find the Lipchitz constant for the following:-
   (a) $f(x) = |x|$ on $-\infty < x < \infty$
   (b) $f(x) = x^2 \log x$ on $2 \leq x \leq 3$
   (c) $f(x, y) = \dfrac{xy}{1 + x^2 + y^2}$; $x^2 + y^2 \leq 4$.

Kul milakar gradient ko bound karna h then we are done

6. Comment on the uniqueness of the following problems:-
   (a) $x'(t) = x^{\frac{1}{3}}$; $x(0) = 0$.
   (b) $x'(t) = x^2$; $x(0) = 1$.

ASSIGNMENT 2 (MTH421A - ODE)

(a) For a square matrix A show $\frac{d}{dt} e^{At} = Ae^{At}$

(b) Find A, B s.t. $e^A e^B \neq e^{A+B}$.

(c) Compute the matrix e^{Bt} when $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$; $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ and $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(d) Solve $X' = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix} X$

(e) Suppose that the square matrix has a negative eigenvalue. Show that the linear system $X' = AX$ has at least one solution (nontrivial) that satisfies $\lim_{t \to \infty} x(t) = 0$.

(d) If $\phi(t, x_0)$ be the solution of $X' = AX$; $X(0) = X_0 (\in \mathbb{R}^n)$. Use the Fundamental theorem to show that for each fixed $t \in \mathbb{R}$, $\lim_{y \to x_0} \phi(t, y) = \phi(t, x_0)$

Assignment 3 :-

1. Is the set of all vector valued function $x(t) = [x_1(t) \ \ x_2(t)]^T$ satisfying $x_1' = x_2 + 1$; $x_2' = x_1 + t$ is not a vector space. Find $x(t)$?

2. Solve the following :- $x' = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} x$

3. Let $x_1(t), x_2(t), \ldots, x_n(t)$ be the solution of $x' = A(t)x$ in $I (\ni 0)$. Then for all $x \in I$, $W(t) = W(0)\exp\left[\int_0^t \operatorname{Tr} A \, ds\right]$
   [W = Wronskian]

Abel's thm (photo)

4. If $\phi(t)$ is the Fundamental solution of $x' = A(t)x$, show that $[\psi^{-1}(t)]^T$ is the Fundamental matrix of the system $x' = -A^T(t)x$. (This equation is called Adjoint System). Find the relation between the two Fundamental solution.

5. Show that $x(t) = \exp\left(\int_0^t A(s)\,ds\right)x_0$ is not a solution of $x' = A(t)x$ unless $A(t)$ and $\int_0^t A(s)\,ds$ commutes for all t.

6. Let $v(t)$ be the solution of the I.V.P $y'' + P_1 y' + P_2 y = 0$; $y(0) = 0, y'(0) = 1$. Show that the function

$$w(t) = \int_0^t v(t-s)\, r(s)\, ds$$

solves, $y'' + P_1 y' + P_2 y = r(x)$ with $y(0) = 0$.

7. True/False :- The DE $y' = \cos^2 x \, y$ admits a non-periodic solution.

8. Comment on the existence of a unique periodic solution for the equation $y' = ay + \sin x$; $a \in \mathbb{R}$.

9. Let $y_1(x)$ and $y_2(x)$ be two solutions of $y'' + p(x)y' = 0$; $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2(0) = 0$; $y_2'(0) = 1$. Further let $p(x)$ be ω-periodic and continuous in $\mathbb{R}$.

later

(a) Show $\exists$ atleast one non-trivial periodic solution of period ω iff $y_1(\omega) + y_2'(\omega) = 2$.

(b) Show $\exists$ atleast one non-trivial anti periodic solution $y(x) = -y(x+\omega)$; $x \in \mathbb{R}$ iff $y_1(\omega) + y_1'(\omega) = -2$.

10. Let $p(x)$ be continuous and $p(x+\pi) = p(x) \not\equiv 0$ $\forall x \in \mathbb{R}$. If $0 \leq \pi \int_0^\pi |p(s)| ds \leq 4$ then show that all solutions of $y'' + p(x)y = 0$ are bounded in $\mathbb{R}$.

11. (Back to a little analysis) :- Can one find a nontrivial function f in $\mathbb{R}$ satisfying $|f(x) - f(y)| \leq |x-y|^\alpha$; $\alpha > 0$ and for all $x, y \in \mathbb{R}$
(Let's call such function space (if it exists) as Hölder continuous space)

both with same example of x^alpha

12. Let $y' = f(x,y)$ in I with f is continuous w.r.t x and Hölder continuous with exponent $\alpha = \frac{1}{2}$ w.r.t y. Does there exists an unique solution in a nbd of $x_0 \in I$.

13. Let $x(t) = e^{\lambda t} v$ be a solution of $x' = Ax$. Prove that $y(t) = \mathrm{Re}\{x(t)\}$ and $z(t) = \mathrm{Im}\{x(t)\}$ are Linearly independent.

14. Solve :-

$$x'(t) = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} x(t) \; ; \; x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Ans: $\left[x(t) = e^{2t} \begin{pmatrix} 1 + 5t - \frac{1}{2}t^2 \\ 2 - t \\ 1 \end{pmatrix} \right]$.

15. Let $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$. Calculate e^{At} ?

Assignment 4 :-

(a) Let $y(x)$ be a solution of $y' = y - y^2$; $y(0) = y_0 \in (0,1)$. Show that $y(x) \in (y_0, 1]$ $\forall x \in (0,\infty)$.

(b) Let $y(x)$ be a solution of $y' = y^2 - x$; $y(0) = 1$. Show $1 + x < y(x) < \frac{1}{1-x}$ $\forall x \in (0,1)$.

(c) Let $f_i(x,y)$ are continuous function in $D := \{(x,y) : x \in I = [x_0, x_0 + a]\}$. such that $f_1(x,y) < f_2(x,y)$ $\forall (x,y) \in D$. Further let y_1 and y_2 are two solutions of $y_i' = f_i(x,y)$ s.t $y_1(x_0) < y_2(x_0)$. Show $y_1(x) < y_2(x)$ $\forall x \in I$.

(d) Comment on the asymptotic behaviour of the equation $y' = -y(1+y^2)$; $y(0) = 1$.

(e) Sketch the phase portrait of the following :-

(i) $X' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X$

(ii) $X' = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} X$

(iii) $X' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} X$

Assignment - 5

(1) Let $u(x)$, $p(x)$ and $q(x)$ are non-negative continuous fxn in $|x-x_0| \le a$ and

$$u(x) \le p(x) + \left|\int_{x_0}^{x} q(t)u(t)\,dt\right| \quad \text{for } |x-x_0| \le a$$

Then the following holds:-

$$u(x) \le p(x) + \left|\int_{x_0}^{x} p(t)q(t)\exp\left(\left|\int_{t}^{x} q(s)\,ds\right|\right)dt\right| \quad \text{for } |x-x_0| \le a$$

(2) Let $f(x,y)$ be continuous and satisfy the Lipchitz condition

$$|f(x,y_1) - f(x,y_2)| \le L(x)|y_1 - y_2| \quad \forall (x,y_1),(x,y_2) \in D$$

where $L(x)$ is such that $\int_{x_0-a}^{x_0+a} L(\xi)\,d\xi$ exists. Show that $y' = f(x,y)$; $y(x_0) = y_0$ admits atmost one solution in $|x-x_0| \le a$.

(3) Find the general solution of

(a) $yy'' + y'^2 = 0$ (b) $(1-x)y'' + xy' + y = 0$, $x \ne 1$.

(4) If $z_1(x)$ and $z_2(x)$ are linearly independent soln of the adjoint eqn of $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$ --- (1), $P_0 > 0$.
Show that $y_1(x) = \frac{z_1(x)}{\sigma(x)}$ and $y_2(x) = \frac{z_2(x)}{\sigma(x)}$ are linearly independent Solution of (1).

(5) True / False :- The equation $y'' + [1 + m(x)]y = 0$ is oscillatory provided $m(x) \to 0$ as $x \to \infty$.

(6) True / False :- Sturm seperation theorem holds for DE's with order ≥ 3.

(7) Show that every solution of the equation $y'' - 2xy' + 2ay = 0$ $(a \geq 0)$ admits finitely many zero in $I = \mathbb{R}$.

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MTH 421A :- Assignment 6

1. Solve the following B.V.P :-

   (a) $y'' - y = 0$; $y(0) = 0$ & $y(1) = 1$ gen. soln then b.c.

   (b) $x^2 y'' + 7xy' + 3y = 0$; $y(1) = 1$; $y(2) = 2$ y=cx^m

   (c) $y'' + y' + y = x$; $y(0) + 2y'(0) = 1$ and $y(1) - y'(1) = 8$. particular soln using y=Ax+B, add into homo. soln and then b.c.

2. Comment on the no of solutions of $y'' + y = x$ s.t $y(0) + y'(0) = \pi/2$ and $y(\pi/2) - y'(\pi/2) = \pi/2$.

   unique soln, gen soln is homo+x and then b.c.

3. Show that the problem $y'' + p(x)y = q(x)$ with Dirichlet Boundary Condition has a unique solution.

4. Show that the set $\left\{ \sqrt{\frac{2}{\pi}} \sin nx \; ; \; n \in \mathbb{N} \right\}$ is orthonormal on $[0, \pi]$ with respect to the weight $r(x) = 1$. easy

5. Find the eigenvalues and eigenfunctions of $y'' + \lambda y = 0$ s.t

   (a) $y'(0) = 0$ and $y(1) = 0$ take cases of lambda, write gen. solns and then b.c.

   (b) $y(0) = 0$ and $y(1) + y'(1) = 0$

   (c) $y(0) - y'(0) = 0$ and $y(1) + y'(1) = 0$.

6. Consider the B.V.P $x^2 y'' + xy' + \lambda y = 0$; $1 < x < e$

   $y(1) = 0$; $y(e) = 0$

   (a) Show that the above B.V.P is equivalent to $(xy')' + \frac{\lambda}{x} y = 0$, $1 < x < e$

   RSLBVP $y(1) = y(e) = 0$

   (b) Find the eigenvalues and eigenfunctions. RSLBVP has only real eig.values, now take cases of +,-,0 lambda and use y=x^m if needed

* 7. Solve :- $y'' + \lambda y = 0$; $y'(0) = 0$, $|y(x)| < \infty$ $\forall x \in (0, \infty)$.

ASSIGNMENT-7 :-

(a) Consider the eigenvalue problem :-

$$y'' + \lambda y = 0 \;;\; y(0) + y'(0) = 0 \text{ and } y(1) = 0$$

(i) Find the eigenvalues and eigenfunctions

(ii) write the Fourier Expansion of $f(x) = x - x^2$ in terms of the above eigenfunctions.

(b) Show that the following cannot be the Fourier Series representation for any piecewise continuous function :

(i) $\sum_{n=1}^{\infty} n^{1/n} \phi_n(x)$ (ii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \phi_n(x)$. check using fourier convg. thm

(c) Verify Bessel's Inequality using $f(x) = 1$ w.r.t $\{\sqrt{2} \sin n\pi x \;;\; n = 1, 2, \ldots\}$. Take [0,1], r(x)=1

(d) Let $f, g \in C_p[\alpha, \beta]$ and have the same Fourier Series w.r.t to an orthonormal set. Show $f = g$ at points of continuity. if both convg. then ok

(e) Find the Fourier Series of $f(x) = |x| \;; -\pi < x < \pi$ w.r.t $\{1, \sin nx, \cos nx \;;\; n \in \mathbb{N}\}$.

(f) Solve: $y'' + 3y = e^x \;;\; y(0) = y(1) = 0$ using eigenfunction expansion. ?

Assignment-9

1. Find the largest interval where the solution of $yy' - 3x^2(1+y^2) = 0$; $y(0) = 1$ is defined

2. If $K \subset \Omega \subset \mathbb{R}^n$, where K is compact and Ω is open, then $\exists$ a larger compact set $K' \subset \Omega$ and a $\delta > 0$ such that for every $x \in K$, the closed ball $\overline{B(x,\delta)}$ is contained in K'.

3. Show that the problem $x' = A(t)x + Y$; $x(0) = b$ admits an unique solution. (Assuming A and Y is continuous.

4. Show that if $g : [0,T] \to \mathbb{R}$ is continuous and if there are non-negative constants C, B, K such that $g(t) \leq C + Bt + K\int_0^t g(s)\,ds$; $0 \leq t \leq T$ then $g(t) \leq Ce^{Kt} + \frac{B}{K}(e^{Kt} - 1)$; $0 \leq t \leq T$.

5. Let $x^* : (-\alpha_*, \beta_*) \to \mathbb{R}^n$ be the maximal solution st $\beta_* < \infty$. Then for any compact set $K \subset \Omega$ $\exists\, \varepsilon > 0$ s.t $x^*(t) \notin K$ for $\beta_* - \varepsilon < t < \beta_*$.

6. Consider the system
$$\left.\begin{aligned} x' &= x^2 - y^2 \\ y' &= 2xy \end{aligned}\right\} \text{—(i)}$$
Find solutions of (i) that blows up but nearby solutions exists for all time t.

ASSIGNMENT-10.

1. Let $\lambda_1, \lambda_2, \ldots, \lambda_k ; k \leq n$ be distinct eigenvalues of the matrix A with multiplicity $r_1, r_2, \ldots, r_k$ resp so that $p(\lambda) = (\lambda-\lambda_1)^{r_1} \cdots (\lambda-\lambda_k)^{r_k}$

then $$e^{Ax} = \sum_{i=1}^{n} \left[e^{\lambda_i x} a_i(A) q_i(A) \sum_{j=0}^{r_k-1} \left\{ \frac{1}{j!} (A-\lambda_i I)^j x^j \right\} \right]$$

where $q_i(\lambda) = p(\lambda)(\lambda-\lambda_i)^{-r_i}$; $1 \leq i \leq k$

and $a_i(\lambda)$; $1 \leq i \leq k$ are polynomials of degree less than r_i in the expansion

$$1/p(\lambda) = \sum_{i=1}^{k} \frac{a_i(\lambda)}{(\lambda-\lambda_i)^{r_i}}$$

2. Prove/Disprove :- Any solution of the equation $x' = Ax$ satisfies $\|u(t)\| \leq ce^{\eta t}$, $t \geq 0$.

3. Let all solutions of $x' = Ax$ be bounded in $[0, \infty)$. Then find conditions on B s.t all solutions of the system $x' = [A + B(t)]x$ is bounded in $[0, \infty)$.

discuss again +photo-2 april

see Tr(A(t)) =0 wala thm

4. Show all solutions $\tilde{x}(t)$ of $x' = Ax + B(t)$ satisfies $\|\tilde{x}(t)\| \le c_2 e^{\gamma t}$ $\forall\, t \ge 0$, provided $\|B(t)\| \le c_1 e^{\gamma t}$ for large γ, where c_1 and c_2 are constants.

5. Consider the problem $y'' + p(t)y = 0$ with $p(t) \to \infty$ monotonically as $t \to \infty$. Show that all solutions are bounded in $[0, \infty)$.

6. Show that $y'' + \left[1 + \dfrac{1}{1+x^4}\right] y = \cos x$; $x \in [0, \infty)$ does not admit a bounded solution.

7. Test the stability / A.S / Un.S of the trivial solution for each of the following system.

(a) $x' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$ (b) $x' = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix} x$

(c) $x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -6 & -5 \end{pmatrix} x$

8. Consider the system $x' = Ax + B(t)$. Show that

(a) All solutions are bounded in $[0, \infty)$ then they are stable.

(b) All solutions are stable and one is bounded $\Rightarrow$ All solutions are bounded in $[0, \infty)$.

Last Assignment

1. Show that $(1,1)$ is a stable equilibrium for $x' = 1 - xy$; $y' = x - y^3$

2. Classify the equilibrium of the system $x' = -y - x^2$; $y' = x$.

3. Prove that all solutions of the following DE are periodic :

   (a) $y'' + a^2 y + by^3 = 0$, $b > 0$

   agar highest power polynomial eqn mei even ho aur uska coeff +ve ho for both x and y then closed curve solution aata h hence periodic

   (b) $y'' + y + y^7 = 0$

4. Consider the DE $y'' + py' + qy = 0$, $q \neq 0$. Determine the nature of critical point $(0,0)$

   for (a) $p^2 > 4q$, $q > 0$, $p < 0$

   (b) $p^2 > 4q$, $q > 0$, $p > 0$

   (c) $p = 0$ and $q > 0$

   (d) $p^2 = 4q$ and $p > 0$.

5. Find and classify the critical point of the equation

$$x_1' = x_1(2x_2 - x_1 + 5)$$
$$x_2' = x_1^2 + x_2^2 - 6x_1 - 8x_2 .$$

6. Show that the D.E $x_1' = x_2 + x_1 \frac{f(r)}{r}$; $x_2' = -x_1 + x_2 \frac{f(r)}{r}$; $r = \sqrt{x^2+y^2}$ has limit cycles corresponding to the zeroes of $f(r)$

note that r* of f(r*)=0 is eqm point of this system

7. Find all limit cycle of $x_1' = x_2 + x_1(x_1^2+x_2^2)^{1/2}(x_1^2+x_2^2-3)$; $x_2' = -x_1 + x_2(x_1^2+x_2^2)^{1/2}(x_1^2+x_2^2-3)$

note: r mujse to na nikla

8. Show ∃ non-trivial periodic solution of the system

$$x_1' = 2x_1 - 2x_2 - x_1[x_1^2 + x_2^2]$$
$$x_2' = 2x_1 + 2x_2 - x_2(x_1^2 + x_2^2).$$

9. Prove / Disprove :- $\left.\begin{aligned} x_1' &= x_1 + 7x_2^2 + 2x_2^3 \\ x_2' &= -x_1 + 3x_2 + x_2 x_1^2 \end{aligned}\right\}$ does not admit admit a non-trivial periodic solution in $\mathbb{R}^2$.

(10) Show that $V(x_1, x_2) = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2$ in $\mathbb{R}^2$ is +ive definite iff $c_1 > 0$ and $c_2^2 - 4c_1c_3 < 0$.

(11) Determine whether the trivial solution is Stable / Asyp. St / Un. St for the eqn

(a) $x_1' = -x_1 + e^{x_1} x_2$; $x_2' = -e^{x_1} x_1 - x_2$

(b) $x_1' = x_1^3 - x_2$; $x_2' = x_1 + x_2^3$.

(12) Find an appropriate Lyapunov function for the eqn $y'' + \omega^2 \sin y = 0$; $-\pi/2 \le y \le \pi/2$ to show the trivial solution is stable.

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