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For finite element methods, the highly localized support of basis functions automatically makes the “nearly” orthogonal resulting in (a) a relatively well-conditioned equations to solve and (b) it also makes the system sparse, so that a much less work and storage is required to solve it.

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Numerical Analysis & Scientific Computing II

Lesson 3

Boundary Value Problems for ODEs

3.2 Shooting Method

3.3 Finite Difference Method

3.4 Variational Methods

- Convergence Analysis for Galerkin Method



Akash Anand
MATH, IIT KANPUR

Boundary Value Problems: Variational Methods

For analyzing the computational error in the Galerkin method, note that the solution u to the boundary value problem

$$\begin{aligned}u'' &= f(t), & a < t < b, \\u(a) &= 0, & u(b) = 0,\end{aligned}$$

and any function of the form

$$v(t, x) = \sum_{i=1}^n x_i \varphi_i(t),$$

with $\varphi_i(a) = \varphi_i(b) = 0$, satisfies

$$\int_a^b u''(t) v(t, x) dt = \int_a^b f(t) v(t, x) dt.$$

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$$\|w\|_{H^1}^2 := \|w\|_{L^2}^2 + \|w'\|_{L^2}^2$$

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As $z \in \mathbb{R}^n$ is arbitrary, we conclude that

$$\|u - v(\cdot, y)\|_{H^1} \leq (2 + c^2(b - a)^2) \inf_{z \in \mathbb{R}^n} \|u - v(\cdot, z)\|_{H^1}$$

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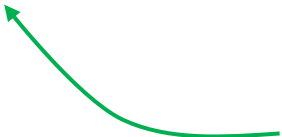
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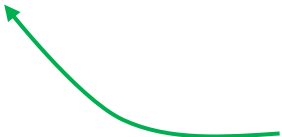
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$$\inf_{z \in \mathbb{R}^n} \|u - v(\cdot, z)\|_{H^1} \leq Ch \|u''\|_{L^2}$$

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