1. The magnetic vector potential due to a surface current distribution $\mathbf{K}(\mathbf{r}')$ at any point $\mathbf{r}$ is given by,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,da'$$

Now consider a thin spherical shell of radius R with center at the origin, carrying a surface charge density σ and rotating with angular velocity $\omega\hat{z}$. Find the vector potential everywhere using the above expression.

$$\vec{K} = \sigma\omega R\sin\theta'\hat{\varphi}'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\sigma\omega R\sin\theta'\hat{\varphi}'}{|\vec{r}-\vec{r}'|}da' \quad , \quad \vec{r}' = \vec{r}'(\theta,\varphi),\ |\vec{r}'| = R$$

$$= \frac{\mu_0\sigma\omega R}{4\pi}\int\frac{\sin\theta'(-\sin\varphi'\hat{x} + \cos\varphi'\hat{y})}{|\vec{r}-\vec{r}'|}da'$$

$$= \frac{\mu_0\sigma\omega R}{4\pi}\left[-\int\frac{\sin\theta'\sin\varphi'}{|\vec{r}-\vec{r}'|}da'\,\hat{x} + \int\frac{\sin\theta'\cos\varphi'}{|\vec{r}-\vec{r}'|}da'\,\hat{y}\right]$$

Let us evaluate the integrals using our knowledge of electrostatics.

Consider the case of a uniformly polarized sphere of polarization P along x direction $\Rightarrow$ $\vec{P} = P\hat{x}$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0 \quad , \quad \sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r} = P\hat{x}\cdot\hat{r} = P\sin\theta\cos\varphi$$

Note that the surface charge distribution has (θ,φ) dependence similar to the numerator in the second integral.

The scalar potential V is given by,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma_b(\vec{r}')}{|\vec{r}-\vec{r}'|}da'$$

$$= \frac{P}{4\pi\epsilon_0}\int\frac{\sin\theta'\cos\varphi'}{|\vec{r}-\vec{r}'|}da'$$

Again, outside the sphere, the electrostatic potential is equal to that of the dipole moment $\vec{p} = \frac{4\pi}{3}R^3\vec{P}$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{3}R^3\frac{\vec{P}\cdot\vec{r}}{r^3} = \frac{1}{3\epsilon_0}R^3\frac{P\sin\theta\cos\varphi}{r^2}$$

comparing, $\boxed{\int\frac{\sin\theta'\cos\varphi'}{|\vec{r}-\vec{r}'|}da' = \frac{4\pi}{3}\frac{R^3}{r^2}\sin\theta\cos\varphi}$ for $r \geq R$

Inside the sphere, the electric field is given by $\vec{E} = \frac{-\vec{P}}{3\epsilon_0} = \frac{-P}{3\epsilon_0}\hat{x}$

$$\Rightarrow V(\vec{r}) = \frac{Px}{3\epsilon_0} + \text{const} = \frac{P}{3\epsilon_0}r\sin\theta\cos\varphi + \text{const.}$$

comparing, $\boxed{\int\frac{\sin\theta'\cos\varphi'}{|\vec{r}-\vec{r}'|}da' = \frac{4\pi}{3}r\sin\theta\cos\varphi}$ for $r \leq R$

Similarly, the other integral can be evaluated by considering a uniformly polarized sphere in the y direction,

Finally, we have,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sigma \omega R \cdot \frac{4\pi}{3} \frac{R^3}{r^2} \left(-\sin\theta \sin\varphi \,\hat{x} + \sin\theta \cos\varphi \,\hat{y}\right)$$

$$= \frac{\mu_0 \sigma \omega R}{3} \frac{R^3}{r^2} \sin\theta \,\hat{\varphi} \qquad \text{for } r \geq R$$

And,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sigma \omega R \frac{4\pi}{3} r \left(-\sin\theta \sin\varphi \,\hat{x} + \sin\theta \cos\varphi \,\hat{y}\right)$$

$$= \frac{\mu_0 \sigma \omega R r \sin\theta}{3} \hat{\varphi} \qquad \text{for } r \leq R$$

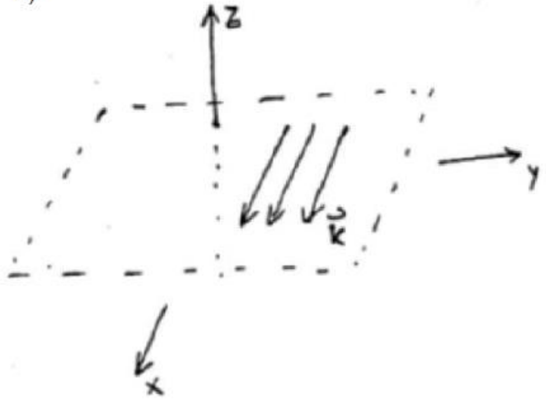
$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 \sigma \omega R^4}{3r^3} \left(2\cos\theta \,\hat{r} + \sin\theta \,\hat{\theta}\right) \qquad r \geq R$$

$$= \frac{2}{3} \mu_0 \sigma \omega R \qquad r \leq R$$

2. (a) Find the magnetic vector potential everywhere for an infinite sheet with a uniform surface current $K\hat{x}$ using the relation $\oint \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \Phi$.

   (b) Find the vector potential everywhere for a long conducting wire of radius R carrying uniform current I along its axis using electrostatic analogy.

a)



Using Ampere's law, we have,

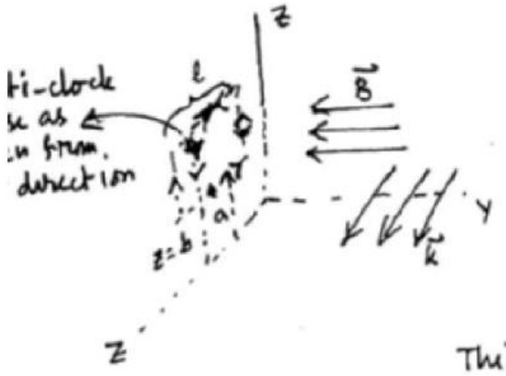
$$\vec{B} = -\frac{\mu_0 K}{2}\hat{y} \quad z>0$$
$$= \frac{\mu_0 K}{2}\hat{y} \quad z<0$$

Now use $\oint \vec{A}\cdot d\vec{l} = \int_s \vec{B}\cdot d\vec{a} = \Phi$

line integral around a loop

surface integral over the area enclosed by the loop

For $z>0$, take the loop in the xz plane with



$$\int_s \vec{B}\cdot d\vec{a} = -\frac{1}{2}\mu_0 K (b-a)\, l$$

and $\oint \vec{A}\cdot d\vec{l} = [A_x(b) - A_x(a)]\, l$

By comparing one can guess,

$$\boxed{\vec{A} = -\frac{\mu_0}{2} K z \hat{x}} \quad , z>0$$

This is a correct guess, as $\vec{\nabla}\cdot\vec{A} = 0$, and $\vec{\nabla}\times\vec{A} = -\frac{\mu_0 K}{2}\hat{y}$ as should be for $z>0$

Similarly for $z<0$, $\vec{A} = \frac{\mu_0 K}{2} z\hat{x}$

b)

outside $s > R$, consider the electrostatic case, uniformly charged cylinder

$$\vec{E} = \frac{\rho R^2}{2\epsilon_0 r}\hat{r}$$

$$V(r) - V(R) = -\int_R^r E\,dr = -\frac{\rho R^2}{2\epsilon_0}\int_R^r \frac{dr}{r} = -\frac{\rho R^2}{2\epsilon_0}\ln\frac{r}{R}$$

soln of $\nabla^2 V = -\frac{\rho}{\epsilon_0}$, $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}d\tau'$

" $\nabla^2\vec{A} = -\mu_0\vec{J}$, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}d\tau'$

since $V(r) = -\frac{\rho R^2}{2\epsilon_0}\ln\frac{r}{R}$, $\vec{A}(\vec{r})$ will have same functional form with ρ/ϵ_0 replaced by $\mu_0 J$

$$\vec{A} = -\frac{\mu_0 J R^2}{2}\ln\frac{r}{R}\hat{z}$$

Inside the cylinder $r < R$, $\vec{E}_{in} = \frac{\rho\vec{r}}{2\epsilon_0}$ for the electrostatic case

$$V(r) - V(R) = -\int_R^r E\,dr = -\int_R^r \frac{\rho r}{2\epsilon_0}dr = \frac{\rho}{2\epsilon_0}\frac{R^2 - r^2}{2}$$

$$V(r) = -\frac{\rho}{2\epsilon_0}\frac{r^2}{2}$$

Hence, $\vec{A} = -\frac{\mu_0 J_0}{4} r^2\hat{z}$

In summary,

$$\boxed{\begin{aligned}\vec{A} &= -\frac{\mu_0 J}{4} r^2\hat{z} & r < R\\ &= -\frac{\mu_0 J}{4}R^2\left(1 + 2\ln\frac{r}{R}\right) & r > R\end{aligned}}$$

extra constant term to make $\vec{A}$ continuous at $r = R$

3. The magnetic dipole moment of a volume current distribution, as discussed in the lecture, is given by,

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, d\tau'$$

Using the above expression find the magnetic dipole moment of a spherical shell of radius R, carrying a surface charge density σ and rotating with angular velocity $\omega\hat{z}$.

$$\vec{K} = \sigma\omega R \sin\theta\, \hat{\varphi} \Rightarrow \omega\sigma r' \sin\theta' \delta(r'-R)\, \hat{\varphi}' = \vec{J}(\vec{r}')$$

$$\vec{m} = \frac{1}{2}\int \vec{r}' \times \vec{J}(\vec{r}')\, d\tau'$$

$$= \frac{1}{2}\int r'(\sigma\omega r' \sin\theta')\, \delta(\vec{r}'-R)(\hat{r}' \times \hat{\varphi}')\, d\tau'$$

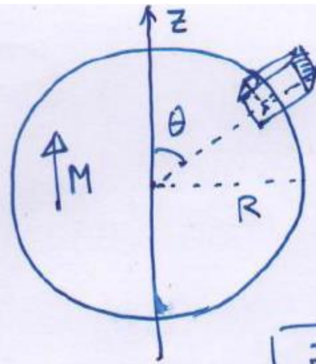
$$= \frac{1}{2}\sigma\omega \int r'^2 \sin\theta'\, \delta(r-R)\, (-\hat{\theta})\, r'^2 \sin\theta' d\theta' d\varphi' dr'$$

$$= \frac{1}{2}\sigma\omega R^4 \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \sin^2\theta' d\theta' (\sin\theta' \hat{z} - \cos\theta' \cos\varphi' \hat{x} - \cos\theta' \sin\varphi' \hat{y})\, d\varphi'$$

Noting, $\int_0^{2\pi} \sin\varphi\, d\varphi = \int_0^{2\pi} \cos\varphi\, d\varphi = 0$, we finally have,

$$\boxed{\vec{m} = \frac{4\pi}{3}\sigma\omega R^4\, \hat{z}} \Rightarrow \text{Magnetization } \vec{M} = \sigma\omega R\, \hat{z}$$

4. Consider a sphere of radius R, having frozen-in uniform magnetization M pointing towards the north pole. Find the 'auxiliary H' field inside the sphere using electrostatic analogy. Find the 'B' field inside the sphere.



$$\boxed{\begin{aligned}\vec{\nabla}\cdot\vec{H} &= -\vec{\nabla}\cdot\vec{M}\\ \vec{\nabla}\times\vec{H} &= 0\end{aligned}}$$

$\int \vec{\nabla}\cdot\vec{M}\, d\tau = \oint \vec{M}\cdot d\vec{a}$

As $l \to 0$,

$\oint \vec{M}\cdot d\vec{a} = -M\cos\theta\, A$

Again, $\displaystyle\int_{R-l}^{R+l} \vec{\nabla}\cdot\vec{M}\, d\tau = \int_{R-l}^{R+l} (\vec{\nabla}\cdot\vec{M})\, A\, dr$

$\Rightarrow \displaystyle\int_{R-l}^{R+l} (\vec{\nabla}\cdot\vec{M})\, dr = -M\cos\theta$ even as $l \to 0$

$\Rightarrow \vec{\nabla}\cdot\vec{M} = -M\cos\theta\, \delta(r-R)$

$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{H} = M\cos\theta\, \delta(r-R)}$

Repeat the same procedure for the bottom half of the sphere $\Rightarrow$ $\vec{\nabla}\cdot\vec{H} = -M\cos\theta\, \delta(r-R)$

Positive 'surface charge' on the top half and negative 'surface charge' on the bottom half.

Recall the problem of uniformly polarized sphere, (along $\hat{z}$) where the surface charge is $P\cos\theta$ and the electric field inside the sphere is $-\frac{1}{3\epsilon_0}\vec{P}\hat{z}$

Here the H field inside the sphere will be $\vec{H} = -\frac{M}{3}\hat{z}$

$\Rightarrow \vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\mu_0\vec{M}$, inside the sphere