Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Complex Analysis (MTH 403)

Hints for Exercise Sheet 7

1. Zeros of analytic functions

Throughout this section, unless otherwise mentioned, U always stands for a region.

1.1. Show that $H(U)$ is an integral domain with respect to pointwise addition and multiplication.

1.2. Let U be an open connected subset of $\mathbb{C}$ and $f \in H(U)$. Assume that for all $z \in U$ there exists $n \geq 0$ such that $f^{(n)}(z) = 0$. What can you conclude about f?

1.3. Let $f : \mathbb{D} \longrightarrow \mathbb{C}$. Show that, if f^2 and f^3 both are holomorphic, then so if f.

Hint. *Observe that $f = \frac{f^3}{f^2}$ at all ponts $z \in \mathbb{D}$ such that $f(z) \neq 0$. So zeros are needed to be taken care of.*

1.4. (L'Hôpital's rule). Let $U \subseteq_{open} \mathbb{C}$ and $f, g \in H(U)$. Suppose that $z_0 \in U$ is such that on some neighbourhood of z_0 in U, none of f and g vanishes identically, but $\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to z_0} g(z) = 0$. Show that $\frac{f(z)}{g(z)}$ approaches to a finite limit or ∞ as $z \to z_0$, and furthermore, $\dfrac{\lim\limits_{z \to z_0} f(z)}{\lim\limits_{z \to z_0} g(z)} = \dfrac{\lim\limits_{z \to z_0} f'(z)}{\lim\limits_{z \to z_0} g'(z)}$.

1.5.* Let U be a region in $\mathbb{C}$. Assume that U is *symmetric* with respect the real axis, i.e., $z \in U \Longrightarrow \bar{z} \in U$. Suppose that $f \in H(U)$ is such that $f(J) \subseteq \mathbb{R}$, for some open interval containied in $U \cap \mathbb{R}$. Show that $f(U \cap \mathbb{R}) \subseteq \mathbb{R}$ and $f(\bar{z}) = \overline{f(z)}$, for all $z \in U$.

1.6. Let f be a nonzero entire function such that $f(0) = 0$ and $f(\mathbb{R}) \subseteq \mathbb{R}$. Show that if the image of the imaginary axis under f is contained in a line, then that line must be either the real axis or the imaginary axis.

Sketch of the solution. *Let L be the line containing the image of the imaginary axis. As $f(0) = 0$, the equation of L must be $ax + by = 0$, for some $a, b \in \mathbb{R}$. We apply 1.5. to obtain that $f(\bar{z}) = \overline{f(z)}$, for all $z \in \mathbb{C}$. Note that, there exists $t \in \mathbb{R}$ such that $f(it) = x' + iy' \neq 0$, otherwise from Identity theorem f will be constantly zero. Then one has $x' - iy' = \overline{x' + iy'} = \overline{f(it)} = f(-it)$. Hence $(x', -y')$ also satisfies $ax + by = 0$. From this, it follows that $a = 0$ or $b = 0$.*

Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$. We say that $w \in \mathbb{C}$ is a *period* of φ if $\varphi(w + z) = \varphi(z)$, for all $z \in \mathbb{C}$.

1.7. Let L_1 and L_2 stand for the lines $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \pi$ respectively. Suppose that f is an entire function such that $f(L_j) \subseteq \mathbb{R}$, for $j = 1, 2$. Show that f is $2\pi i$-periodic.

Sketch of the solution. *From 1.5. we obtain that $f(\bar{z}) = \overline{f(z)}$, for all $z \in \mathbb{C}$. So, for all $x \in \mathbb{R}$, one has $f(x - \pi i) = f(x + \pi i)$. This shows that $f(z + 2\pi i) - f(z) = 0$, whenever $\operatorname{Im} z = -\pi$. Now from identity theorem, one obtains that $f(z + 2\pi i) - f(z) = 0$ for all $z \in \mathbb{C}$.*

1.8. Show that a nonconstant entire function can have at most countably many periods.

Sketch of the solution. *Enough to show that f has finitely many periods on any compact subset K of $\mathbb{C}$ (why?) Suppose f has infinitely many periods in K. Fix $z \in \mathbb{C}$. Consider the function $g(w) \stackrel{def}{=} f(w + z) - f(z)$, for all $w \in \mathbb{C}$. Then g has infinitely many zeros in K. This implies that $g \equiv 0$. This makes f is constant, which is not possible.*

1.9. Let $f, g \in H(\mathbb{D})$ be nowhere vanishing. Assume that $\frac{f'}{f}\left(\frac{1}{n}\right) = \frac{g'}{g}\left(\frac{1}{n}\right)$, for all $n \in \mathbb{N} \setminus \{1\}$. Show that $\frac{f}{g}$ is constant.

Hint. *This is a straightforward application of Identity theorem*

2. Maximum modulus principle

2.1. Formulate and prove the 'Minimum modulus principle'. Conclude that, for any region U in $\mathbb{C}$ and nonconstant holomorphic function $f : U \longrightarrow \mathbb{C}$, $|f|$ can attain a local minima only at zeros of f.

Statement. *Let U be a region in $\mathbb{C}$ and $f \in H(U)$. Let $z_0 \in U$ and $r > 0$ be such that $\overline{D(z_0; r)} \subseteq U$ and f vanishes nowhere in $\overline{D(z_0; r)}$. Then*

$$|f(z_0)| \geq \min_{t\in[0,2\pi]} |f(z_0 + re^{it}|, \tag{2.1}$$

and equality occurs if and only is f is constant.

Proof. Since f vanishes nowhere in $\overline{D(z_0; r)}$, there exists $R > r$ such that $D(z_0; R) \subseteq U$ and f does not have a zero in $D(z_0; R)$. Restrict f to $D(z_0; R)$ and apply Maximum modulus principle on $\frac{1}{f}$ to obtain that

$$\frac{1}{|f(z_0)|} \leq \max_{t\in[0,2\pi]} \frac{1}{|f(z_0 + re^{it})|}. \tag{2.2}$$

Now (2.1) is immediate from (2.2). Furthermore, equality occurs in 2.1 if and only if equality occurs in (2.2) if and only if $\frac{1}{f}$ is constant on $D(z_0; R)$ if and only if f is constant on $D(z_0; R)$ if and only if f is constant on U, in view of Identity theorem. □

Note to the student. *Likewise the Maximum modulus principal, one deduces a couple of corollaries as follows:*

Corollary 1. *Let U be a region and $f \in H(U)$. Suppose that z_0 is a local minima of f and $f(z_0) \neq 0$. Then f is constant.*

Corollary 2. *Let U be a bounded region, $f : \overline{U} \longrightarrow \mathbb{C}$ is continuous and $f \in H(U)$. Assume that f does not have a zero in U. Then*

$$\min_{z\in\overline{U}} |f(z)| = \min_{z\in\partial U} |f(z)|.$$

The proof of the above corollaries are easy exercises.

2.2. Find the maximum and minimum of $|f|$ in each of the following cases:

(a) $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} \frac{z^2}{z+2}$.
(b) $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} z^2 - z$.
(c) $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} e^{z^2}$.
(d) $f : \{z \in \mathbb{C} : |z|^2 \leq 4, \operatorname{Re} z, \operatorname{Im} z \geq 0\} \longrightarrow \mathbb{C},\ f(z) \stackrel{\text{def}}{=} ze^z$.

Sketch of the solution. *These are routine computations. Use Maximum (minimum) modulus principle(s).*

2.3. Let $n \in \mathbb{N}$ and $P(z) \stackrel{\text{def}}{=} z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients.

(a) Choose $r > 1 + 2|a_0| + a_1 + \cdots + |a_{n-1}|$. Show that, for any $t \in [0, 2\pi]$, one has $|P(re^{it})| > |P(0)|$.

Solution. *It is easy to see that, if* $|z| > r$, *then* $2|a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1} < (2|a_0| + |a_1| + \cdots + |a_{n-1}|)|z|^{n-1} < |z|^n$. *From this it now follows that, whenever* $|z| > r$,

$$\begin{aligned} |P(z)| &\geq |z|^n - \left(|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0|\right) \\ &\geq |z|^n - \left(|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|\right) \\ &> |a_0|. \end{aligned}$$

(b) Using Minimum modulus principle, show that P must have a zero.
(c) Conclude the Fundamental theorem of algebra.

Note to the student. *The above exercise yields another proof of the Fundamental theorem of algebra.*

2.4.$*$ (a) Let $U \subseteq \mathbb{C}$ be a bounded region and $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\overline{U}$ converging uniformly on ∂U. Show that, if each $f_n \in H(U)$, then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\overline{U}$.

Hint. *Let* $z \in U$. *From Maximum modulus principle, one observes that, for all* $m, n \in \mathbb{N}$,

$$|f_n(z) - f_m(z)| \leq \max_{w \in \overline{U}} |f_n(w) - f_m(w)|.$$

From this, follows that $\{f_n\}_{n=1}^{\infty}$ *is uniformly Cauchy on* $\overline{U}$, *hence uniformly convergent.*

(b) Find all functions $f : \partial\mathbb{D} \longrightarrow \mathbb{C}$ such that there is a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ which converges uniformly to f (on $\partial\mathbb{D}$).

Hint. *It follows from 2.4.a that there exists a continuous* $g : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ *such that* $g \in H(\mathbb{D})$ *and* $g|_{\partial\mathbb{D}} = f$. *Conversely, let* $g : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ *be continuous and* $g \in H(\mathbb{D})$. *For each* $n \in \mathbb{N}$, *consider* $g_n(z) = g\left(\frac{n}{n+1}z\right)$, *for all* $z \in D\left(0; \frac{n+1}{n}\right)$. *Clearly all* g_n*'s are holomorphic. Now for any* $n \in \mathbb{N}$, *choose a Taylor polynomial* P_n *such that* $|P_n(z) - g_n(z)| < \frac{1}{n}$, *for all* $z \in \overline{\mathbb{D}}$. *Using the uniform continuity of* g, *show that* $\{P_n\}_{n=1}^{\infty}$ *which converges uniformly to* g *on* $\overline{\mathbb{D}}$.

2.5. Let $U \subseteq \mathbb{C}$ be as above in 2.4.a and $f : \overline{U} \longrightarrow \mathbb{C}$ be continuous and holomorphic on U. Show the following:

(a) If f is nonconstant and $|f|$ is constant on ∂U, then f must have a zero in U.
(b) if $f \equiv 0$ on ∂U then f must be identically zero everywhere.
(c) If f is real valued on ∂U, then f is constant. What if f assumes purely imaginary values on ∂U?

Hint. *If* f *is real valued on* ∂U, *then* $|\exp(-if)| \equiv 1$ *on* ∂U. *Then* $\exp(-if)$ *must be constant, otherwise it would have a zero in* U. *Now use connectedness of* U *to conclude from this that* f *is constant. Similar argument works if* f *assumes purely imaginary values on* ∂U.

(d) If $U = \mathbb{D}$, $|f(z)| > 1$ whenever $|z| = 1$, and $f(0) = i$, then f has a zero on $\mathbb{D}$.

Hint. *Assume that* f *does not have a zero in* $\mathbb{D}$. *Then from Muminum modulus principle, one has* $1 = |f(0)| \geq \min_{|z|=1} |f(z)| > 1$, *which is absurd.*

2.6. Let $U \subseteq_{open} \mathbb{C}$ and $f \in H(U)$ be nonconstant. Can $\operatorname{Re} f$ and $\operatorname{Im} f$ have local maxima or minima?

Hint. *Consider* $\exp f$ *and* $\exp(if)$

2.7. Show that, for any finite subset $\{a_1, \ldots, a_n\}$ of the unit circle, $\max\limits_{|z|=1} |z - a_1| \ldots |z - a_n| \geq 1$.

Sketch of the solution. *This is a straightforward application of Maximum modulus principle.*

2.8. (a) Let U be a bounded region in $\mathbb{C}$ and $f \in H(U)$. Suppose that, for every $\{z_n\}_{n=1}^{\infty}$ in U converging to a point of ∂U, $f(z_n) \xrightarrow[n\to\infty]{} 0$. Then show that $f \equiv 0$ on U.

Sketch of the solution. *Let $\mu \overset{def}{=} \{|f(z)| : z \in U\}$. Note that μ can be $+\infty$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in U such that $|f(z_n)| \xrightarrow[n\to\infty]{} \mu$. Now choose a convergent subsequence, say $\{z_{n_k}\}_{k=1}^{\infty}$, converging to z_0. If $z_0 \in U$, then $f(z_0) = \lim_{k\to\infty} |f(z_{n_k})| = \mu$ and hence z_0 will be point of maximum. Hence f is constant, say c. Consequently, for every $\{w_n\}_{n=1}^{\infty}$ in U converging to a point of ∂U, $f(w_n) \xrightarrow[n\to\infty]{} c$. This shows that $c = 0$. So we now assume that $z_0 \in \partial U$. Since $\lim_{k\to\infty} |f(z_{n_k})| = \mu$, it follows that $\mu = 0$. This forces $f \equiv 0$.*

(b)∗ Let $U \overset{\mathrm{def}}{=} \mathbb{D}$ in 2.8.a. Suppose that the hypothesis is weakened as follows: for every $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ converging to a point of an arc $\{e^{it} : \alpha \le t \le \beta\}$, where $\alpha < \beta$, $f(z_n) \xrightarrow[n\to\infty]{} 0$. Show that one can arrive at the same conclusion, i.e., $f \equiv 0$ on $\mathbb{D}$.

(c) Conclude that if $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ is continuous and holomorphic on $\mathbb{D}$ and vanishes identically on an arc of the boundary, then $f \equiv 0$.

2.9.∗ Suppose that $f \in H(\mathbb{D})$ is such that $f(0) = 0$ and $\forall z \in \mathbb{D}$, $|f(z)| \le 1$. Show that, if f has any other fixed point different from 0 then it must be the identity function.

Hint. *Consider the the following function:*

$$g(z) \overset{def}{=} \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Then g is holomorphic. Use Maximum modulus principle to show that $|g(z)| \le 1$ for all $z \in \mathbb{D}$, and from this show that $g \equiv 1$ on $\mathbb{D}$.

2.10. Let $f : \mathbb{D} \longrightarrow \mathbb{C}$ be holomorphic.

(a) Show that there exists $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that $|z_n| \xrightarrow[n\to\infty]{} 1$ and $\{f(z_n)\}_{n=1}^{\infty}$ is convergent.

Hint. *Enough to work with the case f is nonconstant and has finitely many zeros in $\mathbb{D}$ (why?) Then dividing it by a suitable polynomial, we may further assume that f is zero-free (Why would this not make any loss in generality?). For each $n \in \mathbb{N}$, let $M_n \overset{def}{=} \min_{|w|=\frac{n}{n+1}} |f(w)|$. Now choose z_n such that $|z_n| = \frac{n}{n+1}$ and $|f(z_n)| = M_n$. Since f is nonconstant, clearly $\{|f(z_n)|\}_{n=1}^{\infty}$ is a decreasing sequence. Choose a convergent subsequnce of $\{f(z_n)\}_{n=1}^{\infty}$.*

(b)∗ Assume that f is nonconstant. Show that there are sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that $|z_n|, |w_n| \xrightarrow[n\to\infty]{} 1$, both $\{f(z_n)\}_{n=1}^{\infty}$ and $\{f(w_n)\}_{n=1}^{\infty}$ are convergent but limits are not equal.

Hint. *Let $\{z_n\}_{n=1}^{\infty}$ be as obtained in 2.10.a. If necessary, subtracting a constant from f we may assume that $f(z_n) \xrightarrow[n\to\infty]{} 0$. Passing through a subsequence if needed, we may further assume that $\{|z_n|\}_{n=1}^{\infty}$ is strictly increasing. Now, for each $n \in \mathbb{N}$, consider $M_n \overset{def}{=} \max_{|w|=|z_n|} |f(w)|$. What can you say about the sequence $\{M_n\}_{n=1}^{\infty}$? For n sufficiently large, find b_n with $|w_n| = |z_n|$ such that $|f(w_n)| = M_1$. Now choose a convergent subsequence of $\{w_n\}_{n=1}^{\infty}$.*

2.11. Let $P(z)$ and $Q(z)$ be nonconstant complex polynomials of the same degree. Assume that there exists $r > 0$ such that $|P(z)| = |Q(z)|$, whenever $|z| = r$, and all zeros of $P(z)$ and $Q(z)$ lie in $D(0; r)$. Show that there exists $\lambda \in S^1$ such that $P(z) = \lambda Q(z)$, for all $z \in \mathbb{D}$.

3. Open mapping theorem

3.1. Prove that there cannot exist bijective holomorphic map from the punctured disc $\mathbb{D}\setminus\{0\}$ to the annulus $A(1,2) \overset{\mathrm{def}}{=} \{z \in \mathbb{C} : 1 < |z| < 2\}$.

3.2. Let $U \subseteq \mathbb{C}$ be a region and $f \in H(U)$ be nonconstant. Deduce from Open mapping theorem that neither $|f|$ nor $\operatorname{Re} f$ nor $\operatorname{Im} f$ can have a local maxima.

Hint. *Suppose that $|f|$ has a local maxima, say z_0. Then there exists $r > 0$ such that $D(z_0; r) \subseteq U$ and $\forall z \in D(z_0; r)$, $|f(z)| \leq |f(z_0)|$. From Open mapping theorem, it follows that, there exists some $\rho > 0$ such that $D(f(z_0); \rho) \subseteq f(D(z_0; r))$. Now show that there exists $w \in D(f(z_0); \rho)$ such that $|w| > |f(z_0)|$, which leads to a contradiction. Similar argument works for* $\operatorname{Re} f$ *and* $\operatorname{Im} f$.

3.3. Let $U, V \subseteq \mathbb{C}$ be open and connected and $f \in H(U)$ be such that $f(U) \subseteq V$. If the inverse image of every compact subset of V under f is compact, then show that $f(U) = V$. Does the above statement remain true if holomorphic is replaced by continuous in the hypothesis?

Solution. *We first show that f has to be nonconstant. If f is constant, say α, then $f^{-1}(\{\alpha\}) = U$, which is not possible as $f^{-1}(\{\alpha\})$ is supposed to be compact. From Open mapping theorem, one has $f(U)$ open. Hence it now requires to show that $f(U)$ is closed in V. Let $w \in V$ be a limit point of $f(U)$. It follows that there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in U such that $f(z_n) \xrightarrow[n \to \infty]{} w$. Consider the set $K \overset{def}{=} \{f(z_n) : n \in \mathbb{N}\} \cup \{w\}$. Clearly K is compact, whence $f^{-1}(K)$ will be compact. Since $\{z_n\}_{n=1}^{\infty}$ is a sequence in K, it has a convergent subsequence, say $\{z_{n_k}\}_{k=1}^{\infty}$, converging to $z_0 \in K$. From this we obtain that $f(z_0) = w$.*