MTH 421A (ODE)

Consider the ODE $y' = f(x,y)$ s.t $y(x_0) = y_0$.

Question :- (a) Does $\exists$ a solution.

(b) If $\exists$ a solution, is it unique?

(c) Continuous dependence on initial condition?

PICARD-PEANO-LINDELOFF THEOREM :- Define, $D := \{(x,y) \in \mathbb{R}^2 \,/\, |x-x_0| \le a, |y-y_0| \le b\}$

Let $f: D \to \mathbb{R}$ be continuous, then $\exists\, \alpha = \min(a, b/M)$ where $M = \max_D |f|$ and $y \in C^1[x_0-\alpha, x_0+\alpha]$ satisfying (1).

Moreover, if f is $\boxed{\text{Lipschitz continuous}}$ w.r.t the second variable i.e.,

$$|f(x,y_1) - f(x,y_2)| \le M|y_1 - y_2| \quad \forall (x,y) \in D$$

then the solution so obtained is UNIQUE.

Ex:- (a) $f(x) = \log x, \quad x \geq \alpha > 0$.

WLOG, $\alpha \leq x < y$ then

$$|\log y - \log x| = \left|\log \frac{y}{x}\right| = \log\left(1 + \frac{y}{x} - 1\right) \leq \frac{y}{x} - 1 = \frac{y-x}{x} \leq \frac{1}{\alpha}|y-x|$$

Question:- Can one comment on the Lipchitz continuity of $g(x) = x^{\beta} \log x$.

(b) $f(x) = \min\{f_1, f_2\}$ where f_1 and f_2 are Lipchitz continuous in $\mathbb{R}$.

Note, $\min\{f_1, f_2\} = \dfrac{f_1 + f_2 - |f_1 - f_2|}{2}$

Now, If f_1 and f_2 are Lipchitz continuous so is $f_1 \pm f_2$.

Also, $|f_1 - f_2|$ is Lipchitz since

$$\big||f_1(x) - f_2(x)| - |f_1(y) - f_2(y)|\big| \leq |f_1(x) - f_2(x) - f_1(y) + f_2(y)|$$
$$\leq |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)|$$
$$\leq M_1|x-y| + M_2|x-y|$$ where M_1 and M_2 are L.C of f_1 & f_2.

Question :- Can one comment on $\max(f_1, f_2)$ where f_1 and f_2 are Lipchitz in $\mathbb{R}$.

Generalising this concept to vector field :- $F: \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is said to be Lipchitz continuous in Ω if $\exists\, M > 0$ such that $\|F(x) - F(y)\| \leq M\|x-y\| \quad \forall\, x, y \in \Omega$.

[In $\mathbb{R}^2$ this looks like, $\|F(x_1,x_2) - F(y_1,y_2)\|_2 \leq M\left[(x_1-y_1)^2 + (x_2-y_2)^2\right]^{\frac{1}{2}} \quad \forall\, x = (x_1,x_2)$ and $y = (y_1,y_2) \in \Omega$]

Ex: $F(x,y) = (x, \cos y)$

$$\|F(x_1,x_2) - F(y_1,y_2)\| = \|(x_1, \cos x_2) - (y_1, \cos y_2)\|_2 \leq M\|x-y\|$$

[$\because |\cos x_2 - \cos y_2| = |\sin \xi|\,|x_2-y_2| \leq |x_2-y_2|$]

OFTEN ITS HARD TO SHOW LIPCHITZ CONTINUITY OF F

We circumvent this issue by providing a much easier condition. (In the sense that can be easily verified)

THEOREM :- Suppose that $F : \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ is C^1. Then F is locally lipchitz.

O!

[We say F is locally lipchitz if each point in Ω has a neighbourhood $\Omega' (\subseteq \Omega)$ such that $F|_{\Omega'}$ is Lipchitz].

norm wali dikkat h ask after midsem

sup norm -> basically what max it can take

Proof :- Let $x_0 \in \Omega$ and $\overline{B(x_0, \varepsilon)} \subset \Omega$ and $K = \|DF(x_0)\|$ in $\overline{B(x_0,\varepsilon)}$.

For $Y, Z \in \overline{B(x_0, \varepsilon)}$ and define $\psi(\lambda) = F(\lambda Z + (1-\lambda) Y)$. ; $0 \leq \lambda \leq 1$.

Clearly, ψ is continuously differentiable using Chain Rule. and $\psi'(\lambda) = \nabla F(\lambda Z + (1-\lambda)Y) \cdot (Z - Y)$

$\therefore F(Z) - F(Y) = \psi(1) - \psi(0) \overset{F.T.C}{=} \int_0^1 \psi'(s)\, ds = \int_0^1 \nabla F(\lambda Z + (1-\lambda)Y) \cdot (Z-Y)\, ds \leq K \int_0^1 \|Z - Y\| ds$

$\left[A \cdot B \leq \|A\| \, \|B\| \quad \forall A, B \in \mathbb{R}^n \right]$

$\therefore \|F(Z) - F(Y)\| \leq K \|Y - Z\|. \quad \forall Y, Z \in \overline{B(x_0, \varepsilon)}$.

So basically we only need to bound gradient

LINEAR ALGEBRA :- Let $F : \Omega (\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be a continuously differentiable function in Ω i.e., $D_i F_{x_j}$'s exists and are continuous in Ω. for all $i, j \in \{1, 2, \dots, n\}$.

For $x \in \mathbb{R}^n$, define the norm $\|DF(x)\|$ by

$$\|DF(x)\|_* \overset{O.N}{=} \sup_{\substack{\|Y\|=1 \\ Y \in \mathbb{R}^n}} \|DF(x)(Y)\|_n$$

($*$ denotes the matrix norm on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$).

linear transformn Rn to Rn

Note, $\|DF(x)(Y)\| = \left\|DF(x) \cdot \left(\frac{Y}{\|Y\|} \cdot \|Y\|\right)\right\| = \|Y\| \left\|DF(x)\left(\frac{Y}{\|Y\|}\right)\right\| \leq \|DF(x)\|_* \|Y\|$. $\left(\because \|tx\| = |t| \|x\|\right)$

Ex :- (a) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$\|A\|_* = \sup_{\substack{(x,y) \in \mathbb{R}^2 \\ \{x^2+y^2=1\}}} \left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \sup_{\substack{\{x^2+y^2=1\} \\ (x,y) \in \mathbb{R}^2}} \left\| \begin{pmatrix} 2x \\ y \end{pmatrix} \right\| = \sup_{\substack{\{x^2+y^2=1\} \\ (x,y) \in \mathbb{R}^2}} \left(4x^2+y^2\right)^{\frac{1}{2}} = 2.$$

The Fundamental Theorem of ODE :- Consider the I.V.P $X' = F(X)$; $X(0) = X_0$ where $X_0 \in \mathbb{R}^n$. Suppose that $F: \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then there exists a unique solution to the I.V.P. More precisely, $\exists$ $\alpha > 0$ and a unique solution $X : (-\alpha, \alpha) \to \mathbb{R}^n$ of this differential equation satisfying $X(0) = X_0$.

$\boxed{A}$ Consider the system :

$$y_1' = y_2 - \cos y_1$$
$$y_2' = \sin y_1 .$$

subject to $y_1(t_0) = y_{10}$ and $y_2(t_0) = y_{20}$.

Write, $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \Rightarrow Y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}$ and $F(Y) = \begin{pmatrix} y_2 - \cos y_1 \\ \sin y_1 \end{pmatrix}$

Hence, (1) can be written as $Y' = F(Y)$; $Y(t_0) = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}$.

Question :- Can you use the Fundamental theorem to deduce the uniqueness of solution for the system $\boxed{A}$?

ig Yes! as F is C1

What is $DF(X_0)$?

$$\begin{bmatrix} \sin y1 & 1 \\ \cos y1 & 0 \end{bmatrix}$$

$\boxed{B}$ Consider the O.D.E $y''' + y = 0$; $y''(t_0) = y'(t_0) = y(t_0) = \alpha \in \mathbb{R}$.

Write, $y' = v \Rightarrow y'' = v'$ and $y''' = v''$ and hence, $v'' = -y$.

Denote, $Y(t) = \begin{pmatrix} y''(t) \\ y'(t) \\ y(t) \end{pmatrix}$ then $Y'(t) = \begin{pmatrix} -y(t) \\ v'(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v' \\ v \\ y \end{pmatrix}$

$\therefore$ The ODE can be written as $Y'(t) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} Y(t)$; $Y(t_0) = (\alpha, \alpha, \alpha)^T$.

and so, $F(Y) = AY$ where $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

$(-y_3, y_1, y_2)$

Question : Is $F : \mathbb{R}^3 \to \mathbb{R}^3$ is Lipchitz continuous?

Note, $DF(x_0) = A$ and $\|DF(x_0)\| = \|A\|_* = \sup_{\substack{\|Y\|=1 \\ Y \in \mathbb{R}^3}} \|AY\| = \sup_{\|Y\|=1} \|(-y_3, y_1, y_2)\|$ for $Y = (y_1, y_2, y_3)$

$= 1$.

Observe, If $I = (a,b)$ containing zero and $X : I \xrightarrow[C^1]{} \Omega$ satisfies

$$X'(t) = F(X(t)) \;;\; X(0) = X_0$$

Then, $X(t) = X_0 + \int_0^t F(X(t))\,ds.$ ⟵ Integral Form of ODE.

and both this two forms are equivalent.

Proof of Existence :---

Let, (a) $\overline{B(X_0, \rho)}$ is the closed ball of radius $\rho > 0$ and center at X_0.

(b) K denote the Lipchitz constant for F on $\overline{B(X_0, \rho)}$.

(c) $|F(X)| \leq M$ on $\overline{B(X_0, \rho)}$

(d) Let, $\alpha < \min\{\rho/M, 1/K\}$ and $J = [-\alpha, \alpha]$.

We start by defining Picard's Iterate,

$$U_0(t) \equiv X_0.$$

For $t \in J$ define, $U_1(t) = X_0 + \int_0^t F(U_0(s))\,ds = X_0 + tF(X_0).$

$\because |t| \leq \alpha$ and $|F(X_0)| \leq M$, $\|U_1(t) - X_0\| = |t|\|F(X_0)\| \leq \alpha M < \rho.$

so, $U_1(t) \in \overline{B(X_0,\rho)}$ $\forall t \in J.$

By induction, assume $U_m(t)$ is defined and $\|U_m(t) - X_0\| < \rho$ $\forall t \in J.$

let, $U_{m+1}(t) = X_0 + \int_0^t F(U_m(s))\,ds$

We show, $\|U_{m+1}(t) - X_0\| < \rho \Rightarrow U_{m+1}(t) \in \overline{B(X_0,\rho)}$ $\forall t \in J.$

$$\|U_{m+1}(t) - X_0\| = \left\|\int_0^t F(U_m(s))\,ds\right\| \leq M\int_0^t ds \leq M\alpha < \rho.$$

Now we show that $\exists\, L>0$ s.t for all $m \geq 0$,

$$\|U_{m+1}(t) - U_m(t)\| \leq (\alpha K)^m L.$$

Define, $L := \max_{t\in[-\alpha,\alpha]} |U_1(t) - U_0(t)|$. Clearly, $L \leq \alpha M$

and, $\|U_2(t) - U_1(t)\| = \left\| \int_0^t \left(F(U_1(s)) - F(U_0(s))\right) ds \right\| \leq K \int_0^t \|U_1(s) - U_0(s)\|\, ds.$

$$\leq \alpha K L.$$

$U_{s+1} \cdots U_{s+(r-s)}$

By induction it is easy to see,

$$\|U_{m+1}(t) - U_m(t)\| \leq (\alpha K)^m L =: \beta^m L \quad (\beta := \alpha K < 1)$$

$\therefore$ Given $\epsilon > 0$, $\exists\, N$ large s.t for any $r > s > N$ we have,

$$\|U_r(t) - U_s(t)\| \leq \sum_{m=N}^{\infty} \|U_{m+1}(t) - U_m(t)\| \leq \sum_{m=N}^{\infty} \beta^m L \leq \epsilon.$$

Geometric series

$\therefore$ $v_0, v_1, \dots,$ converges uniformly to a continuous function $x : J \to \overline{B(x_0, \rho)}$

$$\therefore \; v_{m+1}(t) = x_0 + \int_0^t F(v_m(s))\,ds$$

Taking limit on both side,

$$x(t) = x_0 + \lim_{m\to\infty} \int_0^t F(v_m(s))\,ds.$$

$$= x_0 + \int_0^t F(x(s))\,ds.$$

$\therefore$ $x : J \to \overline{B(x_0, \rho)}$ satisfies the integral equation and hence the O.D.E system.

In particular $x : J \to \overline{B(x_0, \rho)}$ is C^1.

$\blacksquare$

Uniqueness : Let $X, Y : J \to \overline{B(x_0, \rho)}$ are two solutions s.t $X(0) = Y(0) = x_0$.

We want to show $X(t) = Y(t)\ \forall t \in J$.

Let $Q = \max\limits_{t \in J} \|X(t) - Y(t)\|$ and the maximum is attained at $t_1 \in J$.

$$\therefore Q = \|X(t_1) - Y(t_1)\| = \left\| \int_0^{t_1} (X'(s) - Y'(s))\, ds \right\|$$

$$\leq \int_0^{t_1} \|F(X(s)) - F(Y(s))\|\, ds$$

$$\leq \int_0^{t_1} K \|X(s) - Y(s)\|\, ds$$

$$\leq \alpha K Q$$

$\because \alpha K < 1$, $Q = 0$

$\therefore X(t) = Y(t)$ ∎

CONTINUOUS DEPENDENCE ON INITIAL DATA :-

Theorem :- Let $\Omega (\subseteq \mathbb{R}^n)$ be open and suppose $F : \Omega \to \mathbb{R}^n$ has a lipchitz constant K. Let $Y(t)$ and $Z(t)$ be solutions of $X' = F(X)$ that remains in Ω and are defined on the interval $[t_0, t_1]$.

then $\forall\ t \in [t_0, t_1]$ we have,

$$\| Y(t) - Z(t) \| \leq \| Y(t_0) - Z(t_0) \| \exp(K(t - t_0)).$$

or, $\| Y - Z \|_\infty \leq C \| Y(t_0) - Z(t_0) \|$.

Gronwall Inequality :- Let $u : [0, \alpha] \to \mathbb{R}$ be continuous and non-negative. Suppose $C \geq 0$ and $K \geq 0$ be such that

$$u(t) \leq C + \int_0^t K u(s)\, ds \quad \forall\, t \in [0, \alpha].$$

Then for all $t \in [0, \alpha]$, $u(t) \leq C e^{Kt}$.

Proof :- Let $C > 0$, $U(t) := C + \int_0^t K u(s)\, ds > 0$

$$\therefore u(t) \leq U(t) \Rightarrow U'(t) = K u(t) \Rightarrow \frac{U'(t)}{U(t)} = \frac{K u(t)}{U(t)} \leq K.$$

$\therefore [\log v(t)]' \leq K \Rightarrow \log v(t) \leq \log v(0) + Kt$

$\therefore v(0) = C \Rightarrow v(t) \leq Ce^{Kt} \Rightarrow u(t) \leq Ce^{Kt}$

If $C = 0$, preceding argument for $C_n = \frac{1}{n}$. $\blacksquare$

Cont. dependence wale ka proof

PROOF :- Define, $v(t) = \|Y(t) - Z(t)\|$

$\Rightarrow$ Y and Z soln h X'=F(x) ke so Y'-Z' ko integrate karke we get

$$\therefore Y(t) - Z(t) = Y(t_0) - Z(t_0) + \int_{t_0}^{t} F(Y(s)) - F(Z(s))\, ds$$

we have, $v(t) \leq v(t_0) + \int_{t_0}^{t} K v(s)\, ds$

Use the proof of Gronwall for $v(t) = v(t_0) + \int_{t_0}^{t} \mathcal{K} v(s)\, ds$.

Hence, one has $v(t) \leq v(t_0) \exp[\mathcal{K}(t - t_0)]$; $0 < t_0 \leq t \leq t_1$

$\Rightarrow \| Y(t) - Z(t) \| \leq \| Y(t_0) - Z(t_0) \| \exp[\mathcal{K}(t - t_0)]$ $\blacksquare$

—— x ——

Non-Autonomous system :- Let $\Omega (\subseteq \mathbb{R}\times\mathbb{R}^n)$ be open and $F : \Omega \to \mathbb{R}^n$ is C^1 in X and continuous w.r.t the first variable. Let $(t_0, X_0) \in \Omega$.

Consider, $X'(t) = F(t, X(t))$; $X(t_0) = X_0$. —— $(*)$

A solution of this system is a continuously differentiable curve $X(t)$ in $\mathbb{R}^n$ defined for $t \in J$ s.t

(a) $t_0 \in J$ and $X(t_0) = X_0$

(b) $(t, X(t)) \in \Omega$ and $X'(t) = F(t, X(t))$ $\forall\, t \in J$

LOCAL FUNDAMENTAL THEOREM :- Let $\Omega (\subseteq \mathbb{R}\times\mathbb{R}^n)$ be open and $F: \Omega \to \mathbb{R}^n$ is C^1 in X and continuous w.r.t the first variable.
If $(t_0, x_0) \in \Omega$, $\exists$ an open interval J containing t and a unique solution of $x' = F(t,x)$ defined on J and satisfying $x(t_0) = x_0$.

$\boxed{\text{///}}$

COROLLARY :- If $A(t)$ is a family of continuous $(n\times n)$ matrices. Let $(t_0, x_0) \in J\times\mathbb{R}^n$. Then the I.V.P $x'(t) = A(t)X$; $x(t_0) = x_0$ has a unique solution on all of J.

$\boxed{\text{//}}$

MTH 421A (ODE) Assignment - 1

1. Calculate the norm of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2. Let $F : A (\subseteq \mathbb{R}^n, \text{open}) \to \mathbb{R}^n$ is locally Lipchitz and $B (\subset A)$ is compact, then $F|_B$ is Lipchitz.

3. Let $\Omega (\subseteq \mathbb{R} \times \mathbb{R}^n)$ be an open set containing $(0, x_0)$ and $F, G : \Omega \to \mathbb{R}^n$ are C^1 function w.r.t x and continuous w.r.t t such that $\| F(t, x_n) - G(t, x_n) \| < \epsilon$. $\forall (t, x) \in \Omega$ and K is the Lipchitz constant in x for $F(t,x)$. If $X(t)$ and $Y(t)$ solves $X'(t) = F(t, X)$; $X(0) = X_0$ and $Y'(t) = G(t, Y)$; $Y(0) = Y_0$ resp. Show, $|X(t) - Y(t)| \leq \epsilon/K \exp(K|t| - 1)$ $\forall t \in J$.

4. Let A be a $(n \times n)$ matrix. Use Picard's iterate to solve the problem $X' = AX$; $X(0) = X_0$.

5. Find the Lipchitz constant for the following :-
   (a) $f(x) = |x|$ on $-\infty < x < \infty$ (b) $f(x) = x^2 \log x$ on $2 \leq x \leq 3$ (c) $f(x,y) = \dfrac{xy}{1 + x^2 + y^2}$; $x^2 + y^2 \leq 4$.

6. Comment on the uniqueness of the following problems :-
   (a) $x'(t) = x^{\frac{1}{3}}$; $x(0) = 0$.
   (b) $x'(t) = x^2$; $x(0) = 1$.

Exponential of Operator :-

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator i.e, $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Our aim is to introduce the concept of convergence in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

Recall, $\boxed{\|T\|_* = \max_{\substack{\|x\|=1 \\ x \in \mathbb{R}^n}} \|Tx\|}$ and hence for any $S, T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ one has,

(a) $\|T\|_* \geq 0$ and $\|T\| = 0$ iff $T = 0$

(b) $\|kT\|_* = |k| \, \|T\|$ for $k \in \mathbb{R}$

(c) $\|S+T\| \leq \|S\| + \|T\|$.

Convergence of a sequence of operators :- $T_k \in L(\mathbb{R}^n, \mathbb{R}^n)$ is said to converge to a linear operator $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ as $k \to \infty$ if for all $\epsilon > 0$, $\exists\, N$ s.t for $k \geq N$

$$\|T - T_k\|_* < \epsilon$$

Lemma :- For $S, T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$

(a) $\|Tx\| \leq \|T\|_* \|x\|$

(b) $\|TS\| \leq \|T\| \, \|S\|$

(c) $\|T^k\| \leq \|T\|^k$ for $k = 0, 1, 2, \dots$

Proof :- (a) when $x = 0$, (a) is trues when $x \neq 0$ write $y = x/\|x\|$ and

$$\|T\|_* \geq \|T(y)\| = \|T(x/\|x\|)\| = \frac{1}{\|x\|}\|T(x)\| \Rightarrow \|T(x)\| \leq \|T\|_* \|x\|$$

(b) For $|x| \leq 1$, $\|T(Sx)\| \leq \|T\|_* \|Sx\|$

$$\leq \|T\|_* \|S\|_* \|x\|$$

$$\Rightarrow \|TS\| \leq \|T\| \, \|S\|$$

(c) Follows from (b).

Theorem :- Given $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $t_0 > 0$ the series

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{\lfloor k}$$

is absolutely and uniformly convergent for all $|t| \leq t_0$.

Proof :- Let $\|T\| = a$. For all $|t| \leq t_0$

$$\left\| \frac{T^k t^k}{\lfloor k} \right\| \leq \frac{\|T\|^k |t|^k}{\lfloor k} \leq \frac{a^k |t|^k}{\lfloor k} \leq \frac{a^k t_0^k}{\lfloor k}.$$

But, $\displaystyle\sum_{0}^{\infty} \frac{T^k t^k}{\lfloor k} = e^{a t_0}$

$\therefore$ By M-test $\displaystyle\sum_{0}^{\infty} \frac{T^k t^k}{\lfloor k}$ is absolutely and uniformly convergent for all $|t| \leq t_0$.

The Absolutely convergent series $\displaystyle\sum_{k=0}^{\infty} \frac{T^k}{\lfloor k}$ is then defined as e^T

Definition :- Let A be a $(n \times n)$ matrix. Then for $t \in \mathbb{R}$,

$$e^{At} := \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Lemma :-

(a) $\|e^{At}\| \leq e^{\|A\|\,|t|} \quad \forall\, t \in \mathbb{R}$

(b) Let S and T be such that $S = PTP^{-1}$, then $e^S = Pe^TP^{-1}$ for $S, T \in M(n, \mathbb{R})$

(c) Let $ST = TS$ then $e^{T+S} = e^T e^S$.

Proof :- (a) $\|e^{At}\| = \left\| \sum_0^{\infty} \frac{A^k t^k}{k!} \right\| \leq \sum_0^{\infty} \frac{\|A\|^k |t|^k}{k!} = e^{\|A\|\,|t|}$

[Note, $\because \sum_{k=0}^{\infty} \left\| \frac{A^k t^k}{k!} \right\| < \infty$, for any $n \in \mathbb{N}$, $\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i| \leq \sum_{i=1}^{\infty} |a_i|$]

(b) $S = PTP^{-1}$, $\quad e^{S} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(PTP^{-1})^k}{k!} = P \lim_{n\to\infty} \sum_{0}^{n} \frac{T^k}{k!} P^{-1} = P e^{T} P^{-1}$

(c) $\because ST = TS$,

Check, $(S+T)^n = n! \sum_{j+k=n} \frac{S^j T^k}{j!\,k!}$

$$\therefore e^{S+T} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{S^j T^k}{j!\,k!} = \sum_{j=0}^{\infty} \frac{S^j}{j!} \sum_{k=0}^{\infty} \frac{T^k}{k!} = e^{S} e^{T}.$$

COROLLARY: (a) If $P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ then $e^{At} = P\, \mathrm{diag}(e^{\lambda_j})\, P^{-1}$.

(b) $(e^{T})^{-1} = e^{-T}$ $\left[\because e^{T+(-T)} = I\right]$

Fundamental Theorem of Linear System :

Let A be a $(n\times n)$ matrix. Then for given $x_0 \in \mathbb{R}^n$, the Initial Value Problem (I.V.P)

$$X' = AX$$
$$X(0) = X_0$$

has a unique solution given by $X(t) = e^{At} X_0$.

Proof :- If $X(t) = e^{At} X_0$

$$\Rightarrow X'(t) = \frac{d}{dt}\left(e^{At} X_0\right) = A e^{At} X_0 = A X(t) \quad \forall t \in \mathbb{R}$$

Also, $X(0) = I X_0 = X_0$.

Thus, $X(t) = e^{At} X_0$ is a solution.

To see that this is the only solution, let $Z(t)$ be any other solution of the I.V.P and set, $Y(t) = e^{-At} Z(t)$

$\Rightarrow Y'(t) = -Ae^{-At}Z(t) + e^{-At}Z'(t)$

$= -Ae^{-At}Z(t) + e^{-At}AZ(t)$

$= 0$ $\quad$ ($\because$ A and e^{-At} commutes)

$\therefore Y(t) = \text{constant}$

$\because Y(0) = X_0 \Rightarrow e^{-At}Z(t) = X_0 \Rightarrow Z(t) = e^{At}X_0$. $\boxed{}$

ASSIGNMENT 2 (MTH421A - ODE)

(a) For a square matrix A show $\frac{d}{dt} e^{At} = Ae^{At}$

(b) Find A, B s.t $e^A e^B \neq e^{A+B}$.

(c) Compute the matrix e^{Bt} when $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$; $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ and $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(d) Solve $X' = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix} X$

(e) Suppose that the square matrix has a negative eigenvalue. Show that the linear system $x' = Ax$ has at least one solution (nontrivial) that satisfies $\lim_{t \to \infty} x(t) = 0$.

(d) If $\phi(t, x_0)$ be the solution of $X' = AX$; $X(0) = X_0 (\in \mathbb{R}^n)$. Use the Fundamental theorem to show that for each fixed $t \in \mathbb{R}$, $\lim_{y \to x_0} \phi(t, y) = \phi(t, x_0)$

System of ODE :-

Let $f, g : I \to \mathbb{R}^n$ be continuously differentiable. They are called linearly dependent if there exists $c_1, c_2 \in \mathbb{R}$ with $|c_1| + |c_2| \neq 0$ and $c_1 f(t) + c_2 g(t) = 0 \ \forall\, t \in I$.

Otherwise they are linearly independent.

Ex: (a) $\{\sin t, \cos t\}$ is L.I in $\mathbb{R}$

[$\because$ $c_1 \sin t + c_2 \cos t = 0 \ \forall\, t \in \mathbb{R} \Rightarrow c_1 = c_2 = 0$].

(b) $\{t, t^2\}$ is L.I in $\mathbb{R}$

[$\because$ $c_1 t + c_2 t^2 = 0 \ \forall\, t \in \mathbb{R}$ gives $c_1 + c_2 = 0$ $(t=1)$

again for $t = 2$, $2c_1 + 4c_2 = 0$.

$\Rightarrow c_1 = c_2 = 0$].

Definition :- Given $f_i : I \to \mathbb{R}^n$ are C^1-curves for $i = 1, 2, \dots, n$ one defines the Wronskian as

$$W[f_1, f_2, \dots, f_n](t) = \det [f_1(t)\ f_2(t)\ \dots\ f_n(t)].$$

With the help of Wronskian one defies that if $f_1, f_2, \dots, f_n$ are linearly dependent then $W[f_1, f_2, \dots, f_n](t) = 0 \ \forall t \in I$.

Note :- Converse is not true since for $f_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$ and $f_2(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ one has

$$W(f_1, f_2)(t) = \begin{vmatrix} 1 & t \\ t & t^2 \end{vmatrix} = 0 \quad \forall t \in I$$

But, $c_1 f_1(t) + c_2 f_2(t) = 0 \Rightarrow c_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t^2 \end{pmatrix} = 0 \Rightarrow \begin{cases} c_1 + c_2 t = 0 \\ c_1 t + c_2 t = 0 \end{cases}$

$\Rightarrow c_1 = c_2 = 0$.

and hence they are Linearly independent.

(2x2) system

$c_1 f_1(t) + c_2 f_2(t) = 0$

$$\begin{cases} c_1 f_{11}(t) + c_2 f_{21}(t) = 0 \\ c_1 f_{12}(t) + c_2 f_{22}(t) = 0. \end{cases} \Rightarrow \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$W(f_1, f_2)(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note :- If $W[f_1, f_2, \ldots, f_n](t) \neq 0$ for some $t \in I \Rightarrow f_1, f_2, \ldots, f_n$ are linearly independent.

Theorem :- If $f_i : I \to \mathbb{R}^n$ are solutions of $X' = AX$ for $i = 1, 2, \ldots, n$. Then $W[f_1, \ldots, f_n](t) = 0 \ \forall t \in I \Rightarrow$ $f_1, \ldots, f_n$ are Linearly dependent.

Proof :- Consider the relation, $c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) = 0 \ \forall t \in I$

Let $f_i(t) = (f_{i1}(t) \ldots f_{in}(t))^T$ then,

$$\begin{vmatrix} f_{11}(0) & \cdots & f_{n1}(0) \\ \vdots & & \vdots \\ f_{1n}(0) & & f_{nn}(0) \end{vmatrix} = 0 \Rightarrow$$

The system given by,

$$\left.\begin{aligned} c_1 f_{11}(0) + \ldots + c_n f_{n1}(0) &= 0 \\ \ldots \\ \ldots \\ c_1 f_{1n}(0) + \ldots + c_n f_{nn}(0) &= 0 \end{aligned}\right\} (*)$$

$(*)$ has a nontrivial solution. WLOG lets call it $(\hat{c}_1 \ldots \hat{c}_n) \in \mathbb{R}^n$.

Define $F(t) = \hat{c}_1 f_1(t) + \cdots + \hat{c}_n f_n(t)$

$\Rightarrow F'(t) = \hat{c}_1 f_1'(t) + \cdots + \hat{c}_n f_n'(t)$.

$\Rightarrow F'(t) = \hat{c}_1 A f_1(t) + \cdots + \hat{c}_n . A f_n(t)$

$\Rightarrow F'(t) = A F(t)$

Again, $F(0) = \hat{c}_1 f_1(0) + \cdots + \hat{c}_n f_n(0) = 0$.

$\therefore F \in C^1(I)$ is such that $F'(t) = AF(t)$; $F(0) = 0 \Rightarrow F(t) = 0 \ \forall t \in I$.

Hence, $\hat{c}_1 f_1(t) + \hat{c}_2 f_2(t) + \cdots + \hat{c}_n f_n(t) = 0 \quad \forall t \in I$ with $\sum |c_j| \neq 0$.

$\Rightarrow f_1, \ldots, f_n$ are Linearly dependent.

STRUCTURE OF SOLUTION SPACE OF $X' = AX$:-

Consider the equation $X' = AX$ and define $\mathcal{D} := \{ X \in C^1(I;\mathbb{R}^n) \,/\, X' = AX \}$ where A is a $(n\times n)$ matrix

. It is easy to see that $\mathcal{D}$ forms a vector space over $\mathbb{R}$.

Question :- What is the dimension of $\mathcal{D}$.

Note :- The above question will classify all possible solutions of the linear system.

Theorem :- $\dim \mathcal{D} = n$.

To see this note that the problem $X' = AX$; $X(0) = e_i \,(\in \mathbb{R}^n)$ has a unique solution given by $Z_i(t)$ (say).

We claim $\{ Z_1(t), \ldots, Z_n(t) \}$ forms a basis for $\mathcal{D}$.

Step 1 :- $Z_1(t), \ldots, Z_n(t)$ are Linearly independent.

Consider, $c_1 Z_1(t) + \cdots + c_n Z_n(t) = 0 \;\; \forall t \in \mathbb{R}$ with $\sum_1^n |c_i| \neq 0$.

$\therefore$ For $t = 0$, $c_1 Z_1(0) + \cdots + c_n Z_n(0) = 0$ with $\sum_1^n |c_i| \neq 0$

$\Rightarrow c_1 e_1 + \cdots + c_n e_n = 0$ with $\sum_1^n |c_i| \neq 0$.

— a contradiction.

Step 2 :- $Z_1(t), \ldots, Z_n(t)$ spans $\mathcal{D}$.

Let $Y(t)$ be any solution of $X' = AX$. Clearly $Y(0) = Y_0 \;(\in \mathbb{R}^n$, say$)$

$\therefore Y_0 = \hat{c}_1 e_1 + \hat{c}_2 e_2 + \cdots + \hat{c}_n e_n$.

Define, $Z(t) = \hat{c}_1 Z_1(t) + \cdots + \hat{c}_n Z_n(t)$.

$\therefore Z'(t) = AZ(t)$; $Z(0) = Y_0$.

Hence $R(t) = Y(t) - Z(t)$ satisfies $R'(t) = AR(T)$ and $R(0) = 0$.

$\therefore R(t) \equiv 0 \;\forall t \in I$ and so, $Y(t) = \sum_{i=1}^{n} \hat{c}_i Z_i(t) \;\forall t \in \mathbb{R}$.

SUMMARY:- If one finds n-linearly independent solution of the system $X' = AX$ given by $X_i(t)$ for $i = 1, 2, \ldots, n$ then any solution (General solution) is given by

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + \cdots + c_n X_n(t) \text{ with } c_i \in \mathbb{R} \text{ arbitrary.}$$

$$= \Phi(t) C \quad [\Phi(t) = (X_1(t) \ldots X_n(t)) \text{ and } C \text{ is a constant vector}].$$

We will in future consider $X_i = Z_i$ where Z_i's solves $X' = AX$; $X(0) = e_i \; (\in \mathbb{R}^n)$

Question:- How does one go about solving the inhomogeneous system given by

$$X'(t) = AX(t) + B(t) \;;\; X(0) = X_0 \longrightarrow (**)$$

where A is a constant $(n \times n)$ matrix and B is a continuous matrix function on I

Duhamel's Principle :-

Let $X(t) = c_1(t)X_1(t) + c_2(t)X_2(t) + \dots + c_n(t)X_n(t)$ solves (**) [$c_i(t)$ are unknown].

Write, $\Phi(t) = \begin{pmatrix} X_1(t) & \dots & X_n(t) \end{pmatrix}_{n \times n}$. ($\Phi$ is called the Fundamental Matrix)

So, $X(t) = \Phi(t)\,C(t)$ where $C(t) = \begin{pmatrix} c_1(t) & \dots & c_n(t) \end{pmatrix}^T$.

$$\Rightarrow X'(t) = \Phi'(t)\,C(t) + \Phi(t)\,C'(t)$$

$$\Rightarrow AX(t) + B(t) = A\Phi(t)\,C(t) + \Phi(t)\,C'(t). \quad [\because \Phi'(t) = A\Phi(t)].$$

$$\Rightarrow A\Phi(t)\,C(t) + B(t) = A\Phi(t)\,C(t) + \Phi(t)\,C'(t).$$

$$\Rightarrow \Phi(t)\,C'(t) = B(t)$$

$$\Rightarrow C'(t) = \Phi^{-1}(t)\,B(t).$$

$$\Rightarrow C(t) = \int_0^t \Phi^{-1}(s)\,B(s)\,ds + \Phi^{-1}(0)X(0).$$

$$\therefore\ X(t) = \Phi(t)\Phi^{-1}(0)X(0) + \Phi(t)\int_0^t \Phi^{-1}(s)B(s)\,ds$$

$$\boxed{= \Phi(t)\Phi^{-1}(0)X_0 + \Phi(t)\int_0^t \Phi^{-1}(s)B(s)\,ds.}$$

Recall any solution of $X' = AX$ is given by $X(t) = \Phi(t)C$ where Φ is the Fondamental matrix.

Now if $X(0) = X_0$ is given then $X(t) = \Phi(t)\Phi^{-1}(0)X_0$.

Again by Fondamental theorem of Linear system one has $X(t) = e^{tA}X_0$ is the unique solution of the system $X' = AX$; $X(0) = X_0$ one has, $\Phi(t)\Phi^{-1}(0) = e^{tA}$.

If we choose Φ to be principal Fundamental matrix, then $\Phi(0) = I$ and hence,

$$\Phi(t) = e^{tA}.$$

and, $X(t) = e^{tA}X_0 + e^{tA}\int_0^t e^{-sA}B(s)\,ds = \boxed{e^{tA}X_0 + \int_0^t e^{A(t-s)}B(s)\,ds.}$

Easily Uniqueness can be seen

Ex :- $X' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X + \begin{pmatrix} 0 \\ t \end{pmatrix}$; $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Here $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$

$\therefore$ By Duhamel's Principle :-

$$X(t) = e^{tA} X_0 + \int_0^t e^{A(t-s)} B(s)\, ds$$

$$= \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{t-s} & 0 \\ 0 & e^{2(t-s)} \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds.$$

$$= \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ s e^{2(t-s)} \end{pmatrix} ds.$$

$$= \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \int_0^t s e^{2(t-s)}\, ds \end{pmatrix}$$

$$= \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{t}{2} e^{2t} + \frac{1}{4} e^{2t} - \frac{t}{2} - \frac{1}{4} \end{pmatrix}.$$

Please check the calculation.

How to find solutions of $X' = AX$: Relation with eigenvalue problem :-

One may guess that any solution of $X' = AX$ may look like $X(t) = e^{\lambda t} B$, $B \in \mathbb{R}^n$

$\therefore X'(t) = \lambda e^{\lambda t} B$ and hence, $\lambda e^{\lambda t} B = A e^{\lambda t} B \Rightarrow (A - \lambda I) B = 0$ $(\because e^{\lambda t} \neq 0)$ —— (1)

Note in our ansatz $\lambda \in \mathbb{R}$ and $B \in \mathbb{R}^n$ is unknown which reduces (1) to an eigenvalue problem.

Let, (λ_1, B_1) be an eigenpair of A. Then $X(t) = e^{\lambda_1 t} B_1$ will be a solution.

$\because$ the solution space of $X' = AX$ is n, hence if one finds 'n' independent eigenvectors $\{B_i\}_{i=1}^{n}$ for associated eigenpair $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ one has.

$$X(t) = C_1 e^{\lambda_1 t} B_1 + C_2 e^{\lambda_2 t} B_2 + \dots + C_n e^{\lambda_n t} B_n.$$

▨

Calculating Matrix Exponential :-

Note for a (2×2) diagonal matrix, $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ one has

$$e^{At} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} t + \frac{t^2}{2!} \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} + \dots$$

$$= \begin{pmatrix} 1 + at + \frac{a^2t^2}{2!} + \dots & 0 \\ 0 & 1 + bt + \frac{b^2t^2}{2!} + \dots \end{pmatrix} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}.$$

Hence, one may easily infer for a $(n\times n)$ matrix (diagonal) $A = \begin{pmatrix} \lambda_1 & \dots & 0 \\ 0 & \lambda_2 \dots & 0 \\ \vdots & & \vdots \\ 0 & 0 \dots & \lambda_n \end{pmatrix}$ one has,

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & \dots & 0 \\ 0 & e^{\lambda_2 t} \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{pmatrix}.$$

What happens if A is a diagonalizable Matrix :-

Note that if A is diagonalisable then $\exists\, B$ (invertible) such that $B^{-1}AB = \mathcal{D}$ ($\mathcal{D}$ is a Diagonal Matrix)

Now, consider the equation, $X' = AX$ — (1)

One may define, $Y = B^{-1}X$ and so $Y' = B^{-1}X' \Rightarrow Y' = B^{-1}(AX) = (B^{-1}AB)Y = \mathcal{D}Y$.

So the problem $X' = AX$ then reduces to $Y' = \mathcal{D}Y$ where $\mathcal{D}$ is a diagonal matrix.

Ex :- $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{pmatrix}$

Note that A has eigenvalues $\lambda_{1,2} = 2$ and $\lambda_3 = 6$.

The eigenvectors corresponding to $\lambda_{1,2,3}$ are $\underset{\lambda_1=2}{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}, \underset{\lambda_2=2}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{\lambda_3=6}{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}$.

$\therefore B = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$. and, $\mathcal{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$\therefore$ Solution of $Y' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} Y$ is given by $Y(t) = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{6t} \end{pmatrix} Y(0)$.

Hence, $X(t) = BY(t) = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{6t} \end{pmatrix} B^{-1} X(0)$

$$= \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{6t} \end{pmatrix} \begin{pmatrix} -1/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & -1/4 \end{pmatrix} X(0).$$

$$= \begin{pmatrix} \text{Please.} \\ \text{Calculate:} \\ \ \end{pmatrix} X(0)$$

$\square$

Alternatively,

$$X(t) = c_1 e^{2t}\begin{pmatrix}-2\\1\\0\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3 e^{6t}\begin{pmatrix}0\\1\\-2\end{pmatrix}$$ is a general solution.

$$= \begin{pmatrix}-2e^{2t} & e^{2t} & 0\\ e^{2t} & 0 & e^{6t}\\ 0 & e^{2t} & -2e^{6t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

- Using the eigenpair method.

Calculating e^A when A is not diagonalisable or admits complex eigenvalue.

Case 2 Complex Root :-

If $\lambda = \alpha + i\beta$ is a complex eigenvalue of A with eigenvector $\hat{v} = \overbrace{v_1 + i v_2}^{\text{vectors}}$ then

$\hat{x}(t) = e^{\lambda t}\hat{v}$ is complex valued solution of $X' = AX$ — (1).

Lemma 1 :- Let $x(t) = y(t) + i z(t)$ be a complex valued solution of (1), then both $y(t)$ and $z(t)$ solves (1).

Proof :- If $x(t) = y(t) + i z(t) \Rightarrow x'(t) = y'(t) + i z'(t)$.

$$\Rightarrow A x(t) = y'(t) + i z(t)$$

$$\Rightarrow A(y(t) + i z(t)) = y'(t) + i z'(t)$$

$$\Rightarrow A y(t) + i A z(t) = y'(t) + i z'(t).$$

Separating the real and imaginary parts one has the conclusion. ☑

Note, $x(t) = e^{(\alpha+i\beta)t}(v_1 + i v_2)$ can be written as

$$= e^{\alpha t}\left[\cos\beta t + i\sin\beta t\right]\left[v_1 + i v_2\right]$$

$$= e^{\alpha t}\left[\left(v_1\cos\beta t - v_2\sin\beta t\right) + i\left(v_1\sin\beta t + v_2\cos\beta t\right)\right].$$

Hence if $\lambda = \alpha + i\beta$ is an eigenvalue of A with eigen~~function~~ vector $v_1 + i v_2$, then

$$\boxed{\begin{aligned} y(t) &= e^{\alpha t}\left(v_1\cos\beta t - v_2\sin\beta t\right) \\ \text{and, } z(t) &= e^{\alpha t}\left(v_1\sin\beta t + v_2\cos\beta t\right) \end{aligned}}$$

are two real valued solution.

Also, $W(y,z) \neq 0$ (Can you prove that).

Hence, y and z are linearly independent.

Ex:

$$X' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} X \quad ; \quad X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The eigenvalues of A are 1, $1 \pm i$.

when, $\lambda = 1$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the corresponding eigen~~function~~ vector.

when $\lambda = 1+i$, we look for $v \neq 0$ s.t

$$[A - (1+i)I]v = \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{cases} -iv_1 = 0 \\ -iv_2 - v_3 = 0 \\ v_2 - iv_3 = 0 \end{cases} \Rightarrow v = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$ is an eigen~~function~~ vector. $\Rightarrow X(t) = e^{(1+i)t} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$ is a complex-valued solution.

$$\therefore e^{(1+i)t}\begin{pmatrix}0\\ i\\ 1\end{pmatrix} = e^t\begin{pmatrix}0\\ -\sin t\\ \cos t\end{pmatrix} + i e^t\begin{pmatrix}0\\ \cos t\\ \sin t\end{pmatrix}.$$

$\therefore x_2(t) = e^t\begin{pmatrix}0\\ -\sin t\\ \cos t\end{pmatrix}$ and $x_3(t) = e^t\begin{pmatrix}0\\ \cos t\\ \sin t\end{pmatrix}$ are real valued solution.

Also note, $W(x_1, x_2, x_3)(0) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ is invertible.

Hence, $\{x_1, x_2, x_3\}$ are linearly independent.

$$\therefore x(t) = c_1 e^t\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + c_2 e^t\begin{pmatrix}0\\ -\sin t\\ \cos t\end{pmatrix} + c_3 e^t\begin{pmatrix}0\\ \cos t\\ \sin t\end{pmatrix}$$

At $t=0$, $x(t) = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix} \Rightarrow c_1 = c_2 = c_3 = 1$ and hence,

$$x(t) = e^t\begin{pmatrix}1\\ \cos t - \sin t\\ \cos t + \sin t\end{pmatrix}$$

■

Case 3 :- <u>Equal Root</u> :- Let $A \in M(n, \mathbb{R})$ has only $k < n$ linearly independent eigenfunctions.

Then, $X' = AX$ has k linearly independent solution of the form $\hat{X}_k(t) = e^{\lambda_k t} \hat{v}_k$.

Our problem is to find the additional $(n-k)$ linearly independent solution.

Note, $e^{At} \hat{v} = e^{(A-\lambda I)t} e^{\lambda I t} \hat{v}$. $\left[\because (A-\lambda I) \lambda I = \lambda I (A - \lambda I) \right]$

Moreover,

$$e^{\lambda I t} \hat{v} = \left[I + \lambda I t + \frac{\lambda^2}{2!} I t^2 + \dots \right] \hat{v}.$$

$$= \left[1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \dots + \right] \hat{v}$$

$$= e^{\lambda t} \hat{v}.$$

Hence, $e^{At} \hat{v} = e^{\lambda t} e^{(A-\lambda I)t} \hat{v}$.

Also, if v satisfies $(A-\lambda I)^m \hat{v} = 0$ for some integer m then $e^{(A-\lambda I)t}\hat{v}$ terminates after m terms.

$$\left[\because \text{ If } (A-\lambda I)^m \hat{v} = 0 \Rightarrow (A-\lambda I)^{m+l}\hat{v} = (A-\lambda I)^l \left[(A-\lambda I)^m \hat{v} \right] = 0 \right].$$

Consequently,

$$e^{(A-\lambda I)t}\hat{v} = \hat{v} + t(A-\lambda I)\hat{v} + \frac{t^2}{2!}(A-\lambda I)^2\hat{v} + \dots + \frac{t^{m-1}}{m-1!}(A-\lambda I)^{m-1}\hat{v}.$$

and,
$$e^{At}v = e^{\lambda t} e^{(A-\lambda I)}\hat{v}$$
$$= e^{\lambda t}\left[v + t(A-\lambda I)v + \dots + \frac{t^{m-1}}{\lfloor m-1}(A-\lambda I)^{m-1}v \right].$$

This suggests the following algorithm :-

Algorithm:-

(a) Find all eigenvalues and eigenvectors of A. If A has n-linearly independent eigenvector then $x' = Ax$ has n L.I.S of the form $e^{\lambda t}\hat{v}$.

[Observe that, the infinite series $e^{(A-\lambda I)t}\hat{v}$ terminates after one term if v is an eigenvector of A with eigenvalue λ].

(b) Suppose A has only $K<n$ linearly independent eigenvectors. Then we have only K L.I.S of the form $\{e^{\lambda_i t}\hat{v}_i\}_{i=1}^{K}$.

To find additional solutions we pick an eigenvalue λ of A and find all vectors v for which $(A-\lambda I)^2\hat{v}_* = 0$ but $(A-\lambda I)\hat{v}_* \neq 0$.

For such a vector $\hat{v}_*$,

$$e^{At}\hat{v}_* = e^{\lambda t}\left[\hat{v}_* + t(A-\lambda I)\hat{v}_*\right]$$ is an additional solution of $x' = Ax$.

We do this for all eigenvalues of A.

(c) If we still don't have enough solutions then we find all vectors v for which

$(A-\lambda I)^3 \hat{v}_* = 0$ but $(A-\lambda I)^2 \hat{v}_* \neq 0$.

For each vector $\hat{v}_*$,

$$e^{At}\hat{v}_* = e^{\lambda t}\left[\hat{v}_* + t(A-\lambda I)\hat{v}_* + \frac{t^2}{2!}(A-\lambda I)^2 v_*\right]$$ is an additional solution of

$x' = Ax$

(d) We proceed in this manner until we get n L.I.S.

[Relation b/w any fundamental matrix and principle one, e^At = phi(t) phi(0)^-1]

Guarantee that this yields all possible eigenfunction :-

Lemma 1 :- Let $P(\lambda) = (\lambda-\lambda_1)^{\alpha_1} \cdots (\lambda-\lambda_n)^{\alpha_n}$ be the char. poly of A. Let A has only $\beta_i < \alpha_i$ linearly independent eigenvectors corresponding to λ_i. Then the equation $(A-\lambda_i I)^2 v = 0$ has at least $(\beta_i + 1)$ no of linearly independent solution.

More generally, if the equation $(A-\lambda_i I)^m v = 0$ has $\gamma_i < \alpha_i$ L.I. solutions then $(A-\lambda_i I)^{m+1} v = 0$ has at least $\gamma_i + 1$ independent solution.

how? why? think.

Ex :- Find 3 L.I.S of $x' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$

$\therefore$ Char Poly $P(\lambda) = (1-\lambda)^2 (2-\lambda)$

and the eigenvalues are $\lambda = 2, 1, 1$.

When $\lambda = 2$, $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector.

Hence, $x_1(t) = e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a solution.

When $\lambda = 1$, we seek $v \neq 0$ s.t $(A - I)v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow v_2 = v_3 = 0$ and v_1 is arbitrary.

$\therefore x_2(t) = e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is another solution.

$\because$ A has only one linearly independent eigenvector with eigenvalue 1, we look for solution of $(A-I)^2 v_* = 0$ but $(A - I)v_* \neq 0$.

$$\therefore \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} v_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v_3 = 0$ but v_1 and v_2 are arbitrary.

$$\therefore v_* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ satisfies } (A-I)^2 v_* = 0 \text{ but } (A-I)v_* \neq 0.$$

Hence, $x_3(t) = e^{At}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^t e^{(A-I)t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^t\left[I + t(A-I)\right]\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$= e^t\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] = e^t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + te^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= e^t\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}.$$

Periodic System :- $A(t) \equiv$ Continuous Matrix function.

Consider the problem $X' = A(t)X$; $X(0) = X_0$ — (1).

(1) admits an unique solution $X \in C^1(\mathbb{R}, \mathbb{R}^n)$. (Will be discussed later).

(Look again, I think sir said proof same as picard's one)

Keeping that in mind let $X_i(t)$ be the unique solution of

$$X' = A(t)X$$
$$X(0) = E_i = (0, \ldots, \underset{i^{th}}{1}, 0 \ldots 0)$$

Hence, $\Psi(t) = (X_1(t), \ldots, X_n(t))$ forms a $(n \times n)$ matrix called the Fundamental matrix (Principle).

Note :- (a) $\because \{X_1(0), \ldots, X_n(0)\}$ are Linearly independent hence $\{X_1, X_2 \ldots X_n\}$ are L.I.

So, $\det \Psi(t) \neq 0$. or, Ψ is invertible.

(b) $\Psi(t)$ satisfies the matrix differential equation given by $\Psi'(t) = A(t)\Psi(t)$.

Lemma 1: If $\Psi(t)$ is the Fundamental matrix of the System $x' = A(t)x$ then for any invertible constant matrix C, $\Psi(t)C$ is also a fondamental matrix. Moreover If $\Psi_1(t)$ is any other Fundamental matrix of $x' = Ax$ then $\Psi_1(t) = \Psi(t)C$. for some constant matrix C (which is invertible).

Proof: Note, $[\Psi(t)C]' = \Psi'(t)C = A(t)\Psi(t)C = A(t)[\Psi(t)C]$

$\therefore$ $\Psi(t)C$ satisfies the matrix differential equation.

also, $\det(\Psi(t)C) = \det\Psi(t)\det C \neq 0$.

$\therefore$ $\Psi(t)C$ is also a fundamental matrix.

Now, let $C(t) := \Psi^{-1}(t)\Psi_1(t)$.

$\Rightarrow \Psi_1(t) = \Psi(t)C(t) \Rightarrow \Psi_1'(t) = \Psi(t)C'(t) + \Psi'(t)C(t) \Rightarrow A(t)\Psi_1(t) = \Psi(t)C'(t) + A\Psi(t)C(t)$.

$\Rightarrow A(t)\Psi(t)C(t) = \Psi(t)C'(t) + A\Psi(t)C(t) \Rightarrow \Psi(t)C'(t) = 0 \Rightarrow C'(t) = 0 \Rightarrow C$ is independent of t

Assumption: Let us assume that $A(t)$ and $B(t)$ are continuous function of t and consider the following equation $X'(t) = A(t)X(t) + B(t)$ —— (*)

Theorem: Let $A(t)$ and $B(t)$ are periodic functions of period ω. Then (*) admits a periodic solution $Y(t)$ iff $Y(0) = Y(\omega)$.

Proof: If Y is periodic of period ω then $Y(0) = Y(\omega)$,

Now if $Y(0) = Y(\omega)$ then one needs to show that the solution turns out to be periodic.

Define, $Z(t) = Y(t+\omega)$ where $Y(t)$ satisfies (*)

then, $Z'(t) = Y'(t+\omega) = A(t+\omega)Y(t+\omega) + B(t+\omega) = A(t)Z(t) + B(t)$

[$\because$ A and B are ω-periodic]

Also, $Z(0) = Y(\omega) = Y(0)$ (Given)

Thus, Z and Y both satisfies (*) and $Z(0) = Y(0) \Rightarrow Z(t) = Y(t)$ $\forall t \in I$ due to uniqueness of soln of X'=A(t)X; X(0)=Xo

and hence, $Y(t+\omega) = Y(t)$ $\forall t \in I$.

Which shows that the solution is indeed periodic in $\mathbb{R}$ of period ω.

Corollary :- When can the problem $X' = A(t)X$ admits a non-trivial ω-periodic solution.

N.T Note that if $X(t)$ is any solution of $X' = A(t)X$ then $X(t) = \Psi(t)C$ where $\Psi(t)$ is the Fundamental matrix and $C \in \mathbb{R}^n$ (arbitrary)

If $X(t)$ is ω-periodic then $X(0) = X(\omega)$ and so, $\Psi(0)C = \Psi(\omega)C$.

$$\Leftrightarrow [\Psi(0) - \Psi(\omega)]C = 0.$$

$$\Leftrightarrow \det[\Psi(0) - \Psi(\omega)] = 0$$

$$\boxed{\text{If } \Psi(t) \text{ is a Fundamental Matrix of the system } X' = AX, \text{ then the system admits a non-trivial } \omega\text{-periodic solution iff } \det[\Psi(0) - \Psi(\omega)] = 0.}$$

Corollary (b) :- The system $x' = A(t)x + B(t)$ has an unique ω-periodic solution iff the system $x' = A(t)x$ does not have a ω-periodic solution other than the trivial one.

Proof :- Let $\psi(t)$ be the fundamental matrix of $x' = A(t)x$ then any solution of $x' = A(t)x + B(t)$ can be written as

$$x_{IN}(t) = \psi(t)C + \int_0^t \psi(t)\psi^{-1}(s)B(s)\,ds.$$ where C is arbitrary.

x_{IN} is periodic(ω) iff $x_{IN}(0) = x_{IN}(\omega) \Rightarrow \psi(0)C = \psi(\omega)C + \int_0^{\omega} \psi(\omega)\psi^{-1}(s)B(s)\,ds$

$$\Rightarrow [\psi(0) - \psi(\omega)]C = \int_0^{\omega} \psi(\omega)\psi^{-1}(s)B(s)\,ds. \quad \text{—— (1)}$$

(1) has an unique solution when $\det[\psi(0) - \psi(\omega)] \neq 0$. pura hogya!

Floquet theorem :- Let $\psi(t)$ be the fundamental matrix of $X' = AX$. Then

(A is w-periodic)

(a) The matrix $\Phi(t) = \psi(t+\omega)$ is also a fundamental matrix of $X' = AX$

(b) $\exists$ a periodic non-singular matrix $P(t)$ of period ω and a constant matrix R such that $\psi(t) = P(t)e^{Rt}$.

Proof :- $\Phi'(t) = \psi'(t+\omega) = A(t+\omega)\psi(t+\omega) = A(t)\Phi(t)$

$\therefore$ $\Phi(t)$ is the solution matrix of the $\Phi'(t) = A(t)\Phi(t)$.

Further, $\det \psi(t+\omega) \neq 0$ $\forall t$ and hence, $\det \Phi(t) \neq 0$ $\forall t$.

$\therefore$ Φ is a fundamental matrix of $X' = AX$.

$\because$ $\Psi(t)$ and $\Psi(t+\omega)$ are both Fundamental solution of $x' = Ax$ one

has, $\Psi(t+\omega) = \Psi(t)C$.

[$\exists$ constant matrix R s.t $C = e^{R\omega}$]

$A_{n\times n}$ is a non-singular matrix
Then $\exists$ $B_{n\times n}$ s.t $A = e^{B}$.
"B is called the logarithm of A"

$\therefore$ $\Psi(t+\omega) = \Psi(t)e^{R\omega}$

Define, $P(t) = \Psi(t)e^{-Rt}$

$\therefore$ $P(t+\omega) = \Psi(t+\omega)\, e^{-R(t+\omega)} = \Psi(t)\, e^{R\omega}\, e^{-R(t+\omega)} = \Psi(t)\, e^{-Rt} = P(t)$.

Hence, $P(t)$ is ω-periodic

Also, $\Psi(x)$ and e^{-Rt} are non-singular hence $\det P(t) \neq 0$.

Theorem: Let $\psi(t) = P(t)e^{Rt}$ be the such that P is a non-singular ω-periodic matrix and R is a constant matrix. Then the transformation $X(t) = P(t)v(t)$ reduces $X' = A(t)X$ reduces to the system $v' = Rv$.

Proof: $\because \psi'(t) = A(t)\psi(t)$

$\Rightarrow \left[P(t)e^{Rt}\right]' = A(t)P(t)e^{Rt}$

$\Rightarrow P'(t)e^{Rt} + P(t)Re^{Rt} = A(t)P(t)e^{Rt}$

$\Rightarrow P'(t) + P(t)R - A(t)P(t) = 0$ ——— (1)

Again, $X'(t) = A(t)X(t) \Rightarrow P'(t)v(t) + P(t)v'(t) = A(t)P(t)v(t)$

$\Rightarrow \left[P'(t) - A(t)P(t)\right]v(t) + P(t)v'(t) = 0$ —— (ii)

From (1) $\Rightarrow -P(t)Rv(t) + P(t)v'(t) = 0 \Rightarrow v'(t) = Rv(t)$.

$\boxtimes$

Phase Portrait :-

Consider a (2×2) linear system given by $X' = AX$. — ① We would like to sketch the integral curves of the system.

Observation 1 :- The canonical form of any 2×2 matrix is given by

ⓐ $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, ⓑ $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ or ⓒ $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Why this is important : Given any linear system, one can always change coordinate

So the new system's coefficient matrix is in canonical form.

To elaborate, let $Y(t)$ be the solution of the system $Y' = (T^{-1}AT)Y$ — ⑪ for some $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ and given $A \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$.

If one sets $X(t) = TY(t)$ then

$$X'(t) = TY'(t) = T(T^{-1}AT)Y = (TT^{-1})ATY = AX(t). \quad \text{— ①}$$

i.e, T converts the soln of ⑪ to solutions of ① and T^{-1} takes solutions of ① to solutions of ⑪.

$\therefore$ we hope to be able to find a linear map T that converts $X' = AX$ into a system of the form $Y' = (T^{-1}AT)Y$ that is easily solved.

Question :- How does one go about finding such a $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$.

Case a :- When A admits real eigenvalues :-

Let (λ_1, v_1) and (λ_2, v_2) be two distinct pair of eigenpair in the sense $\lambda_1 \neq \lambda_2$.

Define, $T(e_i) = v_i$ for $i = 1, 2$.

$$\therefore T^{-1}AT(e_i) = T^{-1}A[T(e_i)] = T^{-1}Av_i = T^{-1}(\lambda_i v_i) = \lambda_i T^{-1}(v_i) = \lambda_i e_i = \operatorname{diag}(\lambda_1, \lambda_2)e_i$$

Hence, $T^{-1}AT = \operatorname{diag}(\lambda_1, \lambda_2)$.

Thus, $T^{-1}AT$ assumes the canonical form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Case b: Complex Eigenvalues: Let A admits complex eigenvalues $\alpha \pm i\beta$ $(\beta \neq 0)$ and $v_1 \pm i v_2$ be the corresponding eigenfunction.

Claim 1: v_1 and v_2 are linearly independent in $\mathbb{R}^2$.

If not then $v_1 = c v_2$ and hence

$$A(v_1 + i v_2) = (\alpha + i\beta)(v_1 + i v_2)$$

$$\Rightarrow A[(c+i)v_2] = (c+i)(\alpha+i\beta)v_2$$

$$\Rightarrow (c+i)Av_2 = (c+i)(\alpha+i\beta)v_2$$

$$\Rightarrow Av_2 = (\alpha+i\beta)v_2.$$

Real vector   Complex vector

– a contradiction.

Now, $A(v_1 + i v_2) = (\alpha + i\beta)(v_1 + i v_2)$

$\Rightarrow \quad Av_1 = \alpha v_1 - \beta v_2$

and, $Av_2 = \beta v_1 + \alpha v_2$.

Define, $T(e_i) = v_i$ for $i = 1, 2$

Consider, $(T^{-1}AT)(e_1) = T^{-1}A\,T(e_1) = T^{-1}Av_1 = T^{-1}[\alpha v_1 - \beta v_2] = \alpha T^{-1}(v_1) - \beta T^{-1}(v_2)$

$= \alpha e_1 - \beta e_2 = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$.

And, $(T^{-1}AT)(e_2) = T^{-1}A(\beta v_1 + \alpha v_2) = \beta e_1 + \alpha e_2 = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$.

$\therefore$ the matrix $T^{-1}AT = \left(T^{-1}AT(e_1) \quad T^{-1}AT(e_2) \right)$

$= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Case c : Repeated Eigenvalue : Let (λ, v) be an eigenpair of A and every other eigenvector is multiple of v.

Let w be any vector s.t v and w are linearly independent.

then, $Aw = \alpha v + \beta w$ $(\alpha, \beta \in \mathbb{R})$

Note, If $\alpha = 0$ then β is an eigenvalue of A and hence $\beta = \lambda$.

If $\alpha \neq 0$ then we again claim $\beta = \lambda$, if not then $\beta - \lambda \neq 0$.

note that, $A\left[w + \frac{\alpha}{\beta-\lambda} v\right] = \alpha v + \beta w + \frac{\alpha\lambda}{\beta-\lambda} v$

$$= \frac{\alpha\beta}{\beta-\lambda} v + \beta w.$$

$$= \beta\left[w + \frac{\alpha}{\beta-\lambda} v\right].$$

$\therefore$ β is a second eigenvalue different from λ and hence $\beta = \lambda$.

let, $u = \frac{1}{\alpha} w$.

$$\therefore Au = \frac{1}{\alpha} Aw = \frac{1}{\alpha}\left[\alpha v + \beta w\right] = v + \frac{\beta}{\alpha} w = v + \lambda u.$$

Define, $T(e_1) = v$ and $T(e_2) = u$

then, $T^{-1}AT(e_1) = T^{-1}A(v) = T^{-1}(\lambda v) = \lambda e_1 = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$

and, $T^{-1}AT(e_2) = T^{-1}A(u) = T^{-1}(v + \lambda u) = T^{-1}(v) + \lambda T^{-1}(u)$

$$= e_1 + \lambda e_2 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

$$\therefore T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

IN ALL THE CASES, $\exists\, T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ s.t $X' = AX$ can be put in one of the canonical forms given by (a) $Y' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} Y$, (b) $Y' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} Y$ or (c) $Y' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} Y$.

Hence we will study only the canonical cases :-

Case a:- (i) Real and Distinct eigenvalue $(\lambda_1 < \lambda_2 < 0)$

$Y' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y$. is the system (canonical) given $A \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ and (λ_i, v_i) are the corresponding eigenpairs.

(T^-1 A T ki eigenvector e1, e2 bnayi h apan ne by construction)

$$Y_{G.S}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 .$$

w.l.o.g we may assume $v_i = e_i$ and hence

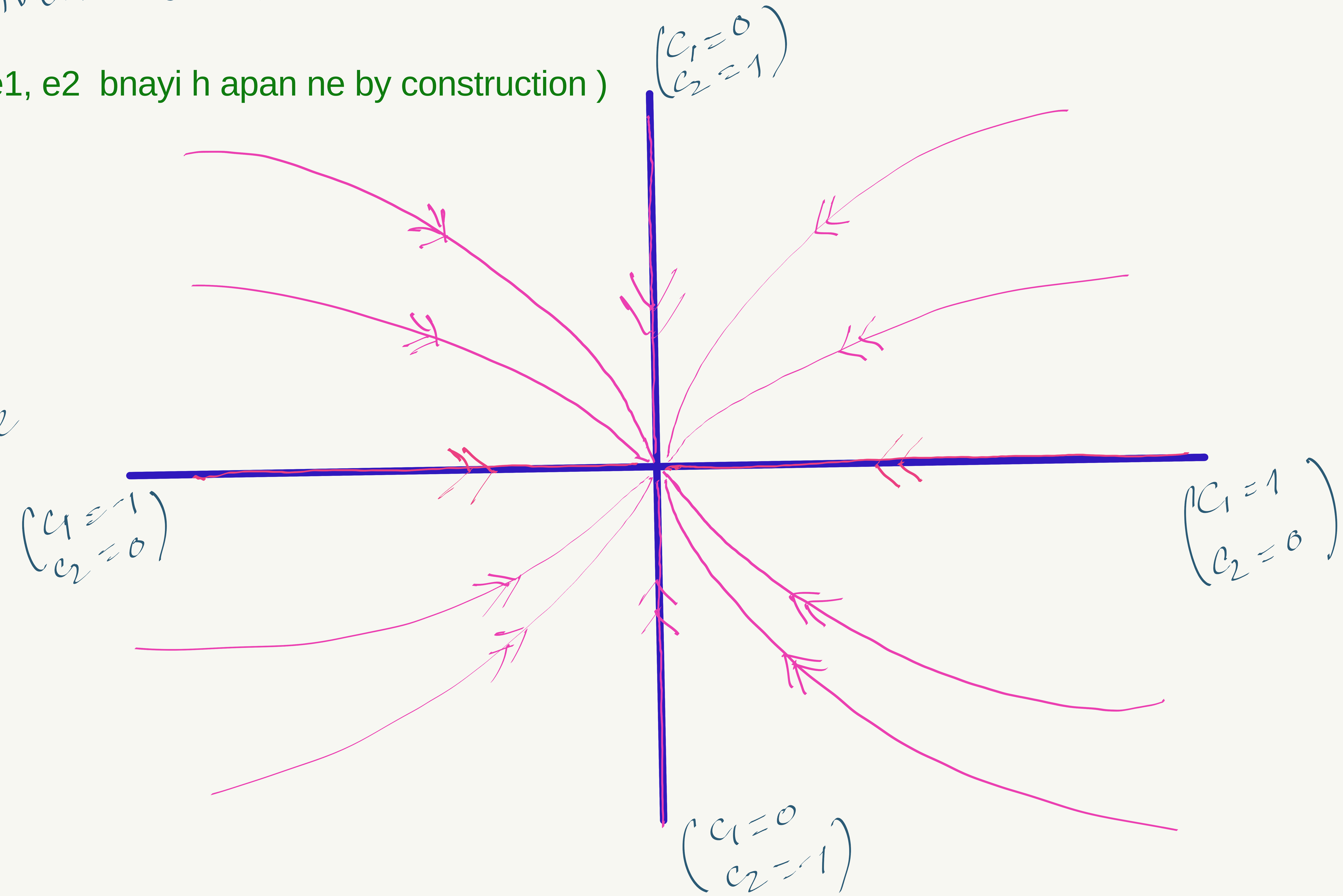
$$Y_{G.S}(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

When $c_1 = \pm 1, c_2 = 0$ then $Y_{G.S}(t) = \pm e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and $\lim_{t \to \infty} Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

When $c_1 = 0, c_2 = \pm 1$ then $Y_{G.S}(t) = \pm e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $\lim_{t \to \infty} Y_{\cdot}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



SINK

Also note, $\lim_{t\to\infty} Y_{G.S}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any $c_1, c_2 \in \mathbb{R}$.

Question :- How do they approach the origin ?

Let's compute the slope of the integral curves for $c_1, c_2 \neq 0$.

write, $x(t) = c_1 e^{\lambda_1 t}$; $y(t) = c_2 e^{\lambda_2 t}$.

and $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{c_2}{c_1} e^{(\lambda_2 - \lambda_1)t}$

$\because \lambda_2 - \lambda_1 > 0$, these slopes approach $\pm\infty$.

Hence, these solutions tends to origin tangentially to the y-axis.

[when, $0 < \lambda_1 < \lambda_2$ the situation reverses with same phase portrait but with reversed direction. we say it source -].

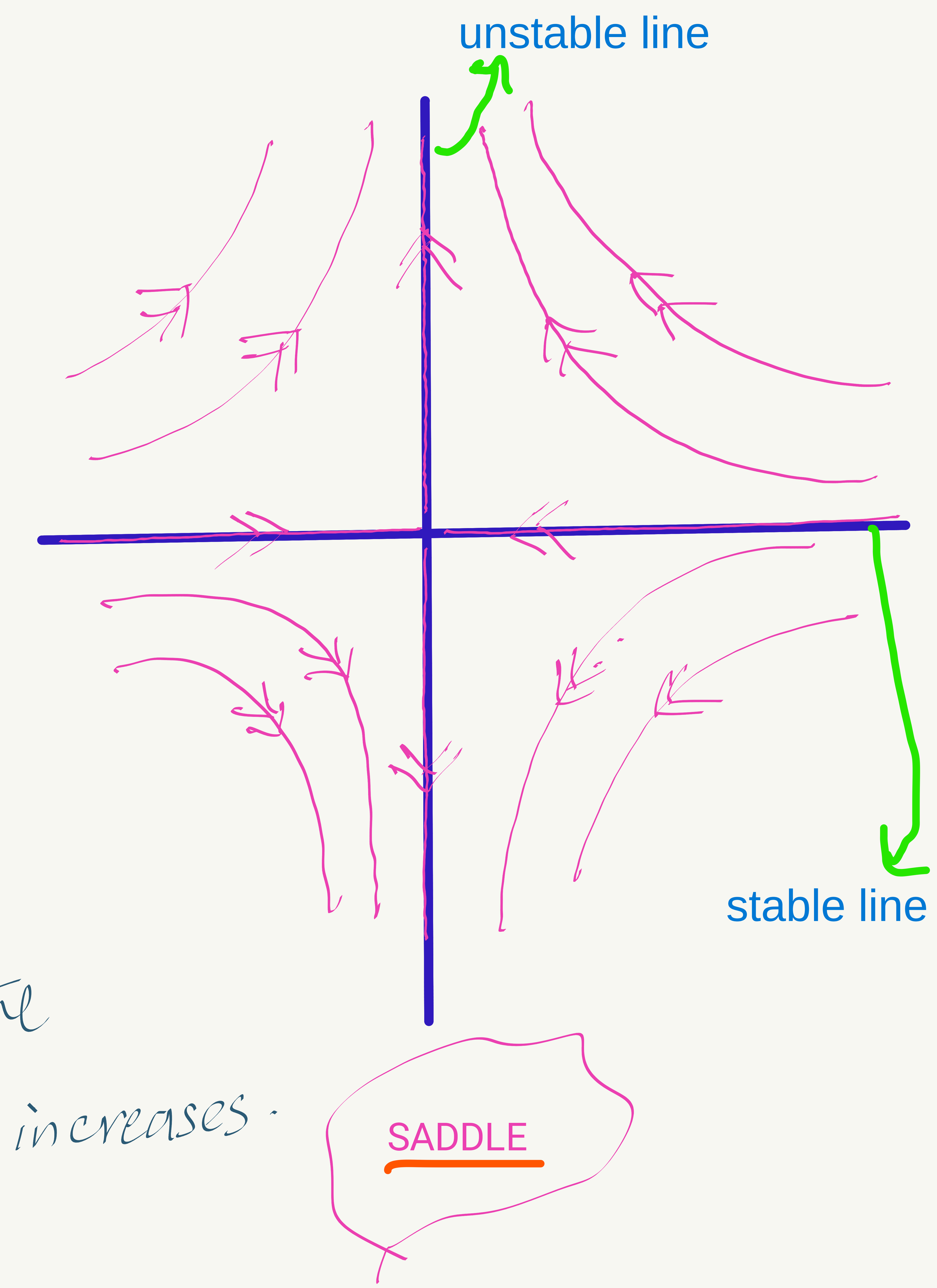
Case a (ii) Consider $x' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x$ with $\lambda_1 < 0 < \lambda_2$.

$\therefore x(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\because \lambda_1 < 0$, the st-line solution of the form $e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ lies on the x-axis and tends to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$. This axis is called stable line.

Again since, $\lambda_2 > 0$, the st-line solution of the form $c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ lies on the y-axis and tends away from origin as $t \to \infty$, hence called the unstable line.

All other solutions ($c_1, c_2 \neq 0$) tends to infinite in the direction of the unstable line as $t \to \infty$ since $x(t)$ comes closer to $(0, c_2 e^{\lambda_2 t})$ as t increases.



line goes towards origin at t->infinity = stable line
line goes away from origin at t->infinity = unstable line

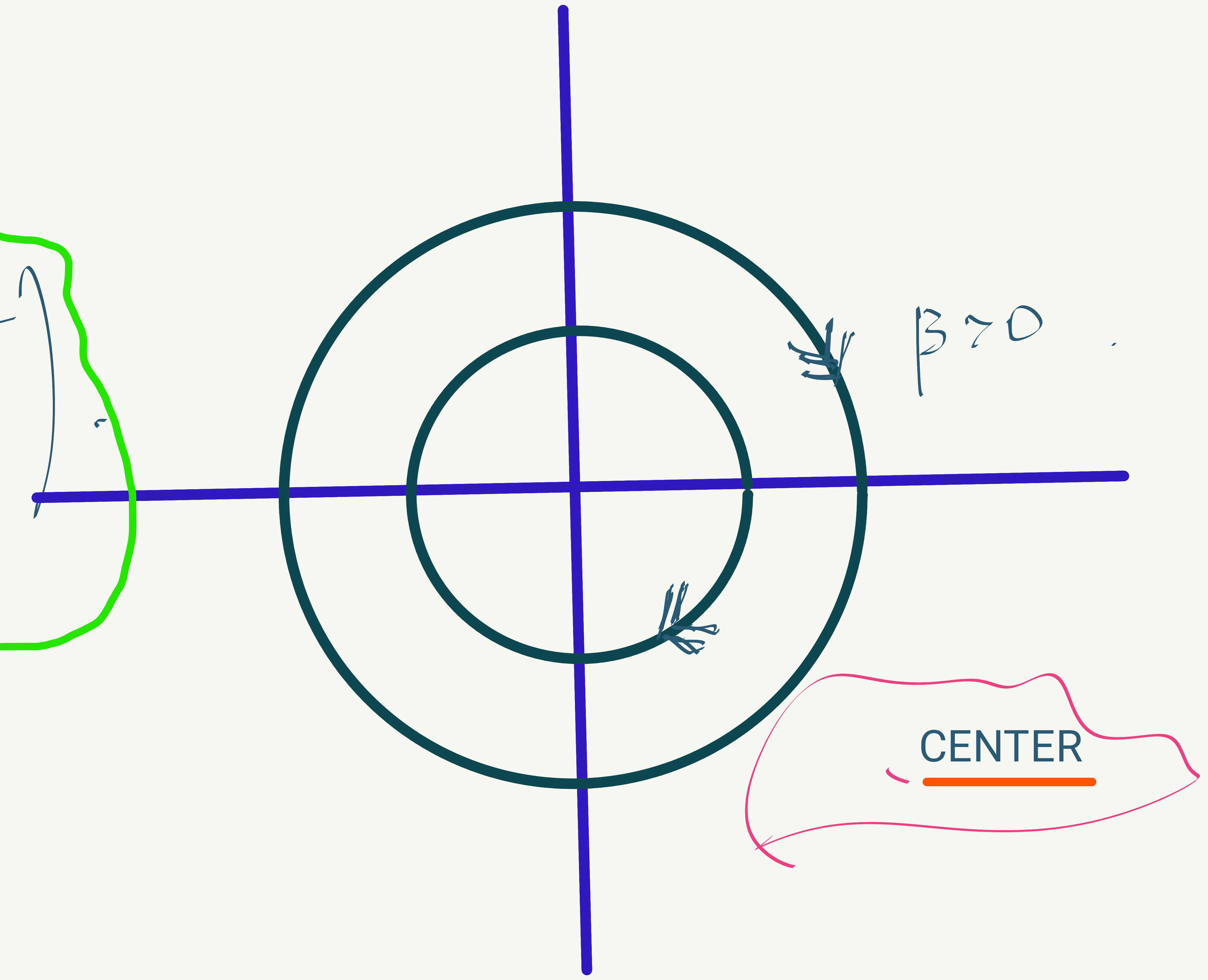
Case b: Complex Eigenvalue:

(a) Consider $X' = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} X$; $\beta \neq 0$

the general solution is given by $\boxed{X(t) = c_1 \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + c_2 \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}}$

Note, the solutions are periodic of period $2\pi/\beta$.

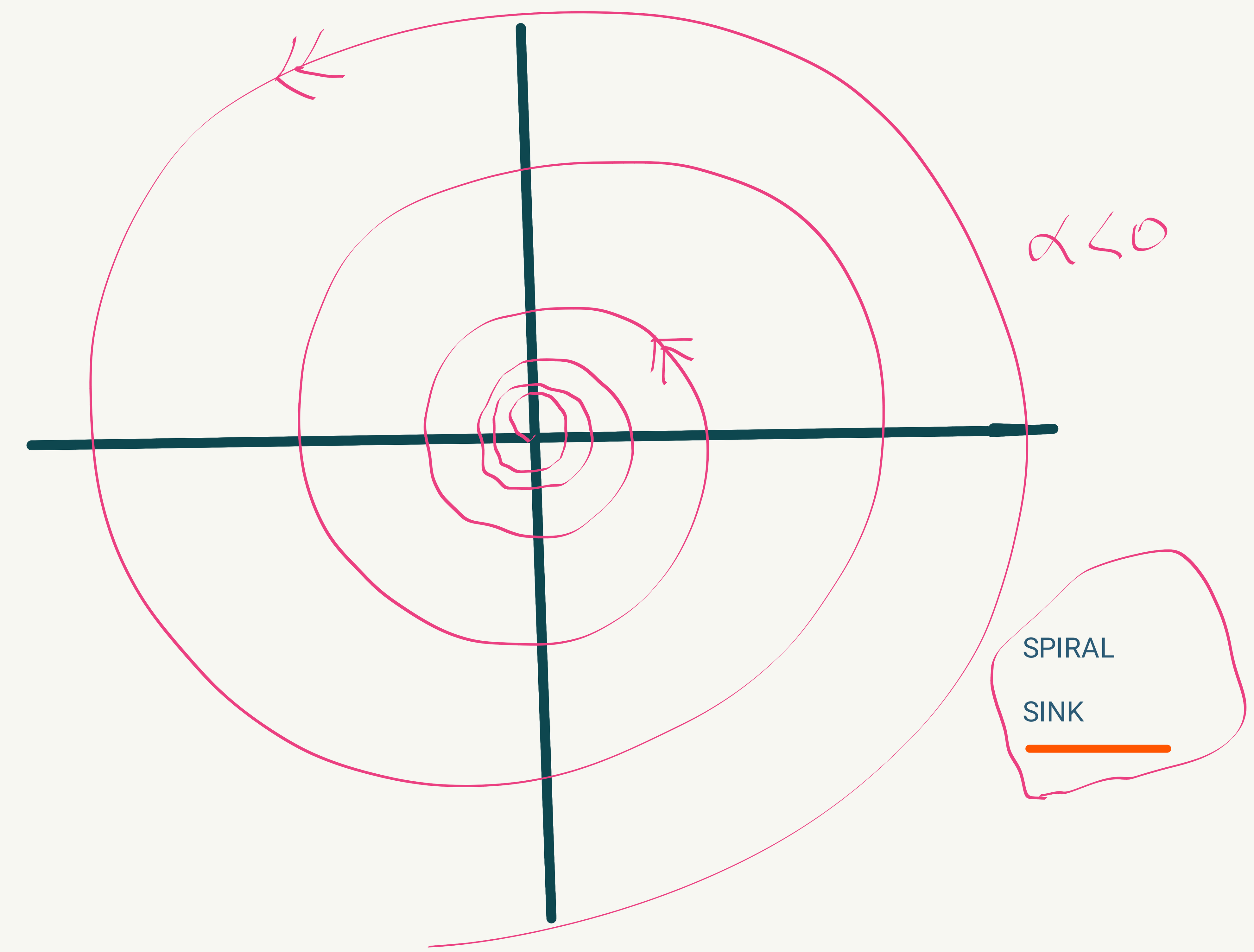
the circles are transversed in clockwise direction if $\beta > 0$ and the other way around for $\beta < 0$.



(b) If $X' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

then $\boxed{X(t) = c_1 e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}}$.

Clearly the solutions spiral into the origin as $t \to \infty$ if $\alpha < 0$ or away from the origin for $\alpha > 0$.



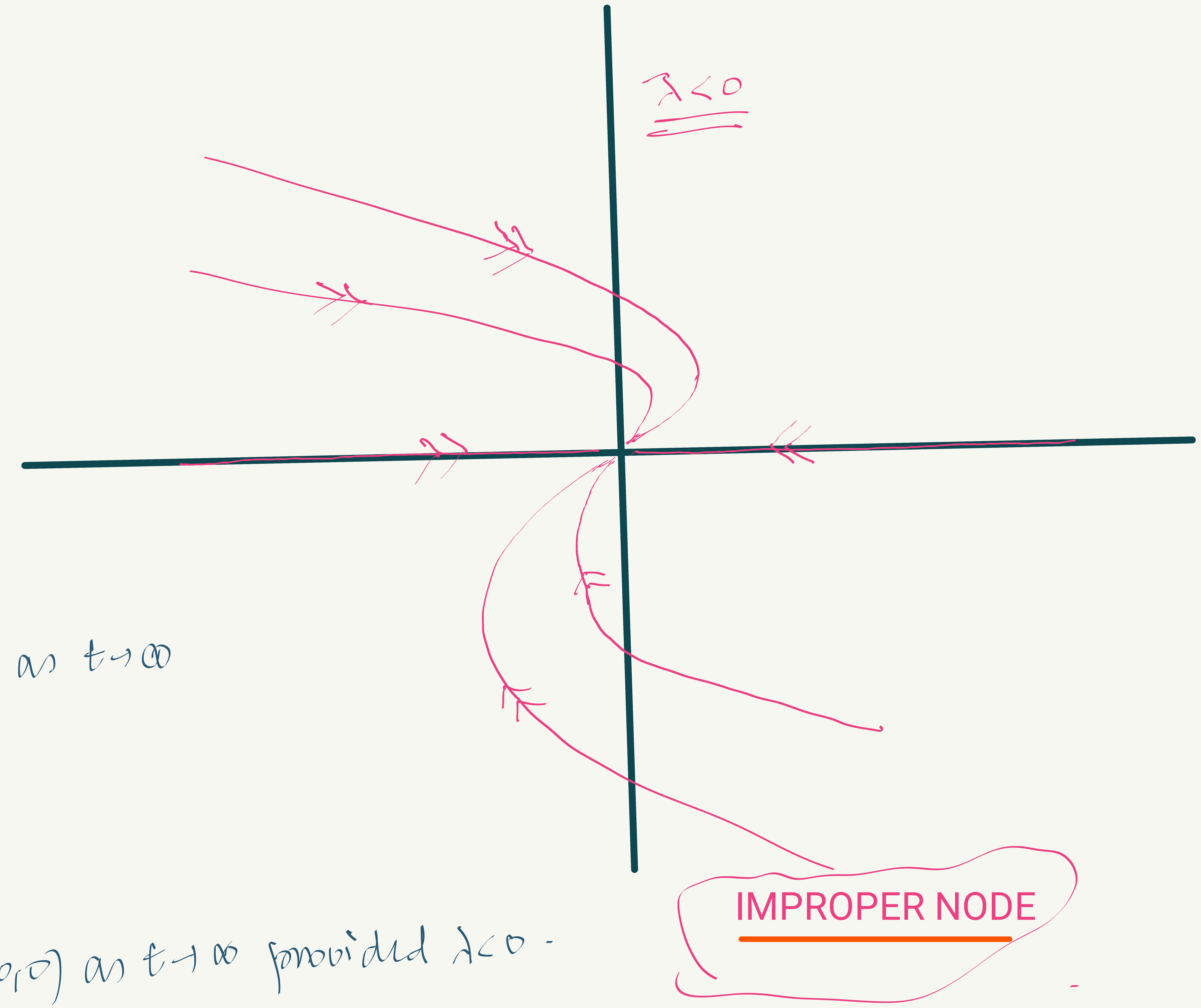
spiral sink if alpha<0
spiral source if alpha>0

Case c: Equal Eigenvalue:

Consider $X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X$.

$$X(t) = c_1 e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda t}\begin{pmatrix} t \\ 1 \end{pmatrix}$$ dhyan se

$$= \begin{bmatrix} c_1 e^{\lambda t} + c_2 t e^{\lambda t} \\ c_2 e^{\lambda t} \end{bmatrix}.$$



When $c_1 = 1$, $c_2 = 0$.

$X(t) = e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which tends to $(0,0)$ as $t \to \infty$ provided $\lambda < 0$.

When $c_1 = 0$, $c_2 = 1$

$X(t) = \begin{pmatrix} t e^{\lambda t} \\ 0 \end{pmatrix}$ which again tends to $(0,0)$ as $t \to \infty$ provided $\lambda < 0$.

Clearly, $X(t) \to (0,0)$ as $t \to \infty$, $\lambda < 0$.

"The solutions tend towards or away from the origin in a direction tangent to the vector $(1,0)$."

Note: In subsequent pages replace(assume) 0 as a or using shifting of origin everywhere.

(Weak) Comparison Principle :-

Theorem 1:- Let u and v are continuously differentiable in $[a,b]$ and $f : D\ (\subseteq \mathbb{R}^2) \to \mathbb{R}$ be continuous. such that $u(0) < v(0)$. and moreover let $u' - f(t,u) < v' - f(t,v)$ on $[a,b]$. Then $u < v$ on $[a,b]$.

Proof:- If $u \not< v$ on $[a,b] \Rightarrow \exists\ z \in (a,b]$ s.t $u(z) \geq v(z)$.

Note that since $u(0) < v(0)$ hence $\exists\ z_0 \in (a,b]$ s.t $u(z_0) = v(z_0)$ and $u(t) < v(t)$ $\forall\ t \in [0, z_0)$.

Hence, $u'(z_0) = \lim_{h \to 0^-} \dfrac{u(z_0+h) - u(z_0)}{h} \geq \lim_{h \to 0^-} \dfrac{v(z_0+h) - v(z_0)}{h} = v'(z_0)$

$\therefore\ u'(z_0) - f(z_0, u(z_0)) \geq v'(z_0) - f(z_0, v(z_0))$.

— a contradiction.

* $D := \{(t, x(t)) : t \in I\}$. (We call this a weak Comparison Principle).

Corollary: Let $f(x,y)$ is continuous in $D := \{(x, y(x)) : x \in I = [a,b]\}$. Further assume

ⓐ y solves $y' = f(x,y)$; $y(0) = y_0$.

ⓑ y_1 and y_2 satisfies $y_1' < f(x,y_1)$ and $y_2' > f(x,y_2)$ in I.

ⓒ $y_1(0) \leq y_0 \leq y_2(0)$

then $y_1(x) < y(x) < y_2(x)$ $\forall x \in (a,b]$.

Proof: We will show $y(x) < y_2(x)$ $\forall x \in (a,b]$.

If $y_0 < y_2(0)$ then the results follow from theorem 1.

If $y_0 = y_2(0)$ define $z(x) = y_2(x) - y(x)$. then $z'(x) = y_2'(x) - y'(x) > f(x, y_2(x)) - f(x, y(x)) \overset{\text{for } x=0}{=} 0$

$\therefore z'(0) > 0 \Rightarrow z$ is strictly increasing in the interval $[0,\epsilon]$ for some $\epsilon > 0$ small.

$\therefore y_2(\epsilon) > y(\epsilon)$, and hence from theorem ①; $y_2(x) > y(x)$ $\forall x \in [\epsilon, a]$.

Ex :- Consider the I.V.P :-

$$y' = x^2 + y^2 \; ; \; y(0) = 1 \text{ and } x \in$$

$[0, \pi/4\rangle$

Define, $y_1(x) = 1 + x^3/3$

$$\therefore y_1'(x) = x^2 < x^2 + \left(1 + x^3/3\right)^2 = x^2 + y_1^2(x)$$

and, $y_1(0) = 1$

Ily, $y_2(x) = \tan(x + \pi/4)$

$$\therefore y_2'(x) = \sec^2(x + \pi/4) = 1 + \tan^2(x + \pi/4) > x^2 + y_2^2(x)$$

and, $y_2(0) = 1$.

$\therefore$ From Corollary one has,

$$1 + \frac{x^3}{3} \le y(x) \le \tan(x + \pi/4) \quad \forall x \in$$

STRONG COMPARISON THEOREM : Let y_1 and y_2 are continuously differentiable in $[a,b]$ and f is continuous in D and Lipchitz w.r.t the second variable. and $y_1(0) < y_2(0)$, $y_1' - f(t,y_1) \le y_2' - f(t,y_2)$ on $(a,b]$.

Then $y_1 < y_2$ on $[a,b]$.

Proof : Define $y_2(0) - y_1(0) = \alpha > 0$

Let, $w = y_1 + \frac{\alpha}{2}\varphi(t)$ where φ is sufficiently smooth (To be chosen later)

$$w' - f(t,w) = y_1' + \frac{\alpha}{2}\varphi'(t) - f(t,w) + f(t,y_1) - f(t,y_1)$$

$$\le y_1' - f(t,y_1) + |f(t,w) - f(t,y_1)| + \frac{\alpha}{2}\varphi'(t)$$

$$\le y_1' - f(t,y_1) + \frac{K\alpha}{2}\varphi(t) + \frac{\alpha}{2}\varphi'(t).$$

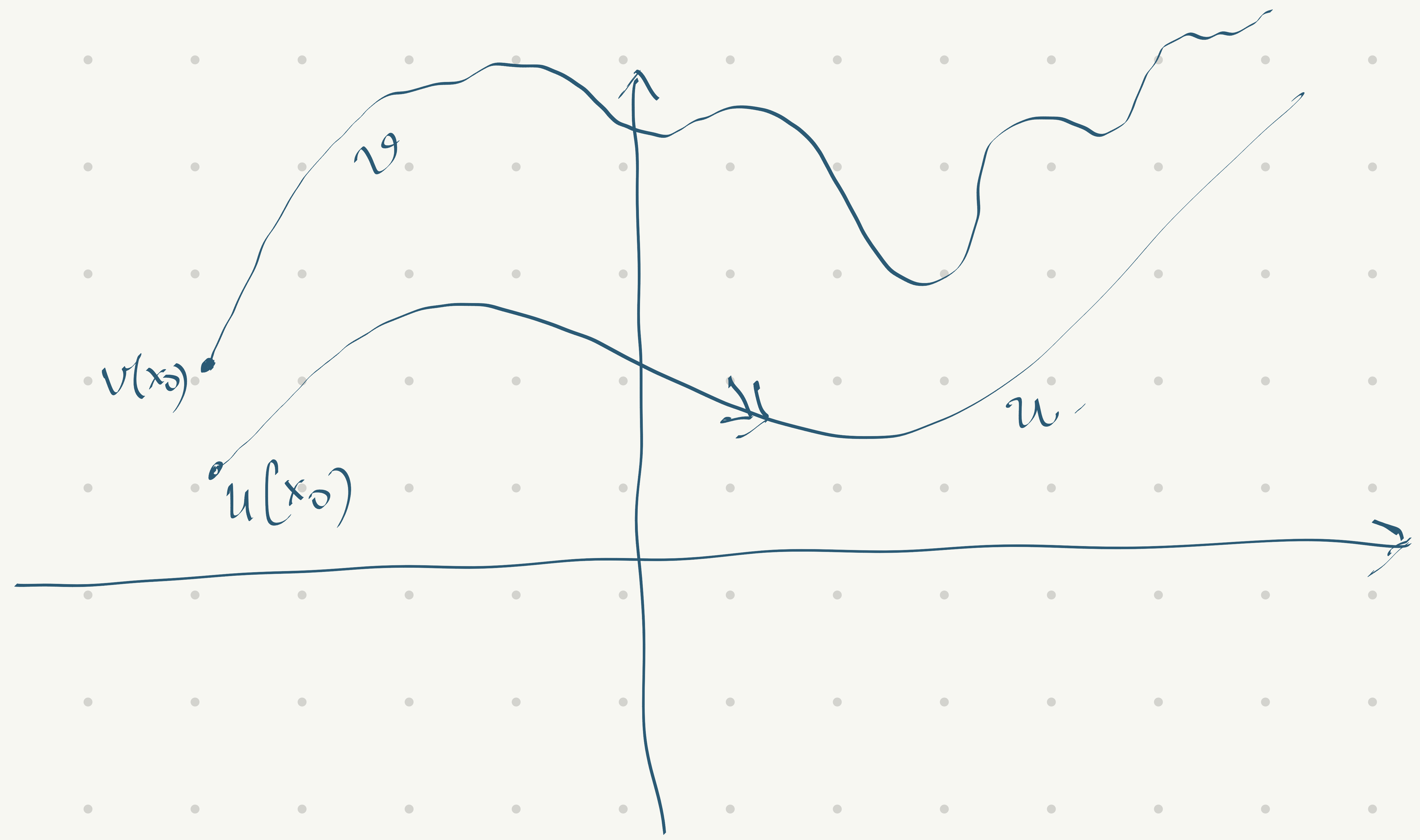
We want, $\varphi'(t) + K\varphi(t) < 0 \;\; \forall t \in [0,a] \Rightarrow [e^{+Kt}\varphi(t)]' < 0 \Rightarrow e^{+Kt}\varphi(t) < \varphi(0) \Rightarrow \varphi(t) < \varphi(0)e^{-Kt}$

$\therefore$ $w' - f(t,w) < y_2' - f(t,y_2)$ for $\varphi(t) = e^{-2Kt}$

Note, $w(0) = y_2(0) - \alpha + \frac{\alpha}{2} = y_2(0) - \frac{\alpha}{2} < y_2(0) \Rightarrow w(t) < y_2(t). \Rightarrow y_1 + \frac{\alpha}{2}\varphi(t) < y_2(t) \Rightarrow y_1(t) < y_2(t) \;\; \forall t \in [0,a].$

Corollary: Let y_1 and y_2 are two solutions of $y' = f(x,y)$ where f is continuous in D and Lipchitz w.r.t y.

Then they do not touch or intersect if they start apart



Exact and Adjoint Equation :-

Consider the equation $P_0(x)y'' + P_1(x)y' + P_2(x)y = r(x)$ — (1)

(1) is said to be exact if $\exists\ p, q \in C^1(I)$ s.t

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = (py' + qy)' \quad \text{— (ii)}$$

Conditions under which (ii) is true :-

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = py'' + p'y' + q'y + qy'$$

$$\Rightarrow (P_0 - p)y'' + [P_1 - (p' + q)]y' + (P_2 - q')y = 0 \quad \text{— (iii)}$$

$\because$ (iii) holds for all $y \in C^2[a,b]$ one has,

$$P_0 = p \ ; \ P_1 = p' + q \ \text{ and } \ P_2 = q'$$

and hence, $P_0'' - P_1' + P_2 = p'' - p'' - q' + q' = 0$

CONCLUSION :-

For (1) to be exact, $\boxed{P_0'' - P_1' + P_2 = 0}$

Ex:- $x^2y'' + xy' - y = x^4$, $x > 0$.

Note, $P_0(x) = x^2$; $P_1(x) = -x$ and $P_2(x) = -1$.

$$\therefore P_0'' - P_1' + P_2 = 2 - 1 - 1 = 0.$$

Hence the above equation is exact.

$$\therefore x^2y'' + xy' - y = (x^2y' - xy)'$$

and, $x^2y' - xy = \frac{x^5}{5} + c_1$

$$\Rightarrow y(x) = \frac{x^4}{15} + \frac{c_1}{x} + c_2 x.$$

Q: What happens if (1) is not exact.

We multiply with integrating factor. i.e, $\exists\, z(x)$ s.t

$$[P_0(x)z(x)]'' - [P_1(x)z(x)]' + P_2(x)z(x) = 0$$

$$\left.\begin{aligned} &\Leftrightarrow P_0 z'' + (2P_0' - P_1)z' + (P_0'' - P_1' + P_2)z = 0 \\ &\boxed{\text{OR}}\ q_0 z'' + q_1 z' + q_2 z = 0 \end{aligned}\right\} \longrightarrow \text{(IV)}$$

where q_i are resp coefficient w.r.t P_0, P_1 and P_2.

Note that, $P_0 y'' + P_1 y' + P_2 y = 0$ is equivalent to (1)

Adjoint system is: X' = - A^T X

$$\Rightarrow \begin{pmatrix} y' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1/P_0 \\ -P_2 & \frac{P_0' - P_1}{P_0} \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} \text{ where, } P_0 y' = v.$$

If one defines $A = \begin{bmatrix} 0 & 1/P_0 \\ -P_2 & \frac{P_0' - P_1}{P_0} \end{bmatrix}$ then, $\begin{pmatrix} y' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & P_2 \\ -1/P_0 & P_1 - P_0'/P_0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$. is the

adjoint system which is same as,

$$y' = P_2 v \text{ and } v' = -1/P_0\, y + P_1 - P_0'/P_0\, v \Rightarrow P_0 v' = -y + (P_1 - P_0') v.$$

$$\therefore P_0 v'' + P_0' v' = -P_2 v + (P_1 - P_0') v' + (P_1 - P_0')' v$$

$$\Rightarrow P_0 v'' + (2P_0' - P_1) v' + (P_0'' - P_1' + P_2) v = 0.$$

which is same as (4).

$\Rightarrow$ (1) and (4) are adjoint system.

When an equation and its adjoint are same they are self-adjoint.

i.e, $2p_0' - p_1 = p_1 \Rightarrow p_1 = p_0'$

$\therefore$ (I) is self-adjoint if it is in the form

$$p_0(x)y'' + p_0'(x)y' + p_2(x)y = 0.$$

which is same as

$$\boxed{(p_0(x)y')' + p_2(x)y = 0}$$ self-adjoint

Hence one may turn (I) into self-adjoint form by multiplying it with $\sigma(x)$ i.e

$$[\sigma(x)p_0(x)]' = \sigma(x)p_1(x).$$

$$\Rightarrow \frac{(\sigma p_0)'}{\sigma p_0} = \frac{p_1}{p_0} \Rightarrow \boxed{\sigma(x) = \frac{1}{p_0}\exp\left(\int \frac{p_1}{p_0}\right).} \longrightarrow \boxed{IF}$$ Integrating Factor

Note if $z(x) = \sigma(x)y(x)$, where σ is given by $\boxed{IF}$ and y solves (I) then z solves (IV) (it's adjoint equation)

Ex: Consider, $xy'' + 2y' + a^2 xy = 0$, $x > 0$ — (*)

The equation is not exact since $P_0'' - P_1' + P_2 = a^2 x \neq 0$.

Now, $v(x) = \frac{1}{x} \exp\left(\int \frac{2}{x}\right) = x$.

and its adjoint equation is

$$xz'' - \cancel{\ \ } + a^2 x z = 0 \Rightarrow z'' + a^2 z = 0 \Rightarrow z(x) = c_1 \cos ax + c_2 \sin ax.$$

and hence, $y(x) = \frac{z(x)}{x} = \frac{c_1}{x} \cos ax + \frac{c_2}{x} \sin ax$

LAGRANGE IDENTITY :-

Define, $P(y) = p_0 y'' + p_1 y' + p_2 y = 0$ — (V)

and, $Q(\tilde{y}) = q_0 \tilde{y}'' + q_1 \tilde{y}' + q_2 \tilde{y} = 0$ — (VI)

Multiplying (V) with $\tilde{y}$ and (VI) with y we have,

$$\tilde{y}P(y) - yQ(\tilde{y}) = \tilde{y}(p_0 y'' + p_1 y' + p_2 y) - y(q_0\tilde{y}'' + q_1\tilde{y}' + q_2\tilde{y})$$

If $\tilde{y}$ satisfies the adjoint equation of (V) one has,

$$\left[\tilde{y}P(y) - yQ(\tilde{y})\right] = \tilde{y}(\underline{p_0 y''} + \underline{p_1 y'} + p_2 y) - y\left[\underline{(p_0\tilde{y})''} - \underline{(p_1\tilde{y})'} + p_2\tilde{y}\right]$$

$$= (p_0\tilde{y})y'' - (p_0\tilde{y})''y + (p_1\tilde{y})y' + (p_1\tilde{y})'y$$

$$= \left[(p_0\tilde{y})y' - (p_0\tilde{y})'y\right]' + \left[p_1 y\tilde{y}\right]'$$

$$\boxed{= \left[p_0(y'\tilde{y} - \tilde{y}'y) + (p_1 - p_0')y\tilde{y}\right]'} \Leftarrow \text{Lagrange Identity.}$$

Integrating above one gets,

$$\boxed{\int_\alpha^\beta \left[\tilde{y}P(y) - yQ(\tilde{y})\right] = \left[p_0(zy' - zy') + (p_1 - p_0')yz\right]\Big|_\alpha^\beta .} \Leftarrow \text{Green's Identity.}$$

Ex :- Solve :-

$$xy'' + (2x-1)y' + (x-1)y = x^2 + 2x - 2.$$

?

Theory of Oscillation :-

Consider the problem $(p(x)y')' + q(x)y = 0$ with $p, q \in C^{\infty}(I)$ and $p(x) > 0 \ \forall x \in I$. — (I)

Definition :- * A non-trivial solution $y(x)$ of (I) is called oscillatory if it has no last zero i.e, if $y(x_1) = 0$, $\exists \ x_2 > x_1$ s.t $y(x_2) = 0$.

* The equation is called oscillatory if every solution is oscillatory.

Ex :- (a) Consider $y'' + y = 0$. — (II)

G.S of (II) is given by $y(x) = c_1 \sin x + c_2 \cos x$; $c_1, c_2 \in \mathbb{R}$.

Clearly all solution of (II) is oscillatory and hence (II) is oscillatory equation in $[0, \infty)$.

(b) $y'' - y = 0$ — (III)

G.S of (III) is $y(x) = c_1 e^{x} + c_2 e^{-x}$ and hence (III) is non-oscillatory in $[0, \infty)$.

Th: (Sturm Comparison Theorem):-

If α, β are consecutive zeroes of a non-trivial solution $y(x)$ of $y''+q(x)y=0$ and if $q_1(x)$ is continuous and $q_1(x) \geq q(x)$; $q_1(x) \not\equiv q(x)$ in $[\alpha,\beta]$, then every non-trivial solution of $z''+q_1(x)z=0$ has a zero in (α,β).

Proof:- Let $y(x)>0 \ \forall x \in (\alpha,\beta)$, then $y'(\alpha)>0$ and $y'(\beta)<0$.

Multiply z by $y''+q(x)y=0$ and y with $z''+q_1(x)z=0$

$$zy''+q(x)yz - yz'' - q_1(x)yz = 0$$

$$\Rightarrow (zy'-yz')' + [q(x)-q_1(x)]yz = 0$$

$\because y(\alpha)=y(\beta)=0$ we have,

$$z(\beta)y'(\beta) - z(\alpha)y'(\alpha) + \int_\alpha^\beta [q(x)-q_1(x)]yz = 0 \Rightarrow -z(\alpha)y'(\alpha)+z(\beta)y'(\beta) = \int_\alpha^\beta [q_1(x)-q(x)]\,yz. \quad \text{— (IV)}$$

If $z(x) \neq 0 \ \forall x \in (\alpha, \beta)$ then $z(x) > 0$ (WLOG)

$\therefore$ From (IV), z can't have fixed sign in (α, β).

$$\left[\because\ z(\beta) y'(\beta) - z(\alpha) y'(\alpha) > 0 \Rightarrow y'(\alpha) z(\alpha) < y'(\beta) z(\beta) \Rightarrow \frac{z(\alpha)}{z(\beta)} < \frac{y'(\beta)}{y'(\alpha)} < 0. \right].$$

Cor. ~~Ex~~: If $q(x) \geq \frac{1+\epsilon}{4x^2}$, $\epsilon > 0 \ \forall x > 0$, then $y'' + q(x) y = 0$ is oscillatory in $[0, \infty)$.

Proof: For every $\epsilon > 0$, $y'' + \frac{\epsilon}{4} y = 0$ is oscillatory

Let $t = e^x$ in above equation, $x = \ln t$

Define, $v(t) = y(x)$

$$\Rightarrow v'(t) = y'(x) \frac{dx}{dt} = y'(x) \cdot \frac{1}{t} \Rightarrow t v'(t) = y'(x)$$

$$\Rightarrow v''(t) = y''(x) \frac{1}{t^2} - \frac{1}{t^2} y'(x) = \frac{y''(x) - y'(x)}{t^2}$$

$$\Rightarrow t^2 v''(t) = y''(x) - y'(x) \Rightarrow y''(x) = t^2 v''(t) + t v'(t)$$

$\therefore\ t^2 v''(t) + t v'(t) + \varepsilon/4\, v(t) = 0$ —— $(*)$

Substitute, $v(t) = \varphi(t)/\sqrt{t}$ we have,

$\Rightarrow \varphi(t) = \sqrt{t}\, v(t) \Rightarrow \varphi'(t) = \sqrt{t}\, v'(t) + \frac{1}{2} t^{-\frac{1}{2}} v(t)$

and, $\varphi''(t) = \sqrt{t}\, v''(t) + \frac{1}{2} t^{-\frac{1}{2}} v'(t) + \frac{1}{2} t^{-\frac{1}{2}} v'(t) - \frac{1}{4} t^{-\frac{3}{2}} v(t)$

$\Rightarrow t^{3/2} \varphi''(t) = t^2 v''(t) + t v'(t) - \frac{1}{4} v(t) = -\varepsilon/4 - 1/4\, v(t) = -\frac{1+\varepsilon}{4} v(t)$

$\Rightarrow \varphi''(t) + \frac{1+\varepsilon}{4t^{3/2}} v(t) = 0 \Rightarrow \varphi''(t) + \frac{1+\varepsilon}{4t^2} \varphi(t) = 0$ —— $(**)$

$(**)$ is oscillatory.

Hence, $y'' + q(x) y = 0$ is oscillatory if $q(x) \geq \frac{1+\varepsilon}{4t^2}$

$\boxtimes$

Or in other words, any solution of y''+q(x)y=0 where q(x)<=0 in I can have atmost one zero in I

Ex:- The DE $y''=0$ is non oscillatory. Thus if $q(x)\le 0$ $(\not\equiv 0)$ in I, then no solution of $y''+q(x)y=0$ can have more than one zero in I.

Ex:- The DE $y''+y=0$ is oscillatory $\Rightarrow$ $y''+(1+\epsilon)y=0$ is oscillatory in $[0,\infty)$.

Picone Identity:- Let p, p_1, y and z are continuously differentiable with $z(x)\neq 0$ $\forall\, x\in I$. Then the following identity holds:-

$$\left[\frac{y}{z}\left(zpy'-yp_1z'\right)\right]' = y\left(py'\right)' - \frac{y^2}{z}\left(p_1z'\right)' + \left(p-p_1\right)y'^2 + p_1\left(y'-\frac{y}{z}z'\right)^2.$$

Sturm-Picone theorem:- If α, β are consecutive zeroes of I of a non-trivial solution $y(x)$ of $(p(x)y')'+q(x)y=0$ and if p_1 and q_1 are continuous and $0<p_1(x)\le p(x)$; $q_1(x)\ge q(x)$ in $[\alpha,\beta]$ then every non-trivial soln $z(x)$ of $(p_1(x)z)'+q_1(x)z=0$ has a zero in $[\alpha,\beta]$.

Proof :- Let $z(x) \neq 0 \ \forall x \in [\alpha, \beta]$, then by Picone Identity,

$$\left[\frac{y}{z}\left(zpy' - yp_1 z'\right)\right]' = (q_1 - q)y^2 + (p - p_1)y'^2 + p_1\left(y' - \frac{y}{z}z'\right)^2$$

Integrating the above relation with $y(\alpha) = y(\beta) = 0$

$$\Rightarrow \int_\alpha^\beta \left[(q_1 - q)y^2 + (p - p_1)y'^2 + p_1\left(y' - \frac{y}{z}z'\right)^2\right]dx = 0$$

- which is a contradiction unless, $q_1 = q$, $p = p_1$ and $y' = \frac{y}{z}z'$

Now, $y' = \frac{y}{z}z' \Rightarrow y = (\text{constant})\,z$

$\because y(\alpha) = 0 \Rightarrow y(x) \equiv 0$

Hence, z must admit a zero in $[\alpha, \beta]$.

Sturm Seperation theorem :- If y_1 and y_2 are two linearly independent solutions of $(py')' + qy = 0$ in $I = [\alpha, \beta]$, then their zeroes are interlaced.

Proof :- Note that if $\exists\, x_0 \in [\alpha, \beta]$ s.t $y_1(x_0) = y_2(x_0) = 0$ then

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$$

$\Rightarrow W(y_1, y_2)(x) = 0 \;\; \forall\, x \in [\alpha, \beta]$ (By Abel's theorem)

$\Downarrow$

y_1 and y_2 are linearly dependent and hence a contradiction.

$\therefore$ y_1 and y_2 can not have common zero, between any two zero of y_1, $\exists$ zero of y_2 (By S-P theorem).

Also, if x_0, x_1 are consecutive zeroes of y_2 between any two consecutive zeroes of y_1, then we have a contradiction.

Theorem :- The only solution of $(py')' + qy = 0$ which vanishes infinitely often in $I = [\alpha, \beta]$ is the trivial solution.

Proof :- Let $\{x_m\}$ be a sequence in $[\alpha, \beta]$ such that $y(x_m) = 0 \ \forall m \in \mathbb{N}$.

Clearly, $x_m \to x_0$ (upto a subsequence)

We show, $y'(x_0) = 0$

$$\because \; y'(x_0) = \lim_{m \to \infty} \frac{y(x_m) - y(x_0)}{x_m - x_0} = \lim_{m \to \infty} \frac{0 - 0}{x_m - x_0} = 0$$

$\therefore$ y solves $(py')' + qy = 0$; $y(x_0) = y'(x_0) = 0$

$\Rightarrow y \equiv 0$ (By Uniqueness). ∎

Conclusion :- Any non-trivial solution of the equation $(py')' + qy = 0$ has atmost finite no of zeroes in $[\alpha, \beta]$.

TEST FOR $(py')' + qy = 0$ TO BE OSCILLATORY :-

LEIGHTON OSCILLATION THEOREM :- If $\int^{\infty} 1/p(x)\, dx = \infty$ and $\int^{\infty} q(x)\, dx = \infty$ then $(py')' + qy = 0$ — (*)

is oscillatory in $(0, \infty)$.

Proof :- Let $y(x)$ be the non-oscillatory solution of (*) which we assume to be positive in $[x_0, \infty)$; $x_0 > 0$.

then from (#) we have,

$$z' + q(x) + \frac{z^2}{p(x)} = 0 \quad \text{(RICCATI EQUATION)}$$

has a solution $z(x)$ in $[x_0, \infty)$ and satisfy

$$z(x) = z(x_0) - \int_{x_0}^{x} q(s)\, ds - \int_{x_0}^{x} \frac{z^2(s)}{p(s)}\, ds$$

$\because \int^{\infty} q(s)\, ds = \infty$ we may find $x_1 > x_0$ s.t. $z(x_0) - \int_{x_0}^{x} q(s)\, ds < 0 \quad \forall x \in [x_1, \infty)$

$$\Rightarrow z(x) < -\int_{x_0}^{x} \frac{z^2(s)}{p(s)}\, ds \quad \forall x \in [x_1, \infty]$$

Define, $w(x) := \int_{x_0}^{x} \frac{z^2(t)}{p(t)}\,dt$, Then $z(x) < -w(x)$ and,

$$\therefore\ w'(x) = \frac{z^2(x)}{p(x)} > \frac{w^2(x)}{p(x)}. \quad \forall\, x \in [x_1, \infty)$$

Integrating, $\int_{x_1}^{\infty} \frac{w'(x)}{w^2(x)}\,dx > \int_{x_1}^{\infty} \frac{dx}{p(x)}$

$$\Rightarrow -\frac{1}{w(\infty)} + \frac{1}{w(x_1)} > \int_{x_1}^{\infty} \frac{dx}{p(x)}$$

$$\Rightarrow \int_{x_1}^{\infty} \frac{dx}{p(x)} < \frac{1}{w(x_1)}$$

— a contradiction.

(#) RICCATI EQUATION :-

$(py')' + qy = 0$ has a solution without zeroes in $[\alpha, \beta]$ iff the Riccati Eqn has a solution defined in $[\alpha, \beta]$.

Proof :- Let $z' + q(x) + \frac{z^2}{p(x)} = 0$ has a solution in $[\alpha, \beta]$, define $py' = zy$ where y solves $(py')' + qy = 0$ and does have a zero at $x_0 \in [\alpha, \beta]$.

$\therefore (zy)' + qy = 0 \Rightarrow zy' + z'y + qy = 0$

$\Rightarrow zy' - \left(q + \frac{z^2}{p}\right)y + qy = 0$

$\Rightarrow zy' - qy - \frac{z^2}{p}y + qy = 0.$

$\Rightarrow zy' - \frac{z^2}{p}y = 0 \Rightarrow y' - \frac{1}{p}y = 0$

Clearly, y is of exponential type and hence doesnot admit a zero.

Conversely, let $y(x) > 0 \ \forall x \in [a, b]$ and solves $(py')' + qy = 0$.

RTP:- The Ricati Eqn has a soln.

Define, $py' = zy$

$$\therefore z' + q + \frac{z^2}{p}$$

$$= \left[\frac{py'}{y}\right]' + q + \frac{1}{p}\left[\frac{py'}{y}\right]^2$$

$$= \frac{y(py')' - y'(py')}{y^2} + q + \frac{1}{\cancel{p}} \cdot \frac{p^{\cancel{2}} y'^2}{y^{2}}$$

$$= -q - \frac{py'^2}{y^2} + q + \frac{py'^2}{y^2} = 0.$$

$\boxed{\square}$

$\therefore z$ satisfies $z' + q + \frac{z^2}{p} = 0$.