Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Complex analysis (MTH 403)

Quiz 4

Date: 06 Nov 2023

Time: 30 minutes

Total marks: 10

Name:	Marks obtained ↓
Roll number:	

Answer to any question must be written within the space provided below the same. Extra sheets can be used for rough works only.

1. Let $f : \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \longrightarrow \mathbb{D}$ be holomorphic. Show that $|f'(z)| \leq \dfrac{1 - |f(z)|^2}{2 \operatorname{Re} z}$, whenever $\operatorname{Re} z > 0$.

[4]

Consider

$$\mathbb{D} \to \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \xrightarrow{f} \mathbb{D}$$

$$w \mapsto \frac{1-w}{1+w} \mapsto f\left(\frac{1-w}{1+w}\right).$$ Clearly this map is holomorphic.

Now, $\forall\, w \in \mathbb{D}$, as a consequence of the Schwarz lemma it follows that

$$\left|f'\left(\frac{1-w}{1+w}\right)\right| \leq \frac{1 - \left|f\left(\frac{1-w}{1+w}\right)\right|^2}{1-|w|^2} \cdot \frac{|1+w|^2}{2} \quad \cdots (*)$$

Observe that, $\forall\, w \in \mathbb{D}$

$$\operatorname{Re}\left(\frac{1-w}{1+w}\right) = \operatorname{Re}\left(\frac{(1-w)(1+\bar{w})}{|1+w|^2}\right) = \operatorname{Re}\left(\frac{1-|w|^2}{|1+w|^2} - \frac{2i \operatorname{Im} w}{|1+w|^2}\right)$$

$$= \frac{1-|w|^2}{|1+w|^2} \quad \cdots (**)$$

From $(*)$ & $(**)$ the conclusion follows, in view of the bijection $\mathbb{D} \to \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

$$w \mapsto \frac{1-w}{1+w}$$

Please go on to the next page...

2. Prove or disprove the following statement: let $U \subseteq_{open} \mathbb{C}$ and $\mathscr{F} \subseteq H(U)$ be relatively compact. Then $\mathscr{F}$ is uniformly bounded on every compact subset of U.

[2]

From the compactness criterion, $\overline{\mathcal{F}}$ is uniformly bounded on compact subsets of U, i.e.,

$\forall K \underset{cpt.}{\subseteq} U$, $\exists M_K > 0$ s.t. $\forall f \in \overline{\mathcal{F}}$,

$$|f(z)| \leq M_K, \ \forall z \in K.$$

As $\mathcal{F} \subseteq \overline{\mathcal{F}}$, $\mathcal{F}$ is clearly uniformly bounded on every compact subset of U.

3. (a) The Möbius transformations on $\hat{\mathbb{C}}$ that fix 0 and ∞ only are precisely all maps of the form $z \mapsto \lambda z$, where $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

(Fill in the blank only, No justification is required.)

(b) Find all Möbius transformations on $\hat{\mathbb{C}}$ that interchange 0 and ∞.

[2+2=4]

Let f_A be a Mobius transformation, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$. Then one must have $c \neq 0$, otherwise $f_A(\infty) = \infty$. Now since $f_A(\infty) = \frac{a}{c} = 0$, so $a = 0$. Similarly ~~$0 = f$~~ $f_A(0) = \infty \Rightarrow -\frac{d}{c} = 0 \Rightarrow d = 0$. Hence f_A is of the form: $f_A(z) = \frac{\lambda}{z}$, where $\lambda \in \mathbb{C} \setminus \{0\}$.

Conversely, it is easy to see that any map of the form $z \mapsto \lambda/z$, where $\lambda \neq 0$ interchanges 0 & ∞.