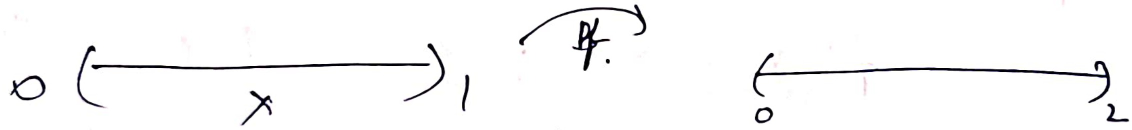
Next lect: Homotopy theory & fundamental Group

Munkers topology Chapter - 9

Recall:- Two topological spaces (subsets of $\mathbb{R}^n$), X & Y are said to be homeomorphic If there is a map $f : X \longrightarrow Y$ s.t., f is 1-1, onto; f is continuous, f^{-1} is continuous. Then X & Y are homeomorphic (topologically equivalent) $X \cong Y$.

Example:- (1) $X = (0,1)$, $Y = (0,2)$.



$$f(x) = 2x$$

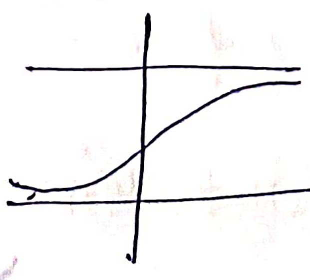
$$f^{-1}(y) = y/2.$$

as metric they are not equivalent. but as topologically they are equivalent.

ex $X = (0,1)$, $Y = R$.

Define $f : \mathbb{R} \longrightarrow (0,1)$ by

$$f(x) = \frac{e^x}{1+e^x}$$



$$f'(x) = 1+ \frac{e^x}{(1+e^x)^2}$$

monotonically increasing.

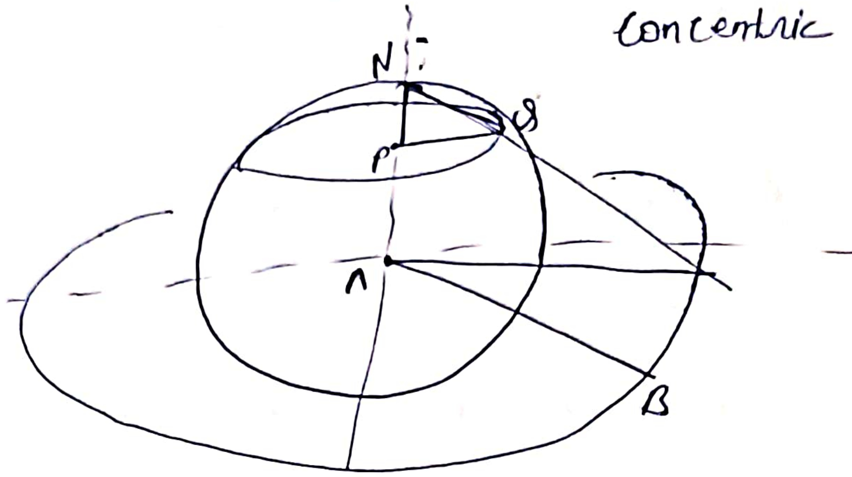
3. $X = S^2 \setminus N$ $\quad S^2 = \{x^2+y^2+z^2=1\}$.

$Y = R \times R$ $\quad N = (0,0,1)$

$\frac{r}{1} = \frac{r'}{1-z}$ $\quad \theta = \theta'$

$x = \frac{x'}{1-z}, \; y = \frac{y'}{1-z}$

concentric



Execise:-

Show $S^2 \setminus N$ is homeomorphic to $R \times R$.

Note:- (1) If $X \cong Y$ we need to find a map

$f: X \to Y$; f is 1-1 onto, Cont

with Continuous inverse.

(2) Suppose $X \ncong Y$ we need to distinguish between then by topological invariants.

Defn:- Given a topological space X, a topological invariant is a quantity $\mu(X)$ Can be a (property, number gp etc)

s.t. if $X \overset{\text{homomeomic}}{\cong} Y \Rightarrow \mu(X) = \mu(Y)$.

If $\mu(X) \neq \mu(Y) \Rightarrow X \ncong Y$

Example:- (1) If $X = (0,1)$ $Y = [0,1]$

If $X \cong Y$? Not homeomorphic since X is not compact Y is compact

$\mathcal{M}(X) :=$ compactness.

(2) $X = (0,1)$ $Y = (0,1) \cup (1,2)$

Is $X \cong Y$?

$\mathcal{M}(X) :=$ connectedness

X is connected Y is not connected.

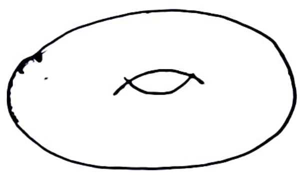
(3) $X = \mathbb{R}^2$; $Y = \mathbb{R}^3$ If $X \cong Y$?

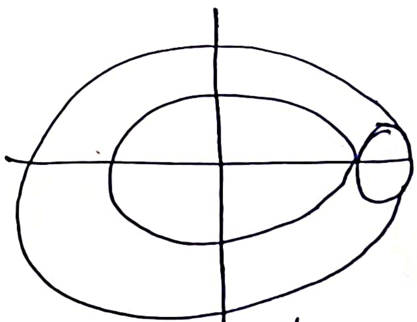
Needs additional topological invariant ?

(3') $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ $m \neq n$, $m, n > 2$.

(4) $X = S^2$, $x^2 + y^2 + z^2 = 1$

$Y = S^1 \times S^1$

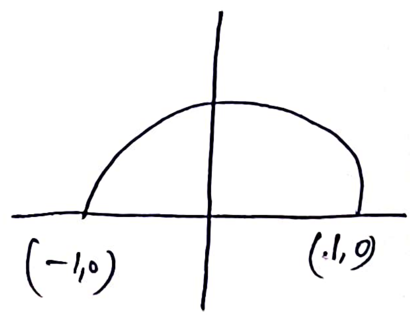




Defn:- (Path Homotopy) :- A path is a continuous map $f : [0,1] \longrightarrow X$

$f : [0,1] \longrightarrow S^1$

$f(s) = (\cos \pi s, \sin \pi s)$



Two paths $\alpha : [0,1] \to X$ are said to
$\beta : [0,1] \to X$
be path homotopic iff

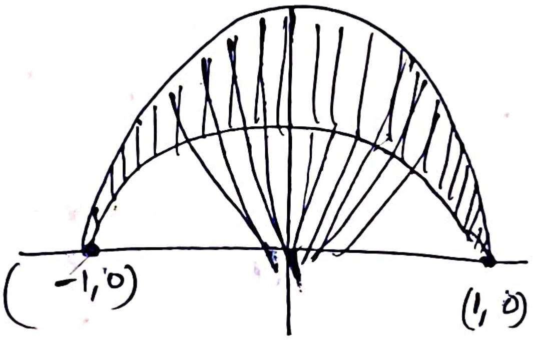
$$\alpha(0) = \beta(0) \qquad \alpha(1) = \beta(1).$$

$H : [0,1] \times [0,1] \to X$ s.t.

$H(S, 0) = \alpha(S)$ $(\cos \pi s, \sin \pi s)$

$H(S, 1) = \beta(S)$. $\beta(s) = (\cos \pi s, 2 \sin \pi s)$

$$H(s,t) = (1-t)\,\alpha(s) + t\,\beta(s).$$



$H(s, 0) = \alpha(s)$

$H(s,1) = \beta(s)$.

section 51, problems 1, 2, 3.

Next lecture :-

Fundamental group :- Munkers 'Topology' - chapter 9.

we defined :- Path Homotopy ; $[0,1] = I$.

$\alpha : [0,1] \longrightarrow X$; Continuous maps are

$\beta : [0,1] \longrightarrow X$

said to be path homotopic if $\exists$ a continuous map

$H : I \times I \longrightarrow X$

H continuous s.t. $H(s,0) = \alpha(s)$ and

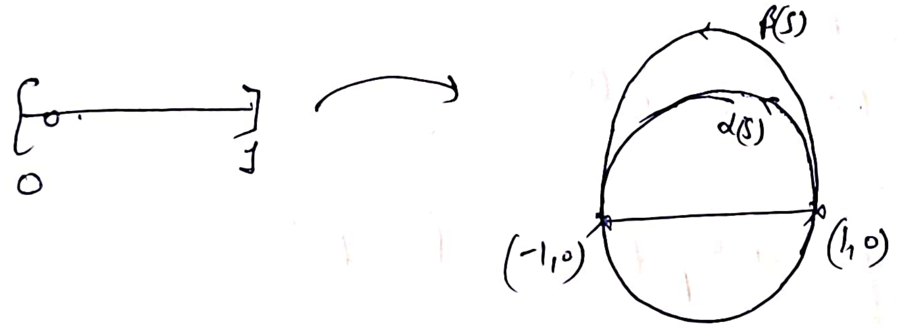
$H(s,1) = \beta(s)$

$H(0,t) = \alpha(0) = \beta(0) = x$ and

$H(1,t) = \alpha(1) = \beta(1) = y$

$\alpha(s) = (\cos \pi s, \sin \pi s)$ $\alpha : [0,1] \longrightarrow S^1$

$\beta(s) = (\cos \pi s, 2 \sin \pi s)$ $\beta : [0,1] \longrightarrow S^1$



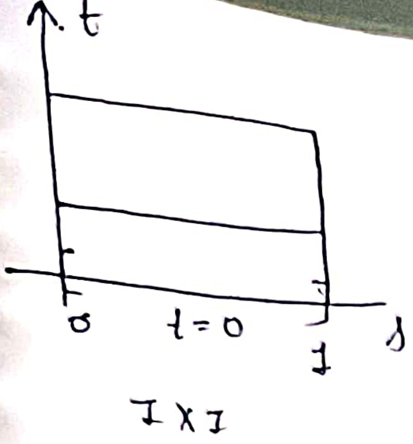
Straight line Homotopy

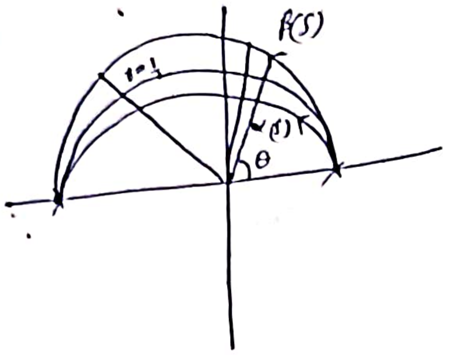
$H(s,t) : I \times I \longrightarrow S^1$

$H(s,t) := (1-t)\,\alpha(s) + t\,\beta(s)$

$H(s,0) = \alpha(s)$

$H(s,1) = \beta(s)$.





Path homotopy
$\alpha \simeq_p \beta$.

<u>Claim</u>:- Path Homotopy is an equivalence relation on the set of all paths $\alpha : [0,1] \to X$ s.t.

$$\alpha(0) = x, \quad \alpha(1) = y$$

<u>Proof</u>:- Given paths $\alpha : [0,1] \to X$ s.t.
$\beta : [0,1] \to X$

$$\alpha(0) = x = \beta(0), \quad \alpha(1) = \beta(1) = y$$

$$H : I \times I \to X \text{ s.t. } H(s,0) = \alpha(s)$$
$$H(s,1) = \beta(s).$$

(1) Reflexive :-

$$\alpha \simeq_p \alpha.$$

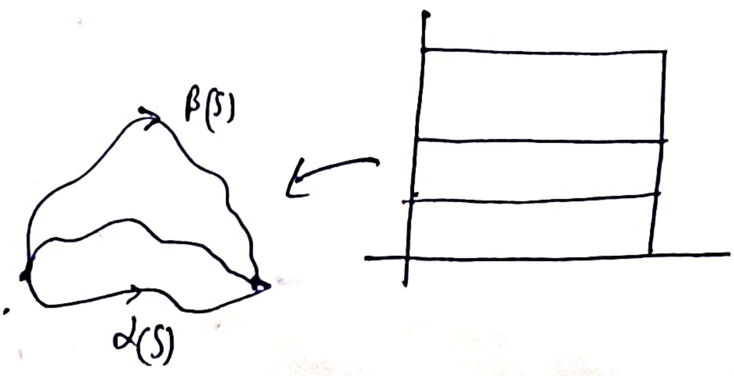
$$H(s,t) = \alpha(s).$$

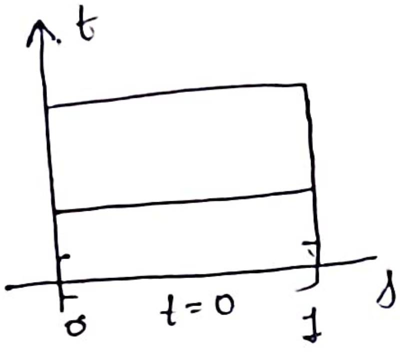
2. Symmetric If $\alpha \simeq_p \beta \Rightarrow \beta \simeq_p \alpha$.

$$H : I \times I \to X; \quad H(s,0) = \alpha(s); \; H(s,1) = \beta(s)$$

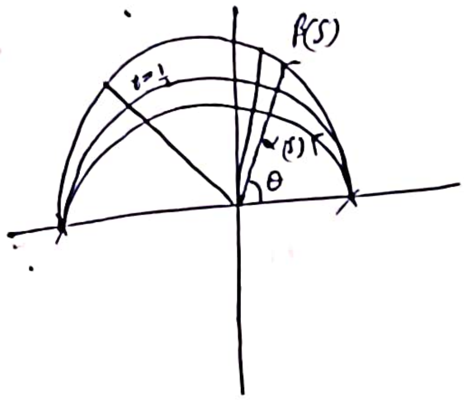
$H : I \times I \to X$

$$H'(s,t) = H(s, 1-t)$$





I X I



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$\beta : [0,1] \to X$

$$\alpha(0) = x = \beta(0), \quad \alpha(1) = \beta(1) = y$$

$H : I \times I \to X$ s.t. $H(s,0) = \alpha(s)$
$H(s,1) = \beta(s)$.

(1) Reflexive :- $\alpha \simeq_p \alpha$.

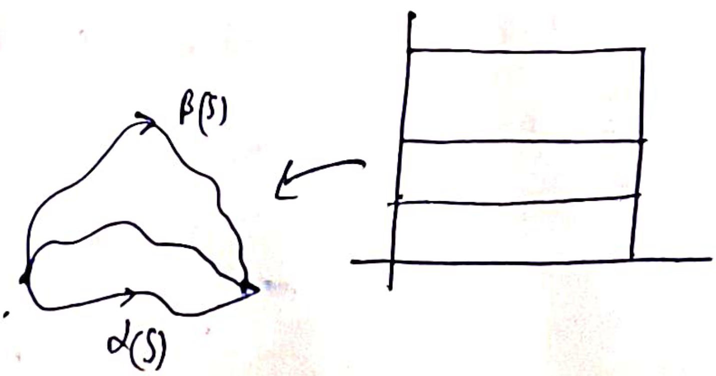
$$H(s,t) = \alpha(s).$$

2. Symmetric iff $\alpha \simeq_p \beta \Rightarrow \beta \simeq_p \alpha$.

$H : I \times I \to X$; $H(s,0) = \alpha(s)$; $H(s,1) = \beta(s)$

$H : I \times I \to X$

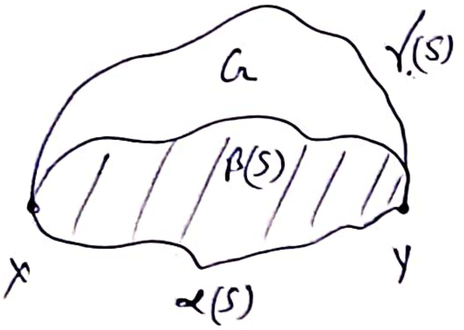
$H'(s,t) = H(s, 1-t)$



3. <u>Transitive</u>:- If $\alpha \simeq_p \beta$, $\beta \simeq_p \gamma \Rightarrow \alpha \simeq_p \gamma$.

$F : I \times I$

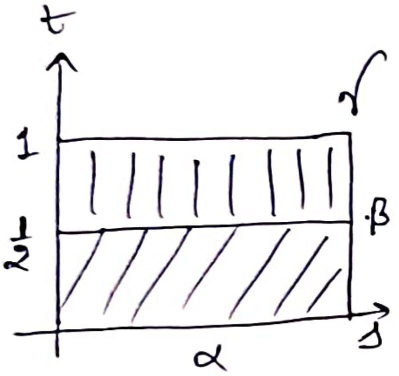
$$F(s,0) = \alpha(s) \qquad G(s,0) = \beta(s)$$
$$F(s,1) = \beta(s) \qquad G(s,1) = \gamma(s).$$



$$H(s,t) = F(s, 2t) \; ; \quad 0 \le t \le \tfrac{1}{2}.$$
$$= G(s, 2t-1) \qquad \tfrac{1}{2} \le t \le 1.$$

$[\alpha]$ denotes the path homotopy class of the path α.

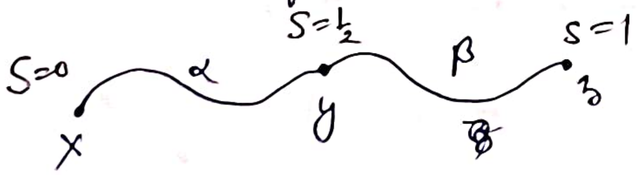
$G(s, 2(t-1))$,



<u>Group Composition of paths</u>:-

Let $\alpha : [0,1] \longrightarrow X$ be a continuous path $\alpha(0) = x$, $\alpha(1) = y$ and let $\beta : [0,1] \to X$ be a continuous path s.t. $\beta(0) = \alpha(1) = y$.

then we form a product $[\alpha] \times [\beta]$



$$[\alpha] \times [\beta] = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Claim :- $([\alpha], *)$ has the following "Groupoid's" properties:

(1) identity :-

$$\alpha : [0,1] \to X \quad \text{s.t.}$$

$$\alpha(0) = x, \quad \alpha(1) = y.$$

Let $e_y : [0,1] = y$.

$$[\alpha] * [e_y] = [\alpha]$$

(2) Let $e_x : [0,1] \to X$

$$[e_x] * [\alpha] = [\alpha].$$

(2) inverse :-

Claim :- $[\alpha] * [\bar{\alpha}] = [e_x]$

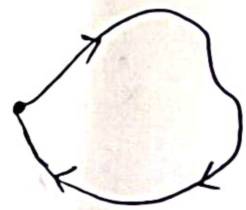
$[\bar{\alpha}] * [\alpha] = [e_y]$.

(exercise).

Let $\alpha^{-1} = \bar{\alpha} = \alpha(1-s)$.

(3) associative :- $([\alpha]*[\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$.

Def^n :- a loop $\alpha : [0,1] \to X$ is a path continuous map s.t $\alpha(0) = x = \alpha(1)$



Claim:- $([\alpha], *)$ has the group prop. $\pi_1(X, x)$.

(1) $[\alpha] * [e_x] = [e_x] * [\alpha] = [\alpha]$ (identity)

(2) $[\alpha] * [\bar{\alpha}] = [e_x] = [\bar{\alpha}] * [\alpha]$ (inverse).

(3) $\left([\alpha] * [\beta]\right) * \left([\gamma]\right) = [\alpha] * \left([\beta] * [\gamma]\right)$.

Next lecture:-

∴ The fundamental group:-

Note:- The set of path homotopy classes does not form a group but a "groupoid"

left ~~inverse~~ identity e_x

we pick the special point $x_0 \in X$ (basepoint)

$e_x : [0,1] \longrightarrow X$

Right ~~inverse~~ identity $e_y : [0,1] \longrightarrow X$

$\alpha : [0,1] \longrightarrow X$

$\alpha(0) = \alpha(1) = x_0$. 

$e_x \neq e_y$.

Munkers section 52 (chapter 9)

Defn:- (Fundamental group $\pi_1(X, x_0)$) Let X be a space (Subset of $\mathbb{R}^n$) ; $x_0 \in X$ a path

$\alpha : [0,1] \longrightarrow X$ s.t. $\alpha(0) = \alpha(1) = x_0$ is called a loop based at x_0.

$[\alpha]$ be the path homotopy class of α, the set of all path homotopy classes ~~α~~ of loops based ~~with~~ at x_0, with the composition $*$ is called the fundamental group of X based at x_0 $\pi_1(X, x_0)$.

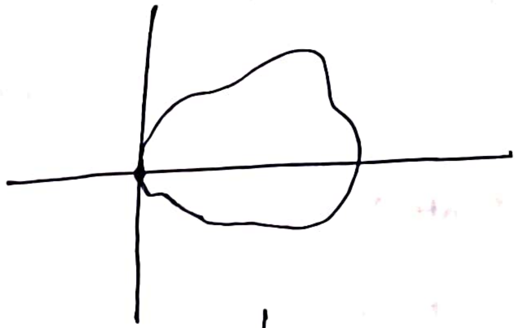
(1) $[f][g] \in \pi_1(X, x_0) \Rightarrow [f * g] \in \pi_1(X, x_0)$ (closure)

(2) $[f] * [e_{x_0}] = [e_{x_0}] * [f] = [f]$ (Identity).

(3) $([f] * [g]) * [h] = [f] * ([g] * [h])$ (associative)

(4) $[f] * [\bar{f}] = [e_{x_0}]$ (Inverse).

<u>Ex</u> $\pi_1(\mathbb{R}^n, x_0)$

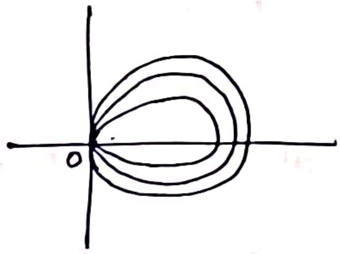


$\pi_1(\mathbb{R}^n, 0) = 0 = e$.

$H(x, t) = (1-t)x$

$t = 0$; $H(x,t) = x$

$t = 1$; $H(x, t) = 0$



$H(x, t)$ is shrinking the loop α to the origin via st. lines.

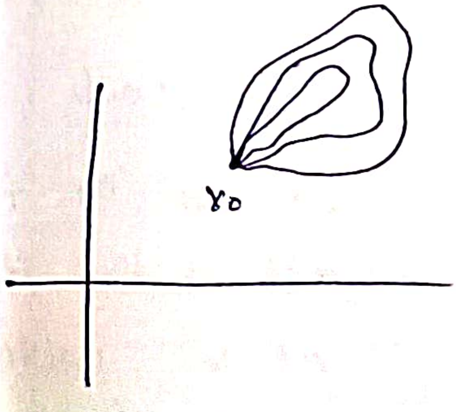
In particular,

$$H(s, t) = (1-t)\,\alpha(s)$$

$t = 0$; ~~$H = \alpha(s)$~~ $\quad H(s, 0) = \alpha(s)$

$H(s, 1) = 0$

$\pi_1(\mathbb{R}^n, x_0)$



$\pi_1(\mathbb{R}^n, x_0) = e$.

$H(x, t) = (1-t)\,\alpha(s) + t\,x_0$

Ex 2. $C \subset \mathbb{R}^n$; C convex

$c_0 \in C$; $\pi_1(C, c_0)$.

If it not convex. we could use straight line homotopy.

$$H(x,t) = (1-t)\,\alpha(x) + t\,c_0.$$

α is path homotopic to $\{e\}$ at c_0.

Problem 1:-

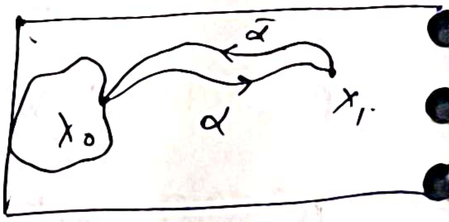
#1 $A \subset \mathbb{R}^n$ is star convex. if for some $a_0 \in A$; the line $a_0 a$ lines in A $\forall$ $a \in A$

(a) Find a star convex set which is not convex.

(b) $\pi_1(A, a_0) = e$ using st. line homotopy.

Thm:- X is space $\subset \mathbb{R}^n$; let $x_0, x_1 \in X$, Let X be path connected. Then

$$\pi_1(X, x_0) \overset{\text{isomorphic}}{\cong} \pi_1(X, x_1)$$



Proof Let α be a path in X s.t.

$\alpha(0) = x_0$ $\quad$ $\alpha(1) = x_1$

Let $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$$\hat{\alpha}[f] = [\bar{\alpha}] * [f] * [\alpha]$$

Claim:- $\hat{\alpha}$ is a group homo a group iso.

$\phi : G_1 \longrightarrow G_2$

ϕ is group homo. $\phi(x+y) = \phi(x) + \phi(y)$. $\forall x, y \in G$.

iso:- ϕ is 1-1 & onto.

$\hat{\alpha}[f] * \hat{\alpha}[g]$.

$\left([\bar{\alpha}] * [f] * [\alpha]\right) * \left([\bar{\alpha}] * [g] * [\alpha]\right)$

$[\bar{\alpha}] * [f] * \left([\alpha]\,[\bar{\alpha}]\right) * [g] * [\alpha]$

$[\bar{\alpha}] * [f] * [g] * [\alpha]$.

ϕ; ψ · gp homo.

$\psi : G_1 \longrightarrow G_2$

$\phi : G_2 \longrightarrow G_1$

$\psi \circ \phi = Id$

$\phi \circ \psi = Id$.

then $\psi \cong \phi$.

$\hat{\beta} : \Pi_1(X, x_1) \longrightarrow \Pi_1(X, x_0)$

~~$\hat{\beta}[g]$~~ Let $\beta = \bar{\alpha} = \alpha(1-s)$

$\hat{\beta}[g] = [\bar{\beta}] * [f] * [\beta]$.

Similarly

$\hat{\beta} * \hat{\alpha}[f] = [f]$

Show $\hat{\alpha} * \hat{\beta} = id$.

$\hat{\beta} * \hat{\alpha} = id$.

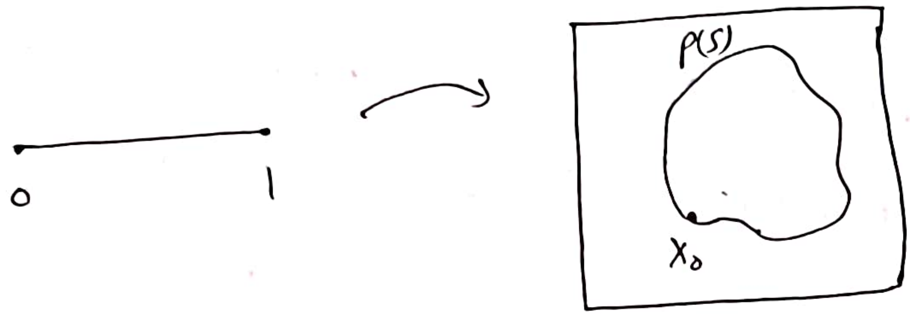
$\hat{\alpha} * \hat{\beta}([g]) = \hat{\alpha}\left([\bar{\beta}] * [g] * [\beta]\right)$

$= [\bar{\alpha}]\left([\bar{\beta}] * [g] * [\beta]\right) * [\alpha]$

$= [\bar{\alpha}]\left([\alpha] * [g] * [\bar{\alpha}]\right)[\alpha] = [g]$.

Munkers - Chapter - 9.

Defn:- Let X be a space (subset of R^n) let $x_0 \in X$. a path in X that begins and ends at X_0, is called a loop based at X_0, we denote its path homotopy class.



$[P]$ the set $\pi_1(X, X_0) = \{ [P] \mid P$ is a loop based at $X_0 \}$

$$P * q = \begin{cases} P(2s) & : 0 \le s \le \frac{1}{2} \\ q(2s-1) & : \frac{1}{2} \le s \le 1 \end{cases}.$$

$e_{x_0} = x_0$, $\quad P^{-1}(S) = P(1-S)$

then (1) $[P], [q] \in \pi_1(X_0, x_0)$

$[P] * [q] \in \pi_1(X, X_0)$

(2) $[P] * [e_{x_0}] = [P] = [e_{x_0}] * [P]$ (identity)

(3) $[P] * [P^{-1}] = [e_{x_0}]$

(4) $([P] * [q]) * [r] = [P] * ([q] * [r])$

Properties:- of $\pi_1(X, x_0)$.

Thm (1) - Let X be a space (subset of $\mathbb{R}^n$) Let X_0 be path - connected . Let $x_0, y_0 \in X$.

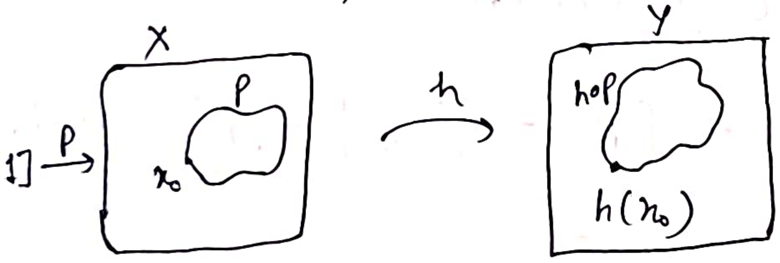
$\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, y_0)$.

Thm (2) $\pi_1(X, x_0)$ is a topological invariant i.e. if X is homeomorphic to Y then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$.

Let $h : X \longrightarrow Y$ be a homeomorphism.

h is continuous function from X to Y s.t. h is 1-1 ; onto & h^{-1} is cts.

Let $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, h(x_0))$.



$h_*[p] = [h \circ p]$

h_* is called the gp homo. associated with h.

(i) h_* is homomorphism. h_* i

$$h_*([p] * [q]) = h_*[p] * h_*[q]$$
$$= [h \circ p] * [h \circ q]$$

$$[h \circ p] * [h \circ q] = \begin{cases} h \circ p\,(s) & 0 \le s \le \frac{1}{2} \\ h \circ q\,(s) & \frac{1}{2} \le s \le 1. \end{cases}$$

$$h_* \left([p] * [q] \right) = \begin{cases} h \circ p\,(2s) & 0 \le s \le \frac{1}{2} \\ h \circ q\,(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

$$G_1 \xrightarrow{\phi_1} G_2 \qquad G_2 \xrightarrow{\phi_2} G_1$$

$$\phi_2 \circ \phi_1 = Id, \qquad \phi_1 \circ \phi_2 = Id.$$

$$\Rightarrow \quad G_1 \cong G_2.$$

<u>Next</u> <u>lecture</u> :-

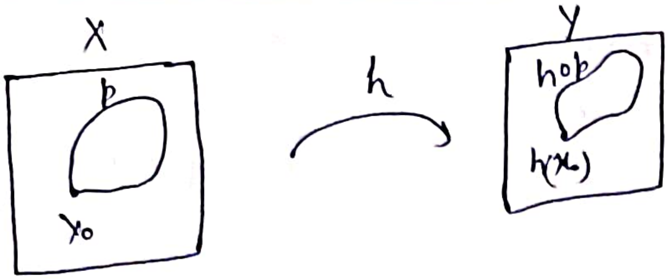
Defn:- A space X is said to be simply connected if it is <u>path connected</u> and if <u>$\pi_1(X, x_0) = e$</u>. for some $x_0 \in X$.

Ex $(\mathbb{R}^n, 0)$

Thm:- $\pi_1(X)$ is a topological invariant i.e.
If X is homeomorphic to $Y \Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, y_0)$

Proof :- Let $h : X \longrightarrow Y$ be a homeomorphism.
h is 1-1, onto, continuous & h^{-1} is cts.

we defined $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, h(x_0))$ is Called the ~~homeomorphism~~ homomorphism induced by h.



$$h_*[f] = [h \circ f].$$

h_* has 2-basic properties "functional properties".

• **Thm** If $h : (X, x_0) \longrightarrow (Y, y_0)$

$k : (Y, y_0) \longrightarrow (Z, z_0)$

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

$$k_* : \pi_1(Y, y_0) \longrightarrow \pi_1(Z, z_0)$$

then (a) $(k \circ h)_* = k_* \circ h_*$.

(b) $i : X \longrightarrow X$ then $(id)_* = id$.

Proof :

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "(k\circ h)_*"'] & \pi_1(X, h(x_0)) \arrow[d, "k_*"] \\
& \pi_1(Z, k\circ h(x_0))
\end{tikzcd}$$

Let $f \in \pi_1(X, x_0)$

$$h_*[f] = [h \circ f]$$

$$k_* \circ h_*[f] = [k \circ [h \circ f]]$$

$$(k \circ h)_*[f] = [k \circ h[f]] = (k \circ [h[f]])$$

Proof (b) $i : X \longrightarrow X$.

$$i_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$$

$$i_*[f] = [id \circ f] = [f].$$

$$i_* = id.$$

Th^m:- $\pi_1(X, x_0)$ is a topo. invariant.

Let $h : X \longrightarrow Y$ be a homeomorphism (h is 1-1, onto cts h^{-1} cts.)

$\cancel{h_* : \pi_1(X, x_0)}$

(1)
$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "Id."'] & Y \arrow[r, "h^{-1}"] & X \end{tikzcd}$$

(2)
$$\begin{tikzcd} Y \arrow[r, "h^{-1}"] \arrow[rr, bend right, "id."'] & X \arrow[r, "h"] & Y \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[rr, bend right] & \pi_1(Y, h(x_0)) \arrow[r, "(h^{-1})_*"] & \pi_1(X, x_0) \end{tikzcd}$$

$$(h^{-1} \circ h)_* = (id)_* = id.$$

$$h_*^{-1} \circ h_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$$

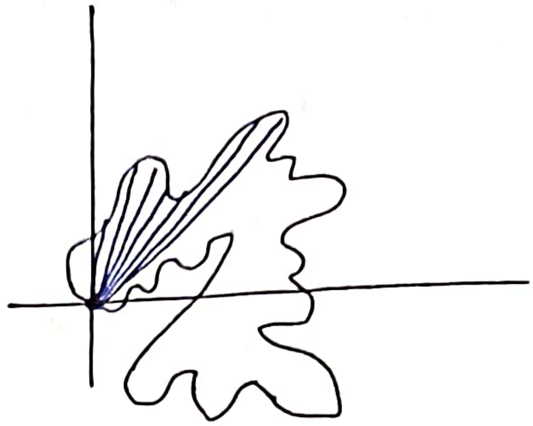
$$h_*^{-1} \circ h_* = id.$$

$$h_* \circ h_*^{-1} : \pi_1(Y, y_0) \longrightarrow \pi_1(Y, y_0)$$

$$h_* \circ h_*^{-1} = id$$

$$\pi_1(X) \cong \pi_1(Y)$$

Ex $\pi_1(\mathbb{R}^n, 0) = e$.



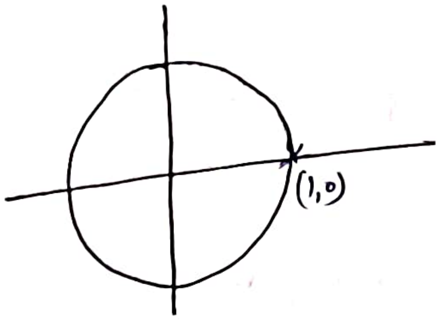
$H(s,t) = (1-t)f(s)$

$t = 0$

$H(s,0) = f(s)$

$H(s,1) = 0$

(2) $X = S^1 = \{x^2+y^2=1\}$.



$\pi_1(S^1, (1,0))$

$f(s) = (\cos 2\pi s, \sin 2\pi s)$

$f*f(s) = (\cos 4\pi s, \sin 4\pi s)$

$f*f*f*\cdots*f = (\cos 2\pi n s, \sin 2\pi n s)$

$e_0 : [0,1] \to (1,0)$

$f^{-1}(s) = (\cos 2\pi s, -\sin 2\pi s)$

$\pi_1(S^1,(1,0)) \cong (\mathbb{Z}, +)$.

$[\underbrace{\cos 2\pi s, \sin 2\pi s}_{f}] \longrightarrow 1$

$f*f \longrightarrow 1+1. = 2.$

$f*f*\cdots*f = \underbrace{1+1+1+\cdots+1}_{n}$

$e_{x_0} \longrightarrow 0.$

$1^{-1} \longrightarrow -1.$

$$\underbrace{1^{-1} \times \cdots \times 1^{-1}}_{n} \Longrightarrow -n$$

Thm:- (Brower's fixed point thm) :-

Let D be the unit disc in R^2 $\{x^2+y^2 \le 1\}$.

Let $h : D \longrightarrow D$ be any cts map. then h has at least one fixed point: $h(x_0) = x_0$.

Proof:- By the method of contradiction

Let $h : D \longrightarrow D$ a cts map with no fixed point.

$h(x) \neq x \quad \forall\, x \in D$. connect $h(x)$ to x by straight lines extend the st. line till it hits the bdry circle, call it $g(x)$.

D 

$D \xrightarrow{g} S^1$ s.t. if $x \in S^1$ $g(x) = x$.

~~of~~ Consider

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "id"'] & D \arrow[r, "g"] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "id_* = id"'] & \pi_1(D) \arrow[r, "g_*"] & \pi_1(S^1) \end{tikzcd}$$

$$\begin{tikzcd} (\mathbb{Z}, +) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & e \arrow[r, "g_*"] & (\mathbb{Z}, +) \end{tikzcd}$$

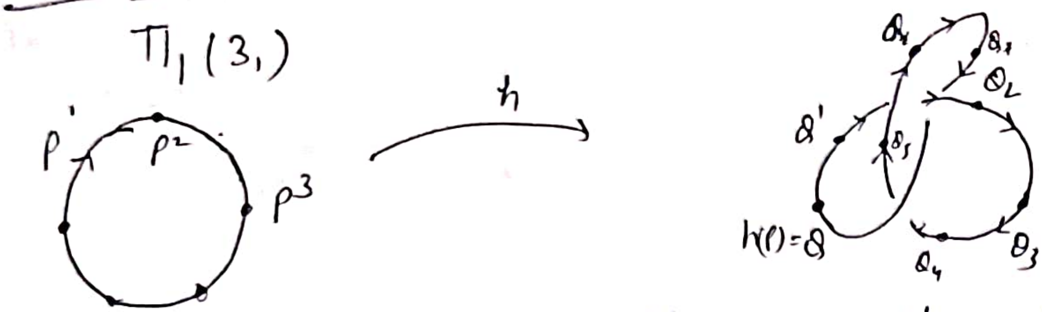
$$\begin{tikzcd} \mathbb{Z} \arrow[r, "j_*"] \arrow[rr, bend right, "g_* \circ i_* = id"'] & 0 \arrow[r, "g_*"] & \mathbb{Z} \end{tikzcd}$$

$g_* \circ i_* (\mathbb{Z}) \longrightarrow 0 \neq id.$ (*)

$h(x) = x$

Next lect:-

$\Pi_1(3_1)$



Any knot K is homeomorphic to $S^1 = \{x^2+y^2=1\}$.

$\exists\, h : S^1 \longrightarrow K$ s.t. h is 1-1, onto cont.

$\Pi_1(K) \overset{iso}{\cong} \Pi_1(S^1 \cong (\mathbb{Z},+)$ with cts inverse.

$\Pi_1(\mathbb{R}^3 \setminus K)$ knot gp is a kont invariant.

Simplical Complexes:-

Defn:- (1) K-simplex Δ_K

In $\mathbb{R}^n$ let $E_0 = (0,0,0,\dots 0)$

$E_1 = (1,0,0,\dots 0)$

$E_2 = (0,1,\dots 0)$

$\vdots$

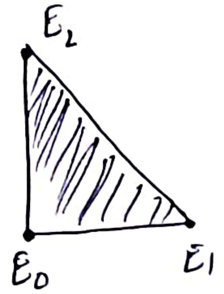
$E_K = (0,0,\dots 1, 0, \dots 0)$

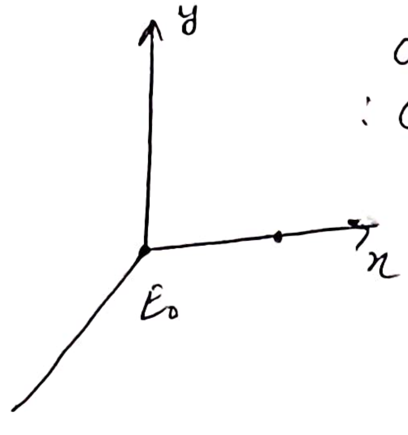
$\downarrow$ Kth simplex.

Defn a K-simplex is the convex span of vartices $E_0\ E_1 \dots E_K$.

Eg: 0-simplex $\overset{\bullet}{E_0}$ a point.

1-simplex E_0 ——— E_1 an interval $[0,1]$





a tetrahedron.
: a convex hull of 4 points E_0, E_1, E_2, E_3.

Defn:- Faces of a k-simplex.

$\Delta_k = (E_0, E_1, \dots E_k)$ is any subcollection

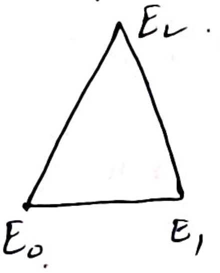
$(E_{\alpha_1}, \dots E_{\alpha_j})$

where $\alpha_j < k$.

2-simplex

$E_0 E_1 E_2$

faces are E_0E_1, E_1E_2, E_2E_0 & E_0, E_1, E_2.



Defn: (more gen).

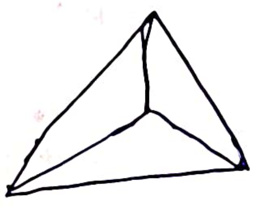
Let $V_0, V_1, \dots V_k$ are points in R^n $(n > k+1)$ s.t. $\overrightarrow{V_1 - V_0}$, $\overrightarrow{V_2 - V_0}$, $\dots$ $\overrightarrow{V_k - V_0}$ are linearly independent. Then the convex span

$\{ \lambda_0 V_0 + \lambda_1 v_1 + \dots + \lambda_k v_k \mid 0 \leq \lambda \leq 1 \ \ \sum \lambda_i = 1 \}$

is called a k-simplex.

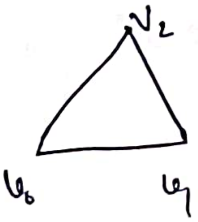
• 0-simplex

1-simplex



3-simplex.

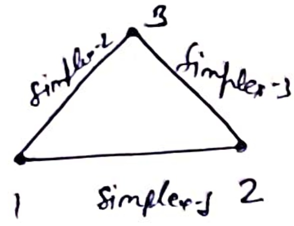
Do more mathematics



2-simplex

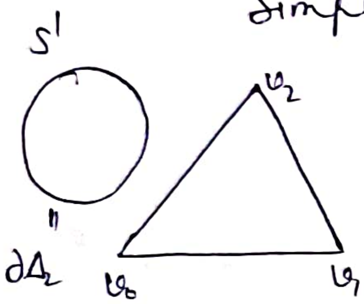
Def^n ③ A simplical complex K^n of dim n, is a collection of simplexes $\Delta_1, \Delta_2, \dots \Delta_n$ s.t. any two simplexes Δ_i & Δ_j intersect in a common face.

Ex :-

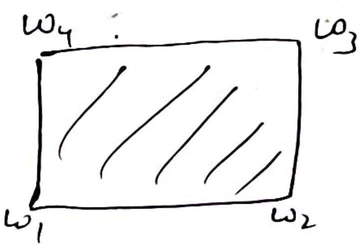


3 line-simplex intersect at point.

Def^n :- A triangulation of a space X is a ~~simplex~~ simplical complex K s.t. X is homeo to K^n?



triangulation of a circle.



2-simplex (Δ_2)

Boundary of Δ_2 consist of

3 · 0-simplices (the vertices & (v_1, v_2, v_3))

3 1-simplices (the edges) $(V_1V_2,\ V_2V_3,\ V_1V_3)$.

Simplical

Def^n (Euler number) A topological invariant of K^n;

Let α_0 = # of vertices 0-simplexes

α_1 = # of edges 1-simplexes.

$\vdots$

α_n = # of edges of n-simplexes.

$$\chi(K^n) = \alpha_0 - \alpha_1 + \alpha_2 + \cdots (-1)^n \alpha_n$$

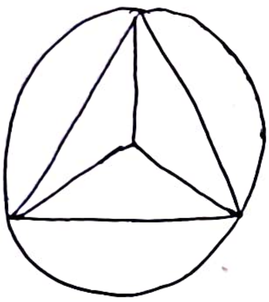
$= \sum_{i=0}^{n} (-1)^i \alpha_i$ Euler number

$G_i \neq G_j$

$\chi(S^1) = 3-3+0+0+\dots \pm 1\,0 = 0.$

$\chi\chi$

S^2:- sphere



$\alpha_0 = 4$

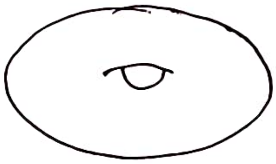
$\alpha_1 = 6.$

$\alpha_2 = 4$

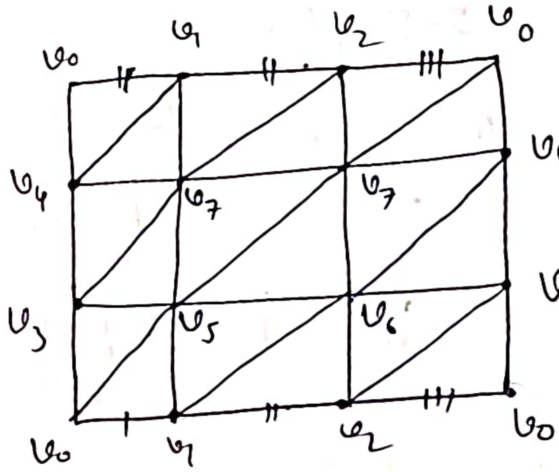
$\chi(S^2) = 4-6+4$

$= 8-6 = 2.$

$S^2 \not\cong S^1.$



$S^1 \times S^1.$ Quotient space.



$\alpha_0 = 9$

$\alpha_1 = 27$

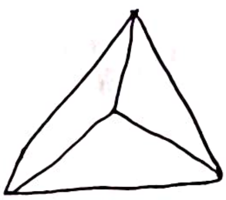
$\alpha_2 = 18$

$\chi(\text{Tours}) = 9-27+18$

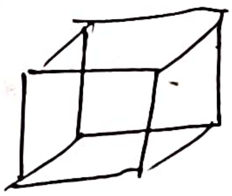
$= 0.$

$\exists$ 5- regular polyhedron.

(1) Tetrahedron



(2) cube



(3) Tetrahedran.

Simplical Complexes.

Meetleet :-

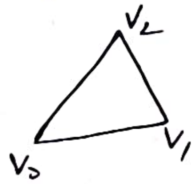
Def^n :- Let $v_0, v_1, \ldots, v_k$ be vertices in R^n, $n > k+1$ s.t. $v_1 - v_0$, $v_2 - v_0$, $\ldots$ $v_k - v_0$ are linearly independent. Then the convex span of $v_0, v_1, \ldots v_k$ is a k-simplex denoted by Δ_k.

Ex :- (1) 0-simplex v_0 . a point

(2) 1-simplex $v_0 v_1$

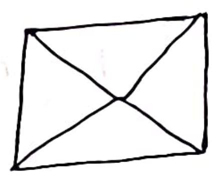
edge.

(3) 2-simplex $v_0 v_1 v_2$



Triangle

(4) 3-simplex $v_0 v_1 v_2 v_3$



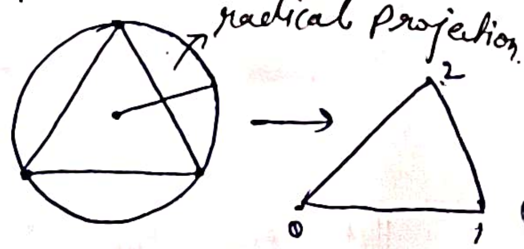
Tetrahedron.

Def^n Let $\Delta_k = v_0 \ldots v_k$ be a k-simplex then any subset $v_{\alpha_1}, v_{\alpha_2} \ldots v_{\alpha_l}$ is

Def a simplicical complex of dim n, K^n consists of finitely many of dim $j \le n$.

Def^n; Triangulation of a space X is a simplicial complex K^n and a homeomorphism.

$h : K^n \longrightarrow X$



$V \to 0, 1, 2$

$e \to 01, 02, 20.$

Def^n <u>Euler Number</u> :- Let K^n be a simplicial complex

Let α_0 = # of 0-simplexes = # of vertices of K^n?

α_1 = # of 1-simplexes. = # of edges of K^n

α_2 = # of 2-simplexes = # of faces of K^n.

$\vdots$

α_k = # of k-simplexes

Then consider the Euler $\chi = \sum (-1)^k \alpha_k$

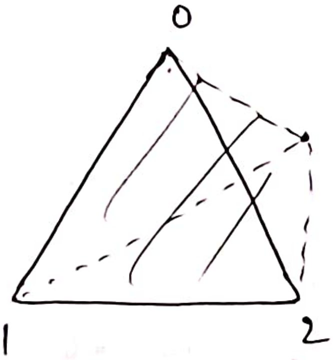
<u>Th^m</u> (Know to Greeks 2000 year ago) There exist 5 and only 5 regular polyhedron in nature.

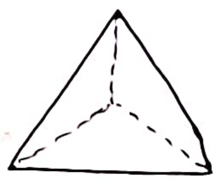
<u>Pf</u> Regular Polyhedron $\longrightarrow$ (1) Polyhedron P is homeomorphic to the Sphere S^2 ; $\chi(P) = 2$.

(2) Polyhedron consists of vertxes, edges & faces s.t

(i) all faces identical.

(ii) at any vertces same no of edges emanate.





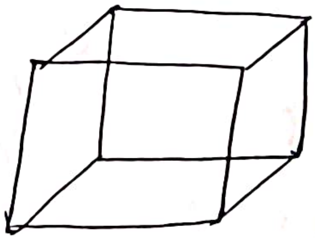
$\alpha_0 = 4$

$\alpha_1 = 6$

$\alpha_2 = 4$

$\chi(T) = 4-6+4 = 2$.

Cube



$\chi(C) = 2$.

Petagonal hydra

Thm:- $\exists$ 5 regular polyhedron in nature.

Proof (Helbert - Geom r th Imag) Let X be a regular polyhedron

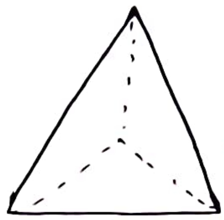
Let V = # of faces F = # of faces of X.

E = # of edges.

Suppose each face has r edges.

Suppose that at each vertex n edges emanate.

$V - E + F = 2.$



$rF = 2E. \Rightarrow F = \frac{2E}{r}$

$n.V = 2E \Rightarrow V = \frac{2E}{n}.$

$V - E + F = 2 \Rightarrow \frac{2E}{n} - E + \frac{2E}{r} = 2.$

$\Rightarrow E\left(\frac{2}{n} - 1 + \frac{2}{r}\right) = 2 \Rightarrow \boxed{\frac{1}{n} + \frac{1}{r} = \frac{1}{E} + \frac{1}{2}} - (*)$

$\frac{1}{n} + \frac{1}{r} = \frac{1}{E} + \frac{1}{2}$ want integer solution.

$r \geq 3$ $n, r \geq 4$

$n \geq 3.$ $\leq \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0 = \frac{1}{E}.$

So either $n = 3$ or $r = 3$.

$\frac{1}{3} + \frac{1}{3} = \frac{1}{E} + \frac{1}{2}$

Solve for E.

$n = 3$	$r = 3$	tetrahedron
$n = 3$	$r = 4$	

$\frac{1}{n} + \frac{1}{r} = \frac{1}{E} + \frac{1}{2}$.

$\frac{1}{3} + \frac{1}{4} = \frac{1}{E}$

$E = 12$, $\quad F = \frac{2E}{r} = 6$.

$n = 3$, $r = 5$. $\quad$ 12-Pent

$\frac{1}{3} +$

$n=3$ $\quad r=3$ $\quad$ Tetrahadron.

$n=3$ $\quad r=4$ $\quad$ cube

$n=3$ $\quad r=5$ $\quad$ 12-pentagonal faces.

$n=4$ $\quad r=3$. $\quad$ octaherron.

5 $\quad 3$ $\quad$ Isohedron.

<u>Note</u> $\quad \frac{1}{n} + \frac{1}{r} = \frac{1}{E} + \frac{1}{2}$

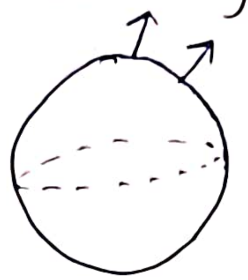
(*) is symmetric in n & r.

Classification theorem:-

Classification theorem for 2-dimensional surfaces which are oriented, compact and without boundary.

(1) Sphere $x^2+y^2+z^2=1$

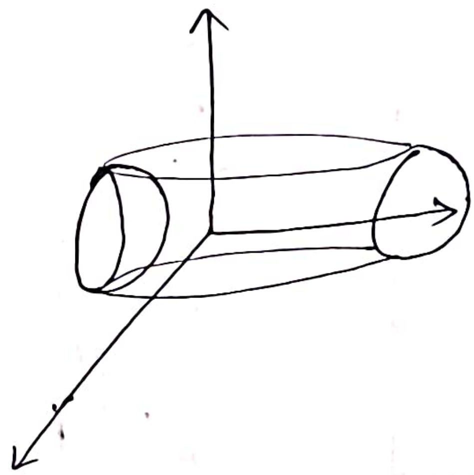
$x\hat{i}+y\hat{j}+z\hat{k}$



(2) Tours.

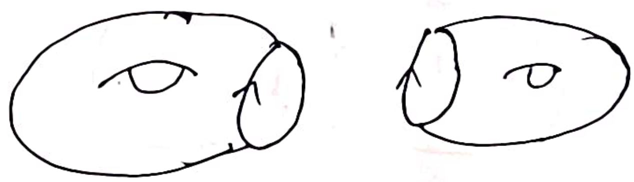
$S^1 \times S^1$

$(x-3)^2+z^2=1.$

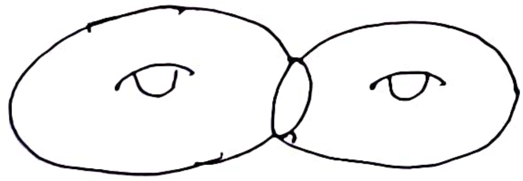


Connected sum:- $S_1 \# S_2$

$T \# T$



T

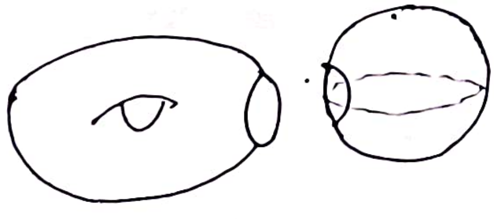


T_2

or double Tours.

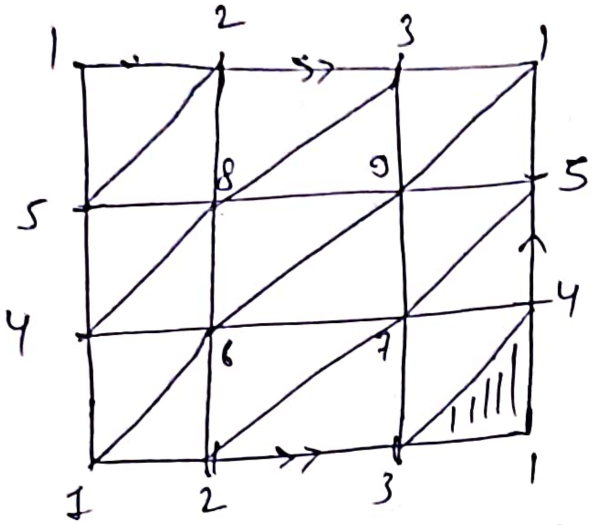
double tours is differnt from a tours. by fundamental group.

they both have differn f.g



$T \# S$

$S^1 \times S^1$



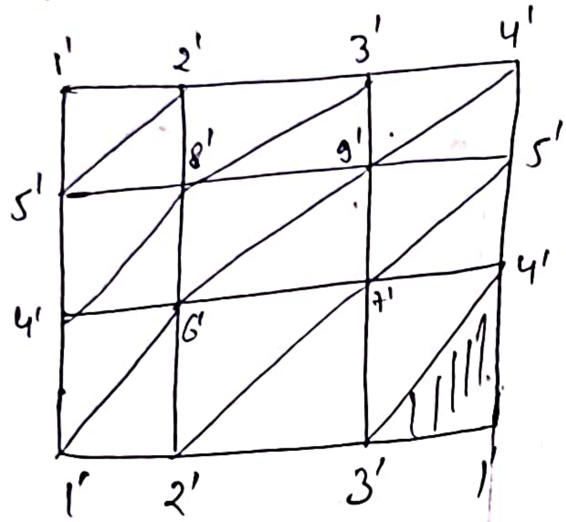
$\alpha_0 = -9$

$\alpha_1 = 27$

$\alpha_2 = 18$

$\alpha_0 - \alpha_1 + \alpha_2$

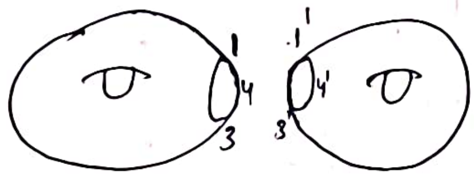
$\chi(T') = 0.$



fr double torus we remove one circle it is equivalent to remove a ~~surface~~ interior of a circle.

Remove ~~all~~ one face 134 from T

1'3'4' from T'



$\chi(T_2)$

$\Downarrow$

double torus.

$\alpha_0(T_2) = 9 + 9 - 3$

$\alpha_1(T_2) = 27 + 27 - 3$

$\alpha_2(T_2) = 18 + 18 - 2$

$\chi(T_2) = 0 + 0 - 3 + 3 - 2 = -2.$

$S_1 = \alpha_0$ vertices $\qquad$ S_2. α_0' vertices

α_1 edges $\qquad$ α_1' edges

α_2 faces $\qquad$ α_2' faces

α_0 of $S_1 \# S_2 = \alpha_0 + \alpha_0' - 3.$

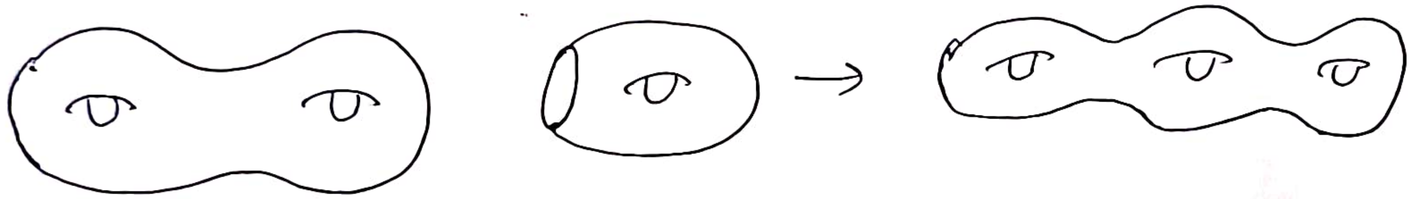
α_1 of $S_1 \# S_2 = \alpha_1 + \alpha_1' - 3$

α_2 of $S_1 \# S_2 = \alpha_2 + \alpha_2' - 2.$

$$\chi(S_1 \# S_2) = (\alpha_0 - \alpha_1 + \alpha_2) + (\alpha_0' - \alpha_1' + \alpha_2') \cancel{+} \cancel{3} + \cancel{3} - 2.$$

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

<u>Mobious</u>:



$$\chi(T_3) = \chi(T_2) + \chi(T_1) - 2$$
$$= -2 + 0 - 2 = -4.$$

<u>Similarly</u> :-

$T_3 \# T. \cong T_4.$

$\vdots$

$T_{g-1} \# T. \cong T_g.$

$$\chi(T_g) = \chi(T_g) + \chi(T_{g-1}) - 2$$
$$= 2 - 2g + 0 - 2 = -2g$$
$$= -2g. = -2(g)$$

$S_1 = \alpha_0$ vertices $\qquad$ S_2: α_0' vertices

α_1 edges $\qquad$ α_1' edges

α_2 faces $\qquad$ α_2' faces

α_0 of $S_1 \# S_2 = \alpha_0 + \alpha_0' - 3$.

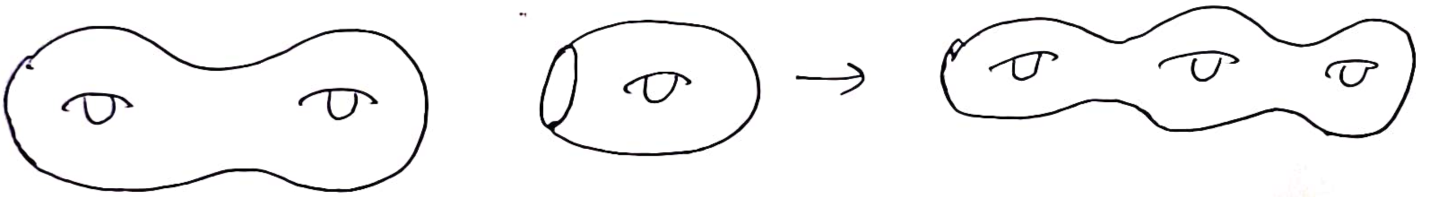
α_1 of $S_1 \# S_2 = \alpha_1 + \alpha_1' - 3$

α_2 of $S_1 \# S_2 = \alpha_2 + \alpha_2' - 2$.

$$\chi(S_1 \# S_2) = (\alpha_0 - \alpha_2 + \alpha_2) + (\alpha_0' - \alpha_1' + \alpha_2') \cancel{-3} + \cancel{3} - 2.$$

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

<u>Mobious</u>:



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: Simplical Homology Theory :-

So far topological invariants compactness, Connectedness

$\pi_1(X, x_0)$, $\chi(K^n)$ $\rightarrow$ $R^n \cong R^m$? $n \neq m$

$S^n \cong S^m$ $(n \neq m)$?

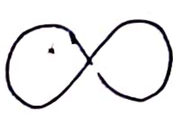
Defn :- Let K^n be an n-dimensional simplical complex, dim $= n$ means simplex σ_i of highest dimension in K^n is σ_n.

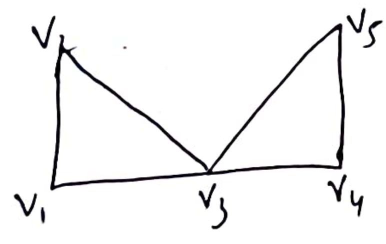
We will associate with K^n, a set of $(n+1)$ abelian groups, denoted by $H_0(K^n)$, $H_1(K^n)$, $H_2(K^n)$ ...

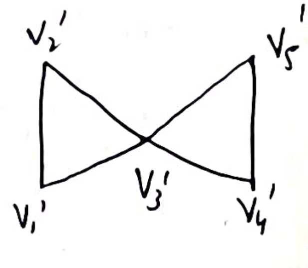
$$H_n(K^n).$$

Called the simplicial homology groups of K^n and each of them is a topological invariant of K^n.

Defn Let K and L be simplical complexes K is isomorphic to L if there is a map ϕ from the vertices of K to the vertices of L which is 1-1, onto s.t. if $v_1, v_2, \ldots v_s$ spans a simplex if and only if $\phi(v_1)\,\phi(v_2) \ldots \phi(v_s)$ span a ~~sum~~ simplic of L.







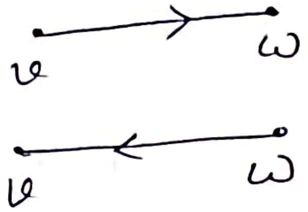
Def^n:- Orientention of a simplex :-

0-simplex only one orient

1-simplex

the symbol $\overrightarrow{vw}$ means vw oriented

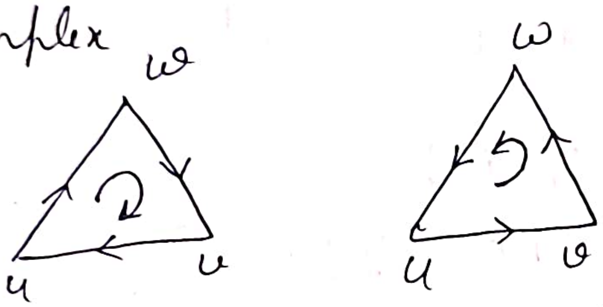
$\overrightarrow{wv}$ means vw oriented



The relation between these two are

$$(w, v) = -(v, w).$$

2-simplex



$(u\cancel{wv})$ is the 2-simplex oriented ~~anti~~clockwise.

$(u v w)$ is the 2-simplex oriented anticlockwise.

$$(w, v) = -(v, w)$$

$$(u, v, w) = -(v, u, w)$$

If we do even number orientation

$\sigma_i = (v_0 v_1 \cdots v_i)$ an i-simplex, with a given orientation

$-\sigma_i = (v_1 v_0 v_2 \cdots v_i)$ is i-simplex ... opp. orientation

Let θ be a permutation of $0, 1, 2, \dots, i$

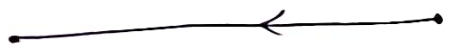
$\sin\theta = +1$ or -1

$$\left(v_{\theta(0)}\, v_{\theta(1)} \cdots v_{\theta(i)}\right) = (\text{Sign of } \theta)\left(v_0, v_1 \cdots v_i\right)$$

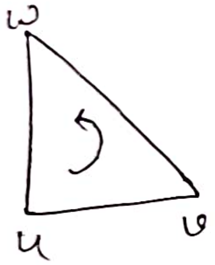
Def^n(3) Boundary of a simplex — oriented 1-simplex $\overrightarrow{vw}$

$$\partial(\overrightarrow{vw}) = w - v.$$

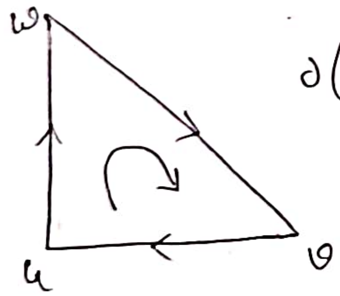
$$\partial(\overrightarrow{wv}) = v - w.$$
$$= -\partial(\overrightarrow{vw})$$



$$-\partial(\overrightarrow{vw}).$$



$$\partial(\overrightarrow{uvw}) = \overrightarrow{uv} + \overrightarrow{vw} + \overrightarrow{wu}$$
$$=$$



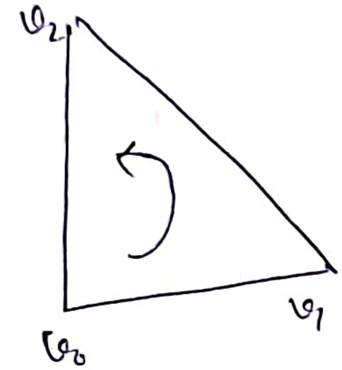
$$\partial(\overrightarrow{uvw}) = \overrightarrow{vu} + \overrightarrow{uw} + \overrightarrow{wv}$$
$$= -\partial(\overrightarrow{vuw})$$

Boundary of an i-simplex $\sigma_i = (v_0 v_1 \cdots v_i)$

$$\partial\sigma_i = \sum_{j=0}^{i} (-1)^j\, v_0\, v_1 \cdots \hat{v}_j \cdots v_i$$

where $\hat{v}_j$ means v_j is omitted / deleted.

$$\partial(v_0 v_1 v_2) = (-1)^0 \hat{v_0} v_1 v_2 + (-1)^1 v_0 \hat{v_1} v_2 + (-1)^2 v_0 v_1 \hat{v_2}$$



$$= v_1 v_2 - v_0 v_2 + v_0 v_1$$

$$= v_1 v_2 + v_2 v_0 + v_0 v_1$$

Def^n (4) We define $C_q(K)$ - the qth chain of group of K^n

Suppose K^n has finitely many q-simplexes $\sigma_1^q\ \sigma_2^q \dots \sigma_p^q$ a chain is a formal finite sum

$$C_q(K^n) = \{d_1\sigma_1^q + d_2\sigma_2^q + \dots + d_p\sigma_p^q \mid d \in \mathbb{Z}\}.$$

Claim:- $C_q(K^n)$ is an abelian group.

(1) $d_1\sigma_1^q + d_2\sigma_2^q + \dots + d_p\sigma_p^q \mid d_i \in \mathbb{Z}$

$\circ$ $d_1'\sigma_1^q + d_2'\sigma_2^q + \dots + d_p'\sigma_p^q$

$(d_1+d_1')\sigma_1^q + (d_2+d_2')\sigma_2^q + \dots + (d_p+d_p')\sigma_p^q.$

identity:- $0.\sigma_1^q + 0.\sigma_2^q + \dots + 0.\sigma_p^q$

inverse $-d_1\sigma_1^q + (-d_2)\sigma_2^q + \dots + (-d_p')\sigma_p^q.$

associative

commutative.

$C_q(K)$ is an abelian group $\cong \underbrace{\mathbb{Z}\times\mathbb{Z}\dots\times\mathbb{Z}}_{p}$

$C_n(K^n) \xrightarrow{\partial} C_{n-1}(K^n)$. (boundary homo).

$C_n(K^n) \xrightarrow{\partial} C_{n-1}(K^n) \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \dots C_1 \to C_0 \to 0$.

(Homological algebra)

$\partial_{j-1} \circ \partial_j = 0$

$C_{j+1}(K^n) \xrightarrow{\partial_{j+1}} C_j(K^n) \xrightarrow{\partial_j} C_{j-1}(K^n)'$

$\partial_j \circ \partial_{j+1} = 0$.

$\ker \partial_j = Z_j$ (subgroup of j cycles)

$\operatorname{Im} \partial_{j+1} = B_j$ = j-boundries.

$B_j \subset Z_j$

H_j' jth homology $= \dfrac{Z_j}{B_j}$.