Lesson 5

Integral Equations

5.1 Some solutions of boundary value problems for PDEs via integral equations

5.2 An Introduction

5.3 Numerical Methods

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Integral Equations: An Introduction

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There are three main ideas for numerical solution of the second kind Fredholm integral equation

$$(I - A)u = f$$

with the linear integral operator

$$(Au)(x) = \int_{\Omega} K(x, y)u(y)dy \, .$$

Approximate the integral operator by

- *approximating the kernel* $K(x, y)$.
- *approximating the solution* $u(x)$.
- *approximating the integral* $\int_{\Omega} f(y)dy$ *by a quadrature.*

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- Degenerate Kernel Method

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Integral Equations: Numerical Methods

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$$K(x,y) \approx K_n(x,y) = \sum_{j=1}^{n} a_j(x)b_j(y)\,.$$

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The solution of the integral equation of the second kind, $u - Au = f$, is then obtained as

$$u_n(x) - \sum_{j=1}^{n} \langle b_j, u_n \rangle a_j(x) = f(x)\,,$$

$$\langle f, g \rangle = \int_{\Omega} f(y) g(y) dy.$$

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where the coefficients $\gamma_1, \gamma_2, \dots, \gamma_n$ satisfy the linear system

$$\gamma_j - \sum_{k=1}^{n} \langle a_k, b_j \rangle \gamma_k = \langle f, b_j \rangle, \qquad j = 1,2, \dots, n .$$

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The idea is to construct K_n *such that* $\|A_n - A\| \to 0$ *as* $n \to \infty$ *where*

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Theorem

Let X *and* Y *be Banach spaces and let* $A: X \to Y$ *be a bounded linear operator with a bounded operator* $A^{-1}: Y \to X$*. Assume the sequence* $A_n: X \to Y$ *of bounded linear operators to be norm convergent, that is,* $\|A_n - A\| \to 0$ *as* $n \to \infty$*. Then, for sufficiently large* n*, more precisely, for all* n *with* $\|A^{-1}(A_n - A)\| < 1$*, the inverse operators* $A_n^{-1}: Y \to X$ *exist and are bounded by*

$$\|A_n^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}.$$

For all solutions of the equations $A\varphi = f$ *and* $A_n\varphi_n = f_n$*, we have the error estimate*

$$\|\varphi_n - \varphi\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}\{\|(A_n - A)\varphi\| + \|f_n - f\|\}.$$

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- Degenerate Kernel Method

- via interpolation

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One idea that works when K is continuous is interpolation -- approximate K by interpolating in x with respect to the points $x_1, x_2, \ldots, x_n$ in $[a, b]$ for each $y \in [a, b]$, we have

$$K_n(x, y) = \sum_{j=1}^{n} K(x_j, y)\, l_j^{(n)}(x).$$

Integral Equations: Numerical Methods

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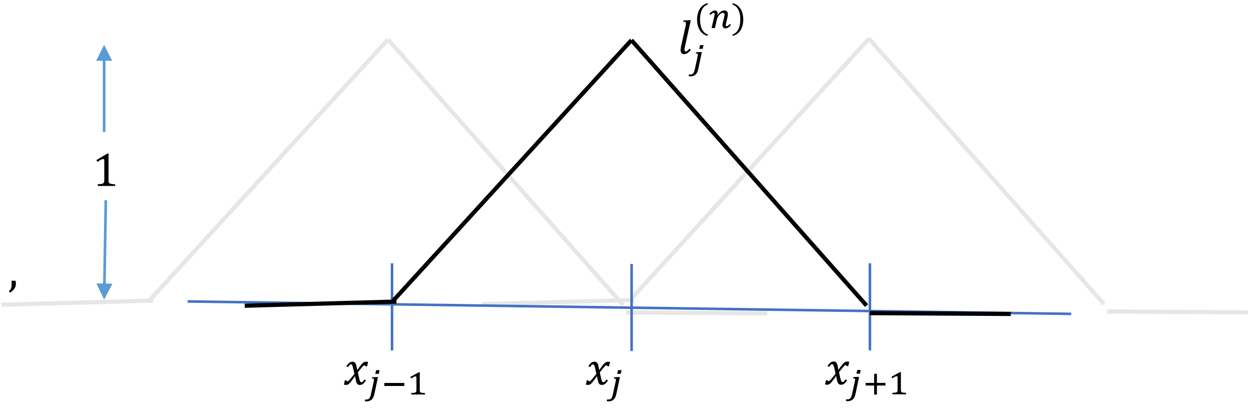
Example

Equidistant piecewise linear interpolation:

$$l_j^{(n)}(x) = \begin{cases} (x - x_{j-1})/h, & x \in [x_{j-1}, x_j], j \geq 1, \\ (x_{j+1} - x)/h, & x \in [x_j, x_{j+1}], j \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$



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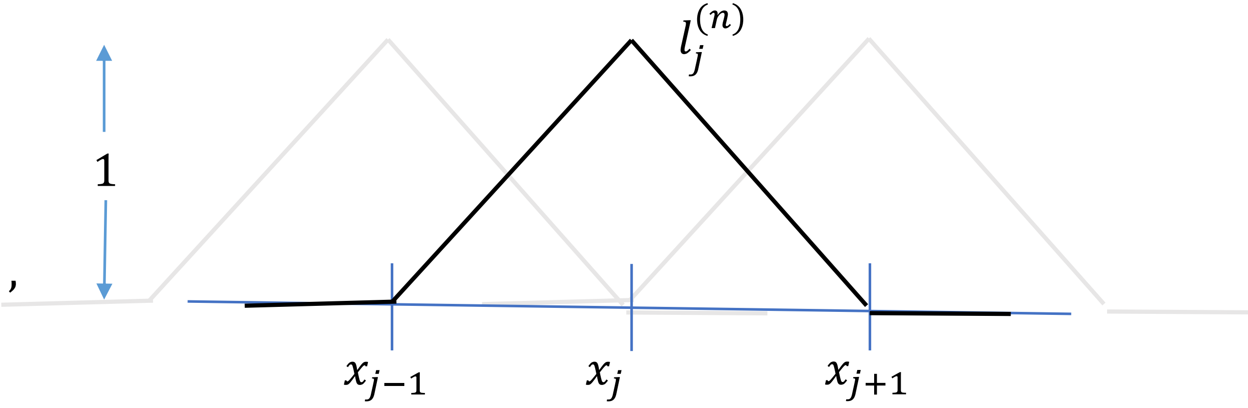

Then, the resulting linear system $(I - M)\gamma = F$ *is given by*

$$m_{ij} = \langle a_j, b_i \rangle = \int_a^b a_j(x) b_i(x) dx$$

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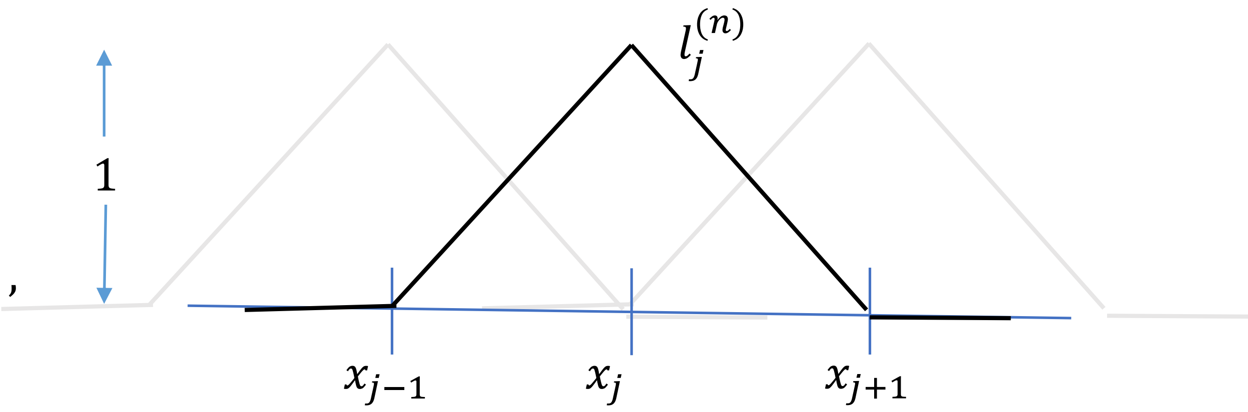

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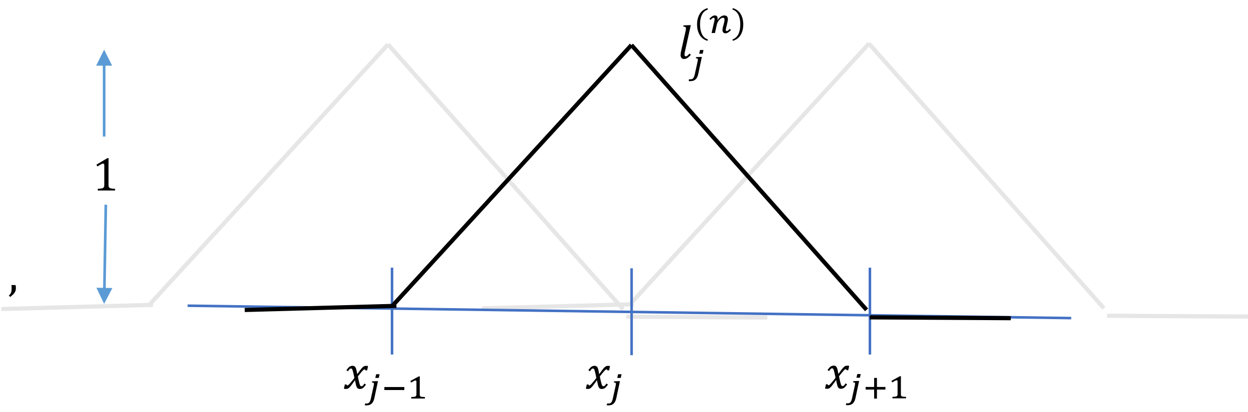

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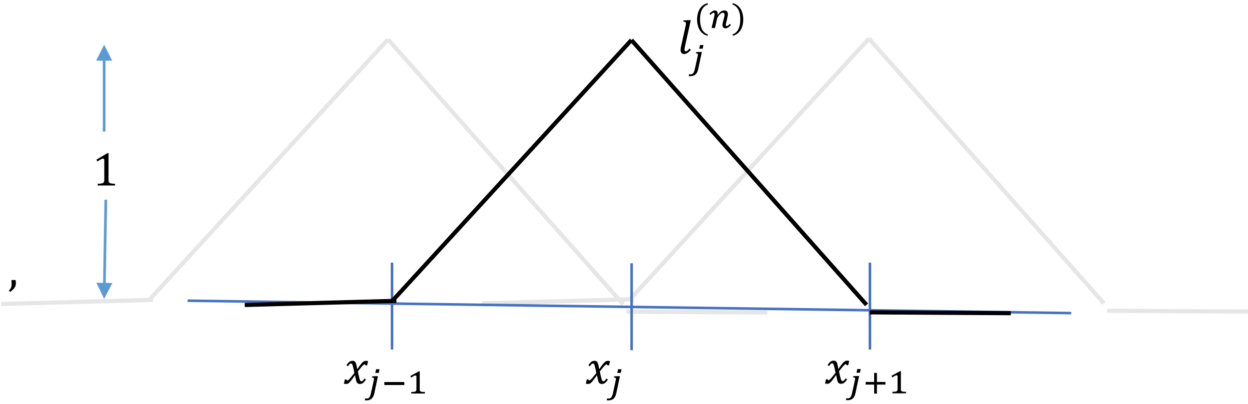

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where

$$W = \frac{h}{6} \begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix}.$$

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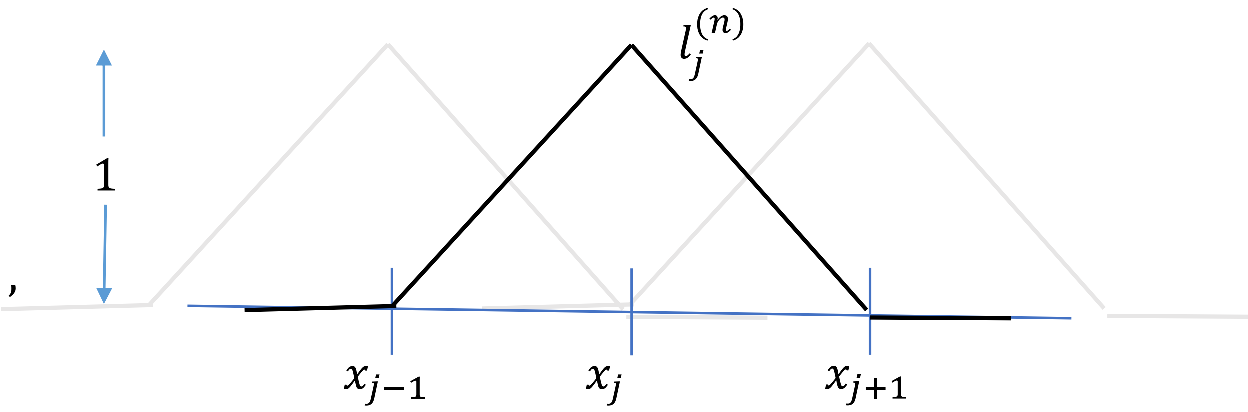

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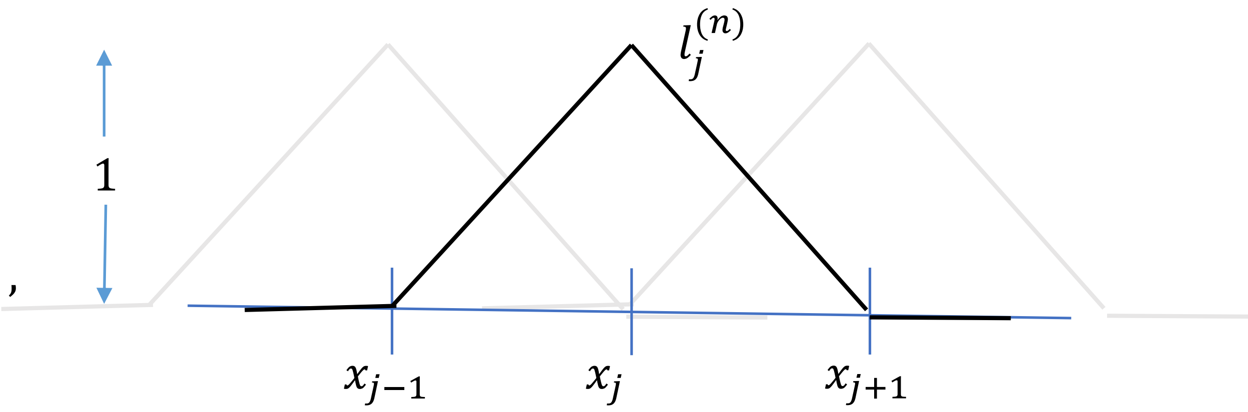

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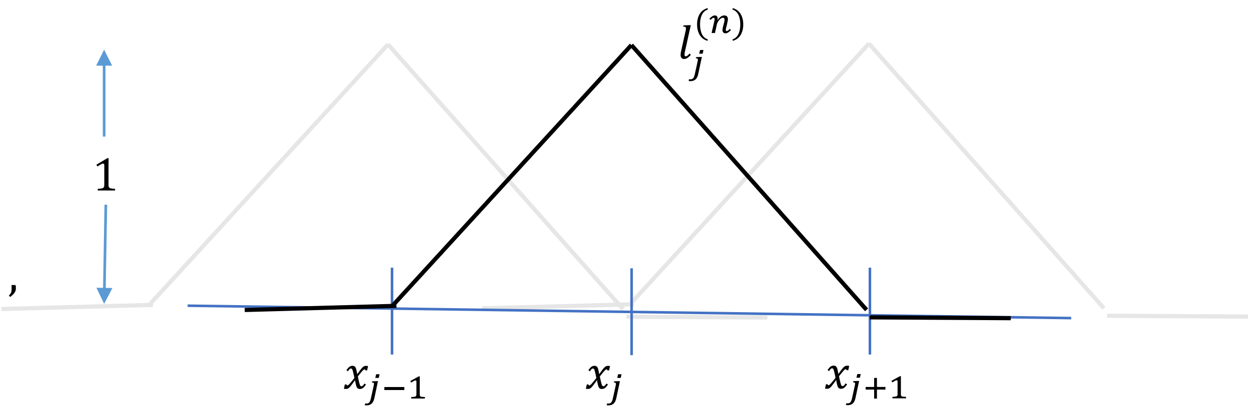

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