Numerical Analysis & Scientific Computing II

Module 2
Initial Value Problems

2.5 Stiffness

2.6 Linear Multistep Method

2.7 Non-Linear Methods

- Consistency and Convergence of one step methods

Akash Anand
MATH, IIT KANPUR

Remark

1. We assume that Ψ is defined for $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, and is continuous there. Moreover, we assume the uniform Lipschitz condition

$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$

whenever (t, y, h) and $(t, \hat{y}, h)$ belong to the domain of Ψ.

Remark

1. We assume that Ψ is defined for $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, and is continuous there. Moreover, we assume the uniform Lipschitz condition
$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$
whenever (t, y, h) and $(t, \hat{y}, h)$ belong to the domain of Ψ.

2. Note that a single step method $y_{n+1} = y_n + h\Psi(f; t_n, y_n, h)$ is consistent if
$$0 = \lim_{h \to 0} \left| \frac{\ell_{n+1}(y, h)}{h} \right|$$

Akash Anand
MATH, IIT KANPUR

Remark

1. *We assume that* Ψ *is defined for* $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, *and is continuous there. Moreover, we assume the uniform Lipschitz condition*
$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$
whenever (t, y, h) *and* $(t, \hat{y}, h)$ *belong to the domain of* Ψ.

2. *Note that a single step method* $y_{n+1} = y_n + h\Psi(f; t_n, y_n, h)$ *is consistent if*
$$0 = \lim_{h \to 0} \left| \frac{\ell_{n+1}(y, h)}{h} \right| = \lim_{h \to 0} \left| \frac{y(t_n) + h\Psi(f; t_n, y, h) - y(t_{n+1})}{h} \right|$$

Remark

1. We assume that Ψ is defined for $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, and is continuous there. Moreover, we assume the uniform Lipschitz condition

$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$

whenever (t, y, h) and $(t, \hat{y}, h)$ belong to the domain of Ψ.

2. Note that a single step method $y_{n+1} = y_n + h\Psi(f; t_n, y_n, h)$ is consistent if

$$0 = \lim_{h \to 0} \left| \frac{\ell_{n+1}(y, h)}{h} \right| = \lim_{h \to 0} \left| \frac{y(t_n) + h\Psi(f; t_n, y, h) - y(t_{n+1})}{h} \right| = \lim_{h \to 0} \left| \Psi(f; t_n, y, h) - \frac{y(t_{n+1}) - y(t_n)}{h} \right|$$

Remark

1. *We assume that* Ψ *is defined for* $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, *and is continuous there. Moreover, we assume the uniform Lipschitz condition*
$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \le K|y - \hat{y}|,$$
whenever (t, y, h) *and* $(t, \hat{y}, h)$ *belong to the domain of* Ψ.

2. *Note that a single step method* $y_{n+1} = y_n + h\Psi(f; t_n, y_n, h)$ *is consistent if*
$$0 = \lim_{h \to 0} \left| \frac{\ell_{n+1}(y, h)}{h} \right| = \lim_{h \to 0} \left| \frac{y(t_n) + h\Psi(f; t_n, y, h) - y(t_{n+1})}{h} \right| = \lim_{h \to 0} \left| \Psi(f; t_n, y, h) - \frac{y(t_{n+1}) - y(t_n)}{h} \right|,$$
for all $y \in C^1$.

Remark

1. *We assume that* Ψ *is defined for* $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$*, and is continuous there. Moreover, we assume the uniform Lipschitz condition*

$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$

whenever (t, y, h) *and* $(t, \hat{y}, h)$ *belong to the domain of* Ψ.

2. *Note that a single step method* $y_{n+1} = y_n + h\Psi(f; t_n, y_n, h)$ *is consistent if*

$$0 = \lim_{h \to 0} \left| \frac{\ell_{n+1}(y, h)}{h} \right| = \lim_{h \to 0} \left| \frac{y(t_n) + h\Psi(f; t_n, y, h) - y(t_{n+1})}{h} \right| = \lim_{h \to 0} \left| \Psi(f; t_n, y, h) - \frac{y(t_{n+1}) - y(t_n)}{h} \right|,$$

for all $y \in C^1$.

3. *In view of the continuity, consistency condition is equivalent to* $\Psi(f; t_n, y, 0) = f(t_n, y)$.

Remark

1. *We assume that* Ψ *is defined for* $t \in [t_0, T], y \in \mathbb{R}, h \in [0, T - t_0]$, *and is continuous there. Moreover, we assume the uniform Lipschitz condition*

$$|\Psi(f; t, y, h) - \Psi(f; t, \hat{y}, h)| \leq K|y - \hat{y}|,$$

whenever (t, y, h) *and* $(t, \hat{y}, h)$ *belong to the domain of* Ψ.

2. *Note that a single step method* $y_{n+1} = y_n + h\Psi(f; t_n, y_n, h)$ *is consistent if*

$$0 = \lim_{h\to 0}\left|\frac{\ell_{n+1}(y,h)}{h}\right| = \lim_{h\to 0}\left|\frac{y(t_n) + h\Psi(f; t_n, y, h) - y(t_{n+1})}{h}\right| = \lim_{h\to 0}\left|\Psi(f; t_n, y, h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right|,$$

for all $y \in C^1$.

3. *In view of the continuity, consistency condition is equivalent to* $\Psi(f; t_n, y, 0) = f(t_n, y)$.

4. *The method has order* p *if*

$$\left|\Psi(f; t, y_n, h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right| \leq Ch^p$$

for all $h \leq h_0$ *for some constants* $C, h_0 > 0$, *for all* $y \in C^{p+1}$.

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

($\Longleftarrow$)

Let $e_n = y_n - y(t_n)$. Then

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})|$$

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. Then

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f; t_n, y_n, h) - y(t_{n+1})|$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. Then

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f; t_n, y_n, h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f; t_n, y_n, h) - \big(y(t_{n+1}) - y(t_n)\big)\right|$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. *Then*

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f; t_n, y_n, h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f; t_n, y_n, h) - \left(y(t_{n+1}) - y(t_n)\right)\right|$$

$$\le |e_n| + h|\Psi(f; t_n, y_n, h) - \Psi(f; t_n, y(t_n), h)| + h\left|\Psi(f; t_n, y(t_n), h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right|$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

($\Longleftarrow$)

Let $e_n = y_n - y(t_n)$. Then

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f; t_n, y_n, h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f; t_n, y_n, h) - \big(y(t_{n+1}) - y(t_n)\big)\right|$$

$$\le |e_n| + h|\Psi(f; t_n, y_n, h) - \Psi(f; t_n, y(t_n), h)| + h\left|\Psi(f; t_n, y(t_n), h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right|$$

$$\le |e_n| + Kh|y_n - y(t_n)| + h\left|\frac{\ell_{n+1}(y, h)}{h}\right|$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. Then

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f;t_n,y_n,h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f;t_n,y_n,h) - \big(y(t_{n+1}) - y(t_n)\big)\right|$$

$$\le |e_n| + h|\Psi(f;t_n,y_n,h) - \Psi(f;t_n,y(t_n),h)| + h\left|\Psi(f;t_n,y(t_n),h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right|$$

$$\le |e_n| + Kh|y_n - y(t_n)| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right| = (1+Kh)|e_n| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right|$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. *Then*

$$\begin{aligned}|e_{n+1}| &= |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f;t_n,y_n,h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f;t_n,y_n,h) - \big(y(t_{n+1}) - y(t_n)\big)\right| \\ &\le |e_n| + h|\Psi(f;t_n,y_n,h) - \Psi(f;t_n,y(t_n),h)| + h\left|\Psi(f;t_n,y(t_n),h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right| \\ &\le |e_n| + Kh|y_n - y(t_n)| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right| = (1+Kh)|e_n| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right|\end{aligned}$$

Now, for all $n \le N$ *where the final time* $T = t_0 + Nh$, *we see that*

$$|e_{n+1}| \le (1+Kh)|e_n| + \max_{0\le n<N} |\ell_{n+1}(y,h)|\,.$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. *Then*

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f; t_n, y_n, h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f; t_n, y_n, h) - \big(y(t_{n+1}) - y(t_n)\big)\right|$$

$$\le |e_n| + h|\Psi(f; t_n, y_n, h) - \Psi(f; t_n, y(t_n), h)| + h\left|\Psi(f; t_n, y(t_n), h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right|$$

$$\le |e_n| + Kh|y_n - y(t_n)| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right| = (1 + Kh)|e_n| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right|$$

Now, for all $n \le N$ *where the final time* $T = t_0 + Nh$, *we see that*

$$|e_{n+1}| \le (1 + Kh)|e_n| + \max_{0 \le n < N} |\ell_{n+1}(y,h)|\,.$$

Therefore, for all $n \le N$, *we have*

$$|e_n| \le (1 + Kh)^n |e_0| + h \max_{0 \le j < N} \left|\frac{\ell_{j+1}(y,h)}{h}\right| \frac{(1 + Kh)^n - 1}{Kh}$$

Initial Value Problems: Non-Linear Methods

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. *Then*

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f;t_n,y_n,h) - y(t_{n+1})| \le |e_n| + \left|h\Psi(f;t_n,y_n,h) - \big(y(t_{n+1}) - y(t_n)\big)\right|$$

$$\le |e_n| + h|\Psi(f;t_n,y_n,h) - \Psi(f;t_n,y(t_n),h)| + h\left|\Psi(f;t_n,y(t_n),h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right|$$

$$\le |e_n| + Kh|y_n - y(t_n)| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right| = (1+Kh)|e_n| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right|$$

Now, for all $n \le N$ *where the final time* $T = t_0 + Nh$, *we see that*

$$|e_{n+1}| \le (1+Kh)|e_n| + \max_{0\le n<N}|\ell_{n+1}(y,h)|.$$

Therefore, for all $n \le N$, *we have*

$$|e_n| \le (1+Kh)^n|e_0| + h\max_{0\le j<N}\left|\frac{\ell_{j+1}(y,h)}{h}\right|\frac{(1+Kh)^n - 1}{Kh} \le (1+Kh)^N|e_0| + \max_{0\le n<N}\left|\frac{\ell_{n+1}(y,h)}{h}\right|\frac{e^{K(T-t_0)} - 1}{K}.$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longleftarrow)$

Let $e_n = y_n - y(t_n)$. *Then*

$$|e_{n+1}| = |y_{n+1} - y(t_{n+1})| = |y_n + h\Psi(f; t_n, y_n, h) - y(t_{n+1})| \leq |e_n| + \left|h\Psi(f; t_n, y_n, h) - \big(y(t_{n+1}) - y(t_n)\big)\right|$$
$$\leq |e_n| + h|\Psi(f; t_n, y_n, h) - \Psi(f; t_n, y(t_n), h)| + h\left|\Psi(f; t_n, y(t_n), h) - \frac{y(t_{n+1}) - y(t_n)}{h}\right|$$
$$\leq |e_n| + Kh|y_n - y(t_n)| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right| = (1 + Kh)|e_n| + h\left|\frac{\ell_{n+1}(y,h)}{h}\right|$$

Now, for all $n \leq N$ *where the final time* $T = t_0 + Nh$, *we see that*

$$|e_{n+1}| \leq (1 + Kh)|e_n| + \max_{0 \leq n < N} |\ell_{n+1}(y,h)|.$$

Therefore, for all $n \leq N$, *we have*

$$|e_n| \leq (1 + Kh)^n |e_0| + h \max_{0 \leq j < N} \left|\frac{\ell_{j+1}(y,h)}{h}\right| \frac{(1 + Kh)^n - 1}{Kh} \leq (1 + Kh)^N |e_0| + \max_{0 \leq n < N} \left|\frac{\ell_{n+1}(y,h)}{h}\right| \frac{e^{K(T - t_0)} - 1}{K}.$$

Thus, we see that consistency implies convergence as $e_0 = 0$ *and the method is consistent!*

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longrightarrow)$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longrightarrow)$

We define y^h *by the single step method*

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longrightarrow)$

We define y^h *by the single step method*

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

As $h \to 0$, y^h *converges to the initial value problem*

$$z'(t) = g\big(t, z(t)\big), \qquad z(t_0) = y_0,$$

where $g(t, y) = \Psi(f; t, y, 0)$.

Initial Value Problems: Non-Linear Methods

Akash Anand
MATH, IIT KANPUR

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longrightarrow)$

We define y^h *by the single step method*

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

As $h \to 0$, y^h *converges to the initial value problem*

$$z'(t) = g\big(t, z(t)\big), \qquad z(t_0) = y_0,$$

where $g(t, y) = \Psi(f; t, y, 0)$. *To see this, note that*

$$z(t_{n+1}) = z(t_n) + hz'(\xi_n) = z(t_n) + hg\big(\xi_n, z(\xi_n)\big),$$

for some $\xi_n \in (t_n, t_{n+1})$.

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longrightarrow)$

We define y^h by the single step method

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

As $h \to 0$, y^h converges to the initial value problem

$$z'(t) = g\big(t, z(t)\big), \qquad z(t_0) = y_0,$$

where $g(t, y) = \Psi(f; t, y, 0)$. To see this, note that

$$z(t_{n+1}) = z(t_n) + hz'(\xi_n) = z(t_n) + hg\big(\xi_n, z(\xi_n)\big),$$

for some $\xi_n \in (t_n, t_{n+1})$. Putting $e_n = z(t_n) - y^h(t_n)$, we get

$$e_{n+1} = e_n + h\Big(g\big(\xi_n, z(\xi_n)\big) - \Psi(f; t_n, y^h(t_n), h)\Big).$$

Theorem

A single step method is convergent if and only if it is consistent.

Proof.

$(\Longrightarrow)$

We define y^h *by the single step method*

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

As $h \to 0$, y^h *converges to the initial value problem*

$$z'(t) = g\big(t, z(t)\big), \qquad z(t_0) = y_0,$$

where $g(t, y) = \Psi(f; t, y, 0)$. *To see this, note that*

$$z(t_{n+1}) = z(t_n) + hz'(\xi_n) = z(t_n) + hg\big(\xi_n, z(\xi_n)\big),$$

for some $\xi_n \in (t_n, t_{n+1})$. *Putting* $e_n = z(t_n) - y^h(t_n)$, *we get*

$$e_{n+1} = e_n + h\Big(g\big(\xi_n, z(\xi_n)\big) - \Psi(f; t_n, y^h(t_n), h)\Big).$$

The term in the brackets may be decomposed as

$$g\big(\xi_n, z(\xi_n)\big) - g\big(t_n, z(t_n)\big) + \Psi(f; t_n, z\,(t_n), 0) - \Psi(f; t_n, z\,(t_n), h) + \Psi(f; t_n, z\,(t_n), h) - \Psi(f; t_n, y^h(t_n), h)$$

where the first two differences tend to 0 *with* h *and the last one is bounded by* $K|e_n|$.

We define y^h by the single step method

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

As $h \to 0$, y^h converges to the initial value problem

$$z'(t) = g\big(t, z(t)\big), \qquad z(t_0) = y_0,$$

where $g(t, y) = \Psi(f; t, y, 0)$. To see this, note that

$$z(t_{n+1}) = z(t_n) + hz'(\xi_n) = z(t_n) + hg\big(\xi_n, z(\xi_n)\big),$$

for some $\xi_n \in (t_n, t_{n+1})$. Putting $e_n = z(t_n) - y^h(t_n)$, we get

$$e_{n+1} = e_n + h\Big(g\big(\xi_n, z(\xi_n)\big) - \Psi(f; t_n, y^h(t_n), h)\Big).$$

The term in the brackets may be decomposed as

$$g\big(\xi_n, z(\xi_n)\big) - g\big(t_n, z(t_n)\big) + \Psi(f; t_n, z\,(t_n), 0) - \Psi(f; t_n, z\,(t_n), h) + \Psi(f; t_n, z\,(t_n), h) - \Psi(f; t_n, y^h(t_n), h)$$

where the first to differences tend to 0 with h and the last one is bounded by $K|e_n|$.

Thus, we have

$$|e_{n+1}| \le (1 + Kh)|e_n| + \omega(h)$$

where $\lim_{h\to 0} \omega(h) = 0$. Since $e_0 = 0$, it follows that e_n tends to 0 with h, that is, $y^h(t_n) \to z(t_n)$ as $h \to 0$.

We define y^h *by the single step method*

$$y^h(t_{n+1}) = y^h(t_n) + h\Psi(f; t_n, y^h(t_n), h),$$

starting from $y^h(t_0) = y_0$.

As $h \to 0$, y^h *converges to the initial value problem*

$$z'(t) = g\big(t, z(t)\big), \qquad z(t_0) = y_0,$$

where $g(t, y) = \Psi(f; t, y, 0)$. *To see this, note that*

$$z(t_{n+1}) = z(t_n) + hz'(\xi_n) = z(t_n) + hg\big(\xi_n, z(\xi_n)\big),$$

for some $\xi_n \in (t_n, t_{n+1})$. *Putting* $e_n = z(t_n) - y^h(t_n)$, *we get*

$$e_{n+1} = e_n + h\Big(g\big(\xi_n, z(\xi_n)\big) - \Psi(f; t_n, y^h(t_n), h)\Big).$$

The term in the brackets may be decomposed as

$$g\big(\xi_n, z(\xi_n)\big) - g\big(t_n, z(t_n)\big) + \Psi(f; t_n, z\,(t_n), 0) - \Psi(f; t_n, z\,(t_n), h) + \Psi(f; t_n, z\,(t_n), h) - \Psi(f; t_n, y^h(t_n), h)$$

where the first to differences tend to 0 *with* h *and the last one is bounded by* $K|e_n|$.

Thus, we have

$$|e_{n+1}| \leq (1 + Kh)|e_n| + \omega(h)$$

where $\lim_{h\to 0} \omega(h) = 0$. *Since* $e_0 = 0$, *it follows that* e_n *tends to* 0 *with* h, *that is,* $y^h(t_n) \to z(t_n)$ *as* $h \to 0$.

We, therefore, have consistency as me must have $f(t, y) = g(t, y) = \Psi(f; t, y, 0)$.

Remark

Recall that, for RK methods

$$\Psi(f;t,y,h) = b_1 f(t + c_1 h, p_1(f;t,y,h)) + b_2 f(t + c_2 h, p_2(f;t,y,h)) + \cdots + b_q f(t + c_q h, p_q(f;t,y,h))$$

where

$$p_1(f;t,y,h) = y$$
$$p_2(f;t,y,h) = y + h(a_{21}k_1)$$
$$p_3(f;t,y,h) = y + h(a_{31}k_1 + a_{32}k_2)$$
$$\vdots$$
$$p_q(f;t,y,h) = y + h\left(a_{q1}k_1 + a_{q2}q_2 + \cdots + a_{q,q-1}k_{q-1}\right)$$

and $k_i = f(t + c_i h, p_i)$.

Remark

Recall that, for RK methods

$$\Psi(f;t,y,h) = b_1 f(t+c_1 h, p_1(f;t,y,h)) + b_2 f(t+c_2 h, p_2(f;t,y,h)) + \cdots + b_q f(t+c_q h, p_q(f;t,y,h))$$

where

$$\begin{aligned} p_1(f;t,y,h) &= y \\ p_2(f;t,y,h) &= y + h(a_{21}k_1) \\ p_3(f;t,y,h) &= y + h(a_{31}k_1 + a_{32}k_2) \\ &\vdots \\ p_q(f;t,y,h) &= y + h\big(a_{q1}k_1 + a_{q2}q_2 + \cdots + a_{q,q-1}k_{q-1}\big) \end{aligned}$$

and $k_i = f(t+c_i h, p_i)$.

If f *is continuous on* $[t_0, T] \times \mathbb{R}$ *and satisfies uniform Lipschitz condition with respect to* y*, then* Ψ *satisfies the assumptions under which the convergence theorem is proved.*

Remark

Recall that, for RK methods

$$\Psi(f;t,y,h) = b_1 f(t + c_1 h, p_1(f;t,y,h)) + b_2 f(t + c_2 h, p_2(f;t,y,h)) + \cdots + b_q f(t + c_q h, p_q(f;t,y,h))$$

where

$$p_1(f;t,y,h) = y$$
$$p_2(f;t,y,h) = y + h(a_{21}k_1)$$
$$p_3(f;t,y,h) = y + h(a_{31}k_1 + a_{32}k_2)$$
$$\vdots$$
$$p_q(f;t,y,h) = y + h\big(a_{q1}k_1 + a_{q2}q_2 + \cdots + a_{q,q-1}k_{q-1}\big)$$

and $k_i = f(t + c_i h, p_i)$.

If f *is continuous on* $[t_0, T] \times \mathbb{R}$ *and satisfies uniform Lipschitz condition with respect to* y*, then* Ψ *satisfies the assumptions under which the convergence theorem is proved.*

To see this, we have

$$|p_1(f;t,y,h) - p_1(f;t,\hat{y},h)| = |y - \hat{y}|$$
$$|k_1(f;t,y,h) - k_1(f;t,\hat{y},h)| = \big|f\big(t + c_1 h, p_1(f;t,y,h)\big) - f\big(t + c_1 h, p_1(f;t,\hat{y},h)\big)\big| \le L|y - \hat{y}|$$

Remark

Recall that, for RK methods

$$\Psi(f;t,y,h) = b_1 f(t + c_1 h, p_1(f;t,y,h)) + b_2 f(t + c_2 h, p_2(f;t,y,h)) + \cdots + b_q f(t + c_q h, p_q(f;t,y,h))$$

where

$$\begin{aligned} p_1(f;t,y,h) &= y \\ p_2(f;t,y,h) &= y + h(a_{21}k_1) \\ p_3(f;t,y,h) &= y + h(a_{31}k_1 + a_{32}k_2) \\ &\vdots \\ p_q(f;t,y,h) &= y + h\big(a_{q1}k_1 + a_{q2}q_2 + \cdots + a_{q,q-1}k_{q-1}\big) \end{aligned}$$

and $k_i = f(t + c_i h, p_i)$.

If f is continuous on $[t_0, T] \times \mathbb{R}$ and satisfies uniform Lipschitz condition with respect to y, then Ψ satisfies the assumptions under which the convergence theorem is proved.

To see this, we have

$$|p_1(f;t,y,h) - p_1(f;t,\hat{y},h)| = |y - \hat{y}|$$

$$|k_1(f;t,y,h) - k_1(f;t,\hat{y},h)| = \big|f\big(t + c_1 h, p_1(f;t,y,h)\big) - f\big(t + c_1 h, p_1(f;t,\hat{y},h)\big)\big| \leq L|y - \hat{y}|$$

$$|p_2(f;t,y,h) - p_2(f;t,\hat{y},h)| \leq (1 + Lha_{21})|y - \hat{y}|$$

$$|k_2(f;t,y,h) - k_2(f;t,\hat{y},h)| \leq L(1 + Lha_{21})|y - \hat{y}|$$

Remark

Recall that, for RK methods

$$\Psi(f;t,y,h) = b_1 f(t + c_1 h, p_1(f;t,y,h)) + b_2 f(t + c_2 h, p_2(f;t,y,h)) + \cdots + b_q f(t + c_q h, p_q(f;t,y,h))$$

where

$$\begin{aligned} p_1(f;t,y,h) &= y \\ p_2(f;t,y,h) &= y + h(a_{21}k_1) \\ p_3(f;t,y,h) &= y + h(a_{31}k_1 + a_{32}k_2) \\ &\vdots \\ p_q(f;t,y,h) &= y + h\big(a_{q1}k_1 + a_{q2}q_2 + \cdots + a_{q,q-1}k_{q-1}\big) \end{aligned}$$

and $k_i = f(t + c_i h, p_i)$.

If f is continuous on $[t_0, T] \times \mathbb{R}$ and satisfies uniform Lipschitz condition with respect to y, then Ψ satisfies the assumptions under which the convergence theorem is proved.

To see this, we have

$$|p_1(f;t,y,h) - p_1(f;t,\hat{y},h)| = |y - \hat{y}|$$

$$|k_1(f;t,y,h) - k_1(f;t,\hat{y},h)| = \big|f\big(t + c_1 h, p_1(f;t,y,h)\big) - f\big(t + c_1 h, p_1(f;t,\hat{y},h)\big)\big| \le L|y - \hat{y}|$$

$$|p_2(f;t,y,h) - p_2(f;t,\hat{y},h)| \le (1 + Lha_{21})|y - \hat{y}|$$

$$|k_2(f;t,y,h) - k_2(f;t,\hat{y},h)| \le L(1 + Lha_{21})|y - \hat{y}|$$

$$|p_3(f;t,y,h) - p_3(f;t,\hat{y},h)| \le (1 + Lh(a_{31} + a_{32}(1 + Lha_{21})))|y - \hat{y}|$$

and, so on …