$P_K(v, z)$

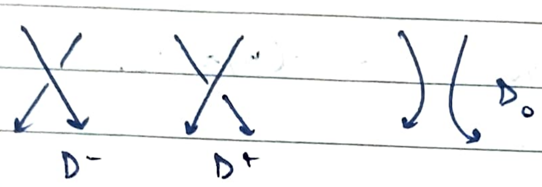
HOMFLY POLYNOMIAL

PAGE NO.
DATE: / /

Given an oriented knot (or link) its HOMFLY poly $P_K(v,z)$ is defined by 2 axioms →

Axiom 1 → If $K \cong O$, $P_K(v,z) = 1$

Axiom 2 →



$\frac{1}{v} P_{D+}(v,z) - v P_{D-}(v,z) = z$

$$\boxed{\frac{1}{v} P_{D+}(v,z) - v P_{D-}(v,z) = z P_{D_0}}$$

$$\Rightarrow \quad P_{D_+}(v,z) = v^2 P_{D_-}(v,z) + vz P_{D_0}$$

$$P_{D_-}(v,z) = \frac{1}{v^2} P_{D_+}(v,z) - \frac{z}{v} P_{D_0}$$

If $v \equiv t$, $z \equiv \sqrt{t} - \frac{1}{\sqrt{t}}$ } Jones polynomial $P_K\left(t, \sqrt{t} - \frac{1}{\sqrt{t}}\right) = V_K(t)$

If $v \equiv 1$, $z \equiv \sqrt{t} - \frac{1}{\sqrt{t}}$ } Alexander polynomial $P_K\left(1, \sqrt{t} - \frac{1}{\sqrt{t}}\right) = \Delta_K(t)$

Ex → Calculate Jones & $P_K(v,z)$ for

(a) 4_1

(b) 5_1

(c) 3_1

(d) O_2

⇒

PAGE NO.
DATE / /

Open Problem

Jones Unknotting Conjecture -.
(Jones 1984)

If $V_K(t) = 1 \Rightarrow K \cong O$, (??)

24/1/24

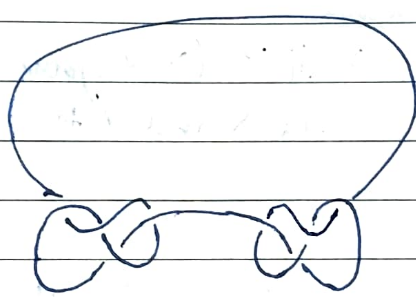
True or False.

Q. If $V_{K_1}(t) = V_{K_2}(t) \overset{(?)}{\Rightarrow} K_1 \cong K_2$

→ Jones polynomial is not complete invariant.

Ans. NO! Counter Example (Murasugi chapter 11)

K_1

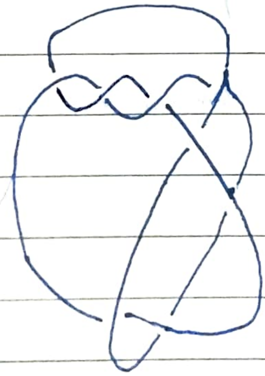


$4_1 \# 4_1$

$$V_{K_1}(t) = \left(\frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2\right)^2$$

$$\Delta_{K_1}(t) = \left(\frac{1}{t} - 3 + t\right)^2$$

K_2
$\cong$
8_9



$$V_{K_2}(t) = V_{K_1}(t) = \left(\frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2\right)^2$$

$$\Delta_{K_2}(t) = \frac{-1}{t^3} + \frac{3}{t^2} - \frac{5}{t} + 7 - 5t - 3t^2 - t^3$$

PAGE NO.:
DATE: / /

Thm • $V_K(t) = V_{K_1}(t) \cdot V_{K_2}(t)$ — Will be proved

$K = K_1 \# K_2$

• $\Delta_K(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$ — No proof available

$K = K_1 \# K_2$

Thm → If $K = K_1 \# K_2$, then we have –

$$V_K(t) = V_{K_1}(t) \cdot V_{K_2}(t)$$

Proof → Requires a lemma –

Let us denote $X \amalg Y$ (X disjoint union Y) the union of 2 sets X and Y that have no point in common

Ex – $O_2 = OO$. $O_2 = O_1 \amalg O_1$

Lemma – $$\boxed{V_{K \amalg O_n}(t) = (-1)^n \underbrace{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^n}_{V_{O_{n+1}}} V_K(t)} \quad -(*)$$

Proof by induction

Proof – By induction –

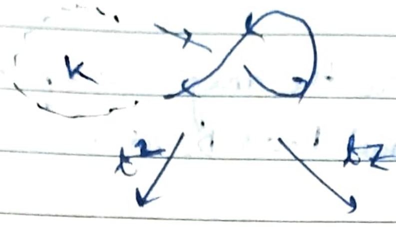
$n=1$

$$V_{K \amalg O}(t) = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_K(t)$$

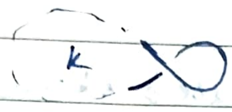
For any K, we can create an artificial extra crossing K^*

$\Longrightarrow$

artificial lexterior crossing created →



$K^+ \simeq K$



$K \sqcup O$

$K^+ \simeq K$

$$\frac{1}{t} V_K - t V_K = z V_{K \sqcup O}$$

$$\Rightarrow \left(\frac{1}{t} - t\right) V_K = z V_{K \sqcup O}$$

$$\Rightarrow \quad V_{K \sqcup O}(t) = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_K(t)$$

$(n=1)$

Let us assume lemma holds for $n-1$;

$$V_{K \sqcup O_{n-1}}(t) = (-1)^{n-1} \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-1} V_K(t)$$

Same idea →

$$\begin{tikzcd} & K \sqcup O_{n-1} \arrow[dl, "t^2"'] \arrow[dr, "tz"] & \\ K \sqcup O_{n-1} & & K \sqcup O_n \end{tikzcd}$$

$$\frac{1}{t} V_{K \sqcup O_{n-1}} - t V_{K \sqcup O_{n-1}} = z V_{K \sqcup O_n}$$

$$\Rightarrow \left(\frac{1}{t} - t\right) V_{K \sqcup O_{n-1}} = z V_{K \sqcup O_n}$$

$$\Rightarrow \boxed{V_{K \sqcup O_n} = (-1)^n \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^n V_K(t)}$$

Lemma proved.

$\Rightarrow$ Pf of thm.

27/1/24

Suppose D_1 and D_2 are the diagrams of k_1 & k_2 respectively.
Let us suppose (for now) that D_2 is a single black spot.

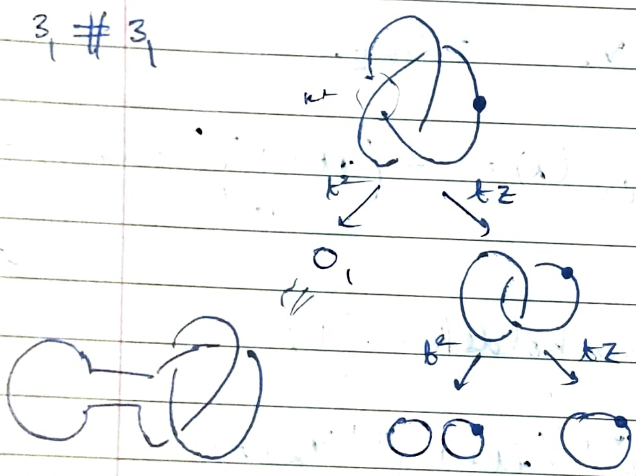
* Assuming D_2 as a single black spot.

So by ignoring D_2, we will create a skein diagram for D_1 as

$$V_{k_1}(t) = f_1(t)\,V_{O_1}(t) + f_2(t)\,V_{O_2}(t) + \dots + f_m(t)\,V_{O_m}(t)$$

(for some m)

$3_1 \# 3_1$



• expanded.

Now we replace the dot by D_2 to get

$$V_{k_1 \# k_2} = f_1(t)\,V_{D_2}(t) + f_2(t)\,V_{D_2 \amalg O_1}(t) + \dots + f_m(t)\,V_{D_2 \amalg O_{m-1}}(t)$$

Since, $V_{D_2 \amalg O_k} = (-1)^k \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^k V_{D_2}$

$$= V_{O_{k+1}} \cdot V_{D_2}$$

$\Rightarrow$

$$V_{K \amalg O_n}(t) = \underbrace{(-1)^n \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^n}_{V_{O_{n+1}}(t)} \underbrace{V_K(t)}_{V_K(t)}$$

$$V_{K_1 \# K_2}(t) = f_1(t)\, V_{O_1}(t)\, V_{D_2}(t) + f_2(t)\, V_{O_2}(t)\, V_{D_2}(t) + \dots + f_m(t)\, V_{O_m}(t)\, V_{D_2}(t)$$

$$= V_{D_2}(t) \left[f_1(t)\, V_{O_1}(t) + f_2(t)\, V_{O_2}(t) + \dots + f_m(t)\, V_{O_m}(t) \right]$$

$$= V_{D_2}(t)\, V_{D_1}(t)$$

$$= V_{K_1}(t)\, V_{K_2}(t)$$

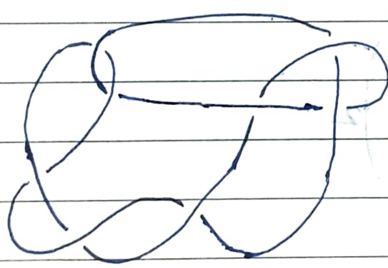
QED ✓ ▨

Knots with the same Jones & Alexander polynomials yet distinct.

(have same HOMFLY poly.)

$8_8 \simeq K_1$

$\boxed{\mu(K_1) = 2}$



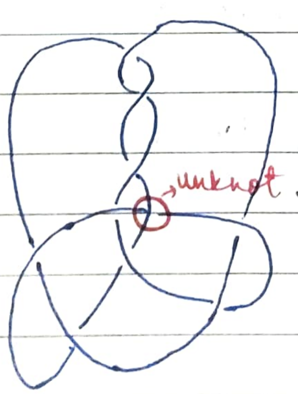
$$V_{K_1} = -t^{-3} + 2t^{-2} - 3t^{-1} + 5 - 4t + 4t^2 - 3t^3 + 2t^4 - t^5$$

$$\Delta_{K_1} = 2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2$$

$10_{129} \simeq K_2$

$\boxed{\mu(K_2) = 1}$

Prime knot

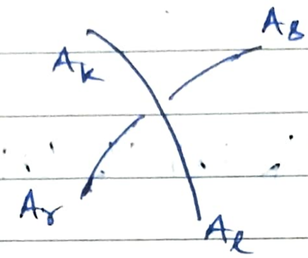


$$V_{K_2} = V_{K_1}$$

$$\Delta_{K_2} = \Delta_{K_1}$$

Classical Knot Invariant → Tricolorability

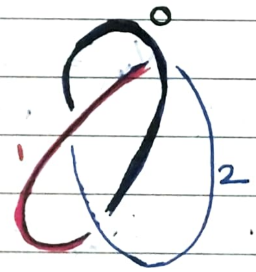
Defn → Suppose K has n crossing points $P_1, P_2, \ldots, P_n$ at each crossing point, we label the 4 segments A_k, A_r, A_s, A_l



We assign 3 colors R, B, Y, s.t.

(1) A_k & A_l have same color.

(2) A_k, A_r, A_s all have diff. colors or all have the same color.



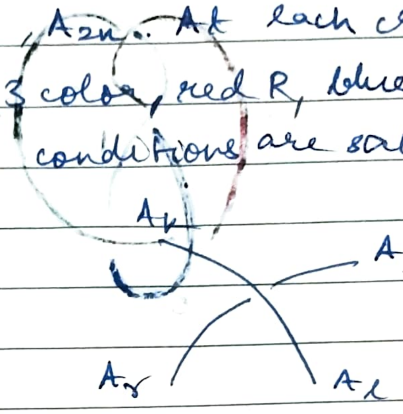
$$\lambda_r + \lambda_s \equiv \lambda_k + \lambda_l \pmod 3$$

If we can do this knot is tricolorable

$$\lambda_r + \lambda_s \equiv 2\lambda_k \pmod 3$$

31/1/24

Tricolorability – Let K be a knot with n crossing pts. $P_1, P_2, \ldots, P_n$. We can use these to divide K into $2n$ segments $A_1, A_2, \ldots, A_{2n}$. At each crossing point, we assign one of 3 color, red R, blue B, yellow Y s.t. the following conditions are satisfied.

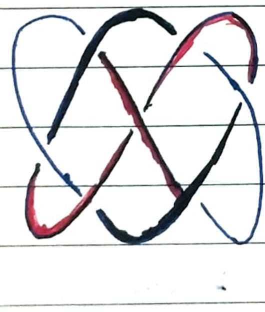


① A_k & A_l have the same colours.

② A_k, A_r, A_s all have the same color or all have different colour (*)

Def^n → A diagram D which satisfies this property (*) is tricolorable or 3-colorable.
(provided atleast 2 difft. colours are used)

Ex →



7_4

Def^n → Given a knot (or link) K is said to be p-colorable (p is prime)

If at each segment we assign an integer λ_i from the set of integers $0, 1, 2, \ldots, p-1$ s.t.

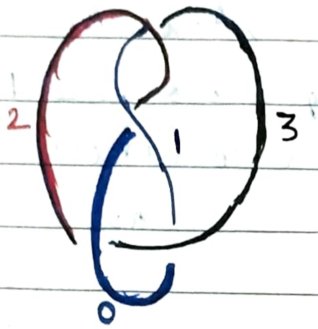
(1) $\lambda_k = \lambda_l$ (2) $\lambda_r + \lambda_s \equiv \lambda_k + \lambda_l \pmod p$.

$\Rightarrow$

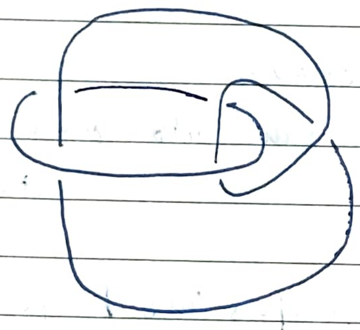
PAGE NO
DATE / /

**Note – Atleast 2 distinct integers must be used (not necessarily all of them)

4_1

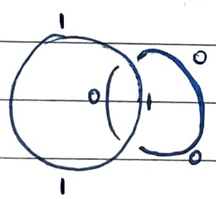


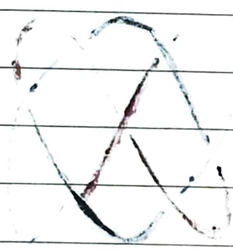
5-colorable ✓



2-colorable ✓

O_2 ⟶ 2-colorable ✓





Def.ⁿ A given knot (or link) may be p-colorable with respect to several different choices of p. The diff. choices of p which works are called the colouring number set of K.

- 3 colourable knots with $c(K) \leq 8$

$6_1, 7_4, 7_9, 8_5, 8_{10}, 8_{11}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}$

Twist
Poke
Slide

The Reidemeister Moves -

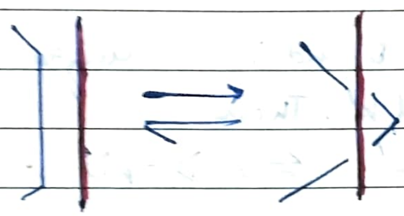
Def^n - 2 knots K_1 and K_2 are equivalent if we can change K_1 to K_2 by finitely many knot moves.

Def^n - Reidemeister Moves -
R moves on a knot diagram D are as follows :-

1) Ω_1

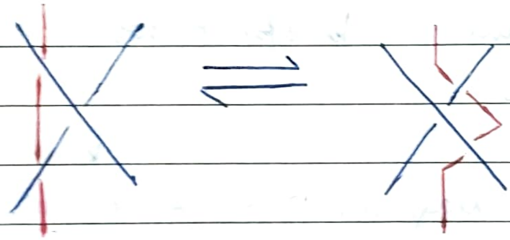
"Twist move"

2) Ω_2



"Poke move"

3) Ω_3



"Slide move"

$D_1 \underset{R}{\simeq} D_2$

Def^n → 2 knot diagrams D_1 and D_2 are R-equivalent if we can get from D_1 to D_2 by finitely many R-moves.

Thm → $\boxed{K_1 \simeq K_2 \iff D_1 \underset{R}{\simeq} D_2}$

2/2/24

Note → A knot (or link) invariant is a property which is unchanged under the elementary knot moves.

Note → It is useful to project the knot onto the plane and study it via its regular diagram.

We ask - What is the effect on the diagram D when we perform elementary knot moves on the knot K.

This was studied in 1920s by German mathematician Reidemeister

Def^n → If we can change a knot diagram D to another D' by applying $\Omega_1, \Omega_2, \Omega_3$ (3 R-moves) finitely many times, then we say

$$D \simeq D' \text{ (equivalent).}$$

Thm (Reidemeister) If D and D' are diagrams of knots (links) K and K'. Then

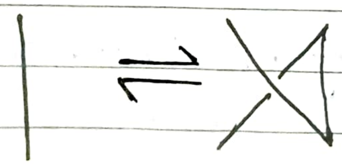
$$K \simeq K' \iff D \simeq_R D'$$

- A knot invariant is a property which is unchanged under the 3 R-moves.

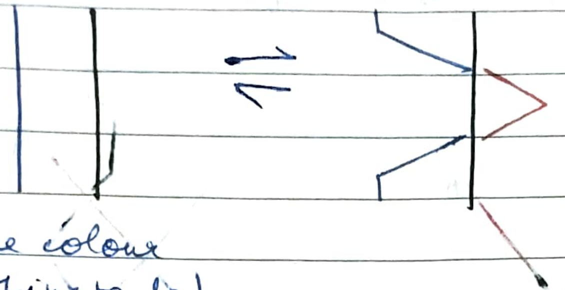
Thm → If there is a diagram D of a knot K which is 3-colourable then every diagram of K is 3-colorable. Such a knot is 3-colorable.

Proof - Suppose D and D' are 2 diagrams of K.
We can get from D to D' by finitely many R-moves $\Omega_1, \Omega_2, \Omega_3$ and their inverses. Thus to prove the thm it is enough to show tricolorability is preserved under Ω_1, Ω_2 and Ω_3.

(1) Ω_1

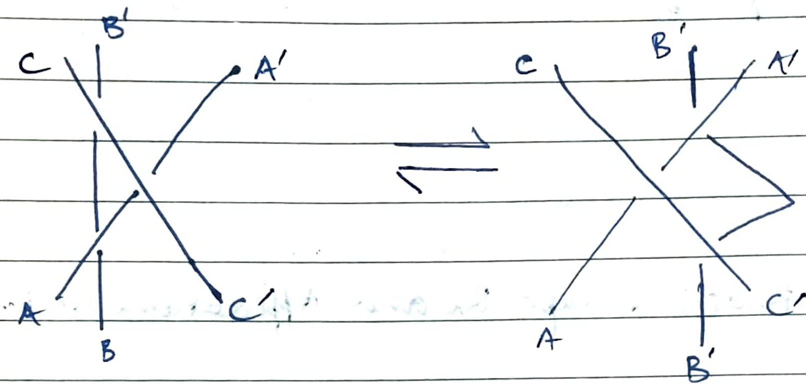


Ω_2



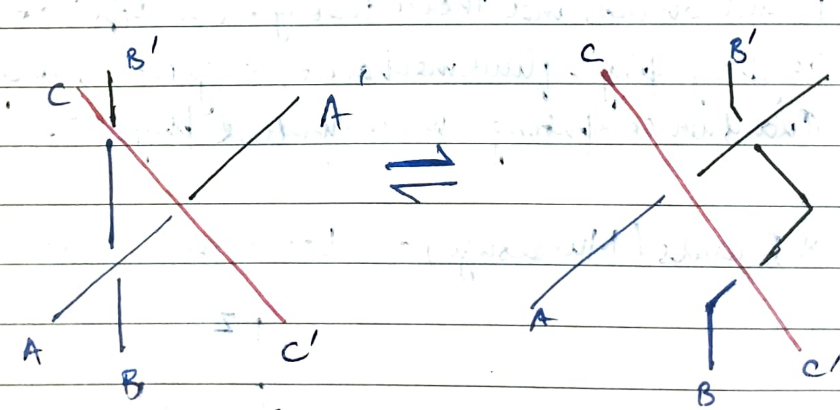
If of same colour
(nothing to do)

Ω_3



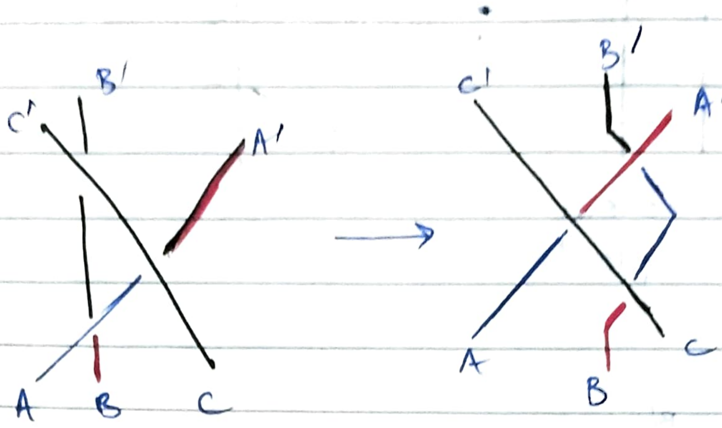
Case 1 All have same color → Trivial.

Case 2 A, B have color α (Blue), C color β (Red)
ya third color γ — (black)



PAGE NO.
DATE / /

Case 3 → A, B, C each has a diff. color.

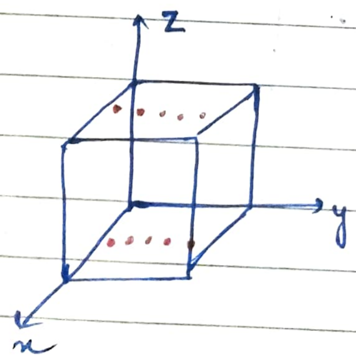


5/2/24 Braid Groups B_n and Applications to Knot Theory

History → In 1930s, Artin introduced the concept of a "mathematical braid" to study knots.
This was the tool used by Jones to discover his "Jones polynomial".

Braid Groups and their theory have applications in cryptography, fluid mechanics, algebra, anyons and Quantum computing, mathematical physics.

\# n Braids (Murasugi) — Let B be a cube $0 \leq x, y, z \leq 1$



PAGE NO.
DATE / /

We mark out n-points.

$A_1, A_2, \ldots, A_n$ at the top of the cube.

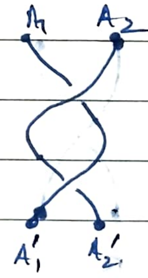
$A_1', A_2', \ldots A_n'$ at the bottom of the cube.

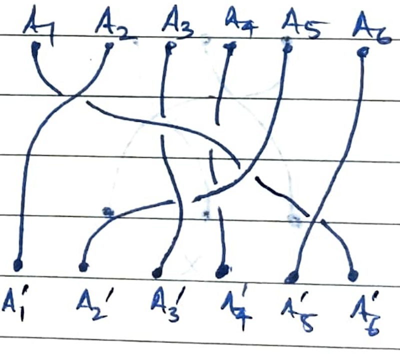
$$A_1 = \left(\frac{1}{2}, \frac{1}{n+1}, 1\right), A_2 = \left(\frac{1}{2}, \frac{2}{n+1}, 1\right) \ldots, A_n = \left(\frac{1}{2}, \frac{n}{n+1}, 1\right)$$

$$A_1' = \left(\frac{1}{2}, \frac{1}{n+1}, 0\right), A_2' = \left(\frac{1}{2}, \frac{2}{n+1}, 0\right), \ldots, A_n' = \left(\frac{1}{2}, \frac{n}{n+1}, 0\right)$$

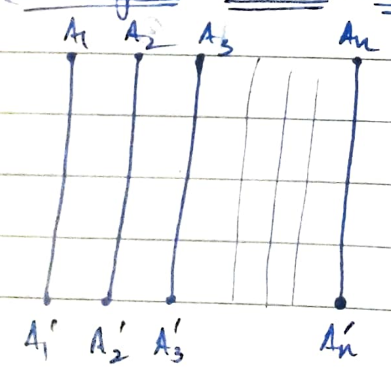
Now join $A_1, A_2, \ldots, A_n$ to $A_1', A_2', \ldots A_n'$.
by means of n-curves s.t. they do not intersect each other. We will call these n-curves as strings, s.t. the n-strings go strictly monotone downwards. (every horizontal plane cuts it only once)

Ex → 1-braid 2-braid n-braid





Identity or Trivial n-Braid →



Def^n → Two n braids α and β are said to be equivalent if we can convert α to β by finitely many elementary knot moves. $(\alpha \simeq \beta)$.

Braid Permutation –

(an invariant for braids)

$A_1 \to A_{j_1}$

$A_2 \to A_{j_2}$

$\vdots$

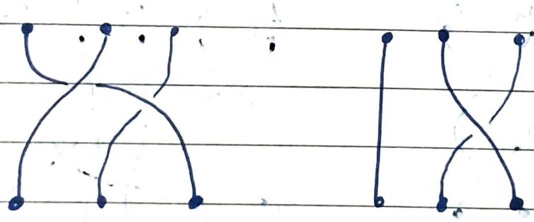
$A_n \to A_{j_n}$

The permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}.$$

Suppose B_n is the set of all n-braids (up to equivalence class). For 2 elements in B_n, α and β, we define their group composition by first doing α, followed by β.

(non-abelian) ✓.

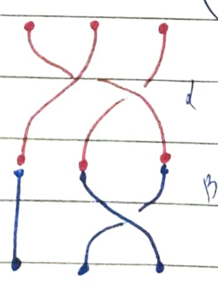


α $\qquad\qquad$ β

First α then β

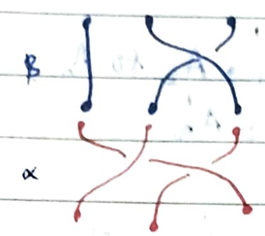
$$\alpha . \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$\alpha.\beta$



$$\beta.\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$\beta.\alpha$



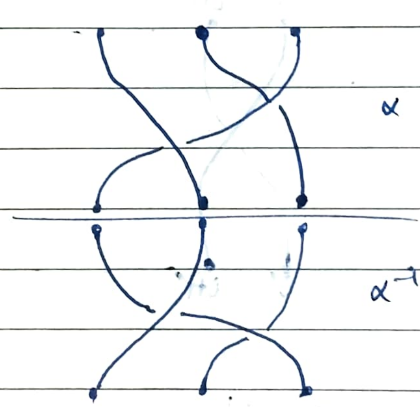
$\alpha.\beta \neq \beta.\alpha$

(Non-abelian)

Braid product is not commutative but always associative.

$$(\alpha.\beta).\gamma = \alpha.(\beta.\gamma)$$ (Intuitive)

- Given α, its inverse α^{-1} the mirror image of α (in the case of cube) will be the inverse.



$\alpha.\alpha^{-1} = 1$

7/2/24

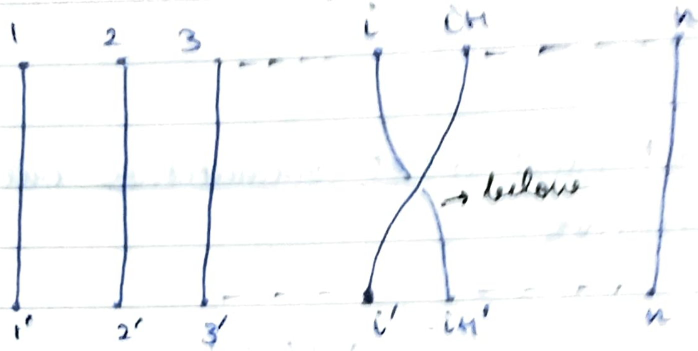
So, since B_n satisfies closure, associativity, identity & inverse properties, hence B_n is a group.

$B_n \longrightarrow$ not abelian

$\Longrightarrow$

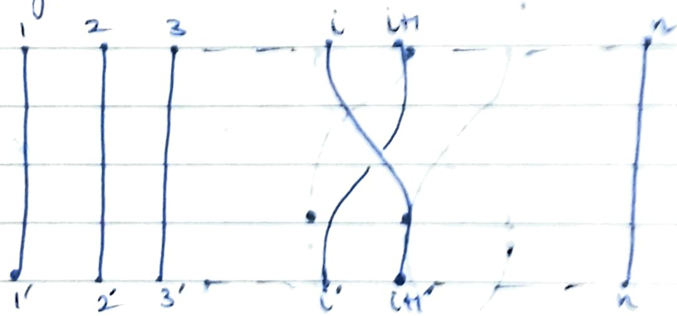
Among the n-Braids, we create certain special n-braids as follows :-

- We connect A_1 to A_1', A_2 to A_2', ..., A_i to A_{i+1}', A_{i+1} to A_i', ..., A_n to A_n'



This braid is called σ_i.

Similarly, we have σ_i^{-1} :-



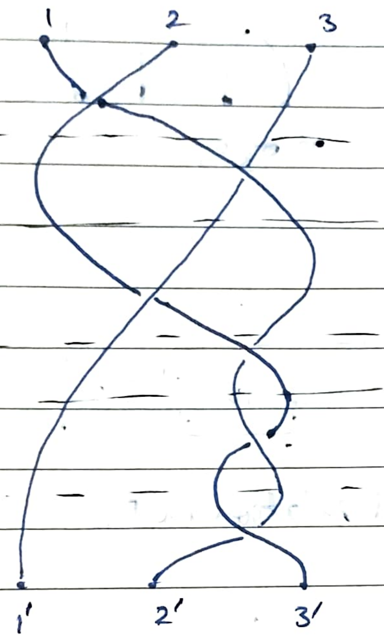
Clearly, we see $(\sigma_i)(\sigma_i)^{-1} = e$.

- In this way, we create in B_n, $n-1$ special braids called $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$

- Now given any $\alpha \in B_n$, we can express α as a finite product of σ_i and σ_j^{-1}

$\rightarrow$ for this reason, the braids $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and $\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}$ are said to generate B_n, and are called the generators of B_n.

Exn $B_3 \equiv$



$\equiv \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_2^{-1} \sigma_2^{-1}$

Consider $B_2 \equiv$ In B_2, generator is only σ_1 & σ_1^{-1}. Any $\alpha \in B_2$

then $\alpha = \sigma_1^{m}$; $m \geq 0$

$\alpha = \sigma_1^{-n}$; $n \geq 0$

$B_2 \equiv$ abelian

$B_i \not\equiv$ not abelian.
$i \geq 2$

$\boxed{B_2 \simeq (\mathbb{Z}, +)}$

Braid Words - B_3, a particular braid

$\alpha = \underbrace{\sigma_1 \sigma_2^{-1} . \sigma_1 \sigma_2^{-1}}$

are called Braid Words.

- But not unique.

Not unique $\rightarrow$

$\Longrightarrow$

By $\sigma_1 \sigma_3$ & $\sigma_3 \sigma_1$

$\Rightarrow \sigma_1 \sigma_3$

$\Rightarrow \sigma_3 \sigma_1$

So $\sigma_1 \sigma_3 \simeq \sigma_3 \sigma_1$

Similarly, In $B_n \rightarrow$

$\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$

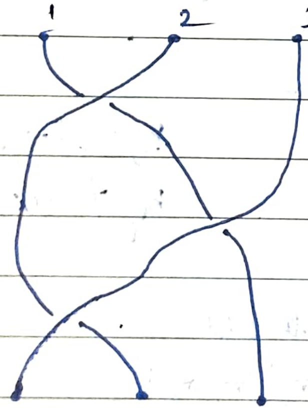
$$\boxed{\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2}$$

$\rightarrow$ 1st fundamental relation in B_n.

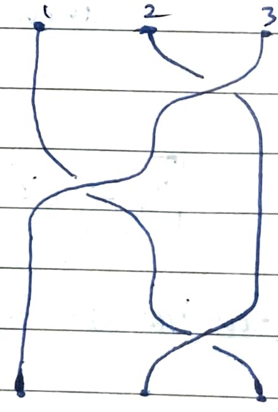
2nd fundamental relation in $B_n \rightarrow$

B_3; consider $\alpha = \sigma_1 \sigma_2 \sigma_1$; $\beta = \sigma_2 \sigma_1 \sigma_2$

$\alpha =$



$\beta =$



$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

By sliding 2 we can get or by 90° rotation.

So, $$\boxed{\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}}$$ $\rightarrow$ 2nd fundamental relation.

PAGE NO.:
DATE: / /

9/2/21

For any n, we have defined B_n - the n^{th} Braid group; which is a free group with $n-1$ generators, and 2 fundamental relations

$$B_n = \{(\sigma_1, \sigma_2, \ldots \sigma_{n-1}) \mid (1)\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2$$
$$(2)\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } i = 1, 2, \ldots, n-2$$

(1) $B_3 = \{(\sigma_1, \sigma_2) \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$

$$B_4 = \left\{(\sigma_1, \sigma_2, \sigma_3) \;\middle|\; \begin{array}{l} \sigma_1\sigma_3 = \sigma_3\sigma_1 \\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \end{array}\right\}$$

Ex → Show that the Braid words →

B_5 $\quad w_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\sigma_2\sigma_4$ are equivalent.

$\quad w_2 = \sigma_2\sigma_1\sigma_2^2$

Ans. $w_1 = \sigma_1\sigma_2\sigma_4^{-1}\sigma_1\underbrace{\sigma_2\sigma_4}$

$= \sigma_1\sigma_2\sigma_4^{-1}\underbrace{\sigma_1\sigma_4}\sigma_2$

$= \sigma_1\sigma_2\cancel{\sigma_4^{-1}\sigma_4}\sigma_1\sigma_2$

$= \underbrace{\sigma_1\sigma_2\sigma_1}\sigma_2$

$= \sigma_2\sigma_1\sigma_2^2 = w_2$

$B_1 = \{e\}$

$B_2 = \{\sigma_1^m \mid m \in \mathbb{Z}\}$ $\quad B_2 \simeq (\mathbb{Z}, +)$

PAGE NO
DATE / /

Knots and Braids (Murasugi)

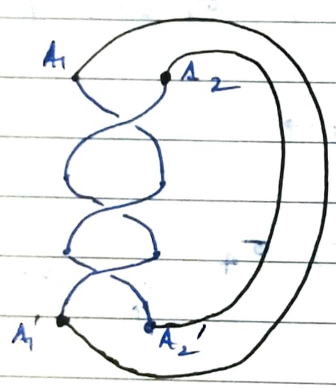
Given an n-braid $\alpha \in B_n$, let us connect by a set of parallel arcs (outside the braid diagram) the points $A_1, A_2, \dots, A_n$ at the top of the braid to $A_1', A_2', \dots A_n'$ respectively at the bottom of the braid diagram.

When we do this, we will get either a knot or link K. K is called a closed braid or the closure of the braid α — $\overline{\alpha}$

Closure α

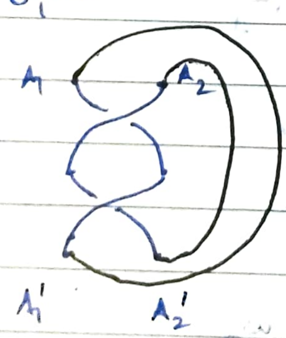
$\overline{\alpha}$

Ex→ B_2 (1) $\alpha = \sigma_1 \sigma_1 \sigma_1$



$= 3_1$

(2) $\alpha = \sigma_1^2$



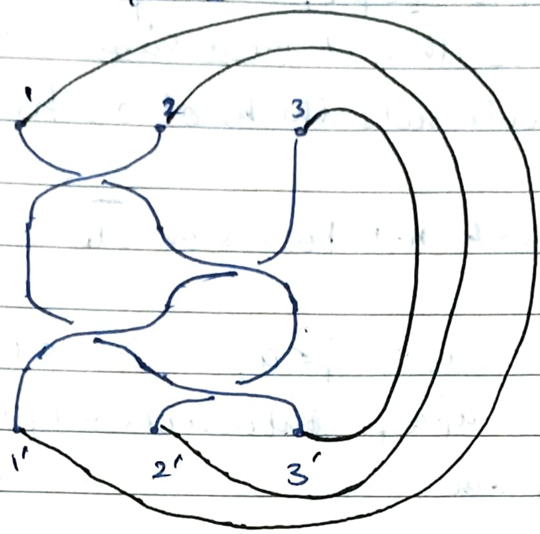
= Hopf Link

(3) $\alpha = \sigma_1^5$

$\overline{\alpha} = 5_1$

B_3

(4) $\alpha = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$



$= 4_1$

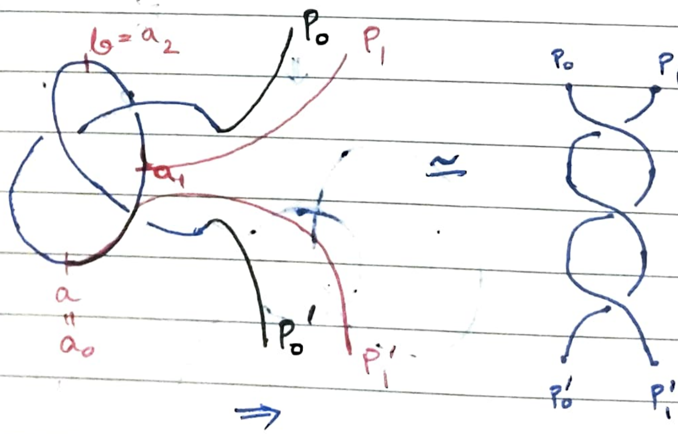
(1923) "Alexander's theorem" – Every knot has a braid diagram.

Given any arbitrary knot or link (K), then it is equivalent to a knot or link that has been formed from some braid $\alpha \in B_n$ i.e. K is the closure of α, sometimes written $\bar{\alpha}$.

by construction

Proof → Step 1 – Suppose D is a diagram of a knot (or link). First we cut D at a point P_0 (P_0 is not a crossing point) and pull the loose ends apart; call them P_0, P_0'

eg. $K = 3_1$



$\sigma = (\sigma_1^{-1})^3$

⇒

Step 2 → Suppose after step 1, the diagram has one local ~~maxima~~ minima a, then it must have a local max b. The arc ab intersects the knot at finitely many ~~places~~ points.

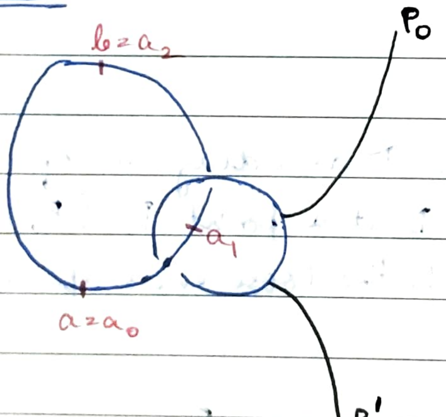
Let us mark $n-1$ points on $\overline{ab}$

$$a = a_0 < a_1 < a_2 < \dots < a_n = b$$

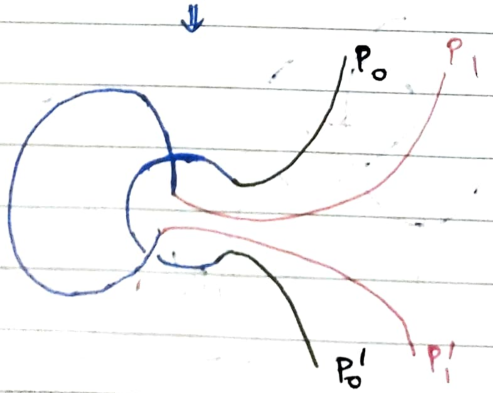
such that $\overline{a_i a_{i+1}}$ intersects only one crossing pt. of the diagram.

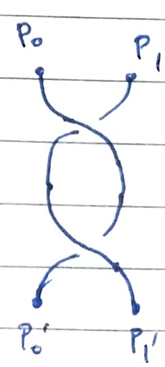
Step 3 → Replace the arc $\overline{a_0 a_1}$ by the larger piece $a_0 P_1' P_1 a_1$ s.t. when we join P_1 to P_1', we get back $\overline{a_0 a_1}$.
We continue in this manner for $\overline{a_1 a_2}, \overline{a_2 a_3}, \dots \overline{a_{n-1} a_n}$ till we get a n-braid α s.t. $\overline{\alpha} = K$.

Ex → **Hopf Link** –



⇓



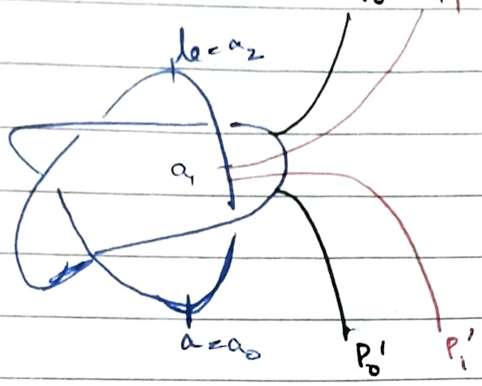


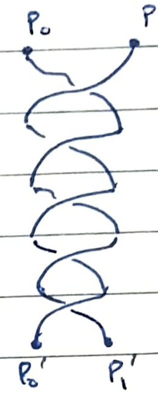
PAGE NO :
DATE : / /

H.W. Find the braid words for 4_1, 5_1, 6_2, 7_1, 6_3

12/2/24

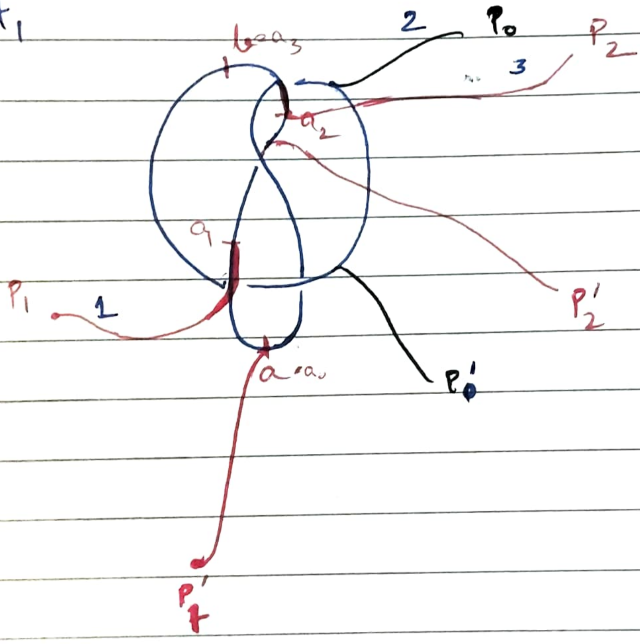
Ex: 5_1

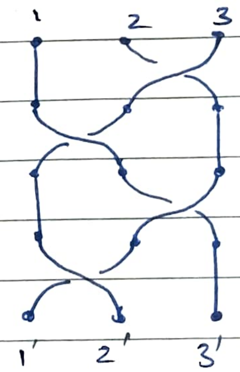




$(\sigma_1)^5$

4_1





$\sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1}$