MTH 101-Calculus

Spring-2021

Assignment 7-Solutions : Improper Integrals, Appl. of Integration, Pappus Theorem

1. (a) Converges by limit comparison test (LCT) with $\frac{1}{\sqrt{x}}$.

   (b) Diverges by LCT with $\frac{1}{x^2}$.

   (c) The integral $-\int_0^1 \frac{\log x}{\sqrt{x}}dx$ converges by LCT with $\frac{1}{x^p}$, where $\frac{1}{2} < p < 1$.

   (d) Since $|\sin\frac{1}{x}| \leq 1$, the integral converges. Note that in this case the integral is a proper integral.

   (e) Converges by LCT with $\frac{1}{x^2}$.

   (f) Converges by LCT with $\frac{1}{x^p}$, where $p \geq 2$.

   (g) Note that the integral converges iff $\int_1^\infty \sin x^2 dx$ converges. By substituting $t = x^2$, we get $\int_1^c \sin x^2 dx = \frac{1}{2}\int_1^{c^2} \frac{\sin t}{\sqrt{t}}dt$. Use Drichlet test.

   (h) $\int_x^{\frac{\pi}{2}} \cot x dx = -\log\sin x \to \infty$ as $x \to 0^+$.

   (i) $\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = \int_0^1 \frac{x\log x}{(1+x^2)^2}dx + \int_1^\infty \frac{x\log x}{(1+x^2)^2}dx = I_1 + I_2$.

   Since, $\lim\limits_{x\to 0} x\log x = 0$, I_1 is a proper integral.

   For large x, $\log x \leq x$. Hence $\frac{x\log x}{(1+x^2)^2} \leq \frac{x^2}{(1+x^2)^2} \leq \frac{1}{1+x^2}$ and I_2 converges.

   Use the substitution $x = \frac{1}{t}$ in I_1 to get $I_1 = -I_2$.

2. (a) Let $f(x) = \frac{1-e^{-x}}{x^p}$ and denote $I_1 = \int_0^1 f(x)dx$ and $I_2 = \int_1^\infty f(x)dx$. By LCT with $\frac{1}{x^{p-1}}$, I_1 converges only for $p < 2$. Similarly by LCT with $\frac{1}{x^p}$, I_2 converges only for $p > 1$. Therefore $\int_0^\infty f(x)dx$ converges if and only if $1 < p < 2$.

   (b) $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^1 \frac{\sin^2 x}{x^2}dx + \int_1^\infty \frac{\sin^2 x}{x^2}dx = I_1 + I_2$.

   I_1 is a proper integral and I_2 converges by a comparison with $\frac{1}{x^2}$.

   Similarly $\int_0^\infty \frac{\sin x}{x}dx$ converges by Drichlet test.

   Using integration by parts we see that
   $\int_0^\infty \frac{\sin^2 x}{x^2}dx = -\frac{\sin^2 x}{x}\Big|_0^\infty + \int_0^\infty \frac{2\sin x\cos x}{x}dx = \int_0^\infty \frac{\sin 2x}{2x}d(2x) = \int_0^\infty \frac{\sin x}{x}dx$.

3. (a) (i) Fix n and let $\epsilon < \frac{1}{n}$. Consider the partitions

$$P_1 = \{\epsilon, \frac{1}{n}, \frac{2}{n}, ..., 1 - \frac{1}{n}\} \quad \text{and} \quad P_2 = \{\frac{1}{n}, \frac{2}{n}, ..., 1 - \frac{1}{n}, 1 - \epsilon\}$$

for the intervals $[\epsilon, 1 - \frac{1}{n}]$ and $[\frac{1}{n}, 1 - \epsilon]$ respectively. Then

$$\int_\epsilon^{1-\frac{1}{n}} f(x)dx \leq U(P_1, f) = (\frac{1}{n} - \epsilon)f(\frac{1}{n}) + \frac{f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n}.$$

Allow $\epsilon \to 0$ to get

$$\int_0^{1-\frac{1}{n}} f(x)dx \leq \frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n}.$$

Similarly, by using the inequality $L(P_2, f) \leq \int_{\frac{1}{n}}^{1-\epsilon} f(x)dx$, we get that

$$\frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n} \leq \int_{\frac{1}{n}}^{1} f(x)dx.$$

(a) (ii) Allow $n \to \infty$.

(b) Take $f(x) = ln(x)$ in a(ii).

(c) Follows from (b).

4. Let $r = \cos 2\theta$. Between $\theta = 0$ to $\theta = \frac{\pi}{4}$, we plot (r, θ) (in polar coordinate) i.e., for each θ we find r. The graph lies in the first quadrant for these $\theta's$.

   Note that, since r is negative for $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$, if we sketch the graph for these $\theta's$, the graph appears in the third quadrant.

   Whenever $(r, \theta) \in G$, the graph, we see that $(r, -\theta), (r, \pi - \theta), (r, \pi + \theta) \in G$. Therefore, there is symmetry about the x-axis, y-axis and the origin.

   Let $r = \sin 2\theta$. Again, we see that there is symmetry about the x-axis, y-axis and the origin.

5. Each cross section is a rectangle of area $A(x) = x.2.\sqrt{9 - x^2}$. Therefore the volume $V = \int_0^3 2x\sqrt{9 - x^2}dx = 18$.

6. (a) Note that the disc is bounded by the curves $x = a + \sqrt{b^2 - y^2}$ and $x = a - \sqrt{b^2 - y^2}$. The volume of the torus, evaluated by the Washer Method, is $\pi \int_{-b}^{b} \left((a + \sqrt{b^2 - y^2})^2 - (a - \sqrt{b^2 - y^2})^2\right) dy = 4a\pi \int_{-b}^{b} \sqrt{b^2 - y^2}dy$. The last integral is the area of the semicircle of radius b. Therefore the volume is $2\pi^2 ab^2$.

   (b) The volume of the torus is same as the volume of the torus generated by revolving the circular disc $x^2 + y^2 \leq b^2$ about the line $x = a$. Using the Shell Method, we find that the volume is $\int_{-b}^{b} 2\pi(a - x)(2\sqrt{b^2 - x^2})dx = 4\pi \left[\int_{-b}^{b} a\sqrt{b^2 - x^2}dx - \int_{-b}^{b} x(\sqrt{b^2 - x^2})dx\right]$ $= 4\pi a \int_{-b}^{b} \sqrt{b^2 - x^2}dx$.

7. (a) The length $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt = \int_0^{\frac{\pi}{2}} 3|\cos t \sin t|dt = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin 2t dt = \frac{3}{2}$.

   (b) The surface area $S = \int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}dt = \int_0^{\frac{\pi}{2}} 2\pi(\sin^3 t)(3 \sin t \cos t)dt = 6\pi \int_0^{\frac{\pi}{2}} \sin^4 t \cos t dt$ $\frac{6\pi}{5}$.

   (c) By Pappus theorem $S = 2\pi\overline{y}L$ which implies that $\frac{6\pi}{5} = 2\pi\overline{y}\frac{3}{2}$. Therefore $\overline{y} = \frac{2}{5}$.

8. By Pappus Theorem $V(\theta) = 2\pi\rho A$, where ρ is the distance of the centroid from the axis and A is the area of the square.

   $V(\theta) = 2\pi a^2 \frac{a}{\sqrt{2}} \sin\theta$. The volume will be largest if $\theta = \frac{\pi}{2}$

9. Let the coordinates of the centroid be (r, y_0).

   By Pappus Theorem, $4\pi r^2 = 2\pi\pi r y_0$. Hence $y_0 = \frac{2r}{\pi}$ and the centroid has coordinates $(r, \frac{2r}{\pi})$.

   Distance of centroid from the line $y = -mx$ is $\rho = \frac{mr + \frac{2r}{\pi}}{\sqrt{1+m^2}}$.

Again, by Pappus Theorem, we see that $A = 2\pi\rho\pi r$.

$\frac{dA}{dm} = 0 \Rightarrow m = \frac{\pi}{2}$. Easy to see that A has a maxima at $\frac{\pi}{2}$.