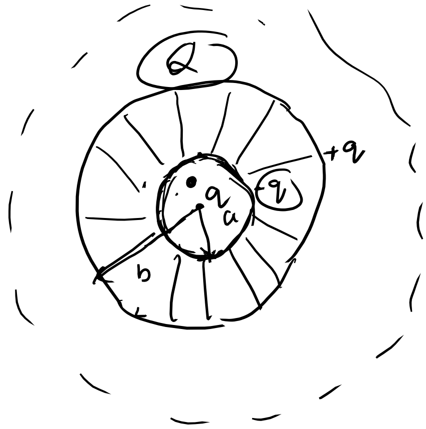
Department of Physics

IIT Kanpur, 2021-22 Ist Semester

PHY103AA (Physics - II) **Tutorial # 4** **28-12-2021**

Problem 4.1: A thick metallic shell of inner radius *a* and outer radius *b* has a charge *Q* on it. A point charge *q* is kept at the center of the shell. Calculate the charge on each surface of the shell. Also, calculate the electric field and potential everywhere.



$\Rightarrow$ $+q$ is at the center of shell

$\Rightarrow$ $-q$ is distributed over surface of radius a.

$\Rightarrow$ $(Q+q)$ is distributed over surface of radius b.

$$\Rightarrow \quad \sigma_{inner} = \frac{-q}{4\pi a^2}$$

$$\sigma_{outer} = \frac{(Q+q)}{4\pi b^2}$$

Electric field $\quad r \leq a \; : \; \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

" " $\quad a < r < b \; : \; \vec{E} = 0$

" " $\quad r > b \; : \; \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{(Q+q)}{r^2}\hat{r}$

at $r = a \Rightarrow E_{in} = \frac{\sigma}{\epsilon_0}$, $E_{out} = 0$ B.C. for E field.

Potential in the region $r > b \Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\frac{(Q+q)}{r}$

" " " " $a < r < b \Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\frac{(Q+q)}{b}$

" " " " $r < a \Rightarrow V(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q+q}{b} - \frac{q}{a} + \frac{q}{r}\right]$

$\Rightarrow$ **CHECK**

(i) $q = 0, \ Q \neq 0$ ✓

(ii) $q \neq 0, \ Q = 0$ ✓

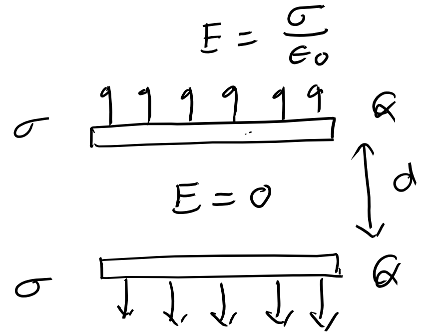
(iii) $(q + Q) = 0$ ✓

(iv) $q \neq 0$, but shell is grounded. ✓

Problem 4.2: Do the following problems explicitly by calculating force and also by energy method.

(a) Two large metal plates (each of area A) are held a small distance d apart. Suppose we put a charge Q on each plate; what is the electrostatic pressure on the plates?

(b) A metallic sphere of radius R carries a total charge Q. what is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

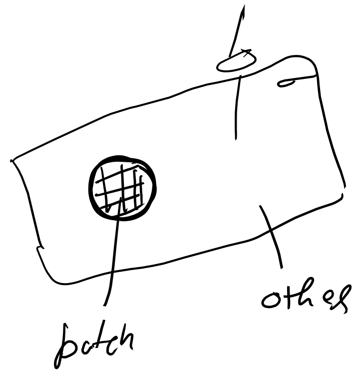


(a) $\vec{E}$ between the plates $= 0$

$|\vec{E}|$ out side " " $= \frac{\sigma}{\epsilon_0}$

Electrostatic pressure

$$f = \frac{F}{A} = \frac{\sigma^2}{2\epsilon_0} = \frac{\epsilon_0 E^2}{2}$$

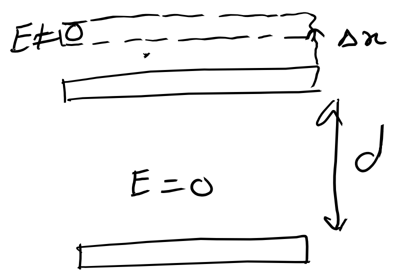


$$\sigma = \frac{Q}{A}, \quad E = \frac{\sigma}{\epsilon_0}$$

$$f = \frac{Q^2}{2\epsilon_0 A^2}$$

$\Rightarrow$ force on the top plate $F = \frac{Q^2}{2\epsilon_0 A}$

Energy method



Energy density of E field

$$u = \frac{U}{\text{volume}} = \frac{1}{2}\epsilon_0 E^2$$

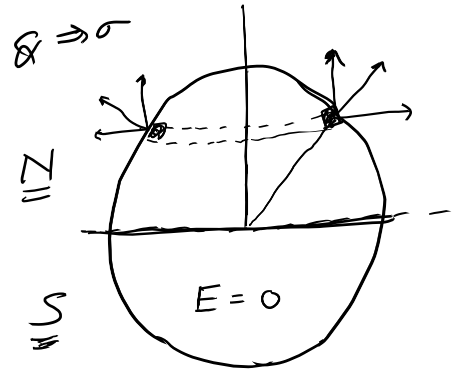
change in the energy $\Delta U = -\frac{1}{2}\epsilon_0 E^2 (A \cdot \Delta x)$ $\qquad E = \frac{\sigma}{\epsilon_0}$

$$\Rightarrow \Delta U = \frac{-1}{2}\epsilon_0 E^2 A \cdot \Delta x$$

$$F = -\frac{\Delta U}{\Delta x} = \frac{1}{2}\epsilon_0 E^2 \cdot A \Rightarrow F = \frac{Q^2}{2\epsilon_0 A}$$

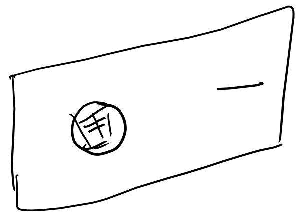
(b)

$$\frac{F}{\text{area}} = \frac{\sigma^2}{2\epsilon_0}$$



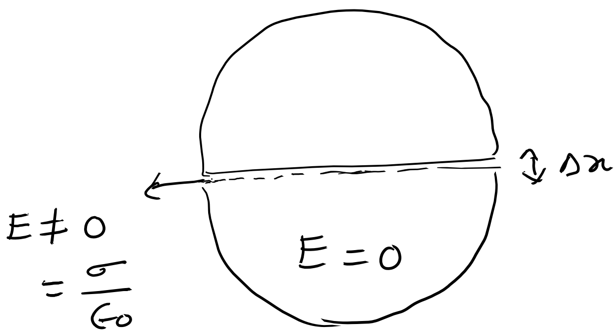
$$F_{net} = \int_{\substack{\text{northern}\\ \text{hemisphere}}} \frac{\sigma^2}{2\epsilon_0} \cos\theta \, R^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{2\epsilon_0}\left(\frac{Q}{4\pi R^2}\right)^2 R^2 \, 2\pi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$



$$\boxed{F = \frac{Q^2}{32\pi\epsilon_0 R^2}}$$

Energy approach



change in volume

$$= \pi R^2 \Delta x$$

change in energy $\Delta U = -\left(\frac{1}{2}\epsilon_0 E^2\right) \cdot \pi R^2 \Delta x$

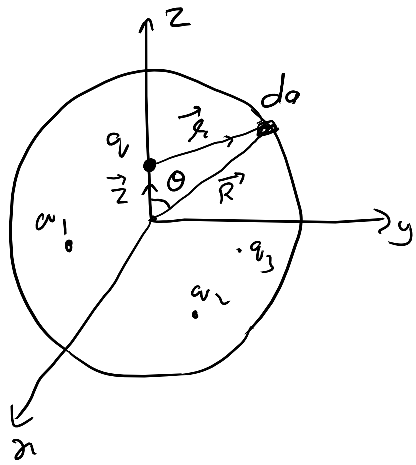
$$F = -\frac{\Delta U}{\Delta x} = \frac{\sigma^2}{2\epsilon_0} \pi R^2 .$$

$$\boxed{F = \frac{Q^2}{32\pi\epsilon_0 R^2}}$$

Problem 4.3: Find the average potential over a spherical surface of radius R due to a point charge q located inside the sphere. Also, show that in general

$$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\varepsilon_0 R}$$

where V_{center} is potential at the center due to all the external charges and Q_{enc} is the total enclosed charge.



V' is the potential at an small area element da,

$$V'(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Average potential on the surface of the sphere due to charge q:

$$\vec{R} = \vec{z} + \vec{r}$$
$$\Rightarrow \vec{r} = \vec{R} - \vec{z}$$

$$V_{ave} = \frac{1}{4\pi R^2}\oint_{sphere} V'\,da = \frac{1}{4\pi R^2}\int \frac{q}{4\pi\epsilon_0}\frac{R^2\sin\theta\, d\theta\, d\phi}{\sqrt{R^2+z^2-2zR\cos\theta}}$$

$$\Rightarrow V_{ave} = \frac{q}{4\pi\epsilon_0}\frac{2\pi R^2}{4\pi R^2}\int_{-1}^{+1}\frac{d(\cos\theta)}{\sqrt{R^2+z^2-2Rz\cos\theta}}$$

$$\boxed{V_{ave} = \frac{1}{4\pi\epsilon_0}\frac{q}{R}}$$

$$V_{ave} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{R}$$

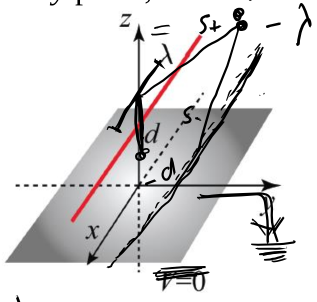
Q_{enc} = total enclosed charge inside the sphere.

$V_{ave} = V_{center}$ due to charge outside

$\Rightarrow$ Total averge potential

$$\boxed{V_{ave} = \underline{\underline{V_{center}}} + \frac{Q_{enc}}{4\pi \epsilon_0 R}}$$

Problem 4.4: An infinite line charge runs parallel to the x-axis at a distance *d* from the *xy* plane, which is an infinite grounded conductor as shown in the figure.
(a) What is the potential in the region above the plane.
(b) Find the charge density σ induced on the conducting plane.



$$\Rightarrow V_+ = -\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_+}{d}\right) \rightarrow \text{due to line charge } \lambda$$

$$V = -\int_{d}^{s_+} E\, dl$$

$$V_- = +\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_-}{d}\right)$$

$\Rightarrow$ Total potential $$V = \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_-^2}{s_+^2}\right)$$

$$\boxed{V(y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]}$$

(b) induced surface charge density

$$\sigma = -\epsilon_0\frac{\partial V}{\partial n}$$

$\hat{n} \rightarrow$ unit surface normal vector

$$\sigma = -\epsilon_0\left.\frac{\partial V}{\partial z}\right|_{z=0}$$

$$\Rightarrow \boxed{\sigma = \frac{\lambda d}{\pi(y^2+d^2)}} \rightarrow \text{induced surface ch}$$

$$\rightarrow E = \frac{\sigma}{2\epsilon_0}$$