MTH 101-Calculus
Spring-2021
Assignment 5 : Series, Power Series, Taylor Series

1. Let $f : [0,1] \to \mathbb{R}$ and $a_n = f(\frac{1}{n}) - f(\frac{1}{n+1})$. Show that if f is continuous then $\sum_{n=1}^{\infty} a_n$ converges and if f is differentiable and $|f'(x)| < 1$ for all $x \in [0,1]$ then $\sum_{n=1}^{\infty} |a_n|$ converges.

2. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$ where a_n equals:

   (a) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$ (b) $1 - \cos\frac{1}{n}$ (c) $2^{-n-(-1)^n}$ (d) $\left(1+\frac{1}{n}\right)^{n(n+1)}$

   (e) $\frac{n \ln n}{2^n}$ (f) $\frac{logn}{n^p}, (p > 1)$ (g) $e^{-n}(\cos n)n^2 \sin\frac{1}{n}$

3. Let $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all $n \geq N$. Show that if $\sum\limits_{n=1}^{\infty} b_n$ converges then $\sum\limits_{n=1}^{\infty} a_n$ also converges. Test the series $\sum\limits_{n=1}^{\infty} \frac{n^{n-2}}{e^n n!}$ for convergence.

4. Show that the series $\frac{1}{4^1} + \frac{1}{5^2} + \frac{3}{4^3} + \frac{1}{5^4} + \frac{5}{4^5} + \frac{1}{5^6} + \frac{7}{4^7} + \cdots$ converges.

5. Show that the series $\sum_{n=1}^{\infty} (-1)^n \sin\frac{1}{n}$ converges but not absolutely.

6. Determine the values of x for which the series $\sum\limits_{n=1}^{\infty} \frac{(x-1)^{2n}}{n^2 3^n}$ converges.

7. Among π^e and e^π which one is bigger ?

8. For a complex number $z = x + iy$, define e^z to be $e^x.e^{iy}$. Assuming the fact that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ converges to e^z absolutely, show that $e^{ix} = cosx + isinx$ for $x \in \mathbb{R}$.