# Indian Institute of Technology Kanpur
## Department of Mathematics and Statistics
Complex Analysis (MTH 403)
Hints for Exercise Sheet 6

### 1. INTEGRALS ALONG REAL LINE

1.1. Let $b > 0$. Show the following:

$$\int_0^\infty e^{-x^2} \cos 2bx \, dx = e^{-b^2} \frac{\sqrt{\pi}}{2}$$

and

$$\int_0^\infty e^{-x^2} \sin 2bx \, dx = e^{-b^2} \int_0^b e^{t^2} \, dt.$$

You may assume that $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$.

**Hint.** *For any $R > 0$, the integral of $e^{-z^2}$ along the positively boundary of the rectangle with vertices $0, R, R + ib$ and $ib$ is $0$.*

1.2. Show that

$$\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

**Hint.** *For any $R > 0$, consider the sector of $C(0; R)$ made of the line segment joining $0$ and $R$ and having the central angle $\frac{\pi}{4}$ in the upper half plane. The integral of $e^{iz^2}$ along the boundary of this sector is $0$ by Cauchy's theorem. You may proceed as follows:*

*(a) First, using Calculus or otherwise, show that $\dfrac{\sin t}{t} \geq \dfrac{2}{\pi}$, for all $t \in (0, \frac{\pi}{2}]$.*

*(b) Show that, as $R \longrightarrow \infty$, the integral along the arc goes to $0$.*

*(c) Show that, as $R \longrightarrow \infty$, the integral along the line segment joining $0$ and $Re^{i\frac{\pi}{4}}$ converges to $e^{i\frac{\pi}{4}} \cdot \frac{\sqrt{\pi}}{2}$*

1.3. Evaluate $\int_0^\infty e^{-x^2} \cos x^2 \, dx$.

**Hint.** *The above integral is the real part of the following:*

$$\int_0^\infty e^{-(1 + i)x^2} \, dx = \int_0^\infty e^{\sqrt{2}e^{\frac{i\pi}{4}}x^2} \, dx$$

$$= \int_0^\infty e^{\sqrt{2}\left(e^{\frac{i\pi}{8}}x\right)^2} \, dx.$$

*Now proceed along the lines exactly similar to 1.2., but the central angle is $\frac{\pi}{8}$ in this case. The final answer is $\dfrac{\cos \frac{\pi}{8}}{2^{\frac{5}{4}}} \sqrt{\pi}$.*

### 2. LIOUVILLE'S THEOREM

2.1. Find all entire functions $f$ with the property that, for all $z \in \mathbb{C}$, $f(z + 1) = f(z) = f(z + i)$.

**Sketch of the solution.** *It is easy to see that, for any $m, n \in \mathbb{Z}$ and $z \in \mathbb{C}$ one has $f(z + m + in) = f(z)$. It follows that $f(\mathbb{C}) = f(\{z \in \mathbb{C} : 0 \leq \mathrm{Re}\, z, \mathrm{Im}\, z \leq 1\})$. Since $f$ is continuous and $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}\, z, \mathrm{Im}\, z \leq 1\}$ is compact, there exists $M > 0$ such that $|f(z)| \leq M$ whenever $0 \leq \mathrm{Re}\, z, \mathrm{Im}\, z \leq 1$. This imples that $f$ is bounded. Hence, from Liouville's theorem, $f$ is constant.*

2.2. Let $f$ and $g$ be two entire functions. Assume that there exists $c \in \mathbb{R}$ such that
$$\mathrm{Re}\, f(z) \le c\, \mathrm{Re}\, g(z), \quad \forall z \in \mathbb{C}.$$
Show that $f = ag + b$, for some $a, b \in \mathbb{C}$. In particular, if $\mathrm{Re}\, f$ is bounded above then $f$ must be a constant function.

**Sketch of the solution.** *Let $h \overset{def}{=} f - cg$. Then $|e^h| = \left|e^{\mathrm{Re}(f-cg)}\right|$ which is bounded from above by $1$. Hence, from Liouville's theorem, $f - cg$ is constant.*

2.3. (a) Let $f$ be an entire function such that $\mathrm{Re}\, f > 0$. Show that $f$ is constant.

   **Hint.** *Consider $\frac{1}{1+f}$.*

   **Sketch of the solution.** *Observe that $|1 + f| \ge |\mathrm{Re}(1 + f)| = |1 + \mathrm{Re}\, f| = 1 + \mathrm{Re}\, f \ge 1$. So $\left|\frac{1}{1+f}\right| \le 1$. This makes $\frac{1}{1+f}$ constant, due to Liouville's theorem, which implies that $f$ is constant.*

   (b) Suppose that $f$ is an entire function such that $\mathrm{Re}\, f$ or $\mathrm{Im}\, f$ has no zeros. What can you conclude about $f$?

   **Sketch of the solution.** *Suppose that $\mathrm{Re}\, f$ has no zero. From intermediate value theorem, it follows that either $\mathrm{Re}\, f > 0$ or $\mathrm{Re}\, f < 0$ (how?). If $\mathrm{Re}\, f > 0$ then from 2.3.a, we get that $f$ is constant. If $\mathrm{Re}\, f < 0$ then $\mathrm{Re}(-f) > 0$, so that $-f$ is constant, due to 2.3.a. Hence in either case $f$ is constant. If $\mathrm{Im}\, f$ does not have a zero then consider the function $if$ and use the previous case.*

   (c) Prove the analogue of 2.3.a for $2 \times 2$ valued functions: If $F : \mathbb{C} \longrightarrow M_2(\mathbb{C})$ is such that $F_{jk}$ is entire for each $j, k = 1, 2$, and $\forall z \in \mathbb{C}$, $F(z) + F(z)^*$ is positive definite. Then show that $F$ is constant.

   **Hint.** *Diagonal entries of a positive definite matrix are positive, and the determinant of a positive definite matrix is positive. So from 2.3.a, it follows that $f_{11}$ and $f_{22}$ are constant. This implies that $f_{12} + \overline{f_{21}}$ is bounded. Thus $f_{12} + f_{21}$ is constant. Similarly, one obtains $f_{12} - f_{21}$ is constant.*

   **Sketch of the solution.** *Follow the hints given above. Observe that, if $f_{12} + \overline{f_{21}}$ is bounded, then so is $\mathrm{Re}(f_{12} + f_{21})$. From 2.2., it follows that $f_{12} + f_{21}$ is constant. Observe that $\mathrm{Re}(f_{21} - f_{12}) = \mathrm{Re}(i(f_{12} + \overline{f_{21}}))$. The latter is bounded as $f_{12} + \overline{f_{21}}$ is bounded. Hence again from 2.2., it follows that $f_{12} - f_{21}$ is constant. The rest is immediate.*

2.4.∗ Let $f$ be an entire function satisfying the following:
$$|f(z)| \le \frac{1}{\sqrt{|\mathrm{Re}\, z|}}, \quad \text{whenever } \mathrm{Re}\, z \ne 0. \tag{2.1}$$

   (a) Suppose that $|\mathrm{Re}\, z_0| < \frac{1}{2}$. Write $z_0 = x_0 + iy_0$. Consider the square $S$ with the following vertices:
   $$-1 + i(y_0 - 1),\ 1 + i(y_0 - 1),\ 1 + i(y_0 + 1) \text{ and } -1 + i(y_0 + 1).$$
   Assume that $S$ is equipped with the counterclockwise orientation. Show that
   $$f(z_0) = \frac{1}{2\pi i} \int_S \frac{f(z)}{z - z_0}\, dz. \tag{2.2}$$

   (b) Using (2.1), show that in the integral appearing in (2.2), the contribution from each horizontal edge of $S$ is at most 8 in absolute value.
   (c) Show that, in the integral appearing in (2.2), the contribution from each vertical edge of $S$ is at most 4 in absolute value.
   (d) Deduce from 2.4.b and 2.4.c that $|f(z_0)| \le \frac{12}{\pi}$.
   (e) Conclude that $f$ must be a constant function.

**3.1.** Is the condition $\text{Ind}_\gamma(z) = 0$, for all $z \in \mathbb{C} \setminus U$, necessary in the Global Cauchy theorem?

**Hint.** *Let $z \in \mathbb{C} \setminus U$. Then the function $w \mapsto \frac{1}{w-z}$ is holomorphic on $U$.*

**3.2.** Let $U \subseteq \mathbb{C}$ be open, $\gamma$ be a cycle in $U$ which is homologous to $0$ in $U$. Consider finitely many distinct points, say $z_1, z_2, \cdots, z_n$, of $U$. For each $j \in \{1, \cdots, n\}$, choose a closed disc $D_j \subseteq U$ centered at $z_j$ such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Denote the boundary of $D_j$ oriented anticlockwise by $\gamma_j$ and let $k_j \overset{\text{def}}{=} \text{Ind}_\gamma(z_j)$ for all $j = 1, \cdots, n$.

  (a) Show that $\gamma$ is homologous to $\displaystyle\sum_{j=1}^{n} k_j \gamma_j$ in $U \setminus \{z_1, z_2, \cdots, z_n\}$.

  (b) Show that, for every $f \in H(U \setminus \{z_1, z_2, \cdots, z_n\})$,

$$\int_\gamma f = \sum_{j=1}^{n} k_j \int_{\gamma_j} f.$$

**3.3.** Let $U, \gamma$ and $f$ be as in Global Cauchy theorem. Show that, for all $n \geq 0$,
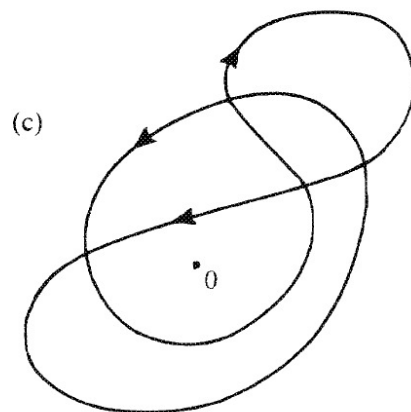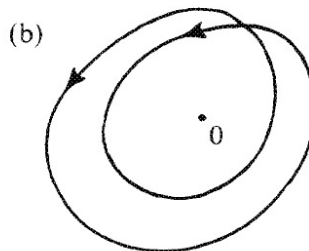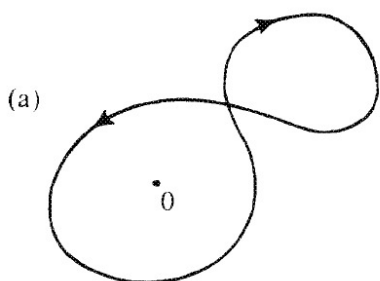
$$\text{Ind}_\gamma(z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, dw, \quad \forall z \in U \setminus \gamma^*.$$

**Sketch of the solution.** *Let $V$ be a connected component of $U \setminus \gamma^*$. being the connected component of an open set, clearly $V$ is open. Note that the function $\text{Ind}_\gamma$ is constant on $V$, say $\alpha$. From Global Cauchy theorem, we obtain that,*

$$\alpha f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)} \, dw, \quad \forall z \in U \cap V.$$

*Now use Exercise 1.3 of Exercise Sheet 5.*

**3.4.** Along each of the following paths, evaluate the integral of $\dfrac{e^z + e^{-z}}{z^4}$:



(a)    (b)    (c)

**Answer.** *(a) $\frac{\pi i}{3}$ (b), (c) $\frac{2\pi i}{3}$ (use Global Cauchy theorem).*

**3.5.** Let $a, b > 0$. Show that $\displaystyle\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}$.

**Hint.** *Consider the path $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ given by $\gamma(t) \overset{\text{def}}{=} a \cos t + ib \sin t$. What is $\displaystyle\int_\gamma \frac{dz}{z}$ ? Can yo use that here?*

3.6. Let $P(z) \in \mathbb{C}[z]$ whose all distinct zeros are $a_1, \ldots, a_k$. Suppose $\gamma$ is a closed curve in $\mathbb{C}$ such that $a_j \notin \gamma^*$, for all $j = 1, \ldots, n$. Find $\dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{P'}{P}$.

**Answer.** *Let $m_1, \ldots, m_k$ be the multiplicities of $a_1, \ldots, a_k$ respectively. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{P'}{P} = m_1 \operatorname{Ind}_\gamma(a_1) + \cdots + m_k \operatorname{Ind}_\gamma(a_k).$$

3.7. Let $U \subseteq_{open} \mathbb{C}$, $h \in H(U)$ be zero-free, $z_0 \in U$, $m \in \mathbb{N}$ and $f(z) \overset{\text{def}}{=} (z-z_0)^m h(z)$, for all $z \in U$. Suppose that $\gamma$ is a closed path in $U$ such that it is homologous to $0$ in $U$. Assume that $z_0 \notin \gamma^*$. Prove that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = m \operatorname{Ind}_\gamma(z_0).$$

**Solution.** *Observe that, for all $z \in U$, one has*

$$\frac{f'(z)}{f(z)} = \frac{m}{(z - z_0)} + \frac{h'(z)}{h(z)}.$$

*Since $\frac{h'}{h} \in H(U)$, it follows from Global Cauchy theorem that $\int_\gamma \frac{h'}{h} = 0$. The rest is obvious.*

3.8. Let $U \subseteq \mathbb{C}$ be open such that every closed path in $U$ is homologous to $0$ in $U$. Consider finitely many distinct points, say $z_1, z_2, \cdots, z_n$, of $U$. For each $j \in \{1, \cdots, n\}$, choose a closed disc $D_j \subseteq U$ centered at $z_j$ such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Denote the boundary of $D_j$ oriented anticlockwise by $\gamma_j$. Suppose that $f \in H(U \setminus \{z_1, \ldots, z_n\})$ and $a_k \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma_k} f$ for all $k = 1, \cdots, n$. Define

$$h(z) = f(z) - \sum_{k=1}^n \frac{a_k}{z - z_k}, \quad \forall z \in U \setminus \{z_1, \ldots, z_n\}. \tag{3.1}$$

Show that the function $h$, defined as above in (3.1), has a primitive.

**Sketch of the solution.** *Let $\gamma$ be a closed path in $U \setminus \{z_1, \ldots, z_n\}$. Let $\sigma \overset{\text{def}}{=} \displaystyle\sum_{k=1}^n \operatorname{Ind}_\gamma(z_k)\gamma_k$. Verify that $\gamma$ is homologous to $\sigma$ in $U$. From Golbal Cauchy theorem, it follows that,*

$$\int_\gamma f = \int_\sigma f$$
$$= \sum_{k=1}^n \operatorname{Ind}_\gamma(z_k) \int_{\gamma_k} f$$
$$= \sum_{k=1}^n 2\pi i \cdot \operatorname{Ind}_\gamma(z_k) \cdot a_k$$
$$= \sum_{k=1}^n \int_\gamma \frac{a_k}{w - z_k} \, dw$$
$$= \int_\gamma \left( \sum_{k=1}^n \frac{a_k}{w - z_k} \right) dw.$$

*Thus $\int_\gamma h \, dw = 0$. Hence $h$ has a primitive.*

## 4. ANALYTIC LOGARITHMS AND $n$-TH ROOTS OF FUNCTIONS

Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C} \setminus \{0\}$ be an analytic function.

4.1. Show that $f$ has an analytic logarithm if and only if its *logarithmic derivative*, i.e., $\frac{f'}{f}$ has a primitive.

4.2. Show that the following are equivalent:

    (a)  $f$ has an analytic logarithm on $U$.

    (b)  For every $n \in \mathbb{N}$, $f$ has an *analytic n-th root on* $U$, i.e., there exists an analytic $g : U \longrightarrow \mathbb{C}$ such that $g^n = f$.

    (c)∗  For infinitely may $n \in \mathbb{N}$, $f$ has an analytic $n$-th root on $U$.

        **Hint.** *Fix a closed path $\gamma$ in $U$.*

        *(i) If $g_n$ is an analytic n-th root of $f$, then observe that $\dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{g_n'}{g_n} = \dfrac{1}{2\pi i n} \displaystyle\int_\gamma \dfrac{f'}{f}$.*

        *(ii) Observe that $\mathrm{Ind}_{g_n \circ \gamma}(0) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{g_n'}{g_n}$.*

        *(iii) Conclude from 4.2.(c)i and 4.2.(c)ii that $\displaystyle\int_\gamma \dfrac{f'}{f} = 0$.*

        **Note:** It is not enough to have an analytic $n$-th root for a particular $n$ in order to conclude that $f$ has an analytic logarithm. The next exercise provides an example.

4.3. Consider the function $f(z) \overset{\text{def}}{=} z^2$, for all $z \in \mathbb{D} \setminus \{0\}$. Show that $f$ does not have an analytic logarithm despite having an analytic square root.

4.4.∗ Let $U \subseteq_{open} \mathbb{C}$. Show that the following are equivalent:

    (a) For every cycle $\gamma$ in $U$ and $z \in \mathbb{C} \setminus U$, $\mathrm{Ind}_\gamma(z) = 0$.

    (b) For every cycle $\gamma$ in $U$ and $f \in H(U)$, $\int_\gamma f = 0$.

    (c) Every analytic function on $U$ has a primitive.

    (d) Every zero-free analytic function on $U$ has an analytic logarithm.

    (e) Every zero-free analytic function on $U$ has an analytic $n$-th root for all $n \in \mathbb{N}$.