Let $U \subseteq \mathbb{C}$ be a region and γ be a closed path in U satisfying $\operatorname{Ind}_\gamma(z) = 0$, $\forall z \in \mathbb{C} \setminus U$. Suppose that $f \in H(U)$. Assume that

(i) both γ and $f \circ \gamma$ have interiors, and

(ii) $f(\operatorname{Int}\gamma) \cap (f\circ\gamma)^* = \emptyset$.

Then $f(\operatorname{Int}\gamma) \subseteq \operatorname{Int} f\circ\gamma$ and $f : \operatorname{Int}\gamma \to \operatorname{Int} f\circ\gamma$ is 1-1. Furthermore, if $\operatorname{Int} f\circ\gamma$ is connected then f establishes a conformal equivalence between $\operatorname{Int}\gamma$ & $\operatorname{Int} f\circ\gamma$.

PROOF:- Let $z \in \operatorname{Int}\gamma$. Then $f(z) \notin (f\circ\gamma)^*$ by (ii)

It is easy to see that

$$\operatorname{Ind}_{f\circ\gamma}(f(z)) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{d\xi}{\xi - f(z)} = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t)) - f(z)}\,dt$$

$$= \frac{1}{2\pi i}\int_a^b \frac{(g\circ\gamma)'(t)}{(g\circ\gamma)(t)}\,dt$$

$$= \cancel{\frac{1}{2\pi i}}\ \operatorname{Ind}_{g\circ\gamma}(0) \geq 1,$$

where $g(w) = f(w) - f(z)$, $\forall w \in U$.

Since $f\circ\gamma$ has an interior ~~and I~~, it follows that $\operatorname{Ind}_{f\circ\gamma}(f(z)) = \operatorname{Ind}_{g\circ\gamma}(0) = 1$. This shows that

$f(z) \in \text{Int } f \circ \gamma$, and $\forall \omega \in \text{Int } \gamma \setminus \{z\}$,
$f(\omega) \neq f(z)$, otherwise no. of zeros of g in
$\text{Int } \gamma = \text{Ind}_{g \circ \gamma}(0) > 1$.

Assume now that $\text{Int } f \circ \gamma$ is connected.
Since $f(\text{Int } \gamma) \neq \emptyset$ is an open subset of
$\text{Int } f \circ \gamma$ by Open mapping theorem, it is
enough to show that $f(\text{Int } \gamma)$ is closed
in $\text{Int } f \circ \gamma$.

Let $\{z_n\}_{n=1}^{\infty}$ be a seqn. in $\text{Int } \gamma$ s.t.
$\{f(z_n)\}_{n=1}^{\infty}$ ~~is~~ converges to a point, say $\omega \in \text{Int } f \circ \gamma$.
Since $\text{Int } \gamma \cup \gamma^*$ is compact, $\{z_n\}_{n=1}^{\infty}$ has a
convergent subsequence, say $\{z_{n_k}\}_{k=1}^{\infty}$. Put
$z_0 := \lim_{k \to \infty} z_{n_k}$. From continuity, $f(z_{n_k}) \xrightarrow[k \to \infty]{} f(z_0)$,
hence $\omega = f(z_0)$. As $\omega \in \text{Int } f \circ \gamma$, $\omega \notin (f \circ \gamma)^*$
by definition. Hence $z_0 \notin \gamma^*$; so that
$\omega = f(z_0) \in f(\text{Int } \gamma)$.