Indian Institute of Technology Kanpur
Department of Mathematics and Statistics
Complex Analysis (MTH 403)
Exercise Sheet 3

1. Exponential and logarithm functions

1.1. Show that, $\forall r > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N$, the polynomial $1 + z + \dfrac{z^2}{2!} + \cdots + \dfrac{z^n}{n!}$ has all zeros outside the closed disc $\overline{D(0;r)}$.

1.2.$*$ (a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative reals such that $a_n \xrightarrow[n\to\infty]{} 0$. Show that, for any $x \in \mathbb{R}$, $\displaystyle\sum_{n=1}^{\infty} a_n \sin nx$ converges.

(**Hint:** Observe that, $\forall x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $\displaystyle\sum_{k=1}^{n} e^{ikx} = e^{ix} \cdot \frac{1 - e^{inx}}{1 - e^{ix}} = e^{i\frac{(n+1)x}{2}} \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}}$. Taking imaginary part, we get that $\left|\displaystyle\sum_{k=1}^{n} \sin kx\right| \leq \dfrac{1}{\left|\sin \frac{x}{2}\right|}$, for all $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.)

(b) Show that the convergence in 1.2.a is uniform on each $[2k\pi + \delta, 2(k+1)\pi - \delta]$, where $\delta > 0$ and $k \in \mathbb{Z}$.

(c) For $z \in \mathbb{C}$, $\sin z \stackrel{\text{def}}{=} \dfrac{e^{iz} - e^{-iz}}{2i} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$. Show that, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nz}{n}$ cannot converge unless z is real.

1.3. Let $\alpha \in \mathbb{R}$. Consider the *ray* $R_\alpha \stackrel{\text{def}}{=} \{e^x(\cos\alpha + i \sin\alpha) : x \in \mathbb{R}\}$. Pick any $z_0 \in \overline{R_\alpha}$. Show that, $\lim\limits_{z \to z_0} \log_\alpha z$ and $\lim\limits_{z \to z_0} \arg_\alpha z$ do not exist. Conclude from this that, the functions $\log_\alpha$ and $\arg_\alpha$ are not continuous at any point R_α.

1.4.$*$ Let $r > 0$. Show that there cannot exist any continuous $\theta : C(0;r) \longrightarrow \mathbb{R}$ with the property

$$z = re^{i\theta(z)}, \text{ whenever } |z| = r.$$

In words, it is not possible to make a continuous choice of arguments on $C(0;r)$.

(**Hint:** First show that, if such a θ exists then $\theta - \arg_{-\pi}$ must be constant on $C(0;r) \setminus \{-r\}$. Let $z_n \stackrel{\text{def}}{=} \log r + i\left(-\pi + \frac{1}{n}\right)$, for all $n \in \mathbb{N}$. Where does the sequence $\{e^{z_n}\}_{n=1}^{\infty}$ converge to? Use this to find $\theta(-r) - \arg_{-\pi}(-r)$.

Alternatively, first show that, if such a θ exists then it must be injective. Next consider the function $f : C(0;r) \longrightarrow \{\pm 1\}$ defined by $f(z) = \dfrac{\theta(z) - \theta(-z)}{|\theta(z) - \theta(-z)|}$, for all $z \in C(0;r)$. Does connectedness of $C(0;r)$ help now?)

1.5.$*$ Give an example of an open connected $U \subseteq \mathbb{C}$ such that $\forall \alpha \in \mathbb{R}$, $U \cap R_\alpha = \emptyset$ but the function $f(z) = z$ has a continuous argument, and also a holomorphic logarithm on U.

1.6. Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C} \setminus \{0\}$ be holomorphic. Show that every continuous logarithm of f is holomorphic.

1.7. Let X be a metric space with the property that every bounded sequence in X admits a convergent subsequence. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with exactly one subseqeuntial limit ℓ, i.e., any convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to ℓ. Show that $x_n \xrightarrow[n\to\infty]{} \ell$.

2. Power function

Let $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$. Choose $\alpha \in \mathbb{R}$ in such a way that $z \notin R_\alpha$. We define

$$z^w = e^{w \log_\alpha z}. \tag{2.1}$$

2.1. Show that, for any $\alpha \in \mathbb{R}$, the function $\mathbb{C} \setminus \overline{R}_\alpha \longrightarrow \mathbb{C}$, $z \mapsto z^w$, where z^w is defined above in (2.1), is holomorphic. What is the derivative?

2.2. Let $x > 0$ and $n \in \mathbb{N}$. Find all possible values of $x^{\frac{1}{n}}$ for $\alpha \neq 0$.

3. Basic properties of index

3.1. Let γ, γ_1 and $\gamma_2 : [a, b] \longrightarrow \mathbb{C}$ be closed curves. Prove the following:

(a) $z \notin \gamma^* \implies \operatorname{Ind}_\gamma(z) = \operatorname{Ind}_{\gamma - z}(0)$.
(b) $0 \notin \gamma_1^* \cup \gamma_2^* \implies \operatorname{Ind}_{\gamma_1\gamma_2}(0) = \operatorname{Ind}_{\gamma_1}(0) + \operatorname{Ind}_{\gamma_2}(0)$ and $\operatorname{Ind}_{\gamma_1/\gamma_2}(0) = \operatorname{Ind}_{\gamma_1}(0) - \operatorname{Ind}_{\gamma_2}(0)$.
(c) If $\gamma^* \subseteq D(z_0; r)$ then $\operatorname{Ind}_\gamma(z) = 0$, for all $z \notin D(z_0; r)$.

3.2.∗ If $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$, for all $t \in [a, b]$, then $0 \notin \gamma_1^* \cup \gamma_2^*$ and $\operatorname{Ind}_{\gamma_1}(0) = \operatorname{Ind}_{\gamma_2}(0)$. (**Hint:** Consider $\gamma \stackrel{\text{def}}{=} \frac{\gamma_1}{\gamma_2}$. Now use 3.1.b and 3.1.c.)

Note: 3.2. has a nice interpretation. Suppose that a man is walking with his pet dog on a leash with variable length. A tree is located in the origin. At time t, the position of the man and his dog are $\gamma_1(t)$ and $\gamma_2(t)$ respectively. Then 3.2. shows that, if the length of the leash is always less than the distance between the man and the tree, then the man and his dog must walk around the tree exactly same number of times. That is why sometimes 3.2. is also referred to as the *Dog-on-a-Leash lemma* or *Dog-walking lemma*.