Initial Value Problems: Linear Multistep Methods

Akash Anand
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Theorem

The linear multistep method is convergent if and only if it is consistent and stable.

Remark

This theorem states that it can be determined whether or not a linear multistep method is convergent simply by checking some algebraic conditions concerning its characteristic polynomials.

Proof. (sketch)

Convergence ⇒ Consistency

Apply the method to $y' = 0, y(0) = 1$ *and* $y' = 1, y(0) = 0$ *for verifying satisfiability of the consistency conditions.*

Convergence ⇒ Stability

Consistency and Stability ⇒ Convergence

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Consistency and Stability ⇒ Convergence

Recall that

$$\begin{bmatrix} y_{n-k+1} \\ y_{n-k+2} \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix} \begin{bmatrix} y_{n-k} \\ y_{n-k+1} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1} f_{n+1} + b_0 f_n + \cdots + b_n f_0) \end{bmatrix}$$

We define

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_0 \end{bmatrix}, \qquad E_n = \begin{bmatrix} y_{n-k} - y(t_{n-k}) \\ y_{n-k+1} - y(t_{n-k+1}) \\ \vdots \\ y_{n-1} - y(t_{n-1}) \\ y_n - y(t_n) \end{bmatrix},$$

where $y(t)$ is the exact solution that satisfies a similar difference equation

$$\begin{bmatrix} y(t_{n-k+1}) \\ y(t_{n-k+2}) \\ \vdots \\ y(t_n) \\ y(t_{n+1}) \end{bmatrix} = A \begin{bmatrix} y(t_{n-k}) \\ y(t_{n-k+1}) \\ \vdots \\ y(t_{n-1}) \\ y(t_n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(b_{-1} f(t_{n+1}, y(t_{n+1})) + b_0 f(t_n, y(t_n)) + \cdots + b_n f(t_0, y(t_0))) - \ell_{n+1}(y, h) \end{bmatrix}.$$

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Then, we have $E_{n+1} = AE_n + Q_n$ where

$$Q_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\big(b_{-1}(f_{n+1} - f(t_{n+1}, y(t_{n+1}))) + b_0(f_n - f(t_n, y(t_n))) + \cdots + b_n(f_0 - f(t_0, y(t_0)))\big) + \ell_{n+1}(y,h) \end{bmatrix}.$$

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So, there is a constant C such that, for $h \le (2C\|b\|_1 L)^{-1}$, we have

$$\|E_{k+n}\| \le C\|E_k\| + C\sum_{j=0}^{n-1} \|Q_{k+j}\| \le 2C\|E_k\| + 4hC\|b\|_1 L \sum_{j=0}^{n-1} \|E_{k+j}\| + 2nC \max_{0\le j<n} |\ell_{k+j+1}(y,h)|.$$

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Therefore,

$$\|E_{k+n}\| \le 2C\left(\|E_k\| + (T-t_0)\max_{0\le j<N}\left|\frac{\ell_j(y,h)}{h}\right|\right)e^{4(T-t_0)C\|b\|_1 L} \ldots$$

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The highest order of a stable k-step method is $k + 1$ if k is odd and $k + 2$ if k is even.

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Remark

The Adams method are linear multistep methods with best possible stability properties, namely, the first characteristic polynomial $\rho(t) = t^{k+1} - t^k$ has all its roots at the origin except for the mandatory root at 1.