Cauchy-Riemann equations

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1. Holomorphic functions

Let U be an open subset of $\mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$.

Definition 1.1. *We say that f is holomorphic at z_0 if the following limit exists:*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{1.1}$$

Since any linear map from $\mathbb{C}$ to $\mathbb{C}$ (here we regard $\mathbb{C}$ as a vector space over $\mathbb{C}$) is just multiplication by a scalar, the existence of the limit mentioned above in (1.1) is equivalent to saying that there exists a linear map $\alpha : \mathbb{C} \longrightarrow \mathbb{C}$ with the following property:

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \,\&\, z \in U \implies \left| \frac{f(z) - f(z_0) - \alpha(z - z_0)}{z - z_0} \right| < \varepsilon. \tag{1.2}$$

Clearly if we now consider $\mathbb{C}$ a vector space of $\mathbb{R}$ instead, α remains linear as well. Recall that, the euclidean plane $\mathbb{R}^2$ can be identified with $\mathbb{C}$ by means of the following map:

$$(x, y) \leftrightarrow x + iy. \tag{1.3}$$

Loosely speaking, we do not distinguish between the point (x, y) and the complex number $x + iy$. Clearly the above-mentioned identification establishes an isomorphism of between the real vector spaces $\mathbb{R}^2$ and $\mathbb{C}$. In view of the identification (1.3), f can now be considered as a map from U to $\mathbb{R}^2$. If $z = x + iy\,(\in U)$ and $z_0 = x_0 + iy_0$, one can immediately rewrite the expression $\left| \frac{f(z) - f(z_0) - \alpha(z - z_0)}{z - z_0} \right|$ as follows:

$$\frac{\|f(x, y) - f(x_0, y_0) - \alpha(x - x_0, y - y_0)\|}{\|(x - x_0, y - y_0)\|}.$$

From this, it follows at once that f is differentiable at (x_0, y_0) (see Definition A.1). The converse is not true. To see this, consider the following example:

$$f(z) \overset{\text{def}}{=} \bar{z}, \ \forall z \in \mathbb{C}. \tag{1.4}$$

It is easy to see that f is not holomorphic at 0. However, it is easy to see that f is differentiable, as it is nothing but the function $(x, y) \mapsto (x, -y)$, once we identify $\mathbb{C}$ with $\mathbb{R}^2$ as above in (1.3). Thus we have shown that being holomorphic is stronger than being differentiable.

2. Cauchy-Riemann equations

It is now quite natural to ask, in addition to the differentiablity, what extra hypothesis one requires so as to make the function holomorphic. Let f be as in the beginning of §1. Assume that it is differentiable at $z_0 = x_0 + iy_0 \leftrightarrow (x_0, y_0) \in U$. So we have

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\|f(x,y) - f(x_0,y_0) - Df(x_0,y_0)(x - x_0, y - y_0)\|}{\|x - x_0, y - y_0\|} = 0. \tag{2.1}$$

A moment's thought reflects that, if $Df(x_0, y_0)$ is a linear map on the complex vector space $\mathbb{C}$, then (1.2) follows immediately from (2.1). It is customary to denote the real and imaginary component of f by u and v. Then the matrix representation of $Df(x_0, y_0)$ with respect to the standard basis of $\mathbb{R}^2$ is as follows:

$$\begin{pmatrix} \dfrac{\partial u}{\partial x}(x_0,y_0) & \dfrac{\partial u}{\partial y}(x_0,y_0) \\ \dfrac{\partial v}{\partial x}(x_0,y_0) & \dfrac{\partial v}{\partial y}(x_0,y_0) \end{pmatrix}. \tag{2.2}$$

Theorem B.1 shows us that $Df(x_0, y_0)$ can be a linear map on $\mathbb{C}$ (over $\mathbb{C}$) when and only when we have the following pair of equations:

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \text{ and } \frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0). \tag{2.3}$$

Hence, differentiability at (x_0, y_0) together with the equations mentioned above in (2.3) forces f to be holomorphic at z_0.

We also note that the additional hypothesis given by (2.3) cannot be replaced by a weaker one, as it is necessary too. To see this, suppose that f is holomorphic at z_0. Then if we regard the linear map $\alpha : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying (1.2)as a linear map on $\mathbb{R}^2$, from the uniqueness of derivative, it must be same as $Df(x_0, y_0)$. This in turn makes $Df(x_0, y_0)$ a linear map on $\mathbb{C}$. As we have seen above, this cannot happen unless one has (2.3).

The pair of equations mentioned above in (2.3) is called the *Cauchy-Riemann equations* at the point (x_0, y_0). When the point (x_0, y_0) is understood, we deliberately drop that from the notation, provided that causes no confusion. Thus we simply rewrite (2.3) as follows:

$$\boxed{u_x = v_y \text{ and } v_x = -u_y.} \tag{2.4}$$

Summarizing the discussion of this section, we conclude the following theorem:

Theorem 2.1. *Let U be an open subset of $\mathbb{C}$, $f : U \longrightarrow \mathbb{C}$ and $z_0 = x_0 + iy_0 \in U$. Then the following are equivalent:*

(H.1) f is holomorphic at z_0.

(H.2) f is differentiable at (x_0, y_0) and the Cauchy-Riemann equations hold at (x_0, y_0).

We close this section with an example showing that a function might satisfy Cauchy-Riemann equations at a given point without being holomorphic at that point. Consider

$$f(x+iy) \overset{\text{def}}{=} \sqrt{|x||y|}, \ \forall (x,y) \in \mathbb{R}^2. \tag{2.5}$$

One can easily verify that the function f defined above in (2.5) satisfies the Cauchy-Riemann equations at the origin, yet it is not holomorphic at 0.

3. Some applications of the Cauchy-Riemann equations

Application 1. Consider $f(z) \overset{\text{def}}{=} |z|, \forall z \in \mathbb{C}$. It is easy to see that f is differentiable everywhere except at the origin. We now show that f cannot holomorphic at any point of $\mathbb{C}$. In this case, one has $u(x,y) = \sqrt{x^2+y^2}$ and $v \equiv 0$. A simple calculation shows that, for any $(x,y) \neq (0,0)$,

$$u_x = \frac{2x}{\sqrt{x^2+y^2}} \text{ and } u_y = \frac{2y}{\sqrt{x^2+y^2}}. \tag{3.1}$$

We can see from (3.1) that both u_x and u_y cannot vanish simultaneously at any $(x,y) \neq (0,0)$. So the Cauchy-Riemann equations cannot hold at any $(x,y) \neq (0,0)$. Therefore f is not holomorphic anywhere. Using exactly similar arguments one can show that the function $|z|^2$ is holomorphic only at 0, and the function mentioned above in (1.4) is nowhere holomorphic.

Application 2. Let $U \subseteq_{open} \mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$. Suppose that f is holomorphic at z_0 and $f'(z_0) = a + ib$. Consider the map $\alpha : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $\alpha(z) = f'(z_0)z$, for all $z \in \mathbb{C}$. Then α satisfies (1.2). When we look on α as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, its matrix representation with respect to the standard basis of $\mathbb{R}^2$ is easily seen to be the following:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{3.2}$$

On the other hand, in §2, we have seen that if we consider α as a linear map on $\mathbb{R}^2$, it is nothing but $Df(x_0, y_0)$. Therefore the matrices given by (2.2) and (3.2) must be equal. From this it follows immediately that

(i) $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$, and

(ii) $\det Df(x_0,y_0) = (u_x(x_0,y_0))^2 + (v_x(x_0,y_0))^2 = |f'(z_0)|^2$. In particular, if $f'(z_0) \neq 0$, then $Df(x_0,y_0)$ is invertible.

Application 3. Let $U \subseteq_{open} \mathbb{C}$ be connected. We leave it to the reader to verify that f is constant if any of $\operatorname{Re} f, \operatorname{Im} f$ and $|f|$ is constant.

4. Cauchy-Riemann equations in polar coordinates

Let $U \subseteq_{open} \mathbb{C}$ and $\alpha \in \mathbb{R}$ be such that $U \cap \overline{R_\alpha} = \emptyset$. Then $V \stackrel{\text{def}}{=} \exp^{-1}(U)$ must be an open subset of $\operatorname{Int} B_\alpha = \mathbb{R} \times (\alpha, \alpha + 2\pi)$. Recall that the usual logarithm function gives a homeomorphism between $(0, \infty)$ and $\mathbb{R}$. Therefore $\varphi : (0, \infty) \times (\alpha, \alpha + 2\pi) \longrightarrow \mathbb{R} \times (\alpha, \alpha + 2\pi)$ defined by $\varphi(r, \theta) = (\log r, \theta)$ is a homeomorphism. Denote $\varphi^{-1}(V)$ by V'. Let $z_0 \in U$ and $(r_0, \theta_0) \in V'$ be such that $z_0 = r_0 e^{i\theta_0}$.

Suppose $f : U \longrightarrow \mathbb{C}$. As usual, denote the real and imaginary component of f by u and v respectively. Consider the composition of $\psi \stackrel{\text{def}}{=} \exp \circ \varphi : V' \longrightarrow U$ and f. Note that $\forall (r, \theta) \in V'$, $\psi(r, \theta) = (r \cos\theta, r \sin\theta)$, hence ψ is differentiable. Assume now that f is holomorphic at $z_0 = x_0 + iy_0$. Theorem A.2 yields that $f \circ \psi$ is differentiable at (r_0, θ_0) and one has the following at the point (r_0, θ_0):

$$\begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}. \tag{4.1}$$

Equating the entries of both sides of (4.1) and using (2.4), we obtain that

$$\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \text{ and } r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}.}$$

Appendices

A. Differentiability of a function

Let $m, n \in \mathbb{N}$, $U \subseteq_{open} \mathbb{R}^n$ and $f : U \longrightarrow \mathbb{R}^m$. Denote by $\|\cdot\|_{\mathbb{R}^m}$ and $\|\cdot\|_{\mathbb{R}^n}$ the euclidean norms of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. When $n = m$, for the simplicity in notation, we use the symbol $\|\cdot\|$ instead.

Definition A.1. *f said to be differentiable at $\mathbf{x}_0 \in U$ if there exists a linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - A(\mathbf{x} - \mathbf{x}_0)\|_{\mathbb{R}^m}}{\|\mathbf{x} - \mathbf{x}_0\|_{\mathbb{R}^n}} = 0,$$

i.e., $\forall \varepsilon > 0 \exists \delta > 0$ *such that* $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta \;\&\; \mathbf{x} \in U \Longrightarrow \dfrac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - A(\mathbf{x} - \mathbf{x}_0)\|_{\mathbb{R}^m}}{\|\mathbf{x} - \mathbf{x}_0\|_{\mathbb{R}^n}} < \varepsilon.$

The linear map A appeared in Definition A.1 is unique ([1, Theorem 9.12]). We call it the *derivative* of f at $\mathbf{x}_0$, and denote by $Df(\mathbf{x}_0)$. Let $f_1, \ldots, f_m$ denote the *components* of f, i.e.,

$$\forall \mathbf{x} \in U,\; f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})).$$

The following theorem shows that, if f is differentiable at $\mathbf{x}_0 \in U$ then all partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ exist at $\mathbf{x}_0$ and thence $Df(\mathbf{x}_0)$ is completely determined:

Theorem A.1. *Let $m, n, U, \mathbf{x}_0$ and f be as above in Definition A.1. If f is differentiable at $\mathbf{x}_0$ then, for all $i = 1, \dots, m$ and $j = 1, \dots, n$, $\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)$ exist, and one has*

$$Df(\mathbf{x}_0)e_i = \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i}(\mathbf{x}_0)e_j, \ \forall i = 1, \dots, n. \tag{A.1}$$

Proof. See [1, Theorem 9.17] for a proof. □

It follows at once from (A.1) that the matrix of $Df(\mathbf{x}_0)$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}_0) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_0) & \dots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}_0) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix}. \tag{A.2}$$

We now talk about the *chain rule* of differentiation:

Theorem A.2. *Let $n.m.k \in \mathbb{N}$, $U \subseteq_{open} \mathbb{R}^n$, $V \subseteq_{open} \mathbb{R}^m$, $f : U \longrightarrow V$ and $g : V \longrightarrow \mathbb{R}^k$. Assume that f is differentiable at $\mathbf{x}_0 \in U$ and g is differentiable at $f(\mathbf{x}_0) \in V$. Then $g \circ f$ is differentiable at $\mathbf{x}_0$, and*

$$D(g \circ f)(\mathbf{x}_0) = Dg(f(\mathbf{x}_0)) \circ Df(\mathbf{x}_0). \tag{A.3}$$

Proof. See [1, Theorem 9.15] □

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B. $\mathbb{R}$ and $\mathbb{C}$ linear maps on the complex plane

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear map and $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be its matrix representation with respect to the standard basis of $\mathbb{R}^2$. The identification, mentioned as above in (1.3), yields the following map from $\mathbb{C}$ to $\mathbb{C}$:

$$x + iy \mapsto (ax + cy) + i(bx + dy). \tag{B.1}$$

It is customary to denote the linear map mentioned above in (B.1) by the same letter T by abuse of notation. Clearly T is a linear map on the real vector space $\mathbb{C}$. It is now natural to ask whether T preserves the *complex structure* of $\mathbb{C}$, i.e., whether T will still be linear if we look on $\mathbb{C}$ as a vector space over $\mathbb{C}$.

If T remains linear on the complex vector space $\mathbb{C}$, then there must exist $\alpha \in \mathbb{C}$ such that $T(z) = \alpha z$, for all $z \in \mathbb{C}$. If $\operatorname{Re} \alpha = a'$ and $\operatorname{Im} \alpha = b'$, then it is easy to see that

$$
\begin{aligned}
(ax + cy) + i(bx + dy) &= T(x + iy) \\
&= (a' + ib')(x + iy) \\
&= (a'x - b'y) + i(b'x + a'y), \ \forall x, y \in \mathbb{R}.
\end{aligned} \tag{B.2}
$$

From (B.2), it follows that $a = a' = d$ and $c = -b' = -b$. Conversely, if $a = d$ and $b = -c$, then it is obvious that $(ax + cy) + i(bx + dy) = (ax - by) + i(by + ax) = (a + ib)(x + iy)$, for all $x, y \in \mathbb{R}$. Thus we prove the following:

Theorem B.1. *Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear map. Then T is $\mathbb{C}$-linear, i.e., a linear map on the complex vector space $\mathbb{C}$, upon identification of $\mathbb{R}^2$ and $\mathbb{C}$ by* (1.3)*, if and only if $a = d$ and $b = -c$, where $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the matrix representation with respect to the standard basis of $\mathbb{R}^2$.*

References

[1] Rudin, Walter; *Principles of mathematical analysis*. Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. x+342 pp.