MTH 101-Calculus

Spring-2021

Assignment-11-Solutions: Double and Triple Integrals

1. (a) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2} dx \right) dy = \int_0^1 (1-y^2) dy = \frac{2}{3}.$$

   (b) $$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} dy dx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y} dx dy = \int_0^{y} \sin y dy = 2.$$

   (c) $$\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy = \int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx = \int_0^1 x(e^{x^2}-1) dx = \frac{e-2}{2}.$$

2. Choose $u = x(1-y)$ and $v = xy$. Then, $1 \le u \le 2$ and $1 \le v \le 2$. Note that $x = u+v$, $y = \frac{v}{u+v}$ and $\mid J(u,v) \mid = \frac{1}{u+v}$. Therefore, $\iint\limits_R x dx dy = \int_1^2 \int_1^2 dv du = 1$

3. Area $= 3\int_0^{\frac{\pi}{3}} \int_0^{sin 3\theta} r dr d\theta = \frac{3}{2}\int_0^{\frac{\pi}{3}} sin^2 3\theta = \frac{3}{4}\int_0^{\frac{\pi}{3}} (1 - cos 6\theta) d\theta = \frac{1}{4}\pi.$

4. (i) Let $D(a) = \{(x,y) : x^2+y^2 \le a\}$. Then
$$\iint\limits_{D(a)} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi}\int_0^{a} e^{-r^2} r dr d\theta = \pi(1-e^{-a^2}).$$
   Therefore, $\lim\limits_{a\to\infty} \iint\limits_{D(a)} e^{-(x^2+y^2)} dx dy = \pi.$

   (ii) Let $D_1(a) = \{(x,y) : x, y \ge 0,\ x^2+y^2 \le a\}$ and $D_2(a) = \{(x,y) : 0 \le x, y \le a\}$. Note that
$$\iint\limits_{D_1(a)} e^{-(x^2+y^2)} dx dy \le \iint\limits_{D_2(a)} e^{-(x^2+y^2)} dx dy \le \iint\limits_{D_1(\sqrt{2}a)} e^{-(x^2+y^2)} dx dy.$$
   Now, use the sandwich theorem. We see that
$$\lim_{a\to\infty} \iint\limits_{D_2(a)} e^{-(x^2+y^2)} dx dy = \lim_{a\to\infty} \iint\limits_{D_1(a)} e^{-(x^2+y^2)} dx dy = \frac{1}{4}\pi.$$

   (a) $$\left(\int_0^{\infty} e^{-x^2} dx\right)^2 = \left(\int_0^{\infty} e^{-x^2} dx\right)\left(\int_0^{\infty} e^{-y^2} dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx dy = \frac{\pi}{4}.$$

   (b) $$\int_0^{\infty} x^2 e^{-x^2} dx = \lim_{t\to\infty} \int_0^{t} -\frac{x}{2} d(e^{-x^2}) = \frac{1}{2}\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{4}.$$

5. The solid is enclosed by the cylinder $x^2+y^2 = 1$ and the surfaces $z = -\sqrt{1-x^2}$ and $z = \sqrt{1-x^2}$. Let $R = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$. The required volume is equal to $\iint\limits_R (\sqrt{1-x^2} - (-\sqrt{1-x^2})) dx dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2} dy dx = \frac{16}{3}.$

6. Use spherical coordinates. Let $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$ and $z = \rho \cos\phi$, where $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

$$\iiint\limits_W \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}} = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \frac{\rho^2 \sin\phi d\rho d\theta d\phi}{\sqrt{1+\rho^2}} = 2\pi(\sqrt{2} - \ln(1+\sqrt{2})).$$