MTH 101-Calculus

Spring-2021

Assignment 9-Solutions: Functions of several variables (Continuity and Differentiability

1. (i) The function is continuous everywhere.

   (ii) The function is continuous on the X-axis and the Y-axis. At other points the function is not continuous.

2. (a) Let $x \neq 0$. $\lim\limits_{y\to 0} \dfrac{x^2y^2}{x^2y^2+(x-y)^2} = \lim\limits_{y\to 0} \dfrac{x^2}{x^2+(\frac{x}{y}-1)^2} = 0.$

   Similarly, $\lim\limits_{x\to 0} f(x,y) = 0$, $\forall y \neq 0$.

   Hence, $\lim\limits_{x\to 0}[\lim\limits_{y\to 0} f(x,y)] = 0 = \lim\limits_{y\to 0}[\lim\limits_{x\to 0} f(x,y)]$.

   (b) Along the line $y = x$, $f(x,y) = 1$. Hence, the limit does not exist.

   (c) From above, the function is not continuous.

   (d) easy.

3. $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

   $\dfrac{|f(h,k)|}{\| (h,k) \|} = \dfrac{h^2+k^2 \sin \frac{1}{h^2+k^2}}{\sqrt{h^2+k^2}} \to 0$ as $(h,k) \to (0,0)$. Hence, $f'(0,0) = (0,0)$.

   $f_x(x,y) = 2x \sin \dfrac{1}{x^2+y^2} - \dfrac{2x}{x^2+y^2} \cos \dfrac{1}{x^2+y^2}.$

   $f_y(x,y) = 2y \sin \dfrac{1}{x^2+y^2} - \dfrac{2y}{x^2+y^2} \cos \dfrac{1}{x^2+y^2}.$

   for $(x,y) \neq (0,0)$.

   Neither f_x nor f_y is continuous (they are not even bounded in any neighborhood of $(0,0)$) at $(0,0)$.

   Since f_x and f_y are continuous at other points f is differentiable everywhere.

4. (a) Observe that $f_x(0,0) = f_y(0,0) = 0$. Now $\epsilon(h,k) = \frac{f(h,k)-f(0,0)-f_x(0,0)\cdot h - f_y(0,0)\cdot k}{\sqrt{h^2+k^2}} = \frac{|hk|}{\sqrt{h^2+k^2}} \leq \frac{\sqrt{h^2+k^2}|k|}{\sqrt{h^2+k^2}} \to 0$ as $(h,k) \to (0,0)$. Therefore f is differentiable at $(0,0)$.

   (b) For $y_0 \neq 0$, $\lim_{t\to 0} \frac{f(t,y_0)-f(0,y_0)}{t} = \lim_{t\to 0} \frac{|t||y_0|}{t}$does not exist.

5. Since $\frac{\partial f}{\partial x} = 0$ we have $f(x,y) = k + g(y)$. Now $\frac{\partial f}{\partial y} = 0$ implies that

   $$\frac{\partial}{\partial y}(k + g(y)) = g'(y) = 0.$$

   So $g(y)$ is a constant and as a result f is a constant as well.

   This also follows immediately from the MVT for functions of several variables.