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Thus,

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Example: Equidistant piecewise linear interpolation

Consider the following integral equation

$$\varphi(x) - \frac{1}{2} \int_0^1 (x+1)e^{-xy}\varphi(y)dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}, \qquad 0 \le x \le 1.$$





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Table showing the error between the approximate and the exact solution

n	x = 0	x = 0.25	x = 0.5	x = 0.75	x = 1
4	0.004808	0.005430	0.006178	0.007128	0.008331
8	0.001199	0.001354	0.001541	0.001778	0.002078
16	0.000300	0.000385	0.000385	0.000444	0.000519
32	0.000075	0.000085	0.000096	0.000111	0.000130





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If the kernel is analytic, then we have

$$||A - A_n||_{\infty} \le O(e^{-ns})$$

for some s > 0.



Example: Trigonometric interpolation

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$$u(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{n-1} [\alpha_k \cos kt + \beta_k \sin kt] + \frac{\alpha_n}{2} \cos nt$$

with the interpolation property
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To check the interpolation property, note that

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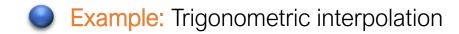
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Exercise





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Lagrange Basis

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Theorem: Let $g: \mathbb{R} \to \mathbb{R}$ be analytic and 2π periodic. Then there exists a strip $D = \mathbb{R} \times (-s,s) \subset \mathbb{C}$ with s > 0 such that g can be extended to a holomorphic and 2π periodic bounded function $g: D \to \mathbb{C}$. The error for the trigonometric interpolation can be extended by

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Thus, we have the estimate

$$||P_n g - g||_{\infty} \le Ce^{-ns}$$

for the trigonometric interpolation of periodic analytic functions, where C and s are some positive constants depending on g.





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Consider the following integral equation ($a \ge b > 0$, c = (a - b)/(a + b))

$$\varphi(t) + \frac{ab}{\pi} \int_0^{2\pi} \frac{\varphi(\tau)d\tau}{a^2 + b^2 - (a^2 - b^2)\cos(t + \tau)} dy = e^{\cos t}\cos(\sin t) + e^{c\cos t}\cos(c\sin t), \qquad 0 \le t \le 2\pi.$$





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Table showing the error between the approximate and the exact solution

	2n	t = 0	$t = \pi/2$	$t=\pi$
a = 1 $b = 0.2$	4	-0.56984945	-0.18357135	0.06022598
	8	-0.14414257	-0.00368787	-0.00571394
	16	-0.00602543	-0.00035953	-0.00045408
	32	-0.00000919	-0.00000055	-0.00000069

Numerical Analysis & Scientific Computing II

Lesson 5 Integral Equations

- 5.2 An Introduction
- **5.3 Numerical Methods**
 - Degenerate Kernel Method
 - via orthogonal expansion



Integral Equations: Numerical Methods

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$$K_n(x,y) = \sum_{j=1}^n u_j(x) \langle K(\cdot,y), u_j \rangle$$

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Note that the setting up the linear system requires a double integration for each coefficient and for each right-hand side.

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defines an inner product $\langle T_m, T_n \rangle = \frac{\pi}{2} (\delta_{mn} + \delta_{n0})$, that is, T_n form an orthogonal system with respect to $\langle \cdot, \cdot \rangle$. The approximate solution u_n to (I - A)u = f is obtained as

$$u_n(x) = f(x) + \sum_{j=1}^{n} \gamma_j T_j(x)$$

where the coefficients γ_i solve the linear system

$$\gamma_{j} - \sum_{k=1}^{n} \gamma_{k} \int_{-1}^{1} T_{k}(y) \langle K(\cdot, y), T_{k} \rangle dy = \int_{-1}^{1} f(y) \langle K(\cdot, y), T_{k} \rangle dy, \qquad j = 1, 2, \dots, n.$$



Theorem

Let $g: [-1,1] \to \mathbb{R}$ be analytic. Then, there exists an ellipse E with foci at -1 and 1 such that g can be extended to a holomorphic and bounded function $g: D \to \mathbb{C}$ where D denotes the open interior of E. The orthonormal expansion with respect to the Chebyshev polynomials

$$g = \frac{a_0}{2}T_0 + \sum_{n=1}^{\infty} a_n T_n$$
, $a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{g(x)T_n(x)}{\sqrt{1-x^2}} dx = \frac{2}{\pi} \langle g, T_n \rangle$

is uniformly convergent with the estimate

$$\left\|g - \frac{a_0}{2}T_0 - \sum_{k=1}^n a_k T_k\right\|_{\infty} \le \frac{2M}{R-1}R^{-n}.$$

Here R is given through the semi-axis a and b of E by R = a + b and M is a bound on g in D.



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Using this result from the "approximation theory", we see that

$$\|A_n - A\|_{\infty} = \max_{x \in [-1,1]} \int_{-1}^{1} \left| \frac{1}{\pi} T_0(x) \langle K(\cdot, y), T_0 \rangle - \frac{2}{\pi} \sum_{k=1}^{n} a_k T_k(x) \langle K_n(\cdot, y), T_k \rangle - K(x, y) \right| dy \le O(R^{-n}).$$

This leads to accurate approximations if the kernel is sufficiently smooth.