Lesson 4

Numerical Solution of PDE

4.1 BVP for 2^{nd} Order Elliptic PDE

Akash Anand
MATH, IIT KANPUR

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- Finite Difference Method

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Numerical Methods for PDE: 2nd Order Elliptic PDE

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A natural generalization to the two-point BVP

$$u'' = f(t), \qquad a < t < b,$$
$$u(a) = 0, \qquad u(b) = 0,$$

to two dimensions is

$$\Delta u := u_{x_1x_1} + u_{x_2x_2} = f, \qquad \text{in } \Omega,$$
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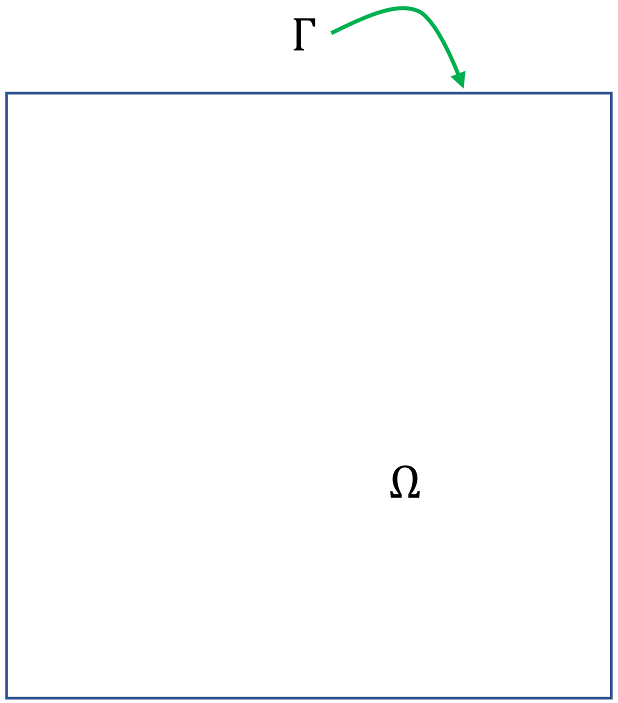

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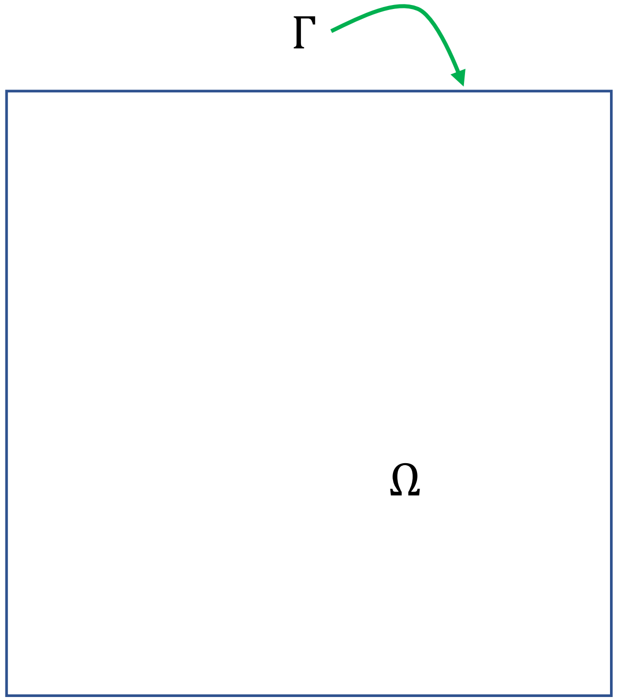

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Let N *be a positive integer and set* $h = 1/N$. *Consider the mesh in* $\mathbb{R}^2$

$$\mathbb{R}_h^2 = \{(mh, nh): m, n \in \mathbb{Z}\}.$$

Let $\Omega_\mathrm{h} = \Omega \cap \mathbb{R}_h^2$, *the set of interior mesh points.*

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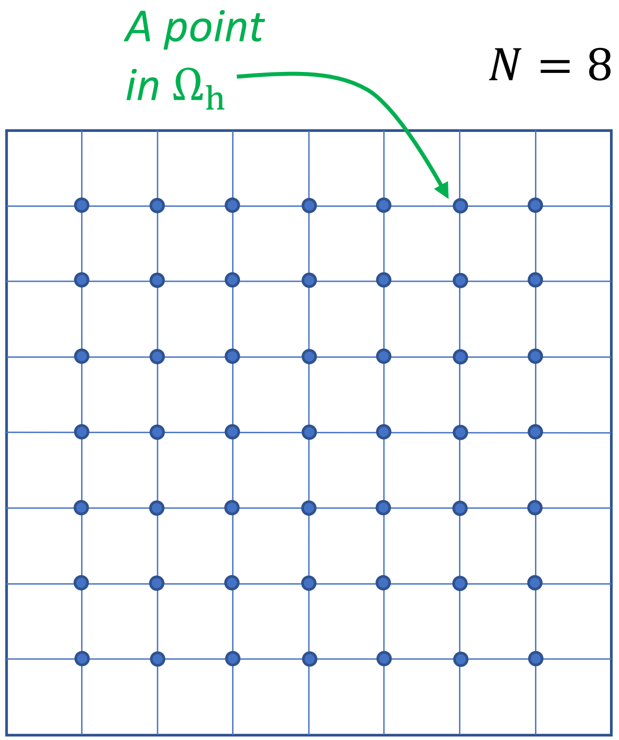

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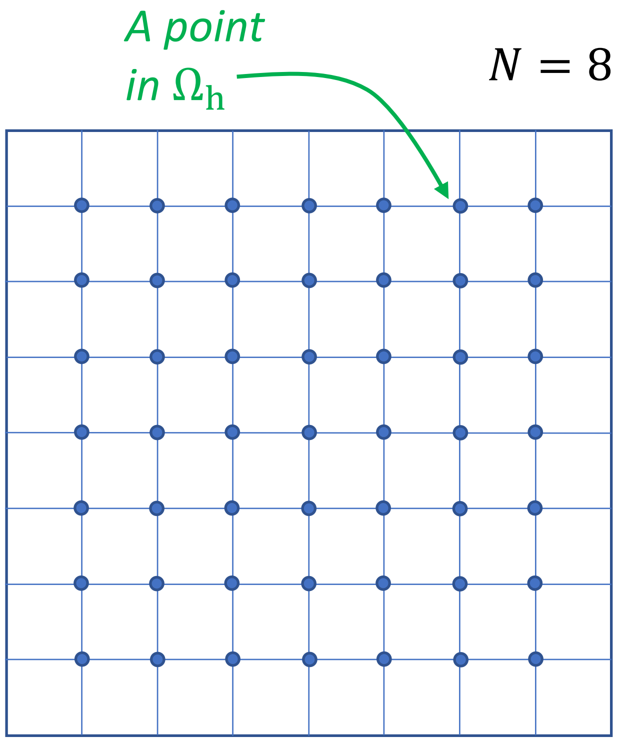

Note that for each $x \in \mathbb{R}_h^2$ *has a set of four nearest neighbors in* $\mathbb{R}_h^2$, *one each to the left, right, above and below. We define* Γ_h *as the set of mesh points in* $\mathbb{R}_h^2$ *which in not in* Ω_h *but has a nearest neighbor in* Ω_h.

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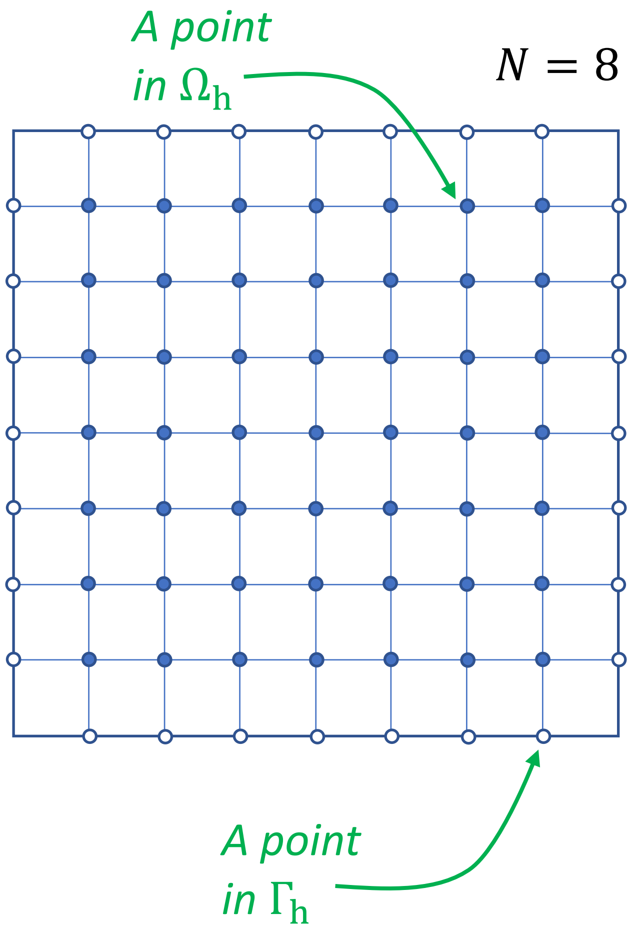

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Let N be a positive integer and set $h = 1/N$. Consider the mesh in $\mathbb{R}^2$

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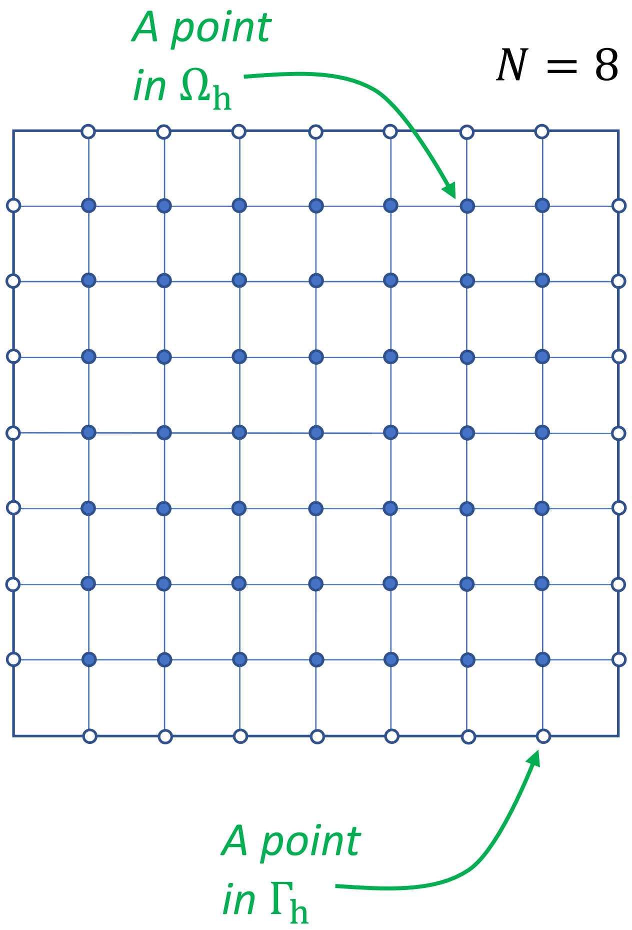

Note that for each $x \in \mathbb{R}_h^2$ has a set of four nearest neighbors in $\mathbb{R}_h^2$, one each to the left, right, above and below. We define Γ_h as the set of mesh points in $\mathbb{R}_h^2$ which in not in Ω_h but has a nearest neighbor in Ω_h.

Also let $\overline{\Omega}_h = \Omega_h \cup \Gamma_h$.

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To discretize

$$\Delta u = f, \quad \text{in } \Omega,$$
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we seek a function $u_h : \overline{\Omega_h} \to \mathbb{R}$ *satisfying*

$$\Delta_h u_h = f, \quad \text{on } \Omega_\mathrm{h},$$
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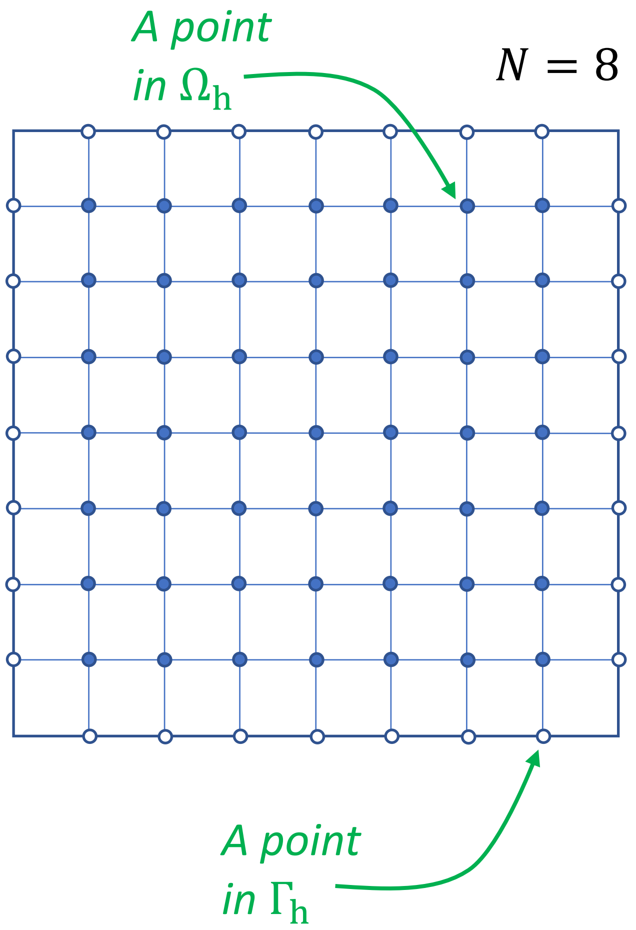
where, writing $v_{m,n} = v(mh, nh)$*, we have the 5-point Laplacian*

$$\Delta_h v(mh, nh) = \frac{v_{m+1,n} - 2v_{m,n} + v_{m-1,n}}{h^2} + \frac{v_{m,n+1} - 2v_{m,n} + v_{m,n-1}}{h^2}$$



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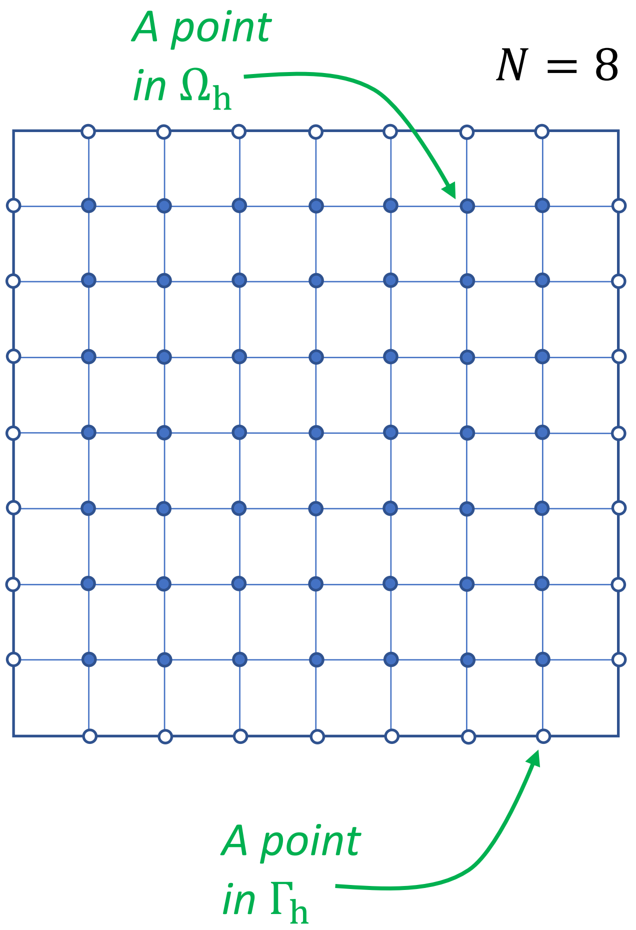
$$= \frac{v_{m+1,n} + v_{m-1,n} + v_{m,n+1} + v_{m,n-1} - 4v_{m,n}}{h^2}.$$



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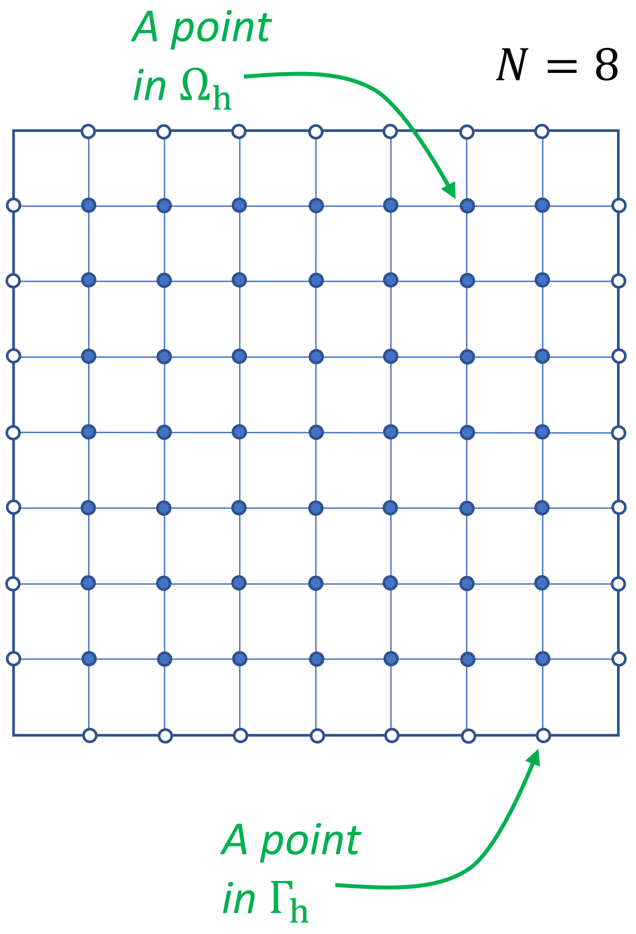
$$= \frac{v_{m+1,n} + v_{m-1,n} + v_{m,n+1} + v_{m,n-1} - 4v_{m,n}}{h^2}.$$

From the error estimate in one-dimensional case, we case easily get that for $v \in C^4(\overline{\Omega})$,

$$\Delta_h v(mh, nh) - \Delta v(mh, nh) = \frac{h^2}{12}\left[\frac{\partial^4 v}{\partial x_1^4}(\xi, nh) + \frac{\partial^4 v}{\partial x_2^4}(mh, \eta)\right]$$

for some ξ, η.



Theorem

If $v \in C^2(\overline{\Omega})$, *then*

$$\lim_{h \to 0} \|\Delta_h v - \Delta v\|_{\infty,\Omega_h} = 0.$$

If $v \in C^4(\overline{\Omega})$, *then*

$$\|\Delta_h v - \Delta v\|_{\infty,\Omega_h} \leq \frac{h^2}{6} M_4,$$

where

$$M_4 = \max\left\{ \left\| \frac{\partial^4 v}{\partial x_1^4} \right\|_{\infty,\overline{\Omega}}, \left\| \frac{\partial^4 v}{\partial x_2^4} \right\|_{\infty,\overline{\Omega}} \right\}.$$

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Note that the discrete PDE $\Delta_h u_h = f,\ \text{on } \Omega_\text{h}$ *is a system of* $(N-1)^2$ *equations in* $(N-1)^2$ *unknowns.*

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Note that the discrete PDE $\Delta_h u_h = f$, on Ω_h *is a system of* $(N-1)^2$ *equations in* $(N-1)^2$ *unknowns.*

Theorem (Discrete Maximum Principle)

Let v be a function on $\overline{\Omega}_h$ satisfying $\Delta_h v \ge 0$ on Ω_h. Then $\max_{\Omega_h} v \le \max_{\Gamma_h} v$. Equality holds if and only if v is constant.

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Let v *be a function on* $\overline{\Omega}_h$ *satisfying* $\Delta_h v \ge 0$ *on* Ω_h. *Then* $\max_{\Omega_h} v \le \max_{\Gamma_h} v$. *Equality holds if and only if* v *is constant.*

Proof: *Exercise. HINT: Use* $4v(x_0) = \sum_{i=1}^4 v(x_i) - h^2\,\Delta_h v(x_0)$ *where* x_1, x_2, x_3, x_4 *are neighbors of* x_0.

Theorem (Discrete Minimum Principle)

Let v be a function on $\overline{\Omega}_h$ satisfying $\Delta_h v \leq 0$ on Ω_h. Then $\min_{\Omega_h} v \geq \min_{\Gamma_h} v$. Equality holds if and only if v is constant.

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Theorem

There is a unique solution to the discrete BVP

$$\begin{aligned} \Delta_h u_h &= f, && \text{on } \Omega_\text{h}, \\ u_h &= g, && \text{on } \Gamma_\text{h}. \end{aligned}$$

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The solution u_h to

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satisfies

$$\|u_h\|_{\infty,\overline{\Omega}_h} \leq \frac{1}{8}\|f\|_{\infty,\Omega_h} + \|g\|_{\infty,\Gamma_h}.$$

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Proof: Exercise. HINT: Use $w(x_1, x_2) = [(x_1 - 1/2)^2 + (x_2 - 1/2)^2]/4.$

Theorem

Let u be the solution to

$$\begin{aligned} \Delta u &= f, && \text{in } \Omega, \\ u &= g, && \text{on } \Gamma, \end{aligned}$$

and u_h be the solution to the corresponding discrete problem

$$\begin{aligned} \Delta_h u_h &= f, && \text{on } \Omega_\mathrm{h}, \\ u_h &= g, && \text{on } \Gamma_\mathrm{h}. \end{aligned}$$

Then,

$$\|u_h - u\|_{\infty,\overline{\Omega}_h} \le \frac{1}{8}\|\Delta u - \Delta_h u\|_{\infty,\overline{\Omega}_h}.$$

Proof: *Exercise.*

Corollary

If $u \in C^2\left(\overline{\Omega}_h\right)$, then

$$\lim_{h\to 0}\|u_h - u\|_{\infty,\overline{\Omega}_h} = 0.$$

If $u \in C^4\left(\overline{\Omega}_h\right)$, then

$$\|u_h - u\|_{\infty,\overline{\Omega}_h} \le \frac{h^2}{48}M_4, \qquad M_4 = \max\left\{\left\|\frac{\partial^4 v}{\partial x_1^4}\right\|_{\infty,\overline{\Omega}}, \left\|\frac{\partial^4 v}{\partial x_2^4}\right\|_{\infty,\overline{\Omega}}\right\}.$$