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Lesson 3

Boundary Value Problems for ODEs

3.2 Shooting Method

3.3 Finite Difference Method

3.4 Variational Methods

- Convergence Analysis for Galerkin Method

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For analyzing the computational error in the Galerkin method, note that the solution u to the boundary value problem

$$\begin{aligned} u'' &= f(t), \qquad && a < t < b, \\ u(a) &= 0, \qquad && u(b) = 0, \end{aligned}$$

and any function of the form

$$v(t,x) = \sum_{i=1}^{n} x_i \varphi_i(t),$$

with $\varphi_i(a) = \varphi_i(b) = 0$, satisfies

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As $z \in \mathbb{R}^n$ is arbitrary, we conclude that

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For piecewise linear elements, we have

$$\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1} \le Ch\|u''\|_{L^2}$$

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and, more generally, for elements of degree r, we have an estimate

$$\inf_{z\in\mathbb{R}^n}\|u-v(\cdot,z)\|_{H^1} \le Ch^r\|u\|_{H^{r+1}}$$