Lesson 3

Boundary Value Problems for ODEs

3.1 Well-posedness

3.2 Shooting Method

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Boundary Value Problems: Shooting Method

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Recall the following discussion in the contest of solvability of two-point BVP

$$y' = f(t, y), \qquad a < t < b,$$

with boundary conditions

$$g(y(a), y(b)) = 0.$$

We noted that if $y(t; x)$ *denotes the solution to the IVP* $y' = f(t, y),\ y(a) = x,\ x \in \mathbb{R}^n$*, then this solution is a solution to the BVP if*

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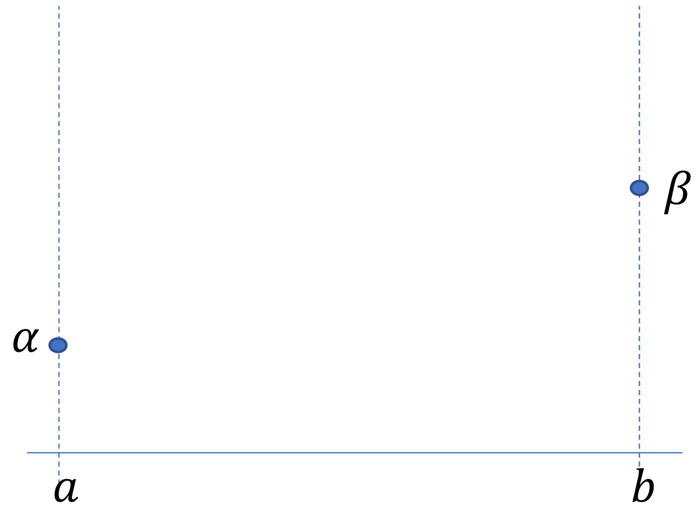

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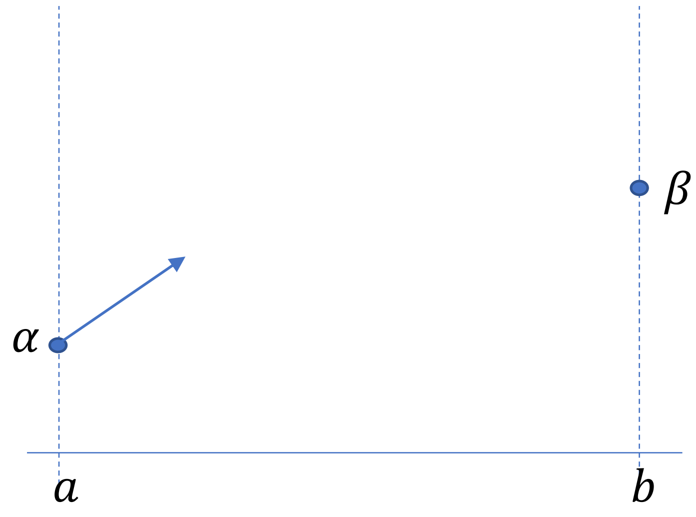

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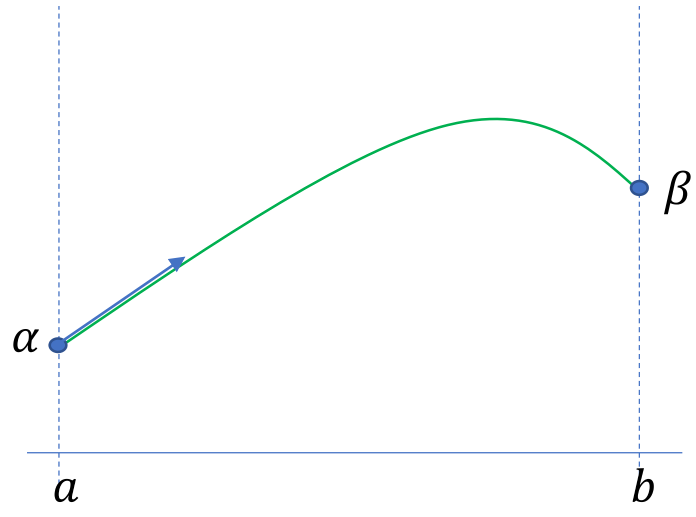

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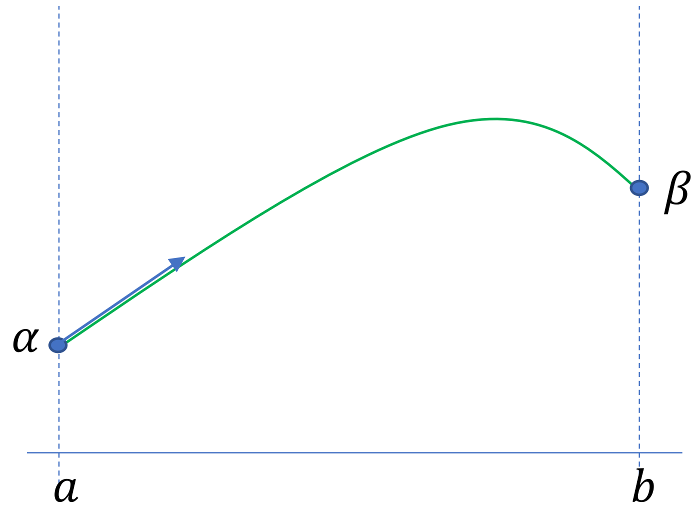

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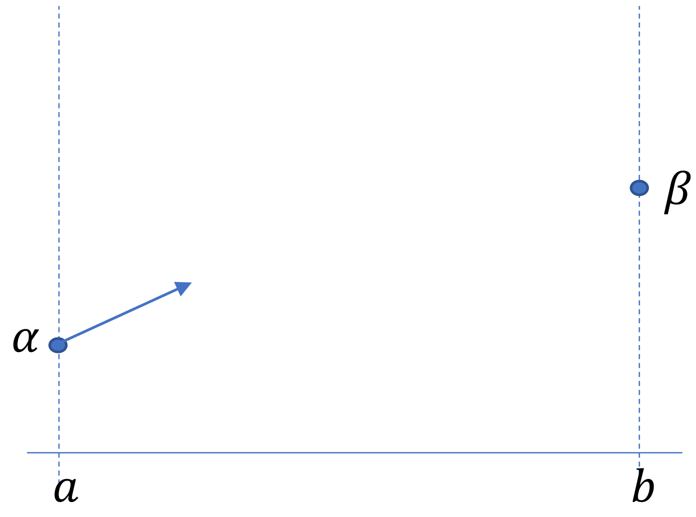

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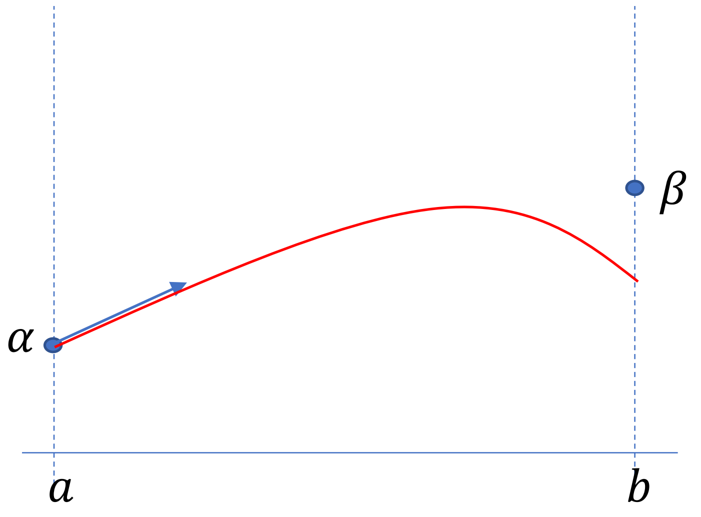

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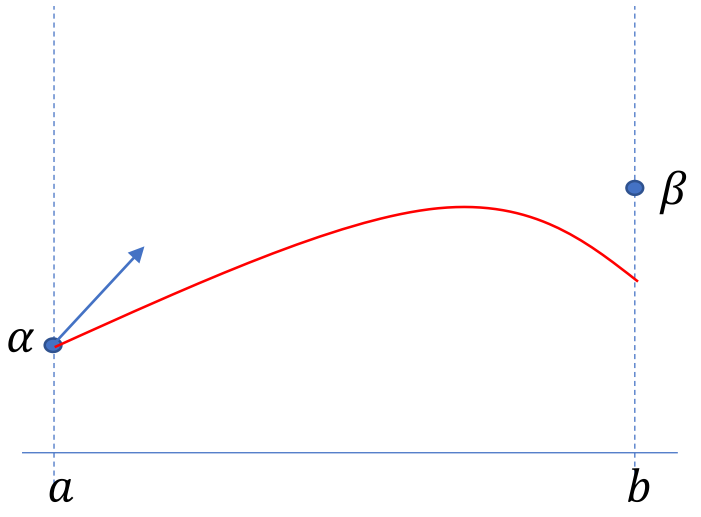

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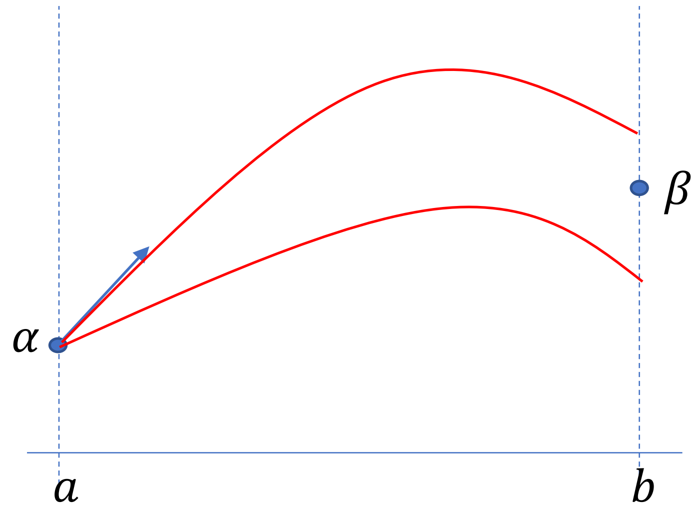

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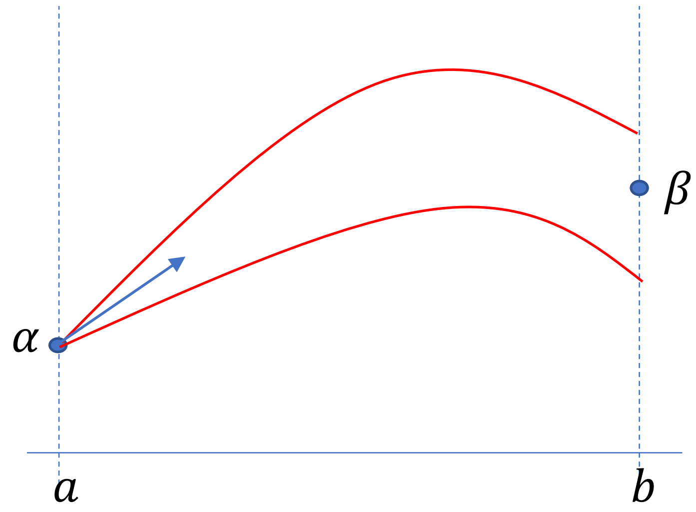

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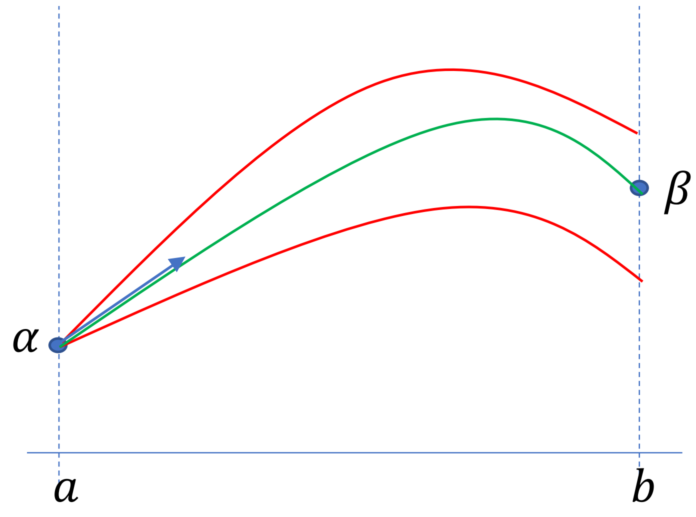

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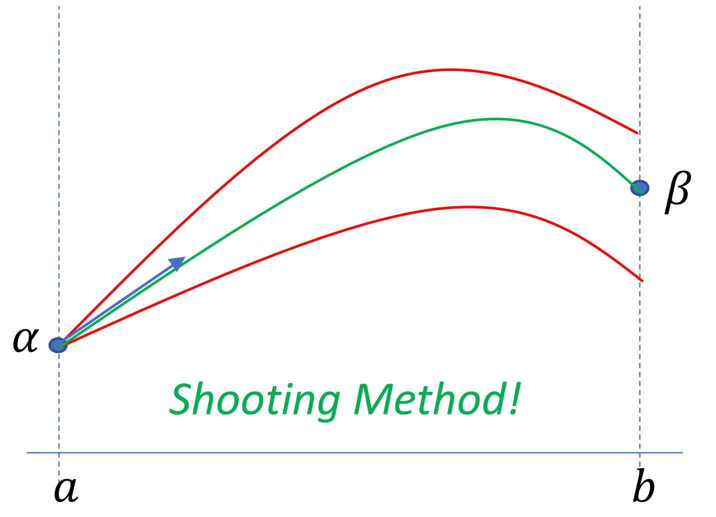

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$$u'' = f(t, u, u'), \qquad a < t < b,$$
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where the function f *is assumed to satisfy the following Lipschitz conditions:*

$$|f(t, u_1, v) - f(t, u_2, v)| \le K|u_1 - u_2|,$$
$$|f(t, u, v_1) - f(t, u, v_2)| \le K|v_1 - v_2|,$$

for all points $(t, u_i, v), \left(t, u, v_j\right) \in R \coloneqq [a, b] \times \mathbb{R} \times \mathbb{R}$. *In addition, assume that on* R, f *satisfies*

$$f_u(t, u, v) = \frac{\partial f(t, u, v)}{\partial u} > 0, |f_v(t, u, v)| = \left|\frac{\partial f(t, u, v)}{\partial v}\right| \le M,$$

for some $M > 0$. *For the boundary conditions, assume*

$$a_0 a_1 \ge 0, \qquad b_0 b_1 \ge 0, \qquad |a_0| + |a_1| \ne 0, \qquad |b_0| + |b_1| \ne 0, \qquad |a_0| + |b_0| \ne 0.$$

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Theorem

The BVP, under these assumptions, has a unique solution.

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How do we solve the BVP using the shooting method?

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Let $y(t;s)$ *be the solution to the IVP*

$$y'' = f(t,y,y'), \qquad a < t < b,$$
$$y(a) = a_1 s - c_1 \alpha, \qquad y'(a) = a_0 s - c_0 \alpha$$

depending on the parameter s, *where* c_0, c_1 *are arbitrary constants satisfying* $a_1 c_0 - a_0 c_1 = 1$. *Note that*

$$a_0 y(a;s) - a_1 y'(a;s) = \alpha.$$

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depending on the parameter s, *where* c_0, c_1 *are arbitrary constants satisfying* $a_1c_0 - a_0c_1 = 1$. *Note that*

$$a_0y(a;s) - a_1y'(a;s) = \alpha.$$

For $y(t;s)$ *to be the solution to the BVP, it must satisfy*

$$h(s) \coloneqq b_0y(b;s) + b_1y'(b;s) - \beta = 0.$$

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$$y(a) = a_1 s - c_1 \alpha, \qquad y'(a) = a_0 s - c_0 \alpha$$

depending on the parameter s, where c_0, c_1 are arbitrary constants satisfying $a_1 c_0 - a_0 c_1 = 1$. Note that

$$a_0 y(a;s) - a_1 y'(a;s) = \alpha.$$

For $y(t;s)$ to be the solution to the BVP, it must satisfy

$$h(s) \coloneqq b_0 y(b;s) + b_1 y'(b;s) - \beta = 0.$$

For solution of this equation using the Newton's method, we have

$$s_{m+1} = s_m - \frac{h(s_m)}{h'(s_m)}, \qquad m = 0,1,2,\ldots$$

Boundary Value Problems: Shooting Method

Akash Anand
MATH, IIT KANPUR

Consider the two-point BVP

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$$z_s''(t) = f_2\big(t, y(t;s), y'(t;s)\big) z_s(t) + f_3\big(t, y(t;s), y'(t;s)\big) z_s'(t), \qquad z_s(a) = a_1, \qquad z_s'(a) = a_0.$$

The functions f_2 and f_3 denote the partial derivatives of $f(t,u,v)$ with respect to u and v respectively.