Discussion Hour 2 (13-12-2021):

Announcement:

- Quiz 1: Result is declared on CT. All reported issues have been taken care of.
- Rules for all subsequent quizzes (exam environment, question types, …….)
- Issues related to CodeTantra (technical problem, difficulty, ………)
- Friday Discussion Hour (some issue with the Zoom License)

Discussion topics:

Q. How to understand vector derivatives (grad, div and curl) in spherical polar and cylindrical coordinate system

Del operator in Spherical polar coordinate system

A scalar field u is a function of the spherical coordinates r, θ, and ϕ.

Small change in u i.e. du can be written as

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \phi}d\phi$$

According to the definition of gradient;

$$du = \vec{\nabla}u \cdot d\vec{r}$$

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \phi}d\phi = \vec{\nabla}u \cdot d\vec{r}$$

$$d\vec{r} = d(r\hat{r})$$

$$= \hat{r}dr + rd\hat{r}$$

$$= \hat{r}dr + r\left(\frac{\partial \hat{r}}{\partial r}dr + \frac{\partial \hat{r}}{\partial \theta}d\theta + \frac{\partial \hat{r}}{\partial \phi}d\phi\right)$$

$$= \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$$

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \phi}d\phi = \left(\vec{\nabla}u\right)_r dr + \left(\vec{\nabla}u\right)_\theta rd\theta + \left(\vec{\nabla}u\right)_\phi r\sin\theta d\phi$$

This holds true for any choice of dr, $d\theta$, and $d\phi$. So,

$$\left(\vec{\nabla}u\right)_r = \frac{\partial u}{\partial r}, \quad \left(\vec{\nabla}u\right)_\theta = \frac{1}{r}\frac{\partial u}{\partial \theta}, \quad \left(\vec{\nabla}u\right)_\phi = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\boxed{\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}}$$

Divergence in Spherical polar coordinate system

The divergence $\vec{\nabla} \cdot \vec{A}$ is carried out taking into account that the unit vectors themselves are functions of the coordinates. So,

$$\vec{\nabla} \cdot \vec{A} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \cdot \left(A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}\right)$$

Here the derivatives must be taken *before* the dot product, so

$$\vec{\nabla} \cdot \vec{A} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \cdot \vec{A}$$

$$= \hat{r} \cdot \frac{\partial \vec{A}}{\partial r} + \frac{\hat{\theta}}{r} \cdot \frac{\partial \vec{A}}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta} \cdot \frac{\partial \vec{A}}{\partial \phi}$$

$$= \hat{r} \cdot \left(\frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\phi}{\partial r}\hat{\phi} + A_r\frac{\partial \hat{r}}{\partial r} + A_\theta\frac{\partial \hat{\theta}}{\partial r} + A_\phi\frac{\partial \hat{\phi}}{\partial r}\right)$$

$$+ \frac{\hat{\theta}}{r} \cdot \left(\frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\phi}{\partial \theta}\hat{\phi} + A_r\frac{\partial \hat{r}}{\partial \theta} + A_\theta\frac{\partial \hat{\theta}}{\partial \theta} + A_\phi\frac{\partial \hat{\phi}}{\partial \theta}\right)$$

$$+ \frac{\hat{\phi}}{r\sin\theta} \cdot \left(\frac{\partial A_r}{\partial \phi}\hat{r} + \frac{\partial A_\theta}{\partial \phi}\hat{\theta} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} + A_r\frac{\partial \hat{r}}{\partial \phi} + A_\theta\frac{\partial \hat{\theta}}{\partial \phi} + A_\phi\frac{\partial \hat{\phi}}{\partial \phi}\right)$$

Using partial derivatives of spherical polar coordinates,

$$\vec{\nabla}\cdot\vec{A} = \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\phi}{\partial r}\hat{\phi} + 0 + 0 + 0\right)$$

$$+\frac{\hat{\theta}}{r}\cdot\left(\frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\phi}{\partial \theta}\hat{\phi} + A_r\hat{\theta} + A_\theta(-\hat{r}) + 0\right)$$

$$+\frac{\hat{\phi}}{r\sin\theta}\cdot\left(\frac{\partial A_r}{\partial \phi}\hat{r} + \frac{\partial A_\theta}{\partial \phi}\hat{\theta} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} + A_r\sin\theta\hat{\phi} + A_\theta\cos\theta\hat{\phi} + A_\phi\left[-\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\right]\right)$$

$$= \left(\frac{\partial A_r}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r}\right) + \left(\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} + \frac{A_r}{r} + \frac{A_\theta\cos\theta}{r\sin\theta}\right)$$

$$= \left(\frac{\partial A_r}{\partial r} + \frac{2A_r}{r}\right) + \left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{A_\theta\cos\theta}{r\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}}$$

Vector derivatives (grad, div and curl) in spherical polar and cylindrical coordinate system will be provided during exams. So, you need not to remember them. But still, you must understand their derivation.

Problem 1.8: A vector $\vec{V}$ is called irrotational if curl $\vec{V} = 0$.

(a) Find constants a, b and c so that following vector is irrotational.

$$\vec{V} = (x + 2y + az)\hat{\mathbf{i}} + (bx - 3y - z)\hat{\mathbf{j}} + (4x + cy + 2z)\hat{\mathbf{k}}$$

(b) Show that $\vec{V}$ can be expressed as the gradient of a scalar function.

(a)

$$\Rightarrow \vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}$$

$$= \hat{i}(c+1) + \hat{j}(a-4) + \hat{k}(b-2) = 0$$

$$\Rightarrow a = 4,\ b = 2,\ c = -1$$

(b) $\vec{\nabla} \times \vec{V} = 0 \quad \Rightarrow \quad \vec{V} = \vec{\nabla}\phi$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla}\phi) = 0 \qquad \boxed{\text{curl of grad} = 0}$$

$$\vec{V} = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = x + 2y + 4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz + f_1(y,z) + C_1$$

$$\frac{\partial \phi}{\partial y} = 2x - 3y - z \Rightarrow \phi = 2xy - \frac{3y^2}{2} - yz + f_2(x,z) + C_2$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2 + f_3(x,y) + C_3$$

$$f_1(y,z) = -\frac{3y^2}{2} + z^2$$

$$f_2(x,z) = \frac{x^2}{2} + z^2$$

$$f_3(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

$$\vec{V} \Rightarrow \vec{\nabla} \times \vec{V} = 0$$

$$\Downarrow$$

$$\vec{V} = \vec{\nabla}\phi$$

$$\Downarrow$$

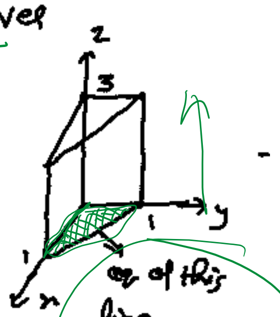
$$\phi = f(x,y,z) + \text{const}$$

$$\Rightarrow \boxed{\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + C}$$

$\vec{V} \longrightarrow \phi$ (choice over C)

Ex. 8, Lecture 4: Volume integration of a vector over a given prism.

calculate the volume integral of $T = xyz^2$ over the shown prism.



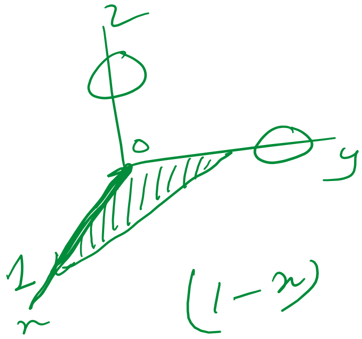
$\int_V T\,d\tau \Rightarrow$ for $x \Rightarrow \int_0^{1-y} x\,dx$

for $y \Rightarrow \int_0^1 y\,dy$

$z \Rightarrow \int_0^3 z^2\,dz$

eq. of this line
$y = -x+1$
$\Rightarrow x = 1-y$

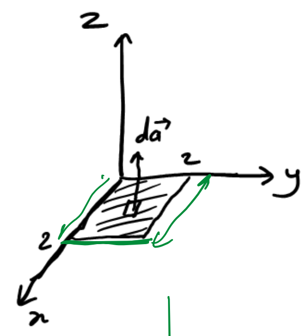
$$\int_V T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left(\int_0^{1-y} x\,dx \right) dy \right\} dz = \cdots = \boxed{\frac{3}{8}}$$



Q. At 28 mins 24 sec, in surface integral example how have they splitted the multiplication terms dx and dy while integration. We do this in addition but is it correct in multiplication as well?

$$\vec{A} = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$$

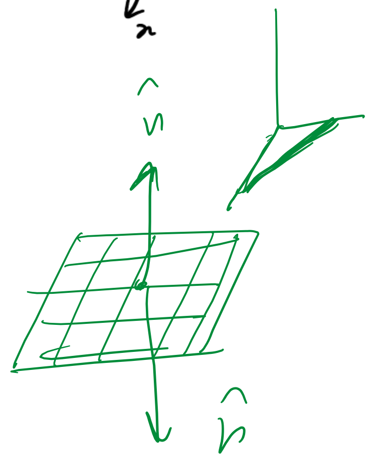
$\int_S \vec{A}\cdot d\vec{a}$ over the shaded area.



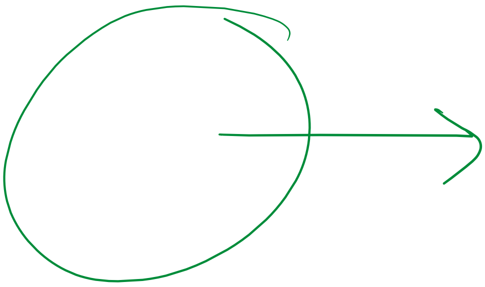
\# $x \,\&\, y : 0 \rightarrow 2$, $z = 0$, $d\vec{a} = dx\,dy\,\hat{z}$

$$\vec{A}\cdot d\vec{a} = y(z^2-3)\,dx\,dy = -3y\,dx\,dy$$

$$\int_S \vec{A}\cdot d\vec{o} = -3\int_0^2 dx \int_0^2 y\,dy = -12 = 12$$



$$d\bar{a} \rightarrow \begin{pmatrix} +\hat{z} \\ -\hat{z} \end{pmatrix}$$



Q. Geometrical Interpretation of Greens theorem as in Griffith

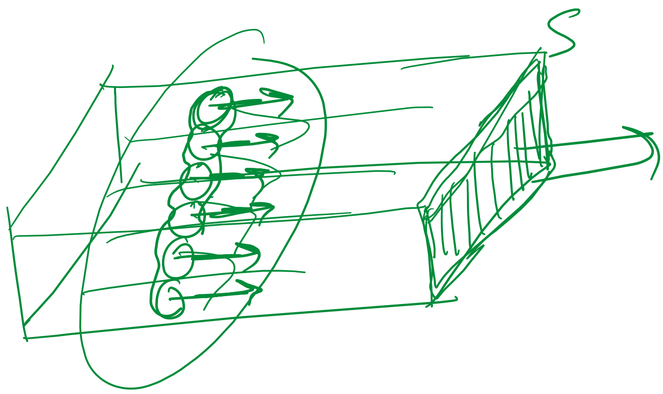
The theorem states that volume integral of divergence of a vector function over a region is equal to the value of the function at the boundary of the surface of that region.

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.$$

Geometrical Interpretation: If $\mathbf{v}$ represents the flow of an incompressible fluid, then the *flux* of $\mathbf{v}$ (the right side of Eq. 1.56) is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the "spreading out" of the vectors from a point—a place of high divergence is like a "faucet," pouring out liquid. If we have a bunch of faucets in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region. In fact, there are *two* ways we could determine how much is being produced: (a) we could count up all the faucets, recording how much each puts out, or (b) we could go around the boundary, measuring the flow at each point, and add it all up. You get the same answer either way:

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface}).$$

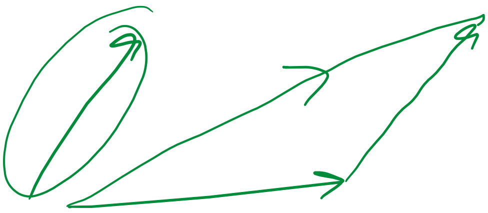
This, in essence, is what the divergence theorem says.

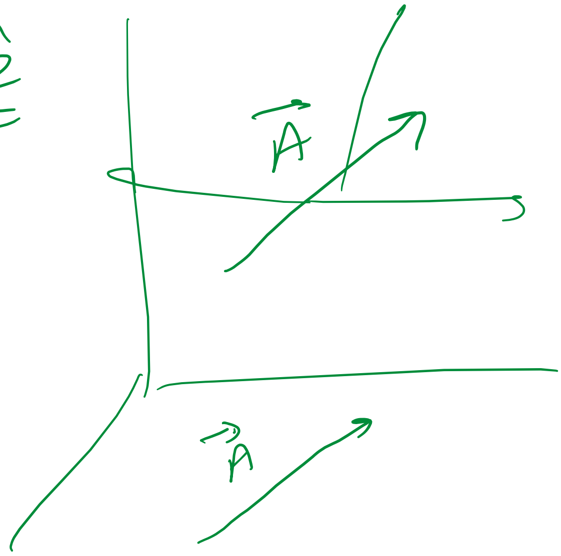


$\int_S \bar{V} \cdot d\bar{a}$

Q. Sir what is the meaning of sentence "vectors does not have location". Please explain.

$$\vec{A} = 5\hat{x} + 6\hat{y} - 2\hat{z}$$

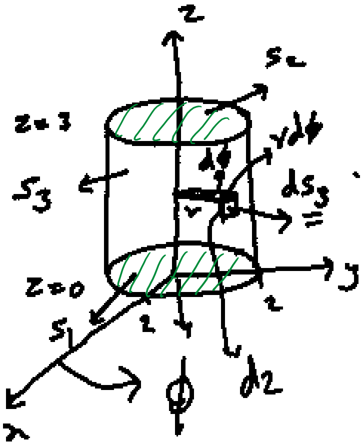




Q. lecture 5 example 3, how is s3 calculated in last?

Verify Divergence theorem for $\vec{A} = 4x\hat{x} - 2y^2\hat{y} + z^2\hat{z}$ over region bounded by $x^2+y^2=4$ between $z=0$ to $z=3$.

$\int_V (\vec{\nabla}\cdot\vec{A})\, d\tau = \oint \vec{A}\cdot d\vec{a} = \oint \vec{A}\cdot\hat{n}\, da$



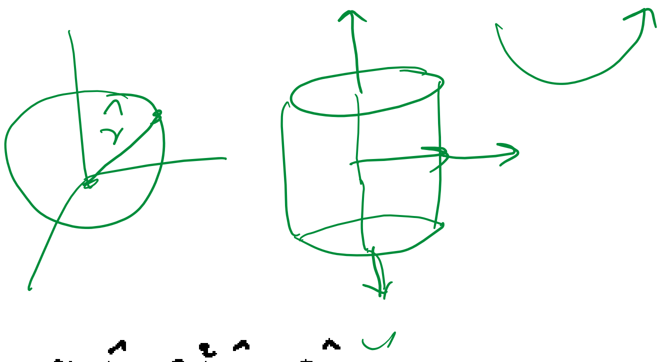
R.H.S. are integration over three surfaces

(i) S_1 $(z=0)$ base, (ii) S_2 $(z=3)$ top

(iii) $S_3 \rightarrow$ curved part of the cylinder.

(i) S_1: $\hat{n} \rightarrow -\hat{z}$, $\vec{A} = 4x\hat{x} - 2y^2\hat{y} + 0\hat{z}$

$\Rightarrow \vec{A}\cdot\hat{n} = 0$



(ii) for S_2: $(z=3)$ $\hat{n} = \hat{z}$ $\Rightarrow \vec{A} = 4x\hat{x} - 2y^2\hat{y} + 9\hat{z}$

$\vec{A}\cdot\hat{n} = 9 \Rightarrow \int_{S_2} (\vec{A}\cdot\hat{n})\,da = \int_{S_2} 9\,da = 9\int_{S_2} da = 9.\pi 2^2$

$= 36\pi$

(iii) for S_3: $ds_3 = r\,d\phi\,dz$ for $\hat{n} \rightarrow$ equation of the ~~Conver~~ curved surface $x^2+y^2=4$

$\downarrow$

perpendicular to the curved surface

$\nabla.(x^2+y^2) = 2x\hat{x} + 2y\hat{y}$

$\Rightarrow \hat{n} = \dfrac{2x\hat{x} + 2y\hat{y}}{\sqrt{4x^2+4y^2}} = \dfrac{x\hat{x} + y\hat{y}}{2}$

$\vec{A}\cdot\hat{n} = (4x\hat{x} - 2y^2\hat{y} + z^2\hat{z}).\left(\dfrac{x\hat{x} + y\hat{y}}{2}\right)$

$= 2x^2 - y^3$

$$\int_{S_3} (\vec{A}\cdot\hat{n})\, da = \int (2x^2 - y^3)\, r\, d\phi\, dz = \int_{\phi=0}^{2\pi}\int_0^3 \left[2.(2\cos\phi)^2 - (2\sin\phi)^3\right] 2\, d\phi\, dz$$

$r = 2$

$x = r\cos\phi = 2\cos\phi$

$y = 2\sin\phi$

$$= \underline{\underline{48\pi}}$$

total R.H.S. $= 0 + 36\pi + 48\pi = \underline{\underline{84\pi}}$

$\underline{\underline{\text{L.H.S.}}}$ $$\int (\vec{\nabla}\cdot\vec{A})\, dz = \int\left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dz = \int (4 - 4y + 2z)\, dx\, dy\, dz$$

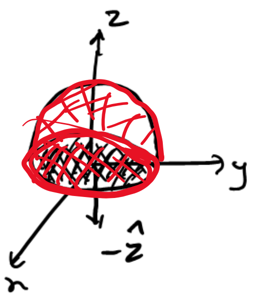
$$= \int_{-2}^{+2}\left[\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\left(\int_0^3 (4 - 4y + 2z)\, dz\right) dy\right] dx = \underline{\underline{84\pi}}$$

Q. In example 11 while calculating LHS why didn't we integrated for surface 2 that is the surface of the hemisphere?
Lecture 5, last example, doubt in verification of stokes theorem
Q. At the end of example 11, why did we ignore the negative sign of da?

verify stokes theorem $\vec{A} = (2x-y)\hat{x} - yz^2\hat{y} - y^2z\hat{z}$

for the surface of upper half of the sphere $x^2+y^2+z^2=1$.

$\int_S (\vec{\nabla}\times\vec{A}).d\vec{a} = \oint_P \vec{A}.d\vec{l}$



R.H.S. $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$, $z=0$

$\vec{A}\cdot d\vec{l} = (2x-y)\,dx$ parametric eq's $x = \cos t$, $y = \sin t$, $z = 0$

$$\int \vec{A}\cdot d\vec{l} = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt$$

$$= \pi$$

L.H.S. $\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{z}$

$d\vec{a} = -dx\,dy\,\hat{z}$

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{A}).d\vec{a} = \int_S \hat{z}.\hat{z}\,da = \int_S da = \pi\cdot 1^2 = \pi = \text{R.H.S}$$

Problem 2.4: The integral $\vec{a} \equiv \int_S d\vec{a}$ is sometimes called the vector area of the surface S. If S happens to be *flat*, then $|\vec{a}|$ is the *ordinary* (scalar) area, obviously.

(a) Find the **vector area** of a hemispherical bowl of radius R.

(b) Show that $\vec{a} = 0$ for any closed surface.

(c) Show that $\vec{a}$ is the same for all surfaces sharing the same boundary.

(d) Show that $\vec{a} = \frac{1}{2}\oint \vec{r}\times d\vec{l}$, where the integral is around the boundary line.

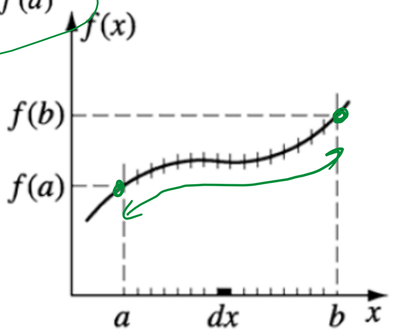
(e) Show that $\oint(\vec{c}\cdot\vec{r})\,d\vec{l} = \vec{a}\times\vec{c}$.

Q. What is the general meaning of Fundamental theorem of calculus?

Fundamental theorem of calculus

Suppose $f(x)$ is a function of one variable, then $\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$

or, $\int_a^b F(x)\, dx = f(b) - f(a)$ where $df/dx = F(x)$



So the fundamental theorem tells you that to integrate a function $F(x)$, figure out a function $f(x)$ whose derivative is equal to $F(x)$.

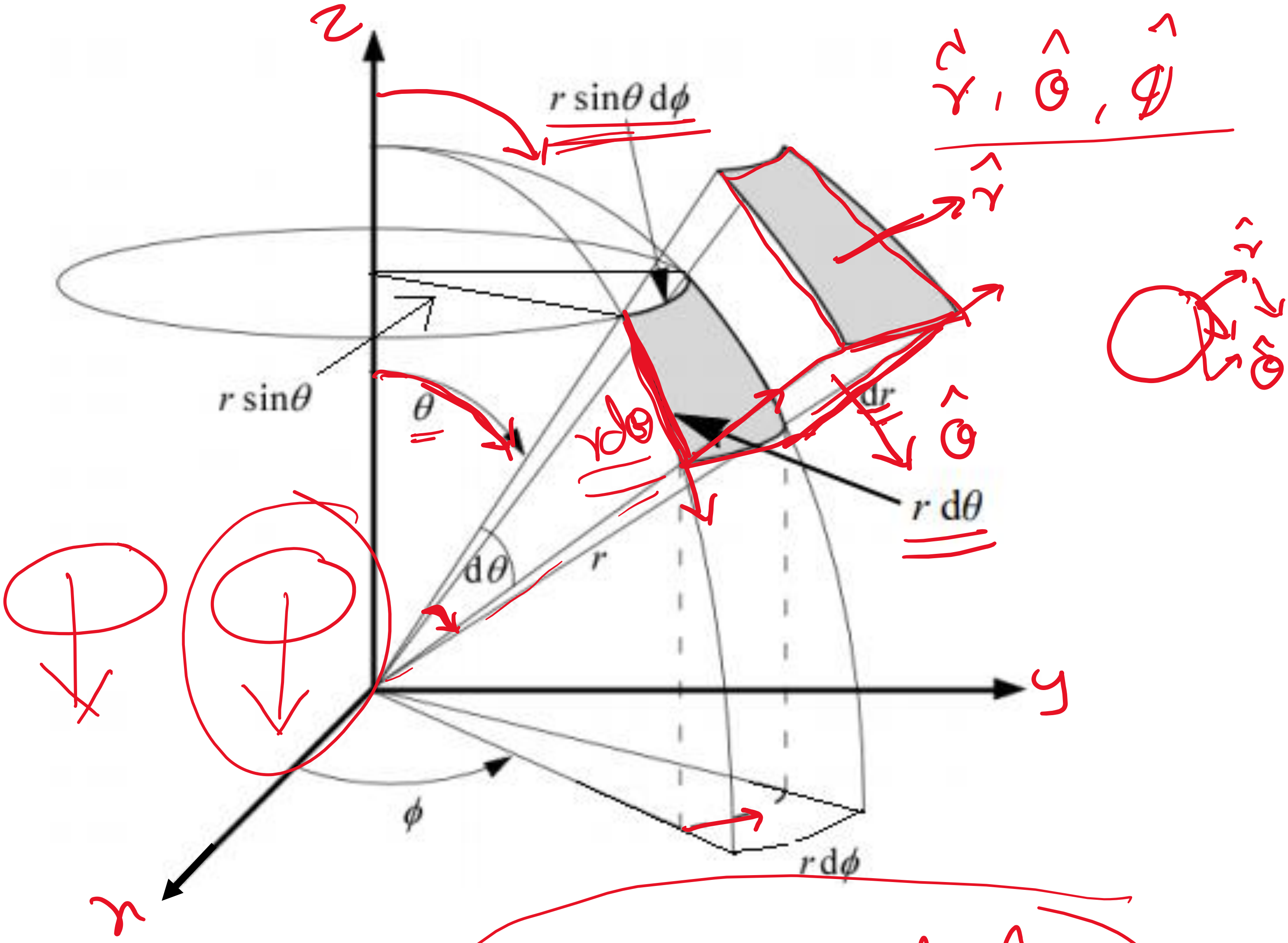
As per the definition $df = (df/dx)dx$ is the infinitesimal change in f as you go from x to $x + dx$.

So the theorem says that if you divide the interval between the limits into many small intervals, dx, and add up the increments df from each interval then it will be equal to the total change in f, which will also be equal to $f(b) - f(a)$.

Therefore there are two ways to determine the total change; either subtract the values at the end or add the function for all small intervals between the limits.

Similar to the ordinary calculus, the fundamental theorem also holds true for vector calculus. Since there are three kinds of derivatives in vector calculus, i.e. gradient, divergence and curl, each of them have their own fundamental theorem.

Q. how the axis are mutually perpendicular in orthogonal/curvilinear coordinates?



$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\hat{r}\cdot\hat{y} = 0$$

$$\hat{r}\cdot r\sin\theta\,\hat{\phi} = 0$$

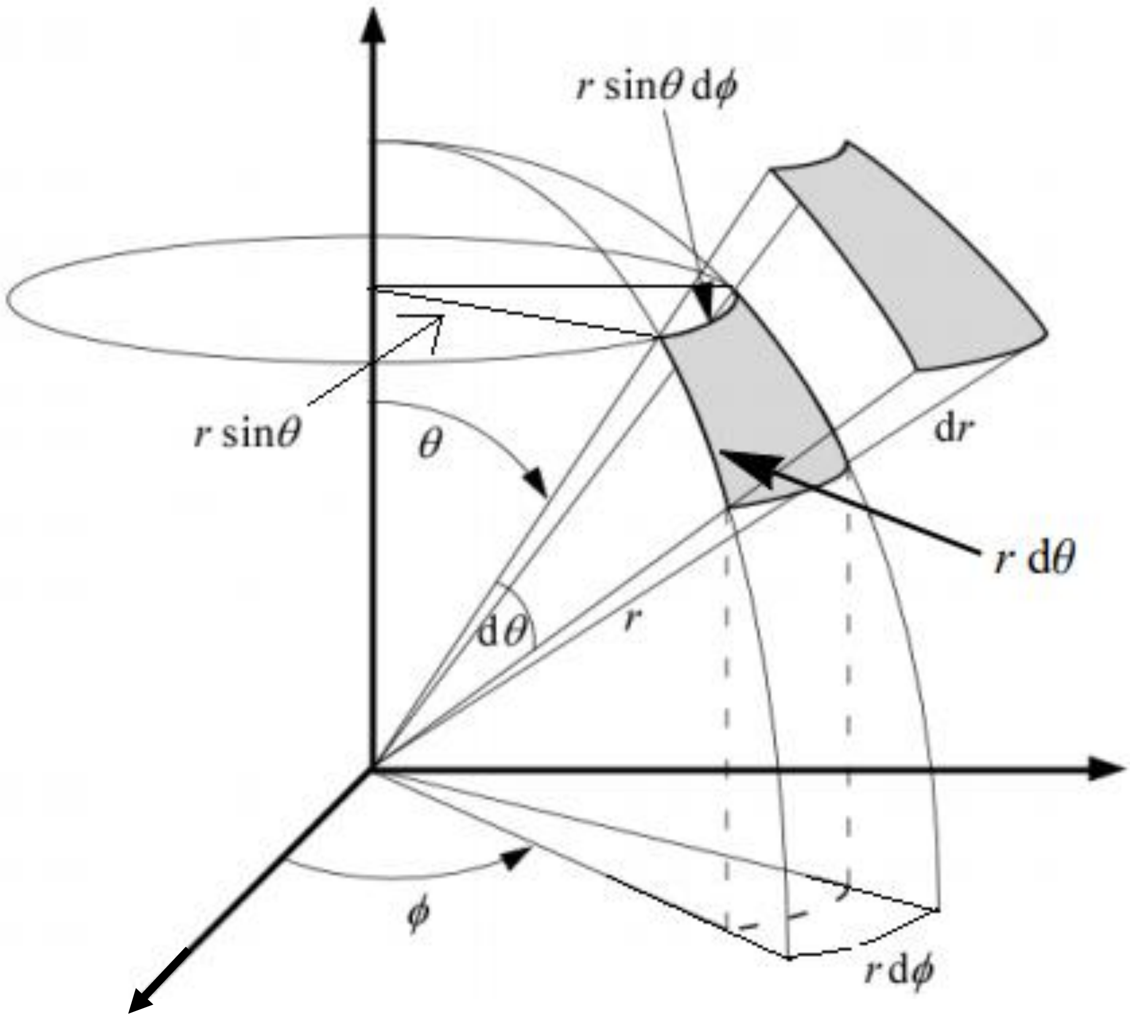
$$,\ dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$

$$d\vec{l}$$

$$d\vec{a}$$

$$dz =$$

Q. sir I am facing difficulty in finding ding length vector can you please guide me?



Q. How to calculate unit vector in spherical and cylindrical coordinate system.