RIEMANN MAPPING THEOREM

Let $\emptyset \neq U \subsetneq \mathbb{C}$ be open & connected. Assume that every zero-free analytic function on U has an analytic square root. Then U is conformally equivalent to $\mathbb{D}$.

Recall that we have proved the existence of an injective holomorphic function $f_0 : U \to \mathbb{D}$. Now consider the family

$$\mathcal{F} := \left\{ f : U \xrightarrow[\text{holo.}]{1-1} \mathbb{D} : |f'(z_0)| \geqslant |f_0'(z_0)| \right\},$$

where $z_0 \in U$ is fixed. Note that $f_0'(z_0) \neq 0$. Since $\mathcal{F}$ is compact, $\exists\, g \in \mathcal{F}$ s.t. $\forall\, f \in \mathcal{F}$ $|g'(z_0)| \geqslant |f'(z_0)|$. We claim that $g(U) = \mathbb{D}$.

Assume contrary, i.e., $g(U) \subsetneq \mathbb{D}$. Then $\exists\, a \in \mathbb{D} \setminus g(U)$ ~~s.t.~~. Consider the function $\varphi_a \circ g$. Clearly, ~~$\varphi_a \circ g$~~ this function is analytic & zero-free, hence admits an analytic square root, say h, i.e., $h^2 = \varphi_a \circ g$.

It is easy to see that h is zero-free, as h^2 is zero-free, and 1-1, otherwise h^2 would not be 1-1.

Put ~~$h = f($~~ $b = h(z_0)$. Consider

$f := \varphi_b \circ h$. ~~By~~ Then $f(z_0) = \varphi_b(h) = 0$.

Now observe that $g = \varphi_a \circ h^2 = \varphi_a \circ (\varphi_b' \circ f)^2$

$$= \varphi_a \circ (\varphi_b^2 \circ f)$$

$$= (\varphi_a \circ \varphi_b^2) \circ f.$$

It follows that

$$|g'(z_0)| = |((\varphi_a \circ \varphi_b^2) \circ f)'(z_0)|$$

$$= |(\varphi_a \circ \varphi_b^2)'(0)|\, |f'(z_0)| \quad \cdots (*)$$

As φ_b^2 is not 1-1, so ~~d~~ is $\varphi_a \circ \varphi_b^2$.

Hence $|(\varphi_a \circ \varphi_b^2)'(0)| < (1 - |\varphi_a \circ \varphi_b^2(0)|^2)$

So from $(*)$ one obtains that

$$|g'(z_0)| < (1 - |\varphi_a \circ \varphi_b^2(0)|^2)\, |f'(z_0)|$$

$$\leq |f'(z_0)|$$

On the other hand f, being the composition of two injective maps, is injective, and furthermore $|f'(z_0)| \geq |g'(z_0)| \geq |f_0(z_0)|$. This implies that $f \in \mathcal{F}$, which challenges the assumption that g is the maximizer.

Remark 1. If g is a maximizer of $\{|f'(z_0)| : f \in \mathcal{F}\}$ then $g(z_0) = 0$. Otherwise, consider $\varphi_a \circ g$, where $a := g(z_0) \neq 0$. Then

$$|(\varphi_a \circ g)'(z_0)| = |\varphi_a'(a)\, g'(z_0)| = \frac{1}{1-|a|^2} |g'(z_0)| > |g'(z_0)| \geq |f'(z_0)|$$

From this, it follows that $\varphi_a \circ g \in \mathcal{F}$, which is not possible.

2. Let g be as above. Suppose $f : U \to \mathbb{D}$ is holomorphic & $f(z_0) = 0$. Then $f \circ g^{-1} : \mathbb{D} \to \mathbb{D}$ fixes origin. From Schwarz's lemma, one has $\forall z \in \mathbb{D}$, $|(f \circ g^{-1})(z)| \leq |z|$, which implies that $|f(\omega)| \leq |g(\omega)|$, $\forall \omega \in U$, and also

$$|(f \circ g^{-1})'(0)| = \left| f'(z_0) \cdot \frac{1}{g'(z_0)} \right| \leq 1 \Rightarrow |f'(z_0)| \leq |g'(z_0)|.$$

Furthermore, if equality occurs in $|f(\omega)| \leq |g(\omega)|$ for some $\omega \neq z_0$ or in $|f'(z_0)| \leq |g'(z_0)|$ iff $f = \lambda g$, for some $|\lambda| = 1$.

NOTE: All that we have used here regarding is $g : U \to \mathbb{D}$ is bijective and holomorphic and $g(z_0) = 0$.

Uniqueness:-

Obs: Suppose $g_1, g_2 : U \to \mathbb{D}$ are bijective holomorphic maps and $g_1(z_0) = g_2(z_0) = 0$. Then $g_1 \circ g_2^{-1} \in \mathrm{Aut}(\mathbb{D})$ and it fixes origin. From this $g_1 = \lambda g_2$, for some $|\lambda| = 1$.

Let U be as before. Then $\exists!$ bijective holomorphic function $g: U \to \mathbb{D}$ satisfying $g(z_0) \neq 0$ & $g'(z_0) > 0$.

<u>Proof</u>:- The existence of such a map has already been established. We now prove the uniqueness. Let g_1 & g_2 are two such maps. From the previous observation, we get $g_1 = \lambda g_2$, for some $|\lambda| = 1$. This yields that $g_1'(z_0) = \lambda g_2'(z_0)$. Since both $g_1'(z_0)$ and $g_2'(z_0)$ are positive, we obtain that $\lambda > 0$. Hence $\lambda = 1$.