MTH 424 - PARTIAL DIFFERENTIAL EQUSTION

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Odd Semester, 2024-25

Assignment 6

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1. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.
   (a) Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.
   (b) Show that $u(x,t) = u(1-x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.
   (c) Use the energy method to show that $\int_0^1 u^2(x,t)dx$ is a strictly decreasing funciton of t.

2. Prove the *comparison principle* for the heat equation: If u and v are two solutions, and if $u(x,0) \leq v(x,0)$, $u(0,t) \leq v(0,t)$ and $u(\ell,t) \leq v(\ell,t)$, then $u \leq v$ for $0 \leq t < \infty$, $0 \leq x \leq \ell$.

3. Let f be a bounded continuous function. Show that if $v(x,t)$ satisfies

$$\begin{cases} v_t = kv_{xx} + f(x,t), & \text{in} \quad \mathbb{R} \times (0,T) \\ v(x,0) = 0 \quad \text{on} \quad \mathbb{R}. \end{cases}$$

   then $v(x,t) \leq T \max\limits_{\mathbb{R}\times(0,T)} f(x,t)$.

4. Given $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, derive the d'Alembert's formula for wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty) \\ u(0,x) = g, u_t(0,x) = h \quad \text{on} \quad \mathbb{R} \end{cases} \tag{1}$$

   by factorizing the equation in terms of 1st order PDEs and solving by method of Characteristic.

5. Assume g and h are odd functions and $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. show that the solution $u(x,t)$ of the wave equation (1) is also odd in x for all t.

6. Let u be the solution of (1) and g, h be compactly supported. Define $k(t) = \frac{1}{2}\int_{\mathbb{R}} u_t^2(t,x)dx$ and $p(t) = \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)dx$. Show
   (a) $k(t) + p(t)$ is constant in t.
   (b) $k(t) = p(t)$ for all large enough time t.

7. Find the formula of the solution of 2 dimensional Klein-Gorodn equation

$$\begin{cases} u_{tt} - \Delta u + m^2 u = 0 \quad \text{in} \quad \mathbb{R}^2 \times (0,\infty) \\ u(0,x) = g, u_t(0,x) = h \quad \text{on} \quad \mathbb{R}^2. \end{cases}$$

explicit form?

Hint: take v(t,x1,x2,x3) = cos(mx3).u(t,x1,x2)

8. For $n = 3$ assume $g, h \in C_c^\infty(\mathbb{R}^3)$ be the initial data of the wave equation. Let u be the solution given by Kirchoff's formula. Show that there exists a constant $C > 0$ such that

$$|u(t,x)| \le \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and $t > 0$.

9. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume $u \in C^2(\mathbb{R}^n)$ is a solution of the wave equation

$$\begin{cases} u_{tt} - \triangle u = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu}(t,x) = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

where ν is the exterior outward unit normal vecotr. Define

$$E(t) = \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)dx,$$

show that $E(t)$ is constant.

10. Find the solution of the problem

explicit form?

$$\begin{cases} u_{tt} - \triangle u = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u(0,x,y,z) = x^2 + y^2, \;\; u_t(0,x,y,z) = 0 \quad \text{on} \quad \mathbb{R}^3. \end{cases}$$