MTH 101-Calculus
Spring-2021
Assignment 4 : Mean Value Theorem, Taylor's Theorem, Curve Sketching

1. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$ and $f(b) = b$. Show that there is $c \in (a, b)$ such that $f'(c) = 1$. Further, show that there are distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

2. Using Cauchy Mean Value Theorem, show that
   (a) $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.
   (b) $x - \frac{x^3}{3!} < \sin x$ for $x > 0$.

3. Let f be the function $f(x) = e^x$. Let $a_1 < a_2$ be two real numbers and set $P = (a_1, f(a_1))$ and $Q = (a_2, f(a_2))$. Let $L = \overline{PQ}$ be the line containing P and Q. Show that there exists a unique real number c such that the tangent line to f at $x = c$ is parallel to the line L.

4. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Then show that f' has intermediate value property.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $|f'(x)| \leq M$. Prove that there is a constant $c \in \mathbb{R}$ such that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x + cf(x)$ is a bijection.

6. Find $\lim\limits_{x \longrightarrow 5} (6 - x)^{\frac{1}{x-5}}$ and $\lim\limits_{x \longrightarrow 0^+} (1 + \frac{1}{x})^x$.

7. Sketch the graphs of $f(x) = x^3 - 6x^2 + 9x + 1$ and $f(x) = \frac{x^2}{x^2-1}$.

8. (a) Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x_0 \in [a, b]$. Show that for any $x \in [a, b]$
   $$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$
   i.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.
   (b) Show that $\cos y - \cos x \geq (x - y) \sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

9. Suppose f is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem show that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.