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Complex Analysis (MTH 403)
Hints for Exercise Sheet 4

1. Integration of complex valued functions

1.1. For each of the following, formulate and prove the analogous statement for complex valued functions:

(a) Linearity property.
(b) First and second fundamental theorems of Calculus.
(c) Integration by parts.
(d) Substitution principle.
(e)$*$ *Triangle inequality*: $\forall$ Riemann integrable $f : [a,b] \longrightarrow \mathbb{C}$, $\left|\int_a^b f\right| \leq \int_a^b |f|$.

(**Hint:** We may assume $\int_a^b f \neq 0$. Write $\int_a^b f = re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi)$. Now observe that $|\int_a^b f| = |e^{-i\theta} \int_a^b f| = |\int_a^b e^{-i\theta} f| = |\int_a^b \mathrm{Re}\,(e^{-i\theta} f)| \leq \int_a^b |\mathrm{Re}\,(e^{-i\theta} f)|$.)

1.2. Let $\{f_n\}_{n=1}^{\infty}$ be sequence of complex valued functions defined over a closed and bounded interval $[a,b]$. Assume that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f : [a,b] \longrightarrow \mathbb{C}$ on $[a,b]$. Show that f is Riemann integrable and furthermore

$$\int_a^b f_n \xrightarrow[n\to\infty]{} \int_a^b f.$$

1.3.$*$ Let $z_0 \in \mathbb{C}$ and $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with radius of convergence $R \in (0, \infty]$. Define

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \forall z \in B(z_0; R).$$

Pick any $r \in (0, R)$. Suppose that $|f(z)| \leq M$, whenever $|z - z_0| = r$.

(a) Show that, for any $n \geq 0$, $\frac{1}{2\pi} \int_0^{2\pi} \left| s_n \left(z_0 + re^{it}\right) \right|^2 dt = \sum_{k=0}^{n} |a_k|^2 r^{2k}$, where s_n denotes the n-th partial sum of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Solution. *Observe that* $\left| s_n \left(z_0 + re^{it}\right) \right|^2 = \sum_{k=0}^{n} \sum_{\ell=0}^{m} a_k \bar{a}_m r^{k+m} e^{it(k-m)}$. *The integrate both sides.*

(b) Deduce from 1.3.a that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f \left(z_0 + re^{it}\right) \right|^2 dt = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

(**Hint:** Recall that, if $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are two uniformly bounded sequences of complex valued functions defined over a set X and converge uniformly (on X) to f and g respectively, then $\{f_n g_n\}_{n=1}^{\infty}$ converges to fg uniformly on X.)

Solution. $s_n\left(z_0 + re^{it}\right) \underset{n\to}{\longrightarrow} f\left(z_0 + re^{it}\right)$ *uniformly in* t *and* $\left|s_n\left(z_0 + re^{it}\right)\right| \le \sum_{k=0}^{n} |a_k| r^k \le \sum_{k=0}^{\infty} |a_k| r^k$. *Similar observatios hold for the sequence of functions* $\left\{\overline{s_n\left(z_0 + re^{it}\right)}\right\}_{n=1}^{\infty}$ *(in* t*). Now use the hint provided above.*

Note to the student. $\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{it}\right)\right|^2 dt = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}$ *is called the Parseval's identity. We shall use this several times later.*

(c) Using 1.3.b show that $\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \le M^2$.

(d) Conclude from 1.3.c that if $R = \infty$ and f is bounded then f must be constant.

(e) Set $M(r) \stackrel{\text{def}}{=} \sup_{|z-z_0|=r} |f(z)|$. Conclude from 1.3.b that, for all $n \ge 0$, $|f^{(n)}(z_0)| \le n!\frac{M(r)}{r^n}$.

(f) Conclude from 1.3.b that, if $|f(z_0)| = M(r)$, for some $0 < r < R$, then f is constant.

2. Integration over paths

In what follows, for $z_0 \in \mathbb{C}$ and $r > 0$, by *positively oriented circle centered at* z_0 *with radius* r we mean the following closed path:

$$t \mapsto z_0 + re^{it}, \ \forall t \in [0, 2\pi].$$

We usually denote this by $C(z_0; r)$. When $z_0 = 0$ and $r = 1$, we simply refer to this as the *positively oriented unit circle*.

2.1. Evaluate the following integrals along the specified paths:

(a) z^n along the path $\gamma(t) = e^{ikt}$, $\forall t \in [0, 2\pi]$, where $n, k \in \mathbb{Z}$.

(b) e^z along the part of the unit circle joining 1 to i in the anticlockwise direction (without using the fundamental theorem).

Solution.

$$\begin{aligned}
\int_\gamma e^z\,dz &= \int_0^{\frac{\pi}{2}} e^{\cos t + i\sin t}(-\sin t + i\cos t)\,dt \\
&= \int_0^{\frac{\pi}{2}} e^{\cos t}(\cos(\sin t) + i\sin(\sin t))(-\sin t + i\cos t)\,dt \\
&= \int_0^{\frac{\pi}{2}} e^{\cos t}(-\sin t\cos(\sin t) - \cos t\sin(\sin t))\,dt \\
&+ i\int_0^{\frac{\pi}{2}} e^{\cos t}(\cos t\cos(\sin t) - \sin t\sin(\sin t))\,dt \\
&= e^{\cos t}\cos(\sin t)|_0^{\pi/2} + ie^{\cos t}\sin(\sin t)|_0^{\pi/2} \\
&= e^{\cos t + i\sin t}\Big|_0^{\pi/2} = e^i - e^1.
\end{aligned}$$

(c) $\log_\pi z$ along the semicircle joining $-i$ to i on the right half place.

2.2. Let $\alpha \in [0, \pi]$ and $n \in \mathbb{N}$. Consider the n-th root function $f(z) = z^{\frac{1}{n}} \stackrel{\text{def}}{=} e^{\frac{1}{n}\log_\alpha z}$, for all $z \in \mathbb{C} \setminus \overline{R_\alpha}$. Can it be integrated along the path $\gamma(t) \stackrel{\text{def}}{=} e^{it}$, $\forall t \in [0, \pi]$? If so, then evaluate the integral.

2.3. (a) Evaluate the integral of the function z^{i-1} (using the principal branch) along the path $\gamma(t) \stackrel{\text{def}}{=} e^{it}$, $\forall t \in [-\pi, \pi]$.

(b) What if the path in 2.3.a would have been changed to the positively oriented unit circle? Do you see any change in the value of the integral with that of 2.3.a? Can you explain this?

2.4. Show that, for any $n \in \mathbb{N}$,

$$\frac{1}{2\pi}\int_0^{2\pi}(2\cos\theta)^{2n}\,d\theta = \frac{(2n)!}{(n!)^2}.$$

(**Hint:** Consider the integral of the function $\frac{1}{z}\cdot(z+\frac{1}{z})^{2n}$ along the positively oriented unit circle. Which one among the terms appearing in the binomial expansion of $\frac{1}{z}\cdot(z+\frac{1}{z})^{2n}$ contribute to the integral?)

3. Applications of ML inequality

3.1. (a) Show that $\left|\int_\gamma \frac{e^z}{z}\,dz\right| \le 2\pi e$, where γ is the positively oriented unit circle.

(b) Let $z_0 \in \mathbb{C}$, $r > 0$ and $f : D(z_0; r) \longrightarrow \mathbb{C}$ be continuous. Find $\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{C(z_0;\varepsilon)}\frac{f(z)}{z-z_0}\,dz$.

Sketch of the solution. *Observe that* $f(z_0) = \frac{1}{2\pi i}\int_{C(z_0;\varepsilon)}\frac{f(z_0)}{z-z_0}\,dz$. *Hence*

$$\left|\frac{1}{2\pi i}\int_{C(z_0;\varepsilon)}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C(z_0;\varepsilon)}\frac{f(z_0)}{z-z_0}\,dz\right| \le \frac{1}{2\pi}\left|\int_{C(z_0;\varepsilon)}\frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \tag{3.1}$$

Let $\eta > 0$. *From the continuity of* f *at* z_0, *there exists* $\delta > 0$ *such that* $|f(z)-f(z_0)| < \eta$, *whenever* $|z-z_0| < \delta$. *Now from* (3.1), *in view of* ML*-inequality, it follows that*

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{C(z_0;\varepsilon)}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_{C(z_0;\varepsilon)}\frac{f(z_0)}{z-z_0}\,dz\right| &\le \frac{1}{2\pi}\left|\int_{C(z_0;\varepsilon)}\frac{f(z)-f(z_0)}{z-z_0}\,dz\right|\\ &\le \frac{1}{2\pi}\cdot\frac{\eta}{\varepsilon}\cdot 2\pi\varepsilon = \eta,\end{aligned}$$

whenever $\varepsilon < \delta$.

(c) Find $\lim_{R\to\infty}\int_{C(0;R)}\frac{z}{z^5+9}\,dz$.

(d) Find $\lim_{R\to\infty}\int_{[-R,-R+i]}\frac{z^3e^z}{z+3}\,dz$.

Sketch of the solution. *This is a straightforward application of* ML*-inequality. Observe that, for* R *sufficiently large, the integrand is at most* $\frac{(r^2+1)^{\frac{3}{2}}}{R-3}\cdot\frac{1}{e^R}$ *in absolute value.*

3.2. Let γ be a closed path in $\mathbb{C}$ and $\{f_n\}_{n=1}^\infty$ be a sequence of complex valued functions defined on γ^*.

(a) Assume that $\{f_n\}_{n=1}^\infty$ converges uniformly to $f : \gamma^* \longrightarrow \mathbb{C}$ on γ^*. Show that $\int_\gamma f_n \underset{n\to\infty}{\longrightarrow} \int_\gamma f$.

Solution. *Let* $\varepsilon > 0$. *Since* $\{f_n\}_{n=1}^\infty$ *converges uniformly to* $f : \gamma^* \longrightarrow \mathbb{C}$ *on* γ^*, *there exists* $N \in \mathbb{N}$ *such that, for all* $n \ge N$, *for all* $w \in \gamma^*$, *we have* $|f_n(w) - f(w)| < \frac{\varepsilon}{2}$. *Now using the* ML*-inequality, we have*

$$\left|\int_\gamma f_n - \int_\gamma f\right| \le \frac{\varepsilon}{2}L_\gamma < \varepsilon L_\gamma,\ \forall n \ge N.$$

(b) Assume that $\sum_{n=1}^\infty f_n$ is uniformly convergent. Show that $\sum_{n=1}^\infty\int_\gamma f_n = \int_\gamma\sum_{n=1}^\infty f_n$.

Sketch of the solution. *This follows from* 3.2.a.

3.3. (a) Using series expansion method, show that

$$\frac{1}{2\pi i}\int_{C(z_0;r)} \frac{dw}{w-z} = \begin{cases} 0 & \text{if } z \notin \overline{D(z_0,r)} \\ 1 & \text{if } z \in D(z_0,r) \end{cases}. \tag{3.2}$$

Sketch of the solution. *Let $|z-z_0| < r$. Then it is clear that, for all $|w-z_0| = r$,*

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\left(\frac{z-z_0}{w-z_0}\right)} = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}.$$

Similarly one can show that, if $|z-z_0| > r$, then

$$\frac{1}{w-z} = \sum_{n=0}^{\infty} -\frac{(w-z_0)^n}{(z-z_0)^{n+1}}.$$

In either case, 3.2.b yields the desired conclusion.

(b) Let $P(z) \in \mathbb{C}[z]$ and $|\alpha - z_0| \neq r$. Evaluate $\displaystyle\frac{1}{2\pi i}\int_{C(z_0;r)} \frac{P(z)}{z-\alpha}\, dz$.

Hint: Write $P(z) = a_0 + a_1(z-\alpha) + \ldots a_d(z-\alpha)^d$.

(c) Let $a \neq b \in \mathbb{C}$. Evaluate the following for every $r \neq |a|, |b|$: $\displaystyle\int_{C(0;r)} \frac{dz}{(z-a)(z-b)}$.

Hint: $\frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)$.

(d) Consider $\gamma_1 \stackrel{\text{def}}{=} [z_0 - r, z_0 + r]$ and $\gamma_2 : [0,\pi] \longrightarrow \mathbb{C}$, $\gamma_2(t) \stackrel{\text{def}}{=} z_0 + re^{it}$. Let $z \notin \gamma_1^* \cup \gamma_2^*$. Evaluate $\displaystyle\frac{1}{2\pi i}\int_{\gamma_1 * \gamma_2} \frac{dw}{w-z} dw$.

Solution. *If $|z| > 1$ then we are done. So assume that z lies strictly inside the upper-semicircle. Let $\gamma_3 : [\pi, 2\pi] \longrightarrow \mathbb{C}$, $\gamma_3(t) \stackrel{def}{=} z_0 + re^{it}$. Then clearly*

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_1*\gamma_2}\frac{dw}{w-z}dw + \frac{1}{2\pi i}\int_{\gamma_3*\tilde{\gamma}_1}\frac{dw}{w-z}dw &= \frac{1}{2\pi i}\int_{\gamma_2}\frac{dw}{w-z}dw + \frac{1}{2\pi i}\int_{\gamma_3}\frac{dw}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{C(0;1)}\frac{dw}{w-z}dw \\ &= 1.\end{aligned}$$

Since z lies in the unbounded component of $\mathbb{C} \setminus (\gamma_3 * \tilde{\gamma}_1)^*$, so $\displaystyle\frac{1}{2\pi i}\int_{\gamma_3*\tilde{\gamma}_1}\frac{dw}{w-z}dw = 0$. Hence $\displaystyle\frac{1}{2\pi i}\int_{\gamma_1*\gamma_2}\frac{dw}{w-z}dw = 1$.

3.4. Let $U \subseteq \mathbb{C}$ be open and connected, and $f : U \longrightarrow \mathbb{C}$ be continuous. Show that the following are equivalent:

(a) Whenever two paths γ_1, γ_2 with $\gamma_1^*, \gamma_2^* \subseteq U$, have same initial and end points, $\int_{\gamma_1} f = \int_{\gamma_2} f$.

(b) $\int_\gamma f = 0$, for every closed path γ such that $\gamma^* \subseteq U$.

(c)$*$ Show that, f has a primitive. (**Hint:** Fix $z_0 \in U$. For any $z \in U$, there exists a path γ joining z_0 and z (why?). Define $F(z) = \int_\gamma f$. First check that this does not depend on the choice of γ. Show that, the function F thus obtained is a primitive of f.)

Can you drop the connectedness assumption from the hypothesis?

Sketch of the solution. *(i) (a)$\Longrightarrow$ (b): Let $\gamma(a) = \gamma(b) = z_0$ and σ be the constant path z_0. Then $\int_\gamma f = \int_\sigma f = 0$.*
(ii) (b)$\Longrightarrow$ (a): Take the path $\gamma_1 * \tilde{\gamma}_2$. This is a closed path having image $\gamma_1^* \cup \gamma_2^* \subseteq U$. Now use (b).

(iii) (c)$\Longrightarrow$ (b): This follows from the fundamental theorem.

(iv) (b) $\Longrightarrow$ (c): Since U is open and connected, it is ***polygonally path connected****. Fix $z_0 \in U$. Then for any $z \in U$, there exists a* ***polygonal path****, say $\gamma_z : [a, b] \longrightarrow U$ such that $\gamma_z(a) = z_0$ and $\gamma_z(b) = z$. Define*

$$F(z) = \int_{\gamma_z} f.$$

Let $a \in U$. Then there exists $r > 0$ such that $D(a; r) \subseteq U$. Suppose that $z \in D(z_0; r)$. Then one has

$$\begin{aligned}\frac{F(z) - F(a)}{z - a} - f(a) &= \frac{\int_{\gamma_z} f - \int_{\gamma_a} f}{z - a} - f(a)\\ &= \frac{\int_{[a,z]} f}{z - a} - \frac{\int_{[a,z]} f(a)}{z - a}\\ &= \frac{1}{z - a} \int_{[a,z]} (f(w) - f(a))\, dw \qquad (3.3)\end{aligned}$$

Let $\varepsilon > 0$. As f is continuous at a, one can choose $0 < \delta < r$ such that $\forall w \in D(a; \delta)$, $|f(w)-f(a)| < \frac{\varepsilon}{2}$. Now if $|z - a| < \delta$, then it is clear that the line segment joining a and z is entirely contained inside $D(a; \delta)$. Hence from (3.3), using ML-inequality, we obtain that,

$$\left|\frac{F(z) - F(a)}{z - a} - f(a)\right| \leq \frac{\varepsilon}{2} < \varepsilon,$$

for all $z \in D(z_0; \delta)$. Finally, if U is not assumed to be connected, we consider each of its connected components and proceed like above. This yields a primitive in each of the connected components. So the connectedness assumption can be dropped.

Note to the student. *We shall use the equivalence of 3.4.b and 3.4.c quite a few times later.*

3.5. Let $U \subseteq \mathbb{C}$ be open and convex and $f : U \longrightarrow \mathbb{C}$. Assume that, there exists $z_0 \in U$ such that, $|f'(z) - f'(z_0)| < |f'(z_0)|$, for all $z \in U$. Show that f is injective. (**Hint:** Suppose that, $f(z_1) = f(z_2)$, for some $z_1 \neq z_2 \in U$. Then observe that $\displaystyle\int_{[z_1,z_2]} f' = 0$. What is $\displaystyle\int_{[z_1,z_2]} (f'(z) - f'(z_0))\, dz$?)

Sketch of the solution. *Follow the hint. It is easy to see that $\displaystyle\int_{[z_1,z_2]} (f'(z) - f'(z_0))\, dz = -f'(z_0)(z_2 - z_1)$. From this, using the $M - L$ inequality, we get that*

$$|f'(z_0)(z_2 - z_1)| < \left|\int_{[z_1,z_2]} (f'(z) - f'(z_0))\, dz\right| < |f'(z_0)||z_2 - z_1|,$$

which is not possible.

4. Continuity of zeros of polynomials

Fix $d \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $P_n(z) \overset{\text{def}}{=} a_{n,d}z^d + a_{n,d-1}z^{d-1} + \ldots a_{n,1}z + a_{n,0} \in \mathbb{C}[z]$ have degree d. Assume that $P(z) \overset{\text{def}}{=} a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$ is of degree d and P_n *converges to P coefficientwise* as $n \to \infty$, i.e., for every $j = 0, \ldots, d$, $a_{n,j} \xrightarrow[n\to\infty]{} a_j$.

4.1. Show that $P_n \xrightarrow[n\to\infty]{} P$ uniformly on every bounded subset of $\mathbb{C}$.

Solution. *Let S be a bounded subset of $\mathbb{C}$. Choose $M > 1$ such that $|z| \leq M$, for all $z \in S$. Observe that, for any $z \in S$ and $n \in \mathbb{N}$, one has*

$$\begin{aligned} |P_n(z) - P(z)| &\leq \left| \sum_{j=0}^{d} (a_{n,j} - a_j) z^j \right| \\ &\leq \sum_{j=0}^{d} |a_{n,j} - a_j| M^j \\ &\leq M^d \left(\sum_{j=0}^{d} |a_{n,j} - a_j| \right). \end{aligned}$$

4.2.$*$ Consider a zero $z_0 \in \mathbb{C}$ $P(z)$ with multiplicity m. Suppose that $R > 0$ is such that $P(z)$ does not vanish anywhere on the closed disc $\overline{D(z_0; R)}$ except the center z_0. Then show that there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $P_n(z)$ does not have a zero on the boundary of the open disc $D(z_0; R)$ but has precisely m zeros inside.

Solution. *Denote by S the circle $\{z \in \mathbb{C} : |z - z_0| = R\}$. Since S is compact and $P(z)$ does not vanish anywhere S, there exists $M_1 > 0$ such that $|P(z)| \geq M$, for all $z \in S$. From 4.1., we have $P_n \xrightarrow[n\to\infty]{} P$ uniformly on S. So there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$ and $z \in S$, $|P_n(z) - P(z)| < \frac{M}{2}$. This shows that, $|P_n(z)| > |P(z)| - \frac{M}{2} \geq \frac{M}{2}$, for all $n \geq N_1$ and $z \in S$. Hence $P_n(z)$ does not have a zero in S, whenever $n \geq N_1$. It now suffices to show that*

$$\frac{1}{2\pi i} \int_{C(z_0;R)} \frac{P_n'(z)}{P_n(z)}\, dz \xrightarrow[n\to\infty]{} \frac{1}{2\pi i} \int_{C(z_0;R)} \frac{P'(z)}{P(z)}\, dz. \tag{4.1}$$

We use 3.2.a to show (4.1). Observe that $\forall n \geq N_1$ and $z \in S$,

$$\begin{aligned} \left| \frac{P_n'(z)}{P_n(z)} - \frac{P'(z)}{P(z)} \right| &= \frac{|P(z)P_n'(z) - P_n(z)P'(z)|}{|P_n(z)||P(z)|} \\ &< \frac{2}{M^2} \left(|P(z)P_n'(z) - P(z)P'(z)| + |P(z)P'(z) - P_n(z)P'(z)| \right) \\ &\leq \frac{2}{M^2} \left(\sup_{w\in S} |P(w)||P_n'(z) - P'(z)| + \sup_{w\in S} |P'(w)||P_n(z) - P(z)| \right). \end{aligned} \tag{4.2}$$

As P_n converges to P coefficientwise as $n \to \infty$, so does P_n' to P'. Consequently $P_n' \xrightarrow[n\to\infty]{} P'$ uniformly on S. In view of (4.2), it is immediate that $\frac{P_n'}{P_n} \xrightarrow[n\to\infty]{} \frac{P'}{P}$ uniformly on S.

Let $\alpha_1, \ldots, \alpha_k$ be all distinct zeros of $P(z)$ with multiplicities $n_1, \ldots, n_k$ respectively. Thus

$$P(z) = a_d \prod_{i=1}^{k} (z - \alpha_i)^{n_i}.$$

Pick any $R > 0$ such that $D(\alpha_i; R) \cap D(\alpha_j; R) = \emptyset$, for all $i \neq j$. For any $i = 1, \ldots, k$ and $n \in \mathbb{N}$, we define

$$\Phi_{i,n,R}(z) = \prod (z - \beta),$$

where β runs over all zeros of $P_n(z)$, counting multiplicities, in $D(\alpha_i; R)$.

4.3. Show that, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\deg \Phi_{i,n,R}(z) = n_i$, for all $i = 1, \ldots, k$.

Solution. *Let $i = 1, \ldots, k$. As $P(z)$ does not vanish anywhere on $C(\alpha_i; R)$, it follows from 4.2. that, there exists $N_i \in \mathbb{N}$ such that $P_n(z)$ has precisely n_i zeros, counting multiplicities, in $D(\alpha_i; R)$, whenever $n \geq N_i$. Thus for all $n \geq N_i$, $\deg \Phi_{i,n,R}(z) = n_i$. Finally, take $N = \max\{N_1, \ldots, N_k\}$.*

4.4. Let N be as above in 4.3. Show that, for all $n \geq N$, $P_n(z) = a_{n,d} \prod_{i=1}^{k} \Phi_{i,n,R}(z)$.

Solution. *As $D(\alpha_i; R) \cap D(\alpha_j; R) = \emptyset$, $\Phi_{i,n,R}(z)$ and $\Phi_{j,n,R}(z)$ are coprime, for all $i \neq j$. Hence $\prod_{i=1}^{k} \Phi_{i,n,R}(z)$ divides $P_n(z)$. Since $\deg \prod_{i=1}^{k} \Phi_{i,n,R}(z) = n_1 + \cdots + n_k = d$, so the conclusion is immediate.*

4.5. Let $0 < \varepsilon < R$. For any $i = 1, \ldots, k$ and $n \in \mathbb{N}$, we define $\Phi_{i,n,\varepsilon}(z)$ in the similar way as before. Show that, there exists $N \in \mathbb{N}$ such that, for any $n \geq N$, one has $\Phi_{i,n,\varepsilon}(z) = \Phi_{i,n,R}(z)$, for all $i = 1, \ldots, k$.

Solution. *Let $i = 1, \ldots, k$ and N be as above in 4.3.. As $P(z)$ does not vanish anywhere on $C(\alpha_i; \varepsilon)$, it follows from 4.2. that, there exists $N_0 \geq N$ such that $P_n(z)$ has precisely n_i zeros, counting multiplicities, in $D(\alpha_i; \varepsilon)$, whenever $n \geq N_0$. Let $n \geq N_0$. Since $P_n(z)$ contains precisely n_i zeros counting multiplicities, in the larger disc $D(\alpha_i; R)$, all zeros of $P_n(z)$, counting multiplicties, in $D(\alpha_i; R)$ must lie in $D(\alpha_i; \varepsilon)$. Hence $\Phi_{i,n,\varepsilon}(z) = \Phi_{i,n,R}(z)$.*

4.6.**Show that, for each $n \in \mathbb{N}$, one can arrange the zero of $P_n(z)$ in an order, say $\alpha_{n,1}, \ldots, \alpha_{n,d}$, such that $\alpha_{n,j} \xrightarrow[n\to\infty]{} \alpha_j$, for all $j = 1, \ldots, d$.

Solution. *Let N be as above in 4.3. and $n \geq N$. For each $i = 1, \ldots, k$, fix an ordering, say $\alpha_{i,1}^{(n)}, \ldots, \alpha_{i,n_i}^{(n)}$, for the zeros of $\Phi_{i,n,R}(z)$. This provides an ordering of zeros of $P_n(z)$ for all $n \geq N$. When $n < N$, put the zeros of $P_n(z)$ in any order. Let $0 < \varepsilon < R$. From 4.5., we obtain $N_0 \geq N$ such that, for any $n \geq N_0$, and $i = 1, \ldots, k$, one has $\Phi_{i,n,\varepsilon}(z) = \Phi_{i,n,R}(z)$. Hence, whenever $n \geq N_0$, $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$, we have $|\alpha_{i,j}^{(n)} - \alpha_i| < \varepsilon$. This shows that, for any $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$, $\alpha_{i,j}^{(n)} \xrightarrow[n\to\infty]{} \alpha_i$.*

Use 4.6. to the following:

4.7.**Let $\{A_n\}_{n\geq 1}$ be a sequence of $d \times d$ complex matrices, where $d \in \mathbb{N}$, and $A \in M_d(\mathbb{C})$. Assume that $A_n \xrightarrow[n\to\infty]{} A$. Denote the eigenvalues of A by $\lambda_1, \ldots, \lambda_d$. Prove that for each $n \in \mathbb{N}$, there exists an ordering $\lambda_{n,1}, \ldots, \lambda_{n,d}$ of the eigenvalues of A_n such that $\lambda_{n,j} \xrightarrow[n\to\infty]{} \lambda_j$, for all $j = 1, \ldots, d$.

Sketch of the solution. *Recall that for $M = (m_{ij})_{1\leq i,j\leq d}$, $\det M$ is a polynomial in the variable $m_{i,j}$'s, consequently $M \mapsto \det M$ is a continuous function on $M_d(\mathbb{C})$. Now if $A_n \xrightarrow[n\to\infty]{} A$, it follows that the χ_{A_n} converges to χ_A coefficientwise.*

Note: The above statement 4.7. can also be proved without using Complex analysis. You may see Section 5.2 of Artin's Algebra book (2nd edition).

4.8.* Let $d \geq 2$. Show that the following subset of $\mathrm{GL}_d(\mathbb{R})$ is not dense in $\mathrm{GL}_d(\mathbb{R})$:

$$\{A \in \mathrm{GL}_d(\mathbb{R}) : \exists P \in \mathrm{GL}_d(\mathbb{R}) \text{ s.t. } PAP^{-1} \text{ is doagonal}\}$$

(**Hint:** First try to prove for $d = 2$. Can $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the limit of a sequence of 2×2 matrices with real entries that are diagonalizable over $\mathbb{R}$?)

Sketch of the solution. *Denote* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *by A. Consider the block matrix*

$$B \stackrel{def}{=} \begin{pmatrix} A & \\ & I_{d-2} \end{pmatrix}.$$

If possible, let $\{B_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathrm{GL}_d(\mathbb{R})$ *such that each* B_n *is diagonaligable over* $\mathbb{R}$ *and* $B_n \xrightarrow[n\to\infty]{} B$*. For all* $n \in \mathbb{N}$*, since* B_n *is diagonalizable over* $\mathbb{R}$*, its all eigenvalues are real. Now considering all* B_n*'s and B as complex matrices, and using 4.7., one obtains that a sequence of real numbers is converging to i, which is absurd.*

Note: The analogous statement holds over $\mathbb{C}$, as every matrix is similar to an upper traingular matrix.