Lesson 4

Numerical Solution of PDE

4.2 Parabolic PDE

- *Semi-discretization*
- *Full finite difference discretization*
- *Fourier Analysis*
- ***Unconditional Stability***

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$$\frac{u_n^{j+1}-u_n^j}{k} = c\,\frac{u_{n+1}^{j+1}-2u_n^{j+1}+u_{n-1}^{j+1}}{h^2} + f_n^{j+1}, \qquad 0<n<N, j=0,1,\dots,M-1,$$
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Thus, u_n^{j+1} *can be obtained by solving a tridiagonal system*

$$-\lambda u_{n+1}^{j+1} + (1+2\lambda)u_n^{j+1} - \lambda u_{n-1}^{j+1} = u_n^j + k f_n^{j+1}, \qquad u_0^{j+1} = u_N^{j+1} = 0.$$

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The operator $I - ckD_h^2$ *has eigenvalues* $1 + ck\lambda_m$ *which are all greater that one, so* $\left\|\left(I-ckD_h^2\right)^{-1}v\right\|_h \le \|v\|_h$.

We, therefore, have

$$\left\|u^{j+1}\right\|_h = \left\|\left(I - ckD_h^2\right)^{-1}\left(u^j + kf^{j+1}\right)\right\|_h \leq \left\|u^j\right\|_h + k\left\|f^j\right\|_h, \qquad j = 0,1,\dots,M-1.$$

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$$\max_{1\leq j\leq M}\left\|e^j\right\|_h = O(k + h^2)$$

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$$\frac{u^{j+1} - u^j}{k} = \frac{c}{2}\left(D_h^2 u^j + D_h^2 u^{j+1}\right) + \frac{1}{2}\left(f^j + f^{j+1}\right).$$

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Using the Taylor expansion about the point $(nh, (j+1/2)k)$, it is straightforward to show that the local truncation error is $O(k^2 + h^2)$, so the Crank-Nicholson method is second order in both space and time.

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Thus

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The eigenvalues of $\left(I-\frac{1}{2}ckD_h^2\right)^{-1}\left(I+\frac{1}{2}ckD_h^2\right)$ is $(1-ck\lambda_m/2)/(1+ck\lambda_m/2)$ which is less than 1. Therefore, we get unconditional stability. The Crank-Nicolson method converges with $O(k^2+h^2)$.