Common mistakes made in Quiz 1

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Question 1.

1. $f(U)$ is contained in a circle means there is circle whose equation must be satisfied for by every $f(z)$. This means that, there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $|f(z) - z_0| = r$, $\forall z \in U$. It has come to the notice of the graders that some students have written

$$f(z) = |z|^2 - z_0\bar{z} - \bar{z}_0 z + |z_0|^2 - r^2.$$

This is certainly wrong, and so are its consequences.

2. Given $f(U)$ is contained in a line, that line can be arbitrary. It need not be x-axis or y-axis or any special one. Similarly $f(U)$ is contained in a circle does not imply that the circle should have center at the origin.

3. Some students have written equations like $f(U) = |z|^2 - z_0\bar{z} - \bar{z}_0 z + |z_0|^2 - r^2$. This does not make any sense as the left hand side is a subset of $\mathbb{C}$.

4. Some of you have given examples like $f(z) = \frac{z}{|z|}$ in this case. Certainly that is not correct.

5. Using results like “Liouville’s theorem" or “Open mapping theorem" are not permitted, as these have not been discussed in the course yet. Indeed yet are nowhere close to them at this moment. Using such results to solve this problem only indicates the inability to solve by elementary means.

Question 2.

1. It is not appropriate to say $R = \lim_{n\to\infty} \left|\frac{P(n)}{P(n+1)}\right|$ in the first place, as the limit need not exist. Indeed, the fact is that if $\lim_{n\to\infty} \left|\frac{P(n)}{P(n+1)}\right|$ exists then it will be equal to R. Hence, to be mathematically correct, first the existence of the limit needs to be established, the conclusion that $R = \lim_{n\to\infty} \left|\frac{P(n)}{P(n+1)}\right|$ follows only after that. Many students have made the same mistake. However this mistake has been excused to an extent if there is no serious mathematical error in the subsequent steps.

2. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ is not true in general, unless it is given that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Hence, if $P(z) = a_k z^k + \cdots + a_1 z + a_0$, then

$$\sum_{n=0}^{\infty} P(n) z^n = \sum_{n=0}^{\infty} a_0 z^n + \sum_{n=0}^{\infty} a_1 z^{n+1} + \cdots + \sum_{n=0}^{\infty} a_k z^{k+n} \tag{$*1$}$$

does presuppose that all $\sum_{n+0}^{\infty} a_r z^{r+n}$, where $r = 0, 1, \ldots, k$. converge. So it is not correct to write $(*1)$ in the first place without showing each of the $\sum_{n+0}^{\infty} a_r z^{r+n}$'s converges first. The order is certainly important.

3. Similarly the radius of convergence of the sum of two power series need not be the 'intersection' or 'minimum' of their radii of convergence.

4. $n^k \geq n^j$, for all $j = 0, \ldots, k$. So one obtains that

$$|P(n)| \leq (|a_0| + |a_1| + \cdots + |a_k|)|z|^k, \tag{$*2$}$$

which implies that $\limsup_{n\to\infty} |P(n)|^{\frac{1}{n}} \leq \limsup_{n\to\infty} (|a_0| + |a_1| + \cdots + |a_k|)^{\frac{1}{n}} |z|^{\frac{k}{n}} = 1$. So from this we get $R \geq 1$ only, not the equality.

5. It's completely wrong to write $\sum_{n=1}^{\infty} P(n) z^n \leq \ldots$ There is no comparison between two complex numbers in general.

6. Some students have directly claimed that $\lim_{n\to\infty} \frac{1}{n} \log |P(n)| = 0$, without any justification. This is not immediate. In fact, mathematically precise reasoning must be provided. I have seen one case where the concerned student has said 'from continuity of log'. It is not clear how 'continuity' of log could be of any help in this regard. The upshot is some correct reasoning must be provided for $\lim_{n\to\infty} \frac{1}{n} \log |P(n)| = 0$.

7. L'Hôpital's rule that you are familiar with applies to real valued functions of a real variable only.