DMS625: Introduction to stochastic processes and their applications

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Renewal Theory

November 9, 2024

1 Introduction

While studying CTMC we saw that for any stochastic system the time spent at a state being independent and exponentially distributed is equivalent to the system possessing the Markov property and vice-versa. For a Poisson process the time spent at a state is the interarrival time, which was also iid exponential. We will now consider a generalization of Poisson process, where the interarrival times will follow an iid arbitrary distribution.

Formally, a counting process $\{N(t), t \geq 0\}$ is said to be a **Renewal process** if the interarrival times, $X_1, X_2, \ldots, X_n \sim F$ iid. The Poisson process is a Renewal process where F is the exponential distribution. Naturally from above if F is not the exponential distribution, then $N(t)$ will not have the Markov property. So in general renewal processes need not be Markovian. Using renewal processes we can model all the phenomena we considered while studying Poisson processes, such as arrival of natural disasters, shocks, people in queues, traffics, etc. but we can model it as a non-Markovian process.

We will now introduce some notation that we also came across during our discussion of Poisson processes. S_n is the waiting time for the $n-th$ arrival. $S_0 = 0$ and,

$$S_n = \sum_{i=1}^{n} X_i, n \geq 1$$

Now say $S_3 \leq t$, i.e., the 3rd arrival happened by time t and $S_4 > t$, i.e., the 4th arrival happened after time t. What is $N(t)$?

Notice that since the third arrival happened before time t and the 4th arrival happened after time t, it implies that at time t, there are only 3 arrivals. Therefore, $N(t) = 3$. We can therefore conclude a much more general result,

$$N(t) = \max\{n : S_n \leq t\}$$

Exercises

1. Verify that $N(t) \geq n \Leftrightarrow S_n \leq t.$, i.e., $N(t) \geq n$ implies $S_n \leq t$ and $S_n \leq t$ implies $N(t) \geq n$.

Suppose after infinite time has passed, the number of arrivals is finite, i.e., $N(\infty) < \infty$. This implies that in an infinite amount of time there were only finitely many arrivals, and therefore it follows that one of the interarrival times, say $X_m, m < \infty$ is infinite. We shall not consider distributions F, where $P(X_m = \infty) > 0$, can occur. Therefore, X_m cannot be infinite and $N(\infty) = \infty$.

The following identity holds for a Renewal process.

$$\begin{aligned}\mathbb{P}(N(t) = n) &= \mathbb{P}(N(t) \geq n) - \mathbb{P}(N(t) \geq n+1) \\ &= \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t)\end{aligned} \tag{1}$$

Example 1.1. *Suppose $X_1, X_2, \ldots, X_n \sim Geo(p)$, i.e., $\mathbb{P}(X_i = k) = p(1-p)^{k-1}, k \geq 1$. That is the interarrival times are geometrically distributed.*

Verify that $\mathbb{P}(S_n = k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}$.

Therefore, from (1),

$$\mathbb{P}(N(t) = n) = \sum_{k=n}^{\lfloor t \rfloor} \binom{k-1}{n-1} p^n (1-p)^{k-n} - \sum_{k=n+1}^{\lfloor t \rfloor} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1}$$

Proposition 1.1. *Let X be a non-negative and integer valued random variable, then,*

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$$

Proof.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{n=0}^{\infty} n\mathbb{P}(X = n) \\ &= 0 \times \mathbb{P}(X = 0) + \sum_{n=1}^{\infty} n\mathbb{P}(X = n) \\ &= \sum_{n=1}^{\infty} n\mathbb{P}(X = n) \\ &= \mathbb{P}(X = 1) + 2\mathbb{P}(X = 2) + 3\mathbb{P}(X = 3) + 4\mathbb{P}(X) + \ldots \\ &= \mathbb{P}(X = 1) \\ &+ \mathbb{P}(X = 2) + \mathbb{P}(X = 2) \\ &+ \mathbb{P}(X = 3) + \mathbb{P}(X = 3) + \mathbb{P}(X = 3) \\ &+ \mathbb{P}(X = 4) + \mathbb{P}(X = 4) + \mathbb{P}(X = 4) + \mathbb{P}(X = 4) \\ &+ \ldots \\ &= \mathbb{P}(X \geq 1) + \mathbb{P}(X \geq 2) + \mathbb{P}(X \geq 3) + \ldots \\ &= \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)\end{aligned}$$

$\square$

We define the **mean renewal function** $m(t)$ as,

$$m(t) = \mathbb{E}[N(t)]$$

It also follows that,

$$\begin{aligned}
m(t) &= \mathbb{E}[N(t)] \\
&= \sum_{n=1}^{\infty} \mathbb{P}(N(t) \geq n) \\
&\text{(Applying the above proposition)} \\
&= \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t)
\end{aligned}$$

We now state two results without proofs,

1. $m(t)$ uniquely determines the renewal process, i.e., each unique renewal process will have a unique mean renewal function.

2. $m(t) < \infty, \forall t < \infty$, i.e., for every renewal process, the mean renewal function is finite in finite time.

Renewal equation

We will derive a key identity for working with renewal processes which is referred to as the renewal equation.

First observe that,

$$m(t) = \mathbb{E}[N(t)] = \int_0^{\infty} \mathbb{E}[N(t)|X_1 = x] f(x) dx$$

In the above X_1 is the first interarrival time and $f(x) = \frac{d}{dx} F(x)$. Now note that if $x < t$, $\mathbb{E}[N(t)|X_1 = x] > 0$, however if $x > t$, it follows that $\mathbb{E}[N(t)|X_1 = x] = 0$. Verify this.

Therefore,

$$m(t) = \mathbb{E}[N(t)] = \int_0^{t} \mathbb{E}[N(t)|X_1 = x] f(x) dx \tag{2}$$

For $x < t$, it follows that (Verify!),

$$\mathbb{E}[N(t)|X_1 = x] = 1 + \mathbb{E}[N(t - x)]$$

Therefore, substituting the above in (2), we obtain the renewal equation,

$$\begin{aligned}
m(t) &= \int_0^{t} [1 + m(t - x)] f(x) dx \\
&= F(t) + \int_0^{t} m(t - x) f(x) dx
\end{aligned} \tag{3}$$

Example 1.2. *Let $X_i \sim$ Unif$(0, 1)$ i.i.d., $0 \leq t \leq 1$. Find $m(t)$.*

Applying the renewal equation,

$$\begin{aligned}
m(t) &= F(t) + \int_0^t m(t-x)f(x)dx \\
&= t + \int_0^t m(t-x)dx \\
&\textit{(Let } y = t - x\textit{)} \\
&= t + \int_0^y m(y)dy \\
&\textit{(Differentiating)} \\
\frac{dm(t)}{dt} &= 1 + m(t) \\
\frac{dm(t)}{1+m(t)} &= dt \\
\log(1+m(t)) &= t \\
&\textit{(Note that } m(0) = 0\textit{)} \\
m(t) &= e^t - 1
\end{aligned}$$

Limit theorems for Renewal processes

Theorem 1.1 (Strong Law of Large numbers (SLLN)). *Let X_i be i.i.d. random variables and $\mathbb{E}[X_i] = \mu < \infty$. Define,*

$$S_n = X_1 + X_2 + \ldots + X_n$$

Then,

$$\mathbb{P}(\lim_{n\to\infty} \frac{S_n}{n} = \mu) = 1$$

This is also alternatively expressed as,

$$\frac{S_n}{n} \to \mu, \text{ with probability 1 (also written as "w.p. 1")}$$

The strong law of large numbers says that the arithmetic mean of a large number of iid random variables converges to the expected value of the random variable.

Theorem 1.2. *Let X_i's denote the interarrival times of a renewal process and $\mathbb{E}[X_i] = \mu$.*

$$\frac{N(t)}{t} \to \frac{1}{\mu} \text{ as } t \to \infty \text{ w.p. 1}$$

Proof. Verify that,

$$S_{N(t)} \le t \le S_{N(t)+1}$$

Now,

$$\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{S_{N(t)+1}}{N(t)+1}\frac{N(t)+1}{N(t)}$$

Consider,

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \lim_{t\to\infty} \frac{X_1 + X_2 + \ldots + X_{N(t)}}{N(t)} = \mu \ (w.p.1, \text{ by SLLN})$$

Similarly,

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1} = \mu \ (w.p.1, \text{ by SLLN})$$

and,

$$\lim_{t\to\infty} \frac{N(t)+1}{N(t)} = \lim_{t\to\infty} 1 + \frac{1}{N(t)} = 1 \ (\text{Since } N(\infty) = \infty)$$

Now using these and applying the sandwich theorem we obtain the result. □

Exercises

1. Suppose the lifetime of a bulb is distributed Unif$(30, 60)$ days. Each time a bulb breaks down, we replace it with another bulb. Find the long-run rate (bulb replaced per day) of replacing the bulbs. $\frac{1}{45}$

2. In the previous question now assume that after the bulb breaks down, you go to the market to buy a bulb. This buying time is distributed as Unif$(0, 1)$ days. Find the long-run rate (bulb replaced per day) of replacing the bulbs. $\frac{1}{46}$?

Theorem 1.3 (Wald's theorem)**.** *Let $X_i's$ be iid random variables. Let N be a positive integer valued random variable. Assume that N and $X_i's$ are independent. Then,*

$$\mathbb{E}[\sum_{i=1}^{N} X_i] = \mathbb{E}[N]\mathbb{E}[X_i]$$

Wald's theorem gives the expectation of a random sum, i.e., the number of terms to be summed are random (depends on N, which is a random variable). We can derive the compound Poisson process result for the expectation using Wald's theorem.

Proposition 1.2. *For a renewal process $N(t)$ where $\mathbb{E}[X_1] = \mu$,*

$$\mathbb{E}[S_{N(t)+1}] = \mu(m(t) + 1)$$

Proof. Apply Wald's theorem. Left as an exercise. □

We will now state a result that is referred to as the elementary renewal theorem,

Theorem 1.4 (Elementary renewal theorem)**.**

$$\frac{m(t)}{t} \to \frac{1}{\mu} \ \text{as } t \to \infty$$

Proof. Recall,

$$t \leq S_{N(t)+1}$$

$S_{N(t)+1} = t + Y(t)$, here $Y(t)$ denotes the excess time after t to get the $N(t) + 1$ arrival. Note that $Y(t) \geq 0$.

$$\mathbb{E}[S_{N(t)+1} = \mathbb{E}[t + Y(t)]]$$
$$\mu(m(t) + 1) = t + \mathbb{E}[Y(t)]$$

Rearranging,

$$\frac{m(t)}{t} = \frac{1}{\mu} + \frac{\mathbb{E}[Y(t)]}{t\mu} - \frac{1}{t}$$

Since, $Y(t) \geq 0$, therefore, $\mathbb{E}[Y(t)] \geq 0$. Hence,

$$\frac{m(t)}{t} \geq \frac{1}{\mu} - \frac{1}{t}$$

Therefore,

$$\lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

Now note that $Y(t) \leq X_{N(t)+1}$. (Thanks to Aditya Kumar for this idea). Therefore,

$$\mathbb{E}[Y(t)] \leq \mathbb{E}[X_{N(t)+1}] = \mu$$

Therefore,

$$\frac{m(t)}{t} \leq \frac{1}{\mu} + \frac{\mathbb{E}[X_{N(t)+1}]}{t\mu} - \frac{1}{t} = \frac{1}{\mu} + \frac{\mu}{t\mu} - \frac{1}{t} = \frac{1}{\mu}$$

and,

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}$$

Now apply the sandwich theorem and conclude the result.

□

Remark 1. *Instead of the proof above you might be tempted to apply expectation directly to Theorem 1.2. However that would be an incorrect application even though in this case you seem to obtain the correct result.*

Let Y_n be a sequence of random variables. In general it is not true that $\lim_{n\to\infty} \mathbb{E}[Y_n]$ and $\lim_{n\to\infty} Y_n$ are equal. To see this, consider the following. Let $U \sim$ Unif$(0,1)$.

$$Y_n = \begin{cases} 0 & U > \frac{1}{n} \\ n & U \leq \frac{1}{n} \end{cases}$$

Now note here that $Y_n \to 0$ as $n \to \infty, w.p.1$ and $\mathbb{E}[Y_n] = 1$. Verify these.

We now state the central limit theorem for renewal processes without proof,

Theorem 1.5 (CLT for renewal processes)**.** *Let $N(t)$ be a renewal process. Then,*

$$\lim_{t\to\infty} N(t) \sim N\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right)$$

Remark 2. *The mean of the limiting distribution is $\frac{t}{\mu}$, i.e.,*

$$\lim_{t\to\infty} \mathbb{E}[N(t)] \to \frac{t}{\mu}$$

Note that this is in agreement with the conclusion of the first limit theorem for renewal processes. The another additional insight we obtain from the CLT for renewal processes is,

$$\lim_{t\to\infty} Var[N(t)] \to \frac{t\sigma^2}{\mu^3}$$

Exercises

1. Find the limiting distribution for large t for a Poisson process. $N\left(\frac{1}{\lambda}, \frac{1}{\lambda}\right)$
2. Let $N_1(t)$ and $N_2(t)$ be two independent renewal processes. The interarrival times for $N_1(t)$ are Unif$(0,1)$ and the interarrival times for $N_2(t)$ are Exp(5). Find the approximate distribution of $N_1(500) + N_2(500)$.

N(t/0.5, t *(1/12)^2 / 0.5^3) + N(t/0.2 , t*(0.2)^4/0.2^3)

Renewal-reward processes

Renewal-reward processes may be considered as the generalization of compound Poisson processes. Think of an insurer that receives claims following a renewal process. But the size of each claim would be different. So there are two sources of randomness, first the arrival of claims itself is random following a renewal process and secondly the size of the claims are also random. We would want to understand at any point in time, what is the total size of claims that the insurer has received.

Formally, let the arrivals follow a renewal process $N(t)$ and the $i-th$ arrival brings with it a reward R_i (think of size of claims from the above example). Then at time t the total rewards received is,

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

We assume that R_i's are iid random variables. Let X_i's denote the interarrival times for $N(t)$. Then $\mathbb{E}[X_i] = \mathbb{E}[X]$ and $\mathbb{E}[R_i] = \mathbb{E}[R]$.

Proposition 1.3. *Let* $\mathbb{E}[R] < \infty$ *and* $\mathbb{E}[X] < \infty$*, then,*

$$\lim_{t\to\infty} \frac{R(t)}{t} \to \frac{\mathbb{E}[R]}{\mathbb{E}[X]} \; w.p.1$$

Proof. Apply SLLN and the first limit theorem. Left as an exercise. □

We state the second result without proof.

Proposition 1.4. *Let* $\mathbb{E}[R] < \infty$ *and* $\mathbb{E}[X] < \infty$*, then,*

$$\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} \to \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

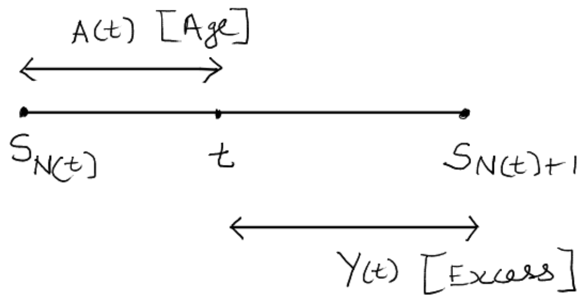
Exercises

1. Suppose the lifetime, X, of a car is distributed via a distribution with CDF $H(x)$ and the density is $h(x)$. The owner buys a new car when it breaks down or the life of the car reaches T. The owner incurs a cost of C_1 if the life of the car reaches T and $C_1 + C_2$ if it breaks down. Show that the long-run time-average cost of replacing a car is $\frac{C_1 + C_2 H(T)}{\int_0^T xh(x)dx + T(1-H(T))}$.

2. In the above problem, $X \sim \text{Unif}(0, 10)$. $C_1 = 3$ and $C_2 = 1/2$. Find T that minimizes long-run average cost of replacing a car.

Age and Excess of a Renewal process

We define the **age** of a renewal process as $A(t) = t - S_{N(t)}$ and the **excess** of a renewal process as $Y(t) = S_{N(t)+1} - t$. The figure below gives a pictorial illustration of the age and excess.



We are in particular interested in understanding the **long-run time-average excess** given by,

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)ds}{t}$$

and the **long-run time-average age** given by,

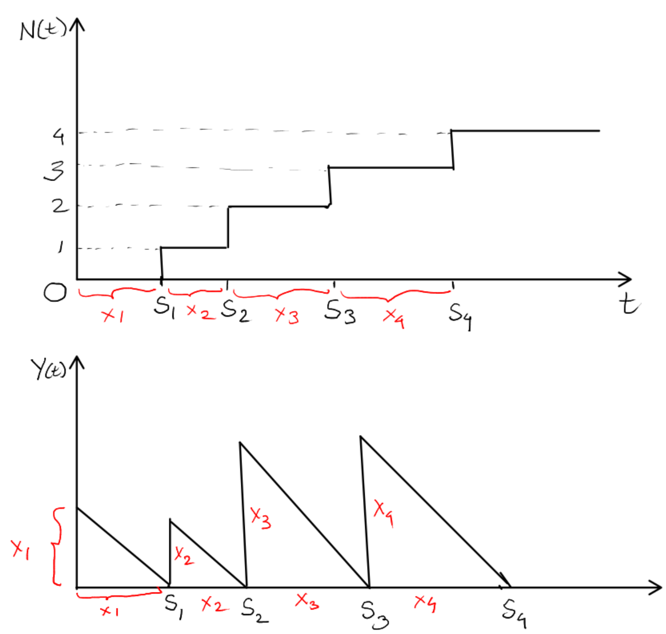
$$\lim_{t\to\infty} \frac{\int_0^t A(s)ds}{t}$$

We will soon see the significance both these quantities hold. The following presentation and proof is adapted from the MIT OCW notes on Renewal processes.

We are interested in first understanding the integral,

$$\int_0^t Y(s)ds$$

Take a look at the figure below,



The top plot has $N(t)$ on the y-axis and t on the x-axis. It shows how $N(t)$ is changing in time with each arrival and the plot below shows the excess time for each arrival on the y-axis and the time on the x-axis. Verify if the plot below makes sense. Each triangle in the figure is a right-angled isosceles triangle. Now note that,

$$\int_0^t Y(u)du = \sum_{i=1}^{N(t)} \frac{X_i^2}{2} + \int_{S_{N(t)}}^t Y(u)du$$

The first term is just the area under the triangles for the first $N(t)$ arrivals. The second term appears because $N(t)+1$'th arrival has not yet occurred and therefore its corresponding triangle will not appear, we denote its area by an integral.

Since $Y(t) \geq 0$, therefore, $\int_{S_{N(t)}}^t Y(u)du \geq 0$ and it follows that,

$$\int_0^t Y(u)du \geq \sum_{i=1}^{N(t)} \frac{X_i^2}{2} \tag{4}$$

Since the $N(t)+1$'th arrival has not yet occurred, therefore,

$$\frac{X_{N(t)+1}^2}{2} \geq \int_{S_{N(t)}}^t Y(u)du$$

Verify this. Hence, we obtain,

$$\int_0^t Y(u)du \leq \sum_{i=1}^{N(t)} \frac{X_i^2}{2} + \frac{X_{N(t)+1}^2}{2} = \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2}$$

Therefore,

$$\sum_{i=1}^{N(t)} \frac{X_i^2}{2} \leq \int_0^t Y(u)du \leq \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2} \tag{5}$$

We are interested in the long-run time-average, therefore, dividing by t and taking limit,

$$\lim_{t\to\infty} \sum_{i=1}^{N(t)} \frac{X_i^2}{2t} \leq \lim_{t\to\infty} \frac{1}{t}\int_0^t Y(u)du \leq \lim_{t\to\infty} \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2t} \tag{6}$$

Verify that (apply SLLN and first limit theorem for renewal processes),

$$\lim_{t\to\infty} \sum_{i=1}^{N(t)} \frac{X_i^2}{2t} \to \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \ w.p.1$$

and,

$$\lim_{t\to\infty} \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2t} \to \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \ w.p.1$$

Here $\mathbb{E}[X_i^2] = \mathbb{E}[X^2]$. Now applying sandwich theorem we conclude that,

Proposition 1.5.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(u)du \to \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \ w.p.1$$

and,

Proposition 1.6.

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t Y(u)du \to \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \ w.p.1$$

Proof. Left as an exercise. Note that the plot of $A(t)$ will be a "reflection" of the plot of $Y(t)$, all else remains same. □

References

1. Sheldon Ross, Introduction to Probability Models, Academic Press, 2024.
2. Rick Durrett, Essentials of Stochastic Processes, Springer, 1999.