Problem Set - 3
MTH-204, MTH-204A
Abstract Algebra

1. Give an example of an infinite group all of whose elements have finite order.

2.If G is a group and H is a subgroup of index 2 in G, prove that H is a normal subgroup of G.

3. Give an example of a non-abelian group all of whose subgroups are normal.

4. Give an example of a group G, subgroup H, and an element $a \in G$ such that $aHa^{-1} \subset H$ but $aHa^{-1} \neq H$.

5. Suppose H is the only subgroup of order $|H|$ in a finite group G. Prove that H is a normal subgroup of G.

6. If H is a subgroup of G, let $N(H) = \{g \in G| \ gHg^{-1} = H\}$. Prove that H is normal in $N(H)$ and $N(H)$ is the largest subgroup of G in which H is normal. Prove that H is normal in G if and only if $N(H) = G$.

7. If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

8. Suppose that N and M are two normal subgroups of G and that $N \cap M = \{e\}$. Show that for any $n \in N, m \in M, nm = mn$.

9. If a cyclic subgroup T of G is normal in G, then show that every subgroup of T is normal in G.

10. Prove, by an example, that we can find three groups $E \subset F \subset G$, where E is normal in F, F is normal in G, but E is not normal in G.

11. If N is normal in G and $a \in G$ is of order $o(a)$, prove that the order, m, of Na in G/N is a divisor of $o(a)$.

12. If N is a normal subgroup in the finite group such that $[G : N]$ and $|N|$ are relatively prime, show that any element $x \in G$ satisfying $x^{|N|} = e$ must be in N.