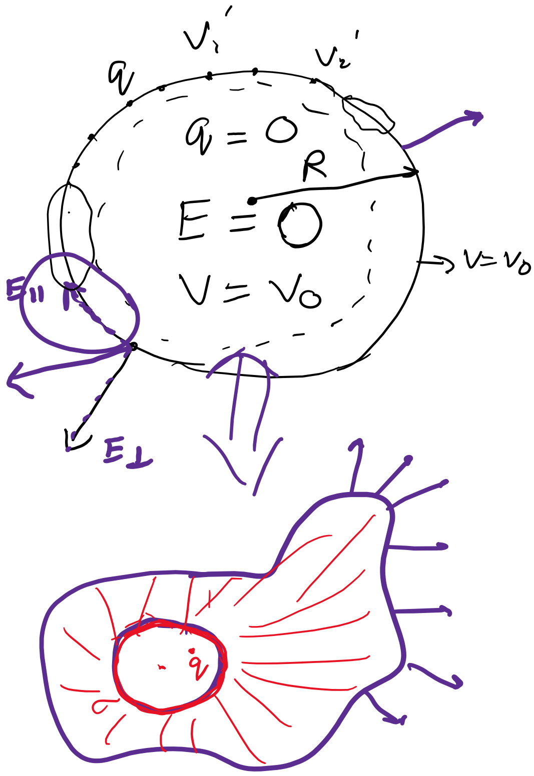
Discussion Hour 4 (27-12-2022)

Conductors:

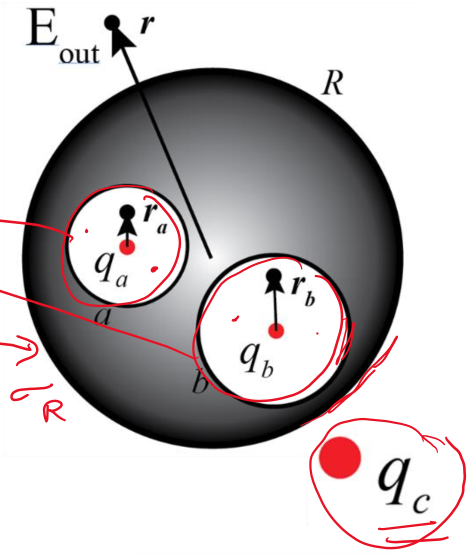
- Why charges reside only on the surface of a conductor, and not in bulk?
- Why surface charge density for a conductor is constant (or charge distribution is uniform)?
- Why conductor's surface is equipotential? ✓
- Why electric field is perpendicular to the surface, just outside the conductor?
- the charge on the conductor is not uniform as the charges inside are not placed on the center, how can we apply the result of a uniform shell?

$$\boxed{\oint \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0}}$$

$$\sigma = \frac{q}{4\pi R^2} = \text{const} \Rightarrow$$



(a) Find the surface charge densities σ_a, σ_b and σ_R.
(b) What is the field witnin each cavity?
(c) What is the field outside the conductor?
(d) What is the force on q_a and q_b.
(e) Which of these answers would change if a third charge, q_c, were brought near the conductor?

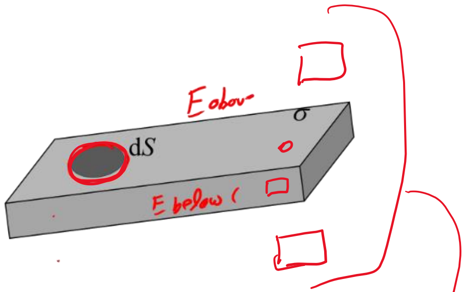


- Surface charge σ_a?	$\sigma_a = -\frac{q_a}{4\pi a^2}$	Same
- Surface charge σ_b?	$\sigma_b = -\frac{q_b}{4\pi b^2}$	Same
Surface charge σ_R?	$\sigma_R = \frac{q_a + q_b}{4\pi R^2}$	**Changes**
- $\mathbf{E}(\mathbf{r_a})$?	$\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{\mathbf{r}}_a$	Same
- $\mathbf{E}(\mathbf{r_b})$?	$\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b$	Same
- $\mathbf{E}_{\text{out}}(\mathbf{r})$?	$\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$	**Changes**
- Force on q_a ?	0	Same
- Force on q_b ?	0	Same

Bring in a third charge q_c

what is the meaning of continuity of E over the pitch if the surface charge is removed from the pitch?

Force on a conductor in external electric field

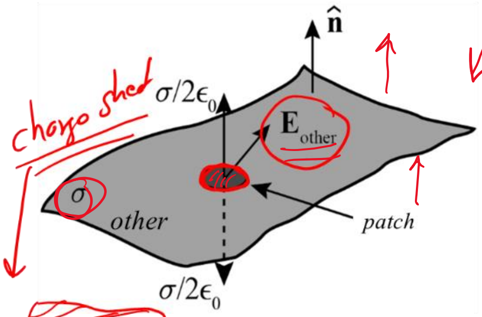


In presence of an electric field, a surface charge (ds*σ) will experience a force, and the force per unit area is $\sigma\mathbf{E}$.

But since $\mathbf{E}$ is discontinuous at a surface, which $\mathbf{E}$ should be use?

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

Field inside a conductor is zero, and the field immediately outside is: $\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$



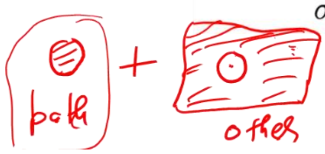
We want to find force experienced by a small region, called "patch".

Using the superposition principle, total field at the patch consists of two parts:

(i) due to patch itself, and

(ii) due to everything else (other regions of the surface)

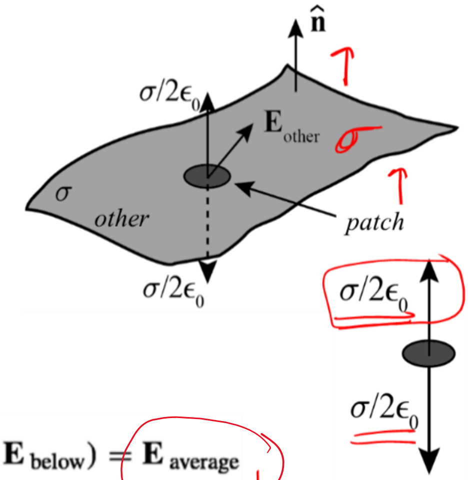
$$\mathbf{E} = \mathbf{E}_{\text{patch}} + \boxed{\mathbf{E}_{\text{other}}}$$



But the patch can not exert force on itself.

So the force on the patch is only due to $\mathbf{E}_{\text{other}}$.

$\mathbf{E}_{\text{other}}$ does not show any discontinuity as you remove patch, because electric field in the hole would be perfectly smooth.

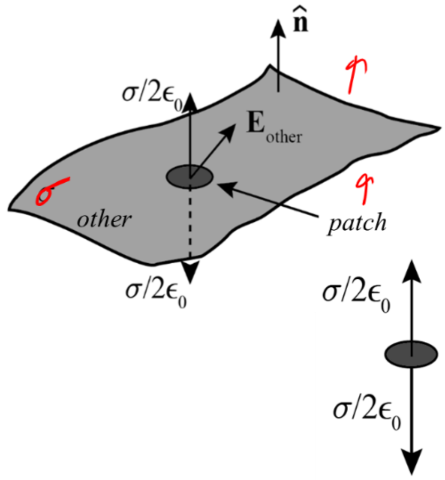


Discontinuity is entirely due to the charged patch which puts a field ($\sigma/2\varepsilon_0$) on either side.

Using the superposition principle, total field below and above the surface charge,

$$\left.\begin{aligned}\mathbf{E}_{\text{above}} &= \mathbf{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}} \\ \mathbf{E}_{\text{below}} &= \mathbf{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}\end{aligned}\right\} \mathbf{E}_{\text{other}} = \frac{1}{2}(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}) = \mathbf{E}_{\text{average}}$$

In case of a conductor, the field is zero inside and σ/ε_0 outside.



So, the average field is $\sigma/2\varepsilon_0$.

So, the force on the patch: (force per unit area)

$$F = \frac{F}{A} = \mathbf{f} = \sigma\mathbf{E}_{\text{average}} \longrightarrow \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{\mathbf{n}}$$

Electrostatic pressure:

$$P = \boxed{\frac{1}{2\epsilon_0}\sigma^2} = \boxed{\frac{\epsilon_0}{2}E^2} \rightarrow E_{ave}$$

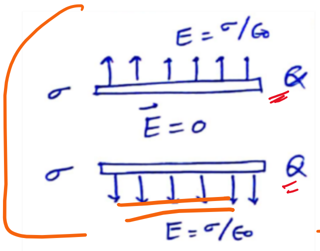
$$\mathbf{f} = \sigma \mathbf{E}_{\text{average}} \longrightarrow \quad \mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2 \hat{\mathbf{n}}$$

$$P = \frac{1}{2\epsilon_0}\sigma^2 \quad = \frac{\epsilon_0}{2}E^2$$

Problem 4.2 (Tutorial 4)

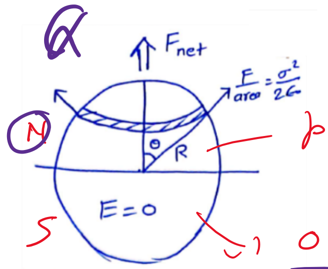
a) Two large metal plates (each of area A) are held a small distance d apart. Suppose we put a charge Q on each plate; what is the electrostatic pressure on the plates?

b) A metallic sphere of radius R carries a total charge Q. what is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?



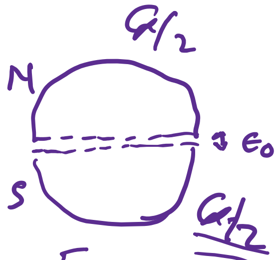
both

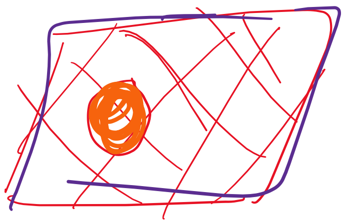
other



both

other = E_{ave}





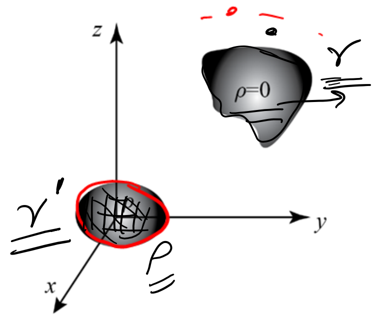
$$E_{oth} = E_{ave}$$
$$= E_{\text{due to lower plate}}.$$

is the only condition for Laplace equation that charge density=0 at the place where we need to find the potential?? If not, then how did sir use that equation when charge density was localized (not equal to zero in space)?

according to laplacian equation if no point of extrema is possible then also ;on being V1=V2 at the surface ,howdoes it apply that V1=V2 is true for entire volume? I mean could it notbe linearly increasing ?

$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$ (Poisson's Equation)

When $\rho = 0$ $\boxed{\nabla^2 V = 0}$ (Laplace's Equation)



If $\rho = 0$ everywhere, $V = 0$ everywhere.

If ρ is localized, what is V away from the charge distribution?

Laplace's equation in one dimension

$\nabla^2 V = 0$ (Laplace's Equation)

In Cartesian coordinates, $\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$

If $V(x, y, z)$ depends on only one variable, x, then $\frac{d^2}{dx^2}V = 0$ (One dimensional Laplace's equation, ordinary differential equation)

$E, V \propto \frac{1}{r^n}$

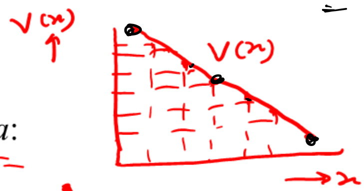
General solution, $V(x) = mx + b$

How to calculate the constants m and b? Using boundary condition

$V \to 0$
$x \to \infty$

What decides the boundary conditions? The charge distribution

General solution, $V(x) = mx + b$



In one dimension, $V(x)$ is an average of $V(x + a)$ and $V(x - a)$, for any a:

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$$

Laplace's equation is a kind of averaging technique.

It tells you to assign to the point x the average of the values to the left and right of x.

Laplace's equation tolerates no maxima or minima, which means that the extreme values of V must occur at the end point.

Because if there were a local maxima, V would be greater at that point than on the either side and therefore could not be the average

Also second derivative is zero so no maxima or minima.

Laplace's equation in two dimensions

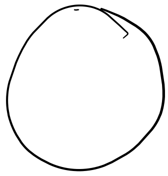
$\nabla^2 V = 0$ (Laplace's Equation)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Similar to 1D case, the value of V at a point (x, y) is the average of those around the point.

So, for a circle of radius R, the value of V at the center (x, y) is equal to the average value of V on the circle.

$$\boxed{V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V\, dl}$$

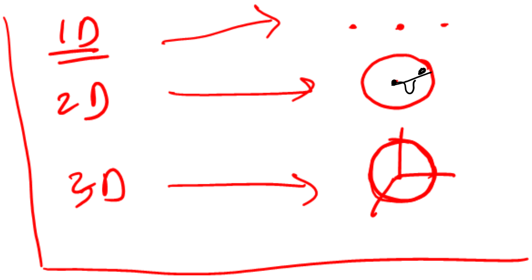


Similar to 1D case, $V(x, y)$ cannot have local maxima or minima as the extreme values of $V(x, y)$ must occur only at the boundaries.

Laplace's equation in three dimensions

$\nabla^2 V = 0$ (Laplace's Equation)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$



Similar to 1 & 2D cases, the value of V at a point **r** is the average value of V over a spherical surface of radius R centered at **r**.

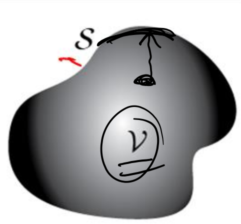
$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V\, da}$$

Again V cannot have local maxima or minima as the extreme values of $V(x, y)$ must occur only at the boundaries.

In this statement, are we saying that V3 would be zero even for boundaries other than the surface S? If yes, then how are we saying this for boundaries other than the one("S") for which we are currently proving?

not able to understand proof of second uniqueness theorem and its application

First uniqueness theorem



"The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface S."

Suppose V_1 and V_2 are two distinct solutions to Laplace's equation within volume $\mathcal{V}$ with the same value on the boundary surface S.

$$\nabla^2 V_1 = 0 \text{ and } \nabla^2 V_2 = 0 \Rightarrow \text{in } \mathcal{V}$$

$$V_3 \equiv V_1 - V_2 \text{ so } \nabla^2 V_3 = \nabla^2(V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

V_3 also satisfies Laplace equation.

$|\mathscr{R}| = |\vec{r} - \vec{r}'|$

$E \propto \frac{\vec{\mathscr{R}}}{|\mathscr{R}|^3}$

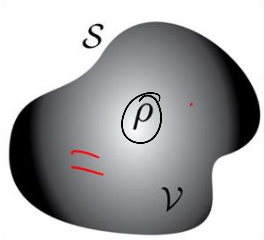
What is the value of V_3 at the boundary S?

0, because at the boundary $V_1 = V_2$, hence $V_3 = V_1 - V_2 = 0$.

But Laplace equation doesn't allow any local extrema, so since $V_3 = 0$ at the boundary, V_3 must be 0 everywhere.

Hence, $V_1 = V_2$ everywhere.

Corollary to First uniqueness theorem



The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density throughout the region, and (b) the value of V at all boundaries, are specified.

Suppose V_1 and V_2 are two distinct solutions to Poisson's equation in a region within volume $\mathcal{V}$ and charge density ρ. V_1 and V_2 have the same value at the boundary surface S.

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \text{ and } \nabla^2 V_2 = -\frac{\rho}{\epsilon_0}$$

$$V_3 \equiv V_1 - V_2 \text{ so } \nabla^2 V_3 = \nabla^2(V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

V_3 also satisfies Laplace's equation.

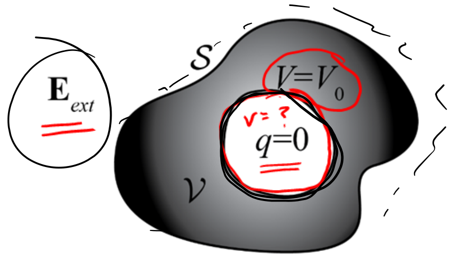
What is the value of V_3 at the boundary S?

0, because at the boundary $V_1 = V_2$, hence $V_3 = V_1 - V_2 = 0$.

So once again, the difference $V_3 = V_1 - V_2$ satisfies Laplace's equation and has the value zero on all boundaries, so $V_3 = 0$ and hence $V_1 = V_2$.

I cannot understand properly why V=V0 is the solution of the problem

Example: What is the potential inside an enclosure with no charge and surrounded completely by a conducting material?



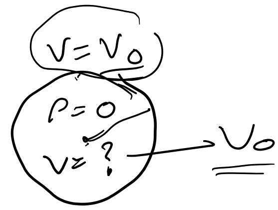
Potential on the boundary of the cavity is const. $= V_0$

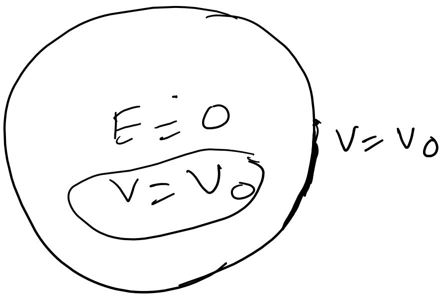
$\Rightarrow$ The potential inside cavity will be a function which satisfies Laplace's eqⁿ & has the value V_0 at the boundary.

$\Rightarrow$ $V = V_0$ is "a" solution inside the cavity
V_0 satisfies the B.C. & also satisfies Laplace's eqⁿ

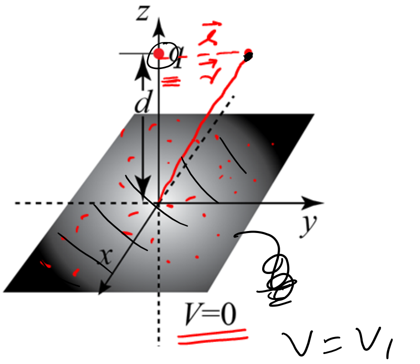
$\Rightarrow$ $V = V_0$ is "the" solution

$V = V_1$





In lecture 12, in the question with image method for plane infinite sheet, the sheet is grounded . So we take potential of sheet as 0.But since the ground as been taken as reference for potential =0 , how can we say that potential at infinity will also be 0(how can we take two references for potential)?



The Method of Images
(Application of Uniqueness theorem)

Let's learn this technique with some examples.

Question: What is the potential in the region above an infinite grounded plane?

Answer: $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$ **NO** **Why?**

$V = V_1$

Because the point charge will induce some charges in the conducting plane which will also contribute to the total potential.

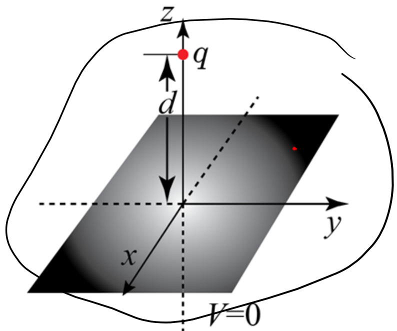
Therefore to get the right answer we will have to solve the Poisson's equation $\boxed{\nabla^2 V = -\rho/\epsilon_0}$

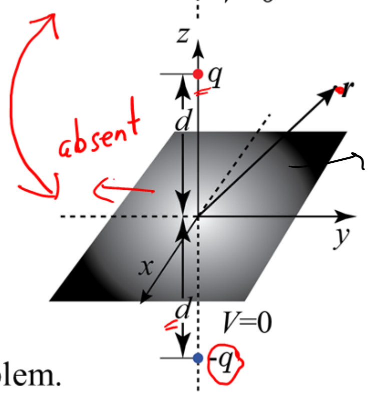
Boundary conditions are:

(i) V = 0 when z = 0

(ii) V → 0 far from the charge (r → ∞) i.e. when $x^2 + y^2 + z^2 >> d^2$ or $r >> d$

$E, V \propto \frac{1}{r^n}$





The Method of Images
(Application of Uniqueness theorem)

Question: What is the potential in the region above an infinite grounded plane?

Trick: Let's first solve the problem by removing the plane and adding –q at z = –d.

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Does this satisfy the boundary conditions?

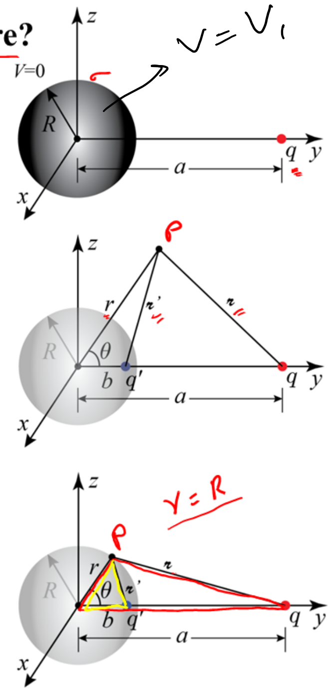
(i) V = 0 when z = 0 **YES**

(ii) V → 0 far from the charge (r → ∞) i.e. $r >> d$ **YES**

So the first uniqueness theorem says that this is ***the*** solution of the problem.

How do we pick the charge and position of the mirror image? Especially around 22:30, Finding the induced charge isn't easy so there must be another technique, and then how to we get to R^2 = ab

Question: What is the potential outside a grounded conducting sphere?



We will have to solve Poisson's equation $\nabla^2 V = -\rho/\epsilon_0$ with following boundary condition:

Boundary Condition: $V = 0$ at $r = R$

$V \to 0$ as $r \to \infty$

Trick: we first solve a different problem, with charge q and another point charge q′ such that

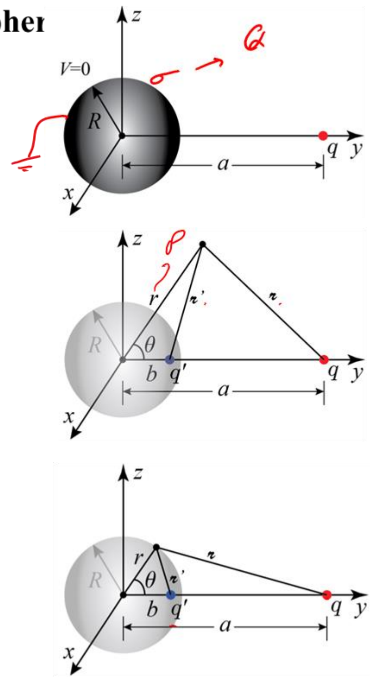
$$q' = -\frac{R}{a}q \qquad b = \frac{R^2}{a} \qquad (R^2 = ab)$$

The potential at r is: $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)}$

At $r = R$, for charge q $\quad \mathscr{r} = \sqrt{R^2 + a^2 - 2aR\cos\theta}$

for charge q′ $\quad \mathscr{r}' = \sqrt{R^2 + b^2 - 2bR\cos\theta} = \frac{R}{a}\mathscr{r}$

Question: What is the potential outside a grounded conducting spher



Trick: we first solve a different problem, with charge q and another point charge q′ such that $\boxed{q' = -\frac{R}{a}q}$ $\boxed{b = \frac{R^2}{a}}$

The potential at r is: $\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)}$

At $r = R$, $\quad \mathscr{r} = \sqrt{R^2 + a^2 - 2aR\cos\theta} \quad \mathscr{r}' = \sqrt{R^2 + b^2 - 2bR\cos\theta} = \frac{R}{a}\mathscr{r}$

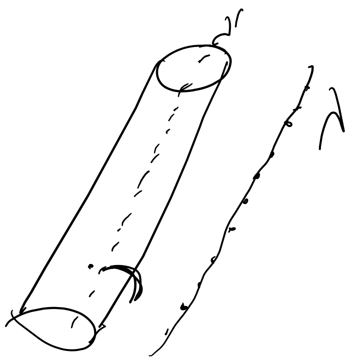
Check Boundary conditions:

$$V(R) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} - \frac{Rq}{a}\frac{a}{R\mathscr{r}}\right) = 0$$

$$V(r) \to 0 \text{ as } r \to \infty$$

So the potential V(r) satisfies the boundary conditions.

Hence, this is *the* solution.



SIR HOW LAST AND FINAL SOLN V(X,Y)=2V_0 /PIE *tan inerse fuxn

$n\pi$

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=odd} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a)}$$

$$\boxed{V(x,y) = \frac{2V_0}{\pi} \tan^{-1}\left[\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right]}$$

Will we have to remember the solution of the double differential equations like R(r)=Ar^l +B/r^l+1 etc. And also do we need to know the sinh(x) cosh (x) tanh(x) functions?

$$\boxed{V(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left[\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right]}$$

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B}{r^{l+1}}\right) P_l(\cos\theta)}$$