MTH 424 - PARTIAL DIFFERENTIAL EQUSTION

IIT KANPUR

Instructor: Indranil Chowdhury

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Assignment *Prep*

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1. Find the eigenvalues and corresponding eigenfunctions of the following problems:
   (i) $-\frac{d^2\phi}{dx^2} = \lambda\phi$ in $(0,1)$ with $\phi(0) = 0$ and $\phi(1) = 0$.
   (ii) $-\frac{d^2\phi}{dx^2} = \lambda\phi$ in $(0,1)$ with $\phi(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$
   Show that directional derivative of f at $(0,0)$ exists for all direction (i,e. $D_v f(0,0)$ exists for all $v \in \mathbb{R}^2$). Check whether f is continuous at $(0,0)$.

3. **Support of a function**: A support of a function f is a closed set defined as the closure of the set $\{x : f(x) \neq 0\}$. We denote the set by $\text{supp}(f)$.
   Compactly Supported function: A function is said to be compactly supported if $\text{supp}(f)$ is a compact set.

   Consider the function $\rho : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$\rho = \begin{cases} c \exp\left(\frac{1}{1-|x|^2}\right) & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geq 1, \end{cases}$$
   where the constant c is chosen such a way that $\int_{\mathbb{R}^2} \rho(x)\, dx = 1$. Verify $\text{supp}(\rho) = \overline{B_1(0)}$ and $\rho \in C^\infty(\mathbb{R}^2)$. Define ρ_ϵ as
$$\rho_\epsilon(x) = \frac{1}{\epsilon^2}\rho\left(\frac{x}{\epsilon}\right).$$
   Show that $\rho_\epsilon \in C^\infty(\mathbb{R}^2)$, $\text{supp}(\rho_\epsilon) = \overline{B_\epsilon(0)}$ and $\int_{\mathbb{R}^2} \rho_\epsilon(x)\, dx = 1$.
   (Similarly, ρ and ρ_ϵ can be defined in $\mathbb{R}^n$)

4. Find directional derivative of the function $f(x,y,z) = xyz$ in the direction of the vector $v = (5,-3,2)$.

5. Let $f(x,y) = \tan(\frac{x^2}{y}); y \neq 0$. Verify the mixed derivatives f_{xy} and f_{yx} are equal.

6. Let R be the region lying inside $x^2 + y^2 = 2x$ and to the right $x = 1$. Use polar co-ordinate to compute the following integration:
$$\iint_R \frac{x\, dx\, dy}{x^2 + y^2}.$$

7. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and suppose that $\partial_x g_2 = \partial_y g_1$. Let

$$f(x, y) = \int_0^x g_1(t, 0)\, dt + \int_0^y g_2(x, t)\, dt$$

Show that $\partial_x f(x, y) = g_1(x, y)$.