MTH 424 - PARTIAL DIFFERENTIAL EQUSTION

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Assignment 5

1. Consider the following function
$$\begin{cases} \triangle u = 0 \quad \text{in} \quad B(0,1) \subset \mathbb{R}^2 \\ \frac{\partial u}{\partial \nu} = g \quad \text{in} \quad \partial B(0,1), \end{cases}$$
where $g \in C(\partial B(0,1))$. Find the solution of the equation using separation of variable principle. Find the compatibility condition on g. [**Hint:** Change it to polar co-ordinate]

2. Solve the following heat equation using separation of variable method:
$$\begin{cases} u_t - u_{xx} = 0 \quad \text{in} \quad 0 < x < 1, t > 0 \\ u(x,0) = g(x) \quad \text{in} \quad 0 < x < 1, \\ u(0,t) = u(1,t) = 0 \quad \text{in} \quad t > 0 \end{cases}$$

3. Show that if u is a smooth solution of the heat equation $u_t - \triangle u = 0$, then $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ is also a solution for each $\lambda \in \mathbb{R}$. Consider $u(x,t) = \frac{1}{t^{n/2}} v(\frac{|x|^2}{t})$ and derive the fundamental solution of heat equation. [**Note:** the dilatation makes $\frac{|x^2|}{t}$ unchanges, and taking $\lambda = t^{-1/2}$ we can see $u_\lambda(x,t) = u(1, \frac{x}{t^{1/2}})$]

4. Let u be a smooth solution of
$$\begin{cases} u_t - u_{xx} = 0 \quad \text{in} \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = g(x) \quad \text{in} \quad x \in \mathbb{R}. \end{cases}$$
Assume $\lim_{|x| \to \infty} u_x(x,t) = 0$. Show that the conservation law for all $t > 0$,
$$\int_{-\infty}^{\infty} u(t,x)dx = \int_{-\infty}^{\infty} g(x)dx.$$

5. Assume $\Omega_T = (0,T) \times \Omega$ and $u \in C^{1,2}(\Omega_T) \cap C(\overline{\Omega_T})$ solves $u_t - \triangle u \leq cu$ in Ω_T, where $c \leq 0$. If $u \geq 0$, show that
$$\max_{\Omega_T} u = \max_{\partial_p \Omega_T} u,$$
where $\partial_p \Omega_T = \{(x,0) : x \in \Omega\} \cup \{(x,t) : x \in \partial\Omega, t \in [0,T]\}$. Give a counter example without the condition $u \geq 0$.

6. Let $u \in C^2((0,\infty) \times \mathbb{R})$ be a solution of the equation
$$u_t - a^2 u_{xx} = bu_x + cu + f(t,x)$$
where a, b, c are constants. Denote $v(t,x) = e^{-ct} u(t, x - bt)$ for $x \in \mathbb{R}$ and $t > 0$. Show that v satisfies the equation
$$v_t = a^2 v_{xx} + e^{-ct} f(t,x).$$

7. Find the explicit formula for a solution of

$$\begin{cases} u_t - \triangle u + cu = f & \text{in} \quad (0,\infty)\times\mathbb{R}^n \\ u(x,0) = g(x) & \text{in} \quad \mathbb{R}^n, \end{cases}$$

where $c \in \mathbb{R}$.

8. Let u be a solution to heat equation $u_t - u_{xx} = 0$ on $\mathbb{R}\times(0,T)$ with the initial condition $u(x,0) = \phi(x)$. If $\phi(x)$ is an odd function, show that the solution $u(x,t)$ is also an odd function of x.

9. Show that $u(x,t) = \frac{1}{\sqrt{1-t}}e^{\frac{x^2}{4(1-t)}}$ is a solution of heat equation on $\mathbb{R}\times(0,1)$ which blows up at $t = 1$.