Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Complex Analysis (MTH 403)
Exercise Sheet 12

1. Applications of Residue theory

1.1.∗ Prove the following Rouché's theorem:

(a) Fundamental theorem of algebra.

Hint. *Let* $P(z) \overset{def}{=} z^n + a_{n-1}z^{n-1} + \cdots + a_z + a_0$. *Show that when* R *is sufficiently large,* $|P(z) - z^n| < |z|^n$ *on* $|z| = R$.

(b) Hurwitz's theorem.

1.2. (a) Suppose that f is analytic on a region containing $\overline{\mathbb{D}}$ and $|f(z)| < 1$ whenever $|z| = 1$. Show that, for any $n \in \mathbb{N}$, the equation $f(z) = z^n$ has exactly n solutions in $\mathbb{D}$.
(b) Let $|a| > e$ and $n \in \mathbb{N}$. Show that the function $\exp z - az^n$ has precisely n zeros in $\mathbb{D}$.
(c) Show that there exists unique $z \in \mathbb{D}$ such that $\exp z = 2z + 1$.
(d) How many zeros (counting multiplicities) does the function $3z^{100} - \exp(z)$ have in $\mathbb{D}$?
(e) How many roots does the polynomial $2z^5 + 4z^2 + 1$ have in $\mathbb{D}$?
(f) Find the number of zeros of the polynomial $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ in $A(0; 1, 2)$.
(g) Find the number of zeros of the polynomial $z^5 + z^3 + 5z^2 + 2$ in $A(0; 1, 2)$.

1.3. Let $z_0 \in U \subseteq_{open} \mathbb{C}$ and $r > 0$ be such that $\overline{D(z_0; r)} \subseteq U$. Suppose that $f, g \in H(U)$ and f does not vanish anywhere on $|z - z_0| = r$. Show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon > 0$, the function f and $f + \varepsilon g$ have the same number of zeros in $D(z_0; r)$.

1.4.∗ Let $0 < r < 1 < R$. Show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon > 0$, $\varepsilon z^7 + z^2 + 1$ has exactly five roots (counting multiplicities) in $A\left(0; \frac{r}{\varepsilon^{\frac{1}{5}}}, \frac{R}{\varepsilon^{\frac{1}{5}}}\right)$.

Hint. *Use* εz^7 *and* $z^2 + 1$ *for comparison.*

1.5. For $z \in \mathbb{C}$, define $A(z) = \begin{pmatrix} 4z^2 & 1 & -1 \\ -1 & 2z^2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$. Find the cardinality of $\{z \in \mathbb{D} : A(z) \text{ is singular}\}$.

1.6.∗ Let $\lambda > 1$. Show that $\exp z - z - \lambda$ has only one zero in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ and furthermore, that zero is real.

Hint. *Some semi-circular contour with end points* $\pm iR$ *will be useful.*

1.7. Let $\overline{\mathbb{D}} \subseteq U \subseteq_{open} \mathbb{C}$ and $f \in H(U)$.

(a) Suppose that $|f| > m$ on $\partial\mathbb{D}$ and $|f(0)| < m$ where $m > 0$. Show that f has at least one zero on $\mathbb{D}$?

Note: This can also be proved using the Maximum modulus principle. Try!
(b) Assume that $|f(z_0)| < 1$, for some $z_0 \in \mathbb{D}$, and $|f| \geq 1$ on $\partial\mathbb{D}$. Show that $f(\mathbb{D})$ contains $\mathbb{D}$.

1.8.∗ Let $a \in \mathbb{C}$. Show that, for all $\varepsilon > 0$, the function $\sin z + \frac{1}{z-a}$ has infinitely many zeros in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \varepsilon\}$.

Hint. *First observe that* $\sin z = 0$ *if and only if* $z = m\pi$, *for some* $m \in \mathbb{Z}$. *Consider any* $m \in \mathbb{Z}$ *and the rectangle with vertices* $m\pi - \frac{\pi}{2} \pm \varepsilon$ *and* $m\pi + \frac{\pi}{2} \pm \varepsilon$. *Use Rouché's theorem to show that, when* m *is sufficiently large, such rectangular regions always contain a zero of* $\sin z + \frac{1}{z-a}$.

1.9.∗ Let U, z_0 and r be as in 1.3. and $f \in H(U)$. Suppose that $z_0, z_1, \ldots, z_n$ are distinct points in $D(z_0; r)$. Consider the polynomial $g(z) \overset{\text{def}}{=} (z - z_0)(z - z_1)\ldots(z - z_n)$. Show that the function P defined as follows is a polynomial of degree n and $P(z_j) = f(z_j)$, for all $j = 0, 1, \ldots, n$:

$$P(z) = \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{f(w)}{g(w)} \cdot \frac{g(w) - g(z)}{w - z}\, dw.$$

Deduce *Lagrange's interpolation formula* from this.

1.10.∗ (a) Let $\mathbb{H} \subseteq U \subseteq_{open} \mathbb{C}$ and $f \in H(U)$. Assume that there exist $M, a > 0$ such that $|f(z)| \le \frac{M}{|z|^a}$ for all $z \in \mathbb{H}$. Prove the following version of Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z}\, dx. \tag{1.1}$$

(b) Can the formula mentioned in (1.1) be proved when $f : \{z \in \mathbb{C} : \operatorname{Im} z \ge 0\} \longrightarrow \mathbb{C}$ is continuous and $f \in H(\mathbb{H})$?

1.11. Show that the function $\frac{z}{(z-1)(z-2)(z-3)}$, defined on $\{z \in \mathbb{C} : |z| > 4\}$ has a primitive.

Hint. *Observe that, for any closed path* γ *in its domain, all of* $1, 2$ *and* 3 *lie on the same component of* $\mathbb{C} \setminus \gamma^*$. *Now calculate the sum of the residues at those points.*

2. Continuity of zeros of a polynomial

Let $f(z) \overset{\text{def}}{=} a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with complex coefficients of degree n, and $z_1, \ldots, z_p$ are the distinct roots of f with multiplicities $m_1, \ldots, m_p$ respectively. For each $\Xi = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{C}^{n+1}$, we consider the following polynomial

$$f_\Xi(z) = (a_0 + \xi_0) + (a_1 + \xi_1)z + \cdots + (a_n + \xi_n)z^n. \tag{2.1}$$

For each $k = 1, \ldots, p$, choose r_k such that $0 < r_k < \min\limits_{\ell \ne k} |z_k - z_\ell|$.

2.1.∗ (a) Show that, for each $k = 1, \ldots, p$ and $z \in \partial D(z_k, r_k)$,

$$|f(z)| = |a_n| \prod_{\ell=1}^{p} |z - z_\ell|^{m_\ell} \ge |a_n| r_k^{m_k} \prod_{\ell \ne k} (|z_\ell - z_k| - r_k)^{m_\ell}.$$

(b) For each $k = 1, \ldots, p$, let $\delta_k \overset{\text{def}}{=} |a_n| r_k^{m_k} \prod\limits_{\ell \ne k} (|z_\ell - z_k| - r_k)^{m_\ell}$ and $M_k \overset{\text{def}}{=} \sum\limits_{j=0}^{n} (|z_k| + r_k)^j$. Choose

$$0 < \varepsilon < \min_{k=1,\ldots,p} \frac{\delta_k}{M_k}.$$

Show that, if all $|\xi_j| < \varepsilon$ then, for all $z \in D(z_k; r_k)$,

$$|f_\Xi(z) - f(z)| \le \sum_{j=0}^{n} \xi_j (|z_k| + r_k)^j < \varepsilon M_k < \delta_k \le |f(z)|.$$

(c) Conclude from Rouché's theorem that, whenever $\max\limits_{0 \le j \le n} |\xi_j| < \varepsilon$, f_Ξ has precisely m_k zeros in $B(z_k; r_k)$, for all $k = 1, \ldots, p$.

2.2.* Suppose that α is a simple root of f. Then show that there exists an open $U \subseteq \mathbb{C}^{n+1}$ containing $(a_0, a_1, \dots, a_n)$ and a unique continuous function $r : U \longrightarrow \mathbb{C}$ such that $r(b_0, b_1, \dots, b_n)$ is a root of f, for all $(b_0, b_1, \dots, b_n) \in U$, and $r(a_0, a_1, \dots, a_n) = \alpha$.

2.3.* Let f be as above. Refine the choices of r_k's so that $\overline{D(z_k; r_k)}$'s are pairwise disjoint. Cosnider $\varepsilon \stackrel{\text{def}}{=} \min\{|f(z)| : |z - z_k| = r_k, k = 1, \dots, n\}$. Show that, whenever $|w| < \varepsilon$, the equation $f(z) = w$ has exactly m_k solutions in $D(z_k; r_k)$, for all $k = 1, \dots, n$.

Note: This shows that the solutions of the equation $f(z) = w$ varies continuously with w.

3. Evaluation of integrals

3.1. Let $P(x), Q(x) \in \mathbb{R}[x]$, $\gcd(P(x), Q(x)) = 1$ and $\deg Q(x) \geq \deg P(x) + 2$. Assume that $Q(x)$ does not have a real root and the complex roots of $Q(x)$ that lie in the upper half plane are $z_1, \dots, z_n$. Then show that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\left(\frac{P}{Q}, z_j\right).$$

3.2. Evaluate the following integrals:

(a) $\displaystyle \frac{1}{2\pi i} \int_{C(0;1)} \frac{(z+2)^2}{z^2(2z-1)}\, dz.$

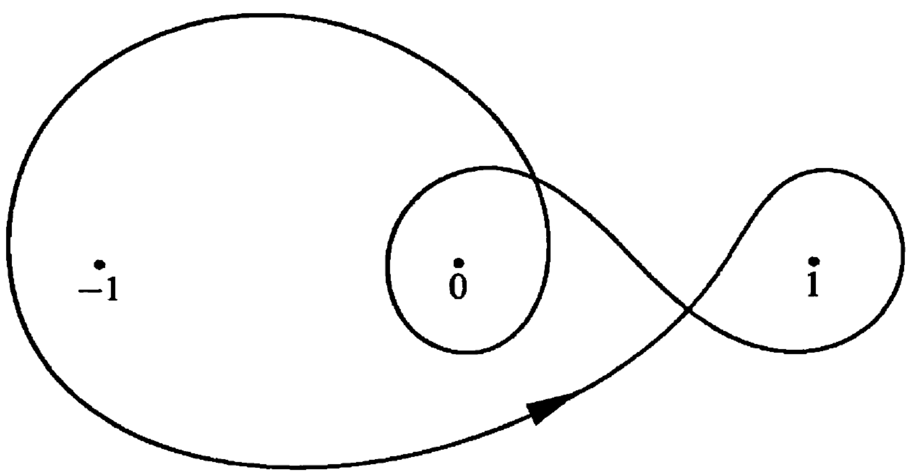
(b) $\displaystyle \int_{C(0;r)} \frac{z^2 + \exp(z)}{z^2(z-2)}\, dz$, where $r \in (0, \infty) \setminus \{2\}$.

(c) $\displaystyle \frac{1}{2\pi i} \int_{C(0;1)} \frac{dz}{\sin 4z}.$

(d) $\displaystyle \int_{C\left(0;(n+\frac{1}{2})\pi\right)} \frac{1}{z^3 \sin z}\, dz.$

(e) $\displaystyle \int_{C(0;1)} \frac{\exp(z)}{z(2z+1)^2}\, dz.$

(f) $\displaystyle \int_{\gamma} \frac{\exp(z)}{z^2(1-z^2)}\, dz$, where γ is depicted as follows:

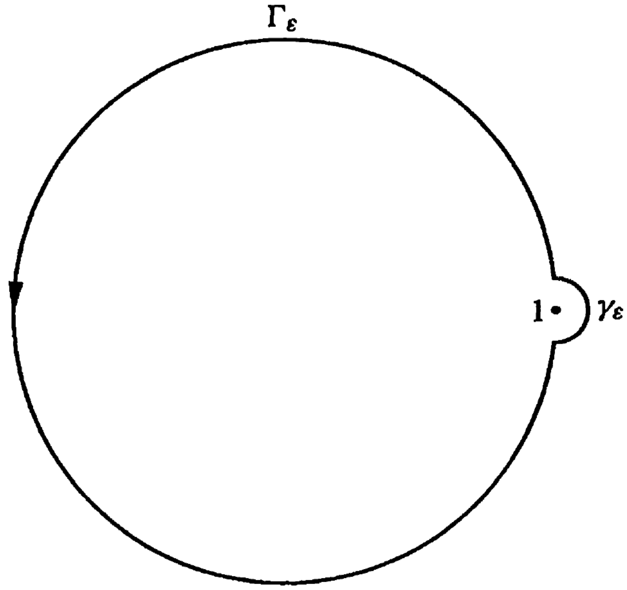


3.3. Evaluate $\displaystyle \int_0^{2\pi} \frac{d\theta}{r^2 - 2r\cos\theta + 1}$, where $r^2 \neq 1$.

Hint. *The function* $\dfrac{i}{(z-r)(rz-1)}$ *might be useful.*

3.4.$*$ Let $n \in \mathbb{N}$. Evaluate $\displaystyle\int_0^{2\pi} \frac{1-\cos n\theta}{1-\cos\theta}\,d\theta$.

Hint. *Consider the following contour:*



Observe that $\displaystyle\int_0^{2\pi} \frac{1-\cos n\theta}{1-\cos\theta}\,d\theta = \lim_{\varepsilon\to 0+}\int_\varepsilon^{2\pi-\varepsilon} \frac{1-\cos n\theta}{1-\cos\theta}\,d\theta = \lim_{\varepsilon\to 0+}\int_\varepsilon^{2\pi-\varepsilon} \frac{1-e^{in\theta}}{1-\cos\theta}\,d\theta$. *The last equality holds because* $\displaystyle\int_\varepsilon^{2\pi-\varepsilon} \frac{i\sin n\theta}{1-\cos\theta}\,d\theta = 0$ *(why?).*

3.5. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin n\theta}{\sin\theta}\,d\theta$.

3.6. For $a > 1$ and $n = 0, 1, 2, \ldots$, evaluate the integrals: $\displaystyle\int_0^{2\pi} \frac{\cos n\theta}{a-\cos\theta}\,d\theta$, and $\displaystyle\int_0^{2\pi} \frac{\sin n\theta}{a-\cos\theta}\,d\theta$.

Hint. *It might be easier to handle* $\displaystyle\int_0^{2\pi} \frac{\cos n\theta}{a-\cos\theta}\,d\theta + i\int_0^{2\pi} \frac{\sin n\theta}{a-\cos\theta}\,d\theta = \int_0^{2\pi} \frac{e^{in\theta}}{a-\cos\theta}\,d\theta$.

3.7. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx$.

Hint. *Observe that* $\displaystyle\frac{1-\exp(2iz)}{z^2}$ *has a simple pole at* 0.

3.8. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{(1+x^2)^2}\,dx$.

Hint. *Consider* $\displaystyle\frac{z\exp(z)}{(1+z^2)^2}$.

3.9. Let $a > 0$. Evaluate $\displaystyle\int_0^{\infty} \frac{\sin x}{x(x^2+a^2)}\,dx$.

Hint. *Observe that* $\displaystyle\frac{\exp(iz)-1}{z(z^2+a^2)}$ *has a removable singularity at* 0 *and a simple pole at* ia.

3.10. Evaluate $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+1}\,dx.$

3.11. Evaluate $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$

Hint. *Does any obvious substitution transform this integral to a familiar one?*

4. Miscellaneous exercises

4.1. Find all $z \in \mathbb{C}$ such that the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{z^n}{n!}+\frac{n^2}{z^n}\right)$ converges.

4.2. Let $\alpha \in \left(0, \frac{\pi}{2}\right)$ and S_α denote the minor sector of the unit circle made of the arc with end points $e^{-i\alpha}$ and $e^{i\alpha}$ along with two radii. Show that the function $\exp\left(-\frac{1}{z}\right)$ is uniformly continuous on $S_\alpha \setminus \{0\}$.

4.3. Let k be a positive integer > 1. Find all entire functions f satisfying $f\left(z^k\right) = (f(z))^k$, for all $z \in \mathbb{C}$.

4.4. Let $U \subseteq_{open} \mathbb{C}$ and $f \in H(U)$ whose derivative vanishes nowhere. Show that the following is an open subset of $\mathbb{R}$:

$$\{\operatorname{Re} f(z) + \operatorname{Im} f(z) : z \in U\}.$$

4.5.∗ Let f be an entire function such that $f(\mathbb{C}) \cap L = \emptyset$, for some line L in $\mathbb{C}$. Show that f is constant.

Hint. *Without loss in generality, we may assume that L is the imaginary axis. Then either* $\operatorname{Re} f > 0$ *or* < 0. *Use Exercise 2.3.(a) of Exercise Sheet 6.*

4.6. Let $f, g \in H(\mathbb{D})$ and $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ and $\displaystyle\sum_{n=0}^{\infty} b_n z^n$ are their power series representations on $\mathbb{D}$.

(a) Let $r \in (0,1)$. Show that, for all $z \in D(0;r)$,

$$\frac{1}{2\pi i}\int_{C(0;r)} \frac{f(w)}{w}\cdot g\left(\frac{z}{w}\right)dw = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{4.1}$$

(b) Show from (4.1) that the integral $\displaystyle\frac{1}{2\pi i}\int_{C(0;r)} \frac{f(w)}{w}\cdot g\left(\frac{z}{w}\right)dw$ is independent of r as long as $z \in D(0;r)$, and thus it defines a holomorphic function on $\mathbb{D}$.

(c) Denote the holomorphic function obtained in 4.6.b by h. Prove or disprove the following: if neither f nor g is identically zero then so is h.

4.7. Show that the following map is onto but not one-one:

$$\mathbb{C}^3 \longrightarrow \mathbb{C}^3,\ (u, v, w) \mapsto (u+v+w, uv+vw+uw, uvw).$$

4.8. Let a and b be positive numbers.

(a) Show that, if a nowhere vanishing entire function f satisfies $|f(z)| \le e^{a|\log|z||+b}$, for all $z \in \mathbb{C}\setminus\{0\}$, then it must be constant.

Hint. *For any positive integer $n > a$, observe that $\dfrac{f(z)}{z^n}$ approaches to* 0 *when $|z| \to \infty$. From this it follows that, at f ha either a removable singularity or a pole at ∞. In the former one* *obtains that f is constant, while in the latter f becomes a polynomial. Now from Fundamental theorem of algebra yields that f is consant.*

(b) Show that any harmonic function u on the complex plane satisfying $u(z) \leq a|\log|z|| + b$, for all $z \in \mathbb{C} \setminus \{0\}$, must be constant.