MTH 101-Calculus
Spring-2021
Assignment 1 : Real Numbers, Sequences

1. Find the supremum of the set $\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\}$.
2. (a) Show that the sequences $(sin(n))$ and $(cos(n))$ are divergent.

   (b) Show that the sequence defined by $a_n = 2cos(n) - sin(n)$ has a convergent subsequence.
3. Let $y \in (1, \infty)$ and $x \in (0, 1)$. Evaluate $\lim_{n\to\infty}(2n)^y x^n$.
4. Prove or disprove that any Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ has a limit in $\mathbb{Q}$.
5. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E dense in $[0, 1]$?
6. Do the following sequences satisfy the Cauchy criterion ?

   (a) $x_1 = \frac{1}{2}$ and $x_{n+1} = \frac{1}{7}\left(x_n^3 + 2\right)$ for $n \in \mathbb{N}$.

   (b) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ for $n \in \mathbb{N}$.
7. If $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + \sqrt{x_n}}$, $n = 1, 2, 3, ...$ Prove that $\{x_n\}$ converges, and that $x_n < 2$ for $n = 1, 2, 3, ...$

$*$8. Let α be an irrational number. Show that the set $S = \{m + n\alpha : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.