Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Complex Analysis (MTH 403)
Exercise Sheet 10

1. Conformal maps

1.1. Let U be a bounded region and $f : \overline{U} \longrightarrow \mathbb{C}$ be a continuous function such that $f \in H(U)$. Assume that $|f| \equiv 1$ on ∂U. Show that $f(U) = \mathbb{D}$. Conclude that, if in addition f is assumed to be in injective, then it is establishes a conformal equivalence between U and $\mathbb{D}$.

1.2. Let $U \subseteq_{open} \mathbb{R}^2$ and $f : U \longrightarrow \mathbb{R}^2$ be orientation preserving, i.e., the linear map $Df(\mathbf{x})$ preserves orientation for every $\mathbf{x} \in U$. Assume that f always maps pairs of orthogonal curves to orthogonal ones. Show that f is holomorphic.

1.3. Find the kernel of the action of $\mathrm{GL}_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ by Möbius transformations.

1.4. Let $z \neq w \in \hat{\mathbb{C}}$. Show that the set of all Möbius transformations that fix z and w form an abelian group. Does this group look familiar to you?

Hint. *First look at the special case $z = 0$ and $w = \infty$.*

1.5. Find all Möbius transformations T in each of the following cases:

(a) $T(\mathbb{D}) = \mathbb{D}$.
(b) $T(\mathbb{H}) = \mathbb{H}$.
(c) $T(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$.
(d) $T(\mathbb{H}) = \mathbb{D}$.
(e)$*$ $T(\partial\mathbb{D}) = \partial\mathbb{D}$.

1.6.$*$ We say that $\mathcal{L} \subseteq \hat{\mathbb{C}}$ is a *circle* if it is either a circle in $\mathbb{C}$ or $\mathcal{L} = L \cup \{\infty\}$, where L is a line in the complex plane.

(a) Show that the image of a circle in $\hat{\mathbb{C}}$ under a Möbius transformation on $\hat{\mathbb{C}}$, is again a circle in $\hat{\mathbb{C}}$.
(b) Show that the inverse image of the real line under a Möbius transformation is a circle in $\hat{\mathbb{C}}$.
(c) Show that the group of all Möbius transformation on $\hat{\mathbb{C}}$ acts freely on the set of all circles in $\hat{\mathbb{C}}$.

Recall. *The action of a group G on a set X is said to be free if for all $g \in G \setminus \{1\}$ and $x \in X$, one always has $gx \neq x$. An example of free action is the action of a group on itself by left multiplication.*

1.7. Let z_1, z_2, z_3, z_4 be four distinct points in $\hat{\mathbb{C}}$. The *cross ratio* of z_1, z_2, z_3 and z_4, denoted by $[z_1, z_2, z_3, z_4]$, is defined by the image of z_1 under the unique Möbius transformation that sends z_2, z_3 and z_4 to $1, 0$ and ∞ respectively.

(a) Show that, for any four distinct points w_1, w_2, w_3 and w_4 in $\hat{\mathbb{C}}$ and Möbius transformation T, $[w_1, w_2, w_3, w_4] = [Tw_1, Tw_2, Tw_3, Tw_4]$.
(b) Show that, $[w_1, w_2, w_3, w_4] \in \mathbb{R}$ if and only if w_1, w_2, w_3 and w_4 line on a circle in $\hat{\mathbb{C}}$.

1.8. Let $A \in \mathrm{GL}_2(\mathbb{C})$ be such that the Möbius transformation f_A has two distinct fixed points in $\mathbb{C}$.

(a) Show that the eigenvalues of A are distinct.
(b) Express the fixed points of f_A using the eigenvectors of A.
(c)$*$ Suppose that the eigenvalues of A have different absolute values. Show that there exists two distinct points w, w' in $\mathbb{C}$ such that, for all $z \in \mathbb{C} \setminus \{w\}$, $f_M^n(z) \xrightarrow[n\to\infty]{} w'$.

1.9. (a) Consider $f(z) \stackrel{\text{def}}{=} \exp\left(\frac{z-1}{z+1}\right)$, for all $z \in \mathbb{D}$. Show that f maps $\mathbb{D}$ conformally onto $\mathbb{D} \setminus \{0\}$.
(b) Show that the following maps $\mathbb{D}$ conformally onto an annulus:

$$f(z) \stackrel{\text{def}}{=} \exp\left(-i \log\left(i\frac{1-z}{1+z}\right)^{\frac{1}{2}}\right), \ \forall z \in \mathbb{D},$$

where the branch of logarithm that has been considered is $\log_0$.

1.10. Exhibit a conformal map from U onto V in the following:

(a) $U \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $V \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| > 1\}$.
(b) $\left\{z \in \mathbb{C} : |z-1| < 1 \text{ and } |z - \frac{1}{2}| > \frac{1}{2}\right\}$ and $V \stackrel{\text{def}}{=} \mathbb{H}$.

2. Conformal metric

Throughout this section, unless otherwise mentioned, we assume that $U \subseteq \mathbb{C}$ is a region and $\rho : U \longrightarrow [0, \infty)$ satisfies the following two properties:

(M.1) ρ is continuous on U, and
(M.2) ρ is C^2 on $\{z \in U : \rho(z) > 0\}$.

2.1. In each of the following, show that ρ satisfies (M.1) and (M.2):

(a) $U \stackrel{\text{def}}{=} \mathbb{C}$ and $\rho \equiv 1$.
(b) $U \stackrel{\text{def}}{=} \mathbb{H}$ and $\rho(z) \stackrel{\text{def}}{=} \frac{1}{\operatorname{Im} z}$.
(c) $U \stackrel{\text{def}}{=} \mathbb{D}$ and $\rho(z) \stackrel{\text{def}}{=} \frac{2}{1-|z|^2}$.
(d) $U \stackrel{\text{def}}{=} \mathbb{D} \setminus \{0\}$ and $\rho(z) \stackrel{\text{def}}{=} -\frac{1}{|z| \log |z|}$.

2.2. Suppose Ω is a region and $f : \Omega \longrightarrow U$ is conformal. Show that the function $f^*(\rho) \stackrel{\text{def}}{=} (\rho \circ f)|f'|$ satisfies the properties similar to (M.1) and (M.2) on Ω.

2.3. Determine $f^*(\rho)$ in each of the following cases:

(a) $f : \mathbb{D} \longrightarrow \mathbb{H}$, $f(z) \stackrel{\text{def}}{=} i\frac{1-z}{1+z}$ and ρ is same as that of 2.1.b.
(b) $f : \mathbb{H} \longrightarrow \mathbb{D}$, $f(z) \stackrel{\text{def}}{=} \frac{i-z}{i+z}$ and ρ is same as that of 2.1.c.
(c) $f : D(z_0; r) \longrightarrow \mathbb{D}$, $f(z) \stackrel{\text{def}}{=} \frac{z-z_0}{r}$, where $z_0 \in \mathbb{C}$ and $r > 0$, and ρ is same as that of 2.1.c.
(d) f and ρ are same as those of 1.9.a and 2.1.d respectively.
(e) $f : \{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\} \longrightarrow \mathbb{D}$, $f(z) \stackrel{\text{def}}{=} \frac{1-e^z}{1+e^z}$, and ρ is same as that of 2.1.c.

2.4. For any real valued C^2 function φ, defined on some open subset of $\mathbb{R}^2$, its *Laplacian* is defined by

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}. \tag{2.1}$$

The symbol $\Delta\varphi$ is commonly used to denote (2.1). If $\Delta\varphi \equiv 0$, then we say that φ is a *harmonic function*. Show that the real part of an analytic function is harmonic.

Let U and ρ be as mentioned above in the beginning of this section. For any $z \in U$ with $\rho(z) > 0$,

$$\kappa_\rho(z) \stackrel{\text{def}}{=} -\frac{\Delta \log \rho(z)}{\rho(z)^2}.$$

2.5. Calculate $\kappa(z;\rho)$, for an arbitrary $z \in U$, in 2.1.a and 2.1.b.

2.6.$*$ Let Ω and f be as in 2.2. Show that, for any $z \in f^{-1}(\{w \in U : \rho(w) > 0\})$, one has the following:

$$\kappa_{f^*(\rho)}(z) = \kappa_\rho(f(z)).$$

Using 2.6. conclude in each of 2.1.c, 2.1.d, 2.3.c and 2.3.e, $\kappa_\rho \equiv -1$.

2.7. Let ρ be as in 2.1.c. Show that, for any holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$, one has $f^*(\rho) \leq \rho$.