MTH 101-Calculus
Spring-2021
Assignment 6 : Integration

1. Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{x}$ is integrable on $[1, 2]$.

2. If f and g are continuous functions on $[a,\ b]$ and if $g(x) \geq 0$ for $a \leq x \leq b$, then show the mean value theorem for integrals : there exists $c \in [a,\ b]$ such that $\int\limits_a^b f(x)g(x)dx = f(c) \int\limits_a^b g(x)dx$.

   (a) Show that there is no continuous function f on $[0, 1]$ such that $\int\limits_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$.

   (b) If f is contiunuous on $[a, b]$ then show that there exists $c \in [a, b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f(t)dt$.

   (c) If f and g are continuous on $[a, b]$ and $\int_a^b f(x)dx = \int_a^b g(x)dx$ then show that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

3. Give an example of a Riemann integrable function having infinitely many (possibly countable) discontinuities.

4. Assume that $f(x)$ is continuous on $[a, b]$ and for any continuous function $g(x)$ if $\int_a^b g(x)dx = 0$ then $\int_a^b f(x)g(x)dx = 0$. Show that $f(x)$ is a constant function.

5. (a) Show with examples that composition of two Riemann integrable functions need not be Riemann integrable.

   (b) Suppose f is a bounded real function on $[a, b]$, and f^2 is Riemann Integrable on $[a, b]$. Does it follow that f is integrable ? Does the answer change if we assume f^3 is integrable ? Here f^2 and f^3 mean the square and cube of f (not the composition).

6. Let $f : [0, 2] \to \mathbb{R}$ be a continuous function such that $\int_0^2 f(x)dx = 2$. Find the value of $\int_0^2 [xf(x) + \int_0^x f(t)dt]dx$.

7. Show that $\int\limits_0^x (\int\limits_0^u f(t)dt)du = \int\limits_0^x f(u)(x - u)du$, assuming f to be continuous.

8. Let $f : [0, 1] \to \mathbb{R}$ be a positive continuous function. Show that $\lim_{n\to\infty}(f(\frac{1}{n})f(\frac{2}{n})\cdots f(\frac{n}{n}))^{\frac{1}{n}} = e^{\int_0^1 lnf(x)}$.