Gauss - Green Formula

- Assume $u \in C^1(\bar{\Omega})$. Then

$$\int_{\Omega} u_{x_i} dx = \int_{\partial\Omega} u \nu^i dS. \quad \text{for } i=1,\dots,n.$$

~~On the other hand;~~ where, $\nu^i(x)$ is the ith component of outward normal vector ν of $\partial\Omega$ at $x \in \partial\Omega$.

Therefore,

$$\int_{\Omega} \operatorname{div}(\nabla u)\, dx = \int_{\partial\Omega} \nabla u \cdot \nu \, ds$$

where $\operatorname{div}(v) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$ for $v = (v_1, \dots, v_n)$

Integration by parts formula:

Let $u, v \in C^1(\bar{\Omega})$. Then

$$\int_{\Omega} u_{x_i} v \, dx = -\int_{\Omega} u v_{x_i} dx + \int_{\partial\Omega} u v \nu_i \, dS$$

for $i = 1, \dots, n$ and $\nu^i(x)$ is the ith component of unit outward normal vector $\nu(x)$ of $\partial\Omega$ at $x \in \partial\Omega$.

Green's Formula: Let $u, v \in C^2(\bar{\Omega})$. Then

(i) $\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS$, where $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$

(ii) $\int_{\Omega} \nabla v \cdot \nabla u \, dx = -\int_{\Omega} u \Delta v \, dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} dS$

(iii) $\int_{\Omega} (u \Delta v - v \Delta u)\, dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}\right) dS$

⊙ Let $f : B(x_0, r) \to \mathbb{R}$ be continuous. Then

$$\int_{B(x_0,r)} f\,dx = \int_0^r \int_{\partial B(x_0,s)} f(y)\,dS(y)\,ds.$$

~~In particular~~ If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\int_{\mathbb{R}^n} |f|\,dx < \infty$,

$$\int_{\mathbb{R}^n} f\,dx = \int_0^\infty \left(\int_{\partial B(x_0,s)} f(y)\,dS(y) \right) ds.$$

⊙ $\Omega \subset \mathbb{R}^n$ open set and assume $f : \Omega \to \mathbb{R}^n$ is C^1; let $f = (f_1, \dots, f_n)$. For any $x_0 \in \Omega$

$$Df(x_0) = \begin{bmatrix} \frac{\partial f_1(x_0)}{\partial x_1} & \cdots & \frac{\partial f_1(x_0)}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_n(x_0)}{\partial x_1} & \cdots & \frac{\partial f_n(x_0)}{\partial x_n} \end{bmatrix}$$

We define, Jacobian of $f = |\det Df| := |J_f|$

- We say a map $\phi : \Omega \to \mathbb{R}^n$ is a diffeomorphism if it is one-one, differentiable and it's inverse $\phi^{-1} : \phi(\Omega) \to \Omega$ is also differentiable.

Change of Variable formula

$$\int_{\phi(\Omega)} f(y)\,dy = \int_{\Omega} f(\phi(x))\,|J_\phi(x)|\,dx.$$

where $\phi : \Omega \to \mathbb{R}^n$ is a diffeomorphism and f is integrable.

(Note: In the change of variable formula 'Jacobian of ϕ' is considered)