Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)
Exercise Sheet 2

1. Basic properties of holomorphic functions

1.1. Suppose that $\Omega \subseteq_{open} \mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic. Assume further that $f(\Omega) \subseteq \mathbb{R}$. Show that, if f is holomorphic at $z_0 \in \Omega$ then $f'(z_0) = 0$.

1.2. Let $z_0 = x_0 + iy_0 \in \Omega \subseteq_{open} \mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$.

(a) Assume that f is holomorphic at z_0. Show that, when f is viewed as a function defined from the open subset Ω of $\mathbb{R}^2$ to $\mathbb{R}^2$, it is differentiable at the point (x_0, y_0).
(b) Conclude from 1.2.a that f must be continuous at z_0.
(c) Does the converse of 1.2.a hold?
(d) If the converse of 1.2.a is false, then find necessary and sufficient conditions on f so that the converse of 1.2.a holds true.
(e) Can you express the conditions obtained above in 1.2.d in polar coordinates?

1.3. In each of the following cases, find all points in $\mathbb{C}$ at which f is holomorphic:

(a) $f(z) \stackrel{\text{def}}{=} \bar{z}$ (b) $f(z) \stackrel{\text{def}}{=} |z|$ (c) $f(z) \stackrel{\text{def}}{=} |z|^2$ (d) $f(z) \stackrel{\text{def}}{=} e^{\operatorname{Re} z}$.

1.4. Assume that $\Omega \subseteq \mathbb{C}$ is open and connected and $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic. In each of the following cases show that f is constant:

(a) $\operatorname{Re} f$ is constant (b) $\operatorname{Im} f$ is constant (c) $|f|$ is constant.

1.5. Let $z_0 \in \Omega \subseteq_{open} \mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$.

(a) Show that f is holomorphic at z_0 if and only if $\exists f^* : \Omega \xrightarrow[\text{cts. at } z_0]{} \mathbb{C}$ satisfying

$$f(z) - f(z_0) = f^*(z)(z - z_0),\ \forall z \in \Omega.$$

(b) Find $f^*(z_0)$ in case of 1.5.a.

1.6. Suppose that $z_0 \in \Omega \subseteq_{open} \mathbb{C}$ and $f, g : \Omega \longrightarrow \mathbb{C}$ are holomorphic at z_0. Show the following:

(a) $f \pm g$ is differentiable at z_0 and $(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$.
(b) fg is differentiable at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.
(c) If $g(z_0) \neq 0$, there exists $r > 0$ such that $D(z_0; r) \subseteq \Omega_1$ and g vanishes nowhere in $D(z_0; r)$.
(d) Let g be as above in 1.6.c. Then $\frac{1}{g} : D(z_0; r) \longrightarrow \mathbb{C}$ is differentiable at z_0 and

$$\left(\frac{1}{g}\right)'(z_0) = -\frac{g'(z_0)}{g(z_0)^2}.$$

(e) Let g be as above in 1.6.c. The function $\dfrac{f}{g} : D(z_0; r) \longrightarrow \mathbb{C}$ is differentiable at a and $\left(\dfrac{f}{g}\right)'(z_0) = \dfrac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$.

In 1.7. and 1.8., we let $\Omega_1, \Omega_2 \subseteq_{open} \mathbb{C}$, $z_0 \in \Omega_1$, $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \mathbb{C}$.

1.7. Show that if f is holomorphic at z_0 and g is holomorphic at $f(z_0)$, then $g \circ f$ is also holomorphic at z_0 and $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

1.8. Assume that f is continuous at z_0, g is holomorphic at $f(z_0)$ and $\forall z \in \Omega_1, g(f(z)) = z$. Show that, if $g'(f(z_0)) \neq 0$ then f must be holomorphic at z_0 and $f'(z_0) = \dfrac{1}{g'(f(z_0))}$.

In 1.9. and 1.10., let $f : [a,b] \longrightarrow \mathbb{C}$ be Riemann integrable and $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a continuous curve. Recall that the function

$$F(z) \stackrel{\text{def}}{=} \int_a^b \frac{f(t)}{\gamma(t) - z}\, dt, \ \forall z \notin \gamma^*,$$

is holomorphic. In fact, if $z_0 \in \mathbb{C} \setminus \gamma^*$ and $r > 0$ such that $D(z_0; r) \cap \gamma^* = \emptyset$, then we see that

$$F(z) = \sum_{n=0}^{\infty} \left(\int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{n+1}}\, dt \right) (z - z_0)^n, \ \forall z \in D(z_0; r).$$

1.9. For any $n \in \mathbb{N}$, define the function $F_n : \mathbb{C} \setminus \gamma^* \longrightarrow \mathbb{C}$ as follows:

$$F_n(z) = \int_a^b \frac{f(t)}{(\gamma(t) - z)^n}\, dt, \ \forall z \notin \gamma^*.$$

Then show that, F_n is holomorphic and

$$F_n'(z) = n \int_a^b \frac{f(t)}{(\gamma(t) - z)^{n+1}}\, dt, \ \forall z \notin \gamma^*.$$

1.10. Consider $z_0 \in \mathbb{C} \setminus \gamma^*$ and $r > 0$ such that $D(z_0; r) \cap \gamma^* = \emptyset$. For any $n \geq 0$, consider the n-th remainder term of the Taylor series of F at z_0, i.e.,

$$R_n(z) \stackrel{\text{def}}{=} F(z) - \sum_{k=0}^{n} \frac{F^{(k)}(z_0)}{k!}(z - z_0)^k, \ \forall z \in D(z_0; r).$$

Show that, for every $z \in D(z_0; r)$,

$$R_n(z) = (z - z_0)^{n+1} \int_a^b \frac{f(t)}{(\gamma(t) - z_0)^{n+1}(\gamma(t) - z)}\, dt.$$

2. Analytic maps on $\mathbb{D}$ and $\mathbb{H}$

In what follows, $\mathbb{D}$ and $\mathbb{H}$ stand for the unit disc $D(0;1)$ and the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ respectively. For $\Omega_{open} \subseteq \mathbb{C}$, by an *(analytic) automorphism* of Ω we mean a bijective holomorphic map from Ω to Ω whose inverse is also holomorphic. It is easy to see that, the set of all automorphisms of Ω forms a group with respect to composition. This group is denoted by $\operatorname{Aut}(\Omega)$.

2.1. Consider the following maps

$$F : \mathbb{H} \longrightarrow \mathbb{C}, \ F(z) \stackrel{\text{def}}{=} \frac{i - z}{i + z},$$

and

$$G : \mathbb{D} \longrightarrow \mathbb{C}, \ G(w) \stackrel{\text{def}}{=} i\frac{1 - w}{1 + w}.$$

Show the following:

(a) F and G are inverse to each other.
(b) Both are holomorphic.

(c) The groups $\mathrm{Aut}(\mathbb{H})$ and $\mathrm{Aut}(\mathbb{D})$ are isomorphic.

2.2. Let $g \overset{\text{def}}{=} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$. Define $gz = \dfrac{az+b}{cz+d}$.

(a) Verify that $gz \in \mathbb{H}$, for all $z \in \mathbb{H}$.
(b) Show that, for any $g \in \mathrm{SL}_2(\mathbb{R})$, the map $\mathbb{H} \longrightarrow \mathbb{H}$, $z \mapsto gz$, is an automorphism of $\mathbb{H}$.

2.3. Let $w \in \mathbb{D}$. Consider the function $\varphi_w : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$, $\varphi_w(z) \overset{\text{def}}{=} \dfrac{w-z}{1-\bar{w}z}$.

(a) Verify that $\varphi_w(z) \in \overline{\mathbb{D}}$, for all $z \in \overline{\mathbb{D}}$.
(b) Show that, φ_w maps $\mathbb{D}$ and $\partial\mathbb{D}$ to $\mathbb{D}$ and $\partial\mathbb{D}$ respectively.
(c) Show that $\varphi_w \in \mathrm{Aut}(\mathbb{D})$.

In what follows, for any $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$, we let gz be as defined above in 2.2.

2.4.$**$ (a) Show that, the following defines an action of the group $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$:

$$(g, z) \mapsto gz, \ \forall (g, z) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{H}. \tag{2.1}$$

(b) Geometrially describe how the following matrices act on a point $z \in \mathbb{H}$:

(i) $\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$, where $a > 0$ (ii) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \in \mathbb{R}$.

(c) Show that the action defined as above in (2.1) is transitive.
(d) Find the stabilizer of the point i under the action defined in (2.1).
(e) Consider the following subgroups:

$$A \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} : a > 0 \right\},$$

$$N \overset{\text{def}}{=} \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{R} \right\},$$

and

$$K \overset{\text{def}}{=} \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Using 2.4.c and 2.4.d, show that, for every $g \in \mathrm{SL}_2(\mathbb{R})$, there exist $a \in A, n \in N$ and $k \in K$ such that $g = nak$. (**Hint:** Do you see that A normalizes N?)
(f) For any $g \in \mathrm{SL}_2(\mathbb{R})$, are the matrices n, a and k obtained in 2.4.e unique?

2.5.$*$ Consider

$$S \overset{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T \overset{\text{def}}{=} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

(a) Find the orders of S, T and ST.
(b) Let $\mathcal{G}$ be the part of either a vertical line or a circle centred on the real axis in $\mathbb{H}$. What can you say about $S(\mathcal{G})$ and $T(\mathcal{G})$? (**Hint:** What are the equations of a line or circle in $\mathbb{C}$?)
(c) Let $D \overset{\text{def}}{=} \left\{ z \in \mathbb{C} : |z| \geq 1 \text{ and } |\mathrm{Re}\, z| \leq \frac{1}{2} \right\}$. Find the image of D under T, S, TS, ST, TS^{-1} and $S^{-1}T$.