Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)
Hints for Exercise Sheet 9

1. Conformal equivalence

1.1. Let $\alpha \in [0,1]$ and $\mathbb{D}_\alpha \stackrel{\text{def}}{=} \mathbb{D} \setminus [\alpha, 1]$.

(a) Show that $\mathbb{D}_\alpha$ is conformally equivalent to $\mathbb{D}_0$.

Hint. *Check with $-\varphi_\alpha$.*

(b) Is $\mathbb{D}_\alpha$ conformally equivalent to the upper-half of the unit disc $\mathbb{D}^+ \stackrel{\text{def}}{=} \mathbb{D} \cap \mathbb{H}$?

Hint. *The map $z \mapsto z^2$ may be useful.*

1.2. Let $f(z) \stackrel{\text{def}}{=} \exp(2\pi i z)$, for all $z \in \mathbb{H}$.

(a) Show that $f(\mathbb{H}) \subseteq \mathbb{D} \setminus \{0\}$.

Sketch of the solution. *Straightforward.*

(b) For $r > 0$, find the image of $\{z \in \mathbb{H} : \operatorname{Im} z > r\}$.

Sketch of the solution. $|z| < \frac{1}{e^{2\pi r}}$.

(c) Is $\{z \in \mathbb{H} : \operatorname{Im} z > r\}$ conformally equivalent to its image under f? If not, what needs to be done so as to obtain a conformal equivalence?

Hint. *Restrictions to appropriate vertical strips might be useful.*

1.3. In each of the following, exhibit a bijective holomorphic map between the given subsets:

(a) The first quadrant $\{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ and $\mathbb{H}$.

Hint. *Check with $z \mapsto z^2$.*

(b) The quarter disc $\{z \in \mathbb{D} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ and $\mathbb{D}^+$.

Hint. *Check with $z \mapsto z^2$.*

(c) $\mathbb{D}^+$ and the first quadrant $\{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\}$.

Hint. *What about the bijective holomorphic map from $\mathbb{D}$ to $\mathbb{H}$?*

(d) The quarter disc $\{z \in \mathbb{D} : \operatorname{Re} z, \operatorname{Im} z > 0\}$ and $\mathbb{H}$.

Hint. *From the quarter disc to half disc $\mathbb{D}^+$, from that to the first quadrant, and finally to $\mathbb{H}$.*

(e) $\mathbb{D}^+$ and the half strip $\{z \in \mathbb{C} : \operatorname{Re} z < 0, 0 < \operatorname{Im} z < \pi\}$.

Hint. *Does $\log_0$ help?*

(f) $\mathbb{H}$ and the strip $\{z \in \mathbb{H} : 0 < \operatorname{Im} z < \pi\}$.

Hint. *Same as before.*

(g) $\{z \in \mathbb{D} : \operatorname{Re} z > 0\}$ and $\mathbb{D}$.

Hint. $\{z \in \mathbb{D} : \operatorname{Re} z > 0\} \longrightarrow \mathbb{D}^+ \longrightarrow \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z > 0\} \longrightarrow \mathbb{H} \longrightarrow \mathbb{D}$.

(h) $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$ and $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$, where r_1, r_2, R_1 and $R_2 > 0$ and $\frac{r_1}{r_2} = \frac{R_1}{R_2}$.

Hint. $z \mapsto \frac{r_2}{r_1} z$.

1.4.* (a) Let $\alpha \in [0, \pi]$. Show that $\mathbb{H}\backslash\left\{e^{it} : t \in [0, \alpha]\right\}$ is conformally equivalent to $\mathbb{H}\backslash\left\{it : 0 \le t \le \frac{1}{2}\tan\frac{\alpha}{2}\right\}$.

Hint. *Get an automorphism of $\mathbb{H}$ that maps the half circle $\{z \in \mathbb{H} : |z| = 1\}$ to the vertical line $\{it : t > 0\}$.*

(b) Let $\beta \ge 0$. Show that $\{z \in \mathbb{C} : \operatorname{Re} z > 0\} \setminus [0, \beta]$ is confornmally equivalent to $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Hint. *What is the image of $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ under the map $z \mapsto z^2$? Can you use an analytic square root function?*

(c) Show that, for any $a > 0$, $\mathbb{H} \setminus \{it : 0 \le t \le a\}$ is conformally equivalent to the right half plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Hint. *Use 1.4.a and 1.4.b.*

1.5. Show that the following map establishes a conformal equivalence between $\{z \in \mathbb{H} : |z| > 1\}$ and $\mathbb{H}$:

$$f : \{z \in \mathbb{H} : |z| > 1\} \longrightarrow \mathbb{H},\ f(z) \stackrel{\text{def}}{=} z + \frac{1}{z}.$$

Hint. *Show that, for every $w \in \mathbb{H}$, the quadratic equation $z^2 - wz + 1 = 0$ has a unique root in $\{z \in \mathbb{H} : |z| > 1\}$.*

2. Families of analytic functions

Recall that, for $U \subseteq_{open} \mathbb{C}$, one has $U = \bigcup_{n=1}^{\infty} K_n$, where

$$K_n \stackrel{\text{def}}{=} \overline{D(0; n)} \cap \left\{z \in U : |w - z| \ge \frac{1}{n},\ \forall w \in \mathbb{C} \setminus U\right\}.$$

These compact sets K_n's have the following properties:

(i) For all $n \in \mathbb{N}$, K_n is contained in the interior of K_{n+1}.
(ii) For every compact subset K of U, there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$.

Let $C(U)$ denote the set of all complex valued continuous functions on U. For $f, r \in C(U)$, define

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}\right), \tag{2.1}$$

where for any $n \in \mathbb{N}$,

$$\|f - g\|_{K_n} \stackrel{\text{def}}{=} \begin{cases} \sup_{z \in K_n} |f(z) - g(z)| & \text{if } K_n \ne \emptyset \\ 0 & \text{if } K_n = \emptyset \end{cases}.$$

2.1. Show that d, defined as above in (2.1), is a metric on $C(U)$.

Hint. *If $a, b, c \ge 0$ with $a \le b + c$ then $\frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}$. This can be seen considering the function $\frac{x}{1+x}$, for all $x \ge 0$.*

2.2. Show that the metric d on $C(U)$ is bounded.

Sketch of the solution. *For any $f, g \in C(U)$, it is clear that $d(f, g) \le \sum_{n=1}^{\infty} \frac{1}{2^n} \le 1$.*

From now on, unless otherwise mentioned, $C(U)$ is will always be endowed with the metric d.

2.3.* Let $\{f_n\}_{n=1}^{\infty}$ in $C(U)$ be a sequence in $C(U)$.

(a) Show that $\{f_n\}_{n=1}^{\infty}$ is convergent (with respect to the metric d) if and only if it is uniformly convergent on each compact subset of U.

(b) Show that $\{f_n\}_{n=1}^{\infty}$ is Cauchy (with respect to the metric d) if and only if it is uniformly Cauchy on each compact subset of U.

Sketch of the solution. *This can be proved exactly in the similar way to that of 2.3.a.*

2.4. (a) Show that $C(U)$ is a complete metric space.

Sketch of the solution. *Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C(U)$. Then it is uniformly Cauchy on every compact subset of U. Hence it converges uniformly on every compact subset of U. Because of the uniform convergence, the limit function has to be continuous. Thus $\{f_n\}_{n=1}^{\infty}$ converges in the metric space $C(U)$.*

(b) Show that $H(U)$ is closed in $C(U)$.

Hint. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $C(U)$. Then it is uniformly convergent on compact subset of U. Hence the limit function is holomorphic. Thus $\{f_n\}_{n=1}^{\infty}$ converges in the metric space $H(U)$.*

(c) Conclude that $H(U)$ is a complete metric space.

2.5. Show that a sequence $\{f_n\}_{n=1}^{\infty}$ in $H(\mathbb{D})$ converges to f if and only if $\displaystyle\int_{C(0;r)} |f_n(z) - f(z)|\,|dz| \xrightarrow[n\to\infty]{} 0$, for all $0 < r < 1$.

Sketch of the solution. *The 'only if' direction is straightforward. Assume that, for all $0 < r < 1$, $\displaystyle\int_{C(0;r)} |f_n(z) - f(z)|\,|dz| \xrightarrow[n\to\infty]{} 0$. Let $r > 0$. We show that $f_n \xrightarrow[n\to\infty]{} f$ uniformly on the closed disc $\overline{D(0;r)}$. Choose $\rho \in (r,1)$. For any $n \in \mathbb{N}$, we have*

$$\begin{aligned}
|f_n(z) - f(z)| &\le \frac{1}{2\pi}\left|\int_{C(0;\rho)} \frac{f_n(w) - f(w)}{w - z}\,dw\right| \\
&\le \frac{1}{2\pi}\int_{C(0;\rho)} \left|\frac{f_n(w) - f(w)}{w - z}\right|\,|dw| \\
&\le \frac{1}{2\pi(\rho - r)}\int_{C(0;\rho)} |f_n(w) - f(w)|\,|dw| \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

2.6. Let U and V are open subsets of $\mathbb{C}$.

(a) Suppose $\varphi : U \longrightarrow V$ is a bijective holomorphic map. Show that, if $\mathscr{F} \subseteq H(V)$ is relatively compact, then so is $\{f \circ \varphi : f \in \mathscr{F}\}$.

Sketch of the solution. *Straightforward. Use that the image of any compact subset under φ is compact.*

(b)$*$ Let $\mathscr{F} \subseteq H(U)$ be relatively compact. Assume that $f(U) \subseteq V$, for all $f \in \mathscr{F}$. Show that, for any $g \in H(V)$, $\{g \circ f : f \in \mathscr{F}\}$ is relatively compact.

Sketch of the solution. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{F}$. It has a convergent subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ converging to $f \in H(U)$. Suppose that $C \subseteq U$ is compact. It follows from the compactness of $f(C)$ that there exists $\eta > 0$ such that $\mathcal{K} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : d(z, f(C)) \le \eta\} \subseteq V$ (why?). It is easy to see that $\mathcal{K}$ is compact since it is closed and bounded. From the uniform convergence of $\{f_{n_k}\}_{k=1}^{\infty}$ to f on C, one obtains an $N \in \mathbb{N}$ such that, for all $k \ge N$, $|f_{n_k}(z) - f(z)| < \eta$, whenever $z \in C$. This shows that, for all $k \ge N$, $f_{n_k}(C) \subseteq \mathcal{K}$. Thus we obtain a compact set, namely $\mathscr{K} \stackrel{\text{def}}{=} \left(\cup_{k=1}^{N} f_{n_k}(C)\right) \cup \mathcal{K}$ that contains all $f_{n_k}(C)$'s and $f(C)$. Let $\varepsilon > 0$. Sine g is uniformly continuous on $\mathscr{K}$, there exists $\delta > 0$ such that $|g(z_1) - g(z_0)| < \varepsilon$, whenever $z_1, z_2 \in \mathscr{K}$ and $|z_1 - z_2| < \delta$. Since $\{f_{n_k}\}_{k=1}^{\infty}$ converges uniformly to f on C, there exists $n_0 \in \mathbb{N}$ such that, for* *all $k \geq n_0$ and $z \in C$, $|f_{n_k}(z) - f(z)| < \delta$. It now follows at once that, for all $k \geq n_0$ and $z \in C$, $|g(f_{n_k}(z)) - g(f(z))| < \varepsilon$.*

2.7. Let $\mathscr{F} \overset{\text{def}}{=} \{f \in H(\mathbb{D}) : \operatorname{Re} f > 0 \text{ and } |f(0)| \leq 1\}$. Show that $\mathscr{F}$ is relatively compact, but not compact.

Sketch of the solution. *Use Exercise 1.7 of Exercise Sheet 8 and Montel's theorem. To see this family is not closed, consider the seqeunce $f_n(z) \overset{def}{=} \frac{n}{n+1}z$, for all $z \in \mathbb{D}$.*

2.8. Let $U \subseteq \mathbb{C}$ be a region, $w \in \mathbb{C}$ and $r > 0$. Consider $\mathscr{F} \overset{\text{def}}{=} \{f \in H(U) : |f(z) - w| \geq r, \forall z \in U\}$. Show that for any sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathscr{F}$, one has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which either converges (in $H(U)$) to some $f \in H(U)$ or diverges to ∞ uniformly on every compact subset of U.

Sketch of the solution. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{F}$. For any $n \in \mathbb{N}$, $g_{n_k}(z) \overset{def}{=} \frac{1}{f_{n_k}(z)-w}$, for all $z \in U$. Clearly the sequence $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded on U, hence it admits a convergent subsequence, say $\{g_{n_k}\}_{k=1}^{\infty}$ in $H(U)$. Assume that $g_{n_k} \xrightarrow[k\to\infty]{} g$ uniformly on every compact subset of U. Since all g_{n_k}'s are zero-free, either g is zero-free or $g \equiv 0$. In the former case, show that $f_{n_k} \xrightarrow[k\to\infty]{} \frac{1}{g} + w$ uniformly on every compact subset of U, while in the latter it is easy to see that $f_{n_k} \xrightarrow[k\to\infty]{} \infty$ unifomrly on every compact subset of U.*

2.9. Let $U \subseteq_{open} \mathbb{C}$ and $\mathscr{F} \subseteq H(U)$. Denote $\mathscr{F}' \overset{\text{def}}{=} \{f' : f \in \mathscr{F}\}$.

(a) Show that, if $\mathscr{F}$ is relatively compact, then so is $\mathscr{F}'$.
(b) Is the converse of 2.9.a true?
(c)$*$ Prove the converse of 2.9.a when U is an open disc under the additional hypothesis that there exists $z_0 \in U$ such that the set $\{f(z_0) : f \in \mathscr{F}\}$ is bounded.

Hint. *Let $U = D(a; R)$. One can choose a convergent subsequence $\{f_{n_k}\}_{n=1}^{\infty}$ in such a way that $\{f_{n_k}(z_0)\}_{n=1}^{\infty}$ also converges. Show that $\{f_{n_k}\}_{n=1}^{\infty}$ is uniformly Cauchy on $\overline{D(a; r)}$, for every $0 < r < R$.*

(d)$*$ Let the additional assumption be as above in 2.9.c. Denote by V the set of all $z \in U$ such that $\{f|_{D(z_0;r)} : f \in \mathscr{F}\}$ is relatively compact in $H(D(z_0; r))$, for some $r > 0$. Show that V is nonempty and both open and closed in U.
(e)$*$ Assume that U is connected. Prove the converse of 2.9.a under the additional assumption mentioned in 2.9.c.

2.10. Show that $\mathscr{F} \subseteq H(\mathbb{D})$ is relatively compact if and only if there exists a sequence $\{M_n\}_{n=0}^{\infty}$ of nonnegative reals such that $\limsup_{n\to\infty} M^{\frac{1}{n}} \leq 1$ and $\left|\frac{f^{(n)}(0)}{n!}\right| \leq M_n$, for all $f \in \mathscr{F}$ and $n = 0, 1, 2, \ldots$.

2.11.$*$ Let $U \subseteq_{open} \mathbb{C}$ and $L : H(U) \longrightarrow \mathbb{C}$ is a linear map. Assume that L is *multiplicative*, i.e., $L(fg) = L(f)L(g)$, for all $f, g \in H(U)$. Suppose that L is nonzero.

(a) Show that, if $f \equiv 1$, then $L(f) = 1$.
(b) Denote the identity map on U by I. Show that $L(I) \in U$.

Hint. *Assume $z_0 \overset{def}{=} L(I) \notin U$. Then the function $I - z_0$ is nowhere vanishing, so that you can consider the holomorphic function $\frac{1}{I-z_0}$ on U.*

(c) Show that, for every $f \in H(U)$, $L(f) = f(z_0)$.

Hint. *Consider $g : U \longrightarrow \mathbb{C}$, $g(z) \overset{def}{=} \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0. \end{cases}$ Is g analytic? $g(I - z_0)$ might be useful.*

(d) Find all linear maps from $H(U)$ to $\mathbb{C}$ that are multiplicative.

3. Riemann mapping theorem

3.1. Let U be a nonempty proper simply connected region in $\mathbb{C}$ and $z_0 \in U$. If f is the Riemann map from U to $\mathbb{D}$, i.e., f is bijective, holomorphic, $f(z_0) = 0$ and $f'(z_0) > 0$. Express any arbitrary bijective holomorphic map $g : U \longrightarrow \mathbb{D}$ in terms of f.

3.2. Let U and V be nonempty proper simply connected open subsets of $\mathbb{C}$. Show that, for any $z_1 \in U$ and $z_2 \in V$, there exists a unique bijective holomorphic map $f : U \longrightarrow V$ such that $f(z_1) = z_2$ and $f'(z_1) > 0$.

Hint. *Use existence and uniqueness of Rieman map.*

3.3. Let U, V, z_1 and z_2 be as above in 3.2. Suppose that $g : U \longrightarrow V$ is a bijective holomorphic map with $g(z_1) = z_2$ and $h : U \longrightarrow V$ be any holomorphic map satisfying $h(z_1) = z_2$. Show that $|h'(z_1)| \leq |g'(z_1)|$. What about the equality case?

Hint. *Let $\varphi : V \longrightarrow \mathbb{D}$ be a bijective holomorphic map sending z_2 to 0. Now work with $\varphi \circ g$ and $\varphi \circ h$.*

3.4. Let $U, V \subseteq \mathbb{C}$ be open and connected. Assume further that $V \neq \mathbb{C}$ and it is simply connected. Show that the family $\{f \in H(U) : f(U) \subseteq V\}$ is relatively compact.