Department of Mathematics

Calculus of Several Variables and Differential Geometry

ASSIGNMENT-II

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and $f(0) = 0$. Show that every point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ around 0 can be written as $f(x) = \sum_{i=1}^n x_i f_i(x)$ where f_i's are continuous around 0.

2. Write the first two terms of the Taylor expansion of the following functions.
   (a) $f(x, y) = \frac{1}{x^2+y^2+1}$ around $(0, 0)$.
   (b) $f(x, y) = \sin(xy)$ around $(0, 0)$.
   (c) $f(x, y) = \frac{xy}{1-x-y}$ around $(0, 0)$.

3. Find the points where the function $f(x, y) = (x^2 + y^2)e^{-(x^2+y^2)}$ attains its (local) maxima/minima.

4. Find the critical points of the following functions and classify them.
   (a) $f(x, y) = xy(x + y)$.
   (b) $f(x, y) = x \sin y$.
   (c) $f(x, y) = (x^2 + y^2)e^{x^2-y^2}$.
   (d) $f(x, y) = \log(2 + \sin(xy))$.
   (e) $f(x, y) = (x^2 + 3y^2)e^{1-(x^2+y^2)}$.

5. Show that origin is a critical point of $f(x, y) = (y - 3x^2)(y - x^2)$ and the function f has a local minimum along every straight line passing through origin. However origin is not a minimum for the function f.

6. **Lagrange Multipliers**
   (a) Show that for any vector a in $\mathbb{R}^n$, $\| a \| = \max\{\langle a, x\rangle : \| x \| = 1\}$.
   (b) Let $n \geq 2$ and $g : \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$. Find the extrema of g on the set $S := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_k \geq 0 \text{ for } 1 \leq k \leq n \text{ and } \sum_{k=1}^n x_k = 1\}$.
   (c) Find the maximum of $g(x) = x_1^2 x_2^2 \cdots x_n^2$ subject to the constraint $\sum_{i=1}^n x_i^2 = 1$.
   (d) Find the extrema of the function $g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$ subject to the constraint $f(x) = x_1 x_2 \cdots x_n = 1$.
   (e) Let $A = (a_{ij}) \in M(n, \mathbb{R})$. Let $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and $d_i = \| A_i \|$. Show that $\det(A)^2 \leq d_1^2 d_2^2 \cdots d_n^2$.