ESO 209: Probability and Statistics
2011-2012: II Semester
End Semester Examination
Instructor: Neeraj Misra

Time Allowed: 180 Minutes **Maximum Marks: 100**

NOTE: (i) Attempt all the parts of a question at one place.
(ii) For each question, provide details of all the steps involved in arriving at the final answer(s)/conclusion(s) and box your final answer(s)/conclusion(s).

1. (a) Let X be a random variable of absolutely continuous type with probability density function

$$f_X(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2}e^{-(x-\frac{1}{2})}, & \text{if } x > \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Find the distribution function of X. $\boxed{\text{5 MARKS}}$

(b) Let A, B and C be events with $P(A) > 0$ and $P(B \cap C) > 0$. Show that

$$P(A \mid B \cap C)P(B \mid C) \geq \frac{P(A) + P(B) + P(C) - 2}{P(C)}.$$

$\boxed{\text{5 MARKS}}$

(c) A number, denoted by X_1, is chosen at random from the set of integers $\{1, \ldots, 5\}$. Then a second number, denoted by X_2, is chosen at random from the set $\{1, \ldots, X_1\}$. Find $P(X_2 \in \{4, 5\})$. $\boxed{\text{5 MARKS}}$

2. Let X be a random variable with distribution function

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{4}(2 - e^{-x}), & \text{if } 0 \leq x < 1, \\ \frac{1}{4}(\frac{5}{2} - e^{-x}), & \text{if } 1 \leq x < 2, \\ \frac{1}{4}(4 - e^{-x}), & \text{if } x \geq 2. \end{cases}$$

(a) Show that X is neither of discrete type nor of absolutely continuous type. $\boxed{\text{5 MARKS}}$

(b) Find the decomposition $F_X(x) = \alpha F_d(x) + (1 - \alpha)F_c(x), x \in \mathbb{R}$, where $F_d(x)$ and $F_c(x)$ are distribution functions of some random variables of discrete type and absolutely continuous type respectively and $\alpha \in [0, 1]$. $\boxed{\text{7 MARKS}}$

(c) Calculate $P(X < 2 \mid X \geq 1)$. $\boxed{\text{3 MARKS}}$

3. (a) Suppose that a retailer has 50 pens in stock and let X denote the number of pens, that the retailer sells in a week. Assume that X follows the poisson distribution with mean 20 and suppose that retailer makes a profit of Rs. 1/- on each pen sold. Let Y denote the net profit made by the retailer during a week. Find the distribution function of Y. Hence, find the p.m.f. of Y. $\boxed{\text{5 MARKS}}$

(b) Let $\Omega = \{(i, j) : i = 1, 2; j = 1, 2\}$ and $\mathcal{A} = \{\phi, \Omega, \{(1, 2)\}, \{(1, 1), (2, 1), (2, 2)\}\}$ be a sigma field of subsets of Ω. Verify whether $X(i, j) = i + j; (i, j) \in \Omega$, is a random variable with respect to $\mathcal{A}$. $\boxed{\text{5 MARKS}}$

4. Let X_1 and X_2 be two independent random variables having uniform distribution on $[0,1]$.

(a) Find the distribution function of $Y = \max\left(\frac{X_2}{X_1}, \frac{X_1}{X_2}\right)$. Hence, find the p.d.f. of Y. $\boxed{\text{9 MARKS}}$

(b) Calculate $E(\mid X_1 - X_2 \mid)$. $\boxed{\text{6 MARKS}}$

5. Let $\underline{X} = (X, Y, Z)$ be an absolutely continuous type random vector having the joint p.d.f.

$$f_{\underline{X}}(x,y,z) = \begin{cases} ke^{-(x+y+z)}, & \text{if } 0 < x < y < z < \infty \\ 0, & \text{otherwise} \end{cases},$$

(a) Find the value of k. $\boxed{\text{2 MARKS}}$

(b) Find the marginal p.d.f of Y. $\boxed{\text{3 MARKS}}$

(c) Find the joint p.d.f. of (X, Z). $\boxed{\text{5 MARKS}}$

(d) Find the conditional expectation $E(e^Y \mid (X, Z) = (1, 2))$. $\boxed{\text{5 MARKS}}$

6. (a) Suppose that $X \sim N(0,1)$. Find $E(\mid X \mid)$. $\boxed{\text{3 MARKS}}$

(b) Let the random vector $\underline{X} = (X, Y) \sim N_2(0, 0, 1, 1, \rho)$. Find $E(\mid X - Y \mid)$ and $\text{Corr}(X - Y, \mid X - Y \mid)$. $\boxed{\text{7 MARKS}}$

(c) Let (X, Y) be a two dimensional random variable with p.m.f.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{55}, & \text{if } y = 1, 2, \ldots, x \text{ and } x = 1, 2, \ldots, 10, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional expectation $E(X \mid Y = 2)$. $\boxed{\text{5 MARKS}}$

7. Suppose that $\underline{X} = (X_1, X_2, X_3, X_4, X_5) \sim \text{Mult}(5, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

(a) Calculate $P(X_1 + X_2 + X_3 \geq 2)$. $\boxed{\text{5 MARKS}}$

(b) Calculate $E(e^{X_1 + 2X_2})$. $\boxed{\text{5 MARKS}}$

(c) Let X and Y be jointly distributed random variables such that $E(X) = 3, E(Y) = 4, E(X^2) = 15, E(Y^2) = 22$, and $E(XY) = 13$. Show that $P(Y = X + 1) = 1$. $\boxed{\text{5 MARKS}}$

(ALL THE BEST)

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Model Solutions

Problem-No. 1

(a) When $x < 0$, $F_X(x) = 0$

When $0 \le x < \frac{1}{2}$, $F_X(x) = \int_0^x 1\, dy = x$

When $\frac{1}{2} \le x < \infty$, $F_X(x) = \int_0^{1/2} 1\, dy + \int_{1/2}^{x} \frac{1}{2} e^{-(y-\frac{1}{2})} dy$

$$= 1 - \frac{1}{2} e^{-(x-\frac{1}{2})}$$

Hence, the distribution function of X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \quad \text{.........} \boxed{\text{1 MARK}} \\ x & \text{if } 0 \le x < \frac{1}{2} \quad \text{---} \boxed{\text{2 MARKS}} \\ 1 - \frac{1}{2} e^{-(x-\frac{1}{2})} & \text{if } x \geq \frac{1}{2} \quad \text{.....} \boxed{\text{2 MARK}} \end{cases}$$

(b) $\text{L.H.S} = P(A \mid B \cap C) \cdot P(B \mid C)$

$$= \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)}$$

$$= \frac{P(A \cap B \cap C)}{P(C)} \quad \text{---} \boxed{\text{2 MARKS}}$$

$$\geq \frac{P(A) + P(B) + P(C) - 2}{P(C)} = \text{R.H.S} \quad \text{---} \boxed{\text{3 MARKS}}$$

(C) The p.m.f of X_1 is

$$P\{X_1 = K\} = \begin{cases} \frac{1}{5} & \text{if } K = 1, 2, \ldots, 5 \\ 0 & \text{o/w} \end{cases}$$

For $K \in \{1, 2, \ldots, 5\}$, the Conditional p.m.f of X_2 given $X_1 = K$ is

$$P\{X_2 = j \mid X_1 = K\} = \begin{cases} \frac{1}{K} & \text{if } j = 1, 2, \ldots K \\ 0 & \text{o/w} \end{cases}$$

$$P\{X_2 = 4\} = \sum_{K=1}^{5} P\{X_2 = 4, X_1 = K\}$$

$$= \sum_{K=1}^{5} P\{X_2 = 4 \mid X_1 = K\} \cdot P\{X_1 = K\}$$

$$= \sum_{K=4}^{5} P\{X_2 = 4 \mid X_1 = K\} \cdot P\{X_1 = K\}$$

$$= \frac{1}{20} + \frac{1}{25} = \frac{9}{100} \quad \text{---} \quad \boxed{\text{2 MARKS}}$$

$$P\{X_2 = 5\} = \sum_{K=1}^{5} P\{X_2 = 5, X_1 = K\}$$

$$= \sum_{K=1}^{5} P\{X_2 = 5 \mid X_1 = K\} \cdot P\{X_1 = K\}$$

$$= P\{X_2 = 5 \mid X_1 = 5\} \cdot P\{X_1 = 5\}$$

$$= \frac{1}{25} \quad \text{-----} \quad \boxed{\text{2 MARKS}}$$

Hence, $P(X_2 \in \{4, 5\}) = P(X_2 = 4) + P(X_2 = 5)$

$$= \frac{13}{100} \quad \text{---} \quad \boxed{\text{1 MARKS}}$$

$\boxed{\text{Problem No-2}}$

$\boxed{(a)}$ $F_X(0) - F_X(0^-) = \frac{1}{4} - 0 = \frac{1}{4} > 0$

$\Rightarrow F_X(x)$ is not a Continuous function.

$\Rightarrow X$ is not an absolutely Continuous type random Variable.

--- $\boxed{\text{2 MARKS}}$

It has jumps at $x \in \{0, 1, 2\}$.

$P\{X=0\} = F_X(0) - F_X(0^-) = \frac{1}{4}$

$P\{X=1\} = F_X(1) - F_X(1^-) = 1/8$

$P\{X=2\} = F_X(2) - F_X(2^-) = 3/8$

Sum of jump size $= \frac{1}{4} + \frac{1}{8} + \frac{3}{8} = \frac{3}{4} < 1$

$\Rightarrow X$ is not a discrete type random variable

— $\boxed{\text{3 MARKS}}$

$\boxed{(b)}$ The set of discontinuity, $D_X = \{0, 1, 2\}$

$p_1 = P\{X=0\} = \frac{1}{4}$, $p_2 = P\{X=1\} = \frac{1}{8}$, $p_3 = P\{X=2\} = \frac{3}{8}$

$\alpha = p_1 + p_2 + p_3 = 3/4$ ----- $\boxed{\text{1 MARKS}}$

$P\{X_d = 0\} = \frac{p_1}{\alpha} = \frac{1}{3}$

$P\{X_d = 1\} = \frac{p_2}{\alpha} = 1/6$

$P\{X_d = 2\} = p_3/\alpha = \frac{1}{2}$

Thus,

$$F_d(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{3} & \text{if } 0 \le x < 1 \\ \frac{1}{2} & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

---- $\boxed{\text{3 MARKS}}$

and

$$F_c(x) = \frac{F_X(x) - \alpha F_d(x)}{(1-\alpha)}$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ 1-e^{-x} & \text{if } x \geqslant 0 \end{cases} \quad \text{- - -} \boxed{\text{3 MARKS}}$$

$\boxed{\text{(C)}}$

$$P(X<2 \mid X \geqslant 1) = \frac{P(X<2, X\geqslant 1)}{P(X \geqslant 1)} = \frac{P(1 \leq X < 2)}{P(X \geqslant 1)}$$

$$= \frac{F_X(2^-) - F_X(1^-)}{1 - F_X(1^-)}$$

$$= \frac{\frac{1}{2} - (e^{-2} - e^{-1})}{2 + e^{-1}}$$

$\boxed{\text{3 MARKS}}$

Problem No-3

(a) X : # of pens that the retailer sells in a week, $X \sim P(20)$.

Y denote the net profit during a week.

i.e., $Y = 1 \cdot \min(X, 50) = \min(X, 50)$

--- 1 MARKS

The support of Y is $S_Y = \{0, 1, 2, \dots 50\}$

For $y < 0$, $F_Y(y) = 0$,

For $0 \le y < 50$,

$$
\begin{aligned}
F_Y(y) &= P\{Y \le y\} \\
&= P\{\min(X, 50) \le y\} \\
&= P\{X \le y\} \\
&= \sum_{K=0}^{[y]} \frac{e^{-20}\, 20^K}{\lfloor K},
\end{aligned}
$$

For $y \geq 50$, $F_Y(y) = 1$

Hence, the distribution function of Y is

$$
F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \displaystyle\sum_{K=0}^{[y]} \frac{e^{-20}\, 20^K}{\lfloor K} & \text{if } 0 \le y < 50 \\ 1 & \text{if } y \geq 50 \end{cases}
$$

--- 2 MARKS

The p.m.f of Y is

$$f_Y(y) = P\{Y=y\}$$
$$= F_Y(y) - F_Y(y^-)$$
$$= \begin{cases} \dfrac{e^{-20}\, 20^y}{\lfloor y} & \text{if } y \in \{0,1,2,\cdots 49\} \\ 1 - \displaystyle\sum_{j=0}^{49} \dfrac{e^{-20}\, 20^j}{\lfloor j} & \text{if } y = 50 \\ 0 & \text{o/w} \end{cases}$$

— $\boxed{\text{2 MARKS}}$

$\boxed{\text{(b)}}$

$$X(i,j) = \begin{cases} 2 & \text{if } (i,j) = (1,1) \\ 3 & \text{if } (i,j) \in \{(1,2),(2,1)\} \\ 4 & \text{if } (i,j) = (2,2) \end{cases}$$

$$X^{-1}((-\infty,a]) = \begin{cases} \phi & \text{if } a < 2 \\ \{(1,1)\} & \text{if } 2 \le a < 3 \\ \{(1,1),(1,2),(2,1)\} & \text{if } 3 \le a < 4 \\ \Omega & \text{if } a \geq 4 \end{cases}$$

$\boxed{\text{3 MARKS}}$

$\{(1,1)\} \notin \mathcal{A} \Rightarrow X$ is not a random variable

$\boxed{\text{2 MARKS}}$

$\boxed{\text{Problem No - 4}}$

$\boxed{(a)}$ $F_Y(y) = P\{Y \le y\}$

$$= P\left\{\max\left(\frac{X_2}{X_1}, \frac{X_1}{X_2}\right) \le y\right\}$$

Clearly for $y \le 0$, $F_Y(y) = 0$

For $y > 0$,

$$F_Y(y) = P\left\{\frac{X_2}{X_1} \le y, \frac{X_1}{X_2} \le y\right\}$$

$$= P\left\{\frac{X_2}{y} \le X_1 \le y X_2, \frac{X_2}{y} \le y X_2\right\}$$

Clearly, for $0 < y < 1$, $F_Y(y) = 0$

Thus, for $y < 1$, $F_Y(y) = 0$ --- $\boxed{\text{2 MARKS}}$

For $y \geq 1$,

$$F_Y(y) = P\left\{\frac{X_2}{y} \le X_1 \le y X_2\right\}$$

$$= \int_0^1 P\left\{\frac{X_2}{y} \le X_1 \le y X_2 \,/\, X_2 = x_2\right\} f_{X_2}(x_2)\, dx_2$$

$$= \int_0^1 P\left\{\frac{x_2}{y} \le X_1 \le y x_2\right\} f_{X_2}(x_2)\, dx_2$$

--- $\boxed{\text{2 MARKS}}$

$$F_{X_1}(x_1) = \begin{cases} 0 & \text{if } x_1 < 0 \\ x_1 & \text{if } 0 \le x_1 < 1 \\ 1 & \text{if } x_1 \geq 1. \end{cases}$$

For $y \geq 1$, and $x_2 \in [0, 1]$

$$P\left\{\frac{x_2}{y} \le X_1 \le y x_2\right\} = F_{X_1}(y x_2) - F_{X_1}\left(\frac{x_2}{y}\right)$$

$$P\left\{\frac{x_2}{y} \le X_1 \le x_2 y\right\} = \begin{cases} \left(x_2 y - \frac{x_2}{y}\right) & \text{if } 0 \le x_2 < \frac{1}{y} \\ \left(1 - \frac{x_2}{y}\right) & \text{if } \frac{1}{y} \le x_2 \le 1 \end{cases}$$

Thus, For $y \geq 1$,

$$F_Y(y) = \int_0^{1/y} \left(x_2 y - \frac{x_2}{y}\right) dx_2 + \int_{1/y}^{1} \left(1 - \frac{x_2}{y}\right) dx_2$$

$$= 1 - \frac{1}{y} \, ,$$

Hence, the distribution function of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 1 \\ 1 - \frac{1}{y} & \text{if } y \geq 1 \end{cases}$$ --- $\boxed{\text{3 MARKS}}$

The p.d.f of Y is

$$f_Y(y) = \begin{cases} \frac{1}{y^2} & \text{if } y \geq 1 \\ 0 & \text{o/w} \end{cases}$$ --- $\boxed{\text{2 MARKS}}$

$\boxed{\text{(b)}}$ $$E(|X_1 - X_2|) = \int_0^1 \int_0^1 |x_1 - x_2| \, dx_1 \, dx_2$$

$$= \iint_{0 < x_1 < x_2 < 1} |x_1 - x_2| \, dx_1 \, dx_2 + \iint_{0 < x_2 < x_1 < 1} |x_1 - x_2| \, dx_1 \, dx_2$$

$$= 2 \iint_{0 < x_1 < x_2 < 1} |x_1 - x_2| \, dx_1 \, dx_2$$ --- $\boxed{\text{3 MARKS}}$

$$E(|X_1 - X_2|) = 2\int_0^1 \left[\int_0^{x_2} (x_2 - x_1)\, dx_1 \right] dx_2$$

$$= 2\int_0^1 \frac{x_2^2}{2}\, dx_2$$

$$= \frac{1}{3}$$ —— $\boxed{\text{3 MARKS}}$

Problem No = 5

(a) We know that (i) $f_{\underline{X}}(\underline{x}) \geq 0 \quad \forall\, \underline{x}$

$$\Rightarrow \quad K \geq 0$$

(ii) $\iiint f_{\underline{X}}(\underline{x})\, d\underline{x} = 1$

$$\Rightarrow \int_0^\infty \int_x^\infty \int_y^\infty K e^{-(x+y+z)}\, dz\, dy\, dx = 1$$

$$\Rightarrow K \int_0^\infty \int_x^\infty e^{-x} e^{-2y}\, dy\, dx = 1$$

$$\Rightarrow \frac{K}{2} \int_0^\infty e^{-3x}\, dx = 1$$

$$\Rightarrow \boxed{K = 6} \quad ---- \quad \boxed{\text{2 MARKS}}$$

(b) The marginal p.d.f of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{X}}(x, y, z)\, dx\, dz, \quad y \in \mathbb{R}$$

$$= \begin{cases} \int_0^y \int_y^\infty 6 e^{-(x+y+z)}\, dz\, dx & \text{if } y > 0 \\ 0 & \text{o/w} \end{cases}$$

$$= \begin{cases} 6(1-e^{-y})e^{-2y} & \text{if } y > 0 \\ 0 & \text{o/w} \end{cases} \quad --- \quad \boxed{\text{3 MARKS}}$$

$\boxed{(c)}$ The joint p.d.f of (X, Z) is

$$f_{X,Z}(x,z) = \int_{-\infty}^{\infty} f_{\underline{X}}(x,y,z)\, dy, \quad (x,z) \in \mathbb{R}^2.$$

For $0 < x < z < \infty$

$$f_{X,Z}(x,z) = \int_x^z 6\, e^{-(x+y+z)}\, dy$$

$$= 6\, e^{-(x+z)} \left(e^{-x} - e^{-z}\right)$$

Hence,

$$f_{X,Z}(x,z) = \begin{cases} 6\, e^{-(x+z)} \left(e^{-x} - e^{-z}\right) & \text{if } 0 < x < z < \infty \\ 0 & \text{o/w} \end{cases}$$

—— $\boxed{\text{5 MARKS}}$

$\boxed{(d)}$ Let $0 < x < z < \infty$. Then the Conditional p.d.f of Y given $(X, Z) = (x, z)$ is

$$f_{Y|(X,Z)}(y|x,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{(X,Z)}(x,z)}$$

$$= \begin{cases} \dfrac{e^{-y}}{(e^{-x} - e^{-z})} & \text{if } x < y < z \\ 0 & \text{o/w} \end{cases}$$

Hence,

$$f_{Y|(X,Z)}\big(y|(1,2)\big) = \begin{cases} \dfrac{e^{-y}}{(e^{-1}-e^{-2})} & \text{if } 1<y<2 \\ 0 & , \ \text{o/w} \end{cases}$$

--- $\boxed{\text{3 MARKS}}$

$$E\big(e^{Y}|(X,Z)=(1,2)\big) = \int_1^2 e^{y}\cdot\frac{e^{-y}}{(e^{-1}-e^{-2})}\,dy$$

$$= \frac{1}{(e^{-1}-e^{-2})}$$

$$= \frac{e^2}{e-1}$$

--- $\boxed{\text{2 MARKS}}$

Problem No - 6

(a) $X \sim N(0,1)$

$$E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x e^{-\frac{x^2}{2}} dx \text{ -- } \boxed{\text{1 MARKS}}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} dt$$

$$= \sqrt{\frac{2}{\pi}} \text{ ----- } \boxed{\text{2 MARKS}}$$

(b) $(X, Y) \sim N_2(0, 0, 1, 1, \rho)$

let $X_1 = X - Y \sim N(0, 2(1-\rho))$ -- $\boxed{\text{2 MARKS}}$

Define $Z = \frac{X_1}{\sqrt{2(1-\rho)}} \sim N(0,1)$

$$\Rightarrow E(|Z|) = E\left(\left|\frac{X_1}{\sqrt{2(1-\rho)}}\right|\right)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} = E\left(\frac{|X_1|}{\sqrt{2(1-\rho)}}\right)$$

$$\Rightarrow E(|X_1|) = 2\sqrt{\frac{1-\rho}{\pi}} \text{ --- } \boxed{\text{2 MARKS}}$$

$$Corr(X_1, |X_1|) = \frac{Cov(X_1, |X_1|)}{\sqrt{V(X_1)}\sqrt{V(|X_1|)}}$$

$$Cov(X_1, |X_1|) = Cov\left(\sqrt{2(1-\rho)}\, Z, \left|\sqrt{2(1-\rho)}\, Z\right|\right)$$

$$Cov(Z, |Z|) = E(Z|Z|) - E(Z).E(|Z|)$$

$$= E(Z|Z|) \text{ --- } \boxed{\text{2 MARKS}}$$

$$= \int_{-\infty}^{\infty} z|z| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= 0$$

$$\Rightarrow Cov(X_1, |X_1|) = 0$$

$$\Rightarrow Corr(X_1, |X_1|) = 0 \text{ --- } \boxed{\text{1 MARKS}}$$

$\boxed{(C)}$ For $y = 1, 2, \ldots, 10$,

$$P\{Y=y\} = \sum_{x=y}^{10} P\{X=x, Y=y\}$$

$$= \sum_{x=y}^{10} \frac{1}{55}$$

$$= \frac{11-y}{55}$$

The p.m.f of Y is

$$f_Y(y) = \begin{cases} \frac{11-y}{55} & \text{If } y = 1, 2, \ldots, 10 \\ 0 & \text{o/w} \end{cases}$$

The Conditional p.m.f of X given Y=2 is

$$P\{X=x \mid Y=2\} = \frac{P\{X=x, Y=2\}}{P\{Y=2\}}$$

$$= \begin{cases} \frac{1}{9} & \text{if } x=2,3,\cdots 10 \\ 0 & \text{o/w} \end{cases} \quad \text{--- } \boxed{\text{3 MARKS}}$$

$$E(X \mid Y=2) = \sum_{x=2}^{10} x \cdot P\{X=x \mid Y=2\}$$

$$= \sum_{x=2}^{10} x \cdot \frac{1}{9}$$

$$= 6 \quad \text{--- } \boxed{\text{2 MARKS}}$$

Problem No-7

(a) We know that

$$Y = X_1 + X_2 + X_3 \sim B\left(5, \tfrac{3}{5}\right) \quad \text{-- } \boxed{2 \text{ MARKS}}$$

$$P\{X_1 + X_2 + X_3 \geq 2\} = P\{Y \geq 2\}$$

$$= 1 - P\{Y \leq 1\}$$

$$= 1 - \{P(Y=0) + P(Y=1)\}$$

$$= 1 - \left\{\binom{5}{0}\left(\tfrac{2}{5}\right)^5 + \binom{5}{1}\left(\tfrac{3}{5}\right)^1\left(\tfrac{2}{5}\right)^4\right\}$$

$$= 1 - \left[\frac{32}{3125} + \frac{48}{625}\right]$$

$$= 1 - \frac{272}{3125} = \frac{2853}{3125} \quad \text{--- } \boxed{3 \text{ MARKS}}$$

(b) $\underline{X} \sim \text{Mult}(n, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$, then the m.g.f of $\underline{X}$ is

$$M_X(\underline{t}) = E\left(e^{t_1 X_1 + t_2 X_2 + \cdots + t_5 X_5}\right) = \left(\theta_1 e^{t_1} + \theta_2 e^{t_2} + \cdots + \theta_5 e^{t_5}\right)^n$$

$\text{----} \boxed{2 \text{ MARKS}}$

We have, $n = 5$, $\theta_1 = \theta_2 = \cdots = \theta_5 = \frac{1}{5}$

let $t_1 = 1$, $t_2 = 2$, $t_3 = t_4 = t_5 = 0$, then

$$E\left(e^{X_1 + 2X_2}\right) = \left(\tfrac{1}{5}e^1 + \tfrac{1}{5}e^2 + \tfrac{1}{5} + \tfrac{1}{5} + \tfrac{1}{5}\right)^5$$

$$= \left(\tfrac{3}{5} + \tfrac{1}{5}e + \tfrac{1}{5}e^2\right)^5 \quad \text{--- } \boxed{3 \text{ MARKS}}$$

$\boxed{16/17}$

$\boxed{1(C)}$ Given data

$E(X) = 3$, $E(Y) = 4$, $E(X^2) = 15$, $E(Y^2) = 22$

$E(XY) = 13$.

$V(X) = E(X^2) - \{E(X)\}^2 = 15 - 9 = 6$

$V(Y) = E(Y^2) - \{E(Y)\}^2 = 22 - 16 = 6$

$Cov(X, Y) = E(XY) - E(X).E(Y)$

$= \cancel{13}\ 18 - 3 \times 4 = \cancel{1}\ 6$

$\Rightarrow$ $Corr(X, Y) = 1$

$\Rightarrow$ X and Y are perfactly positive Correlated.

-- $\boxed{2 \text{ MARKS}}$

$$\Rightarrow p\left(\frac{X - E(X)}{\sqrt{V(X)}} = \frac{Y - E(Y)}{\sqrt{V(Y)}}\right) = 1$$

$$\Rightarrow p\left(\frac{X - 3}{\sqrt{6}} = \frac{Y - 4}{\sqrt{6}}\right) = 1$$

$$\Rightarrow p(Y = X + 1) = 1$$

--- $\boxed{3 \text{ MARKS}}$