2.3. (a) Let $f_n \xrightarrow[n\to\infty]{} f$ in $C(U)$. Suppose $K \underset{\text{cpt.}}{\subseteq} U$. Then $\exists N \in \mathbb{N}$ s.t. $K \subseteq K_N$. We show that $f_n \xrightarrow[n\to\infty]{} f$ uniformly on K_N. Let $\varepsilon > 0$. Then $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$,

$$d(f_n, f) = \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{\|f_n - f\|_{K_r}}{1 + \|f_n - f\|_{K_r}} < \frac{\varepsilon}{1+\varepsilon} \cdot \frac{1}{2^N}$$

$$\Rightarrow \quad \frac{1}{2^N} \frac{\|f_n - f\|_{K_N}}{1 + \|f_n - f\|_{K_N}} < \frac{\varepsilon}{1+\varepsilon} \cdot \frac{1}{2^N} \Rightarrow \|f_n - f\|_{K_N} < \varepsilon,$$

$\forall n \geq n_0$.

Conversely, suppose $f_n \xrightarrow[n\to\infty]{} f$ uniformly on every compact subset of U. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $\sum_{r=N+1}^{\infty} \frac{1}{2^r} < \varepsilon$. Since $f_n \xrightarrow[n\to\infty]{} f$ uniformly on $K_1, \dots, K_N$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, $\|f_n - f\|_{K_j} < \varepsilon$, $\forall j = 1, \dots, N$.

It follows that, $\forall n \geq n_0$,

$$d(f_n, f) = \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{\|f_n - f\|_{K_r}}{1 + \|f_n - f\|_{K_r}}$$

$$\leq \sum_{r=1}^{N} \frac{1}{2^r} \frac{\|f_n - f\|_{K_r}}{1 + \|f_n - f\|_{K_r}} + \sum_{r=N+1}^{\infty} \frac{1}{2^r} \frac{\|f_n - f\|_{K_r}}{1 + \|f_n - f\|_{K_r}}$$

$$\leq \sum_{r=1}^{N} \frac{1}{2^r} \cdot \|f_n - f\|_{K_r} + \sum_{r=N+1}^{\infty} \frac{1}{2^r}$$

$$< \varepsilon \sum_{r=1}^{N} \frac{1}{2^r} + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon.$$