Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

MTH 636M: Game Theory

Quiz 2, Date: April 04, 2024, Thursday

Timing: 12:00 PM to 01:30 PM

- Answer any four questions from the following. The exam is for 20 marks.
- Try not to use any result not done in the class. However, if you use any such result, clearly state and prove it.
- Write your name, roll no., program name, and seat number clearly on the top of your answer sheet.
- For prove or disprove type questions, clearly state whether it's a prove or a disprove and then provide the arguments.
- One A4 sheet with ONLY necessary definitions and results are allowed during the exam. Use of a calculator, mobile, and smart watch is strictly prohibited.
- Be precise in writing the answers. Unnecessary arguments would lead to a deduction in marks.

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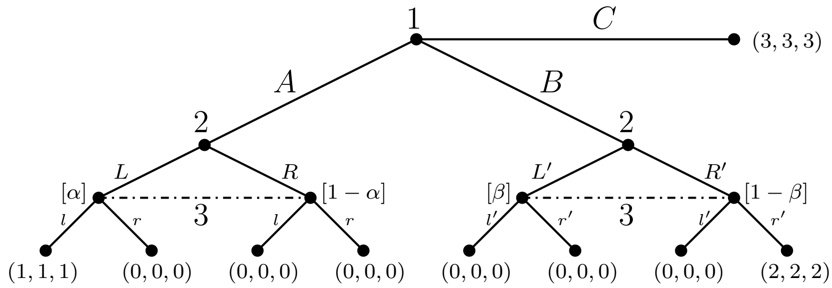
1. Find all the perfect Bayesian equilibria for the following game. (**7 marks**)



Answer: Observe that Player 1 will always play C as she will get maximum pay-off by playing that action. For Player 3, if $\alpha > 0$, Player 3's optimal action is l and if $\alpha = 0$, she is indifferent between l and r. Similarly, if $\beta < 1$, Player 3's optimal action is r' and if $\beta = 1$, she is indifferent between l' and r'.

Coming to Player 2, note that $\alpha = b_2(L)$ and $\beta = b_2(L')$. Further, if Player 3 puts a positive probability to l, Player 2 will prefer playing L implying $\alpha = 1$, and if Player

3 puts a positive probability to r', Player 2 will prefer playing R' implying $\beta = 0$. This will give us the following perfect Bayesian equilibrium

$$b_1(C) = 1, b_2(L) = 1, b_2(R') = 1, b_3(l) = 1, b_3(r') = 1, \alpha = 1, \beta = 0.$$

Moreover, Player 2 will play R only if Player 3 plays r, and Player 3 plays r only if $\alpha = 0$ implying Player 2 plays R. Similarly, Player 2 will play L' only if Player 3 plays l', and Player 3 plays l' only if $\beta = 1$ implying Player 2 plays L'. Thus, we get 3 more Perfect Bayesian equilibria

$$b_1(C) = 1, b_2(R) = 1, b_2(R') = 1, b_3(r) = 1, b_3(r') = 1, \alpha = 0, \beta = 0,$$

$$b_1(C) = 1, b_2(L) = 1, b_2(L') = 1, b_3(l) = 1, b_3(l') = 1, \alpha = 1, \beta = 1, \text{ and}$$
$$b_1(C) = 1, b_2(R) = 1, b_2(L') = 1, b_3(r) = 1, b_3(l') = 1, \alpha = 0, \beta = 1.$$

2. Consider the following game (known as Nim game in the literature). It is a game between two-player where two piles of matches are placed before the players, and each pile has three matches. Each player in turn chooses a pile, and removes any number of matches from the pile he has selected (he must remove at least one match). The player who removes the last match wins the game. Prove or disprove that one of the players has a winning strategy in this game. If you prove the above statement, can you find out the winning strategy? (**5 marks**)

   Answer: Player 2 has an winning strategy. The strategy is for any action of Player 1, follow the same action on the other pile, i.e., if Player 1 removes i matches from the first pile, player 2 in his turn takes away three matches from the second pile. Note that this action will always ensure winning for Player 2 as after both the players make their moves, there will be an equal number of matches in both the piles, implying it is never possible to arise a situation where after Player 1's move, both the piles are empty.

3. Allen and Rick need to divide \$100 between them as follows: first Allen suggests an integer x between 0 and 100 (which is the amount of money he wants for himself). Rick, on hearing the suggested amount, decides whether to accept or reject. If Rick accepts, the payoff of the game is $(x, 100 - x)$: Allen receives x dollars, and Rick receives $100 - x$ dollars. If Rick chooses to reject, neither player receives any money.

   (i) Describe this situation as an extensive-form game. (**2 marks**)

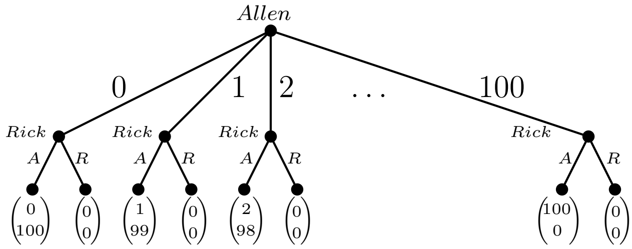
   (ii) What is the set of pure strategies each player has? (**1 mark**)

   (iii) Show that any result $(a, 100 - a)$, $a \in \{0, 1, \ldots, 100\}$, is a Nash equilibrium payoff. What are the corresponding equilibrium strategies? (**2 marks**)

   (iv) Find all the subgame perfect equilibria (in pure strategies) of this game. (**3 marks**)

   Answer:

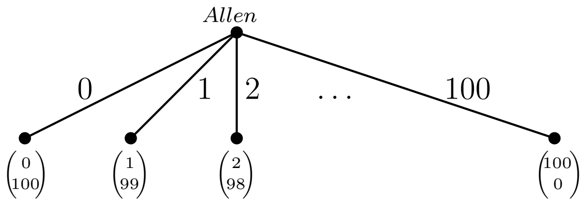
(i) The extensive-form of the game is as follows:



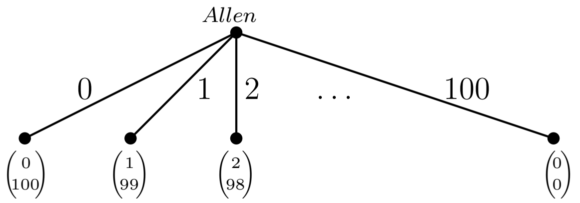
(ii) Allen's pure strategy space is $\{0, 1, \ldots, 100\}$. For Rick, the pure strategy space is $\{(x_0, x_1, \ldots, x_{100}) \mid x_i \in \{\text{A}, \text{R}\} \text{ for all } i \in \{0, \ldots, 100\})\}$.

(iii) For $(a, 100-a)$, consider the strategy profile $(a, (x_0, x_1, \ldots, x_{100}))$ where $x_i = \text{R}$ if $i \neq a$ and $x_a = \text{A}$. No player has a profitable deviation and the outcome is $(a, 100-a)$.

(iv) If we consider the optimal choices of Rick at all his information sets, we get the following two games



and



Therefore, the two SPNEs are $(100, (x_0, x_1, \ldots, x_{100}))$ where $x_i = \text{A}$ for all $i \in \{0, \ldots, 100\}$ and $(99, (x_0, x_1, \ldots, x_{100}))$ where $x_i = \text{A}$ if $i \neq 100$ and $x_{100} = \text{R}$.