Index of a point with respect to a closed curve

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August 20, 2023

Theorem 1. *Let $\gamma : [a,b] \longrightarrow \mathbb{C} \setminus \{0\}$ be a curve. Then γ has a continuous argument and hence a continuous logarithm.*

In fact, we will be proving the following generalized version.

Theorem 2. *Let $f : [a,b] \times [c,d] \longrightarrow \mathbb{C} \setminus \{0\}$ be continuous. Then f has a continuous argument.*

Proof. Let $\varepsilon = \min\{|f(t)| : t \in [a,b] \times [c,d]\}$. From the uniform continuity of f on $[a,b] \times [c,d]$, we can choose partitions $a = a_0 < a_1 < \cdots < a_n = b$ and $c = c_0 < c_1 < \cdots < c_m = d$, such that for each $(k, j) \in \{0, \cdots, n-1\} \times \{0, \cdots, m-1\}$, $f(R_{k,j}) \subseteq B(f(a_k, c_j); \varepsilon)$, where $R_{k,j} \overset{\text{def}}{=} [a_k, a_{k+1}] \times [c_j, c_{j+1}]$. Note that, for all $(k, j) \in \{0, \cdots, n-1\} \times \{0, \cdots, m-1\}$, $0 \notin B(f(a_k, c_j); \varepsilon)$. Clearly each of these balls $B(f(a_k, c_j); \varepsilon)$ avoids some R_α, and consequently $f|_{R_{k,j}}$ will have a continuous argument.

Fix $j \in \{0, \cdots, m-1\}$. Let θ_0 and θ_1 be the continuous arguments of $f|_{R_{0,j}}$ and $f|_{R_{1,j}}$ respectively. Then $\theta_0 - \theta_1 = 2\pi\ell$, for some $\ell \in \mathbb{Z}$, on the connected set $\{a_1\} \times [b_j, b_{j+1}] = R_{0,j} \cap R_{1,j}$. We define

$$R_{0,j} \cup R_{1,j} \longrightarrow \mathbb{R},\ (s,t) \mapsto \begin{cases} \theta_0(s,t) & \text{if } (s,t) \in R_{0,j} \\ \theta_1(s,t) + 2\pi\ell & \text{if } (s,t) \in R_{1,j} \end{cases}. \tag{$*1$}$$

It is easy to see that $(*1)$ defines a continuous argument of f on $R_{0,j} \cup R_{1,j}$. Proceeding, we get a continuous argument of f on the horizontal strip $R_{0,j} \cup R_{1,j} \cup \cdots \cup R_{(n-1),j}$, for each $j \in \{0, \cdots, m-1\}$. Since $[a,b] \times [c,d]$ is the union of these horizontal strips, arguing in exactly similar manner, we can get a continuous argument of f on whole $[a,b] \times [c,d]$. □

Theorem 1 can be deduced from Theorem 2 by considering the map

$$\tilde{\gamma} : [a,b] \times [0,1] \longrightarrow \mathbb{C} \setminus \{0\},\ (s,t) \mapsto \gamma(t)$$

Clearly $\tilde{\gamma}$ is a continuous map. Now, if $\tilde{\theta}$ is a continuous argument of $\tilde{\gamma}$ then, $\theta : [a,b] \longrightarrow \mathbb{R}$ defined as $\theta(x) = \tilde{\theta}(x, 0)$, $\forall x \in [a,b]$, is a continuous argument of γ.

Let us now define the *Index of a point* with respect to a *closed curve*.

Definition 1. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a closed curve. Fix $z_0 \notin \gamma^*$, a continuous argument θ of $\gamma - z_0$ (this exists in view of Theorem 1). Then the Index of of z_0 with respect to γ, denoted by* $\operatorname{Ind}_\gamma(z_0)$*, is defined by*

$$\frac{\theta(b) - \theta(a)}{2\pi}. \tag{$*2$}$$

Since $\gamma(a) = \gamma(b)$, $1 = \dfrac{\gamma(b) - z_0}{|\,\gamma(b) - z_0\,|} \cdot \dfrac{|\,\gamma(a) - z_0\,|}{\gamma(a) - z_0} = e^{i(\theta(a) - \theta(b))}$, whence $\theta(a) - \theta(b)$ is an integral multiple of 2π, consequently $(*2)$ is an integer. If θ_1 be another continuous argument of $\gamma - z_0$, then we get $\theta_1 - \theta = 2\pi\ell$ for some $\ell \in \mathbb{Z}$. So, we see that $\theta(a) - \theta(b) = \theta_1(a) - \theta_1(b)$, which shows that this definition is independent of the choice of θ.

Thus in order to compute the Index of a point z_0, with respect to a closed curve γ such that $z_0 \notin \gamma^*$, from the definition, we need to choose a continuous argument first. But this is unfortunately not feasible in practice. However, if the curve under consideration happens to be a path, i.e., piecewise continuously differentiable curve, then the index can be obtained, in view of our next theorem, just by computing an integral. More often than not, this turns out to be much easier.

Theorem 3. *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a closed path and $z_0 \notin \gamma^*$. Then,*

$$\operatorname{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int\limits_\gamma \frac{dz}{z - z_0}.$$

Proof. Let $\varepsilon = d(z_0, \gamma^*)$. Since γ is uniformly continuous on $[a,b]$, $\exists$ a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$, such that $\gamma[t_j, t_{j+1}] \subseteq D(\gamma(t_j), \varepsilon)$ for all $j = 0, \cdots, n-1$. Fix $j \in \{0, \cdots, n-1\}$. The image of the disc $D(\gamma(t_j); \varepsilon)$ under the holomorphic function $z \mapsto z - z_0$ is $D(\gamma(t_j) - z_0; \varepsilon)$. Since, by definition of ε, $z_0 \notin D(\gamma(t_j); \varepsilon)$, so $0 \notin D(\gamma(t_j) - z_0; \varepsilon)$. Hence the function $z \mapsto z - z_0$ has a holomorphic logarithm, say g_j, on $D(\gamma(t_j); \varepsilon)$. Now from the fundamental theorem for integration along paths, we have, for all $j = 0, \ldots, n-1$,

$$\int\limits_{\gamma_j} \frac{dz}{z - z_0} = g_j(\gamma(t_{j+1})) - g_j(\gamma(t_j)),$$

where $\gamma_j \overset{\text{def}}{=} \gamma|_{[t_j, t_{j+1}]}$. Clearly $\gamma = \gamma_0 * \gamma_1 * \cdots * \gamma_{n-1}$. Hence, one obtains that,

$$\int\limits_\gamma \frac{dz}{z - z_0} = \sum_{j=0}^{n-1} \int_{\gamma_j} \frac{dz}{z - z_0} = \sum_{j=0}^{n-1} (g_j(\gamma_j(t_{j+1})) - g_j(\gamma_j(t_j))). \tag{$*3$}$$

For each $j = 0, \ldots, n-1$, denote by θ_j, the imaginary part of g_j, so that θ_j is a continuous argument of $z \mapsto z - z_0$ on $D(\gamma(t_j); \varepsilon)$. It follows that

$$\begin{aligned}
\sum_{j=0}^{n-1}(g_j(\gamma_j(t_{j+1})) - g_j(\gamma_j(t_j))) &= \sum_{j=0}^{n-1}(\operatorname{Re} g_j(\gamma_j(t_{j+1})) - \operatorname{Re} g_j(\gamma_j(t_j))) \\
&\quad + i\sum_{j=0}^{n-1}(\theta_j(\gamma_j(t_{j+1})) - \theta_j(\gamma_j(t_j))) \\
&= \sum_{j=0}^{n-1}(\log|(\gamma(t_{j+1}) - z_0| - \log|\gamma(t_j) - z_0|) \\
&\quad + i\sum_{j=0}^{n-1}(\theta_j(\gamma_j(t_{j+1})) - \theta_j(\gamma_j(t_j))) \\
&= i\sum_{j=0}^{n-1}(\theta_j(\gamma_j(t_{j+1})) - \theta_j(\gamma_j(t_j))), \qquad (*4)
\end{aligned}$$

as $\sum_{j=0}^{n-1}(\log|(\gamma(t_{j+1}) - z_0| - \log|\gamma(t_j) - z_0|) = \log|(\gamma(b) - z_0| - \log|\gamma(a) - z_0| = 0$, because γ is closed.

If θ be any continuous argument of $\gamma - z_0$, then $\theta|_{[t_j,t_{j+1}]}$ is a continuous argument of $(\gamma - z_0)|_{[t_j,t_{j+1}]}$, for all $j = 0, \cdots, n-1$. Therefore, one must have, for all $j \in \{0, \cdots, n-1\}$,

$$\theta_j(\gamma_j(t_{j+1})) - \theta_j(\gamma_j(t_j)) = \theta(t_{j+1}) - \theta(t_j).$$

In view of $(*3)$ and $(*4)$, we now arrive at

$$\int_\gamma \frac{dz}{z - z_0} = i\sum_{j=0}^{n-1}(\theta(t_{j+1}) - \theta(t_j)) = i(\theta(b) - \theta(a)) = 2\pi i \operatorname{Ind}_\gamma(z_0).$$

This completes our proof. □

Any closed $\gamma : [a,b] \longrightarrow \mathbb{C}$ gives rise to the function $\operatorname{Ind}_\gamma : \mathbb{C} \setminus \gamma^*$, $z \mapsto \operatorname{Ind}_\gamma(z)$. This function is holomorphic, and continuous in particular, in view of Theorem 3. Thus if $S \subseteq \mathbb{C} \setminus \gamma^*$ is connected then $\operatorname{Ind}_\gamma$ must be constant on S. If S is unbounded in addition, then one can further show that $\operatorname{Ind}_\gamma$ vanishes identically on S. To see this, let c be the value taken by $\operatorname{Ind}_\gamma$ on S and R be any positive number. We claim that there exists $z \in S$ such that, for any $w \in \gamma^*$, $|w - z| > R$. Otherwise, for any $z \in S$, we would have a point $w_z \in \gamma^*$ such that $|w_z - z| \leq R$. That would imply,

$$|z| \leq |z - w_z| + |w_z| \leq R + \sup_{a \leq t \leq b} |\gamma(t)|, \ \forall z \in S,$$

which is absurd. Now using ML-inequality, one obtain that $|c| = |\operatorname{Ind}_\gamma(z)| \leq \dfrac{L_\gamma}{2\pi R}$. Hence c must be 0.

Corollary 1. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Then,*

$$\frac{1}{2\pi i}\int_{C(z_0;r)} \frac{dw}{w-z} = \begin{cases} 0 & \text{if } z \notin \overline{D(z_0, r)} \\ 1 & \text{if } z \in D(z_0, r) \end{cases} .$$

Proof. $\mathbb{C}\setminus C(z_0;r)^*$ has precisely two connected components, $D(z_0;r)$ and $\mathbb{C}\setminus\overline{D(z_0;r)}$. So if $|z-z_0| > r$, then clearly one has

$$\frac{1}{2\pi i}\int_{\gamma_r} \frac{dw}{w-z} = \mathrm{Ind}_{\gamma_r}(z) = 0.$$

Now if $|z - z_0| < r$, then

$$\frac{1}{2\pi i}\int_{\gamma_r} \frac{dw}{w-z} = \mathrm{Ind}_{\gamma_r}(z) = \mathrm{Ind}_{\gamma_r}(z_0) = 1.$$

□

Corollary 2. *Let $P(z)$ be a polynomial with complex coefficients wish has no zeros on the circle $|z - z_0| = r$. Then the number of zeros of $P(z)$ inside $D(z_0;r)$ is given by the following integral:*

$$\int_{C(z_0;r)} \frac{P'(z)}{P(z)}\, dz.$$

Proof. Let $P(z) = (z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\dots(z-\alpha_k)^{m_k}$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are all distinct zeros with respective multiplicities $m_1, m_2, \dots m_k$. Then one has

$$\frac{P'(z)}{P(z)} = \frac{m_1}{(z-\alpha_1)} + \frac{m_1}{(z-\alpha_1)} + \cdots + \frac{m_k}{(z-\alpha_k)}.$$

Now from Corollary 1, we obtain that,

$$\int_{C(z_0;r)} \frac{m_j}{(z-\alpha_j)}\, dz = \begin{cases} m_j & \text{if } \alpha_j \in D(z_0;r) \\ 0 & \text{if } \alpha_j \notin D(z_0;r) \end{cases}, \ \forall j = 1, \dots, k. \tag{$*5$}$$

Therefore,

$$\frac{1}{2\pi i}\int_{C(z_0;R)} \frac{P'(z)}{P(z)}\, dz = \sum_{\substack{1 \le j \le k, \\ \alpha_j \in D(z_0;r)}} m_j.$$

□