Lesson 4

Numerical Solution of PDE

4.1 BVP for 2^{nd} Order Elliptic PDE

- *Finite Difference Method*
- ***Finite Element Method***

Akash Anand
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Now, lets try to solve the Poisson's equation with homogeneous boundary condition

$$\Delta u = f, \qquad \text{in } \Omega,$$
$$u = 0, \qquad \text{on } \Gamma.$$

using the finite element method where Ω is a bounded domain.

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We, therefore, see that the Galerkin approximation error is bounded by a constant multiple of the best approximation error for u by functions in V_h!

Lesson 4

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- *Finite Difference Method*
- ***Finite Element Method***
  - ***Construction of FEM Approximation Spaces***

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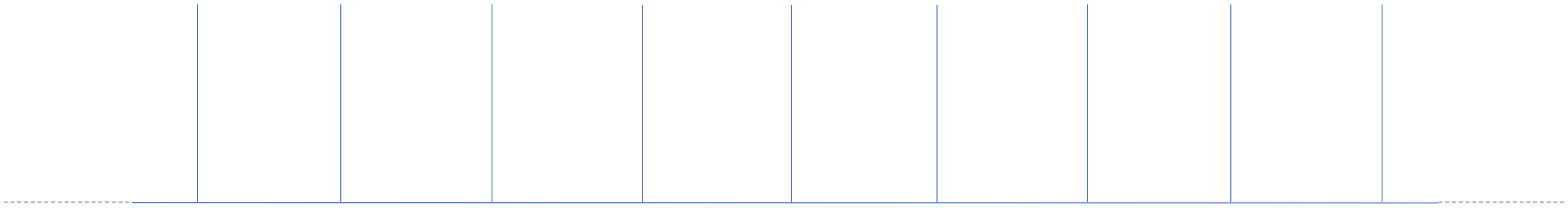

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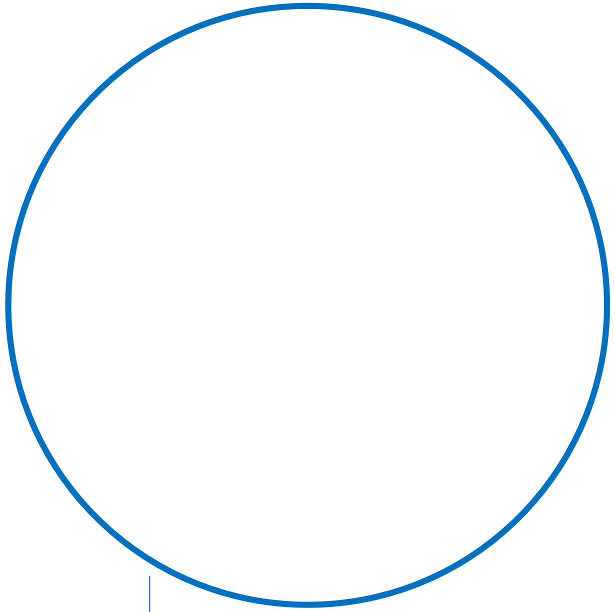

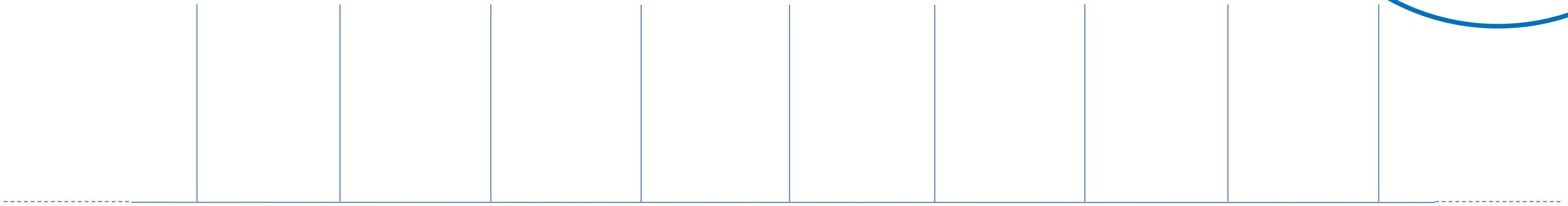

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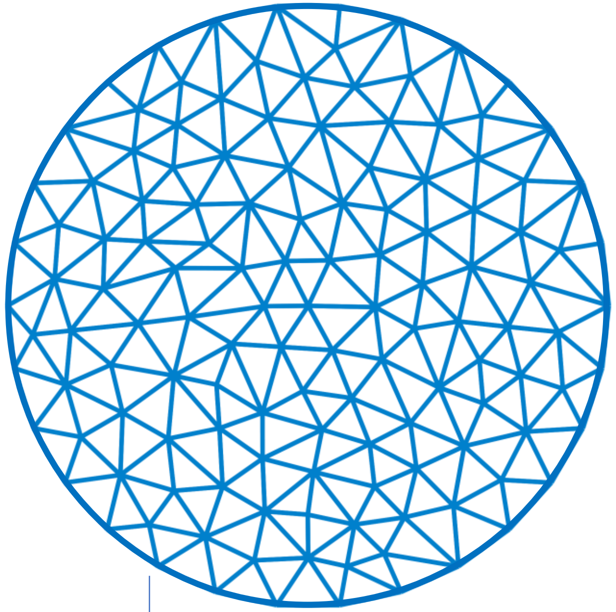

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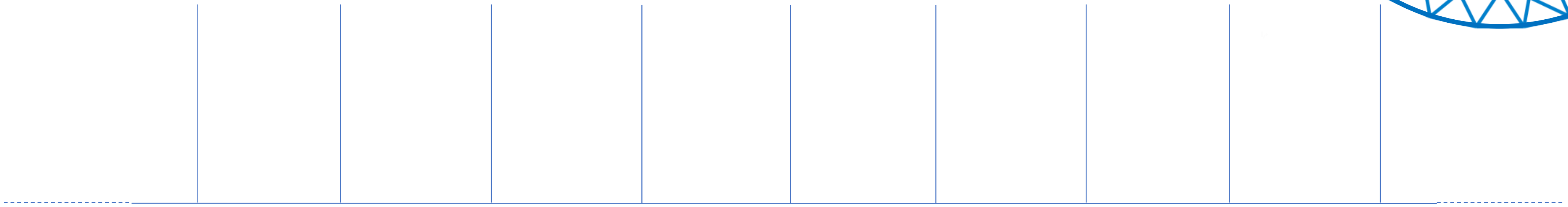

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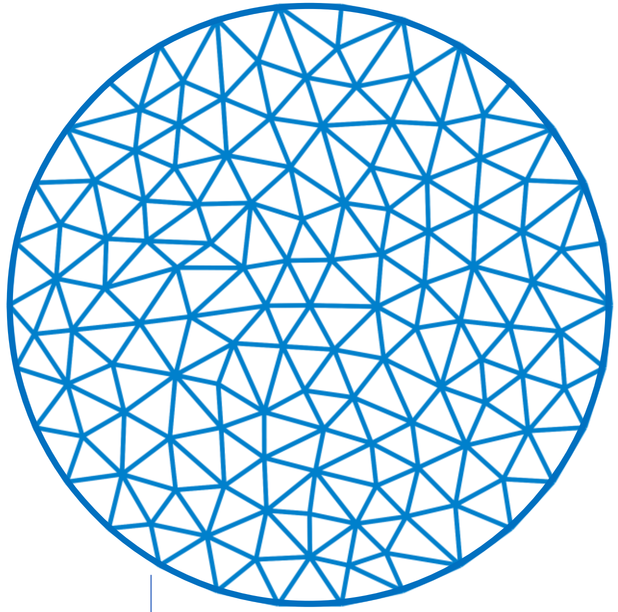

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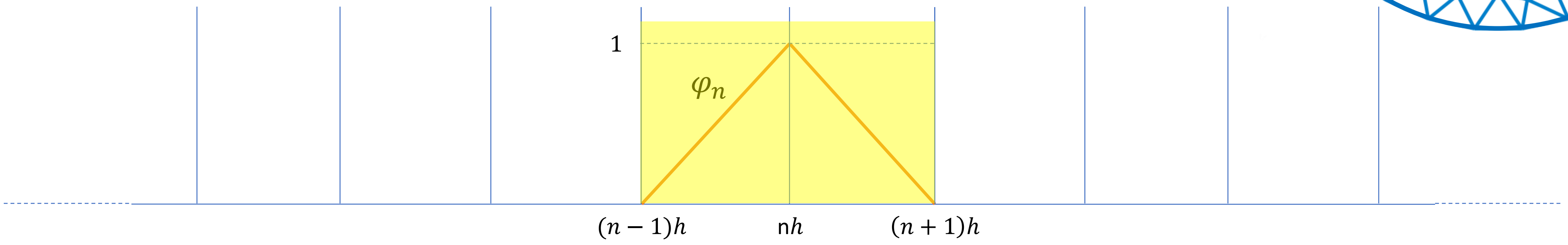

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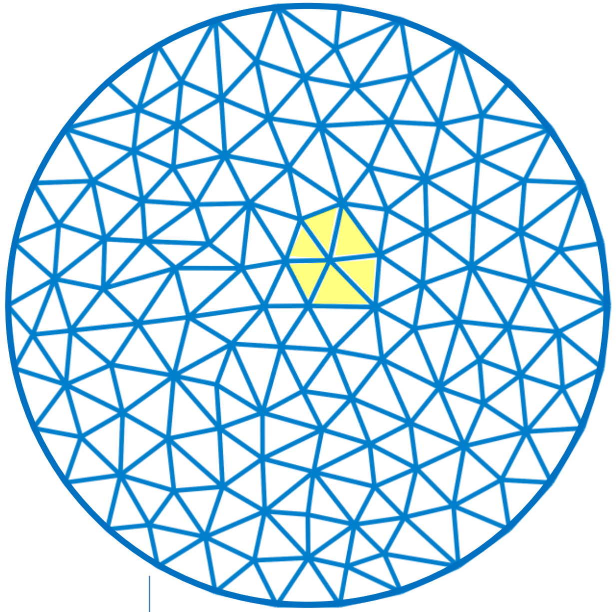

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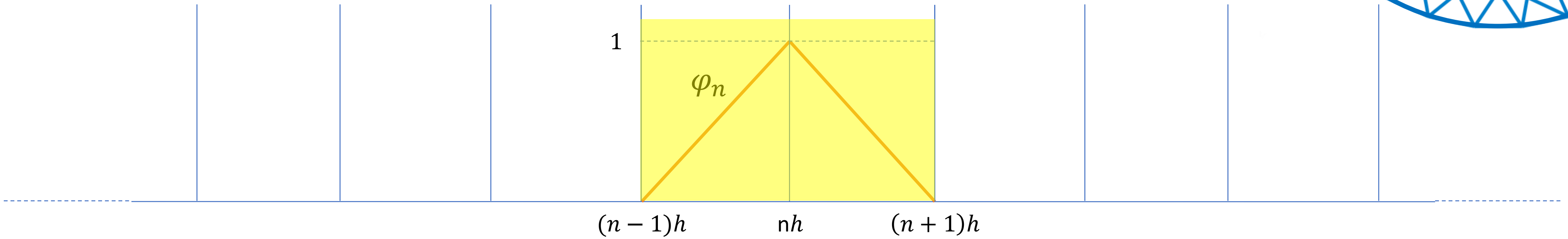

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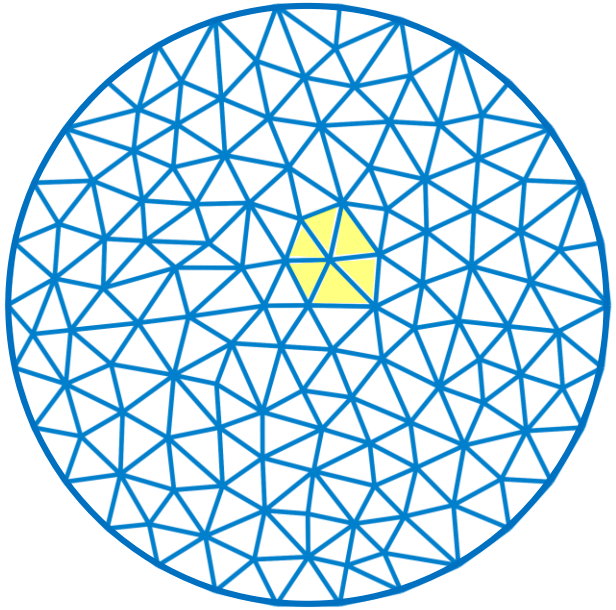

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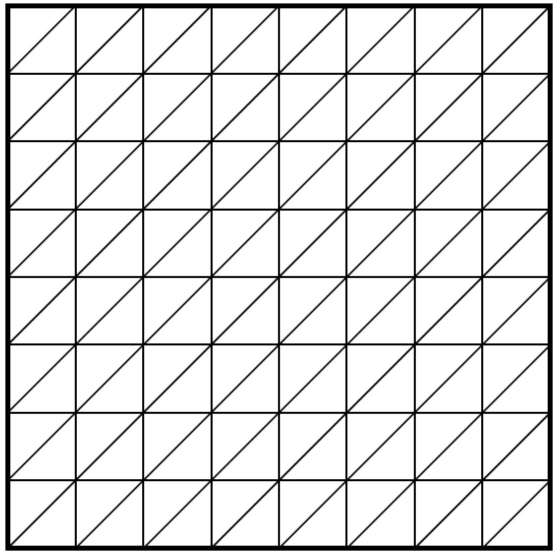

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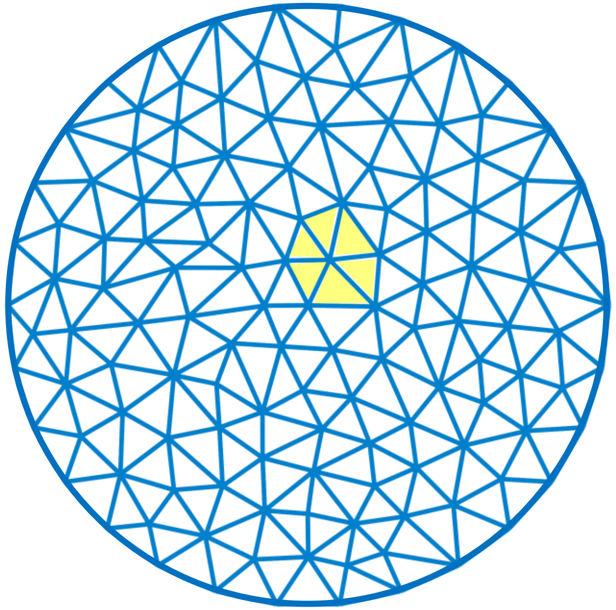

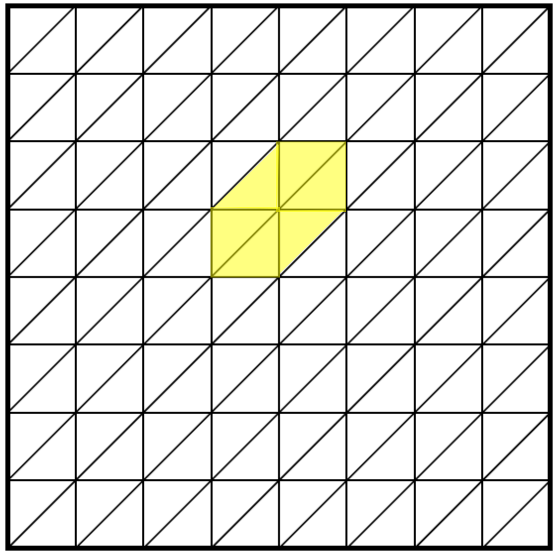

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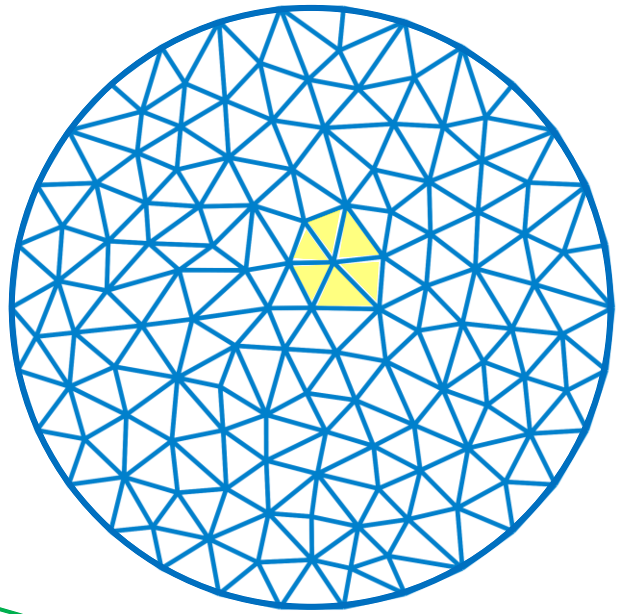

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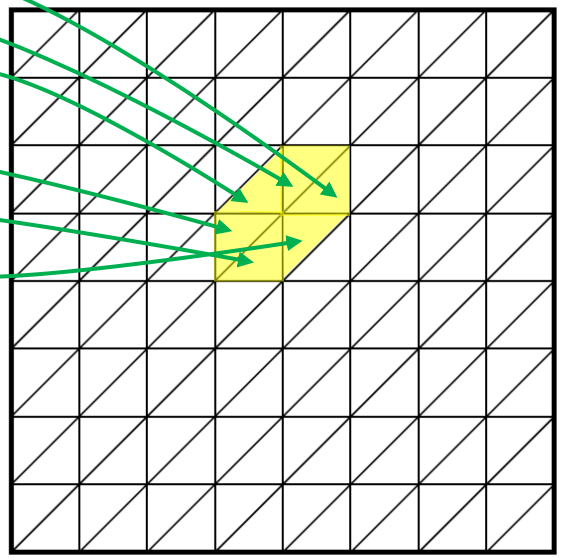
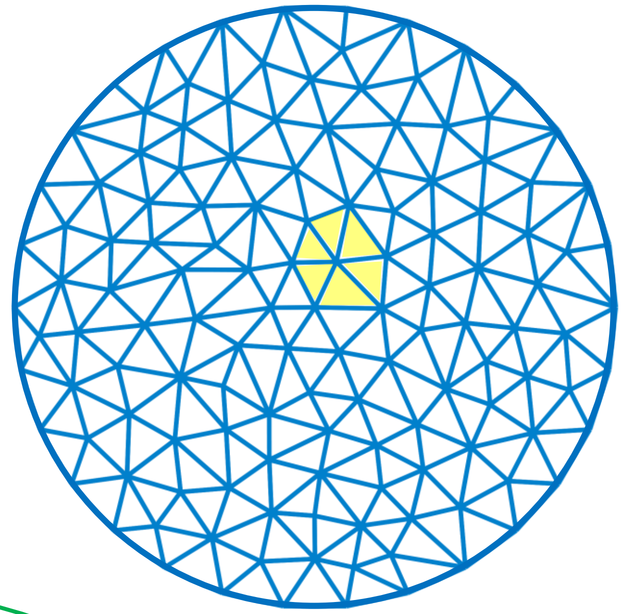
$$\varphi_{mn}(x_1, x_2) = \begin{cases} 1 - (x_1 - mh)/h \\ 1 - (x_2 - nh)/h \\ 1 + (x_1 - mh)/h - (x_2 - nh)/h \\ 1 + (x_1 - mh)/h \\ 1 + (x_2 - nh)/h \\ 1 - (x_1 - mh)/h + (x_2 - nh)/h \end{cases}$$





Back to the Poisson's equation with homogeneous boundary condition

$$\Delta u = f, \qquad \text{in } \Omega,$$
$$u = 0, \qquad \text{on } \Gamma.$$

How do we construct a finite element (that is, locally supported) basis for V_h?

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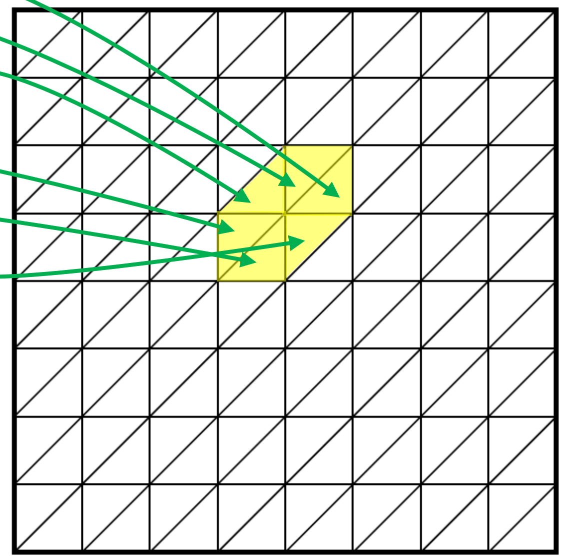

The functions φ_{mn} form a basis for subspace of piecewise linear functions with respect to the given partition/triangulation.

Note that (exercise)

$$\int_\Omega \nabla\varphi_{mn}\cdot\nabla\varphi_{kl} = \begin{cases} 4, & m=k, n=l,\\ -1, & m=k\pm 1, n=l \text{ or } m=k, n=l\pm 1,\\ 0, & \text{otherwise}.\end{cases}$$

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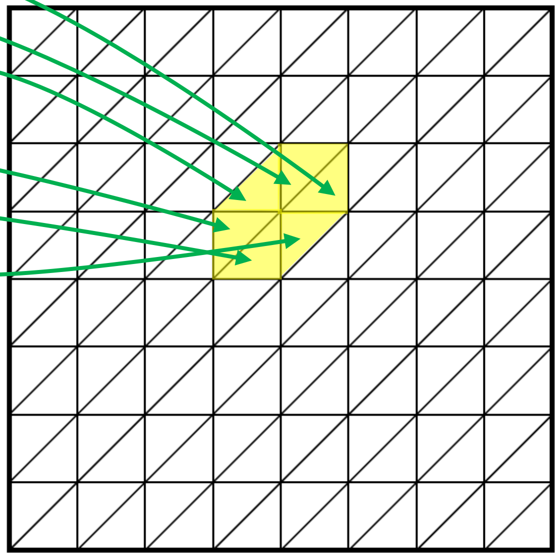

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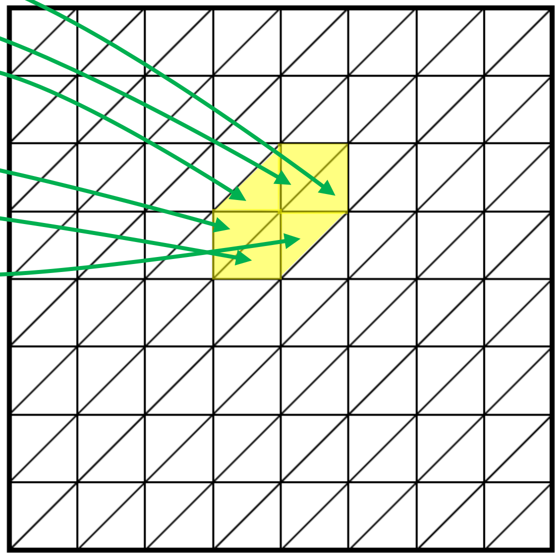

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Recall that, from the general error analysis, we have

$$\|u_h-u\|_{H^1}\le (1+C/\gamma)\inf_{w\in V_h}\|u-w\|_{H^1}.$$

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From the approximation theory, we have the following result on the best approximation error.

Theorem

Let there be given a family of triangulations $\{\mathcal{T}_h\}$ of a polygonal domain Ω and let $h = \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$. Let r be a positive integer. For each h let $P_h: C(\Omega) \to M_0^r(\mathcal{T}_h)$ denote the nodal interpolant, where $M_0^r(\mathcal{T}_h)$ is the space of continuous functions which restrict to polynomials of degree at most r when restricted to any triangle $T \in \mathcal{T}_h$. Then, there is a constant c such that

$$\|u - P_h u\|_{L^\infty(\Omega)} \leq c h^{r+1} \left\| u^{(r+1)} \right\|_{L^\infty(\Omega)}, \qquad u \in C^{r+1}(\overline{\Omega}),$$

$$\|u - P_h u\|_{L^2(\Omega)} \leq c h^{r+1} \left\| u^{(r+1)} \right\|_{L^2(\Omega)}, \qquad u \in H^{r+1}(\Omega).$$

Moreover, if the family of triangulations are shape regular (the minimal angle of each triangulation is bounded below uniformly), then there is a constant C such that

$$\|\nabla(u - P_h u)\|_{L^\infty(\Omega)} \leq C h^{r} \left\| u^{(r+1)} \right\|_{L^\infty(\Omega)}, \qquad u \in C^{r+1}(\overline{\Omega}),$$

$$\|\nabla(u - P_h u)\|_{L^2(\Omega)} \leq C h^{r} \left\| u^{(r+1)} \right\|_{L^2(\Omega)}, \qquad u \in H^{r+1}(\Omega).$$