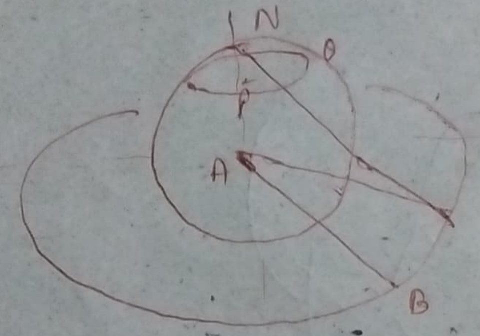
Endsem

Exercise - Show S^2/N is homeomorphic to $\mathbb{R}^2$

A:-



$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$

Let $N = (0,0,1)$ be the North Pole.

$$f : S^2 \setminus \{N\} \to \mathbb{R}^2$$

Given a point $P = (x,y,z) \in S^2 \setminus \{N\}$, draw a line from N through P, and find where it intersects the plane $z=0$.

$$f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

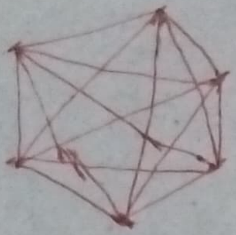
$$f^{-1} : \mathbb{R}^2 \to S^2 \setminus \{N\}$$

$$f^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

f is bijective: Every point $S^2 \setminus \{N\}$ maps uniquely to a point in $\mathbb{R}^2$, and vice versa

$$L(t) = (0,0,1) + t(u,v,-1)$$
$$= (tu, tv, 1-t)$$
$$x^2+y^2+z^2 = 1$$

ans8.



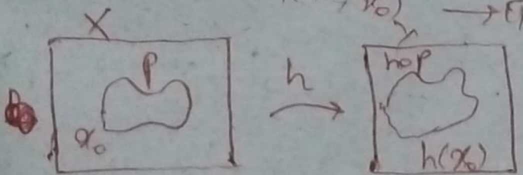
${}^6C_2 = 15$ edges

There are 20 disjoint triangle pairs in K_6.
Define the mod 2 linking number for each pair
Conway-Gordon showed that the sum of these linking numbers is odd
$\Rightarrow$ At least one pair must be linked

Ans4 $\rightarrow$ Let X be a space (subset of R^n); $x_0 \in X$ a path
$\alpha : [0,1] \rightarrow X$ s.t $\alpha(0) = \alpha(1) = x_0$ is called a
loop based at x_0.
$[\alpha]$ be the path homotopy class of α, the set of all path
homotopy classes of loops based at x_0 with the composition
$*$ is called the fundamental group of X based at x_0
$\pi_1(X, x_0)$

Let $h : X \rightarrow Y$ be a homeomorphism
h is continuous function from X to Y s.t h is 1-1, onto
& h^{-1} is cts.

Let $h_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, h(x_0))$



$h_*[P] = [h \circ P]$
h_* is called the gp. homeomorphism associated with h
$[P]$ the set $\pi_1(X, x) = \{ [P] \mid P$ is a loop based at $x_0 \}$
(1) h_* is homeomorphism
$h_*([P] \cdot [Q]) = h_*[P] * h_*[Q]$
$= [h \circ P] * [h \circ Q]$

$$[h\circ p][h\circ q] = \begin{cases} h\circ p(s), & 0\le s\le \frac{1}{2} \\ h\circ q(s), & \frac{1}{2}\le s\le 1 \end{cases}$$

$$h_*([p]\cdot[q]) = \begin{cases} h\circ p(2s), & 0\le s\le \frac{1}{2} \\ h\circ q(2s-1), & \frac{1}{2}\le s\le 1 \end{cases}$$

$\hat{h}_1 : \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$

$\hat{h}_2 : \pi_1(\cancel{Y}, h(x_0)) \to \pi_1(X, x_0)$

$\hat{h}_1 \cdot \hat{h}_2([g]) = \hat{h}_1([\bar{h}_2]\cdot[g][h_2])$

$= [\bar{h}_1]([\bar{h}_2]\cdot[g]\cdot[h_2])[h_1]$

$\therefore h_2 = \bar{h}_1 = h_1(1-s)$

$= [\bar{h}_1]([h][g][\bar{h}_1])[h_1]$

$= [g]$

Similarly

$\hat{h}_2 \cdot \hat{h}_1[b] = [b]$

(1) $X \xrightarrow{h} Y \xrightarrow{h^{-1}} X$

(2) $Y \xrightarrow{h^{-1}} X \xrightarrow{h} Y$

$\pi_1(X, x_0) \xrightarrow{h_*} \pi_1(Y, h(x_0)) \xrightarrow{(h^{-1})_*} \pi_1(X, x_0)$

$(h^{-1}\circ h)_* = (id)_* = id$

$h^{-1}_* \circ h_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$

$h^{-1}_* \circ h_* = id$

$h_* \circ h^{-1}_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$ $\Rightarrow \pi_1(X) \cong \pi_1(Y)$

$h_* \circ h^{-1}_* = id$

Example:-

Let $X = S^1 \cup S^1$ (disjoint)

Y = figure-eight space

Then:

$\pi_1(X) = \pi_1(S^1) * \pi_1(S^1) = \mathbb{Z} * \mathbb{Z}$ (free product)

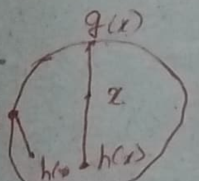
$\pi_1(Y) = \mathbb{Z} * \mathbb{Z}$ also

So: $\pi_1(X) \cong \pi_1(Y)$

but $X \not\cong Y$

Ans(1)→ Let D be the unit disc in $\mathbb{R}^2$ $\{x^2+y^2 \le 1\}$.
Let $h: D \to D$ be any cts map, then h has atleast one fixed point $h(x_0) = x_0$.

Proof:- By the method of contradiction.
Let $h: D \to D$ a cts map with no fixed point
$h(x) \neq x \ \forall x \in D$.



Connect $h(x)$ to x by st. line, extend the st. line till it hits the bdry circle, call it $g(x)$.

$D \xrightarrow{g} S^1$ s.t if $x \in S^1$ $g(x) = x$.

Consider
$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "id"'] & D \arrow[r, "g"] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "id_* = id"'] & \pi_1(D) \arrow[r, "g_*"] & \pi_1(S^1) \end{tikzcd}$$

$$\begin{tikzcd} (\mathbb{Z},+) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & e \arrow[r, "g_*"] & (\mathbb{Z},+) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right] & 0 \arrow[r, "g_*"] & \mathbb{Z} \end{tikzcd}$$

$g_* \circ i_* = id$

g. $0_1(z) \to 0$ I.id (Contraction)

Ans 2 (a) Let X be a space $\subset \mathbb{R}^n$, A be subspace of X. Then A is said to be a strong Deformation Retract of X if there is a continuous map

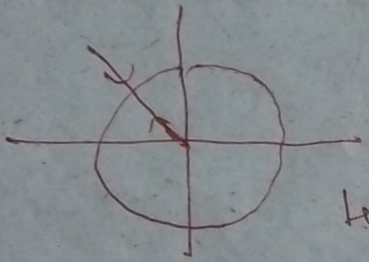
$$H : X \times I \to X \text{ s.t}$$

$$H(x,0) = x \quad \forall x \in A$$

$$H(x,1) \in A, \quad H(a,t) = a \quad \forall a \in A$$

$$\pi_1(X, a_0) \cong \pi_1(A, a_0)$$

Ex:- $X = \mathbb{R}^2 \setminus \{0,0\}$ = Punctured plane



$$\mathbb{R}^2 \setminus 0 \quad S^1 \subset \mathbb{R}^2 \setminus 0$$

$$H : X \times I \to X$$

$$H(\vec{x}, t) = (1-t)\vec{x} + \frac{t\,\vec{x}}{\|\vec{x}\|}$$

$$H(x,0) = \vec{x}$$

$$H(x,1) = \frac{x}{\|\vec{x}\|}$$

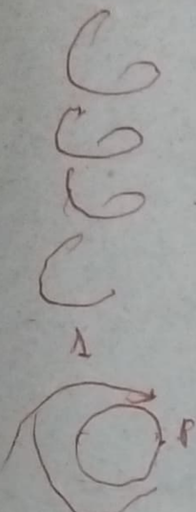
$$H(a,t) = (1-t)\vec{a} + t\,\vec{a} = \vec{a}, \quad a \in S$$

$$\pi_1(\mathbb{R}^2 \setminus 0) \cong \pi_1(S^1, (1,0)) \cong (\mathbb{Z}, +)$$

Ans. $\pi_1(S^1)$ Cover $\pi: \mathbb{R} \to S^1 = \frac{\mathbb{R}}{\mathbb{Z}}$, $p \in$

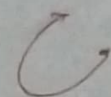
$\pi^{-1}(p) = \mathbb{Z}$

As $\pi^{-1}(S)$ = set



$\mathbb{R} \downarrow S^1$

$\mathbb{Z} \cong \pi_1(S^1)/\pi_1(\mathbb{R}) = \pi_1(S^1)$

$1 \longmapsto$ 

Prop $\pi_1(S^1) \cong \mathbb{Z}$ as groups

~~Fundamental Thm of Algebra~~

~~Thm: Any $p(z) = a_0 + a_1 z + \ldots +$~~

$X = S^1 = \{x^2 + y^2 = 1\}$

$\pi_1(S^1, (1,0))$

$f(s) = (\cos 2\pi s, \sin 2\pi s)$

$f \circ f(s) = (\cos 4\pi s, \sin 4\pi s)$

$\underbrace{f \cdot f \cdots f}_{n} = (\cos 2\pi n s, \sin 2\pi n s)$

$f^{-1}(s) = (\cos 2\pi s, -\sin 2\pi s)$

$\underbrace{[\cos 2\pi s, \sin 2\pi s]}_{f} \to 1$

$f \cdot f \to 2$

$f \cdots f \to 1 + 1 \cdots + 1 = n$

$f^0 \to 0$

$f^{-1} \to -1$

$\underbrace{f^{-1} \cdots f^{-1}}_{n} \to -n$

From above we get that $\pi_1(S^1) \cong \mathbb{Z}$ as groups, as it is homomorphic and bijective

Q(A) This is the Cone over S^1 – a topological space obt by taking a Circle and shrinking it continuously to a point (the apex of the Cone)

Mathematically, the Cone over a space X is defined as

$$CX = (X \times [0,1]) / (X \times \{0\})$$ (Collapse to base the

- $X = S^1$ the Circle
- The Cone $C(S^1)$ is a 2D Surface with one singular point at the

$$\pi_1(C(X)) = \{e\}$$

- Every Point in the Cone can be connected to the apex by a st. line (Contractible path)
- Any loop in the Cone can be Continuously deformed to the apex – hence, it's null-homotopic
- So all loops are equivalent to the constant loop at the

Ans 16 (a) The fundamental group of the 2-sphere S^2 is:

$$\pi_1(S^2) = 0$$

This means that every loop on the 2-sphere can be continuously shrunk to a point – in other words 2 sphere is simply connected

Theorem used: If a topological space is path-connected and every loop can be contracted to a point, then its fundamental group is trivial, i.e. $\pi_1 = 0$

- Path Connected: Any two points on the sphere can be joined by a continuous path.
- Every loop is null homotopic: Any closed curve on S^2 can be continuously deformed into a point

(b) Any knot K homeomorphic to $S^1 = \{(x,y) \mid x^2+y^2=1\}$

$\exists$ $h: S^1 \to K$ s.t. h is 1-1, onto, cont. such

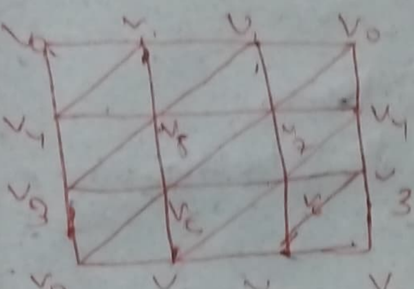
cont. inverse

$\pi_1(K) \cong \pi_1(S^1) \cong (\mathbb{Z}, +)$

AMIQ 1)

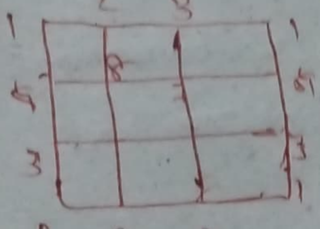
$\chi(C_4) = 9 - 12 = -3$

2) Torus $S^1 \times S^1$



$\chi(S^1 \times S^1) = \alpha_0 - \alpha_1 + \alpha_2 = 9 - 27 + 18 = 0$.

3)



$\chi(S, M) = 15 - 51 + 34$
$= -2$

4).

$\chi(S, M) = 7 - 12 + 8 = 3$

5)

$\chi(S M) = 6 - 9 + 4 = 1$

K^n any simplicial complex of dim n we have defined n+1 abelian hom. grp of K^n.

$H_0(K^n), H_1(K^n), \ldots H_n(K^n)$

Any abelian grp $H_i \cong F_i \oplus T_i$
free grp, torsion grp, where

$H_n(K^n) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$
β_i is betti number
$\Sigma(-1)^i \chi(K^n) = \Sigma(-1)^i \beta_i$

Let K^n be an n-dimensional simplicial complex dim = n means simplex σ_i of highest dimension in K^n is σ_n.
We will associate with K^n a set of (n+1) abelian group denoted $H_0(K^n), H_1(K^n), \ldots \ldots H_n(K^n)$ called the simplicial homology group of K^n and each of them is topological invariant of K^n

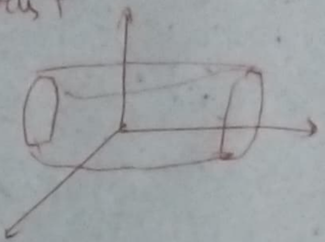
Ans 5→ Classification theorem for 2-dimensional surfaces which are oriented, compact and without boundary.

(1) The sphere S^2, $x^2+y^2+z^2=1$

$x\hat{i}+y\hat{j}+z\hat{k}$

(2) Torus $S^1\times S^1$ the torus T

$(x-2)^2+z^2=1$



(3) A connected sum of g tori T_g.

(4) A connected sum of k projective planes (a non-orientable surface) for $k\geq 1$.

- If the surface is orientable, it is homeomorphic to a sphere with some number g of handles (tori).
- If the surface is non-orientable, it is homeomorphic to a sphere with some number k of cross caps (projective planes)

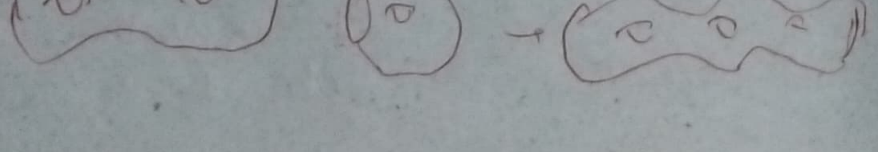
Ans 6→ α_0 of $S_1 \# S_2 = \alpha_0+\alpha_0'-3$

α_1 of $S_1 \# S_2 = \alpha_1+\alpha_1'-3$

α_2 of $S_1 \# S_2 = \alpha_2+\alpha_2'-2$

$\chi(S_1 \# S_2) = (\alpha_0-\alpha_1+\alpha_2)+(\alpha_0'-\alpha_1'+\alpha_2') - \cancel{3+3}\ 2$

$\chi(S_1 \# S_2) = \chi(S_1)+\chi(S_2)-2$



$\chi(T_3) = \chi(T_2) + \chi(T_1) - 2$

$= -2 + 0 - 2 = -4$

$T_3 \# T \cong T_4$

$T_{g-1} \# \;\; \cong T_g$

$\chi(T_g) = \chi(T_g) + \chi(T_{g-1}) - 2$

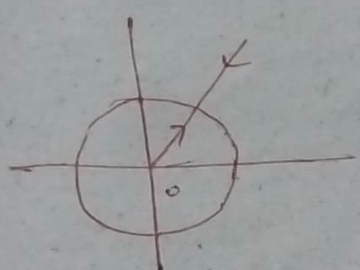
$= \cancel{2} - 2g - \cancel{2}$

$= -2(g-1)$

Ans 13) Suppose $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ via h

$\Rightarrow \mathbb{R}^n \setminus \{0\}$ is homeomorphic to $\mathbb{R}^m \setminus \{h(0)\}$

$\mathbb{R}^n \setminus \{x_0\} \xrightarrow[\text{retract}]{\text{def}} S^{n-1}$



$h(\vec{x}, t) = (1-t)\vec{x} + \dfrac{t\vec{x}}{\|\vec{x}\|}$

S^{n-2} is homeo to S^{m-2}

$\Rightarrow n-2 = m-2$

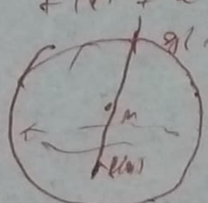
$\Rightarrow n = m$

Ans 18) $D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \le 1\}$

Let $f: D^n \to D^n$ be any cnt. func. Then f has atleast one fixed point in D^n, x_0, i.e $f(x_0) = x_0$.

Proof by the method of Contradiction $\exists\, f: D^n \to D^n$ s.t

$f(x) \neq x \;\; \forall x \in D^n$



$\partial(D^n) = S^{n-1}$

Suppose $f(x) \neq x \;\forall x$ join $f(x)$ to x by st. line and extend it to hit the boundary call that point $g(x)$

$i: S^{n-1} \to D_n$ (inclusion map)

$g: D_n \to S^{n-1}$

$$\begin{tikzcd} S^{n-1} \arrow[r, "i"] \arrow[rr, bend right] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

$i \circ g = I$ (identity)

$$\begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[r, "i"] \arrow[rr, bend right] & H_{n-1}(D^n) \arrow[r, "g'"] & H_{n-1}(S^{n-1}) \end{tikzcd}$$

(homology group)

$$\begin{tikzcd} (\mathbb{Z},+) \arrow[r] \arrow[rr, bend right] & \{0\} \arrow[r] & (\mathbb{Z},+) \end{tikzcd}$$

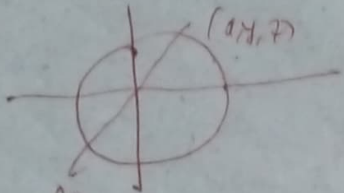
$i \circ g' =$ identity

Contradiction

$\Rightarrow f(x) = x$ for some $x \in D^n$.

Ans7 (a) $f: S^2 \to S^2$

$(x,y,z) \to (-x,y,-z)$ does not have any fixed point



$f(x,y,z)$
$\parallel$
$(-x,y,-z)$

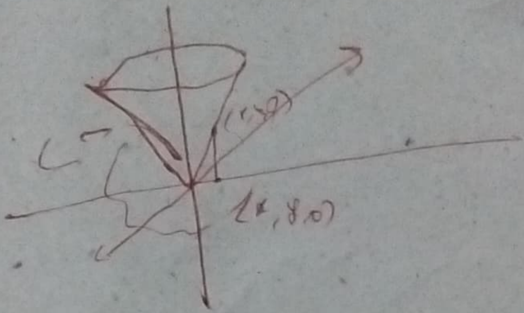
(b) $f: S^1 \times S^1 \to S^1 \times S^1$

$(e^{i\theta_1}, e^{i\theta_2}) \to (e^{i(\theta_1+\pi)}, e^{i\theta_2})$

No fixed point

$x^2+y^2=z^2$, $0 \le z \le 1$

(9)



$f: C \to D^2$

$(x,y,z) \to (x,y,0)$

∴ One has fixed point as it is homeomorphic to a disc.

Ans 1.3, (a) $K_1 = [3, 1, 4, 1]$