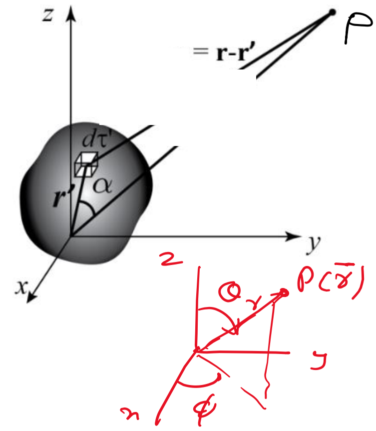
Department of Physics
IIT Kanpur, 2021-22 I[st] Semester

PHY103AA (Physics - II) **Tutorial # 5** **04-01-2022**

Problem 5.1: A thin insulating rod, running from $z = -a$ to $z = +a$, carries the a line charge $\lambda(z)$ as, (a) $\lambda(z) = k\cos(\pi z/2a)$, (b) $\lambda(z) = k\sin(\pi z/a)$ and (c) $\lambda(z) = k\cos(\pi z/a)$, where k is a constant. For each case, find the leading order term in the multipole expansion of the potential.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta)\,\rho(\vec{r}\,')\,dz'$$



For the given line charge,

$\vec{r} \equiv (r,\theta,\phi)$ $\quad \rho\, dz \longrightarrow \lambda\, dz$

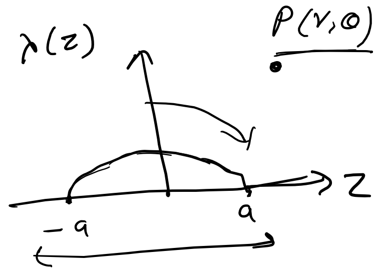
$r' \longrightarrow z$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{-a}^{a} z^n P_n(\cos\theta)\,\lambda(z)\,dz = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{r^{n+1}}\, I_n$$

(a) $\lambda(z) = k\cos\left(\frac{\pi z}{2a}\right)$

$$n=0, \quad I_0 = k\int_{-a}^{a}\cos\left(\frac{\pi z}{2a}\right)dz = \frac{4ak}{\pi}$$

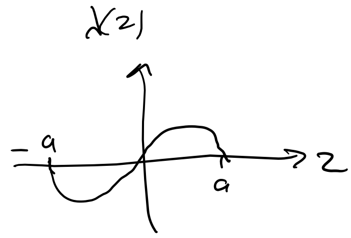
$$\boxed{V(r,\theta) \simeq \frac{1}{4\pi\epsilon_0}\left(\frac{4ak}{\pi}\right)\cdot\frac{1}{r}} \Longrightarrow \text{monopole}$$



(b) $\lambda(z) = k\sin\left(\frac{\pi z}{a}\right) \Longrightarrow I_0 = 0$

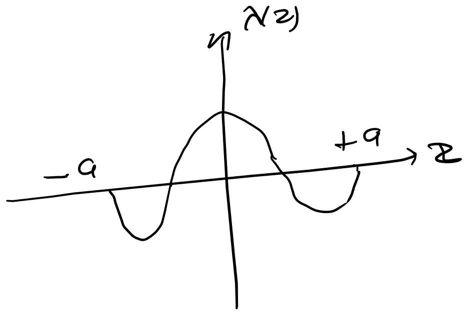
$$I_1 = k\int_{-a}^{a} z\sin\left(\frac{\pi z}{a}\right)dz = k\cdot\frac{2a^2}{\pi}$$

$$\boxed{V(r,\theta) \simeq \frac{1}{4\pi\epsilon_0}\left(\frac{2a^2k}{\pi}\right)\frac{\cos\theta}{r^2}} \Longrightarrow \text{dipole}$$



(c)

$$\lambda(z) = k\cos\left(\frac{\pi z}{a}\right)$$



$$I_0 = I_1 = 0$$

$$I_2 = k\int_{-a}^{a} z^2 \cos\left(\frac{\pi z}{a}\right) dz$$

$$= -\frac{4a^3 k}{\pi^2}$$

$$\boxed{V(r,\theta) = \frac{1}{4\pi\epsilon_0}\,\frac{(3\cos^2\theta - 1)}{2r^3}\left(-\frac{4a^3k}{\pi^2}\right)}$$ $\longrightarrow$ Quadrupole

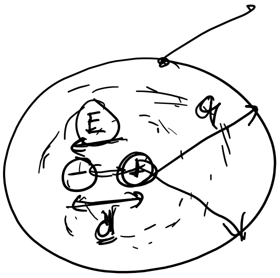
Problem 5.2: According to *Quantum Mechanics*, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a}$$

where q is the electron charge and a is the Bohr radius. Find the atomic polarizability of such an atom.

(electronic)

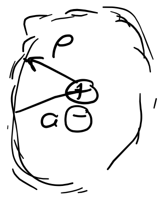
E_{ext}



H^+

$E_{ext} = 0$

$p = q \cdot d = 0$



$r \simeq a$

$d \ll a$

$p = \alpha E_{ext}$, $p = q d.$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q_{enc}}{r^2}$$

$$Q_{enc} = \int \rho \, d\tau = \frac{q}{\pi a^3} \int_0^r \int_0^\pi \int_0^{2\pi} e^{-2r'/a} r'^2 dr' \sin\theta \, d\theta \, d\phi$$

$$= \frac{q \cdot 4\pi}{\pi a^3} \int_0^r e^{-2r'/a} r'^2 dr'$$

E_{ext} d E_{in}

$$Q_{enc} = q\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$$

at $r = d$

$$\Rightarrow E_{ext} = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2}\left[1 - \left(1 - \frac{2d}{a} + \frac{1}{2}\left(\frac{2d}{a}\right)^2 - \frac{1}{6}\left(\frac{2d}{a}\right)^3 \cdots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right]$$

$$\simeq \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \frac{4}{3} \frac{d^3}{a^3} = \frac{1}{3\pi\epsilon_0 a^3} \cdot qd$$

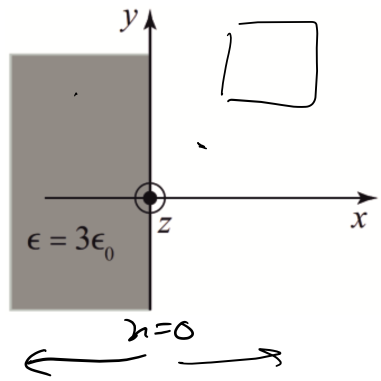
$$\Rightarrow p = \alpha E_{ext} \Rightarrow \boxed{\alpha = 3\pi\epsilon_0 a^3}$$

For uniform charge density $\Rightarrow$ $\boxed{\alpha = 4\pi\epsilon_0 a^3}$

Problem 5.3: Consider the situation shown in the figure. The region of space with $x < 0$ is uniformly filled with a dielectric of permittivity $\varepsilon = 3\varepsilon_0$ and has the electric field in the region given by $\vec{E} = (1 - x)\hat{x} + 2\hat{y} + 3\hat{z}$. $x > 0$ region is the vacuum. Assume no free surface charge on the boundary at $x = 0$.

(i) Calculate the electric field $\vec{E}(0^+, y, z)$ in the vacuum region at $x = 0^+$ (i.e., very close to the boundary at $x = 0$ in the vacuum region).

(ii) Calculate the polarization and find the resulting bound charge densities.



(i) $= (1-x)\hat{x} + 2\hat{y} + 3\hat{z} \qquad x < 0$

B.C. tells that the parallel component of E field is always continuous.

for $x = 0^+$

$$\left.\begin{aligned} E_y(0^+) &= E_y(0^-) = 2 \\ E_z(0^+) &= E_z(0^-) = 3 \end{aligned}\right]$$

In the absence of free charge, the normal component of $\vec{D}$ shows discontinuity.

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f = 0$$

$$\Rightarrow D_x(0^+) = D_x(0^-)$$

$$\Rightarrow \epsilon_0 E_x(0^+) = \epsilon E_x(0^-) = 3\epsilon_0(1-x)\Big|_{x=0}$$

$$\Rightarrow E_x(0^+) = 3$$

$$\Rightarrow \vec{E}(0^+, y, z) = \underline{3}\,\hat{x} + 2\hat{y} + 3\hat{z}$$

B.C. of $\vec{E}$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$$

$$\sigma = \sigma_f + \sigma_b = \sigma_b$$

$$\Rightarrow E_x(0^+) = E_x(0^-) + 2$$

$$= (1-x)\Big|_{x=0} + 2$$

$$= 3$$

(ii) $\vec{P} = \vec{D} - \epsilon_0\vec{E} = \epsilon\vec{E} - \epsilon_0\vec{E} = 2\epsilon_0\vec{E}$

$$\boxed{\vec{P} = 2\epsilon_0(1-x)\hat{x} + 4\epsilon_0\hat{y} + 6\epsilon_0\hat{z}}$$

surface bound charge density $\sigma_b = \vec{P} \cdot \hat{n} \Big|_{(x=0)}$

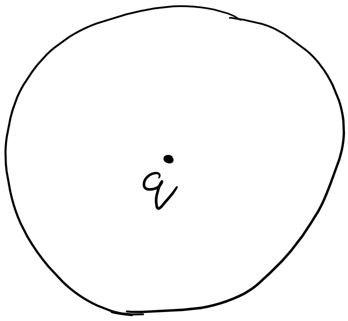
$$= 2 C_0$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{dP}{dx}$$

$$= 2 C_0$$

Problem 5.4: A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Find the electric field, the polarization, and the bound charge densities. What is the total bound charge on the surface? Where is the compensating negative bound charge located?

$+$ at $\vec{r}=0$



$$\int \vec{D}\cdot d\vec{a} = Q_f$$

$$\Rightarrow \vec{D} = \frac{q}{4\pi r^2}\hat{r}$$

$$\vec{E} = \frac{D}{\epsilon} = \frac{q}{4\pi\epsilon_0(1+\chi_e)}\cdot\frac{\hat{r}}{r^2}$$

$$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{q\chi_e}{4\pi(1+\chi_e)}\cdot\frac{\hat{r}}{r^2}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{q\chi_e}{4\pi(1+\chi_e)}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) = -\frac{q\chi_e}{(1+\chi_e)}\delta^3(\vec{r})$$

$-\ \vec{r}=0$

$$\boxed{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})}$$

$$\sigma_b = \vec{P}\cdot\hat{r}\Big|_{r=R} = \frac{q\chi_e}{4\pi(1+\chi_e)R^2}$$

Total surface charge: $Q_{surface} = \sigma_b \cdot 4\pi R^2$

$$= \frac{q\chi_e}{(1+\chi_e)}$$

$$\boxed{\int \rho_b\, d\tau = Q_{surface}}$$

$$\int \delta(\vec{r}-\vec{a})\, d\tau = 4\pi\delta(a)$$