Problem Set - 4
MTH-204, MTH-204A
Abstract Algebra

1. Prove that the additive group $\mathbb{Q}$ of rational numbers is not finitely generated.

2. Let G be a finite abelian group of order n and suppose the integer r is relatively prime to n. Prove that every $g \in G$ can be written as $g = x^n$ with $x \in G$.

3. Describe all homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}$, and determine which are injective, which are surjective, and which are isomorphisms.

4. Let $G = \mathbb{R} \setminus \{0\}$ be the group of nonzero real numbers under multiplication. Suppose r is a positive integer. Show that $x \mapsto x^r$ is a homomoprhism. Determine the kernel, and determine r so that the map is an isomorphism.

5. Let G be the group of polynomials in x with real coefficients. Define the map $p(x) \mapsto P(x) = \int p(x)dx$ such that $P(0) = 0$. Show that f is a homomorphism, and determine its kernel.

6. Determine all homomorphisms from $\mathbb{Z}_n$ to itself.

7. Let G be the group of nonzero complex numbers under multiplication and let N be the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplication.

8. Prove that for a group G if $G/Z(G)$ is cyclic, then G is abelian.

9. Prove that if a group contains exactly one element of order 2, then that element is in the center of the group.

10. Find the center of the following groups.

a. $GL_n(\mathbb{R})$, b. D_n, c. Q_8.

11. What is the largest order of an element of S_{12} ?

12. If $a > 1$ is an integer, prove that n divides $\phi(a^n - 1)$.