Note that ψ_m is an eigenvector for the forward difference operator D_h^+, the backward difference operator D_h^- and the centered difference operator D_h. For example,

$$D_h^-\psi_m(x) = \frac{\psi_m(x) - \psi_m(x-h)}{h} = \frac{1 - e^{-2\pi imh}}{h}\psi_m(x).$$

It is also an eigenvector for the forward and backward shift operators S_h^+ and S_h^- defined as

$$S_h^+v(x) = v(x+h), \qquad S_h^-v(x) = v(x-h),$$

where

$$S_h^+\psi_m(x) = e^{2\pi im(x+h)} = e^{2\pi imh}\psi_m(x),$$
$$S_h^-\psi_m(x) = e^{2\pi im(x-h)} = e^{-2\pi imh}\psi_m(x).$$

Now, consider forward-backward difference method in the operator form:

$$u^{j+1} = (1-\lambda)u^j + \lambda S_h^- u^j = \big((1-\lambda)I + \lambda S_h^-\big)u^j.$$

Thus, for stability of the method, we need that the eigenvalues are less than or equal to one in magnitude, that is

$$\left|1 - \lambda + \lambda e^{-2\pi imh}\right| \leq 1.$$

As $1 - \lambda + \lambda e^{-2\pi imh}$ describes are circle centered at $1-\lambda$ of radius $|\lambda|$, we see that the method is stable if and only if $0 \leq \lambda \leq 1$.

For the forward-centered method for advection equation, the eigen-values are

$$-\frac{\lambda}{2}e^{2\pi imh} + 1 + \frac{\lambda}{2}e^{-2\pi imh} = 1 - i\lambda\sin(2\pi mh).$$

Numerical Methods for PDE: Hyperbolic PDE

Akash Anand
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As every eigenvalue has magnitude greater that one, the method is indeed unconditionally unstable.

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Similarly, for forward-forward difference method, the eigenvalues are

$$-\lambda e^{2\pi imh} + 1 + \lambda$$

that lie on the circle centered at $1+\lambda$ *of radius* $|\lambda|$*. This confirms our earlier observation that, for every* $\lambda > 0$*, the method is unstable.*

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Let us now look at the backward-centered difference method for the advection equation given by

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Thus the method suggests discretization of the equation $\partial u/\partial t + c\partial u/\partial x - (ch/2\lambda)\partial^2 u/\partial x^2 = 0$. This is an advection-diffusion equation.

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Thus the method suggests discretization of the equation $\partial u/\partial t + c\partial u/\partial x - (ch/2\lambda)\partial^2 u/\partial x^2 = 0$. This is an advection-diffusion equation. The method can be viewed as a forward-centered scheme with small diffusion added.