3 fields medals

Topics in Topology

- Knot Theory → cryptography, Quantum computing, Physics
- Fundamental Group
- Simp Hom Group → Statistics & Data Science

  who dis (TDA, eg ayasdi)

Knot Theory

Def^N: Let S^1 be the unit circle $x^2+y^2=1$. A knot K is an embedding of S^1 into R^3.

Eg 1) O_1 - unknot

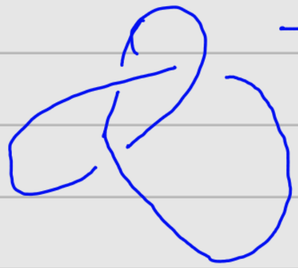
2) Trefoil Knot (Right Hand) 3 crossing points

3_1

Remark:

$A \simeq B$ are said to be equivalent if one can be transformed into the other through cts variation.

This actually exists in R^3, the diagram is a planar representation.



→ Left Hand Trefoil

3_1^*.

$3_1 \not\simeq 3_1^*$

3) Figure-8 knot (because it looks like 8-duh)

=> 4_1 (4 crossing points)

4

=> 5_1.

इतिहास (गांठ सिद्धांत)

Fairly dumb proposal in hindsight.

→ Began with chemists in 1880. William Thomson (a.k.a Lord Kelvin) proposed atomic structures were knots in either.

→ PG Tait decided to make a list of all knots.

→ correctly classified the first 800 knots.

→ Tait also made 4 conjectures creatively called the Tait conjectures. - Solved through Jones Polynomials.

→ Turns out, atomic structures were not knots. Chemists and physicists then dropped the subject.

→ At that point, Mathematicians took over. Developed classical knot theory. (1930-1984)

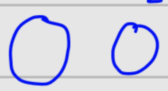
→ In 1984, a mathematician in Operator theory discovered a new knot invariant - Jones Polynomial

↳ Received Fields Medal in 1990.

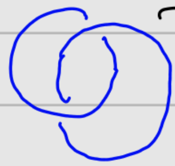
Links

Def^N: A link is a finite ordered collection of knots that do not intersect each other. $L = \{K_1, \dots, K_n\}$

Ex 1) ← O_2 (Unlink)



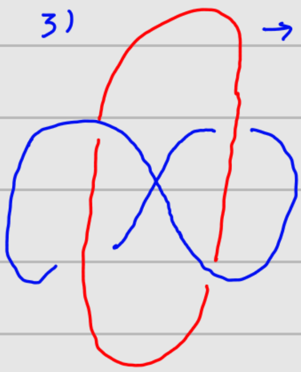
2 2 circles linked



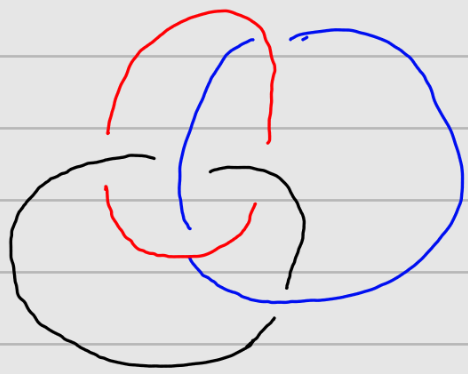
→ Hopf Link

2_1^2 → number of components; → order; number of crossings

3) → 5_1^2



4 Borromean Rings



→ any 2 circles are unlinked, all circles together are linked.

Def^N: (Alternating knots / Links)

Has atleast on diagram that goes over, under, over.

8_{19} 8_{20} 8_{21} are alternating

Def^N (Embedding)

Image of a 1-1 and continuous map. In case of a knot, k is the image of a map from $S^1 \mapsto \mathbb{R}^3$.

Towards Equivalence.

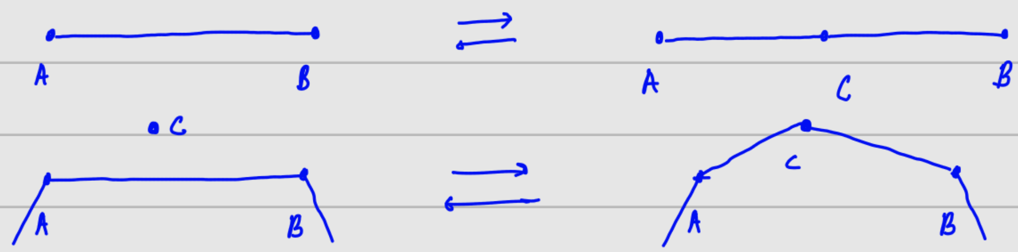
when are two knots K_1 and K_2 equivalent.

when are two links L_1 and L_2 equivalent.

Def^N (Elementary Knot Moves)

(1)

(2)



Def^N knots K_1 and K_2 are equivalent if it is possible to go from K_1 to K_2 by finitely many knot moves.

Note:

✓ If $K_1 \cong K_2$, we would prove this equivalence by demonstrating a finite series of knot moves that take K_1 to K_2.

✓ But if $K_1 \ncong K_2$ how do we prove ?

↳ need knot invariants to distinguish b/w knots

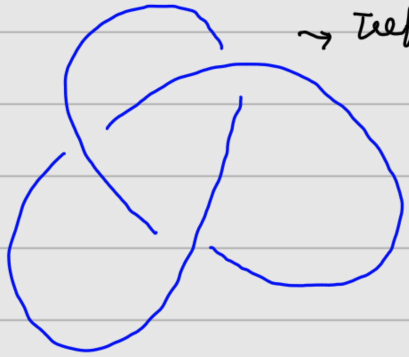
Def^N (Knot Invariant) can be a number, polynomial or property.

A knot invariant $\mu(K)$ such that if $K_1 \cong K_2 \Rightarrow \mu(K_1) = \mu(K_2)$

$\Rightarrow$

$$\mu(K_1) \neq \mu(K_2) \Rightarrow K_1 \ncong K_2$$

1) Crossing Number ($c(K)$): $c(K)$ is the minimum number of crossing points over all possible diagrams.

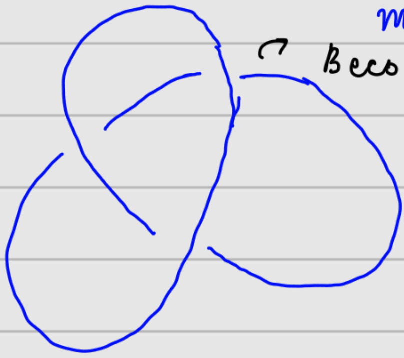


→ Trefoil $c(K) = 3$

also trefoil

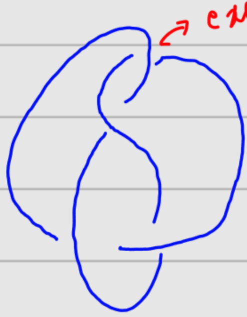
is a reducible crossing.

2) Unknotting : Minimum number of exchanges needed to make K - trivial.



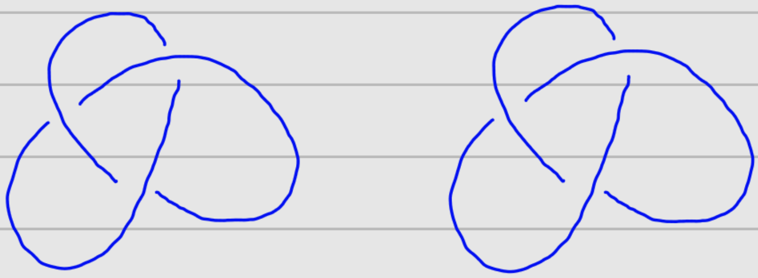
Becomes O_1 after 1 exchange.

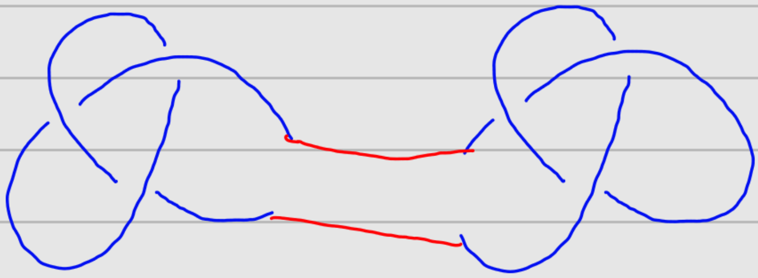
exchange here



Def^n: (connected sum of knots)

cut a small arc away from cross. from K_1, K_2 and glue ends.





Def^n A knot is said to be a prime knot if it is written as a sum of K_1 and K_2, atleast one out of $K_1 \simeq O_1$ or $K_2 \simeq O_2$ is true.

Thm (Prime Decomposition Theorem)

Any knot K has a unique decomposition $K \simeq K_1 \# K_2 \cdots \# K_n$ where each K_i is prime.

open question

Q. $c(K_1 \# K_2) \overset{?}{=} c(K_1) + c(K_2)$

- True for alternative notes.

Q. $u(K_1 \# K_2) \overset{?}{=} u(K_1) + u(K_2)$

- Very open.

HW: Find unknotting numbers for $5_2, 6_2, 6_3, 7_2, \ldots, 8_1$

The unknotting numbers and crossing numbers are not sufficient.

$u(6_1) = u(6_2) \qquad c(6_1) = c(6_2)$

$\therefore$ we need more invariants.

In 1928, a topologist Alexander discovered a new knot invariant – Alexander Polynomial. $\Delta_K(t)$ (-ve powers also allowed, so a formal polynomial)

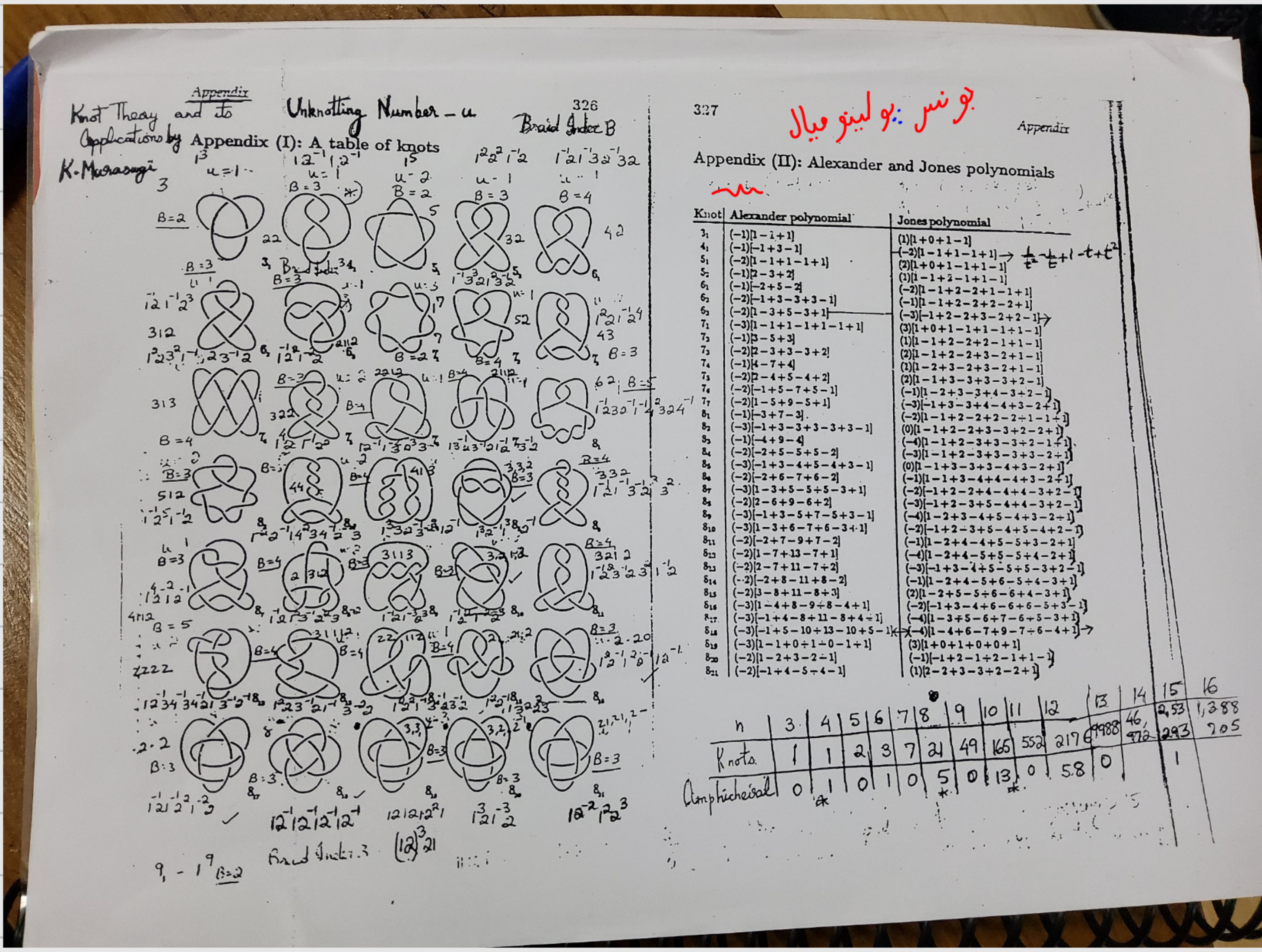
All polynomials are symmetric.

Knot	Alexander Polynomial
3_1	$(-1)[1 - 1 + 1] \equiv t^{-1}(1 - t + t^2) \equiv 1/t - 1 + t$

The first 35 knots (upto crossing number 8) have different polynomials.

However, unfortunately, it is not a complete invariant.

$\exists K_1, K_2 \quad K_1 \not\simeq K_2, \quad \Delta_{K_1}(t) = \Delta_{K_2}(t)$



Knot	Alexander polynomial	Jones polynomial
3_1	(−1)[1 − 1 + 1]	(1)[1 + 0 + 1 − 1]
4_1	(−1)[−1 + 3 − 1]	(−2)[1 − 1 + 1 − 1 + 1] → $\frac{1}{t^2} - \frac{1}{t} + 1 - t + t^2$
5_1	(−2)[1 − 1 + 1 − 1 + 1]	(2)[1 + 0 + 1 − 1 + 1 − 1]
5_2	(−1)[2 − 3 + 2]	(1)[1 − 1 + 2 − 1 + 1 − 1]
6_1	(−1)[−2 + 5 − 2]	(−2)[1 − 1 + 2 − 2 + 1 − 1 + 1]
6_2	(−2)[−1 + 3 − 3 + 3 − 1]	(−1)[1 − 1 + 2 − 2 + 2 − 2 + 1]
6_3	(−2)[1 − 3 + 5 − 3 + 1]	(−3)[−1 + 2 − 2 + 3 − 2 + 2 − 1]
7_1	(−3)[1 − 1 + 1 − 1 + 1 − 1 + 1]	(3)[1 + 0 + 1 − 1 + 1 − 1 + 1 − 1]
7_2	(−1)[3 − 5 + 3]	(1)[1 − 1 + 2 − 2 + 2 − 1 + 1 − 1]
7_3	(−2)[2 − 3 + 3 − 3 + 2]	(2)[1 − 1 + 2 − 2 + 3 − 2 + 1 − 1]
7_4	(−1)[4 − 7 + 4]	(1)[1 − 2 + 3 − 2 + 3 − 2 + 1 − 1]
7_5	(−2)[2 − 4 + 5 − 4 + 2]	(2)[1 − 1 + 3 − 3 + 3 − 3 + 2 − 1]
7_6	(−2)[−1 + 5 − 7 + 5 − 1]	(−1)[1 − 2 + 3 − 3 + 4 − 3 + 2 − 1]
7_7	(−2)[1 − 5 + 9 − 5 + 1]	(−3)[−1 + 3 − 3 + 4 − 4 + 3 − 2 + 1]
8_1	(−1)[−3 + 7 − 3]	(−2)[1 − 1 + 2 − 2 + 2 − 2 + 1 − 1 + 1]
8_2	(−3)[−1 + 3 − 3 + 3 − 3 + 3 − 1]	(0)[1 − 1 + 2 − 2 + 3 − 3 + 2 − 2 + 1]
8_3	(−1)[−4 + 9 − 4]	(−4)[1 − 1 + 2 − 3 + 3 − 3 + 2 − 1 + 1]
8_4	(−2)[−2 + 5 − 5 + 5 − 2]	(−3)[1 − 1 + 2 − 3 + 3 − 3 + 3 − 2 + 1]
8_5	(−3)[−1 + 3 − 4 + 5 − 4 + 3 − 1]	(0)[1 − 1 + 3 − 3 + 3 − 4 + 3 − 2 + 1]
8_6	(−2)[−2 + 6 − 7 + 6 − 2]	(−1)[1 − 1 + 3 − 4 + 4 − 4 + 3 − 2 + 1]
8_7	(−3)[1 − 3 + 5 − 5 + 5 − 3 + 1]	(−2)[−1 + 2 − 2 + 4 − 4 + 4 − 3 + 2 − 1]
8_8	(−2)[2 − 6 + 9 − 6 + 2]	(−3)[−1 + 2 − 3 + 5 − 4 + 4 − 3 + 2 − 1]
8_9	(−3)[−1 + 3 − 5 + 7 − 5 + 3 − 1]	(−4)[1 − 2 + 3 − 4 + 5 − 4 + 3 − 2 + 1]
8_{10}	(−3)[1 − 3 + 6 − 7 + 6 − 3 + 1]	(−2)[−1 + 2 − 3 + 5 − 4 + 5 − 4 + 2 − 1]
8_{11}	(−2)[−2 + 7 − 9 + 7 − 2]	(−1)[1 − 2 + 4 − 4 + 5 − 5 + 3 − 2 + 1]
8_{12}	(−2)[1 − 7 + 13 − 7 + 1]	(−4)[1 − 2 + 4 − 5 + 5 − 5 + 4 − 2 + 1]
8_{13}	(−2)[2 − 7 + 11 − 7 + 2]	(−3)[−1 + 3 − 4 + 5 − 5 + 5 − 3 + 2 − 1]
8_{14}	(−2)[−2 + 8 − 11 + 8 − 2]	(−1)[1 − 2 + 4 − 5 + 6 − 5 + 4 − 3 + 1]
8_{15}	(−2)[3 − 8 + 11 − 8 + 3]	(2)[1 − 2 + 5 − 5 + 6 − 6 + 4 − 3 + 1]
8_{16}	(−3)[1 − 4 + 8 − 9 + 8 − 4 + 1]	(−2)[−1 + 3 − 4 + 6 − 6 + 6 − 5 + 3 − 1]
8_{17}	(−3)[−1 + 4 − 8 + 11 − 8 + 4 − 1]	(−4)[1 − 3 + 5 − 6 + 7 − 6 + 5 − 3 + 1]
8_{18}	(−3)[−1 + 5 − 10 + 13 − 10 + 5 − 1]	(−4)[1 − 4 + 6 − 7 + 9 − 7 + 6 − 4 + 1]
8_{19}	(−3)[1 − 1 + 0 + 1 + 0 − 1 + 1]	(3)[1 + 0 + 1 + 0 + 0 + 1]
8_{20}	(−2)[1 − 2 + 3 − 2 + 1]	(−1)[−1 + 2 − 1 + 2 − 1 + 1 − 1]
8_{21}	(−2)[−1 + 4 − 5 + 4 − 1]	(1)[2 − 2 + 3 − 3 + 2 − 2 + 1]

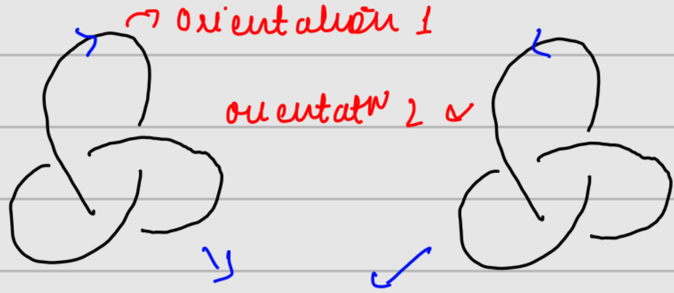
n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Knots	1	1	2	3	7	21	49	165	552	2176	9988	46,972	2,53293	1,388705
Amphicheiral	0	1	0	1	0	5	0	13	0	58	0		1	

In 1950, Conway (of Game of Life fame) found another knot invariant $\nabla_K(z)$ called the Alexander-Conway polynomial.

We can obtain $\Delta_K(t)$ from $\nabla_K(z)$ by the substitution $z = \sqrt{t} - \frac{1}{\sqrt{t}}$

Def^N: Oriented Knot : Orientation is a choice of direction.

Trefoil



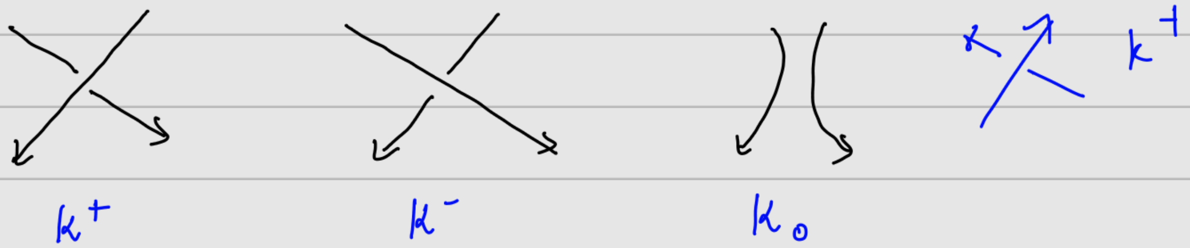
Oriented Trefoil

Def^n: Alexander - Conway Polynomial

Given an oriented knot/link, we assign an AC polynomial $\nabla_K(z)$ by 2-axioms

→ If K is the trivial knot, $\nabla_K z = 1$

→

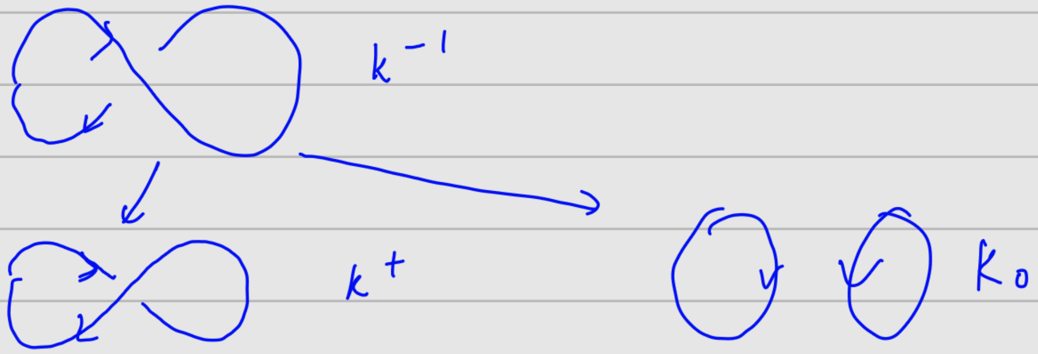


If a knot K differs only at one crossing point, in the manner as above,

$$\nabla_{K^+}(z) - \nabla_{K^-}(z) = z\,\nabla_{K_0}(z)$$

calculation

$\nabla_K(z) = 1$ for O_1

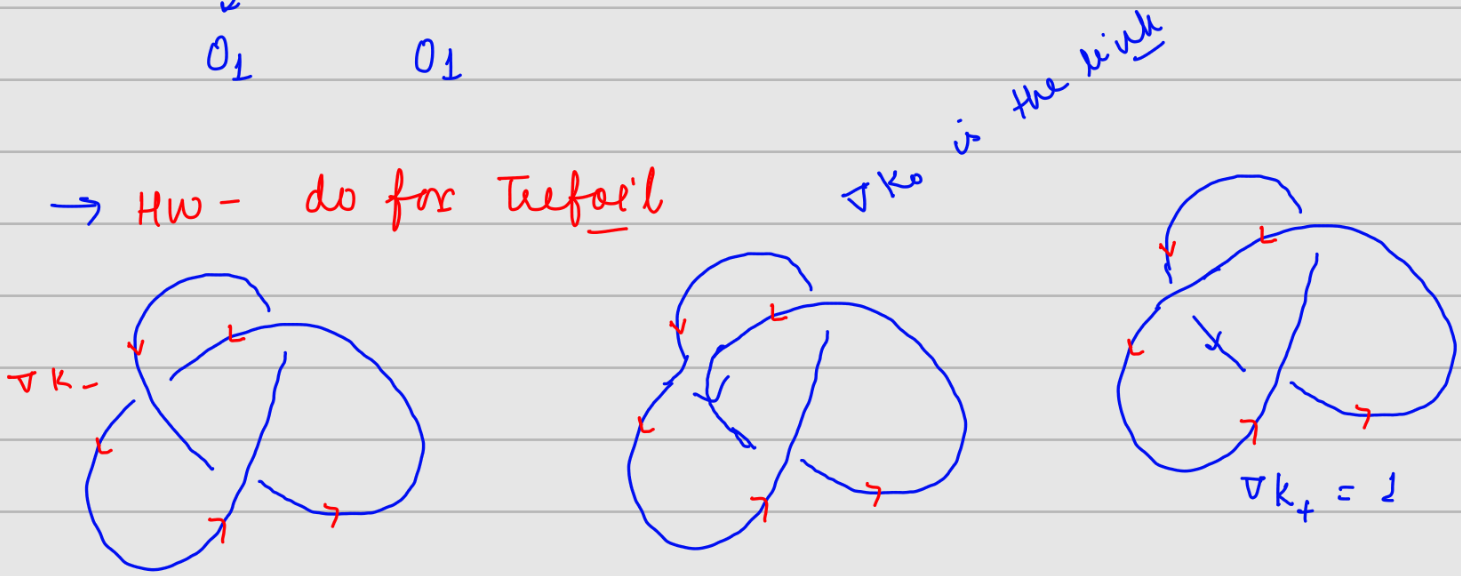


$$\nabla_{K^{-1}} = \nabla_{K^+} - z\,\nabla_{O_2}$$

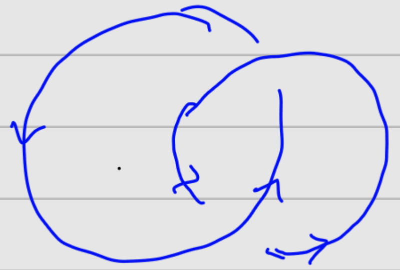
$\wr \quad\quad \wr$

$O_1 \quad\quad O_1$

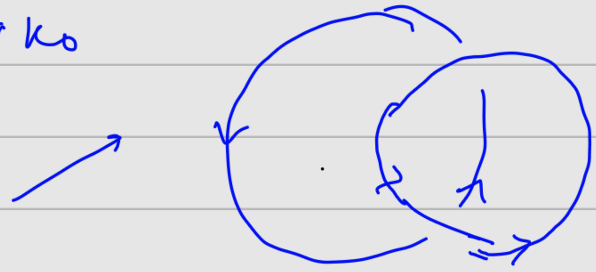
→ HW - do for Trefoil



$\nabla K_- = \nabla K_+ - z \nabla K_0$



K^-



O_2

K^+

O_1

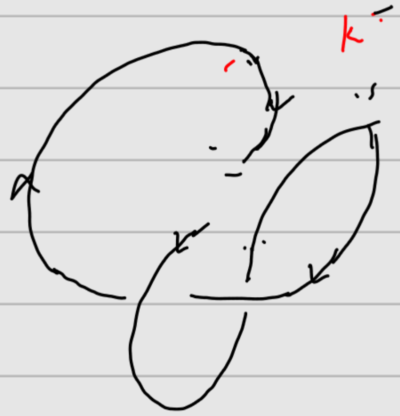
K^0

$\nabla K^- = 0 - z$

$\nabla K_- = 1 + z^2$

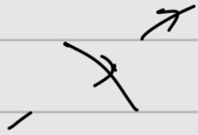
Δ_z Alexa. $= \left(t + \frac{1}{t} - 1\right)$

K^+

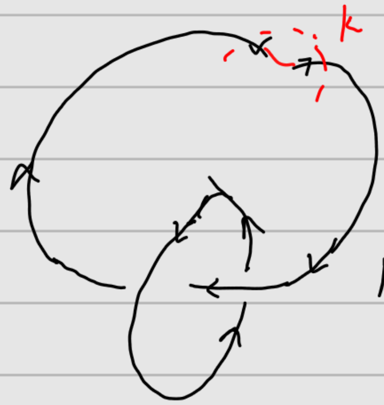


K^-

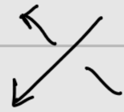
O^1



Hopf link



K^-



K^-

$\nabla K_- = \nabla K_+ + z^2$

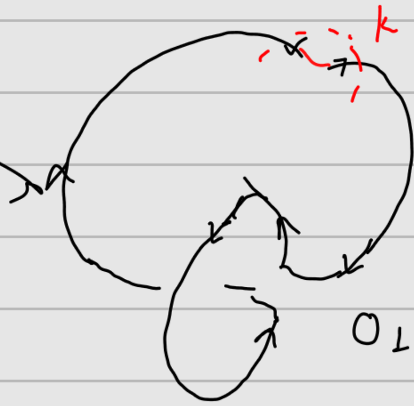
$\nabla K_+ = 1 - z^2$

$= 1 - (t + 1/t - 2)$

$= 3 - t - \frac{1}{t}$

K^+

O_2



K^-

O_1

$\nabla K_- = 0 - z.$

HW

$5_1 \quad 6_1 \quad 6_2$

Aust, Summers

Thm

alexander polynomial

If K is a knot, $\Delta_K(1) = 1$.

Proof:

To go from conway to alexander polynomial, we use the substitution $z = \sqrt{t} - \frac{1}{\sqrt{t}}$, so at $t = 1$, $z = 0$.

To compute conway polynomial, we use the axiom

$$\nabla_{K+}(z) = \nabla_{K-}(z) + z\nabla_{K_0}(z)$$

at $z = 0$,

$$\nabla_{K+}(z) = \nabla_{K-}(z)$$

Now, this means that the value does not change after 1 exchange.

If the knot is unknotted, we are done, for $\nabla_{0_1}(0) = 1$.

If not, we can make finitely many exchanges to get to 0_1.

$\therefore \quad \nabla_K(0) = 1$, i.e $\Delta_K(1) = 1$.

چین سلوتر بار بیرمان سر ط آل.

(≥ 2 components)

Thm: If L is a link, $\Delta_L(1) = 0$

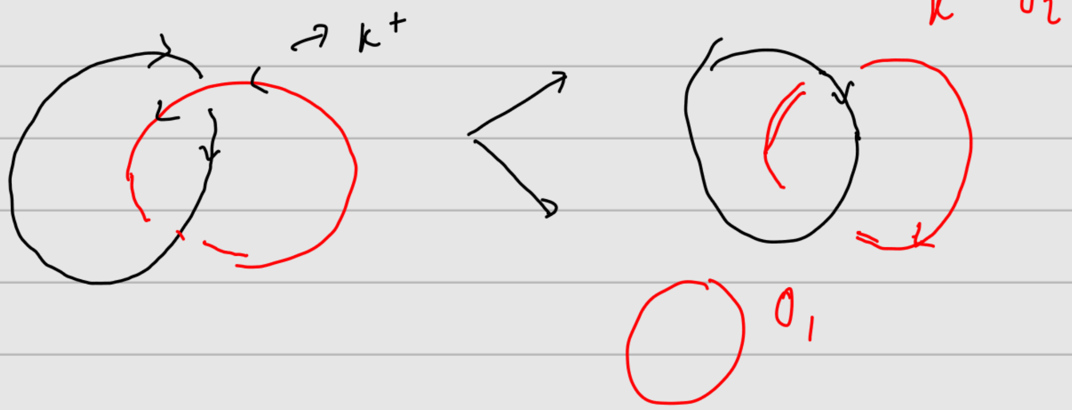
Proof If $z = 0$ $\quad \nabla_{L+}(0) = \nabla_{K-}(0)$

By the previous argument, we can always get to 0_μ.

$\therefore \quad \nabla_L(0) = 0$

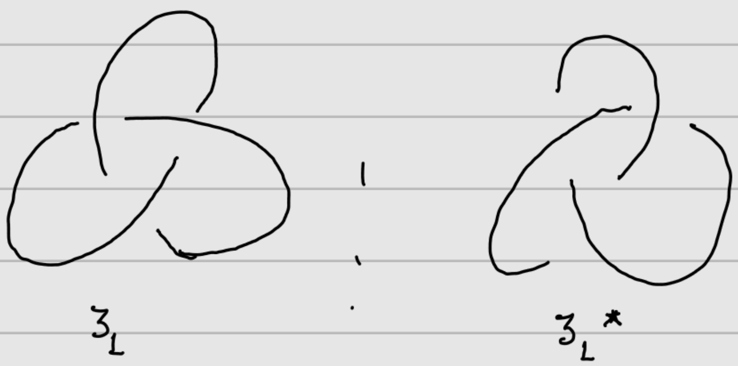
$\therefore \quad \Delta_L(1) = 0$

Example : Alexander Polynomial for links.



$$\nabla_{O_2} = \nabla_{2_1^2} - z(1) \Rightarrow \nabla_{2_1^2} = z$$

The Alexander polynomial, then, is $\sqrt{t} - \frac{1}{\sqrt{t}}$ ⇝ fractional powers oW.



Both have the same Alexander polynomial, however they are not the same.

Jones Polynomial ($V_k(t)$)

In 1984, Jones was studying operator theory and stumbled on a new invariant.

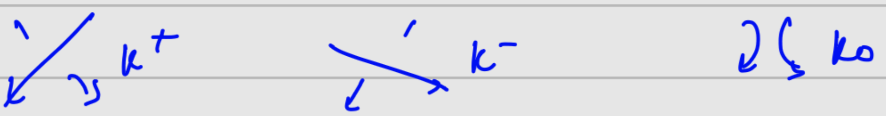
Proved all Tait conjectures.

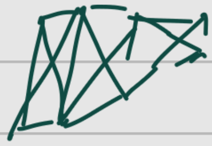
Defn: Jones Polynomial

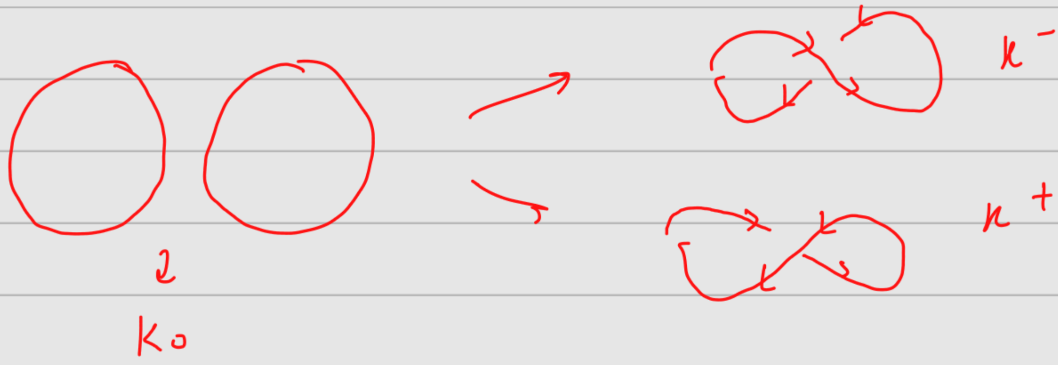
Let K be an oriented knot, then the Jone's polyno. can be uniquely defined from two axioms

a) if K is O_1, $V_{O_1}(t) = 1$

(b) $\frac{1}{t} V_{k^+} - t V_{k^-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{k_0}$





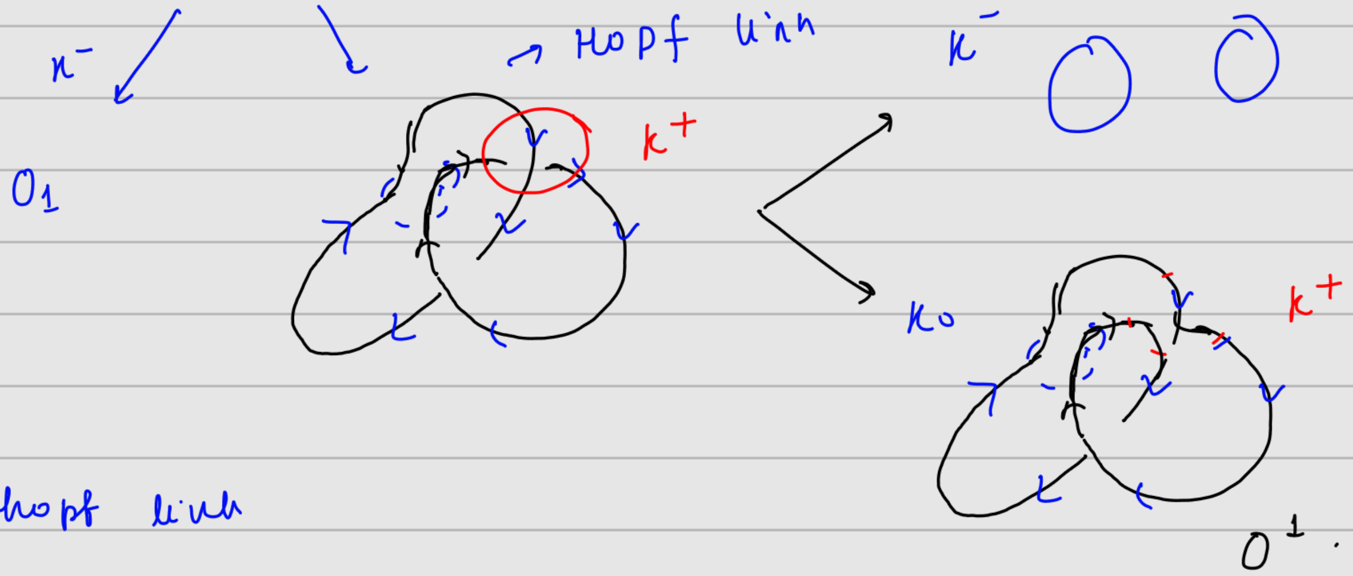


$$\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{O_2} = \frac{1}{t} - t$$

$$V_{O_2} = \left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)\left(\frac{1}{\sqrt{t}} + \sqrt{t}\right)$$

$$V_{O_2} = -\left(\frac{1}{\sqrt{t}} + \sqrt{t}\right)$$

Newsflash: Jone's Polynomial



For hopf link

$$\frac{1}{t} V_{K^+} - t V_{O_2} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) 1$$

$$V_{K^+} = t^2 V_{O_2} + t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)$$

$$= -t^2\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) + t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)$$

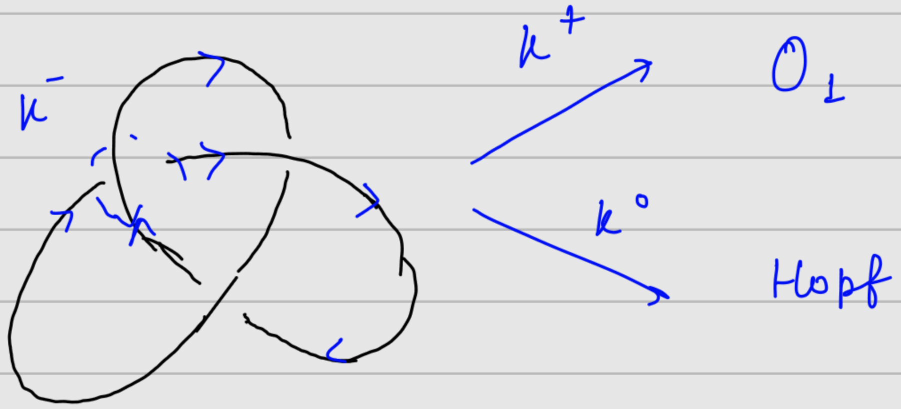
$$\frac{1}{t} V_{K^+} - t V_{K^-} = -t^2\left(t - \frac{1}{t}\right) + t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2$$

$$\frac{1}{t} V_{K^+} - t = -t^3 + t + t\left(t + \frac{1}{t} - 2\right)$$

$$\frac{V_{K^+}}{t} - t = -t^3 + t + t^2 + 1 - 2t$$

$$\frac{V_{K^+}}{t} = -t^3 + t^2 + 1$$

$$= t^3 - t^4 + t$$



$$\frac{1}{t} 1 - t V_{K^-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left[t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) - t^2\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)\right]$$

$$-\cancel{t} V_K = \sqrt{t}\left[t\left(t + \frac{1}{t} - 2\right) - t^3\left[t - \frac{1}{t}\right]\right]$$

Theorem: Suppose K^* is the mirror image of a knot/link K, then $V_{K^*}(t) = V_K(1/t)$.

$\therefore$ Therefore, if $K \simeq K^*$, $\underbrace{V_K(t) = V_K(1/t)}$

This implies palindromic symmetry

Pf: Suppose D is the diagram of K and D^* is its mirror image. If the skein tree diagram of D is R, we form the skein tree diagram of D^*, call it R^* as follows.

When we perform a skein operation at a crossing point c of D to make R, at the corresponding crossing pt of D^*, also perform a skein operation so forming R^*.

$$t^2 \to t^{-2} \qquad t^{-2} \to t^2$$

$$tz \to -\frac{z}{t}$$

$$t^{3/2} - t^{1/2} \to -t^{-1/2} + t^{-3/2}$$

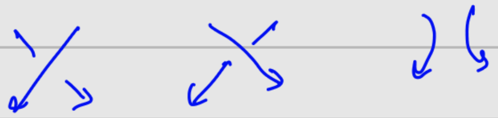
$\therefore$ All co-effs change from t to $1/t$ qed

Within a year of Jones', 8 people found another polynomial invariant. (HOMFLY polynomial (1985))

Def^n: HOMFLY polynomial (1985) is defined as follows.
$P_K(v, z)$ for an oriental knot (or links) K is defined as follows.

Axiom-1 If K is O_1, $P_K(v,z) = 1$

Axiom-2.



K^+ K^- K^0

If 3-knots differ as at one crossing.

$$\frac{1}{v} P_{K^+}(v,z) - v P_{K^-}(v,z) = z P_{K^0}(v,z)$$

Corollary : $v = 1$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$, Homfly is then conway - alexander.

: $v = t$, $z = \sqrt{t} - \frac{1}{\sqrt{t}}$ Jones

Jones Unknotting Conjecture :

does there exist a non-trivial knot with Jones' polynomial 1 ?

Thm : If K is an alternating knot / link and $V_K(t)$ is its Jone's polynomial

If the maximum degree $\Rightarrow n$

minimum $\Rightarrow -m$

span $V_K(t) \Rightarrow n+m$

span $V_K(t) = c(K)$

Corollary: If $K \simeq K^*$, $\Rightarrow c(K) = 2n$ is even.

If knot is not alternating, Taits conjecture is false.

^{15} 2,24 980.

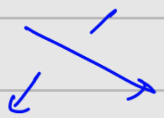
↗ connected sum

Thm : If $K \simeq K_1 \# K_2$, then $V_K(t) = V_{K_1}(t) V_{K_2}(t)$

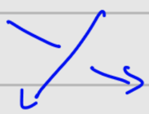
$\Delta_K(t) = \Delta_{K_1}(t) \Delta_{K_2}(t)$

Lemma :

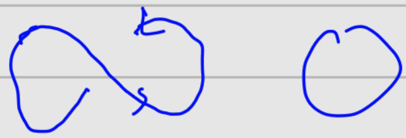
Pf $V_{O_1} = 1$, $V_{O_2} = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)$



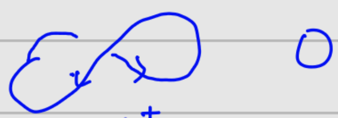
k^-



k^+



k^-



k^+

$$\frac{1}{t} v_{k^+} - t v_{k^-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) v_{O_3}$$

$$\left(\frac{1}{t}\right)\left(-\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)\right) + t\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) v_{O_3}$$

$$\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)\left(t - \frac{1}{t}\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) v_{O_3}$$

$$v_{O_3} = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^2$$

$$\vdots$$

$$\frac{1}{t} (-1)^{n-2}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-2} - t(-1)^{n-2}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-2}$$

$$= \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) v_{O_n}$$

$$= (-1)^{n-2}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-2}\left(\frac{1}{t} - t\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) v_{O_n}$$

$$= (-1)^{n-1}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-1} = \underline{v_{O_n}}$$

QED

Pf:

Def^n: we denote $X \amalg Y \Rightarrow$ disjoint union of X and Y
i.e $X \cup Y$ where $X \cap Y = \Phi$

Lemma $\quad v_{K \amalg O_n} = (-1)^n \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^n v_K(t)$

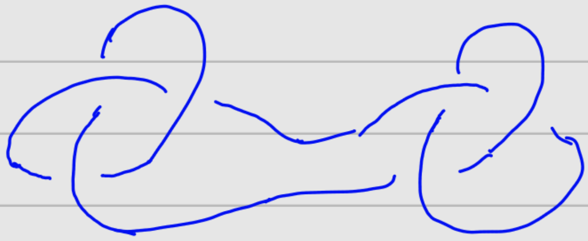
Proof by Induction

$$-\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)\left[(-1)^{n-1}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-1} v_K(t)\right] = \cancel{\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)} v_{K_O}$$

$$(-1)^n \left(\sqrt{t} + \frac{1}{\sqrt{t}} \right)^n V_u(t) .$$

$\square$

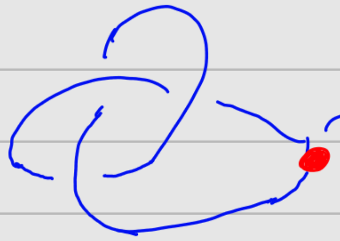
Thm :



$3_1 \# 3_1$

Initially, represent K_2 by a single dot.

(lmao)



Using this representation (ignoring K_2 for a short while) we create a skein diagram for Jones' polynomial as follows.

$$V_{K_1}(t) = f_1(t) V_{O_1}(t) + f_2(t) V_{O_2}(t) + \cdots + \cdots + f_m(t) V_{O_m}(t)$$

[we eventually get to links]

The dot would be in one of the circles.

- - -

we resubstitute K_2 in place of the dot.

$$V_{K_1 \# K_2} = f_1(t) V_{K_2}(t) + f_2(t) V_{O_1 \amalg K_2} + \cdots + f_m(t) V_{O_{m-1} \amalg K_2}$$

$$= \sum f_i(t) (-1)^n \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right)^n V_{K_2}(t)$$

$$= V_{K_2}(t) \sum f_i(t) (-1)^n \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right)^n$$

$$= V_{K_2}(t) V_{K_1}(t)$$

Braid Theory

Began in 1930s by Emil Artin. Introduced the Braid Group B_n as a means of studying knots.

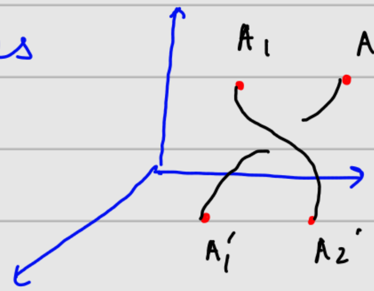
Mathematical braids have applications in Maths, cryptography and Quantum computing.

Braids 2-braid

$A_1 = (1/2, 1/3, 1)$ $A_2 = (1/2, 2/3, 1)$

$A_1' = (1/2, 1/3, 0)$ $A_2' = (1/2, 2/3, 0)$

Join A_1, A_2 to A_1', A_2' by means of curves that



(a) travel monotonically down

(b) do not intersect.

Def^n (n-braid) Let B be a cube $0 \leq x, y, z \leq 1$. We mark n-points $A_1, \dots, A_n$ at the top of the cube and n-points $A_1' \dots A_n'$ at the base of the cube

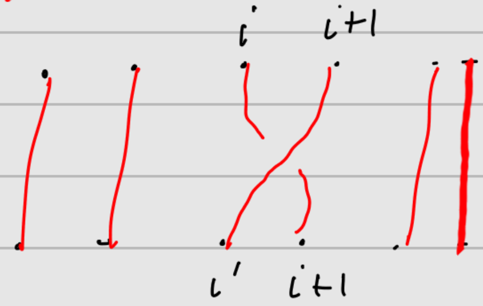
$$A_i = \left(\frac{1}{2}, \frac{i}{n+1}, 1\right) \qquad A_i' = \left(\frac{1}{2}, \frac{i}{n+1}, 0\right)$$

An n-braid is now obtained by joining $A_1, A_2 \dots A_n$ to $A_1', \dots, A_n'$ by means of non-intersecting monotonic curves

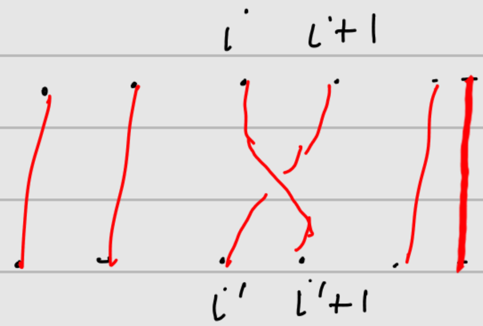
Algebra - Yay

Generators in B_n

σ_i



σ_i^{-1}



Claim: $B_n ::= \{\text{set of all } n\text{-Braids}\}$ is a non-abelian group called the n-th Braid group

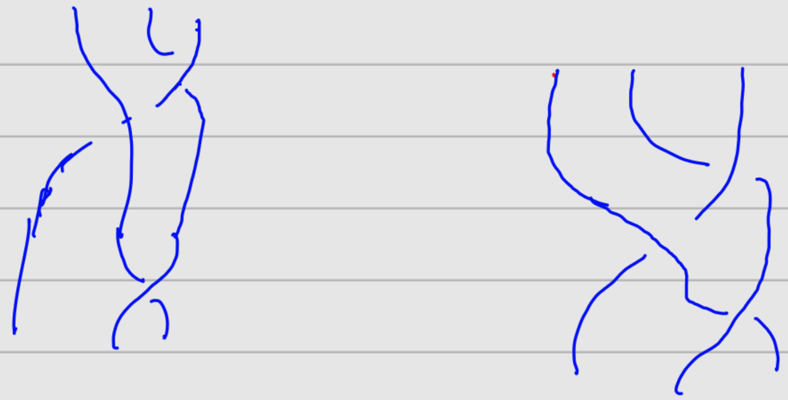
Defn: Let α, β be n-braids we define $\alpha \circ \beta$ as

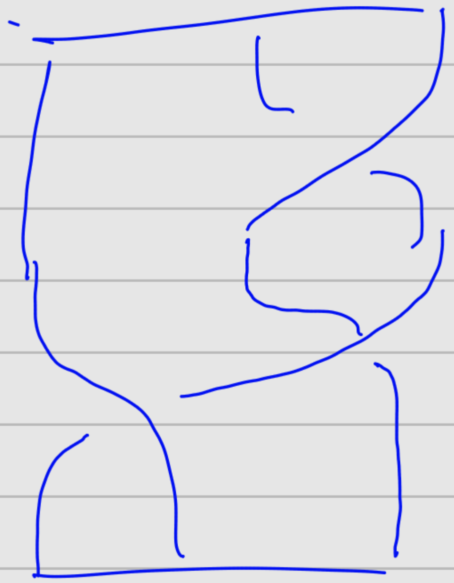
by gluing the base of cube containing α

to topface containing β.

2-nbraids are equivalent if we can move from $\alpha \to \beta$ via finitely many knot moves.

If $\alpha \cong \beta$, $\pi_\alpha = \pi_\beta$





Identity

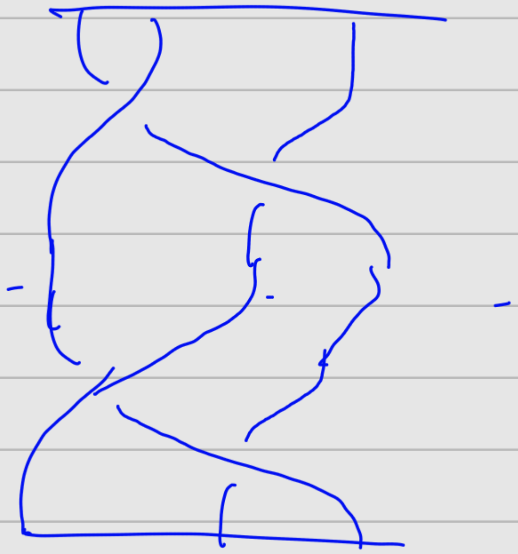
e

$\alpha \cdot e = e \cdot \alpha = \alpha$

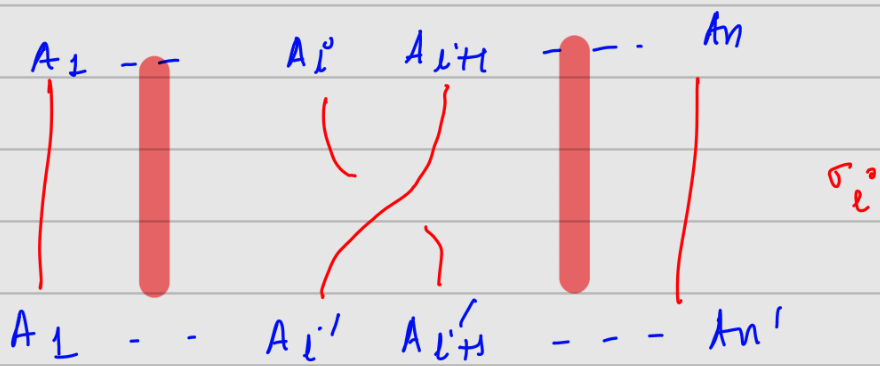
Inverse: Reflection across base.

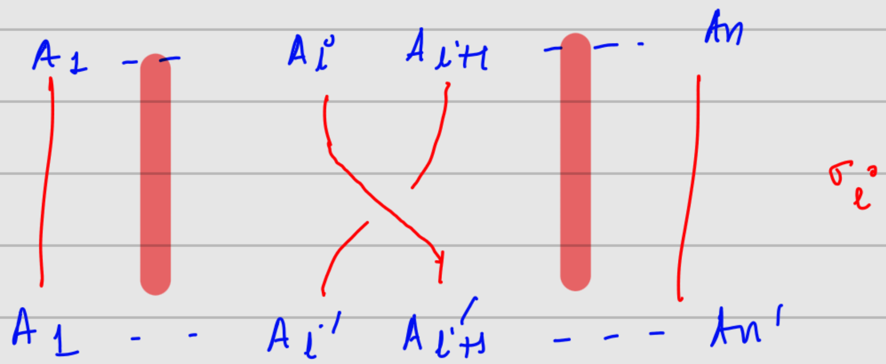
Any braid can be written as a product of generators (finite)

$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$



Among the n-braids (B_n), there exist some special braids.





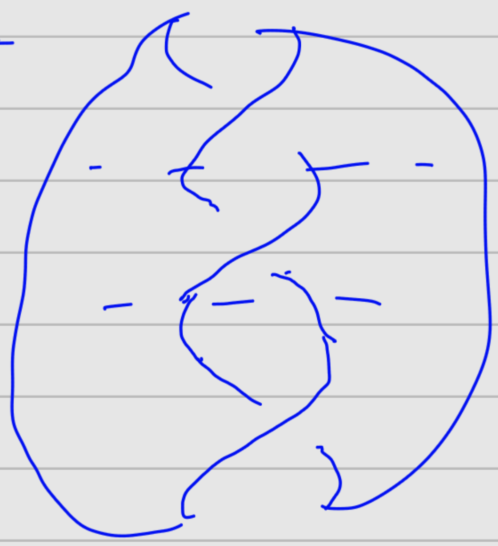
B_n has $\sigma_1, \dots, \sigma_{n-1}, \sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_{n-1}^{-1}$

$\therefore$ $2n-2$ generators.

claim: Any n-braid $\alpha \in B_n$ is a finite product of $\sigma_1, \ldots, \sigma_{n-1}$, & their inverses.

Given an n-braid α, we form its closure, $\bar{\alpha}$ by connecting A_i to A_i' by non intersecting arcs.

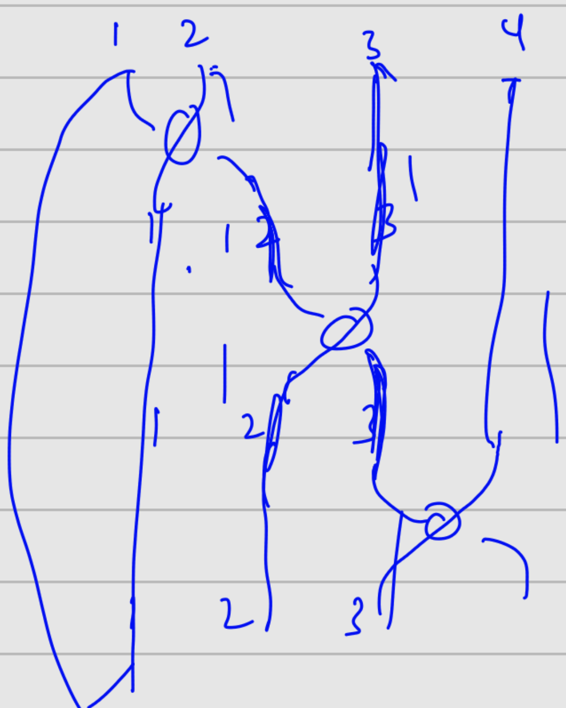
$\overline{\sigma_1^{\,3}} = 3_1$



$B_2 \cong \mathbb{Z}^+$

Thm (Alexander)

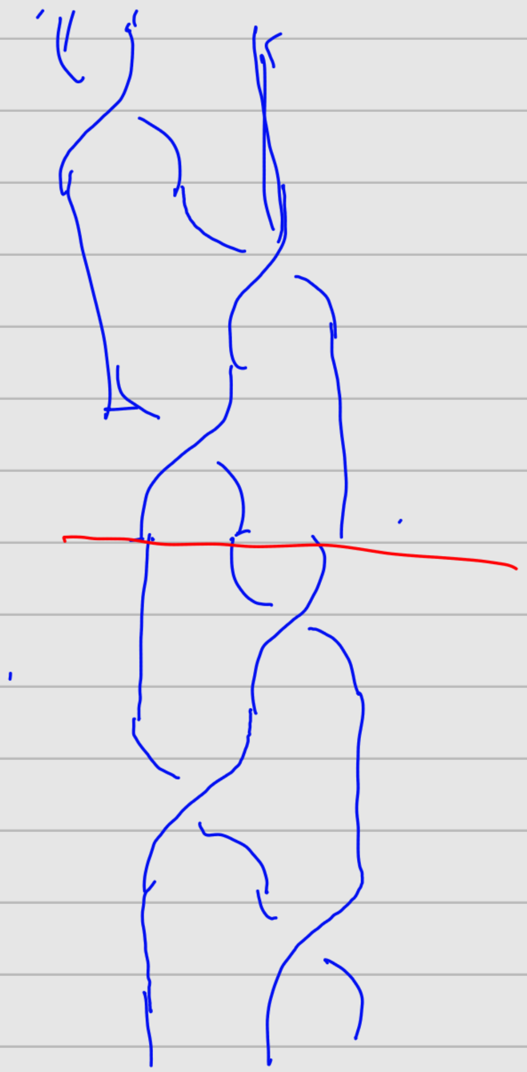
Any knot or link can always be written as the closure of n-braid α.



unknot

$\sigma_i \sigma_j = \sigma_j \sigma_i$ is a braid relation

$|i-j| \geq 2$



Thm: The Braid group B_n is a free group on $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and has 2 fundamental relations.

(a) $\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j| \geq 2$

(b) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

$\therefore \quad B_1 = \{e\} \qquad B_2 = \{\sigma_1^m \mid m \in \mathbb{Z}\} \qquad B_3 = \{\sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$

$B_4 = \{\sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1,\ \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1,\ \sigma_3\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_2\}$

$B_n = \{\sigma_1 \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| \geq 2$
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad i \leq n-2\}$

Exercise:

$w_1 = \sigma_1 \sigma_2 \sigma_4^{-1} \sigma_1 \sigma_2 \sigma_4$

$w_2 = \sigma_2 \sigma_1 \sigma_2^{2}$

$\rightarrow \sigma_1 \sigma_2 \ \sigma_4^{-1} \sigma_1 \ \sigma_4 \ \sigma_2$

$\rightarrow \sigma_1 \sigma_2 \ \sigma_1 \sigma_2$

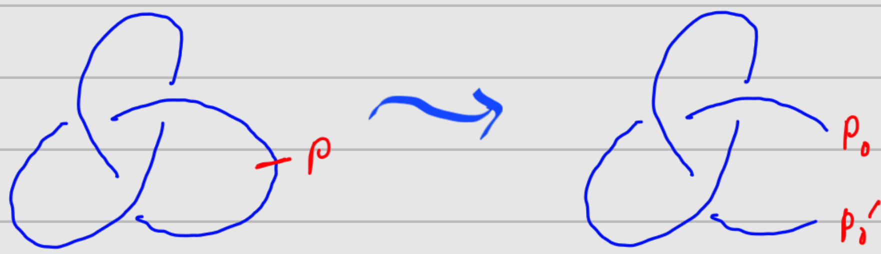
$\rightarrow \sigma_2 \sigma_1 \sigma_2$ ✓

Def^{n}: Given a braid $\alpha \in B_n$, we form its closure $\bar{\alpha}$ by joining A_1 to A_1', A_2 to A_2', ..., A_n to A_n' by large non-intersecting arcs which do not intersect the braid diagram. $\bar{\alpha}$ is a knot/link.

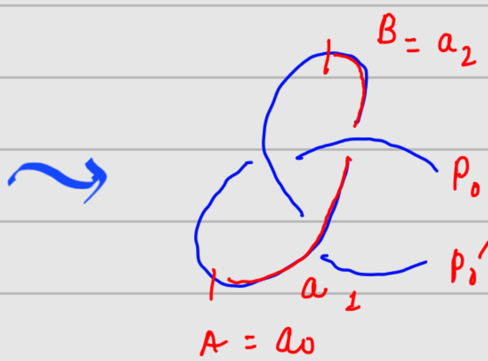
(Alexander) Any knot/link K is the closure of some braid α. $\exists \, \alpha$ st $\bar{\alpha} = K$.

Pf (constructive):

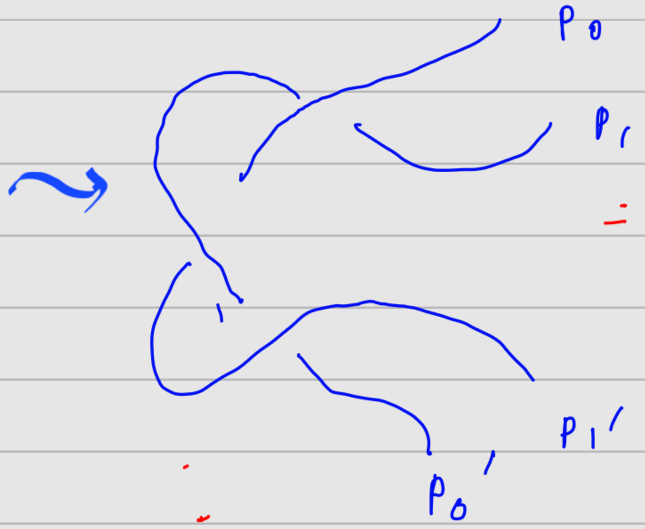
Cut K at a point P (not a crossing point) and pull the loose ends apart ($P_0 - P_0'$)

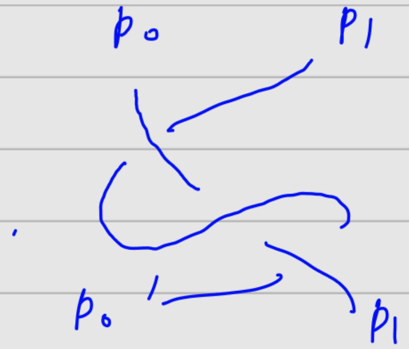


If there is a local maximum, B, $\exists$ a local minimum A. Call $a = a_0$. call crossing points on $\overline{ab}$ $a_1, a_2 \ldots a_n = b$.



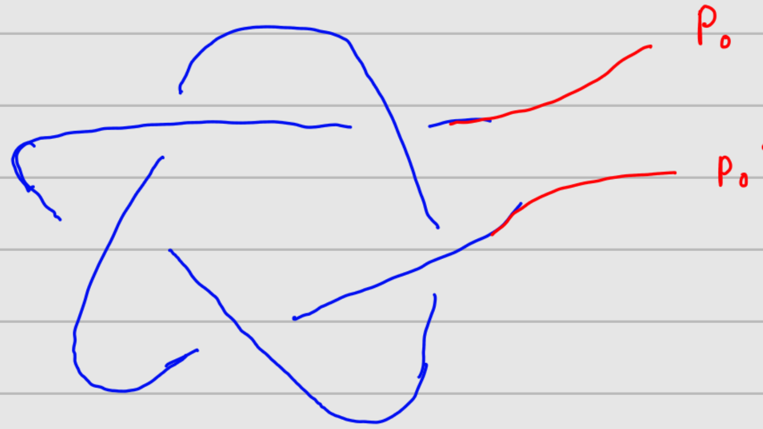
Replace arc $a_0 a_1$ by the larger arc $a_0 P_1' P_1 a_1$

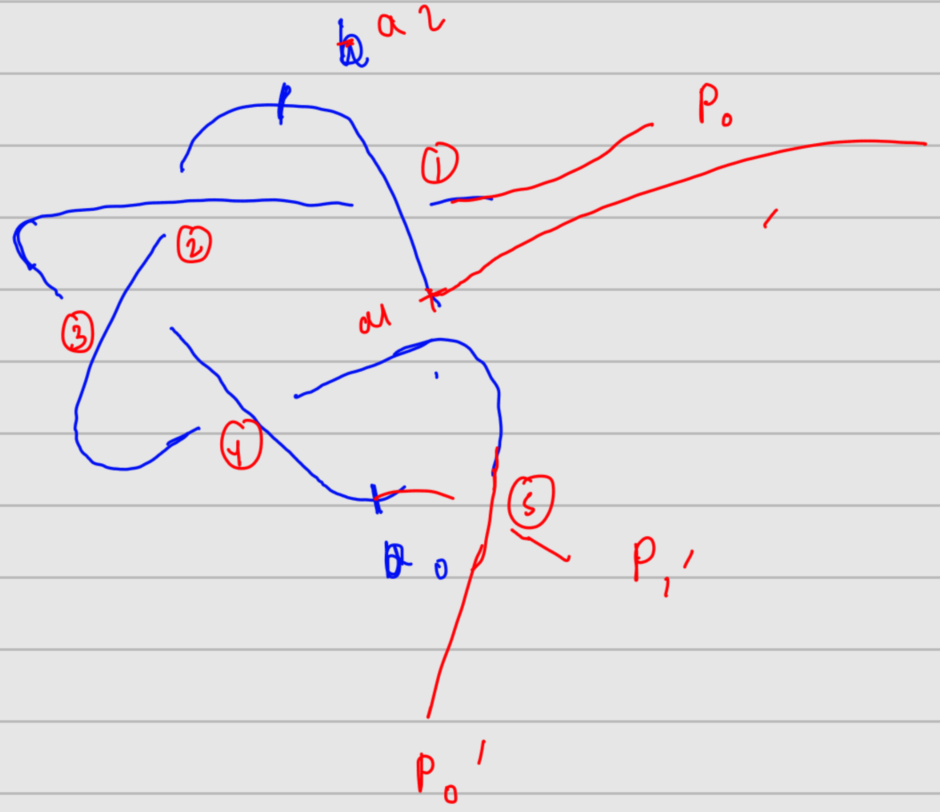


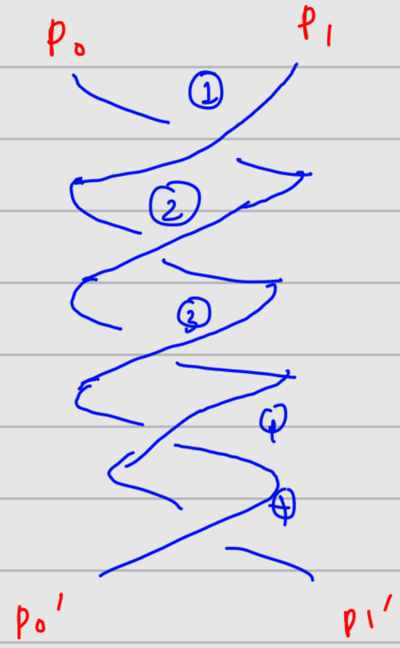


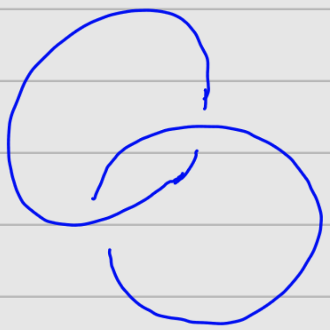
continue in this manner till there is no maximum or minimum.

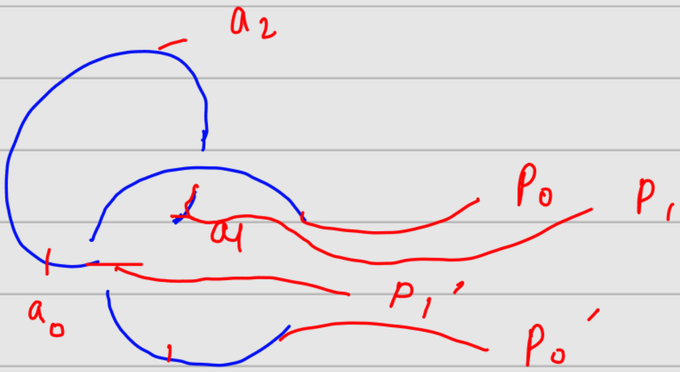
Eg 5_2

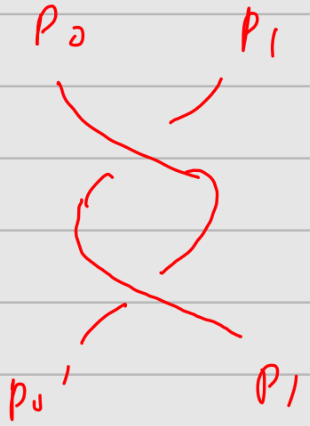




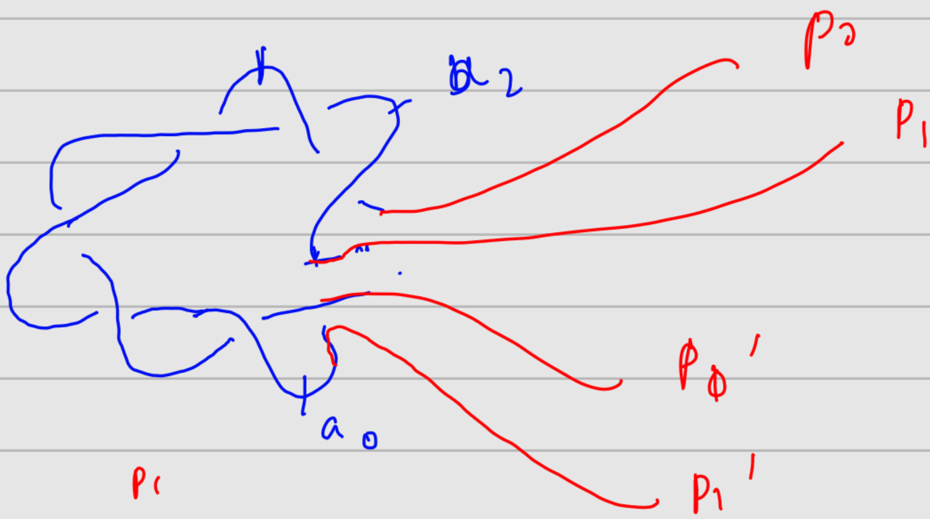


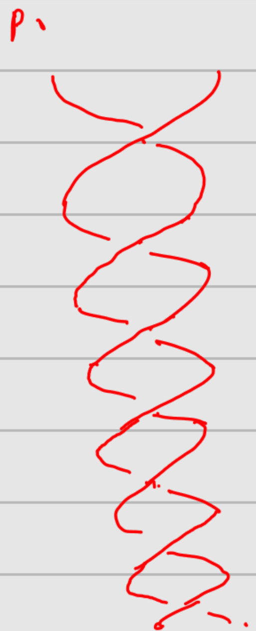




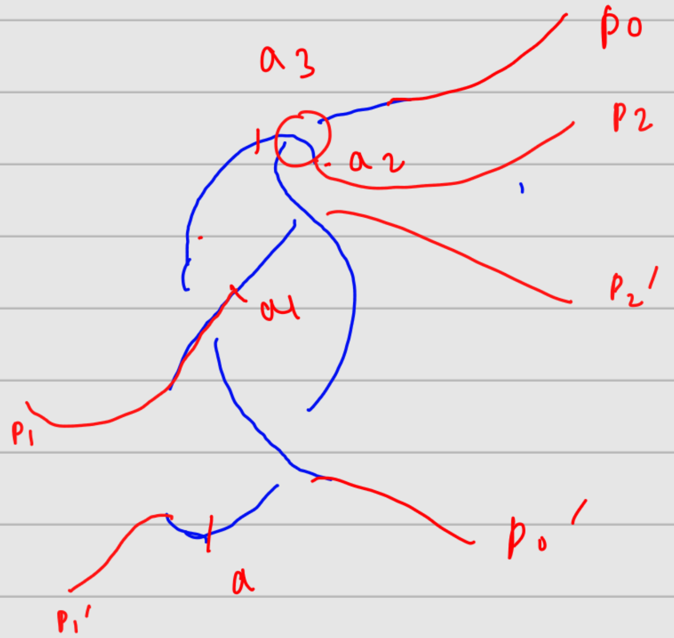


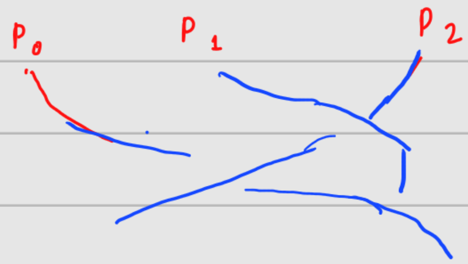
σ_1^{-2}





$\overline{\sigma}^{7}$





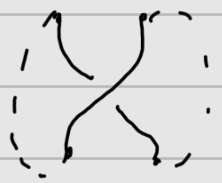
P_2'

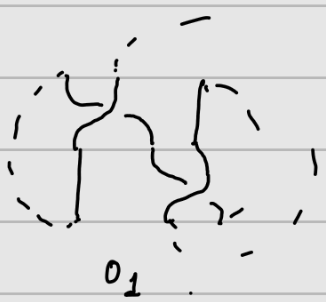
P_0

Note: If $\alpha, \beta \in B_n$ if $\alpha \simeq \beta \Rightarrow \bar{\alpha} \simeq \bar{\beta}$

However, the converse is not true. $\bar{\alpha} \simeq \bar{\beta} \not\Rightarrow \alpha \simeq \beta$.

σ_1 $\sigma_1 \sigma_2$





O_1 O_1

Braids different, closures same.

Markov's theorem (condition under which different braids give the same closure)

Def^n: Let B_∞ be the set which is the union $\bigcup_{i=1}^{\infty} B_i$.
we will perform 2 operations on B_∞ called the two Markov Moves.

1st Markov Move: (M_1) (conjugation)

if $\beta \in B_n$ and $\gamma \in B_n$. Then M_1 is the operation that transforms $\beta \mapsto \gamma \beta \gamma^{-1}$

2nd Markov Move: (M_2) (stabilization)

M_2 is the operation that transforms an n-braid into either of the two $\overset{+1}{n}$-braids

$\beta \sigma_n$ or $\beta \sigma_n^{-}$

Def^n: Two braids are said to be Markov equivalent if $\exists$ a sequence of markov moves $M_1, \ldots M_n$ st $M_1 \ldots M_n$ takes one braid to the other.

Markov's theorem

Suppose K_1 and K_2 are knots or links formed as braid closures β_1 and β_2 respectively. Then $K_1 \cong K_2 \Leftrightarrow \beta_1 \simeq_M \beta_2$.

Braid index

A knot K can be formed from an infinite number of braids. Among them $\exists$ braid α with the least number of strings $b(\alpha)$. we call $b(\alpha)$ the braid index of K.

Also a knot invariant.

✓ If K is alternating span $V_K(t) = c(K)$

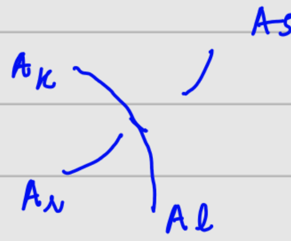
✓ let $P_K(v, z)$ be the HOMFLY polynomial of K, max v deg $P_K(v,z)$ − min v-deg $P_K(v,z)$ $\leq 2(n-1)$.

If $c(K) \leq 8$

$b(K) = \frac{1}{2}(v\text{span } P_K) + 1$.

ToDo: Get DMK notes on Linking Number and Colourability

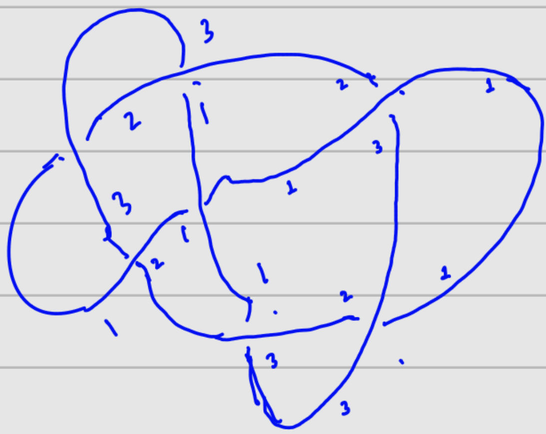
Def^N:



① A_k, A_l have the same colour.

② A_k, A_r, A_s have the same colour or all have different colors

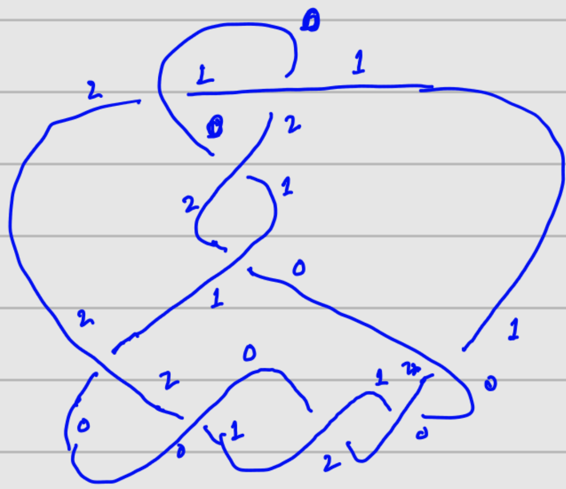
Eg: 7_4



using numbers

0, 1, 2

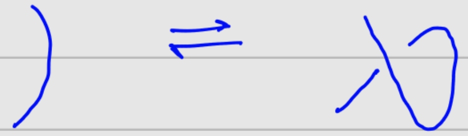
$\lambda_k + \lambda_l = \lambda_r + \lambda_s$ mod 3

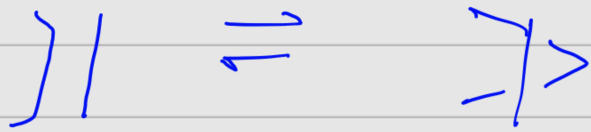


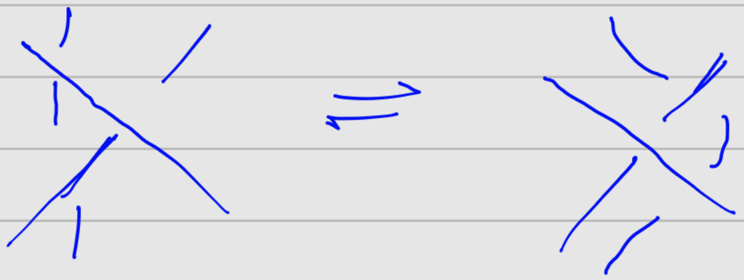
Knots live in 3d, knot moves live in 3d.

Reidemeister ⇒ discovered 3 moves in $\mathbb{R}^2$.

(moves on knot diagrams)

① Ω_1 

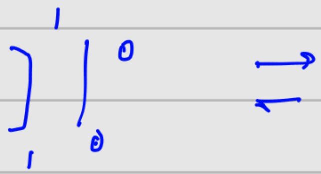
Ω_2 

Ω_3 

Thm: If D and D' are knot diagrams and it is possible to go from D to D' by finitely many Reidemeister moves, $D \simeq D'$

Reidemeister Thm

If K, K' are knots with diagrams D, D', then $K \simeq K' \leftrightarrow D \simeq D'$.



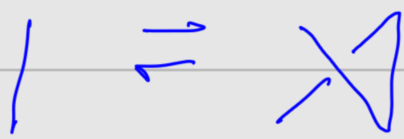
Thm: Let K be a knot/link. If there is a regular diagram of K which is 3 colorable.

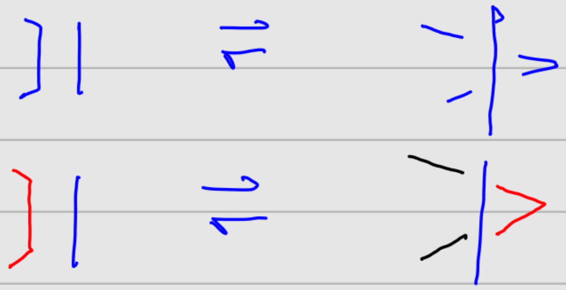
Then every diagram of K is 3-colorable.

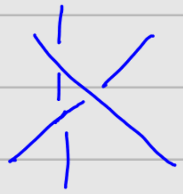
K is 3-colorable.

Pf (Reduces to proving Reidemeister moves preserve 3-colorability)

Let D be a 3-colorable regular diagram.
Any other diagram D' can be obtained using finitely many R-moves (and possibly their inverses).

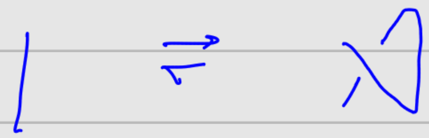




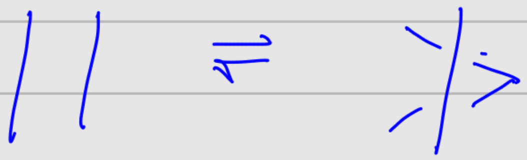
 → Trivial etc

Thm: Let $L = \{K_1, K_2\}$ be a 2-component link. The linking number $lk\, L = \frac{1}{2}(\text{sign } c_1 + \cdots + \text{sign } c_m)$ is a link invariant

Pf: Again, if R-moves preserve linking numbers we are done.

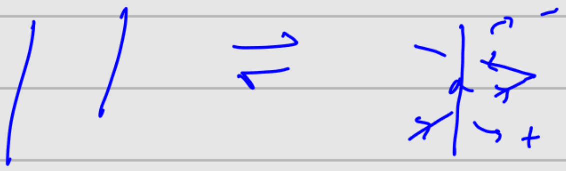


0 0



If $A, B \in K_1$ or $A, B \in K_2$

→ No effect.



K_1 K_2