Indian Institute of Technology Kanpur
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Complex Analysis (MTH 403)

Exercise Sheet 8

1. Analytic maps on $\mathbb{D}$

1.1. Let $f : \mathbb{D} \longrightarrow \mathbb{C}$ be analytic and $f(0) = 0$. Show that $\sum_{n=0}^{\infty} f(z^n)$ converges almost uniformly on $\mathbb{D}$.

1.2. (a) Let $f \in H(\mathbb{H})$ satisfy $|f(z)| \leq 1$, for all $z \in \mathbb{H}$, and $f(i) = 0$. Find the maximum value of $|f(2i)|$.
(b) Let f be a holomorphic function on the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Suppose that, $|f(z)| \leq 1$, whenever $\operatorname{Re} z > 0$. What is the largest possible value of $|f'(1)|$ under the condition $f(1) = 0$?

1.3. Let $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous and $f \in H(\mathbb{D})$. Assume that $|f(z)| \leq |e^z|$, whenever $|z| = 1$. Maximize $|f(\log 2)|$ subject to the condition $f(-\log 2) = 0$.

1.4. Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function with $f(0) = 0$.

(a) Show that $|f(z) + f(-z)| \leq 2|z|^2$, for all $z \in \mathbb{D}$.
(b)$*$ Show that, equality occurs in 1.4.a for some $z \neq 0$ if and only if there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $f(z) = \lambda z^2$.

Hint. *Let $f(z) = a_1 z + a_2 z^2 + \ldots$ be the power series representation of f on $\mathbb{D}$. Then for every $z \in \mathbb{D} \setminus \{0\}$, we have*

$$\frac{f(z) + f(-z)}{2z} = a_2 z + a_4 z^3 + a_6 z^5 + \ldots$$

From this, show that if equality occurs in 1.4.a for some $z \neq 0$, then $|a_2| = 1$ and $a_4 = a_6 = \cdots = 0$. Hence, for some odd function $h \in H(\mathbb{D})$, one has $f(z) = a_2 z^2 + h(z)$, for all $z \in \mathbb{D}$.

1.5.$*$ Let $f \in H(\mathbb{D})$ be such that $|f(z)| \leq 1$, for all $z \in \mathbb{D}$, and $f(0) = 0$. Assume that, there is a number $r \in (0, 1)$ satisfying $f(r) = f(-r) = 0$. Show that, for all $z \in \mathbb{D}$,

$$|f(z)| \leq |z| \left| \frac{z^2 - r^2}{1 - r^2 z^2} \right|.$$

Hint. *The function $z \mapsto \frac{f(z)}{z}$ is bounded on $\mathbb{D} \setminus \{0\}$, hence it can be redefined to an $f_1 \in H(\mathbb{D})$. Applying Schwarz lemma on $f_1 \circ \varphi_r$, one obtains that $|f_1(z)| \leq |\varphi_r(z)|$, for all $z \in \mathbb{D}$. This shows that the function $z \mapsto \frac{f_1(z)}{\varphi_r(z)}$ is bounded on $\mathbb{D} \setminus \{0\}$. Redefine it to an holomorphic function f_3 on $\mathbb{D}$. Now applying Schwarz lemma on $f_3 \circ \varphi_{-r}$, we get that $|f_3(z)| \leq |\varphi_{-r}(z)|$, for all $z \in \mathbb{D}$.*

1.6. Show that, for any analytic $f : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$, one has the following for every $z \in \mathbb{D}$:

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}.$$

1.7. Let $f : \mathbb{D} \longrightarrow \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ be analytic. Show that, for all $z \in \mathbb{D}$,

$$\frac{1 - |z|}{1 + |z|}|f(0)| \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}|f(0)|,$$

and $|f'(0)| \leq |2 \operatorname{Re} f(0)|$.

1.8.∗ (a) Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic. Show that,

$$\left|\frac{f(w)-f(z)}{1-\overline{f(w)}f(z)}\right| \leq \left|\frac{w-z}{1-\bar{w}z}\right|, \ \forall z, w \in \mathbb{D}. \tag{1.1}$$

(b) Suppose that $r, M > 0$ and $f : D(0; r) \longrightarrow D(0; M)$ is holomorphic. What modification does one need to make in (1.1) in this case?

(c) In 1.8.b, show further that, for all $z, w \in D(0; r)$ with $z \neq w$,

$$\left|\frac{f(z)-f(w)}{z-w}\right| \leq \frac{r}{M} \cdot \frac{|M^2 - \overline{f(w)}f(z)|}{|r^2 - \bar{w}z|}.$$

Hint. *Consider* $g : \mathbb{D} \longrightarrow \mathbb{D}$, $g(z) \overset{def}{=} \dfrac{f(rz)}{M}$.

(d) Given $w \in D(0; r)$, how large $|f'(w)|$ can be in 1.8.b?

1.9. Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. Show that, if f has at least two distinct points then it must be the identity function.

1.10.∗ Let $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous and analytic on $\mathbb{D}$. Assume that $|f(z)| = 1$ whenever $|z| = 1$.

(a) Show that f can have at most finitely many zeros.

(b) What happens if f has no zero?

(c) Let $a_1, \ldots, a_n$ be precisely all distinct zeros of f with order $k_1, \ldots, k_n$ respectively. Consider

$$g(z) \overset{\text{def}}{=} \prod_{j=1}^{n} \left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{k_j}, \forall z \in \overline{\mathbb{D}}.$$

Show that $\dfrac{f}{g}$ can be redefined to an holomorphic function on $\mathbb{D}$.

(d) Show that $\dfrac{f}{g}$ is constant on $\mathbb{D} \setminus \{a_1, \ldots, a_n\}$.

(e) Conclude that, there exists $\lambda \in S^1$ such that,

$$\boxed{f(z) = \lambda \prod_{j=1}^{n} \left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{k_j}, \forall z \in \overline{\mathbb{D}}.} \tag{1.2}$$

Note: (1.2) provides a formula for all continuous functions $f : \overline{D} \longrightarrow \mathbb{C}$ with $f \in H(\mathbb{D})$ and $|f(z)| = 1$ whenever $|z| = 1$. The following exercises are easy applications of this.

1.11. Let $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous, $f \in H(\mathbb{D})$ and $|f(z)| = 1$ whenever $|z| = 1$. Assume that f has a zero at $\frac{1+i}{4}$ and a double zero at $\frac{1}{2}$. Can $f(0)$ be $\frac{1}{2}$?

1.12. Let $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ be continuous. Let $a_1, \ldots, a_n$ be precisely all distinct zeros of f with order $k_1, \ldots, k_n$ respectively. Show that, there exists $M > 0$ such that, for all $z \in \overline{\mathbb{D}}$,

$$|f(z)| \leq M \prod_{j=1}^{n} \left|\frac{z-a_j}{1-\overline{a_j}z}\right|^{k_j}. \tag{1.3}$$

What if equality occurs in (1.3) for some $z \in \mathbb{D} \setminus \{a_1, \ldots, a_n\}$?

2. Reflection principle

In 2.1.-2.3. and 2.6. and 2.7., we let $U^+ \subseteq_{open} \mathbb{H}$ and ∂U^+ contains an open interval I of $\mathbb{R}$. Denote $\{\bar{z} : z \in U^+\}$ by U^-. Assume that $U \overset{\text{def}}{=} U^+ \cup I \cup U^-$ is an open subset of $\mathbb{C}$. Suppose that $f : U^+ \cup I \longrightarrow \mathbb{C}$ is holomorphic on U^+, continuous on I and, furthermore $f(I) \subseteq \mathbb{R}$. Consider

$$F(z) \stackrel{\text{def}}{=} \begin{cases} f(z) & \text{if } z \in U^+ \cup I \\ \overline{f(\bar{z})} & \text{if } z \in U^-. \end{cases} \tag{2.1}$$

2.1. Show that the function F defined as above in (2.1) satisfies the following property:

$$F(z) = \overline{F(\bar{z})},\ \forall z \in U.$$

2.2. Show that F is analytic on $U^+ \cup U^-$ and continuous on I.

2.3. Show that the function F is the unique analytic extension of f to U, provided U^+ is connected (note that, this implies U is connected).

Hint. *Use Exercise 1.5 of Exercise Sheet 7.*

2.4. Formulate and prove the analogue of "Schwarz reflection principle" when U^+ is an open subset of the right-half plane and ∂U^+ contains an open segment of the imaginary axis.

2.5.$*$ Let $\mathbb{D} \cap \mathbb{H}$, i.e., open upper-half of the unit disc. Suppose that $f : \overline{U} \setminus [-1, 1] \longrightarrow \mathbb{C}$ is continuous and holomorphic on U. Assume further that $f(z) \in \mathbb{R}$, whenever $z \in \overline{U}$ and $|z| = 1$. Show that f admits a unique holomorphic extension to $\mathbb{H}$.

We now generalize 2.4. and 2.5. to the following. Suppose that $V \subseteq_{open} \mathbb{C}$ is the disjoint union

$$V = V^+ \cup \gamma^* \cup V^{-1},$$

where V^+, V^- are open and γ is a curve. Assume further that there is a bijective holomorphic function $\psi : U \longrightarrow V$ such that $\psi(U^+) = V^+, \psi(U^-) = V^-$ and $\psi(I) = \gamma^*$.

2.6. Let $g : V \longrightarrow \mathbb{C}$ be a continuous function such that it is holomorphic on V^+ and V^-. Show that g is holomorphic on V.

2.7. Let $g : V^+ \cup \gamma^* \longrightarrow \mathbb{C}$ be a continuous function which is holomorphic on V^+ and real valued on γ^*. Then g has a holomorphic extension on V, and furthermore this extension is unique, provided U^+ is connected.

Hint. *For both 2.6. and 2.7., consider* $f \stackrel{def}{=} g \circ \psi$ *and use the "Schwarz reflection principle".*

Note: The assertion 2.7. may be considered as the *reflection principle across a curve*.