Numerical Analysis & Scientific Computing II

Lesson 5

Integral Equations

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Integral Equations

5.1 Some solutions of boundary value problems for PDEs via integral equations

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Integral Equations: Some examples

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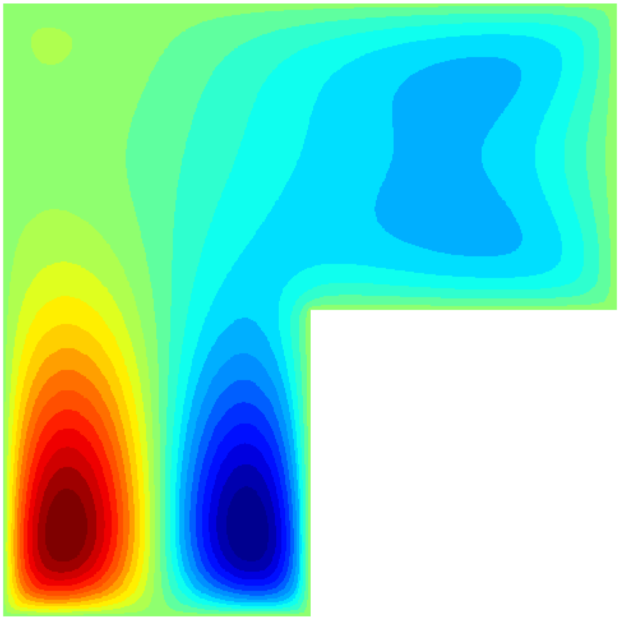
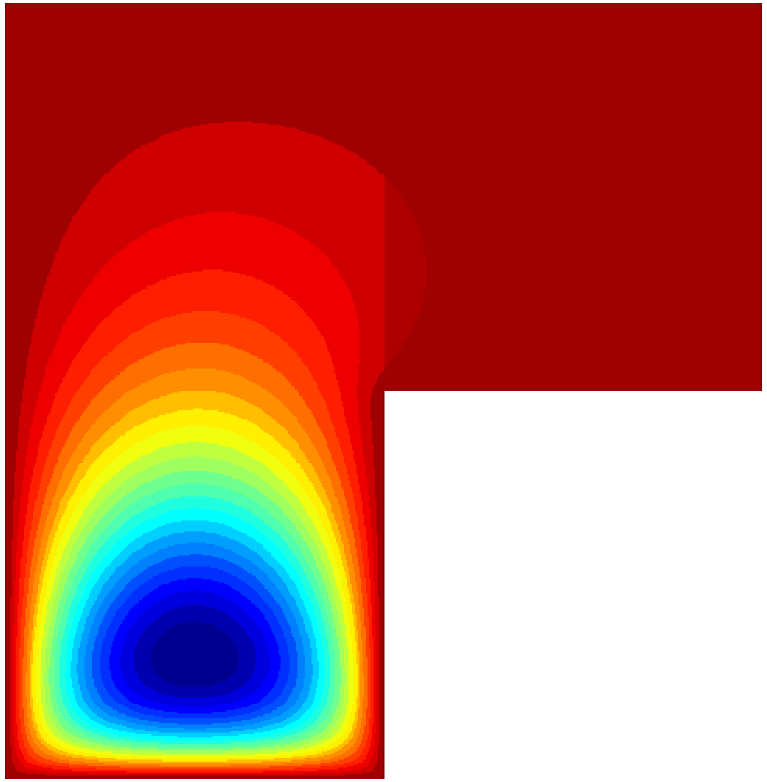
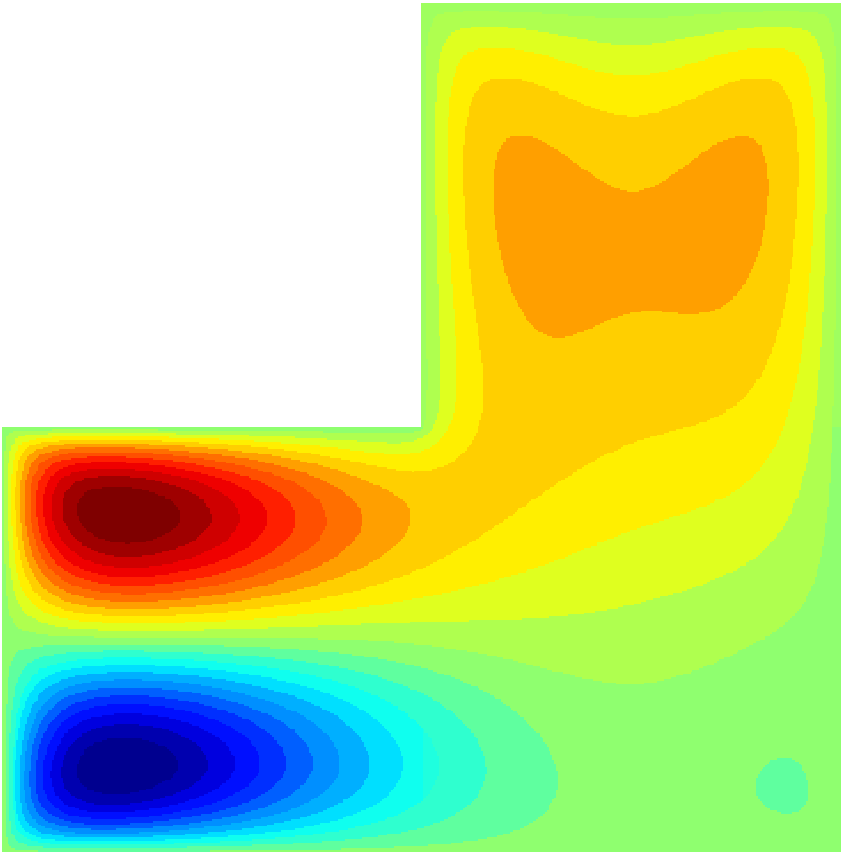
Some solutions of integral Fredholm integral equations







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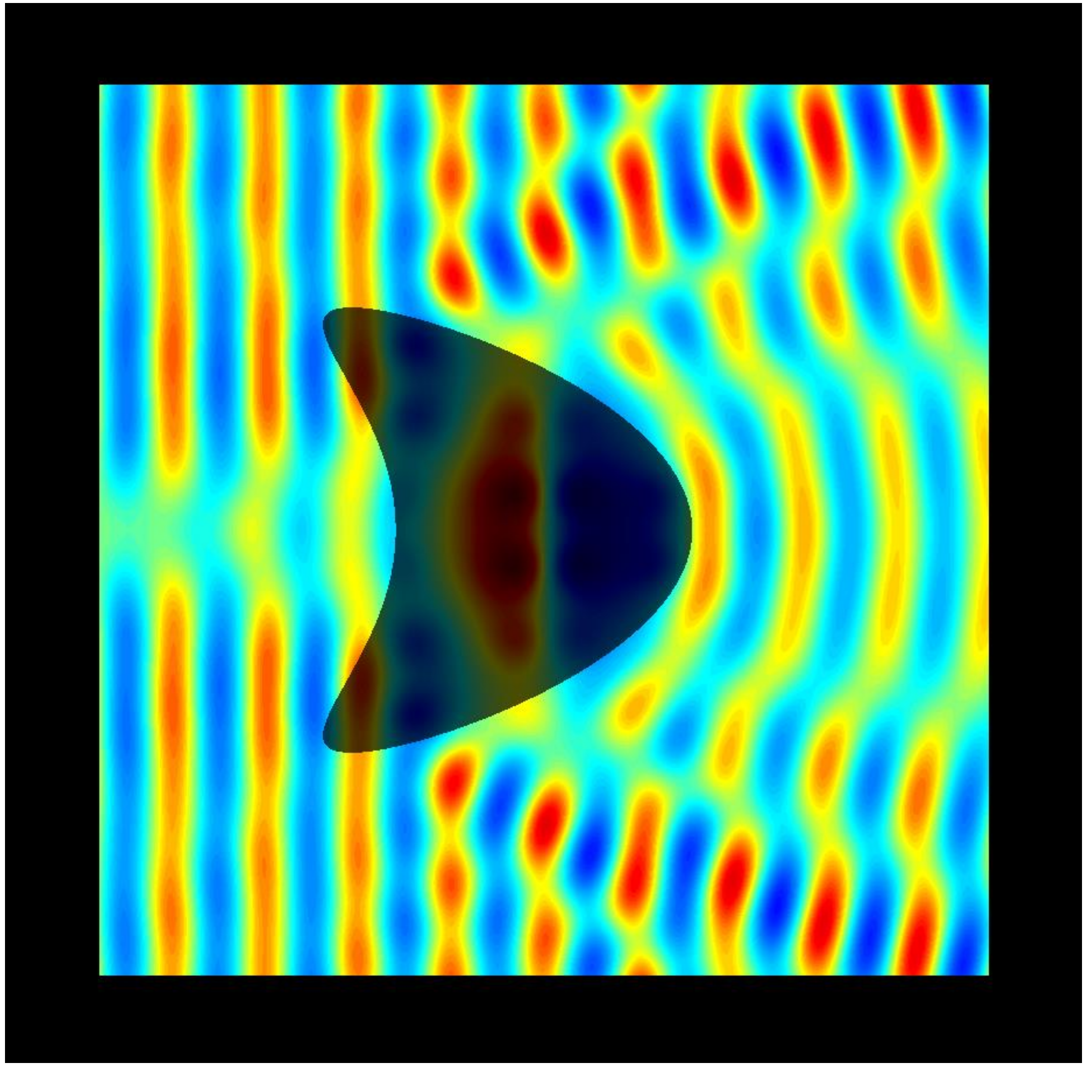
Integral Equations: Some examples

Some solutions of integral Fredholm integral equations in wave scattering



(Ambuj Pandey, Akash Anand)

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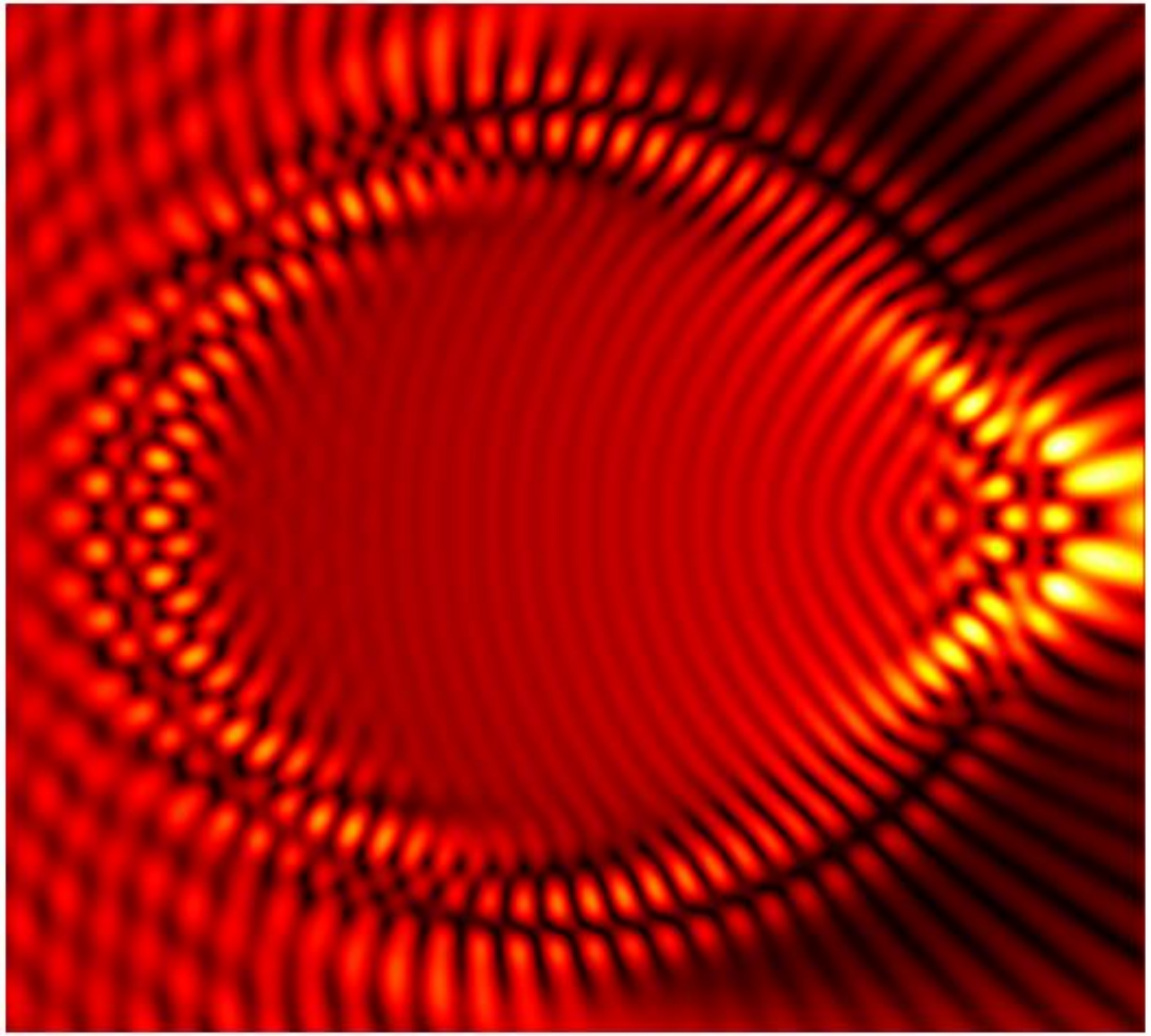

(Awanish Tiwari, Ambuj Pandey, Akash Anand)

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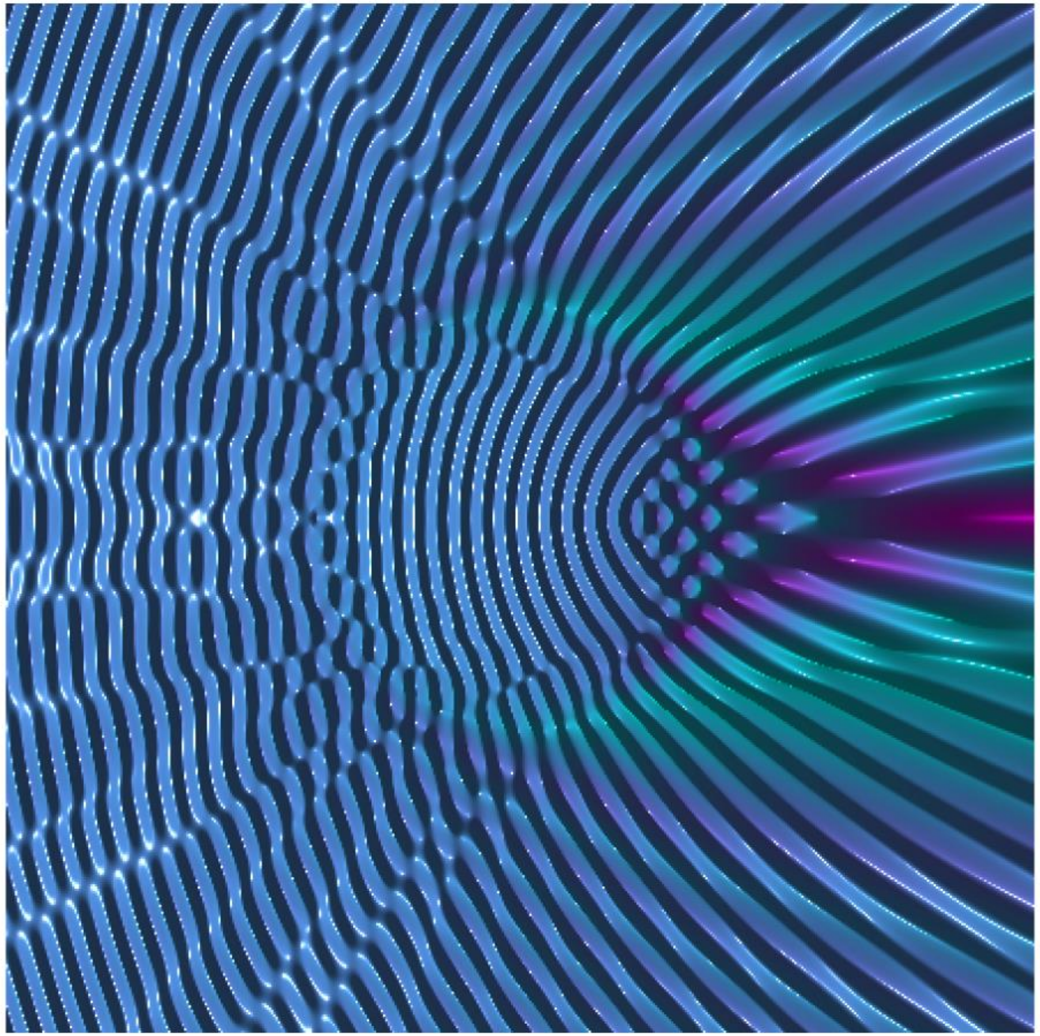

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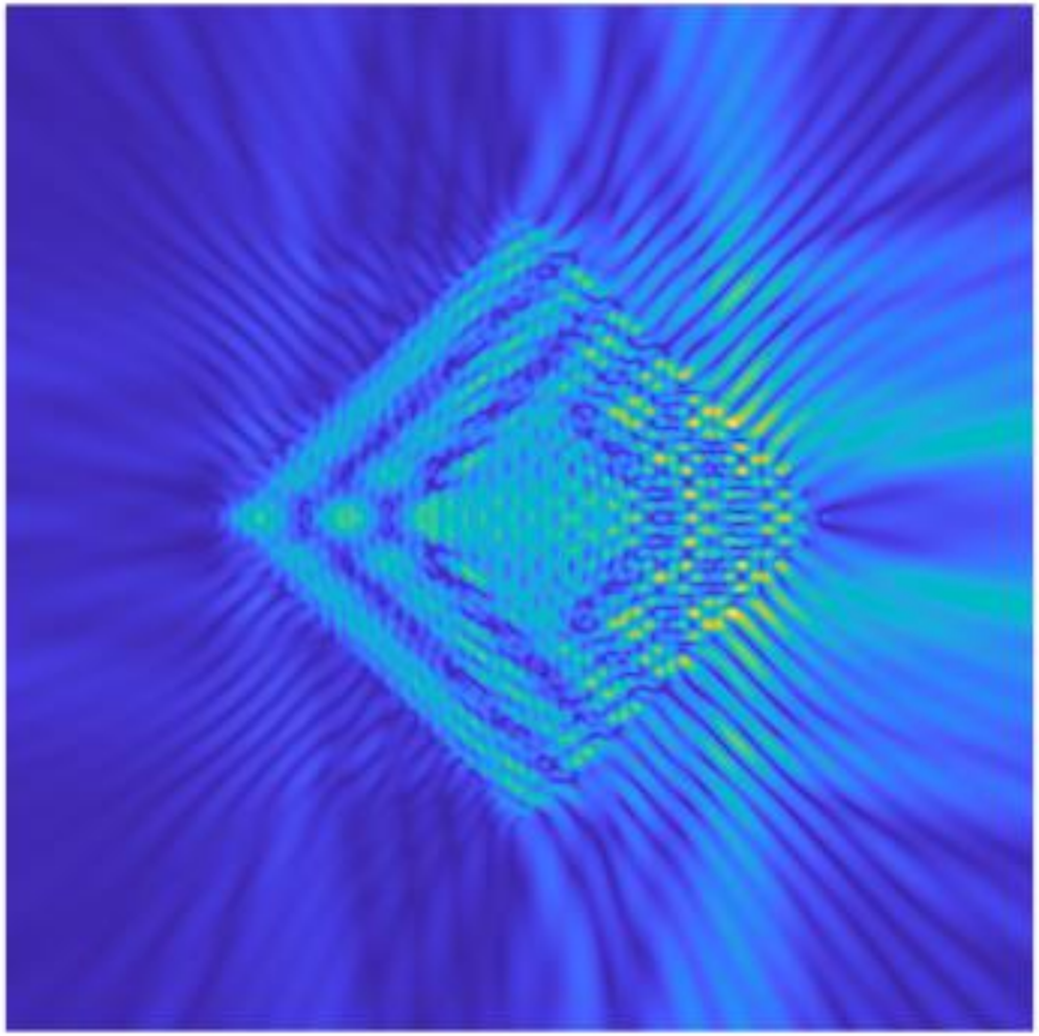

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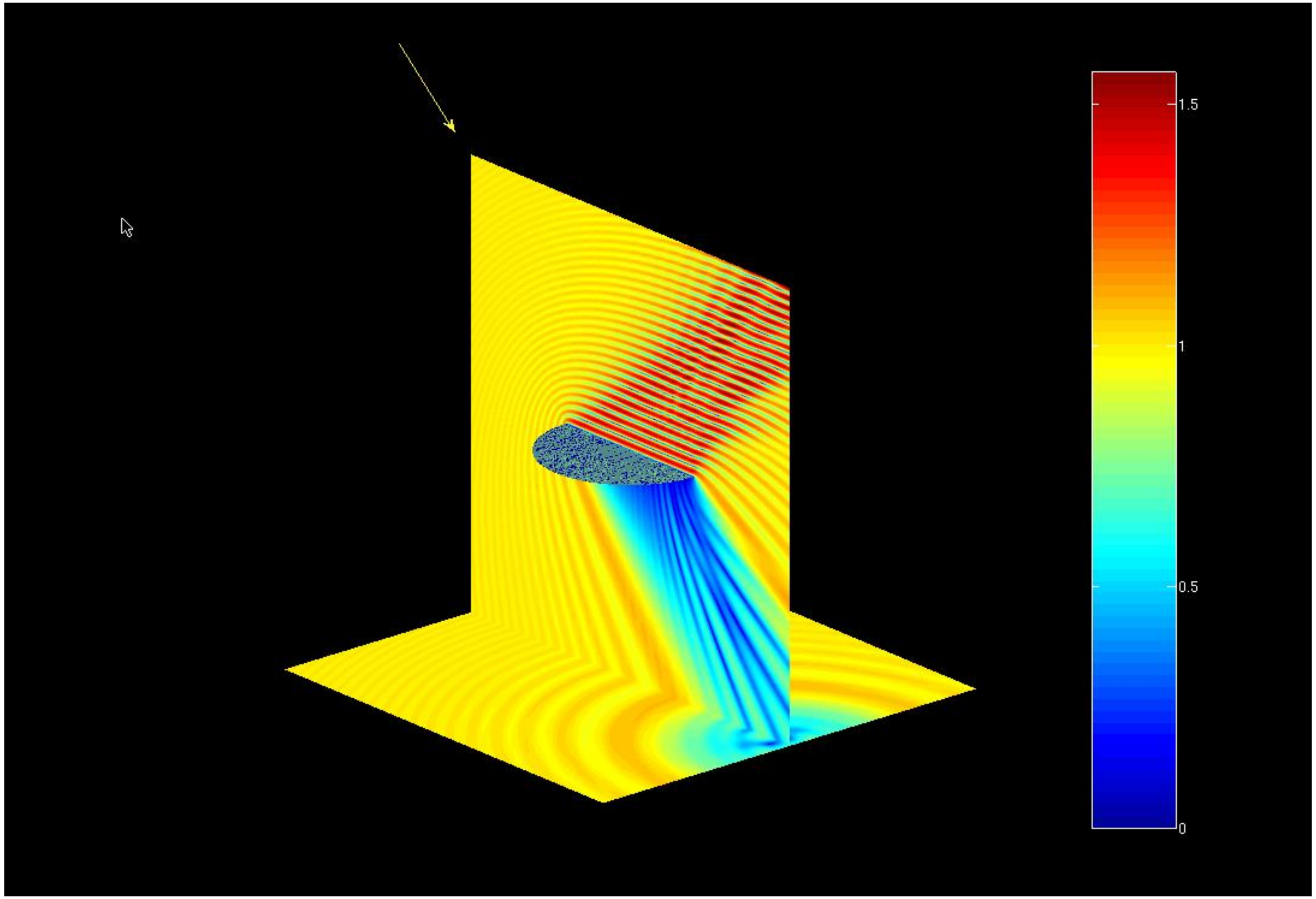

(Stephen Lintner, Oscar Bruno)

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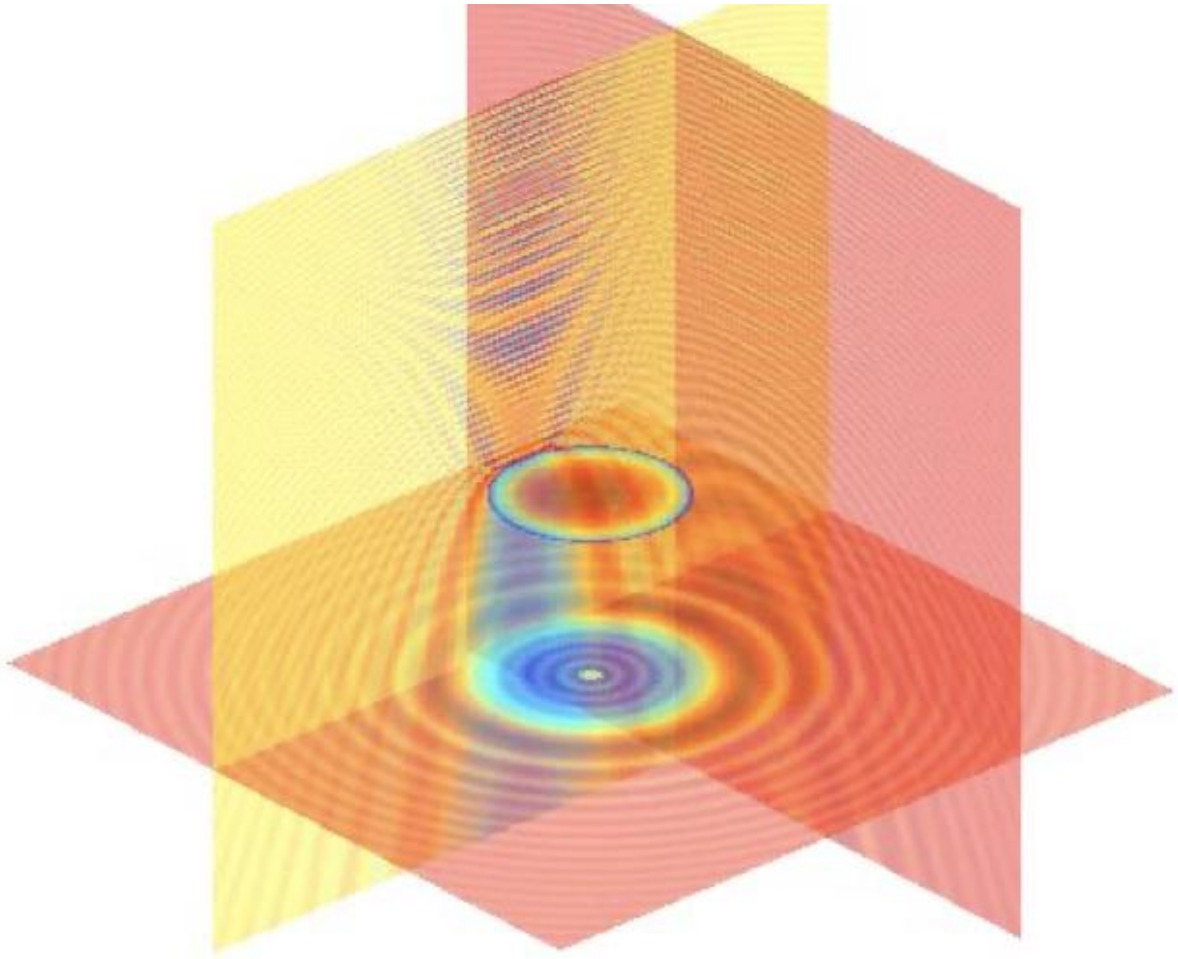

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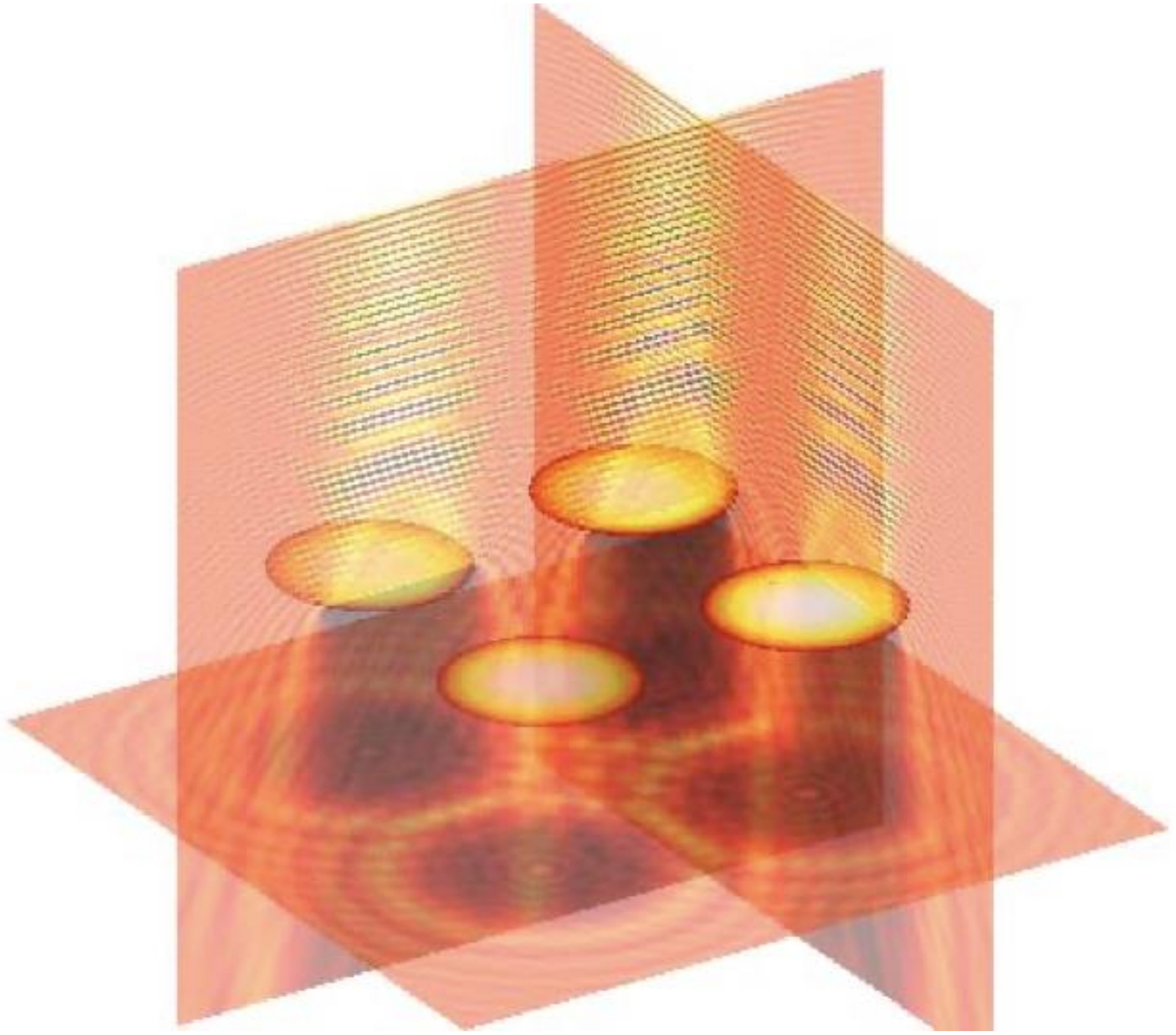

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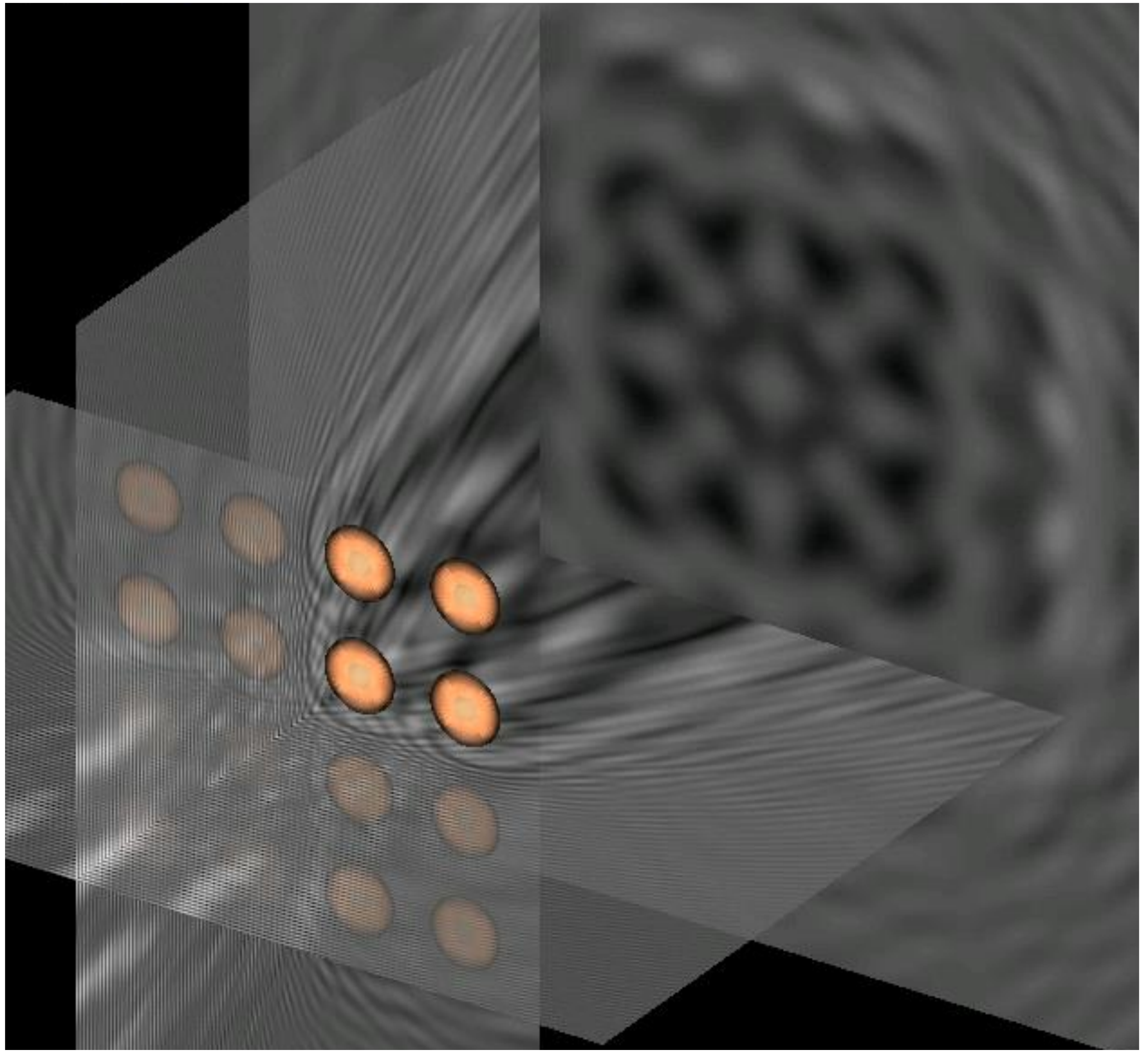

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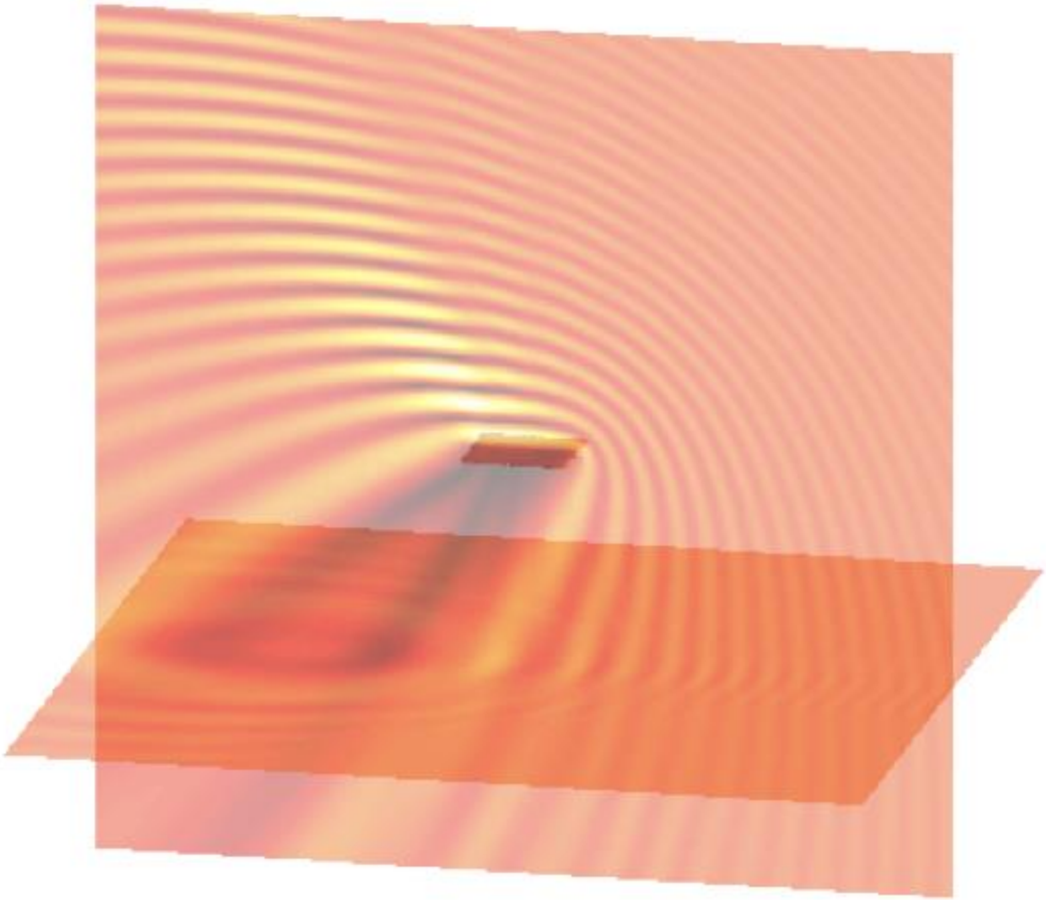

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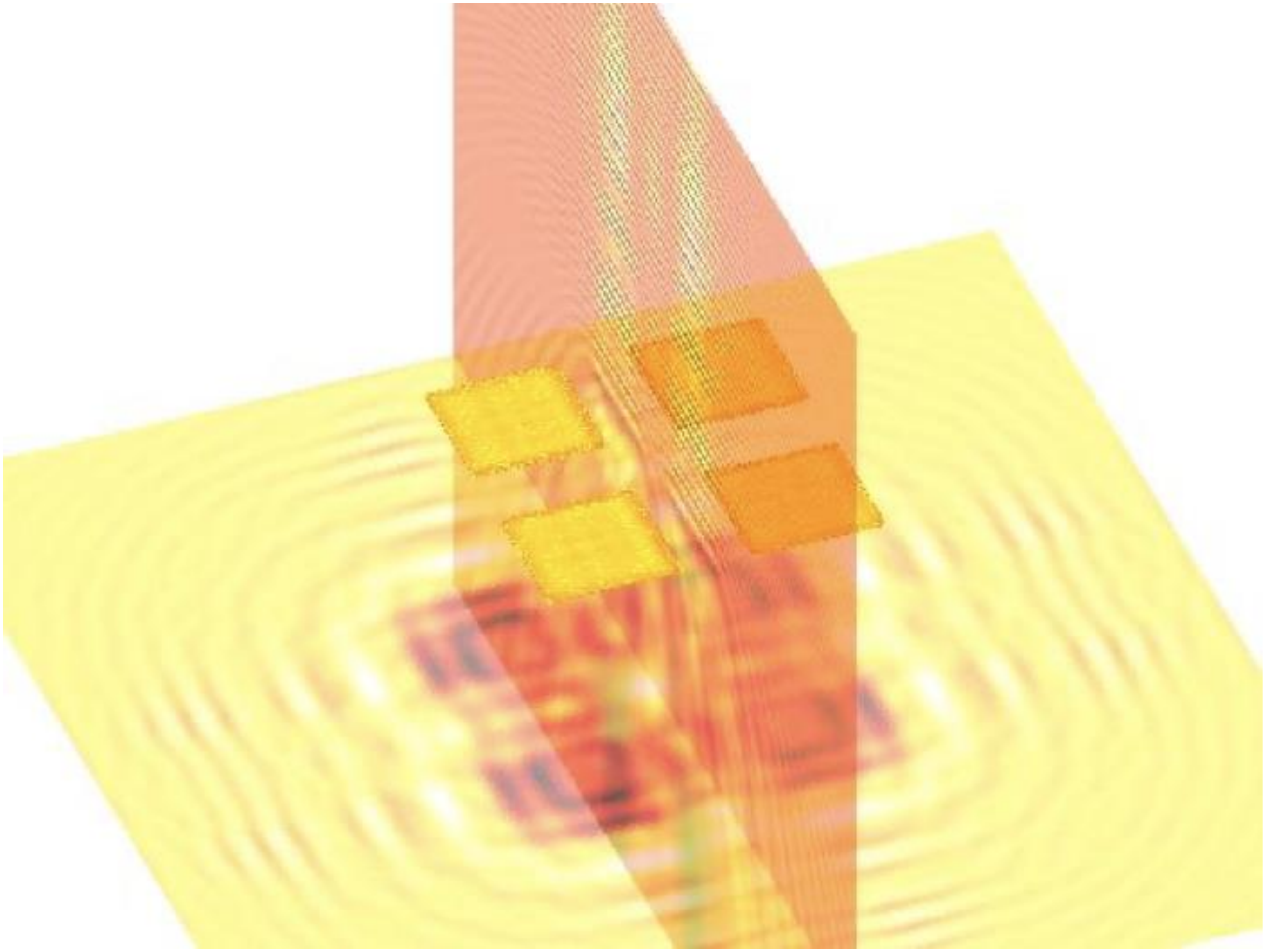

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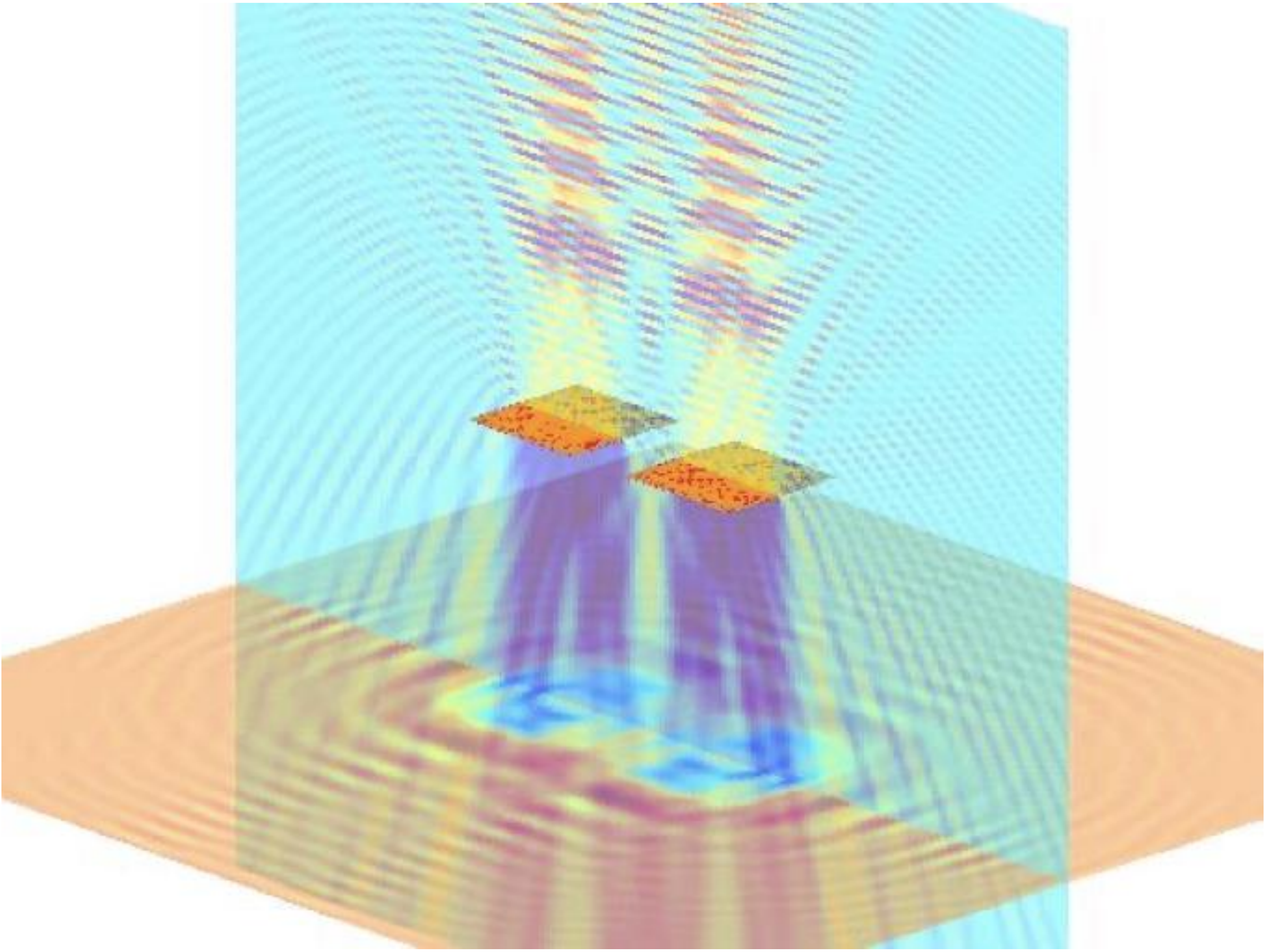

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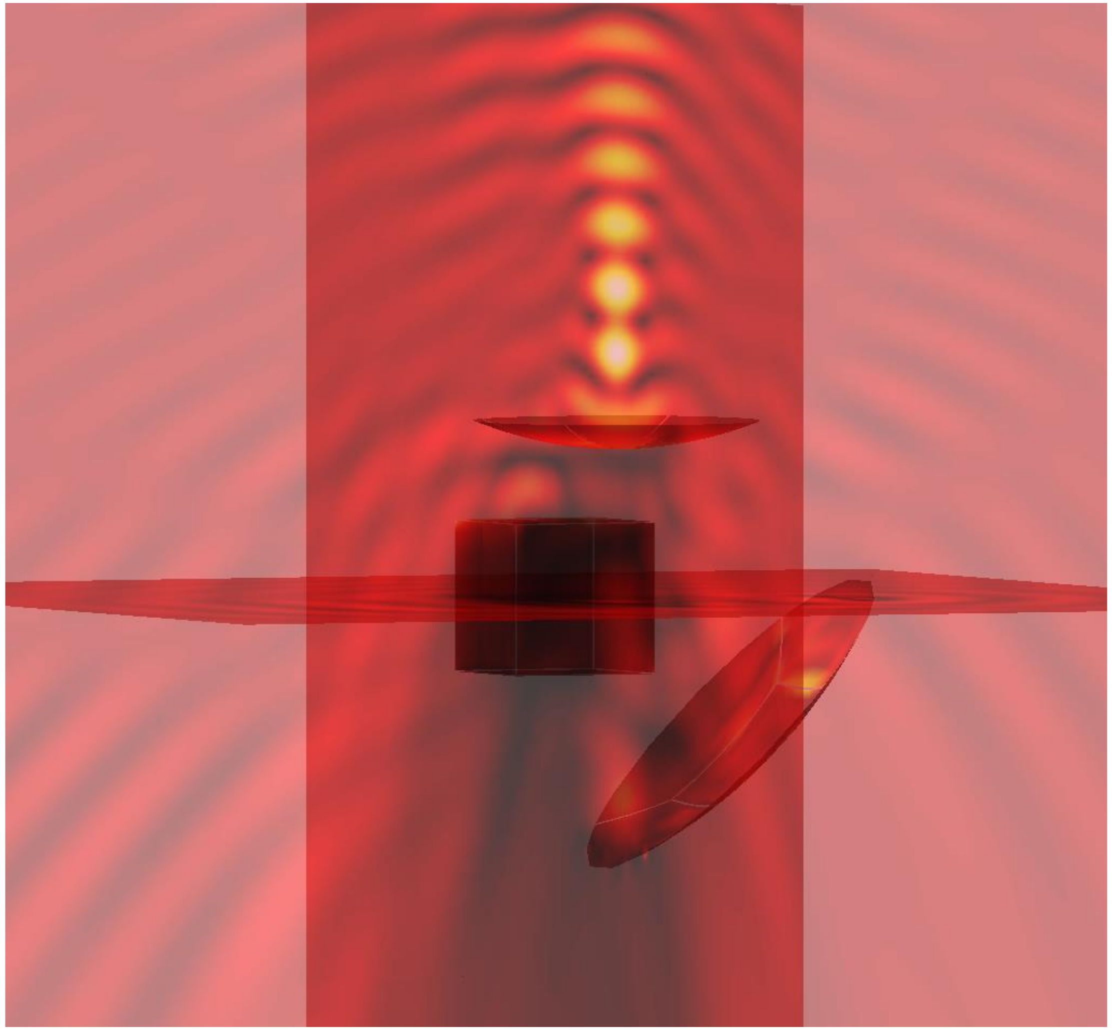

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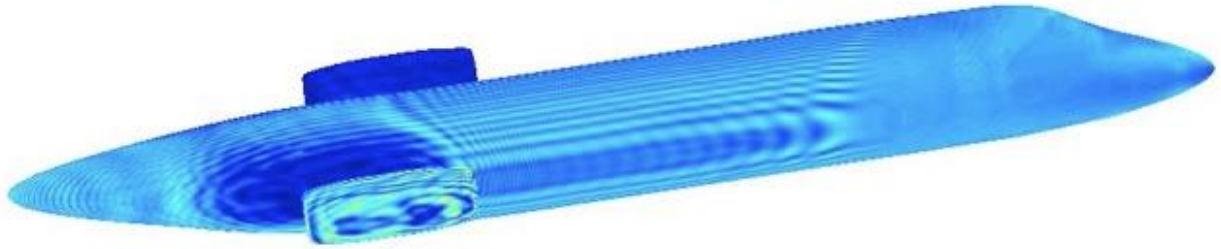

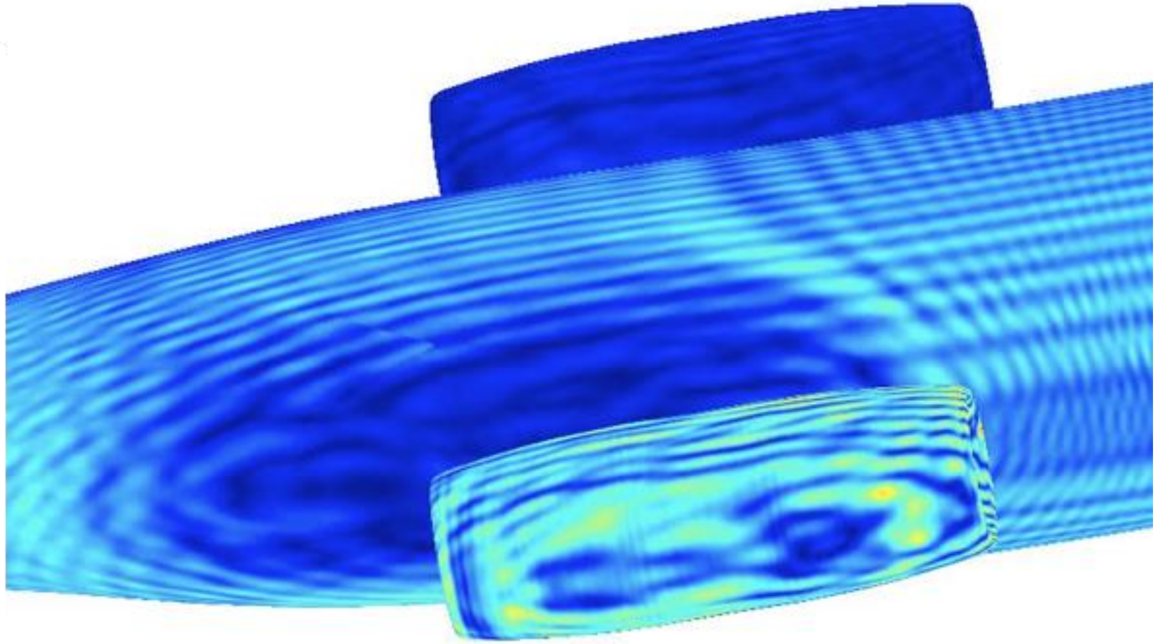

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5.1 Some solutions of boundary value problems for PDEs via integral equations

5.2 An Introduction

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Integral Equations: An Introduction

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We have already encountered integral equations in this course.

Integral Equations: An Introduction

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Recall that the initial value problem

$$y' = f(t, y), y(t_0) = y_0,$$

is equivalent to finding y satisfying

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds\,.$$

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This is a particular case of the more general Volterra integral equations

$$\int_{t_0}^{t} K\big(t, s, y(s)\big) ds = g(t), \qquad t \geq t_0.$$

where K and g are are given functions.

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The numerical solution of these equations are closely related to the initial value problem. We will, however, focus on a different type of integral equations known as Fredholm integral equations, in particular, of the second kind.

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The general form of such an integral equation is

$$u(t) - \int_{\Omega} K(t,s)u(s)ds = f(t), \qquad t \in \Omega.$$

where Ω *is a bounded set in* $\mathbb{R}^m, m \geq 1$.

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Example

Consider solving the problem

$$\begin{aligned} \Delta u(x) &= 0, \qquad x \in \Omega, \\ u(x) &= g(x), \qquad x \in \Gamma, \end{aligned}$$

where Ω is a bounded region in $\mathbb{R}^3$ with non-empty interior and Γ is the boundary of Ω.

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where Ω is a bounded region in $\mathbb{R}^3$ with non-empty interior and Γ is the boundary of Ω. If we seek u in the form of a single layer potential, that is

$$u(x) = \int_{\Gamma} \frac{1}{|x-y|}\rho(y)dy, \qquad x \in \Omega,$$

where $\rho(y)$ is called a single layer density function and it is the unknown in the equation.

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Using the boundary condition (and the fact from "Potential Theory" that single layer potentials are continuous everywhere), it is straightforward to see that ρ satiisfies

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a Fredholm integral equation of the first kind. If we see the solution in the form of a double layer potential

$$u(x) = \int_\Gamma \frac{\partial}{\partial \nu(y)} \left(\frac{1}{|x-y|} \right) \mu(y) dy\,, \qquad x \in \Omega,$$

then $\mu(y)$ satisfies a Fredholm integral equation of the second kind.

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then $\mu(y)$ satisfies a Fredholm integral equation of the second kind. Indeed, the double layer density function $\mu(y)$ satisfies

$$\frac{1}{2}\mu(x) - \int_\Gamma \frac{\partial}{\partial \nu(y)}\left(\frac{1}{|x-y|}\right)\mu(y)dy = -g(x), x \in \Gamma,$$

(another fact from "Potential Theory" know as jump relation for double layer potential).

Integral Equations: An Introduction

We say that a kernel $K: \Omega \times \Omega \to \mathbb{C}$ *is weakly singular if* K *is defined and continuous for all* $x, y \in \Omega \subseteq \mathbb{R}^m$, $x \neq y$, *and there exist positive constants* M *and* $\alpha \in (0, m]$ *such that*

$$|K(x,y)| \leq M|x-y|^{-(m-\alpha)}$$

for all $x, y \in \Omega$, $x \neq y$.

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Remark

One can show that, the integral operator

$$(Au)(x) = \int_{\Omega} K(x,y)u(y)dy\,,$$

with weakly singular kernel K maps continuous functions to continuous function, that is, the operator $A: C(\Omega) \to C(\Omega)$ is well-defined.

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Moreover, it is known that, for such integral operators, the Fredholm alternative holds, that is,

$$(I - A)u = f$$

has a unique solution for every $f \in C(\Omega)$ if and only if the homogeneous equation $(I - A)v = 0$ has only the trivial solution $v = 0$.