Module 2

Initial Value Problems

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Derivation using Taylor series

Initial Value Problems: Euler's method

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Derivation using Taylor series

Consider the Taylor series

$$y(t_{k+1}) = y(t_k) + (t_{k+1} - t_k)y'(t_k) + \frac{(t_{k+1} - t_k)^2}{2} y''(t_k) + \cdots$$

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The Euler's method results from taking $y_k = y(t_k)$, $y'(t_k) = f(t_k, y_k)$ *and dropping terms of second and higher orders!*

Initial Value Problems: Euler’s method

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The Euler's method results from taking $y_k = y(t_k)$, $y'(t_k) = f(t_k, y_k)$ *and dropping terms of second and higher orders!*

Derivation using finite difference approximation

Replacing the $y'(t)$ *in the ODE* $y' = f(t, y)$ *by a first order forward difference approximation, we obtain an algebraic equation*

$$\frac{y_{k+1} - y_k}{t_{k+1} - t_k} = f(t_k, y_k)$$

that yields the Euler's method.

Initial Value Problems: Euler's method

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Derivation using polynomial interpolation

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One point Hermite polynomial $p(t)$ *that matches the function and derivative data at* $t = t_k$, *that is,*

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Derivation using the method of undetermined coefficients

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At $t = t_k$, we know the values y_k and y_k', and based on these values we want to predict the value y_{k+1}.

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Derivation using the method of undetermined coefficients

At $t = t_k$, we know the values y_k and y'_k, and based on these values we want to predict the value y_{k+1}. A predictor based on their linear combination has the form

$$y_{k+1} = \alpha y_k + \beta y'_k$$

where α and β are coefficients to be determined.

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$$e = \alpha e + \beta 0, \qquad\qquad t_{k+1} e = \alpha t_k e + \beta e$$

implying $\alpha = 1$ *and* $\beta = t_{k+1} - t_k$ *resulting in the Euler's method.*

Module 2

Initial Value Problems

2.1 Well-posedness

2.2 Stability

2.3 Euler's method

- Errors and error propagation

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Initial Value Problems: Euler's method

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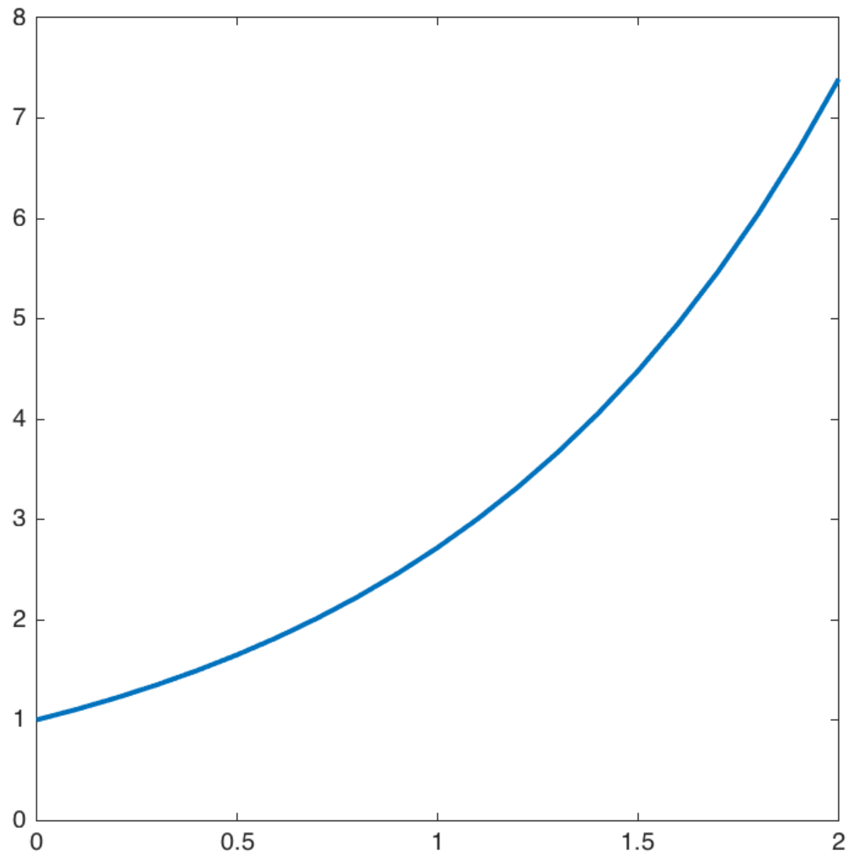
Example

Let us solve $y' = y,\ y(0) = 1$ *using the Euler's method taking the uniform step size* $h = h_k = t_{k+1} - t_k = 0.5$.

$$y_1 = 1 + h,\quad y_2 = y_1 + hy_1 = (1+h)^2,\quad y_3 = y_2 + hy_2 = (1+h)^3, \dots,\quad y_k = (1+h)^k, \dots$$



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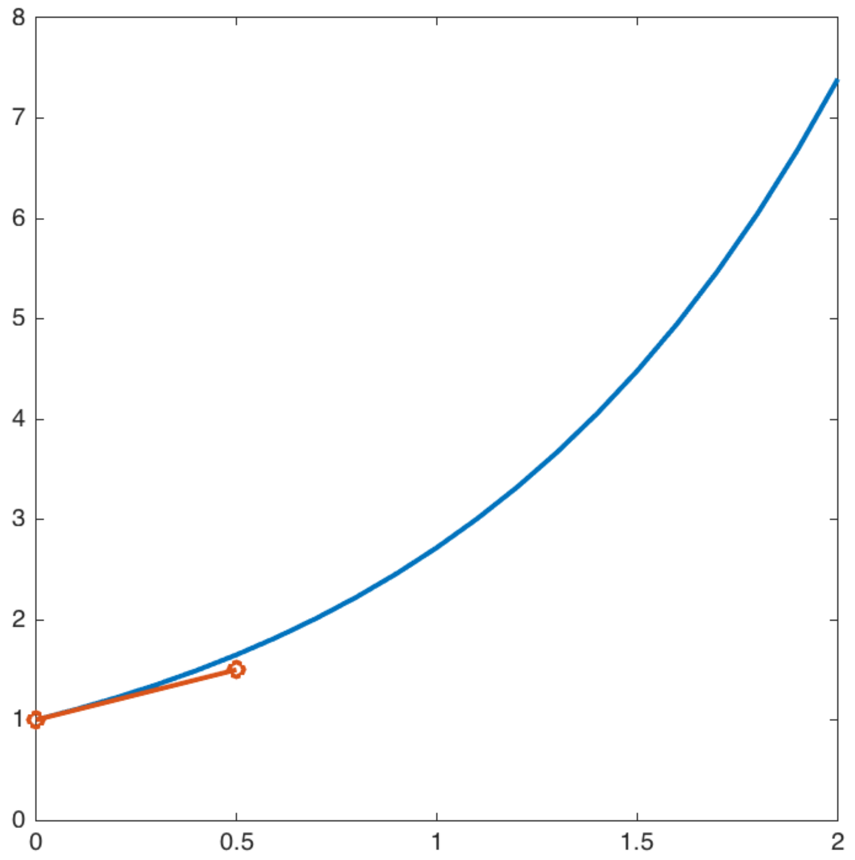

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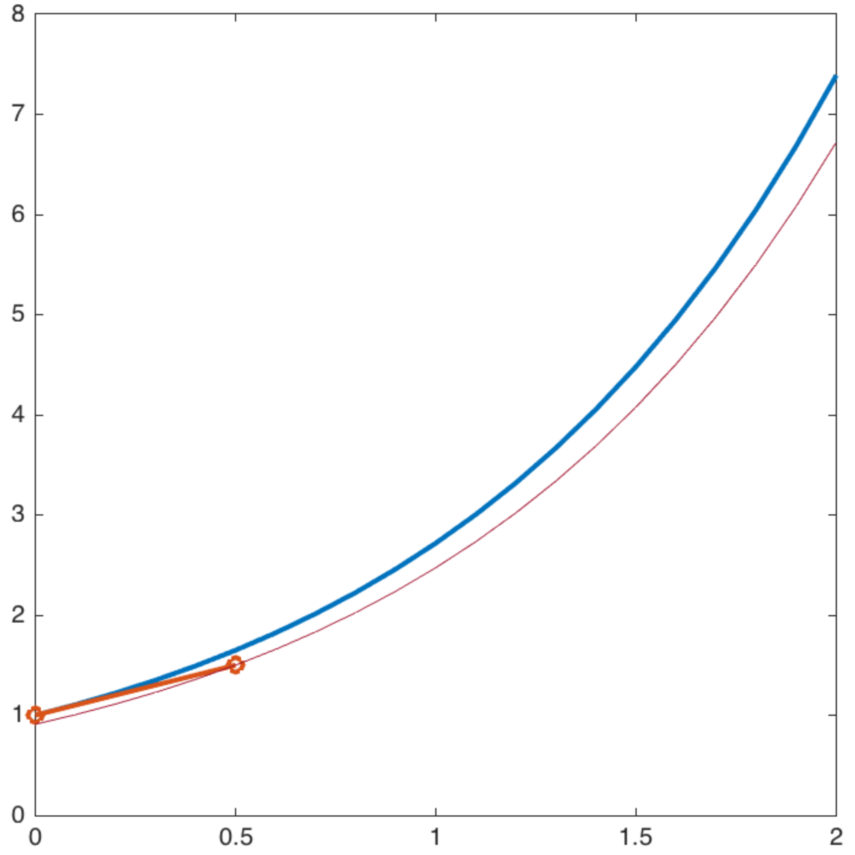

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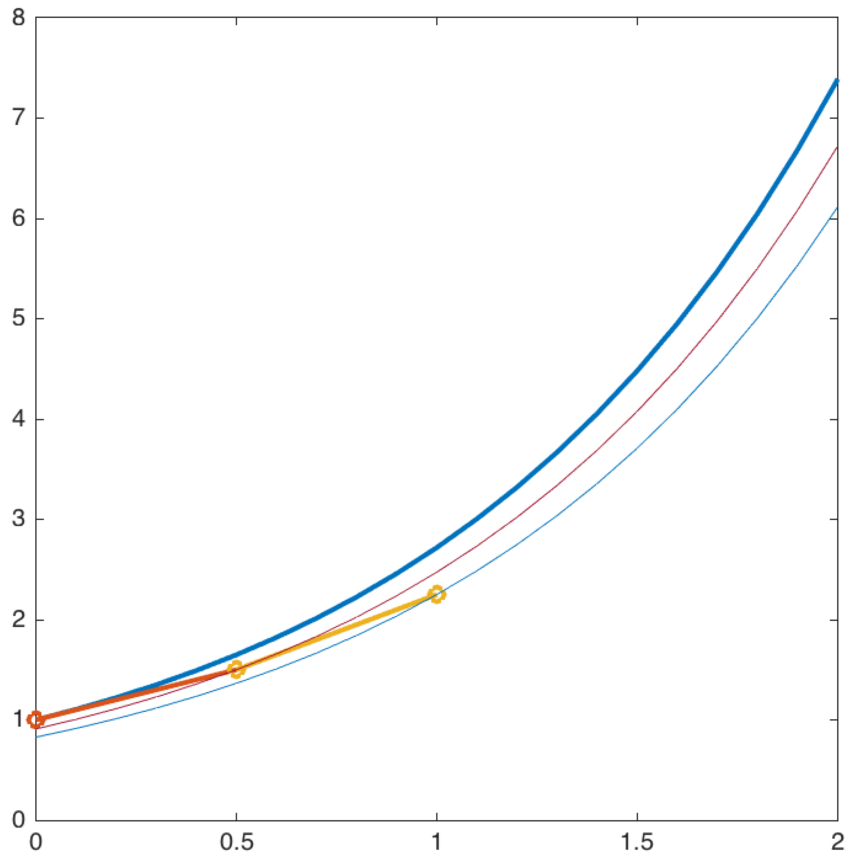

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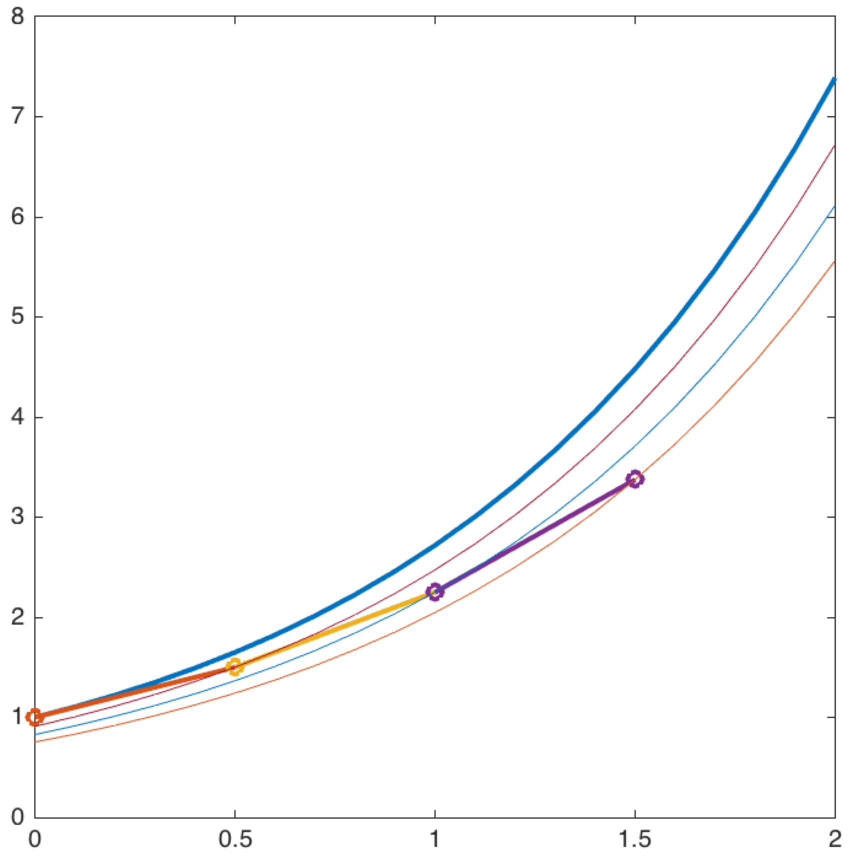

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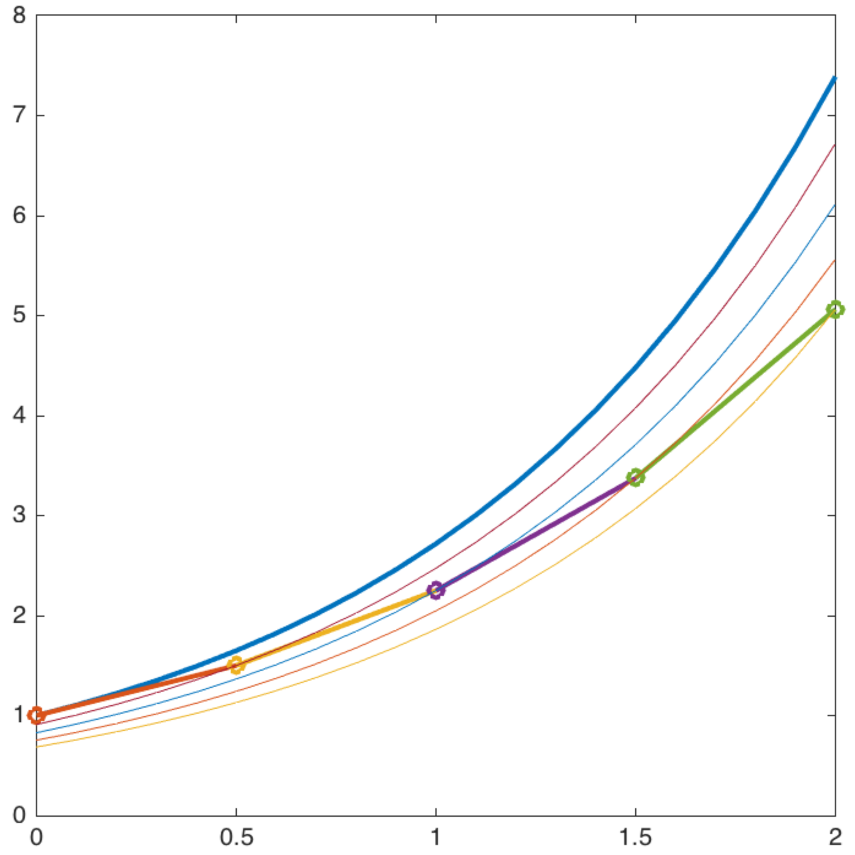

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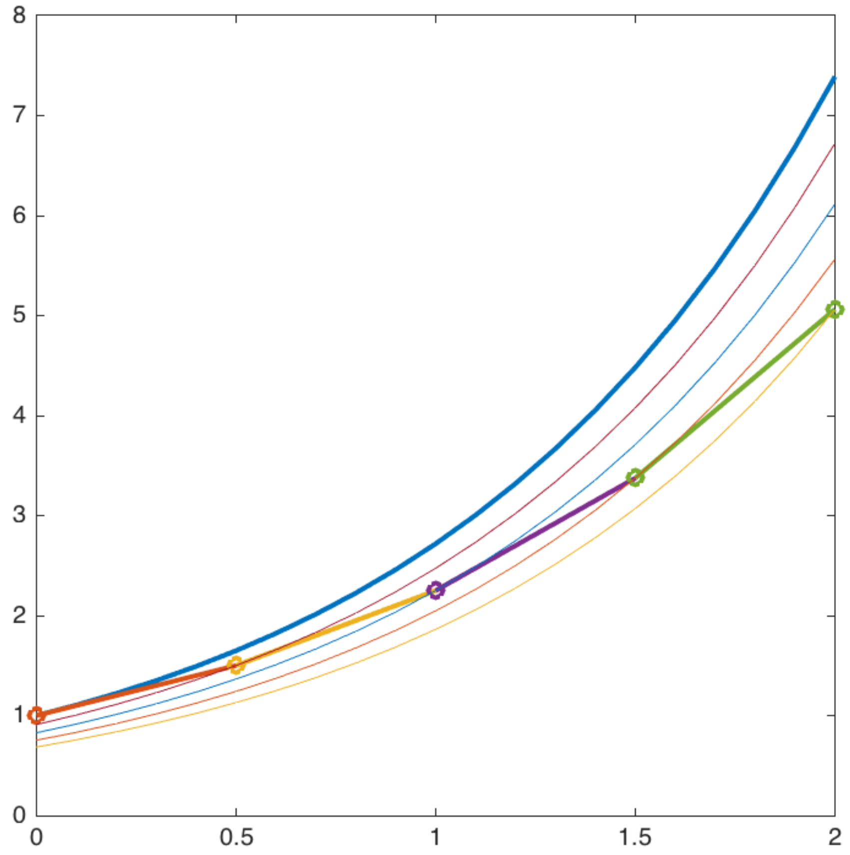

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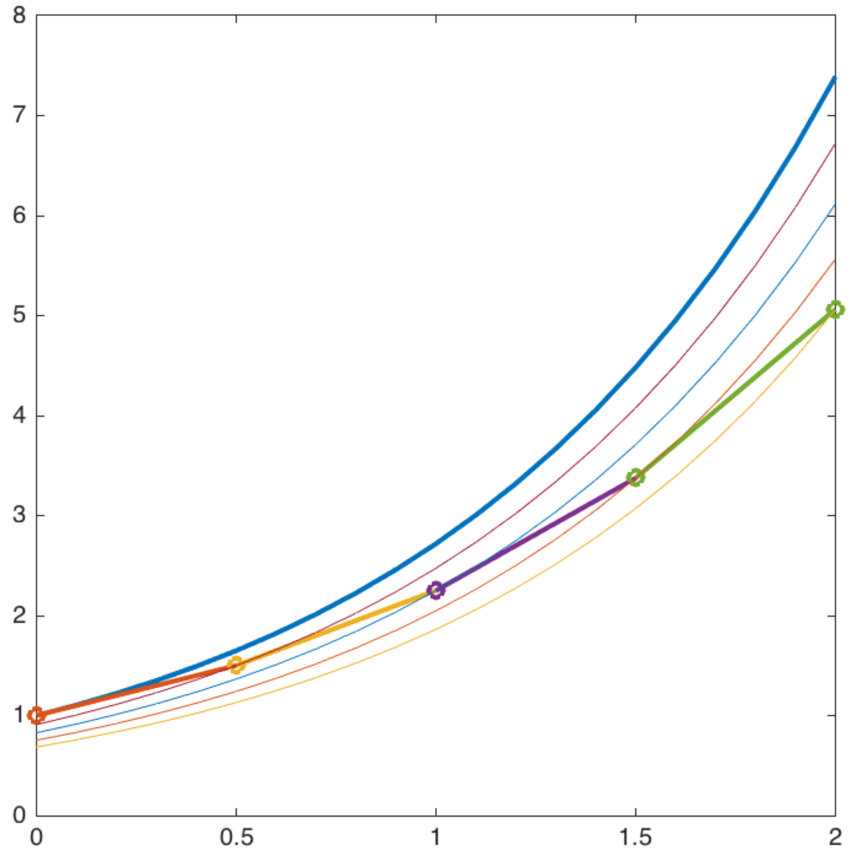

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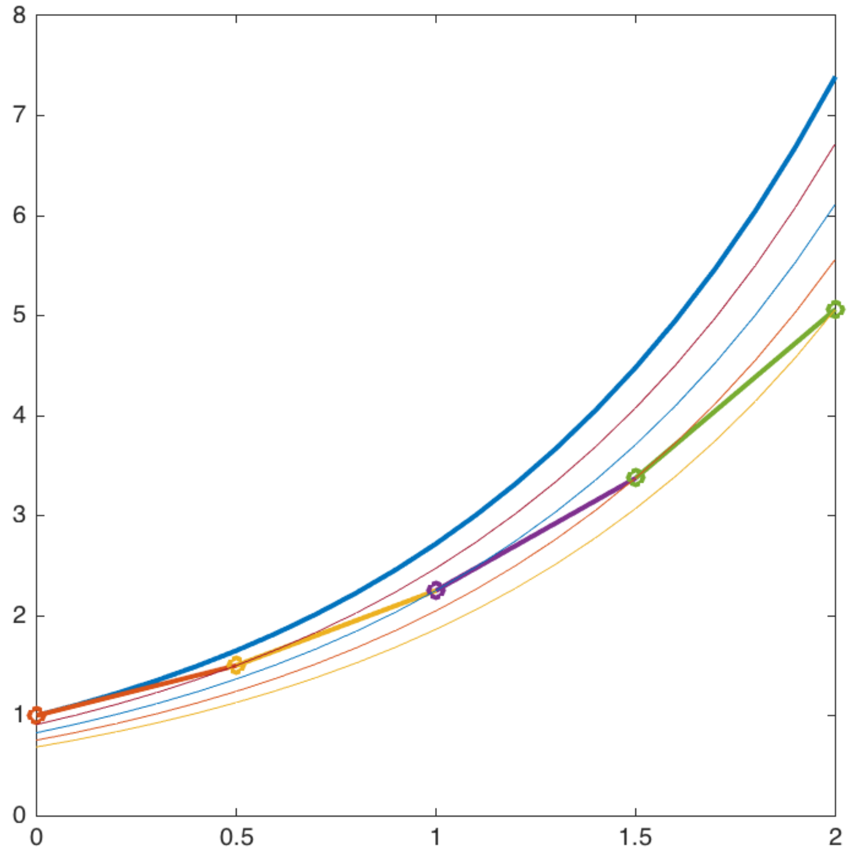

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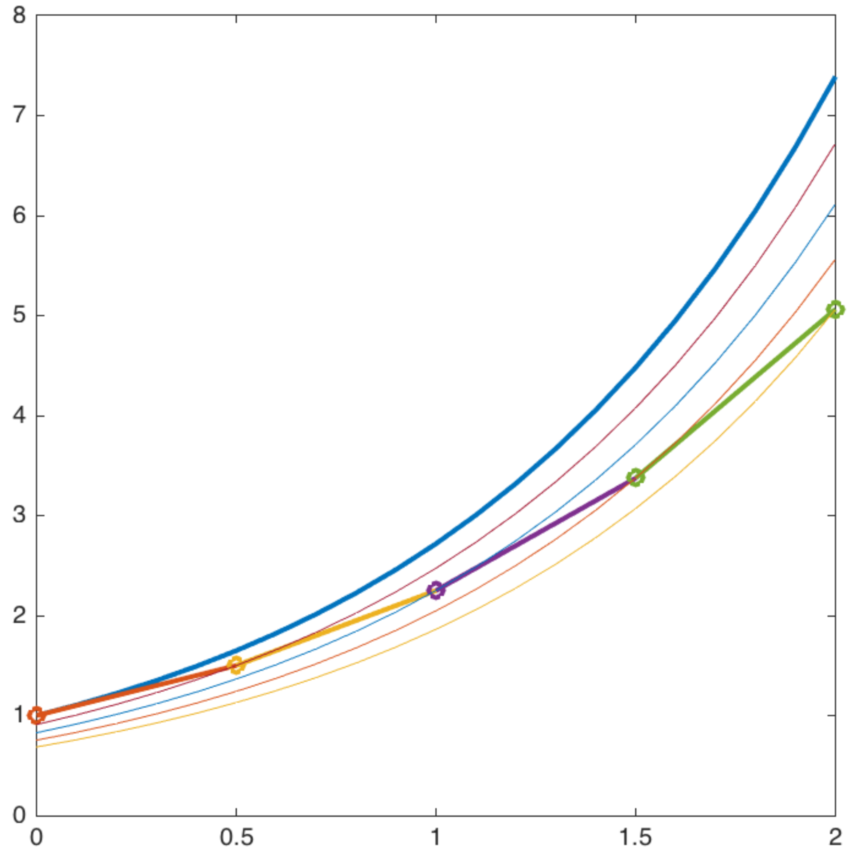

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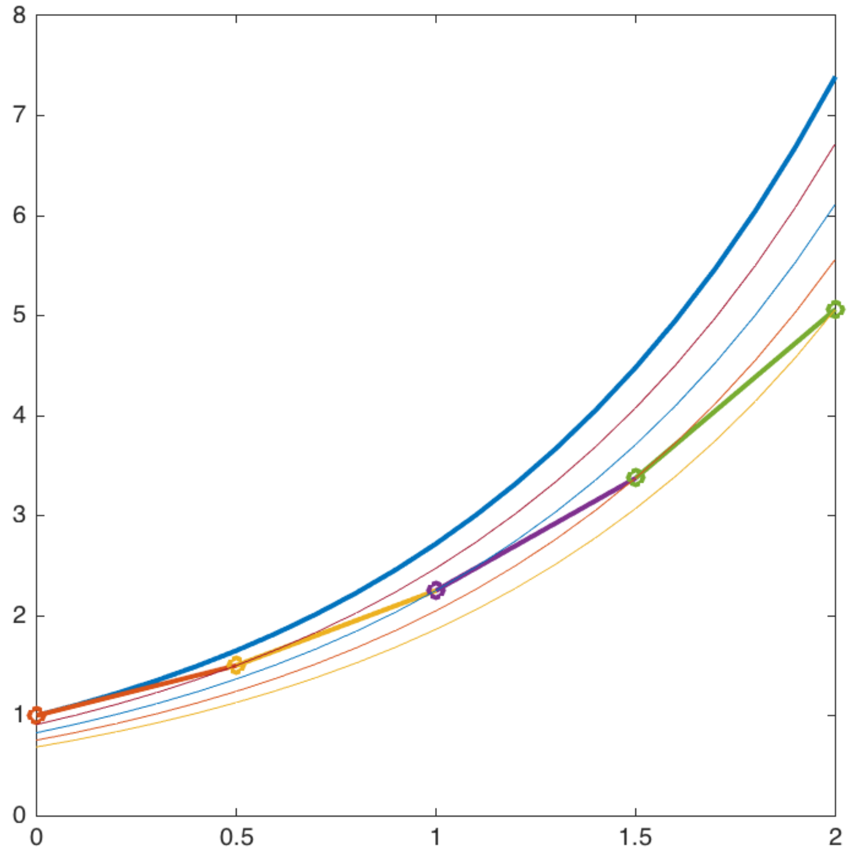

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Local errors get amplified if the solutions to the ODE are unstable.

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Source: https://youtu.be/714HgSO0h7g

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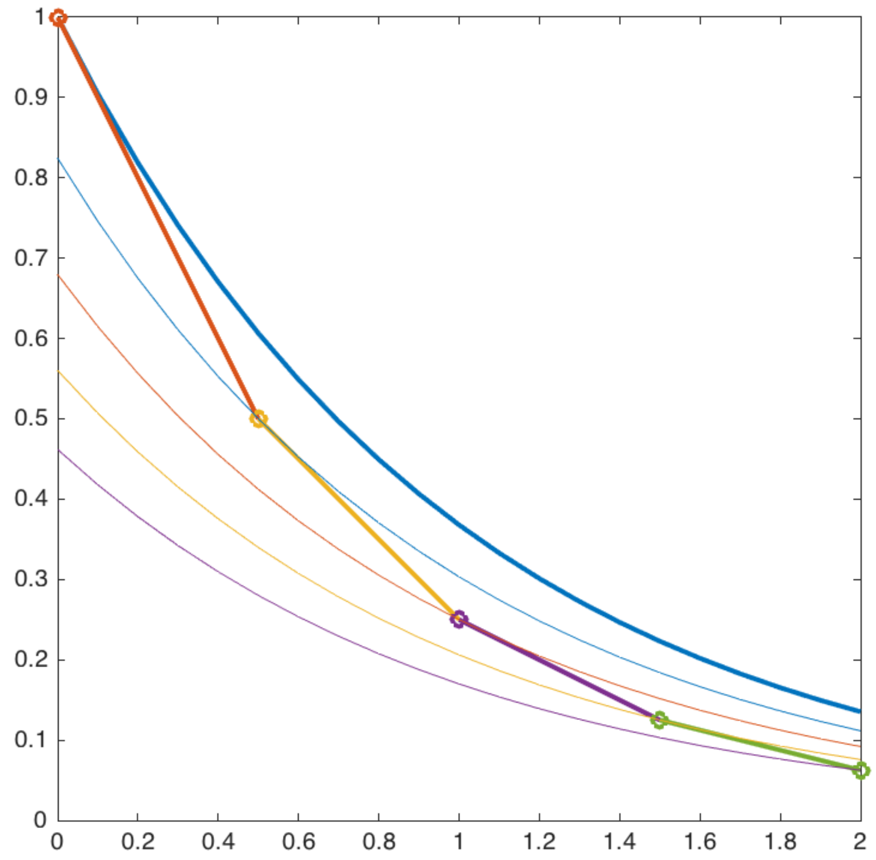

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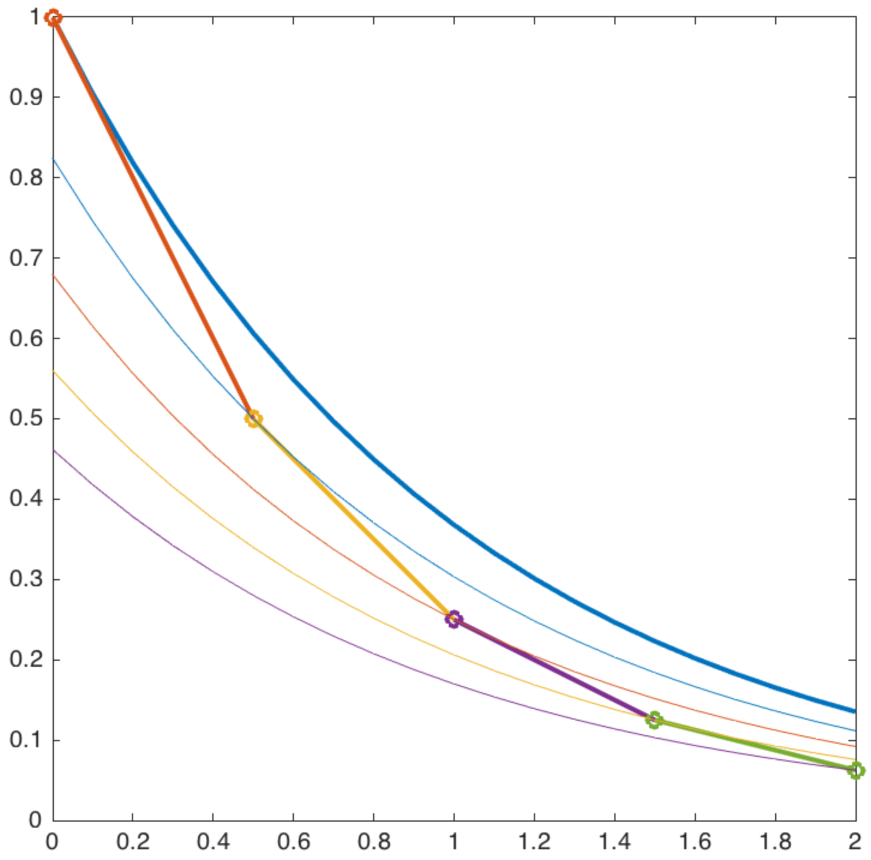

For an equation with stable solutions, the errors in the numerical solution do not grow, and for equations with asymptotically stable solutions, the errors diminish with time.

Initial Value Problems: Accuracy and Stability

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Sources of error

Rounding error –

… due to truncation of data (e.g., real numbers requiring infinite space is represented using a finite amount of space); finite precision of the floating point arithmetic.

Truncation error –

… due to truncation of infinite mathematical processes to finite processes or algorithms (e.g., derivative replaced by a finite difference approximation, or integration replaced by a quadrature).

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Errors propagate from one step to the next. We have seen how single step error in the Euler's method can get amplified over multiple steps.

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In most practical situations, truncation error dominates other sources and, therefore, in analysis of numerical methods, we will focus exclusicely on truncation error.