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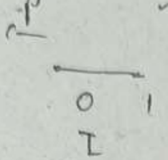
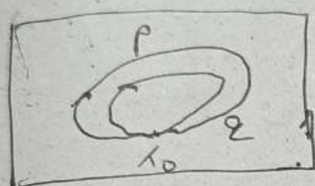
Fundamental Group $\pi_1(X, x_0)$

Defn. Let X be a space (subset of $\mathbb{R}^n$) let $x_0 \in X$. A path p in X that begins and ends at x_0 is called a loop based at x_0. We denote its path homotopy class $[p]$. The set $\pi_1(X, x_0) = \{[p] \mid p$ is a loop based at $x_0\}$.



$e_{x_0}(s) = x_0$, $p^{-1}(s) = p(1-s)$. Then

1) $[p], [q] \in \pi_1(x_0, x_0)$.



$$p \cdot q = \begin{cases} p(2s), & 0 \le s \le \frac{1}{2} \\ q(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

2) $[p] \cdot [e_{x_0}] = [p] = [e_{x_0}] \cdot [p]$ (identity)

3) $[p] \cdot [p^{-1}] = [e_{x_0}]$ (inverse)

4) $([p] \cdot [q]) \cdot [r] = [p] \cdot ([q] \cdot [r])$ (associative).

Properties of $\pi_1(X, x_0)$

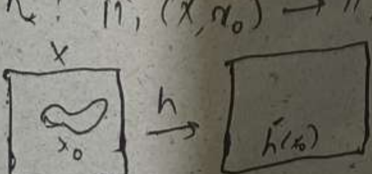
Thm: Let X be a space (subset of $\mathbb{R}^n$). Let X be path-connected. Let $x_0, y_0 \in X$

$\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, y_0)$.

Thm (2) $\pi_1(X, x_0)$ is topological invariant i.e. if X is homeomorphic to Y then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$.

~~Proof~~ Let $h: X \to Y$ be a homeomorphism h is a cont. fn from X to Y s.t. h is 1-1, onto and h^{-1} is cont. Let

$h_*: \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$



$h_*[p] = [h \circ p]$

h_* is called the gp. homomorphism associated with h.

1) h_* is a homomorphism $h_* : \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$

$h_*([p]\cdot[q]) = h_*[p] \cdot h_*[q]$

$[h\circ p]\cdot[h\circ q]$

$h\circ p(s)$

$$h\circ p(2s) \quad 0 \le s \le \tfrac{1}{2}$$
$$h\circ q(2s-1) \quad \tfrac{1}{2} \le s \le 1$$

$h\circ p(2s), \ 0 \le s \le \frac{1}{2}$

$h\circ q(2s-1), \ \frac{1}{2} \le s \le 1$

$G_1 \xrightarrow{\phi_1} G_2$; $G_2 \xrightarrow{\phi_2} G_1$

$\phi_2 \circ \phi_1 = id$; $\phi_1 \circ \phi_2 = id$

$\Rightarrow G_1 \overset{iso}{\cong} G_2$

$x_0) \to \pi_1(Y, h(x_0))$

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Fund Gp.

Defn:- A space X is said to be simply connected if it is path connected and if $\pi_1(X, x_0) = \{e\}$ for some $x_0 \in X$.

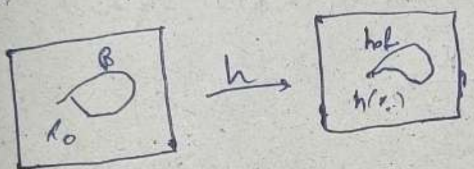
Ex:- $\mathbb{R}^n \; \forall n$. $(\mathbb{R}^n, 0)$

Thm:- $\pi_1(X)$ is a topological invariant i.e if X is homeomorphic to $Y \Rightarrow \pi_1(X, x_0) \overset{\text{isomorphic}}{\cong} \pi_1(Y, y_0)$

Proof:- Let $h : X \longrightarrow Y$ be a homeomorphism.

h is 1-1 onto, continuous & h^{-1} is continuous

we define $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, h(x_0))$, called the homomorphism induced by h.



$h_*[f] = [h \circ f]$.

We showed $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, h(x_0))$ is a homomorphism.

h_* has 2 basic properties "functional properties"

Thm:- If $h : (X, x_0) \to (Y, y_0)$, $k : (Y, y_0) \to (Z, z)$

$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $k_* : \pi_1(Y, y_0) \to \pi_1(Z, z)$

then (a) $(k \circ h)_* = k_* \circ h_*$

(b) $i : X \to X$ then $(id)_* = id$

Proof of thm:

$h_* : \pi_1(X, x_0) \xrightarrow{h_*} \pi_1(Y, h(x_0)) \xrightarrow{k_*}$

$\pi_1(Z, k \circ h(x_0))$

$(k \circ h)_*$

Let $f \in \pi_1(X, x_0)$ $k_* \circ h_*[f] = k_*[h \circ f] = [k \circ (h \circ f)]$

$(k \circ h)_*[f] = [(k \circ h) \circ f] = [k \circ (h \circ f)]$

(b) $i : X \to X$

$i_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$

$i_*[f] = [id \circ f] = [f]$

$i_* = id : \pi_1(X) \to \pi_1(X)$

Thm :- $\pi_1(X, x_0)$ is a top invariant

Proof:- let $h : X \to Y$ be a homeomorphism (h: 1-1, onto, cont, h^{-1} cont).

1) $X \xrightarrow{h} Y \xrightarrow{h^{-1}} X$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "id"'] & Y \arrow[r, "h^{-1}"] & X \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[rr, bend right, "(h^{-1}\circ h)_* = (id)_* = id"'] & \pi_1(Y, h(x_0)) \arrow[r, "(h^{-1})_*"] & \pi_1(X, x_0) \end{tikzcd}$$

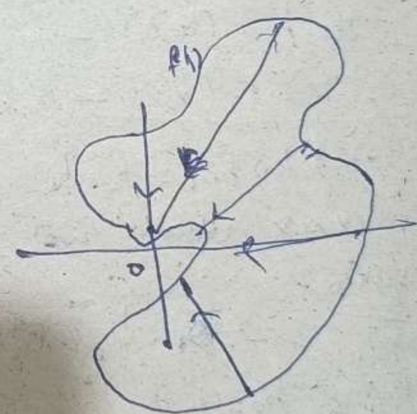
$\Rightarrow$ 1) $h^{-1}_* \circ h_* \quad \pi_1(X, x_0) \to \pi_1(X, x_0)$ $\quad h^{-1}_* \circ h_* = id$

2) $$\begin{tikzcd} Y \arrow[r, "h^{-1}"] \arrow[rr, bend right, "id"'] & X \arrow[r, "h"] & Y \end{tikzcd}$$

2) $h_* \circ h^{-1}_* : \pi_1(Y, y_0) \to \pi_1(Y, y_0)$

$h_* \circ h^{-1}_* = id$

Ex $\quad \pi_1(\mathbb{R}^n, 0) = e$



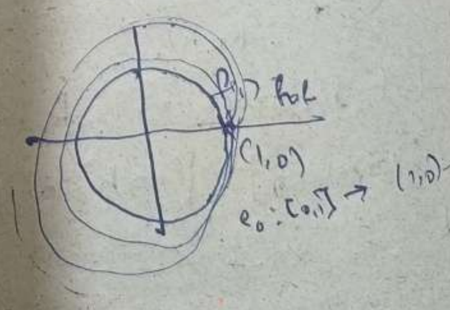
$h(s,t) = (1-t) f(s)$

$t = 0 \quad h(s,0) = f(s)$

$h(s,1) = 0$

2) $X = S^1 = \{x^2 + y^2 = 1\}$

$\pi_1(S^1, (1,0))$



$f(s) = (\cos 2\pi s, \sin 2\pi s)$

$f \circ f(s) = (\cos 4\pi s, \sin 4\pi s)$

$\underbrace{f \cdot f \cdots f}_{n} = (\cos 2\pi ns, \sin 2\pi ns)$

$f^{-1}(s) = (\cos 2\pi s, -\sin 2\pi s)$

$\pi_1(S^1,(1,0)) \overset{\text{isomorphic}}{\cong} (\mathbb{Z}, +)$

$[\underbrace{\cos 2\pi s, \sin 2\pi s}_{f}] \to 1$

$f \cdot f \longrightarrow 1+1=2$

$f \cdots f \longrightarrow \underbrace{1+1+\cdots+1}_{n} = n$

$e_{x_0} \to 0$

$f^{-1} \to 1$

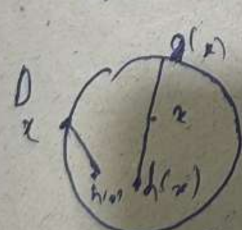
$\underbrace{f^{-1} \cdots f^{-1}}_{n} \longrightarrow -n$

Thm:- (Brower's fixed Point Theorem) Let D be the unit disc in $\mathbb{R}^2$ $\{x^2+y^2 \le 1\}$.
Let $h: D \to D$ be any cont. map. Then h has atleast one fixed point $h(x_0) = x_0$.

Proof:- By the Method of Contradiction.

Let $h: D \to D$ a cont. map with no fixed pt.

$h(x) \neq x$ $\forall x \in D$. Connect $h(x)$ to x by a st. line and extend the st. line till it hits the boundary circle, call it $g(x)$.



$D \xrightarrow{g} S^1$ s.t if $x \in S^1$, $g(x) = x$

Consider

$i(x) = x$

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "id"'] & D \arrow[r, "g"] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend right, "id_* = id"'] & \pi_1(D) \arrow[r, "g_*"] & \pi_1(S^1) \end{tikzcd}$$

$$\begin{tikzcd} (\mathbb{Z},+) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & e \arrow[r, "g_*"] & (\mathbb{Z},+) \end{tikzcd}$$

$g \circ i = id$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i"] \arrow[rr, bend right, "id"'] & 0 \arrow[r, "g"] & \mathbb{Z} \end{tikzcd}$$

But $g \circ i(\mathbb{Z}) \to 0 \neq id$ ⊗
Contradiction

$\Rightarrow h(x) = x$

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Deformation Retraction

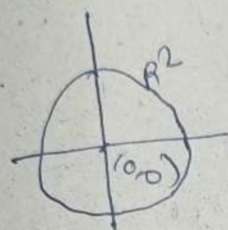
1) $\mathbb{R}^n$: $\pi_1 = e$

2) $A \subset \mathbb{R}^n$: A convex $\pi_1 = e$

3) S^1 : $x^2+y^2=1$ $\quad \pi_1(S^1,(1,0)) \cong (\mathbb{Z},+)$ $\quad \mathbb{R}^2 \setminus \{0,0\}$

Hurlies chapter 9 . sec 5.8

Punctured plane



$\pi_1(\mathbb{R}^2 \setminus (0,0), (1,0))$

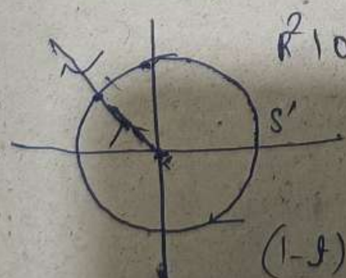
Defn:- Let X be a space $\subset \mathbb{R}^n$, A a subspace of X. Then A is said to be a strong deformation retract of X if there is a cont. map $h: X \times I \to X$ s.t.

$h(x,0) = x \quad \forall x \in A$

$h(x,1) \in A$, $h(a,t) = a \ \forall a \in A$

$\pi_1(X,a_0) = \pi_1(A,a_0)$

Ex1:- $X = \mathbb{R}^2 \setminus \{0,0\}$ = Punctured plane



$\mathbb{R}^2 \setminus 0 \quad S^1 \subset \mathbb{R}^2 \setminus 0$

$H: X \times I \to X$

$h(\vec{x},t) =$

$(1-t)\vec{x} + t \dfrac{\vec{x}}{\|\vec{x}\|}$

$h(x,0) = \vec{x}$

$h(x,1) = \dfrac{x}{\|\vec{x}\|}$

$h(a,t) = (1-t)\vec{a} + t\vec{a} = \vec{a}, \; a \in S^1$

$\pi_1(\mathbb{R}^2\setminus 0) = \pi_1(S^1,(1,0)) \cong (\mathbb{Z},+)$

Ex2:- $\vec{X} = \mathbb{R}^2$, $A = \{(0,0)\}$

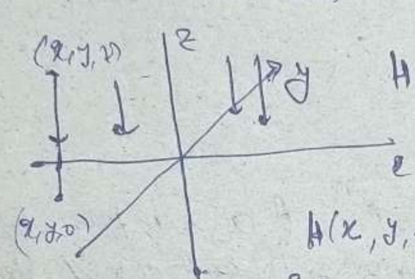
$h(x,t) = (1-t)\vec{x}$

$h(x,0) = \vec{x}$

$h(x,1) = (0,0)$

$\pi_1(\mathbb{R}^2) = \pi_1((0,0)) = e$

Ex3:- $X = \mathbb{R}^3 \setminus z$ axes



$H((x,y,z),t) = \{(x,y,(1-t)z)\}$

$H(x,y,z) = (x,y,0)$

$\mathbb{R}^3 \setminus z$ axis $\xrightarrow{\text{sdr}} \mathbb{R}^2\setminus\{0,0\} \xrightarrow{\text{sdr}} S^1$

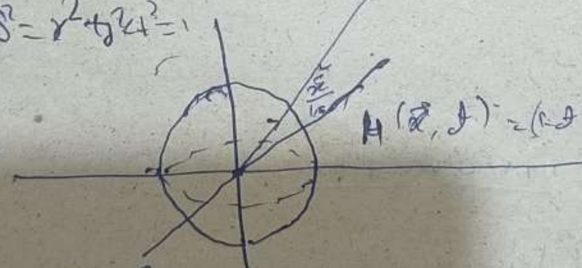
Ex5:- ~~...~~

$\mathbb{R}^3 \setminus \{0,0,0\} \to S^2$

$h(x,0) = x$

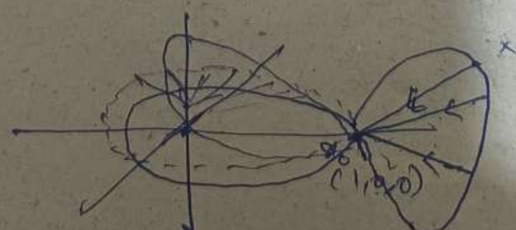
$h(x,1) = \frac{x}{\|x\|} \in S^2$

$S^2 = x^2+y^2+z^2 = 1$



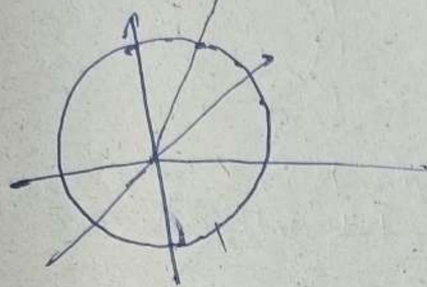
$H(\vec{x},t) = (1-t)\vec{x} + t\frac{\vec{x}}{\|\vec{x}\|}$

$\pi_1(\mathbb{R}^3 \setminus \{0,0,0\}) = \pi_1(S^2,(1,0,0))$



$X = R^{n+1} \setminus \{0, \dots, 0\}$

$S^n = \{ (x_1, \dots, x_{n+1}) \mid \sum x_i^2 = 1 \}$

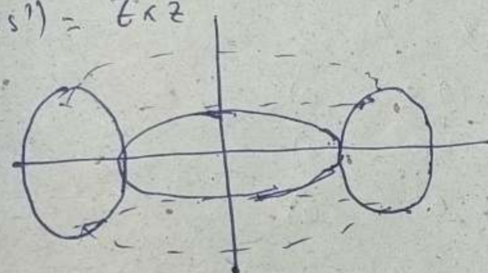


$p = \pi_1(R^{n+1} \setminus 0) \cong \pi_1(S^n)$

Section 60

$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Ex: $\pi(S^1 \times S^1) = Z \times Z$



Simplical Homology

M.A. Armstrong

Basic Top

Q] Is $R^n \underset{\text{homo}}{\cong} R^m$?

Q] $S^n \underset{\text{homo}}{\cong} S^m$?

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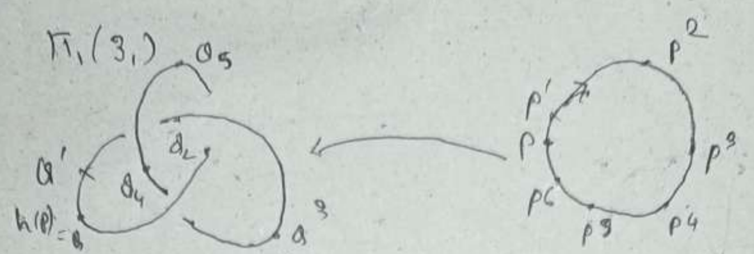


Any knot K is homeomorphic to $S' = \{x^2+y^2=1\}$
$\exists\, h : S' \longrightarrow K$ s.t. h is 1-1, onto, cont. with cont. inverse

$\pi_1(K) \overset{\text{iso}}{\cong} \pi_1(S') \cong (\mathbb{Z}, +)$

$\pi_1(\mathbb{R}^3 \setminus K) =$ knot gp. is a knot invariant.

Simplicial Complexes

Defn: -① k-Simplex Δ_k

In $\mathbb{R}^n$, let $E_0 = (0, 0, \ldots, 0)$

$E_1 = (1, 0, \ldots, 0)$

$E_2 = (0, 1, \ldots, 0)$

$\vdots$

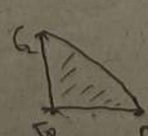
$E_k = (0, \ldots, \underset{k\text{th}}{1}, 0 \ldots 0)$

Defn: - A k-simplex is the convex span of vertices $E_0, E_1, \ldots, E_k$.

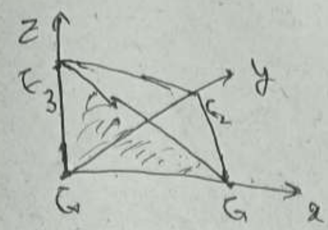
Ex: - 0-Simplex E_0 a Point

1-Simplex an interval $[0,1]$

2-Simplex

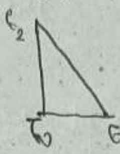


3-Simplex



Defⁿ 2:- Faces of a k-simplex

$\Delta_k = (E_0, E_1, \ldots, E_k)$ is any subcollection $(E_{\alpha_1}, \ldots, E_{\alpha_s})$ where $\alpha_s < k$

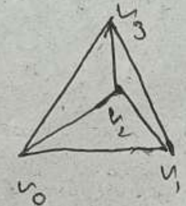


2-simplex
$E_0 E_1 E_2$

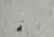
faces are E_0E_1, E_1E_2, E_2E_0 and E_0, E_1, E_2

Defⁿ (more gen) Let $v_0, v_1, \ldots, v_k$ be points in $\mathbb{R}^n$ $(n \geq k+1)$ s.t $\overrightarrow{v_1 - v_0}, \overrightarrow{v_2 - v_0}, \ldots, \overrightarrow{v_k - v_0}$ are linearly indep. Then the convex span

$$\{\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k \mid 0 \leq \lambda \leq 1 \ \sum \lambda_i = 1\}$$ is called a k-simplex.



3-Simp



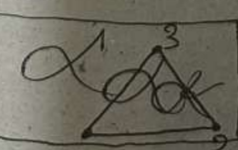
0-Simp

1-Simp

2-Simp

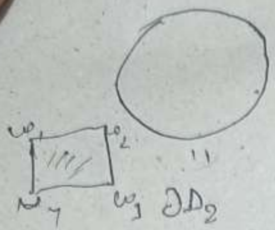
Defⁿ (3):- A simplicial complex K^n of dim n, is a collection of simplexes $\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_n$ s.t any 2 simplexes Δ_i and Δ_j intersect in a common face.

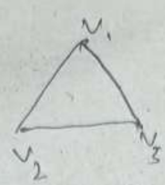
Ex:-



Defⁿ (4):- A triangulation of a space X is a simplicial complex K s.t X is homeo to K^n.

Eg:- S^1





2-Simplex (Δ_2)
Boundary of Δ_2 consist
3-0 simplex ~~...~~ (the vertices) (v_1, v_2, v_3)
3-1 ,, (the edges) (v_1v_2, v_2v_3, v_1v_3)

Defn (Euler No.) A topological invariant of K^n.

Let α_0 = # of vertices 0-simplexes
α_1 = ,, ,, edges 1- ,,
α_2 = ,, ,, faces 2- ,,
⋮
α_n = ,, ,, n-simplexes

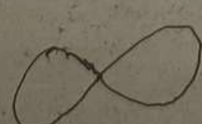
$$\chi(K^n) = \alpha_0 - \alpha_1 + \alpha_2 - \dots (-1)^n \alpha_n$$
$$= \sum_{i=0}^{n} (-1)^i \alpha_i$$

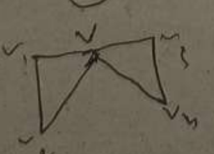
Euler Number
Top Inv.

Eg. 1 $\chi(K^2) = 3 - 3 + 0 = 0$

Eg. 2 Figure 8

$S_1 \vee S_1$





$\chi(F8) = 5 - 6 = -1$

(the vertices &
(v_1, v_2, v_3)
$v_1 v_2,\ v_2 v_3,\ v_1 v_3)$.
$1 < n$

$S_1 \vee S_1 \vee S_1$

G_3

$\chi(G_3) = 7 - 9 = -2$

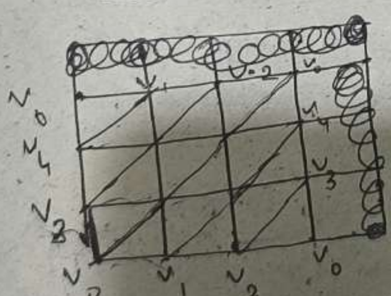
$\chi(G_n) = -(n-1)$

Ex3:- S^2

$= 4 - 6 + 4$

$= -2 + 4 = 2$

Ex4:- $S^1 \times S^1$

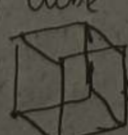


$\chi(\text{Torus}) = 9 - 27 + 18$

$= 0$

1) Tetrahedron 2) Cube (3) Octahedron

8 - Triangular face.



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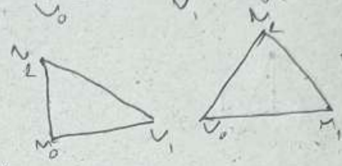
Simplical Complexes

Defn:- Let $v_0, v_1, \dots, v_k$ be vertices in $\mathbb{R}^n$, $n \geq k+1$ s.t. $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$ are linearly independent. Then the convex span of $v_0, v_1, \dots, v_k$ is a k-simplex denoted Δ_k.

Ex:-
1) 0-simplex v_0. a point
2) 1-simplex $v_0 v_1$

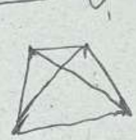
edge

3) 2- ,, $v_0 v_1 v_2$



Triangle

4) 3- ,, $v_0 v_1 v_2 v_3$



Tetrahedron.

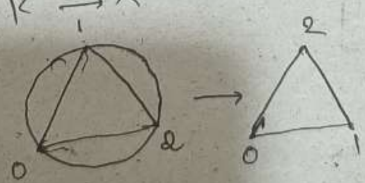
Defn②:- let $\Delta_k = v_0 \dots v_k$ be a k-simplex. Then any strict subset $v_{\alpha_1}, \dots, v_{\alpha_s}$ $i<k$ is a face of the k-simplex.

Defn③:- A simplicial complex of dimn, K^n consists of finitely many simplexes of dim $\leq n$ s.t. $\Delta_i \cap \Delta_j$ is a common face.

Defn④: Triangulation of a space X is a simplicial complex K^n and a homeomorphism

$h: K^n \to X$

S^1



$V \to 0, 1, 2$
$e \to 01, 12, 20$

Defn:- Euler Number:- Let K^n be a simp. complex.

Let α_0 = No. of 0-simplexes = no. of vertices of K^n
α_1 = ,, ,, 1- ,, = ,, ,, edges of K^n
α_2 = ,, ,, 2- ,, = ,, ,, faces ,, ,,
⋮
α_k = ,, ,, k- ,,

Thm consider the Euler Number $= \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \cdots (-1)^k \alpha_k$

$$\chi(K) = \sum_{i=0}^{k} (-1)^i \alpha_i$$

is a top invariant, discovered in 1750.

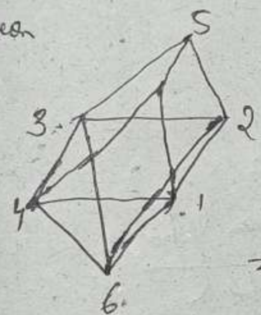
Thm (known to Greeks "2000 years ago") There exist 5 and only 5 regular polyhedron in nature.

Def:- Regular polyhedron → 1) Polyhedron is homeomorphic to the sphere S^2; $\chi(P) = 2$.

2) Polyhedron consists of vertices, edges & faces. st
(1) all face identical
(2) at any vertex same no. of edges emanate.

<u>Regular Polyhedron</u>

3) Octahedron



$v = 6$
$e = 12$
$f = 8$
$v - e + f$
$= 6 - 12 + 8$
$= 2$

4) Dodecahedron
12 Pentagonal face

5) Icosahedron
20 triangular face

Thm $\exists$ 5 regular polyhedron in nature

Proof.- (Hilbert - Geom. with imag) Let X be a regular polyhedron. Let V = # of vertices, E = # of edges, F = # of faces of X.
Suppose each face has r edges.
Suppose that at each vertex n edge emanate.

<u>We know</u>

$$V - E + F = 2$$

$$rF = 2E \Rightarrow F = \frac{2E}{r}$$

$$nV = 2E \Rightarrow V = \frac{2E}{n}$$

$V - E + F = 2$

$\frac{2E}{n} - E + \frac{2E}{r} = 2$

$\Rightarrow E\left(\frac{2}{n} - 1 + \frac{2}{r}\right) = 2$

$\Rightarrow \boxed{\frac{1}{n} + \frac{1}{r} = \frac{1}{E} + \frac{1}{2}}$ — (*)

$r \geq 3 \qquad n, r \geq 4$

$n \geq 3 \qquad$ so $\frac{1}{n} + \frac{1}{r} - \frac{1}{2} = \frac{1}{E}$

$\leq \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0 = \frac{1}{E}$

So either $n = 3$ or $r = 3$

$n = 3$	$r = 3$	Tetrahedron
$n = 3$	$r = 4$	~~Dodecahedron~~ Cube
$n = 3$	$r = 5$	dodechadedron
$n = 4$	$r = 3$	Octahedron

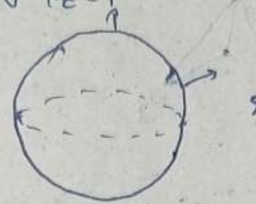
Note (*) is symmetric in $n \times r$.

2/4/25

Classification Theorems

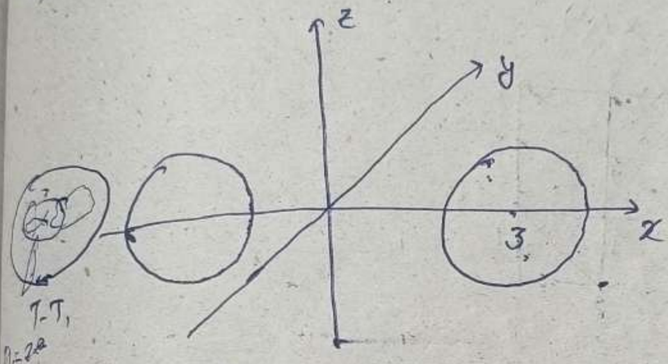
Classification Theorem for 2-dimensional surfaces which are oriented, compact and without boundary. (Mobius 1782)

1) Sphere $x^2+y^2+z^2=1$



$x\hat{i} + y\hat{j} + z\hat{k}$

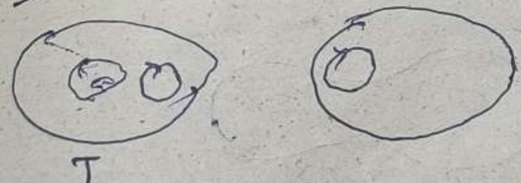
2) Torus $S^1 \times S^1$



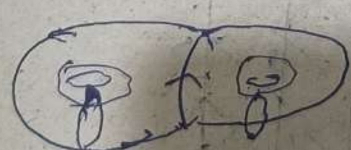
$(x-3)^2+z^2=1$

Connected Sum $S_1 \# S_2$

$T \# T$



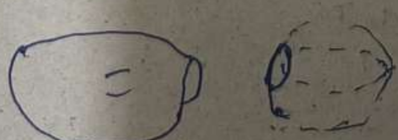
T



T_2 or Double Torus

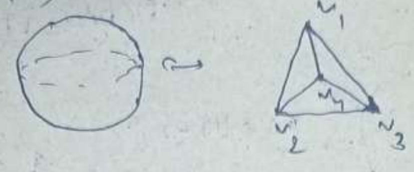
T

Torus # sphere = T (Torus)



$\chi = V - E + F$, a top invariant

1) $\chi(S^2)$



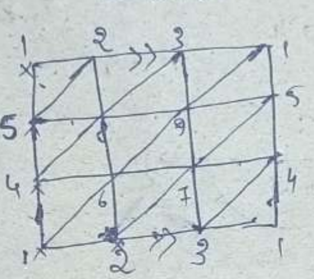
$V = 4$
$E = 6$
$F = 4$

$V - E + F = \chi(S^2)$

$4 - 6 + 4 = \chi(S^2)$

$2 = \chi(S^2)$

2)

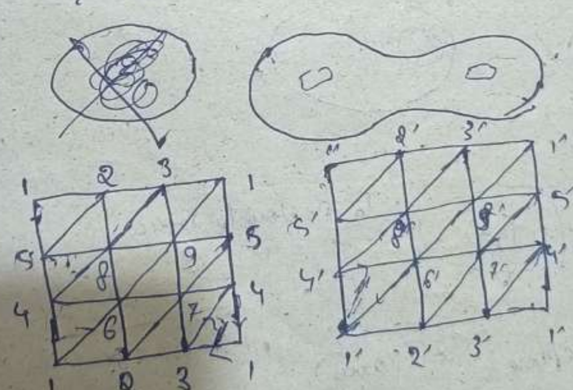


$\alpha_0 = 9$ $\qquad 9 - 27 + 18 = 0.$

$\alpha_1 = 27$

$\alpha_2 = 18$

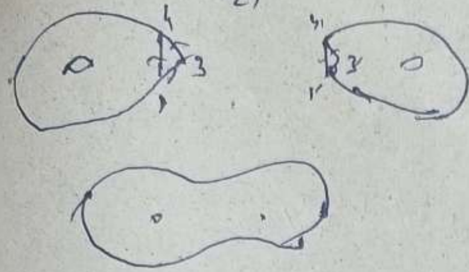
3)



Remove one face 134 from T
$\quad$ 1'3'4' '' T'

Join 13 to 1'3'
$\quad$ 14 to 1'4'
$\quad$ 34 to 3'4'

$\chi(T_2)$



$\alpha_0 = 9$
$\alpha_1 = 27$
$\alpha_2 = 17$

$\alpha_0 = 9+9-3 = 15$
$\alpha_1 = 27+27-3 = 51$
$\alpha_2 = 18+18-2 = 34 \qquad 15-51+34 = -2$

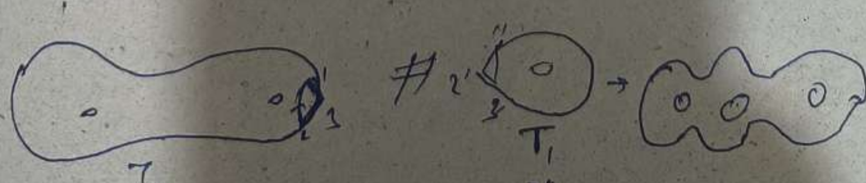
$S_1 \# S_2$

S_1 - α_0 vertices $\qquad$ S_2 - α_0' vertices
α_1 edge $\qquad$ - α_1' edges
α_2 faces $\qquad$ - α_2' faces

$S_1 \# S_2$

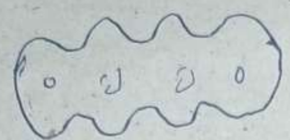
α_0 of $S_1 \# S_2 = \alpha_0 + \alpha_0' - 3$
α_1 " $S_1 \# S_2 = \alpha_1 + \alpha_1' - 3$
α_2 " $S_1 \# S_2 = \alpha_2 + \alpha_2' - 2$

$$\chi(S_1 \# S_2) = (\alpha_0+\alpha_0'-3) - (\alpha_1+\alpha_1'-3) + (\alpha_2+\alpha_2'-2)$$
$$= \boxed{\alpha_0+\alpha_0'-\alpha_1-\alpha_1'+\alpha_2+\alpha_2'-2}$$
$$= (\alpha_0-\alpha_1+\alpha_2)+(\alpha_0'-\alpha_1'+\alpha_2')-2$$
$$= \chi(S_1)+\chi(S_2)-2.$$



$= \qquad -2 \qquad 9-\cancel{27}^{0}+18-2 = -4$

Similarly $T_3 \# T \cong T_4$



$T_{g-1} \# T \cong T_g$

$T_1 \# T_1 = -2$

$T_2 \# T_1 = -4$

$T_3 \# T = -6$

(c) $\chi(T_g) = 2(1-g) = 2-2g$

$\chi(T_{g+1}) = \chi(T_g) + \chi(T) - 2$

$= \cancel{2} - 2g + 0 - \cancel{2}$

$= -2g.$

Note:- Classification theorem says that any 2 dim compact oriented surface without boundary is either

1) The sphere S^2
2) The torus T
3) The connected sum of g torus T_g.

1/12/25

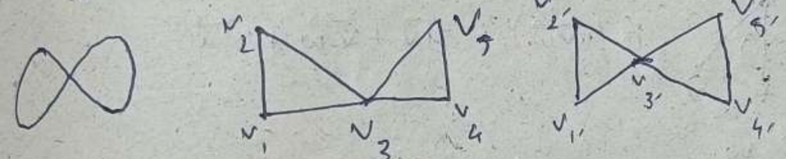
Simplicial Homology Theory

$\chi(S^n) = 2$ if n is even

$\chi(S^n) = 0$ if n is odd

Defn:- Let K^n be a n-dimensional simplicial complex, dim n means simplex σ_i of highest dim in K^n is σ_n.
We will associate with K^n, a set of $(n+1)$ abelian groups denoted $H_0(K^n), H_1(K^n), H_2(K^n), \ldots, H_n(K^n)$ called the simplicial homology gp of K^n and each of them is a ~~topological~~ top invariant of K^n.

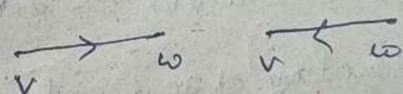
Defn:- (1) Let K and L be simp. Complexes K is isomorphic to L if there is a map ϕ from the vertices of K to the vertices of L which is 1-1, onto s.t. if $v_1, v_2, \ldots v_s$ spans a simplex if and only if $\phi(v_1), \phi(v_2) \ldots \phi(v_s)$ span a simp of L.



Defn (2) Orientation of a simplex

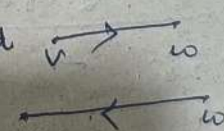
0 simplex only one vertex v

1- simplex v, w



The symbol $\overrightarrow{vw}$ means vw oriented

,, $\overrightarrow{wv}$,, vw ,,



$(w, v) = -(v, w)$

$(u\,v\,w)$ is the 2-simplex oriented anticlockwise

$(v\,u\,w)$ is the 2-simplex oriented clockwise

$(u, v, w) = -(v, u, w)$

$\sigma_i = (v_0, v_1, \dots, v_i)$ an i-simplex with a given orientation.

$-\sigma_i = (v_1, v_0, \dots, v_i)$ an i-simplex with a given opposite orientation.

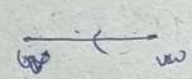
Let θ be a permutation of $0, 1, 2, \dots, i$.
$\text{Sign}\,\theta = +1$ or -1.

$$(v_{\theta(0)}\, v_{\theta(1)} \dots v_{\theta(i)}) = (\text{Sgn}\,\theta)(v_0 v_1 \dots v_i)$$

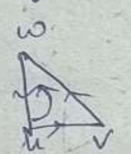
Def ③ Boundary of a simplex

Oriented 1-simplex $\vec{vw}$

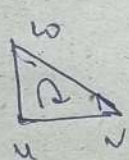
$\partial(\vec{vw}) = w - v$.



$\partial(\vec{wv}) = v - w = -\partial(\vec{vw})$



$\partial(\vec{uvw}) = \vec{uv} + \vec{vw} + \vec{wu}$



$\partial(\vec{vuw}) = \vec{uw} + \vec{wv} + \vec{vu}$

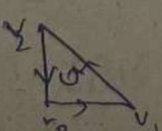
$\partial(\vec{uvw}) = -\partial(\vec{vuw})$

Boundary of an i-simplex $\sigma_i\ (\overrightarrow{v_0 v_1 \dots v_i})$

$$\partial\sigma_i = \sum_{j=0} (-1)^j v_0 v_1 \dots \hat{v_j} \dots v_i$$

where $\hat{v_j}$ means v_j is omitted/deleted.

$$\partial(v_0 v_1 v_2) = (-1)^0 \hat{v_0} v_1 v_2 + (-1) v_0 \hat{v_1} v_2 + (-1)^2 v_0 v_1 \hat{v_2}$$
$$= v_1 v_2 + v_2 v_0 + v_0 v_1$$



(Def): We define $C_q(K)$ - the qth chain group of K^n as follows

Suppose K^n has finitely many q-simplices $\sigma_1^q, \sigma_2^q, \dots \sigma_p^q$.

A chain is a formal finite sum.

$C_q(K^n) = \{\lambda_1 \sigma_1^q + \lambda_2 \sigma_2^q + \dots + \lambda_p \sigma_p^q \mid \lambda_i \in \mathbb{Z}\}$

<u>claim $C_q(K^n)$ is an abelian gp</u>

1) $\lambda_1 \sigma_1^q + \lambda_2 \sigma_2^q + \dots + \lambda_p \sigma_p^q \mid \lambda_i \in \mathbb{Z}$

$+ \lambda_1' \sigma_1^q + \lambda_2' \sigma_2^q + \dots + \lambda_p' \sigma_p^q$

$= (\lambda_1 + \lambda_1')\sigma_1^q + (\lambda_2 + \lambda_2')\sigma_2^q + \dots + (\lambda_p + \lambda_p')\sigma_p^q$

1) Identity $\quad 0 \cdot \sigma_1^q + 0 \cdot \sigma_2^q + \dots + 0 \cdot \sigma_p^q$

2) Inverse of $\lambda_1 \sigma_1^q + \lambda_2 \sigma_2^q + \dots + \lambda_p \sigma_p^q$ is

$\quad - \lambda_1 \sigma_1^q - \lambda_2 \sigma_2^q \dots - \lambda_p \sigma_p^q$

$C_q(K)$ is an abelian gp $\cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}}_{p}$

$C_n(K^n) \xrightarrow{\partial_n} C_{n-1}(K^n) \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_{n-2}} \dots C_1 \to C_0 \to 0.$

$\partial_i \circ \partial_j = 0$

$C_{i+1}(K^n) \xrightarrow{\partial_{i+1}} C_i(K^n) \longrightarrow C_{i-1}(K^n)$

$\partial_i \circ \partial_{i+1} = 0$, $\ker \partial_i = Z_i$ - (subgp of i-cycles)

$\operatorname{Im} \partial_{i+1} = B_i = i$-boundaries.

$B_i \subset Z_i$, H_i ith homgp $= Z_i / B_i$

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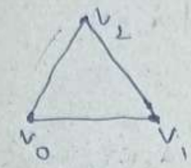
Examples of Simplical Homology groups

Thm: If K^n is a simplical complex which is connected and path-connected then $H_0(X) = \mathbb{Z}$.

Examples: 1) Circle S^1



$C_0 = \{v_0, v_1, v_2\} \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

$C_1 = \{v_0v_1, v_1v_2, v_2v_3\} \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$Z_1 = \ker \partial_1 \cong (\mathbb{Z}, +)$

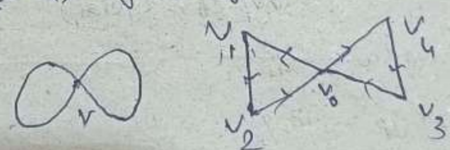
$B_1 = \operatorname{Im} \partial_2 \cong \phi$

$H_1(S^1) \cong Z_1/B_1 \cong Z_1 \cong (\mathbb{Z}, +)$

$Z_1 = \{\lambda(v_0v_1 + v_1v_2 + v_2v_0) \mid \lambda \in \mathbb{Z}\}$

$H_0(S^1) \cong (\mathbb{Z}, +)$.

Example 2) Figure eight



$$\mathcal{C}: C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

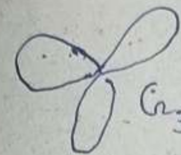
$Z_1 = v_0v_1 + v_1v_2 + v_2v_0$

$\partial(z_1) = \cancel{v_1} - \cancel{v_0} + \cancel{v_2} - \cancel{v_1} + \cancel{v_0} - \cancel{v_2} = 0$

$\ker \partial_1 = Z_1 = \mathbb{Z}\times\mathbb{Z}$

$H_1 = Z_1 = \mathbb{Z}\times\mathbb{Z}$

$H_0 = \mathbb{Z}$



Since $H_1(\text{fig } 8) = \mathbb{Z}\times\mathbb{Z} \neq H_1(S^1) \cong \mathbb{Z}$

$\Rightarrow$ fig 8 not homeo to S^1.

$\text{Im}\, \partial_2 = \phi$

$\ker \partial_2 = \lambda_1(v_0v_1 + v_1v_2 + v_2v_0) + \ker\partial_1$

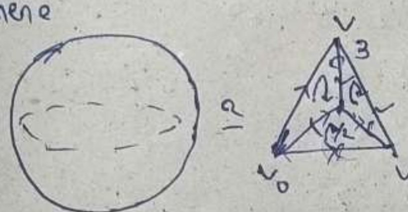
$\lambda_2(v_0v_3 + v_3v_4 + v_4v_0) +$ "

$\lambda_3(v_0v_5 + v_5v_6 + v_6v_0) +$ "

$\Rightarrow \ker \partial_1 \cong \mathbb{Z}^3$

$H_1 = \dfrac{\ker \partial_1}{\text{Im}\, \partial_2}$

Sphere



$\mathcal{C}: C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$

$\partial\{v_0v_3v_2 + v_2v_3v_1 + v_0v_2v_1 + v_3v_0v_1\} = 0$

$Z_2(S^2) \cong \mathbb{Z}$

$H_2(S^2) \cong \dfrac{Z_2}{B_2} \cong Z_2$

$B_2 = \phi$

$Z_1 = v_2v_1 + v_1v_0 + v_0v_2$

$\partial(v_0v_2v_1) = Z_1$

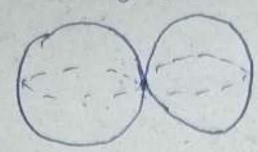
$H_1 = \dfrac{Z_1}{B_1} = \phi$

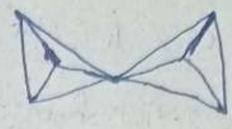
$H_0 = Z$

$H_2 = Z$

Ex:-

$S^2 \vee S^2$





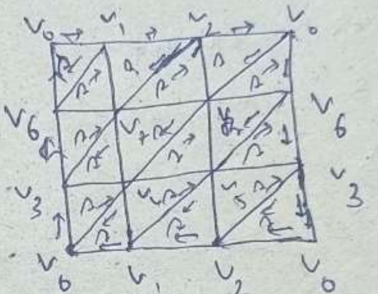
$H_i(S^2 \vee S^2)$

$H_0 = Z$

$H_1 = \phi$

$H_2 = Z \times Z$

Example:-



$H_2 = \frac{Z_2}{B_2}$ $\quad H_2 = Z_2 = Z$

$f =$ [illegible]

$B_2 = \phi$ $\quad \partial(c) = 0$

$Z_2 = Z$

$H_2 = Z$

$H_0 = ?$

$H_1 = Z \times Z$

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Thm:- K^n a simplicial complex of dim n; A^m a sub-complex of K^n. If A^m is a deformation of K^n, then $H_i(K^n) \cong H_i(A^m)$ $\forall i$.

Isomorphism

Examples :-

i) Mobius strip

$H_i(M)$

$K^n = M$

$A^m = S^1$

Mobius strip def retracts to central circle S^1.

$H_0(M) = \mathbb{Z}$

$H_1(M) = \mathbb{Z}$

ii) Cone on a Circle C

$x^2+y^2=z^2$

$0 \le z \le 1$

$H: C \times I \longrightarrow C$

$H(x, y, z, t) = (1-t)(x, y, z)$

$H(x, y, z, 0) = (x, y, z)$

$H(x, y, z, 1) = (0, 0, 0)$

$H_0 = \mathbb{Z}$ Deformation retracts to a point $(0,0,0)$

$H_1(C) = 0$.

f =

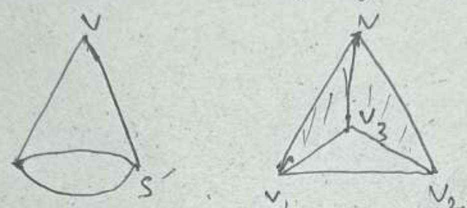
1) $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$

Since S^n is connected, path connected $H_0(S^n) \cong (\mathbb{Z}, +)$

<u>Thm</u> $H_i(S^n) \cong 0$ if $i = 1, 2, \ldots < n$

$H_n(S^n) \cong (\mathbb{Z}, +)$

<u>Proof of Thm</u> goes via the calculation of Hom. Gr. of a Cone Complex Ck^n of any Simplicial Complex k^n



<u>Cone on a Circle</u>
$C(S^1)$

$C(S') = C_0\{v_1, v_2, v_3, V\}$

$C_1\{v_1v_2, v_2v_3, v_3v_1, Vv_1, Vv_2, Vv_3\}$

$C_2 = \{Vv_1v_3, Vv_3v_2, Vv_1v_2\}$

~~Join~~

<u>Defn</u>:- Let L be a Simplicial Complex in R^n. Let $v = (\underbrace{0, 0, 0, \ldots, 1}_{n+1})$. We then construct the Cone complex CL called the Cone on L as follows If A is a k-Simplex of L, with vertices $v_0 v_1 \ldots v_k$, then the points $v - v_0, v - v_1, \ldots v - v_k$ are lin. ind. & thus $vv_0v_1 \ldots v_k$ is a $(k+1)$ simplex in R^{n+1}. Called the join of v with A.

Def:- The Cone Complex CL consists of
1) vertices of L, + one new vertex v,
2) Edges of L, + vv_i for each $v_i \in K$

1) i-simplex of L + v ($i-1$ simplex of L)

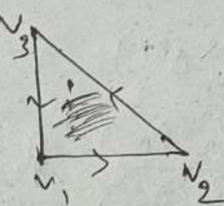
Armstrong Chap 8 P. 181-182 Ex 4, 6

Thm 1 - If CL is any Cone Complex, then $H_0(CL) = \mathbb{Z}$
$H_i(CL) = 0 \quad \forall i \neq 0$

We will use Thm 1 to Calculate $H_i(S^n)$ for every n

Step 1 Δ_2 - 2 Simplex

$\partial(\Delta_2) = S^1$



Δ_3 - 3 Simplex

$\partial(\Delta_3) = S^2$

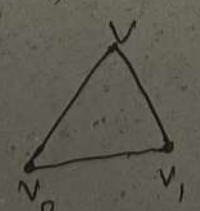
(1) $\boxed{\partial(\Delta_{n+1}) = S^n}$

Step 2 0 - Simplex $\Delta_0 = \cdot\, v_0$

$v = (0, 0, \dots, 1)$ $(\Delta_0 = \Delta_1$

b) $\Delta_1 = v_0 v_1$; $(v = 0, \dots, 0, 1)$

$v\Delta_1 = \Delta_2$



c) $v\Delta_2 = \Delta_3$

(2) $\boxed{v\Delta_n = \Delta_{n+1}}$

$\underline{\underline{S^n}} = \partial(\Delta^{n+1})$

i-simplices of S^n are the same as the i-simplices of Δ_{n+1} for $0 \le i \le n$.

$H_i(S^n) = H_i(\Delta^{n+1})$, $1 \le i \le n-1$

$H_i(S^n) = 0 \quad \forall \; 1 \le i \le n-1$

$H_0(S^n) = \mathbb{Z}$ since S^n is path connected.

$$H_n(S^n) = \frac{Z_n(S^n)}{B_n(S^n)}$$

$Z_n(S^n) = Z_n(\Delta^{n+1})$

$$H_n(\Delta_{n+1}) = \frac{Z_n(\Delta_{n-1})}{B_n(\Delta_{n+1})} = 0.$$

$Z_n(\Delta_{n+1}) = B_n(\Delta_{n+1})$

$B_n(\Delta_{n+1}) \cong \mathbb{Z}$.

$\Rightarrow Z_n(\Delta_{n+1}) \cong \mathbb{Z}$

$Z_n(S^n) \cong (\mathbb{Z}, +)$

$$H_n(S^n) = \frac{Z_n(S^n)}{B_n(S^n) = 0} = (\mathbb{Z}, +)$$

$H_0(S^n) = \mathbb{Z}$

$H_n(S^n) = \mathbb{Z}$

$H_i(S^n) = 0$

$i \ne 0, n$

same as the i-simplexes of

$\leq n+1$

th connected

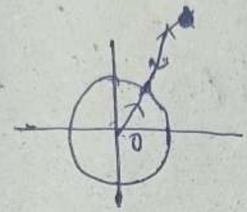
12/4/25

application 2 :- If $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ $\Rightarrow n = m$

proof :- Suppose $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ via h

$\Rightarrow \mathbb{R}^n \setminus \{0\}$ is homeomorphic to $\mathbb{R}^m \setminus h(0)$.

$\mathbb{R}^n \setminus \{0\} \xrightarrow[\text{retract}]{\text{def}} S^{n-1}$ $\quad H(\vec{x}, t) = (1-t)\vec{x} + t\dfrac{\vec{x}}{\|\vec{x}\|}$



S^{n-2} is homeo to S^{m-2}

$\Rightarrow n-2 = m-2$

$\Rightarrow n = m$

application ③ :- Brouwer's Fixed Point Thm. for D^n

proof :- $D^n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \leq 1 \}$

Let $f: D^n \to D^n$ be any cont. fn. Then f has at least one fixed point in D^n, x_0, i.e. $f(x_0) = x_0$.

Proof by the Method of Contradiction $\exists f: D^n \to D^n$ s.t. $f(x) \neq x \ \forall x \in D^n$.

$\partial(D^n) = S^{n-1}$

Suppose $f(x) \neq x \ \forall x$ join $f(x)$ to x by st. line and extend it to hit the boundary call that point $g(x)$

$i: S^{n-1} \to D_n$ (inclusion map)

$g: D_n \to S^{n-1}$

$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{g} S^{n-1}$$

$i \circ g(x) = x$ (identity)

$$\begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[r, "i'"] \arrow[rr, bend right] & H_{n-1}(D^n) \arrow[r, "g'"] & H_{n-1}(S^{n-1}) \end{tikzcd}$$

(homology groups)

$$\begin{tikzcd} (\mathbb{Z},+) \arrow[r] \arrow[rr, bend right] & \{0\} \arrow[r] & (\mathbb{Z},+) \end{tikzcd}$$

$i' \circ g' =$ identity

Contradiction

$\Rightarrow f(x) = x$ for some $x \in D$.

Let K^n be any simp. cplex.
K^n is said to have the fixed point property if every cont. fun. $f: K \to K$ has atleast one fixed point $f(x)=x$.

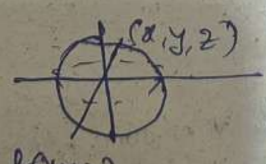
1) D^2, D^n has FPP

Q) Does S^2 have FPP?

A: S^1 does not have a fixed pt.

$f: S^2 \to S^2$

$(x,y,z) \to (-x,y,-z)$ does not have any fixed pt.



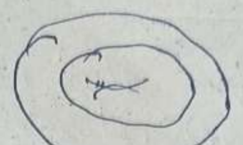
$f(x,y,z)$
$\|$
$(-x,y,-z)$

$f: S^n \to S^n$

$f((x_1, \dots, x_{n+1})) = (-x_1, \dots, -x_{n+1})$ does not have any fixed pt., S^n does not have FPP.

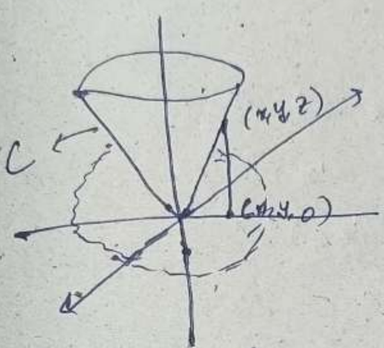
(2) $S^1 \times S^1$

No FPP



$(e^{i\theta_1}, e^{i\theta_2}) \to (e^{i(\theta_1+\alpha)}, e^{i\theta_2})$

(3) $x^2+y^2=z^2, \quad 0 \le z \le 1$



$f: C \to D^2$

$(x,y,z) \to (x,y,0)$

$\therefore$ One has a fixed point as it is homeomorphic to a disc

<u>Thm</u> – (Lipschitz fixed point Thm)

<u>Corollary</u> If K^n is a simp. complex s.t.

$H_0(K^n) = \mathbb{Z}$

$H_i(K^n) = 0$ then K^n has the fixed point property.

16/4/25 **Euler Poincare Thm**

Recall: For a surface Euler number $= v - e + f$

Defn: Let K^n be any simplicial complex of dim n.
(n means the simplex of highest dim in K is Δ_n).
Then we define the Euler Number $\chi(K^n)$.

$$\chi(K^n) = \sum_{i=0}^{n} (-1)^i \alpha_i$$

Let $\alpha_0 =$ # of 0-simplex (vert) in K^n

$\alpha_1 =$ " " 1 " (edge) " "

$\alpha_2 =$ " " 2 " (faces) " "

$\alpha_3 =$ " " 3 " "

⋮

$\alpha_n =$ # of n "

Defn: K^n any simplicial complex of dim n.
We have defined $n+1$ abelian hom grp of K^n.

$H_0(K^n), H_1(K^n), \ldots, H_n(K^n)$

Any abelian gp. $G_i \cong \underbrace{F_i}_{\text{free subgp.}} \oplus \underbrace{T_i}_{\text{Torsion subgroup}}$

$$H_i(K^n) \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{\beta_i} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$$

ith Betti Number

$H_i(K^n)$ $0 \le i \le n$, ith Betti number $\beta_i = \text{rank } F_i(K^n)$

(Euler Poincare Thm) K^n any simplicial complex of dim n. we have defined $n+1$ abelian hom gp. of K^n.

$H_i(K^n)$ $0 \le i \le n$; ith Betti number $\beta_i = \text{rank } F_i(K^n)$

= v - e + f
plex of dim n
in $(-1)^i \beta_i$).
K^n).

K^n

dim n.
grp of K^n

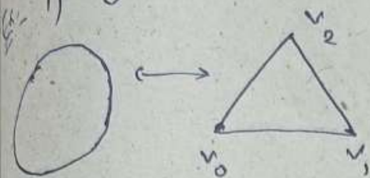
n subgroup
$\mathbb{Z}_5$

$\beta_r = \text{rank } H_r(K^n)$
m n. we have defined

rank $H_r(K^n)$

Thm $\chi(K^n) = \sum_{i=0}^{n} (-1)^i \beta_i$

Ex: 1) S^1



$\chi(S^1) = 3 - 3 = 0.$

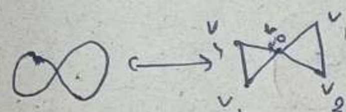
$H_0 \cong \mathbb{Z}$ $\quad \beta_0 = 1$

$H_1 \cong \mathbb{Z}$ $\quad \beta_1 = 1$

$\sum_{i=0} (-1)^i \beta_i = 1 - 1 = 0$

$\chi(S^1) = \sum_{i=0} (-1)^i \beta_i$

2)



$\chi(\infty) = 5 - 6 = -1$

$H_0 \cong \mathbb{Z}$

$H_1 \cong \mathbb{Z} \times \mathbb{Z}$

$\sum_{i=0} (-1)^i \beta_i = 1 - 2 = -1$

$\quad\quad \parallel$

$\quad\ \chi(\infty)$

3)

$\chi(S^r) = (2r+1) - 3r$

$\quad\quad = 1 - r$

$H_0 \cong \mathbb{Z}$ $\quad \beta_0 = 1$

$H_1 \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{r}$ $\quad \beta_1 = r$

$\sum_{i=0} (-1)^i \beta_i = 1 - r.$

2) Sphere S^2

$v-e+f=2$

$e=6$

$v=4$

$f=4$

$\chi(S^2)=2$

$H_0=\mathbb{Z}$, $\beta_0=1$

$H_1=0$ $\beta_1=0$

$H_2=\mathbb{Z}$ $\beta_2=1$

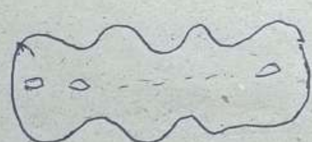
$\chi=1-0+1=2$.

3) Toxus T ; $T \neq T = T_2$

T_1 (i)

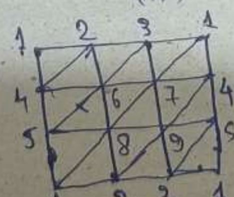
T_2 (ii)

T_g



(iii)

(i)



$V_0=9$

$V_1=27$

$V_2=18$

$V_0-V_1+V_2=9-27+18=0.$

$H_0 \cong \mathbb{Z} \to \beta_0=1$

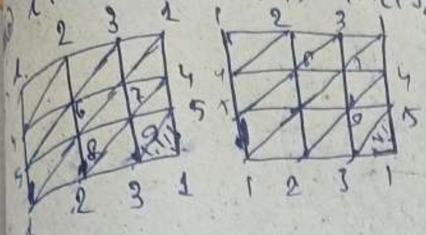
$H_1 \cong \mathbb{Z}\times\mathbb{Z} \to \beta_1=2$

$H_2 \cong \mathbb{Z} \to \beta_2=1$

$\sum_{i=0}^{2}(-1)^i\beta_i = 1-2+1=0.$

$\chi(T)=\sum_{i=0}(-1)^i\beta_i$

$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$



$\chi(T_1 \# T_2) = \chi(T_1) + \chi(T_2) - 2$

$= 0 + 0 - 2 = -2.$

$H_0 = \mathbb{Z}$; $\beta_0 = 1$

$H_1 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$; $\beta_1 = 4$

$H_2 = \mathbb{Z}$; $\beta_2 = 1$

$\Rightarrow \sum_{i=0}^{2} (-1)^i \beta_i = 1 - 4 + 1 = -2.$

(iii) $\chi(T_g) = 2 - 2g$

$H_0 \cong \mathbb{Z}$; $\beta_0 = 1$

$H_1 \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \cdots \times \mathbb{Z}}_{2g}$; $\beta_1 = 2g$

$H_2 \cong \mathbb{Z}$; $\beta_2 = 1$

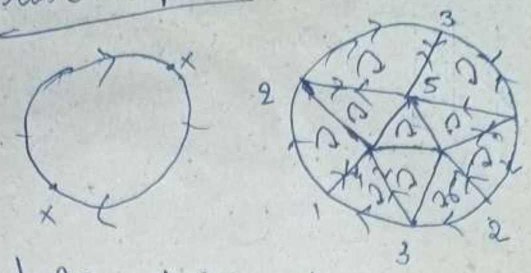
$\Rightarrow \sum_{i=0}^{2} (-1)^i \beta_i = 1 - 2g + 1 = 2 - 2g.$

(iv) $\chi(S^n)$, $H_0 \cong \mathbb{Z}$, $H_1 \cong 0, \dots, H_n \cong \mathbb{Z}$

So betti # $\beta_0 = 1$, $\beta_n = 1$, $\beta_i = 0$

$\chi(S^n) = \begin{cases} (-1)^n + 1 = 0, & n \text{ odd} \\ 2, & n \text{ even} \end{cases}$

Projective plane



$H_0 \cong \mathbb{Z}$, $H_2 \cong 0$, $H_1 \cong \mathbb{Z}_2$

$\partial\left(\qquad \right)$

$= 12 + 23 + 31 = 2(12+23+31)$

$+ 12 + 23 + 31$

$\partial(\underbrace{12+23+31}_{1\text{-cycle}}) = 0$

Thm:- classification of 2 dim compact surfaces without boundary is either

1) S^2 or 2) Torus or
3) Connected sum of g-tori g OR
4) Connected sum of n proj planes

1/4/25

Rational knots

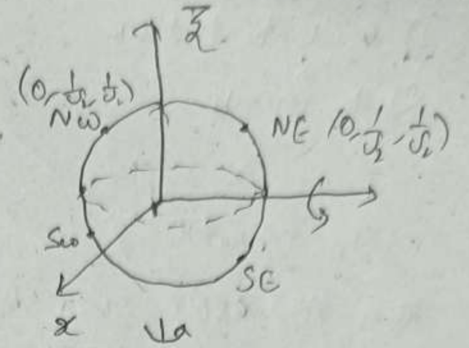
Let S^2 be $x^2+y^2+z^2=1$;

$NE = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$NW = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$SE = \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

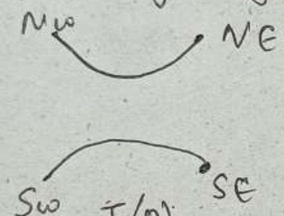
$SN = \left(0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$



Construction of a tangle $T(a_1, a_2, \dots, a_n)$ $a_i \in \mathbb{Z}_+$

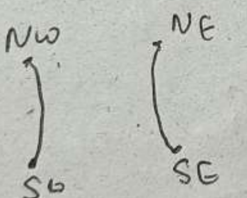
Defn (1). Elementary Tangles.

1)



T(0)

2)



T(0,0)

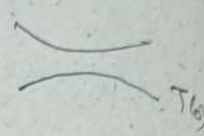
Defn (2):- Vertical Twist Let us rotate S^2 about the z axis keeping north hemisphere & south pole fixed.

SW & SE are exchanged

Horizontal Twist Rotate S^2 180° about y-axis, keeping west hemi fixed and (0,1,0) fixed, NE & SE get exchanged.

Case 1: n is odd Start with T(0)

Perform a_1 horizontal twists



1) a_2 vertical twists

2) a_3 horizontal ''

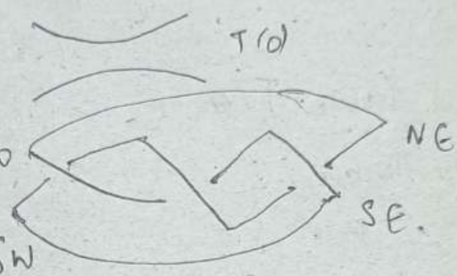
⋮

a_n horizontal ''

Join NW or NE, SW or SE to get a rational knot.

T(3)

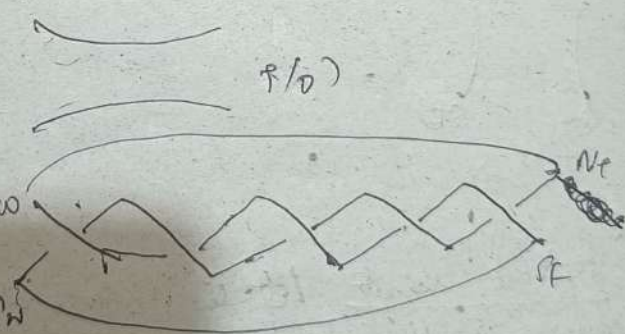
1)



Trefoil

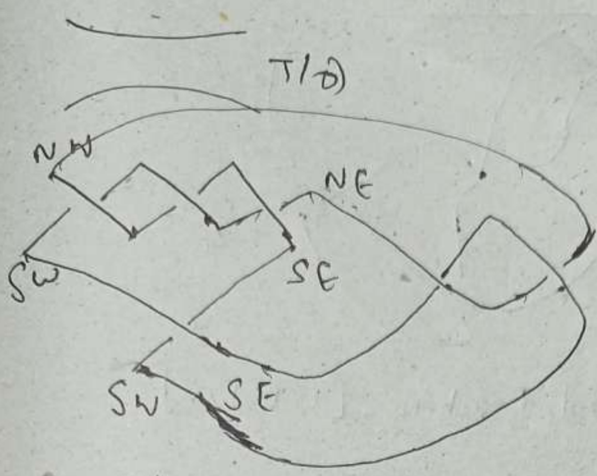
T(5) closure

2)



5'

Eg $T(3,1,2)$

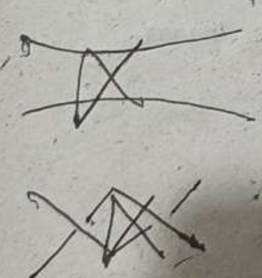


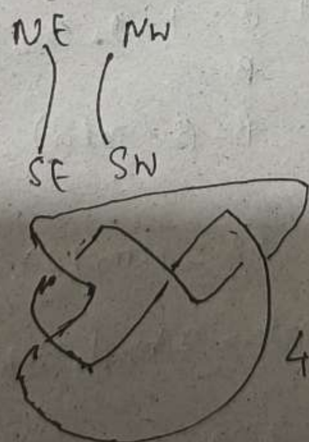
Case 2 n is even

Start with $T(0,0)$
Perform a_1 vertical twists
" a_2 horizontal "
" a_3 vertical "
⋮
" a_n horizontal "

Join NW to NE
SW to SE to get a knot
$T/(a_1 \ldots a_n)$ closure

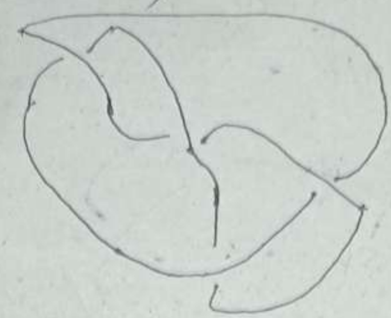
Ex:- $T(2,2)$





4_1

Ex:- $T(1,1,1,1)$



4_1

Continued Fractions $[1,1,1,1]$

$$\frac{\alpha}{\beta} = 1 + \cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}$$

$$[2,2] = 2 + \frac{1}{2} = \frac{5}{2}$$

Thm:- $T(a_1, \ldots, a_n)$ and $T(b_1, \ldots, b_m)$ corresponding to rational numbers $[a_n, a_{n-1}, \ldots, a_1]$ and $[b_m, b_{m-1}, \ldots, b_1] = \frac{\alpha'}{\beta'}$ $= \alpha/\beta$

A) $\frac{\alpha}{\beta}$ and $\frac{\alpha'}{\beta'}$ knots are the same

a) $\alpha = \alpha'$; $\beta = \beta' \pmod{\alpha}$

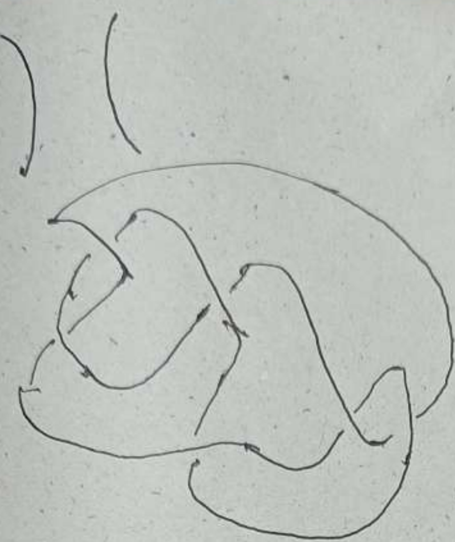
Or

b) $\alpha = \alpha'$; $\beta\beta' = 1 \pmod{\alpha}$

B) $\frac{a}{\beta}$ knot is achiral iff $\beta^2 \equiv -1 \bmod \alpha$

$T(a_1, a_2, \ldots, a_n, a_n, a_{n-1}, \ldots, a_1)$ will all be achiral.

$6_3 = 2\,1\,1\,2$



$$\frac{\alpha}{\beta} = 2 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}} = 2 + \cfrac{1}{1+\cfrac{2}{3}} = 2 + \frac{3}{5}$$

$$= \frac{13}{5}$$

$$\beta^2 = 25 = 12 \bmod 13$$

$$= -1 \bmod 13$$