HWO

EE16BTECH11023
HAVISH

1)

a) $X_n \xrightarrow{a.s.} X$ means that

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

b) $X_n \xrightarrow{p} X$ means that

$$\Pr\left(\lim_{n\to\infty} |X_n - \mu| > \varepsilon\right) = 0$$

c) $$\lim_{n\to\infty} E\left((X_n - X)^2\right) = 0$$

$$\Rightarrow X_n \xrightarrow{m.s.} X$$

d) $$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

$$\Rightarrow X_n \xrightarrow{d} X$$

3) MGF :-

$$E\left(e^{sX}\right) = \int_{-\infty}^{\infty} e^{sx} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{\frac{2\sigma^2 sx - x^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{\frac{-\sigma^4 s^2 + 2\sigma^2 s x - x^2 + \sigma^4 s^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\, dx.$$

$$= \left(e^{\frac{\sigma^2 s^2}{2}}\right) \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-\sigma^2 s)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

$$= 1$$

$$\Rightarrow \boxed{MGF(s) = e^{\sigma^2 s^2/2}}$$

---

4) a) <u>Cheybyshev's Inequality</u> :-

$$\Pr(|X-\mu| > \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$$

<u>Proof</u> :-

Say $Y = |X-\mu|^2 > 0$ is a R

$$\Pr(|X-\mu| > \varepsilon) = \Pr(|X-\mu|^2 > \varepsilon^2)$$

$$= \Pr(Y > \varepsilon^2) = \frac{E(Y)}{\varepsilon^2}$$

$$\Rightarrow \Pr(|x-\mu| > \varepsilon) \leq \frac{E((x-\mu)^2)}{\varepsilon^2}$$

$$\leq \frac{\sigma^2}{\varepsilon^2}$$

$$\boxed{\therefore \Pr(|x-\mu| > \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}}$$

4.) **Chernoff Bound** :-

suppose $t>0$ & ~~x>e~~ X be any R.V.

$$P(X \geq \varepsilon) = P(xt \geq \varepsilon t)$$

$$= P(e^{xt} \geq e^{\varepsilon t})$$

Now, say $Y = e^{xt}$

$$\Rightarrow P(Y \geq e^{\varepsilon t}) \leq \frac{E(Y)}{e^{\varepsilon t}}$$

$$\Rightarrow \boxed{P(e^{xt} \geq e^{\varepsilon t}) \leq \frac{E(e^{xt})}{e^{\varepsilon t}}}$$

6.)

a.) Suppose

$$S_{\cancel{n}} = \sum_{i=1}^{n} X_i \rightarrow \text{iid}.$$

$$E\left(\frac{S}{n}\right) = \cancel{n}\,\mu .$$

$$\text{var}\left(\frac{S}{n}\right) = \frac{\cancel{(n)}\,\sigma^2}{n^{\cancel{2}}}$$

$$\Pr\Big(\underbrace{\Big|\frac{S}{n} - \mu\Big| > \varepsilon}_{>0} \Big) \le \frac{\text{var}\left(\frac{S}{n} - \mu\right)}{\varepsilon^2}$$

$$\Rightarrow \boxed{\Pr\left(\left|\frac{S}{n} - \mu\right| > \varepsilon\right) \le \frac{\sigma^2}{n\varepsilon^2}}$$

(By Cheyshev Inequ

$$\Rightarrow \lim_{n\to\infty} \Pr\left(\left|\frac{S}{n} - \mu\right| > \varepsilon\right)$$

$$\le \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} \rightarrow 0$$

$$\therefore \lim_{n\to\infty} \Pr\left(\left|\frac{S}{n} - \mu\right| > \varepsilon\right) \rightarrow 0$$

$$\Rightarrow \sigma^2 \geq 2\log\left(\frac{e^{\lambda}+e^{-\lambda}}{2}\right)$$

$$\lambda^2$$

$$2\left(\frac{\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}}}{\cancel{2}}\right) \cdot \lambda^2 -$$

7)

b) $E(e^{\lambda x})$ is nothing but MGF of the gaussian $(0, \sigma^2)$

$$\therefore E(e^{\lambda x}) = e^{\frac{\lambda^2\sigma^2}{2}} \leq e^{\frac{\lambda^2\sigma^2}{2}}$$

$\therefore N(0, \sigma^2)$ is sub gaussian with variance same as original distribution a.)

$$E(e^{\lambda x}) = \frac{e^{-\lambda} + e^{\lambda}}{2}$$

$$e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots$$

$$e^{-\lambda} = 1 - \frac{\lambda}{1!} + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \ldots$$

$$\frac{e^{-\lambda} + e^{\lambda}}{2} = \left(\frac{1}{\cancel{2}}\right) \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \ldots$$

$$\boxed{e^{\lambda^2/2} = 1 + \frac{\lambda^2}{2} + \frac{\lambda^4}{2(2!)} + \ldots\ldots}$$

$(\because n(n!) \leq (n+1)!)$

$$\therefore E(e^{\lambda x}) \leq e^{\lambda^2/2} \Rightarrow \boxed{\sigma^2 = 1}$$

$(\sigma^2 = 1$

c.) By hoeffding's lemma,

$$E(e^{\lambda x}) \leq e^{\frac{\lambda^2 (b-a)^2}{8}}$$

$$\Rightarrow \frac{\sigma^2}{\cancel{2}} = \frac{(b-a)^2}{\cancel{8}\,4}$$

$$\Rightarrow \boxed{\sigma^2 = \frac{(b-a)^2}{4}}$$

9.)

$$E(e^{\lambda x_i}) \le e^{\frac{\sigma^2 \lambda^2}{2}}$$

$$E(e^{\lambda \Sigma x_i}) = E(e^{\lambda x_1} \cdot e^{\lambda x_2} \cdots e^{\lambda x_n})$$

$$= E(e^{\lambda x_1}) \cdot E(e^{\lambda x_2}) \cdots E(e^{\lambda x_n})$$

($\because$ independent)

$$\le e^{\frac{\Sigma \sigma^2 \lambda^2}{2}}$$

$\rightarrow$ Hence proved

Scanned by CamScanner

8) $P(|X-\mu| \geq t) = P(X-\mu \geq t) + P(X-\mu \leq -t)$

$$p(X-\mu \geq t) = P\left(e^{\lambda(X-\mu)} \geq e^{\lambda t}\right)$$

$$\leq \frac{E\left(e^{\lambda(X-\mu)}\right)}{e^{\lambda t}}$$

$X-\mu = Y$

$$E\left(e^{\lambda Y}\right) \leq e^{\frac{\sigma^2\lambda^2}{2}} \quad (\because \text{sub gaussian})$$

$$\Rightarrow \boxed{P(X-\mu \geq t) \leq e^{\frac{\sigma^2\lambda^2}{2} - \lambda t}}$$

To make the bound tighter, we would like to have the RHS as small as possible

$\Rightarrow$ minimize $\frac{\sigma^2\lambda^2}{2} - \lambda t$.

$\Rightarrow \lambda = \frac{t}{\sigma^2}$ is a minimum point

~~IIly, $P(X-\mu \leq -t) \leq$~~

$$\Rightarrow \boxed{P(|X-\mu| \geq t) \leq 2 \cdot e^{-\frac{t^2}{2\sigma^2}}}$$

can be proved IIly

$(\because P(X-\mu \leq -t) \leq e^{-t^2/2\sigma^2})$

~~9.~~

11.

a. $p_X(x) = \lambda e^{-\lambda x}$

$F(X) = \frac{1}{\lambda}$

$E\left(e^{s\left(X - \frac{1}{\lambda}\right)}\right)$

$= e^{-\frac{s}{\lambda}} \int_0^\infty e^{s(x)} \cdot \lambda e^{-\lambda x} \, dx.$

$\Rightarrow E\left(e^{s\left(X-\frac{1}{\lambda}\right)}\right) = \lambda e^{-\frac{s}{\lambda}} \int_0^\infty e^{(s-\lambda)x} \, dx.$

finite only if

$|s| < \lambda$

also $\lambda = \frac{1}{b}$ where b is the mean of exponential distribution.

$\Rightarrow E\left(e^{s\left(X-\frac{1}{\lambda}\right)}\right) = \lambda e^{-s/\lambda}\left(\frac{1}{\lambda - s}\right)$

$$\therefore E\left(e^{s\left(x-\frac{1}{\lambda}\right)}\right) = \frac{\lambda e^{-s/\lambda}}{\lambda - s}$$

$$\cancel{\lambda\left(e^{-s/\lambda}\right)\left(\frac{1}{\lambda}\right) - \lambda e^{-s/\lambda}(-1) = 0}$$

$$1+\left(\frac{s}{\lambda}\right)+\left(\frac{s}{\lambda}\right)^2+\left(\frac{s}{\lambda}\right)^4$$

$$= \frac{\lambda e^{-s/\lambda}}{\lambda - s}$$

$$= \frac{\lambda e^{-s/\lambda}}{\lambda}\left(1+\frac{s}{\lambda}+\left(\frac{s}{\lambda}\right)^2+\cdots\infty\right)$$

$$\Rightarrow E\left(e^{s\left(x-\frac{1}{\lambda}\right)}\right) = e^{-s/\lambda}\left(1+\frac{s}{\lambda}+\left(\frac{s}{\lambda}\right)^2+\cdots\infty\right)$$

$$\left(1-\frac{s}{\lambda}+\left(\frac{s}{\lambda}\right)^2\frac{1}{2!}\right)\left(1+\frac{s}{\lambda}-\left(\frac{s}{\lambda}\right)^2+\left(\frac{s}{\lambda}\right)^3\frac{1}{2!}+\left(\frac{s}{\lambda}\right)^2-\left(\frac{s}{\lambda}\right)^3+\left(\frac{s}{\lambda}\right)^4\frac{1}{2!}\right)$$

$$\cancel{1+s+s^2+\cdots}$$

$$\left(\frac{1}{1-\frac{s}{\lambda}}\right)e^{-s/\lambda} = \left(1+\frac{s}{\lambda}+\left(\frac{s}{\lambda}\right)^2+\cdots\right)\left(1-\frac{s}{\lambda}+\left(\frac{s}{\lambda}\right)^2\frac{1}{2!}-\cdots\right)$$

Scanned by CamScanner

$$E\left(e^{s(X-\frac{1}{\lambda})}\right) = \frac{e^{-s/\lambda}}{1-s/\lambda}$$

there is an inequality $\frac{e^{-s}}{\sqrt{1-2s}} \leq e^{2s^2}$ (verified graphically) $|s| < \frac{1}{4}$

$\Rightarrow \frac{e^{-2s}}{1-2s} \leq e^{4s^2} \Rightarrow \frac{e^{-t}}{1-t} \leq e^{t^2}$ $(|t| < \frac{1}{2}$

$\Rightarrow \frac{e^{-s/\lambda}}{1-s/\lambda} \leq e^{s^2/\lambda^2} = e^{\frac{\vartheta^2 s^2}{2}}$, $\boxed{|s| < \frac{1}{2}}$ $\boxed{\vartheta = \frac{\sqrt{2}}{\lambda}}$ $\therefore$ exponential is sub exp with $\left(\frac{\sqrt{2}}{\lambda}, 2\right)$

(2) $X - (\vartheta, b)$

$$\Rightarrow E\left(e^{\lambda(X-\mu)}\right) \leq e^{\frac{\vartheta^2\lambda^2}{2}} \text{ for}$$

$$|\lambda| \leq \frac{1}{b}$$

Say $X - \mu = Y$

$$\Rightarrow E\left(e^{\lambda Y}\right) \leq e^{\frac{\vartheta^2\lambda^2}{2}}$$

$$P(X-\mu \geq t) = P(Y \geq t)$$

$t > 0 \Rightarrow Y > 0$ for probability to ~~exist~~ be non-zero.

$$P(Y \geq t) \leq \frac{E\left(e^{\lambda Y}\right)}{e^{\lambda t}}$$

$\Rightarrow P(Y \geq t) \leq \left(e^{\frac{\lambda^2 \nu^2}{2} - \lambda t}\right)$

Taylor $\Rightarrow \lambda = \frac{t}{\nu^2}$

$\Rightarrow \boxed{P(Y \geq t) \leq e^{-\frac{t^2}{2\nu^2}}}$

~~wh~~ where $\frac{|t|}{\nu^2} < \frac{1}{b}$

$\Rightarrow \boxed{|t| < \frac{\nu^2}{b}}$

13.) X_i - sub-exponential

~~(for each~~

$\Rightarrow E\left(e^{\lambda(X_i - \mu)}\right) \leq e^{\frac{\nu^2 \lambda^2}{2}}$ for

$|\lambda| < \frac{1}{b}$

$E\left(e^{\lambda(\Sigma X_i - n\mu)}\right)$

$= E\left(\prod_{i=1}^{n} e^{\lambda(X_i - \mu)}\right)$

$= \prod_{i=1}^{n} E\left(e^{\lambda(X_i - \mu)}\right)$ ($\because$ Independent)

Now, for the bound to be valid for all random variables, $|\lambda| < \frac{1}{b_i}$, ($b_i$ is the constraint on X_i) should be always true for any i

$$\Rightarrow |\lambda| < \frac{1}{b^*}, \quad \boxed{b^* = \max_i b_i}$$

($\because$ It is the only b_i for which there will be intersection with all b_i's)

Applying the bound

$$\Rightarrow \prod_{i=1}^{n} e^{\frac{\vartheta_i^2 \lambda^2}{2}} = e^{\frac{\lambda^2(\Sigma \vartheta_i^2)}{2}}$$

$$\leq e^{\frac{\lambda^2((\Sigma \vartheta_i)^2)}{2}} \quad (\because$$

$$\Rightarrow E\left(e^{\lambda(\Sigma X_i - n\mu)}\right) \leq e^{\frac{\lambda^2((\Sigma \vartheta_i)^2)}{2}}$$

$\therefore \Sigma X_i$ sub-exponential with $(\Sigma \vartheta_i, b^*)$ as parameters

10.)

$Y, \quad E(Y) = 0$

$\mathrm{var}(Y) = \dfrac{\sigma^2}{n}$

$$P\left(\underbrace{\frac{1}{n}\left(\sum_{i=1}^{n} X_i - E(X_i) \right)}_{= Y} \geq t \right)$$

$$= P\left(e^{\lambda Y} \geq e^{\lambda t} \right)$$

$$\leq e^{-\lambda t} E\left(e^{\lambda Y}\right)$$

$$\cancel{E\left(e^{\lambda Y}\right) \leq e^{\frac{\lambda^2 (b-a)^2}{8}}}$$

$$a_i \leq X_i \leq b_i$$

$$\Rightarrow \quad \Sigma$$

$$E\left(e^{\lambda Y}\right) = E\left(e^{\frac{\lambda}{n}\left(\Sigma(X_i - E(X_i))\right)} \right)$$

$$= \prod_{i=1}^{n} E\left(e^{\frac{\lambda}{n}\left(X_i - E(X_i)\right)} \right) \quad (\because \text{independ}$$

$$E\left(e^{\lambda\frac{(x_i - E(x_i))}{n}}\right)$$

$$\frac{a_i - E(x_i)}{n} \le \frac{x_i - E(x_i)}{n} \le \frac{b_i - E(x_i)}{n}$$

$$\frac{x_i - E(x_i)}{n} = y_i$$

$$\Rightarrow E(e^{\lambda y_i}) \le e^{\frac{\lambda^2\left(\frac{b_i - E(x_i)}{n} - \frac{a_i - E(x_i)}{n}\right)^2}{8}}$$

$$\Rightarrow E(e^{\lambda y_i}) \le e^{\frac{\lambda^2 (b_i - a_i)^2}{8n^2}}$$

(By Hoeffding's Lemma.)

$$E(e^{\lambda y}) \le \prod_{i=1}^{n} e^{\frac{\lambda^2 (b_i - a_i)^2}{8n^2}}$$

$$\Rightarrow P(e^{\lambda y} \ge e^{\lambda t}) \le e^{\frac{\lambda^2 (\Sigma b_i - a_i)^2}{8n^2} - \lambda t}$$

To tighten the bound;

$$\frac{\cancel{2}\lambda}{{}_{4}\cancel{8}n^2}\left(\Sigma (b_i - a_i)^2\right) = t$$

$$\Rightarrow \lambda = \frac{4n^2 t}{\left(\Sigma b_i - a_i\right)^2}$$

$$\therefore P\left(\frac{\sum x_i - E(x_i)}{n} \geq t\right)$$

$$\leq e^{-\frac{2n^2t^2}{\sum(b_i-a_i)^2}}$$

For the other part, ~~$\left(P\left(\frac{\sum E(x_i) - x_i}{n}\right) \geq t\right)$~~

Let $Y_i = \dfrac{E(x_i) - x_i}{n}$

$$\Rightarrow \frac{E(x_i)-b_i}{n} \leq Y_i \leq \frac{E(x_i)-a_i}{n}$$

$\Rightarrow$ By Hoeffdings Lemma,

$$E\left(e^{\lambda Y_i}\right) \leq e^{\frac{\lambda^2(b_i-a_i)^2}{8n^2}}$$

$$\Rightarrow E\left(e^{\lambda Y}\right) = \prod_{i=1}^{n} E\left(e^{\frac{\lambda}{n}\underbrace{\left(E(x_i)-x_i\right)}_{Y_i}}\right)$$

$$\leq \prod_{i=1}^{n} e^{\frac{\lambda^2(b_i-a_i)^2}{8n^2}}$$

$$\Rightarrow E\left(e^{\lambda Y}\right) \leq e^{\frac{\lambda^2\left(\sum (b_i-a_i)^2\right)}{8n^2}}$$

$$\Rightarrow P\left(\frac{\sum X_i - E(X_i))}{n} \leq -t\right)$$

$$= P\left(\frac{\sum (E(X_i) - X_i)}{n} \geq t\right)$$

$$= P\left(\sum Y_i \geq t\right)$$

$$= P\left(e^{\lambda \sum Y_i} \geq e^{\lambda t}\right)$$

$$\leq e^{-\lambda t} \cdot e^{\frac{\lambda^2\left(\sum (b_i-a_i)^2\right)}{8n^2}}$$

$$\leq e^{\frac{\lambda^2\left(\sum (b_i-a_i)^2\right)}{8n^2} - \lambda t}$$

For tightening,

$$\boxed{P\left(\frac{\sum (X_i - E(X_i))}{n} \leq -t\right) \leq e^{-\frac{2n^2t^2}{\sum (b_i-a_i)^2}}}$$

14.

$$E\left(e^{\lambda(X-\mu)}\right) = E\left(1+\frac{\lambda(X-\mu)}{1!}+\frac{\lambda^2(X-\mu)^2}{2!} + \cdots \infty\right)$$

$$= 1+\lambda E(X-\mu) + \frac{\lambda^2 E((X-\mu)^2)}{2!} + \cdots \infty$$

$$\le 1 + \cancel{|\lambda||E(X-\mu)|} + \frac{\lambda^2\left(|E((X-\mu)^2|\right)}{2!} + \cdots \infty$$

$(\because E(X-\mu)=0)$

$$\le 1 + \cancel{(|\lambda|)\frac{1}{2}\cdot\frac{\sigma^2}{b}} + |\lambda|^2\frac{2!}{2\cdot 2!}\sigma^2$$

$$+ |\lambda|^3\frac{3!}{2\cdot 3!}\sigma^2 b + \cdots$$

$$\leq 1 + \frac{\sigma^2}{2} \sum_{k=2}^{\infty} |\lambda|^k \cdot b^{k-2}$$

$r = |\lambda| b$

< 1

$$\leq 1 + \left(\frac{\sigma^2}{2}\right) \frac{\lambda^2}{1-|\lambda| b}$$

$|\lambda| < \frac{1}{b}$

$$\leq e^{\frac{\lambda^2\sigma^2}{2(1-|\lambda| b)}}$$

$\frac{1}{b} > |\lambda|$

$$\boxed{E\left(e^{\lambda(x-\mu)}\right) \leq e^{\frac{\lambda^2\sigma^2}{2(1-|\lambda| b)}}}$$

$\cancel{\lambda^2(x)}$

Scanned by CamScanner

2)

a) $\cos(n\theta)$., $\theta \sim U(0, 2\pi)$

Clearly $\cos(n\theta)$ is periodic function

So, even ~~at~~ as $\lim_{n\to\infty}$, we won

have a fixed real number

$\therefore$ $\cos(n\theta)$ doesen't conver

---

b) $\theta \in [0, 2\pi]$

$\Rightarrow -1 \leq 1 - \frac{\theta}{\pi} \leq 1$

$\Rightarrow \left|1 - \frac{\theta}{\pi}\right| \leq 1$

$\Rightarrow \cancel{\lim_{n\to\infty}} \left|1 - \frac{\theta}{\pi}\right|^n \to 0$ as $n \to \infty$

$\left(\because \left(1 - \frac{\theta}{\pi}\right) \leq 1\right)$

$\cancel{\theta}$

So,

$$\underset{n\to\infty}{\text{lt}}\, Pr\left(\left| \left|1-\frac{\theta}{\pi}\right|^n - 0 \right| > \varepsilon \right)$$

$$= \underset{n\to\infty}{\text{lt}}\, Pr\left(\left|\left|1-\frac{\theta}{\pi}\right|\right|^n > \varepsilon \right)$$

$$= \underset{n\to\infty}{\text{lt}}\, Pr\left(\left|1-\frac{\theta}{\pi}\right| > \sqrt[n]{\varepsilon} \right)$$

$$= Pr\left(\left|1-\frac{\theta}{\pi}\right| > 1 \right)$$

$$= 0$$

$$\left(\because \left|1-\frac{\theta}{\pi}\right| \in [0,1] \right)$$

c.)

$Pr(\bar{x}-\mu > 250)$

$\bar{x}-\mu \quad = \frac{1}{2}\left(Pr\left(|x-\mu| > 250\right)\right)$

($\because$ $X \sim U$ → symmetry)

$= \left(\frac{1}{2}\right)\left(\frac{(16875)}{(250)^2}\right)$ (Cheybyshev)

≤ 0.135

$\therefore$ $Pr(x-\mu > 250) \leq 0.135$

CLT :-

$Pr(X - \mu > 250)$

$X \simeq \Sigma x_i$ → each x_i is uniformly distribut-ed

$\Rightarrow$ $X \sim N(750, 16875)$

(By CLT)

$= Pr\left(\frac{x-\mu}{\sqrt{16875}} > \frac{250}{\sqrt{16875}}\right)$

$= Q\left(\frac{250}{\sqrt{16875}}\right) = 0.027$

~~Ans~~

a) $X - \{0, 1\}$

$Pr(X=1) = \frac{1}{k}$

$\Rightarrow E(X) = \frac{1}{k}$

$Pr(X \geq kE(X)) \leq \frac{\cancel{E(X)}}{k\cancel{E(X)}}$

$\Rightarrow Pr(X \geq kE(X)) \leq \frac{1}{k}$

$\Rightarrow Pr(X \geq 1) \leq \frac{1}{k}$

$Pr(X == 1) = \frac{1}{k}$

$\therefore$ Tighter.

b) Say $X \sim N(0,1)$

$Pr(|X| > \varepsilon) \leq \frac{1}{\varepsilon^2}$ (Cheybyshev)

also, $Pr(e^{Xt} > e^{\varepsilon t}) \leq e^{\frac{\sigma^2 t^2 - 2\varepsilon t}{2}}$ (Chernoff

$\hookrightarrow Pr(X > \varepsilon)$

Chernoff $\rightarrow$ $\Pr(X > \varepsilon) \leq e^{\frac{t^2 - 2\varepsilon t}{2}}$

Chebyshev $\rightarrow$ $\Pr(|X| > \varepsilon) \leq \frac{1}{\varepsilon^2}$

~~Comparison~~ Checking the following :-

$$e^{\frac{t^2 - 2\varepsilon t}{2}} \leq \frac{1}{\varepsilon^2}$$

$$\Rightarrow e^{t^2/2} \leq \frac{e^{\varepsilon t}}{2\varepsilon^2}$$

$\sqrt{e}$, which is true.

$N(0,1)$ is a case where Chernoff is tighter than Chebyshev.

---

say mean of x^2 is k.

1) b.) $$E\left(e^{t(X-k)}\right) = \frac{1}{2^{k/2}\Gamma(k/2)} \int_0^\infty e^{tx} e^{-tk} x^{(k-2)/2} e^{-k/2}\, dx$$

$$\cancel{\Rightarrow E\left(e^{t(X-k)}\right) = \frac{e^{-tk}}{\sqrt[k/2]{1-2t}}}$$

$$\Rightarrow E\left(e^{t(X-k)}\right) = \frac{e^{-tk}}{(1-2t)^{k/2}}$$

$$= \left(\frac{e^{-t}}{\sqrt{1-2t}}\right)^k \quad \left(|t| < \frac{1}{2}\right)$$

$$\leq \left(e^{2t^2}\right)^k \quad \left(|t| < \frac{1}{4}\right)$$

$$\Rightarrow E\left(e^{t(X-k)}\right) \leq e^{\frac{4kt^2}{2}}$$

$$\leq e^{\frac{\nu^2 t^2}{2}}, \quad \nu^2 = 4k$$

$$\& \ |t| < \frac{1}{4}.$$

$\therefore$ χ^2 is sub-exponential with $(\sqrt{4k}, 4)$ as (ν, b)