ig Aaysuh

Exercise 1.1

Find the domain and range of the functions:

1. $y = \frac{1}{x}$

Solution: Given that,

$$y = \frac{1}{x}.$$

Here, the given function y is defined for all the values of x except zero. So, the domain of the function is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$.

And, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

Clearly, x is defined for all the values of y except zero. So, the range of the function is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$.

2. $y = \sqrt{x}$

Solution: Here, the given function y is defined only for $x \geq 0$.

Therefore domain of this function is $[0, \infty)$.

For the value of x, the function y takes any value except negative value. So, the range of the given function is $[0, \infty)$.

3. $y = \sqrt{4 - x}$ **[2009, Spring][2008, Fall][2007, Spring]**

Solution: Here, the given function y is defined only when $(4 - x) \geq 0 \Rightarrow x \leq 4$. Therefore, the domain of the given function is $(-\infty, 4]$.

For all values for x in the domain y takes any value except negative value.

So, the range of that function is $[0, \infty)$.

4. y = sinx

Solution : Here, the given function y is defined for any value of x so domain of this function is $(-\infty, \infty)$.

For the values of x in domain, y takes any value but the sine of function is bounded function and has oscillating value in between -1 to 1.

That means $|\sin x| \leq 1$.

So, range of this function is $[-1, 1]$.

5. $y = \sqrt{x + 4}$ **[2011 Spring, Short]**

Solution: Here, the given function y is defined for all value of x for $(4 + x) \geq 0 \Rightarrow x \geq -4$. So, domain of the function is $[-4, \infty)$.

At $x = -4$, we get $y = 0$ and at $x = \infty$, we get $y = \infty$.

Thus, y takes all non–negative values.

Therefore, range of the given function is $[0, \infty)$.

6. $y = \frac{1}{x - 2}$ **[2004, Spring]**

Solution: Given that,

$$y = \frac{1}{x - 2}.$$

Here, the given function y is defined for each value of x except at $x = 2$. Therefore, the domain of this function is $R - \{2\}$.

And,

$$y = \frac{1}{x - 2} \Rightarrow x = \frac{1}{y} + 2.$$

This shows x is defined for all values of y except at $y = 0$.

So, range of the given function is $R - \{0\}$.

7. y = x sinx

Solution: Here, the given function y is defined for all the values of x. So, the domain of this function is $(-\infty, \infty)$.

For the values of x, we have y takes any value. So, range of the given function is $(-\infty, \infty)$.

8. $y = (\sqrt{x})^2$

Solution: Here, the given function y is defined for $x \geq 0$. So, domain of the given function is $[0, \infty)$.

Since y has square form of x. So, it takes each non–negative values. So, range of the given function is $[0, \infty)$.

9. y = 2cosx

Solution : Here, the given function y is defined for all values of x.

So, domain $= (-\infty, \infty)$.

The cosine function is bounded function having osculating value in $[-1, 1]$.

So, range $= 2[-1, 1] = [-2, 2]$.

10. y = –3sin x. **[2006, Fall]**

Solution : Here, the given function y is defined for all x.

So, domain $= (-\infty, \infty)$.

The sine function is bounded having osculating value in $[-1, 1]$.

So, range $= 3[-1, 1] = [-3, 3]$.

11. $y = x^2 + 1$

Solution : Here, the given function y is defined for all x.

So, domain $= (-\infty, \infty)$.

Since, for $x \in (-\infty, \infty)^2$, x^2 has value in $[0, \infty)$. Thus, value of y is lie in $[0, \infty) + 1 = [1, \infty)$ for any value of x in $(-\infty, \infty)$.

So, range $= [1, \infty)$.

12. $y = -x^2$

Solution: Here, the given function y is defined for all the values of x.

So, domain $= (-\infty, \infty)$.

Since, $(-\infty, \infty)^2$ has value in $[0, \infty)$. So, $y = -[0, \infty) = (-\infty, 0]$ for any value of x in $(-\infty, \infty)$.

So, range $= (-\infty, 0]$.

13. $y = \sqrt{x+1}$

Solution: Since, the function $\sqrt{(.)}$ defines only for non–negative value. So, y is defined for all x with $x \geq -1$.

So, domain $= [-1, \infty)$.

And, $y = 0$ at $x = -1$. Also, $y \to \infty$ as $x \to \infty$.

So, range of $y = [0, \infty)$.

14. $y = 1 + \sqrt{x}$

Solution: Since, the function $\sqrt{(.)}$ defines only for non–negative value. So, the given function y is defined for all the values of x with $x \geq 0$. So, domain of the given function is $[0, \infty)$.

For all the values of $x \geq 0$, we get the value of $\sqrt{x}$ is in $[0, \infty)$. So,

$\therefore$ Range of $y = [0, \infty) + 1 = [1, \infty)$.

15. $y = (\sqrt{2x})^2$

Solution : Here, the given function y is defined for all the values of x only for non–negative values.

So, domain of $y = [0, \infty)$.

For the values of x, the function y takes all non–negative value.

So, range $= [0, \infty)$.

16. $y = -\frac{1}{x}$

Solution: Here,

$$y = -\frac{1}{x}$$

Clearly, y is defined for all the values of x except at $x = 0$.

So, domain $= R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$.

And,

$$y = -\frac{1}{x} \Rightarrow x = -\frac{1}{y}$$

Clearly, x is defined for all the values of y except at $y = 0$.

So, range $= R - \{0\}$.

17. $y = \sin 2x$

Solution : Since, the sine function is defined for all x with x is in $(-\infty, \infty)$. So, sin 2x is defined for all x with x is in $(-\infty, \infty)$.

Therefore, domain $= (-\infty, \infty)$.

We know the sine function has osculating value in $[-1, 1]$. So, sin 2x has osculating value in $[-1, 1]$.

So, range $= [-1, 1]$.

18. $y = \sin^2 x$

Solution: Since, the sine function is defined for all x with x is in $(-\infty, \infty)$.

So, $\sin^2 x$ is defined for all x with x is in $(-\infty, \infty)$.

Therefore, domain of $y = (-\infty, \infty)$.

We know the sine function has osculating value in $[-1, 1]$. So, $\sin^2 x$ has osculating value in $[0, 1]$.

So, Range of $y = [0, 1]$.

19. $y = 1 + \sin x$

Solution : Since, the sine function is defined for all x with x is in $(-\infty, \infty)$.

So, $y = 1 + \sin x$ is defined for $1 + (-\infty, \infty)$ i.e. $(-\infty, \infty)$.

Therefore, domain of $y = (-\infty, \infty)$.

We know the sine function has osculating value in $[-1, 1]$. So, $(1 + \sin x)$ has osculating value in $1 + [-1, 1]$ i.e. $[0, 2]$.

So, range of $y = [0, 2]$.

Exercise 1.2

1. Let $f(x) = x^4 - 5$ and $g(x) = \cos x$. Find (fog) (x) and (gof) (x).

Solution: We have, $f(x) = x^4 - 5$, $g(x) = \cos x$.

Then, $(fog)(x) = f[g(x)] = f(\cos x) = \cos^4 x - 5$

$(gof)(x) = g[f(x)] = g(x^4 - 5) = \cos(x^4 - 5)$.

2. Let $f(x) = x + 1$ and $g(x) = x^2$. Find (fog) (x) and (gof) (x).

Solution: We have, $f(x) = x + 1$ and $g(x) = x^2$

Then, $(fog)(x) = f[g(x)] = f(x^2) = x^2 + 1$

$(gof)(x) = g[f(x)] = g(x + 1) = (x + 1)^2$.

3. Let $f(x) = \tan x$, $g(x) = x^7$. Find (fog) (x) and (gof) (x).

Solution: We have, $f(x) = \tan x$ and $g(x) = x^7$

Then, $(fog)(x) = f[g(x)] = f(x^7) = \tan(x^7)$

$(gof)(x) = g[f(x)] = g(\tan x) = (\tan x)^7$.

Q. Let two functions $f:R \to R$ and $g:R \to R$ defined as $f(x) = x + 2$ and $g(x) = 3x^2$ for $x \in R$. Find fog(x) and gof(x). **[2015 Spring short]**

Exercise 1.3

Find the limit of the following function at specified points (if exists):

1. $f(x) = \begin{cases} 3x^2 - 1 & \text{when } x \leq 2 \\ 4x + 3 & \text{when } x > 2 \end{cases}$ **at x = 2.**

Solution: At x = 2,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} (4x+3) = 11$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} (3x^2 - 1) = 11$$

Thus, L.H.L. = R.H.L. = 11. So the limit of f(x) exits at x = 2 and its limiting value is 11.

2. $f(x) = \begin{cases} 2x + 1 & \text{when } x \geq 1 \\ 4x^2 - 1 & \text{when } x < 1 \end{cases}$ **at x = 1.**

Solution: At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2x+1) = 3$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (4x^2 - 1) = 3$$

Thus, L.H.L. = R.H.L. = 3. So the limit of f(x) exits at x = 1 and its limiting value is 3.

3. $f(x) = \begin{cases} 3x + 2 & \text{when } x \geq 2 \\ 2x^2 + 1 & \text{when } x < 2 \end{cases}$ **at x = 2.**

Solution: At x = 2,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} (3x+2) = 8$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} (2x^2 + 1) = 9$$

Thus, L.H.L. ≠ R.H.L. So the limit of f(x) does exits at x = 2.

4. **A function f(x) defined by** $f(x) = \begin{cases} x^2 & \text{when } x < 1 \\ 2.5 & \text{when } x = 1 \\ x^2 + 2 & \text{when } x > 1 \end{cases}$

Does $\lim_{x\to1} f(x)$ **exist?**

Solution: At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^2 + 2) = 3$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2 = 1.$$

Thus, L.H.L. ≠ R.H., so the limit of f(x) does exits at x = 1.

5. **A function f(x) is defined as follows:** $f(x) = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -x & \text{when } x < 0 \end{cases}$. **Find the value of** $\lim_{x\to0} f(x)$.

Solution: At x = 0, R.H.L. = $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x = 0.$

$$\text{L.H.L.} = \lim_{x\to0^-} f(x) = -x = -0 = 0.$$

Thus, L.H.L. = R.H.L. = 0. So the limit of f(x) exits at x = 0 and its limiting value is 0. Therefore, $\lim_{x\to0} f(x) = 0.$

Exercise 1.4

1. **At what point is the function**

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases} \quad \textbf{continuous?}$$

Solution: Here, the function f(x) is defined on (0, 1] At x = 0,

$$\text{R.H.L.} = \lim_{x\to0^+} f(x) = 1 \quad \text{and} \quad \text{L.H.L.} = \lim_{x\to0^-} f(x) = 0.$$

Here, R.H.L. ≠ L.H.L. is not continuous at x = 0

At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = 1 \quad \text{and} \quad \text{L.H.L.} = \lim_{x\to1^-} f(x) = 1.$$

Also, f(1) = 1

Here, L.H.L. = R.H.L. = f(1), so the given function f(x) is continuous only at x = 1.

2. **Let** $f(x) = \begin{cases} \left(\dfrac{x^2 - 1}{x - 1}\right) & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$. **Is f(x) continuous or not continuous at x = 1? Explain.**

Solution: Given function is

$$f(x) = \begin{cases} \left(\dfrac{x^2 - 1}{x - 1}\right) & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases}$$

And, at x = 1,

f(1) = 2,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to1} f(x) = \lim_{x\to1} (x+1) = 2.$$

Thus, L.H.L. = R.H.L. = f(1) = 2. So, the given function f(x) is continuous at x = 1. This means f(x) is continuous everywhere on R.

3. Define f(3) so that $f(x) = \frac{x^2-9}{x-3}$ is continuous at x = 3.

Solution: At x = 3,

$$\text{R.H.L.} = \lim_{x\to3^+}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to3^+}\left(\frac{(x-3).(x+3)}{x-3}\right) = \lim_{x\to3^+}(x+3) = 6.$$

$$\text{L.H.L.} = \lim_{x\to3^-}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to3^-}\left(\frac{(x-3).(x+3)}{x-3}\right) = \lim_{x\to3^-}(x+3) = 6.$$

In order that f(x) is continuous at x = 3, we must have,

f(3) = L.H.L. = R.H.L. = 6.

4. Let $f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 2-x & \text{for } 1 < x < 2 \end{cases}$. Is f(x) continuous at x = 1?

Solution: Given function is

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 2-x & \text{for } 1 < x < 2 \end{cases}$$

And, at x = 1,

$$f(1) = 1,$$

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+}(2-x) = 1,$$

and, $$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x) = 1.$$

Therefore, L.H.L. = R.H.L. = f(1).

Thus, L.H.L. = R.H.L. = f(1) = 1. So, the given function f(x) is continuous at x = 1. This means f(x) is continuous everywhere on [0, 2).

5. What value should be assigned to 'a' to make the function.

(i) $f(x) = \begin{cases} x^2-1 & \text{for } x < 3 \\ 2ax & \text{for } x \ge 3 \end{cases}$ **is continuous at x = 3.**

Solution: Given function is

$$f(x) = \begin{cases} x^2-1 & \text{for } x < 3 \\ 2ax & \text{for } x \ge 3 \end{cases}$$

And, at x = 3,

$$f(3) = 2a(3) = 6a,$$

$$\text{R.H.L.} = \lim_{x\to3^+} f(x) = \lim_{x\to3^+} 2ax = 6a$$

and, $$\text{L.H.L.} = \lim_{x\to3^-} f(x) = \lim_{x\to3^-}(x^2-1) = 8.$$

By hypothesis, the given function f(x) is continuous at x = 3.

So, L.H.L. = R.H.L. = f(3).

$$\Rightarrow \quad 6a = 8 \quad \Rightarrow a = \frac{8}{6} = \frac{4}{3}.$$

Thus, for $a = \frac{4}{3}$, the given function f(x) is continuous at x = 3.

(ii) $f(x) = \begin{cases} x^3 & \text{for } x < 1/2 \\ ax^2 & \text{for } x \ge 1/2 \end{cases}$ **is continuous at $x = \frac{1}{2}$.**

Solution: Given function is

$$f(x) = \begin{cases} x^3 & \text{for } x < 1/2 \\ ax^2 & \text{for } x \ge 1/2 \end{cases}$$

And, at x = (1/2),

$$f(1/2) = a\left(\frac{1}{2}\right)^2 = \frac{a}{4}.$$

$$\text{R.H.L.} = \lim_{x\to(1/2)^+} f(x) = \lim_{x\to(1/2)^+}(ax^2) = \frac{a}{4}$$

$$\text{L.H.L.} = \lim_{x\to(1/2)^-} f(x) = \lim_{x\to(1/2)^-}(x^3) = \frac{1}{8}$$

By hypothesis, the given function f(x) is continuous at x = 3.

So, L.H.L. = R.H.L. = f(1/2).

$$\Rightarrow \frac{a}{4} = \frac{1}{8} \quad \Rightarrow a = \frac{1}{2}.$$

Thus, for $a = \frac{1}{2}$, the given function f(x) is continuous at x = 3.

6. A function f(x) is defined as follows:

$$f(x) = \begin{cases} (1/2) - x & \text{when } 0 < x < (1/2) \\ 1/2 & \text{when } x = (1/2) \\ (3/2) - 2 & \text{when } (1/2) < x < 1 \end{cases}$$

Show that f(x) is discontinuous at $x = \frac{1}{2}$.

Solution: And, at x = (1/2),

$$\text{R.H.L.} = \lim_{x\to(1/2)^+} f(x) = \lim_{x\to(1/2)^+}\left(\frac{3}{2}-x\right) = \left(\frac{3}{2}-\frac{1}{2}\right) = 1$$

$$\text{L.H.L.} = \lim_{x\to(1/2)^-} f(x) = \lim_{x\to(1/2)^-}\left(\frac{1}{2}-x\right) = 0$$

Here, R.H.L. ≠ L.H.L. So, the limit of f(x) at $x = \frac{1}{2}$, does not exist. So, the given function f(x) is not continuous at $x = \frac{1}{2}$.

7. A function f(x) is defined in (0, 3) such that

$$f(x) = \begin{cases} x^2 & \text{for } 0 < x < 1 \\ x & \text{for } 1 \le x < 2 \\ \dfrac{x^3}{4} & \text{for } 2 \le x < 3 \end{cases}$$

Show that f(x) is continuous at x = 1 and x = 2.

Solution: And, at x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} x = 1$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^2 = 1$$

and, $f(1) = 1$

This shows L.H.L. = R.H.L. = f(1). This means f(x) is continuous at x = 1. Therefore, the given function f(x) is continuous at x = 1.

Also, at x = 2,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} \left(\frac{x^3}{4}\right) = \frac{8}{4} = 2$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} x = 2$$

and, $f(2) = 2$

This shows L.H.L. = R.H.L. = f(2). This means f(x) is continuous at x = 2. Therefore, the given function f(x) is continuous at x = 2.

Thus, f(x) is continuous on (0, 3).

8. A function f(x) is defined by $f(x) = \begin{cases} x^2 & \textbf{when } x \ne 1 \\ 2 & \textbf{when } x = 1 \end{cases}$

Is f(x) continuous at x = 1?

Solution: Here f(x) is in piecewise form. And, clearly f(x) is continuous elsewhere at x = 1.

And, at x = 1,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to1} f(x) = \lim_{x\to1} x^2 = 1$$

and, $f(1) = 2.$

Here, L.H.L. = R.H.L. ≠ f(1). So the given function is not continuous at x = 1.

9. A function f(x) is defined as follows: $f(x) = \begin{cases} x^2 & \textbf{when } x < 1 \\ 2.5 & \textbf{when } x = 1 \\ x^2 + 2 & \textbf{when } x > 1 \end{cases}$

Is f(x) continuous at x = 1?

Solution: At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^2 + 2) = 3$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^2) = 1$$

Here, L.H.L. ≠ R.H.L. So, the given function f(x) is not continuous at x = 1.

10. A function f(x) is defined by $f(x) = \begin{cases} -x & \textbf{when } x \le 0 \\ x & \textbf{when } 0 < x < 1 \\ 2 - x & \textbf{when } x \ge 1 \end{cases}$.

Show that it is continuous at x = 0 and x = 1.

Solution: At x = 0,

$$\text{R.H.L.} = \lim_{x\to0^+} f(x) = \lim_{x\to0^+} x = 0$$

$$\text{L.H.L.} = \lim_{x\to0^-} f(x) = \lim_{x\to0^-} (-x) = 0$$

and, $f(0) = 0$

Here, L.H.L. = R.H.L. = f(0).

So, the given function is continuous at x = 0.

And, at x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2 - x) = 1$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x) = 1$$

and, $f(1) = 1.$

Here, L.H.L. = R.H.L. = f(1).

So, the given function f(x) is continuous at x = 1.

Thus, the given function f(x) is continuous at x = 0 and x = 1.

Exercise 1.5

1. Examine the continuity of the following functions at the specified points.

(i) $f(x) = \begin{cases} (1 + 3x)^{1/x} & \textbf{when for } x \ne 0 \\ e^3 & \textbf{when } x = 0 \end{cases}$ **, at x = 0**

Solution: given that, $f(x) = (1 + 3x)^{1/3}$ for $x \ne 0$.

So, f(x) has same value for R.H.L. and L.H.L. at x = 0.

At x = 0,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to0} f(x) = \lim_{x\to0} \left[(1 + 3x)^{1/3x}\right]^3 = e^3$$

$f(0) = e^3$

Thus L.H.L. = R.H.L. = f(0).

Hence, the given function f(x) is continuous at x = 0.

(ii) $f(x) = \begin{cases} -x & \text{for } x \le 0 \\ x & \text{for } 0 < x < 1 \\ 2-x & \text{for } x \ge 1 \end{cases}$ **, at x = 1.**

Solution: At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2-x) = 1$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x) = 1$$

and, $f(1) = 2 - 1 = 1$

Here, L.H.L. = R.H.L. = f(1).

So, the given function f(x) is continuous at x = 1.

(iii) $f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 1 \end{cases}$ **, at x = 0.**

Solution: At x = 0

$$\text{R.H.L.} = \lim_{x\to0^+} f(x) = 1 \quad \text{and} \quad \text{L.H.L.} = \lim_{x\to0^-} f(x) = -1.$$

Here, L.H.L. ≠ R.H.L. So, f(x) is not continuous at x = 0.

(iv) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$ **, at x = 0.**

Solution : Since $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \ne 0$.

So, f(x) has same value for R.H.L. and L.H.L., if they have finite value.

At x = 0,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to0} f(x)$$

$$= \lim_{x\to0} \left(\sin\frac{1}{x}\right)$$

= a finite quantity oscillates from –1 to 1.

That means RHL has not necessarily equals to LHL.

That means f(x) is not continuous at x = 0.

(v) $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$ **, at x = 0.** **[2007, fall]**

Solution: Given that

$$f(x) = x\sin\frac{1}{x} \quad \text{for } x \ne 0.$$

So, f(x) has same value for R.H.L. and L.H.L, if they have finite value.

At x = 0,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to0} f(x)$$

$$= \lim_{x\to0} \left[x\sin\frac{1}{x}\right]$$

= 0 (a finite quantity oscillating between –1 to 1)

= 0.

And, f(0) = 0.

Here, R.H.L. = L.H.L. = f(0) = 0. So, the given function f(x) is continuous at x = 0.

(vi) $f(x) = \begin{cases} (1+x)^{1/x} & \text{for } x \ne 0 \\ 1 & \text{for } x = 0 \end{cases}$ **, at x = 0.**

Solution: Given that

$$f(x) = (1+x)^{1/x} \quad \text{for } x \ne 0.$$

So, f(x) has same value for R.H.L. and L.H.L. if they have finite value.

At x = 0,

$$\text{R.H.L.} = \text{L.H.L.} = \lim_{x\to0} (1+x)^{1/x} = e.$$

and, f(0) = 1.

Here, L.H.L. = R.H.L. ≠ f(0). So, the given function f(x) is not continuous at x = 0.

(vii) $f(x) = \begin{cases} \dfrac{x-1}{1+e^{1/(x-1)}} & \text{for } x \ne 1 \\ 0 & \text{for } x = 1 \end{cases}$ **, at x = 1.**

Solution: Since $f(x) = \left(\dfrac{x-1}{1+e^{1/(x-1)}}\right)$ for $x \ne 1$.

So, f(x) has same value for R.H.L. and L.H.L., if they have finite value.

At x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} \left(\frac{x-1}{1+e^{1/(x-1)}}\right) = \frac{0}{1+e^{\infty}} = \frac{0}{1+\infty} = 0.$$

$$\text{L.H.L.} = \lim_{x\to1^-} \left(\frac{x-1}{1+e^{1/(x-1)}}\right) = \frac{0}{1+e^{-\infty}} = \frac{0}{1+0} = 0.$$

and, f(1) = 0.

Here, L.H.L. = R.H.L. = f(1) = 0.

So, the given function f(x) is continuous at x = 1.

2. Examine the continuity and derivability of the function f(x) defined as follows: **[2011 Fall] [2017 Fall]**

$$f(x) = \begin{cases} 1 & \text{when } x \in (-\infty, 0) \\ 1 + \sin x & \text{when } x \in (0, \pi/2) \text{ and } x = 0 \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & \text{when } x \in (\pi/2, \infty) \text{ and } x = \pi/2 \end{cases}$$

Solution: Here, we have to examine continuity and differentiability at $x = 0$ and at $x = \frac{\pi}{2}$.

For the continuity at $x = 0$,

$$\text{R.H.L.} = \lim_{x\to 0^+} (1 + \sin x) = 1 + \sin 0 = 1.$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = 1.$$

and, $f(0) = 1 + \sin 0 = 1 + 0 = 1.$

Here, L.H.L. = R.H.L. = f(0).

So, the given function f(x) is continuous at $x = 0$.

For derivability at $x = 0$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(a+h) - f(a)}{h}\right) = \lim_{h\to 0}\left(\frac{f(0+h) - 1}{h}\right) = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(0-h) - f(0)}{-h}\right) = \lim_{h\to 0}\left(\frac{1-1}{-h}\right) = 0.$$

Here LHD ≠ RHD. So, the given function is not differentiable at $x = 0$.

For the continuity at $x = \frac{\pi}{2}$,

$$\text{R.H.L.} = \lim_{x\to \pi/2^+} f(x) = \lim_{x\to \pi/2^+}\left(2 + \left(x - \frac{\pi}{2}\right)^2\right) = 2.$$

$$\text{L.H.L.} = \lim_{x\to \pi/2^-} f(x) = \lim_{x\to \pi/2^-} (1 + \sin x) = 2.$$

and, $f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2.$

Here, L.H.L. = R.H.L. = $f\left(\frac{\pi}{2}\right) = 2.$

So, the given function is continuous at $x = \frac{\pi}{2}$.

For differentiability at $x = \frac{\pi}{2}$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(a+h) - f(a)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f\left(\frac{\pi}{2} + h\right) - 2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2 + \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^2 - 2}{h}\right) = \lim_{h\to 0}\frac{h^2}{h} = 0.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(a-h) - f(a)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 + \sin\left(\frac{\pi}{2} - h\right) - 2}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 + \sin\frac{\pi}{2}.\cos h + \cos\frac{\pi}{2}.\sin h - 2}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 + \cos h - 2}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-1 + \cos h}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1 - \cos h}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\sin^2\left(\frac{h}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\sin(h/2)}{(h/2)}\right)^2 \times \frac{1}{2} = 1 \times 0 = 0.$$

Here, LHD = RHD at $x = \frac{\pi}{2}$.

Thus, the given function f(x) is continuous everywhere and derivable at $x = \frac{\pi}{2}$ but not derivable at $x = 0$.

3. Determine whether f(x) is continuous and has a derivative at the origin, where

$$f(x) = \begin{cases} 2 + x & \text{if } x \geq 0 \\ 2 - x & \text{if } x < 0 \end{cases}.$$

Solution: At $x = 0$,

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2 + x) = 2$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2 - x) = 2$$

and $f(0) = 2 + 0 = 2.$

Thus, L.H.L. = R.H.L. = f(0). So, f(x) is continuous at x = 2.

And,

$$\text{R.H.D.} = \lim_{h\to0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to0}\frac{2+(h)-2}{h} = \lim_{h\to0}\frac{h}{h} = 1.$$

$$\text{L.H.D.} = \lim_{h\to0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to0}\frac{2-(-h)-2}{-h} = \lim_{h\to0}\frac{-h}{h} = -1$$

Thus, the function f(x) is continuous at x = 0 but not derivable at x = 0.

4. Show that the function f(x) defined as below is continuous at x = 1 and x = 2 and it is derivable at x = 2 but not at x = 1.

$$f(x) = \begin{cases} x & \text{for } x < 1 \\ 2 - x & \text{for } 1 \le x < 2 \\ -2 + 3x - x^2 & \text{for } x \ge 2 \end{cases}$$

[2003, fall] [2006, spring] [2012, Fall]

Solution: For continuity at x = 1,

$$\text{R.H.L.} = \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (2 - x) = 1.$$

$$\text{L.H.L.} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1.$$

and, f(1) = 2 − 1 = 1.

Here, L.H.L. = R.H.L. = f(1) = 1. So, the given function f(x) is continuous at x = 1.

For derivability at x = 1,

$$\text{R.H.D.} = \lim_{h\to0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to0}\frac{2-(1+h)-1}{h} = -1$$

$$\text{L.H.D.} = \lim_{h\to0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to0}\frac{1-h-1}{-h} = 1$$

Here, L.H.D. ≠ R.H.D. So, the function is not differentiable at x = 1.

For continuity at x = 2,

$$\text{R.H.L.} = \lim_{x\to2^+} f(x) = \lim_{x\to2^+} (-2 + 3x - x^2) = -2 + 6 - 4 = 0.$$

$$\text{L.H.L.} = \lim_{x\to2^-} f(x) = \lim_{x\to2^-} (2 - x) = 0.$$

and, f(2) = −2 + 6 − 4 = 0.

Here, L.H.L. = R.H.L. = f(2) = 0.

So, the given function f(x) is continuous at x = 2.

For derivability at x = 2,

$$\text{R.H.D.} = \lim_{h\to0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{-2+3(2+h)-(2+h)^2-0}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{-2+6+3h-4-4h-h^2}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h(3-4-h)}{k}\right) = -1.$$

$$\text{L.H.D.} = \lim_{h\to0}\left(\frac{f(2-h)-f(2)}{-h}\right)$$

$$= \lim_{h\to0}\left(\frac{2-(2-h)-0}{-h}\right) = \lim_{h\to0}\left(\frac{2-2+h}{-h}\right) = -1.$$

Here, L.H.D. = R.H.D. So, f(x) is derivable at x = 2.

Thus, the given function f(x) is continuous at x = 1 and x = 2 and it is derivable at x = 2 but not at x = 1.

5. Find f '(0) of $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \textbf{for } x \neq 0 \\ 0 & \textbf{for } x = 0 \end{cases}$.

Solution: Here we have to find the derivative value of f(x) at x = 0. So,

LHL = RHL = f '(0).

At x = 0,

$$\text{R.H.D.} = \lim_{h\to0}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h^2 . \sin\left(\frac{1}{h}\right)-0}{h}\right)$$

$$= \lim_{h\to0}\left(h . \sin\left(\frac{1}{h}\right)\right)$$

= 0 (finite oscillatory quantity lies between −1 to 1).

= 0.

Thus, f '(0) = L.H.D. = R.H.D. = 0.

6. If $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \textbf{for } x \neq 0 \\ 0 & \textbf{for } x = 0 \end{cases}$**, Find f '(0).**

Solution: Here we have to find the derivative value of f(x) at x = 0. So,

LHL = RHL = f '(0).

At x = 0,

$$\text{R.H.D.} = \lim_{h\to0}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h^2 . \cos\left(\frac{1}{h}\right)-0}{h}\right)$$

$$= \lim_{h\to0}\left(h . \cos\left(\frac{1}{h}\right)\right)$$

$= (0)$ (finite oscillatory quantity lies between -1 to 1).

$= 0.$

Thus, $f'(0) = \text{L.H.D.} = \text{R.H.D.} = 0.$

7. If $f(x) = \begin{cases} x & \text{for } 0 \le x < 1/2 \\ 1 - x & \text{for } 1/2 < x < 1 \end{cases}$. **Does** $f'\left(\frac{1}{2}\right)$ **exists?**

Solution: Here we have to find the derivative value of f(x) at $x = \frac{1}{2}$. So,

$$\text{LHL} = \text{RHL} = f'\left(\frac{1}{2}\right).$$

At $x = \frac{1}{2}$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{1-\left(\frac{1}{2}+h\right)-\frac{1}{2}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1-\frac{1}{2}-h-\frac{1}{2}}{h}\right) = -1.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f\left(\frac{1}{2}-h\right) - f\left(\frac{1}{2}\right)}{-h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{2}-h-\frac{1}{2}}{-h}\right) = 1.$$

This shows that, L.H.D. $\neq$ R.H.D. So, $f'\left(\frac{1}{2}\right)$ does not exists.

8. If $f(x) = \begin{cases} 3 + 2x & \text{for } -3/2 < x \le 0 \\ 3 - 2x & \text{for } 0 < x < 3/2 \end{cases}$. **[2013 Spring]**

Show that f(x) is continuous at x = 0 but f '(0) does not exists.

Solution: For continuity at x = 0,

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3 - 2x) = 3.$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (3 + 2x) = 3.$$

and, $f(0) = 3 + 2(0) = 3.$

Here, L.H.L. = R.H.L. = f(0) = 3. This means f(x) is continuous at x = 0.

For differentiability,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(0+h) - f(0)}{h}\right) = \lim_{h\to 0}\left(\frac{3 - 2(h) - 3}{h}\right) = -2.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(0-h) - f(0)}{-h}\right) = \lim_{h\to 0}\left(\frac{3 + 2(-h) - 3}{-h}\right) = 2.$$

Here, L.H.D. $\neq$ R.H.D. This means f(x) is not differentiable at x = 0.

Thus, f '(0) does not exist.

9. If $f(x) = \begin{cases} 5x - 4 & \text{for } 0 < x \le 1 \\ 4x^2 - 3x & \text{for } 1 < x < 2 \\ 3x + 4 & \text{for } x \ge 2 \end{cases}$

Discuss the continuity of f(x) at x = 1 and 2, and the existence of f '(x) for these values.

Solution: For continuity at x = 1,

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x^2 - 3x) = 1$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5x - 4) = 1$$

and, $f(1) = 5(1) - 4 = 1$

Here, L.H.L. = R.H.L.= f(1). So, f(x) is continuous at x = 1.

For derivability at x = 1,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(1+h) - f(1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4(1+h)^2 - 3(1+h) - 1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4(1 + 2h + h^2) - 3 - 3h - 1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4 + 8h + 4h^2 - 3 - 3h - 1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(5 + 4h)}{h}\right) = 5.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(1-h) - f(1)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{5(1-h) - 4 - 1}{-h}\right) = \lim_{h\to 0}\left(\frac{5 - 5h - 5}{-h}\right) = 5.$$

Thus, L.H.D. = R.H.D. This means f '(x) exists at x = 1.

For continuity at x = 2,

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x + 4) = 10$$

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (4x^2 - 3x) = 16 - 6 = 10.$$

and, $f(2) = 3(2) + 4 = 10.$

Here, L.H.L. = R.H.L. = f(2). So, f(x) is continuous at x = 2.

For derivability at x = 2,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(2+h) - f(2)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3(2+h) + 4 - 10}{h}\right) = \lim_{h\to 0}\left(\frac{6 + 3h + 4 - 10}{h}\right) = 3.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(2-h)-f(2)}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{4(2-h)^2-3(2-h)-10}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{16-16h+4h^2-6+3h-10}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(-16+4h+3)}{-h}\right) = 18.$$

Thus, L.H.D. $\neq$ R.H.D. Hence $f'(x)$ does not exist at $x = 2$.

10. If $$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \le x \le 2 \\ x - \dfrac{x^2}{2} & \text{for } x > 2 \end{cases}$$

Is f(x) continuous at x = 1 and 2? Does f '(x) exist for these values?

Solution: For continuity at $x = 1$,

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x) = 1$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1.$$

and, $f(1) = 1$.

Here, R.H.L. = L.H.L. = f(1). So, f(x) is continuous at $x = 1$.

For continuity at $x = 2$,

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}\left(x - \frac{x^2}{2}\right) = 0$$

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 0$$

and, $f(2) = 2 - 2 = 0$.

Here, L.H.L. = R.H.L. = f(2). So, f(x) is continuous at $x = 2$.

For derivability at $x = 1$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2-(1+h)-1}{h}\right) = \lim_{h\to 0}\left(\frac{-h}{h}\right) = -1.$$

$$\text{L.H.D.} = \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{-h}\right) = \lim_{h\to 0}\left(\frac{(1-h)-1}{-h}\right) = 1.$$

Thus, L.H.D. $\neq$ R.H.D. This means $f'(1)$ does not exist.

For derivability at $x = 2$,

$$\text{R.H.D.} = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{(2+h) - \frac{(2+h)^2}{2} - 0}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{4 + 2h - (4 + 2h + h^2) - 0}{2h} \right)$$

$$= \lim_{h \to 0} \left(\frac{4 + 2h - 4 - 4h - h^2}{2h} \right)$$

$$= \lim_{h \to 0} \left(\frac{h(2 - 4 - h)}{2h} \right) = -\frac{2}{2} = -1.$$

$$\text{L.H.D.} = \lim_{h \to 0} \left(\frac{f(2-h) - f(2)}{-h} \right)$$

$$= \lim_{h \to 0} \left(\frac{2 - (2-h) - 0}{-h} \right) = \lim_{h \to 0} \left(\frac{h}{-h} \right) = -1.$$

This shows that, L.H.D. = R.H.D.

So, f(x) is differntiable at x = 2 and f '(2) = −1.

Exercise 2

1. Find y_n in the following:

(i) $y = (a - bx)^m$

Solution: Let, $y = (a - bx)^m$

Differentiating, we get

$$y_1 = m(a - bx)^{m-1} (-b)$$

$$y_2 = m(m - 1) (a - bx)^{m-2} (-b)^2$$

$$y_3 = m(m - 1) (m - 2) (a - bx)^{m-3} (-b)^3$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$y_n = m(m - 1) (m - 2) \ldots (m - (n - 1)) (-b)^n (a - bx)^{m-n}$$

$$= (-1)^n m(m - 1)(m - 2) \ldots (m - n + 1) b^n (a - bx)(a - bx)^{m-n}$$

(ii) $y = \dfrac{1}{a-x} = (a-x)^{-1}$

Solution: We have, $y = \dfrac{1}{a-x} = (a-x)^{-1}$

Differentiating w.r.t. x,

$$y_1 = (-1)(a-x)^{-2}(-1) = (-1)^2 (a-x)^{-2} = (a-x)^{-2}.$$
$$y_2 = (-2)(a-x)^{-3}(-1) = (-1)^2 2!\,(a-x)^{-3} = 2!\,(a-x)^{-3}$$

Continuing the process up to n steps then,

$$y_n = n!\,(a-x)^{-(n+1)} = \frac{n!}{(a-x)^{n+1}}$$

(iii) $y = x^{2n}$

Solution: Let, $y = x^{2n}$

Differentiating, we get

$$y_1 = 2n\,x^{2n-1}$$
$$y_2 = 2n(2n-1)\,x^{2n-2}$$
$$y_3 = 2n(2n-1)(2n-2)\,x^{2n-3}$$
$$\ldots \quad \ldots \quad \ldots$$
$$y_n = 2n(2n-1)(2n-2)\ldots\ldots(2n-(n-1))\,x^{2n-n}$$
$$= 2n(2n-1)(2n-2)\ldots(n+1)\,x^n$$

Multiplying numerator and denominator by $n(n-1)(n-2)\ldots 2.1$ we get

$$y_n = 2n(2n-1)(2n-2)\ldots(n+1)\frac{n(n-1)(n.2)\ \ldots\ 2.1}{n(n-1)(n-2)\ \ldots\ 2.1}x^n$$

Separating even and odd brackets

$$= \frac{\{2n[(2n-2)(2n-4)\ldots 6.4.2][(2n-1)(2n-3)(2n-5)\ldots 3.1]\}}{n!}x^n$$

$$= \frac{2^n\{n(n-1)(n-2)\ldots 3.2.1\}\{(2n-1)(2n-3)(2n-5)\ldots 3.1\}}{n!}x^n$$

$$= 2^n.\frac{n!}{n!}(2n-1)(2n-3)(2n-5)\ldots 3.1\,x^n$$

Thus, $\quad y_n = 2^n\{1.3.5.\ \ldots\ (2n-1)\}\,x^n$

(iv) $y = \sqrt{x}$

Solution: Let, $y = \sqrt{x} = x^{1/2}$

Differentiating w.r.t. x, we get

$$y_1 = \frac{1}{2}x^{-(1/2)}$$

$$y_2 = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-(1/2)-1} = \frac{1}{2^2}(-1)(1)\,x^{-(1/2)-1}$$

$$y_3 = \frac{1}{2^2}(-1)(1)\left(-\frac{1}{2}-1\right)x^{-(1/2)-2} = \frac{1}{2^3}(-1)^2(1.3)\,x^{-(1/2)-2}$$

$$y_4 = \frac{1}{2^3}(-1)^2(1.3)\left(-\frac{1}{2}-2\right)x^{-(1/2)-3} = \frac{1}{2^4}(-1)^3(1.3.5)\,x^{-(1/2)-3}$$

$$\ldots \quad \ldots$$

$$y_n = \frac{1}{2^n}(-1)^{n-1}[1.3.5\ldots(2n-3)]\,x^{-(1/2)-(n-1)}$$

$$= (-1)^{n-1}\left(\frac{1.3.5\ldots(2n-3)}{2^n\,x^{n-1/2}}\right)$$

(v) $y = \dfrac{1}{\sqrt{x}} = (x)^{-1/2}$

Solution: Let, $y = \dfrac{1}{\sqrt{x}} = (x)^{-(1/2)}$

Differentiating w.r.t. x, we get

$$y_1 = \frac{-1}{2}x^{-(1/2)-1} = (-1)\frac{1}{2}x^{-(1/2)-1}$$

$$y_2 = (-1)\frac{1}{2}\left(-\frac{1}{2}-1\right)x^{-(1/2)-2} = \frac{1}{2^2}(-1)^2(1.3)\,x^{-(1/2)-2}$$

$$y_3 = \frac{1}{2^2}(-1)^2(1.3)\left(-\frac{1}{2}-2\right)x^{-(1/2)-3} = \frac{1}{2^3}(-1)^3(1.3.5)\,x^{-(1/2)-3}$$

$$y_4 = \frac{1}{2^3}(-1)^3(1.3.5)\left(-\frac{1}{2}-3\right)x^{-(1/2)-4} = \frac{1}{2^4}(-1)^4(1.3.5.7)\,x^{-(1/2)-4}$$

$$\ldots \quad \ldots$$

$$y_n = \frac{1}{2^n}(-1)^n[1.3.5\ldots(2n-1)]\,x^{-(1/2)-n}$$

$$= (-1)^n\left(\frac{1.3.5\ldots(2n-1)}{2^n\,x^{n+1/2}}\right)$$

(vi) $y = 10^{3-2x}$

Solution: Let,

$$y = 10^{3-2x} = (10^3)10^{-2x} = (10^3)\,e^{-2x\log 10} \qquad [\because a^x = e^{x\log a}]$$

$$\Rightarrow \quad y = 10^3 e^{kx} \quad \ldots\ldots\text{(i)} \qquad \text{where, } -2\log 10 = k$$

Differentiating both sides of (i) then

$$y_1 = (10^3)\,k\,e^{kx}$$
$$y_2 = (10^3)\,k^2 e^{kx}$$
$$\ldots \quad \ldots \quad \ldots$$
$$y_n = (10^3)\,k^n e^{kx}$$
$$\Rightarrow \quad y_n = (10^3)(-2\log 10)^n e^{(-2\log 10)x}$$
$$= (10^3)(-2)^n(\log 10)^n\,10^{-2x} \qquad (\because 10^{-2x} = e^{-2x\log 10})$$
$$= 10^{3-2x}(-2)^n(\log 10)^n$$

(viii) $y = \dfrac{x^n}{x-1}$

Solution: Since,

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \dots + 1)$$

Here, $y = \dfrac{x^n}{x-1} = \dfrac{(x^n - 1) + 1}{x-1} = \dfrac{(x-1)(x^{n-1} + x^{n-2} + \dots + 1) + 1}{(x-1)}$

$$\Rightarrow y = (x^{n-1} + x^{n-2} + \dots + 1) + \frac{1}{x-1} \quad \dots (i)$$

Differentiating w. r. t. x upto n steps

$$y_n = 0 + D^n\left(\frac{1}{x-1}\right) \qquad [\because D^m(x^n) = 0 \text{ if } m > n]$$

$$= D^n\left(\frac{1}{x-1}\right) = \frac{(-1)^n\, n!}{(x-1)^{n+1}} \qquad \left[\because D_n\left(\frac{1}{x+1}\right) = \frac{(-1)^n\, n!}{(x+a)^{n+1}}\right]$$

(ix) $y = e^x$ **sinx. sin2x**

Solution: We know,

$$y = \frac{1}{2} e^x\, [2\sin 2x.\ \sin x] = \frac{1}{2} e^x\, [2\cos(2x - x) - \cos(2x + x)]$$

So, $y = \dfrac{1}{2} e^x \cos x - \dfrac{1}{2} e^x \cos 3x \quad \dots (i)$

Differentiating both sides w.r. t x, upto n steps

$$y_n = \frac{1}{2} D^n (e^x \cos x) - \frac{1}{2} D^n (e^x \cos 3x)$$

Applying $D^n (e^{ax} \cos bx) = (a^2 + b^2)^{n/2} e^{ax} \cos(bx + n\tan^{-1}\frac{b}{a})$

Then

$$y_n = \frac{1}{2}(1^2 + 1^2)^{n/2} e^x \cos(x + n\tan^{-1} 1)$$

$$- \frac{1}{2} . (1^2 + 3^2)^{n/2} e^x \cos\left(3x + n\tan^{-1}\frac{3}{1}\right)$$

$$= \frac{e^x}{2} [2^{n/2} \cos(x + n\tan^{-1} 1) - 10^{n/2} \cos(3x + n\tan^{-1} 3)]$$

$$\Rightarrow y_n = \frac{e^x}{2}\left[2^{n/2} \cos\left(x + \frac{n\pi}{4}\right) - 10^{n/2} \cos(3x + n\tan^{-1} 3)\right]$$

(x) $y = e^{3x}$ **sin 4x** **... (i)**

Solution: We have, $y = e^{3x} \sin 4x \quad \dots (i)$

Applying $D^n (e^{ax} \sin bx) = (a^2 + b^2)^{n/2} e^{ax} \sin\left(x + n\tan^{-1}\frac{b}{a}\right)$

Then, differentiating upto n steps in (i), we get

$$y_n = D^n (e^{3x} \sin 4x)$$

$$= (3^2 + 4^2)^{n/2} e^{3x} \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

$$= e^{3x} (25)^{n/2} \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

$$\therefore \quad y_n = e^{3x}\, 5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

2. Find the n^{th} derivatives of the following functions:

(i) $y = \dfrac{1}{x^2 + 16}$

Solution: Let,

$$y = \frac{1}{x^2 + 16} = \frac{1}{x^2 - (4i)^2} = \frac{1}{(x - 4i)(x + 4i)}$$

$$y = \frac{1}{8i}\left[\frac{1}{x - 4i} - \frac{1}{x + 4i}\right] \quad \dots\dots (i)$$

Differentiating both sides w.r. t x upto n steps

$$y_n = \frac{1}{8i}\left[D^n\left(\frac{1}{x - 4i}\right) - D^n\left(\frac{1}{x + 4i}\right)\right]$$

Applying $D^n\left(\dfrac{1}{x+a}\right) = \dfrac{(-1)^n\, n!}{(x+a)^{n+1}}$ Then,

$$y_n = \frac{1}{8i}\left[\frac{(-1)^n\, n!}{(x - 4i)^{n+1}} - \frac{(-1)^n\, n!}{(x + 4i)^{n+1}}\right]$$

$$= \frac{(-1)^n\, n!}{8i} [(x - 4i)^{-(n+1)} - (x + 4i)^{-(n+1)}]$$

Putting $x = r\cos\theta$, $4 = r\sin\theta$. So that $r = \sqrt{x^2 + 4^2}$, $\theta = \tan^{-1}\left(\dfrac{4}{x}\right)$ then,

$$y_n = \frac{(-1)^n\, n!}{8i\,.\, r^{n+1}} [r^{-(n+1)} (\cos\theta - i\sin\theta)^{-(n+1)} - r^{-(n+1)} (\cos\theta + i\sin\theta)^{-(n+1)}]$$

Applying De Moivre's Theorem, we get,

$$y_n = \frac{(-1)^n\, n!}{8i\,.\, r^{n+1}} [\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta]$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{(-1)^n\, n!}{8i \times r^{n+1}} . 2i \sin(n+1)\theta$$

$$\therefore \quad y_n = \frac{(-i)^n\, n!}{4\,.\,(4/\sin\theta)^{n+1}} \sin(n+1)\theta\,.$$

$$y_n = \frac{(-1)^n\, n!\, \sin^{n+1}\theta\, \sin(n+1)\theta}{4^{n+2}} \qquad \text{where } \theta = \tan^{-1}\left(\frac{4}{x}\right)$$

(ii) $y = \dfrac{x}{x^2 + a^2}$

Solution: Let,

$$y = \frac{x}{x^2+a^2} = \frac{x}{(x-ai)(x+ai)} = \frac{1}{2}\left[\frac{1}{x+ai} + \frac{1}{x-ai}\right]$$

Differentiating w.r.t. x up to n steps, we get

$$y_n = \frac{1}{2}\left[d^n\left(\frac{1}{x+ai}\right) + D^n\left(\frac{1}{x-ai}\right)\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^n n!}{(x+ai)^{n+1}} + \frac{(-1)^n n!}{(x-ai)^{n+1}}\right]$$

$$= \frac{1}{2}(-1)^n n! [(x+a)^{-(n+1)} = (x-ai)^{-(n+1)}]$$

Putting $x = r\cos\theta$, $a = r\sin\theta$ then $\theta = \tan^{-1}\left(\frac{x}{a}\right)$

So, $y_n = \frac{(-1)^n n!}{2}[(r\cos\theta = ir\sin\theta)^{-(n+1)} + (r\cos\theta - ir\sin\theta)^{-(n+1)}]$

$$= \frac{(-1)^n n!}{2} r^{-(n+1)} [(\cos\theta + i\sin\theta)^{-(n+1)} + (\cos\theta - i\sin\theta)^{-(n+1)}]$$

$$= \frac{(-1)^n n!}{2r^{n+1}} [\cos(n+1)\theta - i\sin(n+1)\theta + \cos(n+1)\theta + i\sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{2r^{n+1}} 2\cos(n+1)\theta$$

Since, $r = \frac{a}{\sin\theta}$ therefore,

$$y_n = (-1)^n n! \times \frac{\sin^{n+1}\theta}{a^{n+1}} \cos(n+1)\theta \quad ,\text{where } \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad y_n = \frac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \cos(n+1)\theta.$$

(iii) $y = \frac{1}{x^2+x+1}$

Solution: We know,

$$y = \frac{1}{x^2+x+1} = \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad \text{...... (i)}$$

Applying, $D^n\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.

Differentiating w.r.t. x, upto n steps in (i) we get,

$$y_n = D^n\left(\frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= \frac{(-1)^n n!}{\left(\frac{\sqrt{3}}{2}\right)^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta, \text{ where } \theta = \tan^{-1}\left(\frac{(\sqrt{3}/2)}{(1/2)+x}\right)$$

$$\therefore \quad y_n = \frac{(-1)^n 2^{n+2} n!}{(\sqrt{3^{n+2}})} \sin^{n+1}\theta \sin(n+1)\theta \text{ ,where } \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2x+1}\right)$$

(iv) $y = \frac{1}{(x^2+a^2)(x^2+b^2)}$

Solution: We have,

$$y = \frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right]$$

Differentiating upto n steps, we get

$$y_n = \frac{1}{a^2-b^2}\left[D^n\left(\frac{1}{x^2+b^2}\right) - D^n\left(\frac{1}{x^2+a^2}\right)\right]$$

Now applying $D^n\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$ then,

$$y_n = \frac{(-1)^n n!}{a^{n+2}}\left[\frac{(-1)^n n!}{b^{n+2}} \sin^{n+1}\theta_1 \sin(n+1)\theta_1 - \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta_2 \sin(n+1)\theta_2\right]$$

where, $\theta_1 = \tan^{-1}\left(\frac{b}{x}\right)$, $\theta_2 = \tan^{-1}\left(\frac{a}{x}\right)$

Hence, $y_n = \left[\frac{\sin^{n+1}\theta_1 \sin(n+1)\theta_1}{b^{n+2}} - \frac{\sin^{n+1}\theta_2 \sin(n+1)\theta_2}{a^{n+2}}\right]$

where $\theta_1 = \tan^{-1}\left(\frac{b}{x}\right)$, $\theta_2 = \tan^{-1}\left(\frac{a}{x}\right)$

(v) $y = \cot^{-1}\left(\frac{x}{a}\right)$

Solution: We have, $y = \cot^{-1}\left(\frac{x}{a}\right)$ ... (i)

Differentiating both sides w. r. t. x, we get

$$y_1 = -\frac{1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1}{a}$$

$$= -\frac{a}{x^2+a^2} = -\frac{a}{(x-ia)(x+ia)} = -\frac{1}{2i}\left[\frac{1}{x+ai} - \frac{1}{x-ai}\right]$$

$$\Rightarrow \quad y_1 = \frac{1}{2i}\left[\frac{1}{x+ai} - \frac{1}{x-ai}\right]$$

Differentiating both sides upto (n – 1) steps,

$$y_n = \frac{1}{2i}\left[D^{n-1}\left(\frac{1}{x+ai}\right) - D^{n-1}\left(\frac{1}{x-ai}\right)\right] \quad \text{....... (ii)}$$

Using, $D^n\left(\frac{1}{x+a}\right) = \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}$ in (i) we get,

$$\Rightarrow \quad D^{n-1}\left(\frac{1}{x+a}\right) = \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n} \text{ etc.}$$

Hence from eq^n (i), we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n} - \frac{(-1)^n(n-1)!}{(x-ai)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}[(x+ai)^{-n} - (x-ia)^{-n}]$$

Put $x = r\cos\theta$, $a = r\sin\theta$ then $\tan\theta = \frac{a}{x}$

So, $y_n = \frac{(-1)^{n-1}(n-1)!}{2i}[r^{-n}(\cos\theta + i\sin\theta)^{-n} - r^{-n}(\cos\theta - i\sin\theta)^{-n}]$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}[\cos n\theta - i\sin n\theta - \cos n\theta - i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!\, r^{-n}}{2i}(-2i\sin n\theta)$$

Since, $r = \frac{a}{\sin\theta}$ then, $r^{-n} = \left(\frac{a}{\sin\theta}\right)^{-n} = \frac{\sin^n\theta}{a^n}$

So, $y_n = (-1)^n(n-1)!\frac{\sin^n\theta}{a^n}.\sin n\theta$

$$= \frac{(-1)^n(n-1)!}{a^n}\sin^n\theta\sin n\theta \quad ,\text{where } \theta = \tan^{-1}\left(\frac{a}{x}\right)$$

3. Find y_n if

(i) $y = x^n e^x$

Solution: Let, $y = x^n e^x$

Applying Leibnitz's rule for higher derivative,

$$y_n = e^x x^n + {}^nC_1 e^x nx^{n-1} + {}^nC_2 e^x n(n-1)x^{n-2} + \ldots + {}^nC_n e^x n!$$

$$= e^x x^n + n e^x nx^{n-1} + \left(\frac{n(n-1)}{2!}\right)e^x n(n-1)x^{n-2} + \ldots + e^x n!$$

$$= e^x\left[x^n + \frac{(n)^2}{1!}x^{n-1} + \left(\frac{n^2(n-1)^2}{2!}\right)x^{n-2} + \ldots + n!\right]$$

Thus, $y_n = e^x\left[x^n + \frac{(n)^2}{1!}x^{n-1} + \left(\frac{n^2(n-1)^2}{2!}\right)x^{n-2} + \ldots + n!\right]$.

(ii) $y = e^x \log x$

Solution: Let, $y = e^x\log(x)$

Applying Leibnitz's rule for higher derivative,

$$y_n = D^n(e^x\log x)$$

$$= D^n(e^x)\log x + {}^nC_1 D^{n-1}(e^x).D(\log x) + {}^nc_2 D^{n-2}(e^x)D^2(\log x) + {}^nC_3 D^{n-3}(e^x).D^3(\log x) + \ldots + e^x.D^n(\log x)$$

$$= e^x\log x + {}^nC_1 e^x.\frac{1}{x} + {}^nC_2 e^x\left(\frac{-1}{x^2}\right) + {}^nC_3 e^x\left(\frac{2}{x^3}\right) + \ldots + e^x.\frac{(-1)^{n-1}(n-1)!}{x^n}$$

(Since, $D^n(\log x)] = \frac{(-1)^{n-1}(n-1)!}{x^n}$)

$$= e^x\left[\log x + \frac{{}^nC_1}{x} - \frac{{}^nC_2}{x^2} + \frac{{}^nC_3}{x^3}2! + \ldots + \ldots \frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

(iii) $y = e^{ax}\cos bx$

Solution: Let, $y = e^{ax}\cos bx$

Applying Leibnitz's rule with taking $u = e^{ax}$, $v = \cos bx$, we get

$$y_n = D^n(e^{ax}\cos bx)$$

$$= D^n(e^{ax}).\cos bx + {}^nC_1 D^{n-1}(e^{ax}).D(\cos bx) + {}^nC_2 D^{n-2}(e^{ax}).D^2(\cos bx) + \ldots + e^{ax}D^n(\cos bx)$$

$$= a^n e^{ax}\cos bx + {}^nC_1 a^{n-1}e^{ax}.b\cos\left(bx + \frac{n\pi}{2}\right) + {}^nC_2 a^{n-2}e^{ax}b^2\cos\left(bx + \frac{2\pi}{2}\right) + \ldots + e^{ax}b^n\cos\left(bx + \frac{n\pi}{2}\right)$$

$$= e^{ax}\left[a^n\cos bx + {}^nC_1 a^{n-1}b\cos\left(bx + \frac{n\pi}{2}\right) + {}^nC_2 a^{n-2}b^2\cos\left(bx + \frac{2\pi}{2}\right) + \ldots + b^n\cos\left(bx + \frac{n\pi}{2}\right)\right]$$

4. If $y = \sin mx + \cos mx$, show that $y_n = m^n\{1 + (-1)^n\sin 2mx\}^{1/2}$

Solution: Let, $y = \sin mx + \cos mx$

Differentiating upto n times we get,

$$y_n = D^n(\sin mx) + D^n[\cos mx]$$

$$\Rightarrow y_n = m^n\sin\left(mx + \frac{n\pi}{2}\right) + m^n\cos\left(mx + \frac{n\pi}{2}\right)$$

Let us put $\theta = mx + \frac{n\pi}{2}$ then,

$$y_n = m^n(\sin\theta + \cos\theta) = m^n[(\sin\theta + \cos\theta)^2]^{1/2}$$

$$= m^n(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta]^{1/2}$$

$$= m^n[1 + \sin 2\theta]^{1/2}$$

Since, $2\theta = 2mx + n\pi$ then,

$$y_n = m^n[1 + \sin(2mx + n\pi)]^{1/2}$$

We know that, $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ then,

$$y_n = m^n[1 + \sin 2mx.(-1)^n + \cos 2mx.0]^{1/2}$$

$$y_n = m^n[1 + (-1)^n\sin 2mx]^{1/2}.$$

5. If $y = x^{n-1}\log(x)$, show that $ny_n = (n-1)!$.

Solution: Let, $y = x^{n-1}\log(x)$ ..... (i)

Differentiating y w. r. t. x, we get

$$y_1 = (n-1)x^{n-2}\log(x) + x^{n-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow xy_1 = (n-1)x^{n-1}\log(x) + x^{n-1}$$

$$= (n-1)y + x^{n-1}$$

Applying Leibnitz's rule for higher differentiation then

$$xy_n + {}^{n-1}C_1(1)(y_{n-1}) = (n-1)(y_{n-1}) + (n-1)(n-2)\ldots(2)(1)$$

$$\Rightarrow xy_n + (n-1)(y_{n-1}) = (n-1)(y_{n-1}) + (n-1)!$$

$$\Rightarrow xy_n = (n-1)!$$

6. If $y = e^{x^2}$, show that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$. **[2013 Fall]**

Solution: Let, $y = e^{x^2}$ ..... (i)

Differentiating w.r.t x, we get

$$y_1 = e^{x^2}\, 2x$$

$$\Rightarrow y_1 = y\, 2x = 2xy \quad \ldots \text{(ii)}$$

Applying Leibnitz's rule for higher differentiation then

$$y_{n+1} = 2[(x)(y_n) + {}^nC_1(1)(y_{n-1})]$$

$$y_{n+1} - 2xy_n - 2ny_{n-1} = 0 \qquad [\because {}^nC_1 = n]$$

7. If $y = e^{ax}\sin bx$, show that (i) $y_2 - 2ay_1 + (a^2 + b^2)y = 0$
(ii) $y_{n+1} = 2ay_n - (a^2 + b^2)y_{n-1}$

Solution: Let, $y = e^{ax}\sin bx$ ... (i)

Differentiating both sides w.r.t. x, we get

$$y_1 = a e^{ax}\sin bx + e^{ax} b\cos bx$$

$$\Rightarrow y_1 = ay + be^{ax}\cos bx \quad \ldots \text{(ii)}$$

Again, differentiating w.r.t. x then,

$$y_2 = ay_1 + b[a e^{ax}\cos bx - e^{ax} b\sin bx]$$

$$= ay_1 + a[be^{ax}\cos bx) - b^2(e^{ax}\sin bx)$$

Since from (ii), $be^{ax}\cos bx = y_1 - ay$ therefore,

$$y_2 = ay_1 + a(y_1 - ay) - b^2 y$$

$$\Rightarrow y_2 = ay_1 - a^2y - b^2y$$

$$\Rightarrow y_2 - 2ay_1 + (a^2 + b^2)y = 0 \quad \ldots \text{(iii)}$$

This proves first part.

For second part:

Applying Leibnitz's rule for higher differentiation to (iii) then

$$y_{n+1} - 2ay_n + (a^2 + b^2)y_{n-1} = 0$$

$$\Rightarrow y_{n+1} = 2ay_n - (a^2 + b^2)y_{n-1}.$$

8. If $y = \log\left(x + \sqrt{a^2 + x^2}\right)$, show that
(i) $(a^2 + x^2)y_2 + xy_1 = 0$
(ii) $(a^2 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2y_n = 0$. **[2011 Fall]**

Solution: Let $y = \log\left(x + \sqrt{a^2 + x^2}\right)$

Differentiating we get,

$$y_1 = \left(\frac{1}{x + \sqrt{a^2 + x^2}}\right)\left(1 + \frac{1}{2\sqrt{a^2 + x^2}}(2x)\right)$$

$$= \left(\frac{1}{x + \sqrt{a^2 + x^2}}\right)\left(\frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2}}\right)$$

$$= \frac{1}{\sqrt{a^2 + x^2}}$$

$$\Rightarrow \left(\sqrt{a^2 + x^2}\right) y_1 = 1$$

Again, differentiating we get,

$$\left(\sqrt{a^2 + x^2}\right) y_2 + y_1\left(\frac{x}{\sqrt{a^2 + x^2}}\right) = 0$$

$$\Rightarrow (a^2 + x^2)y_2 + xy_1 = 0$$

This completes the proof of (i)

For second part:

Applying Leibnitz's rule for higher differentiation then

$$(a^2 + x^2)y_{n+2} + {}^nC_1(2x)\, y_{n+1} + {}^nC_2(2)y_n + xy_{n+1} + {}^nC_1y_n = 0$$

$$\Rightarrow (a^2 + x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n = 0$$

$$\Rightarrow (a^2 + x^2)y_{n+2} + (2n + 1)\, xy_{n+1} + n^2y_n = 0$$

This completes the proof of (ii)

9. Find $y_n(0)$, if $y = e^{a\sin^{-1}x}$

Solution: Let $y = e^{a\sin^{-1}x}$ then, $y(0) = e^0 = 1$

Differentiating w. r. t. x then,

$$y_1 = e^{a\sin^{-1}x}\left(\frac{a}{\sqrt{1 - x^2}}\right) = \frac{ay}{\sqrt{1 - x^2}} \quad \ldots\text{(i)}$$

$$\Rightarrow (1 - x^2)\, y_1^2 - a^2y^2 = 0.$$

Again differentiating w. r. t. x then, we get

$$(1 - x^2)\, 2y_1y_2 + y_1^2(-2x) - 2a^2yy_1 = 0$$

$$\Rightarrow (1 - x^2)\, y_2 - xy_1 - a^2y = 0 \quad \ldots \text{(ii)}$$

Putting x = 0 in (i) $\quad y_1(0) = ay(0) = a$

Putting x = 0 in (ii) $\quad y_2(0) - 0 - a^2y(0) = 0$

$$\Rightarrow y_2(0) - a^2 = 0$$

$$\Rightarrow y_2(0) = a^2$$

Applying Leibnitz's Rule for higher differentiation, we get

$$(1 - x^2)\, y_{n+2} + {}^nc_1\, y_{n+1}(-2x) + {}^nc_2 y_n(-2) - [xy_{n+1} + {}^nc_1\, y_{n.1}] - a^2y_n = 0$$

$$\Rightarrow (1 - x^2)\, y_{n+2} - 2nxy_{n+1} - n(n-1)\, y_n - xy^{n+1} - ny_n - a^2y_n = 0$$

$$\Rightarrow (1 - x^2)\, y_{n+2} - (2n + 1)xy_{n+1} - [n(n-1) + n + a^2]\, y_n = 0 \quad \ldots \text{(iii)}$$

Putting x = 0 in (iii) we get,

$$y_{n+2}(0) - (2n + 1)\,.\,0 - (n^2 + a^2)\, y_n(0) = 0$$

$$\Rightarrow y_{n+2}(0) = (n^2 + a^2)\, y_n(0) \quad \ldots \text{(iv)}$$

Also, we have $y_1(0) = a$, $y_2(0) = a^2$

Putting n = 2, 4, 6, ... in (iv) we get,

$$y_4(0) = (2^2 + a^2)\,.\, y_2(0) = (2^2 + a^2)\, a^2$$

$$y_6(0) = (4^2 + a^2) \cdot y_4(0) = (4^2 + a^2)(2^2 + a^2)\, a^2$$
$$y_8(0) = (6^2 + a^2) \cdot y_6(0) = (6^2 + a^2)(4^2 + a^2)(2^2 + a^2)\, a^2$$
$$\dots \quad \dots \quad \dots$$
$$\Rightarrow \; y_n(0) = [(n-2)^2 + a^2]\,[(n-4)^2 + a^2] \dots (4^2 + a^2)(2^2 + a^2)\, a^2$$
if n is even .... (*)

Again putting n = 1, 3, 5 ... in (iv) we get
$$y_3(0) = (1^2 + a^2)\, a$$
$$y_5(0) = (3^2 + a^2)\, y_3(0) = (3^2 + a^2)(1^2 + a^2)\, a$$
$$y_7(0) = (5^2 + a^2)\, y_5(0) = (5^2 + a^2)(3^2 + a^2)(1^2 + a^2) \cdot a$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$y_n(0) = [(n-2)^2 + a^2]\,[(n-4)^2 + a^2] \dots [3^2 + a^2]\,[1^2 + a^2]\, a.$$
if n is odd .... (**)

From (*) and (**), we get
$$y_n(0) = \begin{cases} [(n-2)^2 + a^2]\,[(n-4)^2 + a^2] \dots (4^2 + a^2)(2^2 + a^2)\, a^2 & \text{if n is even} \\ [(n-2)^2 + a^2]\,[(n-4)^2] + a^2] \dots (3^2 + a^2)(1^2 + a^2)\, a & \text{if n is odd} \end{cases}$$

10. If $y = (x + \sqrt{(1+x)^2})^m$, show that

(i) $(1 + x^2)\, y_2 + xy_1 - m^2 y = 0$

(ii) $(1 + x^2)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$

Solution: Let $y = \left(x + \sqrt{(1+x)^2}\right)^m$

Differentiating w.r.t. x, gives
$$y_1 = m\left(x + \sqrt{(1+x)^2}\right)^{m-1}\left(1 + \left(\frac{1}{2}\right)\frac{2x}{\sqrt{1+x^2}}\right)$$
$$= m\left(x + \sqrt{(1+x)^2}\right)^{m-1}\left(\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$
$$\Rightarrow \; y_1 = \frac{m\left(x + \sqrt{1+x^2}\right)^m}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}}$$
$$\Rightarrow \; (1 + x^2)\, y_1^2 - m^2 y^2 = 0$$

Again differentiating w.r.t. x, we get
$$(1 + x^2)\, 2y_1 y_2 + y_1^2(2x) - m^2(2y) - y_1 = 0$$
$$\Rightarrow \; (1 + x^2)\, y_2 + xy_1 - m^2 y = 0 \quad \dots\dots \text{(i)}$$
This proves the first part

For second part:

Applying Leibnitz's Rule for higher differentiation, we get
$$(1 + x^2)\, y_{n+2} + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + xy_{n+1} + {}^nC_1 y_{n.1} - m^2 y_n = 0$$
$$\Rightarrow \; (1 + x^2) y_{n+2} + 2nxy_{n+1} + \left(\frac{n(n-1)}{2}\right) 2y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$
$$\Rightarrow \; (1 + x^2)\, y_{n+2} + (2n + 1)\, xy_{n+1} + [n(n-1) + n - m^2] y_n = 0$$
$$\Rightarrow \; (1 + x^2)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$
This completes the solution.

11. If $y^{1/m} + y^{-1/m} = 2x$, show that **[2003, Fall]**

$(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2) y_n = 0$

Solution: Let, $\quad y^{1/m} + y^{-1/m} = 2x \quad$ .... (i)

Differentiating (i) w.r.t. x then
$$\left(\frac{1}{m}\right) y^{1/m-1}\, y_1 - \left(\frac{1}{m}\right) y^{-(1/m)-1}\, y_1 = 2$$
$$\Rightarrow \; \left(\frac{y_1}{my}\right)(y^{1/m} - y^{-1/m}) = 2$$
$$\Rightarrow \; y_1(y^{1/m} - y^{-1/m}) = 2my$$
$$\Rightarrow \; y_1^2\,(y^{1/m} - y^{-1/m})^2 = 4m^2y^2$$
$$\Rightarrow \; y_1^2\,[(y^{1/m} + y^{-1/m})^2 - 4y^{1/m}\, y^{-1/m}] = 4m^2y^2$$
$$[\because \; (a - b)^2 = (a + b)^2 - 4ab]$$

Since, by (i), $\quad y^{1/m} + y^{-1/m} = 2x$. So,
$$y_1^2\,[(2x)^2 - 4] = 4m^2y^2$$
$$\Rightarrow \; (x^2 - 1)\, 4y_1^2 - 4m^2y^2 = 0$$

Again diff. w.r.t. x, we get
$$(x^2 - 1)\, 2y_1y_2 + 2x \cdot y_1^2 - m^2.2y.y_1 = 0 \quad [\because 4 \neq 0.]$$
$$\Rightarrow \; 2y_1\,[(x^2 - 1)\, y_2 + xy_1 - m^2 y] = 0$$

Since $2y_1 \neq 0$. So,
$$(x^2 - 1)\, y_2 + xy_1 - m^2 y = 0 \quad \dots \text{(ii)}$$

Applying Leibnitz's Rule for higher differentiation, we get
$$(x^2 - 1)y_{n+2} + {}^nC_1\, y_{n+1}(2x) + {}^nC_2\, y_n(2) + xy_{n+1} + {}^nC_1\, y_n(1) - m^2 y_n = 0$$
$$\Rightarrow \; (x^2 - 1)y_{n+2} + 2nxy_{n+1} + n(n-1)\, y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$
$$\Rightarrow \; (x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + [n(n-1) + n - m^2] y_n = 0$$
$$\Rightarrow \; (x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0.$$
This completes the solution.

12. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$ and hence show that $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0.$

[2007, Spring][2005, Spring]

Solution: We have, $y = (\sin^{-1} x)^2$

Differentiating w. r. t. x then, $y_1 = 2\sin^{-1}x\left(\dfrac{1}{\sqrt{1 - x^2}}\right)$
$$\Rightarrow \; \sqrt{1 - x^2}\; y_1 = 2\sin^{-1}x$$
$$\Rightarrow \; (1 - x^2)\, y_1^2 - 4(\sin^{-1} x)^2 = 0$$
$$\Rightarrow \; (1 - x^2)\, y_1^2 - 4y = 0$$

Again differentiating w.r.t. x, we get
$$(1 - x^2) \cdot 2y_1y_2 + y_1^2 \cdot (-2x) - 4y_1 = 0$$
$$\Rightarrow \; 2y_1\,[(1 - x^2)\, y_2 - xy_1 - 2] = 0$$

$$\Rightarrow (1-x^2)\,y_2 - xy_1 - 2 = 0.$$

This proves the first part.

For second part:

Applying Leibnitz's Rule for higher differentiation, we get

$$(1-x^2)\,y_{n+2} + {}^nC_1\,y_{n+1}(-2x) + {}^nC_2\,y_n(-2) - [xy_{n+1} + {}^nC_1\,y_n(1)] - 0 = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - 2nxy_{n+1} - \left(\frac{n(n-1)}{2}\right)2y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - [n^2 - n + n]\,y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} - n^2y_n = 0.$$

13. If $y = \sin(m\sin^{-1}x)$, show that

(i) $(1-x^2)\,y_2 - xy_1 + m^2y = 0$

(ii) $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2 - n^2)\,y_n = 0$

Proof: Let, $y = \sin(m\sin^{-1}x)$

Differentiating w.r.t. x, we get

$$y_1 = \cos(m\sin^{-1}x)\,(m)\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow (1-x^2)\,y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$= m^2\,[1 - \sin^2(m\sin^{-1}x)] = m^2[1-y^2]$$

$$\Rightarrow y_1^2\,(1-x^2) - m^2 + m^2y^2 = 0$$

Again differentiating w.r.t. x, we get

$$(1-x^2)\,2y_1y_2 + y_1^2\,(-2x) - 0 + m^2\,.\,2yy_1 = 0$$

$$\Rightarrow 2y_1\,[(1-x^2)\,y_2 - xy_1 + m^2y] = 0$$

$$\Rightarrow (1-x^2)\,y_2 - xy_1 + m^2y = 0 \qquad \text{.... (i)}$$

This proves the first part.

For second part:

Applying Leibnitz's Rule for higher differentiation, we get

$$(1-x^2)y_{n+2} + {}^nC_1y_{n+1}\,(-2x) + {}^nC_2\,y_n(-2) - [xy_{n+1} + {}^nC_1\,y_n] + m^2y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - 2nxy_{n+1} - \left(\frac{n(n-1)}{2}\right)2y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - [n(n-1) + n - m^2]\,y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2 - n^2)\,y_n = 0.$$

This completes the solution.

14. If $\log(y) = \tan^{-1}x$, show that **[2011 Spring]**

(i) $(1+x^2)\,y_2 + (2x-1)\,y_1 = 0$

(ii) $(1+x^2)\,y_{n+2} + (2nx + 2x - 1)\,y_{n+1} + n(n+1)\,y_n = 0$

Solution: We have, $\log(y) = \tan^{-1}x$

$$\Rightarrow y = e^{\tan^{-1}x} \qquad \text{.... (i)}$$

Differentiating w. r. t. x, then,

$$y_1 = e^{\tan^{-1}x}\left(\frac{1}{1+x^2}\right) = \frac{y}{1+x^2} \qquad \{\text{by (i)}\}$$

$$\Rightarrow y_1(1+x^2) - y = 0$$

Again differentiating w.r.t. x, gives

$$y_2(1+x^2) + y_1(2x) - y_1 = 0$$

$$\Rightarrow (1+x^2)\,y_2 + (2x-1)\,y_1 = 0 \qquad \text{.... (ii)}$$

This proves the first part.

For second part:

Applying Leibnitz's Rule for higher differentiation, we get

$$(1+x^2)\,y_{n+2} + {}^nC_1\,y_{n+1}(2x) + {}^nC_2\,y_n\,(2) + (2x-1)\,y_{n+1} + {}^nC_1\,y_n\,(2) = 0$$

$$\Rightarrow (1+x^2)\,y_{n+2} + 2nxy_{n+1} + \left(\frac{n(n-1)}{2}\right)2y_n + (2x-1)\,y_{n+1} + 2ny_n = 0$$

$$\Rightarrow (1+x^2)\,y_{n+2} + (2nx + 2x - 1)\,y_{n+1} + [n(n-1) + 2n]\,y_n = 0$$

$$\Rightarrow (1+x^2)\,y_{n+2} + (2nx + 2x - 1)\,y_{n+1} + n(n+1)\,y_n = 0.$$

NOTE: Here, log is noted for ln (i.e. natural log, log = ln = $\log_e$).

15. If $y = e^{a\tan^{-1}x}$ and show that

(i) $(1+x^2)\,y_2 + (2x-a)\,y_1 = 0.$

(ii) $(1+x^2)\,y_{n+2} + (2nx + 2x - a)\,y_{n+1} + n(n+1)\,y_n = 0.$

Solution: Let, $y = e^{a\tan^{-1}x}$

Diff. w.r.t. x then,

$$y_1 = e^{a\tan^{-1}x}\left(\frac{a}{1+x^2}\right)$$

$$\Rightarrow (1+x^2)\,y_1 = a\,e^{a\tan^{-1}x} = ay$$

Again diff. w.r.t. x then,

$$(1+x^2)\,y_2 + 2xy_1 = ay_1$$

$$\Rightarrow (1+x^2)\,y_2 + (2x-a)\,y_1 = 0$$

Applying Leibnitz's Rule for higher differentiation, we get

$$(1+x^2)\,y_{n+2} + {}^nC_1\,y_{n+1}(2x) + {}^nC_2y_n(2) + (2x-a)\,y_{n+1} + {}^nC_1y_n(2) = 0.$$

$$\Rightarrow (1+x^2)\,y_{n+2} + 2nx\,y_{n+1} + \left(\frac{n(n-1)}{2}\right)(2)y_n + (2x-a)\,y_{n+1} + 2ny_n = 0$$

$$\Rightarrow (1+x^2)\,y_{n+2} + (2xn + 2x - a)\,y_{n+1} + (n(n-1) + 2n)\,y_n = 0.$$

$$\Rightarrow (1+x^2)\,y_{n+2} = (2n.\,x + 2x - a)\,y_{n+1} + n(n+1)\,y_n = 0.$$

This completes the solution.

16. If $y = \tan^{-1}x$, show that **[2016 Spring][2014 Fall] [2008 Spring]**

(i) $(1+x^2)\,y_1 = 1.$

(ii) $(1+x^2)\,y_{n+1} + 2nxy_n + n(n-1)\,y_{n-1} = 0.$

Solution: Let $y = \tan^{-1}x$

Differentiating w. r. t. x then,

$$y_1 = \frac{1}{1+x^2} \Rightarrow (1+x^2)\,y_1 = 1.$$

Applying Leibnitz's rule for differentiation then,

$$(1+x^2)\,y_{n+1} + {}^nC_1\,y_n\,(2x) + {}^nC_2\,y_{n-1}\,(2) = 0$$

$$\Rightarrow (1+x^2)\,y_{n+1} + (n)(2x)\,y_n + \left(\frac{n(n-1)}{2}\right)(2)\,y_{n-1} = 0$$

$$\Rightarrow (1+x^2)\,y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} = 0.$$

This completes the solution.

17. If $y = \dfrac{ax^2+bx+c}{(1-x)}$, show that $(1-x)\,y_3 = 3y_2$.

Solution: Let, $y(1-x) = ax^2 + bx + c$ ... (i)

Differentiating w.r.t. x then,

$$(1-x)\,y_1 + y(-1) = 2ax + b$$

Again, differentiating w. r. t. x then,

$$(1-x)\,y_2 - y_1 - y_1 = 2a$$

$$\Rightarrow (1-x)\,y_2 - 2y_1 = 2a$$

Again, differentiating w. r. t. x then,

$$(1-x)\,y_3 - y_2 - 2y_2 = 0$$

$$\Rightarrow (1-x)\,y_3 = 3y_2.$$

This completes the solution.

18. If $y = e^{-x}\cos x$, show that $y_4 + 4y = 0$

Solution: Let $y = e^{-x}\cos x$

Diff. w. r. t. x then,

$$y_1 = -e^{-x}\sin x - e^{-x}\cos x$$

$$\Rightarrow y_1 = -e^{-x}\sin x - y.$$

Again differentiating w.r.t. x then,

$$\begin{aligned} y_2 &= -e^{-x}\cos x + e^{-x}\sin x - y_1 \\ &= -y + e^{-x}\sin x + e^{-x}\sin x + y \\ &= 2e^{-x}\sin x. \end{aligned}$$

Again differentiating,

$$\begin{aligned} y_3 &= -2e^{-x}\sin x + 2e^{-x}\cos x \\ &= -2e^{-x}\sin x + 2y. \end{aligned}$$

Again differentiating,

$$\begin{aligned} y_4 &= -2e^{-x}\cos x + 2e^{-x}\sin x + 2y_1 \\ &= -2y + 2e^{-x}\sin x - 2e^{-x}\sin x - 2y \\ &= -4y. \end{aligned}$$

$$\Rightarrow y_4 + 4y = 0.$$

19. Show that $\dfrac{d^n}{dx^n}\left[\dfrac{\log x}{x}\right] = (-1)^n \dfrac{n!}{x^{n+1}}\left(\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \dots \dfrac{1}{n}\right)$

Solution: Let $y = \frac{\log x}{x}$.

Differentiating w. r. t. x then,

$$y_1 = \frac{x\frac{1}{x} - \log x}{x^2} = \frac{(-1).(\log x - 1)}{x^2}$$

Again differentiating w. r. t. x then,

$$y_2 = (-1)\left(\frac{x^2.\frac{1}{x} - 2x(\log x - 1)}{x^3}\right) = \frac{(-1)^2\,2!}{x^3}\left(\log x - 1 - \frac{1}{2}\right)$$

Again differentiating w. r. t. x then,

$$y_3 = (-1)^2\,2!\left[\frac{x^3.\frac{1}{x} - 3x^2\left(\log x - 1 - \frac{1}{2}\right)}{x^6}\right]$$

$$= \frac{(-1)^3.3!}{x^4}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3}\right).$$

Continuing the process upto n^{th} term then,

$$y_n = \frac{(-1)^n\,n!}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right).$$

1. Verify Rolle's Theorem for,

(i) $f(x) = x^2 - 5x + 10$ for $x \in [2, 3]$

Solution: Given function is,

$$f(x) = x^2 - 5x + 10 \text{ for } x \in [2, 3]$$

Clearly f(x) is a polynomial function, so it is continuous on [2, 3].

And, $f'(x) = 2x - 5$

which is again a polynomial. So, f '(x) is continuous on (2, 3).

Therefore, f(x) is differentiable on (2, 3).

Also,

$$f(2) = 2^2 - 5(2) + 10 = 4 - 10 + 10 = 4$$

$$f(3) = 3^2 - 5(3) + 10 = 9 - 15 + 10 = 4$$

Here, f(2) = f(3).

Thus, f(x) satisfies all three conditions of Rolle's theorem, So, by this theorem there is at least one point c in (2, 3) such that,

$$f'(c) = 0 \Rightarrow 2c - 5 = 0 \Rightarrow c = \frac{5}{2} \in (2, 3).$$

Therefore c prescribed by the Rolle's theorem is $c = \frac{5}{2} \in (2, 3)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

(ii) $f(x) = \dfrac{\sin x}{e^x}$ **for** $x \in [0, \pi]$ **[2003, Fall, Short]**

Solution: Given function is,

$$f(x) = \frac{\sin x}{e^x} \text{ for } x \in [0, \pi].$$

Clearly, $\sin x$ and e^x both are continuous and differential on $\mathfrak{R}$. We know the quotient form of continuous functions is continuous provided the denominator is non-zero. Here

$$e^x \neq 0 \text{ for any x in } [0, \pi].$$

So, f(x) is continuous on $[0, \pi]$.

And, $f'(x) = \dfrac{e^x [\cos x - \sin x]}{e^{2x}}$.

Since $e^{2x} \neq 0$ for any x in $(0, \pi)$.

So, f '(x) is continuous on $(0, \pi)$. This means f(x) is differentiable on $(0, \pi)$.

Also, $f(0) = \dfrac{\sin 0}{e^0} = \dfrac{0}{1} = 0$ and $f(\pi) = \dfrac{\sin \pi}{e^\pi} = \dfrac{0}{e^\pi} = 0$.

Here, $f(0) = f(\pi)$.

Thus, f(x) satisfies all three conditions of Rolle's theorem, So, by this theorem there is at least one point c in $(0, \pi)$ such that,

$$f'(c) = 0 \Rightarrow \frac{e^c \cos c - e^c \sin c}{e^{2c}} = 0$$

$$\Rightarrow \cos c = \sin c$$

$$\Rightarrow \tan c = 1 = \tan\left(\frac{\pi}{4}\right) \Rightarrow c = \frac{\pi}{4} \in (0, \pi).$$

Therefore c prescribed by the Rolle's theorem is $c = \dfrac{\pi}{4} \in (0, \pi)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

(iii) $f(x) = x(x + 3)\, e^{-x/2}$ **for** $x \in [-3, 0]$

Solution: Given function is,

$$f(x) = x(x + 3)\, e^{-x/2} \text{ for } x \in [-3, 0]$$

Clearly the linear functions x and (x + 3) are continuous and differentiable on $\mathfrak{R}$ and also the exponential function $e^{-x/2}$ is also continuous and differentiable on $\mathfrak{R}$. And, we know the product of continuous functions is again a continuous function and the product of differentiable functions is again a differentiable function. Since f(x) is the product of polynomial and exponential function. So, f(x) is continuous and differentiable on [–3, 0].

Also, $f(-3) = -3(-3 + 3)\, e^{-3/2} = 0$ and $f(0) = 0\,(0 + 3)\, e^0 = 0$.

Here, $f(-3) = f(0)$.

Thus, f(x) satisfies all the conditions of Rolle's theorem. Then by the theorem there is at least one point $c \in (-3, 0)$ such that

$$f'(c) = 0 \Rightarrow (2c+3)e^{-c/2} + (c^2+3c)e^{-c/2}.\left(-\frac{1}{2}\right) = 0$$

$$\Rightarrow 4c + 6 - c^2 - 3c = 0 \qquad \left[\because \frac{e^{-c/2}}{2} \neq 0\right]$$

$$\Rightarrow c^2 - c - 6 = 0$$

$$\Rightarrow (c-3)(c+2) = 0$$

$$\Rightarrow c = -2 \in (-3, 0) \text{ but } c = 3 \notin (-3, 0)$$

Therefore c prescribed by the Rolle's theorem is $c = -2 \in (-3, 0)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

(iv) $f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right)$ for $x \in [a, b]$ for $a > 0$.

Solution: Given function is,

$$f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right) \text{ for } x \in [a, b] \text{ with } a > 0.$$

Clearly, $\left(\frac{x^2+ab}{(a+b)x}\right) > 0$ for $a > 0$.

Here, $a > 0$ and logarithm function is defined only for positive value. So, f(x) is continuous on [a, b].

And,
$$f'(x) = \left(\frac{(a+b)x}{x^2+ab}\right).\frac{(ax+bx)2x - (x^2+ab)(a+b)}{(a+b)^2x^2}$$

$$= \frac{x.(2x^2 - x^2 - ab)}{x^2(x^2+ab)}$$

$$= \frac{x^2 - ab}{x(x^2+ab)}$$

which is defined for $a > 0$ because $x(x^2 + ab) > 0$ for any x in (a, b).

This means f '(x) is continuous on (a, b) for $a > 0$.

That is f(x) is differentiable on (a, b).

Also,

$$f(a) = \log\left(\frac{x^2+ab}{(a+b)x}\right) = \log(1) = 0$$

and
$$f(b) = \log\left(\frac{b^2+ab}{(a+b)b}\right) = \log(1) = 0$$

Here, $f(a) = f(b)$.

Thus f(x) satisfies all three conditions of Rolle's theorem. Then by the theorem, there is at least one point $c \in (a, b)$ such that

$$f'(c) = 0 \Rightarrow \frac{c^2 - ab}{c(c^2+ab)} = 0$$

$$\Rightarrow c^2 - ab = 0 \Rightarrow c = \pm\sqrt{ab} \in (a, b).$$

Therefore c prescribed by the Rolle's theorem is $c = \sqrt{ab} \in (a, b)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

Note: For this problem we should have $a > 0$. Otherwise, the problem may not verify the theorem. For, if $a = -1$, $b = 1$ then f(x) is undefined.

(v) $f(x) = (x-a)^m (x-b)^n$ where m and n being positive integers in [a, b].

Solution: Given function is,

$$f(x) = (x-a)^m (x-b)^n \qquad \text{for } x \in [a, b] \text{ and } m, n > 0.$$

Clearly $(x-a)^m (x-b)^n$ is a polynomial function. So, it is continuous on [a, b].

And,
$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$$

which is a polynomial function.

So, f '(x) is continuous on (a, b). That means f(x) is differentiable on (a, b).

Also,
$$f(a) = (a-a)^m(a-b)^n = 0 \quad \text{and} \quad f(b) = (b-a)^m(b-b)^n = 0.$$

Here, $f(a) = f(b) = 0$.

Thus, f(x) satisfies all three conditions of Rolle's theorem. Then by the theorem, there is at least one point $c \in (a, b)$ such that

$f'(c) = 0$.

$$\Rightarrow f'(c) = m(c-a)^{m-1}(c-b)^n + n(c-a)^m(c-b)^{n-1} = 0$$

$$\Rightarrow m(c-b) + n(c-a) = 0 \qquad [\because (c-a)^{m-1}(c-b)^{n-1} \neq 0]$$

$$\Rightarrow c(m+n) - (mb+na) = 0 \Rightarrow c = \frac{mb+na}{m+n}.$$

This means c is the internal division point of the line joining a and b in the ratio m:n. So, $c \in (a, b)$.

Therefore c prescribed by the Rolle's theorem is $c = \frac{mb+na}{m+n} \in (a, b)$.

Thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's theorem.

(vi) $f(x) = e^x(\sin x - \cos x)$ in $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Solution: Given function is,

$$f(x) = e^x(\sin x - \cos x) \qquad \text{for } x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right].$$

Clearly sine, cosine and e^x are continuous on $\Re$ and are differentiable on $\Re$. Since the product of continuous functions and differentiable functions are again continuous and differentiable. So, f(x) is also continuous on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and is differentiable on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

Also,

$$f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left[\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right] = e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0.$$

and,

$$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0.$$

Here, $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 0.$

Thus, f(x) satisfies all three conditions of Rolle's Theorem. Then by this theorem, there is at least one point $c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that

$$f'(c) = 0.$$

Since, $f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x),$

So,

$$f'(c) = e^c[\sin c - \cos c + \cos c + \sin c] = 0.$$

$$\Rightarrow 2\sin c = 0 \qquad [\because e^c \neq 0]$$

$$\Rightarrow \sin c = 0 = \sin\pi$$

$$\Rightarrow c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right).$$

Therefore c prescribed by the Rolle's theorem is $c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$, thus, f(x) satisfies all conditions of Rolle's theorem. So, f(x) verifies the Rolle's Theorem.

2. Verify Lagrange's Mean Value Theorem for,

(i) $f(x) = x^2$ in $x \in [1, 2]$

Solution: Given function is,

$$f(x) = x^2 \quad \text{for } x \in [1, 2]$$

Clearly f(x) is a polynomial function which is continuous on [1, 2].

And,

$$f'(x) = 2x, \text{ which is polynomial function.}$$

So, f '(x) is continuous on (1, 2). Therefore, f(x) is differentiable on (1, 2).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem. Then by this theorem there is at least one point $c \in (1, 2)$ such that,

$$f'(c) = \frac{f(2) - f(1)}{2 - 1} \Rightarrow 2c = \frac{4 - 1}{2 - 1} \Rightarrow 2c = 3 \Rightarrow c = \frac{3}{2} \in (1, 2).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = \frac{3}{2} \in (1, 2)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(ii) $f(x) = x(x - 1)(x - 2)$ in $x \in [0, ½]$

Solution: Given function is,

$$f(x) = x(x - 1)(x - 2) \quad \text{for } x \in [0, ½]$$

$$\Rightarrow f(x) = x^3 - 3x^2 + 2x \quad \text{for } x \in [0, ½]$$

Clearly, $f(x) = x^3 - 3x^2 + 2x$ is a polynomial function, is continuous on [0, ½].

And, $f'(x) = 3x^2 - 6x + 2$, which is polynomial. So, f '(x) is continuous.

So, f(x) is differentiable on (0, ½).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem. Then by the theorem there is at least one point $c \in (0, ½)$ such that

$$f'(c) = \frac{f(1/2) - f(0)}{(1/2) - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\frac{1}{2}} = \frac{3}{4} \qquad [\text{since, } f(0) = 0.]$$

$$\Rightarrow 12c^2 - 24c + 8 = 3$$

$$\Rightarrow 12c^2 - 24c + 5 = 0.$$

$$\Rightarrow c = \frac{+24 \pm \sqrt{24^2 - 240}}{24} = 1 \pm \frac{\sqrt{336}}{24} = 1 \pm 0.76$$

$$\Rightarrow c = 0.24 \in (0, ½).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = 0.24 \in (0, ½)$. Thus, f(x) satisfies all the conditions of Lagrange's Mean Value Theorem, so f(x) verifies the theorem.

(iii) $f(x) = Ax^2 + Bx + C$ for $x \in [a, b]$

Solution: Given function is,

$$f(x) = Ax^2 + Bx + C \quad \text{for } x \in [a, b].$$

Clearly $f(x) = Ax^2 + Bx + C$ is a polynomial, is continuous on [a, b].

And, $f'(x) = 2Ax + B$, which is linear polynomial function. This means f(x) is continuous. Therefore, f(x) is differentiable on (a, b).

Thus f(x) satisfies both conditions of Lagrange's Mean Value Theorem then by the theorem there is at least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 2Ac + B = \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b - a}$$

$$= \frac{A(b^2 - a^2) + B(b - a)}{b - a} = A(b + a) + B$$

$$\Rightarrow 2Ac = A(b + a)$$

$$\Rightarrow c = \frac{b + a}{2} \in (a, b).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = \frac{b + a}{2} \in (a, b)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(iv) $f(x) = e^x$ for $x \in [0, 1]$

Solution: Given function is,

$$f(x) = e^x \quad \text{for } x \in [0, 1]$$

Clearly f(x) is an exponential function, is continuous on [0, 1].

And, $f'(x) = e^x$ which is continuous. So, f(x) is differentiable on (0, 1). Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem then by the theorem there is at least one point $c \in (0, 1)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow e^c = \frac{e^b - e^a}{1 - 0} = e^1 - 1 = 2.72 - 1 = 1.72.$$

$$\Rightarrow c = \log(1.72) = 0.54 \in (0, 1).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = 0.54 \in (0, 1)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(v) $f(x) = \sqrt{x - 1}$ for [1, 3].

Solution: Given function is,

$$f(x) = \sqrt{x - 1} \text{ for } [1, 3].$$

Since, the root function is defined only for non–negative value. So, f(x) is defined for $x \geq 1$. Thus, f(x) is defined and is continuous on [1, 3].

And, $$f'(x) = \frac{1}{2\sqrt{x - 1}}.$$

Clearly, $2\sqrt{x - 1} > 0$ for $x > 1$. So, f '(x) is continuous on (1, 3). Therefore, f(x) is differentiable on (1, 3).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem then there is at least one point $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

i.e. $$\frac{1}{2\sqrt{c - 1}} = \frac{\sqrt{2} - 0}{3 - 1} = \frac{\sqrt{2}}{2}.$$

$$\Rightarrow \sqrt{c - 1} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow c - 1 = \frac{1}{2} = 0.5.$$

$$\Rightarrow c = 1.5 \in (1, 3).$$

Therefore c prescribed by the Lagrange's Mean Value Theorem is $c = 1.5 \in (1, 3)$. Thus, f(x) verifies the Lagrange's Mean Value Theorem.

(vi) $f(x) = x + \frac{1}{x}$ for $x \in \left[\frac{-1}{2}, 2\right]$

Solution: Given function is,

$$f(x) = x + \frac{1}{x} \text{ for } x \in \left[\frac{-1}{2}, 2\right].$$

Clearly $0 \in \left(-\frac{1}{2}, 2\right)$ and $f(0) = 0 + \frac{1}{0}$ which is undefined.

So, f(x) is not continuous at x = 0. Therefore, f(x) does not verify the Lagrange's Mean Value Theorem.

Note: The function $f(x) = x + \frac{1}{x}$ does not satisfy the condition(s) of Rolle's theorem and Lagrange's Mean Value Theorem in any interval that includes 0.

Reason: At $x = 0$, we get $f(0) = 0 + \frac{1}{0}$ which is undefined. So, $f(x)$ is not continuous at $x = 0$.

3. Show that $|\sin b - \sin a| \leq |b - a|$, by using Lagrange's Mean Value Theorem.

Solution: Here we have to show $|\sin b - \sin a| \leq |b - a|$.

Clearly sin*b* and sin*a* is a sine function having angle *b* and *a*, respectively. So, assume that,

$$f(b) = \sin b \quad \text{and} \quad f(a) = \sin a$$

Then,

$$f(x) = \sin x.$$

So, $$f'(x) = \cos x.$$

Clearly $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) for $a, b \in \Re$. Thus, $f(x)$ satisfies both conditions of Lagrange's Mean Value Theorem. Then by the theorem there is at least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \cos c = \frac{\sin b - \sin a}{b - a}.$$

Since, the cosine function has oscillatory value in between –1 to 1.

So, $-1 \le \cos c \le 1$, for any c in R.

$\Rightarrow |\cos c| \le 1$, for any c in (a, b).

So,

$$\left|\frac{\sin b - \sin a}{b - a}\right| = |\cos c| \le 1.$$

$$\Rightarrow \frac{|\sin b - \sin a|}{|b - a|} \le 1$$

$$\Rightarrow |\sin b - \sin a| \le |b - a|.$$

4. If $f(x) = \frac{1}{x}$, show that mean value theorem does not exist in [0, 1].

Solution: Let $f(x) = \frac{1}{x}$. At $x = 0$, we see $f(0) = \frac{1}{0}$ which is undefined. So, f(x) is not continuous on [0, 1]. Therefore, f(x) does not satisfy the condition of the mean value theorem on [0, 1].

Note: The function f(x) also does not satisfy the condition the Rolle's Theorem.

5. Does the function $f(x) = x$ in [0, 1] satisfy the hypothesis of Lagrange's Mean Value Theorem? If not, why not? If so, what value or values could c have?

Solution: Let $f(x) = x$ for $x \in [0, 1]$

Clearly f(x) is a polynomial function. So, f(x) is continuous on [0, 1].

And, $f'(x) = 1$ which is a constant function, is continuous on (0, 1).

So, f(x) is differentiable on (0, 1).

Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem then by the theorem there is at least one point $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

i.e. $1 = \frac{1 - 0}{1 - 0} \Rightarrow 1 = 1$ which is always true.

That means for any value of c in (0, 1), f(x) verifies the Lagrange's mean value theorem.

6. If $f(x) = \tan x$ then $f(0) = 0$ and $f(\pi) = 0$. Is Rolle's theorem applicable to $f(x)$ in $[0, \pi]$? [2008, Spring] [2009 Spring]

Solution: Let $f(x) = \tan x$ for $x \in [0, \pi]$.

Since $\frac{\pi}{2} \in [0, \pi]$, $f\left(\frac{\pi}{2}\right) = \tan\left(\frac{\pi}{2}\right) = \infty$.

So, f(x) is not continuous at $\frac{\pi}{2}$. Therefore, f(x) does not verify the Rolle's theorem.

Note: The function f(x) also does not verify the LMVT.

7. Show that $f(x) = x^2$ and $g(x) = 3x - 2$ in $x \in [1, 2]$ verify Cauchy's mean value theorem.

Solution: Given functions are,

$$f(x) = x^2 \quad \text{and} \quad g(x) = 3x - 2 \quad \text{for } x \in [1, 2].$$

Clearly f(x) and g(x) are polynomial functions. Since every polynomial function is continuous and differentiable, so f(x) and g(x) are continuous on [1, 2] and differentiable on (1, 2).

Also, $f'(x) = 2x$ and $g'(x) = 3 \neq 0$ for any x in (1, 2).

Thus, they satisfies all conditions of Cauchy's Mean Value Theorem then by the theorem there is at least one point c in (1, 2) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)} \quad \dots (i)$$

$$\Rightarrow \frac{2c}{3} = \frac{4 - 1}{4 - 1} = \frac{3}{3} = 1.$$

$$\Rightarrow c = \frac{3}{2} = 1.5 \in (1, 2).$$

This means the functions f(x) and g(x) verify the Cauchy's mean value theorem.

8. Show that $f(x) = x^2 - 4x$ is increasing in $(2, \infty)$ and decreasing in $(-\infty, 2)$.

Solution: Clearly $f(x) = x^2 - 4x$ is a polynomial function which is continuous on R and is differentiable on R.

By Lagrange's Mean Value Theorem we know that if f(x) is continuous on [a, b] and is differentiable on an interval (a, b) then there is at least one point c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

If a function is increasing then we should have $f(b) > f(a)$ for $b > a$. Therefore,

$$f'(c) = \frac{f(b) - f(a)}{b - a} > 0.$$

And similarly if f(x) is decreasing then we have

$$f'(c) < 0.$$

Here given function is,

$$f(x) = x^2 - 4x.$$

Then, $f'(x) = 2x - 4$.

So, for any $x \in (2, \infty)$, we observe $f'(x) > 0$. Therefore, f(x) is increasing on $(2, \infty)$.

Also, for any $x \in (-\infty, 2)$, we observe $f'(x) < 0$. Therefore f(x) is decreasing on $(-\infty, 2)$.

9. Show that $f(x) = x$ and $g(x) = x^2 - 2x$ in $[0, 2]$ does not verify the Cauchy's mean value theorem.

Solution: Here, $f(x) = x$ and $g(x) = x^2 - 2x$ for $x \in [0, 2]$.

And, $f'(x) = 1$ and $g'(x) = 2x - 2$.

Since, $1 \in (0, 2)$ and $g'(1) = 2 - 2 = 0$ that contradicts to the condition $g'(c) \neq 0$ for some $c \in (0, 2)$. Hence, the functions do not verify the Cauchy's mean value theorem.

10. The function $f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 0 & \text{for } x = 1 \end{cases}$, is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$ is never zero. How can this be? Does not Rolle's theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

Solution: Given that, $f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 0 & \text{for } x = 1 \end{cases}$

Clearly $f(x) = x$ and $f(x) = 0$ both are polynomial functions, are continuous on their respective domain. Here f(x) has piecewise form, so we need to check the continuity of f(x) at x = 1.

Here,

$$\text{LHL} = \lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x) = 1.$$

and $f(1) = 0$.

This shows f(x) is not continuous at x = 1. So, the given function f(x) is not continuous on [0, 1]. However, $f(x) = x$ is differentiable on (0, 1) being a polynomial function and $f'(x) = 1$ for $x \in (0, 1)$.

That means $f'(x) \neq 0$ for all $x \in (0, 1)$.

That is, f(x) is not continuous on [0, 1] however it is differentiable on (0, 1) and $f(0) = f(1)$. Being the lack of continuity of the function on closed interval, the Rolle's theorem does not work here.

11. For what value of a, m and b does the function,

$$f(x) = \begin{cases} 3 & \text{for } x = 0 \\ -x^2 + 3x + a & \text{for } 0 < x < 1 \\ mx + b & \text{for } 1 \le x \le 2 \end{cases}$$

Satisfy the hypothesis of the mean value theorem on the interval [0, 2]?

Solution: Given that,

$$f(x) = \begin{cases} 3 & \text{for } x = 0 \\ -x^2 + 3x + a & \text{for } 0 < x < 1 \\ mx + b & \text{for } 1 \le x \le 2 \end{cases}$$

satisfies the hypothesis (condition) of mean value theorem.

Therefore, f(x) is continuous on [0, 2] and is differentiable on (0, 2).

Since, f(x) is continuous at x = 0. So,

$$RHL = f(0) \Rightarrow \lim_{x\to0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to0^+} (-x^2 + 3x + a) = 3$$

$$\Rightarrow a = 3$$

And, f(x) is continuous at x = 1. So,

$$\text{LHL} = f(1) \Rightarrow \lim_{x\to1^-} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to1^-} (-x^2 + 3x + a) = m + b$$

$$\Rightarrow -1 + 3 + 3 = m + b \qquad [\because a = 3]$$

$$\Rightarrow m + b = 5 \qquad \dots \text{(i)}$$

Next, f(x) is differential on (0, 2). So, f(x) is differential at x = 1. That means

LHD = RHD at x = 1.

Here, $Lf'(1) = -2(1) + 3 = 1$

and, $Rf'(1) = m$

Since, $Lf'(1) = Rf'(1) \Rightarrow 1 = m$.

Then from (i), $b = 4$.

Thus, a = 3, b = 4 and m = 1.

12. Explain why Rolle's theorem is not applicable to the function $f(x) = 1 - (x - 1)^{2/3}$ in $0 \le x \le 2$.

Solution: Here, $f(x) = 1 - (x - 1)^{2/3} = 1 - \sqrt[3]{(x - 1)^2}$ for $x \in [0, 2]$.

Clearly $(x - 1)^2$ is always positive. So, the cube root has a real positive value on [0, 2]. So, f(x) is continuous in [0, 2], being a linear form of a constant and defined root functions.

Also,

$$f'(x) = \frac{-2}{3(x-1)^{1/3}}$$

Here,

$$f'(1) = \frac{1}{0} \text{ which is undefined.}$$

Since $1 \in (0, 2)$. Therefore f(x) is not differentiable at x = 1 that lies on (0, 2). Therefore, f(x) does not applicable to verify the Rolle's Theorem.

Note: **The function f(x) also does not** applicable to **verify the LMVT.**

13. Show the $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$ (for $a > 0$) by using Lagrange's Mean Value Theorem.

Solution: Here,

$$\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a} \qquad \text{for } a > 0$$

$$\Rightarrow \frac{1}{b} < \frac{\log(b) - \log(a)}{b-a} < \frac{1}{a} \qquad \ldots\ldots\ (i)$$

Comparing the term $\frac{\log(b) - \log(a)}{b-a}$ with $\frac{f(b) - f(a)}{b-a}$ then we get,

$$f(b) = \log(b), \quad f(a) = \log(a)$$

So, $f(x) = \log(x)$ for $x \in [a, b]$.

Clearly $f(x)$ is continuous on $[a, b]$ with $a > 0$.

Also, $f'(x) = \frac{1}{x}$ that is defined for non–zero value of x.

So, $f(x)$ is differentiable on (a, b) with $a > 0$.

Thus, $f(x)$ satisfies both condition of LMVT. So, by this theorem there at least one point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{1}{c} = \frac{\log(b) - \log(a)}{b-a} \qquad \ldots\ldots\ (ii)$$

Since $c \in (a, b)$. So, $a < c$ and $c < b$. Then,

$$\frac{1}{a} > \frac{1}{c} \text{ and } \frac{1}{c} > \frac{1}{b}$$

Therefore, (ii) becomes,

$$\frac{1}{b} < \frac{\log(b) - \log(a)}{b-a} < \frac{1}{a}$$

This is same to (i). So, the condition fulfills.

Exercise 4.1

1. Show that the following limits:

(i) $\lim\limits_{x\to 2}\left(\dfrac{x^3-2x^2+2x-4}{x^2-5x+6}\right) = -6$

Solution: Here,

$$\lim_{x\to 2}\left(\frac{x^3-2x^2+2x-4}{x^2-5x+6}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 2}\left(\frac{3x^2-4x+2}{2x-5}\right) = \frac{3.4-4.2+2}{2\times 2-5} = -6.$$

Thus, $\lim\limits_{x\to 2}\left(\dfrac{x^3-2x^2+2x-4}{x^2-5x+6}\right) = -6.$

(ii) $\lim\limits_{x\to 0}\left(\dfrac{\tan x - x}{x - \sin x}\right) = 2$ **[Short, 2004, Spring]**

Solution: Here,

$$\lim_{x\to 0}\left(\frac{\tan x - x}{x-\sin x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{\sec^2 x - 1}{1-\cos x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{2\sec^2 x \tan x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sec^2 x \sin x}{\cos x \sin x}\right)$$

$$= \lim_{x\to 0}(2\sec^3 x)$$

$$= 2 \qquad [\because \sec 0 = 1]$$

Thus, $\lim\limits_{x\to 0}\left(\dfrac{\tan x - x}{x-\sin x}\right) = 2.$

(iii) $\lim\limits_{x\to a}\left(\dfrac{x^n - a^n}{x-a}\right) = na^{n-1}$, **for n is positive integer.**

Solution: Here,

$$\lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to a}\left(\frac{nx^{n-1} - 0}{1}\right) = na^{n-1}.$$

Thus, $\lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}$.

(iv) $\lim_{x\to 0}\left(\frac{x - \sin^{-1}x}{\sin^3 x}\right) = -\frac{1}{6}$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x - \sin^{-1}x}{\sin^3 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x - \sin^{-1}x}{x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{1 - \frac{1}{\sqrt{1 - x^2}}}{3x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\frac{1}{2}(1 - x^2)^{-3/2} - 2x}{6x}\right) = \lim_{x\to 0}\left(\frac{-1}{6}(1 - x^2)^{-3/2}\right) = -\frac{1}{6}.$$

Thus, $\lim_{x\to 0}\left(\frac{x - \sin^{-1}x}{\sin^{3x}}\right) = -\frac{1}{6}$.

(v) $\lim_{x\to 0}\left(\frac{2\sin x - \sin 2x}{\tan^3 x}\right) = 1.$

Solution : Here,

$$\lim_{x\to 0}\left(\frac{2\sin x - \sin 2x}{\tan^3 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin x - \sin 2x}{x^3}\right) \qquad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin x - 2\sin x\cos x}{x^3}\right)$$

$$= \lim_{x\to 0}\left[2\left(\frac{1 - \cos x}{x^2}\right)\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin x}{2x}\right) = \lim_{x\to 0}(1) = 1. \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

Thus, $\lim_{x\to 0}\left(\frac{2\sin x - \sin 2x}{\tan^3 x}\right) = 1.$

(vi) $\lim_{x\to 1}\left(\frac{1 + \log x - x}{1 - 2x + x^2}\right) = \frac{-1}{2}.$

Solution: Here,

$$\lim_{x\to 1}\left(\frac{1 + \log x - x}{1 - 2x + x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1}\left(\frac{(1/x) - 1}{-2 + 2x}\right)$$

$$= \lim_{x\to 1}\left(\frac{1 - x}{-2x + 2x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1}\left(\frac{-1}{-2 + 4x}\right) = \frac{-1}{2}.$$

Thus, $\lim_{x\to 1}\left(\frac{1 + \log x - x}{1 - 2x + x^2}\right) = \frac{-1}{2}.$

(vii) $\lim_{x\to 0}\left(\frac{x e^x - \log(1 + x)}{x^2}\right) = \frac{3}{2}.$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x e^x - \log(1 + x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + x e^x - \frac{1}{1 + x}}{2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{(e^x + x e^x)(1 + x) - 1}{2x(1 + x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^x + 2x e^x + x^2 e^x - 1}{2x + 2x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + 2e^x + 2xe^x + 2xe^x + x^2e^x}{2 + 4x}\right) = \frac{3}{2}.$$

Thus, $\lim_{x\to 0}\left(\frac{x e^x - \log(1 + x)}{x^2}\right) = \frac{3}{2}.$

(vii) $\lim_{x\to 0}\left(\frac{\cosh x - \cos x}{x\sin x}\right) = 1.$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{\cosh x - \cos x}{x\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\cosh x - \cos x}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \qquad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{\sinh x + \sin x}{2x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{\cosh x + \cos x}{2}\right) = \frac{\cosh 0 + \cos 0}{2} = \frac{2}{2} = 1.$$

Thus, $\lim_{x\to 0}\left(\frac{\cosh x - \cos x}{x\sin x}\right) = 1.$

(ix) $\lim_{t\to 0}\left(\frac{\sin t^2}{t}\right) = 0.$

Solution: Here,

$$\lim_{t\to 0}\left(\frac{\sin t^2}{t}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{t\to 0}\left(\frac{\cos t^2 . 2t}{1}\right) = 0.$$

Thus, $\lim_{t\to 0}\left(\frac{\sin t^2}{t}\right) = 0.$

(x) $\lim_{x\to 0}\left(\frac{\sin 5x}{x}\right) = 5.$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{\sin 5x}{x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(5\,\frac{\sin 5x}{5x}\right) = \lim_{x\to 0} 5 \quad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= 5.$$

Thus, $\lim_{x\to 0}\left(\frac{\sin 5x}{x}\right) = 5.$

(xi) $\lim_{\theta\to \pi}\left(\frac{\sin \theta}{\pi - \theta}\right) = 1.$

Solution : Here,

$$\lim_{\theta\to \pi}\left(\frac{\sin \theta}{\pi - \theta}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{\theta\to \pi}\left(\frac{\cos \theta}{-1}\right) = \frac{\cos \pi}{-1} = \frac{-1}{-1} = 1.$$

Thus, $\lim_{\theta\to \pi}\left(\frac{\sin \theta}{\pi - \theta}\right) = 1.$

(xii) $\lim_{x\to \frac{\pi}{4}}\left(\frac{\sin x - \cos x}{x - \frac{\pi}{4}}\right) = \sqrt{2}.$ **[2007, Fall (Short)]**

Solution : Here,

$$\lim_{x\to \frac{\pi}{4}}\left(\frac{\sin x - \cos x}{x - \frac{\pi}{4}}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to \frac{\pi}{4}}\left(\frac{\cos x + \sin x}{1}\right) = \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Thus, $\lim_{x\to \frac{\pi}{4}}\left(\frac{\sin x - \cos x}{x - \frac{\pi}{4}}\right) = \sqrt{2}.$

2. Prove the following:

(i) $\lim_{x\to 0^+}\left(\frac{\log (\sin x)}{\cot x}\right) = 0.$

Solution : Here,

$$\lim_{x\to 0^+}\left(\frac{\log (\sin x)}{\cot x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{\cos x}{\sin x}}{(-\operatorname{cosec}^2 x)}\right) = \lim_{x\to 0^+} (-\cos x \sin x) = 0.$$

Thus, $\lim_{x\to 0^+}\left(\frac{\log (\sin x)}{\cot x}\right) = 0.$

(ii) $\lim_{x\to 0^+} [x \log (x)] = 0$

Solution: Here,

$$\lim_{x\to 0^+} [x \log (x)] \qquad [\text{in } 0 \times \infty \text{ form}]$$

$$= \lim_{x\to 0^+}\left(\frac{\log x}{1/x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right) = \lim_{x\to 0^+} (-x) = 0.$$

Thus, $\lim_{x\to 0^+} [x \log (x)] = 0.$

(iii) $\lim_{x\to 0^+} [\log_{\tan x} \tan 2x] = 1.$

Solution: Here,

$$\lim_{x\to 0^+} [\log_{\tan x} \tan 2x]$$

$$= \lim_{x\to 0^+}\left(\frac{\log \tan 2x}{\log \tan x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{2\sec^2 2x}{\tan 2x}}{\frac{\sec^2 x}{\tan x}}\right)$$

Let $\log_x(y) = u$.
$\Rightarrow\ y = x^u$.
So, $\log(y) = u \log(x)$.
$\Rightarrow u = \frac{\log(y)}{\log(x)}$.
Thus,
$\log_x(y) = \frac{\log(y)}{\log(x)}$.

$$= \lim_{x\to 0^+}\left(\frac{\dfrac{2}{\cos^2 2x}\times\dfrac{\cos 2x}{\sin 2x}}{\dfrac{1}{\cos^2 x}\times\dfrac{\cos x}{\sin x}}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{2\cos x \sin x}{\cos 2x \sin 2x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{1}{\cos 2x}\right) = \frac{1}{\cos 0} = 1.$$

Thus, $\lim\limits_{x\to 0^+} [\log_{\tan x} \tan 2x] = 1.$

(iv) $\lim\limits_{x\to a}\left[(a-x)\,.\,\tan\left(\dfrac{\pi x}{2a}\right)\right] = \dfrac{2a}{\pi}.$

Solution: Here,

$\lim\limits_{x\to a}\left[(a-x)\,.\,\tan\left(\dfrac{\pi x}{2a}\right)\right]$ (in $0\times\infty$ form)

$$= \lim_{x\to a}\left(\frac{(a-x)}{\cot\left(\dfrac{\pi x}{2a}\right)}\right) \quad \left(\text{in } \frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to a}\left(\frac{-1}{\left(\dfrac{-\pi}{2a}\right).\operatorname{cosec}^2\left(\dfrac{\pi x}{2a}\right)}\right) = (-1).\left(\frac{-2a}{\pi}\right) = \frac{2a}{\pi}.$$

Thus, $\lim\limits_{x\to a}\left[(a-x)\,.\,\tan\left(\dfrac{\pi x}{2a}\right)\right] = \dfrac{2a}{\pi}.$

(v) $\lim\limits_{x\to\infty}\left(\dfrac{x^n}{e^x}\right) = 0.$

Solution: Here,

$$\lim_{x\to\infty}\left(\frac{x^n}{e^x}\right) \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{n.x^{n-1}}{e^x}\right) \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{n(n-1)\,x^{n-2}}{e^x}\right) \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

Continuing the process up to n^{th} times then,

$$= \lim_{x\to\infty}\left(\frac{n(n-1)\ldots.2.1\,x^0}{e^x}\right) = \frac{n!}{e^\infty} = 0. \ [\text{because } \infty \approx \frac{1}{0}]$$

Thus, $\lim\limits_{x\to\infty}\left(\dfrac{x^n}{e^x}\right) = 0.$

(vi) $\lim\limits_{x\to 0^+}\left(\dfrac{\log(x^2)}{\log(\cot^2 x)}\right) = -1.$

Solution: Here,

$$\lim_{x\to 0^+}\left(\frac{\log(x^2)}{\log(\cot^2 x)}\right) \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\dfrac{2x}{x^2}}{\dfrac{1}{\cot^2 x}\,.\,2\cot x\,(-\operatorname{cosec}^2 x)}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cot x}{-x\operatorname{cosec}^2 x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cos x \sin^2 x}{-x \sin x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-\sin 2x}{2x}\right) = -1 \quad \left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

Thus, $\lim\limits_{x\to 0^+}\left(\dfrac{\log(x^2)}{\log(\cot^2 x)}\right) = -1.$

(vii) $\lim\limits_{x\to 0^+}[x\log(\sin^2 x)] = 0.$

Solution: Here,

$\lim\limits_{x\to 0^+}[x\log(\sin^2 x)]$ (in $0\times\infty$ form)

$$= \lim_{x\to 0^+}\left(\frac{\log(\sin^2 x)}{(1/x)}\right) \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+}\left(\frac{\dfrac{2\sin x.\cos x}{\sin^2 x}}{-1/x^2}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-2\cos x.\,x^2}{\sin x}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-2x^2}{\tan x}\right) = (-2)\lim_{x\to 0^+}(x) \quad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= (-2)\times 0 = 0.$$

Thus, $\lim\limits_{x\to 0^+}[x\log(\sin^2 x)] = 0.$

(viii) $\lim\limits_{x\to\frac{\pi}{2}}\left[\sec x.\left(x\sin x - \dfrac{\pi}{2}\right)\right] = -1.$

Solution: Here,

$$\lim_{x\to\frac{\pi}{2}} \left[\sec x \cdot \left(x \sin x - \frac{\pi}{2}\right)\right]$$ [in ∞ × 0 form]

$$= \lim_{x\to\frac{\pi}{2}} \left(\frac{x \sin x - \frac{\pi}{2}}{\cos x}\right)$$ $\left[\text{in } \frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to\frac{\pi}{2}} \left(\frac{\sin x + x \cos x}{-\sin x}\right) = \frac{\sin\frac{\pi}{2} + 0 \cdot \cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = -1.$$

Thus, $\lim_{x\to\frac{\pi}{2}} \left[\sec x \cdot \left(x \sin x - \frac{\pi}{2}\right)\right] = -1.$

(ix) $\lim_{x\to 0^+} [x^m (\log x)^n] = 0,$ **m and n being position integer.**

Solution: Here,

$\lim_{x\to 0^+} [x^m (\log x)^n]$ for m and n being position integer

[in 0×∞ form]

$$= \lim_{x\to 0^+} \frac{(\log x)^n}{x^{-m}}$$ $\left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x\to 0^+} \left(\frac{n(\log x)^{n-1} \cdot \frac{1}{x}}{-m x^{-m-1}}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{n(\log x)^{n-1}}{-m x^{-m}}\right)$$ $\left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x\to 0^+} \left(\frac{n(n-1)(\log x)^{n-2} \cdot \frac{1}{x}}{(-m)(-m) x^{-m-1}}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{n(n-1)(\log x)^{n-2}}{(-m)^2 x^{-m}}\right)$$ $\left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$

Continuing the process up to n^{th} times then,

$$= \lim_{x\to 0^+} \left(\frac{n(n-1) \ldots\ldots 2.1 (\log x)^0}{(-m)^n x^{-m}}\right)$$

$$= \frac{n!}{(-m)^n} \lim_{x\to 0^+} \left(\frac{1}{x^{-m}}\right) = \frac{n!}{(-m)^n} \lim_{x\to 0^+} (x^m) = \frac{n!}{(-m)^n} \cdot 0 = 0.$$

Then, $\lim_{x\to 0^+} [x^m (\log x)^n] = 0.$

3. Prove the following limits:

(i) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = -\frac{1}{3}.$

Solution: Here,

$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ (in ∞ − ∞ form)

$$= \lim_{x\to 0} \left(\frac{\sin^2 x - x^2}{x^4 \left(\frac{\sin^2 x}{x^2}\right)}\right)$$

$$= \lim_{x\to 0} \left(\frac{\sin^2 x - x^2}{x^4}\right)$$ $\left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$

$$= \lim_{x\to 0} \left(\frac{2 \sin x \cos x - 2x}{4x^3}\right)$$

$$= \lim_{x\to 0} \left(\frac{\sin 2x - 2x}{4x^3}\right)$$ $\left[\text{in } \frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \left(\frac{\cancel{2 \cos 2x - 2}}{12x^2}\right)$$ $\left[\text{in } \frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \left(\frac{-4 \sin 2x}{24x}\right)$$

$$= \lim_{x\to 0} \left(\frac{-4}{12}\right)$$ $\left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$

$$= \frac{-1}{3}.$$

Thus, $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \frac{-1}{3}.$

(ii) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{1}{2}.$

Solution: Here,

$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ (in ∞ − ∞ form)

$$= \lim_{x\to 0} \left(\frac{e^x - 1 - x}{xe^x - x}\right)$$ $\left[\text{in } \frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \frac{e^x - 1}{e^x + xe^x - 1}$$ $\left[\text{in } \frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0} \frac{e^x}{e^x + e^x + xe^x} = \frac{e^0}{e^0 + e^0 + 0 \cdot e^0} = \frac{1}{1+1} = \frac{1}{2}.$$

Thus, $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{1}{2}.$

(iii) $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=\frac{2}{3}$.

Solution: Here,

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right) \qquad (\text{in } \infty-\infty \text{ form})$$

$$=\lim_{x\to 0}\left(\frac{\tan^2x-x^2}{x^2\tan^2 x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\tan^2x-x^2}{x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x\,.\,\sec^2x-2x}{4x^3}\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x\,(1+\tan^2x)-2x}{4x^3}\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan x+2\tan^3x-2x}{4x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\left(\frac{2\sec^2x+6\tan^2x.\,\sec^2x-2}{12x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{2\tan^2x+6\tan^2x.\,\sec^2x}{12x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{\tan^2x}{x^2}\right)\left(\frac{2+6\sec^2x}{12}\right)=\left(\frac{2+6\sec^2 0}{12}\right) \qquad \left(\lim_{x\to 0}\frac{\tan x}{x}=1\right)$$

$$=\frac{8}{12}=\frac{2}{3}.$$

Thus, $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=\frac{2}{3}$.

(iv) $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]=\frac{1}{2}$.

Solution: Here,

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right] \qquad (\text{in } \infty-\infty \text{ form})$$

$$=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to 0}\left(\frac{1-\frac{1}{1+x}}{2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1+x-1}{2x\,(1+x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{x}{2x\,(1+x)}\right)=\lim_{x\to 0}\left(\frac{1}{2\,(1+x)}\right)=\frac{1}{2}.$$

Thus, $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]=\frac{1}{2}$.

(v) $\lim_{x\to 0^+}(x^x)=1$.

Solution: Let $y = x^x$

Taking log on both sides, we get

$$\log y = x\log x$$

Taking limit $\lim_{x\to 0^+}$ on both sides, we get

$$\lim_{x\to 0^+}\log y=\lim_{x\to 0^+}[x\log x] \qquad (\text{in } 0\times\infty \text{ form})$$

$$=\lim_{x\to 0^+}\left(\frac{\log x}{\left(\frac{1}{x}\right)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$=\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}(-x)=0.$$

Hence, $\lim_{x\to 0^+}\log y=0 \Rightarrow \lim_{x\to 0^+}y=e^0=1$.

Thus, $\lim_{x\to 0^+}(x^x)=1$.

(vi) $\lim_{x\to \frac{\pi}{2}}(\sin x)^{\tan x}=1$.

Solution: Let $y=(\sin x)^{\tan x}$

Taking log on both sides, we get

$$\log y=\log(\sin x)^{\tan x}$$

$$\log y=\tan x\,.\,\log(\sin x)$$

Taking $\lim_{x\to \frac{\pi}{2}}$ on both sides then we get,

$$\lim_{x\to \frac{\pi}{2}}\log y=\lim_{x\to \frac{\pi}{2}}[\tan x\log(\sin x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$=\lim_{x\to \frac{\pi}{2}}\left[\frac{\log(\sin x)}{\cot x}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$=\lim_{x\to \frac{\pi}{2}}\left[\frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2x}\right]=\lim_{x\to \frac{\pi}{2}}(-\cos x\sin x)=0.$$

Now, $\lim_{x \to \frac{\pi}{2}} \log y = 0 \Rightarrow \lim_{x \to \frac{\pi}{2}} y = e^0 = 1.$

Therefore, $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} = 1.$

(vii) $\lim_{x\to 0} (\cot x)^{\sin 2x} = 1.$

Solution: Let $y = (\cot x)^{\sin 2x}$

Taking log on both sides, we get

$$\log y = \sin 2x \log (\cot x)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0} [\sin 2x \log (\cot x)] \quad (\text{in } 0 \times \infty \text{ form})$$

$$= \lim_{x\to 0}\left[\frac{\log (\cot x)}{\text{cosec}2x}\right] \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{1}{\cot x}(-\text{cosec}^2 x)}{-2 \text{ cosec } 2x \cdot \cot 2x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\tan x\left(\frac{1}{\sin^2 x}\right)}{2\left(\frac{1}{\sin 2x}\right)\times\left(\frac{1}{\tan 2x}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{(x)\times\left(\frac{1}{x^2}\right)}{2\left(\frac{1}{2x}\right)\times\left(\frac{1}{2x}\right)}\right]$$

$$\left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left[\frac{2x}{1}\right] = \frac{0}{1} = 0.$$

Now, $\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Thus, $\lim_{x\to 0} (\cot x)^{\sin 2x} = 1.$

(viii) $\lim_{x\to \pi^-} (\sin x)^{\tan x} = 1.$

Solution: Let $y = (\sin x)^{\tan x}$

Taking log on both sides, we get

$$\log y = \tan x. \log (\sin x)$$

Taking $\lim_{x\to \pi^-}$ on both sides,

$$\lim_{x\to\pi^-} \log y = \lim_{x\to\pi^-} [\tan x. \log (\sin x)] \quad (\text{in } 0 \times \infty \text{ form})$$

$$= \lim_{x\to\pi^-}\left(\frac{\log (\sin x)}{\cot x}\right) \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\pi^-}\left(\frac{\frac{\cos x}{\sin x}}{-\text{cosec}^2 x}\right)$$

$$= \lim_{x\to\pi^-} (-\cos x \sin x) = \lim_{x\to\pi^-}\left(-\frac{1}{2}\cdot \sin 2x\right)$$

$$= \frac{-1}{2}\cdot \sin 2\pi = \frac{-1}{2}\times 0 = 0.$$

Now,

$$\lim_{x\to\pi^-} \log y = 0 \Rightarrow \lim_{x\to\pi^-} y = e^0 = 1.$$

Thus, $\lim_{x\to\pi-} (\sin x)^{\tan x} = 1.$

(ix) $\lim_{x\to 0} (\cot^2 x)^{\sin x} = 1.$

Solution: Let $y = (\cot^2 x)^{\sin x}$

Taking log on both sides, we get

$$\log y = \sin x \log (\cot^2 x) = \sin x \log (\tan x)^{-2} = -2 \sin x \log(\tan x)$$

Taking $\lim_{x\to 0} \log y = (-2) \lim_{x\to 0} [\sin x \log (\tan x)]$

$$= (-2) \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)(x) \log (\tan x)\right]$$

$$= (-2) \lim_{x\to 0} [(x) \log (\tan x)] \quad (\text{in } 0 \times \infty \text{ form})$$

$$= (-2) \lim_{x\to 0}\left[\frac{\log (\tan x)}{1/x}\right] \quad (\text{in } 0 \times \infty \text{ form})$$

$$\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= (-2) \lim_{x\to 0}\left[\frac{\left(\frac{1}{\tan x}\right)\sec^2 x}{-(1/x^2)}\right] \quad (\text{in } 0 \times \infty \text{ form})$$

$$= (-2) \lim_{x\to0}\left[\frac{\sec^2x}{-(1/x)}\right] \qquad \left(\lim_{x\to0}\frac{\tan x}{x}=1\right)$$

$$= (2)\lim_{x\to0}(x\sec^2x)$$

$$= 0$$

Now,

$$\lim_{x\to0}\log y = 0 \Rightarrow \lim_{x\to0} y = e^0 = 1.$$

Therefore, $\lim_{x\to0}(\cot^2x)^{\sin x} = 1.$

(x) $\lim_{x\to0}\left(\frac{1}{x^2}\right)^{\tan x} = 1.$ **[2018 Spring][2001][1999]**

Solution: Let $y = \left(\frac{1}{x^2}\right)^{\tan x}$

Taking log on both sides, we get

$$\log y = \tan x \log\left(\frac{1}{x^2}\right) = \tan x \log(x^{-2}) = (-2)\tan x \log(x).$$

Taking limit $\lim_{x\to0}$ on both sides

$$\lim_{x\to0}\log y = \lim_{x\to0}[(-2)\tan x\log(x)] \qquad (\text{in } 0\times\infty \text{ form})$$

$$= \lim_{x\to0}[(-2)\,x\log(x)]\left(\lim_{x\to0}\frac{\tan x}{x}=1\right)(\text{in } 0\times\infty \text{ form})$$

$$= (-2)\lim_{x\to0}\left[\frac{\log(x)}{(1/x)}\right] \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= (-2)\lim_{x\to0}\left[\frac{1/x}{-(1/x^2)}\right]$$

$$= (2)\lim_{x\to0}(x) = (2)(0) = 0.$$

Now, $\lim_{x\to0}\log y = 0 \Rightarrow \lim_{x\to0} y = e^0 = 1.$

Thus, $\lim_{x\to0}\left(\frac{1}{x^2}\right)^{\tan x} = 1.$

(xi) $\lim_{x\to0}\left(\frac{\tan x}{x}\right)^{1/x} = 1.$ **[2007, Spring]**

Solution: Let $y = \left(\frac{\tan x}{x}\right)^{1/x}$

Taking log on both sides

$$\log y = \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$$

Taking $\lim_{x\to0}$ both sides, we get

$$\lim_{x\to0}\log y = \lim_{x\to0}\left[\frac{1}{x}\log\left(\frac{\tan x}{x}\right)\right]. \qquad [\text{in } \infty\times0 \text{ form}]$$

$$= \lim_{x\to0}\left[\frac{\log\left(\frac{\tan x}{x}\right)}{x}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to0}\left[\left(\frac{x}{\tan x}\right)\frac{(x\sec^2x - \tan x)}{x^2}\right]$$

$$= \lim_{x\to0}\left(\frac{x\sec^2x-\tan x}{x^2}\right)\left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to0}\frac{\tan x}{x}=1\right)$$

$$= \lim_{x\to0}\left(\frac{\sec^2x + 2x\sec^2x\,.\tan x - \sec^2x}{2x}\right)$$

$$= \lim_{x\to0}\left(\frac{2x\sec^2x\,.\tan x}{2x}\right)$$

$$= \lim_{x\to0}[\sec^2x.\tan x] = 1\,.0 = 0.$$

Now, $\lim_{x\to0}\log y = 0 \Rightarrow \lim_{x\to0} y = e^0 = 1.$

Thus, $\lim_{x\to0}\left(\frac{\tan x}{x}\right)^{1/x} = 1.$

(xii) $\lim_{x\to0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$ **[2009, Fall] [2008, Spring] [2006, Spring] [2013 Fall][2012 Fall][2005, Spring] [2004, Fall]**

Solution: Let $y = \lim_{x\to0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

Taking log on both sides

$$\log y = \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$\log y = \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$$

Taking $\lim_{x\to0}$ on both sides,

$$\lim_{x\to0}\log y = \lim_{x\to0}\left[\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)\right] \qquad [\text{in } \infty\times0 \text{ form}]$$

$$= \lim_{x\to0}\left[\frac{\log\left(\frac{\sin x}{x}\right)}{x^2}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left[\frac{\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)}{2x} \right]$$

$$= \lim_{x\to 0} \left(\frac{x\cos x - \sin x}{2x^3}\right) \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0} \left(\frac{\cos x - x\sin x - \cos x}{6x^2}\right)$$

$$= \lim_{x\to 0} \left(\frac{-x\sin x}{6x^2}\right) = \frac{-1}{6} \qquad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

Now, $\lim_{x\to 0} \log y = \frac{-1}{6} \Rightarrow \lim_{x\to 0} y = e^{-1/6}$

Thus, $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$

(xii) $\lim_{x\to 0^+} (\cot x)^{1/\log x} = e^{-1} = \frac{1}{e}$.

Solution: Let $y = (\cot x)^{1/\log x}$

Taking log on both sides,

$$\log y = \frac{1}{\log x} \log(\cot x)$$

Taking $\lim_{x\to 0^+}$, on both sides, we get

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \left(\frac{\log(\cot x)}{\log x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to 0^+} \left(\frac{\frac{1}{\cot x}(-\text{cosec}^2 x)}{1/x}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{-x}{\cos x.\, \sin x}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{-1}{\cos x}\right) \qquad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$= -\frac{1}{\cos 0^\circ} = -1.$$

Now, $\lim_{x\to 0^+} y = e^{-1} \Rightarrow \lim_{x\to 0^+} (\cot x)^{1/\log x} = e^{-1}$.

(xiv) $\lim_{x\to \pi/2} (\sec x - \tan x) = 0$.

Solution: Here,

$$\lim_{x\to \pi/2} (\sec x - \tan x) \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to \pi/2} \left(\frac{1-\sin x}{\cos x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to \pi/2} \left(\frac{-\cos x}{-\sin x}\right) = \lim_{x\to \pi/2} (\cot x) = 0.$$

Thus, $\lim_{x\to \pi/2} (\sec x - \tan x) = 0$.

(xv) $\lim_{x\to 0} (x^{-1} - \cot x) = 0$.

Solution: Here,

$$\lim_{x\to 0} (x^{-1} - \cot x) \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\tan x}\right)$$

$$= \lim_{x\to 0} \left(\frac{\tan x - x}{x \tan x}\right)$$

$$= \lim_{x\to 0} \left(\frac{\tan x - x}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right] \left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0} \left(\frac{\sec^2 x - 1}{2x}\right) = \lim_{x\to 0} \left(\frac{\tan^2 x}{2x^2}.\, x\right) = \lim_{x\to 0} \left(\frac{x}{2}\right) = 0.$$

Thus, $\lim_{x\to 0} (x^{-1} - \cot x) = 0$.

(xvi) $\lim_{x\to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right] = \frac{1}{2}$.

Solution: Here,

$$\lim_{x\to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right] \qquad (\text{in } \infty - \infty \text{ form})$$

$$= \lim_{x\to 1} \left[\frac{x\log x - x + 1}{(x-1)\log x}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1} \left[\frac{\log x + 1 - 1}{\log x + (x-1) \times \frac{1}{x}}\right]$$

$$= \lim_{x\to 1} \left(\frac{x\log x}{x\log x + x - 1}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 1} \left(\frac{\log x + 1}{\log x + 1 + 1}\right) = \frac{1}{2}.$$

Thus, $\lim_{x\to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right] = \frac{1}{2}$.

(xvii) $\lim_{x\to 2} \left[\frac{4}{x^2-4} - \frac{1}{x-2}\right] = -\frac{1}{4}$.

Solution: Here,

$$\lim_{x\to 2} \left[\frac{4}{x^2-4} - \frac{1}{x-2}\right]$$

$$= \lim_{x\to 2} \left[\frac{4}{(x-2)(x+2)} - \frac{1}{x-2}\right]$$

$$= \lim_{x\to2}\left[\frac{4-(x+2)}{(x-2)(x+2)}\right]$$

$$= \lim_{x\to2}\left[\frac{-(x-2)}{(x-2)(x+2)}\right] = \lim_{x\to2}\left[\frac{-1}{(x+2)}\right] = \frac{-1}{4}.$$

Thus, $\lim_{x\to2}\left[\frac{4}{x^2-4} - \frac{1}{x-2}\right] = \frac{-1}{4}.$

(xviii) $\lim_{x\to\infty}(x - \sqrt{x^2-9}) = 0.$

Solution: Put $x = \frac{1}{y}$ for $y \neq 0$, then $y \to 0$ as $x\to\infty$.

Here,

$$\lim_{x\to\infty}(x-\sqrt{x^2-9}) = \lim_{y\to0}\left(\frac{1}{y} - \sqrt{\frac{1}{y^2}-9}\right)$$

$$= \lim_{y\to0}\left(\frac{1-\sqrt{1-9y^2}}{y}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{y\to0}\left(\frac{-\left(\frac{1}{2}\right)\frac{1}{\sqrt{1-9y^2}}(-18y)}{1}\right)$$

$$= \lim_{y\to0}\left(\frac{9y}{\sqrt{1-9y^2}}\right) = \frac{0}{1} = 0.$$

Thus, $\lim_{x\to\infty}(x-\sqrt{x^2-9}) = 0.$

(xix) $\lim_{x\to\infty}[\sqrt{x^2+2x} - x] = 1.$

Solution: Put $x = \frac{1}{y}$ for $y \neq 0$, then $y \to 0$ as $x\to\infty$.

Here,

$$\lim_{x\to\infty}[\sqrt{x^2+2x} - x]$$

$$= \lim_{y\to0}\left[\sqrt{\frac{1}{y^2}+2\left(\frac{1}{y}\right)} - \frac{1}{y}\right]$$

$$= \lim_{y\to0}\left[\frac{\sqrt{1+2y}-1}{y}\right] \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{y\to0}\left[\frac{\frac{1}{2}\frac{1}{\sqrt{1+2y}}(2)}{1}\right] = \lim_{y\to0}\left[\frac{1}{\sqrt{1+2y}}\right] = 1.$$

Thus, $\lim_{x\to\infty}[\sqrt{x^2+2x} - x] = 1.$

(xx) $\lim_{x\to\infty}\frac{\sqrt{3x+4}}{\sqrt{2x+3}} = \sqrt{\frac{3}{2}}.$

Solution: Here,

$$\lim_{x\to\infty}\frac{\sqrt{3x+4}}{\sqrt{2x+3}} = \lim_{x\to\infty}\left[\frac{\sqrt{x}}{\sqrt{x}}\left(\frac{\sqrt{3+4/x}}{\sqrt{2+3/x}}\right)\right] = \frac{\sqrt{3+0}}{\sqrt{2+0}} = \sqrt{\frac{3}{2}}.$$

Thus, $\lim_{x\to\infty}\frac{\sqrt{3x+4}}{\sqrt{2x+3}} = \sqrt{\frac{3}{2}}.$

(xxi) $\lim_{x\to0} x^{2x} = 1.$

Solution: Let $y = x^{2x}$

Taking log on both side, we get

$$\log(y) = 2x \log x$$

Taking $\lim_{x\to0}$ on both sides,

$$\lim_{x\to0}\log y = \lim_{x\to0}[2x\log x] \qquad [\text{ in } 0\times\infty \text{ form}]$$

$$= 2\lim_{x\to0}\left(\frac{\log x}{(1/x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= 2\lim_{x\to0}\left[\frac{1/x}{-1/x^{-2}}\right] = (-2)\lim_{x\to0}\left(\frac{1}{x}\right) = (-2)\times0 = 0.$$

Now, $\lim_{x\to0}\log y = 0 \Rightarrow \lim_{x\to0} y = e^0 = 1.$

Therefore, $\lim_{x\to0} x^{2x} = 1.$

(xxii) $\lim_{x\to0} x^{2\sin x} = 1$

Solution: Let $y = x^{2\sin x}$

Taking log on both sides, we get

$$\log y = 2\sin x \log x$$

Taking $\lim_{x\to0}$ on both sides

$$\lim_{x\to0}\log y = \lim_{x\to0}[2\sin x\log x] \qquad [\text{in } 0\times\infty \text{ form}]$$

$$= (2)\lim_{x\to0}\left(\frac{x\sin x}{x}.\log x\right)$$

$$= (2)\lim_{x\to0}\left(\frac{\log x}{(1/x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]\left(\lim_{x\to0}\frac{\sin x}{x} = 1\right)$$

$$= (2)\lim_{x\to0}\left(\frac{1/x}{-1/x^2}\right) = (2)\lim_{x\to0}(-x) = 0.$$

Now, $\lim_{x\to0}\log y = 0 \Rightarrow \lim_{x\to0} y = e^0 = 1$

Thus, $\lim_{x\to0} x^{2\sin x} = 1.$

(xxiii) $\lim_{x\to0}(\sin x)^{2\tan x} = 1$

Solution: Let $y = (\sin x)^{2\tan x}$

Taking log on both sides, we get

$$\log y = 2\tan x \log(\sin x)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0}\log y = \lim_{x\to 0}[2\tan x \log(\sin x)] \quad [\text{in } 0\times\infty \text{ form}]$$

$$= (2)\lim_{x\to 0}\left(\frac{x\tan x}{x}\cdot\log(\sin x)\right)$$

$$= (2)\lim_{x\to 0}\left(\frac{\log(\sin x)}{(1/x)}\right)\quad\left[\text{in }\frac{\infty}{\infty}\text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$= (2)\lim_{x\to 0}\left(\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}\right)$$

$$= (2)\lim_{x\to 0}(-x\cos x)\left(\lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$= (-2)\times 0$$

$$= 0.$$

Now, $\lim_{x\to 0}\log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Thus, $\lim_{x\to 0}(\sin x)^{2\tan x} = 1.$

(xxiv) $\lim_{x\to 1} x^{[1/(1-x)]} = \frac{1}{e}$

Solution: Let $y = x^{[1/(1-x)]}$

Taking log on both sides, we get

$$\log y = \frac{1}{1-x}\log x$$

Taking $\lim_{x\to 1}$ on both sides, we get

$$\lim_{x\to 1}\log y = \lim_{x\to 1}\left[\left(\frac{1}{1-x}\right)\log x\right]\quad[\text{in } 0\times\infty \text{ form}]$$

$$= \lim_{x\to 1}\left(\frac{\log(x)}{1-x}\right)\quad\left[\text{in }\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1}\left(\frac{1/x}{-1}\right) = -1.$$

Now, $\lim_{x\to 0}\log y = -1 \Rightarrow \lim_{x\to 0} y = e^{-1} = \frac{1}{e}.$

Thus, $\lim_{x\to 0} x^{[1/(1-x)]} = \frac{1}{e}.$

(xxv) $\lim_{x\to\infty}(\log x)^{1/x} = 1.$

Solution: Let $y = (\log x)^{1/x}$

Taking log on both sides, we get

$$\log y = \frac{\log x}{x}.$$

Taking $\lim_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty}\log y = \lim_{x\to\infty}\left(\frac{\log x}{x}\right)\cdot\left[\text{in }\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = 0.$$

Now, $\lim_{x\to\infty}\log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$

Thus, $\lim_{x\to\infty}(\log x)^{1/x} = 1.$

(xxvi) $\lim_{x\to 0}(e^x + x)^{1/x} = e^2.$ **[2009 Spring]**

Solution: Put $y = (e^x + x)^{1/x}$

Taking log on both sides then,

$$\log y = \frac{1}{x}\log(e^x + x) = \frac{\log(e^x + x)}{x}$$

Taking $\lim_{x\to 0}$ on both sides then,

$$\log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}(\log y) = \lim_{x\to 0}\left(\frac{\log(e^x + x)}{x}\right)\quad\left(\text{in }\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{e^x + x}\cdot(e^x + 1)\right]$$

$$= \frac{1+1}{1+0} = \frac{2}{1} = 2.$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0}(\log y) = 2.$$

$$\Rightarrow \left(\lim_{x\to 0} y\right) = e^2.$$

$$\Rightarrow \lim_{x\to 0}(e^x + x)^{1/x} = e^2.$$

(xvii) $\lim_{x\to\infty}\left[1 + \frac{1}{x^2}\right]^x = 1.$ **[2013 Spring]**

Solution :

Let $y = \left[1 + \frac{1}{x^2}\right]^x$

Taking log both sides, we get

$$\log y = x\log\left(1 + \frac{1}{x^2}\right)$$

Taking $\lim_{x\to\infty}$ on both sides

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty}\left[x \log\left(1+\frac{1}{x^2}\right)\right] \quad [\text{in } \infty \times 0 \text{ form}]$$

$$= \lim_{x\to\infty}\left[\frac{\log\left(1+\frac{1}{x^2}\right)}{\left(\frac{1}{x}\right)}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\frac{1}{1+\frac{1}{x^2}}\cdot\left(-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2}{x\left(1+\frac{1}{x^2}\right)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2x}{x^2+1}\right] \quad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2}{2x}\right] = \lim_{x\to\infty}\left[\frac{1}{x}\right] = 0.$$

Now, $\lim_{x\to\infty} \log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$

Thus, $\lim_{x\to\infty}\left[1+\frac{1}{x^2}\right]^x = 1.$

(xxviii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x} = 1.$ **[2016 Fall][2003, Spring]**

Solution:

Let, $y = \left(\frac{\sin x}{x}\right)^{1/x}$

Taking log on both sides, we get

$$\log(y) = \frac{1}{x}\log\left(\frac{\sin x}{x}\right)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0}\left[\frac{\log\left(\frac{\sin x}{x}\right)}{x}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{x\cos x - \sin x}{x^2}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left[\frac{\cos x - x\sin x - \cos x}{2x}\right] = \lim_{x\to 0}\left[\frac{-\sin x}{2}\right] = 0.$$

Now, $\lim_{x\to 0} \log y = 0 \Rightarrow \lim_{x\to 0} y = e^0 = 1.$

Thus, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x} = 1.$

(xxix) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$ **[2015 Fall][2002]**

Solution:

Let $y = \left(\frac{\tan x}{x}\right)^{1/x^2}$

Taking log both sides, we get

$$\log y = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$$

Taking $\lim_{x\to 0}$ on both sides

$$\lim_{x\to 0} \log y = \lim_{x\to 0}\left[\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{x}{\tan x}\cdot\left[\frac{x\sec^2 x - \tan x}{x^2}\right]}{2x}\right]$$

$$= \lim_{x\to 0}\left[\frac{x\sec^2 x - \tan x}{2x^3}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{x}{\tan x} = 1\right)$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x + 2\sec^2 x \cdot \tan x \cdot x - \sec^2 x}{6x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x\sec^2 x \cdot \tan x}{6x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x}{3}\cdot\frac{\tan x}{x}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sec^2 x}{3}\right]. \quad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= \frac{1}{3}.$$

Now,

$$\lim_{x\to 0} \log y = \frac{1}{3} \Rightarrow \lim_{x\to 0} y = e^{1/3}$$

Thus, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}.$

4. Show that

(i) $\lim_{x\to\infty}(1+2x)^{1/2\log(x)} = \infty.$

Solution: Let $y = (1+2x)^{1/2\log(x)}$

Taking log both sides, we get

$$\log y = \frac{1+2x}{2\log(x)}$$

Taking $\lim_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty}\left(\frac{1+2x}{2\log(x)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{2}{2/x}\right) = = \lim_{x\to\infty}(x) = \infty.$$

Now, $\lim_{x\to\infty} \log y = \infty \Rightarrow \lim_{x\to\infty} y = e^{\infty} = \infty.$

Thus, $\lim_{x\to\infty} (1+2x)^{1/2\log(x)} = \infty.$

Answer to be corrected in the book.

(ii) $\lim_{x\to\infty} x^{1/x} = 1.$

Solution: Let $y = x^{1/x}$

Taking log on both sides, we get

$$\log y = \frac{1}{x}\log x$$

Taking $\lim_{x\to\infty}$ on both sides,

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty}\left(\frac{\log(x)}{x}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0.$$

Now, $\lim_{x\to\infty} \log y = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1.$

Thus, $\lim_{x\to\infty} x^{1/x} = 1.$

(iii) $\lim_{x\to 0}\left(\frac{x e^x - (1+x)\log(1+x)}{x^2}\right) = \frac{1}{2}$

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x e^x - (1+x)\log(1+x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + xe^x - \log(1+x) - 1}{2x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{e^x + e^x + xe^x - \frac{1}{1+x}}{2}\right)$$

$$= \frac{2e^0 + 0\,.\,e^0 - \frac{1}{1+0}}{2} = \frac{1}{2}.$$

Thus, $\lim_{x\to 0}\left(\frac{x e^x - (1+x)\log(1+x)}{x^2}\right) = \frac{1}{2}.$

(iv) $\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$

Solution: Put, $y = \lim_{x\to 0} (\cos x)^{1/x^2}$

Taking log on both sides, we get

$$\log y = \frac{1}{x^2}\log(\cos x)$$

Taking $\lim_{x\to 0}$ on both sides,

$$\lim_{x\to 0} \log y = \lim_{x\to 0}\left(\frac{\log(\cos x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-\frac{\sin x}{\cos x}}{2x}\right)$$

$$= \lim_{x\to 0}\left(-\frac{\tan x}{2x}\right) = \frac{-1}{2} \qquad \left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right).$$

Now,

$$\lim_{x\to 0} \log y = -\frac{1}{2} \Rightarrow \lim_{x\to 0} y = e^{-1/2}.$$

Thus, $\lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}.$

(v) $\lim_{x\to a}\left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)} = e^{2/\pi}$

Solution: Let $y = \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}$

Taking log on both sides, we get

$$\log y = \tan\left(\frac{\pi x}{2a}\right)\log\left(2 - \frac{x}{a}\right)$$

Taking $\lim_{x\to a}$ on both sides

$$\lim_{x\to a} \log y = \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right)\log\left(2 - \frac{x}{a}\right) \qquad [\text{in } \infty \times 0 \text{ form}]$$

$$= \lim_{x\to a}\left(\frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)}\right) \qquad \left[\text{in } \frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x\to a}\frac{\left(\frac{1}{2-(x/a)}\right)\cdot\left(-\frac{1}{a}\right)}{\left(-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\right)\cdot\frac{\pi}{2a}}$$

$$= \lim_{x\to a} \left(\frac{2a \sin^2\left(\frac{\pi x}{2a}\right)}{\pi (2a - x)} \right) = \frac{2a}{\pi a} = \frac{2}{\pi}.$$

Now,

$$\lim_{x\to a} \log y = \frac{2}{\pi} \Rightarrow \lim_{x\to a} y = e^{2/\pi}$$

Thus, $\lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)} = e^{2/\pi}$.

(vi) $\lim_{x\to 0} \left(\frac{x \cos x - \log(1+x)}{x^2}\right) = \frac{1}{2}$.

Solution : Here,

$$\lim_{x\to 0} \left(\frac{x \cos x - \log(1+x)}{x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{- x \sin x + \cos x - \frac{1}{1+x}}{2x} \right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{- \sin x - x \cos x - \sin x + \frac{1}{(1+x)^2}}{2} \right)$$

$$= \frac{0-0-0+1}{2} = \frac{1}{2}.$$

Thus, $\lim_{x\to 0} \left(\frac{x \cos x - \log(1+x)}{x^2}\right) = \frac{1}{2}$.

(vii) $\lim_{x\to 0} \left(\frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5}\right) = \frac{1}{30}$

Solution: Here,

$$\lim_{x\to 0} \left(\frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{e^x + e^{-x} + 2 \cos x - 4}{5x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{e^x - e^{-x} - 2 \sin x}{20x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{e^x + e^{-x} - 2 \cos x}{60x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{e^x - e^{-x} + 2 \sin x}{120x}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{e^x + e^{-x} + 2 \cos x}{120}\right) = \frac{e^0 + e^{-0} + 2 \cos 0}{120} = \frac{4}{120} = \frac{1}{30}$$

Thus, $\lim_{x\to 0} \left(\frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5}\right) = \frac{1}{30}$.

Alternative Method

Here,

$$\lim_{x\to 0} \left(\frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5}\right)$$

$$= \lim_{x\to 0} \left(\frac{2 \sinh x + 2 \sin x - 4x}{x^5}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{2 \cosh x + 2 \cos x - 4}{5x^4}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{2 \sinh x - 2 \sin x}{20x^3}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{2 \cosh x - 2 \cos x}{60x^2}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{2 \sinh x + 2 \sin x}{120x}\right)$$

$$= \lim_{x\to 0} \left(\frac{2 \cosh x + 2 \cos x}{120}\right) = \frac{2+2}{120} = \frac{4}{120} = \frac{1}{30}.$$

Thus, $\lim_{x\to 0} \left(\frac{e^x - e^{-x} + 2 \sin x - 4x}{x^5}\right) = \frac{1}{30}$.

(viii) $\lim_{x\to 0} \left(\frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}\right) = 0.$

Solution: Here,

$$\lim_{x\to 0} \left(\frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{- \sin x - \frac{1}{1+x} + \cos x}{e^x - 1}\right) \qquad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{- \cos x + \frac{1}{(1+x)^2} - \sin x}{e^x}\right) = \frac{-1 + \frac{1}{(1+0)^2} - 0}{1} = \frac{-1+1}{1} = 0.$$

Thus, $\lim_{x\to 0} \left(\frac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}\right) = 0.$

(ix) $\lim_{x\to 2} \left(\frac{x^2 - 4}{\sqrt{x+2} - \sqrt{3x-2}}\right) = -8$

Solution: Here,

$$\lim_{x\to 2} \left(\frac{x^2 - 4}{\sqrt{x+2} - \sqrt{3x-2}}\right)$$

$$= \lim_{x\to 2} \left(\frac{x^2 - 4}{(\sqrt{x+2} - \sqrt{3x+2})} \times \frac{(\sqrt{x+2} + \sqrt{3x-2})}{(\sqrt{x+2} + \sqrt{3x-2})} \right)$$

$$= \lim_{x\to 2} \left(\frac{(x^2 - 4)(\sqrt{x+2} + \sqrt{3x-2})}{x + 2 - 3x + 2} \right)$$

$$= \lim_{x\to 2} \left(\frac{(x-2)(x+2)(\sqrt{x+2} + \sqrt{3x-2})}{-2(x-2)} \right)$$

$$= \lim_{x\to 2} \left[\left(-\frac{1}{2}\right)(x+2)\left(\sqrt{x+2}+\sqrt{3x-2}\right)\right]$$

$$= \left(-\frac{1}{2}\right) \times 4 \times (2+2) = -8.$$

Thus, $\lim_{x\to 2}\left(\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right) = -8.$

(x) $\lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right] = \frac{1}{2}$ **[2017 Fall]**

Solution: Here,

$$\lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right]$$

$$= \lim_{y\to 0}\left[\frac{1}{y} - \frac{1}{y^2}\log(1+y)\right] \quad \left[\text{put } y = \frac{1}{x} \text{ then as } x\to\infty \Rightarrow y\to 0.\right]$$

$$= \lim_{y\to 0}\left[\frac{y-\log(1+y)}{y^2}\right] \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{y\to 0}\left[\frac{1-1/(1+y)}{2y}\right]$$

$$= \lim_{y\to 0}\left[\frac{1+y-1}{(1+y)\,2y}\right] = \lim_{y\to 0}\left[\frac{1}{(1+y)\,2}\right] = \frac{1}{2}.$$

Thus, $\lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right] = \frac{1}{2}.$

5. If $\lim_{x\to 0}\left(\dfrac{a\sin x - \sin 2x}{\tan^3 x}\right)$ is finite show that a = 2 and limit is 1.

[2011 Spring][2006, Fall]

Solution: Here,

$$\lim_{x\to 0}\left(\frac{a\sin x - \sin 2x}{\tan^3 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{a\sin x - \sin 2x}{x^3 \cdot \left(\frac{\tan x}{x}\right)^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{a\sin x - \sin 2x}{x^3}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]\left(\lim_{x\to 0}\frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0}\left(\frac{a\cos x - 2\cos x}{3x^2}\right) \quad \left[\frac{a-2}{0}\right]$$

[Givan that the limit is finite so we must have a − 2 = 0 ⇒ a = 2]. Then,

$$= \lim_{x\to 0}\left(\frac{2\cos x - 2\cos 2x}{3x^2}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-2\sin x + 4\sin 2x}{6x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-2\cos x + 8\cos 2x}{6}\right) = \frac{-2+8}{6} = 1.$$

Thus, the value of a is 2 and limit of the form is 1.

6. If $\lim_{x\to 0}\left(\dfrac{x(1+a\cos x) - b\sin x}{x^3}\right) = 1$. Show that $a = -\frac{5}{2}$, and $b = -\frac{3}{2}$.

Solution: Here,

$$\lim_{x\to 0}\left(\frac{x(1+a\cos x) - b\sin x}{x^3}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{1 + (a-b)\cos x - ax\sin x}{3x^2}\right)\left(\text{has } \frac{1+a-b}{0} = \frac{\text{finite}}{0} \text{ form}\right)$$

As the limit exist we must have a − b + 1 = c

$$\Rightarrow a - b = -1 \quad \text{...... (i)}$$

So, suppose that the above form has $\frac{0}{0}$ form. Then,

$$= \lim_{x\to 0}\left(\frac{-(a-b)\sin x - a\sin x - ax\cos x}{6x}\right)$$

$$= \lim_{x\to 0}\left(\frac{-(2a-b)\sin x - ax\cos x}{6x}\right) \quad \left[\text{in } \frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0}\left(\frac{-(2a-b)\cos x - a\cos x + ax\sin x}{6}\right)$$

$$= \lim_{x\to 0}\left(\frac{-(3a-b)\cos x + ax\sin x}{6}\right) = \left(\frac{-(3a-b)}{6}\right) = \frac{-3a+b}{6}$$

We have,

$$\lim_{x\to 0}\left(\frac{x(1+a\cos x) - b\sin x}{x^3}\right) = 1 \Rightarrow \frac{-3a+b}{6} = 1$$

From (i), $a = b - 1$

So, $-3(b-1) + b = 6. \Rightarrow -3b + 3 + b = 6$

$$\Rightarrow -2b = 3 \Rightarrow b = -\frac{3}{2}.$$

Then, $a = -\frac{3}{2} - 1 = -\frac{5}{2} \Rightarrow a = -\frac{5}{2}.$

Thus, $a = -\frac{5}{2}$ and $b = -\frac{3}{2}$.

7. Show that a = 1, b = 2, c = 1, when $\lim_{x\to 0}\left[\dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x}\right] = 2.$

Solution: Here,

$$\lim_{x\to 0}\left[\frac{ae^x - b\cos x + ce^{-x}}{x\sin x}\right]$$

$$= \lim_{x\to 0}\left(\frac{ae^{2x} - be^x\cos x + ce^{-x}}{x^2}\right)\left[\text{in } \frac{a-b+c}{0} \text{ form}\right]\left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

As the limit exists, we must have

$$a - b + c = 0 \quad \text{...... (i)}$$

then the above form becomes as $\frac{0}{0}$ form. So,

$$= \lim_{x \to 0} \frac{ae^x + b\sin x - ce^{-x}}{2x} \qquad \left[\text{in } \frac{a-c}{0} \text{ form}\right]$$

As the limit exist, we must have get

$$a - c = 0, \quad a = c \qquad \text{........ (ii)}$$

then the above form becomes as $\frac{0}{0}$ form. So,

$$= \lim_{x \to 0} \frac{ae^x + b\cos x + ce^{-x}}{2}$$

$$= \frac{a+b+c}{2} \qquad \text{......... (iii)}$$

Given that,

$$\lim_{x \to 0} \left[\frac{ae^x - b\cos x + ce^{-x}}{x \sin x}\right] = 2.$$

$$\Rightarrow \frac{a+b+c}{2} = 2 \Rightarrow a + b + c = 4 \qquad \text{......(iv)} \quad \text{[by (iii)]}$$

From (i) and (ii), we get

$$2a - b = 0 \Rightarrow b = 2a \qquad \text{... (v)}$$

From equations (ii), (iv) and (v) we get,

$$a + 2a + a = 4 \Rightarrow a = 1,$$

Therefore, $b = 2a = 2$ and $c = a = 1$.

Thus, $a = 1$, $b = 2$ and $c = 1$.

Exercise 5.1

1. Find the maximum and minimum values of the following functions:

(i) $f(x) = x + \frac{1}{x}$

Solution: Given, $f(x) = x + \frac{1}{x}$

So, $f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$.

For extreme point, set

$$f'(x) = 0 \Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1.$$

At $x = 1$,

$$f''(1) = \frac{2}{(1)^3} = 2 > 0.$$

So f(x) has minimum value at x = 1. And minimum value is f(1) = 2.

At x = –1

$$f''(-1) = \frac{2}{(-1)^3} = -2 < 0.$$

So f(x) has maximum value at x = –1. And maximum value is

$$f(-1) = -1 - 1 = -2.$$

Thus, f(x) has maximum value 2 at x = 1 and minimum value at x = –1.

(ii) $f(x) = x^3 - 3x^2 + 6x + 3$ (Question wrong)

Solution: Let $f(x) = x^3 - 3x^2 + 6x + 3$

So, $f'(x) = 3x^2 - 6x + 6$ and $f''(x) = 6x - 6$.

For extreme point, set

$$f'(x) = 0$$

$$\Rightarrow 3x^2 - 6x + 6 = 0$$

$$\Rightarrow x^2 - 2x + 2 = 0$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

Thus, x has imaginary value. So, the solution is impossible.

(iii) $f(x) = \cos x$

Solution: Let, $f(x) = \cos x$

So, $f'(x) = -\sin x$ and $f''(x) = -\cos x$.

For extreme point, set

$f'(x) = 0 \Rightarrow -\sin x = 0 = \sin n\pi \Rightarrow x = n\pi$ for n is integer.

At $x = 0$,

$f''(x) = -1 < 0$, so f(x) has maximum value at $x = 0$.

If n is even integer i.e. at $x = 2n\pi$,

$f''(2n\pi) = -\cos(2n\pi) = -1 < 0$.

So, f(x) has maxima at $x = 2n\pi$ and maximum value is

$f(2n\pi) = \cos 2n\pi = 1$.

If n is odd integer i.e. at $x = (2n+1)\pi$,

$$\begin{aligned} f''((2n+1)\pi) &= -\cos((2n+1)\pi) \\ &= -\cos 2n\pi \cos\pi + \sin 2n\pi \sin\pi \\ &= -(1)(-1) + 0 \\ &= 1 > 0. \end{aligned}$$

So, f(x) has minima at $x = (2n+1)\pi$ and maximum value is

$$\begin{aligned} f((2n+1)\pi) &= \cos(2n+1)\pi \\ &= \cos 2n\pi \cos\pi - \sin 2n\pi \sin\pi \\ &= (1)(-1) + 0 \\ &= -1 \end{aligned}$$

(iv) $f(x) = (1 + \cos x)\sin x$ **at** $x = \frac{\pi}{3}$

Solution: Given, $f(x) = (1 + \cos x)\sin x = \sin x + \frac{\sin 2x}{2}$

So, $f'(x) = \cos x + \cos 2x$ and $f''(x) = -\sin x - 2\sin 2x$.

At $x = \frac{\pi}{3}$,

$$f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2} - \frac{1}{2} = 0.$$

and

$$f''\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right) < 0.$$

So, f(x) has maximum value at $x = \frac{\pi}{3}$. And maximum value is,

$$f\left(\frac{\pi}{3}\right) = \left(1 + \cos\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) = \left(1 + \frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}.$$

2. Find the maximum value of xy and minimum value of $x^2 + y^2$ such that $\frac{x}{2} + \frac{y}{3} = 1$.

Solution: Here, $\frac{x}{2} + \frac{y}{3} = 1 \Rightarrow 3x + 2y = 6$

$$\Rightarrow y = \frac{6-3x}{2} \quad \dots (i)$$

Part I: We have,

$$f = xy = x\left(\frac{6-3x}{2}\right) = \frac{6x - 3x^2}{2} \quad \dots (ii)$$

So, $f'(x) = 3 - 3x$ and $f''(x) = -3 < 0$.

For extreme point, set

$f'(x) = 0 \Rightarrow 3 - 3x = 0 \Rightarrow x = 1$.

Also, $f''(x) = -6 < 0$.

So f has maxima at $x = 1$ and maximum value is

$$f = \frac{6-3}{2} = \frac{3}{2}. \quad [\because \text{By (ii)}]$$

Part II: We have,

$$\begin{aligned} f(x) = x^2 + y^2 &= x^2 + \left(\frac{6-3x}{2}\right)^2 \quad \text{being } y = \frac{6-3x}{2}. \\ &= x^2 + \frac{36 - 36x + 9x^2}{4} \\ &= x^2 + 9 - 9x + \frac{9}{4}x^2 \\ &= \frac{13}{4}x^2 - 9x + 9 \end{aligned}$$

So,

$$f'(x) = \frac{13}{2}x - 9 \quad \text{and} \quad f''(x) = \frac{13}{2}.$$

For extreme point,

$$f'(x) = 0 \Rightarrow \frac{13x}{2} - 9 = 0 \Rightarrow x = \frac{18}{13}.$$

Also, $f''(x) = \frac{13}{2} > 0$.

So f(x) has minima at $x = \frac{18}{13}$ and minimum value is

$$f = \frac{13}{4}\left(\frac{18}{13}\right)^2 - 9\left(\frac{18}{13}\right) + 9 = \left(\frac{81}{13}\right) - \frac{162}{13} + 9 = \frac{36}{13}.$$

3. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

Solution: Let $y = \left(\frac{1}{x}\right)^x$

Taking log on both sides, we get

$$\log y = x \log\left(\frac{1}{x}\right) = x \log(x^{-1}) = -x \log(x).$$

Now, differentiating w. r. t. x then we get

$$\frac{1}{y} y' = -x\left(\frac{1}{x}\right) - \log(x) = -\log(x) - 1$$

$$y' = -y \log(x) - y \quad \ldots (i)$$

Again differentiating w. r. t. x, then

$$y'' = -y' \log(x) - \frac{y}{x} - y' \quad \ldots (ii)$$

For extreme point, set

$$y' = 0 \Rightarrow -y \log(x) - y = 0$$
$$\Rightarrow \log(x) = 1.$$
$$\Rightarrow x = e.$$

So, $y = \left(\frac{1}{e}\right)^e = (e^{-1})^e$

Put x = e in equation (ii) then,

$$y'' = -0 - \frac{(e^{-1})^e}{e} - 0 \qquad [\because y' = 0]$$

$$\Rightarrow y'' = -\frac{(e^{-1})^e}{e} < 0 \qquad [\because e > 0]$$

So f(x) = y has maxima at x = e and maximum value is,

$$f = \left(\frac{1}{e}\right)^e.$$

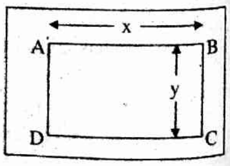
4. Show that of all rectangles of given area the square has the smallest perimeter.

Solution: Let x is the length and y is the breadth of rectangle. Let A be the area and P be the perimeter which is to be minimized.

Since the area of the rectangle is given. So, the area is fixed. Therefore,

$$A = xy \Rightarrow y = \frac{A}{x}$$

and $P = 2(x + y) = 2\left(x + \frac{A}{x}\right) \quad \left[\because y = \frac{A}{x}\right]$



Then,

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} \quad \text{and} \quad \frac{d^2P}{dx^2} = \frac{4A}{x^3}$$

For extreme point, set

$$\frac{dP}{dx} = 0 \Rightarrow 2 - \frac{2A}{x^2} = 0 \Rightarrow x = \sqrt{A} \quad \text{[Being x is positive]}$$

and $y = \frac{A}{\sqrt{A}} = \sqrt{A}$.

At $x = \sqrt{A}$ $\quad \frac{d^2P}{dx^2} = \frac{4A}{(\sqrt{A})^3} > 0.$

So, P is minimum when $x = \sqrt{A} = y$. Thus, the rectangle is square when P is minimum.

Hence, the square has the smallest perimeter of all rectangles of given area.

5. Show that largest rectangle with a given perimeter is square.

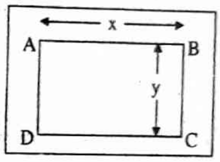
Solution: Let P be the given perimeter. Also, let x be the length and y be the breadth of the rectangle.

And, let A be the area which is to be maximized.

Here, the perimeter is given. So, the perimeter is fixed. Since,

$$P = 2(x + y) \Rightarrow y = \frac{P - 2x}{2} \quad \ldots (i)$$

and $A = xy \Rightarrow A = \frac{x(P - 2x)}{2}$ [from (i)]



$$\Rightarrow A = \frac{Px}{2} - x^2$$

Differentiating w. r. t. x then,

$$\frac{dA}{dx} = \frac{P}{2} - 2x \quad \text{and} \quad \frac{d^2A}{dx^2} = -2 < 0.$$

For extreme point, set

$$\frac{dA}{dx} = 0 \Rightarrow \frac{P}{2} - 2x = 0 \Rightarrow x = \frac{P}{4}.$$

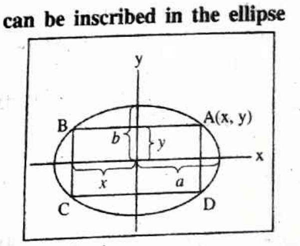
Then $\quad y = \frac{P - 2\left(\frac{P}{4}\right)}{2} = \frac{P}{4}.$

Since, $\frac{d^2A}{dx^2} = -2 < 0$. So A is maximum when $x = \frac{P}{4}$ and $y = \frac{P}{4}$.

Here $x = \frac{P}{4} = y$, so the rectangle is a square.

Thus the largest rectangle is square with a given perimeter.

6. Prove that the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with area 2ab.



Solution: Given equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = \frac{a^2 b^2 - b^2 x^2}{a^2} \quad \ldots (i)$$

So, the centre of ellipse is (0, 0).

Let A, B, C and D be the vertices of rectangle when co–ordinate of A is (x, y) which lies on the ellipse i.e. sides of rectangle are 2x and 2y. Then,

Area of the rectangle (A) = 4xy.

$$\Rightarrow A^2 = 16x^2y^2 = 16x^2\left(\frac{a^2b^2 - b^2x^2}{a^2}\right) \quad \text{[Using (i)]}$$

$$= 16b^2x^2 - 16\left(\frac{b^2}{a^2}\right)x^4 \quad \ldots\text{(ii)}$$

Here we have to show the rectangle is greatest. This means we have to observe that A^2 is maximum.

So, differentiating A^2 w. r. t. x then

$$\frac{dA^2}{dx} = 32\,b^2x - 64\left(\frac{b^2}{a^2}\right)x^3 \quad \text{and} \quad \frac{d^2A^2}{dx^2} = 32b^2 - 192\left(\frac{b^2}{a^2}\right)x^2$$

For extreme point of A^2 (i.e. A), set

$$\frac{dA^2}{dx} = 0 \Rightarrow 32b^2x - 64\left(\frac{b^2}{a^2}\right)x^3 = 0$$

$$\Rightarrow 1 - \frac{2x^2}{a^2} = 0 \Rightarrow x^2 = \frac{a^2}{2}.$$

And at $x^2 = \frac{a^2}{2}$,

$$\frac{d^2A^2}{dx^2} = 32b^2 - 192\left(\frac{b^2}{a^2}\right)\cdot\frac{a^2}{2} = -64\,b^2 < 0.$$

This means the area is maximum.

And the maximum area is,

$$A^2 = 16b^2x^2 - 16\left(\frac{b^2}{a^2}\right)x^4 = 16b^2\left(\frac{a^2}{2}\right)\left[1 - \left(\frac{1}{a^2}\right)\frac{a^2}{2}\right]$$

$$= 8a^2b^2\left(\frac{1}{2}\right) = 4\,a^2b^2.$$

$$\Rightarrow A = 2ab$$

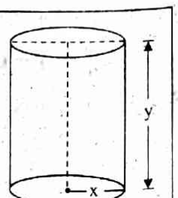
Thus the area of greatest rectangle is 2ab.

7. A cylindrical tin can closed at both ends of given capacity has to be constructed. Show that the amount of tin required will be minimum when the height is equal to the diameter.

[2016 Spring][2015 Fall][2006, Fall] [2008, Fall]

Solution: Let, x = radius of the cylinder.

y = height of the cylinder.



Given that the capacity of the cylinder is given. So, the volume (i.e. capacity) is constant.

Since,

$$V = \pi x^2 y \quad \text{(which is constant)}$$

$$\Rightarrow \quad y = \frac{V}{\pi x^2} \quad \ldots\text{(i)}$$

Let S be the surface area (tin required) of the cylinder which is to be minimized.

We know that,

$$S = 2\pi xy + 2\pi x^2 \quad [\because \text{cylindrical tin is closed at both ends}]$$

$$= 2\pi x\left(\frac{V}{\pi x^2}\right) + 2\pi x^2$$

$$\Rightarrow \quad S = \frac{2V}{x} + 2\pi x^2$$

Differentiating w. r. t. x then

$$\frac{d(S)}{dx} = -\frac{2V}{x^2} + 4\pi x \quad \text{and} \quad \frac{d^2(S)}{dx^2} = \frac{4V}{x^3} + 4\pi$$

For extreme value of S, set

$$\frac{d(S)}{dx} = 0 \Rightarrow \frac{-2V}{x^2} + 4\pi x = 0 \Rightarrow \frac{2V}{x^2} = 4\pi x \Rightarrow x^3 = \frac{V}{2\pi}$$

And, at $x^3 = \frac{V}{2\pi}$,

$$\frac{d^2(S)}{dx^2} = \frac{4V}{V/2\pi} + 4\pi = 8\pi + 4\pi = 12\pi > 0.$$

So, the surface area will be minimum (tin required) when $x^3 = \frac{V}{2\pi}$.

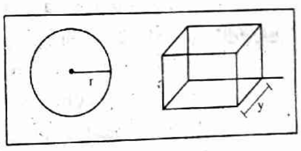
Here, $x^3 = \frac{V}{2\pi} \Rightarrow x^3 = \frac{\pi x^2 y}{2\pi} \Rightarrow 2x = y$

This shows that the diameter of the cylinder is equal to its height.

8. The sum of the surfaces of a cube and a sphere is given, when the sum of their volume is least, show that the diameter of the sphere is equal to the edge of the cube.

Solution: Let x = radius of the sphere

and y = edge of the cube.



Given,

S = sum of the surface area

which is given. So, S is fixed.

Let V be the sum of the volume which is to be minimized.

We know that,

$$S = 4\pi x^2 + 6y^2 \quad \ldots\ldots\text{(i)}$$

So, $$\frac{dS}{dx} = 8\pi x + 12y\frac{dy}{dx}.$$

Since S is fixed, so $\frac{dS}{dx} = 0$. Therefore,

$$0 = 8\pi x + 12y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{8\pi x}{12y} = -\frac{2\pi}{3}\left(\frac{x}{y}\right) \qquad \ldots \text{(ii)}$$

Let V is sum of volume of the sphere and cube. That is,

$$V = \frac{4\pi x^3}{3} + y^3$$

So,

$$\frac{dV}{dx} = 4\pi x^2 + 3y^2\frac{dy}{dx} = 4\pi x^2 - 2\pi xy \qquad [\because \text{using (ii)}]$$

and,

$$\frac{d^2V}{dx^2} = 8\pi x - 2\pi y - 2\pi x\frac{dy}{dx}$$

$$= 8\pi x - 2\pi y + \frac{4\pi^2x^2}{3y} \qquad [\because \text{using (ii)}]$$

For critical point set,

$$\frac{dV}{dx} = 0 \Rightarrow 4\pi x^2 - 2\pi xy = 0$$

$$\Rightarrow \quad y = 2x \qquad [\because 2\pi x \neq 0]$$

At y = 2x,

$$\frac{d^2V}{dx^2} = 8\pi x - 4\pi x + \frac{4\pi^2x^2}{6x}$$

$$= 4\pi x + \frac{2\pi^2 x}{3} > 0.$$

This means the sum of volume is minimum when 2x = y i.e. the sum of volume of a cube and a sphere is least when edge of cube is equal to diameter (twice of radius) of sphere.

9. Show that the semi–vertical angle of the cone of maximum volume and given slant height, is $\tan^{-1}(\sqrt{2})$. [2005, Spring]

Solution: Let ABC be a cone.

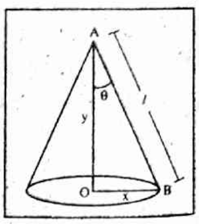
Let l = slant height, x = radius of base

y = height of cone

θ = semi–vertical angle of the cone.

Now, from ΔAOB

$y = l\cos\theta$ and $x = l\sin\theta$



Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi\, l^2\sin^2\theta\; l\cos\theta$$

$$\Rightarrow \quad V = \frac{\pi}{3} l^3 \sin^2\theta\cos\theta = \left(\frac{\pi}{6}\right) l^3 \sin 2\theta \sin\theta$$

Differentiating w.r.t. θ then

$$\frac{dV}{d\theta} = \left(\frac{\pi}{6}\right) l^3 [2\cos 2\theta \sin\theta + \sin 2\theta\cos\theta]$$

and,

$$\frac{d^2V}{d\theta^2} = \left(\frac{\pi}{6}\right) l^3 [-4\sin 2\theta\sin\theta + 2\cos 2\theta\cos\theta + 2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta]$$

$$= \left(\frac{\pi}{6}\right) l^3 [4\cos 2\theta\cos\theta - 5\sin 2\theta\sin\theta]$$

For extreme point, set

$$\frac{dV}{d\theta} = 0 \quad \Rightarrow \quad 2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta = 0.$$

$$\Rightarrow \quad 2\sin\theta(\cos^2\theta - \sin^2\theta + \cos^2\theta) = 0.$$

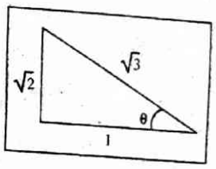
$$\Rightarrow \quad \sin\theta(\cos^2\theta - \sin^2\theta + \cos^2\theta) = 0.$$

So, either $\sin\theta = 0 \Rightarrow \theta = 0$, that is not possible.

or, $2\cos^2\theta - \sin^2\theta = 0.$

$\Rightarrow \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2}.$



Then, $\sin\theta = \dfrac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \dfrac{1}{\sqrt{3}}$.

At $\tan\theta = \sqrt{2}$,

$$\frac{d^2V}{d\theta^2} = \left(\frac{\pi}{6}\right) l^3 [4(\cos^3\theta - \sin^2\theta\cos\theta) - 10\sin^2\theta\cos\theta]$$

$$= \left(\frac{\pi}{6}\right) l^3 \left[4\left(\frac{1}{\sqrt{3}}\right)^3 - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) - 10\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \left(\frac{\pi}{6}\right) l^3 \left[\left(\frac{4}{3\sqrt{3}}\right) - \left(\frac{2}{3\sqrt{3}}\right) - \left(\frac{20}{3\sqrt{3}}\right)\right]$$

$$= \left(\frac{\pi}{6}\right) l^3 \left(\frac{-18}{3\sqrt{3}}\right) < 0$$

So, the volume is maximum when $\tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}(\sqrt{2})$.

10. Find the surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius r.

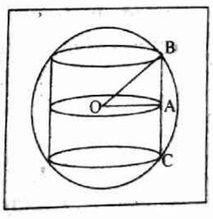
[2013 Spring][2013 Fall][1999] [2001]

Solution: Let O be the centre and r be the radius of the sphere. Let h be the height and R be the radius of the base of the cylinder.

Then let OB = r, OA = R and BC = h. Let ∠BOA = θ.

Then in ΔOAB, $\cos\theta = \dfrac{OA}{OB} \Rightarrow R = r\cos\theta.$

Also, $\sin\theta = \dfrac{AB}{OB} \Rightarrow h = 2r\sin\theta.$



Let S be the surface area of the cylinder.

Then $S = 2\pi Rh + 2\pi R^2$

$= 2\pi r\cos\theta\, 2r\sin\theta + 2\pi r^2\cos^2\theta$

$= 2\pi r^2(\sin 2\theta + \cos^2\theta) \qquad \ldots \text{(i)}$

Differentiating (i) w. r. t. θ,

$$\frac{dS}{d\theta} = 2\pi\,(2\cos 2\theta - 2\sin\theta\cos\theta) = 2\pi\,(2\cos 2\theta - \sin 2\theta)$$

And, $$\frac{d^2S}{d\theta^2} = 2\pi\,(-4\sin 2\theta - 2\cos 2\theta)$$

For extreme value, set

$$\frac{dS}{d\theta} = 0$$

$$\Rightarrow 2\pi\,(2\cos 2\theta - \sin 2\theta) = 0$$

$$\Rightarrow 2\cos 2\theta - \sin 2\theta = 0$$

$$\Rightarrow \tan 2\theta = 2.$$

By triangular law,

Tan 2θ = 2 $\Rightarrow \frac{p}{b} = \frac{2}{1}$

$\Rightarrow$ **p = 2, b = 1.**

Then, h = $\sqrt{5}$.

So that Sin 2θ = $\frac{p}{h} = \frac{2}{\sqrt{5}}$

and Cos 2θ = $\frac{b}{h} = \frac{1}{\sqrt{5}}$.

Then

$$\sin 2\theta = \frac{2}{\sqrt{5}},\quad \cos 2\theta = \frac{1}{\sqrt{5}}.$$

Then, at tan 2θ = 2,

$$\frac{d^2S}{d\theta^2} = 2\pi\left(-\frac{8}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right) = \frac{-20\pi}{\sqrt{5}} < 0.$$

So, the function has maximum value at tan 2θ = 2. And at that point,

$$S = 2\pi r^2(\sin 2\theta + \cos^2\theta)$$

$$= 2\pi r^2\left(\sin 2\theta + \frac{1+\cos 2\theta}{2}\right)$$

$$= 2\pi r^2\left(\frac{2}{\sqrt{5}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\right) = \pi r^2\left(\frac{5+\sqrt{5}}{\sqrt{5}}\right).$$

This is the required surface.

11. Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a core is half that of the cone. **[2007, Fall]**

Solution: Given that a right circular cylinder is inscribed in a cone.

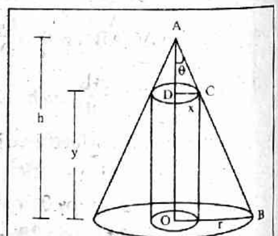
Let s = curved surface area of cylinder
x = radius of the cylinder
y = height of the cylinder
h = height of the core
r = radius of the cone.
and, θ = semi–vertical angle of the cone

From the corresponding figure in ΔADC,

$$x = AD\tan\theta = (h - y)\tan\theta$$

and,

$$s = 2\pi y\,(h - y)\tan\theta = 2\pi\,(hy - y^2)\tan\theta$$



So, $$\frac{ds}{dy} = 2\pi\,(h - 2y)\tan\theta$$

and $$\frac{d^2s}{dy^2} = -4\pi\tan\theta < 0.$$

This means the curved surface area will be minimum at the critical point.

For the critical point, set

$$\frac{ds}{dy} = 0 \Rightarrow 2\pi\,(h - 2y)\tan\theta = 0$$

$$\Rightarrow h - 2y = 0$$

[Being tanθ = 0 ⇒ θ = 0 which is impossible.]

$$\Rightarrow h = 2y.$$

This shows that the curved surface area will be minimum when height of the cylinder is half of height of the cone.

12. Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h. **[2014 Fall][2002][2007, Spring] [2000]**

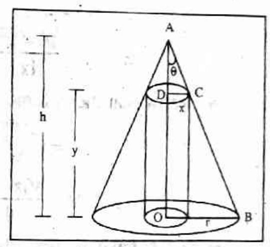
OR A cylinder is inscribed in a given cone of height h. Find the height of the cylinder for which the volume is maximum. **[2006, Spring]**

Solution:

Let x = radius of the cylinder
y = height of the cylinder
r = radius of the cone
V = volume of the cylinder
h = height of the cone.
and, θ = semi–vertical angle of the cone.

Form the corresponding figure in ΔADC,

$$x = (h - y)\tan\theta$$



Let v be the volume of the cylinder. So,

$$V = \pi x^2 y = \pi(h - y)^2\tan^2\theta\; y$$

$$\Rightarrow V = \pi\tan^2\theta\,(h^2y - 2hy^2 + y^3)$$

Then,

$$\frac{dV}{dy} = \pi\tan^2\theta\,(h^2 - 4hy + 3y^2)$$

and $$\frac{d^2V}{dy^2} = \pi\tan^2\theta\,(-4h + 6y)$$

For critical point, set,

$$\frac{dV}{dy} = 0 \Rightarrow \pi\tan^2\theta\,(h^2 - 4hy + 3y^2) = 0$$

$$\Rightarrow 3y^2 - 4hy + h^2 = 0$$

$$\Rightarrow 3y^2 - 3hy - hy + h^2 = 0$$

$$\Rightarrow (3y - h)(y - h) = 0$$

$$\Rightarrow \quad y = h, \frac{h}{3}$$

When y = h, the cylinder cannot be inscribed in a given cone of height h.

When $y = \frac{h}{3}$,

$$\frac{d^2V}{dy^2} = \pi \tan^2\theta \left(-4h + 6\left(\frac{h}{3}\right)\right) = -2h\pi \tan^2\theta < 0$$

Thus when $y = \frac{h}{3}$, the volume of the cylinder is maximum.

13. If 40 sq. feet of sheet metal are to be used in the construction of an open tank with a square base. Find the dimension in order that its capacity is greatest. **[2008, Spring]**

<u>**Solution:**</u> Let the given open tank has square base.

Let $\quad x$ = length = breadth of base

$\quad y$ = height of the tank

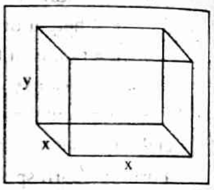
Given that 40 sq ft of steel metal are to be used in the construction of an open tank. So,

$$A = 40 \text{ sq. feet}$$

So, $\quad A = x^2 + 4xy \quad$ [being given tank is open tank]

$$\Rightarrow \quad x^2 + 4xy = 40$$

$$\Rightarrow \quad y = \frac{40 - x^2}{4x} \quad \ldots\ldots \text{(i)}$$



We know that the volume of the tank is,

$$V = x^2y = x^2\left(\frac{40 - x^2}{4x}\right)$$

$$\Rightarrow \quad V = \frac{40x - x^3}{4} \quad \ldots \text{(ii)}$$

Here we have to find the dimension of the tank when the capacity is greatest i.e. maximum.

Differentiating (ii) w.r.t. x,

$$\frac{dV}{dx} = \frac{40 - 3x^2}{4} \quad \text{and} \quad \frac{d^2V}{dx^2} = \frac{-6x}{4} = \frac{-3x}{2}$$

For the extreme point, set

$$\frac{dV}{dx} = 0 \Rightarrow 40 - 3x^2 = 0 \quad \Rightarrow x = \sqrt{\frac{40}{3}}.$$

At $x = \sqrt{\frac{40}{3}}$,

$$\frac{d^2V}{dx^2} = \left(\frac{-3}{2}\right)\left(\sqrt{\frac{40}{3}}\right) < 0.$$

This means the volume of the tank is maximum.

Also,

$$y = \frac{40 - x^2}{4x} = \left(\sqrt{\frac{3}{40}}\right)\left(\frac{40 - \frac{40}{3}}{4}\right) = \left(\sqrt{\frac{3}{40}}\right)\left(\frac{20}{3}\right) = \sqrt{\frac{10}{3}}$$

Thus, capacity of the tank is greatest when its dimensions are

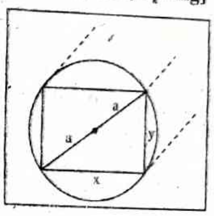
$$\text{length} = \text{breadth} = \sqrt{\frac{40}{3}} \text{ feet, and height} = \sqrt{\frac{10}{3}} \text{ feet.}$$

14. The strength of a beam varies jointly as its breadth and the square of the depth. Find the dimension of the strongest beam that can be cut from a circular wooden log of raidus a.

[2012 Fall][2011 Fall][2003, Spring]

<u>**Solution:**</u> Let $\quad x$ = breadth of the beam

$\quad y$ = depth of the beam



So, given that the radius of circular wood a.

Then,

$$x^2 + y^2 = (2a)^2$$

$$\Rightarrow \quad y^2 = 4a^2 - x^2 \quad \ldots\ldots \text{(i)}$$

Let,

$$S = \text{strength at the beam}$$

Given that the strength of the beam varies jointly as its breadth and the square of the depth. So,

$$S = \lambda xy^2 \quad \text{where } \lambda \text{ be a constant value.}$$

$$= \lambda x(4a^2 - x^2) = 4\lambda a^2x^2 - \lambda x^3 \quad \text{[Using (i)]}$$

Differentiating w.r.t. x then,

$$\frac{dS}{dx} = 4\lambda a^2 - 3\lambda x^2 \quad \text{and} \quad \frac{d^2S}{dx} = -6\lambda x.$$

For extreme point, set

$$\frac{dS}{dx} = 0 \Rightarrow 4\lambda a^2 = 3\lambda x^2 \Rightarrow x = \frac{2a}{\sqrt{3}}.$$

Now, from (i),

$$y^2 = 4a^2 - \frac{4a^2}{3} \Rightarrow y^2 = \frac{8a^2}{3} \Rightarrow y = \frac{2a\sqrt{2}}{\sqrt{3}}.$$

At $x = \frac{2a}{\sqrt{3}}$,

$$\frac{d^2S}{dx^2} = -6\lambda\frac{2a}{\sqrt{3}} < 0.$$

So, S will be maximum when $x = \frac{2a}{\sqrt{3}}$.

Thus, the dimension of the beam is,

$$\text{breadth} = \frac{2a}{\sqrt{3}} \text{ and depth} = \frac{2a\sqrt{2}}{\sqrt{3}}.$$

15. A cone is circumscribed to a sphere of radius r, show that when the volume of the cone is least its altitude is 4r and its semi–vertical angle is $\sin^{-1}\left(\frac{1}{3}\right)$.

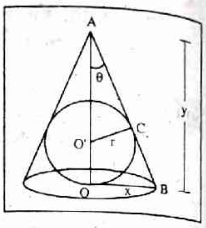
Solution: Let the cone is circumscribed to a sphere of radius r.

Let x = radius of base of the cone

y = height of the cone

which is circumscribed to a sphere of radius r.

Let θ = semi–vertical angle of the cone.



We know that volume of cone $(V) = \frac{1}{3}\pi x^2 y$

From figure, ΔACO' and ΔAOB are similar. So,

$$\frac{HC}{OB} = \frac{AC}{AB} \Rightarrow \frac{r}{x} = \frac{y-r}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \frac{r^2(x^2+y^2)}{x^2} = (y-r)^2$$

$$\Rightarrow r^2 + \frac{(yr)^2}{x^2} = y^2 - 2ry + r^2$$

$$\Rightarrow x^2 = \frac{y^2r^2}{(y^2-2ry)} = \frac{yr^2}{y-2r}$$

Since the volume of the cone is,

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi\left(\frac{yr^2}{y-2r}\right)y = \frac{\pi}{3}\left(\frac{y^2r^2}{y-2r}\right).$$

Differentiating w.r.t. y then,

$$\frac{dV}{dy} = \frac{\pi}{3}\left[\frac{2yr^2(y-2r) - y^2r^2}{(y-2r)^2}\right]$$

$$= \frac{\pi}{3}\left[\frac{2y^2r^2 - 4yr^3 - y^2r^2}{(y-2r)^2}\right] = \frac{\pi}{3}\left[\frac{y^2r^2 - 4yr^3}{(y-2r)^2}\right].$$

and,

$$\frac{d^2V}{dy^2} = \frac{\pi}{3}\left(\frac{(2yr^2 - 4r^3)(y-2r)^2 - (y^2r^2 - 4yr^3).2(y-2r)}{(y-2r)^4}\right).$$

For extreme point,

$$\frac{dV}{dy} = 0 \Rightarrow \frac{\pi}{3}\left[\frac{y^2r^2 - 4yr^3}{(y-2r)^2}\right] = 0$$

$$\Rightarrow y^2r^2 - 4yr^3 = 0$$

$$\Rightarrow y = 4r. \quad \text{[being y = 0 is impossible]}$$

Therefore at y = 4r,

$$\frac{d^2V}{dy^2} = \frac{\pi}{3}\left[\frac{(8r^3 - 4r^3)(4r-2r)^2 - (16r^4 - 16r^4).2(4r-2r)}{(4r-2r)^4}\right]$$

$$= \frac{\pi}{3}\left(\frac{16r^5}{(2r)^4}\right) > 0.$$

So volume (V) is least.

Thus, the cone has attitude 4r when its volume is least.

Also, from ΔO'AC,

$$\sin\theta = \frac{O'C}{OA} \Rightarrow \sin\theta = \frac{r}{y-r} = \frac{r}{3r} = \frac{1}{3}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{3}\right).$$

16. For a given curved surface of right circular cone when the volume is maximum, prove that the semi–vertical angle is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

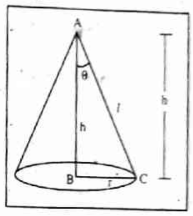
Solution: Let θ be the semi–vertical angle and l be the slant height of the cone ABC with vertex at A.

Then radius of the cone = r = l sin θ.

And height of the cone = h = l cos θ.

Let S be the surface area of the cone which is given. So, S is constant.

Since,



$$S = \pi l.r = \pi l^2 \sin\theta = \text{constant (given)}.$$

$$\Rightarrow l^2 = \frac{S}{\pi \sin\theta}$$

And volume of the cone is,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi l^3 \sin^2\theta \cos\theta.$$

$$\Rightarrow V^2 = \frac{1}{9}\pi^2 l^6 \sin^4\theta \cos^2\theta$$

$$= \frac{1}{9}\pi^2\left(\frac{S^3}{\pi^3\sin^3\theta}\right)\sin^4\theta\cos^2\theta.$$

$$= \frac{1}{9\pi}S^3 \sin\theta\cos^2\theta$$

$$= \frac{1}{9\pi}S^3[\sin\theta - \sin^3\theta].$$

So,

$$\frac{d(V^2)}{d\theta} = \frac{1}{9\pi}S^3[\cos\theta - 3\sin^2\theta\cos\theta]$$

$$= \frac{1}{9\pi}S^3[-2\cos\theta + 3\cos^3\theta].$$

And

$$\frac{d^2(V^2)}{d\theta^2} = \left(\frac{-1}{9\pi}\right)S^3[2\sin\theta - 9\cos^2\theta\sin\theta]$$

$$= \left(\frac{-1}{9\pi}\right)S^3[2\sin\theta - 9(1-\sin^2\theta)\sin\theta]$$

$$= \left(\frac{-1}{9\pi}\right)S^3[9\sin^3\theta - 7\sin\theta]$$

For extreme point, set.

$$\frac{d(V^2)}{d\theta} = 0$$

$$\Rightarrow \cos\theta\,(-2 + 3\cos^2\theta) = 0.$$

$$\Rightarrow \cos\theta = 0 = \cos\frac{\pi}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

$$\text{or} \quad -2 + 3\cos^2\theta = 0 \quad \Rightarrow \quad -2 + 3 - 3\sin^2\theta = 0.$$

$$\Rightarrow \sin^2\theta = \frac{1}{3}$$

$$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$$

At $\theta = \sin^{-1}\left(\frac{-1}{\sqrt{3}}\right)$,

$$\frac{d^2(V^2)}{d\theta^2} = \left(\frac{-1}{9\pi}\right)S^3\left[9\left(\frac{-1}{\sqrt{3}}\right)^3 - 7\left(\frac{-1}{\sqrt{3}}\right)\right] = \left(\frac{-1}{9\pi}\right)S^3\left(\frac{-2}{3\sqrt{3}}\right) > 0.$$

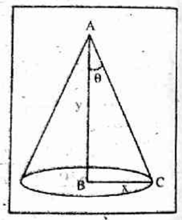
This means V is minimum, which is not of our interest.

At $\theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$,

$$\frac{d^2(V^2)}{d\theta^2} = \left(\frac{-1}{9\pi}\right)S^3\left[9\left(\frac{1}{\sqrt{3}}\right)^3 - 7\left(\frac{1}{\sqrt{3}}\right)\right] = \left(\frac{-1}{9\pi}\right)S^3\left(\frac{2}{3\sqrt{3}}\right) < 0.$$

So, the volume is maximum when $\theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

17. For a given volume of a right cone, show that when the curve surface is minimum, semi-vertical angle is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$.



Solution: Let θ = semi-vertical angle
x = radius of the cone
y = height of the cone
V= volume of the cone
S = curved surface area of the cone.

Since the volume of the right circular cone is

$$V = \frac{\pi x^2 y}{3} \quad \text{which is given, so is constant.}$$

$$\Rightarrow y = \frac{3V}{\pi x^2}.$$

And the curved surface area of the cone is

$$S = \pi x\sqrt{x^2 + y^2}$$

$$\Rightarrow S^2 = \pi^2 x^2 (x^2 + y^2)$$

$$= \pi^2\left(x^4 + x^2 \cdot \frac{9V^2}{\pi^2 x^4}\right)$$

$$\Rightarrow S^2 = \pi^2\left(x^4 + \frac{9V^2}{\pi x^2}\right)$$

Then

$$\frac{d(S^2)}{dx} = \pi^2\left(4x^3 - \frac{18V^2}{\pi^2 x^3}\right) \quad \text{and} \quad \frac{d^2(S^2)}{dx^2} = \pi^2\left(12x^2 + \frac{54V^2}{\pi^2 x^4}\right)$$

For extreme point, set

$$\frac{d(S^2)}{dx} = 0 \quad \Rightarrow \pi^2\left(4x^3 - \frac{18V^2}{\pi^2 x^3}\right) = 0$$

$$\Rightarrow 4x^3 = \frac{18V^2}{\pi^2 x^3}$$

$$\Rightarrow x^6 = \frac{9V^2}{2\pi^2} \quad \Rightarrow x = \left(\frac{3V}{\pi\sqrt{2}}\right)^{1/3}$$

At $x = \left(\frac{3V}{\pi\sqrt{2}}\right)^{1/3}$

$$\frac{d^2(S^2)}{dx^2} = \pi^2\left(12x^2 + \frac{54V^2}{\pi^2 x^4}\right) > 0, \text{ being x is positive.}$$

So S^2 (i.e. (S) is minimum.

Also,

$$y = \frac{3V}{\pi x^2} = \frac{3V}{\pi\left(\frac{3V}{\pi\sqrt{2}}\right)^{2/3}} = \left(\frac{6V}{\pi}\right)^{1/3}.$$

Here,

$$x^2 + y^2 = \left(\frac{3V}{\pi\sqrt{2}}\right)^{2/3} + \left(\frac{6V}{\pi}\right)^{2/3}$$

$$= \left(\frac{3V}{\pi}\right)^{2/3}\left(\frac{1}{2^{1/3}} + 2^{2/3}\right) = \left(\frac{3V}{\pi}\right)^{2/3}\left(\frac{1+2}{2^{1/3}}\right) = \left(\frac{3V}{\pi}\right)^{2/3}\left(\frac{3}{2^{1/3}}\right)$$

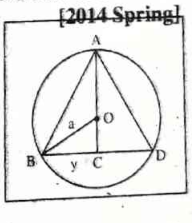
Now, from figure,

$$\sin\theta = \frac{x}{\sqrt{x^2 + y^2}} = \left(\frac{3V}{\pi\sqrt{2}}\right)^{1/3}\left(\frac{\pi}{3V}\right)^{1/3}\left(\frac{2^{1/3}}{3}\right)^{1/2} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

18. Find the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius a. **[2014 Spring]**



Solution: Let ABC is a cone with vertex at A that is inscribed in a sphere of radius a. Let,
y = radius of cone
x = height of the cone
V = volume of the cone

a = radius of the cone.

Now, from figure,

$OC = AC - AO = x - a$

And, $BC^2 = OB^2 - OC^2$

$\Rightarrow y^2 = a^2 - (x - a)^2$

$\Rightarrow y^2 = 2ax - x^2$

Since, the volume of the cone ABC is,

$$V = \frac{\pi}{3} y^2 x = \frac{\pi}{3}(2ax^2 - x^3)$$

Differentiating w.r.t. x,

$$\frac{dV}{dx} = \frac{\pi}{3}(4ax - 3x^2) \text{ and } \frac{d^2V}{dx^2} = \frac{\pi}{3}(4a - 6x).$$

For extreme point, set

$$\frac{dV}{dx} = 0 \Rightarrow \frac{\pi}{3}(4ax - 3x^2) = 0 \Rightarrow 4ax = 3x^2 \Rightarrow x = \frac{4a}{3}$$

being x = 0 is not possible.

Then at $x = \frac{4a}{3}$,

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}\left(4a - \frac{24a}{3}\right) = -\frac{4a\pi}{3} < 0.$$

So, the volume of the cone is maximum when $x = \frac{4a}{3}$.

Thus, the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius a, is $\frac{4a}{3}$.

19. Show that the maximum rectangular that can be inscribed in a circle is a square.

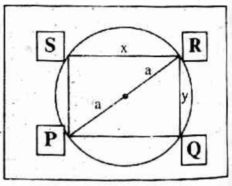
Solution: Let PQRS is a rectangle that can be inscribed in a circle. Let,

x = length of the rectangle

y = breadth of the rectangle

A = area of the rectangle

a = radius of the rectangle.



Now, from the figure in ΔPQR,

$(2a)^2 = x^2 + y^2$

$\Rightarrow y^2 = 4a^2 - x^2$

Since the area of the rectangle PQRS is xy. That is,

$A = xy$

$\Rightarrow A^2 = x^2y^2$

$\Rightarrow A^2 = x^2(4a^2 - x^2) = 4a^2x^2 - x^4$

Then,

$$\frac{dA^2}{dx} = 8a^2x - 4x^3 \text{ and } \frac{d^2A^2}{dx^2} = 8a^2 - 12x^2.$$

For extreme point, set

$$\frac{dA^2}{dx} = 0 \Rightarrow 8a^2x - 4x^3 = 0$$

$$\Rightarrow 8a^2 = 4x^2 \Rightarrow x^2 = 2a^2 \Rightarrow x = a\sqrt{2}.$$

At $x = a\sqrt{2}$,

$$\frac{d^2A^2}{dx^2} = 8a^2 - 12a^2 = -4a^2 < 0.$$

So, A^2 (so A) is maximum when $x = a\sqrt{2}$.

And, at $x = \sqrt{2}a$

$$y = \sqrt{4a^2 - 2a^2} = a\sqrt{2} = x.$$

Therefore, the rectangle is square when area is maximum.

20. An open tank of a given volume consists of a square base with vertical sides. Show that the expense of lining the tank with lead will be least, if the height of the tank is half the width. **[2016 Fall][2009 Spring]**

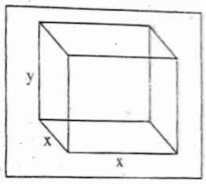
Solution: Let, the open tank of a given volume consists of a square base with vertical sides. Let,

x = length = breadth of a base

y = height of the tank

V = volume of the tank (which is given, so is constant)

L = lead required for lining



We know the volume of the tank is,

$$V = x^2y \Rightarrow y = \frac{c}{x^2} = \frac{V}{x^2}$$

Since the lining is used in corner lines.

So, $L = 4x + 4y = 4x + \frac{4V}{x^2}$

Differentiating w.r.t. x then,

$$\frac{dL}{dx} = 4 - \frac{8V}{x^3} \text{ and } \frac{d^2L}{dx} = \frac{24V}{x^4}.$$

For extreme point, set

$$\frac{dL}{dx} = 0 \Rightarrow 4 - \frac{8V}{x^3} = 0 \Rightarrow x^3 = 2V \Rightarrow x = (2V)^{1/3}$$

At $x = (2V)^{1/3}$

$$\frac{d^2V}{dx^2} = \frac{24V}{(2V)^{1/3}} > 0,$$

So L is minimum when $x = (2V)^{1/3}$

Also,

$x = (2V)^{1/3}$

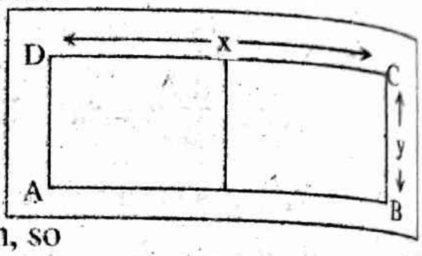
$\Rightarrow x = (2x^2y)^{1/3}$ [Being $V = x^2y$]

$\Rightarrow x = x^{2/3}(2y)^{1/3}$

$\Rightarrow x^{1/3} = (2y)^{1/3} \Rightarrow x = 2y \Rightarrow y = \frac{x}{2}$.

This shows the height of the tank is half the width.

21. The gardener having 120m of fancing wishes to endssed a rectangular plot of land and also ertest a fence across the land parallel to two of the sides what is the maximum area be can enclose?



Solution: Let ABCD is a garden that has

x = length of the garden

y = breadth of the garden

A = area of the garden

Given that the perimeter of the garden is 120m, so

$$2x + 2y + y = 120$$

$$\Rightarrow 2x + 3y = 120$$

$$\Rightarrow y = \frac{120 - 2x}{3}.$$

And the area of the garden is

$$A = xy = x\left(\frac{120 - 2x}{3}\right) = \frac{120x - 2x^2}{3}$$

Then,

$$\frac{dA}{dx} = \frac{120 - 4x}{3} \quad \text{and} \quad \frac{d^2A}{dx^2} = \frac{-4}{3} < 0.$$

So A is maximum at the critical point.

For extreme point, set

$$\frac{dA}{dx} = 0 \Rightarrow \frac{120 - 4x}{3} = 0 \Rightarrow x = 30.$$

When $x = 30$m we get $y = 20$m.

Hence the maximum value of A is,

$$A = 30 \times 20 = 600 \text{ m}^2.$$

22. The electric current i in a circuits varies according to the equation $i = 2t + \frac{200}{t}$, where i is in amperes and $t > 0$ is in seconds. Determine the minimum current.

Solution: Given that the current equation is,

$$i = 2t + \frac{200}{t}.$$

Then,

$$\frac{di}{dt} = 2 - \frac{200}{t^2}, \quad \text{and} \quad \frac{d^2i}{dt^2} = \frac{400}{t^3}.$$

For extreme point, set

$$\frac{di}{dt} = 0 \Rightarrow 2 - \frac{200}{t^2} = 0 \Rightarrow t^2 = 100 \Rightarrow t = 10.$$

Then at $t = 10$,

$$\frac{d^2i}{dt^2} = \frac{400}{1000} > 0.$$

So, the current is minimum when $t = 10$ and minimum current is,

$$i = 20 + 20 = 40.$$

23. In driving a motor boat, the petrol burnt per hour varies directly as the cube of its velocity. Find the most economical trip of the boat when going against a current of 2 km/hr.

Solution: Let, velocity of the boat relative to the water = v.

Current of water = 2 km/hr (given).

Distance to be traveled = d

Then, time required while going against the current $= \dfrac{d}{v-2}$

Also, given that the petrol burnt per hour varies directly as cube of its velocity. i.e. $P \propto v^3$

Then, $P = kv^3$ (let) for some constant k.

Then total amount of total petrol burnt is,

$$P_T = \frac{d}{(v-2)} \times kv^3$$

Now, we have to find the most economical trip of the boat. We know the trip will be most economical when the total amount of total petrol burnt will be least. That is, we should find P_T is minimum.

Here,

$$\frac{d(P_T)}{dv} = kd\left[\frac{(v-2)\,3v^2 - v^3}{(v-2)^2}\right] = kd\left(\frac{2v^3 - 6v^2}{(v-2)^2}\right) = 2kd\left(\frac{v^3 - 3v^2}{(v-2)^2}\right)$$

And,

$$\frac{d^2(P_T)}{dv^2} = 2kd\left[\frac{(v-2)^2(3v^2 - 6v) - (v^3 - 3v^2)\,2(v-2)}{(v-2)^4}\right]$$

For extreme point, set

$$\frac{d(P_T)}{dv} = 0 \quad \Rightarrow \frac{2v^3 - 6v^2}{(v-2)^2} = 0$$

$$\Rightarrow v - 3 = 0 \qquad [\because 2v^2 \neq 0]$$

$$\Rightarrow v = 3.$$

At $v = 3$,

$$\frac{d^2(P_T)}{dv^2} = 2kd\left[\frac{(1)(9) - (0)\,2(1)}{(1)}\right] = 18kd > 0.$$

This means the least amount of petrol will burnt if the boat has velocity v = 3km/hr, which is the most economical trip of the boat.

Chapter 6

ASYMPTOTES

Exercise 6.1

1. Find the asymptote of following curves:

(i) $y = \dfrac{3}{x-7}$

Solution: Here,

$$y = \frac{3}{x-7} \Rightarrow xy - 7y - 3 = 0.$$

Clearly the given equation is of degree 2. So, it may have at most 2 asymptotes.

Here, both x^2 and y^2 are absent. So, there may exist asymptotes parallel to x–axis and y–axis.

For horizontal asymptotes, equate the coefficient of highest power of x to zero.

i.e. coeff. of $x = 0$

or, $y = 0$

And for vertical asymptotes, equate the coefficient of highest power of y to zero.

i.e. coeff. of $y = 0$

or, $x - 7 = 0$.

Since the given curve cannot have more than two asymptotes, so the given curve has no oblique asymptotes.

Therefore, $y = 0$ and $x = 7$ are the required asymptotes of the given curve.

Alternative Method:

Here,

$$y = \frac{3}{x-7}$$

Here,

$$\lim_{x\to 7}(y) = \lim_{x\to 7}\left(\frac{3}{x-7}\right) = \infty.$$

This means $x = 7$ is vertical asymptotes to the given curve.

And,

$$\lim_{x\to\infty}(y) = \lim_{x\to\infty}\left(\frac{3}{x-7}\right) = 0$$

also, $$\lim_{x\to-\infty}(y) = \lim_{x\to-\infty}\left(\frac{3}{x-7}\right) = 0$$

This means $y = 0$ be horizontal asymptotes to the given curve.

(ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution: Here,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x^2 - a^2y^2 - a^2b^2 = 0 \quad \text{... (i)}$$

Clearly the given is of degree 2. So, it may have at most 2–asymptotes.

Here both x^2 and y^2 are present. So, there is no asymptote parallel to x–axis and y–axis.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = b^2 - a^2m^2, \qquad \phi_1(m) = 0, \qquad \phi_0(m) = -a^2b^2.$$

For value m, set $\phi_2(m) = 0$

$$\Rightarrow \quad b^2 - a^2m^2 = 0 \qquad \Rightarrow m = \pm\frac{b}{a}.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = \frac{0}{-2a^2} = 0$$

Thus, $y = \pm\frac{b}{a}x \Rightarrow ay = \pm bx$ are the oblique asymptotes to the given curve.

Therefore, $ay = \pm bx$ are the asymptotes to the given curve.

(iii) $x^2y = 1$ **(for** $x \neq 0$**)**

Solution: Given curve is $x^2y - 1 = 0$ with $x \neq 0$.

Clearly the given curve is of third degree. So, it may have at most 3–asymptotes.

Also, x^3 and y^3 are absent. So, the vertical and horizontal asymptotes may exist.

For horizontal asymptotes, equate the coefficient of highest power of y with zero.

i.e. coeff. of $x^2 = 0$

or, $y = 0$

and for vertical asymptotes, equate the coefficient of highest power of y with zero.

i.e. coeff. of $y = 0$

or, $x^2 = 0 \Rightarrow x = 0$

which is not possible because $x \neq 0$.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = m, \qquad \phi_2(m) = 0, \qquad \phi_1(m) = -1.$$

For value of m. equate $\phi_3(m)$ to zero.

i.e. $\phi_3(m) = 0 \Rightarrow m = 0$.

Here, m has non–repeated value. So, for the value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{0}{1} = 0$$

Thus $y = mx + c \Rightarrow y = 0$ be an oblique asymptotes.

Therefore, $y = 0$ is the asymptotes to the given curve.

(iv) $y = \dfrac{1}{(x+2)^2}$

Solution: Here,

$$y = \frac{1}{(x+2)^2} \quad \Rightarrow x^2y + 4xy + 4y - 1 = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3–asymptotes.

Here x^3 and y^3 are absent, so there may exist vertical and horizontal asymptotes.

For vertical asymptotes, equate the coefficient of highest power of y, to zero. That is, coeff. of $y = 0$.

or, $x^2 + 4x + 4 = 0$

$\Rightarrow \quad (x + 2)^2 = 0.$

$\Rightarrow \quad x = -2.$

So the vertical asymptote is $x + 2 = 0$.

And, for horizontal asymptotes, equate the coefficient of highest power of x, to zero. That is, coeff. of $x^2 = 0$

or $\quad y = 0$

So, the horizontal asymptotes to the given curve is $y = 0$.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = m, \qquad \phi_2(m) = 2m, \qquad \phi_1(m) = 4.$$

For value of m, set $\phi_3(m) = 0$

$\Rightarrow \quad m = 0$

Thus, m has non–repeated value. So, for value of corresponding c.

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2m}{1} = 0 \qquad [\text{since } m = 0]$$

So, $y = mx + c \Rightarrow y = 0$ is an oblique asymptote.

Therefore, $x + 2 = 0$, $y = 0$ are the asymptotes to the given curve.

(v) $x^2 - y^2 = 1$.

Solution: Given curve is,

$$x^2 - y^2 - 1 = 0.$$

Clearly this curve is of second degree. So, it may have at most 2–asymptotes.

Here x^2, y^2 are present so, there is no asymptotes parallel to x–axis and y–axis.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = 1 - m^2, \qquad \phi_1(m) = 0, \qquad \phi_0(m) = -1.$$

For value of m, set $\phi_2(m) = 0$.

$\Rightarrow \quad 1 - m^2 = 0 \quad \Rightarrow \quad m = \pm 1.$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = \frac{0}{-2m} = 0 \quad \text{for } m = \pm 1.$$

Thus, $y = mx + c \Rightarrow y = \pm x$ are oblique asymptotes.

Therefore, $y = \pm x$ are the asymptotes to the given curve.

(vi) $y = e^{1/x}$

Solved in the section, asymptotes of non–algebra functions.

(vii) $x^3 + y^3 = 3xy$.

Solution: Given curve is

$$x^3 + y^3 - 3xy = 0.$$

Clearly, given curve is of third degree. So, it may have at most three asymptotes.

Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + m^3, \quad \phi_2(m) = -3m, \quad \phi_1(m) = 0.$$

For value of m, set $\phi_3(m) = 0$

$$\Rightarrow 1 + m^3 = 0 \quad \Rightarrow m = -1$$

Thus, m has non–repeated value. So, for value of corresponding c is found by,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(-3m)}{3m^2} = -1$$

Thus, $y = -x - 1$ is an oblique asymptotes.

Therefore, $y = -x - 1$ is the asymptotes to the given curve.

(viii) $x^3 - y^3 = 6x^2$.

Solution: Given curve is

$$x^3 - y^3 - 6x^2 = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3–asymptotes.

Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 - m^3, \qquad \phi_2(m) = -6, \qquad \phi_1(m) = 0.$$

For value of m, let $\phi_3(m) = 0$

$$\Rightarrow 1 - m^3 = 0 \quad \Rightarrow m = 1.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{6}{-3m^2} = -2. \qquad \text{[Being } m = 1\text{]}$$

Thus, $y = x - 2$ be an oblique asymptote.

Therefore, $x - y - 2 = 0$ be the asymptotes to the given curve.

(ix) $x^3 + y^3 = a^2x$

Solution: Given curve is

$$x^3 + y^3 - a^2x = 0.$$

Clearly, the given curve is of degree 3. So, it may have at most 3–asymptotes.

Here, both x^3 and y^3 are present. So, there is no vertical and horizontal asymptotes exist.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + m^3, \qquad \phi_2(m) = 0, \qquad \phi_1(m) = -a^2.$$

For value of m, set $\phi_3(m) = 0 \Rightarrow 1 + m^3 = 0$

$$\Rightarrow m = 1$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(0)}{(3m^2)} = \frac{-0}{3} = 0.$$

Thus, $y = mx + c \Rightarrow y = x$ is an oblique asymptotes.

Therefore, $x = y$ is the asymptotes to the given curve.

(x) $x^2 - y^2 - 6x + 4y + 1 = 0.$

Solution: Given curve is

$$x^2 - y^2 - 6x + 4y + 1 = 0.$$

Clearly, the given curve is of degree 2. So, it may have at most 2–asymptotes.

Here, both x^2 and y^2 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = 1 - m^2, \qquad \phi_1(m) = -6 + 4m, \qquad \phi_0(m) = 0.$$

For value of m, set $\phi_2(m) = 0 \Rightarrow 1 - m^2 = 0$

$$\Rightarrow m = \pm 1.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = -\frac{(-6 + 4m)}{(-2m)} = \frac{-3 + 2m}{m}$$

For $m = 1$, the corresponding value of c is,

$$c = \frac{-3 + 2}{1} = -1,$$

and for $m = -1$, the corresponding value of c is,

$$c = \frac{-3 - 2}{-1} = 5.$$

Thus, $y = mx + c \Rightarrow y = x - 1, y = -x + 5$ are oblique asymptotes.

Therefore, $x - y + 1 = 0, x + y = 5$ are the asymptotes to the given curve.

(xi) $x^3 - 4y^3 + 3x^2 + y - x + 3 = 0.$

Solution: Given curve be $x^3 - 4y^3 + 3x^2 + y - x + 3 = 0.$

Clearly, the given curve is of third degree. So, it may have at most 3–asymptotes.

Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 - 4m^3, \qquad \phi_2(m) = 3, \quad \phi_1(m) = m - 1.$$

For value of m, let $\phi_3(m) = 0 \Rightarrow 1 - 4m^3 = 0$

$$\Rightarrow m^3 = \frac{1}{4} \quad \Rightarrow m = \left(\frac{1}{4}\right)^{1/3} = (2)^{-2/3}$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{3}{-12m^2} = \frac{1}{4(2)^{-4/3}} = \frac{2^{4/3}}{4} = \frac{2(2)^{1/3}}{4} = \frac{1}{2^{2/3}} = (2)^{-2/3}$$

Thus, $y = mx + c \Rightarrow y = (2)^{-2/3}x + (2)^{-2/3}$ be an oblique asymptotes.

Therefore, $y = (2)^{-2/3}x + (2)^{-2/3}$ be the asymptotes to the given curve.

(xii) $y = x^2 - 1$.

Solution: Given curve is

$$y - x^2 + 1 = 0.$$

Clearly, the given curve is of second degree. So, it may have at most 2–asymptotes.

Here, x^2 is present but y^2 is absent. So, there is no horizontal asymptote exist.

And for vertical asymptotes, equate the coefficient of highest power of y, to zero. That is, coeff. of $y = 0$

$\Rightarrow \quad 1 = 0$ which is impossible.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = -1.$$

And for value of m, set

$\phi_2(m) = 0 \Rightarrow -1 = 0,$ which is impossible.

Therefore, the given curve has no asymptotes.

2. Find the asymptotes of the following curves

(i) $y^2 - x^2 - 2x - 2y - 3 = 0$.

Solution: Given curve is

$$y^2 - x^2 - 2x - 2y - 3 = 0.$$

Clearly, the given curve is of second degree. So, it may have at most 2–asymptotes.

And, the equation includes both x^2 and y^2, so there is no asymptotes exist parallel to x–axis and y–axis.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_2(m) = m^2 - 1, \quad \phi_1(m) = -2 - 2m, \quad \phi_0(m) = -3.$$

For value of m, set

$$\phi_2(m) = 0 \Rightarrow m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = -\frac{-2-2m}{2m} = \frac{1+m}{m}$$

For, $m = 1$, the corresponding value of c is,

$$c = 2.$$

And, for, $m = -1$, the corresponding value of c is,

$$c = 0.$$

Thus, $y = mx + c \Rightarrow y = x + 2,\ y = -x$ be the oblique asymptotes.

Therefore, $x - y + 2 = 0,\ x + y = 0$ are the asymptotes to the given curve.

(ii) $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0$.

Solution: Given curve is

$$x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3–asymptotes.

And the curve includes both x^3 and y^3. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + 3m - m^2 - 3m^3, \quad \phi_2(m) = 1 - 2m + 2m^2, \quad \phi_1(m) = 4.$$

For value of m, set

$$\phi_3(m) = 0 \quad \Rightarrow 1 + 3m - m^2 - 3m^3 = 0$$

$$\Rightarrow (1 + 3m)(1 - m^2) = 0$$

$$\Rightarrow m = \pm 1, -\frac{1}{3}$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(1 - 2m + 2m^2)}{-3 - 2m - 9m^2} = \frac{1 - 2m + 2m^2}{3 + 2m + 9m^2}$$

For $m = 1$, the corresponding value of c is,

$$c = \frac{1 - 2 + 3}{3 + 2 + 9} = \frac{4}{14} = \frac{2}{7}$$

And, for $m = -1$, the corresponding value of c is,

$$c = \frac{1 + 2 + 3}{3 + 2 + 9} = \frac{6}{14} = \frac{3}{7}$$

Also, for $m = \frac{1}{3}$, the corresponding value of c is,

$$c = \frac{1 - \frac{2}{3} + \frac{2}{9}}{3 + \frac{2}{3} + \frac{9}{9}} = \frac{9 - 6 + 2}{27 + 6 + 9} = \frac{6}{42} = \frac{1}{7}.$$

Thus, the oblique asymptotes are

$$y = x + \frac{1}{14},\ y = -x + \frac{6}{14} \text{ and } \quad y = \frac{x}{3} + \frac{6}{42}.$$

Therefore, the asymptotes to the given curve are

$$7y = 7x + 1,\ 7y + 7x = 3,\ 21y = 7x + 3.$$

(iii) $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$.

Solution: Given curve is

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

Clearly, the given curve is of third degree. So, it may have at most 3–asymptotes. Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 3 + 2m - 7m^2 + 2m^3,$$

$\phi_2(m) = -14m + 7m^2$

$\phi_1(m) = 4 + 5m.$

For the value of m, set

$$\phi_3(m) = 0 \Rightarrow 3 + 2m - 7m^2 + 2m^3 = 0$$
$$\Rightarrow 2m^3 - 2m^2 - 5m^2 + 5m - 3m + 3 = 0$$
$$\Rightarrow (m-1)(2m^2 - 5m - 3) = 0$$
$$\Rightarrow (m-1)(2m+1)(m-3) = 0$$
$$\Rightarrow m = 1, 3, -\frac{1}{2}$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{7m(2-m)}{2 - 14m + 6m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{7(2-1)}{2 - 14 + 6} = \frac{7}{-6}$$

And, for m = 3, the corresponding value of c is,

$$c = \frac{21(2-3)}{2 - 42 + 54} = \frac{-21}{14} = -\frac{3}{2}$$

Also, for $m = \frac{-1}{2}$, the corresponding value of c is,

$$c = \frac{(-7/2)(2 + 1/2)}{2 + 7 + 3/2} = -\frac{35}{42} = -\frac{5}{6}$$

Thus, y = mx + c ⇒ 6y = 6x – 7, 2y = 6x – 3, 6y = – 3x – 5 are oblique asymptotes to the given curve.

Therefore, 6y – 6x + 7 = 0; 2y – 6x + 3 = 0, 3x + 6y + 5 = 0 are the asymptotes to the given curve.

(iv) $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0.$

Solution: Given curve is

$$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0.$$

Clearly given curve is of degree 3. So, it may have at most three asymptotes. Here, x^3 and y^3 are present. So, there is no vertical and horizontal asymptotes exist.

For oblique asymptotes, let y = mx + c be required asymptotes.

For this, set y = m and x = 1. Then with the help of given curve set,

$$\phi_3(m) = 1 + 2m - m^2 - 2m^3, \quad \phi_2(m) = -4m^2 + 2m, \quad \phi_1(m) = -5m.$$

For the value of m, set

$$\phi_3(m) = 0 \Rightarrow 1 + 2m - m^2 - 2m^3 = 0$$
$$\Rightarrow 1 - m + 3m - 3m^2 + 2m^2 - 2m^3 = 0$$
$$\Rightarrow (1-m)(1 + 3m + 2m^2) = 0$$
$$\Rightarrow (1-m)(1+m)(1+2m) = 0$$

$$\Rightarrow m = 1, -1, -\frac{1}{2}.$$

Here, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{4m^2 - 2m}{2 - 2m - 6m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{4-2}{2-2-6} = \frac{2}{-6} = \frac{1}{-3}.$$

And, for m = –1, the corresponding value of c is,

$$c = \frac{4+2}{2+2-6} = \frac{6}{-2} = -3.$$

Also, for $m = \frac{-1}{2}$, the corresponding value of c is,

$$c = \frac{1+1}{2+1-3/2} = \frac{4}{-3}$$

Thus, y = mx + c ⇒ 3y = 3x – 1, y = –x – 3, 3x + 6y + 4 = 0 are oblique asymptotes to the given curve.

Therefore, 3y = 3x – 1, y = –x – 3, 3x + 6y + 4 = 0 are the asymptotes to the given curve.

(v) $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$

Solution: Given curve is

$$x^4 - y^4 - 3x^2y + 3xy^2 + xy = 0.$$

Clearly, the given curve is of fourth degree. So, it may have at most 4–asymptotes.

Here, x^4 and y^4 are present. So, there is no vertical and horizontal asymptote.

Let, y = mx + c be an oblique asymptotes with value of m and c. For this with the help of given equation, set y = m and x = 1 then,

$$\phi_4(m) = 1 - m^4, \quad \phi_3(m) = 3m + 3m^2, \quad \phi_2(m) = m.$$

For value of m, let $\phi_3(m) = 0$

$$\Rightarrow 1 - m^4 = 0$$
$$\Rightarrow 1 - m^2 = 0 \quad [\text{since } m^2 = -1 \text{ does not give any real value of m}]$$
$$\Rightarrow m = \pm 1$$

Thus, m has non–repeated value. So, for value of corresponding c,

$$c = -\frac{\phi_3(m)}{\phi_4'(m)} = \frac{-3m(1+m)}{-4m^3} = \frac{3(1+m)}{4m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{3(1+1)}{4(1)} = \frac{6}{4} = \frac{3}{2}$$

And, for m = –1, the corresponding value of c is,

$$c = \frac{3(1-1)}{4(1)} = 0$$

Thus, $y = mx + c \Rightarrow 2y = 2x + 3,\ y = -x$ are oblique asymptotes.

Therefore, $2x - 2y + 3 = 0,\ x + y = 0$ are the asymptotes to given curve.

(vi) $4x^4 - 5x^2y^2 + y^3 - 3x^2y + 5x - 8 + y^4 = 0$

Solution: Given curve be $4x^4 - 5x^2y^2 + y^3 - 3x^2y + 5x - 8 + y^4 = 0$.

Clearly, the given curve is of fourth degree. So, it may have at most 4–asymptotes.

Here, x^4 and y^4 are present. So, there does not exist any vertical and horizontal asymptote.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_4(m) = 4 - 5m^2 + m^4,\quad \phi_3(m) = m^3 - 3m,\quad \phi_2(m) = 0.$$

For value of m, set

$$\phi_4(m) = 0 \quad\Rightarrow\quad 4 - 5m^2 + m^4 = 0$$
$$\Rightarrow\quad (4 - m^2)(1 - m^2) = 0$$
$$\Rightarrow\quad m = \pm 1, \pm 2$$

Here, m has non–repeated value. So, for the value of corresponding c,

$$c = -\frac{\phi_3(m)}{\phi_4'(m)} = -\frac{(m^3 - 3m)}{-10m + 4m^3} = \frac{-m^2 + 3}{-10 + 4m^2}$$

For m = 1, the corresponding value of c is,

$$c = \frac{3-1}{-10+4} = \frac{2}{-6} = -\frac{1}{3}.$$

And, for m = –1, the corresponding value of c is,

$$c = \frac{3-1}{-10+4} = -\frac{1}{3}$$

Also, for m = 2, the corresponding value of c is,

$$c = \frac{3-4}{-10+16} = -\frac{1}{6}$$

Also, for m = –2, the corresponding value of c is,

$$c = \frac{3-4}{-10+16} = -\frac{1}{6}$$

Therefore, $y = mx + c \Rightarrow 3y = \pm 3x - 1,\ 6y = \pm 12x - 1$ are oblique asymptotes.

Hence, $3y \pm 3x + 1 = 0,\ 6y \pm 12x + 1 = 0$ are the asymptotes to the given curve.

(vii) $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$ **[2011 Fall] [2003, Fall][2004, Spring] [2016 Fall][2018 Fall][2015, Spring]**

Solution: Given curve is

$$x^2(x - y)^2 - a^2(x^2 + y^2) = 0.$$
$$\Rightarrow\quad x^4 - 2x^3y + x^2y^2 - a^2x^2 - a^2y^2 = 0.$$

Clearly, the given curve is of fourth degree. So, it may have at most 4-asymptotes.

Here x^4 is present. So, there does not exist any horizontal asymptotes. But y^4 is absent. So, there may exist vertical asymptotes. Therefore, for vertical asymptotes, equate the coefficient of highest power of y, to zero. i.e. coeff. of $y^2 = 0$

$$\Rightarrow\quad x^2 - a^2 = 0 \quad\Rightarrow\quad x = \pm a$$

These are required vertical asymptotes.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_4(m) = 1 - 2m + m^2,\quad \phi_3(m) = 0,\quad \phi_2(m) = -a^2(1 + m^2)$$

For value of m, set $\phi_4(m) = 0 \Rightarrow 1 - 2m + m^2 = 0$

$$\Rightarrow\quad (m-1)^2 = 0$$
$$\Rightarrow\quad m = 1, 1$$

Here, m has repeated value. So, for value of the corresponding c,

$$\frac{c^2}{2!}\phi_4''(m) + \frac{c}{1!}\phi_3'(m) + \phi_2(m) = 0$$
$$\Rightarrow\quad \frac{c^2}{2}(2) + c(0) - a^2(1 + m^2) = 0$$

For m = 1, the corresponding value of c is,

$$c^2 - 2a^2 = 0 \quad\Rightarrow\quad c = \pm a\sqrt{2}$$

So, $y = mx + c \Rightarrow y = x \pm a\sqrt{2}$ are oblique asymptotes.

Hence, $x = \pm a,\ y = x \pm a\sqrt{2}$ are the asymptotes to the given curve.

(viii) $x^2y^2 - 4(x - y)^2 + 2y - 3 = 0$

Solution: Given curve is

$$x^2y^2 - 4(x - y)^2 + 2y - 3 = 0.$$
$$\Rightarrow\quad x^2y^2 - 4x^2 - 4y^2 + 8xy + 2y - 3 = 0$$

Clearly, the given curve is of fourth degree. So, it may have at most 4–asymptotes.

Here both x^4 and y^4 are absent. So, there may exist horizontal and vertical asymptotes.

For horizontal asymptotes, equate the coefficient of highest power of x, to zero. That is, coeff. of $x^2 = 0$.

$$\Rightarrow\quad y^2 - 4 = 0 \quad\Rightarrow\quad y = \pm 2.$$

And for vertical asymptotes, equate the coefficient of highest power of y to zero. That is, coeff. of $y^2 = 0$.

$$\Rightarrow\quad x^2 - 4 = 0 \quad\Rightarrow\quad x = \pm 2.$$

Since the curve does not have more than 4–asymptotes and we have already found $x = \pm 2,\ y = \pm 2$ as the asymptotes to the given curve, so the oblique asymptote does not exist.

Hence $x = \pm 2,\ y = \pm 2$ are the asymptotes to the given curve.

(ix) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$ **[2017 Spring][2018 Spring]**

Solution: Given curve is

$$x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$$

Clearly given curve is of degree 3. So, it may have at most 3–asymptotes.

Here both x^3 and y^3 are present. So, there does not exist any horizontal and vertical asymptotes.

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = 1 + 3m - 4m^3, \quad \phi_2(m) = 0, \quad \phi_1(m) = -1 + m$$

For value of m, set

$$\phi_3(m) = 0 \quad \Rightarrow \quad 1 + 3m - 4m^3 = 0$$
$$\Rightarrow \quad 1 - m + 4m - 4m^2 + 4m^2 - 4m^3 = 0$$
$$\Rightarrow \quad (1 - m)(1 + 4m + 4m^2) = 0$$
$$\Rightarrow \quad (1 - m)(1 + 2m)^2 = 0$$
$$\Rightarrow \quad m = 1, -\frac{1}{2}, -\frac{1}{2}$$

For m = 1 (non–repeated value of m), the corresponding value of c is,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3 - 12m^2} = -\frac{0}{-9} = 0.$$

And, for $m = \frac{-1}{2}$ (repeated value of m), to find the corresponding value of c,

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$$

$$\Rightarrow \quad \frac{c^2}{2}(-24m) + c(0) + (-1 + m) = 0$$

$$\Rightarrow \quad 6c^2 - \frac{3}{2} = 0.$$

$$\Rightarrow \quad c^2 = \frac{3}{12} = \frac{1}{4} \quad \Rightarrow c = \pm\frac{1}{2}$$

Thus, $y = mx + c \Rightarrow y = x,\ y = -\frac{1}{2}x \pm \frac{1}{2}$ are oblique asymptotes.

Hence, $x = y$, $x + 2y \pm 1 = 0$ are asymptotes to the given curve.

(x) $\mathbf{y^3 + x^2y + 2xy^2 - y + 1 = 0}$ **[2013 Fall][2012 Fall]**

Solution: Given curve is

$$y^3 + x^2y + 2xy^2 - y + 1 = 0.$$

Clearly given curve is of degree 3. So, it may have at most 3–asymptotes.

Here y^3 is present. So, the vertical horizontal do not exist.

And, x^3 is absent. So, there may exist horizontal asymptotes. For this, we equate the coefficient of highest power of x, to zero. That is,

$$\text{coeff. of } x^2 = 0$$
$$\Rightarrow \quad y = 0.$$

Let, $y = mx + c$ be an oblique asymptotes with value of m and c. For this with the help of given equation, set $y = m$ and $x = 1$ then,

$$\phi_3(m) = m^3 + m + 2m^2, \qquad \phi_2(m) = 0, \quad \phi_1(m) = -m$$

For value of m, set

$$\begin{aligned} \phi_3(m) = 0 \quad &\Rightarrow \quad m^3 + m + 2m^2 = 0 \\ &\Rightarrow \quad m(1 + 2m + m^2) = 0 \\ &\Rightarrow \quad m(m+1)^2 = 0 \\ &\Rightarrow \quad m = 0, -1, -1 \end{aligned}$$

Thus, m has non–repeated value 0 and repeated value –1.

For m = 0 (non–repeated value of m), the corresponding value of c is found by,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3m^2 + 1 + 4m} = -\frac{0}{1} = 0$$

For m = –1 (repeated value of m), to find the corresponding value of c,

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$$

$$\Rightarrow \frac{c^2}{2}(6m + 4) + 0 + (-m) = 0 \qquad [\text{since } \phi_2(m) = 0]$$

$$\Rightarrow -c^2 + 1 = 0$$

$$\Rightarrow c = \pm 1$$

Thus, $y = mx + c \Rightarrow y = 0,\ y = -x \pm 1$ are oblique asymptotes.

Hence, $y = 0,\ x + y \pm 1 = 0$ are asymptotes to the given curve.

Chapter 7

CURVATURE

Exercise 7.1

1. Find the radius of curvature at (s, ψ) on the following curve

(i) $s = c\tan\psi$

Solution: Here, the given curve is

$$s = c\tan\psi$$

So, $$\frac{ds}{d\psi} = c\sec^2\psi$$

Thus, radius of curvature of the given curve is,

$$\rho = c\sec^2\psi.$$

(ii) $s = 8a\sin^2\frac{\psi}{6}$ **[2018 Spring (Short)] [2014 Spring (Short)]**

Solution: Here, the given curve is

$$s = 8a\sin^2\frac{\psi}{6}$$

So, $$\frac{ds}{d\psi} = 8a\,2\sin\frac{\psi}{6}.\cos\frac{\psi}{6}\cdot\frac{1}{6} \Rightarrow \frac{ds}{d\psi} = \frac{8a}{6}\sin\frac{2\psi}{6} = \frac{4a}{3}\sin\frac{\psi}{3}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{4a}{3}\sin\frac{\psi}{3}.$$

(iii) $s = c\log(\sec\psi)$

Solution: Here, the given curve is

$$s = c\log(\sec\psi)$$

So, $$\frac{ds}{d\psi} = \frac{d}{d\psi}(c\log(\sec\psi)) \Rightarrow \frac{ds}{d\psi} = c\frac{1}{\sec\psi}\times\sec\psi.\tan\psi$$

Thus, radius of curvature of the given curve is,

$$\rho = c \tan\psi.$$

(iv) $s = a \log\left(\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)\right)$

Solution: Here,

$$s = a \log\left(\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)\right)$$

So, $$\rho = \frac{ds}{d\psi} = \frac{a \cdot \sec^2\left(\frac{\pi}{4} + \frac{\psi}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)} \frac{1}{2}$$

$$= \frac{a}{2 \sin\left(\frac{\pi}{4} + \frac{\psi}{2}\right) \cos\left(\frac{\pi}{4} + \frac{\psi}{2}\right)}$$

$$= \frac{a}{\sin\left(2\left(\frac{\pi}{4} + \frac{\psi}{2}\right)\right)} = \frac{a}{\sin\left(\frac{\pi}{2} + \psi\right)} = \frac{a}{\cos\psi} = a \sec\psi.$$

Thus, radius of curvature of the given curve is,

$$\rho = a \sec\psi.$$

2. Find the radius of curvature at (x, y) for the curves.

(i) $y^2 = 4ax$ **[2014 Fall (short)]**

Solution: Here, $4ax = y^2$ (given).

$$\therefore 4a \frac{dx}{dy} = 2y \qquad \Rightarrow x_1 = \frac{dx}{dy} = \frac{y}{2a}$$

Again, Differentiating w. r. t. x, we get

$$x_2 = \frac{d^2y}{dx^2} = \frac{1}{2a}$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{\left(1 + \frac{y^2}{4a^2}\right)^{3/2}}{1/2a}$$

$$= \frac{2a\,(y^2 + 4a^2)^{3/2}}{8a^3}$$

$$= \frac{(y^2 + 4a^2)^{3/2}}{4a^2}$$

$$= \frac{(4ax + 4a^2)^{3/2}}{4a^2} = \frac{8a^{3/2}}{4a^2}(a + x)^{3/2} = \frac{2}{a^{1/2}}(a + x)^{3/2}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{2}{a^{1/2}}(a + x)^{3/2}.$$

Q. Find the radius of curvature of curve $y^2 = 4ax$ at (0, 0).
[2016 Spring (short)] [2017 Spring (short)]

(ii) $y = \log(\sin x)$.

Solution: Here, $y = \log(\sin x)$

Differentiating w. r. t. x, then

$$y_1 = \frac{1}{\sin x}\cos x = \cot x \qquad \text{and} \qquad y_2 = -\operatorname{cosec}^2 x$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cot^2 x)^{3/2}}{-\operatorname{cosec}^2 x} = \frac{\operatorname{cosec}^3 x}{-\operatorname{cosec}^2 x} = -\operatorname{cosec} x$$

Since ρ is non–negative.

Thus, radius of curvature of the given curve is,

$$\rho = \operatorname{cosec} x.$$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Here,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

Differentiating w. r. t. x, then

$$2b^2x + 2a^2y.y_1 = 0$$

$$\Rightarrow y_1 = -\frac{b^2x}{a^2y}$$

Again differentiating w. r. t. x, then

$$2b^2 + 2a^2y_1^2 + 2a^2y \cdot y_2 = 0.$$

$$\Rightarrow y_2 = \frac{-b^2 - a^2y_1^2}{a^2y} = -\frac{1}{a^2y}\left[b^2 + a^2\frac{b^4x^2}{a^4y^2}\right]$$

$$= -\frac{1 \cdot b^2}{a^2y \cdot a^4y^2}[a^4y^2 + a^2b^2x^2]$$

$$= -\frac{a^2b^2}{a^6y^3}(a^2y^2 + b^2x^2)$$

$$= -\frac{a^2b^2}{a^6y^3} \cdot a^2b^2 \quad = -\frac{b^4}{a^2y^3}$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = -\frac{a^2y^3}{b^4}\left[1 + \frac{b^4x^2}{a^4y^2}\right]^{3/2} = -\frac{a^2y^3}{b^4a^6y^3}[a^4y^2 + b^4x^2]^{3/2}$$

$$\Rightarrow \rho = \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4} \qquad [\because \rho \text{ is not negative}]$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4}$$

(iv) $x^{2/3} + y^{2/3} = a^{2/3}$

Solution: Here, $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w. r. t. x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$$

Now, $1 + y_1^2 = 1 + \frac{y^{2/3}}{x^{2/3}} = \frac{1}{x} \Rightarrow 1 + y_1^2 = a^{2/3}x^{-2/3}$ ... (A)

Differentiating (A) w. r. t. x, we get

$$0 + 2y_1y_2 = a^{2/3}\left(\frac{-2}{3}\right)x^{-5/3}$$

$$\therefore \quad y_2 = -a^{2/3}\frac{x^{-5/3}}{3y_1}$$

$$= -a^{2/3}\frac{x^{-5/3} \cdot x^{1/3}}{y^{1/3}} \qquad \left[\because y_1 = \frac{-y^{1/3}}{x^{1/3}}\right]$$

$$= \frac{a^{2/3}}{3}x^{-4/3}y^{-1/3}$$

Now the radius of the curvature of the given curve at (x, y) is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(a^{2/3}x^{-2/3})^{3/2}}{\frac{a^{2/3}}{3}x^{-4/3}y^{-1/3}} \qquad \text{[Using (A)]}$$

$$= \frac{3ax^{-1}}{a^{2/3}x^{-4/3}y^{-1/3}} = 3a^{1/3}x^{1/3}y^{1/3} = 3(axy)^{1/3}.$$

Thus, radius of curvature of the given curve is,

$$\rho = 3(axy)^{1/3}.$$

(v) $y = 4\sin x - \sin 2x$ at $x = \pi/2$ **[2008, Fall(short)]**

Solution: Here, $y = 4\sin x - \sin 2x$

So, $y_1 = \frac{dy}{dx} = 4\cos x - 2\cos 2x$

And, $y_2 = \frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x$

Then at $x = \frac{\pi}{2}$, $\quad y_1 = 0 - 2(-1) = 2$ and, $\quad y_2 = 4\sin\frac{\pi}{2} = -4.$

Now the radius of the curvature of the given curve at $x = \pi/2$ is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(5)^{3/2}}{4}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{(5)^{3/2}}{4}$$

(vi) $9x^2 + 4y^2 = 36x$ at (2, 3) **[2006, Fall(short)]**

Solution: Here, $9x^2 + 4y^2 = 36x$

Now, differentiating w. r. t. x, then

$$18x + 8y \cdot y_1 = 36 \quad \Rightarrow y_1 = \frac{36 - 18x}{8y}$$

and $18 + 8y_1 \cdot y_1 + 8y \cdot y_2 = 0 \Rightarrow y_2 = -\dfrac{18 + 8y_1^2}{8y}$

At (2, 3)

$$y_1 = \frac{36 - 36}{24} = 0 \quad \text{and} \quad y_2 = -\frac{18 + 8 \cdot 0}{24} = -\frac{3}{4}$$

Now the radius of the curvature of the given curve at (2, 3) is

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{1}{-3/4} = -\frac{4}{3}$$

Since ρ is always non–negative. So, $\rho = \frac{4}{3}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{4}{3}$$

(vii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where y = x cuts it. **[2003, Spring] [2007, Spring]**

Solution: Here, $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

i.e. $y = a + x - 2\sqrt{ax}$.

Differentiating w. r. t. x, then

$$y_1 = 1 - 2\sqrt{a} \cdot \frac{1}{2\sqrt{x}} = 1 - \frac{\sqrt{a}}{\sqrt{x}}$$

and, $$y_2 = \frac{\sqrt{a}}{2(x)^{3/2}}$$

Solving the given line y = x and the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then we get x = a/4 and y = a/4.

Therefore, at (a/4, a/4),

$$y_1 = 1 - \frac{\sqrt{a}}{\sqrt{a/4}} = 1 - 2 = -1 \quad \text{and} \quad y_2 = \frac{\sqrt{a}}{2(a/4)^{3/2}} = \frac{4}{a}$$

Now, the radius of curvature of the given curve at (a/4, a/4) is,

$$\rho = \frac{\{1 + (y_1)^2\}^{3/2}}{y_2} = \frac{(1 + 1)^{3/2}}{4/a} = \frac{2a\sqrt{2}}{4} = \frac{a}{\sqrt{2}}$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{a}{\sqrt{2}}$$

3. Find the radius of curvature at any point to the following curves:

(i) $x = a\cos\theta, y = a\sin\theta$

Solution: Here, $x = a\cos\theta,\ y = a\sin\theta$.

So, $x' = -a\sin\theta,\ y' = a\cos\theta$

And, $x'' = -a\cos\theta,\ y'' = -a\sin\theta$

Now, the radius of curvature of the given curve is,

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(a^2\sin^2\theta + a^2\cos^2\theta)^{3/2}}{-a^2\cos^2\theta - a^2\sin^2\theta} = \frac{(a^2)^{3/2}}{-a^2} = \frac{a^3}{-a^2} = a.$$

Thus, radius of curvature of the given curve is,

$$\rho = a.$$

Q. **Find the radius of curvature of the curves: $x = r\cos\theta, y = r\sin\theta$.** [2017 Spring Short]

Solution: See above solution with replacing a by r.

(ii) $x = at^2, y = 2at$

Solution: Here, $x = at^2$, $y = 2at$.

So, $y' = 2a$, $x' = 2at$

And, $y'' = 0$, $x'' = 2a$.

Now, the radius of curvature of the given curve is,

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(4a^2 + 4a^2t^2)^{3/2}}{2a \cdot 2a} = \frac{8a^3}{4a^2}(1 + t^2)^{3/2} = 2a(1 + t^2)^{3/2}.$$

Thus, radius of curvature of the given curve is,

$$\rho = 2a(1 + t^2)^{3/2}.$$

(iii) $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$

Solution: Here,

$x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

Differentiating we get,

$\dot{x} = a(-\sin t + \sin t + t\cos t)$, and $\dot{y} = a(\cos t - \cos t + t\sin t)$

$\Rightarrow \dot{x} = at\cos t \qquad \Rightarrow \dot{y} = at\sin t$

and $\ddot{x} = a\cos t - at\sin t \qquad \Rightarrow \ddot{y} = a\sin t + at\cos t$

Now, radius of curvature of given curve is,

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$$

$$= \frac{(a^2t^2\cos^2 t + a^2t^2\sin^2 t)^{3/2}}{at\cos t\,(a\sin t + at\cos t) - (a\cos t - at\sin t)\,at\sin t}$$

$$= \frac{(a^2t^2)^{3/2}}{a^2 t[\cos t\sin t + t\cos^2 t - \cos t\sin t + t\sin^2 t]}$$

$$= \frac{a^3t^3}{a^2t^2(\cos^2 t + \sin^2 t)}$$

$$= at$$

Thus, radius of curvature of the given curve is,

$$\rho = at.$$

(iv) $x = a\cos^3\theta$, $y = a\sin^3\theta$, at $\theta = \pi/4$

Solution: Here, $x = a\cos^3\theta$, $y = a\sin^3\theta$

Differentiating w. r. t. θ then,

$$\dot{x} = -3a\cos^2\theta\sin\theta \qquad \dot{y} = 3a\sin^2\theta\cos\theta$$

and $\ddot{x} = +6a\cos\theta\sin^2\theta - 3a\cos^3\theta$

$$\ddot{y} = 6a\sin\theta\cos^2\theta - 3a\sin^3\theta$$

at $\theta = \dfrac{\pi}{4}$,

$$\dot{x} = -3a\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}} = -\frac{3a}{2\sqrt{2}} \qquad \dot{y} = \frac{3a}{2\sqrt{2}}.$$

$$\ddot{x} = \frac{6a}{2\sqrt{2}} - \frac{3a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}} \qquad \ddot{y} = \frac{6a}{2\sqrt{2}} - \frac{3a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}}$$

Now, radius of curvature of given curve at $\theta = \dfrac{\pi}{4}$ is,

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}} = \frac{\left[\left(-\frac{3a}{2\sqrt{2}}\right)^2 + \left(\frac{3a}{2\sqrt{2}}\right)^2\right]^{3/2}}{-\frac{9a^2}{8} - \frac{9a^2}{8}}$$

$$= -\frac{8}{18a^2}\left[2\left(\frac{3a}{2\sqrt{2}}\right)^2\right]^{3/2}$$

$$= -\frac{8}{18a^2}\cdot\frac{27a^3}{16\sqrt{2}}\cdot 2^{3/2}$$

$$= -\frac{3a}{2}$$

Since ρ is non–negative. So, $\rho = \dfrac{3a}{2}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{3a}{2}.$$

4. Find the radius of curvature at any point (r, θ) for the following curves.

(i) $r = a(1 - \cos\theta)$

Solution: Here, $r = a(1 - \cos\theta)$

Differentiating w. r. t. θ, then,

$$r_1 = a\sin\theta \quad \text{and,} \quad r_2 = a\cos\theta$$

Now, radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta)^{3/2}}{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)}$$

$$= \frac{(a^2 - 2a^2\cos\theta + a^2)^{3/2}}{a^2 + a^2\cos^2\theta + 2a^2\sin^2\theta - 3a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{(2a^2)^{1/2}(1-\cos\theta)^{1/2}}{a^2 + 2a^2 - 3a^2\cos\theta}$$

$$= \frac{a\sqrt{2}\,.\sqrt{r/a}}{3a^2(1-\cos\theta)}$$

$$= \frac{\sqrt{2r}}{3r\sqrt{a}} = \frac{1}{3}\sqrt{\frac{2}{ar}}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{3}\sqrt{\frac{2}{ar}}.$$

(ii) $r^2 = a^2\cos 2\theta$

Solution: Here, $r^2 = a^2\cos 2\theta$

Differentiating w.r.t. θ then,

$$2r.\,r_1 = -2a^2\sin 2\theta$$

$$\Rightarrow r_1 = -\frac{a^2\sin 2\theta}{r} = -\frac{ra^2\sin 2\theta}{r^2} = -\frac{r\,a^2\sin 2\theta}{a^2\cos 2\theta} = -r\tan 2\theta$$

and,

$$r_2 = -r_1\tan 2\theta - 2r\sec^2 2\theta$$

$$= r\tan^2 2\theta - 2r\sec^2 2\theta$$

$$= r\tan^2 2\theta - 2r - 2r\tan^2 2\theta$$

$$= -r\tan^2 2\theta - 2r$$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(r^2 + r^2\tan^2 2\theta)^{3/2}}{r^2 + 2r^2\tan^2 2\theta + r^2(2 + \tan^2 2\theta)}$$

$$= \frac{r(1 + \tan^2 2\theta)^{3/2}}{1 + 2\tan^2 2\theta + 2 + 2\tan^2 2\theta}$$

$$= \frac{r\sec^3 2\theta}{3 + 3\tan^2 2\theta}$$

$$= \frac{r\sec^3 2\theta}{3\sec^2 2\theta}$$

$$= \frac{r\sec 2\theta}{3}$$

$$= \frac{a}{3}\sec 2\theta\sqrt{\cos 2\theta}$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{a}{3}\sec 2\theta\sqrt{\cos 2\theta}.$$

(iii) $r = ae^{\theta\cot\alpha}$ [2018 Fall (Short)] [2011 Fall (Short)] [2011 Spring (Short)]

Solution: Here, $r = ae^{\theta\cot\alpha}$

So, $r_1 = a\cot\alpha\, e^{\theta\cot\alpha} = r\cot\alpha$

And, $r_2 = a\cot^2\alpha\, e^{\theta\cot\alpha} = r\cot^2\alpha$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr^2}$$

$$= \frac{(r^2 + r^2\cot^2\alpha)^{3/2}}{r^2 + 2r^2\cot^2\alpha - r^2\cot^2\alpha} = \frac{r^3(1+\cot^2\alpha)^{3/2}}{r^2(1+\cot^2\alpha)} = r\operatorname{cosec}\alpha.$$

Thus, radius of curvature of the given curve is,

$$\rho = r\operatorname{cosec}\alpha.$$

(iv) $r^2\cos 2\theta = a^2$

Solution: Here, $r^2\cos 2\theta = a^2$

So, $-2r^2\sin 2\theta + 2\cos 2\theta\, r\,.\, r_1 = 0$

$\Rightarrow r_1 = r\tan 2\theta.$

And, $r_2 = 2r\sec^2 2\theta + \tan 2\theta\,.\, r\tan 2\theta$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr^2} = \frac{(r^2 + r^2\tan^2 2\theta)^{3/2}}{r^2 + 2r^2\tan^2 2\theta - 2r^2\sec^2 2\theta - r^2\tan^2 2\theta}$$

$$= \frac{r^3(\sec^2 2\theta)^{3/2}}{r^2(1 + 2\tan^2 2\theta - 2\sec^2 2\theta)}$$

$$= r\frac{\sec^2 2\theta}{-\sec^2 2\theta}$$

$$= -r\sec 2\theta$$

$$= -r/\cos 2\theta = -r/a^2/r^2 = -r^3/a^2$$

Since $\rho \neq -$ ve. So, $\rho = r^3/a^3$.

Thus, radius of curvature of the given curve is,

$$\rho = r^3/a^3.$$

(v) $r = a\sin n\theta$, at origin

Solution: Given that, $r = a\sin n\theta$

So, $r_1 = an\cos n\theta$ and $r_2 = -an^2\sin n\theta$

At origin,

$$r = 0,\ r_1 = an,\ r_2 = 0$$

Now, the radius of curvature of given curve is,

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{(a^2n^2)^{3/2}}{2a^2n^2} = \frac{a^3n^3}{2a^2n^2} = \frac{an}{2}$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{an}{2}.$$

5. Find the radius of curvature at origin of the following curves as:

(i) $y = x^4 - 4x^3 - 18x^2$ [2013 Spring (Short)]

Solution: Here, equating the lowest order term in the equation, to zero.

That is, $y = 0$ i.e. the x–axis is tangent at the origin. So the radius of curvature of the given curve is,

$$\rho = \lim_{y\to 0}\left\{\frac{x^2}{2y}\right\}.$$

Now, dividing both sides of given equation $y = x^4 - 4x^3 - 18x^2$ by $2y$ and then taking limit as $x\to 0$, $y\to 0$ both sides, we get,

$$\lim_{\substack{x\to 0\\ y\to 0}}(2) = \lim_{\substack{x\to 0\\ y\to 0}}\left(x^2\left\{\frac{x^2}{2y}\right\}\right) - \lim_{\substack{x\to 0\\ y\to 0}}\left(4x\left\{\frac{x^2}{2y}\right\}\right) - 18\lim_{\substack{x\to 0\\ y\to 0}}\left\{\frac{x^2}{2y}\right\}.$$

$$\Rightarrow 2 = 0(\rho) - 0(\rho) - 18\rho.$$

$$\Rightarrow \rho = \frac{1}{-36}.$$

Since $\rho \neq -$ve. So, $\rho = \frac{1}{36}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{36}.$$

Alternative Method:

Here, $y = x^4 - 4x^3 - 18x^2$

So, $\frac{dy}{dx} = 4x^3 - 12x^2 - 36x$.

And, $\frac{d^2y}{dx^2} = 12x^2 - 24x - 36.$

At origin, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = -36$

Now, the radius of the given curve is,

$$\rho = \left\{\frac{1 + \left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^{2/3}}\right\}^{3/2} = \left\{\frac{1-0^2}{(-36)^{2/3}}\right\}^{3/2} = \frac{1}{-36}.$$

Since $\rho \neq -$ve. So, $\rho = \frac{1}{36}$.

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{36}.$$

(ii) $3x^2 + 4y^2 = 2x$

Solution: Here, equating the lowest order term in the equation, to zero.

That is, $x = 0$ i.e. the y–axis is tangent at the origin. So the radius of curvature of the given curve is,

$$\rho = \lim_{y\to 0}\left\{\frac{y^2}{2x}\right\}$$

Now, dividing both sides of given equation $3x^2 + 4y^2 = 2x$ by 2x and then taking limit as $x\to 0$, $y\to 0$ both sides, we get,

$$\frac{3}{2}\lim_{\substack{x\to 0\\ y\to 0}}(x) + 4\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^2}{2x}\right) - 1 = 0.$$

$$\Rightarrow 0 + 4\rho - 1 = 0.$$

$$\Rightarrow 4\rho = 1$$

$$\Rightarrow \rho = \frac{1}{4}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{1}{4}.$$

(iv) $x^3 + y^3 = 3axy$ **[2017 Fall Short]**

Solution: Here, equating the lowest order term in the equation, to zero.

That is, $xy = 0$. That is the x–axis and y–axis are tangent at the origin.

Thus,

$$\lim_{x\to 0}\left\{\frac{x^2}{2y}\right\} = \rho \quad\text{and}\quad \lim_{y\to 0}\left\{\frac{y^2}{2y}\right\} = \rho$$

Given equation is,

$$x^3 + y^3 = 3axy$$

Dividing both sides of this equation by 2xy we get

$$\left\{\frac{x^2}{2y}\right\} + \left\{\frac{y^2}{2x}\right\} = \frac{3a}{2}$$

Taking limit at $x\to 0$, $y\to 0$ both side, we get

$$\lim_{\substack{x\to 0\\ y\to 0}}\left\{\frac{x^2}{2y}\right\} + \lim_{\substack{x\to 0\\ y\to 0}}\left\{\frac{y^2}{2x}\right\} = \frac{3a}{2}$$

$$\Rightarrow \rho + \rho = \frac{3a}{2}$$

$$\Rightarrow 2\rho = \frac{3a}{2} \quad \Rightarrow \rho = \frac{3a}{4}.$$

Thus, radius of curvature of the given curve is,

$$\rho = \frac{3a}{4}.$$

(v) $x^2 + y^2 + 6x + 8y = 0$

Solution: Here, $x^2 + y^2 + 6x + 8y = 0$ ... (i)

Putting $y = xp + \frac{x^2}{2}q + \ldots$ in equation (i), we get

$$\text{or}\quad x^2 + \left(xp + \frac{x^2}{2}q + \ldots\right)^2 + 6x + 8\left(xp + \frac{x^2}{2}q + \ldots\right) = 0.$$

$$\text{or}\quad x(6 + 8p) + x^2(1 + p^2 + 4q) + \ldots = 0.$$

Equating the coefficient of x and x^2, we get

$$6 + 8p = 0 \quad \Rightarrow p = \frac{-3}{4}$$

and

$$1 + p^2 + 4q = 0 \Rightarrow 4q = (-1 - p^2) = \left(-1 - \frac{9}{16}\right) = \frac{-25}{16}$$

$$\Rightarrow q = \frac{-25}{64}$$

Now, the radius of curvature of the given curve is,

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \left|\frac{(1 + 9/16)^{3/2}}{-25/64}\right| = \left|\frac{125\times 64}{-25\times 64}\right| = 5.$$

(vi) $y^2 - 3xy - 4x^2 + 5x^3 + x^4y - y^5 = 0$

Solution: Here,

$$y^2 - 3xy - 4x^2 + 5x^3 + x^4y - y^5 = 0 \quad \ldots (i)$$

Putting $y = xp + \frac{x^2}{2}q + \ldots$ in equation (i), we get

$$\left(xp + \frac{x^2}{2}q + \ldots\right)^2 - 3x\left(xp + \frac{x^2}{2}q + \ldots\right) - 4x^2 + 5x^3 + x^4\left(xp + \frac{x^2}{2}q + \ldots\right) - \left(xp + \frac{x^2}{2}q + \ldots\right)^5 = 0.$$

$$\Rightarrow \quad x^2(p^2 - 3p - 4) + x^3\left(pq - \frac{3q}{2} + 5\right) + \ldots = 0$$

Equating the coefficient of x^2 and x^3, we get

$$p^2 - 3p - 4 = 0$$

$$\Rightarrow (p - 4)(p + 1) = 0$$

$$\Rightarrow p = 4, -1.$$

and,

$$pq - \frac{3q}{2} + 5 = 0$$

At $p = 4$, $\quad 4q - \frac{3q}{2} + 5 = 0 \Rightarrow 5q = -10 \Rightarrow q = -2.$

At $p = -1$, $\quad -q - \frac{3q}{2} + 5 = 0 \Rightarrow 5q = 10 \Rightarrow q = 2$

When $(p, q) = (4, -2)$ then

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \frac{(1 + 15)^{3/2}}{-2} = -\frac{(17)^{3/2}}{2} = \frac{17\sqrt{17}}{2}.$$

When $(p, q) = (-1, 2)$ then

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \frac{(1 + 1)^{3/2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

Since, $\rho \neq -\text{ve}$. Then, $\rho = \sqrt{17}$ and $\sqrt{2}$

(vii) $3x^2 + xy + y^2 - 4x = 0$

Solution: Here, $3x^2 + xy + y^2 - 4x = 0$ ... (i)

Here, equating the lowest order term in the equation, to zero. That means, $x = 0$ i.e., the x-axis is tangent at the origin.

Therefore,

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \left\{\frac{y^2}{2x}\right\}$$

Dividing both sides of (i) by 2x we get,

$$\frac{3}{2}x + \frac{y}{2} + \left\{\frac{y^2}{2x}\right\} - 2 = 0$$

Taking limit at $x \to 0$ and $y \to 0$ both side we get

$$\frac{3}{2}\lim_{\substack{x\to 0\\ y\to 0}}(x) + \frac{1}{2}\lim_{\substack{x\to 0\\ y\to 0}}(y) + \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^2}{2x}\right) - 2$$

Or, $0 + 0 + \rho - 2 = 0$

$\Rightarrow \rho = 2.$

Thus, radius of curvature of the given curve is,

$\rho = 2.$

(viii) $3x^3 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$

Solution: Here, $3x^3 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$ ... (i)

Here, equating the lowest order term in the equation, to zero. That means, $x = 0$ i.e., the y-axis is tangent at the origin.

Thus,

$$\rho = \lim_{y\to 0}\left\{\frac{y^2}{2x}\right\}$$

Dividing both side of (i) by 2x we get

Taking limit at $x \to 0$, $y \to 0$ both side we get,

$$\frac{3}{2}\lim_{\substack{x\to 0\\ y\to 0}}(x^3) - \lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^4}{x}\right) + \frac{5}{2}\lim_{\substack{x\to 0\\ y\to 0}}(xy) + \lim_{\substack{x\to 0\\ y\to 0}}(y) - 2\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^2}{2x}\right) + 2 = 0$$

$\Rightarrow 0 - 0 + 0 + 0 - 2\rho + 2 = 0$

$\Rightarrow -2\rho + 2 = 0$

$\Rightarrow \rho = 1.$

Thus, radius of curvature of the given curve is,

$\rho = 1.$

(ix) $x^2 + 6y^2 + 2x - y = 0$

Solution: Here, $x^2 + 6y^2 + 2x - y = 0$ ... (i)

Putting $y = xp + \frac{x^2}{2}q + \ldots$ in equation (i), we get

or $x^2 + 6\left(xp + \frac{x^2}{2}q + \ldots\right)^2 + 2x - \left(xp + \frac{x^2}{2}q + \ldots\right) = 0.$

or $x(2 - p) + x^2\left(1 + 6p^2 - \frac{q}{2}\right) + \ldots = 0.$

Equating the coefficient of x and x^2, we get

$2 - p = 0 \Rightarrow p = 2.$

And, $1 + 6p^2 - \frac{q}{2} = 0 \Rightarrow q = 2(1 + 6p^2) = 2(1 + 24) = 50.$

Now the radius of curvature of the given curve is,

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \frac{(1 + 4)^{3/2}}{50} = \frac{5\sqrt{5}}{50} = \frac{1}{2\sqrt{5}}.$$

(x) $x^4 + y^2 = 6a\,(x + y)$

Solution: Here, $x^4 + y^2 = 6a\,(x + y)$

$x^4 + y^2 - 6a\,(x + y) = 0$ ... (i)

Putting $y = xp + \frac{x^2}{2}q + \ldots$ in equation (i), we get

$$x^4 + \left(xp + \frac{x^2}{2}q + \ldots\right)^2 - 6a\left(x + xp + \frac{x^2}{2}q + \ldots\right) = 0.$$

$\Rightarrow x(-6a - 6ap) + x^2\left(p^2 - 6a\frac{q}{2}\right) + \ldots = 0.$

Equating the coefficient of x and x^2, we get

$-6a - 6ap = 0 \Rightarrow p = -1.$

And, $p^2 - 6a\frac{q}{2} = 0 \Rightarrow q = \frac{2p^2}{6a} = \frac{1}{3a}.$

Now the radius of curvature of the given curve is,

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \frac{(1 + 1)^{3/2}}{1/3a} = 6\sqrt{2}\,a.$$

6. Show that the radius of curvature of the curve $y = x^2(x - 3)$ at the points, where the tangent is parallel to x-axis are $\pm\frac{1}{6}$.

Solution: Given curve be

$y = x^2(x - 3)$

$\Rightarrow y = x^3 - 3x^2$ ... (i)

Diff. w.r.t. x then,

$y_1 = 3x^2 - 6x$ and $y_2 = 6x - 6.$

Since by hypothesis, the tangent is parallel to x-axis.

So, $y_1 = 0 \Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x - 2) = 0$

$\Rightarrow x = 0, 2$

Then by (i), $y = 0, -4.$

Thus, the tangents are at (0, 0) and (2, –4).

At (0, 0), $y_1 = 0$ and $y_2 = -6.$

At (2, –4), $y_1 = 0$ and $y_2 = 6.$

Now, the radius f curvature of (i) is,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Therefore, at (0, 0), $\rho = \frac{(1+0^2)^{3/2}}{-6} = \frac{-1}{6}$

And at (2, –4), $\rho = \frac{(1+0^2)^{3/2}}{6} = \frac{1}{6}$

Thus (i) has radius of curvature $\frac{-1}{6}$ at (0, 0) and $\frac{1}{6}$ at (2, –4).

7. **If ρ_1 and ρ_2 be the radius of curvature at the ends of focal chord of the parabola $y^2 = 4ax$, prove that $\rho_1^{(-2/3)} + \rho_2^{(-2/3)} = (2a)^{-2/3}$.**

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Solution: Given curve is, $y^2 = 4ax$ ... (i)

Since the general point of the parabola (i) be $x = at^2$

$y = 2at$

So, $\dot{x} = 2at$, $\dot{y} = 2a$

and $\ddot{x} = 2a$, $\ddot{y} = 0$

Now, radius of curvature of (i) is

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}} = \frac{(4a^2t^2+4a^2)^{3/2}}{0-4a^2} = \frac{8a^3(t^2+1)^{3/2}}{-4a^2}$$

$\Rightarrow \rho = 2a(1+t^2)^{3/2}$ ..... (ii)

Let P(at_1^2, $2at_1$) and Q(at_2^2, $2at_2$) are ends of the chord of the parabola (i), then, $t_1.t_2 = -1$.

Therefore, (ii) become

at P, $\rho_1 = 2a(1+t_1^2)^{3/2}$ and at Q, $\rho_2 = 2a(1+t_2^2)^{3/2}$

Now,

$$\frac{1}{\rho_1^{2/3}} + \frac{1}{\rho_2^{2/3}} = (2a)^{-2/3}\left[(1+t_1^2)^{-1} + (1+t_2^2)^{-1}\right]$$

$$= (2a)^{-2/3}\left(\frac{1}{1+t_1^2} + \frac{1}{1+t_2^2}\right)$$

$$= (2a)^{-2/3}\left(\frac{1+t_1^2+1+t_2^2}{1+t_1^2+t_2^2+t_1^2t_2^2}\right)$$

$$= (2a)^{-2/3}\left(\frac{2+t_1^2+t_2^2}{1+t_1^2+t_2^2+1}\right) \quad \text{[since } t_1.t_2 = -1]$$

$$= (2a)^{-2/3}\left(\frac{2+t_1^2+t_2^2}{2+t_1^2+t_2^2}\right)$$

$$= (2a)^{-2/3}$$

8. Show that of the curvature at any point of a circle is constant.

Solution: Let the equation of circle be

$x^2 + y^2 = a^2$ ... (i)

Differentiating (i) w. r. t. x, then

$2x + 2y \cdot y_1 = 0 \Rightarrow y_1 = -\frac{x}{y}$

and $2 + 2y_1 \cdot y_1 + 2y \cdot y_2 = 0$

$\Rightarrow y_2 = \frac{-1-y_1^2}{y_1} = \frac{-y^2-x^2}{y^3} = -\frac{a^2}{y^3}$.

Now,

$$\rho = \left|\frac{(1+y_1^2)^{3/2}}{y_2}\right| = \left|\frac{(1+x^2/y^2)^{3/2}}{-a^2/y^3}\right| = \left|\frac{(x^2+y^2)^{3/2}}{-a^2}\right| = \left|\frac{a^3}{-a^2}\right| = a.$$

$\Rightarrow \rho = a$

Hence, the curvature be $\frac{1}{\rho} = \frac{1}{a}$ which is a constant.

9. **Find ρ at (r, θ) on r = a(1 – cosθ) and show that it varies as $\sqrt{r}$.**

Solution: Given that r = a(1 – cosθ).

So, $r_1 = a\sin\theta$, $r_2 = a\cos\theta$

Now,

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{a^3(1-2\cos\theta+\cos^2\theta+\sin^2\theta)^{3/2}}{a^2[1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{a\,2\sqrt{2}\,(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}$$

$$= \frac{\sqrt{a}\,.\,2\sqrt{2}\,r^{3/2}}{3r} = \left(\frac{2\sqrt{2a}}{3}\right)\sqrt{r}$$

This shows that ρ varies as $\sqrt{r}$.

10. **Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y–axis.**

Solution: Given that, $y = e^x$

So, $y_1 = e^x = y_2$. Thus, $y = y_1 = y_2 = e^x$

At y–axis, the x–ordinate is 0. And, $y = e^0 = 1$.

At x = 0, the curve $y = e^x$ crosses the y–axis at (0, 1).

At (0, 1),

$y = y_1 = y_2 = e^0 = 1$.

Now, the radius of curvature of $y = e^x$ at the point where y cross the y-axis i.e. at (0, 1) is,

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}.$$

Chapter 8

SPACE IN COORDINATE

Exercise 8.1

1. **Find cylindrical and spherical coordinates system of a point whose cartesian coordinate is (1, 0, 0).**

Solution: Given Cartesian coordinate is

$$(x, y, z) = (1, 0, 0).$$

So, $x = 1, y = 0, z = 0.$

Then,.

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 0} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$z = 0$$

So, the cylindrical coordinate is

$$(r, \theta, z) = (1, 0, 0)$$

and

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 0 + 0} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = 0$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{0}{1}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

So, the spherical coordinate is

$$(\rho, \phi, \theta) = \left(1, \frac{\pi}{2}, 0\right)$$

2. Find cylindrical and spherical coordinate system of a point whose cartesian coordinate is (0, 0, 1).

Solution: Similar to Q1.

3. Find cartesian and spherical coordinate system of a point whose cylindrical coordinate is $(\sqrt{2}, 0, 1)$.

Solution: Given cylindrical coordinate is

$$(r, \theta, z) = (\sqrt{2}, 0, 1)$$

Then,

$$r = \sqrt{2}, \ \theta = 0, \ z = 1$$

Here

$$x = r\cos\theta = \sqrt{2}\cos 0 = \sqrt{2}$$
$$y = r\sin\theta = \sqrt{2}\sin 0 = 0$$
$$z = 1$$

So, Cartesian coordinate is

$$(x, y, z) = (\sqrt{2}, 0, 1)$$

And

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2 + 0 + 1} = \sqrt{3}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{\sqrt{2}}\right) = \tan^{-1}(0) = 0$$

So spherical coordinate is

$$(\rho, \phi, \theta) = \left(\sqrt{3}, \cos^{-1}\left(\frac{1}{\sqrt{3}}\right), 0\right)$$

4. Find cartesian and spherical coordinate system of a point whose cylindrical coordinate is [illegible].

Solution: Similar to Q3.

5. Find cartesian and cylindrical coordinate system of a point whose spherical coordinate is $\left(\sqrt{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$.

Solution: Given spherical coordinate is

$$(\rho, \phi, \theta) = \left(\sqrt{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$$

$$\rho = \sqrt{3}, \ \phi = \frac{\pi}{3}, \ \theta = \frac{\pi}{2}$$

So,

then

$$x = \rho\sin\phi\cos\theta = \sqrt{3}\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) = (\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)(0) = 0$$

$$y = \rho\sin\phi\sin\theta = (\sqrt{3})\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{2}\right) = (\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)(1) = \frac{3}{2}$$

$$z = \rho\cos\phi = \sqrt{3}\cos\left(\frac{\pi}{3}\right) = \sqrt{3}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

So, the cartesian coordinate is

$$(x, y, z) = \left(0, \frac{3}{2}, \frac{\sqrt{3}}{2}\right)$$

and

$$r = \sqrt{x^2 + y^2} = \sqrt{0 + \frac{9}{4}} = \frac{3}{2}$$

So, the cylindrical coordinate is

$$(r, \theta, z) = \left(\frac{3}{2}, \frac{\pi}{2}, \frac{\sqrt{3}}{2}\right)$$

6. Find cartesian and cylindrical coordinate system of a point whose spherical coordinate is $\left(\sqrt{3}, \pi, \frac{2\pi}{3}\right)$.

Solution: Similar to Q5.

7. Translate the equations from the given coordinate system (cartesian, cylindrical or spherical) into equations in the other two systems.

i) $x^2 + y^2 - z^2 = 4$

Solution: Given Cartesian equation is

$$x^2 + y^2 - z^2 = 4 \quad \ldots (i)$$

$$\Rightarrow r^2 - z^2 = 4$$

$$\Rightarrow r^2 - 4 = z^2$$

Also, we have

$$r^2 = x^2 + y^2 \quad .....(ii)$$

Then (i) gives,

$$r^2 + z^2 = 4 \quad .....(iii)$$

Clearly (ii) in spherical equation of (i) and (iii) is cylindrical equation of (i)

(ii) $z^2 = r^2$

Solution: Given cylindrical equation is,

$$z^2 = r^2 \quad .....(i)$$

$$\Rightarrow z^2 = x^2 + y^2$$

$$\Rightarrow 2z^2 = x^2 + y^2 + z^2 = \rho^2$$

$$\Rightarrow \sqrt{2}\, z = \rho$$

$$\Rightarrow \sqrt{z}\ \rho \cos\phi = \rho$$

$$\Rightarrow \cos\phi = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \phi = \frac{\pi}{4} \quad ......(ii)$$

Also (i) implies

$$z^2 = r^2 = x^2 + y^2$$

$$\Rightarrow x^2 + y^2 - z^2 = 0 \quad .....(iii)$$

Here (ii) is spherical equation of (i) and (iii) is Cartesian equation of (i).

(iii) x = y

Solution: Given Cartesian equation is

$$x = y \quad ......(i)$$

$$\Rightarrow \rho \sin\phi \cos\theta = \rho \sin\phi \sin\theta$$

$$\Rightarrow \cos\theta = \sin\theta$$

$$\Rightarrow \tan\theta = 1 = \tan(\pi/4)$$

$$\Rightarrow \theta = \frac{\pi}{4} \quad ......(ii)$$

And $x = y$

$$\Rightarrow r\cos\theta = r\sin\theta$$

$$\Rightarrow \tan\theta = 1 = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \theta = \frac{\pi}{4} \quad .....(ii)$$

Here (ii) is spherical equation of (i) and (iii) is cylindrical equation of (i).

(iv) $\theta = 0$

Solution: Given equation is

$$\theta = 0 \quad .....(i)$$

Since, θ is coordinate element of cylindrical and spherical coordinate. So, (i) is cylindrical as well as spherical equation.

And,

$$\theta = 0 \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{y}{x} = \tan 0 = 0 \Rightarrow y = 0.$$

This is cartesian equation of (i).

(v) $\rho = 1$

Solution: Given spherical equation is

$$\rho = 1 \quad .....(i)$$

$$\Rightarrow \sqrt{x^2 + y^2 + z^2} = 1$$

$$\Rightarrow x^2 + y^2 + z^2 = 1 \quad .....(ii)$$

This is cartesian equation of (i).

Since we know

$$r^2 = x^2 + y^2$$

So, (ii) gives

$$r^2 + z^2 = 1 \quad ......(iii)$$

This is in cylindrical coordinate system.

(vi) $\rho \sin\phi \cos\theta = 2$

Solution: Given spherical equation is

$$\rho \sin\phi \cos\theta = 2 \quad ....(i)$$

$$\Rightarrow y = 2 \quad ....(ii)$$

This is in Cartesian coordinate system

and form (ii)

$$r\cos\theta = 2 \quad ....(iii)$$

This is in cylindrical coordinate system.

(vii) $\rho = 2\cos\phi$

Solution: Given equation is spherical coordinate system is

$$\rho = 2\cos\phi \quad ......(i)$$

$$\Rightarrow \rho = 2\left(\frac{z}{\rho}\right)$$

$$\Rightarrow \rho^2 = 2z$$

$$\therefore x^2 + y^2 + z^2 = 2z \quad .....(ii)$$

This is in Cartesian coordinate system.

(viii) $x^2 + 6xy + y^2 + z^2 = 1$

Solution: Given equation in cartesian coordinate system is

$$x^2 + 6xy + y^2 + z^2 = 1 \quad .....(i)$$

$$\Rightarrow (x^2 + y^2 + z^2) + 6xy = 1$$

$$\Rightarrow \rho^2 + 6\rho \sin\phi \cos\theta \, \rho \sin\phi \sin\theta = 1$$
$$\Rightarrow \rho^2 (1 + 6 \sin^2\phi \cos\theta \sin\theta) = 1$$
$$\Rightarrow \delta^2(1 + 3\sin^2\phi \sin 2\theta) = 1 \qquad \text{......(ii)}$$

This is in spherical coordinate system. And

$$x^2 + 6xy + y^2 + z^2 = 1$$
$$\Rightarrow (x^2 + y^2) + 6xy + z^2 = 1$$
$$\Rightarrow r^2 + 6r^2 \cos\theta \sin\theta + z^2 = 1$$
$$\Rightarrow r^2 (1 + 3 \sin 2\theta) + z^2 = 1$$

This is in cylindrical coordinate system.

8. Transform the equations (by using spherical polar coordinates)

(i) $x^2 + y^2 + z^2 = 2z$

Solution: Given equation is

$$x^2 + y^2 + z^2 = 2z$$
$$\Rightarrow \rho^2 + 2\rho \cos\phi$$
$$\Rightarrow \rho = 2\cos\phi$$

This is spherical polar coordinate system.

(ii) $x^2 + y^2 - 3z^2 = 0$

Solution: Given equation is

$$x^2 + y^2 - 3z^2 = 0$$
$$\Rightarrow x^2 + y^2 + z^2 - 4z^2 = 0$$
$$\Rightarrow \rho^2 - 4\rho^2 \cos^2\phi = 0$$
$$\Rightarrow \rho^2 (1 - 4\cos^2\phi) = 0$$

This is in spherical polar coordinate system.

9. Change to spherical polar and cylindrical polar coordinates:

(i) $x^2 + y^2 = 5$ (ii) $x^2 - z^2 = 4$.

Solution: Similar to Q.7 (i)

10. Transform equation $x^2 + y^2 = x$ to cylindrical coordinates.

Solution: Given equation is

$$x^2 + y^2 = x \Rightarrow r^2 = r\cos\theta$$
$$\Rightarrow r = \cos\theta$$

This is in cylindrical coordinate system.

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Chapter 9

TWO DIMENSIONAL GEOMETRY

List of formulae for transform

1. If the axes are shifted parallel to the axes then
$$x = x_1 + h, \quad y = y_1 + k$$
where (x, y) be coordinate of given axes, (x_1, y_1) be coordinate of new axes and (h, k) be the origin of new (shifted) axes.

2. If the axes are translated with an angle θ remaining same origin (θ, 0) then
$$x = x_1 \cos\theta - y_1 \sin\theta, \quad y = x_1 \sin\theta + y_1 \cos\theta$$
where (x, y) be coordinate of given axes, (x_1, y_1) be coordinate of new axes and (h, k) be the origin of new (shifted) axes.

Exercise 9.1

1. Transform the equation $x^2 - y^2 + 2x + 4y = 0$ by transferring the origin to (–1, 2) the coordinate axes remaining parallel.

Solution: Given equation is
$$x^2 - y^2 + 2x + 4y = 0 \qquad \dots (i)$$
Here, origin transformed to point (–1, 2). Then, x is replaced by x – 1 and y is replaced by y + 2 in equation (i) then,
$$(x-1)^2 - (y+2)^2 + 2(x-1) + 4(y+2) = 0$$
$$\Rightarrow x^2 - 2x + 1 - y^2 - 4y - 4 + 2x - 2 + 4y + 8 = 0$$
$$\Rightarrow x^2 - y^2 + 3 = 0$$
This is the required equation.

2. Transform the equation $x^2 - 3y^2 + 4x + 6y = 0$ by transferring the origin to the point (–2, 1) the co–ordinate axes remaining parallel.

Solution: Given equation is
$$x^2 - 3y^2 + 4x + 6y = 0 \qquad \dots (i)$$
Here, origin transformed to point (–2, 1). Then, x is replaced by x – 2 and y is replaced by y + 1 in equation (i) then,
$$(x-2)^2 - 3(y+1)^2 + 4(x-2) + 6(y+1) = 0$$
$$\Rightarrow x^2 - 4x + 4 - 3y^2 - 6y - 3 + 4x - 8 + 6y + 6 = 0$$
$$\Rightarrow x^2 - 3y^2 - 1 = 0$$
This is the required equation.

3. Translate the axes so as to change the equation $3x^2 - 2xy + 4y^2 + 8x - 10y + 8 = 0$ into an equation with linear terms missing.

Solution: Given equation is
$$3x^2 - 2xy + 4y^2 + 8x - 10y + 8 = 0 \qquad \dots (i)$$
Transform origin to (h, k) when axes remaining parallel, then x becomes (x + h) and y becomes (y + k) in equation (i) then,

$$3(x+h)^2 - 2(x+h)(y+k) + 4(y+k)^2 + 8(x+h) - 10(y+k) + 8 = 0$$

$$\Rightarrow 3(x^2 + 2xh + h^2) - 2xy - 2hy - 2kx - 2hk + 4y^2 + 8yk + 4k^2 + 8x + 8h - 10y - 10k + 8 = 0$$

$$\Rightarrow 3x^2 - 2xy + 4y^2 + x(6h - 2k + 8) + y(-2h + 8k - 10) + 3h^2 - 2hk + 4k^2 + 8h - 10k + 8 = 0 \quad \ldots \text{(ii)}$$

Since, in this equation linear term is missing. So, coefficient of $x = 0$ and coefficient of y is 0.

i.e. $6h - 2k + 8 = 0$ and $-2h + 8k - 10 = 0$

Solving we get, $k = 1$ and $h = -1$.

Substituting these values in equation (ii) then,

$$3x^2 - 2xy + 4y^2 + 3 + 2 + 4 - 8 - 10 + 8 = 0.$$

$$\Rightarrow 3x^2 - 2xy + 4y^2 - 1 = 0.$$

This is the required equation.

4. Transform the equation $x^2 + 2cxy + y^2 = a^2$, by turning the rectangular axes through the angle $\frac{\pi}{4}$.

Solution: Given, $\theta = \frac{\pi}{4}$ and given equation is

$$x^2 + 2cxy + y^2 = a^2 \quad \ldots \text{(i)}$$

By hypothesis, the axes are turned with an angle $\frac{\pi}{4}$. So, replace, x by $x\cos\frac{\pi}{4} - y\sin\frac{\pi}{4} = \frac{x-y}{\sqrt{2}}$ and y by $x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4} = \frac{x+y}{\sqrt{2}}$ in equation (i) then,

$$\left(\frac{x-y}{\sqrt{2}}\right)^2 + 2c\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + \left(\frac{x+y}{\sqrt{2}}\right)^2 = a^2$$

$$\Rightarrow \frac{(x-y)^2}{2} + c(x^2 - y^2) + \frac{(x+y)^2}{2} = a^2$$

$$\Rightarrow (x^2 - 2xy + y^2) + 2cx^2 - 2cy^2 + (x^2 + 2xy + y^2) = 2a^2$$

$$\Rightarrow x^2(1+c) + y^2(1-c) = a^2$$

This is the required equation.

5. What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to rectangular axes through the point $(-2, -3)$, the new axes being inclined at an angle $\frac{\pi}{4}$, with the old?

Solution: When the axes turn by an angle θ without change in origin, then we replace,

$$x = x\cos\theta - y\sin\theta \quad \text{and} \quad y = x\sin\theta + y\cos\theta.$$

Again when axes are transferred from origin to point (h, k) we replace,

$$x = x + h \quad \text{and} \quad y = y + k.$$

Here, after changing origin and turning the axes,

$$x = (x+h)\cos\theta - (y+k)\sin\theta$$

$$y = (x+h)\sin\theta + (y+k)\cos\theta$$

Given equation is

$$2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0 \quad \ldots \text{(i)}$$

This is transforms the axes parallel with $(h, k) = (-2, -3)$ and turned with an angle $\theta = \frac{\pi}{4}$. Therefore the equation (i) transforms as,

$$x = (x-2)\cos\frac{\pi}{4} - (y-3)\sin\frac{\pi}{4} = \frac{x-2-y+3}{\sqrt{2}} = \frac{x-y+1}{\sqrt{2}}$$

$$y = (x-2)\sin\frac{\pi}{4} + (y-3)\cos\frac{\pi}{4} = \frac{x-2+y-3}{\sqrt{2}} = \frac{x+y-1}{\sqrt{2}}$$

Then equation (ii) becomes,

$$2\left(\frac{x-y+1}{\sqrt{2}}\right)^2 + 4\left(\frac{x-y+1}{\sqrt{2}}\right)\left(\frac{x+y-5}{\sqrt{2}}\right) - 5\left(\frac{x+y-5}{\sqrt{2}}\right)^2 + 20\left(\frac{x-y+1}{\sqrt{2}}\right) - 22\left(\frac{x+y-5}{\sqrt{2}}\right) - 14 = 0.$$

$$\Rightarrow (x-y+1)^2 + 2(x-y+1)(x+y-5) - \frac{5}{2}(x+y-5)^2 + 10\sqrt{2}(x-y+1) - 11\sqrt{2}(x+y-5) - 14 = 0.$$

$$\Rightarrow 2(x-y+1)^2 + 4(x-y+1)(x+y-5) - 5(x+y-5)^2 + 20\sqrt{2}(x-y+1) - 22\sqrt{2}(x+y-5) - 28 = 0.$$

$$\Rightarrow 2(x^2 + y^2 + 1 - 2xy + 2x - 2y) + 4(x^2 + xy - 5x - xy - y^2 + 5y + x + y - 5) - 5(x^2 + y^2 + 25 + xy - 5y - 5x) + 20\sqrt{2}x - 20\sqrt{2}y + 20\sqrt{2} - 22\sqrt{2}x - 22\sqrt{2}y + 110\sqrt{2} - 28 = 0.$$

$$\Rightarrow x^2 - 7y^2 - 9xy + (3 - 2\sqrt{2})x + (45 - 42\sqrt{2})y - 171 + 130\sqrt{2} = 0.$$

This is required equation.

6. By transforming to parallel axes through a properly chosen point (h, k) prove that the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ can be reduced to one containing only terms of second degree.

Solution: Given equation is

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0 \quad \ldots \text{(i)}$$

Let $x = x + h$ and $y = y + k$.

Then equation (i) then,

$$12(x+h)^2 - 10(x+h)(y+k) + 2(y+k)^2 + 11(x+h) - 5(y+k) + 2 = 0.$$

$$\Rightarrow 12[x^2 + 2xh + h^2] - 10(xy + xk + hy + hk) + 2(y^2 + 2yk + k^2) + (11x + 11h) - (5y + 5k) + 2 = 0.$$

$$\Rightarrow 12x^2 + 24xh + 12h^2 - 10xy - 10xk - 10hy - 10hk + 2y^2 + 4yk + 2k^2 + 11x + 11h - 5y - 5k + 2 = 0.$$

$$\Rightarrow 12x^2 + 2y^2 - 10xy + x(24h - 10k + 11) - y(10h + 4k - 5) + (12h^2 - 10hk + 2k^2 + 11h - 5k + 2) = 0.$$

We have, in this equation linear term and constant term is absent. If it has only the term of second degree then,

Coefficient of x = 0, , constant term = 0

$$\Rightarrow \quad 24h - 10k + 11 = 0$$

Coefficient of y = 0

$$\Rightarrow \quad 10h + 4k - 5 = 0$$

Solving these two equations we get

$$h = -\frac{3}{2}. \text{ and } \qquad k = -\frac{5}{2}.$$

Hence by transforming the axes to origin (h, k) i.e. $\left(-\frac{3}{2}, -\frac{5}{2}\right)$, the equation becomes,

$$12\left(x-\frac{3}{2}\right)^2 - \left(x-\frac{3}{2}\right)\left(y-\frac{5}{2}\right) + 2\left(y-\frac{5}{2}\right) + 11\left(x-\frac{3}{2}\right) - 5\left(y-\frac{5}{2}\right) + 2 = 0$$

$$\Rightarrow 12\left(x^2 - 3x + \frac{9}{4}\right) - 10\left(xy - \frac{15x}{2} - \frac{3y}{2} + \frac{15}{4}\right) + 2\left(y^2 - 5y + \frac{25}{4}\right) + 11x - \frac{33}{2} - 5y + \frac{25}{2} + 2 = 0$$

$$\Rightarrow \quad 12x^2 - 10xy + 2y^2 - 36x + 25x + 11x + 15y - 10y - 5y + \frac{54}{2} - \frac{150}{4} + \frac{50}{4} - \frac{33}{2} + \frac{25}{2} + 2 = 0$$

$$\Rightarrow \quad 12x^2 - 10xy + 2y^2 = 0.$$

This is the required equation.

7. What does the equation of the straight lines $7x^2 + 4xy + 4y^2 = 0$ become when the axes are turned through 45°, the origin remaining fixed?

Solution: Here, given equation is

$$7x^2 + 4xy + 4y^2 = 0 \qquad \ldots \text{(i)}$$

When the axes are turned through $\theta = 45°$ then, x and y becomes as,

$$x = x\cos\theta - y\sin\theta = \frac{x-y}{\sqrt{2}} \quad \text{and} \quad y = x\sin\theta + y\cos\theta = \frac{x+y}{\sqrt{2}}$$

Putting values of x and y in equation (i) then

$$7\left(\frac{x-y}{\sqrt{2}}\right)^2 + 4\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + 4\left(\frac{x+y}{\sqrt{2}}\right)^2 = 0$$

$$\Rightarrow \quad \frac{7}{2}(x-y)^2 + 2(x^2 - y^2) + 2(x+y)^2 = 0$$

$$\Rightarrow \quad 7x^2 - 14xy + 7y^2 + 4x^2 - 4y^2 + 4x^2 + 8xy + 4y^2 = 0$$

$$\Rightarrow \quad 15x^2 - 6xy + 7y^2 = 0.$$

This is the required equation.

Exercise 9.2

1. Sketch the parabola, showing the focus, vertex and directrix:

(i) V(0, 0), F(0, 2)

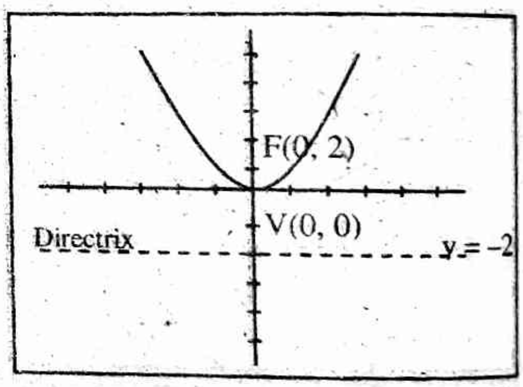
Solution: Here, the x ordinate has same value and y varies. So, the focus lies on the line parallel to y–axis.

Here, V(h, k) = V(0, 0).

And F(h, k + a) = F(0, 2).

Then, a = 2.

Since, y ordinate varies and a has positive value, so the parabola has up openward.

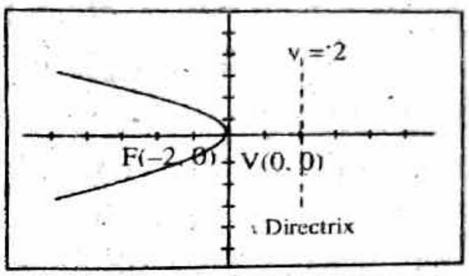


Also, equation of directrix be, $y = -a \Rightarrow y = -2$.

With the help of these information, the sketch of the parabola is as:

(ii) **V(0, 0), F(–2, 0)**

Solution: Here, the y ordinate has same value and x varies. So, the focus lies on the line parallel to x–axis.



Here, $V(h, k) = V(0, 0)$.

And $F(h + a, k) = F(-2, 0)$.

Then, $a = -2$.

Since, x ordinate varies and a has negative value, so the parabola has left openward.

Also, equation of directrix be, $x = -a \Rightarrow x = -(-2) \Rightarrow x = 2$.

With the help of these information, the sketch of the parabola is as:

(iii) V(0, 0), F(0, –4) (iv) V(–2, 3), F(–2, 4) (vii) V(1, –3), F(1, 0)

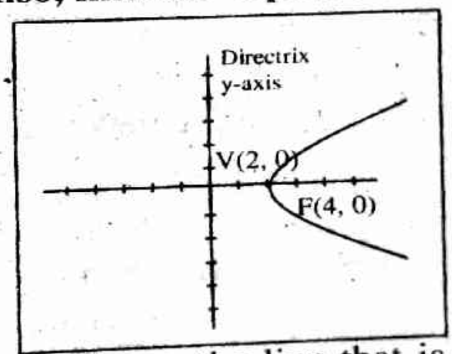
Process as (i)

(v) V(0, 3), F(–1, 3) (vi) V(–3, 2), F(0, 1)

Process as (ii)

2. **Sketch the following parabola with showing the focus, vertex and directrix where V is vertex and L is directrix. Also, find the equation of the parabola.**

(i) **V(2, 0), L is the y–axis.**



Solution: Here, vertex $V(h, k) = V(2, 0)$.

and directrix be y–axis (i.e. $x = 0$).

Since we know 'a' is the distance between V and directrix.

So, $a = 2$.

Also, the directrix be y–axis, so the vertex and focus lie on the line that is parallel to x–axis.

Therefore, $F(h + a, k) = F(2 + 2, 0) = F(4, 0)$.

Now, the equation of parabola is

$$(y - k)^2 = 4a(x - h) \Rightarrow y^2 = 8(x - 2).$$

With the help of these information, the sketch of the parabola is as:

(ii) V(1, –2), L is the x–axis

(iii) V(–3, 1), L is the line x = 1.

(iv) V(–2, –2), L is the line y = – 3,

(v) V(0, 1), L is the line x = –1.

(vi) V(0, 1), L is the line y = 2.

Process to solve as (i).

3. Find the focus, vertex and directrix of the parabola and sketch.

(i) $y^2 = 8x$

Solution: We have, $y^2 = 8x$.

Comparing this equation with $(y-k)^2 = 4a(x-h)$ we get

$h = 0, k = 0$ and $a = 2$.

Now, vertex be, $V(h, k) = V(0, 0)$.

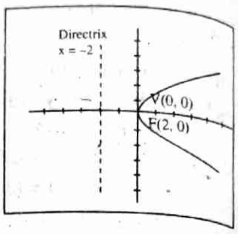
Focus be, $F(h + a, k) = F(2, 0)$.

And the directrix is,

$x - h = -a$

$\Rightarrow x = -2 \Rightarrow x + 2 = 0.$

With the help of these information, the sketch of the parabola is as:



(ii) $x^2 = 100y$ (iii) $y^2 + 36x = 0$

Process as in (i).

4. Find the vertex, axis of symmetry, focus and directrix of the gives parabola and sketch.

(i) $x^2 + 8y - 2x = 7$

Solution: Given equation is

$x^2 + 8y - 2x = 7$

$\Rightarrow x^2 - 2.1.x + (1)^2 = -8y + 7 + 1$

$\Rightarrow (x-1)^2 = -8(y-1).$

Comparing the equation with $(x-h)^2 = 4a(y-k)$

we get,

$h = 1, k = 1$ and $a = -2$.

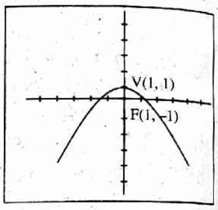
Now, vertex $V(h, k) = V(1, 1)$

axis of symmetry is, $(x-1)^2 = 0 \Rightarrow x = 1$.

Equation of directrix is, $y - 1 = -a \Rightarrow y - 1 = 2 \Rightarrow y = 3$.

and focus $= F(h, k + a) = F(1, -1)$.

With the help of these information, the sketch of the parabola is as:



(ii) – (x) Process as in (i).

5. Find the length of latus rectum of the curve $y^2 = 4px$.

Solution: Here, $y^2 = 4px$

Comparing the equation with $y^2 = 4ax$ then we get,

$4a = 4p \Rightarrow a = p$.

We know, the length of latus rectum $= 4a = 4p$.

6. A double ordinate of the parabola $y^2 = 2ax$ is of length 4a, prove that the lines joining the vertex to its ends are at right angles. [2008, Spring]

Solution: Given parabola is $y^2 = 2ax$.

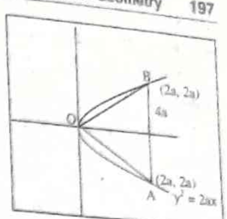
And length of double ordinate is 4a.

So, $2y = 4a \Rightarrow y = 2a$.

For ends of double ordinate, put $y = 2a$ in $y^2 = 2ax \Rightarrow (2a)^2 = 2ax$.

$\Rightarrow x = 2a$.

Thus, the co-ordinate of end points of double ordinate are $A(2a, -2a)$ and $B(2a, 2a)$.



Since the vertex of given parabola is (0, 0). So, the line joining vertex and ends of ordinate are OA and OB. Here,

$$\text{Slope of OA } (m_1 \text{ say}) = \frac{-2a - 0}{2a - 0} = -1$$

$$\text{Slope of OB } (m_2 \text{ say}) = \frac{2a - 0}{2a - 0} = 1$$

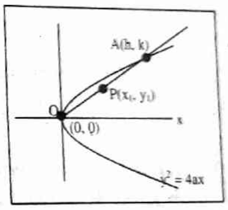
Now, $m_1.m_2 = -1$. This proves that OA and OB are perpendicular to each other.

7. Find the locus of mid-points of the chord on $y^2 = 4ax$ through vertex. Prove that it is parabola. Find its latus rectum.

Solution: Let $P(x_1, y_1)$ be the mid-point of the chord on $y^2 = 4ax$ passing through vertex (0, 0).

Let, A is the point where the line OP cuts the parabola, say the coordinate of A is (h, k). Then the slope of OA is,

$$\text{Slope of OA} = \frac{k - 0}{h - 0} = \frac{k}{h}$$



Here slope of the chord joining (0, 0) and $P(x_1, y_1)$ is,

$$\text{Slope of OP} = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1} x \quad \text{.... (i)}$$

Since the point A lies on the parabola $y^2 = 4ax$. Therefore, it satisfies the equation of parabola $y^2 = 4ax$ then

$k^2 = 4ah$.

Thus,

$$\text{Slope of OA} = \frac{k}{h} = \frac{k}{k^2/4a} = \frac{4a}{k} = \frac{4a}{2y_1} \quad \text{[Being P be mid-point of OA]}$$

Since the line OP and OA are segments of a line. So, their slope should be equal. That is,

Slope of OP = Slope of OA

$$\Rightarrow \frac{y_1}{x_1} = \frac{4a}{2y_1}$$

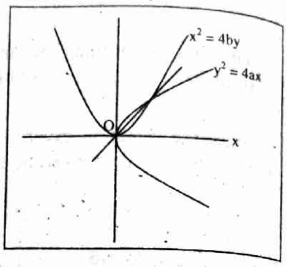
$\Rightarrow y_1^2 = 2ax_1$.

Hence, the locus of x_1 and y_1 is $y^2 = 2ax$.

And, the length of latus rectum of $y^2 = 2ax$ is 2a.

8. Find the points common to the parabolas $y^2 = 4ax$ and $x^2 = 4by$. Find the equation of the common chord that passes through the common points.

Solution: For point of intersection between $y^2 = 4ax$ and $x^2 = 4by$ eliminating y we get,



$$\left[\frac{x^2}{4b}\right]^2 = 4ax$$

$$\Rightarrow \quad x^4 = 64ab^2x$$

$$\Rightarrow \quad x^4 - 64ab^2x = 0$$

$$\Rightarrow \quad x(x^3 - 64ab^2) = 0$$

Either, $x = 0$ or $x^3 - 64ab^2 = 0$

$$\Rightarrow x = 4a^{1/3}b^{2/3}$$

Then, $y = \frac{x^2}{4b} = \frac{(4a^{1/3}b^{2/3})^2}{4b} = 4b^{1/3}a^{2/3}$

Hence the common point to the both curves are $(0, 0)$ and $(4a^{1/3}b^{2/3}, 4b^{1/3}a^{2/3})$.

Now, equation of chord is,

$$y - 0 = \frac{4b^{1/3}a^{2/3} - 0}{4a^{1/3}b^{2/3} - 0}(x - 0)$$

$$\Rightarrow \quad y = a^{1/3}b^{-1/3}x$$

$$\Rightarrow \quad b^{1/3}y = a^{1/3}x$$

This is the required equation of chord.

Thus, the equation of the common chord that passes through the common points is $b^{1/3}y = a^{1/3}x$.

9. Obtain the co–ordinate of the focus and the equation of the directrix of the parabola $x^2 - 8x + 2y - 10 = 0$.

Solution: Given parabola is,

$$x^2 - 8x + 2y - 10 = 0$$

$$\Rightarrow x^2 - 8x + 16 = -2y + 10 + 16$$

$$\Rightarrow (x - 4)^2 = -2(y - 13)$$

Comparing the equation with $(x - h)^2 = -4a(y - k)$ we get

$h = 4$, $k = 13$ and $a = -½$

Now, vertex of the parabola is $V(h, k) = V(4, 13)$

focus is $F(h, k + a) = F(4, 13 - ½) = F(4, 25/2)$.

and equation of directrix is,

$$y - k = -a$$

$$\Rightarrow \quad y - 13 = ½ \Rightarrow 2y - 27 = 0.$$

Thus, the co–ordinate of the focus is $F(4, 25/2)$ and the equation of the directrix of the parabola $x^2 - 8x + 2y - 10 = 0$ is $2y - 27 = 0$.

10. Find the vertex, focus, latus rectum, axis and directrix of the parabola $x^2 - y - 2x = 0$.

Solution: Given equation is,

$$x^2 - y - 2x = 0 \quad \Rightarrow x^2 - 2x + 1 = y + 1$$

$$\Rightarrow (x - 1)^2 = (y + 1)$$

Comparing the equation with $(x - h)^2 = 4a(y - k)$, we get

$h = 1$, $k = -1$ and $a = ¼$

Now, vertex is $V(h, k) = V(1, -1)$

focus, $F(h, k + a) = F(1, -1 + ¼) = F(1, -3/4)$

length of latus rectum $= 4a = 1$

axis of symmetry be, $x = 1$

And equation of directrix is,

$$y - k = -a$$

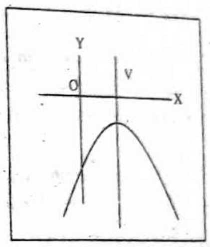
$$\Rightarrow y = -1 - ¼$$

$$\Rightarrow y = -5/4$$

$$\Rightarrow 4y + 5 = 0.$$

11. Discuss the equation $2x^2 + 5y - 3x + 4 = 0$ and sketch the curve.

Solution: Given equation is,



$$2x^2 + 5y - 3x + 4 = 0$$

$$\Rightarrow x^2 + \frac{5y}{2} - \frac{3}{2}x + 2 = 0$$

$$\Rightarrow x^2 - \frac{3}{2}x + \frac{9}{16} = -5/2\,y - 2 + \frac{9}{16}$$

$$\Rightarrow \left(x - \frac{3}{4}\right)^2 = -\frac{5}{2}y - \frac{23}{16}$$

$$\Rightarrow \left(x - \frac{3}{4}\right)^2 = -\frac{5}{2}\left[y + \frac{23}{40}\right] \quad \ldots \text{(i)}$$

Comparing equation (i) with $(x - h)^2 = -4a(y - k)$, we get

$h = ¾$, $k = -23/40$ and $a = -5/8$

Now, vertex $V(h, k) = V(3/4, -23/40)$.

axis of symmetry be, $(x - ¾)^2 = 0 \Rightarrow x = ¾$

equation of directrix is, $y = k - a$

$$\Rightarrow \quad y = -\frac{23}{40} - \frac{5}{8} = \frac{-23 - 25}{40} = \frac{-6}{5}$$

Focus is $(h, k - a) = [3/4, -6/5]$

Then, latus rectum $= 4a = 4 \times \frac{5}{8} = \frac{5}{2}$.

With these information, the sketch of the parabola is as:

12. Find the vertex, axis, focus and latus rectum of the parabola $4y^2 + 12x - 20y + 6y = 0$.

Solution: Given equation is,

$$4y^2 + 12x - 20y + 67 = 0$$

$$\Rightarrow y^2 - 5y = -3x - \frac{67}{4} \quad \Rightarrow y^2 - 2\times\frac{5}{2}y + \frac{25}{4}$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -3x - \frac{21}{2}$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right) \quad \dots \text{(i)}$$

Comparing equation (i) with $(y-k)^2 = -4a(x-h)$, we get

$$h = \frac{-7}{2},\ k = \frac{5}{2} \text{ and } a = \frac{-3}{4}.$$

Now, vertex $V(h, k) = \left(-\frac{7}{2}, \frac{5}{2}\right)$

axis of symmetry is, $\left(y - \frac{5}{2}\right) = 0 \Rightarrow y = \frac{5}{2}$

focus $= F(h-a, k) = \left(-\frac{7}{2} - \frac{3}{4}, \frac{5}{2}\right) = \left(-\frac{17}{4}, \frac{5}{2}\right)$

length of latus rectum $= 4a = 4 \times \frac{3}{4} = 3.$

13. Find the equation of the parabola having focus (–3, 0), directrix $x + 5 = 0$. **[2011 Spring, Short]**

Solution: Given that the focus of the parabola is,

$$F(-3, 0)$$

and the equation of directrix of the parabola is,

$$x + 5 = 0$$
$$\Rightarrow x = -5.$$

This tells us the symmetry line of the parabola is parallel to x-axis. Therefore, the focus of this parabola is

$$F(h + a, k) = F(-3, 0)$$

This implies $h + a = -3,\ k = 0.$

We know the distance between the focus and directrix is,

$$2a = |(-3) - (-5)| = 2$$
$$\Rightarrow a = 1.$$

Therefore, $h = -3 - a = -3 - 1 = -4,\ k = 0.$

Then, the vertex of the parabola is

$$V(h, k) = V(-4, 0).$$

Since the line of symmetry line is parallel to x-axis and $a > 0$. So, the equation of parabola is

$$(y-k)^2 = 4a(x-h)$$
$$\Rightarrow (y-0)^2 = 4 \times 1\,(x+4)$$
$$\Rightarrow y^2 = 4(x+4).$$

This is the equation of the parabola.

14. Find equation of parabola with ends of latus rectum (–1, 5), (–1, –11) and vertex at (–5, –3).

Solution: Given that the vertex is,

$$V(h, k) = (-5, -3)$$

Therefore, $h = -5$ and $k = -3.$ ... (i)

Also given that the ends of latus rectum are (–1, 5), (–1, –11). We know the focus is the mid-point of the end points of the latus rectum. So,

Focus = Mid-point of ends of latus rectum

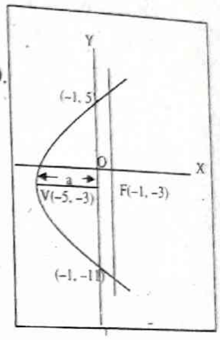
$$= \left(\frac{-1-1}{2}, \frac{5-11}{2}\right) = F(-1, -3)$$

Here, both vertex and focus have same y value, so

$$F(h + a, k) = F(-1, -3) \quad \dots \text{(ii)}$$

Therefore,

$$h + a = -1 \Rightarrow -5 + a = -1 \Rightarrow a = 4 > 0.$$



Since the y value of the vertex and focus is same. So the symmetry line of the parabola is parallel to x axis. Therefore the equation of the parabola is,

$$(y-k)^2 = 4a(x-h)$$
$$\Rightarrow (y+3)^2 = 4 \times 4(x+5)$$
$$\Rightarrow y^2 + 6y + 9 = 16x + 80$$
$$\Rightarrow y^2 - 16x + 6y - 71 = 0$$

Therefore, $y^2 + 6y - 16x - 71 = 0$ is the equation of the parabola.

15. Find equation of parabola passing through (3, 3), (6, 5) and (6, –3), and its axis being parallel to the x–axis.

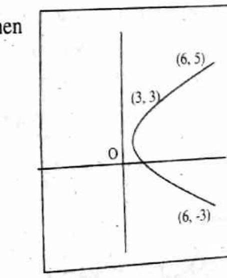
Solution: Given that the parabola is passing through the points (3, 3), (6, 5) and (6, –3). So, while plotting them the parabola was found to be concave right.

Now, the equation of parabola with vertex (h, k) and concave right is

$$(y-k)^2 = 4a(x-h) \quad \dots (1)$$

that passes through (3, 3), (6, 5) and (6, –3), then

$$(3-k)^2 = 4a(3-h) \quad \dots (2)$$
$$(5-k)^2 = 4a(6-h) \quad \dots (3)$$
$$(-3-k)^2 = 4a(6-h) \quad \dots (4)$$



Dividing equation (3) by equation (4), we get

$$(5-k)^2 = (3+k)^2$$
$$\Rightarrow 25 - 10k + k^2 = 9 + 6k + k^2$$
$$\Rightarrow 25 - 9 = 6k + 10k \quad \Rightarrow 16 = 16k$$
$$\Rightarrow k = 1.$$

Putting the value of k in equation (2) and (3)

$$4 = 4a(3-h) \quad \dots (5)$$
$$16 = 4a(6-h) \quad \dots (6)$$

Solving (5) and (6), then

$$h = 2 \quad \text{and} \quad a = 1.$$

Hence, the equation (i) becomes

$$(y - 1)^2 = 4 \times 1(x - 2)$$

$$\Rightarrow y^2 - 2y - 4x + 9 = 0$$

This is the equation of the parabola.

16. Find the equation of parabola whose focus (a, b) and directrix is $\frac{x}{a} + \frac{y}{b} = 1$.

Solution: Given that the focus is F(a, b) and directrix is $bx + ay = ab$.

Let P(x, y) be any point on the parabola. Then by definition,

length of PF = perpendicular distance from P to directrix.

$$\Rightarrow \sqrt{(x - a)^2 + (y - b)^2} = \pm \frac{bx + ay - ab}{\sqrt{b^2 + a^2}}$$

$$\Rightarrow (b^2 + a^2)(x^2 + y^2 - 2ax - 2by + a^2 + b^2)$$
$$= b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2ab^2x - 2a^2by$$

$$\Rightarrow b^2x^2 + a^2x^2 + b^2y^2 + a^2y^2 - 2ab^2x - 2a^3x - 2b^3y - 2a^2by + a^2b^2 + a^4 + b^4 + a^2b^2 = b^2x^2 + a^2y^2 + a^2b^2 + 2abxy - 2ab^2x - 2a^2by$$

$$\Rightarrow a^2x^2 - 2abxy + b^2y^2 - 2a^3x - 2b^3y + a^4 + b^4 + a^2b^2 = 0$$

$$\Rightarrow (ax - by)^2 - 2(a^3x + b^3y) + a^4 + b^3 + a^2b = 0.$$

This is the equation of required parabola.

Exercise 9.3

1. Find the equation of tangent and normal at the extremities of the latus rectum of the parabola $y^2 = 12x$. **[2004, Spring]**

Solution: Since the coordinate of end points of latus rectum of the parabola $y^2 = 4ax$ be $(a, \pm 2a)$. Therefore, the end points of latus rectum of the parabola $y^2 = 12x$ are $(3, 6)$ and $(3, -6)$.

Now, equation of tangent at extremity (3, 6) is,

$$\boxed{yy_1 = 2a(x + x_1)}$$

i.e. $y \cdot 6 = 2 \cdot 3(x + 3) \Rightarrow y = x + 3.$

And, the equation of normal at extremity (3, 6) is,

$$\boxed{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

i.e. $y - 6 = -\frac{6}{2 \times 3}(x - 3) \Rightarrow y - 6 = -x + 3 \Rightarrow x + y = 9.$

Also, the equation of tangent at extremity (3, –6) is,

$$\boxed{yy_1 = 2a(x + x_1)}$$

i.e. $y(-6) = 2 \cdot 3(x + 3) \Rightarrow -y = x + 3 \Rightarrow x + y + 3 = 0.$

And, the equation of normal at extremity (3, –6) is,

$$\boxed{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

i.e. $y + 6 = -\frac{(-6)}{2 \times 3}(x - 3) \Rightarrow y + 6 = x - 3 \Rightarrow x - y = 9.$

Thus, the equation of tangent to the parabola $y^2 = 12x$ at the extremities of the latus rectum (3, 6) is $y = x + 3$ and at (3, −6) is $x + y + 3 = 0$ and the equation of normal is at (3, 6) is $x + y = 9$ and at (3, −6) is $x - y = 9$.

2. **A tangent to $y^2 = 4x$ makes an angle of 45° with $2x + y = 0$. Find its equation and the point of contact.**

Solution: Since, the equation of tangent to $y^2 = 4x$ is

$$y = mx + \frac{1}{m} \quad \ldots (i) \qquad [\text{Since } a = 1]$$

which makes an angle 45° with $y = -2x$ ... (ii)

Since, by definition $\tan\theta = \frac{m_1 - m_2}{1 + m_1.m_2}$ where m_1 and m_2 are slope of two intersecting lines at an angle θ. Therefore

$$\tan 45° = \frac{m - (-2)}{1 + m(-2)} \Rightarrow 1 = \frac{m + 2}{1 - 2m} \Rightarrow 1 - 2m = m + 2 \Rightarrow m = \frac{-1}{3}.$$

Then (i) becomes,

$$y = \left(-\frac{1}{3}\right)x + \frac{1}{-1/3} \Rightarrow y = -\frac{1}{3}x - 3 \quad \ldots (iii)$$

For point of contact between equation (iii) and $y^2 = 4x$, eliminating y from these equation then,

$$\left(-\frac{x}{3} - 3\right)^2 = 4x$$

$$\Rightarrow (x + 9)^2 = 36x$$

$$\Rightarrow x^2 + 18x + 81 = 36x$$

$$\Rightarrow x^2 - 18x + 81 = 0 \Rightarrow (x - 9)^2 = 0 \Rightarrow x = 9.$$

Putting the value of x in (iii),

$$y = -\frac{1}{3}(9) - 3 \quad \text{then} \quad y = -6.$$

Hence, the point of contact is (9, − 6).

3. **Find the condition that the line $y = mx + c$ may touch the parabola $y^2 = 4a(x + a)$.**

Solution: We have, given equation of line and parabola is

$$y = mx + c \quad \ldots (i) \qquad \text{and} \qquad y^2 = 4a(x + a) \quad \ldots (ii)$$

Since (i) touches (ii) so

$$(mx + c)^2 = 4a(x + a)$$

$$\Rightarrow m^2x^2 + 2mxc + c^2 = 4ax + 4a^2$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + (c^2 - 4a^2) = 0 \quad \ldots (iii)$$

which is quadratic in x. Since (i) touches (ii) so the discriminate term of (iii) is equal to zero.

$$\boxed{b^2 - 4ac = 0}$$

i.e. $(2mc - 4a)^2 - 4m^2(c^2 - 4a^2) = 0$

$\Rightarrow\ 4m^2c^2 - 16mca + 16a^2 - 4m^2c^2 + 16m^2a^2 = 0$

$\Rightarrow\ 16mca = 16a^2 + 16m^2a^2.$

$\Rightarrow\ c = \dfrac{-16a^2(1+m^2)}{16ma} \quad \Rightarrow\ c = \dfrac{a(1+m^2)}{m} \quad \Rightarrow\ c = am + \dfrac{a}{m}.$

Thus the line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = am + \dfrac{a}{m}$

4. Find the equation of the common tangent of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution: We have, equation of tangent on $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \qquad \dots (i)$$

Since (i) is tangent to the parabola $y^2 = 4ax$ and also the tangent is common to the parabola $x^2 = 4by$. So, the tangent (i) must satisfy the parabola

$$x^2 = 4by. \qquad \dots (ii)$$

Therefore, eliminating y from (1) and (2), we get,

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow\ x^2 - 4bmx - \frac{4ab}{m} = 0 \qquad \dots (iii)$$

which is quadratic in x,

Since the line (i) is tangent on (ii), so the discriminate value of (iii) is zero.

i.e. $(-4bm)^2 - 4 \,.\, 1\left(-\dfrac{4ab}{m}\right) = 0$

$\Rightarrow\ 16\,b^2 m^2 = -\dfrac{16ab}{m}$

$\Rightarrow\ m^3 = -\dfrac{a}{b} \quad \Rightarrow\ m = -\left(\dfrac{a}{b}\right)^{1/3}$

Substituting the value of m in equation (i), then

$$y = -\left(\frac{a}{b}\right)^{1/3} x + \frac{a}{(-a/b)^{1/3}}$$

$$y = -\frac{a^{1/3}}{b^{1/3}}x - b^{1/3}a^{2/3}$$

$\Rightarrow\ y\,b^{1/3} = -a^{1/3}x - b^{2/3}\,.\,a^{2/3}$

$\Rightarrow\ a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0.$

Thus, the equation of the common tangent of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0.$

5. Prove that the line $lx + my + n = 0$ touches $y^2 = 4ax$ if $ln = am^2$.

Solution: See the theory part before Exercise 9.2.

6. Find the equation of tangent on $y^2 = 25x$ through (4, 10).

Solution: Given equation of parabola is,

$$y^2 = 25x \qquad \dots (i)$$

which has a tangent at (4, 10).

Comparing equation (i) with $y^2 = 4ax$ we get $a = \dfrac{25}{4}$.

Now the equation of tangent on $y^2 = 25x$ at (4, 10) is,

$$yy_1 = 2a(x + x_1)$$

$\Rightarrow\ y\,.\,10 = 2 \times \dfrac{25}{4}(x + 4)$

$\Rightarrow\ 40y = 50(x + 4) \quad \Rightarrow\ 4y = 5x + 20.$

Thus, the equation of tangent on $y^2 = 25x$ that passes through the point (4, 10) is $5x - 4y + 20 = 0$.

7. Find the equations of the tangents to the parabola $y^2 = 9x$ which passes through (4, 10).

Solution: Given parabola is

$$y^2 = 9x \qquad \dots (i)$$

Comparing equation (i) with $y^2 = 4ax$ then we get, $a = \dfrac{9}{4}$.

Now, the equation of tangent to $y^2 = 9x$ is

$$y = mx + \frac{9/4}{m} \qquad \dots (ii)$$

Since the tangent line (ii) passes through (4, 10), then

$$10 = 4m + \frac{9}{4m}$$

$\Rightarrow\ 40m = 16m^2 + 9$

$\Rightarrow\ 16m^2 - 40m + 9 = 0 \ \Rightarrow (4m - 9)(4m - 1) = 0.$

Either, $4m - 9 = 0 \ \Rightarrow m = \dfrac{9}{4}$

or $4m - 1 = 0 \Rightarrow m = \dfrac{1}{4}.$

Thus, the equation of tangents is,

$$y = \frac{9}{4}x + \frac{9/4}{9/4} \quad \Rightarrow\ 4y = 9x + 4.$$

and $4y = x + 36.$

Thus, the equations of the tangents to the parabola $y^2 = 9x$ which passes through (4, 10) are $4y = 9x + 4$ and $4y = x + 36$.

8. A tangent to the parabola $y^2 = 16x$ makes an angle 60° with x–axis. Find its point of contact.

Solution: Given equation of parabola is,

$$y^2 = 16x \quad \text{... (i)}$$

Comparing equation (i) with $y^2 = 4ax$ then we get, $a = 4$.

Now, the equation of tangent to $y^2 = 16x$ is

$$y = mx + \frac{4}{m} \quad \text{... (ii)}$$

Given that the parabola makes an angle 60° with x–axis i.e. $y = 0$. So, $m = \tan 60^\circ = \sqrt{3}$.

Then, the equation (ii) becomes,

$$y = \sqrt{3}\,x + \frac{4}{\sqrt{3}} \Rightarrow \sqrt{3}y = \sqrt{3}x + 4 \quad \text{... (iii)}$$

This is the equation of required tangent to the given parabola satisfying the given conditions.

Now, from equation (i) and (ii)

$$\left(\frac{3x+4}{\sqrt{3}}\right)^2 = 16x$$

$$\Rightarrow 9x^2 + 24x + 16 = 48x$$

$$\Rightarrow 9x^2 - 24x + 16 = 0 \quad \Rightarrow (3x-4)^2 = 0$$

$$\Rightarrow 3x - 4 = 0 \quad \Rightarrow x = \frac{4}{3}.$$

And, $y^2 = 16\left(\frac{4}{3}\right) \Rightarrow y = \pm\frac{8}{\sqrt{3}}$

Thus, the tangent to the parabola $y^2 = 16x$ makes an angle 60° with x–axis, has its point of contact is $\left(\frac{4}{3}, \pm\frac{8}{\sqrt{3}}\right)$.

9. Find the equation of the tangents and normal at the ends of the latus rectum of $y^2 = 4ax$.

Solution: Given equation of parabola is,

$$y^2 = 4ax \quad \text{... (i)}$$

whose length of latus rectum = 4a

and the coordinates of its ends of latus rectum are (a, 2a) and (a, –2a).

Now the equation of tangent at (a, 2a) is,

$$y \cdot 2a = 2a(x + a) \quad \Rightarrow x - y + a = 0,$$

and equation of tangent at (a, –2a) is,

$$y(-2a) = 2a(x + a)$$

$$\Rightarrow -2ay = 2a(x + a) \Rightarrow -y = x + a \Rightarrow x + y + a = 0.$$

Again, equation of normal at (a, 2a) is

$$(y - 2a) = \frac{-2a}{2a}(x - a)$$

$$\Rightarrow y - 2a = -(x - a) \quad \Rightarrow x + y - 3a = 0.$$

and equation of normal at (a, –2a) is,

$$(y + 2a) = -\frac{-2a}{2a}(x - a)$$

$$\Rightarrow y + 2a = (x - a) \quad \Rightarrow x - y - 3a = 0.$$

Thus, the equation of the tangents at the ends of the latus rectum of $y^2 = 4ax$ at (a, 2a) is $x - y + a = 0$ and at (a, –2a) is $x + y + a = 0$ and the equation of the normal at (a, 2a) is $x + y - 3a = 0$ and at (a, –2a) is $x - y - 3a = 0$.

10. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution: Given equation of parabola is

$$y^2 = 8x \quad \text{... (i)}$$

Comparing it with $y^2 = 4ax$ we get $a = 2$.

Now the equation of tangent to (i) is,

$$y = m_1x + a/m_1$$

$$\Rightarrow y = m_1x + 2/m_1 \quad \text{... (ii)}$$

Given that the tangent (ii) makes an angle of 45° with the straight line,

$$y = 3x + 5 \quad \text{... (iii)}$$

Here, using the formula of angle between two lines,

$$\tan\theta = \frac{m - m_1}{1 + (m)(m_1)}$$

where m_1 be the slope of required line.

$$\Rightarrow \tan 45^\circ = \frac{3 - m_1}{1 + (3)(m_1)}$$

$$\Rightarrow 1 + 3m_1 = 3 - m_1 \quad \Rightarrow 4m_1 = 2 \quad \Rightarrow m_1 = \frac{1}{2}.$$

Thus, the equation of tangent is,

$$y = \frac{1}{2}x + \frac{2}{1/2}$$

$$\Rightarrow 2y - x = 8 \quad \Rightarrow x = 2y - 8 \quad \text{... (iv)}$$

And, for the point of contact, solving (i) and (iv),

$$y^2 = 4(2y - 8)$$

$$\Rightarrow y^2 - 16y + 64 = 0$$

$$\Rightarrow (y - 8)^2 = 0 \quad \Rightarrow y = 8.$$

Then, $x = 2 \times 8 - 8 = 8$

Hence, the point of contact is (8, 8).

11. Show that the straight line $7x + 6y - 13 = 0$ is a tangent to the parabola $y^2 - 7x - 8y + 14 = 0$. Find the point of contact.

Solution: Given equation of tangent is

$$7x + 6y - 13 = 0 \quad \text{... (i)}$$

and given equation of parabola is

$$y^2 - 7x - 8y + 14 = 0$$
$$\Rightarrow y^2 - (-6y + 13) - 8y + 14 = 0$$
$$\Rightarrow y^2 - 2y + 1 = 0$$
$$\Rightarrow (y-1)^2 = 0$$
$$\Rightarrow y = 1$$

Then (i) gives

$$7x + 6 - 13 = 0$$
$$\Rightarrow 7x = 7$$
$$\Rightarrow x = 1$$

This shows the given line (i) and curve (ii) meet at a single point (1, 1). This means line (i) is tangent to curve (ii) and their point of contact is (1, 1).

12. Find the value of λ, when the line $x - y + 1 = 0$ is a tangent to the parabola $y^2 = \lambda x$.

Solution: Here, given equation of tangent is,

$$x - y + 1 = 0 \Rightarrow y = x + 1 \quad \dots (i)$$

and equation of parabola is, $y^2 = \lambda x$ ... (ii)

Comparing the equation (i) with $y = mx + c$, we get

$$m = 1 \text{ and } c = 1.$$

and comparing the equation (ii) with $y^2 = 4ax$, then we get

$$a = \frac{\lambda}{4}.$$

Since the line (i) is tangent to the parabola (ii). So, we must have,

$$c = \frac{a}{m} \Rightarrow 1 = \frac{\lambda}{4} \Rightarrow \lambda = 4.$$

Thus, for $\lambda = 4$, the line $x - y + 1 = 0$ is a tangent to the parabola $y^2 = \lambda x$.

13. Find the equation of the tangents to the parabola $y^2 = 7x$, which is perpendicular to the line $4x + y = 0$. Also, find the point of contact.

[2014 Spring]

Solution: Here, given equation of parabola is

$$y^2 = 7x \quad \dots (i)$$

Comparing (i) with $y^2 = 4ax$ we get

$$a = \frac{7}{4}$$

and given that the equation of straight line is

$$4x + y = 0 \Rightarrow y = -4x \quad \dots (ii)$$

Comparing (ii) with $y = m_1x + c$ we get

$$m_1 = -4, c = 0.$$

Since the equation of tangent to (i) is,

$$y = mx + \frac{a}{m}$$
$$\Rightarrow y = mx + \frac{7}{4m} \quad \dots (iii)$$

By hypothesis the line (ii) is perpendicular to the tangent (iii). So, the product of their slopes should equal to -1.

i.e. $m(-4) = -1 \Rightarrow m = ¼$.

Substituting this value in (iii) then it becomes,

$$\Rightarrow y = \frac{1}{4}x + 7.$$
$$\Rightarrow 4y = x + 28.$$
$$\Rightarrow x - 4y + 28 = 0 \quad \dots (iv)$$

This is the required equation of tangent to (i).

For the point of contact, solving equation (i) and (iii) we get,

$$y^2 = 7(4y - 28)$$
$$\Rightarrow y^2 - 28y + 196 = 0$$
$$\Rightarrow (y - 14)^2 = 0$$
$$\Rightarrow y = 14.$$

And $x = 4y - 28 = 4 \times 14 - 28 = 28.$

Hence, the point of contact is (28, 14).

14. Find the equation of tangents to the parabola $y^2 = 5x$ passing through the point (5, 13). Also, find the point of contact of the tangents.

Solution: Given parabola is

$$y^2 = 5x \quad \dots (i)$$

Comparing the equation (i) with $y^2 = 4ax$, we get

$$a = 5/4.$$

And, the equation of tangent to (i) is,

$$y = mx + \frac{a}{m}$$
$$\Rightarrow y = mx + \frac{5}{4m} \quad \dots (ii)$$

Now the equation of a line that passes through the point (5, 13) be,

$$y - 13 = m(x - 5)$$
$$\Rightarrow y = mx + 13 - 5m \quad \dots (iii)$$

Since line (ii) and (iii) both are tangent line to (i) so they must be identical. That is,

$$13 - 5m = \frac{5}{4m}$$
$$\Rightarrow 20m^2 - 52m + 5 = 0$$
$$\Rightarrow 10m(2m - 5) - 1(2m - 5) = 0$$
$$\Rightarrow (2m - 5)(10m - 1) = 0$$

$\Rightarrow m = \frac{5}{2}, \frac{1}{10}$.

Substituting value in equation (ii) then

$y - 13 = \frac{5}{2}(x - 5)$ and $y - 13 = \frac{1}{10}(x - 5)$

$\Rightarrow 5x - 2y + 1 = 0$ $\Rightarrow x - 10y + 125 = 0$

$\Rightarrow x = \frac{2y - 1}{5}$ ... (iv) $\Rightarrow x = 10y - 125$ ... (v)

For the point of contact, solving equation (i), (iv) and (i), (v) then,

$y^2 = 5\left(\frac{2y-1}{5}\right)$ and $y^2 = 5(10y - 125)$

$\Rightarrow y^2 = 2y - 1$ $\Rightarrow y^2 - 50y + 625 = 0$

$\Rightarrow y^2 - 2y + 1 = 0$ $\Rightarrow (y - 25)^2 = 0$

$\Rightarrow (y - 1)^2 = 0$ $\Rightarrow y = 25$

$\Rightarrow y = 1$.

Putting value y = 1 in equation (iv) and y = 25 in (v) then,

$x = \frac{2y-1}{5} = \frac{2-1}{5} = \frac{1}{5}$ and $x = 10 \times 25 - 125 = 125$.

Thus the point of contact are (1/5, 1) and (125, 25) for tangent to $5x - 2y + 1 = 0$ and $x - 10y + 125 = 0$, respectively.

15. Show that the normal to the parabola $y^2 = 8x$ at (2, 4) meets the parabola again in (18, –12).

Solution: Given parabola is

$$y^2 = 8x \quad \text{... (i)}$$

Comparing equation (i) with $y^2 = 4ax$, we get

$$a = 2.$$

Now, equation of normal to parabola at (2, 4) is

$$(y - y_1) = -\frac{y_1}{2a}(x - x_1)$$

$$\Rightarrow (y - 4) = \frac{-4}{2 \times 2}(x - 2)$$

$$\Rightarrow y - 4 = -x + 2 \quad \Rightarrow x + y = 6 \quad \text{... (ii)}$$

If the normal again meet the parabola then the point (18, –12) should satisfy the equation (i) and (ii), we get

$y^2 = 8x$ and $x + y = 6$

$\Rightarrow (12)^2 = 8 \times 18$ $\Rightarrow 18 - 12 = 6$

$\Rightarrow 144 = 144$ (true) $\Rightarrow 6 = 6$ (true).

Hence, the normal again meets the parabola at (18, –12).

16. Show that the line $lx + my + n = 0$ touches the parabola $y^2 = 4a(x - b)$ if $am^2 = bl^2 + nl$.

Solution: Given parabola is,

$$y^2 = 4a(x - 6) \quad \ldots (i)$$

and given line is,

$$lx + my + n = 0$$

$$\Rightarrow \quad y = -\frac{(lx + n)}{m} \quad \ldots (ii)$$

Solving equation (i) and (ii) by eliminating 'y',

$$\left[-\frac{(lx + n)}{m}\right]^2 = 4a(x - 6)$$

$$\Rightarrow \quad l^2x^2 + 2nlx + n^2 = 4am^2x - 4bm^2a$$

$$\Rightarrow \quad l^2x^2 + (2nl - 4am^2)x + (4abm^2 + n^2) = 0 \quad \ldots (iii)$$

This is a quadratic in x.

Since equation (ii) is tangent to (i), so discriminate of (3) must be equal to zero. Therefore,

$$(2nl - 4am^2)^2 - 4l^2\,(4abm^2 + n^2) = 0$$

$$\Rightarrow \quad 4n^2l^2 - 16alm^2n + 16a^2m^4 - 16abl^2m^2 - 4l^2n^2 = 0$$

$$\Rightarrow \quad a^2m^4 = abl^2m^2 + alm^2n. \qquad [\text{Since } 16 \neq 0]$$

$$\Rightarrow \quad a^2m^4 = am^2\,(bl^2 + ln)$$

$$\Rightarrow \quad am^2 = bl^2 + ln$$

Hence, the line (ii) touches the parabola (i) only if $am^2 = bl^2 + nl$.

Exercise 9.4

1. **Find the equation of ellipse which has centre C, focus F and semi-major axis a or b and calculate eccentricity.**

(i) C(0, 0), F(0, 2), b = 4

Solution: Given that, C(0, 0) and F(0, 2). Here the y value in centre and focus, is vary. So, C(h, k) = C(0, 0) and F(h, k + c) = F(0, 2) then,.

$$h = 0,\ k = 0 \text{ and } c = 2 \text{ along y–axis.}$$

Therefore, the major axis is parallel to y–axis. So, b > a

Since, $c^2 = b^2 - a^2$ and given that b = 4.

Therefore, $a^2 = b^2 - c^2 = 16 - 4 = 12 \quad \Rightarrow a = \sqrt{12}$

Now the equation of ellipse with centre (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{x^2}{12} + \frac{y^2}{16} = 1.$$

and its eccentricity is, $\quad e = \frac{c}{b} = \frac{2}{4} = \frac{1}{2}$.

(ii) C(0, 2), b = 3, F(0, 0)

Solution: Given that, C(0, 2) and F(0, 0). Here the y value in centre and focus, is vary. So, C(h, k) = C(0, 2) and F(h, k + c) = F(0, 0) then,

$$h = 0,\ k = 2 \text{ and } c = -2 \quad \text{along y–axis.}$$

Therefore, the major axis is parallel to y–axis. So, b > a.

Since, $c^2 = b^2 - a^2$ and given that b = 4.

Therefore, $a^2 = b^2 - c^2 = 9 - 4 = 5 \quad \Rightarrow a = \sqrt{5}$.

Now the equation of ellipse with centre (0, 0) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

$$\Rightarrow \frac{(x-0)^2}{5} + \frac{(y-2)^2}{1} = 1$$

$$\Rightarrow \frac{x^2}{5} + \frac{(y-2)^2}{9} = 1.$$

and its eccentricity is, $\quad e = \frac{c}{b} = \frac{2}{3}$.

(iii) C(2, 2), F(–1, 2), $a = \sqrt{10}$.

Solution: Given that, C(2, 2) and F(–1, 2). Here the x value in centre and focus, is vary. So, C(h, k) = C(0, 2) and F(h, k + c) = F(0, 0) then,

Solution: Here, C(h, k) = C(2, 2) and F(h + c, k) = F(−1, 2). Then, from centre and focus, we have

$$h = 2, k = 2 \text{ and } c = 3 \text{ along x–axis.}$$

Therefore, the major axis is parallel to x–axis. So, $a > b$.

Since, $c^2 = a^2 - b^2$ and given that $a = \sqrt{10}$.

Therefore, $b^2 = a^2 - c^2 = 10 - 9 = 1 \quad \Rightarrow b = 1$.

Now the equation of ellipse with centre (2, 2) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

$$\Rightarrow \quad \frac{(x-2)^2}{10} + \frac{(y-2)^2}{1} = 1$$

and its eccentricity is, $e = \frac{c}{a} = \frac{3}{\sqrt{10}}$.

2. **The end points of the major and minor axes of ellipse are (1, 1), (3, 4), (1, 7) and (−1, 4). Find equation of ellipse and find its focus.**

Solution: Given that the end points of major and minor axes of ellipse are (1, 1), (3, 4), (1, 7) and (−1, 4). Therefore,

$$2b = |7 - 1| = 6 \quad \Rightarrow \quad b = 3 \quad \text{and} \quad 2a = |-1 - 3| = 4 \Rightarrow a = 2.$$

Here, $b > a$. Then, $c^2 = \sqrt{b^2 - a^2} = \sqrt{9-4} = \sqrt{5}$.

Since the center of the ellipse is the mid-point of the axis. So,

$$h = \frac{1+1}{2} = 1, \quad k = \frac{1+7}{2} = 4$$

That is $\quad C(h, k) = C(1, 4)$.

Therefore, equation of ellipse is,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \Rightarrow \quad \frac{(x-1)^2}{4} + \frac{(y-4)^2}{9} = 1.$$

Here, $b = 3 > 2 = a$. So, the foci lie on the line parallel to y–axis. Therefore,

$$F(h, k \pm c) = F(1, 4 \pm \sqrt{5}).$$

3. **Find center, vertices and foci of the ellipse**

(i) $\mathbf{x^2 + 5y^2 + 4x = 1}$

Solution: Given equation is,

$$x^2 + 5y^2 + 4x = 1.$$

$$\Rightarrow \quad (x+2)^2 + 5y^2 = 5.$$

$$\Rightarrow \quad \frac{(x+2)^2}{5} + \frac{y^2}{1} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$$h = -2, \quad k = 0, \quad a^2 = 5 \text{ and } b^2 = 1.$$

Here, $a > b$. So, $c^2 = a^2 - b^2 = 5 - 1 \Rightarrow c^2 = 4$.

Therefore, centre (h, k) = C(−2, 0).

Since a > b. And, the major axis is parallel to x–axis. Therefore, the foci lie on the line that is parallel to x–axis. So,

Foci are at $F(h \pm c, k) = F(-2 \pm 2, 0)$.

Vertices are at $V(h \pm a, k) = V(-2 \pm \sqrt{5}, 0)$.

(ii) $x^2 + 2y^2 - x - 4y + 1 = 0$

Solution: Given equation is,

$$x^2 + 2y^2 - x - 4y + 1 = 0.$$

$$\Rightarrow \frac{(x - 1/2)^2}{5/4} + \frac{(y - 1)^2}{5/2} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$h = 1/2, k = 1, \ a^2 = 5/4$ and $b^2 = 5/2$.

Here, b > a. So, $c = \sqrt{b^2 - a^2} = \sqrt{5/4}$.

Therefore, centre $(h, k) = C(-1, -4)$.

Since b > a. And, the major axis is parallel to y–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

Foci are at $F(h, k \pm c) = F(1/2, 1 \pm \sqrt{5/4})$.

Vertices are at $V(h, k \pm b) = V(1/2, 1 \pm \sqrt{5/2})$.

(iii) $25(x - 3)^2 + 4(y - 1)^2 = 100$

Solution: The given equation becomes as,

$$25(x - 3)^2 + 4(y - 1)^2 = 100$$

$$\frac{(x-3)^2}{4} + \frac{(y-1)^2}{25} = 1$$

Comparing it with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$h = 3, \ k = 1, \ a^2 = 4 \Rightarrow a = 2$ and $b^2 = 25 \Rightarrow b = 5$.

Here, b > a, so $c = \sqrt{b^2 - a^2} = \sqrt{5^2 - 2^2} = \sqrt{21}$

Therefore, centre of the ellipse is $C(h, k) \Rightarrow C(3, 1)$

Since b > a. And, the major axis is parallel to y–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

foci of the ellipse are at $F(h, k \pm c) = F(3, 1 \pm \sqrt{21})$.

Vertices are $V(h, k \pm b) = V(3, 1 \pm 5)$.

(iv) $x^2 + 10x + 25y^2 = 0$ [2018 Spring Short] [2018 Fall Short][2012 Fall]

Solution: The given equation is,

$$x^2 + 10x + 25y^2 = 0$$

$$\Rightarrow x^2 + 2(5)x + (5)^2 + (5y)^2 - 25 = 0$$

$$\Rightarrow (x + 5)^2 + (5y)^2 = 25$$

$$\Rightarrow \frac{(x+5)^2}{25} + \frac{y^2}{1} = 1$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$h = -5, \ k = 0, \ a^2 = 25 \Rightarrow a = 5$ and $b^2 = 1 \Rightarrow b = 1$.

Here, a > b. So, $c^2 = a^2 - b^2 = 25 - 1 = 24. \Rightarrow c = 2\sqrt{6}$

Therefore, Center of the ellipse is $C(h, k) \Rightarrow C(-5, 0)$

Since a > b. And, the major axis is parallel to x–axis. Therefore, the foci lie on the line that is parallel to x–axis. So,

foci of the ellipse are at $F(h \pm c, k) = (-5 \pm 2\sqrt{6}, 0)$.

Vertices of the ellipse is at $V(h \pm a, k) = F(-5 \pm 5, 0)$.

(v) $x^2 + 9y^2 - 4x + 18y + 4 = 0$

Solution: The given equation is,

$$x^2 + 9y^2 - 4x + 18y + 4 = 0o$$

$$\Rightarrow (x)^2 - 2.2\,x + (2)^2 + 9\{y^2 + 2.1y + (1)^2\} + 4 - 4 - 1 = 0$$

$$\Rightarrow (x - 2)^2 + 9(y + 1)^2 = 1$$

$$\Rightarrow \frac{(x-2)^2}{1} + \frac{(y+1)^2}{1/9} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then we get,

$h = 2, \ k = -1, \ a^2 = 1 \Rightarrow a = 1$ and $b^2 = \frac{1}{9} \Rightarrow b = \frac{1}{3}$.

So, a > b, then $c = \sqrt{a^2 - b^2} = \sqrt{\frac{8}{9}} = 2\frac{\sqrt{2}}{3}$.

Therefore, Center of the ellipse is $C(h, k) = C(2, -1)$

Since a > b. And, the major axis is parallel to x–axis. Therefore, the foci lie on the line that is parallel to x–axis. So,

foci of the ellipse are $F(h, k \pm c) = F(2, \pm 2\frac{\sqrt{2}}{3}, -1)$

Vertices is at $V(h, k \pm b) = V(2, \pm\frac{1}{3}, -1)$.

(vi) $4x^2 + y^2 - 16x + 4y + 16 = 0$

Solution: The given equation is,

$$4x^2 + y^2 - 16x + 4y + 16 = 0$$

$$\Rightarrow 4(x^2 - 4x + 4) + (y^2 + 4y + 4) - y = 0$$

$$\Rightarrow 4(x - 1)^2 + (y + 2)^2 = 4$$

$$\Rightarrow \frac{(x-2)^2}{1} + \frac{(y+2)^2}{4} = 1$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then,

$h = 2, \ k = -2, \ a^2 = 1 \Rightarrow a = 1$ and $b^2 = 4 \Rightarrow b = 2$.

Here, b > a, so $c^2 = b^2 - a^2 = 4 - 1 \Rightarrow c = \sqrt{3}$.

Therefore, Center of the ellipse, C(h, k) = C(2, –2)

Since b > a. And, the major axis is parallel to y–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

foci of the ellipse are $F(h, k \pm c) = F(2, -2 \pm \sqrt{5})$

Vertices of the ellipse $V(h, k \pm b) = V(2, -2 \pm 2)$.

(vii) $9x^2 + 16y^2 + 18x - 96y + 4 = 0$ **[2017 Spring]**

Solution: The given equation is,

$$9x^2 + 16y^2 + 18x - 96y + 4 = 0$$

$$\Rightarrow 9(x^2 + 8x + 1) + 16(y^2 - 2.y.3 + 3^2) = 144$$

$$\Rightarrow 9(x+1)^2 + 16(y-3)^2 = 144$$

$$\Rightarrow \frac{(x+1)^2}{16} + \frac{(y-3)^2}{9} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then,

$h = -1, k = 3, a^2 = 16$ and $b^2 = 9$.

Here, a > b. So, $c^2 = a^2 - b^2 = 16 - 9 = 7 \Rightarrow c = \sqrt{7}$.

Hence, centre, C(h, k) = C(–1, 3)

Since a > b. And, the major axis is parallel to x–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

foci of the ellipse are at $F(h \pm c, k) = F(-1 \pm \sqrt{7}, 3)$

Vertices of the ellipse are at $V(h \pm a, k) = V(-1 \pm 4, 3)$.

(viii) $16(x-2)^2 + 9(y+3)^2 = 144$ **[2016 Fall]**

Solution: The given equation is,

$$16(x-2)^2 + 9(y+3)^2 = 144$$

$$\Rightarrow \frac{(x-2)^2}{9} + \frac{(y+3)^2}{16} = 1.$$

Comparing the equation with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Then, we get

$h = 2,\ k = -3,\ a^2 = 9 \Rightarrow a = 3$ and $b^2 = 16 \Rightarrow b = 4$.

Here, b > a. So $c = \sqrt{b^2 - a^2} = \sqrt{4^2 - 3^2} = \sqrt{7}$.

Therefore, center of the ellipse, C(h, h) = C(2, –3).

Since b > a. And, the major axis is parallel to y–axis. Therefore, the foci lie on the line that is parallel to y–axis. So,

foci of the ellipse are at $F(h, k \pm c) = F(2, -3 \pm \sqrt{7})$.

Vertices of the ellipse are at $V(h, k \pm b) = V(2, -3 \pm 4)$.

4. **Find the equation of an ellipse whose axes lies along the co–ordinate axes and which posses through (4, 3) and (–1, 4).**

Solution: Given that the ellipse has axes along the coordinate axes. So, the centre of ellipse should be at origin.

Therefore, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots (1)$$

Since the given ellipse (1) passes through the point (4, 3). So,

$$\frac{16}{a^2} + \frac{9}{b^2} = 1$$

$$\Rightarrow \frac{1}{a^2} = \frac{1}{16}\left(1 - \frac{9}{b^2}\right) \quad \ldots (2)$$

Again the ellipse (1) passes through the point (–1, 4). So,

$$\frac{1}{a^2} + \frac{16}{b^2} = 1$$

$$\Rightarrow \frac{1}{16} - \frac{9}{16b^2} + \frac{16}{b^2} = 1 \quad \text{[using (2)]}$$

$$\Rightarrow \frac{1}{b^2}\left(\frac{-9}{16} + 16\right) = 1 - \frac{1}{16}$$

$$\Rightarrow b^2 = \frac{247}{15}$$

Therefore, (2) gives,

$$\frac{1}{a^2} = \frac{1}{16}\left(1 - \frac{9 \times 15}{247}\right) = \frac{112}{16 \times 247} = \frac{7}{247}$$

$$\Rightarrow a^2 = \frac{247}{7}.$$

Hence (1) becomes,

$$\frac{7x^2}{247} + \frac{15y^2}{247} = 1.$$

$$\Rightarrow 7x^2 + 15y^2 = 247.$$

5. **Find the eccentricity and the co–ordinate of the foci of the ellipse $2x^2 + 3y^2 - 1 = 0$.**

Solution: Given ellipse is

$$2x^2 + 3y^2 = 1$$

$$\Rightarrow \frac{x^2}{1/2} + \frac{y^2}{1/3} = 1 \quad \ldots (i)$$

Comparing the equation (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, we get,

$h = 0,\ k = 0,\ a^2 = \frac{1}{2}$ and $b^2 = \frac{1}{3}$.

Here, a > b. So, $c = \sqrt{a^2 - b^2} = \sqrt{\frac{1}{2} - \frac{1}{3}} = \frac{1}{\sqrt{6}}$.

Therefore, eccentricity (e) $= \frac{c}{a} = \frac{\frac{1}{\sqrt{6}}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$.

Since, $a > b$, so the foci lie on the line parallel to x–axis. Therefore,

foci of the ellipse is $F(h \pm c, k) = F\left(0 \pm \frac{1}{\sqrt{6}}, 0\right) = F\left(\pm\frac{1}{\sqrt{6}}, 0\right)$.

6. **Show that $2x^2 + y^2 = 3x$ represents an ellipse; find its eccentricity and co–ordinates of foci.**

Solution: Given equation is,

$$2x^2 + y^2 = 3x$$

$$\Rightarrow 2\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) - \frac{9}{8} + y^2 = 0$$

$$\Rightarrow \frac{(x - 3/4)^2}{1/2} + y^2 = \frac{9}{8}$$

$$\Rightarrow \frac{(x - 3/4)^2}{9/16} + \frac{y^2}{9/8} = 1$$

$$h = \frac{3}{4}, k = 0, a^2 = \frac{9}{16}, b^2 = \frac{9}{8}$$

Here,

$b > a$. So, $c^2 = b^2 - a^2 = \frac{9}{8} - \frac{9}{16} = \frac{9}{16}$

$$f(h, k \pm c) = \left(\frac{3}{4}, \frac{3}{4}\right)$$

$$e = \frac{c}{b} = \frac{9/16}{9/8} = \frac{1}{2}$$

7. **Find the equation of the ellipse whose focus, directrix and eccentricity are given as:**

a. **$F(0, 3)$, $x + 7 = 0$ and $e = \frac{1}{3}$**

b. **$F(-1, 1)$, $x - y + 3 = 0$ and $e = \frac{1}{2}$**

c. **$F(2, 5)$, $x + y = 1$ and $e = \frac{2}{3}$**

Solution:

a. Given that the ellipse has focus $F(0, 3)$, equation of directrix is $x + 7 = 0$ and eccentricity is $e = \frac{1}{3}$. Let $P(x, y)$ be any point on the ellipse. Then by definition of eccentricity,

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

$$\Rightarrow \frac{1}{3} = \frac{\sqrt{(x-0)^2 + (y-3)^2}}{\pm\frac{x+7}{1}}$$

Squaring on both sides we get,

$$\Rightarrow \frac{1}{9}(x + 7)^2 = x^2 + (y - 3)^2$$

$$\Rightarrow x^2 + 14x + 49 = 9x^2 + 9(y^2 - 6y + 9)$$

$$\Rightarrow 8x^2 + 9y^2 - 14x - 54y + 32 = 0.$$

This is the equation of the required ellipse.

b. Given that the ellipse has focus $F(-1, 1)$, equation of directrix is $x - y + 3 = 0$ and eccentricity is $e = \frac{1}{2}$. Let $P(x, y)$ be any point on the ellipse. Then by definition of eccentricity,

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

$$\Rightarrow \frac{1}{2} = \frac{\sqrt{(x+1)^2 + (y-1)^2}}{\pm\frac{x-y+3}{\sqrt{2}}}$$

Squaring both sides, we get

$$\frac{1}{4} \times \frac{1}{2}(x - y + 3)^2 = (x + 1)^2 + (y - 1)^2$$

$$\Rightarrow \frac{1}{8}(x^2 + y^2 + 6x - 6y - 2xy + 9) = (x^2 + 2x + 1) + (y^2 - 2y + 1)$$

$$\Rightarrow x^2 + y^2 + 6x - 6y - 2xy + 9 = 8x^2 + 16x + 8y^2 - 16y + 16.$$

$$\Rightarrow 7x^2 + 7y^2 + 10x - 10y + 2xy + 7 = 0.$$

This is the equation of the required ellipse.

c. Given that the ellipse has focus $F(2, 5)$, equation of directrix is $x + y - 1 = 0$ and eccentricity is $e = \frac{2}{3}$. Let $P(x, y)$ be any point on the ellipse. Then by definition of eccentricity,

$$e = \frac{\text{Distance between P and F}}{\text{Perpendicular distance from P to directrix}}$$

$$\Rightarrow \frac{2}{3} = \frac{\sqrt{(x-2)^2 + (y-5)^2}}{\pm\frac{x+y-1}{\sqrt{2}}}$$

Squaring both sides, we get

$$\frac{4}{9} \times \frac{1}{2}(x + y - 1)^2 = (x - 2)^2 + (y - 5)^2$$

$$\Rightarrow 2(x^2 + y^2 + 1 + 2xy - 2x - 2y) = 9[(x^2 - 4x + 4) + (y^2 - 10y + 25)]$$

$$\Rightarrow 2x^2 + 2y^2 + 2 + 4xy - 4x - 4y = 9x^2 - 36x + 225 + 36 + 9y^2 - 90y.$$

$$\Rightarrow 7x^2 + 7y^2 - 4xy - 32x - 86y + 259 = 0$$

This is the equation of required ellipse.

8. **Find the equation of ellipse whose foci are at $(-2, 4)$ and $(4, 4)$; length of major axis is 10. Also, find the eccentricity.**

Solution: Here, given two foci of an ellipse are (–2, 4) and (4, 4).

Since, the centre is the mid–point of foci, so

$$\text{Centre } (h, k) = \left(\frac{-2+4}{2}, \frac{4+4}{2}\right) = (1, 4).$$

This implies, $h = 1$ and $k = 4$.

Since the foci have same y value therefore, the foci lie on the line parallel to x–axis. That is the major axis of the ellipse is the axis parallel to x–axis.

Therefore,

$$F(h + c, k) = F(4, 4).$$

That means, $h + c = 4 \Rightarrow c = 3$ $[\because h = 1.]$

Clearly the major axis of the ellipse is the axis. So,

$$\text{major axis} = 2a.$$

Given that length of major axis = 10.

Therefore,

$$2a = 10 \Rightarrow a = 5.$$

Then, $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$

Now, the equation of ellipse is,

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \frac{(x-1)^2}{25} + \frac{(y-4)^2}{16} = 1$$

$$\Rightarrow 16x^2 - 32x + 16 + 25y^2 - 200y + 400 = 400$$

$$\Rightarrow 16x^2 + 25y^2 - 32x - 200y + 16 = 0$$

This is the equation of the required ellipse.

And, eccentricity $(e) = \frac{c}{a} = \frac{3}{5}$.

9. Find the equation of ellipse referred to its axes as the axes of co-ordinates and foci along x–axis with latus rectum of length 4 and distance between foci is $4\sqrt{2}$.

Solution: Since the ellipse whose axes are co–ordinates axis has centre at origin. So, the equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots (i)$$

Also, given that the distance between foci is $4\sqrt{2}$. Therefore,

$$2c = 4\sqrt{2} \Rightarrow c = 2\sqrt{2}.$$

Given that the foci lie along x–axis, so major axis is x–axis. Therefore, $a > b$.

Again given that the length of latus rectum is 4.

i.e. $\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a.$

Since $a > b$. So, we have,

$$c^2 = a^2 - b^2$$

$$\Rightarrow (2\sqrt{2})^2 = a^2 - (2a)$$

$$\Rightarrow a^2 - 2a - 8 = 0$$

$$\Rightarrow (a + 2)(a - 4) = 0$$

Since a can not measure in negative. So, $a - 4 = 0 \Rightarrow a = 4$.

Then, $b^2 = 2a = 8$.

Hence (i) becomes,

$$\frac{x^2}{16} + \frac{y^2}{8} = 1.$$

This is the equation of required ellipse.

10. **Find the equation of the ellipse having origin at centre, major axis as x–axis, latus rectum is 3 and eccentricity is $\frac{1}{\sqrt{2}}$.**

Solution: The equation of ellipse having origin at centre is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \dots (i)$$

Given that the major axis as x–axis, therefore, $a > b$.

And, length of latus rectum is 3. So, $\frac{2b^2}{a} = 3. \Rightarrow b^2 = \frac{3a}{2}$.

Also, being $a > b$, the eccentricity is,

$$e = \frac{c}{a} = \frac{1}{\sqrt{2}}. \Rightarrow c = \frac{a}{\sqrt{2}}.$$

Here, the major axis as x–axis, so we have $c^2 = a^2 - b^2$.

$$b^2 = a^2 - c^2$$

$$\Rightarrow \frac{3a}{2} = a^2 - \frac{a^2}{2} \Rightarrow \frac{3a}{2} = \frac{a^2}{2} \Rightarrow a = 3.$$

Therefore, $b^2 = \frac{3a}{2} = \frac{3 \times 3}{2} = \frac{9}{2}$.

Hence (i) becomes,

$$\frac{x^2}{9} + \frac{2y^2}{9} = 1.$$

This is the equation of the required ellipse.

Exercise 9.5

1. **Find the equation of tangent and normal at the point (4, 3) on the ellipse $3x^2 + 4y^2 = 84$.**

Solution: Given, ellipse is

$$3x^2 + 4y^2 = 84 \quad \ldots \text{(i)}$$

Differentiating (i) w. r. t. x, we get

$$6x + 8y.\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-6x}{8y} = -\frac{3x}{4y}.$$

At (4, 3)

$$\frac{dy}{dx} = m = -\frac{3 \times 4}{4 \times 3} = -1.$$

Now, equation of tangent to (i) that passes through point (4, 3) is

$$y - 3 = -1(x - 4)$$

$$\left[\because \text{eq}^n \text{ of normal at } (x_1, y_1) \text{ is } y - y_1 = \frac{-1}{m}(x - x_1)\right]$$

$$\Rightarrow x + y = 7.$$

And equation of normal to (i) at (4, 3) is,

$$y - 3 = -(-1)(x - 4). \ [\because \text{eq}^n \text{ of tangent at } (x_1, y_1) \text{ is } y - y_1 = m(x - x_1)]$$

$$\Rightarrow y - 3 = x - 4$$

$$\Rightarrow x - y = 1.$$

2. **Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the point of contact.** **[2006, Fall] [2008, Fall]**

Solution: See theory part of ellipse before Ex. 9.4

3. **Show that the line $y = x \pm \frac{\sqrt{5}}{\sqrt{6}}$ touches the ellipse $2x^2 + 3y^2 = 1$. Find the point of contact.**

Solution: Given line is, $y = x \pm \frac{\sqrt{5}}{\sqrt{6}}$ ... (i)

Comparing the equation (i) with $lx + my + n = 0$ we get,

$$l = 1, m = -1, n = \pm\frac{\sqrt{5}}{\sqrt{6}}$$

And, the equation of ellipse is, $2x^2 + 3y^2 = 1$

$$\Rightarrow \frac{x^2}{1/2} + \frac{y^2}{1/3} = 1 \quad \text{... (ii)}$$

Comparing eq^n. (ii) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$a^2 = \frac{1}{2} \text{ and } b^2 = \frac{1}{3}$$

For line (i) to be tangent on (ii) is

$$a^2l^2 + b^2m^2 = n^2$$

$$\Rightarrow \frac{1}{2} \times (1)^2 + \frac{1}{3} \times (-1)^2 = \frac{5}{6}$$

$$\Rightarrow \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \quad \Rightarrow \frac{5}{6} = \frac{5}{6} \quad \text{This is true.}$$

So, (i) is tangent to (ii).

Also, for point of contact,

$$x = -\frac{a^2 l}{n} = \frac{-\frac{1}{2} \cdot 1}{\pm\frac{\sqrt{5}}{\sqrt{6}}} = \pm\frac{\sqrt{6}}{2\sqrt{5}} \pm \sqrt{\frac{6}{20}} = \pm\sqrt{\frac{3}{10}}$$

$$y = -\frac{b^2 m}{n} = \frac{-\frac{1}{3} \cdot (-1)}{\pm\frac{\sqrt{5}}{\sqrt{6}}} = \pm\sqrt{\frac{6}{45}} = \pm\sqrt{\frac{2}{15}}$$

Hence, point of contact is $\left(\pm\sqrt{\frac{3}{10}}, \pm\sqrt{\frac{2}{15}}\right)$.

4. **Find the equation of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the line $y = 3x + 7$.**

Solution: Given ellipse is

$$4x^2 + 3y^2 = 5 \quad \text{... (i)}$$

Since the tangent line to (i) is parallel to $y = 3x + 7$. So, the equation of tangent to (i) is,

$$3x - y + k = 0 \quad \text{... (ii)}$$

where k is a scalar.

Since (ii) is tangent on (i). So,

$$4x^2 + 3(3x + k)^2 = 5$$

$$\Rightarrow 4x^2 + 27x^2 + 18xk + 3k^2 = 5$$

$$\Rightarrow 31x^2 + 18k + 3k^2 - 5 = 0$$

which is quadratic in x and its discriminate term is zero, being (ii) is tangent to (i). That is,

$$(18k)^2 - 4(31)(3k^2 - 5) = 0$$

$$\Rightarrow 324k^2 - 372k^2 + 620 = 0$$

$$\Rightarrow 48k^2 = 620$$

$$\Rightarrow k^2 = \frac{620}{48} \quad \Rightarrow k = \pm\sqrt{\frac{155}{12}}$$

Therefore, the equation of tangents are

$$3x - y \pm \sqrt{\frac{155}{12}} = 0.$$

5. **Find the condition for the line $y = mx + c$ is tangent on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.** **[2012 Fall][2011 Spring][2009, Fall][2002] [2005, Fall]**

Solution: See the theory part, before 9.4.

6. **Show that $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$ is normal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).**

Solution: Given ellipse is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow b^2x^2+a^2y^2=a^2b^2 \qquad \ldots (i)$$

Differentiating (i) w.r.t. x, we get

$$\Rightarrow 2b^2x+2a^2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

So, slope of tangent on (i) at (x_1, y_1) is,

$$m=\frac{dy_1}{dx_1}=-\frac{b^2x_1}{a^2y_1}$$

Then, the slope of normal on (i) at (x_1, y_1) is $m=\frac{a^2y_1}{b^2x_1}$

Hence, the equation of normal to (i) at (x_1, y_1) is,

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

$$\Rightarrow \left(\frac{b^2}{y_1}\right)y-b^2=\left(\frac{a^2}{x_1}\right)x-a^2$$

$$\Rightarrow \frac{xa^2}{x_1}-\frac{yb^2}{y_1}=a^2-b^2.$$

This completes the requirement.

7. **Show that the line $3x + 4y + \sqrt{7} = 0$ touches the ellipse $3x^2 + 4y^2 = 1$. Also find the point of contact.**

Solution: Given line is

$$3x+4y+\sqrt{7}=0$$

$$\Rightarrow 4y=-3x-\sqrt{7} \Rightarrow y=-\frac{3x}{4}-\frac{\sqrt{7}}{4} \qquad \ldots (i)$$

Comparing this equation with $y = mx + c$, we get

$$m=-\frac{3}{4} \quad \text{and} \quad c=-\frac{\sqrt{7}}{4}$$

Again, given equation of ellipse is

$$3x^2+4y^2=1$$

$$\Rightarrow \frac{x^2}{1/3}+\frac{y^2}{1/4}=1 \qquad \ldots (ii)$$

Comparing the equation (ii) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$a^2=\frac{1}{3}, \quad \text{and } b^2=\frac{1}{4}$$

We have the condition to touch the ellipse by the line is,

$$a^2m^2+b^2=c^2$$

$$\Rightarrow \frac{1}{3}\left(-\frac{3}{4}\right)^2+\frac{1}{4}=\left(-\frac{\sqrt{7}}{4}\right)^2$$

$$\Rightarrow \frac{1}{3}\times\frac{9}{16}+\frac{1}{4}=\frac{7}{16}$$

$$\Rightarrow \frac{3+4}{16}=\frac{7}{16} \Rightarrow \frac{7}{16}=\frac{7}{16}.$$ This is true.

This shows that (i) is tangent to (ii).

And, the point of contact is,

$$(x_1, y_1)=\left(-\frac{ma^2}{c}, \frac{b^2}{c}\right)=\left(\frac{-\frac{1}{3}\cdot\frac{3}{4}}{-\frac{\sqrt{7}}{4}}, \frac{\frac{1}{4}}{-\frac{\sqrt{7}}{4}}\right)=\left(-\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right).$$

8. **Find the value of λ, when the straight line $y = x + \lambda$ touches the ellipse $2x^2 + 3y^2 = 6$.**

Solution: Given equation of ellipse is

$$2x^2+3y^2=6$$

$$\Rightarrow \frac{x^2}{3}+\frac{y^2}{2}=1 \qquad \ldots (i)$$

Comparing eq^n. (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$a^2=3 \quad \text{and} \quad b^2=2$$

Given straight line is $y = x + \lambda$ ... (ii)

Comparing with $y = mx + c$, we get

$$m=1, \quad c=\lambda$$

For the line to touch the ellipse is,

$$a^2m^2+b^2=c^2$$

$$\Rightarrow 3\,.\,1+2=\lambda^2 \Rightarrow 5=\lambda^2 \Rightarrow \lambda=\pm\sqrt{5}.$$

9. **Find the equation of tangents to the ellipse $x^2 + 3y^2 = 3$, which are parallel to the line $4x - y + 8 = 0$. Also, find the point of contact.**

Solution: Given ellipse is,

$$x^2+3y^2=3 \qquad \ldots (i)$$

And, the equation of line parallel to $4x - y + 8 = 0$ is

$$4x-y+k=0 \qquad \ldots (ii)$$

where k is a constant.

Since (ii) is tangent on (i). So,

$$x^2+3(4x+k)^2=3$$

$$\Rightarrow x^2+48x^2+24xk+3k^2=3$$

$$\Rightarrow 49x^2+24k\,.\,x+(3k^2-3)=0 \qquad \ldots (iii)$$

which is quadratic in x. Since (i) touches the line (ii). So, the discriminate term of (iii) should be equal to zero.

i.e. $(24k)^2=4\,.\,49\,(3k^2-3)$

$$\Rightarrow 576k^2-588k^2+588=0$$

$\Rightarrow\ 32k^2 = 588 \quad \Rightarrow k^2 = 49 \quad \Rightarrow k = \pm 7$

Therefore (ii) becomes

$4x - y \pm 7 = 0.$

For points of contact

a. When the line $4x - y + 7 = 0$ touches the ellipse $x^2 + 3y^2 = 3$ the point of contact is,

$$x = -\frac{B}{2A} = \frac{-24k}{2 \times 49} = -\frac{12}{7}$$

and, $$y = -4x + 7 = -\frac{48}{7} + 7 = \frac{1}{7}$$

Thus the point of contact be $\left(-\frac{12}{7}, \frac{1}{7}\right)$.

b. When the line $4x - y + 7 = 0$ touches the ellipse $x^2 + 3y^2 = 3$ the point of contact is,

$$x = -\frac{B}{2A} = -\frac{24k}{2 \times 49} = -\frac{24 \times (-7)}{2 \times 49} = \frac{12}{7}$$

and, $$y = 4x - 7 = \frac{48}{7} - 7 = -\frac{1}{7}$$

Thus the point of contact be $\left(\frac{12}{7}, -\frac{1}{7}\right)$.

Exercise 9.6

1. **Sketch each of the following hyperbolas:**

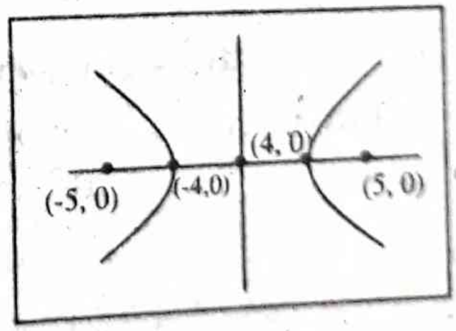
(i) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (ii) $\frac{y^2}{9} - \frac{x^2}{16} = 1$ (iii) $\frac{x^2}{9} - \frac{y^2}{16} = -1$

Solution:

(i) Given hyperbola is,

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \ldots(i)$$

Comparing the equation (i) with the standard equation of hyperbola, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ then we get,



$$a = 4, b = 3.$$

Thus, a > b. Then

$$c = \sqrt{a^2 + b^2} = \sqrt{25} = 5$$

Here, centre (h, k) = (0, 0)

foci are $(\pm c, 0) = (\pm 5, 0)$

vertices are $(\pm a, 0) = (\pm 4, 0)$.

(ii) Given hyperbola is

$$\frac{y^2}{9} - \frac{x^2}{16} = 1 \qquad \ldots (i)$$

Comparing the equation (i) with standard equation of hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ then we get,

$a = 3, b = 4.$

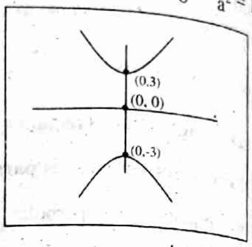
Thus, a < b. Then

$$c = \sqrt{a^2 + b^2} = \sqrt{25} = 5$$

Now, centre (h, k) = (0, 0)

foci are $(0, \pm c) = (0, \pm 5)$

vertices are $(0, \pm b) = (0, \pm 3)$.



(iii) Given hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{16} = -1 \Rightarrow \frac{y^2}{9} - \frac{x^2}{16} = 1$$

The question is reduced to (ii).

2. Find the center, vertices, foci, eccentricity of the following hyperbola and sketch curve

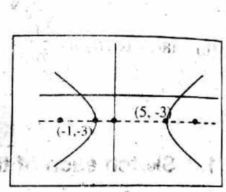
(i) $4(x-2)^2 - 9(y+3)^2 = 36$. [2017 Spring Short][2016 Fall Short]

Solution: Here, the given equation of hyperbola is

$$4(x-2)^2 - 9(y+5)^2 = 36$$

$$\Rightarrow \frac{(x-2)^2}{9} - \frac{(y+3)^2}{4} = 1 \qquad \ldots (i)$$

Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get,



$h = 2, k = 3, a = 3$ and $b = 2$.

Here, a > b. Then,

$$c = \sqrt{a^2 + b^2} = \sqrt{13}.$$

Now, centre (h, k) = (2, –3); foci $(h \pm c, k) = (2 \pm \sqrt{13}, -3)$;

vertices $(h \pm a, k) = (2 \pm 3, -3)$; eccentricity $(e) = \frac{c}{a} = \frac{\sqrt{13}}{3}$.

(ii) $5x^2 - 4y^2 + 20x + 8y = 4$

Solution: Given equation of hyperbola is

$$5x^2 - 4y^2 + 20x + 8y = 4$$

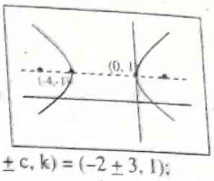
$$\Rightarrow 5(x^2 + 4x + 4) - 4(y^2 - 2y + 1) = 20 + 4 - 4$$

$$\Rightarrow 5(x+2)^2 - 4(y-1)^2 = 20$$

$$\Rightarrow \frac{(x+2)^2}{4} - \frac{(y-1)^2}{5} = 1 \qquad \ldots (i)$$

Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get,

$h = -2, k = 1, a = \sqrt{4} = 2, b = \sqrt{5}$.



Here, a > b. Then,

$$c = \sqrt{a^2 + b^2} = \sqrt{5+3} = 3.$$

Then, centre (h, k) = (–2, 1); foci $(h \pm c, k) = (-2 \pm 3, 1)$;

vertices $(h \pm a, k) = (-2 \pm 2, 1)$; eccentricity $(e) = \frac{c}{a} = \frac{3}{2}$.

Q. $4y^2 = x^2 - 4x$ [2013 Fall]

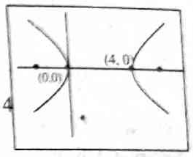
Solution: Given equation of hyperbola is

$$4y^2 = x^2 - 4x$$

$$\Rightarrow x^2 - 4x - 4y^2 = 0$$

$$\Rightarrow x^2 - 4x + 4 - 4y^2 = 4 \quad \Rightarrow (x-2)^2 - 4(y^2) = 4$$

$$\Rightarrow \frac{(x-2)^2}{4} - \frac{y^2}{1} = 1.$$



Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get,

$h = 2, k = 0, a = 2, b = 1.$

Here, a > b. Then,

$$c = \sqrt{a^2 + b^2} = \sqrt{4+1} = \sqrt{5}.$$

Now, centre (h, k) = (2, 0); foci $(h \pm c, k) = (2 \pm \sqrt{5}, 0)$;

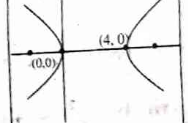
vertices $(h \pm a, k) = (2 \pm \sqrt{5}, 0)$; eccentricity, $e = \frac{c}{a} = \frac{\sqrt{5}}{2}$.

(iii) $x^2 - y^2 - 2x + 4y = 4$ [2015 Spring Short]

Solution: Given equation of conic is

$$x^2 - y^2 - 2x + 4y = 4$$

$$\Rightarrow (x-1)^2 - (y-2)^2 = 1$$



Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get,

$h = 1, k = 2, a = 1, b = 1.$

Here, a = b. Then,

$$c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2}.$$

Now, centre (h, k) = (1, 2); foci $(h \pm c, k) = (1 \pm \sqrt{2}, 2)$;

vertices $(h \pm a, k) = (1 \pm 1, 2)$; eccentricity, $e = \frac{c}{a} = \frac{\sqrt{2}}{1} = \sqrt{2}$.

(iv) $9x^2 - 16y^2 - 18x - 32y - 151 = 0$

Solution: Given equation of hyperbola is

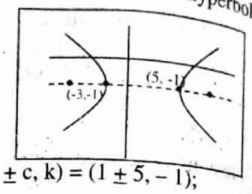
$$9x^2 - 16y^2 - 18x - 32y - 151 = 0$$

$$\Rightarrow 9(x^2 - 2x + 1) - 16(y^2 + 2y + 1) = 151 - 16 + 9$$

$$\Rightarrow 9(x - 1)^2 - 16(y + 1)^2 = 144$$

$$\Rightarrow \frac{(x-1)^2}{4^2} - \frac{(y+1)^2}{3^2} = 1$$

Comparing the equation (i) with the equation of hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get,



$$h = 1, k = -1, a = 4, b = 3.$$

Here, a > b. Then,

$$c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5.$$

Now, centre (h, k) = (1, – 1); foci $(h \pm c, k) = (1 \pm 5, -1)$;

vertices $(h \pm a, k) = (1 \pm 4, -1)$; eccentricity $(e) = \frac{c}{a} = \frac{5}{4}$.

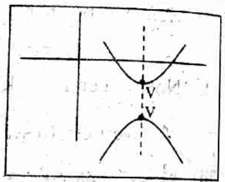
(v) $4(y + 3)^2 - 9(x - 2)^2 = 1$

Solution: Given equation of hyperbola is

$$4(y + 3)^2 - 9(x - 2)^2 = 1$$

$$\Rightarrow \frac{(y+3)^2}{1/4} - \frac{(x-2)^2}{1/9} = 1. \quad \dots (i)$$

Comparing the equation (i) with the equation of hyperbola $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$, we get



$$h = -3, k = 2, b^2 = \frac{1}{4} \text{ and } a^2 = \frac{1}{9}.$$

Then, $c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{4} + \frac{1}{9}} = \frac{\sqrt{13}}{6}$

Hence, Centre (h, k) = (2, –3) Vertices $(h, k \pm b) = (2, -3 \pm \frac{1}{2})$.

Foci $(h, k \pm c) = (2, -3 \pm \frac{\sqrt{13}}{6})$ Eccentricity $(e) = \frac{c}{b} = \frac{\sqrt{13}/6}{1/2} = \frac{\sqrt{13}}{3}$

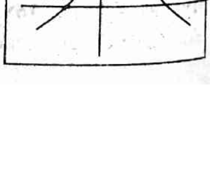
(vi) $4x^2 = y^2 - 4y + 8$

Solution: Given equation of hyperbola is

$$4x^2 = y^2 - 4y + 8 \Rightarrow 4x^2 - (y - 2)^2 = 4$$

$$\Rightarrow \frac{x^2}{1^2} - \frac{(y-2)^2}{2^2} = 1 \quad \dots (i)$$

Comparing the equation (i) with the equation of hyperbola $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$, we get



$$h = 0, k = 2, a = 1, b = 2.$$

Then, $c = \sqrt{a^2 + b^2} = \sqrt{5}$.

Now, centre (h, k) = (0, 1) vertices $(h \pm a, k) = (\pm 1, 2)$

foci $(h \pm a, k) = (\pm\sqrt{5}, 2)$ eccentricity $(e) = \frac{c}{a} = \frac{\sqrt{5}}{1} = \sqrt{5}$

3. **Find the equation of the straight lines which are tangents both to the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$.**

Solution: Here, equation of parabola is

$$y^2 = 8x \quad \dots (i)$$

Comparing (i) with $y^2 = 4ax$ then we get

$$a = 2$$

Therefore, the equation of tangent to the parabola (i) is

$$y = mx + \frac{2}{m} \quad \dots (ii)$$

And, given that the equation of hyperbola is

$$3x^2 - y^2 = 3 \Rightarrow \frac{x^2}{1} - \frac{y^2}{3} = 1 \quad \dots (iii)$$

Comparing equation (iii) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $a^2 = 1, b^2 = 3$.

Now, equation of tangent to hyperbola (iii) is

$$y = mx \pm \sqrt{m^2 - 3} \quad \dots (iv) \quad [\because c = \sqrt{a^2m^2 - b^2}]$$

According to question, the tangent to (i) is also a tangent is (iii). This means the line (ii) and (iv) are identical, so

$$mx + \frac{2}{m} = mx \pm \sqrt{m^2 - 3}$$

$$\Rightarrow \frac{2}{m} = \pm\sqrt{m^2 - 3}$$

$$\Rightarrow \frac{4}{m^2} = m^2 - 3 \quad \text{[Squaring both sides.]}$$

$$\Rightarrow 4 = m^4 - 3m^2$$

$$\Rightarrow m^4 - 3m^2 - 4 = 0$$

$$\Rightarrow m^4 - 4m^2 + m^2 - 4 = 0$$

$$\Rightarrow m^2(m^2 - 4) + 1(m^2 - 4) = 0$$

$$\Rightarrow (m^2 + 1)(m^2 - 4) = 0.$$

i.e. either $m^2 - 4 = 0$ or $m^2 + 1 = 0$

$\Rightarrow m = \pm 2$ otherwise m gives imaginary values.

So, putting the value of m in equation (ii) then we get,

$$y = 2x + 1$$

and $y = -2x - 1 \Rightarrow y + 2x + 1 = 0$

Thus, $y = 2x + 2$ and $y + 2x + 1 = 0$

are equation of tangent lines.

4. Find the equation of hyperbola whose focus, directrix and eccentricity respectively are

(a) (2, 1), $x + 2y = 1$, $e = \sqrt{2}$

Solution: Given that he equation of directrix of the hyperbola is

$$x + 2y = 1 \quad \dots (i)$$

Also given that the focus of the hyperbola is F(2, 1) and eccentricity is $e = \sqrt{2}$. Let P(x, y) be any point on the locus of the hyperbola. Then we know the eccentricity is the ratio of the distance between hyperbola and focus point and perpendicular distance from hyperbola to its directrix. That is,

$$e = \left|\frac{\text{distance form P to F}}{\text{Perpendicular distance from P to } x + 2y = 1}\right|$$

i.e. (e) (Perpendicular distance from P to $x + 2y = 1$) = distance form P to F.

$$\Rightarrow \sqrt{2}\left|\frac{x + 2y - 1}{\sqrt{1^2 + 2^2}}\right| = \sqrt{(x-2)^2 + (y-1)^2}$$

$$\Rightarrow 2(x + 2y - 1)^2 = 5[(x-2)^2 + (y-1)^2]$$

$$\Rightarrow 2x^2 + 8y^2 + 2 + 8xy - 4x - 8y = 5x^2 - 20x + 20 + 5y^2 - 10y + 5$$

$$\Rightarrow 3x^2 - 3y^2 - 8xy - 16x - 2y + 3 = 0.$$

(b) (6, 0), $4x - 3y = 6$, $e = \frac{5}{4}$

(c) (0, 4), $y + 3 = 0$, $e = \frac{4}{3}$

Solution: Similar to (a)

5. Find equation of hyperbola with center origin, conjugate axis is 3 and distance between two foci is 5.

Solution: Let the transverse axis (i.e. conjugate axis) be y–axis.

Hence, equation of hyperbola with centre at origin is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots (i)$$

Given that the length of conjugate axis is 3. That is,

$$2b = 3 \Rightarrow b = \frac{3}{2}.$$

And, distance between two foci is 5. That is,

$$2c = 5 \Rightarrow c = \frac{5}{2}$$

Then, $$a^2 = c^2 - b^2 = \frac{25}{4} - \frac{9}{4} = 4$$

So, (i) becomes

$$\frac{x^2}{4} - \frac{y^2}{9/4} = 1 \Rightarrow 9x^2 - 16y^2 = 36.$$

Therefore, $9x^2 - 16y^2 = 36$ is the equation of hyperbola.

6. Find the equation of hyperbola whose foci (4, 2), (8, 2) and eccentricity 2.

Solution: Given foci of the hyperbola are (4, 2), (8, 2).

Since, centre is the mid–point of foci, so

$$C(h, k) = \left(\frac{4+8}{2}, \frac{2+2}{2}\right) = (6, 2).$$

This gives, $h = 6, k = 2$.

And, the distance between two foci is 4. That is,

$$2c = |4 - 8| = 4 \Rightarrow c = 2.$$

And, given that the eccentricity is,

$$e = 2.$$

The hyperbola has foci with fixed y value. So, the transverse axis is parallel to x–axis. So,

$$e = \frac{c}{a} \Rightarrow 2 = \frac{2}{a} \Rightarrow a = 1.$$

Therefore, $b = \sqrt{c^2 - a^2} = \sqrt{3}$

Hence, the equation of the hyperbola is,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

i.e. $$\frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1.$$

7. Find the equation of the hyperbola with vertices are at (0, $\pm$ 6) and eccentricity is $e = \frac{5}{3}$. Also, find its foci.

Solution: Given, vertices of the hyperbola are V(0, $\pm$ 6).

Since the centre is the mid–point of the vertices. Therefore, the centre of the hyperbola is,

$$C(h, k) = C\left(\frac{0+0}{2}, \frac{6-6}{2}\right) = C(0, 0).$$

That is, $h = 0, k = 0$.

Here x–coordinate of the vertices are fixed. So, the transverse axis is parallel to y–axis.

Therefore,

$$V(h, k \pm b) = V(0, 0 \pm 6)$$

This implies $\pm b = \pm 6 \Rightarrow b = 6.$ $[\because h = k = 0]$

Also, given that eccentricity is,

$$e = \frac{5}{3} \Rightarrow \frac{c}{b} = \frac{5}{3} \Rightarrow \frac{5}{3} = \frac{c}{6} \Rightarrow c = 10.$$

Then, $$a = \sqrt{c^2 - b^2} = \sqrt{64} = 8.$$

Now, the equation of hyperbola is

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

i.e. $\frac{(y-0)^2}{36} - \frac{(x-0)^2}{64} = 1 \Rightarrow \frac{x^2}{64} - \frac{y^2}{36} + 1 = 0.$

And, the foci of the hyperbola are $F(h, k \pm c) = F(0, \pm 10)$.

8. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Find the equation of hyperbola having eccentricity 2.

Solution: Given ellipse is

$$\frac{x^2}{26} + \frac{y^2}{9} = 1 \qquad \dots (i)$$

Comparing equation (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ then we get

$$a = 5, \ b = 3, \ h = 0 \ \text{ and } \ k = 0.$$

Then, centre $C(h, k) = C(0, 0)$.

In the given ellipse (i) we get $a > b$. So, the foci lie on the line parallel to x-axis. So, the foci of the ellipse is $F(h \pm c, k) = F(\pm 5, 0)$.

Then, $c = 5$.

Given that the foci of required hyperbola coincide with the foci of (i). So, the foci of the hyperbola are

$$F(\pm 5, 0) = F(h \pm c, k).$$

This implies $h = 0, k = 0$ and $c = 5$.

Clearly these foci have fixed y value. So, the transverse axis of the hyperbola is the axis parallel to x-axis.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \dots (ii)$$

and its eccentricity is

$$e = \frac{c}{a} \qquad \dots (iii)$$

Given that the eccentricity of the hyperbola is 2. That is,

$$e = \frac{c}{a}. \Rightarrow 2 = \frac{5}{a} \Rightarrow a = \frac{5}{2} \Rightarrow a^2 = \frac{25}{4}.$$

Then, $b^2 = c^2 - a^2 = 25 - \frac{25}{4} = \frac{75}{4}$.

Now the equation (ii) becomes

$$\frac{4x^2}{25} - \frac{4y^2}{75} = 1$$

$$\Rightarrow 12x^2 - 4y^2 = 75$$

This is equation of the hyperbola.

9. Show that the line $y = x + 2$ touches to the hyperbola $5x^2 - 9y^2 = 45$. Find the point of contact.

Solution: Given that the equation of hyperbola is,

$$5x^2 - 9y^2 = 45 \qquad \dots (i)$$

And the line is, $y = x + 2 \qquad \dots (ii)$

Eliminating y from equation (i) and (ii) then

$$5x^2 - 9(x+2)^2 = 45$$

$$\Rightarrow 5x^2 - 9x^2 - 36x - 36 = 45$$

$$\Rightarrow 4x^2 + 36x + 81 = 0$$

$$\Rightarrow x = \frac{-36 \pm \sqrt{(36)^2 - 4(4)(81)}}{8} = \frac{-36}{8} = \frac{-9}{2}.$$

Then (ii) gives

$$y = \frac{-9}{2} + 2 = \frac{-5}{2}.$$

This shows (ii) touches (i) at a single point $\left(-\frac{9}{2}, -\frac{5}{2}\right)$. This means (ii) is tangent to (i) and the point of contact between them is $\left(-\frac{9}{2}, -\frac{5}{2}\right)$.

10. Find the equation of hyperbola with center origin and passing through (2, 1) and (4, 3).

Solution: Given that the center of hyperbola is (0, 0).

Let the equation of hyperbola having center at origin is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \dots (i)$$

Given that the hyperbola (ii) is passing through points (2, 1) and (4, 3). Therefore,

$$\frac{4}{a^2} - \frac{1}{b^2} = 1 \quad \dots (ii) \qquad \text{and} \qquad \frac{16}{a^2} - \frac{9}{b^2} = 1 \quad \dots (iii)$$

Solving equation (ii) and (iii) then we get,

$$a^2 = \frac{5}{2} \ \text{ and } \ b^2 = \frac{5}{8}.$$

Therefore (i) becomes,

$$\frac{2x^2}{5} - \frac{8y^2}{5} = 1 \Rightarrow 2x^2 - 8y^2 = 5.$$

11. Find the value of λ, when the line $y = 2x + \lambda$ is tangent to the hyperbola $3x^2 - y^2 = 3$.

Solution: Given hyperbola is,

$$3x^2 - y^2 = 3 \qquad \dots (i)$$

and given line is

$$y = 2x + \lambda \qquad \dots (ii)$$

Eliminating y from equation (i) and (ii), we get

$$3x^2 - (2x + \lambda)^2 = 3$$

$$\Rightarrow 3x^2 - 4x^2 - 4xs\lambda - \lambda^2 = 0$$

$$\Rightarrow -x^2 - 4\lambda x - \lambda^2 = 3$$

$$\Rightarrow x^2 + 4\lambda x + \lambda^2 + 3 = 0 \qquad \dots (iii)$$

which is quadratic in x. Since, the equation (ii) is tangent to (i). So, the discriminate value of (iii), is zero. That is,

$$(4\lambda)^2 - 4(1)(\lambda^2 + 3) = 0$$

$$\Rightarrow 16\lambda^2 - 4\lambda^2 - 12 = 0 \Rightarrow 12\lambda^2 = 12 \quad \Rightarrow \lambda = \pm 1.$$

Thus for $\lambda = \pm 1$, the line (ii) is tangent to the hyperbola (i).

12. Find the equation of the tangents to the hyperbola $3x^2 - 4y^2 = 12$, which are perpendicular to the line $y = x + 2$. Also, find the point of contact.

[2017 Spring]

Solution: Given that the equation of line is,

$$y = x + 2 \quad \ldots (i)$$

Comparing it with the line $y = mx + c$ then we get

$$m = 1, c = 2.$$

Since we have the equation to tangent on hyperbola is,

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad \ldots (ii)$$

By given condition the line (i) is perpendicular to (ii). So, by condition of normality we get,

$$m_1 . m_2 = -1 \Rightarrow m_1 = -\frac{1}{1} = -1.$$

Also, given hyperbola is,

$$3x^2 - 4y^2 = 12$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{3} = 1 \quad \ldots (iii)$$

Comparing it with the standard equation of hyperbola then we get

$$a^2 = 4, \; b^2 = 3, h = 0, k = 0.$$

Then, the equation (ii) becomes,

$$y = -1(x) \pm \sqrt{4 . 1 - 3}$$

$$\Rightarrow y = -x \pm 1. \quad \ldots (iv)$$

Therefore, $x + y = \pm 1$ are the equation of required tangents to (iii).

For point of contact of $x + y \pm 1 = 0$ and curve $3x^2 - 4y^2 = 12$ is,

$$x = -\frac{B}{2A} = 4 \qquad x = -\frac{B}{2A} = -4$$

So, $y = -3$ $\qquad$ $y = 3$

This, the point of contact (4, –3) and (–4, 3).

13. Show that the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a^2l^2 - b^2m^2 = n^2$.

[2016 Spring][2011 Fall]

OR

Find the condition the line $lx + my + n = 0$ may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

[2015 Fall]

Solution: See theory part of hyperbola before Ex. 9.6.

14. **Let e and e_1 are the eccentricities of a hyperbola and its conjugate, show that $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$.**

Solution: Let e and e_1 are the eccentricities of a hyperbola and its conjugate, so

$$e = \frac{c}{a} \quad \text{and } e_1 = \frac{c}{b}$$

Now, $$\frac{1}{e^2} + \frac{1}{e_1^2} = \frac{1}{(c/a)^2} + \frac{1}{(c/b)^2} = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1.$$

Chapter 11

INTEGRAL CALCULUS

Exercise 11.1

A. Evaluate the following integrals

1. $\int \left(\frac{x^3 + 5x^2 - 3}{x + 2}\right) dx$

Solution: Here,

$$\int \left(\frac{x^3 + 5x^2 - 3}{x + 2}\right) dx$$

$$= \int \left[x^2 + 3x - 6 + \frac{9}{x + 2}\right] dx$$

$$= \int x^2\, dx + 3 \int x\, dx - 6 \int dx + 9 \int \frac{dx}{x + 2}$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} - 6x + 9 \log |x + 2| + C.$$

2. $\int \left(t + \frac{1}{t}\right)^2 dt$

Solution: Here,

$$\int \left(t + \frac{1}{t}\right)^2 dt = \int t^2\, dt + 2 \int dt + \int t^{-2}\, dt$$

$$= \frac{t^3}{3} + 2t + \frac{t^{-1}}{-1} + c \quad = \frac{t^3}{3} + 2t - \frac{1}{t} + C.$$

3. $\int \left(6\,\text{cosec}^2x - \frac{1}{x^2}\right) dx$

Solution: Here,

$$\int \left(6\,\text{cosec}^2x - \frac{1}{x^2}\right) dx$$
$$= 6\int \text{cosec}^2 x\, dx - \int x^{-2}\, dx$$
$$= -6 \cot x - \frac{x^{-1}}{-1} + C \quad = -6 \cot x + \frac{1}{x} + C.$$

4. $\int \left(\frac{x^4 + x^2 + 1}{x+1}\right) dx$

Solution: Here,

$$\int \left(\frac{x^4 + x^2 + 1}{x+1}\right) dx$$
$$= \int \left(x^3 - x^2 + 2x - 2 + \frac{3}{x+1}\right) dx$$
$$= \int x^3\, dx - \int x^2\, dx + 2\int x\, dx - 2\int dx + 3\int \frac{dx}{x+1}$$
$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{2x^2}{2} - 2x + 3 \log|x+1| + C$$
$$= \frac{x^4}{4} - \frac{x^3}{3} + x^2 - 2x + 3 \log|x+1| + C$$

5. $\int (\tan^2x - 3x^2)\, dx$

Solution: Here,

$$\int (\tan^2x - 3x^2)\, dx.$$
$$= \int (\sec^2x - 1)\, dx - 3\int x^2\, dx$$
$$= \int \sec^2x\, dx - \int dx - 3\int x^2\, dx$$
$$= \tan x - x - \frac{3x^3}{3} + C$$
$$= \tan x - x - x^3 + C.$$

6. $\int \frac{\cos 2x\, dx}{\cos^2x \sin^2x}$

Solution: Here,

$$\int \frac{\cos 2x\, dx}{\cos^2x \sin^2x}$$
$$= \int \left(\frac{\cos^2x - \sin^2x}{\cos^2x \cdot \sin^2x}\right) dx$$

$$= \int \frac{\cos^2x}{\cos^2 x \cdot \sin^2x}\, dx - \int \frac{\sin^2x}{\cos^2x \cdot \sin^2x}\, dx$$
$$= \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2x}$$
$$= \int \text{cosec}^2 x\, dx - \int \sec^2x\, dx$$
$$= -\cot x - \tan x + C.$$

7. $\int \sin^2x \cos^2x\, dx$

Solution: Here,

$$\int \sin^2x \cos^2x\, dx$$
$$= \frac{1}{4}\int (2\sin x \cos x)^2\, dx$$
$$= \frac{1}{4}\int \sin^2 2x\, dx$$
$$= \frac{1}{4}\int \left(\frac{1 - \cos 4x}{2}\right) dx \qquad \left[\because \sin^2x = \frac{1-\cos 2x}{2}\right]$$
$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x dx \quad = \frac{1}{8} \cdot x - \frac{1}{8} \cdot \frac{\sin 4x}{4} + C$$
$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$
$$= \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$$

B. Evaluate the following integrals:

1. $\int \left(\frac{16x}{8x^2 + 2}\right) dx$

Solution: Here,

$$I = \int \left(\frac{16x}{8x^2 + 2}\right) dx$$

Put $8x^2 + 2 = y$ then $16x.dx = dy$. Therefore,

$$I = \int \frac{dy}{y} = \log|y| + c \quad = \log|8x^2 + 2| + C.$$

2. $\int \frac{\cos x}{\sqrt{1 + \sin x}}\, dx$

Solution: Here,

$$I = \int \frac{\cos x}{\sqrt{1 + \sin x}}\, dx$$

Put, $1 + \sin x = y$ then $\cos x.\, dx = dy$

Therefore,

$$I = \int \frac{dy}{\sqrt{y}} = \int y^{-1/2}\, dy = \frac{y^{-1/2+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{y^{1/2}}{\frac{1}{2}} + C$$

$$= 2\sqrt{y} + C$$

$$= 2\sqrt{1 + \sin x} + C.$$

3. $\int \left(x\, e^{x^2}\right) dx$

Solution: Here,

$$I = \int \left(x\, e^{x^2}\right) dx$$

Put $x^2 = y$ then $2x.dx = dy \Rightarrow x\, dx = \frac{dy}{2}$. Therefore,

$$I = \frac{1}{2}\int e^y\, dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{x^2} + C.$$

4. $\int \left(\frac{x}{\sqrt{8x^2+1}}\right) dx$

Solution: Here,

$$I = \int \left(\frac{x}{\sqrt{8x^2+1}}\right) dx$$

Put, $8x^2 + 1 = y$ then $16x.dx = dy \Rightarrow x\, dx = \frac{dy}{16}$. Therefore,

$$I = \frac{1}{16}\int \frac{dy}{\sqrt{y}} = \frac{1}{16}\int y^{-1/2}\, dy$$

$$= \frac{y^{1/2}}{16 \cdot \frac{1}{2}} + C = \frac{\sqrt{y}}{8} + C = \frac{\sqrt{8x^2+1}}{8} + C.$$

5. $\int \frac{\sin x}{3 + 4\cos x}\, dx$

Solution: Here,

$$I = \int \frac{\sin x}{3 + 4\cos x}\, dx$$

Put $3 + 4\cos x = y$. Then $-4 \sin x.dx = dy \Rightarrow \sin x.dx = -\frac{dy}{4}$

Therefore,

$$I = -\frac{1}{4}\int \frac{dy}{y} = -\frac{1}{4}\log|y| + C = -\frac{1}{4}\log|3 + 4\cos x| + C.$$

6. $\int e^x \sec^2(e^x)\, dx$

Solution: Here,

$$I = \int e^x \sec^2(e^x)\, dx$$

Put $e^x = y$ then $e^x.\, dx = dy$. Therefore,

$$I = \int \sec^2 y\, dy = \tan y + C = \tan(e^x) + C.$$

7. $\int \frac{dx}{\sqrt{x}\,(1+x)}$

Solution: Here,

$$I = \int \frac{dx}{\sqrt{x}\,(1+x)}$$

Put $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}\, dx = dy$. Therefore,

$$I = 2\int \frac{dy}{1+y^2} = 2\tan^{-1}(y) + C = 2\tan^{-1}(\sqrt{x}) + C$$

8. $\int e^x \sqrt{3 + 4e^x}\, dx$

Solution: Here,

$$I = \int e^x \sqrt{3 + 4e^x}\, dx$$

Put, $3 + 4e^x = y$. Then $4e^x.dx = dy$. Then,

$$I = \int \sqrt{y}\,.\frac{dy}{4} = \frac{1}{4}\int y^{1/2}\, dy = \frac{1}{4}\cdot\frac{y^{3/2}}{\frac{3}{2}} + C.$$

$$\Rightarrow I = \frac{(3 + 4e^x)^{3/2}}{6} + C.$$

9. $\int \frac{dx}{x - \sqrt{x}}$

Solution: Here,

$$I = \int \frac{dx}{x - \sqrt{x}}$$

Put, $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}\, dx = dy \Rightarrow dx = 2y\, dy$. Then,

$$I = \int \frac{2y\, dy}{y^2 - y} = 2\int \frac{dy}{y - 1}$$

Again, put $y - 1 = z$ then $dy = dz$. Therefore,

$$I = 2\int \frac{dz}{z} = 2\log|z| + C$$

$$= 2\log|y-1| + C = 2\log\left|\sqrt{x} - 1\right| + C.$$

10. $\displaystyle\int \frac{dx}{x\log x}$ **for x > 0.**

Solution: Here,

$$I = \int \frac{dx}{x\log x} \qquad \text{for } x > 0$$

Put, $\log x = y$ then $\frac{1}{x}dx = dy$. Therefore,

$$I = \int \frac{dy}{y} = \log|y| + C = \log|\log x| + C.$$

Note: ***If x ≤ 0 then the function is not integrable.***

11. $\displaystyle\int \frac{e^x}{1+e^x}dx$

Solution: Here,

$$I = \int \frac{e^x}{1+e^x}dx$$

Put, $1 + e^x = y$. Then $e^x.dx = dy$. Then,

$$I = \int \frac{dy}{y} = \log|y| + C$$

$$= \log|e^x + 1| + C = \log(e^x + 1) + C.$$

12. $\displaystyle\int e^{3x}dx$

Solution: Here,

$$I = \int e^{3x}dx$$

Put $3x = y$. Then $3.dx = dy \Rightarrow dx = \frac{dy}{3}$. Then,

$$I = \frac{1}{3}\int e^y dy = \frac{e^y}{3} + C = \frac{e^{3x}}{3} + C.$$

13. $\displaystyle\int \left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)dx$

Solution: Here,

$$I = \int \left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)dx.$$

Put, $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}dx = dy \Rightarrow \frac{dx}{\sqrt{x}} = 2\,dy$. Then,

$$I = 2\int \tan y\,dy = 2\log|\sec y| + c = 2\log\left|\sec(\sqrt{x})\right| + c$$

$$\Rightarrow I = \log\left|\sec^2(\sqrt{x})\right| + c.$$

14. $\displaystyle\int \frac{x\,dx}{\sqrt{1-4x^2}}$

Solution: Here,

$$I = \int \frac{x\,dx}{\sqrt{1-4x^2}} \quad \text{for } |x| < ½.$$

Put $1 - 4x^2 = y$ then $-8x.dx = dy \Rightarrow x\,dx = -\frac{dy}{8}$. Then,

$$I = -\frac{1}{8}\int \frac{dy}{\sqrt{y}} = -\frac{1}{8}\int y^{-1/2}dy = -\frac{1}{8}\left(\frac{y^{1/2}}{1/2}\right) + C.$$

$$\Rightarrow I = -\frac{(1-4x)^{1/2}}{4} + C.$$

15. $\displaystyle\int \left(\frac{\sin^2 2x}{1+\cos 2x}\right)dx$

Solution: Here,

$$I = \int \left(\frac{\sin^2 2x}{1+\cos 2x}\right)dx = \int\left(\frac{4\sin^2 x\cos^2 x}{2\cos^2 x}\right)dx = 2\int \sin^2 x\,dx$$

$$\Rightarrow I = 2\int\left(\frac{1-\cos 2x}{2}\right)dx = \int dx - \int \cos 2x.dx$$

$$= x - \frac{\sin 2x}{2} + C.$$

16. $\displaystyle\int \frac{dy}{y\sqrt{y^2-1}}$

Solution: Here,

$$I = \int \frac{dy}{y\sqrt{y^2-1}}$$

Put $y = \sec\theta$, then $dy = \sec\theta.\tan\theta\,d\theta$. So,

$$I = \int \frac{\sec\theta\,.\,\tan\theta\,.\,d\theta}{\sec\theta\,.\,\tan\theta} = \int 1\,.\,d\theta = \theta + C = \sec^{-1}y + C.$$

17. $\displaystyle\int \frac{x\,dx}{(3x^2+4)^3}$

Solution: Here,

$$I = \int \frac{x\,dx}{(3x^2+4)^3}$$

Put, $3x^2 + 4 = y$, then $6x\,dx = dy$. So,

$$I = \frac{1}{6}\int \frac{dy}{y^3} = \frac{1}{-12}\cdot\frac{1}{y^2} + C = \frac{-1}{12\,(3x^2+4)^2} + C.$$

18. $\int \frac{x^2\,dx}{\sqrt{x^3+5}}$

Solution: Here,

$$I = \int \frac{x^2\,dx}{\sqrt{x^3+5}}$$

Put, $x^3 + 5 = y$, then $3x^2.dx = dy$. So,

$$I = \frac{1}{3}\int \frac{dy}{\sqrt{y}} = \frac{2}{3}\sqrt{y} + c = \frac{2}{3}\sqrt{x^3+5} + C.$$

19. $\int 3^{4x}\,dx$

Solution: Here,

$$I = \int 3^{4x}\,dx$$

Put, $4x = y$, then $4\,dx = dy$. So,

$$I = \frac{1}{4}\int 3^y\,dy = \frac{1}{4}\left(\frac{3^y}{\log 3}\right) + C = \frac{1}{4}\left(\frac{3^{4x}}{\log 3}\right) + C.$$

20. $\int \cos^2 x \sin x\,dx$

Solution: Here,

$$I = \int \cos^2 x.\sin x\,dx$$

Put $\cos x = y$, then $-\sin x\,dx = dy$. So,

$$I = \int (y^2)(-dy) = -\frac{y^3}{3} + C = -\frac{\cos^3 x}{3} + C.$$

21. $\int \cot^3 x\,\mathrm{cosec}^2 x\,dx$

Solution: Here,

$$I = \int \cot^3 x.\,\mathrm{cosec}^2 x.\,dx$$

Put, $\cot x = y$, then $\mathrm{cosec}^2 x.\,dx = -\,dy$. So,

$$I = -\int y^3\,dy = -\frac{y^4}{4} + C = -\frac{\cot^4 x}{4} + C.$$

22. $\int \frac{dx}{\sqrt{e^{2x}-1}}$ **[2011 Fall]**

Solution: Here,

$$I = \int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{\sqrt{(e^x)^2-1}}$$

Put, $e^x = \sec\theta$ then $e^x.\,dx = \sec\theta.\tan\theta\,d\theta \Rightarrow dx = \frac{\sec\theta.\tan\theta\,d\theta}{\sec\theta} = \tan\theta\,d\theta.$

Then,

$$I = \int \frac{\sec\theta\,.\,\tan\theta\,.\,d\theta}{\sqrt{\sec^2\theta - 1}}$$

$$= \int \frac{\tan\theta\,.\,d\theta}{\tan\theta} = \int d\theta = \theta + C = \sec^{-1} y + C = \sec^{-1}(e^x) + C.$$

23. $\int \frac{x^8 dx}{(1-x^3)^{1/3}}$

Solution: Here,

$$I = \int \frac{x^8\,dx}{(1-x^3)^{1/3}}$$

Put, $(1 - x^3) = y^3$ then, $(-3x^2)\,dx = 3y^2dy \Rightarrow x^2dx = -y^2dy$. So,

$$I = \int \frac{(1-y^3)^2\,.\,(-y^2\,dy)}{y}$$

$$= -\int y\,(1-y^3)^2\,.\,dy$$

$$= -\int y\,dy + 2\int y^4dy - \int y^7\,dy$$

$$= -\frac{y^2}{2} + \frac{2}{5}y^5 - \frac{y^8}{8} + C.$$

$$= \frac{1}{2}(1-x^3)^{2/3} + \frac{2}{5}(1-x^3)^{5/3} - \frac{1}{8}(1-x^3)^{8/3} + C.$$

24. $\int e^{\tan 3x}\sec^2 3x\,dx$

Solution: Here,

$$I = \int e^{\tan 3x}\,.\,\sec^2 3x\,dx$$

Put, $\tan 3x = y$ then $3\sec^2 3x\,dx = dy$. Then,

$$I = \frac{1}{3}\int e^y\,dy = \frac{1}{3}e^y + C = \frac{1}{3}e^{\tan 3x} + C.$$

25. $\int x^{1/3}\sqrt{x^{4/3}-1}\,dx$

Solution: Here,

$$I = \int x^{1/3}\sqrt{x^{4/3}-1}\,dx$$

Put, $x^{4/3} - 1 = y$ then $\frac{4}{3}x^{1/3}.dx = dy \Rightarrow x^{1/3}.dx = \frac{3}{4}dy$. Then,

$$I = \frac{3}{4}\int \sqrt{y}\,dy = \frac{3}{4}\cdot\frac{2}{3}y^{3/2} + C = \frac{1}{2}(x^{4/3} - 1)^{3/2} + C.$$

26. $\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$

Solution: Here,

$$I = \int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}dx$$

Let, $\sqrt{x+1} = y$. Then, $\frac{1}{2\sqrt{x+1}}dx = dy \Rightarrow \frac{1}{\sqrt{x+1}}dx = 2\,dy$.

Then,

$$I = 2\int e^y\,dy = 2e^y + C = 2e^{\sqrt{x+1}} + C.$$

27. $\int \frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$

Solution: Here,

$$I = \int \frac{\sin\theta}{\sqrt{1+\cos\theta}}d\theta$$

Put, $1 + \cos\theta = y$ then $-\sin\theta\,d\theta = dy$. Then,

$$I = -\int \frac{dy}{\sqrt{y}} = -2\sqrt{y} + C = -2\sqrt{1+\cos\theta} + C.$$

28. $\int \frac{\sqrt{\tan^{-1}x}}{2(1+x^2)}dx$

Solution: Here,

$$I = \int \frac{\sqrt{\tan^{-1}x}}{2(1+x^2)}dx$$

Put, $\tan^{-1}x = y$ then $\frac{1}{1+x^2}dx = dy$. Then,

$$I = \int \left(\frac{\sqrt{y}}{2}\right)dy = \frac{1}{3}(y)^{3/2} + C = \frac{1}{3}(\tan^{-1}x)^{3/2} + C.$$

29. $\int \left(\sin x\sqrt{1-\cos 2x}\right)dx$

Solution: Here,

$$I = \int \left(\sin x\sqrt{1-\cos 2x}\right)dx$$

$$= \int \left(\sin x\sqrt{2\sin^2 x}\right)dx$$

$$= \sqrt{2}\int \sin x\sin x\,dx$$

$$= \sqrt{2}\int \sin^2 x\,dx$$

$$= \sqrt{2}\int\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{\sqrt{2}}\left(x - \frac{\sin 2x}{2}\right) + C.$$

30. $\int x\sin^3(x^2)\cos(x^2)\,dx$

Solution: Here,

$$I = \int x\sin^3(x^2)\cos(x^2)\,dx$$

Put $\sin(x^2) = y$ then $2x.\cos(x^2).dx = dy$. Then,

$$I = \int y^3\frac{dy}{2} = \frac{y^4}{8} + C = \frac{(\sin(x^2))^4}{8} + C.$$

31. $\int \sin^3 x\cos^4 x\,dx$

Solution: Here,

$$I = \int \sin^3 x\cos^4 x.\,dx = \int (1-\cos^2 x)\cos^4 x.\sin x\,dx$$

Let, $\cos x = y$ then $(-\sin x).dx = dy$. Then,

$$I = -\int (y^4 - y^6)\,dx = \frac{y^7}{7} - \frac{y^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C.$$

32. $\int \frac{\cos(\log x)}{x}dx$

Solution: Here,

$$I = \int \frac{\cos(\log x)}{x}dx$$

Let, $\log x = y$ then, $\frac{1}{x}dx = dy$. Then,

$$I = \int \cos y\,.\,dy = \sin y + C = \sin(\log x) + C.$$

33. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx$

Solution: Here,

[illegible]

34. $\int \frac{3e^{x}+3e^{x}}{e^{x}+e^{x}}\,dx$ [illegible]

Solution: Here,

[illegible]

35. $\int \log(x^2+1)\,2x\,(1+x^2)^{-1}\,dx$ [illegible]

Solution: Here,

$I = \int 2x\,\frac{\log(x^2+1)}{(x^2+1)}\,dx$

Let $\log(x^2+1) = y$, then $\frac{2x}{x^2+1}\,dx = dy$. Then

$I = \int y\,dy = \frac{y^2}{2} + C = \frac{[\log(x^2+1)]^2}{2} + C$

36. $\int \frac{\sin 2x}{(\sin^2 x + \cos^2 x)}\,dx$ [illegible]

Solution: Here,

[illegible]

Let $\tan x = y$ then $\sec^2 x\,dx = dy$. Then,

$I = 2\int \frac{y\,dy}{y^4+1}$

Again let $y^2 = z$ then $2y\,dy = dz$. Then,

$I = \int \frac{dz}{z^2+1}$

$= \tan^{-1}(z) + C = \tan^{-1}(y^2) + C = \tan^{-1}(\tan^2 x) + C$

[illegible] $\int \frac{1}{(3\tan x + 1)\cos^2 x}\,dx$

Solution: Here,

$I = \int \frac{\sec^2 x}{(3\tan x + 1)}\,dx$

Let $3\tan x + 1 = y$, then $3\sec^2 x\,dx = dy$. Then,

$I = \frac{1}{3}\int \frac{dy}{y} = \frac{1}{3}\log|y| + C = \frac{1}{3}\log|3\tan x + 1| + C$

[illegible] $\int \frac{1}{x\log x\,\log(\log x)}\,dx$ for $x > 0$.

Solution: Here,

$I = \int \frac{1}{x\log x\,\log(\log x)}\,dx$ for $x > 0$.

Let $\log x = y$, then $\frac{1}{x}\,dx = dy$. Then

$I = \int \frac{1}{y\log y}\,dy$

Again put $\log(y) = z$, then $\frac{1}{y}\,dy = dz$. Then

$I = \int \frac{dz}{z} = \log|z| + C$

$= \log(\log y) + C = \log(\log(\log x)) + C$

[illegible] $\int \sec^4 x\,dx$

Solution: Here,

$I = \int \sec^4 x\,dx = \int \sec^2 x\,\sec^2 x\,dx = \int (1+\tan^2 x)\sec^2 x\,dx$

[illegible]

Put $\tan x = y$ then $\sec^2 x\,dx = dy$. Then,

$$I = \int dy + \int y^2 dy = y + \frac{y^3}{3} + C \quad = \tan x + \frac{\tan^3 x}{3} + C.$$

40. $\int \tan^5 x \, dx$

Solution: Here,

$$\begin{aligned}
I &= \int \tan^5 x \, dx \\
&= \int \tan^2 x \, . \, \tan^3 x \, dx \\
&= \int (\sec^2 x - 1) \tan^3 x \, dx \\
&= \int \tan^3 x \, . \, \sec^2 x \, dx - \int \tan^3 x \, dx \\
&= \int \tan^3 x . \sec^2 x \, dx - \int (\sec^2 x - 1) \tan x \, dx \\
&= \int \tan^3 x \, . \, \sec^2 x \, dx - \int \tan x . \sec^2 x \, dx + \int \tan x \, dx \\
&= \int \tan^3 x \, . \, \sec^2 x \, dx - \int \tan x \, . \, \sec^2 x \, dx + \log|\sec x| + C_1
\end{aligned}$$

Put, $\tan x = y$ then $\sec^2 x \, dx = dy$. Then,

$$\begin{aligned}
I &= \int y^3 dy - \int y \, dy + \log|\sec x| + C_1 \\
&= \frac{y^4}{4} - \frac{y^2}{2} + \log|\sec x| + C_1 + C_2 \\
&= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + C_3 \qquad \text{for } C_1 + C_2 = C_3 \\
&= \frac{(\sec^2 x - 1)^2}{4} - \left(\frac{\sec^2 x - 1}{2}\right) + \log|\sec x| + C_3 \\
&= \frac{\sec^4 x + 1 - 2\sec^2 x - 2\sec^2 x + 2}{4} + \log|\sec x| + C_3 \\
&= \frac{\sec^4 x}{4} - \sec^2 x + \log|\sec x| + C_3 + \frac{3}{4} \\
&= \frac{\sec^4 x}{4} - \sec^2 x + \log|\sec x| + C \qquad \text{for } C = C_3 + \frac{3}{4}
\end{aligned}$$

Exercise 11.2

A. Evaluate the integrals:

1. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 2x + 5}}$

Solution: Here,

$$I = \int \frac{dx}{\sqrt{x^2 - 2x + 5}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 2x + 1 + 4}} = \int \frac{dx}{\sqrt{(x-1)^2 + 9^2}}$$

Put, $x - 1 = y$ then $dx = dy$. Then,

$$I = \int \frac{dy}{\sqrt{y^2 + 2^2}}$$

$$= \log\left|y + \sqrt{y^2 + 2^2}\right| + C = \log\left|x - 1 + \sqrt{x^2 - 2x + 5}\right| + C.$$

2. $\displaystyle\int \left(\frac{x}{x^2 - 2x + 5}\right) dx$

Solution: Here,

$$I = \int \left(\frac{x}{x^2 - 2x + 5}\right) dx = \int \left(\frac{\frac{1}{2}(2x - 2 + 2)}{x^2 - 2x + 5}\right) dx$$

$$= \frac{1}{2} \int \left(\frac{(2x - 2)}{x^2 - 2x + 5}\right) dx + \frac{2}{2}\int \frac{dx}{x^2 - 2x + 1 + 4}$$

Put, $x^2 - 2x + 5 = y$ then $(2x - 1)\, dx = dy$. Then,

$$I = \frac{1}{2} \int \frac{dy}{y} + \int \frac{dx}{(x-1)^2 + 2^2}$$

$$= \frac{1}{2} \log |y| + \int \frac{dx}{\alpha^2 + 2^2} + C_1 \qquad [\text{setting } x - 1 = \alpha]$$

$$= \frac{1}{2} \log |x^2 - 2x + 5| + \frac{1}{2} \tan^{-1}\left(\frac{\alpha}{2}\right) + C$$

$$= \log\left(\sqrt{x^2 - 2x + 5}\right) + 2 \tan^{-1}\left(\frac{x-1}{2}\right) + C.$$

$$= \log\left(\sqrt{x^2 - 2x + 5}\right) + 2 \tan^{-1}\left(\frac{x-1}{2}\right) + C.$$

3. $\int \left(\frac{x}{\sqrt{x^2 - 2x + 5}}\right) dx$

Solution: Here,

$$I = \int \left(\frac{x}{\sqrt{x^2 - 2x + 5}}\right) dx$$

$$= \frac{1}{2} \int \left(\frac{2x - 2 + 2}{\sqrt{x^2 - 2x + 5}}\right) dx$$

$$= \frac{1}{2} \int \left(\frac{2x - 2}{\sqrt{x^2 - 2x + 5}}\right) dx + \frac{2}{2} \int \frac{dx}{\sqrt{x^2 - 2x + 1 + 4}}$$

$$= \frac{1}{2} \int \left(\frac{2x - 2}{\sqrt{x^2 - 2x + 5}}\right) dx + \int \frac{dx}{(x-1)^2 + 2^2}$$

For first integral, put $x^2 - 2x + 5 = y$ then $(2x - 2)\, dx = dy$.
And for second integral, put $x - 1 = \alpha$ then $dx = d\alpha$. Then,

$$I = \frac{1}{2} \int \frac{dy}{\sqrt{y}} + \int \frac{d\alpha}{\sqrt{\alpha^2 + 2^2}}$$

$$= \frac{1}{2} \cdot \frac{y^{1/2}}{1/2} + \log\left|\alpha + \sqrt{\alpha^2 + 2^2}\right| + C$$

$$= \sqrt{x^2 - 2x + 5} + \log\left|x - 1 + \sqrt{x^2 - 2x + 5}\right| + C.$$

4. $\int \left(\frac{x+1}{3 + 2x - x^2}\right) dx$

Solution: Here,

$$I = \int \left(\frac{x+1}{3+2x-x^2}\right)dx = \int \left(\frac{\frac{1}{2}(2x+2-2)+1}{3+2x-x^2}\right)dx$$

$$= \left(\frac{-1}{2}\right)\int \left(\frac{2-2x}{3+2x-x^2}\right)dx + 2\int \frac{dx}{3+2x-x^2}$$

Put $3+2x-x^2 = y$ then $(2-2x)\,dx = dy$. Then,

$$I = \left(\frac{-1}{2}\right)\int \frac{dy}{y} + 2\int \frac{dx}{4-(x-1)^2}$$

Put $x - 1 = \alpha$ then $dx = d\alpha$. Then,

$$I = \left(\frac{-1}{2}\right)\int \frac{dy}{y} + 2\int \frac{dy}{2^2-\alpha^2}$$

$$= \left(\frac{-1}{2}\right)\log|y| + 2\left(\frac{1}{2(2)}\right)\log\left|\frac{2+\alpha}{2-\alpha}\right| + C$$

$$= \left(\frac{-1}{2}\right)\log|3+2x-x^2| + \frac{1}{2}\log\left|\frac{2+x-1}{2-x+1}\right| + C$$

$$= \left(\frac{-1}{2}\right)\log|3+2x-x^2| + \frac{1}{2}\log\left|\frac{x+1}{3-x}\right| + C.$$

5. $\displaystyle\int \left(\frac{3}{\sqrt{15-6x-x^2}}\right)dx$

Solution: Here,

$$I = \int \left(\frac{3}{\sqrt{15-6x-x^2}}\right)dx = (3)\int \frac{dx}{\sqrt{24-(x+3)^2}}$$

Put $x + 3 = y$ then $dx = dy$. Then,

$$I = (3)\int \frac{dy}{\sqrt{24-y^2}} = 3\sin^{-1}\left(\frac{y}{\sqrt{24}}\right) + C = 3\sin^{-1}\left(\frac{x+3}{\sqrt{24}}\right) + C.$$

6. $\displaystyle\int \left(\frac{a-x}{\sqrt{2ax-x^2}}\right)dx$

Solution: Here,

$$I = \int \left(\frac{a-x}{\sqrt{2ax-x^2}}\right)dx = \left(\frac{1}{2}\right)\int \left(\frac{2a-2x}{\sqrt{2ax-x^2}}\right)dx$$

Put, $2ax - x^2 = y$ then $(2a-2x)\,dx = dy$. Then,

$$I = \left(\frac{1}{2}\right)\int \frac{dy}{\sqrt{y}} = \left(\frac{1}{2}\right)\left(\frac{y^{1/2}}{1/2}\right) + C = \sqrt{2ax-x^2} + C.$$

7. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x}}$

Solution: Here,

$$I = \int \frac{dx}{\sqrt{x^2+2x}} = \int \frac{dx}{\sqrt{(x+1)^2-1^2}}$$

Put, $x + 1 = y$ then $dx = dy$. Then,

$$I = \int \frac{dy}{\sqrt{y^2-1^2}}$$

$$= \log\left|y + \sqrt{y^2-1}\right| + C$$

$$= \log\left|x+1+\sqrt{(x+1)^2-1}\right| + C$$

$$= \log\left|x+1+\sqrt{x^2+2x}\right| + C.$$

8. $\displaystyle\int \left(\frac{(1-x)}{\sqrt{8+2x-x^2}}\right)dx$

Solution: Here,

$$I = \int \left(\frac{(1-x)}{\sqrt{8+2x-x^2}}\right)dx = \frac{1}{2}\int \left(\frac{(2-2x)}{\sqrt{8+2x-x^2}}\right)dx$$

Put $8 + 2x - x^2 = y$ then $(2-2x)\,dx = dy$. Then,

$$I = \frac{1}{2}\int \frac{dy}{\sqrt{y}}$$

$$= \frac{1}{2}\int y^{-1/2}\,dy = \frac{1}{2}\left(\frac{y^{1/2}}{\frac{1}{2}}\right) + C = \sqrt{8+2x-x^2} + C.$$

9. $\displaystyle\int \left(\frac{x}{\sqrt{x^2+4x+5}}\right)dx$ **[2002] [2009, Fall]**

Solution: Here,

$$I = \int \left(\frac{x}{\sqrt{x^2+4x+5}}\right)dx = \frac{1}{2}\int \left(\frac{2x+4-4}{\sqrt{x^2+4x+5}}\right)dx$$

$$= \int \left(\frac{2x+4}{\sqrt{x^2+4x+5}}\right)dx - \frac{4}{2}\int \frac{dx}{\sqrt{(x+2)^2+1}}$$

For first integral, put $x^2 + 4x + 5 = y$ then $(2x + 4)\,dx = dy$. Also, for second integral, put $x + 2 = \alpha$ then $dx = d\alpha$. Then,

$$I = \frac{1}{2}\int \frac{dy}{\sqrt{y}} - 2\int \frac{dy}{\sqrt{\alpha^2+1^2}}$$

$$= \frac{1}{2}\cdot\frac{y^{1/2}}{\frac{1}{2}} - 2\,.\log\left|\alpha+\sqrt{\alpha^2+1^2}\right| + C$$

$$= \sqrt{y} - 2\log\left|x+2+\sqrt{(x+2)^2+1}\right| + C$$

$$= \sqrt{x^2+4x+5} - 2\log\left|x+2+\sqrt{x^2+4x+5}\right| + C.$$

10. $\displaystyle\int \frac{dx}{(x+3)\sqrt{x^2+6x+10}}$

Solution: Here,

$$I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+10}}$$

Put $x + 3 = \frac{1}{y}$ then $dx = -\frac{1}{y^2}dy$. Then,

$$I = \int \frac{(-1)/y^2}{\frac{1}{y}\sqrt{\left(\frac{1-3y}{y}\right)^2 + 6\left(\frac{1-3y}{y}\right) + 10}}dy$$

$$= \int \frac{dy}{y\left(\frac{1}{y}\right)\sqrt{(1-3y)^2 + 6y(1-3y) + 10y^2}}$$

$$= -\int \frac{dy}{\sqrt{1 - 6y + 9y^2 + 6y - 18y^2 + 10y^2}}$$

$$= -\int \frac{dy}{\sqrt{y^2+1}} = -\log\left|y + \sqrt{y^2+1}\right| + C$$

$$= -\log\left|\frac{1}{x+3} + \sqrt{\left(\frac{1}{x+3}\right)^2 + 1}\right| + C$$

$$= -\log\left|\frac{1}{x+3} + \frac{1}{x+3}\sqrt{1 + (x+3)^2}\right| + C$$

$$= -\log\left|\frac{1 + \sqrt{x^2 + 6x + 10}}{x+3}\right| + C.$$

11. $\int \left(\frac{(2x+3)}{4x^2 + 4x + 5}\right)dx$

Solution: Here,

$$I = \int \left(\frac{(2x+3)}{4x^2+4x+5}\right)dx$$

$$= \frac{1}{4}\int \left(\frac{8x+4+8}{4x^2+4x+5}\right)dx$$

$$= \frac{1}{4}\int \left(\frac{8x+4}{4x^2+4x+5}\right)dx + \frac{8}{4}\int \frac{dx}{(2x)^2 + 2.2x.1 + 1 + 4}$$

$$= \frac{1}{4}\int \left(\frac{8x+4}{4x^2+4x+5}\right)dx + 2\int \frac{dx}{(2x+1)^2 + 2^2}$$

For first integral, put $4x^2 + 4x + 5 = y$ then $(8x + 4)\,dx = dy$. Also, for second integral, put $2x + 1 = \alpha$ then $2dx = d\alpha$. Then,

$$I = \frac{1}{4}\int \frac{dy}{y} + \int \frac{d\alpha}{\alpha^2 + 2^2}$$

$$= \frac{1}{4}\log|y| + \frac{1}{2}\tan^{-1}\left(\frac{\alpha}{2}\right) + C$$

$$= \frac{1}{4}\log|4x^2 + 4x + 5| + \frac{1}{2}\tan^{-1}\left(\frac{2x+1}{2}\right) + C.$$

12. $\int \left(\frac{x}{\sqrt{x^2 + 4x + 13}}\right)dx$

Solution: Here,

$$I = \int \left(\frac{x}{\sqrt{x^2+4x+13}}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{2x+4-4}{\sqrt{x^2+4x+13}}\right)dx$$

$$= \frac{1}{2}\int \left(\frac{2x+4}{\sqrt{x^2+4x+13}}\right)dx - \frac{4}{2}\int \frac{dx}{\sqrt{(x+2)^2+3^2}}$$

For first integral, put $x^2 + 4x + 13 = y$ then $(2x + 4)\,dx = dy$. Also, for second integral, put $x + 2 = \alpha$ then $dx = d\alpha$. Then,

$$I = \frac{1}{2}\int \frac{dy}{\sqrt{y}} - 2\int \frac{d\alpha}{\sqrt{\alpha^2+3^2}}$$

$$= \frac{1}{2}\cdot\frac{y^{1/2}}{1/2} - 2\log\left|\alpha + \sqrt{\alpha^2+3^2}\right| + C$$

$$= \sqrt{y} - 2\log\left|x + 2 + \sqrt{x^2+4x+13}\right| + C$$

$$= \sqrt{x^2+4x+13} - 2\log\left|x + 2 + \sqrt{x^2+4x+13}\right| + C.$$

13. $\int \left(\frac{2x^2 + 3x + 4}{x^2 + 6x + 10}\right)dx$

Solution: Here,

$$I = \int \left(\frac{2x^2+3x+4}{x^2+6x+10}\right)dx$$

$$= \int \left(\frac{2(x^2+6x+10) - 9x - 16}{x^2+6x+10}\right)dx$$

$$= 2\int dx - \int \left(\frac{9x+16}{x^2+6x+10}\right)dx$$

$$= 2\int dx - \int \left(\frac{\frac{9}{2}(2x+6) - 11}{x^2+6x+10}\right)dx$$

$$= 2\int dx - \frac{9}{2}\int \left(\frac{2x+6}{x^2+6x+10}\right)dx + 11\int \frac{dx}{(x+3)^2+1^2}$$

Put $x^2 + 6x + 10 = y$ then $(2x + 6)\,dx = dy$. Then,

$$\int \left(\frac{2x+6}{x^2+6x+10}\right)dx = \int \frac{dy}{y} = \log|y| + C_1$$

$$= \log|x^2 + 6x + 10| + C_1.$$

Therefore,

$$I = 2\int dx - \frac{9}{2}\log|x^2+6x+10| + 11\int \frac{dx}{(x+3)^2+1^2} + C_1$$

$$= 2x - \frac{9}{2}\log|x^2+6x+10| + 11\tan^{-1}(x+3) + C.$$

14. $\int \frac{dx}{\sqrt{3x^2+4x+5}}$

Solution: Here,

$$I = \int \frac{dx}{\sqrt{3x^2+4x+5}}$$

$$= \int \frac{dx}{\sqrt{3x^2 + 2.\frac{2}{\sqrt{3}}\sqrt{3}\,x + \frac{4}{3} - \frac{4}{3} + 5}}$$

$$= \int \frac{dx}{\sqrt{\left(\sqrt{3}\,x + \frac{2}{\sqrt{3}}\right)^2 - \left(\sqrt{\frac{11}{3}}\right)^2}}$$

Put $\sqrt{3}x + \frac{2}{\sqrt{3}} = y$ then $\sqrt{3}\,dx = dy$. Then,

$$I = \int \frac{dy}{\sqrt{3}.\sqrt{y^2 - \left(\sqrt{\frac{11}{3}}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \log \left| y + \sqrt{y^2 - \left(\frac{11}{3}\right)^2} \right| + C$$

$$= \frac{1}{\sqrt{3}} \log \left| \sqrt{3}\left(x + \frac{2}{3}\right) + \frac{1}{\sqrt{3}}\sqrt{3x^2+4x+5} \right| + C.$$

B. Evaluate the following integrals:

1. $\int \frac{1}{5-4\cos x}\,dx$

Solution: Here,

$$I = \int \frac{1}{5-4\cos x}\,dx$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy \Rightarrow dx = \frac{2dy}{1+y^2}$

Also,

$$\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-y^2}{1+y^2}$$

Then,

$$I = \int \frac{2dy/(1+y^2)}{5 - 4.\left(\frac{1-y^2}{1+y^2}\right)} = 2\int \frac{dy}{5+5y^2-4+4y^2}$$

$$= 2\int \frac{dy}{9y^2+1}$$

$$= \frac{2}{9}\int \frac{dy}{y^2+(1/3)^2}$$

$$= \frac{2}{9}\cdot\frac{1}{1/3}\tan^{-1}\left(\frac{y}{1/3}\right) + C$$

$$= \frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right) + C.$$

2. $\int \frac{dx}{2-3\sin 2x}$ [2014 Spring]

Solution: Here,

$$I = \int \frac{dx}{2-3\sin 2x}$$

Put, $\tan x = y$ then $\sec^2 x.dx = dy \Rightarrow dx = \frac{dy}{1+y^2}$

Also,

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x} = \frac{2y}{1+y^2}$$

Then,

$$I = \int \frac{dy/(1+y^2)}{2 - 3.\frac{2y}{1+y^2}} = \int \frac{dy}{2+2y^2-6y} = \frac{1}{2}\int \frac{dy}{y^2-3y+1}$$

$$\Rightarrow I = \frac{1}{2}\int \frac{dy}{\left(y-\frac{3}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

Put, $y - \frac{3}{2} = t$ then $dy = dt$. Then,

$$I = \frac{1}{2}\int \frac{dt}{t^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2.\frac{\sqrt{5}}{2}}\log\left|\frac{t-\frac{\sqrt{5}}{2}}{t+\frac{\sqrt{5}}{2}}\right| + C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{y-\frac{3}{2}-\frac{\sqrt{5}}{2}}{y-\frac{3}{2}+\frac{\sqrt{5}}{2}}\right| + C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{2\tan x-(3+\sqrt{5})}{2\tan x-(3-\sqrt{5})}\right| + C.$$

3. $\int \frac{dx}{4+5\sin x}$ [2015 Spring][2012 Fall][2008, Fall] [2009, Fall][1999][2001]

Solution: Here,

$$I = \int \frac{dx}{4 + 5\sin x}$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\sec^2\left(\frac{x}{2}\right) \cdot \frac{dx}{2} = dy \Rightarrow dx = \frac{2\,dy}{1 + y^2}$.

Also, $\sin x = \frac{2y}{1 + y^2}$.

Then,

$$I = \int \frac{2\,dy/(1 + y^2)}{4 + 5 \cdot \frac{2y}{1 + y^2}} = 2\int \frac{dy}{4y^2 + 10y + 4} = \frac{1}{2}\int \frac{dy}{y^2 + \frac{5}{2}y + 1}$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{dy}{\left(y + \frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

Put, $y + \frac{5}{4} = t$ then $dy = dt$. Then,

$$I = \frac{1}{2}\int \frac{dt}{t^2 - \left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2 \cdot \frac{3}{4}} \log \left| \frac{t - \frac{3}{4}}{t + \frac{3}{4}} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{y + \frac{5}{4} - \frac{3}{4}}{y + \frac{5}{4} + \frac{3}{4}} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{\tan(x/2) + 1}{2\,[\tan(x/2) + 2]} \right| + C.$$

4. $\displaystyle\int \frac{dx}{1 - \cos x + \sin x}$ **[2018 Spring][2013 Spring][2004, Spring]**

Solution: Here,

$$I = \int \frac{dx}{1 - \cos x + \sin x}$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\sec^2\left(\frac{x}{2}\right) \cdot \frac{dx}{2} = dy \Rightarrow dx = \frac{2\,dy}{1 + y^2}$

Also,

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \frac{1 - y^2}{1 + y^2} \text{ and } \sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} = \frac{2y}{1 + y^2}$$

Now,

$$I = \int \frac{2\,dy/(1 + y^2)}{1 - \frac{1 - y^2}{1 + y^2} + \frac{2y}{1 + y^2}}$$

$$= 2\int \frac{dy}{1 + y^2 - 1 + y^2 + 2y}$$

$$= 2\int \frac{dy}{2y^2 + 2y} = \frac{2}{2}\int \frac{dy}{y^2 + y} = \int \frac{dy}{\left(y + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$\Rightarrow \quad I = \frac{1}{2 \cdot \frac{1}{2}} \log \left| \frac{y + \frac{1}{2} - \frac{1}{2}}{y + \frac{1}{2} + \frac{1}{2}} \right| + C$$

$$= \log \left| \frac{y}{y + 1} \right| + C = \log \left| \frac{\tan\frac{x}{2}}{\tan\frac{x}{2} + 1} \right| + C$$

$$= -\log \left| \frac{\tan\frac{x}{2} + 1}{\tan\frac{x}{2}} \right| + C = -\log \left| 1 + \cot\frac{x}{2} \right| + C.$$

5. $\displaystyle\int \frac{dx}{4\sin x + 3\cos x + 13}$ **[2017 Spring]**

Solution: Here,

$$I = \int \frac{dx}{4\sin x + 3\cos x + 13}$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\sec^2\left(\frac{x}{2}\right) \cdot \frac{dx}{2} = dy \Rightarrow dx = \frac{2\,dy}{1 + y^2}$

Also,

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \frac{1 - y^2}{1 + y^2} \text{ and } \sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} = \frac{2y}{1 + y^2}$$

Now,

$$I = \int \frac{2\,dy/(1 + y^2)}{4\frac{2y}{1 + y^2} + 3\frac{1 - y^2}{1 + y^2} + 13}$$

$$= 2\int \frac{dy}{8y + 3 - y^2 + 13 + 13y^2}$$

$$= 2\int \frac{dy}{12y^2 + 8y + 16} = \frac{1}{6}\int \frac{dy}{y^2 + \frac{2}{3}y + \frac{4}{3}}$$

$$= \int \frac{dy}{(y + 1/3)^2 + (\sqrt{11}/3)^2}$$

$$\Rightarrow \quad I = \frac{3}{\sqrt{11}} \tan^{-1}\left(\frac{3y+1}{\sqrt{11}}\right) + C$$

$$\Rightarrow \quad I = \frac{3}{\sqrt{11}} \tan^{-1}\left(\frac{1}{\sqrt{11}}\left(3\tan\left(\frac{x}{2}\right)+1\right)\right) + C.$$

6. $\displaystyle\int \frac{dx}{5+4\cos x}$ **[2015 Fall][2008, Spring][2009 Spring] [2003, Spring]**

Solution: Here,

$$I = \int \left(\frac{1}{5+4\cos x}\right) dx$$

Put, $\tan\left(\frac{x}{2}\right) = y$ then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy \Rightarrow dx = \frac{2dy}{1+y^2}$

Also,

$$\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \frac{1-y^2}{1+y^2}$$

Then,

$$I = \int \frac{2dy/(1+y^2)}{5+4.\left(\frac{1-y^2}{1+y^2}\right)} = 2\int \frac{dy}{y^2+9}$$

$$= 2.\frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right) + C$$

$$= 2\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right) + C.$$

7. $\displaystyle\int \frac{dx}{3\sin x + 4\cos x}$

Solution: Here,

$$I = \int \frac{dx}{3\sin x + 4\cos x}$$

Put, $3 = r\cos\theta$, $4 = r\sin\theta$. Then, $r^2 = 3^2 + 4^2 = 25$. And, $\tan\theta = \frac{4}{3}$.

Now,

$$I = \int \frac{dx}{r(\sin x\cos\theta + \cos x\sin\theta)}$$

$$= \frac{1}{5}\int \frac{dx}{\sin(x+\theta)} = \frac{1}{5}\int \operatorname{cosec}(x+\theta)\,dx$$

Put, $x + \theta = y$ then $dx = dy$. So,

$$I = \frac{1}{5}\int \operatorname{cosec} y\,dy$$

$$= \frac{1}{5}\log\left|\tan\frac{y}{2}\right| + C = \frac{1}{5}\log\left|\tan\left(x+\frac{\theta}{2}\right)\right| + C$$

$$= \frac{1}{5}\log\left|\tan\left[\frac{x}{2}+\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right]\right| + C.$$

8. $\int \left(\frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\right) dx$ [2017 Fall]

Solution: Let

$$I = \int \left(\frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\right) dx \quad \text{... (i)}$$

Here,

$$\begin{aligned} 2\sin x + 3\cos x &= A.\ d(3\sin x + 4\cos x) + B(3\sin x + 4\cos x) \\ &= A(3\cos x - 4\sin x) + B(3\sin x + 4\cos x) \\ &= (-4A + 3B)\sin x + (3A + 4B)\cos x \end{aligned}$$

Equating the coefficient of sinx and cosx then,

$-A + 3B = 2$ and $3A + 4B = 3$

Solving we get,

$A = \frac{1}{25}$ and $B = \frac{18}{25}$

Then (i) becomes,

$$\begin{aligned} I &= \frac{1}{25}\int \frac{d(3\sin x + 4\cos x)}{3\sin x + 4\cos x} dx + \frac{18}{25}\int \frac{3\sin x + 4\cos x}{3\sin x + 4\cos x} dx \\ &= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}\int dx + C_1 \\ &= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}x + C_1 + C_2. \\ &= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}x + C. \end{aligned}$$

Thus, normally we use the rule ILATEC.

Generalization of the rule:

$\int f.\, g\, dx = f\, g - f_1 g'' + f_2 g''' - f_3 g^{iv} + f_4 g^{v} + \ldots$

where f is the first function and g is the second function. Also, the suffix denotes for derivative and 'prime of g (i.e. power of g)' denotes number of times of integration of g.

B. $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$

Exercise 11.3

Evaluate the integrals:

1. $\int x \sin x\, dx$

Solution: Here,

$I = \int x \sin x\, dx.$

Taking x as 1st function and sinx as 2nd function and we use the integration by parts then,

$$I = uv_1 - u'v_2 + u^2v_3 - \ldots + C$$
$$= -x \cos x + 1 . \sin x + C$$
$$= -x \cos x + \sin x + C.$$

2. $\int x^2 \sin x\, dx$

Solution: Here,

$I = \int x^2 \sin x\, dx$

Taking x^2 as 1st function and sinx as 2nd function and we use the integration by parts then,

$$I = uv_1 - u^1v_2 + u^2v_3 - u^3v_4 + \ldots + C$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

Q. Integrate: $\int x \sin^2 x\, dx$ **[2016 Spring Short]**

3. $\int x \log x\, dx$

Solution: Here,

$I = \int x \log x\, dx$

Taking log x as 1st function and x as 2nd function and we use the integration by parts then,

$$I = uv_1 - u'v_2 + u^2v_3 - \ldots + C$$
$$= \log x . \frac{x^2}{2} - \frac{1}{x} . \frac{x^3}{6} + C \;\; = \log x . \frac{x^2}{2} - \frac{x^2}{6} + C.$$

4. $\int x^n \log(ax)\, dx,\ n \neq -1$

Solution: Here,

$I = \int x^n \log(ax)\, dx$ for $n \neq -1$

Taking log (ax) as 1st function and x^n as 2nd function and we use the integration by parts then,

$$I = \log(ax) \int x^n\, dx - \int \left[\frac{d}{dx}(\log ax) \int x^n\, dx\right] dx$$

$$= \log(ax) . \left(\frac{x^{n+1}}{n+1}\right) - \int \frac{a}{ax} \left(\frac{x^{n+1}}{n+1}\right) dx + C_1$$
$$= \left(\frac{1}{n+1}\right) [x^{n+1} . \log(ax) - \int x^n\, dx] + C_1$$
$$= \left(\frac{1}{n+1}\right)\left[x^{n+1} . \log(ax) - \frac{x^{n+1}}{n+1}\right] + C$$
$$= \left(\frac{x^{n+1}}{n+1}\right)\left[\log(ax) - \frac{1}{n+1}\right] + C.$$

Q. **Integrate: $\int x^3 \log(x)\, dx$,** **[2018 Spring short]**

5. $\int \log(x)\, dx$

Solution: Here,

$I = \int \log x\, dx \;\; = \int 1 . \log x\, dx$

Taking log(x) as 1st function and 1 as 2nd function, and we use the integration by parts then,

$$I = \log(x) \int 1\, dx - \int \left[\frac{d}{dx}(\log x) \int 1\, dx\right] dx$$
$$= \log(x) . x - \int \frac{1}{x} . x\, dx + C_1$$
$$= x \log(x) - x + C$$
$$= x [\log(x) - 1] + C.$$

6. $\int x^5 e^x\, dx$ **[2017 Fall Short]**

Solution: Here,

$I = \int x^5 e^x\, dx$

Taking x^5 as 1st function and e^x as 2nd function and then applying integration by parts then,

$$I = uv_1 - u'v_2 + u^2v_3 - u^3v_4 + u^4v_5 - u^5v_6 + \ldots + C$$
$$= x^5.e^x - 5x^4e^x + 20x^3e^x - 60x^2e^x + 120xe^x - 120e^x + C$$
$$= e^x[x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120] + C.$$

7. $\int x^5 \sin x\, dx$

Solution: Here,

$I = \int x^5 \sin x\, dx$

Taking x^5 as 1st function and sin x as 2nd function and then applying integration by parts then,

$$I = uv_1 - u^1v_2 + u^2v_3 - u^3v_4 + u^4v_5 - u^5v_6 + \ldots + C$$
$$= -x^5 \cos x + 5x^4 \sin x + 20x^3 \cos x - 60x^2 \sin x - 120x \cos x + 120\sin x + C$$
$$= \cos x(-x^5 + 20x^3 - 120x) + \sin x(5x^4 - 6x^2 + 120) + C.$$

8. $\int \frac{(1 - x^2)^{3/2}}{x^6}\, dx$

Solution: Here,

$$I = \int \frac{(1-x^2)^{3/2}}{x^6}\,dx$$

Put $x = \sin\theta$ then $dx = \cos\theta\, d\theta$. Then,

$$I = \int \frac{\cos^3\theta}{\sin^6\theta}\cos\theta\, d\theta = \int \cot^4\theta\, \mathrm{cosec}^2\theta\, d\theta$$

Set $\cot\theta = y$ then $(-\mathrm{cosec}\theta)\, d\theta = dy$. Then,

$$I = -\int y^4\, dy$$

$$= -\frac{y^5}{5} + C$$

$$= -\frac{1}{5}\cot^5\theta + C$$

$$= -\frac{(1-x^2)^{3/2}}{x^6} + C.$$

$\sin\theta = x \Rightarrow \frac{p}{h} = x$
$\Rightarrow p = x,\ h = 1.$
Then $b = \sqrt{h^2 - p^2} = \sqrt{1-x^2}$
So, $\cot\theta = \frac{b}{p} = \frac{\sqrt{1-x^2}}{x}$

9. $\int \left(\frac{\log(x)}{x^2}\right) dx$ **for $x > 0$.**

Solution: Here,

$$I = \int \left(\frac{\log(x)}{x^2}\right) dx \qquad \text{for } x > 0.$$

Taking $\log(x)$ as 1st function and x^{-2} as 2nd function, and then applying integration by parts then,

$$I = \log(x)\int \frac{1}{x^2}\,dx - \int\left(\frac{d}{dx}\log x \int x^{-2}\,dx\right)dx$$

$$= -\log(x)\cdot\frac{1}{x} + \int \frac{1}{x}\cdot(x^{-1})\,dx + C_1$$

$$= -\frac{\log x}{x} - x^{-1} + C$$

$$= -\frac{1}{x}(\log x + 1) + C.$$

10. $\int \sin^{-1}x\, dx$

Solution: Here,

$$I = \int \sin^{-1}x\, dx = \int 1.\sin^{-1}x\, dx$$

Taking $\sin^{-1}x$ as 1st function and 1 as 2nd function and then applying integration by parts then,

$$I = \sin^{-1}x\int 1\, dx - \int\left(\frac{d}{dx}\sin^{-1}x\int dx\right)dx$$

$$= \sin^{-1}x.\,x - \int \frac{1}{\sqrt{1-x^2}}\cdot x\, dx + C_1$$

$$= x\sin^{-1}x + \frac{1}{2}\int\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx + C_1$$

Put, $1 - x^2 = y^2$ then $(-x)dx = y\,dy$. Then,

$$\int\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx = \int \frac{2y\,dy}{y}$$

$$= 2\int dy = 2y + C_2 = 2\sqrt{1-x^2} + C_2$$

Therefore,

$$I = x\sin^{-1}x + \frac{1}{2}\cdot 2\sqrt{1-x^2} + C_1 + C_2$$

$$= x.\sin^{-1}x + \sqrt{1-x^2} + C \qquad \text{,for } C_1 + C_2 = C.$$

Q. **Integrate: $\int \tan^{-1}x\, dx$** **[2016 Fall Short]**

11. $\int x \sin x \sin(2x)\sin(3x)\, dx$

Solution: Here,

$$I = \int x\sin x\sin(2x)\sin(3x)\,dx$$

Since, $2\sin(2x)\sin(3x) = \cos(2x - 3x) - \cos(2x + 3x)$

$$= \cos x - \cos(5x) \qquad [\because \cos(-\theta) = \cos\theta]$$

Then,

$$I = \frac{1}{2}\int x\sin x\,[\cos x - \cos(5x)]\,dx$$

$$= \frac{1}{4}\int x\sin(2x)\,dx - \frac{1}{2}\int x\sin x\cos(5x)\,dx$$

Also, $2\sin x\cos(5x) = \sin(x + 5x) + \sin(x - 5x)$

$$= \sin(6x) - \sin(4x)$$

Therefore,

$$I = \frac{1}{4}\int x\sin(2x)\,dx - \frac{1}{4}\int x\sin(6x)\,dx + \frac{1}{4}\int x\sin(4x)\,dx.$$

Taking x as 1st function and sine as 2nd function and then applying integration by parts then,

$$I = \frac{1}{4}\left[x\int \sin 2x\,dx - \int\left(\frac{d}{dx}x\int \sin 2x\,dx\right)dx\right] +$$

$$\frac{1}{4}\left[x\int \sin 6x\,dx - \int\left(\frac{d}{dx}x\int \sin 6x\,dx\right)dx\right] +$$

$$\frac{1}{4}\left[x\int \sin 4x\,dx - \int\left(\frac{d}{dx}x\int \sin 4x\,dx\right)dx\right]$$

$$= \frac{1}{4}\left[-x\frac{\cos 2x}{2} + \int\frac{\cos 2x}{2}\,dx\right] - \frac{1}{4}\left[-x\frac{\cos 6x}{6} + \int\frac{\cos 6x}{6}\,dx\right] + \frac{1}{4}$$

$$\left[-x\frac{\cos 4x}{4} + \int\frac{\cos 4x}{4}\,dx\right]$$

$$=\frac{1}{4}\left[-\frac{x\cos(2x)}{2}+\frac{\sin(2x)}{4}+\frac{x\cos(6x)}{6}-\frac{\sin(6x)}{36}-\frac{x\cos(4x)}{4}+\frac{\sin(4x)}{16}\right]+C$$

$$=-\frac{x}{8}\left[\cos(2x)+\frac{\cos(4x)}{2}-\frac{\cos(6x)}{3}\right]+\frac{1}{16}\left[\sin(2x)+\frac{\sin(4x)}{4}-\frac{\sin(6x)}{9}\right]+C.$$

12. $\int\left(\frac{x+\sin x}{1+\cos x}\right)dx$ **[2003, Fall] [2002]**

Solution: Here,

$$I=\int\left(\frac{x+\sin x}{1+\cos x}\right)dx=\int\frac{x+2\sin(x/2).\cos(x/2)}{2\cos^2(x/2)}dx$$

$$=\frac{1}{2}\int x\sec^2\left(\frac{x}{2}\right)dx+\frac{2}{2}\int\frac{\sin(x/2).\cos(x/2)}{\cos^2(x/2)}dx$$

$$=\frac{1}{2}\int x\sec^2\left(\frac{x}{2}\right)dx+\int\tan\left(\frac{x}{2}\right)dx.$$

Taking x as 1st function and $\sec^2\frac{x}{2}$ as 2nd function and then applying integration by parts then,

$$I=\frac{1}{2}\left[x\int\sec^2\frac{x}{2}dx-\int\left(\frac{d}{dx}x\int\sec^2\frac{x}{2}dx\right)dx\right]^2+\log|\sec(x/2)|+C_1$$

$$=\frac{1}{2}\left[x.\frac{\tan(x/2)}{(1/2)}-2\int\tan(x/2).dx\right]^2+\log|\sec(x/2)|+C_1$$

$$=x\tan\left(\frac{x}{2}\right)-2\log|\sec(x/2)|+2\log|\sec(x/2)|+C$$

$$=x.\tan\left(\frac{x}{2}\right)+C.$$

13. $\int e^x(\tan x-\log\cos x)\,dx$

Solution: Here,

$$I=\int e^x(\tan x-\log\cos x)\,dx$$

Put $-\log(\cos x)=f(x)$ then $\frac{\sin x}{\cos x}dx=f'(x)\,dx\Rightarrow\tan x\,dx=f'(x)\,dx.$

Therefore,

$$I=\int e^x[f(x)+f'(x)]\,dx=e^x f(x)+C$$

$$=e^x(-\log\cos x)+C$$

$$=e^x.\log\sec x+C.$$

14. $\int\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$

Solution: Here,

$$I=\int\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}dx$$

Put $\tan^{-1}x=y$ then $\frac{1}{1+x^2}dx=dy$. Then,

$$I=\int\frac{\tan y.y}{\sqrt{1+\tan^2 y}}dy$$

$$=\int\frac{\tan y.y}{\sec y}dy$$

$$=\int y\sin y\,dy$$

$$=-y\cos y+\sin y+C \quad [\because \text{ using Q. 1}]$$

$$=-\frac{\tan^{-1}x}{\sqrt{1+x^2}}+\frac{x}{\sqrt{1+x^2}}+C=\frac{1}{\sqrt{1+x^2}}(x-\tan^{-1}x)+C.$$

15. $\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Solution: Here,

$$I=\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Put, $x=\tan y$ then $dx=\sec^2y\,dy$, then,

$$\frac{2x}{1+x^2}=\frac{2\tan y}{1+\tan^2 y}=\frac{2\tan y}{\sec^2 y}=2\sin y\cos y=\sin 2y$$

Now,

$$I=\int\sin^{-1}(\sin 2y).\sec^2y\,dy$$

$$=2\int y.\sec^2y\,dy$$

Taking y as 1st function and $\sec^2 y$ as 2nd function, and then applying integration by parts then,

$$I=2[uv_1-u'v_2]+C$$

$$=2[y\tan y-\log\sec y]+C=2y\tan y+2\log\cos y+C$$

$$=2x.\tan^{-1}x+2\log\left(\frac{1}{\sqrt{1+x^2}}\right)+C$$

$$=2x\tan^{-1}x-\log(\sqrt{1+x^2})^2+C$$

$$=2x\tan^{-1}x-\log(1+x^2)+C.$$

16. $\int x\cos^3 x\sin x\,dx$

Solution: Here,

$$I=\int x\cos^3 x\sin x\,dx$$

$$=\int x\left(\frac{\cos 3x+3\cos x}{4}\right)\sin x\,dx$$

$$=\frac{1}{4}\int x\cos 3x\sin x\,dx+\frac{3}{4}\int x\cos x\sin x\,dx$$

$$= \frac{1}{8}\int x\,[\sin(3x + x) - \sin(3x - x)]\,dx + \frac{3}{8}\int x\sin(2x)\,dx$$

$$= \frac{1}{8}\int x\sin(4x)\,dx - \frac{1}{8}\int x\sin(2x)\,dx + \frac{3}{8}\int x\sin(2x)\,dx$$

$$= \frac{1}{8}\int x\sin(4x) + \frac{2}{8}\int x\sin(2x)\,dx$$

Taking x as 1st function and sine function as 2nd function, and then applying integration by parts then,

$$= \frac{1}{8}\left[x.\int \sin 4x\,dx - \int\left(\frac{d}{dx}x\int \sin 4x\,dx\right)dx\right] +$$

$$\frac{2}{8}\left[x\int \sin 2x\,dx - \int\left(\frac{d}{dx}x\int \sin 2x\,dx\right)dx\right]$$

$$= \frac{1}{8}\left[\frac{-x\cos 4x}{4} + \frac{\sin 4x}{16} - x\cos 2x + \frac{\sin 2x}{2}\right] + C.$$

17. $\int \cos(\log x)\,dx$

Solution: Let, $I = \int \cos(\log x)\,dx$

Put $\log x = y$ then $\frac{1}{x}dx = dy \Rightarrow dx = e^y\,dy$. Then,

$$I = \int \cos y \,.\, e^y\,dy$$

Taking cos y as 1st function and e^y as 2nd function, and then applying integration by parts then,

$$I = \cos y\int e^y\,dy - \int\left(\frac{d}{dy}\cos y\int e^y\,dy\right)dy$$

$$= \cos y \,.\, e^y + \int \sin y\, e^y\,dy + C_1$$

$$= e^y\cos y + \sin y\int e^y\,dy - \int\left(\frac{d}{dy}\sin y\int e^y\,dy\right)dy + C_1$$

$$= e^y\cos y + \sin y\, e^y - \int \cos y \,.\, e^y\,dy + C_2$$

$$= e^y(\cos y + \sin y) + C_2 - I$$

$$\Rightarrow I = \frac{1}{2}e^y(\cos y + \sin y) + C$$

$$= \frac{1}{2}x\,[\cos(\log x) + \sin(\log x)] + C.$$

18. $\int e^x \sin x\,dx$

Solution: Let,

$$I = \int \sin x\, e^x\,dx$$

$$= e^x\sin x - \int\left(\frac{d}{dx}\sin x\int e^x\,dx\right)dx + C_1$$

$$= e^x\sin x - \int \cos x \,.\, e^x\,dx + C_1$$

$$= e^x\sin x - \cos x \,.\, e^x - \int \sin x \,.\, e^x\,dx + C_2$$

$$= e^x(\sin x - \cos x) + C_2 - I$$

$$\Rightarrow I = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

19. $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Solution: Here,

$$I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

Put $x = \cos\theta$ then $dx = -\sin\theta\,d\theta$.

Since,

$$\frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}$$

So that,

$$I = \int \tan^{-1}\sqrt{\tan^2\frac{\theta}{2}}\,.\,(-\sin\theta)\,d\theta$$

$$= -\int \tan^{-1}\left(\tan\frac{\theta}{2}\right).\sin\theta\,d\theta = -\frac{1}{2}\int \theta\sin\theta\,d\theta.$$

And then process as Q. 1.

20. $\int \sin\sqrt{x}\,dx$

Solution: Here,

$$I = \int \sin\sqrt{x}\,dx$$

Put $\sqrt{x} = y$ then $\frac{1}{2\sqrt{x}}dx = dy \Rightarrow dx = 2y\,dy$. Then,

$$I = 2\int y\sin y\,dy.$$

And then process a Q.1.

Exercise 11.4

Integrate the following functions w.r.t. x:

1. (i) $\dfrac{x}{(x-3)(x+1)}$ **[2018 Fall (short)][2015 Fall (short)]**

Solution: Here,

$$\frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$\Rightarrow \quad x = A(x+1) + B(x-3) = (A+B)x + (A-3B)$$

Equating the coefficient of x and the constant term then,

$$A + B = 1, \quad A - 3B = 0$$

Solving we get, $A = \frac{3}{4}, B = \frac{1}{4}$

Now,

$$\int \frac{x}{(x-3)(x+1)} dx = \frac{3}{4}\int \frac{dx}{x-3} + \frac{1}{4}\int \frac{dx}{x+1}$$

$$= \frac{3}{4}\log|x-3| + \frac{1}{4}\log|x+1| + C$$

$$= \frac{1}{4}\log|(x-3)^3| + \frac{1}{4}\log|x+1| + C$$

$$= \frac{1}{4}\log|(x+1)(x-3)^3| + C.$$

(ii) $\dfrac{5x-3}{(x+1)(x-3)}$

Solution: Here,

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$\Rightarrow \quad 5x - 3 = A(x+3) + B(x+1)$$

$$= (A+B)x + (B-3A)$$

Equating the coefficient of x and the constant term then,

$$A + B = 5, \quad B - 3A = -3$$

Solving we get, $A = 2$ and $B = 3$

Now,

$$\int \frac{5x-3}{(x+1)(x-3)} dx = 2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3}$$

$$= 2\log|x+1| + 3\log|x-3| + C.$$

(iii) $\dfrac{(x-1)(x-2)}{(x+3)(x+4)(x+5)}$

Solution: Here,

$$\frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)} = \frac{A}{x+3} + \frac{B}{x+4} + \frac{C}{x+5}$$

$\Rightarrow (x-1)(x-2) = A(x+4)(x+5) + B(x+3)(x+5) + C(x+3)(x+4)$

$\Rightarrow x^2 - 3x + 2 = (A + B + C)x^2 + (9a + 8B + 7C)x + (20A + 15BG + 12C)$

Equating the coefficient of x^2, x and the constant term then,

$A + B + C = 1$, $9A + 8B + 7C = -3$, $20A + 15B + 12C = 2$

Solving we get, $A = 10$, $B = -30$, $C = 21$.

Now,

$$\int \frac{(x-1)(x-2)}{(x+3)(x+4)(x+5)} dx = 10\int \frac{dx}{x+3} - 30\int \frac{dx}{x+4} + 21\int \frac{dx}{x+5}$$

$$= 10\log|x+3| - 30\log|x+4| + 21\log|x+5 + C.$$

(iv) $\dfrac{x+1}{(x^2 - 13x + 42)}$

Solution: Here,

$$\frac{x+1}{(x^2-13x+42)} = \frac{x+12}{x^2 - 7x - 6x + 42} = \frac{x+12}{(x-7)(x-6)}$$

$$\Rightarrow \frac{x+12}{(x-6)(x-7)} = \frac{A}{x-6} + \frac{B}{x-7}$$

$\Rightarrow x + 12 = A(x-7) + B(x-6) = (A+B)x + (-7A - 6B)$

Equating the coefficient of x and the constant term then,

$A + B = 1$ and $-7A - 6B = 12$

Solving we get, $A = -18$ and $B = +19$

Now,

$$\int \frac{x+1}{(x^2 - 13x + 42)} dx = -18\int \frac{dx}{x-6} + 19\int \frac{dx}{x-7}$$

$$= -18\log|x-6| + 19\log|x-7| + C.$$

2. (i) $\dfrac{x+4}{(x+1)^2}$

Solution: Here,

$$\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$\Rightarrow x + 4 = A(x+1) + B = Ax + (A + B)$

Equating the coefficient of x and the constant term then,

$A = 1$, $A + B = 4$

Solving we get, $A = 1$, $B = 3$

Now,

$$\int \frac{x+4}{(x+1)^2} dx = \int \frac{dx}{x+1} + 3\int \frac{dx}{(x+1)^2}$$

$$= \int \frac{dx}{x+1} + 3\int (x+1)^{-2} dx$$

$$= \log|x+1| + 3\left(\frac{-1}{x+1}\right) + C = \log|x+1| - \frac{3}{x+1} + C$$

(ii) $\dfrac{x^2-4}{(x^2+1)(x^2+3)}$

Solution: Here,

$$\frac{x^2-4}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

$\Rightarrow x^2 - 4 = (Ax + B)(x^2 + 3) + (Cx + D)(x^2 + 1)$

$= (A + C)x^3 + (B + D)x^2 + (3A + C)x + (3B + D).$

Equating the coefficient of x^3, x^2, x and the constant term then,

$A + C = 0$, $3A + C = 0$, $B + D = 1$, $3B + D = -4$

Solving we get,

$$A = 0 = C, B = -\frac{5}{2}, D = \frac{7}{2}$$

Now,

$$\int \frac{x^2-4}{(x^2+1)(x^2+3)} dx = -\frac{5}{2}\int \frac{dx}{x^2+1} + \frac{7}{2}\int \frac{dx}{x^2+3}$$

$$= -\frac{5}{2}\tan^{-1}(x) + \frac{7}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

$$= -\frac{5}{2}\tan^{-1}(x) + \frac{7}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c.$$

(iii) $\dfrac{x^2}{(x-a)(x-b)}$

Solution: Here,

$\dfrac{x^2}{(x-a)(x-b)}$, which is improper quotient form. That is the variable has same power in numerator as in denominator. So,

$$\frac{x^2}{(x-a)(x-b)} = \frac{x^2 - x(a+b) + ab}{x^2 - x(a+b) + ab} + \frac{x(a+b) - ab}{(x-a)(x-b)}$$

$$= 1 + \frac{x(a+b) - ab}{(x-a)(x-b)}$$

Here,

$$\frac{x(a+b) - ab}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$\Rightarrow x(a+b) - ab = A(x-b) + B(x-a)$

$= (A + B)x + (-Ab - aB)$

Equating the coefficient of x and the constant term then,

$A + B = a + b$, $Ab + Ba = ab$

Solving we get,

$$A = \frac{a^2}{a-b}, \qquad B = -\frac{b^2}{a-b}$$

Now,

$$\int \frac{x^2}{(x-a)(x-b)}dx = \int dx + \frac{a^2}{a-b}\int \frac{dx}{x-a} - \frac{b^2}{a-b}\int \frac{dx}{x-b}$$

$$= x + \frac{a^2}{a-b}|x-a| - \frac{b^2}{a-b}\log|x-b| + C.$$

(iv) $\dfrac{2x}{(x^2+1)(x^2+3)}$

Solution: Here,

$$\frac{2x}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

$$\Rightarrow 2x = (Ax+B)(x^2+3) + (Cx+D)(x^2+1)$$

$$= (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D).$$

Equating the coefficient of x^3, x^2, x and the constant term then,

$A + C = 0$, $3A + C = 2$, $B + D = 0$ and $3B + D = 0$

Solving we get, $B = 0 = D$, $A = 1$ and $C = -1$.

Now,

$$\int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{x^2+3}$$

$$= \frac{1}{2}\int \frac{2x\,dx}{x^2+1} - \frac{1}{2}\int \frac{2x\,dx}{x^2+3}$$

$$= \frac{1}{2}(\log|x^2+1| - \log|x^2+3|) + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C.$$

3. (i) $\dfrac{1}{1+3e^x+2e^{2x}}$

Solution: Here,

$$I = \int \frac{dx}{1+3e^x+2e^{2x}}$$

Put, $e^x = y$ then $e^x\,dx = dy \Rightarrow dx = \dfrac{dy}{y}$. Then,

$$I = \int \frac{dy}{y(1+3y+2y^2)} = \int \frac{dy}{y(y+1)(2y+1)} \qquad \dots (1)$$

Here,

$$\frac{1}{y(y+1)(2y+1)} = \frac{A}{y} + \frac{B}{y+1} + \frac{C}{2y+1}$$

$$\Rightarrow 1 = A(y+1)(2y+1) + By(2y+1) + C(y+1)y$$

$$= (2A+2B+C)y^2 + (3A+B+C)y + A$$

Equating the coefficient of y^2, y and the constant term then,

$2A + 2B + C = 0$, $3A + B + C = 0$ and $A = 1$.

Solving we get, $A = 1$, $B = 1$ and $C = -4$.

Now, (i) becomes

$$I = \int \frac{dy}{y} + \int \frac{dy}{y+1} - 4\int \frac{dy}{2y+1}$$

$$= \int \frac{dy}{y} + \int \frac{dy}{y+1} - 2\int \frac{2dy}{2y+1}$$

$$= \log|y| + \log|y+1| - 2\log|2y+1| + C$$

$$= \log(e^x) + \log(e^x+1) - 2\log(2e^x+1) + C$$

$$= x + \log(e^x+1) - 2\log(1+2e^x) + C.$$

(ii) $\dfrac{x}{(x-1)^2(x+2)}$

Solution: Here,

$$I = \int \frac{x}{(x-1)^2(x+2)}dx$$

Set,

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$= (A+C)x^2 + (A+B-2C)x + (-2A+2B+C)$$

Equating the coefficient of x^2, x and the constant term then,

$A + C = 0$, $A + B - 2C = 1$, $-2A + 2B + C = 0$.

Solving we get,

$$A = \frac{2}{9}, \quad B = \frac{1}{3}, \quad C = -\frac{2}{9}$$

Now,

$$I = \frac{2}{9}\int \frac{dy}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2}$$

$$= \frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int (x-1)^{-2}dx - \frac{2}{9}\int \frac{dx}{x+2}$$

$$= \frac{2}{9}\log(x-1) - \frac{1}{3(x-1)} - \frac{2}{9}\log(x+2) + c$$

$$\Rightarrow I = \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c.$$

(iii) $\dfrac{x^2+8}{x^2-5x+6}$

Solution: Here,

$$\frac{x^2+8}{x^2-5x+6}$$

This is the improper fraction. So, divide the term at first. So that,

$$\frac{x^2+8}{x^2-5x+6} = \frac{x^2-5x+6}{x^2-5x+6} + \frac{5x+2}{x^2-5x+6}$$
$$= 1 + \frac{5x+2}{(x-2)(x-3)}$$

Here,
$$\frac{5x+2}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$
$$\Rightarrow \quad 5x + 2 = A(x-3) + B(x-2) = (A+B)x - (3A+2B)$$

Equating the coefficient of x and the constant term then,

$A + B = 5$ and $-(3A + 2B) = 2$

Solving we get, $A = -12$ and $B = 17$

Now,
$$\int \frac{x^2+8}{x^2-5x+6}dx = \int dx - 12\int \frac{dx}{x-2} + 17\int \frac{dx}{x-3}$$
$$= x - 12\log|x-2| + 17\log|x-3| + c.$$

(iv) $\dfrac{1}{x(x^2+x+1)}$

Solution: Here,
$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$
$$\Rightarrow \quad 1 = A(x^2+x+1) + (Bx+C)x = (A+B)x^2 + (A+C)x + A$$

Equating the coefficient of x^2, x and the constant term then,

$A = 1, \quad A + B = 0, \quad A + C = 0$

Solving we get, $A = 1, B = C = -1$

Now,
$$\int \frac{dx}{x(x^2+x+1)} = \int \frac{dx}{x} + \int \frac{-x-1}{x^2+x+1}dx$$
$$= \int \frac{dx}{x} - \int \frac{\frac{1}{2}(2x+1) + \frac{1}{2}}{x^2+x+1}dx$$
$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{d\,x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$
$$= \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \left(\frac{1}{2}\right)\log\left|\frac{x^2}{x^2+x+1}\right| - \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{2}}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$
$$= \left(\frac{1}{2}\right)\log\left|\frac{x^2}{x^2+x+1}\right| - \frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

4. (i) $\dfrac{x^2}{(x^2+1)(x^2+4)}$.

Solution: Here,
$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$
$$\Rightarrow \quad x^2 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$
$$= (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D).$$

Equating the coefficient of x^3, x^2, x and the constant term then,

$A + C = 0, \quad 4A + C = 0, \quad B + D = 1$ and $4B + D = 0$

Solving we get, $A = 0 = B, \quad B = \frac{-1}{3}$ and $D = \frac{4}{3}$.

Now,
$$\int \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{-1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$
$$= \frac{-1}{3}\tan^{-1}(x) + \frac{4}{3}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$
$$= \frac{-1}{3}\tan^{-1}(x) + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) + c.$$
$$= \frac{1}{3}\left(2\tan^{-1}\left(\frac{x}{2}\right) - \tan^{-1}(x)\right) + c.$$

(ii) $\dfrac{1}{x[6(\log x)^2 + 7\log(x) + 2]}$

Solution: Here,
$$I = \int \frac{1}{x[6(\log x)^2 + 7\log(x) + 2]}dx$$

Put $\log x = y$ then $\frac{1}{x}dx = dy$. Then,
$$I = \int \frac{dy}{6y^2+7y+2} = \int \frac{dy}{6y^2+3y+4y+2}$$
$$\Rightarrow \quad I = \int \frac{dy}{(3y+2)(2y+1)}$$

Here,
$$\frac{1}{(3y+2)(2y+1)} = \frac{A}{3y+2} + \frac{B}{2y+1}$$
$$\Rightarrow \quad 1 = A(2y+1) + B(3y+2) = (2A+3B)y + (A+2B)$$

Comparing the coefficient of y and the constant term then,

$2A + 3B = 0, \quad A + 2B = 1$

Solving we get, $A = -3$ and $B = +2$

Therefore,
$$I = -3\int \frac{dy}{3y+2} + 2\int \frac{dy}{2y+1}$$

$$= -\log|3y+2| + 2\log|2y+1| + c$$
$$= \log\left|\frac{2y+1}{3y+2}\right| + c$$
$$= \log\left|\frac{2\log x+1}{3\log x+2}\right| + c.$$

(iii) $\dfrac{\cos x}{(\sin x+2)(2\sin x+3)}$

Solution: Here,

$$I = \int \frac{\cos x}{(\sin x+2)(2\sin x+3)}\,dx$$

Put $\sin x = y$ then $\cos x\,dx = dy$. So that,

$$I = \int \frac{dy}{(y+2)(2y+3)}$$

Here,

$$\frac{1}{(y+2)(2y+3)} = \frac{A}{y+2} + \frac{B}{2y+3}$$
$$\Rightarrow\ 1 = A(2y+3) + B(y+2) = (2A+B)\,y + (3A+2B)$$

Equating the coefficient of y and the constant term then

$$2A + B = 0,\quad 3A + 2B = 1$$

Solving we get, $A = -1$ and $B = 2$

Then,

$$I = -\int \frac{dy}{y+2} + \int \frac{2\,dy}{2y+3}$$
$$= -\log|y+2| + \log|2y+3| + c$$
$$= \log\left|\frac{2y+3}{y+2}\right| + c$$
$$= \log\left|\frac{2\sin x+3}{\sin x+2}\right| + c.$$

(iv) $\dfrac{x^3-5x}{(x^2-9)(x^2+1)}$

Solution: Let,

$$I = \int \frac{x^3-5x}{(x^2-9)(x^2+1)}\,dx$$

Here,

$$\frac{x^3-5x}{(x^2-9)(x^2+1)} = \frac{Ax+B}{x^2-9} + \frac{Cx+D}{x^2+1}$$
$$\Rightarrow\ x^3-5x = (Ax+B)(x^2+1) + (Cx+D)(x_2-9)$$
$$= (A+C)x^3 + (B-9D)x^2 + (A-9C)x + (B-9D).$$

Equating the coefficient of x^3, x^2, x and the constant term from sides then,

$$A + C = 1,\quad B + D = 0$$

$$A - 9C = -5,\quad B - 9D = 0$$

Solving we get,

$$A = \frac{2}{5},\ C = \frac{3}{5},\ B = 0,\ D = 0$$

Thus,

$$I = \frac{2}{5}\int \frac{x}{x^2-9}\,dx\,dx + \frac{3}{5}\int \frac{x}{x^2+1}\,dx$$
$$= \frac{2}{5}\cdot\frac{1}{2}\log|x^2-9| + \frac{3}{5}\cdot\frac{1}{2}\log|x^2+1| + c$$
$$= \frac{1}{5}\log|x^2-9| + \frac{3}{10}\log|x^2+1| + c.$$

(v) $\dfrac{1}{(e^x-1)^2}$

Solution: Here, $I = \displaystyle\int \frac{dx}{(e^x-1)^2}$

Put $e^x = y$ then $e^x\,dx = dy \Rightarrow dx = \dfrac{dy}{y}$. Then,

$$I = \int \frac{dy}{y(y-1)^2}$$

Here,

$$\frac{1}{y(y-1)^2} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{(y-1)^2}$$
$$\Rightarrow\ 1 = A(y-1)^2 + By(y-1) + Cy.$$
$$= (A+B)\,y^2 + (-2A-B+C)\,y + A$$

Equating the coefficient of y^2, y and the constant term then,

$$A + B = 0,\quad -2A - B + C = 0,\quad A = 1.$$

Solving we get, $A = 1$, $B = -1$ and $C = 1$

Therefore,

$$I = \int \frac{dy}{y} - \int \frac{dy}{y-1} + \int \frac{dy}{(y-1)^2}$$
$$= \log|y| - \log|y-1| - (y-1)^{-1} + c$$
$$= \log\left|\frac{y}{y-1}\right| - \left|\frac{1}{y-1}\right| + c$$
$$= \log\left|\frac{e^x}{e^x-1}\right| - \frac{1}{e^x-1} + c.$$

(vi) $\dfrac{7x^2+3x+1}{x^2+x}$

Solution: Here,

$$I = \int \left(\frac{7x^2+3x+1}{x^2+x}\right) dx$$

$$= \int \left(7 + \frac{-4x + 1}{x^2 + x}\right) dx$$

$$= 7 \int dx - \int \left(\frac{4x + 2 - 2 - 1}{x^2 + x}\right) dx$$

$$= 7 \int dx - 2 \int \left(\frac{2x + 1}{x^2 + x}\right) dx + 3 \int \frac{dx}{x(x + 1)}$$

Here,

$$\frac{1}{x(x + 1)} = \frac{A}{x} + \frac{B}{x + 1}$$

$$\Rightarrow \quad 1 = A(x + 1) + Bx = (A + B)\, x + A$$

Equating the coefficient of x and the constant term then,

$$A + B = 0,\ A = 1$$

So that, $A = 1, B = -1.$

Therefore,

$$I \quad = 7 \int dx - 2 \int \frac{2x + 1}{x^2 + x} dx + 3 \int \frac{dx}{x} - 3 \int \frac{dx}{x + 1}$$

$$= 7x - 2 \log |x^2 + x| + 3 \log |x| - 3 \log |x + 1| + c.$$

Exercise 11.5

Show that:

1. $\displaystyle\int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\sin\theta + \cos\theta} = \frac{\pi}{4}$ **[2013 Fall Short]**

Solution: Here,

$$I = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\sin\theta + \cos\theta} \quad \ldots \text{(i)}$$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)} d\theta \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta+\sin\theta} d\theta \quad \ldots \text{(ii)}$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\sin\theta+\cos\theta} + \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta+\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta} d\theta = \int_0^{\pi/2} d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2}.$$

Thus, $I = \frac{\pi}{4}$.

2. $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{4}$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx \quad \ldots \text{(i)}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} \quad \ldots \text{(ii)}$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Thus, $I = \frac{\pi}{4}$.

3. $\displaystyle\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$ [2017 Spring][2009 Spring] [2003, Spring] [2007, Fall]

Solution: Here,

$$I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

Put $x = a\sin\theta$ then $dx = a\cos\theta\, d\theta$.

When $x = 0 \Rightarrow \theta = 0$, and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_0^{\pi/2} \frac{a\cos\theta}{a\sin\theta + a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta \quad \ldots (i)$$

$$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-\theta\right) d\theta}{\sin\left(\frac{\pi}{2}-\theta\right) + \cos\left(\frac{\pi}{2}-\theta\right)} \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta \quad \ldots (ii)$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta} d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

Thus, $I = \frac{\pi}{4}$.

4. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cot x}\,dx}{1+\sqrt{\cot x}} = \frac{\pi}{4}$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{\sqrt{\cot x}\,dx}{1+\sqrt{\cot x}} dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

Same to Q. No. 2 equation (i).

Thus from the Q. No 2, we have

$$I = \frac{\pi}{4}.$$

5. $\displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$

Solution: Here,

$$I = \int_0^{\pi} \frac{x\tan x}{\sec x + \cos x} dx$$

$$= \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx \quad \ldots (i)$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx \quad \ldots (ii)$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x} dx$$

Put, $\cos x = y$ then $-\sin x\, dx = dy$.

When $x = 0 \Rightarrow y = 1$ and $x = \pi \Rightarrow y = -1$. Then,

$$2I = \pi \int_1^{-1} \frac{-dy}{1+y^2}$$

$$= \pi \int_{-1}^{1} \frac{dy}{1+y^2} = \pi\left[\tan^{-1} y\right]_{-1}^{1}$$

$$= \pi\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= \pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi^2}{2}.$$

$$\Rightarrow I = \frac{\pi^2}{4}.$$

6. $\displaystyle\int_0^a \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{a-x}} = \frac{a}{2}$ [2012 Fall]

Solution: Here,

$$I = \int_0^a \frac{\sqrt{x}\,dx}{\sqrt{x}+\sqrt{a-x}} \qquad \ldots \text{(i)}$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\,dx \quad \ldots \text{(ii)} \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

Adding (i) and (ii) then,

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\,dx = \int_0^a dx = [x]_0^a = a.$$

$$\Rightarrow I = \frac{a}{2}.$$

7. $\displaystyle\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log 2$ **[2002] [2003, Spring] [2004, Fall]**

[2011 Spring][2011 Fall][2004, Spring][2006, Fall][2008, Spring] [2016 Spring][2016 Fall][2014 Fall][2013 Fall]

Solution: Here,

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/4} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan\theta}{\tan\frac{\pi}{4}\cdot\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta = \int_0^{\pi/4} \log(2)\,d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \log(2)\int_0^{\pi/4} d\theta - I.$$

$$\Rightarrow 2I = \log(2)\int_0^{\pi/4} d\theta = \log(2)\,[\theta]_0^{\pi/4} = \frac{\pi}{4}\cdot\log(2)$$

$$\Rightarrow I = \frac{\pi}{8}\log(2).$$

8. $\displaystyle\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x} = \frac{\pi^2}{4}$ **(Repeated Question to Q. No. 5)**

[2015 Fall][2013 Spring][2011 Spring]

9. $\displaystyle\int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x} = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$ **[2018 Fall]**

Solution: Here,

$$I = \int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x} \qquad \ldots \text{(i)}$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)dx}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)}{\cos x+\sin x}\,dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii) then,

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{\cos x+\sin x}$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x}$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\sin\frac{\pi}{4}\cdot\cos x+\cos\frac{\pi}{4}\cdot\sin x}$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \operatorname{cosec}\left(\frac{\pi}{4}+x\right)dx$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left(\operatorname{cosec}\left(\frac{\pi}{4}+x\right)-\cot\left(\frac{\pi}{4}+x\right)\right)\right]_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log\left(\operatorname{cosec}\frac{3\pi}{4}-\cot\frac{3\pi}{4}\right)-\log\left(\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{2\sqrt{2}}\left[\log(-\sqrt{2}-1) - \log(\sqrt{2}-1)\right]$$

$$= \frac{\pi}{2\sqrt{2}}\log\left(\frac{-\sqrt{2}-1}{\sqrt{2}-1}\right)$$

$$= \frac{\pi}{2\sqrt{2}}\log\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$$

$$= \frac{\pi}{2\sqrt{2}}\log\left\{\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)\times\frac{(1+\sqrt{2})}{(1+\sqrt{2})}\right\}$$

$$= \frac{\pi}{2\sqrt{2}}\log\frac{(1+\sqrt{2})^2}{1}$$

$$= \frac{2\pi}{2\sqrt{2}}\log(1+\sqrt{2})$$

$$\Rightarrow I = \frac{\pi}{2\sqrt{2}}\log(1+\sqrt{2}).$$

10. $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log(2).$ **[2017 Fall]**

Solution: Here,

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

Put $x = \tan\theta$ then $dx = \sec^2\theta\, d\theta$.

When $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{4}$. Then,

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\cdot\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

(Same Question to Q. No. 7)

11. Show that $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x.\cos x}\,dx = 0$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x.\cos x}\,dx \quad \ldots (i)$$

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right).\cos\left(\frac{\pi}{2}-x\right)}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x.\sin x}\,dx \quad \ldots (ii)$$

Now, adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x.\cos x}\,dx = 0.\int_0^{\pi/2}\frac{dx}{1+\sin x.\cos x} = 0$$

$$\Rightarrow I = 0.$$

12. $\displaystyle\int_0^{\pi/2} \log(\tan x)\,dx = 0$

Solution: Here,

$$I = \int_0^{\pi/2} \log(\tan x)\,dx \quad \ldots (i) \quad \text{for } \tan x > 0$$

$$= \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \log(\cot x)\,dx \quad \ldots (ii)$$

Now, adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\,dx$$

$$= \int_0^{\pi/2} \log(\tan x.\cot x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log(1)\,dx = \log(1)\int_0^{\pi/2} dx = 0.\int_0^{\pi/2} dx = 0.$$

$$\Rightarrow I = 0.$$

13. $\int_0^{\pi/2} \sin 2x (\log \tan x)\, dx = 0.$

Solution: Here,

$$I = \int_0^{\pi/2} \sin 2x \log(\tan x)\, dx \quad \ldots (i)$$

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\pi/2} \sin 2x \log(\cot x)\, dx \quad \ldots (ii)$$

Adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \sin 2x \log(\tan x . \cot x)\, dx = \int_0^{\pi/2} \sin 2x \log(1)\, dx.$$

$$\Rightarrow \quad 2I = \log(1) \int_0^{\pi/2} \sin 2x\, dx = 0. \int_0^{\pi/2} \sin 2x\, dx \quad [\because \log(1) = 0]$$

$$\Rightarrow \quad I = 0.$$

14. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \log\left(\frac{1}{2}\right)$ **[2006, Spring] [2007, Spring]**

Solution: Here,

$$I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$$

Put, $x = \sin\theta$ then $dx = \cos\theta\, d\theta$.

When $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_0^{\pi/2} \frac{\log(\sin\theta)}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta = \int_0^{\pi/2} \log(\sin\theta)\, d\theta \quad \ldots (i)$$

$$= \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - \theta\right)\right] d\theta$$

$$= \int_0^{\pi/2} \log(\cos\theta)\, d\theta \quad \ldots (ii)$$

Now, adding (i) and (ii) then,

$$2I = \int_0^{\pi/2} \log(\sin\theta . \cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2\theta}{2}\right) d\theta = \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta - \log(2) \int_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta + \frac{\pi}{2} . \log\left(\frac{1}{2}\right)$$

Put $2\theta = y$ then $2d\theta = dy$. When $\theta = 0 \Rightarrow y = 0$ and $\theta = \frac{\pi}{2} \Rightarrow y = \pi$.

Then,

$$2I = \frac{1}{2} \int_0^{\pi} \log(\sin y)\, dy + \frac{\pi}{2} . \log\left(\frac{1}{2}\right)$$

$$= \int_0^{\pi/2} \log(\sin y)\, dy + \frac{\pi}{2} \log\left(\frac{1}{2}\right)$$

$$= I + \frac{\pi}{2} \log\left(\frac{1}{2}\right) \quad [\because \text{using (i)}]$$

$$\Rightarrow \quad I = \frac{\pi}{2} \log\left(\frac{1}{2}\right).$$

15. $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx = \frac{\pi^2}{16}$

Solution: Here,

$$I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx \quad \ldots (i)$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\cos^4\left(\frac{\pi}{2} - x\right) + \sin^4\left(\frac{\pi}{2} - x\right)} dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\sin^4 x + \cos^4 x} dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii) then,

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx \qquad [\text{dividing by } \cos^4 x]$$

Put $\tan^2 x = y$ then $2\tan x. \sec^2 x \, dx = dy$.

When $x = 0 \Rightarrow y = 0$ and $x = \frac{\pi}{2} \Rightarrow y = \infty$. Then,

$$2I = \frac{\pi}{4} \int_0^{\infty} \frac{dy}{y^2 + 1}$$

$$= \frac{\pi}{4} [\tan^{-1} y]_0^{\infty} = \frac{\pi}{4} (\tan^{-1} \infty - \tan^{-1} 0) = \frac{\pi}{4} \left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{8}.$$

$$\Rightarrow I = \frac{\pi^2}{16}.$$

16. $\displaystyle\int_0^1 \cot^{-1} (1 - x + x^2) \, dx = \frac{\pi}{2} - \log(2).$ **[2018 Spring][2014 Spring]**

Solution: Here,

$$I = \int_0^1 \cot^{-1} (1 - x + x^2) \, dx$$

$$= \int_0^1 \tan^{-1} \left(\frac{1}{1 - x + x^2}\right) dx \qquad \left[\because \cot^{-1} x = \tan^{-1} \left(\frac{1}{x}\right)\right]$$

$$= \int_0^1 \tan^{-1} \left(\frac{x + (1 - x)}{1 - x(1 - x)}\right) dx$$

$$= \int_0^1 [\tan^{-1} x + \tan^{-1} (1 - x)] \, dx$$

$$\left[\because \tan^{-1} \left(\frac{A + B}{1 - AB}\right) = \tan^{-1} A + \tan^{-1} B\right]$$

$$= \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1} (1 - x) \, dx$$

$$= I_1 + I_2$$

Here,

$$I_2 = \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \tan^{-1}(1-(1-x))\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 \tan^{-1}x\,dx$$

Then,

$$I = I_1 + I_2 = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx$$

$$= 2\int_0^1 1.\tan^{-1}x\,dx$$

$$= 2.\left\{[\tan^{-1}x\,.\,x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\right\} [\because \text{integrating by parts}]$$

$$= 2\,.\tan^{-1}(1) - \int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= 2\,.\frac{\pi}{4} - [\log(1+x^2)]_0^1 \quad = \frac{\pi}{2} - \log(2) \qquad [\because \log(1) = 0]$$

$$\Rightarrow \quad I = \int_0^1 \cot^{-1}(1-x+x^2)\,dx = \frac{\pi}{2} - \log 2.$$

Exercise 11.6

Evaluate the following improper integral if it exists.

1. $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$ **[2014 Spring (Short)]**

Solution: Let,

$$I = \int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} [\tan^{-1}(x)]_0^b = \lim_{b\to\infty} \tan^{-1}(b) = \frac{\pi}{2}.$$

2. $\displaystyle\int_0^\infty \frac{x\,dx}{x^2+4}$

Solution: Here,

$$I = \int_0^\infty \frac{x\,dx}{x^2+4} = \lim_{b\to\infty} \int_0^b \frac{x\,dx}{x^2+4}$$

$$= \lim_{b\to\infty} \frac{1}{2} \int_0^b \frac{d(x^2)}{x^2+4}$$

$$= \frac{1}{2} \lim_{b\to\infty} [\log(x^2+4)]_0^b$$

$$= \frac{1}{2} \lim_{b\to\infty} [\log (b^2 + 4) - \log (4)]$$

Since $\lim_{b\to\infty} \log (b^2 + 4)$ does not exist. So, $\int_0^\infty \frac{x\,dx}{x^2 + 4}$ does not exist.

3. $\int_2^\infty \frac{x\,dx}{x^2 - 1}$

Solution: Here,

$$I = \int_2^\infty \frac{x\,dx}{x^2 - 1} = \lim_{b\to\infty} \frac{1}{2} \int_2^b \frac{2x\,dx}{x^2 - 1}$$

$$= \frac{1}{2} \lim_{b\to\infty} [\log (x^2 - 1)]_2^b$$

$$= \frac{1}{2}\left[\lim_{b\to\infty} \log (b^2 - 1) - \log(3)\right]$$

Since $\lim_{b\to\infty} \log (b^2 - 1)$ does not exist. So, $\int_2^\infty \frac{x\,dx}{x^2 - 1}$ does not exist.

4. $\int_0^\infty xe^{-x^2}\,dx$

Solution: Here,

$$I = \int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty} \int_0^b xe^{-x^2}\,dx$$

Put $x^2 = t$ then $2x\,dx = dt$. And $x = 0 \Rightarrow t = 0$, $x = b \Rightarrow t = b^2$. Then,

$$I = \lim_{b\to\infty} \frac{1}{2} \int_0^{b^2} e^{-t}\,dt = \frac{1}{2} \lim_{b\to\infty} \left[\frac{e^{-t}}{-1}\right]_0^{b^2} = \frac{1}{2}\left(\frac{e^{-\infty} - e^0}{-1}\right) = \frac{1}{2}.$$

[since $e^{-\infty} = 0$]

5. $\int_{-1}^1 \frac{dx}{x^3}$ **[2013 Fall (Short)]**

Solution: Here,

$$I = \int_{-1}^1 \frac{dx}{x^3} = \lim_{h\to0} \int_{-1}^{-h} \frac{dx}{x^3} + \lim_{h\to0} \int_h^1 \frac{dx}{x^3}$$

$$= \lim_{h\to0} \left[\frac{x^{-4}}{-4}\right]_{-1}^{-h} + \lim_{h\to0} \left[\frac{x^{-4}}{-4}\right]_h^1$$

$$= \lim_{h\to0} \left[\frac{(-h)^{-4} - (-1)^{-4}}{-4} + \frac{(1)^{-4} - (h)^{-4}}{-4}\right]$$

$$= \lim_{h\to0} \left(\frac{1}{-4}\right)\left[\frac{1}{(-h)^4} - \frac{1}{(-1)^4} + \frac{1}{(1)^4} - \frac{1}{(h)^4}\right]$$

$$= -\frac{1}{4} \lim_{h\to0} \left(\frac{1}{h^4} - 1 + 1 - \frac{1}{h^4}\right)$$

$$= -\frac{1}{4} \lim_{h\to0} (0)$$

$$= 0.$$

Thus, $\int_{-1}^1 \frac{dx}{x^3} = 0.$

6. $\int_{-\infty}^\infty \frac{x\,dx}{x^4 + 1}$

Solution: Let,

$$I = \int_{-\infty}^\infty \frac{x\,dx}{x^4 + 1}$$

Since the integral does not exist only at $x = \infty$ and $x = -\infty$. Then,

$$I = \lim_{a\to\infty} \int_{-a}^a \frac{x\,dx}{x^4 + 1} = \lim_{a\to\infty} \int_{-a}^a \frac{x\,dx}{x^4 + 1}$$

Put $x^2 = y$ then $2x\,dx = dy$. Also, $x = 0 \Rightarrow y = 0$ and $x = a \Rightarrow y = a^2$. Then,

$$I = \lim_{a\to\infty} \int_{-a^2}^{a^2} \frac{dy}{y^2 + 1}$$

$$= \lim_{a\to\infty} [\tan^{-1} y]_{-a^2}^{a^2} = \lim_{a\to\infty} [\tan^{-1}(a^2) - \tan^{-1}(-a^2)]$$

$$= \lim_{a\to\infty} [2 \tan^{-1}(a^2)] = 2\tan^{-1}(\infty) = \pi.$$

Thus, $\int_{-\infty}^\infty \frac{x\,dx}{x^4 + 1} = \pi.$

7. $\int_1^\infty \frac{\log x}{x^2}\,dx$

Solution: Let,

$$I = \int_1^\infty \frac{\log(x)}{x^2}\,dx$$

Put $\log(x) = y$ then $\frac{dx}{x} = dy$. Also, $x = 1 \Rightarrow y = 0$ and $x = \infty \Rightarrow y = \infty$. Then,

$$I = \int_0^\infty \frac{y}{e^y}\,dy = \lim_{b\to\infty} \int_0^b \frac{y}{e^y}\,dy$$

$$= \lim_{b\to\infty} \int_0^b y\,e^{-y}\,dy$$

$$= \lim_{b\to\infty} \left[y\left(\frac{e^{-y}}{-1}\right) - \frac{e^{-y}}{(-1)^2}\right]_0^b$$

$$= \lim_{b\to\infty} [-b\,e^{-b} + 0 - e^{-b} + 1]$$

$$= 1.$$

8. $\int_1^\infty \frac{x\,dx}{(1 + x^2)^2}$ **[2012 Fall (Short)]**

Solution: Let,

$$I = \int_1^\infty \frac{x\,dx}{(1 + x^2)^2}$$

Put $x = \tan\theta$ then $dx = \sec^2\theta \, d\theta$.

Also, $x = 1 \Rightarrow \theta = \frac{\pi}{4}$ and $x = \infty \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_{\pi/4}^{\pi/2} \frac{\tan\theta}{(1+\tan^2\theta)^2} \sec^2\theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (\sin\theta \cos\theta) \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{\sin 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\frac{-\cos 2\theta}{2}\right]_{\pi/4}^{\pi/2} = \frac{-1}{4}[\cos(\pi) - \cos(\pi/2)] = \frac{-1}{4}[-1-0] = \frac{1}{4}$$

Thus, $\int_1^\infty \frac{x\,dx}{(1+x^2)^2} = \frac{1}{4}$.

9. $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution: Let,

$$I = \int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$$

Put $x = \sin\theta$ then $dx = \cos\theta \, dy$.

Also, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$I = \int_0^{\pi/2} \frac{\theta}{\cos\theta} \cos\theta \, d\theta = \int_0^{\pi/2} \theta \, d\theta = \left[\frac{\theta^2}{2}\right]_0^{\pi/2} = \frac{\pi^2}{8}.$$

10. $\int_0^\pi \frac{\sin x}{\cos^2 x} dx$

Solution: Let,

$$I = \int_0^\pi \frac{\sin x}{\cos^2 x} dx$$

Since $\cos\frac{\pi}{2} = 0$. So, the integrand is undefined at $x = \frac{\pi}{2}$. Therefore,

$$I = \lim_{h\to 0}\left[\int_0^{\frac{\pi}{2}-h} \frac{\sin x}{\cos^2 x} dx + \int_{\frac{\pi}{2}+h}^{\pi} \frac{\sin x}{\cos^2 x} dx\right]$$

Put $\cos x = y$ then $(-\sin x)\,dx = dy$.

Also, $x = 0 \Rightarrow y = 1$, $x = \frac{\pi}{2} - h \Rightarrow y = \cos\left(\frac{\pi}{2} - h\right) = \sin h$.

And, $x = \frac{\pi}{2} + h \Rightarrow y = \cos\left(\frac{\pi}{2} + h\right) = -\sin h$, $x = \pi \Rightarrow y = 0$. Then,

$$I = \lim_{h\to 0}\left[\int_1^{\sin h} \left(\frac{-dy}{y^2}\right) + \int_{-\sin h}^{0} \left(\frac{-1}{y^2} dy\right)\right]$$

$$= \lim_{h\to 0}\left\{\left[-\frac{y^{-1}}{-1}\right]_1^{\sin h} + \left[\frac{-y^{-1}}{-1}\right]_{-\sin h}^{0}\right\}$$

$$= \lim_{h\to 0}\left[\left(\frac{1}{\sin h} - 1\right) + \left(\frac{1}{0} + \frac{1}{\sin h}\right)\right]$$

Since $\frac{1}{0}$ is undefined. Therefore, I does not give fixed and acceptable value.

11. $\int_0^\pi \frac{dx}{1+\cos x}$

Solution: Let,

$$I = \int_0^\pi \frac{dx}{1+\cos x}$$

Clearly $\cos\pi = -1$. So, the integrand is undefined at $x = \pi$. Therefore,

$$I = \lim_{h\to 0} \int_0^{\pi-h} \frac{dx}{1+\cos x}$$

$$= \lim_{h\to 0} \int_0^{\pi-h} \frac{dx}{2\cos^2(x/2)} = \frac{1}{2}\lim_{h\to 0} \int_0^{\pi-h} \sec^2\left(\frac{x}{2}\right) dx$$

$$= \frac{1}{2}\lim_{h\to 0}\left[\frac{\tan x/2}{1/2}\right]_0^{\pi-h}$$

$$= \lim_{h\to 0} \tan\left(\frac{\pi}{2} - \frac{h}{2}\right)$$

$$= \lim_{h\to 0} \cot\left(\frac{h}{2}\right)$$

$= \cot 0$

$=$ doesnot exist.

Thus, the value of I doesnot exist.

12. $\int_0^a \sqrt{\frac{a-x}{x}} dx$

Solution: Let,

$$I = \int_0^a \sqrt{\frac{a-x}{x}} dx$$

Clearly, the integrand is undefined at $x = 0$. Therefore,

$$I = \lim_{h\to 0} \int_h^a \sqrt{\frac{a-x}{x}} dx.$$

Put $x = t^2$ then $dx = 2t\,dt$. Also, $x = h \Rightarrow t = \sqrt{h}$, $x = a \Rightarrow t = \sqrt{a}$.

Then,

$$I = \lim_{h\to 0} \int_{\sqrt{h}}^{\sqrt{a}} \sqrt{\frac{a-t^2}{t^2}}\, 2t\, dt$$

$$= \lim_{h\to 0} 2 \int_{\sqrt{h}}^{\sqrt{a}} \sqrt{a-t^2}\, dt$$

$$= 2 \lim_{h\to 0} \left[\frac{t\sqrt{a-t^2}}{2} + \left(\frac{a}{2}\right) \sin^{-1}\left(\frac{t}{\sqrt{a}}\right) \right]_{\sqrt{h}}^{\sqrt{a}}$$

$$= 2 \lim_{h\to 0} \left[\left(0 + \left(\frac{a}{2}\right) \sin^{-1}(1)\right) - \left(\frac{\sqrt{h}\sqrt{a-h}}{2} + \left(\frac{a}{2}\right) \sin^{-1}\left(\sqrt{\frac{h}{a}}\right) \right) \right]$$

$$= 2\left[\left(0 + \left(\frac{a}{2}\right)\left(\frac{\pi}{2}\right)\right) - \left(\frac{0}{2} + 0\right) \right]$$

$$= 2\left[\frac{a\pi}{2}\right] = a\pi.$$

Thus, $\int_0^a \sqrt{\frac{a-x}{x}}\, dx = a\pi.$

13. $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx$

Solution: Let,

$$I = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx$$

Clearly the integrand is undefined at x = 1. Therefore,

$$I = \lim_{h\to 0} \int_{-1}^{1-h} \sqrt{\frac{1+x}{1-x}}\, dx.$$

Put x = cosθ then dx = – sinθ dθ.

Since,

$$\frac{1+x}{1-x} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{2\cos^2(\theta/2)}{2\sin^2(\theta/2)} = \cot^2(\theta/2).$$

So that,

$$I = \lim_{h\to 0} \int_{-1}^{1-h} \cot(\theta/2)\,(-\sin\theta)\, d\theta$$

$$= \lim_{h\to 0} \int_{-1}^{1-h} \cot(\theta/2)\,(-2)\sin(\theta/2)\cos(\theta/2)\, d\theta$$

$$= (-1) \lim_{h\to 0} \int_{-1}^{1-h} 2\cos^2(\theta/2)\, d\theta$$

$$= (-1) \lim_{h\to 0} \int_{-1}^{1-h} [1+\cos\theta]\, d\theta$$

$$= (-1) \lim_{h\to 0} [\theta + \sin\theta]_{-1}^{1-h}$$

$$= (-1) \lim_{h\to 0} [\{1-h+\sin(1-h)\} - \{-1+\sin(-1)\}]$$

$$= (-1)\ [\{1+\sin(1)\} - \{-1-\sin(1)\}]$$

$$= (-2)\ [1+\sin(1)].$$

Thus, $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx = (-2)\ [1+\sin(1)].$

14. $\int_0^2 \frac{dx}{(1-x)^2}$

Solution: Let,

$$I = \int_0^2 \frac{dx}{(1-x)^2}$$

Clearly the integrand is undefined at x = 1. Therefore,

$$I = \lim_{h\to 0} \left[\int_0^{1-h} \frac{dx}{(1-x)^2} + \int_{1+h}^{2} \frac{dx}{(1-x)^2} \right]$$

$$= -\lim_{h\to 0} \left[\int_0^{1-h} (1-x)^{-2}(-dx) + \int_{1+h}^{2} (1-x)^{-2}(-dx) \right]$$

$$= -\lim_{h\to 0} \left\{ \left[\frac{(1-x)^{-1}}{-1}\right]_0^{1-h} + \left[\frac{(1-x)^{-1}}{-1}\right]_{1+h}^{2} \right\}$$

$$= -\lim_{h\to 0} \left\{ \left[\frac{1}{x-1}\right]_0^{1-h} + \left[\frac{1}{x-1}\right]_{1+h}^{2} \right\}$$

$$= -\lim_{h\to 0} \left\{ \left(\frac{1}{-h}\right) - \left(\frac{1}{-1}\right) + \left(\frac{1}{1}\right) - \left(\frac{1}{h}\right) \right\}$$

$$= -\lim_{h\to 0} \left(2 - \frac{2}{h}\right) \quad \text{which does not exist.}$$

Therefore, the given integral I does not exist.

Q. Define Improper integral and evaluate the improper integral $\int_0^2 \frac{dx}{(1-x)^2}$

Solution: See definition and Q. 14.

15. $\int_0^1 \log(x)\, dx$

Solution: Let,

$$I = \int_0^1 \log(x)\, dx$$

Since log(0) is undefined. Therefore, the integrand is undefined at x = 0. Therefore,

$$I = \lim_{h\to 0} \int_h^1 1.\log x\, dx$$

$$= \lim_{h\to 0} \left\{ [\log x.\ dx]_h^1 - \int_h^1 \frac{1}{x}.\ x dx \right\}$$

$$= \lim_{h\to 0}\left[-h\log h - [x]_h^1\right] \quad [\because \log(1) = 0]$$

$$= \lim_{h\to 0}[-h\log h - 1 + h] \quad = -0 - 1 + 0 \quad = -1.$$

16. $\int_0^\infty e^{-ax}\cos bx\,dx$ **for** $a > 0$

Solution: Let,

$$I = \int_0^\infty e^{-ax}\cos bx\,dx \quad \text{for } a > 0$$

We know,

$$\int e^{ax}\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx) + C$$

Clearly,

$$I = \lim_{m\to\infty}\int_0^m e^{-ax}\cos bx\,dx$$

$$= \lim_{m\to\infty}\left[\frac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx)\right]_0^m$$

$$= \lim_{m\to\infty}\left[\frac{e^{-am}}{a^2+b^2}(-a\cos bm + b\sin bm) - \frac{1}{a^2+b^2}(-a)\right]$$

$$= \frac{a}{a^2+b^2} \quad \text{as } e^{-am} \to 0,\ m\to\infty$$

Hence, $I = \dfrac{a}{a^2+b^2}$

Q. Find the formula for $\int \sec^n n\,dx$ and then evaluate $\int \sec^3 x\,dx$

[2013, Spring]

Solution: Let,

$$I_n = \int \sec^n x\,dx$$

$$= \int \sec^{n-2} x \sec^2 x\,d$$

$$= \sec^{n-2} x \tan - (n-2)\int (\sec^{n-3} x \sec x \tan x \times \tan x\,dx)$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\tan^2 x)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\left(\int \sec^n x\,dx - \int \sec^{n-2} x\,dx\right)$$

$$= \sec^{n-2} x \tan x - (n-2)I_n + (n-1)I_{n-2}$$

$$\Rightarrow (1 + n - 2)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$\therefore \quad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}I_{n-2}$$

This is the required reduction formula for $\int \sec^n x\,dx$

Again, $\int \sec^3 x\,dx = I_3 = \dfrac{\sec^3 x \tan x}{4} + \dfrac{3}{4}I_1$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{4}\log(\sec x + \tan x) + C$$

Exercise 11.7

1. **Find the reduction formula for $\int \cos^n x\, dx$ and then evaluate $\int \cos^7 x\, dx$.** **[2015 Spring][2012 Fall]**

Solution: Let $I_n = \int \cos^n x\, dx$

$$= \int \cos^{n-1} x \cos x\, dx$$

Taking $\cos^{n-1}x$ as first function and cosx as second function and then applying by parts then,

$$= \cos^{n-1} x.\sin x - (n-1)\int \cos^{n-2} x\,(-\sin x)\sin x\, dx$$

$$= \sin x \cos^{n-1}x + (n-1)\int \cos^{n-2}x\,(1-\cos^2 x)\, dx$$

$$= \sin x \cos^{n-1}x + (n-1)\left[\int \cos^{n-2}x\, dx - \int \cos^n x\, dx\right]$$

$$= \sin x \cos^{n-1}x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow (1+n-1) I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{\sin x \cos^{n-1} x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}$$

For n = 7,

$$I_7 = \int \cos^7 x\, dx$$

$$= \frac{\sin x \cos^6 x}{7} + \frac{6}{7} I_5$$

$$= \frac{\sin x \cos^6 x}{7} + \frac{6}{7}\left[\frac{\sin x \cos^4 x}{5} + \left(\frac{4}{5}\right) I_3\right]$$

$$= \frac{\sin x \cos^6 x}{7} + \frac{6 \sin x \cos^4 x}{35} + \frac{24}{35}\left[\frac{\sin x \cos x}{3} + \frac{2}{3} I_1\right]$$

$$= \frac{\sin x \cos^6 x}{7} + \frac{6 \sin x \cos^4 x}{35} + \frac{24}{105} \sin x \cos x + \frac{48}{105} \sin x + C.$$

2. **Find the reduction formula for $\int_0^{\pi/2} \cos^n x\, dx$ and then evaluate $\int_0^{\pi/2} \cos^7 x\, dx$.**

Solution: Let $J_n = \int_0^{\pi/2} \cos^n x\, dx$

By Q.1., $J_n = \left[\frac{\sin x \cos^{n-1} x}{n}\right]_0^{\pi/2} + \left(\frac{n-1}{n}\right) J_{n-2}$

$$= \left(\frac{n-1}{n}\right) J_{n-2} \quad [\because \cos(\pi/2) = 0 \text{ and } \sin 0 = 0]$$

For n = 7,

$$J_7 = \frac{6}{7} J_5 = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot J_1$$

$$= \frac{48}{105}\int_0^{\pi/2} \cos x\, dx$$

$$= \frac{48}{105}[\sin x]_0^{\pi/2} = \frac{48}{105} = \frac{16}{35}.$$

3. Find the reduction formula for $\int \cot^n x\, dx$ and then evaluate $\int \cot^7 x\, dx$.

Solution: Let $I_n = \int \cot^n x\, dx$

$$= \int \cot^{n-2}x \cot^2 x\, dx$$
$$= \int \cot^{n-2}x (\operatorname{cosec}^2 x - 1)\, dx$$
$$= \int \cot^{n-2}x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2}x\, dx$$
$$= \cot^{n-2}x(-\cot x) - (n-2)\int \cot^{n-3}x(-\operatorname{cosec}^2 x)(-\cot x)\, dx - I_{n-2}$$
$$= -\cot^{n-1}x - (n-2)\int \cot^{n-2}x(\cot^2 x + 1)\, dx - I_{n-2}$$
$$= -\cot^{n-1}x - (n-2)[I_n + I_{n-2}] - I_{n-2}$$
$$\Rightarrow (1 + n - 2) I_n = -\cot^{n-1}x - (n - 2 + 1) I_{n-2}$$
$$\Rightarrow (n-1) I_n = -\cot^{n-1}x - (n-1) I_{n-2}$$
$$\Rightarrow I_n = -\frac{\cot^{n-1}x}{n-1} - I_{n-2}$$

In particular, for n = 7,

$$I_7 = -\frac{\cot^6 x}{6} - I_5 = -\frac{\cot^6 x}{6} - \left[-\frac{\cot^4 x}{4} - \left[-\frac{\cot^2 x}{2} - I_1\right]\right]$$
$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \int \cot x\, dx$$
$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \log(\sin x) + C.$$

4. Find the reduction formula for $\int \operatorname{cosec}^n x\, dx$ and then evaluate $\int \operatorname{cosec}^5 x\, dx$.

Solution: Let,

$$I_n = \int \operatorname{cosec}^n x\, dx$$
$$= \int \operatorname{cosec}^{n-2}x \operatorname{cosec}^2 x\, dx$$
$$= \operatorname{cosec}^{n-2}x(-\cot x) - (n-2)\int \operatorname{cosec}^{n-3}x(-\operatorname{cosec}x \cot x)(-\cos x)dx$$
$$= -\operatorname{cosec}^{-2}x \cot x - (n-2)\int \operatorname{cosec}^{n-2}x \cot^2 x\, dx$$
$$= -\cot x \operatorname{cosec}^{n-2}x - (n-2)\int \operatorname{cosec}^{n-2}x(\operatorname{cosec}^2 x - 1)\, dx$$
$$= -\cot x \operatorname{cosec}^{n-2}x - (n-2) I_n + (n-2) I_{n-2}$$
$$\Rightarrow I_n = -\frac{\cot x \operatorname{cosec}^{n-2}x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

In particular for n = 5,

$$I_5 = \int \operatorname{cosec}^5 x\, dx$$
$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} + \frac{3}{4} I_3$$
$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} + \frac{3}{4}\left[-\frac{\cot x \operatorname{cosec} x}{2} + \frac{1}{2} I_1\right]$$

$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\int \operatorname{cosec} x\, dx$$
$$= -\frac{\cot x \operatorname{cosec}^3 x}{4} - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\log\left(\tan\frac{x}{2}\right) + C.$$

5. If $J_n = \int_0^{\pi/4} \tan^n x\, dx$ then show that $J_n = \frac{1}{n-1} - J_{n-2}$ and then find the value of J_5.

Solution: Let $I_n = \int \tan^n x\, dx$

$$= \int \tan^{n-2}x \tan^2 x\, dx$$
$$= \int \tan^{n-2}x(\sec^2 x - 1)\, dx$$
$$= \int \tan^{n-2}x \sec^2 x\, dx - \int \tan^{n-2}x\, dx$$
$$\Rightarrow I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

This is the reduction formula for $I_n = \int \tan^n x\, dx$

Again, let

$$J_n = \int_0^{\pi/4} \tan^n x\, dx$$
$$= \left(\frac{\tan^{n-1}x}{n-1}\right)_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2}x\, dx$$
$$J_n = \frac{1}{n-1} - J_{n-2}$$

This is the required reduction formula for $J_n = \int_0^{\pi/4} \tan^n x\, dx$.

Also at n = 5,

$$I_5 = \int_0^{\pi/4} \tan^5 x\, dx$$
$$= \frac{1}{4} - I_3 = \frac{1}{4} - \frac{1}{2} + I_1 = \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \tan x\, dx$$
$$= \frac{1}{4} - \frac{1}{2} + [\log(\sec x)]_0^{\pi/4}$$
$$= -\frac{1}{4} + \left[\log\left(\sec\frac{\pi}{4}\right) - \log(\sec 0)\right]$$
$$= -\frac{1}{4} + [\log\sqrt{2} - 0]$$
$$= \frac{1}{2}\log(2) - \frac{1}{4}$$

6. Find the reduction formula for $\int \cos^m x \cos nx\, dx$ and show that $\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$.

Solution: Let, $I_{m,n} = \int \cos^m x \cos nx\, dx$

$$= \cos^m x \frac{\sin nx}{n} - m\int \cos^{m-1} x \frac{\sin nx}{n}(-\sin x)\, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin nx \sin x\, dx$$

Since, $2\sin nx \sin x = \cos(n-1)x - \cos(n+1)x$

$$= \cos(n-1)x - \cos nx \cos x + \sin nx \sin x$$

$\Rightarrow \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$

So,

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\, [\cos(n-1)x - \cos nx \cos x]dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}[I_{m-1,n-1} - I_{m,n}]$$

$$\Rightarrow \left(1 + \frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

For m = n then,

$$I_{n,n} = \frac{\cos^n x \sin nx}{2n} + \frac{n}{2n} I_{n-1,n-1}$$

$$\Rightarrow I_{n,n} = \frac{\cos^n x \sin nx}{2n} + \frac{1}{2} I_{n-1,n-1}$$

And, let

$$J_{n,n} = \int_0^{\pi/2} \cos^n x \cos nx\, dx$$

$$= [I_{n,n}]_0^{\pi/2} = \frac{1}{2}[I_{n-1,n-1}]_0^{\pi/2} \qquad [\text{Being } \cos(\pi/2) = 0]$$

$$= \frac{1}{2^n}[I_{0,0}]_0^{\pi/2}$$

$$= \frac{1}{2^n}\int_0^{\pi/2} \cos^0 x \cos(0x)\, dx = \frac{1}{2^n}[x]_0^{\pi/2} = \frac{\pi}{2^{n+1}}$$

7. Find the reduction formula for $\int \cos^m x \sin nx\, dx$ and then evaluate $\cos^2 x \sin 3x\, dx$.

Solution: Let, $I_{m,n} = \int \cos^m x \sin nx\, dx$

$$= \cos^m x\left(\frac{\cos nx}{-n}\right) - \int m \cos^{m-1} x\, (-\sin x)\left(\frac{\cos nx}{-n}\right) dx$$

$$= -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n}\int \cos^{m-1} x\, (\sin x \cos nx)\, dx \qquad \ldots (i)$$

We have,

$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$

$\Rightarrow \quad \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$

From (i),

$$I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n}\int \cos^{m-1} x\, [\sin nx \cos x - \sin(n-1)x]\, dx$$

$$\Rightarrow I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n}\int \sin nx \cos^m x\, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

$$\Rightarrow \quad I_{m,n} = -\left(\frac{\cos^m x \cos nx}{n}\right) - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad \left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

This is the required reduction formula for

$I_{m,n} = \int \cos^m x \sin nx\, dx$.

And, in particular, let m = 2, n = 3 then,

$$I_{2,3} = -\frac{\cos^2 x \cos 3x}{5} + \frac{2}{5} I_{1,2}$$

$$= -\frac{\cos^2 x \cos 3x}{5} + \frac{2}{5}\left[-\frac{\cos x \cos 2x}{3} + \frac{1}{3} I_{0,1}\right]$$

$$= -\frac{\cos^2 x \cos 3x}{5} - \frac{2\cos x \cos 2x}{15} + \frac{2}{15}\int \sin x\, dx$$

$$= -\frac{\cos^2 x \cos 3x}{5} - \frac{2\cos x \cos 2x}{15} + \frac{2}{15}\cos x + C.$$

8. Show that $\int_0^{\pi/2} \cos^5 x \sin 3x\, dx = \frac{1}{3} + \frac{5\pi}{64}$.

Solution: By Q.No. 7, We have,

$$I_{m,n} = \int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

So, $\quad J_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$

$$= -\left[\frac{\cos^m x \cos nx}{m+n}\right]_0^{\pi/2} + \frac{m}{m+n}[I_{m-1,n-1}]_0^{\pi/2}$$

$$= \frac{1}{m+n} + \frac{m}{m+n}[I_{m-1,n-1}]_0^{\pi/2}$$

In particular, let m = 5, n = 3 then,

$$J_{5,3} = \frac{1}{8} + \frac{5}{8}[I_{4,2}]_0^{\pi/2}$$

$$= \frac{1}{8} + \frac{5}{8}\left[\frac{1}{6} + \frac{4}{6}\left[\frac{1}{4} + \frac{3}{4}[I_{2,0}]_0^{\pi/2}\right]\right]$$

$$= \frac{1}{8} + \frac{5}{48} + \frac{5}{48} + \frac{15}{48}\int_0^{\pi/2} \cos^2 x\, dx$$

$$= \frac{16}{48} + \frac{15}{48}\int_0^{\pi/2}\left(\frac{1 + \cos 2x}{2}\right) dx$$

$$= \frac{1}{3} + \frac{5}{16}\left[\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{3} + \frac{5}{32} \cdot \frac{x}{2} \qquad = \frac{1}{3} + \frac{5\pi}{64}.$$

10. Find the reduction formula for $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$.

Solution: Already contained in Q.No. 8.

11. If $I_n = \int \sinh^n x\, dx$ then show that $I_n = \frac{\sinh^{n-1} x \cosh x}{n} - \left(\frac{n-1}{n}\right) I_{n-2}$

Solution: Let, $I_n = \int \sinh^n x\, dx$

$$= \int \sinh^{n-1} x \sinh x\, dx$$

$$= \sinh^{n-1} x \cosh x - (n-1) \int \sinh^{n-2} x \cosh x \cosh x\, dx$$

$$= \sin h^{n-1} x \cosh x - (n-1) \int \sinh^{n-2} x \cosh^2 x\, dx$$

$$= \sinh^{n-1} x \cosh x - (n-1) \int \sinh^{n-2} x (1 + \sinh^2 x)\, dx$$

$$= \sin h^{n-1} x \cosh x - (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow \quad I_n = \frac{\sinh^{n-1} x \cosh x}{n} - \left(\frac{n-1}{n}\right) I_{n-2}$$

Exercise 11.8

1. Evaluate the followings:

(i) $\Gamma(9)$

Solution: Here,

$$\Gamma(9) = \Gamma(8-1) = 8\Gamma(8) = 8.7.6.5.4.3.2.1\,\Gamma(1)$$
$$= 40320.1 \qquad [\because \Gamma(1) = 1]$$
$$= 40320$$

(ii) $\Gamma\left(\frac{11}{2}\right)$

Solution: Here,

$$\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{945}{32}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

(iii) $\dfrac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(7)}$

Solution: Here,

$$\frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(7)} = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{6.5.4.3.2.1} = \frac{7\pi}{1024}\cdot \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

2. Evaluate

(i) $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$

Solution: Here,

$$\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$$
$$= \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\Gamma\left(\frac{4+3+2}{2}\right)} \qquad \left[\because \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}\right]$$
$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{2\Gamma\left(\frac{9}{2}\right)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\,1!}{2.\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{2}{35}.$$

Thus, $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx = \frac{2}{35}$

(ii) $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx$

Solution: Here,

$$\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx = \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{8+1}{2}\right)}{2\Gamma\left(\frac{6+8+2}{2}\right)}$$
$$= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right)}{2\Gamma(8)}$$
$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2(7!)}$$
$$= \frac{5\pi}{4096}$$

Thus, $\int_0^{\pi/2} \sin^6 x \cos^8 x\, dx = \frac{5\pi}{4096}$.

3. Evaluate the following integrals using beta and gamma functions:

(i) $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx$

Solution: Here,

$$I = \int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx$$

Put $x = \sin\theta$ then $dx = \cos\theta\, d\theta$. Also, $x = 0 \Rightarrow \theta = 0$, $x = 1 \Rightarrow \theta = \frac{\pi}{2}$.

Then,

$$I = \int_0^{\pi/2} \frac{\sin^6\theta}{\cos\theta}\cos\theta\, d\theta$$
$$= \int_0^{\pi/2} \sin^6\theta\, d\theta$$
$$= \int_0^{\pi/2} \sin^6\theta \cos^0\theta\, d\theta$$
$$= \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)}$$
$$= \frac{\Gamma\left(\frac{7}{2}\right)\left(\frac{1}{2}\right)}{2\Gamma(4)} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{2(3!)} = \frac{5\pi}{32}$$

$$\left[\because \Gamma(m-1) = (m)! \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Thus, $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx = \frac{5\pi}{32}$

(ii) $\int_0^{2a} x^5\sqrt{(2a-x)x}\,dx$

Solution: Let, $I = \int_0^{2a} x^5\sqrt{(2a-x)x}\,dx$

Put $x = 2a\sin^2\theta$ then $dx = 2a.\,2\sin\theta\cos\theta\,d\theta$.

Also, $x = 0 \Rightarrow \theta = 0$, $x = 2a \Rightarrow \theta = \frac{\pi}{2}$.

Therefore,

$$I = \int_0^{\pi/2} (2a)^5 \sin^{10}\theta\sqrt{2a(1-\sin^2\theta)\,2a\sin^2\theta}\;2a.\,2\sin\theta\cdot\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2^8a^7\sin^{12}\theta\cos^2\theta\,d\theta$$

$$= 2^8a^7\frac{\Gamma\left(\frac{12+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{12+2+2}{2}\right)}$$

$$= 2^8a^7\frac{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\Gamma(8)}$$

$$= a^7\frac{11.9.7.5.3.\pi}{7!} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{33a^7\pi}{16}$$

Thus, $\int_0^{2a} x^5\sqrt{(2a-x)x}\,dx = \frac{33a^7\pi}{16}$.

(iii) — (viii) Process as in (ii) with substituting the value of x as recommended in book.

4. Evaluate the following integrals by using beta and gamma functions:

(i) $\int_0^{\pi/8} \cos^3 4x\,dx$

Solution: Let, $I = \int_0^{\pi/8} \cos^3 4x\,dx$

Put $4x = y$ then $dx = \frac{dy}{4}$. Also, $x = 0 \Rightarrow y = 0$, $x = \frac{\pi}{8} \Rightarrow y = \frac{\pi}{2}$. So that,

$$I = \int_0^{\pi/2}\cos^3y\frac{dy}{4} = \frac{1}{4}\int_0^{\pi/2}\cos^3y\sin^0y\,dy$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{3+0+2}{2}\right)}$$

$$= \frac{1}{8}\cdot\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{1}{8}\cdot\frac{1.\Gamma\left(\frac{1}{2}\right)}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{1}{12}$$

Thus, $\int_0^{\pi/8}\cos^3 4x\,dx = \frac{1}{12}$

(ii) — (iv) Process as in (i)

5. Evaluate $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$.

Solution: Here,

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} \qquad \left[\because \Gamma m\Gamma(1-m) = \frac{\pi}{\sin m\pi} \text{ for } 0< m>1\right]$$

$$= \frac{\pi}{1/\sqrt{2}} = \pi\sqrt{2}.$$

6. Show that B(n, m) B(m + n. *l*) = B(m, *l*) B(m + *l*, n).

Solution: Here,

B(n, m) B(m + n. *l*) = B(m, *l*) B(m + *l*, n)

$$\Rightarrow \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}\cdot\frac{\Gamma(m+n)\Gamma(l)}{\Gamma(m+n+l)} = \frac{\Gamma(m)\Gamma(l)}{\Gamma(m+l)}\cdot\frac{\Gamma(m+l)\Gamma(n)}{\Gamma(m+l+n)}$$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} = \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)}$$

This is true.

So, B(n, m) B(m + n, *l*) = B(m, *l*) B(m + *l*, n).

> **Note:** Similarly,
> B(n, m) B(m + n, *l*) = B(*l*, n) B(*l* + n, m)
> Thus, the required form is proved.

7. Show that $\Gamma\left(\frac{2n+1}{2}\right) = \frac{1}{2^n}(2n-1)(2n-3)\ldots5.3.1\sqrt{\pi}$

Solution: Here,

$$\Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n-1}{2}\right)\Gamma\left(\frac{2n-1}{2}\right) \qquad [\because \Gamma(m+1) = m\Gamma(m)]$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\ldots\left(\frac{2(3)+1}{2}\right)\left(\frac{2(2)+1}{2}\right)\left(\frac{2(1)+1}{2}\right)\Gamma\left(\frac{2(1)+1}{2}\right)$$

$$= \frac{1}{2^{n-1}}(2n-1)(2n-3)\ldots7.5.3\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{2^{n-1}} (2n-1)(2n-3) \ldots 5.\,3.\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2^n} (2n-1)(2n-3) \ldots 5.3.1\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

This completes the solution.

8. Show that $\int_{-1}^{1} (1+x)^p (1-x)^q \, dx = 2^{p+q+1} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$ **for p>–1, q>–1.**

Solution: Here, $I = \int_{-1}^{1} (1+x)^p (1-x)^q \, dx$

Set $1 + x = 2y$ then $dx = 2dy$. Also, $x = -1 \Rightarrow y = 0$, $x = 1 \Rightarrow y = 1$.
Then,

$$I = \int_0^1 (2y)^p (1 - (2y-1))^q (2dy)$$

$$= \int_0^1 (2y)^p (2-2y)^q \, 2dy$$

$$= 2^{p+q+1} \int_0^1 y^p (1-y)^q \, dy$$

$$= 2^{p+q+1} \beta(p+1, q+1) \qquad \text{for } (p+1) > 0, (q+1) > 0$$

$$= 2^{p+q+1} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} \qquad \text{for } p > -1, q > -1$$

This is required solution.

9. Show that $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$ **for m > –1, n > –1.**

Solution: Here, $I = \int_a^b (x-a)^m (b-x)^n \, dx$

Set, $x - a = (b - a)y$. Then $x = a \Rightarrow y = 0$, $x = b \Rightarrow y = 1$.
And, $dx = (b - a)\, dy$. Then,

$$I = \int_0^1 (b-a)^m y^m (b - a - (b-a)y)^n (b-a)\, dy$$

$$= \int_0^1 (b-a)^m y^m (b-a)^n (1-y)^n (b-a)\, dy$$

$$= (b-a)^{m+n+1} \int_0^1 y^m (1-y)^n \, dy$$

$$= (b-a)^{m+n+1} \beta(m+1, n+1) \qquad \text{for } (m+1) > 0, (n+1) > 0$$

$$= (b-a)^{m+n+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \qquad \text{for } m > -1, n > -1.$$

10. Show that $\int_0^{\infty} e^{-x^2} x^{\lambda} \, dx = \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right)$ **for λ > –1.**

Solution: Here, $I = \int_0^{\infty} e^{-x^2} x^{\lambda} \, dx$

Set $x^2 = t$ then $2x\, dx = dt \Rightarrow dx = \frac{dt}{2\sqrt{t}}$.

Also, $x = 0 \Rightarrow t = 0$, $x = \infty \Rightarrow t = \infty$. Then,

$$I = \int_0^{\infty} e^{-t} t^{\lambda/2} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t} t^{((\lambda-1)/2)} \, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t} t^{((\lambda+1)/2 + 1)} \, dt = \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right) \qquad \text{for } \frac{\lambda+1}{2} > 0$$

$$= \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right) \qquad \text{for } \lambda > -1.$$

11. Show that $\int_0^{\infty} e^{-x^4} x^2 \, dx \int_0^{\infty} e^{-x^4} \, dx = = \frac{\pi}{8\sqrt{2}}$.

Solution: Here,

$$I = \int_0^{\infty} e^{-x^4} x^2 \, dx \int_0^{\infty} e^{-x^4} \, dx$$

Set, $x^4 = t$ then $4x^3\, dx = dt \Rightarrow dx = \frac{dt}{4\, t^{3/4}}$.

Also, $x = 0 \Rightarrow t = 0$, $x \to \infty \Rightarrow t \to \infty$. Then,

$$I = \int_0^{\infty} e^{-t} t^{1/2} \frac{dt}{4t^{3/4}} \int_0^{\infty} e^{-t} \frac{dt}{4\, t^{3/4}}$$

$$= \frac{1}{16} \int_0^{\infty} e^{-t} t^{-1/4} \, dt \int_0^{\infty} e^{-t} t^{-3/4} \, dt$$

$$= \frac{1}{16} \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \qquad [\because \text{using definition of gamma function}]$$

$$= \frac{1}{16} \pi\sqrt{2} \qquad [\because \text{By Q.5}]$$

$$= \frac{\pi}{8\sqrt{2}}.$$

12. Evaluate $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)$.

Solution: Here,

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)$$

$$= \left[\Gamma\left(\frac{8}{9}\right)\Gamma\left(\frac{1}{9}\right)\right]\left[\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{2}{9}\right)\right]\left[\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{3}{9}\right)\right]\left[\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{4}{9}\right)\right]$$

$$= \left[\Gamma\left(\frac{8}{9}\right)\Gamma\left(1-\frac{8}{9}\right)\right]\left[\Gamma\left(\frac{7}{9}\right)\Gamma\left(1-\frac{7}{9}\right)\right]\left[\Gamma\left(\frac{6}{9}\right)\Gamma\left(1-\frac{6}{9}\right)\right]\left[\Gamma\left(\frac{5}{9}\right)\Gamma\left(1-\frac{5}{9}\right)\right]$$

$$= \frac{\pi}{\sin\left(\frac{8\pi}{9}\right)} \cdot \frac{\pi}{\sin\left(\frac{7\pi}{9}\right)} \cdot \frac{\pi}{\sin\left(\frac{6\pi}{9}\right)} \cdot \frac{\pi}{\sin\left(\frac{5\pi}{9}\right)}$$

$$\left[\because \text{ applying } \Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi} \text{ for } 0 < m > 1\right]$$

$$= \frac{\pi}{\sin 160^\circ} \cdot \frac{\pi}{\sin 140^\circ} \cdot \frac{\pi}{\sin 120^\circ} \cdot \frac{\pi}{\sin 100^\circ}$$

Exercise 11.9

Evaluate the following integrals by the method of summation (i.e. by definition)

1. $\int_{1}^{2} x^2 \, dx$ **[2017 Spring]**

Solution:

Comparing the given integral $\int_{1}^{2} x^2 \, dx$ with the integral $\int_{a}^{b} f(x) \, dx$ then we get,

$f(x) = x^2, a = 1, b = 2$. So, $nh = b - a = 1$.

Then, $f(a + rh) = (a + rh)^2$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_1^2 x^2\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (1 + rh)^2$$

$$= \lim_{h\to 0} h\left[\sum_{r=1}^{n} 1 + 2h\sum_{r=1}^{n} r + h^2\sum_{r=1}^{n} r^2\right]$$

$$= \lim_{h\to 0} h\left[n + 2h \cdot \frac{n(n+1)}{2} + h^2\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to 0}\left[nh + nh\,(nh + h) + \frac{nh\,(nh + b)\,(2nh + h)}{6}\right]$$

$$= \lim_{h\to 0}\left[1 + 1\,(1 + h) + \frac{1(1 + h)\,(2.1 + h)}{6}\right].$$

$$= 1 + 1\,(1 + 0) + \frac{1(1 + 0)\,(2 + 0)}{6}$$

$$= \frac{3 + 3 + 1}{3} = \frac{7}{3}.$$

Thus, $\int_1^2 x^2\,dx = \frac{7}{3}$.

2. $\int_a^b e^{-x}\,dx$ **[2018 Fall][2017 Fall]**

Solution:

Comparing the given integral $\int_a^b e^{-x}\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = e^{-x}, a = a, b = b$. So, $nh = b - a$

Also, $f(a + rh) = e^{-(a + rh)} = e^{-a} \cdot e^{-rh}$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_a^b e^x\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (e^{-a} \cdot e^{-rh})$$

$$= e^{-a} \lim_{h\to 0} h \sum_{r=1}^{n} (e^{-h})^r$$

$$= e^{-a} \lim_{h\to 0} h \frac{e^{-h}[(e^{-h})^n - 1]}{e^{-h} - 1}$$

$$= e^{-a} \lim_{h\to 0} h \frac{e^{-h}(e^{-nh} - 1)}{e^{-h} - 1}$$

$$= e^{-a} \lim_{h\to 0} \frac{e^{-h}(e^{-(b-a)} - 1)}{e^{-h} - 1} . h$$

$$= e^{-a}(e^{a-b} - 1) \lim_{h\to 0} \frac{e^{-h}}{e^{-h} - 1} . h$$

$$= (e^{-b} - e^{-a}) . \lim_{h\to 0} \frac{-h}{e^{-h} - 1} (-1) e^{-h}$$

$$= (e^{-a} - e^{-b}) . 1 \qquad \left[\text{Being } \frac{e^x - 1}{x} = 1\right]$$

$$= e^{-a} - e^{-b}.$$

Thus, $\int_a^b e^{-x} dx = e^{-a} - e^{-b}$.

3. $\int_1^4 (x^2 - x)\,dx$

Solution:

Comparing the given integral $\int_1^4 (x^2 - x)\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$f(x) = (x^2 - x)$, $a = 1$, $b = 4$. So, $nh = b - a = 3$

Also, $f(a + rh) = f(1 + rh)$

$= (1 + rh)^2 - (1 + rh) = r^2h^2 + rh.$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_1^4 (x^2 - x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(1 + rh)$$

$$= \lim_{h\to 0} h \left[\sum_{r=1}^{n} rh + \sum_{r=1}^{n} r^2h^2\right]$$

$$= \lim_{h\to 0} h \left[h \sum_{r=1}^{n} r + h^2 \sum_{r=1}^{n} r^2\right]$$

$$= \lim_{h\to 0}\left[h \frac{n(n+1)}{2} + h^2 \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to 0}\left[\frac{2h(nh+h)}{2} + \frac{nh(nh+h)(2nh+h)}{6}\right]$$

$$= \lim_{h\to 0}\left[\frac{3(3+h)}{2} + \frac{3(3+h)(6+h)}{6}\right]$$

$$= \frac{3(3+0)}{2} + \frac{(3+0)(6+0)}{2} = \frac{9+18}{2} = \frac{27}{2}.$$

Thus, $\int_1^4 (x^2 - x)\,dx = \frac{27}{2}$.

4. $\int_0^1 x^{1/2}\,dx$ **[2018 Spring][2016 Spring]**

Solution: Given integral is,

$$\int_0^1 x^{1/2}\,dx \qquad \text{... (i)}$$

Put, $x = y^2$ then $dx = 2y\,dy$. Also, $x = 0 \Rightarrow y = 0$, $x = 1 \Rightarrow y = 1$.

Then,

$$\int_0^1 \sqrt{x}\,dx = 2\int_0^1 y^2\,dy \qquad \text{... (ii)}$$

Comparing the given integral $\int_0^1 y^2\,dy$ with the integral $\int_a^b f(y)\,dy$ then we get,

$f(y) = y^2$, $a = 0$, $b = 1$. So, $nh = b - a = 1$

Also, $f(a + rh) = f(rh) = (rh)^2$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$2\int_0^1 y^2\,dy = 2 \lim_{h\to 0} h \sum_{r=1}^{n} r^2h^2$$

$$= 2 \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= 2 \lim_{h\to 0} \frac{nh(nh+h)(2nh+h)}{6}$$

$$= 2 \lim_{h\to 0} \frac{1(1+h)(2+h)}{6} = 2\frac{(1+0)(2+0)}{6} = \frac{2}{3}.$$

Thus, $\int_0^1 x^{1/2}\,dx = \frac{2}{3}$.

Note: The ab–initio method does not work here being a = 0.

5. $\int_0^1 (a_1x + b_1)\,dx$

Solution:

Comparing the given integral $\int_0^1 (a_1x+b_1)dx$ with the integral $\int_a^b f(x)dx$ then we get,

$$f(x) = a_1x + b_1,\ a = 0,\ b = 1. \text{ So, } nh = b - a = 1.$$

Also, $f(a + rh) = a_1(a+rh) + b_1 = a_1rh + b_1$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_0^1 (a_1x + b_1)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (a_1rh + b_1)$$

$$= \lim_{h\to 0} h\left(a_1h \sum_{r=1}^{n} r + b_1 \sum_{r=1}^{n} 1\right)$$

$$= \lim_{h\to 0}\left[a_1h^2 \cdot \frac{n(n+1)}{2} + b_1h\cdot n\right]$$

$$= \lim_{h\to 0}\left(\frac{a_1}{2}\cdot nh\,(nh + h) + b_1nh\right)$$

$$= \lim_{h\to 0}\left(\frac{a_1}{2}\cdot 1\,(1 + h) + b_1\cdot 1\right)$$

$$= \frac{a_1}{2}(1 + 0) + b_1 = \frac{a_1}{2} + b_1.$$

Thus, $\int_0^1 (a_1x + b_1)\,dx = \frac{a_1}{2} + b_1$.

6. $\int_a^b \cos x\,dx$ **for a > 0.**

Solution:

Comparing the given integral $\int_a^b \cos x\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$$f(x) = \cos x,\ a = a,\ b = b. \text{ So, } nh = b - a.$$

Also, $f(a + rh) = \cos(a + rh)$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_a^b \cos x\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} \cos(a + rh)$$

$$= \lim_{h\to 0} h\cos\left(a + \frac{nh}{2}\right)\cdot\frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} h\cos\left(a + \frac{b-a}{2}\right)\cdot\frac{\sin\frac{b-a}{2}}{\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} \cos\left(\frac{a+b}{2}\right)\cdot\sin\left(\frac{b-a}{2}\right)\cdot\frac{h}{\sin\frac{h}{2}}$$

$$= \frac{1}{2}\cdot 2\cos\left(\frac{a+b}{2}\right)\cdot\sin\left(\frac{b-a}{2}\right)\lim_{h\to 0}\frac{\frac{h}{2}}{\sin\frac{h}{2}}\cdot 2$$

$$= \left[\sin\left(\frac{a+b}{2} + \frac{b-a}{2}\right) + \sin\left(\frac{a+b}{2} - \frac{b-a}{2}\right)\right]\cdot 1$$

$$= \sin b + \sin a.$$

Thus, $\int_a^b \cos x\,dx = \sin b + \sin a$.

7. $\int_0^{\pi/2} \cos x\,dx$

Solution:

Comparing the given integral $\int_0^{\pi/2} \cos x\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

$$f(x) = \cos x,\ a = 0,\ b = \frac{\pi}{2}. \text{ So, } nh = b - a = \frac{\pi}{2}$$

Also, $f(a + rh) = \cos(a + rh)$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$\int_0^{\pi/2} \cos x\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} \cos(a + rh)$$

$$= \lim_{h\to 0} h \, . \cos\left(\frac{a+nh}{2}\right) \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} \cos\frac{\pi}{4} . \sin\frac{\pi}{4} . \frac{h}{\sin\frac{h}{2}} \qquad [\because a = 0]$$

$$= \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}} \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} . 2 = \frac{1}{2} . 2 . 1 = 1.$$

Thus, $\int_0^{\pi/2} \cos x \, dx = 1.$

8. $\int_0^{\pi/2}$ **sinx dx**

Solution:

Comparing the given integral $\int_0^{\pi/2} \sin x \, dx$ with the integral $\int_a^b f(x) \, dx$ then we get,

$$f(x) = \sin x, \; a = 0, \; b = \frac{\pi}{2}. \text{ So, } nh = b - a = \frac{\pi}{2}.$$

Also, $\quad f(a + rh) = \sin (a + rh).$

Now, by definition of limit as a sum we have

$$\int_a^b f(x) \, dx = f(a + rh).$$

Therefore,

$$\int_0^{\pi/2} \sin x \, dx = \lim_{h\to 0} h \sum_{r=1}^{n} \sin(a + rh)$$

$$= \lim_{h\to 0} h \, . \sin\left(a + \frac{nh}{2}\right) \frac{\sin\frac{nh}{2}}{\sin(h/2)}$$

$$= \lim_{h\to 0} h \sin\frac{\pi}{4} . \sin\frac{\pi}{4} . \frac{1}{\sin(h/2)} \left[\because a = 0, nh = \frac{\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}} \lim_{h\to 0} \frac{(h/2)}{\sin(h/2)} . 2$$

$$= \frac{1}{2} . 2 . 1 \qquad \left[\because \lim_{h\to 0} \frac{\sinh}{h} = 1\right]$$

$$= 1.$$

Thus, $\int_0^{\pi/2} \sin x \, dx = 1.$

9. $\int_a^b \frac{dx}{x^2}$ **(a > 0)**

Solution: Put $x^{-1} = y$ then $-\frac{dx}{x^2} = dy.$

Also, $x = a \Rightarrow y = \frac{1}{a}$ and $x = b \Rightarrow y = \frac{1}{b}$. Then,

$$I = \int_a^b \frac{dx}{x^2} = -\int_{1/a}^{1/b} dy$$

Comparing the given integral $\int_{1/a}^{1/b} dy$ with the integral $\int_a^b f(y) \, dy$ then we get,

$$f(y) = 1, \; a_1 = \frac{1}{a} \text{ and } b_1 = \frac{1}{b}. \text{ So, } nh = b_1 - a_1 = \frac{1}{b} - \frac{1}{a}$$

Now, by definition of limit as a sum we have

$$\int_a^b f(x) \, dx = f(a + rh).$$

Therefore,

$$I = -\int_{1/a}^{1/b} dy = -\lim_{h\to 0} h . \sum_{r=1}^{n} . 1$$

$$= -\lim_{h\to 0} hn = -\lim_{h\to 0} \left(\frac{1}{b} - \frac{1}{a}\right) = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}.$$

Thus, $\int_a^b \frac{dx}{x^2} = \frac{b-a}{ab}.$

10. $\int_a^b \frac{dx}{\sqrt{x}}$ **(a > 0).**

Solution: Put, $\sqrt{x} = y$ then $\frac{1}{2} x^{-1/2} dx = dy.$

When $x = a \Rightarrow y = \sqrt{a}$ and $x = b \Rightarrow y = \sqrt{b}$. Then,

$$I = \int_a^b \frac{dx}{\sqrt{x}} = \int_{\sqrt{a}}^{\sqrt{b}} 2 \, dy = 2 \int_{\sqrt{a}}^{\sqrt{b}} dy$$

Comparing the given integral $\int_{\sqrt{a}}^{\sqrt{b}} . \, dy$ with the integral $\int_a^b f(y) \, dy$ then we get,

$$f(y) = 1, \; a_1 = \sqrt{a}, \; b_1 = \sqrt{b}. \text{ So, } nh = b_1 - a_1 = \sqrt{b} - \sqrt{a}.$$

Also, $\quad f(a_1 + rh) = 1.$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$I = 2\int_{\sqrt{a}}^{\sqrt{b}} dy = 2\lim_{h\to 0} h \cdot \sum_{r=1}^{n} f(a_1 + rh)$$

$$= 2\lim_{h\to 0} \cdot h \cdot \sum_{r=1}^{n} \cdot 1$$

$$= 2\lim_{h\to 0} \cdot h \cdot n$$

$$= 2\lim_{h\to 0} (\sqrt{b} - \sqrt{a}) = 2(\sqrt{b} - \sqrt{a}).$$

Thus, $\int_a^b \frac{dx}{\sqrt{x}} = 2(\sqrt{b} - \sqrt{a}).$

11. $\int_a^b \sqrt{x}\,dx \;(a > 0).$

Solution: Put $x^{3/2} = y$. Then $\frac{3}{2}x^{1/2}\,dx = dy \Rightarrow \sqrt{x}\,dx = \frac{2dy}{3}$.

And, $x = a \Rightarrow y = a^{3/2}$, $x = b \Rightarrow y = b^{3/2}$. Then,

$$I = \int_a^b \sqrt{x}\,dx = \frac{2}{3}\int_{a^{3/2}}^{b^{3/2}} dy$$

Comparing the given integral $\int_{a^{3/2}}^{b^{3/2}} dy$ with the integral $\int_a^b f(y)\,dy$ then we get,

$f(x) = 1$, $a_1 = a^{3/2}$, $b_1 = b^{3/2}$. So, $nh = b_1 - a_1 = b^{3/2} - a^{3/2}$

Also, $f(a_1 + rh) = 1$.

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$I = \frac{2}{3}\int_{a^{3/2}}^{b^{3/2}} dy = \frac{2}{3}\lim_{h\to 0} h \sum_{r=1}^{n} f(a_1 + rh)$$

$$= \frac{2}{3}\lim_{h\to 0} h \cdot \sum_{r=1}^{n} \cdot 1$$

$$= \frac{2}{3}\lim_{h\to 0} h \cdot n$$

$$= \frac{2}{3}\lim_{h\to 0} (b^{3/2} - a^{3/2}) = \frac{2}{3}(b^{3/2} - a^{3/2}).$$

Thus, $\int_a^b \sqrt{x}\,dx = \frac{2}{3}(b^{3/2} - a^{3/2}).$

12. $\int_0^{\pi/4} \sin^2 x\,dx$

Solution: Here,

$$I = \int_0^{\pi/4}\left(\frac{1 - \cos 2x}{2}\right)dx = \frac{1}{2}\int_0^{\pi/4} dx - \frac{1}{2}\int_0^{\pi/4} \cos 2x\,dx$$

$$= \frac{1}{2}(I_1 - I_2)$$

Comparing the given integral $\int_0^{\pi/4} dx$ and $\int_0^{\pi/4} \cos 2x\,dx$ with the integral $\int_a^b f(x)\,dx$ then we get,

for I_1, $f(x) = 1$, $a = 0$, $b = \frac{\pi}{4}$. So, $nh = b - a = \frac{\pi}{4}$.

Also, $f(a + rh) = 1$.

And for I_2, $f(x) = \cos 2x$, $a = 0$, $b = \frac{\pi}{4}$. So, $nh = b - a = \frac{\pi}{4}$.

Also, $f(a + rh) = \cos 2(a + rh)$

Now, by definition of limit as a sum we have

$$\int_a^b f(x)\,dx = f(a + rh).$$

Therefore,

$$I = \frac{1}{2}(I_1 - I_2) = \frac{1}{2}\int_0^{\pi/4} dx - \frac{1}{2}\int_0^{\pi/4} \cos 2x\,dx$$

$$= \frac{1}{2}\lim_{h\to 0} h \sum_{r=1}^{n} 1 - \frac{1}{2}\lim_{h\to 0} h \sum_{r=1}^{n} \cos 2(a + rh)$$

$$= \frac{1}{2}\lim_{h\to 0} h \cdot n - \frac{1}{2}\lim_{h\to 0} h \cos 2\left(a + \frac{nh}{2}\right)\frac{\sin 2\left(\frac{nh}{2}\right)}{\sin 2\left(\frac{h}{2}\right)}$$

$$= \frac{1}{2}\lim_{h\to 0} \cdot \frac{\pi}{4} - \frac{1}{2}\lim_{h\to 0} h \cdot \cos\frac{\pi}{4} \cdot \sin\frac{\pi}{4} \cdot \frac{1}{\sinh}$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \lim_{h\to 0} \frac{h}{\sinh}$$

$$= \frac{\pi}{8} - \frac{1}{4} \cdot 1 = \frac{\pi}{8} - \frac{1}{4}.$$

Thus, $\displaystyle\int_0^{\pi/4} \sin^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}.$

13. $\displaystyle\int_0^{\pi/4} \cos^2 x \, dx$

Solution: Here,

$$I = \int_0^{\pi/4} \cos^2 x \, dx$$

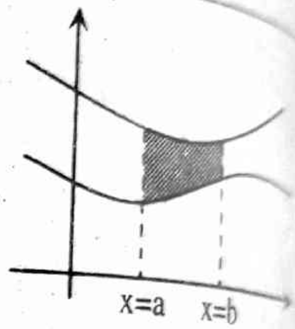
$$= \frac{1}{2}\int_0^{\pi/4} dx + \frac{1}{2}\int_0^{\pi/4} \cos 2x \, dx$$

Process as Q. 12 then will obtain $I = \frac{\pi}{8} + \frac{1}{4}.$

Chapter 12 Application of Integration
AREA

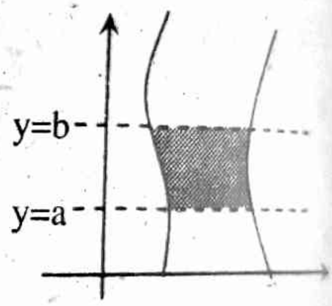
List of Formulae

The area of the region bounded by two curves $y_1 = f(x)$, $y_2 = g(x)$ in between $x = a$ and $x = b$ then,



$$A = \int_a^b (y_1 - y_2)\, dx.$$

The area of the region bounded by two curves $x_1 = f(y)$, $x_2 = g(y)$ in between $y = a$ and $y = b$ then,



$$A = \int_a^b (x_1 - x_2)\, dy.$$

Exercise 12.1

A. Find the area bounded by the x–axis and the following curve and ordinates.

1. $y^2 = 4x, x = 4, x = 9$.

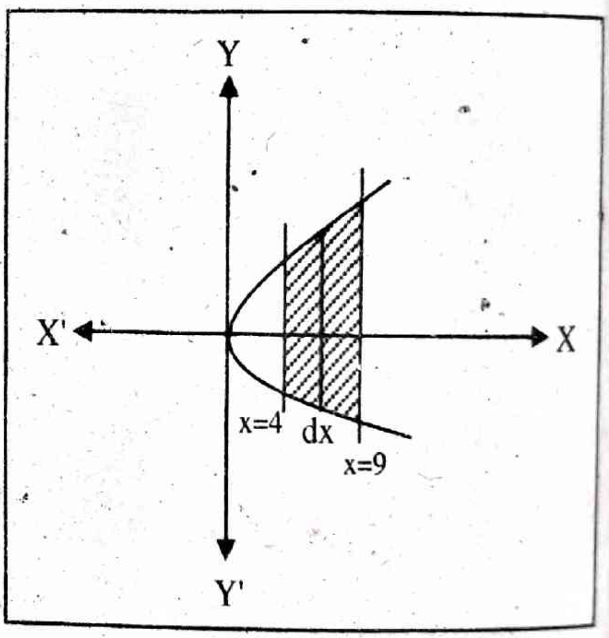
Solution: Given parabola is, $y^2 = 4x$

which is has vertex (0, 0) and a = 1 > 0 and line of symmetry is $y = 0$.

Also, x = 4 and x = 9.

Graph of given curve and ordinates are shown in figure.

Now,



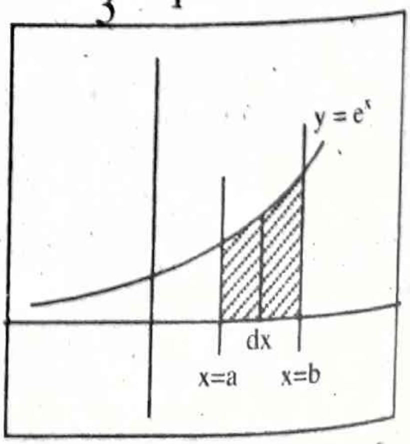
$$\text{Required area, } A = \int_{x=4}^{9} y\,dx$$

$$= \int_{4}^{9} 2\sqrt{x}\,dx$$

$$= \left[2 \cdot \frac{x^{3/2}}{3/2}\right]_4^9$$

$$= \frac{4}{3}(9^{3/2} - 4^{3/2}) = \frac{4}{3}[27-8] = \frac{4}{3} \times 19 = \frac{76}{3} = 25\frac{1}{3} \text{ sq. unit.}$$

Thus, the area of the area bounded by the given curves is $\frac{76}{3}$ sq. units.



2. $y = e^x, x = a, x = b$

Solution: Given curves are,

$$y = e^x, x = a, x = b$$

Curve of given function and ordinate can be drawn as, The region bounded by the curves $y = e^x$, $x = a$, $x = b$ is the shaded portion. Taking vertical strip in the region of integration and taking limit from $x = a$ to $x = b$ we get the area of the region is,

$$A = \int_a^b e^x\,dx = [e^x]_a^b = (e^b - e^a) \text{ sq. unit}$$

Thus, the area bounded by the given curves $y = e^x$, $x = a$, $x = b$ is $(e^b - e^a)$ sq. unit.

3. $y = x^2 - 4$, $x = 3$, $x = 5$

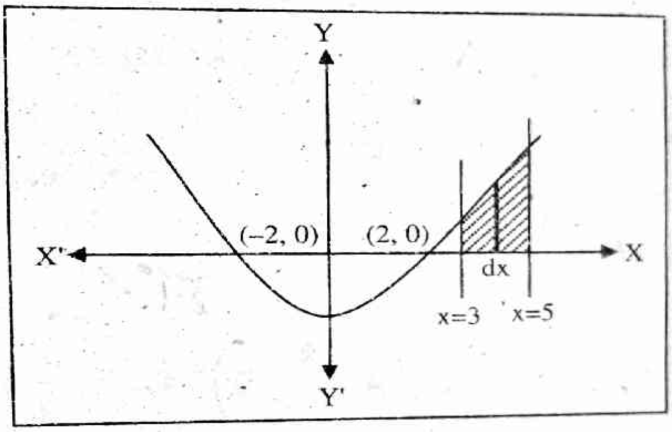
Solution: Given curves are,

$$y = x^2 - 4$$

$$\Rightarrow \quad x^2 = (y + 4)$$

which is a parabola having vertex $(0, -4)$ and $a = \frac{1}{4} > 0$. So its curve is shown in figure including given ordinates.

The region bounded by the curves $y = x^2 - 4$, $x = 3$, $x = 5$ is the shaded portion. Taking vertical strip as dx and taking limits from $x = 3$ to $x = 5$, we get the area of the region is,



$$A = \int_3^5 (x^2 - 4)\,dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_3^5$$

$$= \left[\frac{125}{3} - 20\right] - \left[\frac{27}{3} - 12\right]$$

$$= \frac{65}{3} + 3 = \frac{74}{3}$$

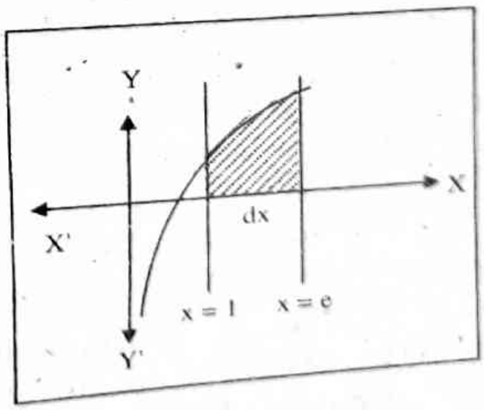
Thus, the area bounded by the given curves $y = x^2 - 4$, $x = 3$, $x = 5$ is $\frac{74}{3}$ sq. unit.

4. $y = \log x$, $x = 1$, $x = e$.

Solution: Given curves are,

$$y = \log x, \; x = 1, \; x = e$$

Tracing the curves the region bounded by the curves $y = \log x$, $x = 1$, $x = e$ is the shaded portion. Taking, vertical strip dx as shown in figure are limits from $x = 1$ to $x = e$ then we get the area of the region is,



$$A = \int_1^e \log x\,dx = [\log x \,.\, x - x]_1^e$$

$$= (e \log e - e) - [(\log 1)(1) - 1]$$

$$= (e - e) - (-1) = 1.$$

Thus, the area bounded by the given curves $y = x^2 - 4$, $x = 3$, $x = 5$ is 1 sq. unit.

B. Find the area bounded by

1. The curve $y = 2 - x^2$ and the line $y = -x$.

Solution: Given curves are,

$$y = 2 - x^2$$

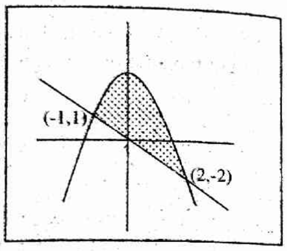
$$\Rightarrow x^2 = -(y - 2) \quad \dots (i)$$

$$\text{and} \quad y = -x \quad \dots (ii)$$

Clearly the curve (i) is a parabola having vertex at (0, 2) whose line of symmetry is $x = 0$ i.e. y-axis and having down openward.

Also, the line (ii) passes through (0, 0) and (1, –1).

With this information, the sketch of the curves is shown in figure. Therefore the region bounded by the curves (i) and (ii) is the shaded portion in the figure. Solving the equations (i) and (ii) we get the point of the intersections are (–1, 1) and (2, –2).



Now, area of the region is

$$A = \int_{x=-1}^{2} (y_1 - y_2)\,dx$$

$$= \int_{x=-1}^{2} (2 - x^2 + x)\,dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) = \frac{10}{3} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2} = 4.5$$

Thus, the area bounded by the given curves $y = 2 - x^2$ and the line $y = -x$ is 4.5 sq. units.

2. $x + y^2 = 0$ and $x + 3y^2 = 2$.

Solution: Given curves are,

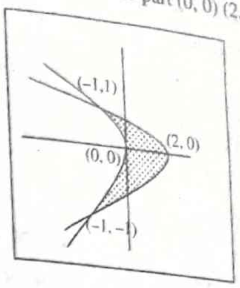
$$y^2 = -x \quad \dots (i)$$

$$\text{and} \quad y^2 = -\frac{1}{3}(x - 2) \quad \dots (ii)$$

Clearly the curves are parabolas in which (i) has vertex at (0, 0) with line of symmetry is $y = 0$ i.e. x-axis and having left open ward. Also, the parabola (ii) has vertex at (2, 0) with line of symmetry is $y = 0$ i.e. x-axis and having left open ward.

With this information, the sketch of the curves is shown in figure. Therefore the region bounded by the curves (i) and (ii) is the shaded portion in the figure. Solving the equations (i) and (ii) we get the point of the intersections are (–1, 1) and (–1, –1).

From figure it is clear that the required region is 2 times of the part (0, 0) (2, 0) and (–1, 1)



Now, the area of the required region be

$$A = 2\left|\int_{y=0}^{1} (x_1 - x_2)\,dy\right|$$

$$= 2\left|\int_{y=0}^{1} (-y^2 - 2 + 3y^2)\,dy\right|$$

$$= 2\left|\left[\frac{-y^3}{3} - 2y + y^3\right]_0^1\right|$$

$$= 2\left|-\frac{1}{3} - 2 + 1\right| = \frac{8}{3}$$

Thus, the area bounded by the given curves $x + y^2 = 0$ and $x + 3y^2 = 2$ is $\frac{8}{3}$ sq. units.

3. The curve $y^2 = 4x$ and the line $y = x$.

Solution: Given curves are,

$$y^2 = 4y \quad \dots (i)$$

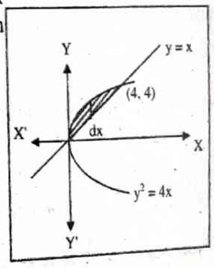
$$\text{and} \quad y = x \quad \dots (ii)$$

Clearly the curve (i) is a parabola having vertex at (0, 0) whose line of symmetry is $y = 0$ i.e. x-axis and having right open ward.

Also, the line (ii) passes through (0, 0) and (1, 1).

With this information, the sketch of the curves is shown in figure. Therefore the region bounded by the curves (i) and (ii) is the shaded portion in the figure. Solving the equations (i) and (ii) we get the point of the intersections are (0, 0) and (4, 4). Clearly the region has no symmetrical parts.

Now, Taking vertical strip dx and limit from line $x = 0$ to $x = 4$, we get the required area of the region is



$$A = \int_0^4 (y_1 - y_2)\,dx$$

$$= \int_0^4 \left(2\sqrt{x} - x\right)\,dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^4 = \left[\frac{32}{3} - 8\right] = \frac{8}{3}$$

Thus, the area bounded by the given curves $y^2 = 4x$ and the line $y = x$ is $\frac{8}{3}$ sq. units.

4. The curve $x = y^2$ and $x = -2y^2 + 3$. **[2006, Spring]**

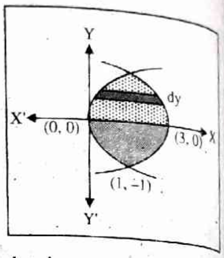
Solution: Given, $x = y^2 \Rightarrow y^2 = x$ ... (i)

which is a parabola with vertex (0, 0) and $a = \frac{1}{4} > 0$, and its line of symmetry is $y = 0$.

Also, $x = -2y^2 + 3$

$\Rightarrow 2y^2 = -x + 3 \Rightarrow y^2 = -\frac{1}{2}(x - 3)$ ... (ii)

which is a of parabola with vertex (3, 0) and $a = -\frac{1}{8} < 0$, and its line of symmetry $y = 0$.



Solving equation (i) and (ii) we get the points of intersection of equation (i) and (ii) are (1, 1) and (1, −1).

With these information, the sketch of the region is the shaded portion in the figure. Clearly the region has two symmetrical parts.

So the area of the region is the twice of the region having extreme points at (0, 0), (3, 0) and (1, 1).

Therefore, the area of the region is,

$$A = 2\int_{y=0}^{1} (x_1 - x_2)\, dy$$

$$\text{i.e. } A = 2\int_{y=0}^{1} \{(-2y^2 + 3) - y^2\}\, dy$$

$$= 6\int_{y=0}^{1} (-3y^2 + 3)\, dy$$

$$= 6\int_{y=0}^{1} (1 - y^2)\, dy$$

$$= 6\left[y - \frac{y^3}{3}\right]_0^1 = 6\left[\left(1 - \frac{1}{3}\right) - 0\right] = 6 \times \frac{2}{3} = 4.$$

Thus, the area bounded by the given curves $x = y^2$ and $x = -2y^2 + 3$ is 4 sq. units.

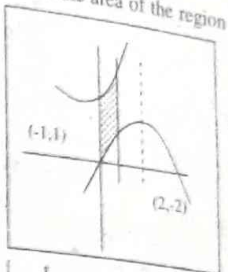
5. $y = \sec^2 x$, $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Solution: The required region is bounded by the curves

$$y = \sec^2 x,\ y = \sin x,\ x = 0 \text{ and } x = \frac{\pi}{4}$$

The sketch of the region bounded by given curves is the shaded portion in the figure.

Clearly the region has no symmetrical part. Therefore the area of the region is,



$$A = \left|\int_{x=0}^{\pi/4} (y_1 - y_2)\, dx\right|$$

$$\text{i.e. } A = \left|\int_{x=0}^{\pi/4} [\sec^2 x - \sin x]\, dx\right|$$

$$= \left|[\tan x + \cos x]_0^{\pi/4}\right|$$

$$= \left|\left(1 - \frac{1}{\sqrt{2}}\right) - (0 + 1)\right| = \left|-\frac{1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}.$$

Thus, the area bounded by the given curves $y = \sec^2 x$, $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$ is $\frac{1}{\sqrt{2}}$ sq. units.

6. the curve $x = y^3$ and $x = y^2$.

Solution: The required region is bounded by the curves $x = y^3$ and $x = y^2$.

Travelling path for $x = y^3$

x	1	−1	8	−8
y	1	−1	2	−2

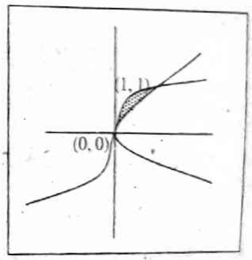
Travelling path for $x = y^2$

x	1	1	4	4
y	1	−1	2	−2

The region bounded by the given curves is the shaded portion. Clearly, the required region is the shaded portion that has corners (0, 0) and (1, 1).

From figure it is clear that the shaded region has no symmetrical part.

Now, the area of the bounded region is,



$$A = \int_{y=0}^{1} (x_1 - x_2)\, dy$$

$$= \int_{y=0}^{1} (y^2 - y^3)\, dy$$

$$= \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Thus, the area bounded by the given curves $x = y^3$ and $x = y^2$ is $\frac{1}{12}$ sq. units.

7. The curve $y = 2x - x^2$ and the line $y = -3$.

Solution: Given curves are,

$$y = 2x - x^2$$

$$\Rightarrow -y = x^2 - 2x$$

$$\Rightarrow -y + 1 = x^2 - 2x + 1$$

$$\Rightarrow (x - 1)^2 = -(y - 1) \quad \text{... (i)}$$

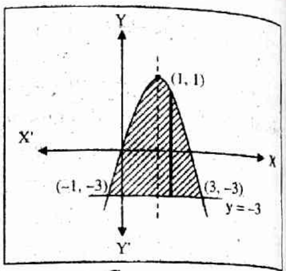
This is a parabola having vertex (1, 1), $a = -\frac{1}{4} < 0$, line of symmetry is $x = 1$ and which is passing through origin.

And a given line is,

$$y = -3 \qquad \text{... (ii)}$$

Solving (i) and (ii) we get the point of intersection are (–1, –3) and (3, –3). With these information, the trace of the curves (i) and (ii) is shown in figure. The bounded region by curve (i) and curve (ii) is the shaded portion in the figure.

Clearly the region has no symmetrical part while taking the vertical strip. Now, taking the vertical strip as dx and limit from x = –1 to x = 3, then the area of bounded region by equation (i) and equation (ii) is



$$A = \left| \int_{y=-1}^{3} (y_1 - y_2)\, dx \right|$$

$$\text{i.e. } A = \left| \int_{-1}^{3} \{-3 - (2x - x^2)\}\, dx \right|$$

$$= \left| \int_{-1}^{3} (x^2 - 2x - 3)\, dx \right|$$

$$= \left| \left[\frac{x^3}{3} - \frac{2x^2}{2} - 3x \right]_{-1}^{3} \right|$$

$$= \left[\frac{3^3}{3} - 3^2 - 3 \times 3 \right] - \left[\frac{(-1)^3}{3} - (-1)^2 - 3 \times -1 \right]$$

$$= \left| (9 - 9 - 9) - \left(-\frac{1}{3} - 1 + 3 \right) \right|$$

$$= \left| -9 + \frac{1}{3} - 2 \right|$$

$$= \left| -11 + \frac{1}{3} \right| = \left| \frac{-33 + 1}{3} \right| = \left| \frac{-32}{3} \right| = \frac{32}{3}.$$

Thus, the area bounded by the given curve $y = 2x - x^2$ and the line $y = -3$ is $\frac{32}{3}$ sq. units.

8. The curve x + y = 2, on the left by y = x² and below by x–axis. **[2009 Spring]**

Solution: Given line is,

$$x + y = 2 \qquad \text{... (i)}$$

Since the line (i) intersects the axes at (0, 2) to y-axis and (2, 0) at x-axis.

And the given curve is,

$$y = x^2 \qquad \text{... (ii)}$$

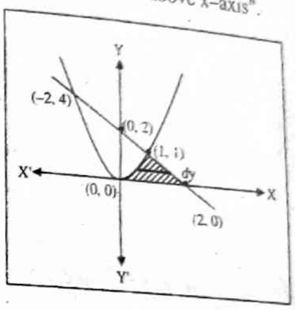
This is parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is $x = 0$.

Now, solving the equation (i) and (ii) we get the point of intersection of curve (i) and (ii) are (1, 1) and (–2, 4).

Now, tracing curve of equation (i) and (ii) below is given aside. Thus the region bounded by the curves is shown by the shaded in the figure.

Clearly the region has no symmetrical parts. The region bounded on "Left by $y = x^2$" means "along right" "below by x–axis" means "above x–axis".

Now, taking horizontal strip dy in which the strip moves from y = 0 to y = 1, area of shaded region is given by



$$A = \int_{y=0}^{1} (y_1 - y_2)\, dy$$

$$\text{i.e. } A = \int_{y=0}^{1} \{(2 - y) - y^{1/2}\}\, dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{2y^{3/2}}{3} \right]_0^1$$

$$= \left(2 - \frac{1}{2} - \frac{2}{3} \right) - 0 = \frac{12 - 3 - 4}{6} = \frac{5}{6} \text{ sq. unit}$$

Thus, the area bounded by the given curve x + y = 2, on the left by $y = x^2$ and below by x–axis is $\frac{5}{6}$ sq. units.

9. The curve $y = \sin\left(\frac{\pi x}{2}\right)$ and the line y = x.

Solution: Given curve is,

$$y = \sin\left(\frac{\pi x}{2}\right) \qquad \text{... (i)}$$

and the line is,

$$y = x \qquad \text{... (ii)}$$

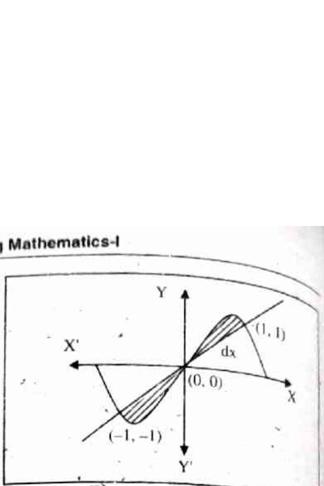
Clearly the curve (i) is the sine curve. And, the line (ii) passes through the point (0, 0) and (1, 1). Solving (i) and (ii) then we get the point of contact of (i) and (ii) are (0, 0), (1, 1) and (–1, –1).

Therefore, the graph of the region bounded by (i) and (ii) is sketch below. The bounded region is the shaded portion in the figure.

Clearly, it has two symmetrical parts. So, taking, vertical strip dx in region that has limits from x = 0 to x = 1.

Now the area of the region bounded by the given curves is,

$$A = \int_{x=0}^{1} (y_1 - y_2)\, dx.$$



i.e. $A = 2\int_0^1 \left\{\sin\left(\frac{\pi x}{2}\right) - x\right\} dx$

$$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1$$

$$= 2\left\{\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{1}{2}\right] - \left[-\frac{2}{\pi}\cos 0 - 0\right]\right\}$$

$$= 2\left\{-\frac{1}{2} + \frac{2}{\pi}\right\}$$

$$= -1 + \frac{4}{\pi} = \frac{4}{\pi} - 1 \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = \sin\left(\frac{\pi x}{2}\right)$ and the line $y = x$ is $\left(\frac{4}{\pi} - 1\right)$ sq. units.

10. The curves $y = x^2$ and $x = y^2$.

Solution: Given curve is,

$$y = x^2 \qquad \dots \text{(i)}$$

and the curve is,

$$y^2 = x \qquad \dots \text{(ii)}$$

Since the curve (i) is a parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is $x = 0$.

And, the curve (ii) is also a parabola having vertex (0, 0), $a = \frac{1}{4} > 0$ and line of symmetry is $y = 0$.

The sketch of the curve is shown in the aside figure. Clearly the region bounded by the curves (i) and (ii) is the shaded portion in the figure, which has extreme points (0, 0) and (1, 1).

Clearly, the region has no symmetrical parts.

Now, the area of the region bounded by the curves (i) and (ii) is,

$$A = \int_{x=0}^{1} (y_1 - y_2)\, dx$$

i.e. $A = \int_0^1 (\sqrt{x} - x^2)\, dx$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \frac{1}{3} \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = x^2$ and $x = y^2$ is $\frac{1}{3}$ sq. units.

11. The curve $y = 3x$, the x-axis and the ordinate $x = 2$.

Solution: Given curve are,

$$y = 3x,\ x\text{-axis i.e. } y = 0 \text{ and } x = 2.$$

Clearly the line $y = 3x$ passes through (0, 0) and (1, 3).

Then the sketch of the bounded region by given curve, is shown in the figure by shaded. Clearly the region has no symmetrical part.

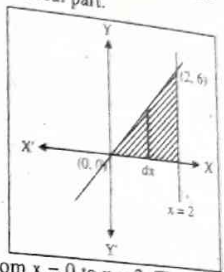
From first and second curves, $y = 0 \Rightarrow x = 0$.

From second and third curves $x = 2 \Rightarrow y = 0$.

From third and first curves $y = 0$, $x = 0$.

When $x = 2$ we get $y = 6$.

So, intersection point of $y = 3x$ and x-axis is (0, 0) and intersection point of $y = 3x$ and $x = 2$ is (2, 6).



Now, taking vertical region dx that has limit from $x = 0$ to $x = 2$. Then the area of the region is,

$$A = \int_0^2 (y_1 - y_2)\, dx$$

i.e. $A = \int_0^2 3x\, dx = \left[3 \cdot \frac{x^2}{2}\right]_0^2 = 3 \times \frac{2^2}{2} - 0 = 6$ sq. unit

Thus, the area bounded by the given curve $y = 3x$, the x-axis and the ordinate $x = 2$ is 6 sq. units.

12. The curve y-axis and the curve $x = y^2 - y^3$.

Solution: For y-axis, $x = 0$.

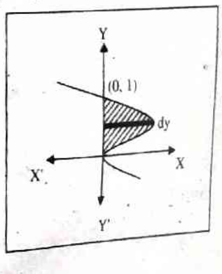
Also, $x = y^2 - y^3$

X	0	0	0.125	2	–4
Y	0	1	0.5	–1	2

The required area bounded by the curve $x = y^2 - y^3$ and y-axis is shown in figure by shaded. Clearly the region has no symmetrical parts. Taking horizontal strip dy that has limits from $y = 0$ to $y = 1$.

Now, the area of the region bounded by y-axis and the curve $x = y^2 - y^3$ is,

$$A = \int_0^1 (x_1 - x_2)\, dy$$



i.e. $A = \int_0^1 (y^2 - y^3)\, dy$

$= \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$

$= \left(\frac{1}{3} - \frac{1}{4}\right) - 0 = \frac{4-3}{12} = \frac{1}{12}$ sq. unit.

Thus, the area bounded by the given curve y–axis and the curve $x = y^2 - y^3$ is $\frac{1}{12}$ sq. units.

13. The curve $y^2 = 12x$ the line $x = 12$.

Solution: Given curve is,

$y^2 = 12x$ ... (i)

which is a parabola having vertex at (0, 0), a = 3 > 0 and the line of symmetry is y = 0.

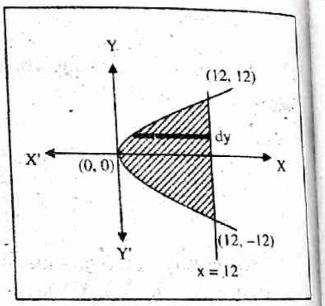
And, the given line is,

$x = 12$ ... (ii)

Solving the equations (i) and (ii) then we get the points of intersection are (12, –12) and (12, 12).

The sketch of the region bounded by (i) and (ii) is the shaded portion in the figure. Clearly the region has two symmetrical parts from the x-axis. So, the area of the bounded region is the twice of the region having extreme points (0, 0), (12, 0) and (12, 12).

Now the area bounded by given curves is



$A = 2\int_0^{12} (x_1 - x_2)\, dy$

i.e. $A = 2\int_0^{12} \left(12 - \frac{y^2}{12}\right) dy$

$= 2\left[12y - \frac{y^3}{36}\right]_0^{12}$

$= 2\left[\left(12 \times 12 - \frac{(12)^3}{36}\right) - 0\right]$

$= 2\,(144 - 48) = 2 \times 96 = 192$ sq. unit.

Thus, the area bounded by the given curve $y^2 = 12x$ the line $x = 12$ is 192 sq. units.

14. The x–axis and the curve $y = 5x - x^2 - 6$.

Solution: For x–axis, $y = 0$ ... (i)

And the given curve is,

$y = 5x - x^2 - 6$

$\Rightarrow -y = x^2 - 2.x.\frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6$

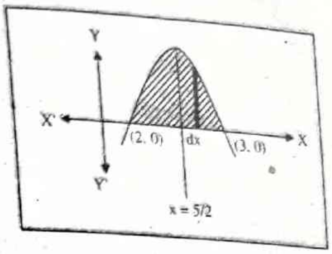
$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -y + \frac{25}{4} - 6$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{-4y + 25 - 24}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{-4y + 1}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -y + \frac{1}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -1\left(y - \frac{1}{4}\right)$ ... (ii)



This is a parabola having vertex at $\left(\frac{5}{2}, \frac{1}{4}\right)$, $a = -\frac{1}{4} < 0$ and the line of symmetry is $x = \frac{5}{2}$.

Solving the equations (i) and (ii) then we get the point of intersection of x–axis and parabola are (2, 0) and (3, 0).

With this information the trace of the region is given in the aside figure. Clearly the region has no symmetrical parts.

Now, taking vertical strip dx that has the limits from x = 2 to x = 3, required area bounded by x–axis and given parabola is

$A = \int_{x=2}^{1} (y_1 - y_2)\, dx$

i.e. $A = \int_{x=2}^{1} (5x - x^2 - 6)\, dx$

$= \left[5\frac{x^2}{2} - \frac{x^3}{3} - 6x\right]_2^3$

$= \left(\frac{45}{2} - 9 - 18\right) - \left(10 - \frac{8}{3} - 12\right)$

$= \frac{45}{2} - 27 + 2 + \frac{8}{3}$

$= \frac{45}{2} + \frac{8}{3} - 25 = \frac{135 + 16 - 150}{6} = \frac{151 - 150}{6} = \frac{1}{6}$ sq. unit

Thus, the area bounded by the given curve x–axis and the curve $y = 5x - x^2 - 6$ is $\frac{1}{6}$ sq. units.

15. The curves $y = x^4 - 2x^2$ and $y = 2x^2$

Solution: Given curve is,

$$y = x^4 - 2x^2 \quad \dots (i)$$

x	0	$\sqrt{2}$	$-\sqrt{2}$	1	-1	2	-2
y	0	0	0	-1	-1	8	8

And the given curve is,

$$y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y \quad \dots (ii)$$

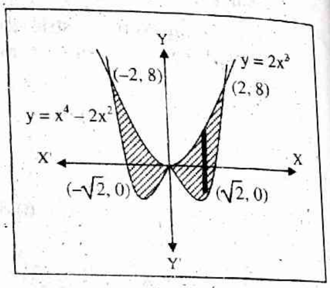
This is a parabola having vertex (0, 0), $a = \frac{1}{8} > 0$ and the line of symmetry is $x = 0$.

Solving the equations (i) and (ii) then we get the points of intersection of the curves (i) and (ii) are (0, 0), (2, 8) and (–2, 8).

With this information the trace of the graph of the given curves are shown in the figure. Clearly the figure has two symmetrical parts. So, the area of the bounded region is the twice of the region having extremity at (0, 0) and (2, 8).

Now, taking vertical strip dx over the limits from $x = 0$ to $x = 2$. Then the area of region bounded by given curves is,

$$A = 2\int_0^2 (y_1 - y_2)\,dx$$



$$\text{i.e. } A = 2\int_0^2 \{2x^2 - (x^4 - 2x^2)\}\,dx$$

$$= 2\int_0^2 (4x^2 - x^4)\,dx$$

$$= 2\left[4 \cdot \frac{x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= 2\left\{\left(\frac{32}{3} - \frac{32}{5}\right) - 0\right\}$$

$$= 2\left(\frac{32 \times 5 - 32 \times 3}{15}\right) = 2 \times \frac{2 \times 32}{15} = \frac{128}{15} \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = x^4 - 2x^2$ and $y = 2x^2$ is $\frac{128}{15}$ sq. units.

16. The curves $y = \sqrt{4 - x}$, $x = 0$, $y = 0$ in the first quadrant.

Solution: Given curve is,

$$y = \sqrt{4 - x} \Rightarrow y^2 = 4 - x \Rightarrow y^2 = -1\,(x - 4).$$

This is a parabola having vertex (4, 0), $a = -\frac{1}{4} < 0$, line of symmetry is $y = 0$.

Also given that, $x = 0$, $y = 0$.

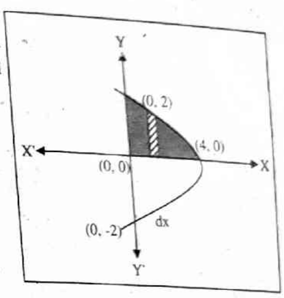
And, $x = 0 \Rightarrow y = \sqrt{4} = \pm 2$ and $y = 0 \Rightarrow x = 4$

Thus, the points of intersection of coordinates and given parabola are (0, 2), (0, –2) and (4, 0).

Therefore, the bounded region by the given curves $y = \sqrt{4 - x}$, $x = 0$, $y = 0$ in the first quadrant is shaded portion that has extremity at (0, 0), (0, –2) and (4, 0).

Clearly, the bounded region has no symmetrical parts.

Now, taking vertical strip dx in region bounded by curves and taking limits from $x = 0$ to $x = 4$, then the area of bounded region is,



$$A = \int_0^4 (y_1 - y_2)\,dx$$

$$\text{i.e. } A = \int_0^4 \left[\sqrt{4 - x} - 0\right] dx$$

$$= \left[\frac{-(4 - x)^{3/2}}{3/2}\right]_0^4 = 0 - \left(-\frac{2 \times 4^{3/2}}{3}\right) = \frac{2 \times 8}{3} = \frac{16}{3} \text{ sq. unit.}$$

Thus, the area bounded by the given curve $y = \sqrt{4 - x}$, $x = 0$, $y = 0$ in the first quadrant is $\frac{16}{3}$ sq. units.

17. The curve $y = x^2 + 1$ and the line $y = -x + 3$. **[2011 Spring][2002]**

Solution: Given curve is,

$$y = x^2 + 1$$

$$\Rightarrow x^2 = \pm (y - 1) \quad \dots (i)$$

This is a parabola having vertex (0, 1), $a = \frac{1}{4} > 0$ and the line of symmetry is $x = 0$.

Also the given line is,

$$y = -x + 3 \quad \dots (ii)$$

which cuts x–axis at (3, 0) and y–axis at (0, 3).

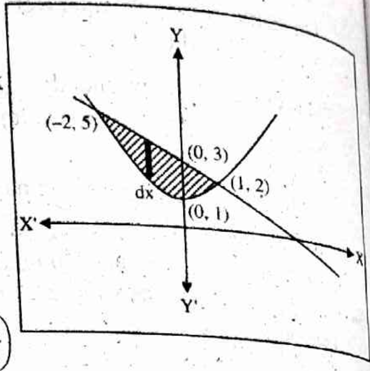
Solving given curves (i) and (ii) then we get the point of intersection of the curves are (1, 2) and (–2, 5).

With this information, the sketch of the region bounded by the curves (i) and (ii) is shown in the figure by shaded.

Clearly the region has no symmetrical parts.

Now, taking vertical strip dx and limits from x = –2, to x = 1 in bounded region, then the area of the region is,

$$A = 2\int_0^2 (y_1 - y_2)\,dx$$



$$\text{i.e. } A = \int_{-2}^{1} [(-x+3) - (x^2+1)]\,dx$$

$$= \int_{-2}^{1} (2 - x - x^2)\,dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)$$

$$= \frac{3}{2} - \frac{1}{3} + 6 - \frac{8}{3} = \frac{45-18}{6} = \frac{9}{2} = 4.5 \text{ sq. unit}$$

Thus, the area bounded by the given curve $y = x^2 + 1$ and the line $y = -x + 3$ is 4.5 sq. units.

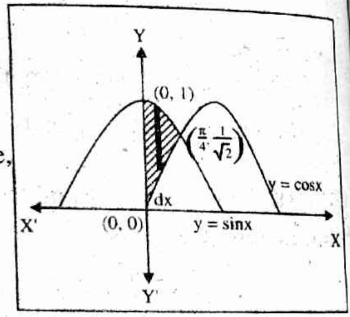
18. The curve y = sinx, y = cosx and the line y–axis in the first quadrant.

Solution: Given curves are,

$y = \sin x$ … (i)

$y = \cos x$ … (ii)

and, y–axis i.e. $x = 0$



Tracing graph of given curves shown aside.

Solving the curves (i) and (ii) then we get the points of intersection of curves is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Therefore, the region bounded by the curves (i), (ii) and y-axis is the shaded portion in the figure that has extreme points at (0, 1), (0, 0) and $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

Clearly, the bounded portion has no symmetrical part.

Now, the area of the bounded region is,

$$A = \int_0^{\pi/4} (y_1 - y_2)\,dx$$

$$\text{i.e. } A = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= [\sin x + \cos x]_0^{\pi/4}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) = \frac{2}{\sqrt{2}} - 1 = (\sqrt{2} - 1) \text{ sq. unit}$$

Thus, the area bounded by the given curve y = sinx, y = cosx and the line y–axis in the first quadrant is $(\sqrt{2} - 1)$ sq. units.

19. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Given curve is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \frac{b}{a}\sqrt{(a^2 - x^2)} \quad \text{… (i)}$$

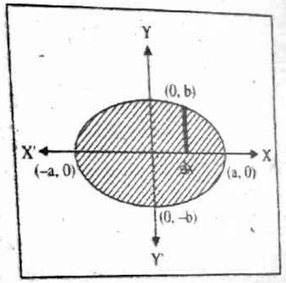
This is an ellipse having centre at (0, 0) and suppose a > b.

The sketch of the ellipse is shown in the figure aside with a > b.

Clearly it has four symmetrical parts.

Taking vertical strip dx in 1st quadrant and limits from x = 0 to x = a, then total area of ellipse is,

A = 4 × Area of ellipse in 1st quadrant



$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx$$

$$= \frac{4b}{a}\int_0^a \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left\{\left(0 + \frac{a^2}{2}.\sin^{-1}(1)\right) - \left(0 + \frac{a^2}{2}\sin^{-1}0\right)\right\}$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2} - 0\right] = \frac{4b}{a}\cdot\frac{a^2\pi}{4} = \pi ab \text{ sq. unit}$$

Thus, the area bounded by the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq. units.

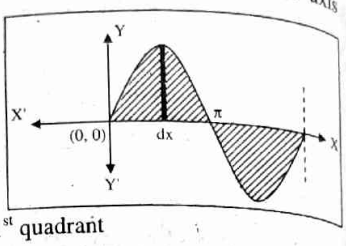
20. The curve y = sinx and x–axis between x = 0 and x = 2π.

Solution: Given curve is,

$y = \sin x$

Given that the region is bounded by the curve y = sinx, x-axis (i.e. y = 0), x = 0 and x = 2π.

The region is the shaded portion in the aside figure.



Clearly the portion has two symmetrical parts. So, the area of the portion is the twice of the portion bounded from x = 0 to x = 2π is,

$$A = 2 \times \text{Area bounded in } 1^{st} \text{ quadrant}$$

$$\text{i.e. } A = 2\int_0^{\pi} (y_1 - y_2)\, dx$$

$$\text{i.e. } A = 2\int_0^{\pi} (\sin x - 0)\, dx$$

$$= 2\int_0^{\pi} \sin x\, dx$$

$$= 2[-\cos x]_0^{\pi} = 2[\cos\pi + \cos 0] = 2 \text{ sq. units}$$

Thus, the area bounded by the given curve y = sinx and x-axis between x = 0 and x = 2π is 2 sq. units.

C 1. Find the area of the region of the circle $x^2 + y^2 = 4$ cut off by the line x–2y = –2 in the first two quadrants. [2016 Fall][2013 Fall][2006, Fall]

Solution: Given curve is,

$$x^2 + y^2 = 4 \quad \dots (i)$$

which is circle having centre at (0, 0) and radius = 2. Clearly, it is symmetry about both axis

$$x - 2y = -2 \quad \dots (ii)$$

$$\Rightarrow \frac{x}{(-2)} + \frac{y}{1} = 1$$

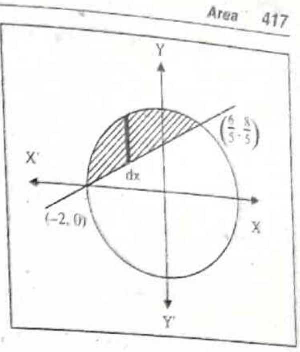
which is a line, cuts x and y-axis on (–2, 0) and (0, 1).

Also, solving the equations (i) and (ii) then we get the point of intersection are (–2, 0) and $\left(\frac{6}{5}, \frac{8}{5}\right)$.

Now, trace of given curve and line is shown in the figure.

Clearly the region bounded by the given curve and the line (i.e. the shaded part) has no symmetrical parts.

Now, taking any vertical strip dx and limits from x = –2 to x = $\frac{6}{5}$. Then the area bounded by the curves is,



$$A = \int_{-2}^{6/5} (y_1 - y_2)\, dx$$

$$\text{i.e. } A = \int_{-2}^{6/5} \left\{\sqrt{4 - x^2} - \frac{x+2}{2}\right\} dx$$

$$= \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\left(\frac{x^2}{2} + 2x\right)\right]_{-2}^{6/5}$$

$$= \left\{\frac{6}{10}\sqrt{4 - \frac{36}{25}} + 2\sin^{-1}\left(\frac{6}{10}\right) - \frac{1}{2}\left(\frac{36}{50} + \frac{12}{5}\right)\right\} - \left\{0 + 2\sin^{-1}(-1) - \frac{1}{2}(2 - 4)\right\}$$

$$= (0.96 + 2\sin^{-1}(0.6) - 3.12) - \left(2 \times \frac{\pi}{2} + 1\right)$$

$$= 0.96 + 2\sin^{-1}(0.6) - 3.12 + \pi - 1$$

$$= 2\sin^{-1}(0.6) - 0.02 \text{ sq. unit}$$

$$= 73.72 \text{ sq. units.}$$

Thus, the area of the region of the circle $x^2 + y^2 = 4$ cut off by the line x – 2y = –2 in the first two quadrants is 73.72 sq. units.

2. Find the area bounded between the parabola $x^2 = 4y$ and the curve y = |x|. [2016 Spring][2014 Fall][2008, Spring][2005, Spring]

Solution: Given curve is,

$$x^2 = 4y \quad \dots (i)$$

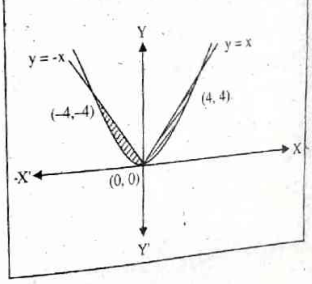
which is a parabola having vertex at (0, 0), a = 1 > 0 and the line of symmetry is x = 0.

Also given line is,

$$y = |x| = \begin{cases} x & x > 0 \\ -x & x < 0 \\ 0 & x = 0 \end{cases} \quad \dots (ii)$$

It passes through origin.

Now trace the curve of above curve is shown in the figure.



Solving the equations (i) and y = x, then we get the point of intersections are (0, 0) and (4, 4).

And, solving the equations (i) and $y = -x$, then we get the point of intersections are (0, 0) and (−4, −4).

Therefore, the point of intersection of curve (i) and (ii) are (0, 0), (4, 4) and (−4, −4).

The bounded region by (i) is the shaded portion that has two symmetrical parts. Therefore, the area of the bounded region is the twice of the region having extreme points (0, 0) and (4, 4).

Now, taking the vertical strip of width dx that has limits from $x = 0$ to $x = 4$. Therefore, the area of the bounded region is,

$$A = 2\int_0^4 (y_1 - y_2)\,dx$$

$$\text{i.e. } A = 2\int_0^4 \left\{|x| - \frac{x^2}{4}\right\} dx$$

$$= 2\int_0^4 \left(x - \frac{x^2}{4}\right) dx = 2$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{12}\right]_0^4$$

$$= 2\left(8 - \frac{16}{3}\right) = 2 \times \frac{(24-16)}{3} = 2 \times \frac{8}{3} = \frac{16}{3} \text{ sq. unit}$$

Thus, the area bounded between the parabola $x^2 = 4y$ and the curve $y = |x|$ is $\frac{16}{3}$ sq. units.

3. Find the area bounded by the parabola $y = 16(x - 1)(4 - x)$ and the x-axis.

Solution: Given parabola is,

$$y = 16(x-1)(4-x)$$

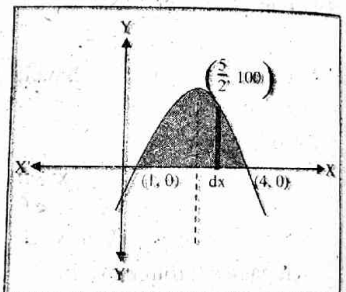
$$\Rightarrow \frac{y}{16} = 4x - x^2 - 4 + x$$

$$\Rightarrow \frac{y}{16} = -x^2 + 5x - 4$$

$$\Rightarrow -\frac{y}{16} = x^2 - 2.x.\frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \frac{25}{4} - \frac{y}{16} = \left(x - \frac{5}{2}\right)^2 \Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{100 - y}{16}$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 = -\frac{1}{16}(y - 100) \quad \text{.... (i)}$$



which is a parabola having vertex at $\left(\frac{5}{2}, 100\right)$, $a = -\frac{1}{64} < 0$ and line of symmetry is $x = \frac{5}{2}$.

And the given line is x–axis i.e. $y = 0$ ... (ii)

Solving (i) and (ii) then we get the points of intersection are (1, 0) and (4, 0).

Now, the trace of the curve is shown in the figure aside. In the figure the region bounded by the curves (i) and (ii) is shaded portion.

Clearly the bounded region has no symmetrical parts.

Now, taking vertical strip of width dx that has limits from $x = 1$ to $x = 4$. Therefore, the area bounded by curves is given by

$$A = \int_1^4 (y_1 - y_2)\,dx$$

$$\text{i.e. } A = \int_1^4 (-16x^2 + 80x - 64)\,dx$$

$$= \left[-16.\frac{x^3}{4} + 40x^2 - 64x\right]_1^4$$

$$= \left(-\frac{1024}{3} + 640 - 256\right) - \left(-\frac{16}{3} + 40 - 64\right)$$

$$= \frac{128}{3} - \left(-\frac{88}{3}\right) = \frac{128}{3} + \frac{88}{3} = \frac{216}{3} = 72 \text{ sq. unit}$$

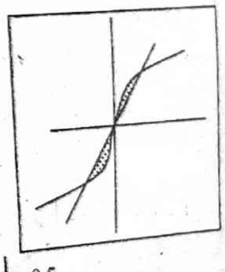
Thus, the area bounded by the parabola $y = 16(x - 1)(4 - x)$ and the x-axis is 72 sq. units.

4. Find the area of the propeller shaped region enclosed by the curve $x = y^3 = 0$ and the line $x - y = 0$.

Solution: The required region is bounded by the curves $x = y^3$ and $x = y$. That is shaded and have corners (1, 1), (0, 0) and (−1,−1).

Clearly the region has 2-symmetrical parts that are 2-times of the region between (0, 0) and (1, 1).

Now, the area of the bounded region is,



$$A = 2\left|\int_{y=0}^{1} (y_1 - y_2)\,dx\right|$$

$$\text{i.e. } A = 2\left|\int_{y=0}^{1} (y^3 - y)dy\right|$$

$$= 2\left|\left[\frac{y^4}{4} - \frac{y^2}{2}\right]_0^1\right| = 2\left|\frac{1}{4} - \frac{1}{2}\right| = \frac{1}{2} = 0.5$$

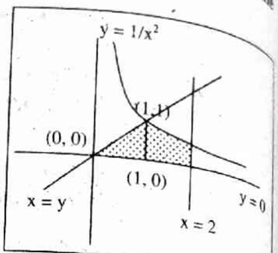
Thus, the area of the propeller shaped region enclosed by the curve $x = y^3 = 0$ and the line $x - y = 0$ is 0.5 sq. units.

5. Find the area of the region in the first quadrant bounded by the line $y = x$, $x = 2$ and the curve $y = \frac{1}{x^2}$ and x–axis.

Solution: The required region is bounded by

$y = x$, $x = 2$, $y = \frac{1}{x^2}$ and x–axis i.e. $y = 0$.

The region is shown by shaded in the figure.



Now, the area of the bounded region is,

$$A = \int_{x=0}^{2} (y_1 - y_2)\, dx$$

$$\text{i.e. } A = \int_{x=0}^{1} (y_1 - y_2)\, dx + \int_{x=1}^{2} (y_1 - y_2)\, dx$$

$$= \int_{x=0}^{1} x\, dx + \int_{x=1}^{2} x^{-2}\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^{-1}}{-1}\right]_1^2 = \frac{1}{2} - \frac{1}{2} + 1 = 1.$$

Thus, the area of the region in the first quadrant bounded by the line $y = x$, $x = 2$ and the curve $y = \frac{1}{x^2}$ and x–axis is 1 sq. units.

D. Show that the area of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{2a^2\pi}{8}$.

Solution: Given that the asteroid is,

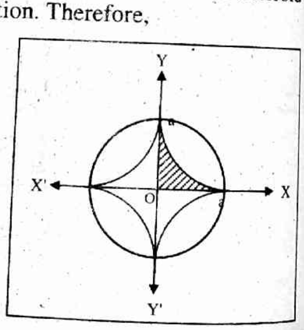
$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Comparing it with $(x - h)^{2/3} + (y - k)^{2/3} = r^{2/3}$ then we get, centre $(h, k) = (0, 0)$ and $r = a$.

Since the asteroid has symmetrical as in the figure. So, area of the asteroid is equal to the four time area of shaded portion. Therefore,

$$A = 4\int_{x=0}^{a} (y_1 - y_2)\, dx$$

$$\text{i.e. } A = 4\int_{x=0}^{a} [(a^{2/3} - x^{2/3})^{3/2} - 0]\, dx$$



$$= 4\int_{x=0}^{a} (a^{2/3} - x^{2/3})^{3/2}\, dx$$

Put, $x = a\sin^3\theta$ then $dx = 3a\sin^2\theta\cos\theta\, d\theta$.

Also, $x = 0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$A = 4\int_{\theta=0}^{\pi/2} a \cdot (1 - \sin^2\theta)^{3/2} \cdot 3a\sin^2\theta\cos\theta\, d\theta$$

$$= 12a^2 \int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta$$

$$= 12a^2 \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{2+4+2}{2}\right)}$$

$$= 12a^2 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\Gamma(4)}$$

$$= 12a^2 \frac{\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{2\cdot 3\cdot 2\cdot 1}$$

$$= \frac{12a^2}{48}\sqrt{\pi}\cdot\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{a^2\pi}{4}.$$

Thus, the area of asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{2a^2\pi}{8}$ i.e. $\frac{a^2\pi}{4}$ sq. units.

E. Show that the area bounded by the circle $x^2 + y^2 = a^2$ is πa^2. **[2011 Fall]**

Solution: Given circle is,

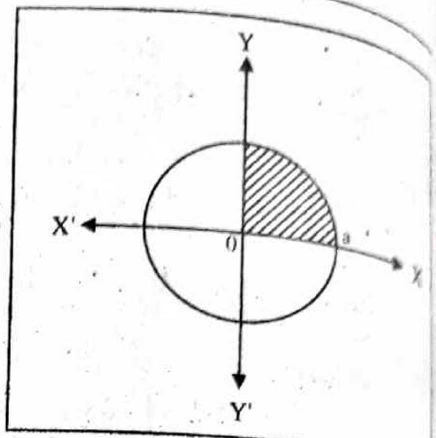
$$x^2 + y^2 = a^2.$$

Compare it with $(x - h)^2 + (y - k)^2 = r^2$ then we get

$(h, k) = (0, 0)$ and $r = a$.

Since the circle has symmetrical figure that is divided into four equal parts. So, the area of circle is equal to 4 times of the area of the shaded portion.

Now, the area of the circle is,



$$A = 4\int_0^a (y_1 - y_2)\, dx$$

$$= 4\int_{x=0}^{a} [(a^2 - x^2)^{1/2} - 0]\, dx$$

$$= 4\int_{x=0}^{a} (a^2 - x^2)^{1/2}\, dx$$

Put $x = a\sin\theta$ then $dx = a\cos\theta\, d\theta$.

Also, when $x = 0 \Rightarrow \theta = 0$ and $x = a \Rightarrow \theta = \frac{\pi}{2}$. Then,

$$A = 4\int_0^{\pi/2} a(1 - \sin^2\theta)^{1/2} . a\cos\theta\, d\theta$$

$$= 4a^2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 4a^2\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \frac{4a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{4a^2}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2}\right] = \frac{4a^2\pi}{4} = a^2\pi.$$

Thus, the area bounded by the circle $x^2 + y^2 = a^2$ is πa^2.

F. Show that the area common to the circle $x^2 + y^2 = 1$ and the parabola $y^2 = 1 - x$ is $\frac{4}{3} + \frac{\pi}{2}$. [2014 Spring]

Solution: Given region be the common part of $x^2 + y^2 = 1$ and $y^2 = 1 - x$. Clearly, the common part is half circle and twice of the shaded portion. Therefore, area of common region be,

$$A = \frac{1}{2}\text{ area of circle} + 2\text{ Area of shaded portion} \quad \dots (i)$$

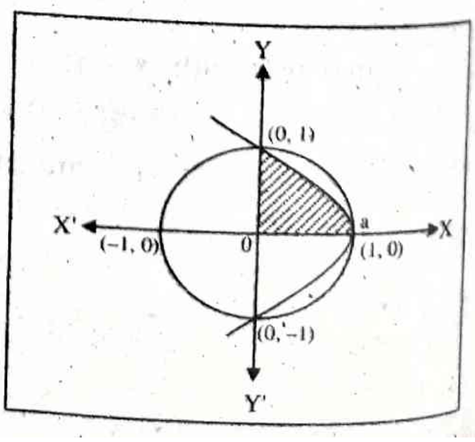
Here,

A_1 = Area of circle

$\Rightarrow A_1 = \pi$ [a = 1] [by (E)]

A_2 = Area of shaded portion

$$= \int_{y=0}^{1} (x_1 - x_2)\, dy$$



$$= \int_{y=0}^{1} (1 - y^2 - 0)\, dy = \left[y - \frac{y^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence, (i) becomes,

$$A = \frac{1}{2}\pi + 2 \cdot \frac{2}{3} = \frac{4}{3} + \frac{\pi}{2}$$

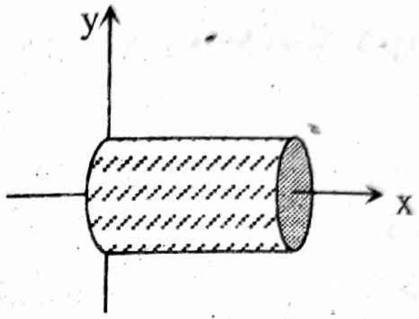
Thus, the area common to the circle $x^2 + y^2 = 1$ and the parabola $y^2 = 1 - x$ is $\frac{4}{3} + \frac{\pi}{2}$ sq. units.

Chapter 12

Application of Integration
VOLUME

List of Formulae

The volume of the region bounded by two curves $y_1 = f(x)$, $y_2 = g(x)$ and revolving about x–axis then the limit should be as $x = a$ and $x = b$ then,



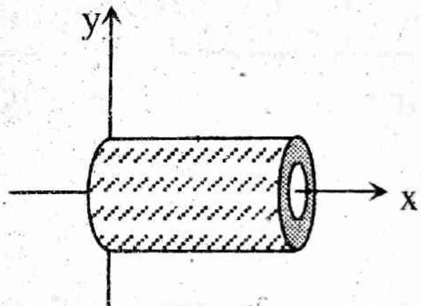
$$V = \pi \int_a^b [(y_1)^2 - (y_2)^2]\, dx.$$

The volume of the region bounded by two curves $x_1 = f(y)$, $x_2 = g(y)$ and revolving about y–axis then the limit should be as $y = a$ and $y = b$ then,

$$V = \pi \int_a^b [(x_1)^2 - (x_2)^2]\, dy.$$

For Washer method

The volume of the region bounded by two curves $y_1 = f(x)$, $y_2 = g(x)$ and revolving about the line $y = m$ then the limit should be as $x = a$ and $x = b$ then,



Washer

$$V = \pi \int_a^b [(y_1 - y)^2 - (y_2 - y)^2]\, dx.$$

The volume of the region bounded by two curves $x_1 = f(y)$, $x_2 = g(y)$ and revolving about the line $x = n$ then the limit should be as $y = a$ and $y = b$ then,

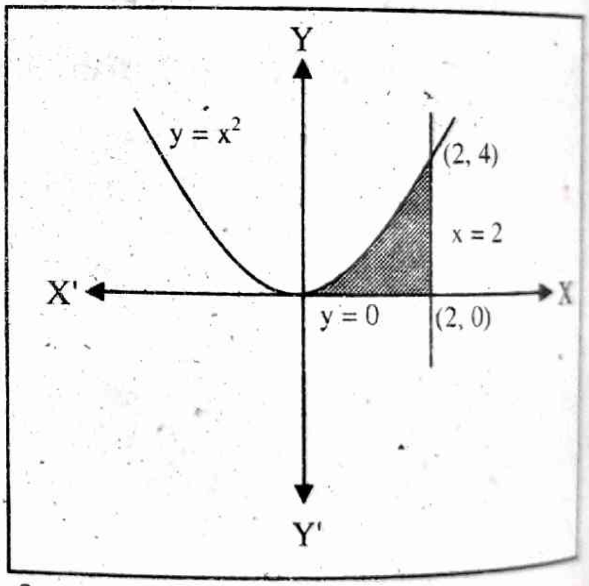
$$V = \pi \int_a^b [(x_1 - x)^2 - (x_2 - x)^2]\, dy.$$

Exercise 12.2

1. Find the volume of the solids generated by revolving the regions bounded by the lines and the curves about the x–axis.

(i) $y = x^2, y = 0, x = 2$

Solution: First we trace the curves from the figure, $R(x) = y = x^2$.



Since the curve $y = x^2$ is a parabola having vertex at (0, 0) and having up-openward. Therefore, the region bounded by the curve $y = x^2$ and the line $y = 0$, $x = 2$, is the shaded portion in the figure.

The limit of integration is $x = 0$ to $x = 2$. Then the volume of the solid thus generated by revolving the region bounded by the given curves is,

$$V = \int_0^2 \pi\{R(x)\}^2\,dx = \int_0^2 \pi\,x^4\,dx = \pi\left|\frac{x^5}{5}\right|_0^2 = \frac{2^5\pi}{5} = \frac{32\pi}{5}.$$

Thus, the volume of the circle is $\frac{32\pi}{5}$ cubic units.

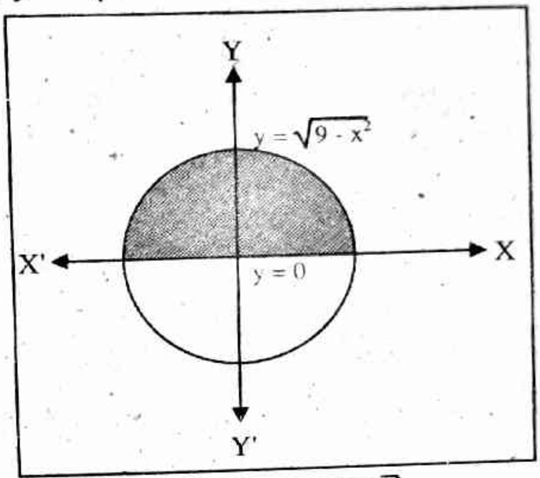
(ii) $y = \sqrt{9 - x^2}$, $y = 0$.

Solution: Here, by the curve $y = \sqrt{9 - x^2}$.

Clearly the given curve is a half circle having centre at (0, 0) and radius 3.

But y takes only the non-negative value being $y = \sqrt{9 - x^2}$.

First we trace the curves from the figure.



$$R(x) = y = \sqrt{9 - x^2}$$

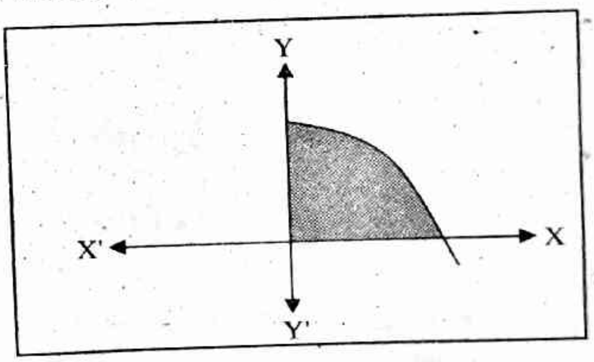
The limit of integration is x = 0 to x = 3.

Then the volume of the solid thus generated by revolving the region bounded by the given curves is, $V = 2\int_0^3 \pi\{R(x)\}^2\,dx$

$$= 2\pi \int_0^3 (9 - x^2)\,dx = 2\pi \left| 9x - \frac{x^3}{3} \right|_0^3 = 2\pi\left[27 - \frac{27}{3}\right] = 36\pi$$

Thus, the volume of the circle is 36π cubic units.

(iii) $y = \sqrt{\cos x}$, $0 \le x \le \frac{\pi}{2}$, $y = 0$, $x = 0$.



Solution: Here,

$$R(x) = y = \sqrt{\cos x}$$

The limit of integration is $x = 0$ to $x = \frac{\pi}{2}$.

Then the volume of the solid thus generated by revolving the region bounded by the given curves is,

$$V = \int_0^{\pi/2} \pi\{R(x)\}^2\,dx = \pi \int_0^{\pi/2} \cos x\,dx = \pi \left| \sin x \right|_0^{\pi/2} = \pi.$$

Thus, the volume of the circle is π cubic units.

2. **Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines below by $y = 1$, $x = 4$ about the line $y = 1$.**
[2018 Spring][2004, Fall] [2009, Fall]

OR Use the Washer to find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$, and the lines below by $y = 1$, $x = 4$ about the line $y = 1$. **[2004, Spring]**

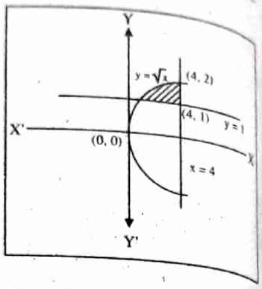
Solution: Here, given curve is $y = \sqrt{x}$ and bounded below by $y = 1$ and $x = 4$.

First we trace the curves from figure.

$R(x) = \sqrt{x}$ and $(x) = 1$

The limit of integration is $x = 1$ to $x = 4$.

Then the volume of the solid thus generated by revolving the region bounded by the given curves is,



$$V = \int_1^4 \pi\,[\{R(x) - 1\}^2 - \{r(x) - 1\}^2]\,dx$$

$$= \pi \int_1^4 [(\sqrt{x} - 1)^2 - (1 - 1)^2]\,dx \quad = \pi \int_1^4 (x - 2\sqrt{x} + 1)\,dx$$

$$\Rightarrow V = \pi \left| \frac{x^2}{2} - \frac{4}{3}x^{3/2} + x \right|_1^4 = \pi \left| 8 - \frac{32}{3} + 4 - \frac{1}{2} + \frac{4}{3} - 1 \right|$$

$$= \pi \left| 11 - \frac{32}{3} - \frac{1}{2} + \frac{4}{3} \right|$$

$$= \pi \left| \frac{66 - 64 - 3 + 8}{6} \right| = \frac{7\pi}{6}.$$

Thus, the volume of the circle is $\frac{7\pi}{6}$ cubic units.

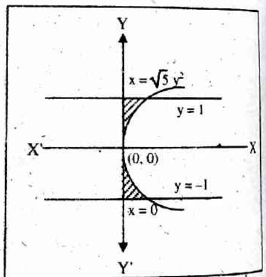
3. Find the volumes of the solids generated by revolving the regions bounded by the lines and the curves about y–axis of the following.

(i) $x = \sqrt{5}\,y^2, x = 0, y = -1, y = 1$

Solution: Here, the region bounded by the curves

$x = \sqrt{5}\,y^2, x = 0, y = -1, y = 1.$

First we traces the figure with the help of given curves,



$R(y) = x = \sqrt{5}\,y^2$

The limit of the integration is $y = 0$ to $y = 1$.

Then the volume of the solid thus generated by revolving the region bounded by the given curves is,

$$V = 2\int_0^1 \pi\{R(y)\}^2\,dy \qquad \text{[because the region has 2 symmetrical parts]}$$

$$= 2\pi \int_0^1 5y^4\,dx = 2\pi \left| y^5 \right|_0^1 = 2\pi.$$

Thus, the volume of the circle is 2π cubic units.

(ii) $x = \sqrt{2 \sin 2y}, 0 \le y \le \frac{\pi}{2}, x = 0$

Solution: Here, the region bounded by the curves

$x = \sqrt{2 \sin 2y}, 0 \le y \le \frac{\pi}{2}, x = 0.$

The limit of the integration is $y = 0$ to $\frac{\pi}{2}$. Then the volume bounded by the given curves be,

$$V = \int_0^{\pi/2} \pi(x)^2\,dy = \pi \int_0^{\pi/2} 2\sin 2y\,dy$$

$$= 2\pi \left| \frac{-\cos 2y}{2} \right|_0^{\pi/2} = \pi(-1) - (-1) = 2\pi.$$

Thus, the volume of the circle is 2π cubic units.

(iii) $x = \frac{2}{y + 1}, x = 0, y = 0, y = 3.$

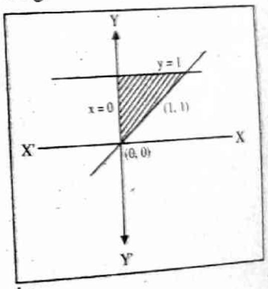
Similarly to Q.No. 2 (ii)

4. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x–axis of the following:

(i) $y = x, y = 1, x = 0$

Solution: The region bounded by the curves $y = x, y = 1, x = 0$.

Here, first we traces the lines from figure, the limit of the integration is $x = 0$ to $x = 1$. Then the volume bounded by the given curves be,



$$V = \int_0^1 (\pi[-R(x)] + [r(x)]^2)\,dx$$

$$= \pi \int_0^1 (1 - x^2)\,dx$$

$$= \pi \left| x - \frac{x^3}{3} \right|_0^1 = \frac{3 - 1}{3}\pi = \frac{2\pi}{3}.$$

Thus, the volume of the circle is $\frac{2\pi}{3}$ cubic units.

(ii) $y = 2\sqrt{x}, y = 2, x = 0$

Solution: *Similarly Q. No. 4(i).*

(iii) $y = x^2 + 1, y = x + 3$

Solution: *Similarly Q. No. 4(i).*

(iv) $y = \sec x, y = \sqrt{2}, -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

Solution: Here, $R(x) = y = \sqrt{2}$ and $r(x) = y = \sec x$

The limit of the function $x = -\pi/4$ to $x = \pi/3$. Then the volume bounded by the given curves be,

$$V = \int_{-\pi/4}^{\pi/4} \pi[\{R(x)\}^2 - \{r(x)\}^2]\, dx$$

$$= \pi \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\, dx = \pi \int_{-\pi/4}^{\pi/4} (2x - \sec^2 x)\, dx$$

$$= \pi \left|2x - \tan x\right|_{-\pi/4}^{\pi/4}$$

$$= \pi(2(\pi/4) - 1 + 2(\pi/4) - 1)$$

$$= \pi(\pi - 2).$$

Thus, the volume of the circle is $\pi(\pi - 2)$ cubic units.

5. Find the volume of the solid generated by revolving the region enclosed by the triangle with vertices (1, 0), (2, 1) and (1, 1) about the y-axis.

Solution: Here, the equation of line having co-ordinates (1, 0) and (2, 1) is,

$$y - 0 = \frac{1-0}{2-1}(x - 1)$$

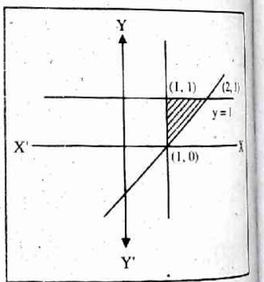
or $y = x - 1 \quad \Rightarrow x = y + 1.$

The equation line having co-ordinate (1, 1) and (1, 0) is $x = 1$.

The equation of line having co-ordinate (1, 1) and (2, 1) is $y = 1$.

From figure, the limit of the integration is $y = 0$ to $y = 1$.

Then the volume bounded by the given curves be,



$$V = \int_2^1 [x_1^2 - x_2^2]\, dy$$

$$= \pi \int_2^1 [(y+1)^2 - 1^2]\, dy$$

$$= \pi \left|\frac{(y+3)^3}{3} - y\right|_0^1 = \pi\left|\frac{8}{3} - 1 - \frac{1}{3}\right|$$

$$= \frac{4\pi}{3}.$$

Thus, the volume of the circle is $\frac{4\pi}{3}$ cubic units.

6. Find the volume of the solid generated by revolving the region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis and on the right by the line $x = 2$ about y-axis.

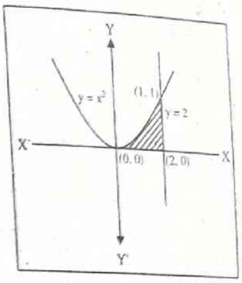
[2017 Fall][2005, Spring]

Solution: The region bounded by the given curves in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis and on the right by the line $x = 2$.

Here, first we trace the curve,

$R(y) = x - 2$ and $r(y) = x - \sqrt{y}$

From figure, the limit of the integration is $y = 0$ to $y = 4$. Then the volume bounded by the given curves be,



$$V = \int_0^4 \pi[\{R(y)\}^2 - \{r(y)\}^2]\, dy$$

$$= \pi \int_0^4 (4 - y)\, dy = \pi\left|4y - \frac{y^2}{2}\right|_0^4$$

$$= \pi(16 - 8) = 8\pi.$$

Thus, the volume of the circle is 8π cubic units.

7. Find the volume of the solid generated by revolving the region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$ about y-axis.

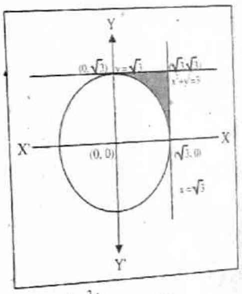
Solution: The region bounded by the given curves in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$.

Here, first we trace the circle and lines

$R(y) = x = \sqrt{3 - y^2}$

$r(y) = x = \sqrt{3}$

From figure, the limit of the integration is $y = 0$ to $y = \sqrt{3}$. Then the volume bounded by the given curves be,



$$V = \int_0^{\sqrt{3}} \pi[\{R(y)\}^2 - \{r(y)^2]\, dy$$

$$= \pi \int_0^{\sqrt{3}} [3 + 3 + y^2]\, dy = \pi \int_0^{\sqrt{3}} [y^2]\, dy = \pi\left|\frac{y^3}{3}\right|_0^{\sqrt{3}} = \sqrt{3}\,\pi.$$

Thus, the volume of the circle is $\sqrt{3}\,\pi$ cubic units.

8. Find the volume of the solid in the region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ revolved about the x-axis. [2003, Fall]

Solution: The region bounded by the given curves $y = x^2 + 1$ and the line $y = -x + 3$.

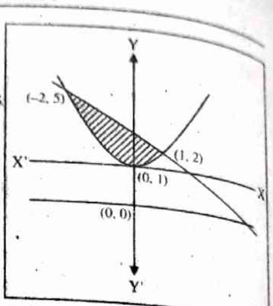
Here, first we trace the curve,

$R(x) = y = x^2 + 1$

$r(x) = y = -x + 3$

From figure, the limit is $x = -2$ to $x = 1$.

Then the volume bounded by the given curves be,



$$V = \int_{-2}^{1} \pi [\{R(x) - 0\}^2 - \{r(x) - 0\}^2]\, dx$$

$$= \pi \int_{-2}^{1} (x^2 + 1)^2 - (-x + 3)^2\}$$

$$= \pi \int_{-2}^{1} (x^4 + 2x^2 + 1 - x^2 + 6x - 9)$$

$$= \pi \left| \frac{x^5}{5} + \frac{x^3}{3} + \frac{6x^2}{2} - 8x \right|_{-2}^{1}$$

$$= \pi \left| \frac{1}{5} + \frac{1}{3} + 3 - 8 + \frac{32}{5} + \frac{8}{3} - 12 - 16 \right|$$

$$= \pi \left| \frac{3 + 5 + 96 - 495 + 40}{15} \right| = \left| -\frac{351}{15} \right| = \frac{351}{15}\pi = \frac{117\pi}{5}.$$

Thus, the volume of the circle is $\frac{117\pi}{5}$ cubic units.

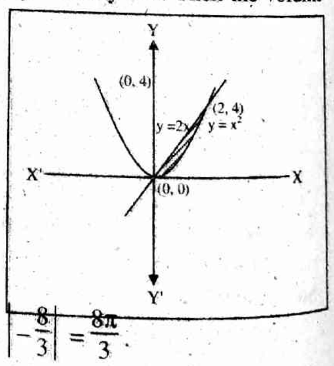
9. Find the volume of the solid in the region in the region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant about y-axis.
[2016 Fall][2006, Fall][2007, Spring]

Solution: The region bounded by the given curves $y = x^2$ and $y = 2x$.

Here, first we trace the curves,

$$R(y) = x = \frac{y}{2} \quad \text{and} \quad r(y) = x = \sqrt{y}$$

From figure, the limit of the integration is $y = 0$ to $y = 4$. Then the volume bounded the curves be,



$$V = \int_0^4 [\{R(y)\}^2 - \{r(y)\}^2]\, dy$$

$$= \pi \int_0^4 \left\{ \left(\frac{y}{2}\right)^2 - (\sqrt{y})^2 \right\} dy$$

$$= \pi \left| \frac{y^3}{12} - \frac{y^2}{2} \right|_0^4$$

$$= \pi \left| \frac{64}{12} - \frac{16}{2} \right| = \pi \left| \frac{16}{3} - 8 \right| = \pi \left| -\frac{8}{3} \right| = \frac{8\pi}{3}.$$

Thus, the volume of the circle is $\frac{8\pi}{3}$ cubic units.

10. Find the volume of the solid in the region in the first quadrant evolved by the parabola $y = x^2$ the y-axis and the line $y = 1$ revolved about the line $x = \frac{3}{2}$. [2014 Fall]

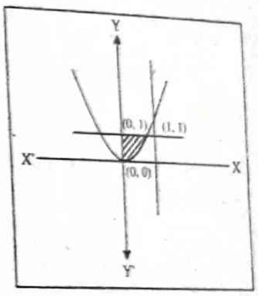
Solution: The region bounded by the given curves $y = x^2$ the y-axis and the line $y = 1$ in the first quadrant.

Here, first we trace the curves,

$$R(y) = x = \sqrt{y} \quad \text{and} \quad r(y) = x = 0.$$

From figure, the limit of the integration is $y = 0$ to $y = 1$. Then the volume bounded the curves be,

$$V = \left| \int_0^1 \pi [(R(y) - 3/2)]^2 - (r(y) - 3/2)^2]\, dy \right|$$



$$= \pi \left| \int_0^1 \left\{ \left(0 - \frac{3}{2}\right)^2 - \left(\sqrt{y} - \frac{3}{2}\right)^2 \right\} dy \right|$$

$$= \pi \left| \int_0^1 \left(\frac{9}{4} - y + 3\sqrt{y} - \frac{9}{4} \right) dy \right|$$

$$= \pi \left| \int_0^1 \left\{ -\sqrt{y^2 + 3\sqrt{y}} \right\} dy \right|$$

$$= \pi \left| \frac{y^2}{2} - 2y^{3/2} \right|_0^1 = \pi \left| -\frac{1}{2} + 2 \right| = \pi \left| \frac{3}{2} \right| = \frac{3\pi}{2}.$$

Thus, the volume of the circle is $\frac{3\pi}{2}$ cubic units.

11. Find the volume of the solid in the region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis and on the right by the line $x = 1$ about the line $x = -1$.
[2014 Spring][2008, Spring][2009 Spring]

OR Find the area bounded by $y = x^2$, below by the x-axis and on the right by the line $x = 1$ about the line $x = -1$. Find the volume thus generated. [2013 Spring]

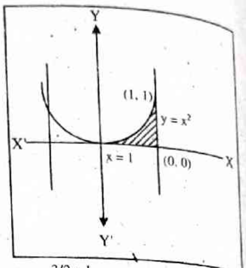
Solution: Since the given region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis and on the right by the line $x = 1$.

Here, first we trace the curves,

$$R(y) = x = 1 \quad \text{and} \quad r(y) = x = \sqrt{y}$$

From figure, the limit of the integration is $y = 0$ to $y = 1$. Then the volume bounded by the curves be,

$$V = \int_0^1 \pi[\{R(y) - (-1)\}^2 - \{r(y) - (-1)\}^2]\, dy$$



$$= \pi \int_0^1 [(2)^2 - (\sqrt{y} + 1)^2]$$

$$= \pi \int_0^1 [4 - y - 2\sqrt{y} - 1]\, dy$$

$$= \pi \int_0^1 [3 - y - 2\sqrt{y})\, dy \quad = \pi \left| -\frac{y^2}{2} + 3y - \frac{4y^{3/2}}{3} \right|_0^1$$

$$= \pi \left| 3 - \frac{1}{2} - \frac{4}{3} \right| = \frac{7\pi}{6}.$$

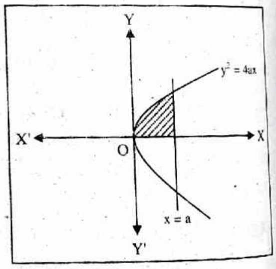
Thus, the volume of the circle is $\frac{7\pi}{6}$ cubic units.

12. Show that the volume of the paraboloid formed by revolving the parabola $y^2 = 4ax$ and the line $x = a$, about x–axis is $2\pi a^3$.

Solution: The region bounded by the given curves are $y^2 = 4ax$ and the line $x = a$. The sketch of the curves is trace below.

Clearly, the half–part region (i.e. shaded region) has same form as given by the rest region. So, we take the shaded region as required region and revolve to it about x–axis i.e. $y = 0$ with $x = 0$ to $x = a$. Then the required volume be,

$$V = \pi \int_{x=a}^{b} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$



$$= \pi \int_{x=0}^{a} [(\sqrt{4ax} - 0)^2 - (0 - 0)^2]\, dx$$

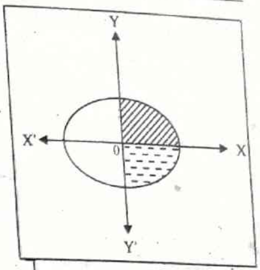
$$= \pi (\sqrt{4a})^2 \left[\frac{x^2}{2}\right]_0^a = \frac{4a^3\pi}{2} = 2a^3\pi.$$

Thus, the volume of the circle is $2a^3\pi$ cubic units.

13. Show that the volume of the solid generated by revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about x–axis is $\frac{4}{3}\pi ab^2$.

Solution: Here, in the corresponding ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the shaded portion by lines, has same volume as given by the shaded portion shaded by dots. Thus the volume of the solid ellipse is equal to twice of line–shaded region.

Clearly, the shaded portion by lines, is bounded by the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $y = 0$. Therefore volume of ellipse revolving about x–axis i.e. $y = 0$ be,



$$V = 2\pi \int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$= 2\pi \int_{x=0}^{a} \left[\left(\sqrt{\frac{a^2b^2 - b^2x^2}{a^2}} - 0\right)^2 - (0 - 0)^2\right] dx$$

$$= 2\pi \int_{x=0}^{a} \left(\frac{a^2b^2 - b^2x^2}{a^2}\right) dx = \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2)\, dx$$

$$\Rightarrow V = \frac{2\pi b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{2\pi b^2}{a^2} \cdot \frac{2a^3}{3} = \frac{4\pi ab^2}{3}.$$

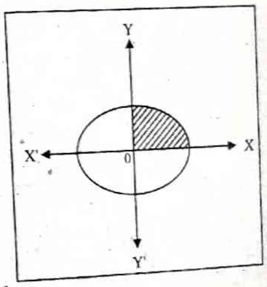
Thus, the volume of the circle is $\frac{4\pi ab^2}{3}$ cubic units.

14. Show that the volume of the sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution: Since the volume of the sphere is obtain by twice of revolving the shaded portion about x–axis or y–axis. Let the region has boundary $x^2 + y^2 = r^2$ and $y = 0$ and it revolve about x–axis i.e. $y = 0$ with limits $x = 0$ to $x = r$.

Therefore, volume of sphere be,

$$V = 2\pi \int_{x=0}^{r} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$



$$= 2\pi \int_{x=0}^{r} [(y_1 - 0)^2 - (y_2 - 0)^2]\, dx$$

$$= 2\pi \int_{x=0}^{r} [(\sqrt{r^2 - x^2 - 0^2}) - (0 - 0)^2]\, dx$$

$$= 2\pi \int_{x=0}^{r} (r^2 - x^2)^2\, dx = 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4\pi r^3}{3}$$

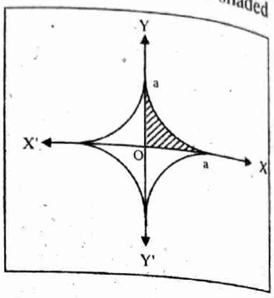
Thus, the volume of the circle is $\frac{4\pi r^3}{3}$ cubic units.

15. Show that the volume of the solid generated by revolving the asteroid $x^{2/3} + y^{2/3} = a^{2/3}$ about x–axis is $\frac{32}{105}\pi a^3$.

Solution: Given asteroid be

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Clearly, if the half part of asteroid is revolved about x–axis then it gives the whole asteroid. Therefore, the volume is twice of revalued form of shaded portion.



Since the shaded portion has limits $x = 0$ to $x = a$ and is bounded by the curves $y = (a^{2/3} - x^{2/3})^{3/2}$ and $y = 0$.

Now, the volume of asteroid revolving about x–axis i.e. $y = 0$ be,

$$V = 2\pi \int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$= 2\pi \int_{x=0}^{a} \{(a^{2/3} - x^{2/3})^{3/2} - 0\}^2 - (0 - 0)^2\}\, dx$$

$$= 2\pi \int_{x=0}^{a} (a^{2/3} - x^{2/3})^3\, dx$$

$$= 2\pi \int_{x=0}^{a} [(a^{2/3})^3 - 3(a^{2/3})^2 . x^{2/3} + 3a^{2/3} (x^{2/3})^2 - (x^{2/3})^3]\, dx$$

$$[\because (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3]$$

$$= 2\pi \int_{x=0}^{a} (a^2 - 3a^{4/3} x^{2/3} + 3a^{2/3} x^{4/3} - x^2)\, dx$$

$$= 2\pi \left[a^2x - 3a^{4/3} \frac{x^{5/3}}{5/3} + 3a^{2/3} \frac{x^{7/3}}{7/3} - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^{4/3} . a^{5/3}}{5} + \frac{a^{2/3} . a^{7/3}}{7} - \frac{a^3}{3}\right]$$

$$= 2\pi \left(a^3 - \frac{a^3}{5} + \frac{a^3}{7} - \frac{a^3}{3}\right) = \frac{2\pi}{105} a^3 (105 - 21 + 15 - 35)$$

$$= \frac{2\pi a^3}{105} . 64 = \frac{128\, \pi a^3}{105}.$$

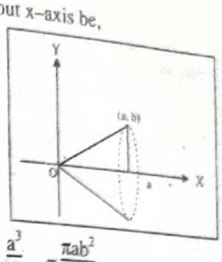
Thus, the volume of the circle is $\frac{128\, \pi a^3}{105}$ cubic units.

16. Show that the volume of the solid general by revolving the line joining origin and the points (a, b) about x–axis is $\frac{1}{3} \pi ab^2$.

Solution: Let (a, b) be a point. Then the equation of line joining (0, 0) and (a, b) be, $y = \frac{b}{a} x$.

Thus, the region is bounded by the curves $y = \frac{b}{a} x$ and x–axis i.e. $y = 0$, the solid has limits $x = 0$ and $x = a$.

Then, the volume of the solid by revolving about x–axis be,



$$V = \pi \int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$= \pi \int_{x=0}^{a} \left[\left(\frac{b}{a} x - 0\right)^2 - (0 - 0)^2\right] dx$$

$$= \frac{\pi b^2}{a^2} \int_{x=0}^{a} x^2\, dx = \frac{\pi b^2}{a^2} \left(\frac{x^3}{3}\right)_0^a = \frac{\pi b^2}{a^2} \frac{a^3}{3} = \frac{\pi ab^2}{3}.$$

Thus, the volume of the circle is $\frac{\pi ab^2}{3}$ cubic units.

17. Show that the volume of the solid generated by revolving the catenary $y = c\cosh \left(\frac{x}{c}\right)$, ordinates $x = 0$, $x = a$, about x–axis, $\frac{\pi}{2} c^2 (a + c \sinh \frac{a}{c} \cosh \frac{a}{c})$

Solution: Given curves are $y = c \cosh \left(\frac{x}{c}\right)$, $y = 0$ with $x = 0$, $x = a$.

Now, the volume of solid bounded by the curves about x–axis be,

$$V = \pi \int_{x=0}^{a} [(y_1 - y)^2 - (y_2 - y)^2]\, dx$$

$$= \pi \int_{x=0}^{a} \left[\left(c \cosh \frac{x}{c} - 0\right)^2 - (0 - 0)^2\right] dx$$

$$= \pi c^2 \int_{x=0}^{a} \cosh^2 \frac{x}{c}\, dx = \frac{\pi c^2}{2} \int_{x=0}^{a} \left(1 + \cosh \frac{2x}{c}\right) dx$$

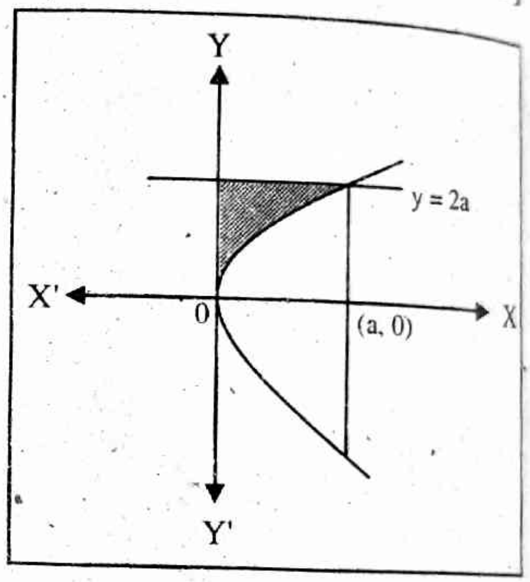
$$= \frac{\pi c^2}{2} \left[x + \sinh \frac{2x}{c} . \frac{c}{2}\right]_0^a$$

$$= \frac{\pi c^2}{2} \left[a + \frac{c}{2} \sinh \frac{2a}{c}\right]$$

$$= \frac{\pi c^2}{2} \left[a + c \sin \left(\frac{a}{c}\right) . \cos h \left(\frac{a}{c}\right)\right]$$

Thus, the volume of the circle is $\frac{\pi c^2}{2} \left[a + c \sin \left(\frac{a}{c}\right) . \cos h \left(\frac{a}{c}\right)\right]$ cubic units.

18. **Show that the volume of the solid generated by revolving the area bounded by parabola $y^2 = 4ax$, and y–axis, $y = 2a$ about y–axis is $\left(\frac{2}{5}\pi a^3\right)$.** [2002]



Solution: By hypothesis the revolved area is the bounded region by $y^2 = 4ax$ and its latus rectum. Since, the region revolved about y–axis. So, the volume of solid is twice of volume of double shaded portion. Since the solide revolve about y–axis, so we need limits in y.

By solving $y^2 = 4ax$ and its latus rectum then we get $y = 0$ and $y = 2abc$ the limits of double shaded portion.

Now, volume of solid be,

$$V = \pi \int_{y=0}^{2a} [(x_1 - x)^2 - (x_2 - x)^2]\, dy$$

$$= \pi \int_{y=0}^{2a} \left[\left(\frac{y^2}{4a} - 0\right)^2 - (0-0)^2\right] dy$$

$$= \frac{\pi}{16a^2}\left[\frac{y^5}{5}\right]_0^{2a} = \frac{\pi}{16a^2 \cdot 5}(2a)^5 = \frac{\pi \cdot 32a^5}{16a^2 . 5} = \frac{2\pi a^3}{5}.$$

Thus, the volume of the circle is $\frac{2\pi a^3}{5}$ cubic units.

Chapter 12 **Application of Integration**
ARC LENGTH

List of Formulae

(i) If the curve is given in variables x and y then

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{OR} \quad L = \int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

(ii) If the curve is given in parametric form then such as if the curve is in x and y in the form of θ, independently. Then,

$$L = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If the curve is in polar form i.e. in (r, θ) then

(i) the length of the arc from $\theta = \theta_1$ to $\theta = \theta_2$ is

$$L = \int_{\theta_1}^{\theta_2} \left(\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right) d\theta$$

(ii) the length of the arc from $r = r_1$ to $r = r_2$ is

$$L = \int_{r_1}^{r_2} \left(1 + \left(r\frac{d\theta}{dr}\right)^2\right) dr$$

Exercise 12.3

1. Find the arc length of the curves.

(i) $y = x^2$, $-1 \le x \le 2$. **[2017 Spring short]**
[2011 Fall (Short)][2003 Spring(Short)][2007 Fall(Short)]

Solution: Here, $y = x^2$ for $-1 \le x \le 2$

Different w. r. t. x, then

$$\frac{dy}{dx} = 2x.$$

Now, arc length of given curve for $-1 \le x \le 2$ be

$$L = \int_{-1}^{2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-1}^{2} \sqrt{1+4x^2}\, dx$$

Put $2x = t$ then $2\,dx = dt$. Also, $x = -1 \Rightarrow t = -2$, $x = 2 \Rightarrow t = 4$. Then,

$$L = \frac{1}{2}\int_{-2}^{4} (\sqrt{1+t^2})\, dx$$

$$= \frac{1}{2}\left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right]_{-2}^{4}$$

$$= \frac{1}{2}\left[\left(2\sqrt{17} + \frac{1}{2}\log(4+\sqrt{17})\right) - \left(-\sqrt{5} + \frac{1}{2}\log(-2+\sqrt{5})\right)\right]$$

$$= \frac{1}{2}\left[2\sqrt{17} + \sqrt{5} + \frac{1}{2}\log\left(\frac{4+\sqrt{17}}{-2+\sqrt{5}}\right)\right]$$

Thus, the length of the curve $y = x^2$ for $-1 \le x \le 2$ is

$$\frac{1}{2}\left[2\sqrt{17} + \sqrt{5} + \frac{1}{2}\log\left(\frac{4+\sqrt{17}}{-2+\sqrt{5}}\right)\right] \text{ unit.}$$

Q. Find the arc length of the curve $y = x^2 + 1$, from $x = 1$ to $x = 2$.
[2016 Spring Short]

(ii) $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$.

Solution: Here, $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$.

So, differentiating w. r. t. x then,

$$2y\frac{dy}{dx} + 2\frac{dy}{dx} = 2 \quad \Rightarrow \frac{dx}{dy} = y + 1.$$

Now, arc length of given curve from $(-1, -1)$ to $(7, 3)$ is,

$$L = \int_{-1}^{3} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_{-1}^{3} \sqrt{1+(y+1)^2}\, dy.$$

Put $y + 1 = t$ then $dy = dt$. And $y = -1 \Rightarrow t = 0$, $y = 3 \Rightarrow t = 4$.

Then,

$$L = \int_0^4 \sqrt{1+t^2}\, dt$$

$$= \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log(t+\sqrt{1+t^2})\right]_0^4$$

$$= 2\sqrt{17} + \frac{1}{2}\log(4+\sqrt{17}) \qquad [\log(1) = 0]$$

$$= 9.29.$$

Thus, the length of the curve $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$ is 9.29 unit.

(iii) $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$

Solution: Here, $x = \frac{y^3}{3} + \frac{1}{4y}$ from $y = 1$ to $y = 3$.

So, differentiating w. r. t. y then,

$$\frac{dx}{dy} = \frac{3y^2}{3} + \frac{1}{4}\left(\frac{-1}{y^2}\right) \quad \Rightarrow \frac{dx}{dy} = y^2 - \frac{1}{4y^2} = \frac{4y^4-1}{4y^2}.$$

Now, arc length of given curve from $y = 1$ to $y = 3$ be

$$L = \int_1^3 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_1^3 \sqrt{1+\left(\frac{4y^4-1}{4y^2}\right)^2}\,dy$$

$$\Rightarrow L = \int_1^3 \sqrt{\frac{16y^4+16y^8-8y^4+1}{16y^4}}\,dy$$

$$\Rightarrow L = \int_1^3 \sqrt{\frac{16y^8+8y^4+1}{18y^4}}\,dy$$

$$\Rightarrow L = \int_1^3 \sqrt{\left(\frac{4y^4+1}{4y^2}\right)^2}\,dy = \int_1^3 \frac{4y^4+1}{4y^2}\,dy$$

$$\Rightarrow L = \int_1^3 \left(y^2+\frac{1}{4y^2}\right)dy = \left[\frac{y^3}{3}+\frac{1}{4}\,\frac{y^{-1}}{-1}\right]_1^3$$

$$\Rightarrow L = 9-\frac{1}{12}-\frac{1}{3}+\frac{1}{4} = \frac{108-1-4+3}{12} = \frac{106}{12} = \frac{53}{6}.$$

Thus, the length of the curve $x = \frac{y^3}{3}+\frac{1}{4y}$ from $y = 1$ to $y = 3$ is $\frac{53}{6}$ unit.

(iv) $x = \frac{y^4}{4}+\frac{1}{8y^2}$, from $y = 1$ to $y = 2$

Solution: Here, $x = \frac{y^4}{4}+\frac{1}{8y^2}$ from $y = 1$ to $y = 2$.

So, differentiating w. r. t. y then,

$$\frac{dx}{dy} = y^3 - \frac{1}{4y^3} = \frac{4y^6-1}{4y^3}$$

Now, arc length of the given curve from $y = 1$ to $y = 2$ be,

$$L = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_1^2 \sqrt{1+\left(\frac{4y^6-1}{4y^3}\right)^2}\,dy$$

$$= \int_1^2 \sqrt{\frac{16y^6+16y^{12}-8y^6+1}{16y^6}}\,dy$$

$$= \int_1^2 \sqrt{\left(\frac{4y^6+1}{4y^3}\right)^2}\,dy = \int_1^2 \left(y^3+\frac{1}{4}y^{-3}\right)dy$$

$$= \left[\frac{y^4}{4}-\frac{1}{8y^2}\right]_1^2$$

$$= 4-\frac{1}{32}-\frac{1}{4}+\frac{1}{8}$$

$$= \frac{128-1-8+4}{32} = \frac{123}{32}.$$

Thus, the length of the curve $x = \frac{y^4}{4}+\frac{1}{8y^2}$ from $y = 1$ to $y = 2$ is $\frac{123}{32}$ unit.

(v) $y = \frac{3}{4}x^{2/3} - \frac{3}{8}x^{2/3} + 5$, for $-1 \le x \le 8$.

Solution: Here, $y = \frac{3x^{2/3}}{4} - \frac{3x^{2/3}}{8} + 5$ for $-1 \le x \le 8$

$$\Rightarrow y = \frac{3x^{2/3}}{8} + 5$$

So, differentiating w. r. t. x then,

$$\frac{dy}{dx} = \frac{3}{8}\cdot\frac{2}{3}x^{-1/3} = \frac{x^{-1/3}}{4}$$

Now, arc length of given curve from $x = -1$ to $x = 8$ be,

$$L = \int_{-1}^8 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_{-1}^8 \sqrt{1+\frac{x^{-2/3}}{16}}\,dx$$

$$= \int_{-1}^8 \sqrt{\frac{x^{2/3}+\frac{1}{16}}{x^{2/3}}}\,dx$$

$$= \int_{-1}^8 x^{-1/3}\sqrt{x^{2/3}+\frac{1}{16}}\,dx.$$

Put $x^{2/3}+\frac{1}{16} = t^2$ then $\frac{2}{3}x^{-1/3}\,dx = dt \Rightarrow x^{-1/3}\,dx = \frac{3}{2}dt$.

Also, $x = -1 \Rightarrow t = \frac{17}{16}$ and $x = 8 \Rightarrow t = \frac{65}{16}$. Then,

$$L = \int_{17/16}^{65/16} \frac{3}{2}t\,dt$$

$$= \frac{3}{2}\left[t^2\right]_{17/16}^{65/16} = \frac{3}{2}\left[\frac{1}{256}(65^2-17^2)\right] = \frac{369}{16} = 23.0625$$

Thus, the length of the curve $y = \frac{3x^{2/3}}{4} - \frac{3x^{2/3}}{8} + 5$ for $-1 \le x \le 8$ is 23.0625 unit.

2. **Find the distance traveled between $t = 0$ and $t = \pi$ by the particle $p(x, y)$ whose position at time t is $x = \cos t$, $y = t + \sin t$.**

Solution: Here, $x = \cos t$, $y = t + \sin t$ for $t = 0$ to $t = \pi$.

Then, differentiating w. r. t. t then,

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = 1 + \cos t$$

Now, arc length of the particle p(x, y) from t = 0 to t = π is,

$$L = \int_{t=0}^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\text{i.e. } L = \int_{t=0}^{\pi} \sqrt{\sin^2 t + (1+\cos t)^2}\, dt$$

$$= \int_{t=0}^{\pi} \sqrt{\sin^2 t + 1 + \cos^2 t + 2\cos t}\, dt$$

$$= \sqrt{2}\int_{t=0}^{\pi} \sqrt{1+\cos t}\, dt$$

$$= \sqrt{2}\int_{t=0}^{\pi} \sqrt{2\cos^2\frac{t}{2}}\, dt$$

$$= 2\int_{t=0}^{\pi} \cos\frac{t}{2}\, dt = 2\left[\frac{\sin t/2}{1/2}\right]_0^{\pi} = 4\left(\sin\frac{\pi}{2} - \sin 0\right) = 4.$$

Thus, the distance travéled between t = 0 and t = π by the particle p(x, y) whose position at time t is x = cos t, y = t + sin t. is 4 unit.

3. Find the length of the arc of the parabola $y^2 = 4x$ cut off by the line y = 2x.

Solution: Given curves is,

$$y^2 = 4x \quad \dots (i)$$

So, $\frac{dx}{dy} = \frac{y}{2}$.

And the curve (i) is cut by the line y = 2x. Therefore, solving (i) and y = 2x then we get, x = 0, 1 and so y = 0, 2.

Now, the arc length of $y^2 = 4x$ from y = 0 to 2 is,

$$L = \int_{y=0}^{2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_{y=0}^{2} \sqrt{1 + \frac{y^2}{4}}\, dy$$

$$= \frac{1}{2}\int_{y=0}^{2} \sqrt{4+y^2}\, dy$$

$$= \frac{1}{2}\left[\frac{y}{2}\sqrt{4+y^2} + \frac{4}{2}\log\left(y + \sqrt{4+y^2}\right)\right]_0^2$$

$$= \frac{1}{2}\left[\sqrt{8} + 2\log(2+\sqrt{8}) - 2\log 2\right]$$

$$= \frac{1}{2}\left[2\sqrt{2} + 2\log\left(\frac{2+2\sqrt{2}}{2}\right)\right]$$

$$= \sqrt{2} + \log(1+\sqrt{2}).$$

Thus, the length of the arc of the parabola $y^2 = 4x$ cut off by the line y = 2x is $\sqrt{2} + \log(1+\sqrt{2})$.

4. Show that the length of perimeter of the circle $x^2 + y^2 = a^2$ is 2πa.

Solution: Here, $x^2 + y^2 = a^2$

Put, x = a cosθ, y = a sinθ

So, differentiating w. r. t. θ then,

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = a\cos\theta.$$

Clearly for a circle, θ varies from 0 to 2π.

Now, arc length of the perimeter of $x^2 + y^2 = a^2$ then

$$L = \int_{\theta=0}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\, d\theta = a\int_0^{2\pi} d\theta = 2\pi a.$$

Thus, the length of perimeter of the circle $x^2 + y^2 = a^2$ is 2πa.

5. Show that the entire length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is 6a.

Solution: Here, $x^{2/3} + y^{2/3} = a^{2/3}$.

Then, differentiating w. r. t. x then,

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

Clearly this asteroid has radius a with having 4-symmetrical parts.

Now, the arc length of whole asteroid is,

$$L = 4\int_0^a \sqrt{1 + \left(-\frac{y^{1/3}}{x^{1/3}}\right)^2}\, dx$$

$$= 4\int_0^a \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx = 4a^{1/3}\left[\frac{x^{-1/3+1}}{-(1/3)+1}\right]_0^a = 4a^{1/3}\cdot\frac{a^{2/3}}{2/3} = 6a.$$

Thus, the entire length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is 6a.

6. Show that the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a) is $\frac{a}{27}(13\sqrt{13} - 8)$.

Solution: Given that $ay^2 = x^3$.

Then, differentiating w.r.t. x then,

$$2ay\frac{dy}{dx} = 3x^2. \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2a\sqrt{x^3/a}} = \frac{3\sqrt{x}}{2\sqrt{a}}.$$

So, $\left(\frac{dy}{dx}\right)^2 = \frac{9x}{4a}$

Clearly the semi–cubical parabola (i) has vertex (0, 0) and radius from (0, 0) to (a, a).

Now, arc length from (0, 0) to (a, a) is,

$$L = \int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^a \sqrt{1+\frac{9x}{4a}}\,dx$$

$$= \frac{3}{2\sqrt{a}}\int_0^a \sqrt{x+\frac{4a}{9}}\,dx.$$

Put, $x + \frac{4a}{9} = t^2$ then $dx = 2t\,dt$. Also, $x = 0 \Rightarrow t = \sqrt{4a/9}$ and $x = a \Rightarrow t = \sqrt{13a/9}$. Then,

$$L = \frac{3}{2\sqrt{a}}\int_{\sqrt{\frac{4a}{9}}}^{\sqrt{\frac{13a}{9}}} 2t^2\,dt = \frac{3}{\sqrt{a}}\left[\frac{t^3}{3}\right]_{\sqrt{\frac{4a}{9}}}^{\sqrt{\frac{13a}{9}}}$$

$$= \frac{1}{\sqrt{a}}\left[\left(\sqrt{\frac{13a}{9}}\right)^3 - \left(\sqrt{\frac{4a}{9}}\right)^3\right]$$

$$= \frac{1}{27\sqrt{a}}a\sqrt{a}\,[13\sqrt{13} - 4\sqrt{4}]$$

$$= \frac{a}{27}[13\sqrt{13} - 8].$$

Thus, the length of the arc of the semi–cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a) is $\frac{a}{27}(13\sqrt{13} - 8)$.

7. Show that the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is $\left[a\left(\log 2 + \frac{15}{16}\right)\right]$.

Solution: Given curve is $y^2 = 4ax$ ... (1)

and the line is, $3y = 8x$ ... (2)

Since we have to find the length of curve segment of (1) that is cut off by the line (2).

Here, solving (1) and (2) then the points of contact, are (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$.

Here, differentiating (1) w. r. t. y then

$$2y = 4a\frac{dx}{dy} \Rightarrow \frac{dx}{dy} = \frac{y}{2a}.$$

Now, arc length of (1) from (0, 0) to $\left(\frac{9a}{16}, \frac{3a}{2}\right)$ is

$$L = \int_0^{3a/2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_0^{3a/2} \sqrt{1+\frac{y^2}{4a^2}}\,dy = \frac{1}{2a}\int_0^{3a/2} \sqrt{y^2+4a^2}\,dy$$

$$\Rightarrow L = \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2}\left\{\log\left(y+\sqrt{y^2+4a^2}\right)\right\}\right]_0^{3a/2}$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\cdot\frac{5a}{2} + \frac{4a^2}{2}\log\left(\frac{3a}{2}+\frac{5a}{2}\right) - \frac{4a^2}{2}\log(2a)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + \frac{4a^2}{2}\left\{\log\left(\frac{8a}{2}\right) - \log(2a)\right\}\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + \frac{4a^2}{2}\log\left(\frac{8a}{2.2a}\right)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + \frac{4a^2}{2}\log(2)\right]$$

$$= \frac{15a}{16} + a\log(2).$$

Thus, the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is $\left[a\left(\log 2 + \frac{15}{16}\right)\right]$.

8. Show that the length of the arc of the parabola $x^2 = 4ay$, the vertex to an extremity of the latus rectum $a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$.

Solution: Given parabola is, $x^2 = 4ay$ ... (1)

Clearly the parabola has vertex (0, 0) and the extremities of the latus rectum are ($\pm$ 2a, a).

Here, differentiating (1) w. r. t. x, $2x = 4a\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$

Now, arc length of the parabola (1) from vertex (0, 0) to the extremity (2a, 0) be,

$$L = \int_0^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^{2a} \sqrt{1+\frac{x^2}{4a^2}}\,dx$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{x^2+4a^2}\,dx$$

$$= \frac{1}{2a}\left[\frac{x}{2}\sqrt{x^2 + 4a^2} + \frac{4a^2}{2}\log\left(x + \sqrt{x^2 + 4a^2}\right)\right]_0^{2a}$$

$$= \frac{1}{2a}\left[a.2a.\sqrt{2} + 2a^2 \log\left(2a + 2a\sqrt{2}\right) - 2a^2 \log(2a)\right]$$

$$= a\sqrt{2} + a \log\left(1 + \sqrt{2}\right)$$

$$= a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right].$$

Thus, the length of the arc of the parabola $x^2 = 4ay$, the vertex to an extremity of the latus rectum $a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$.

Chapter 12

APPLICATION OF INTEGRATION
Trapezoidal and Simpson Rule

List of Formulae

(i) For approximate area by Simpson's rule is obtained by using the formula,

$$S = \frac{b - a}{3n}[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})]$$

$$= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})]$$

When n = 4,

$$S = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

Note: If n is odd then Simpson's process can not be applied. So n should be even.

(ii) For approximate area by Trapezoidal's rule is obtained by using the formula,

$$T = \frac{b - a}{2n}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})] \text{ (n is odd or even)}$$

$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})] \text{ (n is odd or even)}$$

When n = 4,

$$T = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

(iii) For exact value, the integral, is solve by the process of definite integral.

For Error Estimation

(i) With the area determined by Simpson's rule,

$$\text{Error}_s\ \% = \frac{|S - E|}{E} \times 100\%$$

(ii) With the area determined by Trapezoidal's rule,

$$\text{Error}_T\ \% = \frac{|T - E|}{E} \times 100\%$$

Where E denotes the exact value of the integral.

Exercise 12.4

1. (i) $\int_0^2 x\, dx$

Solution: Let

$$I = \int_0^2 x\, dx \qquad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\, dx$ then we get

$f(x) = x,\ a = 0,\ b = 2,\ n = 4.$

Then

and $\quad h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}.$

Here,

$x_0 = a = 0$ and $y_0 = f(x_0) = f(0) = 0$

$x_1 = a + h = \frac{1}{2}$ and $y_1 = f(x_1) = f\left(\frac{1}{2}\right) = \frac{1}{2}$

$x_2 = a + 2h = 1$ and $y_2 = f(x_2) = f(1) = 1$

$x_3 = a + 3h = \frac{3}{2}$ and $y_3 = f(x_3) = f\left(\frac{3}{2}\right) = \frac{3}{2}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = f(2) = 2$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2] \quad \text{... (i)}$$

$$\Rightarrow S \approx \frac{1/2}{3}\left[(0+2) + 4\left(\frac{1}{2} + \frac{3}{2}\right) + 2\cdot 1\right]$$

$$= \frac{1}{6}(2 + 8 + 2) = \frac{1}{6}(12) = 2 \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1/2}{2}\left[(0+2) + 2\left(\frac{1}{2} + 1 + \frac{3}{2}\right)\right] = \frac{1}{4}[8] = 2 \text{ sq. units.}$$

(c) Exact value:

$$E = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_2^0 = \left[\frac{2^2}{2} - 0\right] = 2 \text{ sq. units.}$$

Error percentage with the value given by Simpson rule is,

$$\text{Error}_s\ \% = \frac{|S-E|}{E} \times 100\% = \frac{|2-2|}{2} \times 100\% = 0\%.$$

Error percentage with the value given by trapezoidal rule is,

$$\text{Error}_T\ \% = \frac{|T-E|}{E} \times 100\% = \frac{|2-2|}{2} \times 100\% = 0\%.$$

Hence, Simpson's approximate value and trapezoidal approximate value is equal with exact area bounded by given curve.

(ii) $\int_0^2 x^2\,dx$

Solution: Let

$$I = \int_0^2 x^2\,dx \quad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$f(x) = x^2,\ a = 0,\ b = 2,\ n = 4$

and $\quad h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}.$

Then,

$x_0 = a = 0$ and $y_0 = f(x_0) = x_0^2 = 0$

$x_1 = a + h = \frac{1}{2}$ and $y_1 = f(x_1) = x_1^2 = \frac{1}{4}$

$x_2 = a + 2h = 1$ and $y_2 = f(x_2) = x_2^2 = 1$

$x_3 = a + 3h = \frac{3}{2}$ and $y_3 = f(x_3) = x_3^2 = \frac{9}{4}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = x_4^2 = 4$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1/2}{3}\left[(0+4) + 4\left(\frac{1}{4} + \frac{9}{4}\right) + 2\cdot 1\right]$$

$$= \frac{1}{6}[4 + 1 + 9 + 2] = \frac{1}{6} \times 16 = \frac{8}{3} \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1/2}{2}\left[(0+4) + 2\left(\frac{1}{4} + 1 + \frac{9}{4}\right)\right]$$

$$= \frac{1}{4}\left[4 + \frac{1}{2} + 2 + \frac{9}{2}\right]$$

$$= \frac{1}{4}\left[\frac{8+1+4+9}{2}\right] = \frac{1}{4} \times 11 = 2.75 \text{ sq. units.}$$

(c) Also the exact value is,

$$E = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \left[\frac{2^3}{3} - 0\right] = \frac{8}{3}$$

Error percentage with the value given by Simpson's rule is,

$$\text{Error}_s\ \% = \frac{|S-E|}{E} \times 100\% = \frac{\left|\frac{8}{3} - \frac{8}{3}\right|}{8/3} \times 100\% = 0\%.$$

Error percentage with the value given by Trapezoidal ruleis

$$\text{Error}_T\ \% = \frac{|S-E|}{E} \times 100\% = \frac{\left|\frac{11}{4} - \frac{8}{3}\right|}{8/3} \times 100\% = 3.125\%.$$

Thus, Simpson's approximate is more consistence than trapezoidal approximation with exact area bounded by given curves.

(iii) $\int_0^2 x^3\,dx$ **[2003, Spring]**

OR Find the approximate area by using Simpson's and trapezoidal rule for the region bounded by the curve $y = x^3$, the x–axis, $x = 0$ and $x = 2$, with $n = 4$ and compare with exact value. **[2008, Fall]**

Solution: Let

$$I = \int_0^2 x^3\,dx \quad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$$f(x) = x^3,\ a = 0,\ b = 2,\ n = 4$$

and, $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$.

Here,

$x_0 = a = 0$ and $y_0 = f(x_0) = f(0) = 0$

$x_1 = a + h = \frac{1}{2}$ and $y_1 = f(x_1) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$x_2 = a + 2h = 1$ and $y_2 = f(x_2) = f(1) = (1)^3 = 1$

$x_3 = a + 3h = \frac{3}{2}$ and $y_3 = f(x_3) = f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = f(2) = (2)^3 = 8$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1/2}{3}\left[(0+8) + 4\left(\frac{1}{8} + \frac{27}{8}\right) + 2.1\right]$$

$$= \frac{1}{6}\left[8 + \frac{1}{2} + \frac{27}{2} + 2\right]$$

$$= \frac{1}{6}\left[\frac{16+1+27+4}{2}\right] = \frac{1}{6} \times \frac{48}{2} = 4 \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{\frac{1}{2}}{2}\left[(0+8) + 2\left(\frac{1}{8} + 1 + \frac{27}{8}\right)\right]$$

$$= \frac{1}{4}\left[8 + \frac{1}{4} + 2 + \frac{27}{4}\right] = \frac{17}{4} \text{ sq. unit.}$$

(c) Also, the exact value is,

$$E = \int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = \left[\frac{2^4}{4} - 0\right] = \frac{16}{4} = 4 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule

$$\text{Error}_s\ \% = \frac{|S-E|}{E} \times 100\% = \frac{|4-4|}{4} \times 100\% = 0\%$$

Error percentage with the value given by trapezoidal rule:

$$\text{Error}_T\ \% = \frac{|T-E|}{E} \times 100\% = \frac{\left|\frac{17}{4} - 4\right|}{4} \times 100\% = \frac{1/4}{4} \times 100\% = 6.25\%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

(iv) $\int_1^2 \left(\frac{1}{x^2}\right) dx$ **[2011 Fall]**

Solution: Let

$$I = \int_1^2 \left(\frac{1}{x^2}\right) dx \quad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$$f(x) = \frac{1}{x^2},\ a = 1,\ b = 2,\ n = 4$$

and, $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$

Here,

$x_0 = a = 1$ and $y_0 = f(x_0) = \frac{1}{x_0^2} = 1$

$x_1 = a + h = \frac{5}{4}$ and $y_1 = f(x_1) = \frac{1}{x_1^2} = \frac{16}{25}$

$x_2 = a + 2h = \frac{3}{2}$ and $y_2 = f(x_2) = \frac{1}{x_2^2} = \frac{4}{9}$

$x_3 = a + 3h = \frac{7}{4}$ and $y_3 = f(x_3) = \frac{1}{x_3^2} = \frac{16}{49}$

$x_4 = a + 4h = 2$ and $y_4 = f(x_4) = \frac{1}{x_4^2} = \frac{1}{4}$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1/4}{3}\left[\left(1 + \frac{1}{4}\right) + 4\left(\frac{16}{25} + \frac{16}{49}\right) + 2\left(\frac{4}{9}\right)\right]$$

$$= \frac{1}{12}[(1 + 0.25) + 4(0.64 + 0.33) + 2(0.44)]$$

$= 0.5008$ sq. units.

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \left(\frac{1/4}{2}\right)\left[\left(1 + \frac{1}{4}\right) + 2\left(\frac{16}{25} + \frac{4}{9} + \frac{16}{49}\right)\right]$$

$$= \frac{1}{8}[(1 + 0.25) + 2(0.64 + 0.44 + 0.33)]$$

$= 0.5088$ sq. unit.

(c) Also, the exact value is,

$$E = \int_1^2 \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^2 = -\left[\frac{1}{2} - 1\right] = 0.5 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule is

$$\text{Error}_S \% = \left|\frac{S - E}{E}\right| \times 100\%$$

$$= \left|\frac{0.5008 - 0.5}{0.5}\right| \times 100\% = 0.16\,\%$$

Error percentage with the value given by trapezoidal rule:

$$\text{Error}_T \% = \frac{|T - E|}{E} \times 100\% = \frac{|0.5088 - 0.5|}{0.5} \times 100\% = 1.76\,\%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

(v) $\int_1^4 \sqrt{x}\,dx$

OR Find approximate area bounded by given curves $y = \sqrt{x}$ from $x = 1$ to $x = 4$, by using Simpson's and Trapezoidal rule with $n = 4$. Compare these values with exact value. **[2009 Spring]**

OR Evaluate $\int_1^4 \sqrt{x}\,dx$ with $n = 4$ by Simpson's and Trapezoidal rule and compare this with the exact value of the integral. **[2004, Spring]**

Solution: Let

$$I = \int_1^4 \sqrt{x}\,dx \quad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$$f(x) = \sqrt{x},\ a = 1,\ b = 4,\ n = 4$$

and, $h = \frac{b - a}{n} = \frac{4 - 1}{4} = \frac{3}{4}$.

Here,

$x_0 = a = 1$ and $y_0 = f(x_0) = 1$

$x_1 = a + h = \frac{7}{4}$ and $y_1 = f(x_1) = 1.32$

$x_2 = a + 2h = \frac{5}{2}$ and $y_2 = f(x_2) = 1.58$

$x_3 = a + 3h = \frac{13}{4}$ and $y_3 = f(x_3) = 1.80$

$x_4 = a + 4h = 4$ and $y_4 = f(x_4) = 2$

(a) Approximate area by using Simpson's rule.

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{3/4}{3}[(1 + 2) + 4(1.32 + 1.80) + 2(1.58)]$$

$= (0.25)[3 + 12.48 + 3.16]$

$= 4.66$ sq. units.

(b) Approximate area by using trapezoidal rule

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{3/4}{2}[(1 + 2) + 2(1.32 + 1.58 + 1.80)]$$

$$= (0.375)[3 + 9.4] \quad = 3.525 \text{ sq. unit.}$$

(c) Exact value;

$$E = \int_1^4 \sqrt{x}\,dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2\left[\frac{8-1}{3}\right] = 4.66 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule

$$\text{Error}_s\ \% = \frac{|S - E|}{E} \times 100\% = \frac{|4.66 - 4.66|}{4.66} \times 100\% = 0\%$$

Error percentage with the value given by trapezoidal rule:

$$\text{Error}_T\ \% = \frac{|T - E|}{E} \times 100\% = \frac{|3.525 - 4.66|}{4.66} \times 100\% = 24.35\%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

(vi) $\int_0^{\pi} \sin x\,dx$ **[2011 Spring]**

Solution: Let

$$I = \int_0^{\pi} \sin x\,dx \quad \text{... (i)}$$

Comparing it with the integral $\int_a^b f(x)\,dx$ then we get

$$f(x) = \sin x,\ a = 0,\ b = \pi,\ n = 4$$

Then, $h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$

Here,

$x_0 = a = 0$ and $y_0 = f(x_0) = \sin 0 = 0$

$x_1 = a + h = \frac{\pi}{4}$ and $y_1 = f(x_1) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$x_2 = a + 2h = \frac{\pi}{2}$ and $y_2 = f(x_2) = \sin\frac{\pi}{2} = 1$

$x_3 = a + 3h = \frac{3\pi}{4}$ and $y_3 = f(x_3) = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$x_4 = a + 4h = \pi$ and $y_4 = f(x_4) = \sin\pi = 0.$

(a) Now, the approximate area determined by using Simpson's rule is,

$$S \approx \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{\pi/4}{3}\left[(0 + 0) + 4\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 2(1)\right]$$

$$= \frac{\pi}{12}\left[0 + 4(\sqrt{2}) + 2\right]$$

$$= 0.6381\,\pi \text{ sq. units.} = 2.0047 \text{ sq. units.}$$

(b) And, the approximate area determined by using Trapezoidal rule is,

$$T \approx \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \left(\frac{\pi/4}{2}\right)\left[(0 + 0) + 2\left(\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}\right)\right]$$

$$= 0.6036\,\pi \text{ sq. unit.} = 1.8963 \text{ sq units.}$$

(c) Also, the exact value is,

$$E = \int_0^{\pi} \sin x\,dx$$

$$= [-\cos x]_0^{\pi} = [-\cos\pi + \cos 0] = 1 + 1 = 2 \text{ sq. units}$$

Error percentage with the value given by Simpson's rule is

$$\text{Error}_s\ \% = \left|\frac{S - E}{E}\right| \times 100\%$$

$$= \left|\frac{2.0047 - 2}{2}\right| \times 100\% = 0.235\%$$

Error percentage with the value given by trapezoidal rule is

$$\text{Error}_T\ \% = \frac{|T - E|}{E} \times 100\% = \frac{|1.8963 - 2|}{2} \times 100\% = 5.185\%$$

Thus, Simpson's approximate is more consistence than trapezoidal approximate with exact area bounded by the given curve.

OTHER QUESTION AS AN EXERCISE FROM FINAL EXAM

1. Evaluate $\int_0^4 \left(\frac{1}{x^2 + 1}\right) dx$ by using Trapezoid Rule, Simpson's Rule and compare the result with the exact value taking n = 4. **[2009, Fall]**
2. Use Simpson's rule with n = 4 to approximate the area between the curve $y = (2x + 1)^2$, ordinates at x = 1, x = 3 and x-axis. Also evaluate the same by using trapezoidal rule. **[2007, Spring] [1999] [2001]**

Chapter 13

VECTOR ALGEBRA

Exercise 13.1

1. **If $\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} - \vec{k}$, find $\vec{a} \cdot \vec{b}$ and the angle between them.** **[2011 Spring, Short]**

<u>Solution:</u>

Let, $\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} - \vec{k}$.

Then, $\vec{a} \cdot \vec{b} = (2\vec{i} + \vec{j} - 3\vec{k}) \cdot (3\vec{i} - 2\vec{j} - \vec{k})$

$= 2 \times 3 + 1 \times (-2) + (-3) \times (-1)$

$= 6 - 2 + 3 \quad = 9 - 2 \quad = 7.$

Again,

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$$

Let θ be angle between $\vec{a}$ and $\vec{b}$ then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{7}{\sqrt{14} \cdot \sqrt{14}} = \frac{7}{14} = \frac{1}{2} = \cos 60°.$$

$\Rightarrow \quad \cos\theta = \cos 60°.$

$\Rightarrow \quad \theta = 60°.$

2. **Determine the value of λ, so that $\vec{a} = 2\vec{i} + \lambda\vec{j} + \vec{k}$ and $\vec{b} = 4\vec{i} - 2\vec{j} - 2\vec{k}$ are perpendicular.** **[2017 Spring short]**

<u>Solution:</u> Let,

$$\vec{a} = 2\vec{i} + \lambda\vec{j} + \vec{k} \quad \text{and} \quad \vec{b} = 4\vec{i} - 2\vec{j} - 2\vec{k}$$

Since $\vec{a}$ is perpendicular to $\vec{b}$. So, $\vec{a} \cdot \vec{b} = 0$.

$$\Rightarrow (2\vec{i} + \lambda\vec{j} + \vec{k}) \cdot (4\vec{i} - 2\vec{j} - 2\vec{k}) = 0$$

$\Rightarrow \quad 2(4) + \lambda(-2) + 1(-2) = 0$

$\Rightarrow \quad 8 - 2\lambda - 2 = 0$

$\Rightarrow \quad 6 - 2\lambda = 0$

$\Rightarrow \lambda = 3.$

3. If $\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$, find a unit vector $\hat{n}$ perpendicular to $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$, $\hat{n}$ form a right handed system. Also, find the angle between the vectors $\vec{a}$ and $\vec{b}$.

Solution: Let,

$$\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k} \quad \text{and} \quad \vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$$

So, $|\vec{a}| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{16 + 9 + 1} = \sqrt{26}$

$|\vec{b}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3.$

And, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix}$

$= \vec{i}\,(6 + 1) - \vec{j}\,(8 - 2) + \vec{k}\,(-4 - 6)$

$= 7\vec{i} - 6\vec{j} - 10\vec{k}$

So, $|\vec{a} \times \vec{b}| = \sqrt{7^2 + (-6)^2 + (-10)^2} = \sqrt{49 + 36 + 100} = \sqrt{185}.$

Now,

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{1}{\sqrt{185}}\left(7\vec{i} - 6\vec{j} - 10\vec{k}\right).$$

Let θ be an angle between $\vec{a}$ and $\vec{b}$ then,

$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta \qquad$ [being $\hat{n} = 1$]

$\Rightarrow \quad \sqrt{185} = (\sqrt{26})\ 3 \sin\theta$

$\Rightarrow \quad \sin\theta = \dfrac{\sqrt{185}}{3\sqrt{26}}$

$\Rightarrow \theta = \sin^{-1}\left(\dfrac{\sqrt{185}}{3\sqrt{26}}\right)$

4. (i) Find a unit vector normal to the plane containing $\vec{a} = 3\vec{i} - 2\vec{j} + 4\vec{k}$, $\vec{b} = \vec{i} + \vec{j} - 2\vec{k}$.

Solution: Given that,

$$\vec{a} = 3\vec{i} - 2\vec{j} + 4\vec{k} \quad \text{and} \quad \vec{b} = \vec{i} + \vec{j} - 2\vec{k}.$$

Let the plane contains $\vec{a}$ and $\vec{b}$, so $\vec{a} \times \vec{b}$ is normal to the plane.

Here, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix}$

$= \vec{i}\,(4 - 4) - \vec{j}\,(-6 - 4) + \vec{k}\,(3 + 2)$

$= 0.\vec{i} + 10\vec{j} + 5\vec{k}$

So, $|\vec{a} \times \vec{b}| = \sqrt{0^2 + 10^2 + 5^2} = \sqrt{100 + 25} = \sqrt{125} = 5\sqrt{5}.$

Now, $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$, where $\hat{n}$ be unit vector normal to plane containing $\vec{a}$ and $\vec{b}$.

$$= \frac{0\vec{i} + 10\vec{j} + 5\vec{k}}{5\sqrt{5}} = \frac{2\vec{j} + \vec{k}}{\sqrt{5}}.$$

This is required unit normal vector.

(ii) Find the vector whose length is 7 and which is perpendicular to each of the vectors $2\vec{i} - 3\vec{j} + 6\vec{k}$ and $\vec{i} + \vec{j} - \vec{k}$.

Solution: Let

$$\vec{a} = 2\vec{i} - 3\vec{j} + 6\vec{k}, \quad \vec{b} = \vec{i} + \vec{j} - 3\vec{k}$$

and the length of the vector (p) = 7.

Let $\vec{v}$ be a vector perpendicular to $\vec{a}$ and $\vec{b}$ then

$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$

$= \vec{i}\,(3 - 6) - \vec{j}\,(-2 - 6) + \vec{k}\,(2 + 3)$

$= -3\vec{i} + 8\vec{j} + 5\vec{k}$

So, $|\vec{v}| = \sqrt{(-3)^2 + 8^2 + 5^2} = \sqrt{9 + 64 + 25} = \sqrt{98} = 7\sqrt{2}.$

Then, unit vector along $\vec{v}$ is $\hat{n} = \dfrac{\vec{v}}{|\vec{v}|}$

$\Rightarrow \quad \hat{n} = \dfrac{7}{7\sqrt{2}}\left(-3\vec{i} + 8\vec{j} + 5\vec{k}\right).$

Now the vector which is perpendicular to $\vec{a}$ and $\vec{b}$, having length 7 is

$$= p.\hat{n} = \frac{1}{7\sqrt{2}}\left(-3\vec{i}+8\vec{j}+5\vec{k}\right)$$

$$= \frac{1}{\sqrt{2}}\left(-3\vec{i}+8\vec{j}+5\vec{k}\right).$$

This is the required vector.

5. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vector such that $\vec{a}.\vec{b} = \vec{a}.\vec{c}$; $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$; $\vec{a} \neq \vec{0}$. Show that $\vec{b} = \vec{c}$.

Solution: Given that,

$$\vec{a}.\vec{b} = \vec{a}.\vec{c} \quad \dots (i)$$

$$\vec{a}\times\vec{b} = \vec{a}\times\vec{c} \quad \dots (ii) \text{ with } \vec{a} \neq \vec{0}.$$

From (ii),

$$\vec{a}\times\vec{b} = \vec{a}\times\vec{c}.$$

$$\Rightarrow \vec{a}\times(\vec{a}\times\vec{b}) = \vec{a}\times(\vec{a}\times\vec{c}) \text{ with } \vec{a}\neq 0$$

$$\Rightarrow (\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b} = (\vec{a}.\vec{c})\vec{a} - (\vec{a}.\vec{a})\vec{c}$$

$$\Rightarrow (\vec{a}.\vec{b})\vec{a} - a^2\vec{b} = (\vec{a}.\vec{b})\vec{a} - a^2 \quad [\because \vec{a}.\vec{a} = a^2 \text{ and using (i)}]$$

$$\Rightarrow a^2\vec{b} = a^2\vec{c}$$

$$\Rightarrow \vec{b} = \vec{c}.$$

6. Find the area of the parallelogram having adjacent sides are $\vec{i}+2\vec{j}+3\vec{k}$ and $3\vec{i}-2\vec{j}+\vec{k}$.

Solution: Let $\vec{a} = \vec{i}+2\vec{j}+3\vec{k}$ and $\vec{b} = 3\vec{i}-2\vec{j}+\vec{k}$.

$$\text{Then } \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \vec{i}(2+6) - \vec{j}(1-9) + \vec{k}(-2-6)$$

$$= 8\vec{i}+8\vec{j}-8\vec{k}$$

So, $|\vec{a}\times\vec{b}| = \sqrt{8^2+8^2+(-8)^2} = \sqrt{3\times 8^2} = 8\sqrt{3}$.

Thus, the area of parallelogram whose adjacent sides represents by the given vectors, is $8\sqrt{3}$ sq. unit.

7. Calculate $\vec{a}.\vec{b}$, $|\vec{a}|$, $|\vec{b}|$ and find the cosine of the angle between $\vec{a}$ and $\vec{b}$ of

(i) $\vec{a} = 3\vec{i}+2\vec{j}$, $\vec{b} = 5\vec{j}+\vec{k}$

Solution: Let,

$$\vec{a} = 3\vec{i}+2\vec{j} \text{ and } \vec{b} = 5\vec{j}+\vec{k}$$

Then, $\vec{a}.\vec{b} = \left(3\vec{i}+2\vec{j}\right).\left(5\vec{j}+\vec{k}\right)$

$$= \left(3\vec{i}+2\vec{j}+0\vec{k}\right).\left(0\vec{i}+5\vec{j}+\vec{k}\right)$$

$$= 3(0)+2(5)+0(1) = 10.$$

And, $|\vec{a}| = \sqrt{3^2+2^2+0^2} = \sqrt{13}$, $|\vec{b}| = \sqrt{0^2+5^2+1^2} = \sqrt{26}$.

Let θ be angle between $\vec{a}$ and $\vec{b}$ then,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{10}{\sqrt{13}\sqrt{26}} = \frac{10}{13\sqrt{2}} = \frac{10\sqrt{2}}{13\times 2} = \frac{5\sqrt{2}}{13}.$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5\sqrt{2}}{13}\right).$$

(ii) $\vec{a} = 3\vec{i}-2\vec{j}-\vec{k}$, $\vec{b} = -2\vec{j}$

Solution: Let,

$$\vec{a} = 3\vec{i}-2\vec{j}-\vec{k} \text{ and } \vec{b} = -2\vec{j}$$

Then, $\vec{a}.\vec{b} = \left(3\vec{i}-2\vec{j}-\vec{k}\right).\left(-2\vec{j}\right)$

$$= (3)(0)+(-2)(-2)+(-1)(0) = 0+4+0 = 4.$$

And, $|\vec{a}| = \sqrt{3^2+(-2)^2+(-1)^2} = \sqrt{14}$, $|\vec{b}| = \sqrt{(-2)^2} = 2$.

Let θ be angle between $\vec{a}$ and $\vec{b}$ then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{4}{\sqrt{14}.2} = \frac{2}{\sqrt{14}} = \sqrt{\frac{2}{7}}.$$

(iii) $\vec{a} = 5\vec{j}-3\vec{k}$, $\vec{b} = \vec{i}+\vec{j}+\vec{k}$

Solution: Let,

$$\vec{a} = 5\vec{j}-3\vec{k}, \vec{b} = \vec{i}+\vec{j}+\vec{k}$$

Then, $\vec{a}.\vec{b} = \left(5\vec{j}-3\vec{k}\right).\left(\vec{i}+\vec{j}+\vec{k}\right)$

$$= (0)(1)+(5)(1)+(-3)(1) = 5-3 = 2.$$

So, $|\vec{a}| = \sqrt{5^2 + (-3)^2} = \sqrt{25 + 9} = \sqrt{34}$

$|\vec{b}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Let θ be angle between $\vec{a}$ and $\vec{b}$ then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2}{\sqrt{34}\cdot\sqrt{3}} = \sqrt{\frac{4}{34\times 3}} = \sqrt{\frac{2}{51}}.$$

8. Show that the vectors $\vec{a} = \frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k})$, $\vec{b} = \frac{1}{7}(3\vec{i} - 6\vec{j} + 2\vec{k})$ and $\vec{c} = \frac{1}{7}(6\vec{i} + 2\vec{j} - 3\vec{k})$ are orthogonal.

<u>**Solution:**</u> Here, $\vec{a} = \frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k})$, $\vec{b} = \frac{1}{7}(3\vec{i} - 6\vec{j} + 2\vec{k})$

and, $\vec{c} = \frac{1}{7}(6\vec{i} + 2\vec{j} - 3\vec{k})$.

Now,

$$\vec{a}\cdot\vec{b} = \frac{1}{49}[(2)(3) + (3)(-6) + (6)(2)] = \frac{1}{49}[6 - 18 + 12] = \frac{1}{49}[0] = 0.$$

$\Rightarrow \vec{a}\cdot\vec{b} = 0$ ... (i)

Again,

$$\vec{b}\cdot\vec{c} = \frac{1}{49}[(3)(6) + (-6)(2) + (2)(-3)] = \frac{1}{49}[18 - 12 - 6] = \frac{1}{49}[0] = 0$$

$\Rightarrow \vec{b}\cdot\vec{c} = 0$ ... (ii)

Again,

$$\vec{c}\cdot\vec{a} = \frac{1}{49}[(6)(2) + (2)(3) + (-3)(6)] = \frac{1}{49}[12 + 6 - 18] = \frac{1}{49}[0] = 0$$

$\Rightarrow \vec{c}\cdot\vec{a} = 0$ ... (iii)

From equation (i), (ii) and (iii) we observe,

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$$

Hence, $\vec{a}$, $\vec{b}$, $\vec{c}$ are orthogonal to each other.

9. Find $\vec{a}\times\vec{b}$, when $\vec{a} = 2\vec{i} - 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} + \vec{j} + \vec{k}$.

<u>**Solution:**</u> Here,

$\vec{a} = 2\vec{i} - 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} + \vec{j} + \vec{k}$.

Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}(-2+1) - \vec{j}(2+1) + \vec{k}(2+2)$$

$$= -\vec{i} - 3\vec{j} + 4\vec{k}.$$

10. If $\vec{a} = 5\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{j} - 5\vec{k}$, $\vec{c} = -15\vec{i} + 3\vec{j} - 3\vec{k}$ which pairs of vectors are (a) perpendicular? (b) parallel ?

<u>**Solution:**</u> Here,

$\vec{a} = 5\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{j} - 5\vec{k}$, $\vec{c} = -15\vec{i} + 3\vec{j} - 3\vec{k}$

Then,

$|\vec{a}| = \sqrt{(5)^2 + (-1)^2 + (1)^2} = \sqrt{25 + 1 + 1} = 3\sqrt{3}$

$|\vec{b}| = \sqrt{(0)^2 + (1)^2 + (-5)^2} = \sqrt{26}$

$|\vec{c}| = \sqrt{(-15)^2 + (3)^2 + (-3)^2} = \sqrt{225 + 9 + 9} = \sqrt{243}$.

Now,

$\vec{a}\cdot\vec{b} = (5\vec{i} - \vec{j} + \vec{k})\cdot(\vec{j} - 5\vec{k})$

$= (5)(0) + (-1)(1) + (1)(-5) = 0 - 1 - 5 = -6 \neq 0.$

$\vec{b}\cdot\vec{c} = (\vec{j} - 5\vec{k})\cdot(-15\vec{i} + 3\vec{j} - 3\vec{k})$

$= (0)(-15) + (1)(3) + (-5)(-3) = 3 + 15 = 18 \neq 0.$

$\vec{a}\cdot\vec{c} = (5\vec{i} - \vec{j} + \vec{k})\cdot(-15\vec{i} + 3\vec{j} - 3\vec{k})$

$= (5)(-15) + (-1)(3) + (1)(-3) = -75 - 3 - 3 = -81 \neq 0.$

Thus neither of the pair $\vec{a}\cdot\vec{b}$; $\vec{b}\cdot\vec{c}$; $\vec{a}\cdot\vec{c}$ are perpendicular to each other.

Again, let θ be angle between $\vec{a}$ and $\vec{c}$ then

$$\cos\theta = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} = \frac{-81}{(3\sqrt{3})(\sqrt{243})} = -\frac{81}{81} = -1 = \cos\pi.$$

$\Rightarrow \theta = \pi.$

So, $\vec{a}$ and $\vec{c}$ are parallel to each other.

And, let β be angle between $\vec{a}$ and b then

$$\cos\beta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-6}{(3\sqrt{3})(\sqrt{26})}$$

$\Rightarrow \beta = 113.09°.$

So, $\vec{a}$ and $\vec{b}$ are not parallel to each other.

Also, let γ be the angle between $\vec{b}$ and $\vec{c}$ then

$$\cos\gamma = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{18}{(\sqrt{26})(\sqrt{243})} = \sqrt{\frac{2}{39}}.$$

So, $\vec{b}$ and $\vec{c}$ are not parallel to each other.

11. Find the unit vector perpendicular to both vector $\vec{a} = 2\vec{j} - 3\vec{k}$ and $\vec{b} = 2\vec{i}$.

Solution: Let, $\vec{a} = 2\vec{j} - 3\vec{k}$ and $\vec{b} = 2\vec{i}$.

Since $\vec{a}\times\vec{b}$ be the vector perpendicular to $\vec{a}$ and $\vec{b}$.

Here, $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -3 \\ 2 & 0 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0+6) + \vec{k}(0-4)$$

$$= -6\vec{j} - 4\vec{k}.$$

So, $$\left|\vec{a}\times\vec{b}\right| = \sqrt{(0)^2 + (-6)^2 + (-4)^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}.$$

Let $\hat{n}$ be unit vector along $\vec{a}\times\vec{b}$ then

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|} = \frac{-6\vec{j} - 4\vec{k}}{2\sqrt{13}} = \frac{-(3\vec{j} + 2\vec{k})}{\sqrt{13}}.$$

Thus the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{-(3\vec{j} + 2\vec{k})}{\sqrt{13}}$.

12. Find the vector projection of $\vec{a}$ onto $\vec{b}$ if $\vec{a} = 3\vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$. **[2011 Fall, Short]**

Solution: Let,

$$\vec{a} = 3\vec{i} - \vec{j} + \vec{k} \quad \text{and} \quad \vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}.$$

Then, $$\vec{a}\cdot\vec{b} = (3)(2) + (-1)(1) + (1)(-2) = 6 - 1 - 2 = 3.$$

And, $$\left|\vec{b}\right| = \sqrt{4+1+4} = 3.$$

Now, vector projection of $\vec{a}$ onto $\vec{b}$ is,

$$\text{Proj}_{\vec{b}}(\vec{a}) = \left(\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\right)\vec{b} = \left(\frac{3}{3^2}\right)\left(2\vec{i} + \vec{j} - 2\vec{k}\right)$$

$$= \left(\frac{2}{3}\right)\vec{i} + \left(\frac{1}{3}\right)\vec{j} - \left(\frac{2}{3}\right)\vec{k}$$

And, the scalar projection of $\vec{a}$ onto $\vec{b}$ is,

$$\left|\text{Proj}_{\vec{b}}(\vec{a})\right| = \left|\left(\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\right)\vec{b}\right| = \left(\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\right)\left|\vec{b}\right| = \frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|} = \frac{3}{3} = 1.$$

13. If $\vec{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\vec{OB} = \vec{j} + \vec{k}$, find (i) the projection of $\vec{OA}$ on $\vec{OB}$ and (ii) the projection of $\vec{OB}$ on $\vec{OA}$.

Solution: Let,

$$\vec{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k} \quad \text{and} \quad \vec{OB} = \vec{j} + \vec{k}.$$

Then,

i) The projection of $\vec{OA}$ on $\vec{OB}$ is,

$$\text{Proj}_{\vec{OB}}\left(\vec{OA}\right) = \left(\frac{\vec{OA}\cdot\vec{OB}}{\left|\vec{OB}\right|^2}\right)\vec{OB}$$

$$= \left(\frac{\left(2\vec{i} + 3\vec{j} - 4\vec{k}\right)\cdot\left(\vec{j} + \vec{k}\right)}{\left|\vec{j} + \vec{k}\right|^2}\right)\left(\vec{j} + \vec{k}\right)$$

$$= \frac{(2)(0) + (3)(1) + (-4)(1)}{\left(\sqrt{1^2 + 1}\right)^2}\left(\vec{j} + \vec{k}\right)$$

$$= \left(\frac{3-4}{2}\right)\left(\vec{j} + \vec{k}\right)$$

$$= \left(\frac{-1}{2}\right)\vec{j} + \left(\frac{-1}{2}\right)\vec{k}.$$

(ii) The projection of $\vec{OB}$ on $\vec{OA}$ is

$$\text{Proj}_{\vec{OA}}\left(\vec{OB}\right) = \left(\frac{\vec{OB}\cdot\vec{OA}}{\left|\vec{OA}\right|^2}\right)\vec{OA}$$

$$= \left(\frac{\left(\vec{j} + \vec{k}\right)\cdot\left(2\vec{i} + 3\vec{j} - 4\vec{k}\right)}{\left|2\vec{i} + 3\vec{j} - 4\vec{k}\right|^2}\right)\left(2\vec{i} + 3\vec{j} - 4\vec{k}\right)$$

$$= \left(\frac{(0)(2)+(1)(3)+(1)(-4)}{\left(\sqrt{4+9+16}\right)^2}\right)\left(2\vec{i}+3\vec{j}-4\vec{k}\right)$$

$$= \left(\frac{-1}{29}\right)\left(2\vec{i}+3\vec{j}-4\vec{k}\right)$$

$$= \left(\frac{-2}{29}\right)\vec{i}+\left(\frac{-3}{29}\right)\vec{j}+\left(\frac{1}{29}\right)\vec{k}.$$

14. The perpendicular distance of a plane from the origin is 3 units and the vector $2\vec{i}+2\vec{j}-\vec{k}$ is normal to the plane. Find the equation to the plane.

Solution: Let $\vec{v} = 2\vec{i}+2\vec{j}-\vec{k}$

and the distance of equation plane from origin is 3 i.e. $p = 3$.

Let $\hat{n}$ be unit vector perpendicular to plane along $\vec{v}$ then

$$\hat{n} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{\sqrt{2^2+2^2+(-1)^2}} = \frac{1}{3}\left(2\vec{i}+2\vec{j}-\vec{k}\right).$$

Let $\vec{OP} = \vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$ where P(x, y, z) be only point on the plane. Then the equation of plane is

$$\vec{r}\,.\,\hat{n} = p$$

$$\Rightarrow \left(x\vec{i}+y\vec{j}+z\vec{k}\right).\frac{1}{3}\left(2\vec{i}+2\vec{j}-\vec{k}\right) = 3$$

$$\Rightarrow 2x+2y-z = 9.$$

This is the equation of the required plane.

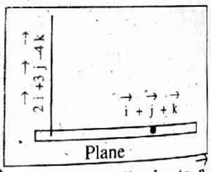
15. Find the equation of the plane passing through the point $\vec{i}+\vec{j}+\vec{k}$ and is perpendicular to the vector $2\vec{i}+3\vec{j}-4\vec{k}$.

Solution: Let P(x, y, z) be any point on the plane and let the given point is,

$$\vec{a} = \vec{i}+\vec{j}+\vec{k}.$$

and the given vector is, $\vec{n} = 2\vec{i}+3\vec{j}-4\vec{k}$.

Also, let $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$.



Now, equation of plane which is passes through $\vec{a}$ and perpendicular to $\vec{n}$ is,

$$\left(\vec{r}-\vec{a}\right).\,\vec{n} = 0$$

$$\Rightarrow \left\{\left(x\vec{i}+y\vec{j}+z\vec{k}\right)-\left(\vec{i}+\vec{j}+\vec{k}\right)\right\}.\left(2\vec{i}+3\vec{j}-4\vec{k}\right) = 0.$$

$$\left\{(x-1)\vec{i}+(y-1)\vec{j}+(z-1)\vec{k}\right\}.\left(2\vec{i}+3\vec{j}-4\vec{k}\right) = 0.$$

$$\Rightarrow (x-1)(2)+(y-1)(3)+(z-1)(-4) = 0.$$

$$\Rightarrow 2x-2+3y-3-4z+4 = 0.$$

$$\Rightarrow 2x+3y-4z-1 = 0.$$

$$\Rightarrow 2x+3y-4z = 1.$$

This is the equation of the required plane.

16. Find the equation of line through (2, –9, 5) and is parallel to $2\vec{i}+5\vec{j}+6\vec{k}$.

Solution: Given point is A(2, –9, 5) and the given vector is

$$\vec{v} = 2\vec{i}+5\vec{j}+6\vec{k}.$$

Let P(x, y, z) be any point on line and A(2, –9, 5) which is passing through A. Then,

$$\vec{AP} = \vec{OP}-\vec{OA} = (x-2)\vec{i}+(y+9)\vec{j}+(z-5)\vec{k}.$$

Given that $\vec{v}$ and $\vec{AP}$ are parallel. So,

$$\vec{AP} = \lambda\vec{v} \qquad \text{for some scalar } \lambda.$$

$$\Rightarrow (x-2)\vec{i}+(y+9)\vec{j}+(z-5)\vec{k} = \lambda(2\vec{i}+5\vec{j}+6\vec{k}).$$

Equating the coefficient of $\vec{i}, \vec{j}, \vec{k}$ on both sides then we get,

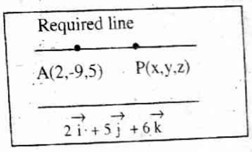
$$x-2 = 2\lambda, \quad y+9 = 5\lambda, \quad z-5 = 6\lambda$$

$$\Rightarrow \frac{x-2}{2} = \lambda, \quad \frac{y+9}{5} = \lambda, \quad \frac{z-5}{6} = \lambda$$

$$\Rightarrow \frac{x-2}{2} = \frac{y+9}{5} = \frac{z-5}{6} = \lambda$$

$$\Rightarrow \frac{x-2}{2} = \frac{y+9}{5} = \frac{z-5}{6}.$$

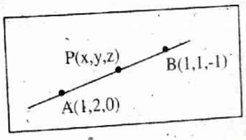
This is the equation of required line.



17. Find the equation of line through (1, 2, 0) and (1, 1, –1).

Solution: Given points are A(1, 2, 0) and B(1, 1, –1). Then their position vector be, $\vec{OA} = \vec{i}+2\vec{j}+0.\vec{k}$, $\vec{OB} = \vec{i}+\vec{j}-\vec{k}$.

Then,



$$\vec{AB} = \vec{OB}-\vec{OA} = 0\vec{i}-\vec{j}-\vec{k}.$$

Let, P(x, y, z) be any point on the line then $\vec{OP} = x\vec{i}+y\vec{j}+z\vec{k}$.

And,

$$\vec{AP} = \vec{OP}-\vec{OA} = \left(x\vec{i}+y\vec{j}+z\vec{k}\right)-\left(\vec{i}+2\vec{j}+0.\vec{k}\right)$$

$$= (x-1)\vec{i} + (y-2)\vec{j} + (z-0)\vec{k}.$$

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are the line segments of same line, so both $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear. Therefore,

$$\overrightarrow{AP} = \lambda\overrightarrow{AB} \quad \text{for some scalar } \lambda.$$

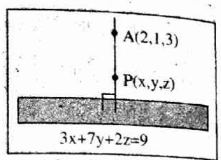
$$\Rightarrow (x-1)\vec{i} + (y-2)\vec{j} + (z-0)\vec{k} = \lambda\left(0.\vec{i} - 1.\vec{j} - 1.\vec{k}\right).$$

$$\Rightarrow \frac{x-1}{0} = \frac{y-2}{-1} = \frac{z-0}{-1} = \lambda.$$

$$\Rightarrow \frac{x-1}{0} = \frac{y-2}{-1} = \frac{z}{-1}$$

This is the equation of required line.

18. Find the equation of line through (2, 1, 3) and is perpendicular to 3x + 7y + 2z = 9.



Solution: Given equation of plane is

$$3x + 7y + 2z = 9 \quad \ldots (i)$$

Then the vector normal to plane (i) is

$$\vec{n} = 3\vec{i} + 7\vec{j} + 2\vec{k}$$

Again, the given point is A(2, 1, 3) and let P(x, y, z) be the general point on the line which is passing through A. Then the position vector of P and A are,

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + 2\vec{k} \quad \text{and} \quad \overrightarrow{OA} = 2\vec{i} + \vec{j} + 3\vec{k}.$$

Then, $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \left(2\vec{i} + \vec{j} + 3\vec{k}\right).$

$$\Rightarrow \overrightarrow{AP} = (x-2)\vec{i} + (y-1)\vec{j} + (z-3)\vec{k}.$$

As given condition, $\overrightarrow{AP}$ is perpendicular to the plane (i) and $\vec{n}$ is perpendicular to (i).

Therefore, $\overrightarrow{AP}$ and $\vec{n}$ both are perpendicular to same plane.

This means the vectors $\overrightarrow{AP}$ and $\vec{n}$ are parallel to each other. So,

$$\overrightarrow{AP} = \lambda\vec{n} \quad \text{where } \lambda \text{ be some scalar.}$$

$$\Rightarrow (x-2)\vec{i} + (y-1)\vec{j} + (z-3)\vec{k} = \lambda\left(3\vec{i} + 7\vec{j} + 2\vec{k}\right).$$

$$\Rightarrow \frac{x-2}{3} = \frac{y-1}{7} = \frac{z-3}{2} = \lambda$$

$$\Rightarrow \frac{x-2}{3} = \frac{y-1}{7} = \frac{z-3}{2}$$

This is the equation of required line.

19. Find the equation of plane through (0, 2, 5) and is normal to $2\vec{i} + 4\vec{j} + \vec{k}$.

Solution: Given point is A(0, 2, 5), so $\vec{a} = \overrightarrow{OA} = 0.\vec{i} + \vec{j} + 5\vec{k}$.

Let P(x, y, z) be general point of the line which passing through A. Then,

$$\vec{r} = \overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Therefore,

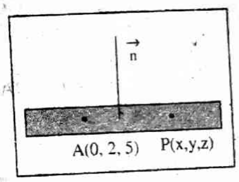
$$\vec{r} - \vec{a} = (x-0)\vec{i} + (y-2)\vec{j} + (z-5)\vec{k}.$$

Also, given vector is, $\vec{n} = 2\vec{i} + 4\vec{j} + \vec{k}$.

Now, the equation of plane which is passing through $\vec{a}$ and is normal to $\vec{n}$, is,

$$\left(\vec{r} - \vec{a}\right) . \vec{n} = 0.$$

$$\Rightarrow \left\{(x-0)\vec{i} + (y-2)\vec{j} + (z-5)\vec{k}\right\} . \left(2\vec{i} + 4\vec{j} + \vec{k}\right) = 0.$$



$$\Rightarrow (x-0)(2) + (y-2)(4) + (2-5)(1) = 0.$$

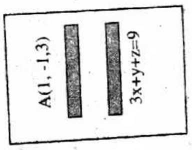
$$\Rightarrow 2x + 4y - 8 + z - 5 = 0.$$

$$\Rightarrow 2x + 4y + z - 13 = 0.$$

$$\Rightarrow 2x + 4y + z = 13.$$

This is the equation of required plane.

20. Find the equation of the plane through (1, –1, 3), parallel to the plane 3x + y + z = 7.



Solution: Given that equation of plane is

$$3x + y + z = 7 \quad \ldots (i)$$

Clearly the plane (i) is normal to the vector, so

$$\vec{n} = 3\vec{i} + \vec{j} + \vec{k} \quad \ldots (ii)$$

Given that A(1, –1, 3) is a point on required plane is,

$$\overrightarrow{OA} = \vec{i} - \vec{j} + 3\vec{k}.$$

Let P(x, y, z) be any point on required plane. Then

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}.$$

Then,

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x-1)\vec{i} + (y+1)\vec{j} + (z-3)\vec{k}.$$

Given that the plane (i) is normal to the required plane, so the vector (ii) is normal to $\overrightarrow{AP}$. Therefore,

$$\overrightarrow{AP}\cdot\vec{n} = 0$$

$$\Rightarrow \left\{(x-1)\vec{i} + (y+1)\vec{j} + (2z-3)\vec{k}\right\}.\left(3\vec{i} + \vec{j} + \vec{k}\right) = 0.$$

$$\Rightarrow (x-1)(3) + (y+1)(1) + (z-3)(1) = 0$$

$$\Rightarrow 3x - 3 + y + 1 + z - 3 = 0$$

$$\Rightarrow 3x + y + z = 5$$

Therefore, the plane $3x + y + z = 5$ is the equation of required plane.

21. Find the equation of the plane through the points (2, 4, 5), (1, 5, 7), (–1, 6, 8). **[2015 Spring][2015 Fall] [2013 Spring][2011 Fall][2008, Spring] [2004, Spring] [2003, Fall]**

Solution: The given points A(2, 4, 5), B(1, 5, 7), C(–1, 6, 8).

Then the position vectors of the points are

$$\overrightarrow{OA} = 2\vec{i} + 4\vec{j} + 5\vec{k},\quad \overrightarrow{OB} = \vec{i} + 5\vec{j} + 7\vec{k},$$

and, $\overrightarrow{OC} = -\vec{i} + 6\vec{j} + 8\vec{k}.$

Then,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(\vec{i} + 5\vec{j} + 7\vec{k}\right) - \left(2\vec{i} + 4\vec{j} + 5\vec{k}\right)$$

$$= -\vec{i} + \vec{j} + 2\vec{k}.$$

And,

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(-\vec{i} + 6\vec{j} + 8\vec{k}\right) - \left(\vec{i} + 5\vec{j} + 7\vec{k}\right)$$

$$= -2\vec{i} + \vec{j} + \vec{k}.$$

Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lies on the same plane. So, $\overrightarrow{AB} \times \overrightarrow{BC}$ is normal to the plane. Here,

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -2 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}(1-2) - \vec{j}(-1+4) + \vec{k}(-1+2)$$

$$= -\vec{i} - 3\vec{j} + \vec{k}.$$

Let P(x, y, z) be any point on the plane whose position vector is,

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$

Then,

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \left(2\vec{i} + 4\vec{j} + 5\vec{k}\right)$$

$$= (x-2)\vec{i} + (y-4)\vec{j} + (2-5)\vec{k}.$$

Since both A and P are two points on the plane, so $\overrightarrow{AP}$ lies on the plane and $\vec{n}$ is normal to the plane. Therefore, $\overrightarrow{AP}$ is normal to $\vec{n}$.

Therefore,

$$\overrightarrow{AP}\cdot\vec{n} = 0$$

$$\Rightarrow \left\{(x-2)\vec{i} + (y-4)\vec{j} + (2-5)\vec{k}\right\}\cdot\left(-\vec{i} - 3\vec{j} + \vec{k}\right) = 0.$$

$$\Rightarrow (x-2)(-1) + (y-4)(-3) + (2-5)(1) = 0$$

$$\Rightarrow -x + 2 - 3y + 12 + 2 - 5 = 0$$

$$\Rightarrow -x - 3y + z + 9 = 0$$

$$\Rightarrow x + 3y - z = 9.$$

Therefore, $x - 3y - z = 9$ be the equation of required plane.

22. Find the equation for the plane through the points (1, 0, 0), (0, 1, 0), (0, 0, 1).

Solution: The given points A(1, 0, 0), B(0, 1, 0), C(0, 0, 1).

Then the position vectors of the points are

$$\overrightarrow{OA} = \vec{i},\quad \overrightarrow{OB} = \vec{j} \quad \text{and } \overrightarrow{OC} = \vec{k}.$$

Then,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{j} - \vec{i} = -\vec{i} + \vec{j}.$$

And, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{k} - \vec{j} = -\vec{j} + \vec{k}.$

Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lies on the same plane. So, $\overrightarrow{AB} \times \overrightarrow{BC}$ is normal to the plane. Here,

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= \vec{i}(1+0) - \vec{j}(-1-0) + \vec{k}(1-0).$$

$$\Rightarrow \vec{n} = \vec{i} + \vec{j} + \vec{k}.$$

Let P(x, y, z) be any point on the plane whose position vector will be,

$$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$$

Then,

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \left(\vec{i}\right)$$

$$= -x\vec{i}.$$

Since both A and P are two points on the plane, so $\overrightarrow{AP}$ lies on the plane and $\vec{n}$ is normal to the plane. Therefore, $\overrightarrow{AP}$ is normal to $\vec{n}$.

Therefore,

$$\overrightarrow{AP}\cdot\vec{n}=0 \Rightarrow \{(x-1)\vec{i}+y\vec{j}+z\vec{k}\}\cdot\{\vec{i}+\vec{j}+\vec{k}\}=0$$

$$\Rightarrow (x-1)\times 1+y\times 1+z\times 1=0$$

$$\Rightarrow x+y+z=1.$$

This is the equation of required plane.

23. Find the equation for the plane through (2, 4, 5) and is perpendicular to the line $\frac{x-5}{1}=\frac{y-1}{3}=\frac{z}{4}$. **[2018 Fall]**

Solution: Given equation of line is

$$\frac{x-5}{1}=\frac{y-1}{3}=\frac{z}{4}=\lambda(\text{say}) \quad \ldots(i)$$

$$\Rightarrow (x-5)\vec{i}+(y-1)\vec{j}+z\vec{k}=\lambda(\vec{i}+3\vec{j}+4\vec{k})$$

$$=\lambda\vec{n}$$

where $\vec{n}=\vec{i}+3\vec{j}+4\vec{k}$

which is parallel to given line.

Given that the plane passes through the point A(2, 4, 5) and let P(x, y, z) be a any point on the plane. So,

$$\overrightarrow{OP}=x\vec{i}+y\vec{j}+z\vec{k} \text{ and } \overrightarrow{OA}=2\vec{i}+4\vec{j}+5\vec{k}.$$

Then,

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\left(x\vec{i}+y\vec{j}+z\vec{k}\right)-\left(2\vec{i}+4\vec{j}+5\vec{k}\right).$$

$$=(x-2)\vec{i}+(y-4)\vec{j}+(2-5)\vec{k}.$$

lies on the plane.

Given that the line (i) is perpendicular to the plane that is through (2, 4, 5).

So, $\vec{n}$ is perpendicular to $\overrightarrow{AP}$. Therefore,

$$\overrightarrow{AP}\cdot\vec{n}=0.$$

$$\Rightarrow \left\{(x-2)\vec{i}+(y-4)\vec{j}+(2-5)\vec{k}\right\}\cdot\left(\vec{i}+3\vec{j}+4\vec{k}\right)=0.$$

$$\Rightarrow (x-2)(1)+(y-4)(3)+(2-5)(4)=0$$

$$\Rightarrow x-2+3y-12+4z-20=0.$$

$$\Rightarrow x+3y+4z=34.$$

This is the equation of required plane.

24. By vector method, find the equation of the plane through the origin and that contains the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{4}$. **[2013 Fall]**

Solution: Given line is,

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{4}=\lambda(\text{say}) \quad \ldots(i)$$

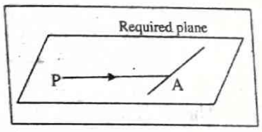
$$\Rightarrow (x-1)\vec{i}+(y+1)\vec{j}+z\vec{k}=\lambda\left(2\vec{i}+3\vec{j}+4\vec{k}\right)$$

$$\Rightarrow \vec{r}=\lambda\vec{v}.$$

where, $\vec{r}=(x-1)\vec{i}+(y+1)\vec{j}+z\vec{k}$ and $\vec{v}=2\vec{i}+3\vec{j}+4\vec{k}$.

Clearly the vector $\vec{v}$ is parallel to (i) and $\vec{r}$ lies on (i).

Let, $\vec{n}=a\vec{i}+b\vec{j}+c\vec{k}$



is normal to (i). So, $\vec{n}$ is normal to both $\vec{r}$ and $\vec{v}$. Therefore,

$$\vec{r}\cdot\vec{n}=0$$

$$\Rightarrow \left\{(x-1)\vec{i}+(y+1)\vec{j}+z\vec{k}\right\}\cdot\left(a\vec{i}+b\vec{j}+c\vec{k}\right)=0$$

$$\Rightarrow (x-1)a+(y+1)b+zc=0 \quad \ldots(i)$$

By given condition (i) passes through origin. So,

$$-a+b=0.$$

$$\Rightarrow a=b. \quad \ldots(ii)$$

Also, $\vec{r}\cdot\vec{v}=0$

$$\Rightarrow (2\vec{i}+3\vec{j}+4\vec{k})\cdot(a\vec{i}+b\vec{j}+c\vec{k})=0.$$

$$\Rightarrow 2a+3b+4c=0 \quad \ldots(iii)$$

Now, putting a = b in equation (iii) then it becomes

$$2b+3b+4c=0$$

$$\Rightarrow 5b+4c=0 \Rightarrow b=-\frac{4}{5}c.$$

Thus, $a=b=-\frac{4}{5}c=k. \Rightarrow a=k, b=k, c=-\frac{5}{4}k.$

Then equation (i) becomes,

$$(x-1)k+(y+1)k+z\left(-\frac{5}{4}\right)k=0.$$

$$\Rightarrow 4(x-1)+4(y+1)-5z=0.$$

$$\Rightarrow 4x+4y-5z=0.$$

This is the equation of required plane.

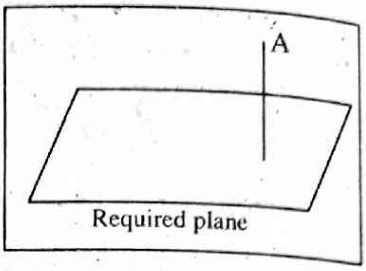
25. Find the plane through A(1, −2, 1) perpendicular to the vector from the origin to A.

Solution: Let A(1, −2, 1) be a point on the plane. Also, let P(x, y, z) be any point on the plane. Then,

$$\overrightarrow{OA}=\vec{i}-2\vec{j}+\vec{k} \text{ and } \overrightarrow{OP}=x\vec{i}+y\vec{j}+z\vec{k}.$$

Then,

$$\overrightarrow{AP} = (x-1)\vec{i} + (y+2)\vec{j} + (z-1)\vec{k}.$$



Clearly the vector $\overrightarrow{AP}$ lies on the plane.
By given hypothesis, the plane is perpendicular to $\overrightarrow{OA}$. So, $\overrightarrow{AP}$ are perpendicular to $\overrightarrow{OA}$. Therefore,

$$\overrightarrow{AP} \cdot \overrightarrow{OA} = 0$$

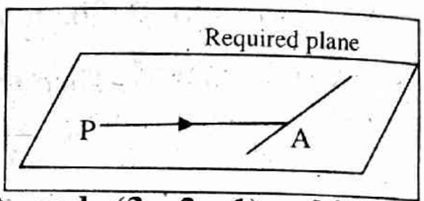
$$\Rightarrow \left\{(x-1)\vec{i} + (y+2)\vec{j} + (2-1)\vec{k}\right\} \cdot \left(\vec{i} - 2\vec{j} + \vec{k}\right) = 0$$

$$\Rightarrow (x-1)(1) + (y+2)(-2) + (z-1)(1) = 0$$

$$\Rightarrow x - 1 - 2y - 4 + z - 1 = 0$$

$$\Rightarrow x - 2y + z = 6.$$

This is the equation of required plane.



26. Find plane through the points (1, 2, 3) and (3, 2, 1) which is perpendicular to the plane $4x - y + 2z = 7$. **[2016 Spring][2016 Fall] [2014 Spring][2008, Fall][2005, Spring][2005, Fall][2009, Fall]**

Solution: Given that the required plane passes through A(1, 2, 3), B(3, 2, 1).

So, $\overrightarrow{AB} = (2, 0, -2)$ lies on the plane.

Let $\vec{n} = (a, b, c)$ be a vector perpendicular to the required plane. Then $\vec{n}$ is perpendicular to $\overrightarrow{AB}$. So,

$$\overrightarrow{AB} \cdot \vec{n} = 0$$

$$\Rightarrow \quad 2a - 2c = 0 \qquad \ldots \text{(i)}$$

Also, given that the plane $4x - y + 2z = 7$ is perpendicular to the required plane. So, the vector $(4, -1, 2)$ is parallel to the required plane. Therefore, $(4, -1, 2)$ is normal to $\vec{n}$. So,

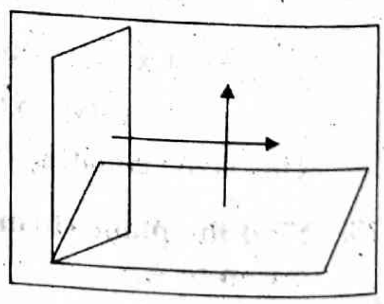
$$(4, -1, 2) \cdot \vec{n} = 0$$

$$\Rightarrow \quad 4a - b + 2c = 0 \qquad \ldots \text{(ii)}$$

Solving equation (i) and (ii), we get

$$a = c = \frac{b}{6} = k \text{ (say)}$$

$$\Rightarrow \quad a = k,\ b = 6k,\ c = k$$



Thus, $\vec{n} = (k, 6k, k)$ is normal to the plane.

Hence, the equation of plane is passing through $\vec{a}$ and normal to $\vec{n}$ be

$$\left(\vec{r} - \vec{a}\right) \cdot \vec{n} = 0 \qquad \text{for } \vec{a} = \overrightarrow{OA}$$

$\Rightarrow \quad (x - 1, y - 2, z - 3) \cdot (1, 6, 1) = 0 \quad$ for $k \neq 0$

$\Rightarrow \quad x + 6y + z = 16$

This is the equation of required plane.

Exercise 13.2

1. **Find the volume of a parallelepiped whose concurrent edges are represented by the vectors.**

> **Note:** Let a parallelepiped (i.e. 3D figure standing upon a parallelogram) has congruent edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ (like as l, b, h) then the volume of the parallelepiped is
>
> $$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.(\vec{b} \times \vec{c})$$
>
> [See also geometrical meaning of scalar product of three vectors]

(i) $\vec{i} + \vec{j} + \vec{k}$; $\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} - \vec{k}$ **[2013 Fall, Short]**

Solution: Let,

$$\vec{a} = \vec{i} + \vec{j} + \vec{k}, \quad \vec{b} = \vec{i} - \vec{j} + \vec{k}, \quad \vec{c} = \vec{i} + 2\vec{j} - \vec{k}$$

Now the volume of the parallelopiped whose current edges represented by $\vec{a}, \vec{b}, \vec{c}$ is

$$V = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= (1)(1-2) - (1)(-1-1) + (1)(2+1)$$
$$= -1 + 2 + 3$$
$$= 4.$$

Thus the volume of the parallelepiped whose concurrent edges are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is 4 cubic units.

(ii) $2\vec{i} - 3\vec{j} + 4\vec{k}$, $\vec{i} + 2\vec{j} - \vec{k}$ and $3\vec{i} - \vec{j} + 2\vec{k}$

Solution: Let,

$$\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k} = (2, -3, 4), \qquad \vec{b} = \vec{i} + 2\vec{j} - \vec{k} = (1, 2, -1)$$
$$\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k} = (3, -1, 2).$$

Now the volume of the parallelopiped whose current edges represented by $\vec{a}, \vec{b}, \vec{c}$ is

$$V = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$
$$= (2)(4-1) - (-3)(2+3) + (4)(-1-6)$$
$$= 6 + 15 - 28$$
$$= 7.$$

Thus the volume of the parallelepiped whose concurrent edges are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is 7 cubic units.

(iii) $\vec{i} + 2\vec{j} + 2\vec{k}$, $3\vec{i} + 7\vec{j} - 4\vec{k}$ and $\vec{i} - 5\vec{j} + 3\vec{k}$.

Solution: Process as above.

Q. **Find the volume of a parallelopiped whose concurrent edges are represented by the vectors $\vec{i} + \vec{j} + \vec{k}$, $2\vec{i} + \vec{j} - 2\vec{k}$ and $3\vec{i} + 2\vec{j} - \vec{k}$.** **[2018 Spring short]**

Solution: Process as above.

2. **Prove that the following four points are coplanar;**

> **Note:** If the volume of any parallelepiped is 0 then the vectors that determined the concurrent edges of the parallelepiped lie on the same plane. So, the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie on the same plane (i.e. they are coplanar) only if
>
> $$[\vec{a}\ \vec{b}\ \vec{c}] \Leftrightarrow \vec{a}.(\vec{b} \times \vec{c}) = 0.$$
>
> The four points A, B, C and D are coplanar if and only if
>
> $$[\vec{a}\ \vec{b}\ \vec{c}] = 0 \text{ or } [\vec{AB}\ \vec{BC}\ \vec{CD}] = 0$$
>
> where $\vec{a} = \vec{AB} = \vec{OB} - \vec{OA}$, $\vec{b} = \vec{BC} = \vec{OC} - \vec{OB}$ and $\vec{c} = \vec{CD} = \vec{OD} - \vec{OC}$.

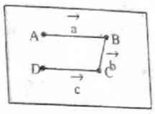
(i) $2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{i} - 2\vec{j} + 3\vec{k}$, $3\vec{i} + 4\vec{j} - 2\vec{k}$ and $\vec{i} - 6\vec{j} + 6\vec{k}$

Solution: Let, the given points are A, B, C and D. So, their position vectors are,

$$\vec{OA} = 2\vec{i} + 3\vec{j} - \vec{k}, \qquad \vec{OB} = \vec{i} - 2\vec{j} + 3\vec{k}$$
$$\vec{OC} = 3\vec{i} + 4\vec{j} - 2\vec{k}, \qquad \vec{OD} = \vec{i} - 6\vec{j} + 6\vec{k}$$

Then,

$$\vec{AB} = \vec{OB} - \vec{OA} = -\vec{i} - 5\vec{j} + 4\vec{k}$$
$$\vec{BC} = \vec{OC} - \vec{OB} = 2\vec{i} + 6\vec{j} - 5\vec{k}$$
$$\vec{CD} = \vec{OD} - \vec{OC} = -2\vec{i} - 10\vec{j} + 8\vec{k}$$



Now,

$$[\vec{AB}\ \vec{BC}\ \vec{CD}] = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 8 \end{vmatrix}$$
$$= (-1)(48-50) - (-5)(16-10) + (4)(-20+12)$$
$$= 2 + 30 - 32$$
$$= 0.$$

Thus, $[\vec{AB}\ \vec{BC}\ \vec{CD}] = 0$. This means the given points are coplanar.

(ii) $-\vec{i} + 4\vec{j} - 3\vec{k}$, $3\vec{i} + 2\vec{j} - 5\vec{k}$, $3\vec{i} + 8\vec{j} - 5\vec{k}$ and $-3\vec{i} + 2\vec{j} + \vec{k}$.

Solution: Let, the given points are A, B, C and D. So, their position vectors are,

$$\vec{OA} = -\vec{i} + 4\vec{j} - 3\vec{k} \qquad \vec{OB} = 3\vec{i} + 2\vec{j} - 5\vec{k}$$
$$\vec{OC} = 3\vec{i} + 8\vec{j} - 5\vec{k} \qquad \vec{OD} = -3\vec{i} + 2\vec{j} + \vec{k}$$

Then,

$$\vec{AB} = \vec{OB} - \vec{OA} = 4\vec{i} - 2\vec{j} - 2\vec{k}$$
$$\vec{BC} = \vec{OC} - \vec{OB} = 0\vec{i} + 6\vec{j} + 0\vec{k}$$
$$\vec{CD} = \vec{OD} - \vec{OC} = -6\vec{i} - 6\vec{j} + 6\vec{k}$$

Now,

$$[\vec{AB}\ \vec{BC}\ \vec{CD}] = \begin{vmatrix} 4 & -2 & -2 \\ 0 & 6 & 0 \\ -6 & -6 & 6 \end{vmatrix}$$
$$= (4)(36-0) - (-2)(0+0) + (-2)(0+36)$$
$$= 144 + 0 - 72$$
$$= 72.$$

Here, $\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right] = 72 \neq 0$. This means the given points are non-coplanar.

(iii) $-\vec{i} + 2\vec{j} - 4\vec{k}, 2\vec{i} - \vec{j} + 3\vec{k}, 6\vec{i} + 2\vec{j} - \vec{k}$ **and** $-12\vec{j} - \vec{k} - 3\vec{k}$. **[2017 Fall]**

Solution: Let, the given points are A, B, C and D. So, their position vector will be,

$$\overrightarrow{OA} = -\vec{i} + 2\vec{j} - 4\vec{k}, \quad \overrightarrow{OB} = 2\vec{i} - \vec{j} + 3\vec{k}$$

$$\overrightarrow{OC} = 6\vec{i} + 2\vec{j} - \vec{k}, \quad \overrightarrow{OD} = -12\vec{i} - \vec{j} - 3\vec{k}$$

Then,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\vec{i} - 3\vec{j} + 7\vec{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 4\vec{i} + 3\vec{j} - 4\vec{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -18\vec{i} - 3\vec{j} - 2\vec{k}$$

Now,

$$\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right] = \begin{vmatrix} 3 & -3 & 7 \\ 4 & 3 & -4 \\ -18 & -3 & -2 \end{vmatrix}$$

$$= (3)(-6 - 12) - (-3)(-8 - 72) + (7)(-12 + 54)$$

$$= -54 - 240 + 294$$

$$= -100.$$

Here, $\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right] = 72 \neq 0$. This means the given points are non-coplanar.

(iv) $-6\vec{i} + 3\vec{j} + 2\vec{k}, 3\vec{i} - 2\vec{j} + 4\vec{k}, 5\vec{i} + 7\vec{j} + 3\vec{k}$ **and** $-13\vec{i} + 17\vec{j} - \vec{k}$ **[2007, Spring]**

Solution: Let, the given points are A, B, C and D. So, their position vector are,

$$\overrightarrow{OA} = -6\vec{i} + 3\vec{j} + 2\vec{k}, \quad \overrightarrow{OB} = 3\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\overrightarrow{OC} = 5\vec{i} + 7\vec{j} + 3\vec{k}, \quad \overrightarrow{OD} = -13\vec{i} + 17\vec{j} - \vec{k}$$

Then,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 9\vec{i} - 5\vec{j} + 2\vec{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\vec{i} + 9\vec{j} - \vec{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -18\vec{i} + 10\vec{j} - 4\vec{k}$$

Now,

$$\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right] = \begin{vmatrix} 9 & -5 & 2 \\ 2 & 9 & -1 \\ -18 & 10 & -4 \end{vmatrix}$$

$$= (9)(-36 + 10) - (-5)(-8 - 18) + (2)(20 + 162)$$

$$= -136 - 130 + 364$$

$$= 0.$$

Here, $\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right] = 0$. This means the given points are coplanar.

3. Find the constant λ such that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} + \lambda\vec{j} + 5\vec{k}$ are coplanar.

Solution: Given vectors are

$$\vec{a} = 2\vec{i} - \vec{j} + \vec{k}, \quad \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}, \quad \vec{c} = 3\vec{i} + \lambda\vec{j} + 5\vec{k}$$

are coplanar. So, scalar triple product of these vectors is zero.

So, $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

$$\Rightarrow (2)(10 + 3\lambda) - (-1)(5 + 9) + (1)(\lambda - 6) = 0$$

$$\Rightarrow 20 + 6\lambda + 14 + \lambda - 6 = 0$$

$$\Rightarrow 28 + 7\lambda = 0$$

$$\Rightarrow \lambda = -4.$$

Thus, for $\lambda = -4$ the given vectors will be coplanar.

4. The position vectors of the points A, B, C and D are $3\vec{i} - 2\vec{j} - \vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$, $-\vec{i} + \vec{j} + 2\vec{k}$ and $4\vec{i} + 5\vec{j} + \lambda\vec{k}$ respectively. If the points A, B, C and D are coplanar, find the value of λ.

Solution: Let the position vector of the given points A, B, C and D are,

$$\overrightarrow{OA} = 3\vec{i} - 2\vec{j} - \vec{k}, \quad \overrightarrow{OB} = 2\vec{i} + 3\vec{j} - 4\vec{k},$$

$$\overrightarrow{OC} = -\vec{i} + \vec{j} + 2\vec{k}, \quad \overrightarrow{OD} = 4\vec{i} + 5\vec{j} + \lambda\vec{k}.$$

Then,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} + 5\vec{j} - 3\vec{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -3\vec{i} - 2\vec{j} + 6\vec{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = 5\vec{i} + 4\vec{j} + (\lambda - 2)\vec{k}$$

Let given points are coplanar, so

$$\left[\vec{AB}\ \ \vec{BC}\ \ \vec{CD}\right] = 0.$$

$$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ -3 & -2 & 6 \\ 5 & 4 & (\lambda-2) \end{vmatrix} = 0$$

$$\Rightarrow (-1)(4-2\lambda-24) - (5)(6-3\lambda-30) + (-3)(-12+10) = 0$$

$$\Rightarrow -4 + 2\lambda + 24 - 30 + 15\lambda + 150 + 6 = 0$$

$$\Rightarrow 146 + 17\lambda = 0$$

$$\Rightarrow \lambda = \frac{-146}{17}.$$

Thus, the value of λ is $\frac{-146}{17}$, so that the position vectors of the given points are coplanar.

5. Find the value of λ, if the volume of parallelepiped whose edges are represented by $-12\vec{i} + \lambda\vec{k}$; $3\vec{j} - \vec{k}$ and $2\vec{i} + \vec{j} - 15\vec{k}$ is 546.

Solution: Let $\vec{a} = -12\vec{i} + \lambda\vec{k}$, $\vec{b} = 3\vec{j} - \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} - 15\vec{k}$

Given that the volume of the parallelopiped whose edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$ is 546. So,

$$\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = 546$$

$$\Rightarrow \begin{vmatrix} -12 & 0 & 2 \\ 0 & 3 & 1 \\ \lambda & -1 & -15 \end{vmatrix} = 546$$

$$\Rightarrow (-12)(-45+1) - 0 + (\lambda)(0-6) = 546$$

$$\Rightarrow 528 - 6\lambda = 546$$

$$\Rightarrow -6\lambda = 18$$

$$\Rightarrow \lambda = -3.$$

Thus for $\lambda = -3$, the vectors determined a parallelepiped with volume 546 cubic units.

6. Show that the vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to the vector $\vec{a}$.

Solution: Here we have to show the vector

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) \quad \ldots \text{(i)}$$

is parallel to the vector $\vec{a}$. For this it sufficient to show the vector (i) is equal to scalar times $\vec{a}$.

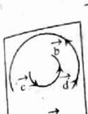
Since we have,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

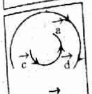
or

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a}$$

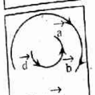
Here,



For $\vec{a}$



For $\vec{b}$



For $\vec{c}$

$$\vec{r} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

$$= [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a} + [\vec{d}\ \vec{b}\ \vec{a}]\vec{c} - [\vec{d}\ \vec{b}\ \vec{c}]\vec{a} + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

$$= [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a} + [\vec{d}\ \vec{b}\ \vec{a}]\vec{c} - [\vec{d}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{a}\ \vec{d}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{d}\ \vec{b}]\vec{c}$$

$$= [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a} + [\vec{d}\ \vec{b}\ \vec{a}]\vec{c} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a} - [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{d}\ \vec{b}\ \vec{a}]\vec{c}$$

$$= -2[\vec{c}\ \vec{d}\ \vec{b}]\vec{a}$$

$$\Rightarrow \vec{r} = \lambda\vec{a} \qquad \text{where } \lambda = 2[\vec{c}\ \vec{d}\ \vec{b}] = \text{scalar}.$$

Hence $\vec{r}$ is parallel to $\vec{a}$.

7. If $\vec{c} = \vec{a}\times\vec{b}$ and $\vec{b} = \vec{a}\times\vec{c}$ with $\vec{a} \neq 0$ then show that $\vec{b} = \vec{0}$ and $\vec{c} = \vec{0}$.

Solution: Let,

$$\vec{c} = \vec{a}\times\vec{b} \quad \ldots \text{(i)}$$

and, $$\vec{b} = \vec{a}\times\vec{c} \quad \ldots \text{(ii)}$$

Here,

$$\vec{c} = \vec{a}\times\vec{b} = \vec{a}\times(\vec{a}\times\vec{c}) \qquad [\because \text{using (ii)}]$$

$$= (\vec{a}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{a})\vec{c}$$

$$\Rightarrow \{1 + (\vec{a}\cdot\vec{a})\}\vec{c} = (\vec{a}\cdot\vec{c})\vec{a}$$

$$\Rightarrow (1+a^2)\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{a}$$

$$\Rightarrow \lambda_1\vec{c} = \lambda_2\vec{a}, \quad \text{where } \lambda_1 = 1 + a^2, \lambda_2 = \vec{a}\cdot\vec{c}$$

So, $\vec{a}$ is parallel to $\vec{c}$. This implies that $\vec{a}\times\vec{c} = 0 \Rightarrow \vec{b} = 0$.

Also, $\vec{c} = \vec{a}\times\vec{b} = \vec{a}\times\vec{0} = 0$.

Thus, $\vec{b} = 0$ and $\vec{c} = 0$.

NOTE: If $\vec{a} = \vec{0}$ in above problem then the problem has trivial solution i.e. $\vec{c} = \vec{0}\times\vec{b} = \vec{0}$ and $\vec{b} = \vec{0}\times\vec{c} = \vec{0}$. So, we should mention $\vec{a} \neq \vec{0}$ in the problem.

8. Show that the vector $\vec{a} = 4\vec{i} - 3\vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} - 4\vec{j} - 4\vec{k}$ and $\vec{c} = 3\vec{i} + 2\vec{j} - \vec{k}$ are linearly independent.

NOTE: If the vectors are non-coplanar then the vectors are linearly independent. That means, if $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) \neq 0$ then the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent.

Solution: Let,

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k};\ \vec{b} = 3\vec{i} + 7\vec{j} - 4\vec{k};\ \vec{c} = \vec{i} - 5\vec{j} + 3\vec{k}$$

Now,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix}$$

$$= (1)(21-20) - (2)(9+4) + (2)(-15-7)$$

$$= 1 - 26 - 44$$

$$= -69 \neq 0.$$

This shows that the vectors are linearly independent.

9. Prove that the four points $\vec{a} = 4\vec{i} + 5\vec{j} + \vec{k}$, $\vec{b} = -\vec{j} - \vec{k}$, $\vec{c} = 3\vec{i} + 9\vec{j} + 4\vec{k}$ and $\vec{d} = -4\vec{i} + 4\vec{j} + 4\vec{k}$ are non coplanar.

Solution: Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are position vector of A, B, C and D with reference to origin O. So,

Let, $\vec{OA} = 4\vec{i} + 5\vec{j} + \vec{k}$, $\quad \vec{OB} = 0\vec{i} - \vec{j} - \vec{k}$,

$\vec{OC} = 3\vec{i} + 9\vec{j} + 4\vec{k}$, $\quad \vec{OD} = -4\vec{i} + 4\vec{j} + 4\vec{k}$.

Then,

$$\vec{AB} = \vec{OB} - \vec{OA} = -4\vec{i} - 6\vec{j} - 2\vec{k}$$

$$\vec{BC} = \vec{OC} - \vec{OB} = 3\vec{i} + 10\vec{j} + 8\vec{k}$$

$$\vec{CD} = \vec{OD} - \vec{OC} = -7\vec{i} - 5\vec{j} + 0\vec{k}$$

Now,

$$[\vec{AB}\ \vec{BC}\ \vec{CD}] = \begin{vmatrix} -4 & -6 & -2 \\ 3 & 10 & 8 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= (-4)(0+40) - (-6)(0+56) + (-2)(-15+70)$$

$$= -160 + 336 - 110$$

$$= 66.$$

Since $[\vec{AB}\ \vec{BC}\ \vec{CD}] = 66 \neq 0$, so given points are non–coplanar.

10. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then show that $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{3}{2}$.

Solution: Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors. So,

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1.$$

Therefore,

$$a^2 = |\vec{a}|^2 = b^2 = |\vec{b}|^2 = c^2 = |\vec{c}|^2 = 1$$

$$\Rightarrow \vec{a}\cdot\vec{a} = \vec{b}\cdot\vec{b} = \vec{c}\cdot\vec{c} = 1.$$

Also, given that

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}.$$

$$\Rightarrow \vec{c} = -(\vec{a} + \vec{b}) \quad \ldots (i)$$

Then taking dot product with $\vec{a}$ then,

$$\vec{a}\cdot\vec{c} = -\vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b}$$

$$\Rightarrow \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = -1 \quad \ldots (ii)$$

And taking dot product of (i) with $\vec{b}$ and $\vec{c}$ respectively then as (ii) we get,

$$\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} = -1 \quad \ldots (iii)$$

and, $$\vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -1 \quad \ldots (iv)$$

Now, adding (ii), (iii) and (iv) then we get,

$$2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -3.$$

$$\Rightarrow \vec a\cdot\vec b+\vec b\cdot\vec c+\vec c\cdot\vec a=\frac{-3}{2}$$

11. Show that $\vec a\times(\vec b\times\vec c)+\vec b\times(\vec c\times\vec a)+\vec c\times(\vec a\times\vec b)=0$.

Solution: Here,

$$\vec a\times(\vec b\times\vec c)+\vec b\times(\vec c\times\vec a)+\vec c\times(\vec a\times\vec b)$$

$$=(\vec a\cdot\vec c)\vec b-(\vec a\cdot\vec b)\vec c+(\vec b\cdot\vec a)\vec c-(\vec b\cdot\vec c)\vec a+(\vec c\cdot\vec b)\vec a-(\vec c\cdot\vec a)\vec b$$

$$=(\vec a\cdot\vec c)\vec b-(\vec a\cdot\vec b)\vec c+(\vec b\cdot\vec a)\vec c-(\vec b\cdot\vec c)\vec a+(\vec b\cdot\vec c)\vec a-(\vec a\cdot\vec c)\vec b$$

$$=0.$$

Thus, $\vec a\times(\vec b\times\vec c)+\vec b\times(\vec c\times\vec a)+\vec c\times(\vec a\times\vec b)=0$.

12. Show that $(\vec b\times\vec c)\times(\vec c\times\vec a)=[\vec a\ \vec b\ \vec c]\vec c$. [2009 Spring]

Solution: Here,

$$(\vec b\times\vec c)\times(\vec c\times\vec a)$$

$$=[\vec b\ \vec c\ \vec a]\vec c-[\vec b\ \vec c\ \vec c]\vec a$$

$$=[\vec b\ \vec c\ \vec a]\vec c-0 \qquad [\because [\vec b\ \vec c\ \vec c]=0]$$

$$=[\vec a\ \vec b\ \vec c]\vec c.$$

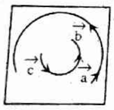
Thus, $(\vec b\times\vec c)\times(\vec c\times\vec a)=[\vec a\ \vec b\ \vec c]\vec c$.

13. Show that $[\vec b\times\vec c\ \ \vec c\times\vec a\ \ \vec a\times\vec b]=[\vec a\ \vec b\ \vec c]^2$. [2002]

Solution: Here,

$$[\vec b\times\vec c\ \ \vec c\times\vec a\ \ \vec a\times\vec b]$$

$$=(\vec b\times\vec c)\cdot\{(\vec c\times\vec a)\times(\vec a\times\vec b)\} \qquad [\because \text{By definition}]$$



$$=(\vec b\times\vec c)\cdot\{[\vec c\ \vec a\ \vec b]\vec a-[\vec c\ \vec a\ \vec a]\vec b\}$$

$$=(\vec b\times\vec c)\cdot\{[\vec c\ \vec a\ \vec b]\vec a-0\}$$

$[\because [\vec c = (\vec a\ \vec b\ \vec a\ \vec c) = 0]$ $[\vec c\ \vec a\ \vec b]\vec a$

$$=\{(\vec b\times\vec c)\cdot\vec a\}\cdot[\vec c\ \vec a\ \vec b]$$

$$=[\vec a\ \vec b\ \vec c]\cdot[\vec a\ \vec b\ \vec c]$$

$$=[\vec a\ \vec b\ \vec c]^2.$$

Thus, $[\vec b\times\vec c\ \ \vec c\times\vec a\ \ \vec a\times\vec b]=[\vec a\ \vec b\ \vec c]^2$.

14. Show that $(\vec a\times\vec b)\times(\vec c\times\vec d)+(\vec a\times\vec c)\times(\vec d\times\vec b)+(\vec a\times\vec d)\times(\vec b\times\vec c)=2[\vec b\ \vec d\ \vec c]\vec a$.

Solution: Since we have,

$$(\vec a\times\vec b)\times(\vec c\times\vec d)=[\vec a\ \vec b\ \vec d]\vec c-[\vec a\ \vec b\ \vec c]\vec d$$

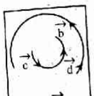
or

$$(\vec a\times\vec b)\times(\vec c\times\vec d)=[\vec c\ \vec d\ \vec a]\vec b-[\vec c\ \vec d\ \vec b]\vec a$$

Here,

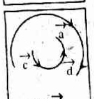
$$\vec r=(\vec a\times\vec b)\times(\vec c\times\vec d)+(\vec a\times\vec c)\times(\vec d\times\vec b)+(\vec a\times\vec d)\times(\vec b\times\vec c)$$

$$=[\vec c\ \vec d\ \vec a]\vec b-[\vec c\ \vec d\ \vec b]\vec a+[\vec d\ \vec b\ \vec a]\vec c-[\vec d\ \vec b\ \vec c]\vec a+(\vec a\times\vec d)\times(\vec b\times\vec c)$$



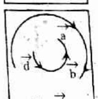
For $\vec a$

$$=[\vec c\ \vec d\ \vec a]\vec b-[\vec c\ \vec d\ \vec b]\vec a+[\vec d\ \vec b\ \vec a]\vec c-[\vec d\ \vec b\ \vec c]\vec a+[\vec a\ \vec d\ \vec c]\vec b-[\vec a\ \vec d\ \vec b]\vec c$$



For $\vec b$

$$=[\vec c\ \vec d\ \vec a]\vec b-[\vec c\ \vec d\ \vec b]\vec a+[\vec d\ \vec b\ \vec a]\vec c-[\vec c\ \vec d\ \vec b]\vec a-[\vec c\ \vec d\ \vec a]\vec b-[\vec d\ \vec b\ \vec a]\vec c$$



For $\vec c$

$$=-2[\vec c\ \vec d\ \vec b]\vec a$$

$$=2[\vec b\ \vec d\ \vec c]\vec a.$$

15. Show that $(\vec a\times\vec b)\times\vec c=\vec a\times(\vec b\times\vec c)$ if and only if the vectors $\vec a$ and $\vec c$ are collinear. [2007, Spring]

Solution: Suppose that,

$$(\vec a\times\vec b)\times\vec c=\vec a\times(\vec b\times\vec c)$$

$$\Leftrightarrow (\vec c\cdot\vec a)\vec b-(\vec c\cdot\vec b)\vec a=(\vec a\cdot\vec c)\vec b-(\vec a\cdot\vec b)\vec c$$

$$\Leftrightarrow (\vec c\cdot\vec b)\vec a-(\vec a\cdot\vec b)\vec c=0$$

$$\Leftrightarrow (\vec{c}\cdot\vec{b})\vec{a} = (\vec{a}\cdot\vec{b})\vec{c}$$

$$\Leftrightarrow \vec{a} = \frac{(\vec{a}\cdot\vec{b})}{(\vec{c}\cdot\vec{b})}\vec{c}$$

$$\Leftrightarrow \vec{a} = \lambda\vec{c} \qquad \text{where } \lambda = \frac{(\vec{a}\cdot\vec{b})}{(\vec{c}\cdot\vec{b})} = \text{some scalar quantity}$$

Thus, $\vec{a}$ and $\vec{c}$ are collinear.

Hence, if $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ then the vector $\vec{a}$ and $\vec{c}$ are collinear.

If $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$, prove that $\vec{a}$ and $\vec{c}$ are collinear. [2013 Fall]

Define scalar and vector triple product vectors. Show that $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$, if $\vec{a}$ and $\vec{c}$ are collinear. [2014 Spring]

16. Show that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

Solution: Here,

$$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$$

$$= [\vec{a}+\vec{b}\ \ \vec{b}\ \ \vec{c}+\vec{a}] + [\vec{a}+\vec{b}\ \ \vec{c}\ \ \vec{c}+\vec{a}]$$

[∵ by using distributive property]

$$= [\vec{a}+\vec{b}\ \ \vec{b}\ \ \vec{c}] + [\vec{a}+\vec{b}\ \ \vec{b}\ \ \vec{a}] + [\vec{a}+\vec{b}\ \ \vec{c}\ \ \vec{c}] + [\vec{a}+\vec{b}\ \ \vec{c}\ \ \vec{a}]$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}] + [\vec{b}\ \ \vec{b}\ \ \vec{c}] + [\vec{a}\ \ \vec{b}\ \ \vec{a}] + [\vec{b}\ \ \vec{b}\ \ \vec{a}] + [\vec{a}+\vec{b}\ \ \vec{c}\ \ \vec{c}] + [\vec{a}\ \ \vec{c}\ \ \vec{a}] + [\vec{b}\ \ \vec{c}\ \ \vec{a}]$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}] + 0 + 0 + 0 + 0 + 0 + [\vec{b}\ \ \vec{c}\ \ \vec{a}]$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}] + [\vec{b}\ \ \vec{c}\ \ \vec{a}]$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}] + [\vec{a}\ \ \vec{b}\ \ \vec{c}] = 2[\vec{a}\ \ \vec{b}\ \ \vec{c}]$$

Thus, $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \ \vec{b}\ \ \vec{c}]$.

17. Find a set of reciprocal vector of $\vec{a} = 2\vec{i}+3\vec{j}-\vec{k}$, $\vec{b} = \vec{i}-\vec{j}-2\vec{k}$ and $\vec{c} = -\vec{i}+2\vec{j}+2\vec{k}$. [2012 Fall]

Solution: Let,

$$\vec{a} = 2\vec{i}+3\vec{j}-\vec{k},\quad \vec{b} = \vec{i}-\vec{j}-2\vec{k},\quad \vec{c} = -\vec{i}+2\vec{j}+2\vec{k}$$

Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = \vec{i}(-6-1) - \vec{j}(-4+1) + \vec{k}(-2-3)$$

$$= -7 + 3\vec{j} - 5\vec{k}.$$

And,

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = \vec{i}(-2+4) - \vec{j}(2-2) + \vec{k}(2-1)$$

$$= 2\vec{i} + \vec{k}.$$

Also,

$$\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix} = \vec{i}(-2-6) - \vec{j}(1-4) + \vec{k}(-3-4)$$

$$= -8\vec{i} + 3\vec{j} + -\vec{k}$$

Then,

$$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2(-2+4) - 3(2-2) - 1(2-1)$$

$$= 4 - 0 - 1$$

$$= 3.$$

Now, reciprocal of $\vec{a}$, $\vec{b}$ and $\vec{c}$ are

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = \frac{2\vec{i}+\vec{k}}{3};$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = \frac{-8\vec{i}+3\vec{j}-7\vec{k}}{3};$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} = \frac{-7\vec{i}+3\vec{j}-5\vec{k}}{3}.$$

18. Show that $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ iff $(\vec{a}\times\vec{c})\times\vec{b} = 0$.

Solution: Here,

$$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$$

$$\Leftrightarrow (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a} = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$\Leftrightarrow (\vec{a}.\vec{b})\vec{c} - (\vec{c}.\vec{b})\vec{a} = 0$$

$$\Leftrightarrow \vec{b}\times(\vec{c}\times\vec{a}) = 0$$

$$\Leftrightarrow (\vec{a}\times\vec{c})\times\vec{b} = 0$$

Hence, $(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$ iff $(\vec{a}\times\vec{c})\times\vec{b} = 0$.

19. Show that $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = 0$.

Solution:

Here,

$$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$$

$$= \begin{vmatrix} \vec{b}.\vec{a} & \vec{b}.\vec{d} \\ \vec{c}.\vec{a} & \vec{c}.\vec{d} \end{vmatrix} + \begin{vmatrix} \vec{c}.\vec{b} & \vec{c}.\vec{d} \\ \vec{a}.\vec{b} & \vec{a}.\vec{d} \end{vmatrix} + \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$$

$$= (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{c}.\vec{a})(\vec{b}.\vec{d}) + (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{a}.\vec{b})(\vec{c}.\vec{d}) + (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$$

$$= 0.$$

Thus, $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = 0$.

20. Show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d}) = -2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$.

Solution:

Since we have,

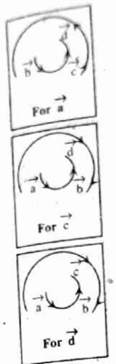
$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

or

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} - [\vec{c}\ \vec{d}\ \vec{b}]\vec{a}$$

Here,

$$\vec{r} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})$$



$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} + [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{b}\ \vec{c}\ \vec{a}]\vec{d} + (\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} + [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{b}\ \vec{c}\ \vec{a}]\vec{d} + [\vec{b}\ \vec{d}\ \vec{c}]\vec{a} - [\vec{b}\ \vec{d}\ \vec{a}]\vec{c}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} + [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$$

$$= -2[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

21. Prove that $2\vec{a} = \vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k})$

[2004, Spring (Short)]

Solution: Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

Here,

$$\vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k})$$

$$= (\vec{i}.\vec{i})\vec{a} - (\vec{i}.\vec{a})\vec{i} + (\vec{j}.\vec{j})\vec{a} - (\vec{j}.\vec{a})\vec{j} + (\vec{k}.\vec{k})\vec{a} - (\vec{k}.\vec{a})\vec{k}.$$

$$= \vec{a} - (\vec{i}.\vec{a})\vec{i} + \vec{a} - (\vec{j}.\vec{a})\vec{j} + \vec{a} - (\vec{k}.\vec{a})\vec{k}.$$

$$= 3\vec{a} - (\vec{i}.(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}))\vec{i} - (\vec{j}.(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}))\vec{j} - (\vec{k}.(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}))\vec{k}.$$

$$= 3\vec{a} - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}.$$

$$= 3\vec{a} - \vec{a}.$$

$$= 2\vec{a}.$$

Thus, $\vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k}) = 2\vec{a}$.

22. Show that $\{(\vec{a}+\vec{b}+\vec{c})\times(\vec{b}+\vec{c})\}.\vec{c} = \vec{a}.(\vec{b}\times\vec{c})$.

Solution: Here,

$$\{(\vec{a}+\vec{b}+\vec{c})\times(\vec{b}+\vec{c})\}.\vec{c}$$

$$= [(\vec{a}+\vec{b}+\vec{c})\ \ (\vec{b}+\vec{c})\ \ \vec{c}]$$

$$= \left[\vec{a}+\left(\vec{b}+\vec{c}\right) \quad \left(\vec{b}+\vec{c}\right) \quad \vec{c}\right]$$

$$= \left[\vec{a} \quad \vec{b}+\vec{c} \quad \vec{c}\right] + \left[\vec{b}+\vec{c} \quad \vec{b}+\vec{c} \quad \vec{c}\right]$$

[∵ by using distributive law]

$$= \left[\vec{a} \quad \vec{b}+\vec{c} \quad \vec{c}\right] + 0.$$ [If any two value in a determinant has same value then its result will be 0.]

$$= \left[\vec{a} \quad \vec{b} \quad \vec{c}\right] + \left[\vec{a} \quad \vec{c} \quad \vec{c}\right]$$

$$= \left[\vec{a} \quad \vec{b} \quad \vec{c}\right] + 0$$

$$= \vec{a}.\left(\vec{b}\times\vec{c}\right).$$

Thus, $\left\{\left(\vec{a}+\vec{b}+\vec{c}\right)\times\left(\vec{b}+\vec{c}\right)\right\}.\vec{c} = \vec{a}.\left(\vec{b}\times\vec{c}\right).$

23. Show that $[\vec{l} \quad \vec{m} \quad \vec{n}][\vec{a} \quad \vec{b} \quad \vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$

Solution: Let

$$l = l_1\vec{i} + l_2\vec{j} + l_3\vec{k} \qquad \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$m = m_1\vec{i} + m_2\vec{j} + m_3\vec{k} \qquad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$n = n_1\vec{i} + n_2\vec{j} + n_3\vec{k} \qquad \vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

Here

$$[\vec{l} \quad \vec{m} \quad \vec{n}][\vec{a} \quad \vec{b} \quad \vec{c}]$$

$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1a_1 + l_2a_2 + l_3a_3 & l_1b_1 + l_2b_2 + l_3b_3 & l_1c_1 + l_2c_2 + l_3c_3 \\ m_1a_1 + m_2a_2 + m_3a_3 & m_1b_1 + m_2b_2 + m_3b_3 & m_1c_1 + m_2c_2 + m_3c_3 \\ n_1a_1 + n_2a_2 + n_3a_3 & n_1b_1 + n_2b_2 + n_3b_3 & n_1c_1 + n_2c_2 + n_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{c} & \vec{n}.\vec{c} \end{vmatrix}$$

24. Show that the vectors $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar. **[2018 Spring]**

Solution: We know that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if $\vec{a}+\vec{b}+\vec{c} = 0$.

Here,

$$\begin{aligned}\vec{r} &= \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})\\ &= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a}\\ &\quad - (\vec{c}.\vec{a})\vec{b}\\ &= 0.\end{aligned}$$

Hence, $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar.