European Journal of Operational Research 275 (2019) 139–154

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Production, Manufacturing, Transportation and Logistics

A hybrid Lagrangian metaheuristic for the cross-docking flow shop scheduling problem

Gabriela B. Fonseca[a], Thiago H. Nogueira[b], Martín Gómez Ravetti[a,*]

[a] Department of Production Engineering, Universidade Federal de Minas Gerais, Av. Antônio Carlos, 6627 CEP, Belo Horizonte 31270-901, MG, Brazil
[b] Department of Production Engineering, Universidade Federal de Viçosa, Rodovia MG-230, CEP, Rio Paranaíba 38810-000, MG, Brazil

ARTICLE INFO

Article history:
Received 24 September 2017
Accepted 6 November 2018
Available online 16 November 2018

Keywords:
Logistics
Truck scheduling
Cross-docking
Lagrangian relaxation

ABSTRACT

Cross-docking is a logistics strategy that minimizes the storage and picking functions of conventional warehouses. The objective is to unload the cargo from inbound trucks and directly load it into outbound trucks, with little or no storage. The success of the strategy depends on an efficient transshipment operation. This work undertakes a study of truck scheduling in a parallel dock cross-docking center. The problem is first modeled as a two-machine flow shop scheduling problem with precedence constraints, with the objective of minimizing the makespan, and later we generalize it to the parallel-dock case. We propose a hybrid method based on a Lagrangian relaxation technique through the volume algorithm. Using information from the Lagrangian multipliers, constructive heuristics with local search procedures generates good feasible solutions. With a series of cuts, the methodology finds tight bounds for small and large instance sizes, outperforming current results.

1. Introduction

The current market environment, characterized by increasingly fierce competition, the globalization of the economy and an accelerated technological revolution has led companies to improve their production, logistics and distribution systems. Customers are demanding better and faster services, direct-to-customers strategies and many other new requirements, that expects more efficient and integrated logistic operations. Cross-docking is one logistic technique that can reduce inventory costs while increasing the goods flow, improving the efficiency of the supply chain.

A Cross-docking Distribution Center (CDC) is a logistics technique widespread throughout the world. Several well-known companies such as retail chains (WalMart Stalk, Evans, & Shulman, 1991), mailing companies (UPS Forger, 1995), automobile manufacturers (Toyota Witt, 1998) and logistics providers (Gue, 1999; Kim, Yang, & Kim, 2008) have gained considerable competitive advantages by using CDC.

The main idea behind cross-docking centers is to receive products from different suppliers or manufacturers and consolidate them to common final delivery destinations. In comparison to traditional warehouses, a CDC is managed with minimal handling and with little or no storage between unloading and loading of goods. This practice can serve different goals: the consolidation of shipments, a shorter delivery lead time, the reduction of costs, etc. Readers are referred to Ladier and Alpan (2016), for a survey discussing industry practices and CDC problem characterization.

Cross-docks raise numerous optimization questions, either strategic, tactical or operational. In a Cross-docking Distribution Center (CDC), scheduling decisions are particularly relevant to ensure a rapid turnover and on-time deliveries. Due to its real-world importance, several truck-scheduling works and procedures have been introduced during recent years, treating specific cross-dock settings.

A schematic representation of processes at a CDC is illustrated in Fig. 1. Firstly, incoming trucks arrive at the yard of the CDC, if the number of trucks is higher than the number of docks, some of them have to wait in a queue until further assignment. Secondly, after being docked, goods of the inbound trucks are unloaded, scanned, sorted, moved across the dock and loaded into outbound trucks for an immediate delivery elsewhere in the distribution chain. Once an outbound (inbound) truck is completely loaded (unloaded), the truck is removed from the dock, replaced by another truck and the course of action repeats.

In a conventional distribution center, five operations are usually carried out when managing products: receiving, sorting, storing, picking and shipping operations. All of which can be done consecutively or in a particular order. As a result of applying a cross-docking system, a reduction of the cost of both storage and

* Corresponding author.
E-mail addresses: gabrielabf@eng-pro.mest.ufmg.br (G.B. Fonseca), thiagoh.nogueira@ufv.br (T.H. Nogueira), martin.ravetti@dep.ufmg.br, gravetti.martin@gmail.com (M.G. Ravetti).

https://doi.org/10.1016/j.ejor.2018.11.033
0377-2217/© 2018 Elsevier B.V. All rights reserved.

Table 1
Previous specific research works for the truck scheduling problems. Based on Boysen and Fliedner (2010).

Publication	Notation	Complexity	Contribution
Miao, Lim, and Ma (2009)	$[M\|limit,\ t_{io}\|\star]$	NP-hard	MM, HM, P
Chen and Lee (2009)	$[E2\|t_j=0\|C_{\max}]$	NP-hard	B, ES, P
Chen and Lee (2009)	$[E2\|t_j=0\|C_{\max}]$	NP-hard	P
Chen and Song (2009)	$[E\|t_j=0\|C_{\max}]$	NP-hard	MM, HS, B
Boysen (2010)	$[E\|p_j=p, no-wait, t_j=0\|\sum T_o]$	Open	MM, HM, ES
Boysen and Fliedner (2010)	$[E\|t_{io}, f_{ix}\|\Sigma W_s U_s]$	NP-hard	MM, P
Boysen and Fliedner (2010)	$[E\|t_i=0,\ f_{ix}\ \|\ \Sigma W_s U_s]$	NP-hard	P
Boysen, Fliedner, and Scholl (2010)	$[E2\|p_j=p, change\|C_{\max}]$	NP-hard	MM, HS, HI, ES, P
McWilliams (2010)	$[E\|no-wait\|*]$	Open	HM
Vahdani and Zandieh (2010)	$[E2\|change\|C_{\max}]$	NP-hard	MM, HM
Araújo and Melo (2010)	$[E\|t_j=0\|C_{\max}]$	NP-hard	HM
Arabani, Ghomi, and Zandieh (2011)	$[E2\|change\|C_{\max}]$	Open	HM
Larbi, Alpan, Baptiste, and Penz (2011)	$[E2\|pmtn\|^*]$	NP-hard	MM, ES
Alpan et al. (2011)	$[E\|pmtn\|^*]$	Open	MM, ES
Lima (2014)	$[E2\|t_j=0\|\sum C_j^2]$	NP-hard	MM, HS, HM
Cota et al. (2016)	$[E\|t_j=0\|C_{\max}]$	NP-hard	MM, HS

The notation used in column “Contribution” is stated as: MM (mathematical model), HI (heuristic improvement procedure), HM (meta-heuristic), B (bound computation), HS (start heuristic for initial solution), ES (exact solution procedure), P (properties [e.g., complexity] of problem). In the Boysen (2010) tuple notation, column “Notation”, the fields are door environment, operational characteristics and the objective, respectively.

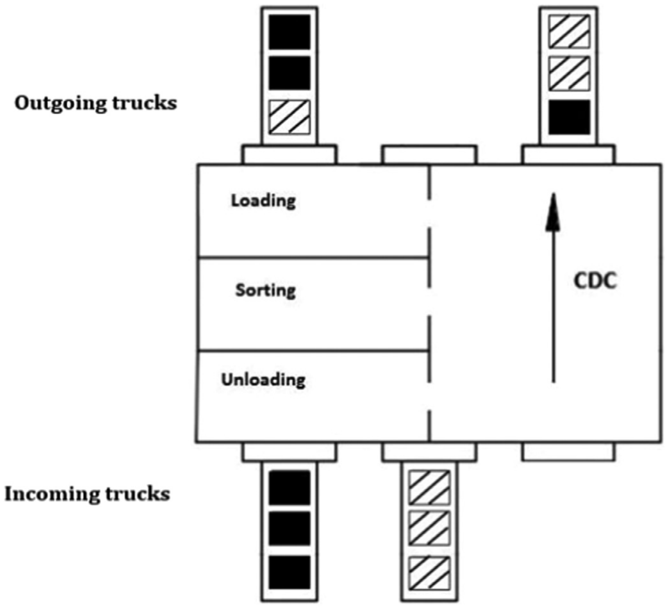


Fig. 1. Schematic representation of a CDC.

products picking can be achieved by synchronizing the inbound and outbound trucks flows. There are many benefits with the cross-docking: costs reduction (warehousing, inventory-holding, handling, and labor costs), shorter delivery lead times (from supplier to customer, increase of throughput), improved customer service and customer satisfaction, reduction of storage space, faster inventory turnover, fewer overstocks, reduced risk of loss and damage, etc. However, efficient transshipment processes and careful operations planning become indispensable within a CDC, where inbound and outbound flows need to be synchronized to keep the terminal storage as low as possible, and on-time deliveries are ensured. Many articles in the literature develop computerized scheduling procedures, which have achieved good results (see examples in Table 1).

In this work, we consider a time-indexed integer programming formulation for a flow shop scheduling problem with cross-docking precedence constraints. We first consider the 2-dock case denoted as $F2|CD|C_{\max}$, and strongly NP-hard (see Chen & Lee, 2009), and later we deal with the parallel dock generalization, $F2(P)\ |CD|C_{\max}$. Computational experiments show that the model is efficient when solving small problems.

A Hybrid Lagrangian Metaheuristic approach is also proposed and tested. The method can efficiently generate strong lower and upper bounds. Results are compared to the best heuristics in the literature introduced by Chen and Lee (2009) (based on Johnson’s algorithm), with the CDH heuristic developed by Cota et al. (2016) for a parallel machines scenario, and to the best heuristics proposed by Chen and Song (2009) for the multiple parallel processors (multiple docks) case. Better solutions are consistently obtained for all instances tested. The article is organized as follows, an overview of related works is presented in Section 2. The mathematical model and the Hybrid Lagrangian Metaheuristic Framework are described in Section 3 and 4, respectively. Section 5 generalizes the algorithm for the parallel machine case. Computational experiments are reported in Section 6. Finally, discussions and conclusions are drawn in Section 7.

In this manuscript, we propose a Hybrid Lagrangian Metaheuristic Framework for the cross-docking flow shop scheduling problem. The work has a similar line of thought than Boschetti and Maniezzo (2009). However, it emphasizes the use of the information of the Lagrangian multipliers as discussed in Pirkwieser, Raidl, and Puchinger (2007), and Paula, Mateus, and Ravetti (2010).

For the 2-dock case, although the subproblems of the Lagrangian relaxation are polynomial, through a series of cuts on the makespan value, we improve the linear lower bounds. The proposed algorithm uses the information obtained from the Lagrangian multipliers to construct feasible solutions through a NEH based heuristic (Nawaz, Enscore, & Ham, 1983), and performs search procedure through a Local Search framework. Furthermore, the proposed hybrid algorithm proves optimality in several instances.

We generalize our approach to the parallel-dock case. In this scenario, the subproblems are NP-hard, with a focus on improving the feasible solutions we solve the subproblem’s linear relaxations, to feed the Lagrangian metaheuristic. Results show that the technique outperforms current heuristic methods.

2. Literature review

The increasing use of cross-docking techniques has motivated several authors to investigate new ways to improve the design and tactical operations of CDCs. In the literature, several decision problems are studied concerning strategical, tactical, and operational decisions.

Boysen and Fliedner (2010), and Belle, Valckenaers, and Cattrysse (2012) focus on the review of the existing literature of cross-docking problems. According to Boysen (2010), decision problems in a cross-docking center can be allocated according to the following classification, ordered from strategic to operational levels: location of cross-docking centers, layout, vehicle routing, dock's assignments of destinations, truck scheduling and resource scheduling inside the center. Here we focus on truck scheduling problems, where the aim is deciding where and when should the trucks be processed.

Examples of strategic decisions in a CDC can be found in many works. Location of a cross-docking is analyzed in Campbell (1994), Klose and Drexl (2005), Chen, Guo, Lim, and Rodrigues (2006), and Chen (2007). The layout of cross-docking centers is studied in Gue (1999), Bartholdi and Gue (2002) and Lee, Lee, and Jung (2008).

Some other works deal with the vehicle routing as Oh, Hwang, Chab, and Lee (2006) and Clausen, Cordeau, Laporte, Wen, and J. (2009). Regarding operational decisions, some works consider the dock assignment problem: Belle et al. (2012), Boysen and Fliedner (2010), Tsui and Chang (1990, 1992), Bozer and Carlo (2008) and Gue (1999).

In a CDC, scheduling decisions are particularly important to ensure a rapid turnover and on-time deliveries. Due to its real-world importance, several truck-scheduling works and procedures have been introduced during recent years, treating specific cross-dock settings. For that reason, we highlight in Table 1, some outstanding works dealing with truck scheduling problems in a CDC. As proposed by Boysen (2010), a classification scheme for deterministic truck scheduling problems is presented for each work, as well as, their contribution.

Chen and Lee (2009) study a two-machine cross-docking flow shop scheduling problem, in which a job at the second machine can be processed only after finalizing some jobs at the first one, with the objective of minimizing the makespan. The authors show the problem is strongly NP-hard. They develop a polynomial approximation algorithm with an error-bound analysis and a branch-and-bound algorithm. Computational results show that the branch-and-bound algorithm can optimally solve problems with up to 60 jobs in a reasonable time.

Chen and Song (2009) extended the Chen and Lee (2009) problem to the two-stage hybrid cross-docking scheduling problem, by considering multiple parallel processors (multiple docks) per stage (inbound and outbound), allowing simultaneous loading and unloading operations. They propose a mixed integer programming model and four constructive heuristics, based on Johnson's rule, to investigate the performance for moderate and large scale instances.

Another studies considering multiple dock environment are proposed by Alpan, Bauchau, Larbi, and Penz (2008), but with a different aim, a dynamic program based model is presented to find the schedule of transshipment operations minimizing the operations total cost, solving only small size instances. Larbi, Alpan, and Penz (2009) presented some heuristics to obtain near optimal solutions for large instances considering the same scenario than Alpan et al. (2008).

Cota et al. (2016) deals with the operational decision problem of scheduling the trucks on multiple inbound and outbound docks, that is, they consider the same setting than Chen and Song (2009). They propose a time-indexed mixed integer linear programming formulation and a constructive polynomial heuristic. Results are compared to the best heuristics proposed in Chen and Song (2009) and Cota et al. (2016) on medium and large instance, and better solutions were consistently obtained for instances with greater number of machines and jobs.

The hybrid metaheuristics arise as an option to achieve solutions within a reasonable computational time in very difficult combinatorial optimization problems. The hybrid method consists of a cooperative combination of methods, exact and/or approximated, and aims to absorb to the limit the potentialities of all the approaches (see surveys Alba (2005); Puchinger and Raidl (2005) and Blum, Puchinger, Raidl, and Roli (2011)).

Paula et al. (2010) propose a variant of the Lagrangian relaxation to obtain good bounds to the problem of parallel machines scheduling with sequence-dependent setup times. In this effort, the Lagrangian relaxation is improved by the use of a metaheuristic as an internal procedure of the Lagrangian relaxation. The heuristic is used to generate feasible solutions during the execution of the non-delayed relax-and-cut algorithm.

Pirkwieser et al. (2007) present a similar work, using the Lagrangian relaxation applied to the Knapsack Constrained Maximum Spanning Tree problem, in which the Lagrangian multipliers influence the procedures executed by the Genetic Algorithm.

Boschetti and Maniezzo (2009) propose a Lagrangian metaheuristic procedure. Their approach consists in using metaheuristic techniques to obtain feasible solutions using the information of Lagrangian multipliers. The solutions obtained tend to be good bounds. We remark that, in these efforts, the potential of the information of the Lagrangian multipliers is still not completely used. Part of our goals is to improve the use of this information. Readers are referred to Blum et al. (2011) for a survey on hybrid metaheuristics.

As discussed here, a significant number of CDC articles are dedicated to the study of strategical and tactical aspects. However, not always aligned with industry practice. More recently, Alpan, Larbi, and Penz (2011) discuss the relationship between the industry needs and academic research topics, pointing out the gap between both worlds. They highlight some gaps between theory and practice which would need to be filled in order to get closer to real-life constraints, for example: researchers should aim towards removing simplifying assumptions which are difficult to justify in an industrial context, or, it would be necessary to take deadlines into account for outbound truck departures.

In this work, we consider the problem of scheduling the trucks as described in Chen and Lee (2009). As Chen and Song (2009) and Cota et al. (2016) generalize their problem, we compare our results to the best algorithms in all these works considering a 2-dock version and a parallel-dock case.

3. Mathematical model

In this section, we present a mathematical formulation for a 2-dock version based on scheduling problem initially proposed by Chen and Lee (2009). The integer programming model considered in this work adopts a time-indexed formulation proposed by Lima (2014) but adopting the makespan as objective function, as proposed by Chen and Lee (2009).

Considering a 2-dock cross-docking center where n loaded trucks arrive with products demanded by one or more customers. In the center, each truck must unload its cargo and load it into one or more outbound trucks, responsible for delivery to specific destinations in the supply chain. Each departing vehicle may leave the center only after the cargo is fully loaded, and it can begin the loading process only after all needed products were available in the center. Here we consider the existence of two docks in the center, one dock dedicated to unload the trucks and the other one for loading purposes, machine 1 (M1) and machine 2 (M2), respectively. The problem corresponds to define the truck sequence in the inbound and outbound dock, minimizing the completion time of the last job processed by machine 2.

The following parameters, sets and variables are considered:
Input parameters

- n: number of jobs to be processed on M1.
- m: number of jobs to be processed on M2.
- p_{ij}: processing time of job j on machine i.
- T_f: size of time horizon. A first estimate for the time horizon is the sum of the processing times of all jobs.
- $T_0 \geq 0$: beginning of time horizon for the jobs $j \in J^2$, i.e., minimum starting date for processing of the output trucks, initially set as $T_0 = \min_{j\in J^2} \{\sum_{i\in J^1, i\in S_j} p_{1i}\}$, then $T = \{T_0, \ldots, T_f\}$ $\forall j \in J^2$.

Sets

The set of periods is defined as $T = \{0, \ldots, T_f\}$. To represent arrivals and truck departures in cross-docking center, two sets of jobs are created:

- $J^1 = \{j^1_1, j^1_2, \ldots, j^1_n\}$, set of n jobs (inbound trucks), which must be processed on M1.
- $J^2 = \{j^2_1, j^2_2, \ldots, j^2_m\}$, set of m jobs (outbound trucks), which must be processed on M2.
- S_j, set of precedent jobs. For each job $j^2_j \in J^2$, there is a corresponding subset of J^1 that must be completed before beginning its process. It is considered that each element job $j \in J^2$ has at least one job S_j as precedent.

Decision variables

- x_{jt} ($\forall j \in J^1$, $\forall t \in T$), binary variable, x_{jt} is equal to 1 if job j starts its process at time t and equal to 0 otherwise.
- y_{jt} ($\forall j \in J^2$, $\forall t \in T$), binary variable, y_{jt} is equal to 1 if job j starts its process at time t and equal to 0 otherwise.
- $C_{\max}$: makespan. Maximum completion time in M2.

The complete mathematical model (F) is presented as follows:

$$(F) \qquad \text{Minimize} \quad C_{\max} \tag{1}$$

subject to

$$\sum_{t=0}^{T_f - p_{1j}} x_{jt} = 1, \quad \forall j \in J^1, \tag{2}$$

$$\sum_{t=T_0}^{T_f - p_{2j}} y_{jt} = 1, \quad \forall j \in J^2, \tag{3}$$

$$\sum_{j\in J^1} \sum_{s=\max(0;t-p_{1j}+1)}^{t} x_{js} \leq 1, \quad \forall t \in T, \tag{4}$$

$$\sum_{j\in J^2} \sum_{s=\max(T_0;t-p_{2j}+1)}^{t} y_{js} \leq 1, \quad \forall t \in T, \tag{5}$$

$$\sum_{t=T_0}^{T_f - p_{2j}} t y_{jt} - \sum_{t=0}^{T_f - p_{1k}} (t + p_{1k}) x_{kt} \geq 0, \quad \forall j \in J^2, \forall k \in S_j, \tag{6}$$

$$C_{\max} \geq p_{2j} + \sum_{t=T_0}^{T_f - p_{2j}} t y_{jt}, \quad \forall j \in J^2, \tag{7}$$

$$x_{jt} \in \{0, 1\}, \quad \forall j \in J^1, \forall t \in T, \tag{8}$$

$$y_{jt} \in \{0, 1\}, \quad \forall j \in J^2, \forall t \in T, \tag{9}$$

$$C_{\max} \geq 0. \tag{10}$$

The objective function (1) minimizes the makespan. The set of constraints (2) ensures that each job $j^1_j \in J^1$ should start its processing in one and only one period within the time horizon. The set (3) works with the same reasoning on M2. Constraints (4) ensure that a job $j^1_j \in J^1$ does not start its processing while another job is being processed in the same machine M1. The constraints set (5) works, in the same way, but applied to jobs $j^2_j \in J^2$. The set (6) controls the precedence relationships. For every existing precedence relation, the start date of the job $j^2_j \in J^2$ should be greater than or equal to the completion time of its precedent $j^1_k \in S_j$. The set of constraints (7) indicates that the variable $C_{\max}$ should be the maximum completion time of jobs in J^2. Finally, sets (8)–(10) define the variable's domain.

3.1. Lagrangian relaxation

Let L(λ) be the relaxation of formulation F, when constraints 6 are dualized and its violation penalized in the objective function. For each constraint is associated one Lagrangian multiplier λ representing the weight given to the violation. The set of Lagrangian multipliers will be represented by λ_{jk}, with $j \in J^2$ and $k \in S_j$. Notice that, these constraints are the cross-docking precedence relations, coupling decisions in M1 with decisions in M2. Thus, the subproblem L(λ) is defined as:

$$L(\lambda) = \text{Minimize} \quad C_{\max} + \sum_{j\in J^2} \sum_{k\in S_j} \lambda_{jk} \left(\sum_{t=0}^{T_f - p_{1k}} (t + p_{1k}) x_{kt} - \sum_{t=T_0}^{T_f - p_{2j}} t y_{jt} \right) \tag{11}$$

s.t. (2)–(5), (7)–(10), $\lambda_{jk} \geq 0$.

Now we are able to uncouple the problem into two new scheduling problems, one for each machine. Subproblem X considers M1 while subproblem Y considers M2. The value of the lower bound is obtained by adding the subproblems objective functions L(λ)$_x$ and L(λ)$_y$.

Subproblem *X*

Isolating the terms of the objective function that contain variables x, we reach to:

$$L(\lambda)_x = \text{Minimize} \sum_{j\in J^2} \sum_{k\in S_j} \lambda_{jk} \sum_{t=0}^{T_f - p_{1k}} (t + p_{1k}) x_{kt} \tag{12}$$

s.t. (2), (4), (8) and $\lambda_{jk} \geq 0$.

Rewriting the above objective function from the perspective of jobs in J^1, we have:

$$L(\lambda)_x = \text{Minimize} \sum_{j\in J^1} \sum_{t=0}^{T_f - p_{1j}} (t + p_{1j}) x_{jt} w^1_j \tag{13}$$

where $\quad w^1_j = \sum_{i\in J^2} \lambda_{ij}$, where $\lambda_{ij} = 0$ if $j \notin S_i$, .

The problem mentioned above is known as The Total Weighted Completion Time (Pinedo, 2008), denoted by $1||\Sigma C_j W_j$. This problem can be solved by the WSPT rule (Weighted Shortest Processing Time First) proposed by Smith (1956). According to this rule, the optimal solution is obtained by ordering the jobs in descending order of $w^1_j \ / \ p_{1j}$, and can be obtained in O($n$log(n)).

Subproblem *Y*

With the terms associated to variables y, subproblem Y can be defined as:

$$L(\lambda)_y = \text{Minimize} \quad C_{\max} - \sum_{j\in J^2} \sum_{k\in S_j} \lambda_{jk} \sum_{t=T_0}^{T_f - p_{2j}} t y_{jt} \tag{14}$$

subject to

$$\sum_{t=T_0}^{T_f - p_{2j}} y_{jt} = 1, \quad \forall j \in J^2, \tag{15}$$

$$\sum_{j\in J^2}\sum_{s=\max(T_0;t-p_{2j}+1)}^{t} y_{js} \le 1, \ \ \forall t \in T, \tag{16}$$

$$C_{\max} \ge p_{2j} + \sum_{t=T_0}^{T_f-p_{2j}} t y_{jt}, \ \ \forall j \in J^2, \tag{17}$$

$$y_{jt} \in 0, 1, \ \ \forall j \in J^2, \forall t \in T, \tag{18}$$

$$\lambda_{jk} \ge 0, \ \ \forall j \in J^2, k \in S_j, \tag{19}$$

$$C_{\max} \ge 0. \tag{20}$$

Defining $w_j^2 = -\sum_{k\in S_j} \lambda_{jk}\ \forall j \in J^2$, we can rewrite the problem as follows:

$$L(\lambda)_y = \text{Minimize} \ \ C_{\max} + \sum_{j\in J^2}\sum_{t=T_0}^{T_f-p_{2j}} t.y_{jt} w_j^2 \tag{21}$$

s.t. (15), (16), (17), (18) e (20).

The first term of the objective function (21) is the makespan. The second term represents the sum of the weighted starting times of the jobs on M2, also known as The Total Weighted Starting Time, denoted $\Sigma I_j W_j$. Thus, the Y subproblem can be defined as $1||C_{\max} + \sum I_j W_j$. To solve this subproblem, we proposed a rule named WSPT-TRD.

The WSPT-TRD rule works as follows. First, the WSPT rule is used to generate a sequence of jobs on M2. The second step is to define the correct allocation of jobs in the time horizon. For that, we evaluate the increase of the makespan when the weights associated with the jobs are negative. When the weights are negative, it creates a trade-off situation, as explained next.

When considering a general, weights w_j^2 may have positive or negative values. Let us first consider schedule decisions for jobs with $w_j^2 \ge 0$, both $C_{\max}$ and $\Sigma I_j W_j$ have similar behavior since the objective function is minimized by allocating the jobs at the beginning of the time horizon. In our particular case, as $\lambda_{jk} \ge 0$, we do not have the case of $w_j^2 > 0$; however, our algorithm here proposed considers this possibility.

Considering jobs with negative weights ($w_j^2 < 0$), the objective function terms lead to different solutions. On the one hand, by considering the makespan $C_{\max}$ one tends to allocate the jobs at the beginning of time horizon, as the weights do not influence its final value. On the other hand, when analyzing the $\Sigma I_j W_j$ one tends to allocate the jobs as late as possible, taking advantage of the negative weights.

We notice that if we delay all jobs with negative weights one-time unit, the $C_{\max}$ increases one unit. Thus, our algorithm computes the sum of the negative weights, if the sum of the negative weights is less than -1, the displacement of the jobs towards the end of the time horizon compensates the increase of $C_{\max}$, defining in this way the best allocation of jobs.

Pseudocode of algorithm WSPT-TRD:

```
Step 1: Compute the weights $w_j^2$ on the machine 2 ($w_j^2 = -\sum_{k\in S_j} \lambda_{jk} \forall j \in J^2$).
Step 2: Sort the jobs in $J^2$ in decreasing order of $w_j^2 / p_{2j}$.
Step 3: Compute the sum of the negative weights as $\delta = \sum_{j\in J^2} w_j^2 \forall j \in J^2 \wedge w_j^2 < 0$.
Step 4: If $w_j^2 = 0\ \forall j \in J^2$ or $\delta \ge -1$, allocate the jobs from $t = T_0$ by sequence generated by WSPT rule.
Step 5: If there exists $w_j^2 < 0$ and $\delta \le -1$, allocate the jobs from t $= T_f$ by the reverse sequence generated by WSPT rule.
```

Theorem 1. *Considering the problem* $1||C_{\max} + \sum I_j W_j$*, the algorithm WSPT-TRD obtains an optimal solution.*

Proof 1. The proof is provided in Appendix A.

3.2. Lower bounds

Besides the lower bound obtained from the Lagrangian relaxation, we consider here two other lower bounds. The first one, called LB_1 is defined as proposed by Chen and Lee (2009), as follows:

$$LB_1 = \sum_{i\in J^2} p_{2i} + \min_{j\in J^2}\left\{ \sum_{i\in J^1, i\in S_j} p_{1i} \right\}. \tag{22}$$

LB_1 basically computes the sum of processing time on $M2$, plus the minimum amount of time necessary to begin the process of the first job in $M2$, that is, the minimum set of precedent jobs in $M1$.

The second lower bound, LB_2, proposed in this article, considers a complementary condition:

$$LB_2 = \sum_{i\in J^1} p_{1i} + \min_{j\in J^1}\left\{ \sum_{i\in J^2, i\in P_j} p_{2i} \right\}. \tag{23}$$

Where, P_j represents the successors subsets jobs $j \in J^1$ corresponding to job $i \in J^2$, i.e., successors are all jobs that must be processed after completion for a given job in the first stage.

In this case, LB_2 computes the sum of processing times in $M1$ plus the minimum set of common successors of $j \in J^1$. Finally, the Lower Bound is defined as the maximum lower bound, $LB = \max\{LB_1, LB_2\}$.

It is worth noticing that with a valid LB we can redefine a new date for the beginning of the processing time of jobs in the $WSPT-TRD$ algorithm. Particularly, in *Step 4*, $T_0 = LB - \sum_{i\in J^2} p_{2i}$, this is valid for the jobs with $w_j^2 = 0$ or for all jobs when $\delta \ge -1$.

4. Hybrid Lagrangian Metaheuristic

In the Hybrid Lagrangian Metaheuristic Framework proposed in this work, the Lagrangian Dual is solved by the volume algorithm, as proposed in Barahona and Anbil (2000) and Nogueira (2014). Readers are referred to Barahona and Ladányi (2006) and Fukuda (2007) for a discussion on the volume algorithm and its performance. In our case, the Lagrangian relaxation incorporates heuristics as internal procedures with a focus on obtaining feasible solutions. These heuristics are based on a constructive method and a Local Search. Both procedures are executed sequentially under the Lagrangian relaxation.

The volume algorithm is an extension of the subgradient algorithm, which produces a sequence of primal and dual solutions, thus being able to prove optimality. This algorithm has similar computational effort than the subgradient algorithm, and it presents similarities with the Conjugate Subgradient method (Lemaréchal, 1975; Wolfe, 1975) and the Bundle method (Lemaréchal, 2001; Lemaréchal, 529–572). A discussion of its main features and a global convergence analysis can be found in Bahiense, Maculan, and Sagastizabal (2002).

In the proposed algorithm the precedence constraints are dualized. The Lagrangian multipliers define a penalty to a given job allocated at a given position if the job has a large associated value, it means that it has a greater impact on the objective function, i.e., it is allocated in a disadvantageous position. This information can be used to decide when to schedule the jobs. Following, we describe two constructive heuristics used to obtain feasible solutions.

4.1. Constructive heuristics

The heuristic H1 uses information from Lagrangian multipliers to sort jobs on M1, while H2 uses information from Lagrangian multipliers to sort jobs on M2. In the final step both heuristics use an NEH-like heuristic (Nawaz et al., 1983) to construct the feasible solution. For a given list of jobs, the NEH function construct a solution by scheduling the jobs on the list one by one, in the best position of the partial schedule. The two heuristics are described below:

H1

Step 1: For each job on the machine 1 ($k \in J^1$), compute $\beta_k = \frac{\sum_{j\in J^2} \lambda_{jk}}{Nprec_k}$.
Where $Nprec_k$ corresponds to the number of jobs in the second stage that are waiting for the completion of job k on M1.
Step 2: Get the sequence on M1 by ordering jobs in decreasing order of β_k.
Step 3: Calculate the new release dates (r_j, $j \in J^2$) of jobs on M2.
Step 4: Get the sequence on M2 by ordering jobs in non-decreasing order of release dates r_j and apply NEH.

H2

Step 1: For each job on M2 ($j \in J^2$), compute $\beta_j = \frac{\sum_{k\in S_j} \lambda_{jk}}{Nprec_j}$.
Where $Nprec_j$ corresponds to the number of precedents that job j has.
Step 2: Get the sequence on M2 by ordering jobs in increasing order of β_j.
Step 3: Sequence the jobs on the M1 according the sequence on M2, respecting the precedence relations of cross-docking.
If there is a job on M1 without precedence relationship in the machine 2, schedule this job last.
Step 4: Calculate the new release dates (r_j, $j \in J^2$) of jobs on M2, from the sequence on M1.
Step 5: Get the sequence on M2 by ordering jobs in non-decreasing order of release dates r_j and apply NEH.

4.2. The Lagrangian algorithm

Let x and y be solutions of subproblems X and Y, respectively, with objective function value z. Then λ are the Lagrangian multipliers obtained by the volume algorithm, and $\nu(\lambda, x, y)$ is the subgradient. Let UB be the upper bound or feasible solution. The initial UB is generated by a Lagrangian metaheuristic. The algorithm is described below:

Step 0: We start with a null vector λ_{jk} (λ_{jk}=0). Solve the Lagrangian subproblem to obtain the initial solution of X subproblem (x^0), Y subproblem (y^0) and objective function value (z^0). Let UB be the feasible solution obtained by Lagrangian metaheuristic. Set $\bar{x} = x^0$, $\bar{y} = y^0$, $\bar{z} = z^0$ and k=1.
Step 1: Compute the subgradient $\nu(\lambda_{jk-1}, \bar{x}, \bar{y})$ and $\lambda_{jk} = \bar{\lambda}_{jk} + s\nu(\lambda_{jk-1})$, the calculation of the step size s is given by the Eq. (27). Solve the Lagrangian subproblem with the new λ_{jk} and let x^k, y^k and z^k be the solutions obtained. Then $\bar{x} = \alpha x^k + (1\text{-}\alpha)\bar{x}$, $\bar{y} = \alpha y^k + (1\text{-}\alpha)\bar{y}$ and $\bar{\lambda} = \alpha\lambda^k + (1\text{-}\alpha)\bar{\lambda}$, where α is a number between 0 and 1, defined by convex combination such as defined in (24).
Step 2: If $z^k > \bar{z}$ update $\bar{\lambda} = \lambda_{jk}$ and $\bar{z} = z^k$. If $\bar{z}$ improves by 10% since the last run of the Lagrangian metaheuristic then go to Step 3. Else go to Step 4.
Step 3: Lagrangian heuristics (H1, H2):

1. $UB^k \leftarrow$ Constructive Heuristics (H1 and H2);
2. $UB^k \leftarrow$ Local Search (list);

If $UB^k < UB$ update UB $= UB^k$ and add a cut in order to reduce the time horizon, improving the limits found (update T_f=UB).
Step 4: Stop criteria: If satisfied stop. Else let $k = k + 1$ and go to Step 1.

The step size s and the parameter α used to define $\bar{x}$ and $\bar{y}$, are computed as proposed in Barahona and Anbil (2000) and Fukuda (2007). First, we define α_{opt} as:

$$\alpha_{opt} = argmin \parallel \alpha \nu'^k_{(\lambda_{jk-1}, x^k, y^k)} + (1-\alpha)\nu^k_{(\lambda_{jk-1}, \bar{x}, \bar{y})} \parallel^2 \tag{24}$$

The parameter values π, *MaxWaste*, *factor*, $\alpha_{\max}$, *st*, α, *yellow* and *green*, some of them yet to be introduced, were defined using the SPOT method, as explained in Appendix B. The parameters $\alpha_{\max}$ and α are initially defined as 0.3034 and 0.0830 respectively, based on computational tests performed by SPOT. And α is computed as:

$$\alpha = \alpha_{\max} * \alpha \ if \ \alpha_{opt} < 0 \tag{25}$$

$$\alpha = \min\{\alpha_{opt}, \alpha_{\max}\} \ if \ \alpha_{opt} \geq 0 \tag{26}$$

Before setting the value of the step s, we need to define the parameter π. Therefore, to set the value of π we define three types of algorithms iterations as defined in Barahona and Anbil (2000) and Barahona and Ladányi (2006).

Each iteration with no improvement is named *red*. Then, the parameter *MaxWaste* represents the maximum number of iterations without a lower bound improvement. If $z^k > \bar{z}$ and $\nu'^k_{(\lambda_{jk-1}, x^k, y^k)} \nu^k_{(\lambda_{jk-1}, \bar{x}, \bar{y})} < 0$, it means that a longer step in the direction to ν^k would have given a smaller value for z^k. Those iterations are denominated *yellow*, otherwise, the iteration is denominated *green*. At each *yellow* iteration we would multiply π by the *yellow* parameter. At each *green* iteration we would multiply π by the *green* parameter. After a sequence of 24 consecutive *red* iterations, we would multiply π by *factor* parameter. Thus, the step size s at iteration k is defined as:

$$s = \frac{\pi * (st * UB - \bar{z}))}{\parallel \nu_{(\lambda_{jk-1}, \bar{x}, \bar{y})} \parallel^2} \tag{27}$$

In the “Lagrangian heuristics step” two constructive heuristics and a Local Search are executed sequentially. The Local Search is implemented based on the proposals of Arroyo, Nunes, and Kamke (2009) and Stutzle (1998). The procedure adopted is composed by “swap” and “insertion” moves in the sequence on machine 2 generated by H1 and H2. The former consists of interchanging all pairs of jobs. The latter consists of removing a job from its original position and inserting it on one of the $n-1$ remaining positions. The local search procedure stops when it is unable to improve the solution further.

Finally, the stop criterion is determined if one of the following criteria is satisfied:

1. Maximum number of iterations, in this case 1000 iterations, or;
2. Relative tolerance GAP defined as: $\frac{z^k - \bar{z}}{\bar{z}} < 0.1$, or;
3. Null Subgradient module or negligible: $\parallel \nu_{(\lambda_{jk-1}, \bar{x}, \bar{y})} \parallel^2 \leq 0.000001$.

4.3. JB heuristic

We compare the results of heuristics H1 and H2 with the best heuristic introduced by Chen and Lee (2009), which is based on Johnson’s algorithm. The heuristic JB is proposed in two steps.

Firstly, for an instance of $F2|CD|C_{\max}$, the authors construct an instance of $F2||C_{\max}$ with n jobs. The first stage is maintained unchangeable and the second stage is converted into n jobs. Secondly, Johnson's algorithm is applied to obtain a sequence in the first stage, generating a lower bound. From the sequence in the first stage, jobs $j \in J_2$ at the second stage are sequenced as soon as possible in the time horizon, respecting the completion time of its predecessors. We run the above algorithm for the primary problem and its reverse problem and select the best solution. As a result, an Upper bound for the problem is generated (see Chen & Lee, 2009 for a more detailed explanation), so we calculated the relative GAP. With the objective of a fair comparison, the NEH algorithm and Local search are used to refine the results of JB in the same way than used in H1 and H2.

4.4. CDH heuristic

The proposed H1 and H2 algorithms are also compared with the heuristic in Cota et al. (2016), called CDH. The CDH heuristic starts by creating a preliminary schedule in the second machine. With this, a schedule is defined in the first machine and finally the second machine is rescheduled. The heuristic initially creates fictitious processing times $TF_j = (\sum_{i \in J^1, i \in S_j} p_{1i}) + p_{2j}$ for each job $j \in J^2$. Then, a preliminary schedule on M2 is obtained by ordering jobs in increasing order of fictitious processing times. Then the jobs on the M1 are sequence according the preliminary schedule on M2, respecting the precedence relations of cross-docking. The sequence on M1 imposes release dates to the jobs in the second machine due to precedence relationships. So, the ERD rule (earliest release date first) is applied to reschedule jobs in the second machine, obtaining a sequence on the M2. Let UB be the upper bound obtained with the heuristic and LB be the lower bound calculated as proposed in Chen and Song (2009), we generate the percentage GAP for all tested instances.

5. Generalization for parallel-docks CDC

In this section, we present a time-indexed mixed integer formulation for $F2(P)|CD|C_{\max}$ based on the proposal of Cota et al. (2016). The cross-docking problem is modeled as a hybrid two-stage flow shop scheduling problem with identical machines and cross-docking constraints, with the objective of minimizing the makespan. The problem is denoted as $F2(P)|CD|C_{\max}$, since $F2|CD|C_{\max}$ is strongly NP-hard, as showed by Chen and Lee (2009), it is not difficult to see that $F2(P)|CD|C_{\max}$ is also strongly NP-hard.

5.1. Time-indexed mixed integer linear programming model for $F2(P)|CD|C_{max}$

For the parallel-docks formulation, we use the following notation:

Input parameters

- m_1: number of parallel processors in stage 1.
- m_2: number of parallel processors in stage 2.
- n_1: number of jobs in stage 1.
- n_2: number of jobs in stage 2.
- p_j: processing time of job j.
- T: Time horizon considered initially. In practice, we used $T = \sum_{j \in J_1} p_j + \sum_{j \in J_2} p_j$.

Sets

- $T = \{0, \dots, T\}$, set of discrete periods considered.
- $J_1 = \{1, 2, \dots, n_1\}$, set of jobs in stage 1.
- $J_2 = \{1, 2, \dots, n_2\}$, set of jobs in stage 2.
- S_j: a set of precedent subset jobs of J_1 corresponding to job $j \in J_2$, $S_j \subset S = \{S_1, \dots, S_{n_2}\}$.

Decision variables

- $C_{\max}$: makespan.
- $x_{jt} = 1$, if job j starts to be processed at time t, 0, otherwise.

The multiple dock formulation is defined as:

$$\text{Minimize} \quad C_{\max} \tag{28}$$

subject to

$$\sum_{t=0}^{T-p_j} x_{jt} = 1, \quad \forall j \in J_1, \tag{29}$$

$$\sum_{t=0}^{T-p_j} x_{jt} = 1, \quad \forall j \in J_2, \tag{30}$$

$$\sum_{j \in J_1} \sum_{s=\max(0;t-p_j+1)}^{t} x_{js} \le m_1, \quad \forall t \in T, \tag{31}$$

$$\sum_{j \in J_2} \sum_{s=\max(0;t-p_j+1)}^{t} x_{js} \le m_2, \quad \forall t \in T, \tag{32}$$

$$\sum_{t=0}^{T-p_j} t x_{jt} \ge \sum_{t=0}^{T} (t+p_i) x_{it}, \quad \forall j \in J_2, \forall i \in S_j, \tag{33}$$

$$C_{\max} \ge p_j + \sum_{t=0}^{T-p_j} t x_{jt}, \quad \forall j \in J_2, \tag{34}$$

$$x_{jt} \in \{0,1\}, \quad \forall j \in J_1 \wedge \forall j \in J_2, \forall t \in T, \tag{35}$$

$$C_{\max} \ge 0. \tag{36}$$

The objective function remains as the minimization of the makespan. Constraints (29) and (30) ensure that in every stage, each job should start its processing on one, and only one, time slot within the time horizon T. Constraints set (31) and (32) indicates that the number of jobs that can be process simultaneously is less than or equal to the number of existing parallel processors in the stage considered. The constraint set (33) represents the cross-docking constraints, in which the release date of each job $j \in J^2$, must be greater than or equal to the completion date of each task belonging to its precedence set. Besides that, the constraint set ensures that each job $j \in J^2$, starts processing only after its release date. Constraints (34) compute the makespan, analyzing the maximum completion time. Constraints (35) and (36) specify the domains of each decision variable.

5.2. Lagrangian relaxation for $F2(P)|CD|C_{max}$

Let $L_{MD}(\lambda)$ be the relaxation of multiple dock formulation, when the cross-docking constraints (33) are dualized and its violation penalized in the objective function. We define the subproblem $L_{MD}(\lambda)$ as:

$$L_{MD}(\lambda) = \text{Minimize } C_{\max} - \sum_{j \in J_2} \sum_{i \in S_j} \lambda_{ij} \left(\sum_{t=0}^{T-p_j} t x_{jt} - \sum_{t=0}^{T} (t+p_i) x_{it} \right) \tag{37}$$

s.t. (29)–(32), (34)–(36), $\lambda_{ij} \ge 0$.

Just as we did for the 2-dock case, we are able to uncouple the problem into two new scheduling problems, one for each stage.

Subproblem 1

$$L'_{MD}(\lambda) = \text{Min} \sum_{j \in J_2} \sum_{i \in S_j} \lambda_{ij} \sum_{t=0}^{T} (t+p_i) x_{it} \tag{38}$$

subject to

$$\sum_{t=0}^{T-p_j} x_{jt} = 1, \quad \forall j \in J_1, \tag{39}$$

$$\sum_{j \in J_1} \sum_{s=\max(0;t-p_j+1)}^{t} x_{js} \le m_1, \quad \forall t \in T, \tag{40}$$

$$x_{jt} \in \{0,1\}, \quad \forall j \in J_1, \forall t \in T, \tag{41}$$

$$\lambda_{ij} \ge 0, \quad \forall j \in J_2, i \in S_j. \tag{42}$$

Subproblem 2

$$L''_{MD}(\lambda) = \text{Min} \quad C_{\max} - \sum_{j \in J_2} \sum_{i \in S_j} \lambda_{ij} \sum_{t=0}^{T-p_j} t x_{jt} \tag{43}$$

subject to

$$\sum_{t=0}^{T-p_j} x_{jt} = 1, \quad \forall j \in J_2, \tag{44}$$

$$\sum_{j \in J_2} \sum_{s=\max(0;t-p_j+1)}^{t} x_{js} \le m_2, \quad \forall t \in T, \tag{45}$$

$$C_{\max} \ge p_j + \sum_{t=0}^{T-p_j} t x_{jt}, \quad \forall j \in J_2, \tag{46}$$

$$x_{jt} \in \{0,1\}, \quad \forall j \in J_2, \forall t \in T, \tag{47}$$

$$\lambda_{ij} \ge 0, \quad \forall j \in J_2, i \in S_j, \tag{48}$$

$$C_{\max} \ge 0. \tag{49}$$

5.3. Constructive heuristic for parallel-dock CDC

The first subproblem can be defined as $Pm||\Sigma C_j W_j$, while the second as $Pm||C_{\max} - \sum I_j W_j$, therefore, it is not difficult to see that both subproblem are strongly NP-hard (see Pinedo, 2008). In this case, we focus on improving the problem's upper bounds, thus, in each iteration, we solve the linear relaxations of the subproblems, obtain integer solutions and then compute their Lagrangian multipliers. The logic of the procedure is described in Algorithm 1, maintaining the same reasoning proposed in Section 4.

For comparison purposes, we limited the computational time in 60 seconds, and we compute the LB precisely as proposed by Chen and Song (2009).

6. Computational experiments

6.1. Results for the 2-dock problem

To investigate the performance of the Complete Model and Hybrid Lagrangian Metaheuristic Framework, artificial instances are generated varying the processing time of jobs and the number of jobs on machines 1 and 2, according to Chen and Lee (2009). We ran the tests on a single thread in the Intel Xeon-Silver 4110 (2.1 gigahertz/8-core/85 watt) with 64 gigabyte memory and Linux operational system. The programming language used is C++ with the optimization software CPLEX 12.4.

6.1.1. Instances generation

The instances used in this work are generated through the software MATLAB, following the description found in Chen and Lee (2009). Table 2 presents a summary of the generated instances and its characteristics. It is divided into two groups, each row informs the features of a sub-group of instances. n and m indicate the number of jobs in each stage. (NP) the discrete uniform distribution to select the number predecessors of jobs $j \in J^2$ and (TP) the distribution to generate all processing times.

Algorithm 1 Algorithm MD.

```
1:  Set Lagrangian multipliers as null.
2:  Compute a Lower Bound (LB), as proposed by Chen and Song (2009).
3:  Compute an initial Upper Bound (UB).
    $UB = \sum_{i \in J_1} p_i/m_1 - \max_{i \in J_1} p_i/m_1 + \max_{i \in J_1} p_i +$
    $+ \sum_{j \in J_2} p_j/m_2 - \max_{j \in J_2} p_j/m_2 + \max_{j \in J_2} p_j.$
4:  while ($Time < TimeLimit$)
5:      Solve the Linear relaxation.
6:      Allocate the job as long as its variable is closer to 1.
7:      Calculate the Lagrangian multipliers.
8:      Compute the weights $\beta_1$ and $\beta_2$ for the heuristics H1 and H2 (the ordered weights define the entry list for H1 and H2).
9:      Apply Heuristics H1 and H2.
10:     Apply a First-Best Local Search.
11:     if UB has improved
12:         Update the UB and add a cut to reduce the time horizon.
13:     else
14:         end algorithm.
15:         end while.
16:     end if
17: end while
```

Table 2
Summary of the artificial benchmark.

Group	Jobs M1(n)	Jobs M2(m)	NP	TP
1	5	3–7	U(1,4)	U(1,10)
	10	6-8-10-12-14	U(1,9)	U(1,10)
	20	12-16-20-24-28	U(1,19)	U(1,10)
	40	24-32-40-48-56	U(1,39)	U(1,10)
	60	36-48-60-72-84	U(1,59)	U(1,10)
2	5	3–7	U(1,4)	U(10,100)
	10	6-8-10-12-14	U(1,9)	U(10,100)
	20	12-16-20-24-28	U(1,19)	U(10,100)
	40	24-32-40-48-56	U(1,39)	U(10,100)
	60	36-48-60-72-84	U(1,59)	U(10,100)

In each group of instances, n is fixed and the number m is randomly chosen from the range of [0.6n, 0.8n, 1.0n, 1.2n, 1.4n]. For each n, five instances with different m values are considered. And for each pair (n, m), we generate 10 different problems, therefore the benchmark consists of 500 instances, 50 instances per row of Table 2[1].

6.1.2. Computational results

In Table 3 we report computational results for the Complete Model (MIP) and heuristics JB, CDH, H1, and H2. Columns n and m show respectively the number of jobs in the first and second machine. For each instance size (n, m) we report the best result and the average of 10 cases tested. We also presented the subgroup's average results.

The GAP(%) is computed as $\frac{\text{(Upper Bound - Lower Bound)}}{\text{Upper Bound}}$ and the column T(s) refers to CPU time expended to solve the problem

[1] All instances and codes are available at https://github.com/GabrielaBragaFonseca/Cross-docking-Problems.

Table 3
Computational results for complete model and constructive heuristics.

			Group 1 - Processing time [1, 10]										Group 2 - Processing time [10, 100]									
n	m		MIP		JB		CDH		H1		H2		MIP		JB		CDH		H1		H2	
			GAP	T(s)	GAP	T(s)	GAP	T(s)	GAP	T(s)	GAP	T(s)	GAP	T(s)	GAP	T(s)	GAP	T(s)	GAP	T(s)	GAP	T(s)
5	3	Best	0.0%	0.1	**0.0%**	0.0	15.2%	0.0	**0.0%**	0.0	**0.0%**	0.0	0.0%	6.7	**0.0%**	0.0	7.8%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	0.0%	0.2	6.8%	0.0	26.3%	0.0	**4.8%**	0.0	**4.8%**	0.0	0.0%	21.0	2.4%	0.0	28.8%	0.0	**2.3%**	0.0	**2.3%**	0.0
	4	Best	0.0%	0.1	**0.0%**	0.0	2.7%	0.0	**0.0%**	0.0	**0.0%**	0.0	0.0%	3.9	**0.0%**	0.0	3.0%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	0.0%	0.3	7.7%	0.0	22.3%	0.0	**5.9%**	0.0	**5.9%**	0.0	0.0%	354.9	7.5%	0.0	24.2%	0.0	**6.2%**	0.0	**6.2%**	0.0
	5	Best	0.0%	0.3	4.5%	0.0	17.1%	0.0	**0.0%**	0.0	**0.0%**	0.0	0.0%	32.6	3.8%	0.0	19.3%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	0.0%	0.5	9.8%	0.0	32.8%	0.0	**2.5%**	0.0	3.3%	0.0	0.8%	1570.1	9.9%	0.0	33.4%	0.0	**2.6%**	0.0	**2.6%**	0.0
	6	Best	0.0%	0.1	4.8%	0.0	13.0%	0.0	**0.0%**	0.0	**0.0%**	0.0	0.0%	80.9	5.7%	0.0	13.5%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	0.0%	1.3	16.6%	0.0	29.4%	0.0	**5.2%**	0.0	5.5%	0.0	11.8%	2388.3	16.1%	0.0	30.7%	0.0	**4.8%**	0.0	6.2%	0.0
	7	Best	0.0%	0.2	**0.0%**	0.0	16.7%	0.0	**0.0%**	0.0	**0.0%**	0.0	0.0%	94.9	**0.0%**	0.0	16.6%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	0.0%	3.7	14.1%	0.0	36.4%	0.0	6.3%	0.0	**5.4%**	0.0	13.1%	2882.5	13.9%	0.0	38.0%	0.0	**5.1%**	0.0	6.7%	0.0
Subgroup Average			0.0%	1.2	11.0%	0.0	29.4%	0.0	**4.9%**	0.0	5.0%	0.0	5.1%	1443.4	10.0%	0.0	31.0%	0.0	**4.2%**	0.0	4.8%	0.0
10	6	Best	0.0%	5.1	**0.0%**	0.0	7.5%	0.0	**0.0%**	0.0	**0.0%**	0.0	28.7%	2801.1	0.7%	0.0	8.9%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	0.0%	20.2	11.8%	0.0	24.8%	0.0	1.8%	0.0	**1.7%**	0.0	28.2%	3509.9	11.3%	0.0	23.0%	0.0	**1.4%**	0.0	**1.4%**	0.0
	8	Best	0.0%	10.2	2.7%	0.0	9.6%	0.0	**0.0%**	0.0	**0.0%**	0.0	23.6%	3597.6	2.3%	0.0	15.0%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	0.0%	167.5	13.3%	0.0	25.8%	0.0	6.0%	0.0	**5.4%**	0.0	36.7%	3600.0	17.0%	0.0	30.5%	0.0	**3.7%**	0.0	4.2%	0.0
	10	Best	0.0%	14.1	1.4%	0.0	14.8%	0.0	**0.0%**	0.0	2.2%	0.0	35.7%	3600.0	2.2%	0.0	26.4%	0.0	4.6%	0.0	**1.0%**	0.0
		Average	0.0%	369.6	15.9%	0.0	28.8%	0.0	6.5%	0.0	**6.4%**	0.0	45.2%	3600.0	18.4%	0.0	36.8%	0.0	**7.6%**	0.0	7.9%	0.0
	12	Best	0.0%	42.1	3.9%	0.0	17.6%	0.0	**0.0%**	0.0	**0.0%**	0.0	51.7%	3600.0	3.1%	0.0	19.3%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	5.0%	2440.5	14.0%	0.0	41.0%	0.0	5.4%	0.0	**5.3%**	0.0	65.2%	3600.0	13.9%	0.0	40.8%	0.0	**5.4%**	0.0	**5.4%**	0.0
	14	Best	0.0%	21.9	1.3%	0.0	27.9%	0.0	**0.0%**	0.0	1.0%	0.0	53.2%	3600.0	2.0%	0.0	28.9%	0.0	**0.0%**	0.0	**0.0%**	0.0
		Average	4.6%	3141.6	10.8%	0.0	45.5%	0.0	**7.0%**	0.0	8.1%	0.0	73.6%	3600.0	11.4%	0.0	45.2%	0.0	6.7%	0.0	**5.8%**	0.0
Subgroup Average			1.9%	1227.9	13.1%	0.0	33.2%	0.0	**5.3%**	0.0	5.4%	0.0	49.8%	3582.0	14.4%	0.0	35.2%	0.0	5.0%	0.0	**4.9%**	0.0
20	12	Best	1.5%	3600.0	10.0%	0.0	12.4%	0.0	1.5%	0.0	**0.7%**	0.0	–	3600.0	10.2%	0.0	13.7%	0.0	**0.6%**	0.0	**0.6%**	0.0
		Average	14.8%	3600.0	18.1%	0.0	26.1%	0.0	**6.6%**	0.0	7.1%	0.0	–	3600.0	17.9%	0.0	26.0%	0.0	**6.4%**	0.0	6.5%	0.0
	16	Best	14.5%	3600.0	16.4%	0.0	21.4%	0.0	**1.6%**	0.0	5.4%	0.0	–	3600.0	17.4%	0.0	22.7%	0.0	**2.3%**	0.0	4.2%	0.0
		Average	32.4%	3600.0	23.0%	0.0	32.6%	0.0	**9.8%**	0.0	10.5%	0.0	–	3600.0	23.3%	0.0	32.6%	0.0	**8.2%**	0.0	10.4%	0.0
	20	Best	24.5%	3600.0	14.3%	0.0	31.9%	0.0	**2.5%**	0.0	4.4%	0.0	–	3600.0	15.5%	0.0	31.2%	0.0	3.7%	0.0	**1.3%**	0.0
		Average	36.0%	3600.0	24.2%	0.0	38.5%	0.0	11.1%	0.0	**10.9%**	0.0	–	3600.0	24.8%	0.0	38.0%	0.0	11.4%	0.0	**10.3%**	0.0
	24	Best	19.9%	3600.0	2.4%	0.0	35.0%	0.0	**0.0%**	0.0	2.6%	0.0	–	3600.0	1.5%	0.0	35.6%	0.0	**0.6%**	0.0	5.2%	0.0
		Average	38.4%	3600.0	21.2%	0.0	44.4%	0.0	**12.8%**	0.0	14.0%	0.0	–	3600.0	21.7%	0.0	44.1%	0.0	**12.2%**	0.0	13.0%	0.0
	28	Best	19.4%	3600.0	1.6%	0.0	39.2%	0.0	**1.1%**	0.0	4.9%	0.0	–	3600.0	3.1%	0.0	41.0%	0.0	**0.9%**	0.0	**0.9%**	0.0
		Average	36.5%	3600.0	16.7%	0.0	49.3%	0.0	15.2%	0.0	**15.0%**	0.0	-	3600.0	17.4%	0.0	48.9%	0.0	14.4%	0.0	**13.5%**	0.0
Subgroup Average			31.6%	3600.0	20.6%	0.0	38.2%	0.0	**11.1%**	0.0	11.5%	0.0	*	3600.0	21.0%	0.0	37.9%	0.0	**10.5%**	0.0	10.7%	0.0
40	24	Best	40.2%	3600.0	15.7%	0.0	20.9%	0.0	**3.9%**	0.0	4.5%	0.0	-	3600.0	2.3%	0.0	50.0%	0.0	**0.0%**	0.0	1.8%	0.0
		Average	46.3%	3600.0	20.7%	0.0	26.2%	0.0	9.6%	0.0	**9.0%**	0.0	-	3600.0	8.6%	0.0	56.0%	0.0	6.2%	0.0	**5.6%**	0.0
	32	Best	52.6%	3600.0	21.3%	0.0	25.2%	0.0	9.0%	0.0	**8.3%**	0.0	-	3600.0	22.4%	0.0	27.4%	0.0	**6.0%**	0.0	**6.0%**	0.0
		Average	57.0%	3600.0	26.5%	0.0	31.8%	0.0	12.8%	0.0	**12.5%**	0.0	-	3600.0	28.5%	0.0	32.7%	0.0	11.8%	0.0	**11.0%**	0.0
	40	Best	57.3%	3600.0	26.1%	0.0	31.7%	0.0	9.1%	0.0	**8.3%**	0.0	-	3600.0	24.4%	0.0	32.6%	0.0	**9.0%**	0.0	10.4%	0.0
		Average	69.2%	3600.0	31.1%	0.0	36.1%	0.0	12.5%	0.0	**12.3%**	0.0	-	3600.0	32.4%	0.0	37.7%	0.0	**13.3%**	0.0	13.6%	0.0
	48	Best	-	3600.0	23.9%	0.0	41.6%	0.0	**9.5%**	0.0	11.6%	0.0	-	3600.0	23.0%	0.0	41.7%	0.0	**10.8%**	0.0	11.6%	0.0
		Average	-	3600.0	29.6%	0.0	45.2%	0.0	**16.6%**	0.0	16.8%	0.0	-	3600.0	29.1%	0.0	45.3%	0.0	16.6%	0.0	**16.1%**	0.0
	56	Best	-	3600.0	16.8%	0.0	44.4%	0.0	**13.2%**	0.0	15.5%	0.0	-	3600.0	18.4%	0.0	44.8%	0.0	**13.8%**	0.0	14.8%	0.0
		Average	-	3600.0	23.3%	0.0	48.3%	0.0	19.7%	0.0	**19.3%**	0.0	-	3600.0	23.3%	0.0	48.3%	0.0	18.7%	0.0	**18.1%**	0.0
Subgroup Average			*	3600.0	26.2%	0.0	37.5%	0.0	14.2%	0.0	**14.0%**	0.0	*	3600.0	24.4%	0.0	44.0%	0.0	13.3%	0.0	**12.9%**	0.0
60	36	Best	-	3600.0	16.9%	0.0	20.9%	0.0	5.5%	0.0	**4.5%**	0.0	-	3600.0	17.8%	0.0	21.4%	0.0	5.9%	0.0	**3.4%**	0.0
		Average	-	3600.0	22.8%	0.0	26.7%	0.0	8.6%	0.0	**8.4%**	0.0	-	3600.0	22.7%	0.0	26.6%	0.0	8.4%	0.0	**7.6%**	0.0
	48	Best	-	3600.0	26.0%	0.0	30.4%	0.0	**9.6%**	0.0	10.7%	0.0	-	3600.0	26.5%	0.0	30.6%	0.0	9.4%	0.0	**8.5%**	0.0
		Average	-	3600.0	29.9%	0.0	33.0%	0.0	**12.0%**	0.0	12.8%	0.0	-	3600.0	29.6%	0.0	32.9%	0.0	**11.2%**	0.0	12.0%	0.0
	60	Best	-	3600.0	26.6%	0.0	35.3%	0.0	14.2%	0.0	**13.9%**	0.0	-	3600.0	28.0%	0.0	36.1%	0.0	13.3%	0.0	**12.9%**	0.0
		Average	-	3600.0	33.5%	0.0	40.6%	0.0	**18.2%**	0.0	18.3%	0.0	-	3600.0	33.6%	0.0	40.7%	0.0	**16.9%**	0.0	17.0%	0.0
	72	Best	-	3600.0	25.7%	0.0	42.6%	0.0	15.5%	0.0	**14.8%**	0.0	-	3600.0	26.0%	0.0	42.8%	0.0	**13.5%**	0.0	**13.5%**	0.0
		Average	-	3600.0	30.6%	0.0	46.2%	0.0	19.1%	0.0	**18.9%**	0.0	-	3600.0	30.5%	0.0	46.0%	0.0	18.2%	0.0	**17.6%**	0.0
	84	Best	-	3600.0	20.3%	0.0	44.0%	0.0	19.3%	0.0	**18.4%**	0.0	-	3600.0	20.2%	0.0	44.4%	0.0	17.8%	0.0	**17.4%**	0.0
		Average	-	3600.0	26.0%	0.0	50.4%	0.0	21.9%	0.0	**21.4%**	0.0	-	3600.0	25.9%	0.0	50.3%	0.0	21.4%	0.0	**20.5%**	0.0
Subgroup Average			*	3600.0	28.6%	0.0	39.4%	0.0	**16.0%**	0.0	**16.0%**	0.0	*	3600.0	28.5%	0.0	39.3%	0.0	15.2%	0.0	**14.8%**	0.0

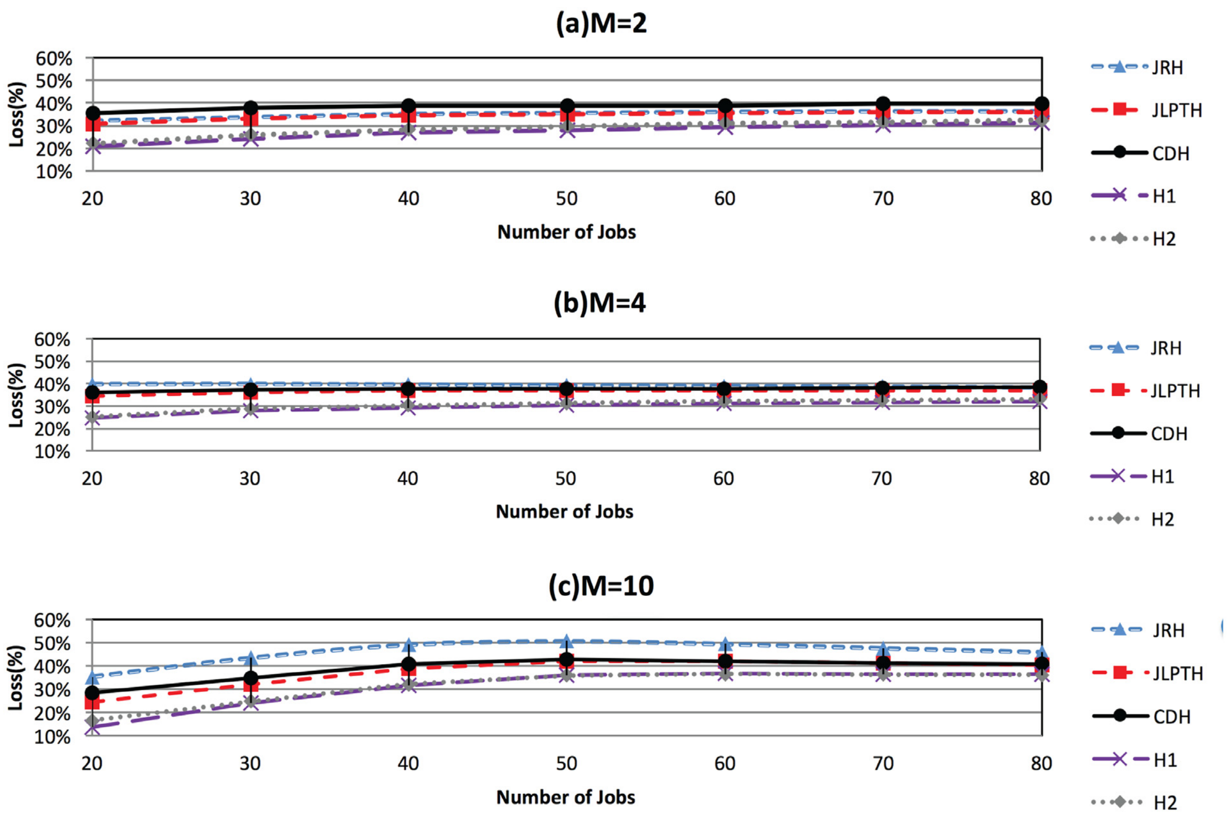


Fig. 2. Average loss value of each heuristic considering the same number of machines at each stage: (a) $M = 2$, (b)$M = 4$, (c) $M = 10$.

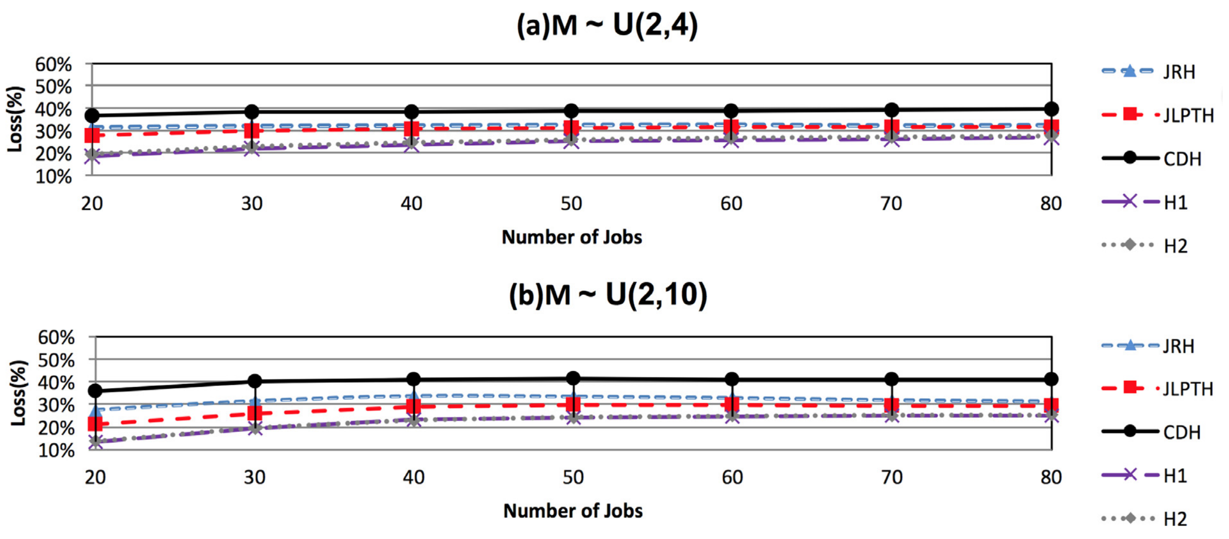


Fig. 3. Average loss value of each heuristic considering a random number of machines at each stage: (a) $M \sim U(2, 4)$, (b) $M \sim U(2, 10)$.

Table 4
Computational results for lower bounds.

	Group 1 - Processing time [1, 10]				Group 2 - Processing time [10, 100]			
n	LR	LB_1	LB_2	LB_{HgR}	LR	LB_1	LB_2	LB_{HgR}
5	23.7	29.3	33.6	**34.7**	232.8	288.9	337.4	**347.0**
10	39.4	60.9	66.2	**70.0**	379.1	635.9	667.6	**706.9**
20	70.7	119.7	140.2	**143.4**	705.7	1203.0	1412.7	**1442.4**
40	132.9	225.5	293.5	**294.9**	–	2253.2	2871.8	**2883.2**
60	193.9	338.0	**444.3**	**444.3**	–	3373.5	**4452.3**	**4452.3**

Table 5
Computational results for large instances. The best, the average and the worst percentage loss values are presented for each instance configuration.

m_i	n_1	n_2		Loss(%)				
				JRH	JLPTH	CDH	H1	H2
2	20	[16,24]	Best	16.92	16.76	17.69	**9.02**	9.76
			Average	32.34	30.91	35.37	**20.93**	22.20
			Worst	53.77	51.24	64.08	36.36	**35.08**
			Std Dev	6.73	6.49	8.67	4.84	**4.74**
	30	[24,36]	Best	17.83	17.14	21.31	**11.62**	12.46
			Average	33.94	32.98	37.59	**24.25**	25.88
			Worst	46.49	45.84	64.72	37.11	**36.38**
			Std Dev	5.39	5.18	6.96	4.35	**4.13**
	40	[32,48]	Best	22.80	22.29	24.60	**15.54**	17.96
			Average	35.29	34.52	38.67	**26.75**	28.36
			Worst	47.91	45.72	59.40	**37.13**	39.12
			Std Dev	4.34	4.25	6.41	3.80	**3.62**
	50	[40,60]	Best	25.58	25.18	23.65	19.44	**19.36**
			Average	35.77	35.03	38.94	**28.01**	29.62
			Worst	46.22	45.58	59.36	**35.91**	38.07
			Std Dev	3.96	3.91	6.28	3.32	**3.29**
	60	[48,72]	Best	25.93	26.12	26.65	**18.83**	21.02
			Average	36.25	35.67	38.93	**29.20**	31.01
			Worst	46.69	45.47	56.3	**36.72**	38.43
			Std Dev	3.62	3.62	5.33	3.24	**3.12**
	70	[56,84]	Best	29.25	28.84	28.02	**22.52**	23.22
			Average	36.52	36.01	39.81	**30.35**	31.51
			Worst	44.62	44.49	56.35	39.09	**39.07**
			Std Dev	3.24	3.20	5.30	3.02	**2.88**
	80	[64,96]	Best	28.84	28.07	28.01	**22.46**	25.42
			Average	36.52	36.12	39.58	**30.99**	32.18
			Worst	43.73	43.52	55.55	**38.98**	39.38
			Std Dev	2.97	2.94	5.13	2.89	**2.73**
Average 2 machines				35.23	34.46	38.41	**27.21**	28.68
4	20	[16,24]	Best	20.90	17.36	17.56	**11.57**	12.99
			Average	39.87	34.38	35.89	**24.78**	25.31
			Worst	61.61	52.48	59.75	35.31	**35.00**
			Std Dev	7.27	6.15	7.43	4.65	**4.33**
	30	[24,36]	Best	26.60	24.04	24.02	17.80	**17.47**
			Average	39.92	36.17	37.29	**28.05**	28.88
			Worst	51.71	46.49	59.00	**36.70**	39.66
			Std Dev	5.17	4.54	6.28	3.75	**3.67**
	40	[32,48]	Best	26.79	24.49	22.00	**19.42**	20.38
			Average	39.68	36.79	37.64	**29.35**	30.25
			Worst	50.72	46.13	54.92	**38.28**	39.28
			Std Dev	4.31	3.87	5.63	3.47	**3.16**
	50	[40,60]	Best	27.85	26.69	21.9	**20.73**	21.36
			Average	39.27	36.71	37.66	**30.29**	31.25
			Worst	49.65	47.95	57.31	39.26	**38.59**
			Std Dev	3.89	3.68	5.43	3.29	**3.06**
	60	[48,72]	Best	28.57	27.55	26.52	**21.08**	24.53
			Average	39.01	36.96	37.7	**31.07**	32.05
			Worst	49.20	45.64	51.43	39.82	**38.62**
			Std Dev	3.44	3.35	4.68	2.94	**2.83**
	70	[56,84]	Best	30.86	30.08	26.71	**23.55**	24.86
			Average	38.75	36.99	38.21	**31.52**	32.49
			Worst	47.80	44.42	52.69	**38.48**	38.67
			Std Dev	3.14	2.90	4.57	2.82	**2.62**
	80	[64,96]	Best	30.27	29.09	27.63	**22.95**	26.10
			Average	38.39	36.94	38.33	**32.17**	33.12
			Worst	44.81	43.51	50.97	**38.69**	39.38
			Std Dev	2.86	2.77	4.47	2.65	**2.60**
Average 4 machines				39.27	36.42	37.53	**29.60**	30.48
10	20	[16,24]	Best	12.54	11.59	5.69	**0.00**	4.10
			Average	35.38	24.56	28.23	**13.50**	16.49
			Worst	55.41	45.50	61.03	**27.01**	27.27
			Std Dev	7.52	5.74	8.44	4.79	**4.24**
	30	[24,36]	Best	26.91	18.44	18.53	**12.24**	13.86
			Average	43.46	32.08	34.65	**24.14**	24.71
			Worst	59.20	45.91	53.51	**34.19**	35.02
			Std Dev	6.31	5.16	6.64	4.60	**4.46**
	40	[32,48]	Best	28.06	22.54	27.20	**15.96**	18.67
			Average	49.17	38.74	40.87	**31.79**	32.10
			Worst	63.17	48.35	57.14	41.04	**40.96**
			Std Dev	6.04	4.87	6.21	4.33	**4.21**
	50	[40,60]	Best	36.50	28.39	27.36	26.04	**25.41**
			Average	50.62	42.04	42.83	35.96	**35.81**

(continued on next page)

Table 5 (*continued*)

m_i	n_1	n_2		Loss(%)				
				JRH	JLPTH	CDH	H1	H2
			Worst	63.10	51.89	57.84	43.15	**41.23**
			Std Dev	4.79	3.77	5.27	3.28	**3.20**
	60	[48,72]	Best	38.27	32.56	30.42	**27.74**	27.91
			Average	49.40	41.85	41.97	36.70	**36.44**
			Worst	59.34	48.44	57.24	42.09	**41.97**
			Std Dev	4.04	3.11	4.79	2.71	**2.67**
	70	[56,84]	Best	37.66	32.57	29.75	**28.49**	29.02
			Average	47.62	41.21	41.24	36.53	**36.24**
			Worst	59.36	46.98	52.51	42.02	**41.12**
			Std Dev	3.55	2.63	4.57	2.50	**2.34**
	80	[64,96]	Best	37.61	33.20	30.17	28.94	**28.30**
			Average	45.92	40.44	40.82	36.34	**36.12**
			Worst	53.71	45.87	51.81	41.30	**41.29**
			Std Dev	3.11	2.50	4.38	2.46	**2.32**
Average 10 machines				45.94	37.28	38.66	**30.71**	31.13
$U(2, 4)$	20	[16,24]	Best	4.57	4.25	9.02	3.85	**2.32**
			Average	31.33	27.94	36.48	**18.62**	19.36
			Worst	59.86	46.01	86.91	**33.61**	35.40
			Std Dev	9.89	8.82	14.55	6.75	**6.69**
	30	[24,36]	Best	9.57	6.79	10.14	**4.82**	5.28
			Average	32.07	29.70	38.14	**21.86**	22.68
			Worst	51.89	47.81	85.47	38.32	**36.28**
			Std Dev	8.26	7.63	13.76	6.60	**6.40**
	40	[32,48]	Best	10.90	9.54	10.61	**3.00**	6.57
			Average	32.33	30.58	38.18	**23.72**	24.67
			Worst	47.42	44.74	79.10	37.64	**36.84**
			Std Dev	7.81	7.34	13.05	**6.52**	6.57
	50	[40,60]	Best	13.26	12.21	12.64	**8.33**	10.21
			Average	32.65	31.27	38.69	**25.04**	26.04
			Worst	48.53	45.11	76.62	38.96	**37.57**
			Std Dev	7.70	7.34	12.63	**6.53**	6.75
	60	[48,72]	Best	11.15	11.04	11.83	**8.34**	9.44
			Average	32.74	31.51	38.74	**25.72**	26.84
			Worst	47.14	43.42	81.55	**36.36**	38.58
			Std Dev	6.97	6.75	12.09	**6.17**	6.28
	70	[56,84]	Best	16.49	14.56	16.39	**9.94**	11.67
			Average	32.36	31.40	39.3	**26.16**	27.32
			Worst	45.60	43.95	77.17	**37.74**	39.01
			Std Dev	6.81	6.58	11.96	**6.01**	6.19
	80	[64,96]	Best	15.01	14.13	15.48	**11.03**	13.05
			Average	32.53	31.67	39.57	**26.95**	27.71
			Worst	43.92	43.37	77.62	**37.52**	37.56
			Std Dev	6.57	6.41	12.05	**5.94**	6.02
Average $U(2, 4)$ machines				32.29	30.58	38.44	**24.01**	24.94
$U(2, 10)$	20	[16,24]	Best	3.40	3.40	2.11	**0.15**	0.46
			Average	27.78	21.25	35.94	**13.15**	13.77
			Worst	55.42	43.67	103.55	34.38	**31.48**
			Std Dev	11.45	9.18	22.69	7.57	**7.24**
	30	[24,36]	Best	4.35	1.97	4.09	**1.18**	**1.18**
			Average	31.67	26.05	40.02	19.44	**19.43**
			Worst	61.61	45.32	97.89	38.78	**37.32**
			Std Dev	12.13	10.24	21.41	8.95	**8.75**
	40	[32,48]	Best	7.67	4.04	4.50	**2.14**	2.72
			Average	33.90	29.08	40.85	23.07	**22.96**
			Worst	60.57	48.93	96.09	40.26	**39.74**
			Std Dev	13.05	10.85	19.21	9.63	**9.58**
	50	[40,60]	Best	8.09	6.88	4.61	**3.58**	3.77
			Average	33.64	29.66	41.28	**24.30**	24.33
			Worst	60.51	50.56	100.21	**40.83**	40.86
			Std Dev	12.11	10.53	18.97	9.46	**9.41**
	60	[48,72]	Best	8.81	6.33	6.36	4.54	**4.45**
			Average	20.78	29.63	41.03	**24.64**	24.87
			Worst	55.67	47.11	98.85	**40.11**	41.00
			Std Dev	11.55	10.18	18.76	**9.38**	9.39
	70	[56,84]	Best	7.49	6.18	6.63	**4.98**	5.18
			Average	32.01	29.25	40.85	**24.92**	25.08
			Worst	51.53	47.36	93.43	40.86	**40.08**
			Std Dev	11.07	9.87	18.56	**9.28**	**9.28**
	80	[64,96]	Best	8.26	7.51	5.94	**5.05**	6.07
			Average	31.49	29.10	41.10	**25.16**	25.32
			Worst	50.58	45.83	92.11	41.67	**40.27**
			Std Dev	10.57	9.63	18.53	**9.01**	9.10
Average $U(2, 10)$ machines				31.93	27.72	40.15	**22.10**	22.25
Total average				36.93	33.29	38.64	**26.73**	27.50

in seconds. The run time is limited to one hour of CPU time (3.600 seconds), and results are depicted in Table 3 are registered. The dash (-) means that the corresponding value is not found. Finally, after comparing and analyzing the MIP, JB, CDH, H1 and H2 the best results obtained are highlighted in the Table.

The GAP is calculated considering the best lower bound found when comparing the lower bounds of Complete Model and Hybrid Lagrangian Metaheuristic Framework. It is worth highlighting that the constructive heuristic JB is proposed by Chen and Lee (2009) and CDH heuristic is proposed by Cota et al. (2016).

The MIP results show that the proposed model is efficient to solve to optimality only small instances. Furthermore, for the larger instances the model found no solution. These results expose the difficulty of solving time-indexed models. In the time-indexed problems, the number of variables is proportional to the time horizon, and the higher the number of jobs the greater the horizon of time to sequence them, increasing the computational effort to solve these problems. This fact justifies the proposal of the hybrid constructive heuristics to solve instances with a greater number of jobs.

Results show that H1 and H2 performs better than heuristics JB and CDH. Based on the Table 3, the heuristic H2 showed GAP results more efficients. Comparing the average GAPs we can observe that the heuristic H2 has its better performance when $n < m$ for small and large instances. When $n > m$, the heuristic H2 gets better for small cases, while H1 presents better results for large instances. The same behavior is noticed when $n = m$, H2 is best for small cases, and H1 for larger ones. The heuristic JB had no best GAP in any instance. As for the CDH heuristic, the GAPs found were high, which was already expected, since the heuristic was developed for a more general machine's scenario.

Table 4 compares the lower bounds obtained for each method. As it is possible to see, not only the hybrid approach achieves the best upper bounds, but also it systematically generates the best lower bounds, proving to be a very efficient and complete method. It is important to point out, that the Hybrid Lagrangian approach significantly improves the linear-relaxation bounds (column LR), even having polynomially solvable subproblems. This occurs because each time we improve the upper bound (makespan) we can transform the original problem, reducing the number of variables, allowing to improve the previous linear bound.

6.2. Results for the multi-dock problem

6.2.1. Instances generation

To investigate the performance of the proposed heuristics and the heuristics presented in literature, we generate artificial instances varying the number of jobs and machines, as performed in Chen and Song (2009). The heuristics are coded in C++ and solved using AMPL and CPLEX 12.4[2]. Tests were performed on a single thread in Intel Xeon-Silver 4110 (2.1 gigahertz/8-core/85 watt) with 64 gigabyte memory and Linux operational system.

The proposed heuristics H1 and H2 are compared with the best heuristics in Chen and Song (2009), called JRH and JLPTH, and with the CDH heuristic (Cota et al., 2016). We consider values of n_1 equal to 20, 30, 40, 50, 60, 70 and 80, and values of n_2 equal to integers in $[0.8n_1, 1.2n_1]$. We examine five groups of instances. The first three groups consider the same number of machines in each stage, 2, 4, and 10. The last two groups of instances use a random number of machines in each stage, selected from a discrete uniform distribution $U(2, 4)$ and $U(2, 10)$, respectively. Processing times were generated with a discrete uniform distribution $U(10, 100)$. A job $j \in J_1$, $J_1 = \{1, 2, \dots, n_1\}$ has a probability of 50% to belong to the set S_j of each job $j \in J_2$, $J_2 = \{1, 2, \dots, n_2\}$. We generate 300 instances for each combination of number of jobs and number of machines, resulting in a total of 10.500 instances.

6.2.2. Computational results

In Table 5 we report computational results obtained with heuristics JRH, JLPTH, CDH, H1 and H2 for large instances. We coded the JRH and JLPTH heuristics in C++ following the description found in Chen and Song (2009). For each combination of number of jobs and number of machines and for each heuristic we report best, average and worst percentage Loss over 300 instances, along with standard deviations. In order to evaluate the heuristics proposed we used 'Loss' as the criterion to balance the quality of each heuristic algorithm, Loss = (makespan-lower bound)/lower bound, as defined by Chen and Song (2009). Boldface indicates which algorithm obtained the best average result for each set of instances.

Results show that for all instances sizes, the proposed heuristics H1 and H2 performs better than the heuristics JRH, JLPTH and CDH, for all tested cases (Best, Average, Worst and Std Dev). It is noted that the total average Loss value is considerably lower for H1 and H2.

In this experiment, we are comparing two types of methodologies, on the one hand, we have simple polynomial time heuristics, they are very easy to code, and they are very fast. On the other hand, we have a more complicated method, that will undoubtedly need specific expertise to be implemented. Still, this methodology is very fast, and able to present an average of 10% of makespan benefit independently of the instance tested, reaching in some cases near 20%. It is important to quantify that in an annual operation of 250 days, a 10% difference in the makespan could represent a benefit of 25 days per year.

The heuristic H1 presents a better average result than H2. Furthermore, when the number of machines increases (easier instances) the H2 heuristic performs slightly better than H1 (10 machines). However, when the number of machines decreases (more difficult instances) the performance of the H1 becomes better.

Figs. 2 and 3 summarizes the results. As previously mentioned, we can see that H1 and H2 present the lowest values in all cases. In the Figure, it is possible to see that H1 dominates the results having better Loss values in most cases. Another interesting point is that the proposed heuristics H1 and H2 have similar behavior to JRH heuristic, this may be because these heuristics arrange jobs in the second stage in increasing order of ready times.

7. Conclusions

In this work, we develop efficient ways to solve the cross-docking flow shop scheduling problem. We analyze a time-indexed formulation and a Hybrid Lagrangian Metaheuristic Framework. As expected, the performance of the MIP is strongly dependent on the instances size, being not able to solve the problem with medium and large dimensions.

For the 2-dock case, even having polynomially solvable Lagrangian subproblems, through a series of cuts, the method improves the model linear-relaxation bound. The Hybrid Lagrangian Metaheuristic shows an efficient performance, obtaining tight bounds in reduced computational time.

The heuristics proved to work very well with the Lagrangian approach, finding good solutions, and outperforming previous results. The heuristic H2 gives the best average GAP and Loss results followed by the heuristic H1.

In the generalized version, the subproblems of the Lagrangian relaxation are NP-hard, and we work with their linear relaxations.

[2] All instances and codes are available at https://github.com/GabrielaBragaFonseca/Cross-docking-Problems.

The Lagrangian heuristics obtain excellent results improving the performance of previous efforts. As future research directions, we are working on new strategies for solving their subproblems and how to integrate them into the Lagrangian framework.

Acknowledgments

This research was partially funded by FAPEMIG, CAPES and CNPq, Brazil. MGR acknowledges support from FUNDEP.

Appendix A. Considerations about WSPT-TRD

Theorem 2. *Considering the problem* $1||C_{\max}+\sum I_jW_j$, *the algorithm WSPT-TRD obtains an optimal solution.*

Proof 2. The objective function has two criteria, C_{max} and ΣI_jW_j. Regardless of the situation, $C_{\max}$, will always aim to allocate the jobs as soon as possible. However, the ΣI_jW_j depend on the weights values. Therefore, the proof is divided into two parts.

Part I: If $w_j^2 \geq 0$ (see Fig. 4).

In this case, the two criteria of the objective function are not in conflict. The criteria $C_{\max}$ and ΣI_jW_j aims to schedule the jobs as soon as possible. As there is no idle time between jobs, for $C_{\max}$ any sequence starting at beginning of time horizon (t_i) is optimal. In this way, the only existing criteria becomes ΣI_jW_j and its optimal WSPT rule is optimal for the problem.

Part II: If $w_j^2 < 0$.

In this case, the criteria are in conflict. $C_{\max}$ aim to schedule the jobs at beginning (t_i) of the time horizon, while ΣI_jW_j at the end (t_f). Consider an optimal sequence generated by WSPT rule with one subset S' of jobs with $-1 < \sum w_j^2 < 0$. Suppose, by contradiction, that this subset must be allocated at the end (t_f) of the time horizon. Initially the sequence is all allocated at the beginning (t_i) of the time (see Fig. 4a). When the schedule S' with $-1 < \sum w_j^2 < 0$ is moved in one unit of the time horizon, its $C_{\max}$ increase one unit, but ΣI_jW_j decrease $\sum w_j^2$. As $C_{\max}$ - $\sum w_j^2 > 0$ the objective function increase and the optimality is contradicted. Therefore, the $\sum w_j^2$ must be less or equal to 1 for allocate the jobs at the end (see Fig. 4b). It should be noted that for the case $\sum w_j^2 = -1$, these jobs are indifferent. □

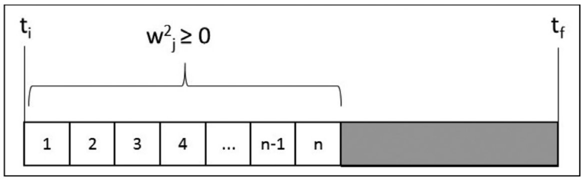
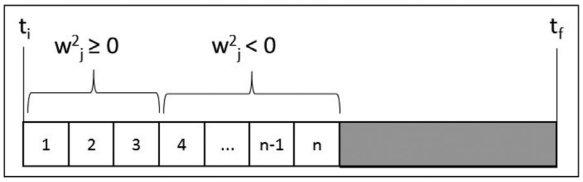
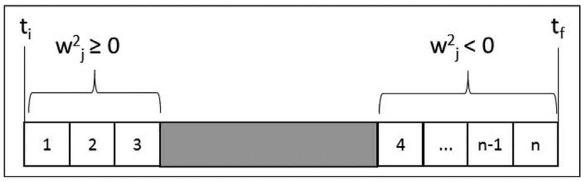


Fig. 4. Sequence with positive weights.



(a) Initial Schedule



(b) Optimal Schedule

Fig. 5. Sequence with negative and positive weights.

Appendix B. Sequential Parameter Optimization Toolbox (SPOT)

Due to its high efficiency, the Sequential Parameter Optimization Toolbox (SPOT) is the defined method to determine the parameters vector of volume algorithm. A parameters vector comprises the values of different parameters of the volume algorithm, i.e. it represents a candidate solution of a parameter setting. SPOT is an implementation of the Sequential Parameter Optimization (SPO), which is an iterative model-based method of tuning algorithms. The tuning process is based on the parameters' data, and their utility delivered by the performance of the volume algorithm. SPO performs a multi-stage procedure where the model is updated at each iteration with a set of new vectors and new predictions of utility in order to improve the algorithm's efficiency.

The goal of SPOT is the determination of good parameters settings for heuristic algorithms. It provides statistical tools for analyzing and understanding algorithm's performance. SPOT is implemented as a R package, and is available in the R archive network at http://cran.r-project.org/web/packages/SPOT/index.html. Further explanation about SPOT can be obtained from Bartz-Beielstein (2010), which provides an exemplification on how SPOT can be used for automatic and iterative tuning. Bartz-Beielstein and Zaefferer (2012) also give an introductory overview about tuning with SPOT.

The key elements of the SPOT methodology are algorithm design (D_a) and problem design (D_p). The first one defines ranges of parameter values that influence the behavior of an algorithm, such as crossover rate. These parameters are treated as variables $p_a \in D_a$ in the tuning algorithm, where p_a represents the vector of parameters settings. D_p refers to variables related to the tuning optimization problem, e.g., the search space dimension.

SPOT is composed by two phases: the build of the model and its sequential improvement. Phase 1 determinates an initial designs population from the algorithm's parameter space. The observed algorithm is run k times for each design, where k is the number of repetitions performed for each parameter setting and is increased in each run. Phase 2 leads to the efficiency of the approach and is characterized by the following loop:

- Update the model given the obtained data;
- Generate design points and predict their utility by sampling the model;
- Choose the best design vectors and run $k+1$ times the algorithm for each of them;
- New design points are added to the population and the loop restarts if the termination criteria is not reached.

The loop simulates the use of different parameters settings, it is subject to interactions between parameters and random effects into the experiment. SPOT uses results from algorithm runs to build up a meta model to tune algorithms in a reproducible way.

B1. Experimental setup

Classical works about SPOT define three different layers to analyze parameter tuning. The first one is the *application layer*, considered in our case as the two-machine flow shop problem with cross-docking constraints. The objective function and the problem parameters are defined at this layer. The *algorithm layer*, is related to the representation of the heuristic algorithm and its required parameters are the ones that determine the algorithm's performance. And finally, the *design layer*, where is the tuning

Table 6
Sampling of instances.

Type	Instances
Y1	5-3-4-1
Y2	5-4-4-1
Y3	5-5-4-1
Y4	5-6-4-1
Y5	5-7-4-1
Y6	10-6-9-1
Y7	10-8-9-1
Y8	10-10-9-1
Y9	10-12-9-1
Y10	10-14-9-1
Y11	5-3-4-2
Y12	5-4-4-2
Y13	5-5-4-2
Y14	5-6-4-2
Y15	5-7-4-2
Y16	10-6-9-2
Y17	10-8-9-2
Y18	10-10-9-2
Y19	10-12-9-2
Y20	10-14-9-2

Table 7
Parameter bounds for tuning the volume algorithm.

Parameter	Lower Bound	Upper Bound	Type
π	0.0005	0.0100	FLOAT
MaxWaste	5	30	INT
factor	0.1	1.0	FLOAT
$\alpha_{\max}$	0.20	0.95	FLOAT
st	0.6	1.8	FLOAT
α	0.05	0.90	FLOAT
yellow	0.20	0.95	FLOAT
green	1.0	2.0	FLOAT

method, tries to find good parameter settings for the algorithm layer.

In this way we face two optimization problems: problem solving and parameter tuning. The problem solving covers the application layer and the volume algorithm of the algorithm layer and aims to find an optimal solution for the problem. The parameters tuning uses a tuning method to find the best parameter values for the volume algorithm based on lower bounds found. The fitness value is the quality measure of the first optimization, which depends on the problem instance to be solved. Utility is the quality measure for parameter tuning, which reflects the performance of the volume algorithm for a vector of parameters.

Thereby, the experimental setup consists of an application layer represented by twenty instances of the two-machine flow shop problem, showed on Table 6. To define the region of interest (ROI) of the tuning algorithm, the type and the lower and upper bounds of the volume algorithm's parameters are summarized on Table 7.

B2. Results of the experiment

The SPOT algorithm, implemented in R package, is connected with the volume algorithm implemented on C++ with the aid of one callString, as presented below. The computer used in the tests is an Intel (R) Xeon (R) CPU X5690 @ 3.47 GHz with 24 processors, 132 GB of RAM, and Ubuntu Linux operating system.

```
callString ← paste("./HgR ", π, MaxWaste, factor, α_max, st, α,
yellow, green)
call ⟵ system(callString, intern = TRUE)
#read the results
y = read.table("Results.txt")
```

Table 8
Four best parameters settings for the volume algorithm.

Iteration	Parameter								Solution value of the instances																			
	π	MaxWaste	factor	$\alpha_{\max}$	st	α	yellow	green	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8	Y9	Y10	Y11	Y12	Y13	Y14	Y15	Y16	Y17	Y18	Y19	Y20
501	0.00708	28	0.22871	0.33265	1.01875	0.21709	0.80291	1.13994	24	14	30	20	30	36	46	56	60	66	155	137	191	328	414	337	537	558	629	793
502	0.00527	9	0.34752	0.75030	1.71397	0.73405	0.58396	1.56980	24	15	29	21	30	37	46	56	60	66	157	154	191	341	414	337	563	558	629	793
503	0.00688	20	0.62855	0.83046	1.17734	0.28130	0.60572	1.40758	25	15	30	21	30	37	46	56	60	66	118	136	191	307	414	337	531	558	629	793
504	0.00679	24	0.87753	0.30337	1.41179	0.08300	0.48159	1.56487	26	15	30	20	30	32	46	56	60	66	95	136	191	307	414	337	531	558	629	793

The Table 8 presents the four best results found by SPOT method for the volume algorithm. The first column indicates the iteration of SPOT, the second one presents the indicated parameters vector and the third column indicate the solution value obtained by the tested instances (Lower Bounds).

The best parameters vector indicated for the volume algorithm, that generated the best lower bounds is $\pi = 0.00679$, MaxWaste $= 24$, factor $= 0.87753$, $\alpha_{\max} = 0.30337$, st $= 1.41179$, $\alpha = 0.08300$, yellow $= 0.48159$ and green $= 1.56487$.

References

Alba, E. (2005). *Parallel metaheuristics: A new class of algorithms*: 47. Wiley.

Alpan, G., Bauchau, S., Larbi, R., & Penz, B. (2008). Optimal operations scheduling in a crossdock with multi strip and multi stack doors. In *Proceedings of the thirty-eight international conference on computers and industrial engineering (CIE38): 2* (pp. 1168–1176).

Alpan, G., Larbi, R., & Penz, B. (2011). A bounded dynamic programming approach to schedule operations in a cross docking platform. *Computers & Industrial Engineering, 60, (3)*, 385–396.

Arabani, A. B., Ghomi, S. F., & Zandieh, M. (2011). Meta-heuristics implementation for scheduling of trucks in a cross-docking system with temporary storage. *Expert Systems with Applications, 38*(3), 1964–1979.

Araújo, D. P. M., & Melo, B. M. R. (2010). Heurísticas construtivas para o sequenciamento de caminhões em centros de cross-docking Master's thesisUniversidade Federal de Minas Gerais.

Arroyo, J., Nunes, G., & Kamke, E. (2009). Iterative local search heuristic for the single machine scheduling problem with sequence dependent setup times and due datesIn Proceedings of the ninth international conference on hybrid intelligent systems, 1, 505–510.

Bahiense, L., Maculan, N., & Sagastizabal, C. (2002). The volume algorithm revised: Relation with bundle methods. *Mathematical Programming, 94*, 41–69.

Barahona, F., & Anbil, R. (2000). The volume algorithm: Producing primal solutions with a subgradient method. *Mathematical Programming, 87*(3), 385–399.

Barahona, F., & Ladányi, L. (2006). Branch and cut based on the volume algorithm: Steiner trees in graphs and max-cut. *RAIRO Operations Research, 40*, 53–73.

Bartholdi, J., & Gue, K. (2002). Reducing labor costs in an LTL crossdocking terminal. *Operations Research, 48*(6), 823–832.

Bartz-Beielstein, T. (2010). SPOT: An R package for automatic and interactive tuning of optimization algorithms by sequential parameter optimization. arXiv:1006.4645v1.

Bartz-Beielstein, T., & Zaefferer, M. (2012). A gentle introduction to sequential parameter optimization. *Technical Report 2*. Bibliothek der Fachhochschule Koeln.

Belle, J. V., Valckenaers, P., & Cattrysse, D. (2012). Cross-docking: State of the art. *Omega, 40*, 827–846.

Blum, C., Puchinger, J., Raidl, G., & Roli, A. (2011). Hybrid metaheuristics in combinatorial optimization: A survey. *Applied Soft Computing, 11*(6), 4135–4151.

Boschetti, M., & Maniezzo, V. (2009). Benders decomposition, Lagrangean relaxation and metaheuristic design. *Journal of Heuristics, 15*(3), 283–312.

Boysen, N. (2010). Truck scheduling at zero-inventory cross docking terminals. *Computers & Operations Research, 37*(1), 32–41.

Boysen, N., & Fliedner, M. (2010). Cross dock scheduling: Classification, literature review and research agenda. *Omega, 38*(6), 413–422.

Boysen, N., Fliedner, M., & Scholl, A. (2010). Scheduling inbound and outbound trucks at crossdocking terminals. *OR Spectrum, 32*(1), 135–161.

Bozer, Y., & Carlo, H. (2008). Optimizing inbound and outbound door assignments in less-than-truckload crossdocks. *IIE Transactions, 40*, 1007–1018.

Campbell, J. (1994). A survey of network hub location. *Studies in Locational Analysis, 6*, 31–49.

Chen, F., & Lee, C.-Y. (2009). Minimizing the makespan in a two-machine cross–docking flow shop problem. *European Journal of Operational Research, 193*(1), 59–72.

Chen, F., & Song, K. (2009). Minimizing makespan in two-stage hybrid cross docking scheduling problem. *Computers & Operations Research, 36*(6), 2066–2073.

Chen, J. (2007). A hybrid heuristic for the uncapacited single allocation hub location problem. *Omega, 35*, 211–220.

Chen, P., Guo, Y., Lim, A., & Rodrigues, B. (2006). Multiple crossdocks with inventory and time windows. *Computers and Operations Research, 33*, 43–46.

Clausen, J., Cordeau, J., Laporte, G., Wen, M., & J. , L. (2009). Vehicle routing scheduling with cross-docking. *Journal of the Operational Research Society, 60*, 1708–1718.

Cota, P. M., Gimenez, B. M., Araújo, D. P., Nogueira, T. H., de Souza, M. C., & Ravetti, M. G. (2016). Time-indexed formulation and polynomial time heuristic for a multi-dock truck scheduling problem in a cross-docking centre. *Computers & Industrial Engineering, 95*, 135–143.

Forger, G. (1995). Ups starts worlds premiere cross-docking operation. *Modern material handling, 36*(8), 36–38.

Fukuda, E. H. (2007). Algoritmo de volume e otimização não diferenciável Master's thesis. USP.

Gue, K. R. (1999). The effects of trailer scheduling on the layout of freight terminals. *Transportation Science, 33, (4)*, 419–428.

Kim, C., Yang, K. H., & Kim, J. (2008). A strategy for third-party logistics systems: a case analysis using the blue ocean strategy. *Omega, 36*(4), 522–534.

Klose, A., & Drexl, A. (2005). Facility location models for distribution system design. *European Journal of Operational Research, 162*, 4–29.

Ladier, A.-L., & Alpan, G. (2016). Cross-docking operations: Current research versus industry practice. *Omega, 62*, 145–162.

Larbi, R., Alpan, G., Baptiste, P., & Penz, B. (2011). Scheduling cross docking operations under full, partial and no information on inbound arrivals. *Computers & Operations Research, 38*(6), 889–900.

Larbi, R., Alpan, G., & Penz, B. (2009). Scheduling transshipment operations in a multiple inbound and outbound door crossdock. In *Proceedings of the thirty-ninth international conference on computers and industrial engineering (CIE39)*. Troyes, France.

Lee, K., Lee, Y., & Jung, J. (2008). Positioning of goods in a cross-docking environment. *Computers & Industrial Engineering, 54*(3), 677–689.

Lemaréchal, C. (1975). An extension of Davidon methods to non differentiable problems. In *Nondifferentiable optimization* (pp. 95–109). Springer.

Lemaréchal, C. (2001). Lagrangean relaxation. In M. Jünger, & D. Naddef (Eds.), *Computational combinatorial optimization. Lecture notes in computer science*: 2241. Berlin, Heidelberg: Springer.

Lemaréchal, C. (1989). Chapter VII Nondifferentiable optimization. *Handbooks in Operations Research and Management Science, 1*, 529–572.

Lima, M. F. (2014). O problema de sequenciamento de caminhões numa estação de cross-docking com duas máquinas: Formulação indexada no tempo, relaxação lagrangeana e geração de colunas Master's thesis. Universidade Federal de Minas Gerais.

McWilliams, D. L. (2010). Iterative improvement to solve the parcel hub scheduling problem. *Computers & Operations Research, 59*(1), 136–144.

Miao, Z., Lim, A., & Ma, H. (2009). Truck dock assignment problem with operational time constraint within crossdocks. *European Journal of Operational Research, 192*(1), 105–115.

Nawaz, M., Enscore, J., & Ham, I. (1983). A heuristic algorithm for the m-machine, n-job flow-shop sequencing problem. *Omega, 11*(1), 91–95.

Nogueira, T. H. (2014). Single machine scheduling problems with sequence-dependent setup times Ph.D. thesis. Universidade Federal de Minas Gerais.

Oh, Y., Hwang, H., Chab, C., & Lee, S. (2006). A dock-door assignment problem for the Korean mail distribution center. *Computers & Industrial Engineering, 51*(2), 288–296.

Paula, M., Mateus, G., & Ravetti, M. (2010). A non-delayed relax-and-cut algorithm for scheduling problems with parallel machines, due dates and sequence-dependent setup times. *Computers & Operations Research, 37*(5), 938–949.

Pinedo, M. L. (2008). *Scheduling: theory, algorithms, and systems*. Springer.

Pirkwieser, S., Raidl, G., & Puchinger, J. (2007). Combining Lagrangian decomposition with an evolutionary algorithm for the knapsack constrained maximum spanning tree problem. In *Evolutionary computation in combinatorial optimization* (pp. 176–187). Springer.

Puchinger, J., & Raidl, G. (2005). Combining metaheuristics and exact algorithms in combinatorial optimization: A survey and classification. *Artificial intelligence and knowledge engineering applications: A bioinspired approach* (pp. 41–53). Springer.

Smith, W. E. (1956). Various optimizers for single-stage production. *Naval Research Logistics, 3*, 59–66.

Stalk, G., Evans, P., & Shulman, L. E. (1991). Competing on capabilities: The new rules of corporate strategy. *Harvard Business Review, 70*(2), 57–69.

Stutzle, T. (1998). *Local search algorithms for combinatorial problems*. Darmstadt University of Technology Ph.D. thesis.

Tsui, L., & Chang, C. (1990). A microcomputer based decision support tool for assigning dock doors in freight yards. *Computers & Industrial Engineering, 19*, 309–312.

Tsui, L., & Chang, C. (1992). An optimal solution to a dock door assignment problem. *Computers & Industrial Engineering, 23*, 283–286.

Vahdani, B., & Zandieh, M. (2010). Scheduling trucks in cross-docking systems: Robust metaheuristics. *Computers & Operations Research, 58*(1), 12–24.

Witt, C. E. (1998). Crossdocking: Concepts demand choice. *Material Handling Engineering, 53*(7), 44–49.

Wolfe, P. (1975). A method of conjugate subgradients for minimizing nondifferentiable functions. *Nondifferentiable optimization* (pp. 145–173). Springer.