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Barrier-mediated predator-prey dynamics

von

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Contents

1 Introduction

The ability to find food and escape from a predator is crucial for an animal to survive. Aside from stamina, the speed of both prey and predator determines the outcome of a chasing process. Ideally, catching happens when the prey is slower than the predator, and if the prey is faster than the predator, it will escape. However, things are much more complex in an inhomogeneous landscape, where their local speed depends on the details of the environment.

One of the few examples in the macroscopic world is the wolf and the deer. A rich forest with rivers would make it harder for the wolf to chase the deer. As the deer will have no problem crossing a river, the wolf might have to swim slowly to reach the other side. The obstacle couples differently with the predator and the prey, deciding all the outcomes of the chase. Other examples of predator and prey can also be found in the microscopic world, such as droplets following each other. This involves an unanimated prey-predator system, that is designed to use synthetic colloidal particles that interact in a non-reciprocal way. [1]

Here, we observe this phenomenon by determining the equation for the problem and then solving it numerically to determine a catching, which involves time and position. This will be discussed further in this report.

2 Ideal Predator-Prey Model

An ideal model of both prey and predator is proposed here through a one-dimensional model, which is enough to show an ideal framework that can classify different characteristic states for both catching and escape in the presence of an obstacle. Our starting point is Newton's equation, which reads

$$m\frac{d\mathbf{u}(t)}{dt} = -\gamma\mathbf{u}(t) + \gamma v_0\vec{e} + \mathbf{F}(\mathbf{r}, t). \tag{1}$$

Since the particle is moving in an inhomogeneous landscape, friction between the particle and its environment can occur. This friction model is represented by $-\gamma\mathbf{u}(t)$, where γ is the friction coefficient and $\mathbf{u}(t)$ the velocity of the particle given by its time. The self-propulsion $\gamma v_0\vec{e}$ describes what happens after the particle is influenced by the friction, as it can move towards different directions at a different speed. Therefore, v_0 is its self-propulsion velocity and $\vec{e}$ the particle orientation. Further, m is the mass of the particle, $\mathbf{r}$ its position, and $\mathbf{F}(\mathbf{r}, t)$ is an external force. Micron-sized objects, such as colloids, bacteria, or microalgae, have a relatively low Reynolds number that compares the inertial and viscous effects. In our case, this means that the mass m is much smaller than the friction γ such that $m/\gamma \ll 1$. Hence, the left-side inertial term in Eq.(1) can be neglected and written as

$$\mathbf{u}(t) = v_0\vec{e} + \frac{1}{\gamma}\mathbf{F}(\mathbf{r}, t), \tag{2}$$

which is an overdamped equation of motion. The external force $\frac{1}{\gamma}\mathbf{F}(\mathbf{r}, t)$ is given by $\alpha_i x_i$ in the Eq.(4)-(5), which is the gradient of a potential, $F_i = -\nabla U_i$ of a parabolic shape, that is

$$U_i(x) = \alpha_i x_i^2. \tag{3}$$

We now model that motion of a predator and prey using Eq.(2), where both are subjected to a barrier. Explicitly, we have for the position of the prey x_1 and predator x_2 the following

$$\dot{x_1} = v_1 + \alpha_1 x_1, \tag{4}$$

$$\dot{x_2} = v_2 + \alpha_2 x_2, \tag{5}$$

where v_i with $i = 1, 2$ are the self-propulsion speed and α_i are coupling constants to the respective potential barrier. A schematic of the potential barrier is shown in Fig.(1).

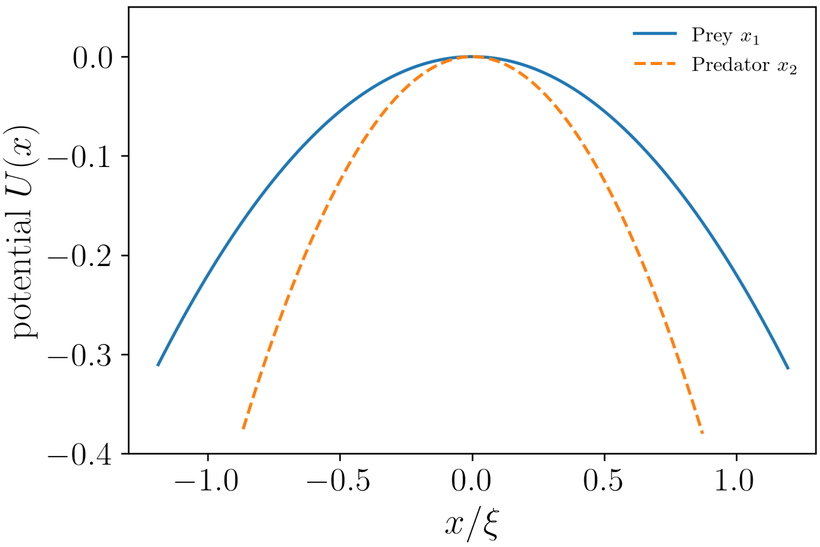

Figure 1: Potentials of the prey and the predator, respectively by their positions x/ξ

2.1 Analytic Solution

To determine the positions x_i with $i = 1, 2$ of the prey and predator, we analytically solve the linear differential equations Eq.(4)-(5). The equations are first-order equations with variable coefficients. Therefore, these are the steps that are used to find $x(t)$:

1. The first step is to rewrite the equation into a homogeneous equation. Here we can assume that $v_i = 0$ and $\dot{x}_i = \alpha_i x_i$. Integration gives:

$$\int \frac{1}{x_i} dx_i = \int \alpha_i \, dt,$$
$$\ln x_i = \alpha_i t + c_i,$$
$$x_i = e^{\alpha_i t + c} = \tilde{C} \cdot e^{\alpha_i t}.$$

Thus, the general solution of the homogenous equation is

$$x_{i,hom}(t) = \tilde{C} \cdot e^{\alpha t}. \tag{6}$$

2. We can determine the special solution of the non-homogenous equation by assuming that $x_{i,sp}(t) = \tilde{C}(t) \cdot e^{\alpha_i t}$. The first derivative of this equation is equal to the original equation. As a result, the function $\tilde{C}(t)$ can be calculated by

$$\dot{x}_i(t) = \dot{\tilde{C}}(t) \cdot e^{\alpha_i t} + \alpha_i \tilde{C} \cdot e^{\alpha_i t} \overset{!}{=} v_i + \alpha_i x_i.$$

The result from Eq.(6) gives

$$\begin{aligned}\dot{\tilde{C}}(t)\cdot e^{\alpha_i t}+\cancel{\alpha_i\tilde{C}\cdot e^{\alpha_i t}} &= v_i+\cancel{\alpha_i\tilde{C}(t)\cdot e^{\alpha_i t}}\\ \tilde{C}(t) &= v_i\int e^{-\alpha_i t}\,dt=-\frac{v_i}{\alpha_i}\cdot e^{-\alpha_i t},\end{aligned}$$

which yields the special solution

$$x_{i,sp}(t)=\tilde{C}(t)\cdot e^{\alpha_i t}=-\frac{v_i}{\alpha}\cdot e^{-\alpha_i t}\cdot e^{\alpha_i t}=-\frac{v_i}{\alpha_i}. \tag{7}$$

3. Therefore, the general solutions to the non-homogeneous equation is the sum of the solution from Eq.(6) and Eq.(7), which reads

$$x_i(t)=\tilde{C}\cdot e^{\alpha_i t}-\frac{v_i}{\alpha_i}=\frac{1}{\alpha_i}\Big(\alpha_i\cdot\tilde{C}\cdot e^{\alpha_i t}-v_i\Big). \tag{8}$$

4. The initial condition $x(0)=\frac{1}{\alpha_i}(\alpha_i\cdot\tilde{C}-v_i)$ is required to be fulfilled in this equation, that is

$$\tilde{C}=\frac{1}{\alpha_i}(v_i+\alpha_i x_i(0)).$$

We arrive at the full solution to the Eq(4)-(5) for both prey and predator

$$x_1(t)=\frac{1}{\alpha_1}\big((v_1+\alpha_1 x_1(0))e^{\alpha_1 t}-v_1\big), \tag{9}$$

$$x_2(t)=\frac{1}{\alpha_2}\big((v_2+\alpha_2 x_2(0))e^{\alpha_2 t}-v_2\big). \tag{10}$$

2.2 Catching Time and Position

To determine whether the predator has caught the prey or not, we need to be able to determine the time and position of the catch. The catching time t^* can be determined by the condition

$$x_1(t^*)=x_2(t^*). \tag{11}$$

Knowing the catching time gives us the catching position $x^*=x_{1,2}(t^*)$, which is determined through Eqs.(9)-(10). This enables us to categorise five different catching cases and five different escaping cases by considering the initial conditions and the long time limits of Eq.(9)-(10), which are summarized in Table (1).

case	description	initial conditions	catching condition
Ca. I	caught while summiting the barrier	$x_1(0) < 0, x_2(0) < x_1(0)$	$x^* > x_1(0)$
Ca. II	caught after summiting the barrier	$x_1(0) < 0, x_2(0) < x_1(0)$	$x^* > 0$
Ca. III	caught after prey summits	$x_1(0) > 0, x_2(0) \geq 0$	$x^* > 0$
Ca. IV	caught descending the barrier	$x_1(0) > 0, x_2(0) \leq 0$	$x^* > 0$
Ca. V	caught descending the barrier without summiting	$x_1(0) < 0, x_2(0) < x_1(0)$	$x^* < x_1(0)$
Es. I	both are summiting the barrier	$x_1(0) < 0, x_2(0) < x_1(0)$	$x_1(\infty) = x_2(\infty) = \infty$
Es. II	both descending in positive direction	$x_1(0) > 0, 0 < x_2(0) < x_1(0)$	$x_1(\infty) = x_2(\infty) = \infty$
Es. III	both descending in opposite directions	$x_1(0) > 0, x_2(0) < 0$	$x_1(\infty) = \infty, x_2(\infty) = -\infty$
Es. IV	both descending in negative directions	$x_1(0) < 0, x_2(0) < x_1(0)$	$x_1(\infty) = x_2(\infty) = -\infty$
Es. V	only prey is summiting the barrier	$x_1(0) < 0, x_2(0) < 0$	$x_1(\infty) = \infty, x_2(\infty) = -\infty$

Table 1: Classification of catching and escape scenarios determined by initial conditions and catching conditions. [1]

In the following, we define $\tau = 1/\alpha_2$ as a natural unit of time and $\xi = v_2/\alpha_2$ a natural length scale. These represent the physical time and length scales related to the predator.

initial condition x_1	initial condition x_2	cases	
		catching	escape
$x_1(0) = -\xi/3$	$x_2(0) = -\xi/2$	I, II, V	I
$x_1(0) = -\xi/3$	$x_2(0) = -3\xi/2$	V	V, IV
$x_1(0) = \xi/4$	$x_2(0) = -\xi/4$	III	III
$x_1(0) = \xi/2$	$x_2(0) = \xi/4$	IV	II

Table 2: Initial conditions for different catching and escape cases.

3 Numerical Results

Despite the fact that we have solved the Eqs.(4)-(5) as shown in section (2.1), it is not possible to solve the catching condition Eq.(11) analytically. Hence, we numerically evaluate Eq.(11) using Halley's method. Halley's method is the second class of the Householder's method, a root-finding algorithm, which is a higher order of Newton's Method. With Halley's method, we can find better approximations to the roots of the Eq.(11), which will give us the catching time. The method consists of an iteration sequence

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}. \tag{12}$$

The initial guess $x_{n=0}$ is crucial for determining the next updated value, which is $x_{(n=0)+1} = x_1$, which is then inserted into the Eq.(12) to find the next $(x_2, x_3, ..., x_n)$. Furthermore, $f(x)$ is the function whose root we seek, and $f'(x)$ is is the function's first derivative. The iteration is executed until the value of x_n in the function $f(x)$ reaches approximately zero. The method Eq.(12) can be found using a Taylor expansion. To evaluate Eq.(12) we need the first and second derivative of Eq.(4)-(5), which are

$$\dot{x}_i(t) = (v_i + \alpha_i x(0))e^{\alpha_i t}, \tag{13}$$

$$\ddot{x}_i(t) = \alpha_i(v_i + \alpha_i x(0))e^{\alpha_i t}. \tag{14}$$

With different parameters of velocity v_i and alpha α_i, respectively to the time, the outcome of the catching differs greatly and, in some cases, the prey manages to escape. To make things easier in the calculation, we fixate the predator's velocity v_2 and alpha α_2 as 1, which gives $\xi = v_2/\alpha_2 = 1$. However, we use a range of parameters for the prey's velocity and its coupling constant.

3.1 Predator-Prey Positions

The Figs.(2)-(5) in this section show the position of the prey and predator x/ξ, respectively by the time t. The sub-figures are classified based on their initial condition, as shown in Table (2). The parameters v_1 and α_1 are changed accordingly to the specific regions following the Table (1) to show at which time the predator catches the prey. If the predator catches the prey, they will meet at a specific time and this event is marked by the black dot. If the Eq.(9) and Eq.(10)

from the initial time $t = 0$ does not cross each other, the prey has escaped the predator and no catching occurs.

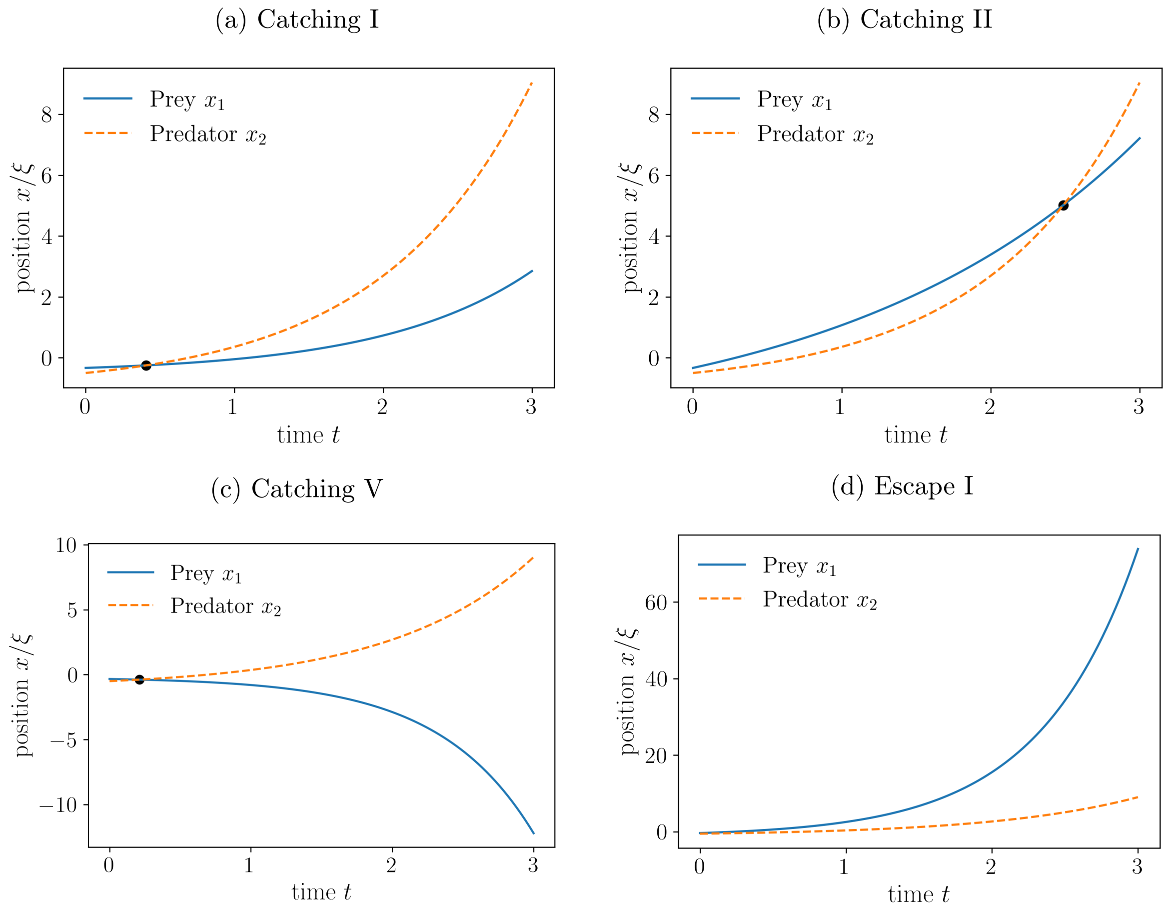

Figure 2: Predator-prey positions in with the condition $x_1(0) = -\xi/3$ and $x_2(0) = -\xi/2$. The black dot marks the catching event.

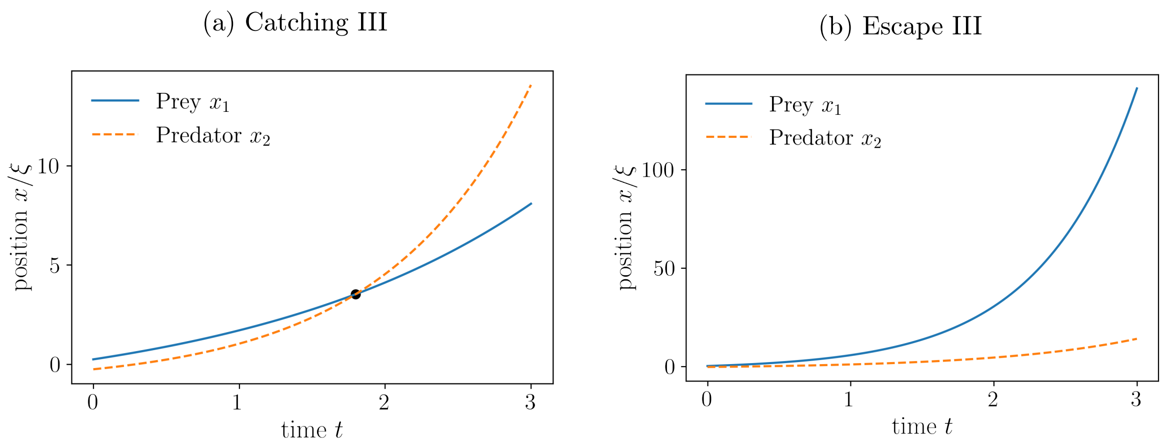

Figure 3: Predator-prey positions in with the condition $x_1(0) = \xi/4$ and $x_2(0) = -\xi/4$. The black dot marks the catching event.

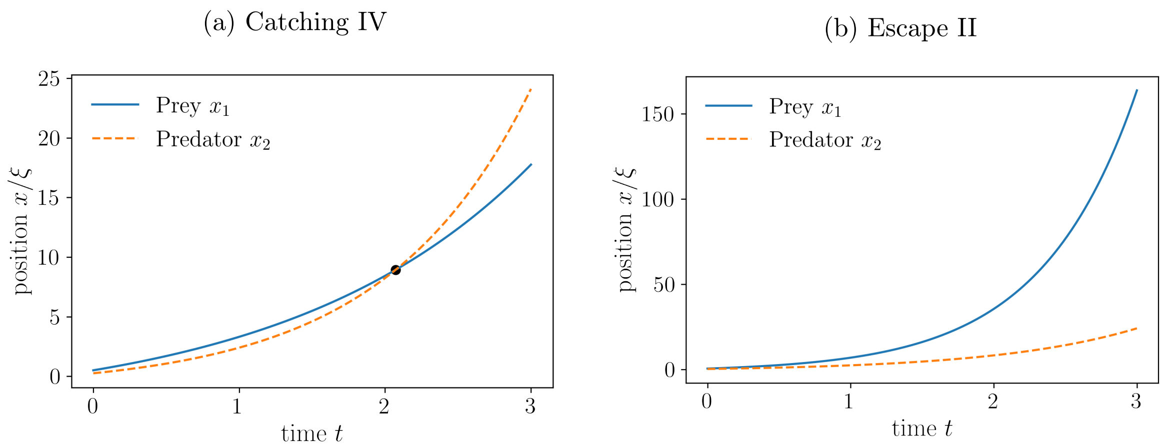

Figure 4: Predator-prey positions with the condition $x_1(0) = \xi/2$ and $x_2(0) = \xi/4$. The black dot marks the catching event.

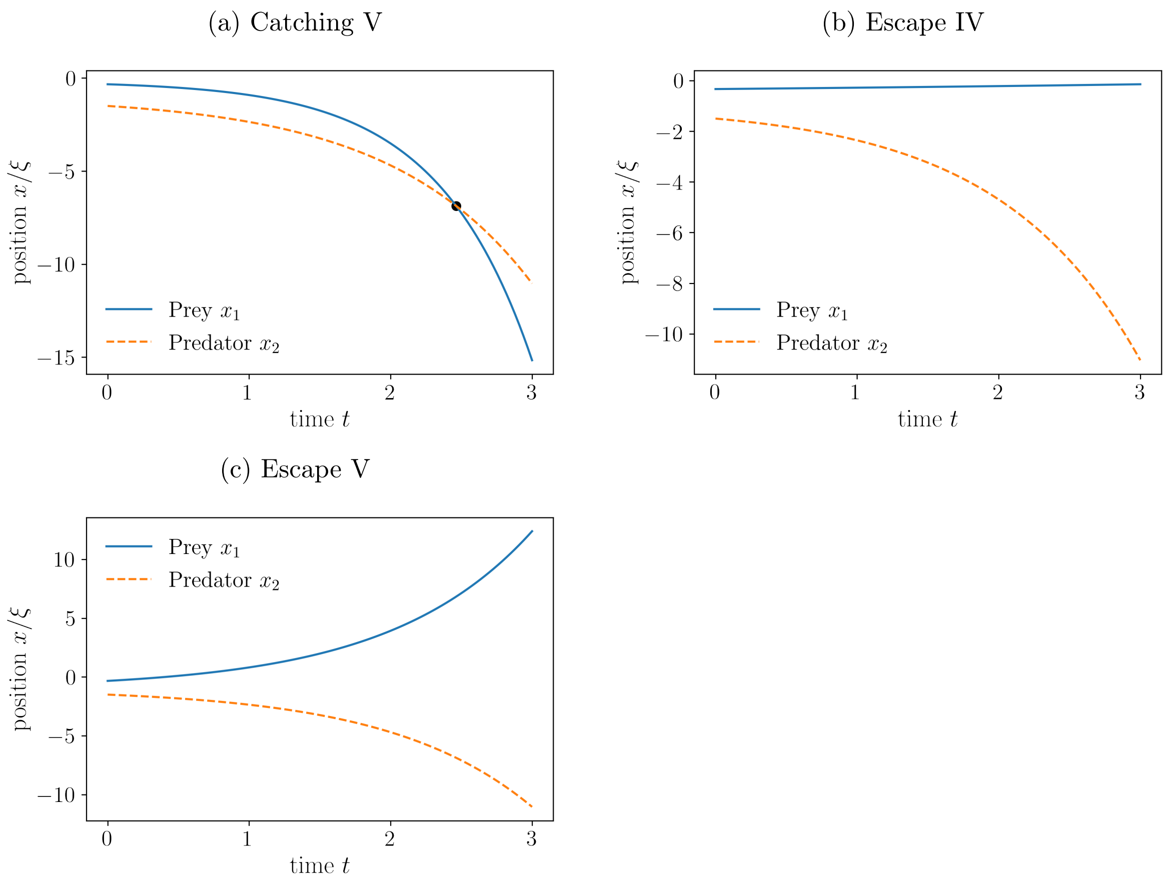

Figure 5: Predator-prey positions with the condition $x_1(0) = -\xi/3$ and $x_2(0) = -3\xi/2$. The black dot marks the catching event.

3.2 Catching and Escape

In this section, we will concentrate on catching and escaping cases with the initial conditions $x_1(0) = -\xi/3$, $x_2(0) = -\xi/2$, as shown in Table 3.

case	description	catching condition
Ca. I	caught while summiting the barrier	$x^* > x_1(0)$
Ca. II	caught after summiting the barrier	$x^* > 0$
Ca. V	caught descending the barrier without summiting	$x^* < x_1(0)$
Es. I	both are summiting the barrier	$x_1(\infty) = x_2(\infty) = \infty$

Table 3: Catching and escape with the initial condition $x_1(0) = -\xi/3$, $x_2(0) = -\xi/2$

3.2.1 Catching Time

We now examine the catching time determined by Eq.(12) for different coupling constants and self-propulsion velocity.

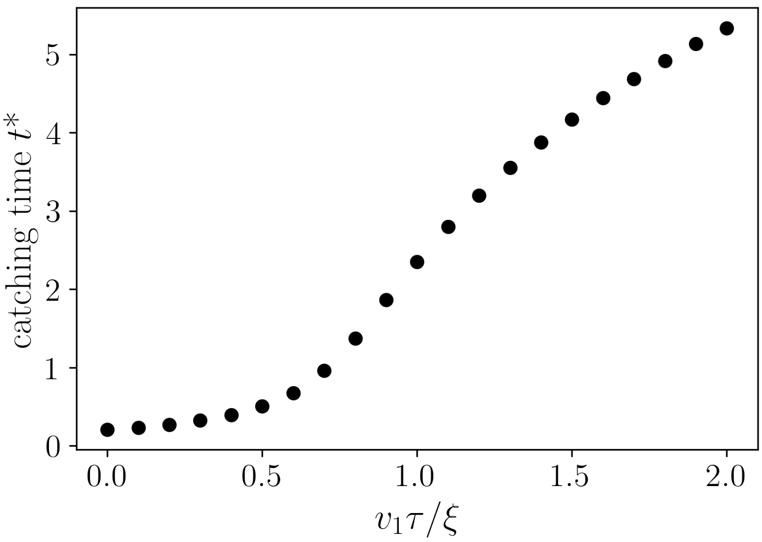

Figure 6: Catching time of prey and predator, respectively to its self-propulsion velocities, where $\alpha_1 < \alpha_2$

Fig.(6) shows the catching time, when the prey has a smaller coupling constant than the predator $\alpha_1 < \alpha_2$ with a wide range of self-propulsion velocity, where the prey is slower $v_1 < v_2$ or faster $v_1 > v_2$ than the predator. The small coupling constant α_1 value ensures that the prey will definitely be caught by the predator. The first few values of the self-propulsion velocity, when the prey is slower than the predator $v_1 < v_2$, show that catching occurs much faster. When the prey's self-propulsion velocity increases, it also increases the catching time since the predator needs to catch up to the prey.

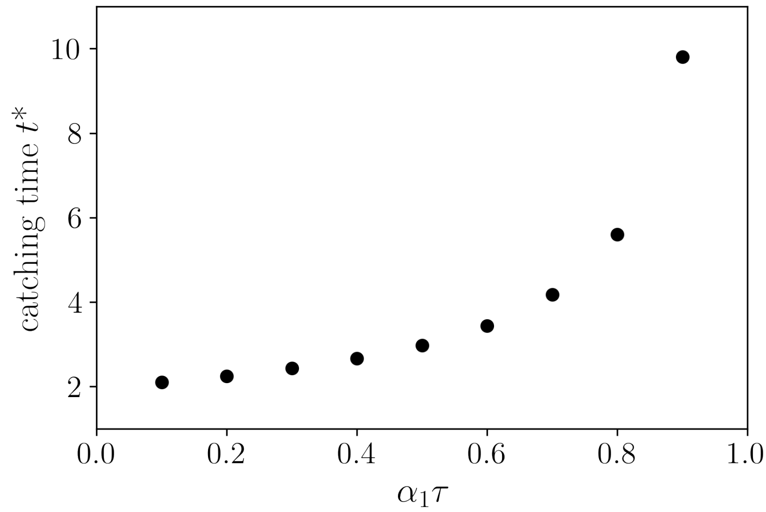

Figure 7: Catching times of prey and predator, respectively to its coupling constants, where $v_1 > v_2$

Fig.(7) shows the catching time, when we approach the coupling constant $\alpha_1 \to \alpha_2$, where the prey is faster than the predator $v_1 > v_2$. Catching occurs much faster with smaller coupling constant values, but only when $\alpha_1 < \alpha_2$, because the catching time begins to diverge at $\alpha_1 = \alpha_2 = \alpha$. An approximation of the solution Eq.(4)-(5) for barrier dominated motion is used to understand the scaling of this divergence, which yields the catching time

$$t^* \sim 1/(\alpha_1 - \alpha_2).[1] \tag{15}$$

For $\alpha_1 > \alpha_2$, the self-propulsion velocity of the predator is not sufficient anymore to catch the prey since the motion of both predator and prey is dominated by them descending the barrier. As we get closer to the divergence, the catching dynamics are dominated by the potential barrier, and the importance of self-propulsion decreases. After $\alpha_1 > \alpha_2$ the predator cannot catch up to the prey, so the prey is able to escape.

3.2.2 Catching Position

Every catching happens at a certain position around the barrier. Using the numerically calculated catching time, we can find the position of a catching. In this section, we can see through the figures that at a certain coupling constant value and velocity, where the catching happens. To see the position of the prey and predator, we compare Figs.(8)-(11) to Fig.(1) which shows the catching position on the potential barrier. A negative value represents a catching that occurs before the summit ($x^* = 0$), while a positive value represents a catching that occurs after the summit.

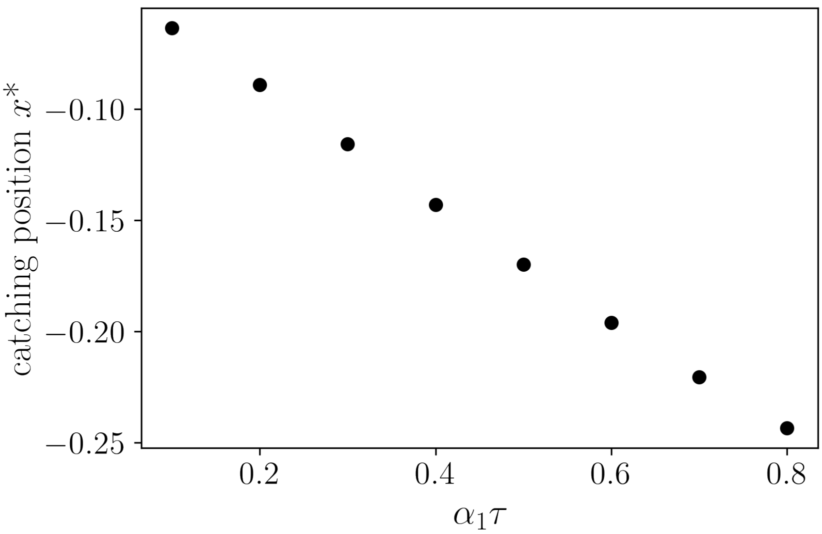

Figure 8: Catching I with increasing values of coupling constants, respectively to its catching positions and $v_1 < v_2$

The self-propulsion velocity and the coupling constant of the prey is smaller than the predator in the following Fig.(8)-(9). The catching position decreases with increasing coupling constants, as shown in Fig.(8). Furthermore, each catching happens before the initial position of the prey ($x^* > x_1(0)$). This means that the catching happens while the prey is moving towards the summit. The first catching with the smallest coupling constant happens long after the prey almost reaches the summit, and the last catching with the biggest coupling constant happens near the initial position. This catching is classified as Catching I.

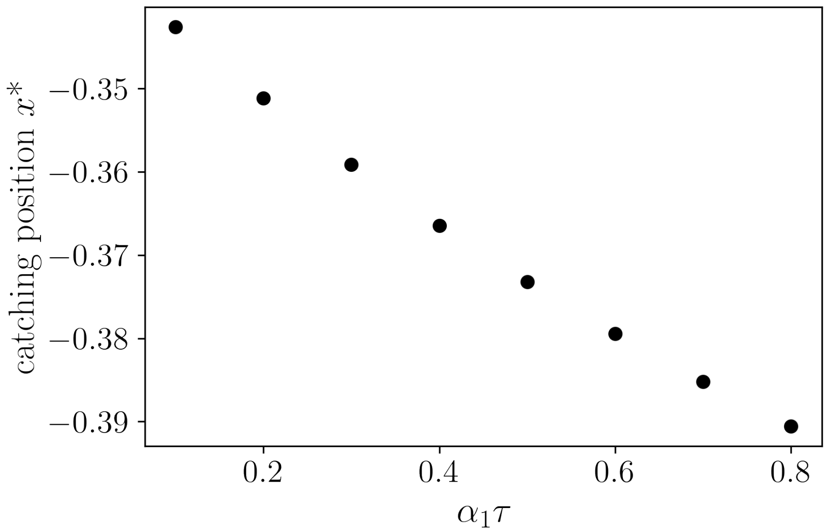

Figure 9: Catching V with increasing values of coupling constants, respectively to its catching position and $v_1 < v_2$

Fig.(9) shows that the catching position also decreases with increasing coupling constants. However, all catching occurs for values $x^* < x_1(0)$. The prey is therefore caught while moving away from the summit in the negative x-direction. The first catching in the figure happens not far from the initial position with a small coupling constant. This catching is classified into Catching V.

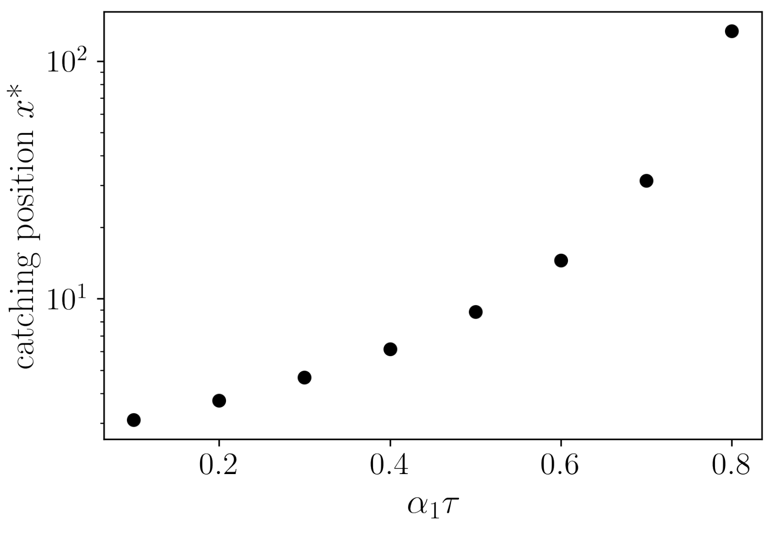

Figure 10: Catching II with increasing values of coupling constants, respectively to its position and $v_1 > v_2$

We can see, however, that in the Fig.(10), all the catching has a positive value. This shows that all catching happens after the summit. Here, the self-propulsion velocity of the prey is higher than the predator, but its coupling constant is still smaller. This is referred to as Catching II. Similar to the explanation in in (3.2.1), when the prey reaches an equal coupling constant with the predator $\alpha_1 = \alpha_2 = \alpha$, it diverges. The scaling of the relative distance between the catching as it begins to diverge $\alpha_1 \to \alpha_2$, which to first order reads

$$x_1(t) - x_2(t) \approx A(t - t^*) + \ldots, \tag{16}$$

with the prefactor A scales in the limit $\alpha_1 \to \alpha_2$ as $\ln A \sim 1/(\alpha_1 - \alpha_2)$. [1]

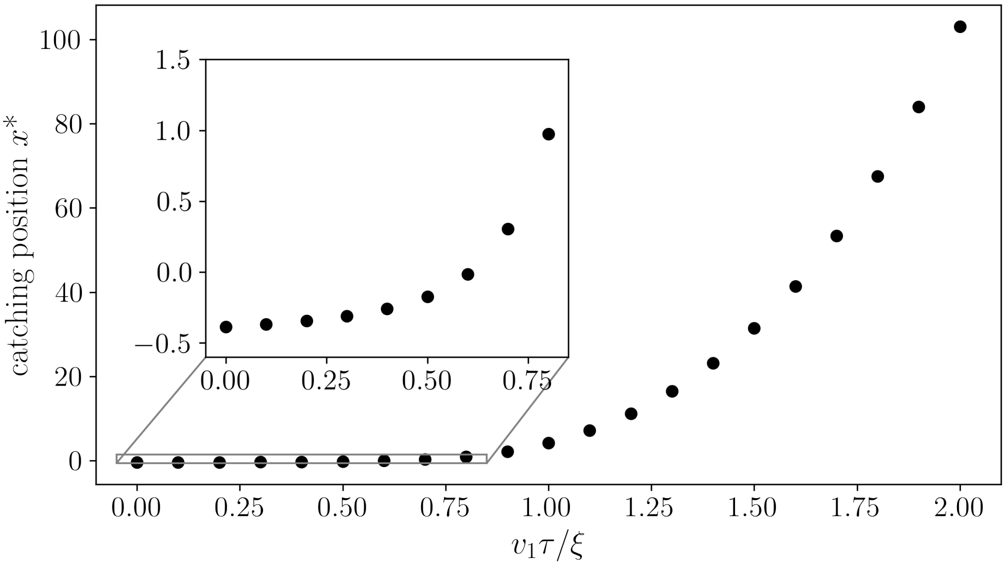

Figure 11: Increasing self-propulsion velocities, respectively to its catching position

Now we have observed all conditions with different conditions of coupling constant values. Finally, in Fig.(11) we can see the catching position in a wide range of self-propulsion velocity of the prey, except that the coupling constant of the prey is smaller than the predator. This ensures that no prey is able to escape. With smaller self-propulsion velocity, it is clear that the catching happens before the prey is able to go over the summit, and some of them are smaller than the initial condition, so the prey is moving away from the summit. Gradually, when the velocity increases, the predator will take some time to catch the prey, and it will be caught after both the prey and the predator have overcome the barrier.

3.2.3 Regions

All the information from sections (3.2.1) and (3.2.2) are now plotted on a state diagram. Fig.(12) shows the catching regions of prey and predator through the coupling constants, respectively to their self-propulsion velocities. We can see that the cases change depending on the self-propulsion velocity and coupling constant values, thus creating regions of catching and escape.

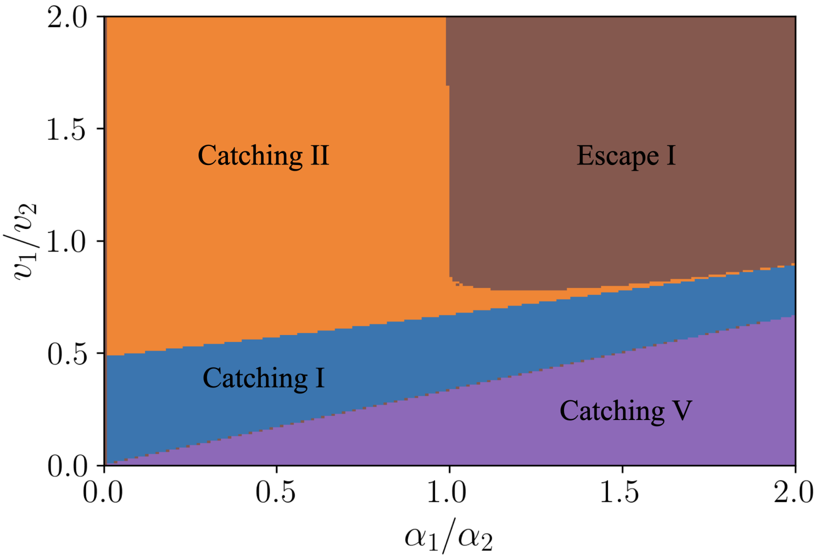

Figure 12: State diagram with initial conditions $x_1(0) = -\xi/3$ and $x_2(0) = -\xi/2$

4 Summary

The survival of prey depends not only on its coupling constants and self-propulsion velocity but also on the complexity of its environment. This obstacle is represented by a potential barrier. Analytically, we can solve the one-dimensional model for each prey and predator, which shows the ideal framework that classifies the characteristic states for both catching and escape in the presence of an obstacle. However, the time of catching can only be numerically solved, which leads us to find the prey-predator positions. The catching time and catching position are shown in the figures, respectively, to either the self-propulsion velocity or the coupling constant. These parameters, therefore, affect the catching or escape greatly. With this knowledge, we can create a state diagram that shows us the complexity that defines the different cases of catching and escape.

5 References

[1] Fabian Jan Schwarzendal and Harmut Löwen. Barrier-mediated predator-prey dynamics. *EPL*, 134(48005), 2021.