# Implementation of the cooled cylinder in FEniCS

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**A brief documentation of equations and methods**

## 1 Low-Mach equations

All flow variables in the compressible Navier–Stokes equations are expanded in orders of the Mach number, e.g. pressure (all others exactly the same):

$$p = p^{(0)} + \gamma Ma^2 p^{(1)} + \gamma^2 Ma^4 p^{(2)} + \cdots$$

. The equations are then taken with terms at the same order. The main results are:

- At leading order, the momentum equation becomes $\nabla p^{(0)} = 0$. We call $p^{(0)}$ the thermodynamic pressure.
- At next order, the momentum equation involves $u^{(0)}$, $\rho^{(0)}$ and $p^{(1)}$. The latter is therefore called the hydrodynamic pressure, because it acts on the dynamics by its gradient. $p^{(1)}$ is the only first-order variable that appears in the final system of equations.
- $p^{(0)}$ governs the relation between $T$ and $\rho$ through the equation of state: $p \sim \rho T$. At leading order, because $p^{(0)}$ is constant in space (also in time, except if we have heat release), we find the extremely convenient relation $\rho^{(0)} T^{(0)} = \rho_\infty T_\infty$, or after nondimensionalisation,

  $$\rho^{(0)} = \frac{1}{T^{(0)}}.$$

  This relation allows us to eliminate temperature from the equations.
- The energy equation, at first order, becomes a simple advection-diffusion equation for temperature (or enthalpy, internal energy etc.). Viscous heating, compression heating, and dilatation stresses disappear; these are higher-order effects in terms of Ma.

Here's our system:

$$\partial_t \rho + u_i \partial_{x_i} \rho + \rho \partial_{x_i} u_i = 0, \tag{1a}$$

$$\partial_t(\rho u_i) + \partial_{x_j}(\rho u_i u_j) + \partial_{x_i} p - Re^{-1} \partial_{x_j x_j} u_i = 0, \tag{1b}$$

$$\rho \partial_t T + \rho u_i \partial_{x_i} T - (Re\, Pr)^{-1} \partial_{x_j x_j} T = 0, \tag{1c}$$

$$T = \rho^{-1}. \tag{1d}$$

Superscripts 0 and 1 are not written. Every quantity has a (0), except for pressure, which has (1), but this has no relevance from here on. The temperature equation has been derived from enthalpy. I made the assumption that diffusion coefficients are independent of temperature. This may have a slight effect on the critical Reynolds number, but that's how I like my equations.

Temperature is now replaced with $\rho^{-1}$ in (1d). When combined with (1a), the material derivative is eliminated, and we obtain a correction to the incompressible $\mathrm{div} u = 0$ condition. We could work with (1b) in conservative form, but since we need $u$ anyway, it is simpler to write the system as

$$\partial_t \rho + u_i \partial_{x_i} \rho - (Re\, Pr)^{-1} \rho \partial_{x_i}(\rho^{-2} \partial_{x_i} \rho) = 0, \tag{2a}$$

$$\rho \partial_t u_i + \rho u_j \partial_{x_j} u_i + \partial_{x_i} p - Re^{-1} \partial_{x_j x_j} u_i = 0, \tag{2b}$$

$$\rho \partial_{x_i} u_i + (Re\, Pr)^{-1} \rho \partial_{x_i}(\rho^{-2} \partial_{x_i} \rho) = 0. \tag{2c}$$

The viscous term in (2a,2c) is what comes from the Laplacian of $T$. Without much harm, we could replace it simply with the Laplacian of $\rho$, which would correspond to a scenario with species mixing instead of temperature. But we do temperature here, so be it.

## 2 Variational formulation

For the finite-element method, we need to write system (2) in variational form. The only technicality is the integration by parts that must be performed on the $(Re\, Pr)^{-1}$ terms. Let $s$ be the test function for density. Then the integration by parts becomes

$$\int \rho \mathrm{div}(\rho^{-2} \mathrm{grad} \rho)\, s \, \mathrm{d}x\mathrm{d}y = [\rho\, s\, \partial_n \rho]_\Gamma - \int \rho^{-2} \mathrm{grad} \rho \cdot \mathrm{grad}(\rho s) \, \mathrm{d}x\mathrm{d}y.$$

By neglecting the boundary term, we prescribe a Neumann condition for $\rho$ at the outflow. Dirichlet conditions are prescribed on $\rho$ at all other boundaries (isothermal).

## 3 Timestepping

Timestepping is tedious, because no linear system can be constructed from our system that would allow the LHS system matrix to be time-invariant. A new matrix must be

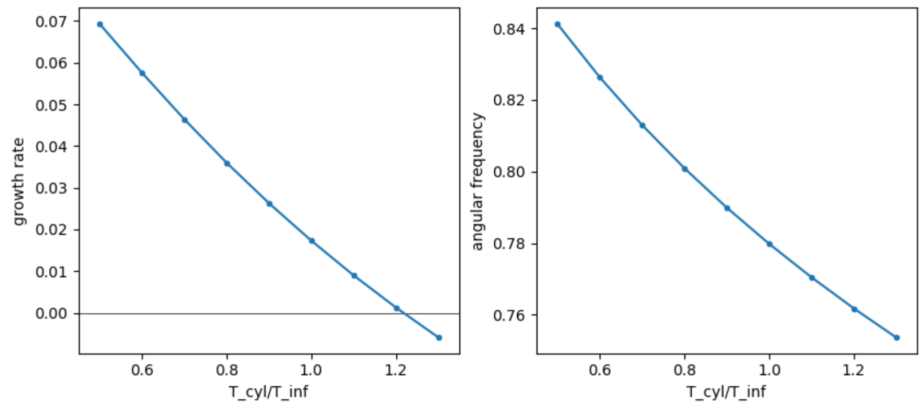


Figure 1: Variations of the dominant eigenvalue with cylinder temperature. Steady base flow at $Re = 50$.

constructed at each time step. Furthermore, I have not been able to make FEniCS solve a linear system iteratively; therefore, a full LU decomposition is performed at each time step. My best option under these circumstances is to use a full Newton solver at each time step, which at least is as accurate as we want it to be. The nonlinear system (2) is written with a Crank–Nicolson time discretisation, and solved with Newton.

## 4 Eigenvalues

The EVP is implemented such that it takes the nonlinear form (2) as input, and computes the Jacobian matrix analytically (maximum niftiness). The dominant eigenvalue in a laminar steady base flow, at $Re = 50$ and with varying cylinder temperature values, is shown in figure 1. Note that the value at $T = 1$ is identical to the standard result for a constant-density incompressible cylinder wake. The trend of the growth rate is as expected.