Development of the resolvent problem for Felics

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We consider the discrete linear system

$$(-i\omega + L)\hat{q} = B\hat{f} \quad \Leftrightarrow \quad \hat{q} = RB\hat{f}. \tag{1}$$

The matrix B is of dimension $m \times n$, with $m = \dim(\hat{q})$ and $n = \dim(\hat{f})$, where $m \geq n$. In general, B will have only 1 and 0 elements, where the ones select the permitted forcing quantities and locations, and the zeros fill up those elements where no forcing is applied. The vector $\hat{f}$ contains the degrees of freedom of the forcing.

The amplitude of the response vector is measured by a semi-norm $\|\hat{q}\|^2 = \hat{q}^H M_q \hat{q}$, where M_q is a positive semi-definite matrix of dimension $m \times m$. This means that M_q is permitted to be singular; perhaps it only takes into account certain quantities in certain regions.

The amplitude of the forcing vector is measured by a true norm $\|\hat{f}\|^2 = \hat{f}^H M_f \hat{f}$, where M_f is a positive definite matrix of dimension $n \times n$. M_f cannot be singular, it takes into account all the degrees of freedom contained in $\hat{f}$. The matrix M_f has the Cholesky decomposition $M_f = N_f^H N_f$.

The energy gain σ^2 of the linear system is defined by

$$\begin{aligned} \sigma^2 = \frac{\|\hat{q}\|^2}{\|\hat{f}\|^2} = \frac{\hat{q}^H M_q \hat{q}}{\hat{f}^H M_f \hat{f}} &= \frac{\hat{f}^H B^H R^H M_q R B \hat{f}}{\hat{f}^H N_f^H N_f \hat{f}} \\ &= \frac{v^H N_f^{H,-1} B^H R^H M_q R B N_f^{-1} v}{v^H v}, \end{aligned} \tag{2}$$

with the substitution $v = N_f \hat{f}$. It follows that the optimal energy gain is given by the largest (real) eigenvalue of a Hermitian operator, according to the equivalent eigenvalue problems

$$N_f^{H,-1} B^H R^H M_q R B N_f^{-1} v = \sigma^2 v, \tag{3a}$$

$$N_f^{-1} N_f^{H,-1} B^H R^H M_q R B N_f^{-1} N_f \hat{f} = \sigma^2 \hat{f}, \tag{3b}$$

$$M_f^{-1} B^H R^H M_q R B \hat{f} = \sigma^2 \hat{f}. \tag{3c}$$

The matrix $M_f^{-1}B^H R^H M_q RB$ is of dimension $n \times n$. The smaller the number of degrees of freedom of our forcing, the cheaper it is to solve the EVP.

For numerical efficiency, it is probably important to make sure that we use a Lanczos algorithm for the iterative solution of this eigenvalue problem. Lanczos is a variant of the implicitly restarted Arnoldi algorithm, adapted for Hermitian operators. It may therefore be more efficient to solve the SVD of $N_q RBN_f^{-1}$ instead of the EVP of $M_f^{-1}B^H R^H M_q RB$. The two are equivalent.