1. What is the formula for the sum of all integers from 5 to n, where $n \geq 5$?

1 / 1 point

- (selected) $\frac{(n+5)(n-4)}{2}$
- $\frac{n(n+1)}{2}$
- $\frac{(n+5)(n-5)}{2}$
- $\frac{(n+5)(n-6)}{2}$
- $\frac{(n+5)(n+1)}{2}$

Correct

Correct! One way to prove this is to remember how we came up with formula for the sum of all integers from 1 to n. We just need to write the sum forward and backward:

$$S = 5 + 6 + 7 + \cdots + n$$

$$S = n + (n-1) + (n-2) + \cdots + 5$$

Notice that the $5 + n = 6 + (n-1) = 7 + (n-2) = \cdots = n + 5$, and there are $n - 4$ summands in the sum, so $2S = (n+5)(n-4)$, and \(S = \frac{(n + 5)(n-4)}{2}.

Another way is to prove this by induction. Indeed, for $n = 5$ (induction base), the sum is 5, and $\frac{(n+5)(n-4)}{2} = \frac{10 \cdot 1}{2} = 5$. If the sum of all integers from 5 to n is $\frac{(n+5)(n-4)}{2}$, then the sum of all integers from 5 to $n+1$ is $= \frac{(n+5)(n-4)}{2} + (n+1) = \frac{n^2+n-20}{2} + (n+1) = \frac{n^2+n-20+2n+2}{2} = \frac{(n^2+2n+1)+(n+1)-20}{2} = \frac{(n+1)^2+(n+1)-20}{2} = \frac{(n+1+5)(n+1-4)}{2}$.

2. Find the formula for the sum of all odd numbers from 1 to $2n-1$:
$1+3+5+\cdots+(2n-3)+(2n-1)$.

1 / 1 point

- ○ $(2n-1)(n)$
- ○ $n(n+1)$
- ◉ n^2
- ○ $3n-2$

Correct

Correct! If we use Gauss's method again, we will see that $1+(2n-1)=3+(2n-3)=5+(2n-5)=\cdots=2n$, and there are exactly n summands, so the sum is $\frac{2n\cdot n}{2}=n^2$.

3. Function T is defined by $T(0) = a, T(n+1) = T(n) + b$. What is the correct formula for $T(n)$?

1 / 1 point

- ○ $T(n) = ab^n$
- ◉ $T(n) = a + nb$
- ○ $T(n) = b + na$
- ○ $T(n) = a + (n-1)b$
- ○ $T(n) = a + (n+1)b$

Correct

Correct! Indeed, for $n = 0$ (induction base), $T(n) = T(0) = a = a + 0b$, and if $T(n) = a + nb$, then $T(n+1) = T(n) + b = a + nb + b = a + (n+1)b$ (induction step). Thus, we can prove this formula by induction.

4. Function $T(n)$ is defined by $T(0) = a, T(n+1) = b \cdot T(n)$. What is the correct formula for $T(n)$?

1 / 1 point

- ○ $T(n) = a^n b$
- ○ $T(n) = b + na$
- ○ $T(n) = a + bn$
- ◉ $T(n) = ab^n$

Correct

Correct! Indeed, for $n = 0, T(0) = a = ab^0$ (induction base), and if $T(n) = ab^n$, then $T(n+1) = b \cdot T(n) = b \cdot ab^n = ab^{n+1}$ (induction step). So, you can prove this formula by induction.

5. Recall the recursive formula for the minimum number of moves in the Hanoi Towers problem: $T(1) = 1, T(n+1) = 2T(n) + 1$, where $T(n)$ is the minimum number of moves for Hanoi Towers with n disks. What is the correct formula for $T(n)$?

1 / 1 point

- ○ $T(n) = \frac{n(n+1)}{2}$
- ○ $T(n) = 2n - 1$
- ◉ $T(n) = 2^n - 1$
- ○ $T(n) = n$

Correct

Correct! Indeed, for $n = 1$ (induction base), $T(1) = 1 = 2^1 - 1$. If $T(n) = 2^n - 1$, then $T(n+1) = 2T(n) + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$.

6. You have $\$1\ 000$ on day 1, and every day you earn 10% of what you already get, so that on day 2 you have $\$1\ 000 + 10\% \cdot \$1\ 000 = \$1\ 100$, and on day 3 you have $\$1\ 100 + 10\% \cdot \$1\ 100 = \$1\ 210$. On which day you will have more than $\$1\ 000\ 000$ for the first time? Feel free to use the notebook about Bernoulli's inequality that we provided in this lesson.

1 / 1 point

- (●) On day 74.
- ( ) On day 76.
- ( ) On day 10000.
- ( ) On day 56.

Correct

On day 73 you will have a little more than $\$955\ 593$, and on day 74 you will have a little more than $\$1\ 051\ 153$.

7. What is the flaw in the following proof?

1 / 1 point

Claim: For any integer $n \geq 0$ and any positive integer a, $a^n = 1$.

Proof by Induction.

Induction base: $n = 0$, so $a^n = a^0 = 1$.

Induction step: If the statement is true for all n up to m, then it is true for $n = m + 1$.

$$a^{m+1} = \frac{a^m \cdot a^m}{a^{m-1}} \stackrel{!}{=} \frac{1 \cdot 1}{1} = 1$$

Where we marked by "!" the assumption of the induction that $a^m = 1$ and $a^{m-1} = 1$.

- ○ The induction step is wrong: $\frac{1 \cdot 1}{1} \neq 1$.
- ○ The induction step must prove something for $n + 1$ based just on the fact that something is true for n.
- ○ The induction base is wrong: a^0 is not always equal to 1.
- ◉ We are using the assumption of the induction for both m and $m - 1$, so we must have proved induction base for at least two different values of n - 0 and 1.

Correct

Correct! The proof breaks on $m = 0 \to m + 1 = 1$, because $a^{m-1} = a^{-1} \neq 1$.

8. Function $T(n)$ is defined as $T(0) = 0, T(2n) = T(n) + 1, T(2n+1) = T(n) + 1$. What is the correct formula for $T(n)$?

2 / 2 points

○ $T(n) = n$

○ $T(n) = 2n$

○ $T(n) = 2^n$

◉ $T(n) = k$, where k is the smallest non-negative integer such that $2^k > n$.

Correct

Correct! You can prove this using complete induction.

Induction base: $n = 0$.

If $n = 0, T(n) = 0$, and $2^0 = 1 > 0 = n$.

Induction step: assume that for all n less than some integer $m > 0$ the formula is correct.

If $m = 2l$, then $T(m) = T(2l) = T(l) + 1$, and $l < m$. Let $T(l) = k$. Then by the assumption of induction, k is the smallest non-negative integer such that $2^k > l$. Then $2^{k+1} > 2l = m$. Also, $m = 2l \geq 2^k$, because otherwise $2l < 2^k$, and so $l < 2^{k-1}$, but this contradicts the fact that $T(l) = k$, because by assumption of induction $T(l)$ if $l < 2^{k-1}$, then $T(l) \leq k - 1$. Thus $T(m) = T(l) + 1 = k + 1$, and $2^k \leq m < 2^{k+1}$, so the formula is also correct for m.

If $m = 2l + 1$, then $T(m) = T(2l + 1) = T(l) + 1$, and $l < m$. Let $T(l) = k$. Again, by the assumption of induction, k is the smallest non-negative integer such that $2^k > l$. Then $2^k \geq l + 1$, and $2^{k+1} \geq 2(l + 1) = 2l + 2 > 2l + 1 = m$. Also, $m = 2l + 1 \geq 2^k$, because otherwise $2l < 2l + 1 < 2^k$, and so $l < 2^{k-1}$. Thus $T(m) = T(l) + 1 = k + 1$, and $2^k \leq m < 2^{k+1}$, so the formula is also correct for m.

9. What is the value of the sum $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}$?

3 / 3 points

- (selected) 0.99
- 0.9
- 1
- 0.5

Correct

Correct!

First, let us generalize the problem: what is the sum of $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}$?

To come up with the answer, note that $\frac{1}{k\cdot(k+1)}=\frac{1}{k}-\frac{1}{k+1}$. Then

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}=$$
$$=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{n-1}-\frac{1}{n}.$$

All the summands in the middle cancel out, and the sum becomes $\frac{1}{1}-\frac{1}{n}=1-\frac{1}{n}$. For $n=100, 1-\frac{1}{n}=1-\frac{1}{100}=1-0.01=0.99.$

We can also prove the formula

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}=1-\frac{1}{n}$$

by induction.

Induction base: $n=2.$

$$\frac{1}{1\cdot 2}=\frac{1}{2}=1-\frac{1}{2}.$$

Induction step: $n\to n+1.$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}+\frac{1}{n\cdot(n+1)}\overset{!}{=}$$
$$\overset{!}{=}1-\frac{1}{n}+\frac{1}{n\cdot(n+1)}=1-\frac{1}{n}+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}.$$

We used the assumption of the induction for n in the equality marked by "!".