

1. What is the formula for the sum of all integers from 5 to n , where $n \geq 5$?

☒ $\frac{(n+5)(n-4)}{2}$

☐ $\frac{n(n+1)}{2}$

☐ $\frac{(n+5)(n-5)}{2}$

☐ $\frac{(n+5)(n-6)}{2}$

☐ $\frac{(n+5)(n+1)}{2}$



Correct

Correct! One way to prove this is to remember how we came up with formula for the sum of all integers from 1 to n . We just need to write the sum forward and backward:

$$S = 5 + 6 + 7 + \cdots + n$$

$$S = n + (n - 1) + (n - 2) + \cdots + 5$$

Notice that the $5 + n = 6 + (n - 1) = 7 + (n - 2) = \cdots = n + 5$, and there are $n - 4$ summands in the sum, so $2S = (n + 5)(n - 4)$, and $S = \frac{(n + 5)(n - 4)}{2}$.

Another way is to prove this by induction. Indeed, for $n = 5$ (induction base), the sum is 5, and $\frac{(n+5)(n-4)}{2} = \frac{10 \cdot 1}{2} = 5$. If the sum of all integers from 5 to n is $\frac{(n+5)(n-4)}{2}$, then the sum of all integers from 5 to $n + 1$ is $\frac{(n+5)(n-4)}{2} + (n + 1) = \frac{n^2 + n - 20}{2} + (n + 1) = \frac{n^2 + n - 20 + 2n + 2}{2} = \frac{(n+1)^2 + (n+1) - 20}{2} = \frac{(n+1+5)(n+1-4)}{2}$.

2. Find the formula for the sum of all odd numbers from 1 to $2n - 1$:

$$1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1).$$

☐ $(2n - 1)(n)$

☐ $n(n + 1)$

☒ n^2

☐ $3n - 2$

✓ **Correct**

Correct! If we use Gauss's method again, we will see that

$1 + (2n - 1) = 3 + (2n - 3) = 5 + (2n - 5) = \cdots = 2n$, and there are exactly n summands, so the sum is $\frac{2n \cdot n}{2} = n^2$.

3. Function T is defined by $T(0) = a, T(n+1) = T(n) + b$. What is the correct formula for $T(n)$?

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☐ $T(n) = ab^n$

☒ $T(n) = a + nb$

☐ $T(n) = b + na$

☐ $T(n) = a + (n-1)b$

☐ $T(n) = a + (n+1)b$

✓ **Correct**

Correct! Indeed, for $n = 0$ (induction base), $T(n) = T(0) = a = a + 0b$, and if $T(n) = a + nb$, then $T(n+1) = T(n) + b = a + nb + b = a + (n+1)b$ (induction step). Thus, we can prove this formula by induction.

4. Function $T(n)$ is defined by $T(0) = a, T(n + 1) = b \cdot T(n)$. What is the correct formula for $T(n)$?

☐ $T(n) = a^n b$

☐ $T(n) = b + na$

☐ $T(n) = a + bn$

☒ $T(n) = ab^n$

✓ **Correct**

Correct! Indeed, for $n = 0, T(0) = a = ab^0$ (induction base), and if $T(n) = ab^n$, then $T(n + 1) = b \cdot T(n) = b \cdot ab^n = ab^{n+1}$ (induction step). So, you can prove this formula by induction.

5. Recall the recursive formula for the minimum number of moves in the Hanoi Towers problem:

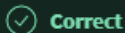
$T(1) = 1$, $T(n + 1) = 2T(n) + 1$, where $T(n)$ is the minimum number of moves for Hanoi Towers with n disks. What is the correct formula for $T(n)$?

☐ $T(n) = \frac{n(n+1)}{2}$

☐ $T(n) = 2n - 1$

☒ $T(n) = 2^n - 1$

☐ $T(n) = n$



Correct! Indeed, for $n = 1$ (induction base), $T(1) = 1 = 2^1 - 1$. If $T(n) = 2^n - 1$, then $T(n + 1) = 2T(n) + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$.

6. You have \$1 000 on day 1, and every day you earn 10% of what you already get, so that on day 2 you have $\$1\,000 + 10\% \cdot \$1\,000 = \$1\,100$, and on day 3 you have $\$1\,100 + 10\% \cdot \$1\,100 = \$1\,210$. On which day you will have more than \$1 000 000 for the first time? Feel free to use the notebook about Bernoulli's inequality that we provided in this lesson.

☒ On day 74.

☐ On day 76.

☐ On day 10000.

☐ On day 56.



Correct

On day 73 you will have a little more than \$955 593, and on day 74 you will have a little more than \$1 051 153.

7. What is the flaw in the following proof?

Claim: For any integer $n \geq 0$ and any positive integer a , $a^n = 1$.

Proof by Induction.

Induction base: $n = 0$, so $a^n = a^0 = 1$.

Induction step: If the statement is true for all n up to m , then it is true for $n = m + 1$.

$$a^{m+1} = \frac{a^m \cdot a^m}{a^{m-1}} \stackrel{!}{=} \frac{1 \cdot 1}{1} = 1$$

Where we marked by "!" the assumption of the induction that $a^m = 1$ and $a^{m-1} = 1$.

- ☐ The induction step is wrong: $\frac{1 \cdot 1}{1} \neq 1$.
- ☐ The induction step must prove something for $n + 1$ based just on the fact that something is true for n .
- ☐ The induction base is wrong: a^0 is not always equal to 1.
- ☒ We are using the assumption of the induction for both m and $m - 1$, so we must have proved induction base for at least two different values of n - 0 and 1.



Correct

Correct! The proof breaks on $m = 0 \rightarrow m + 1 = 1$, because $a^{m-1} = a^{-1} \neq 1$.

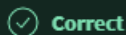
8. Function $T(n)$ is defined as $T(0) = 0$, $T(2n) = T(n) + 1$, $T(2n + 1) = T(n) + 1$. What is the correct formula for $T(n)$?

☐ $T(n) = n$

☐ $T(n) = 2n$

☐ $T(n) = 2^n$

☒ $T(n) = k$, where k is the smallest non-negative integer such that $2^k > n$.



Correct! You can prove this using complete induction.

Induction base: $n = 0$.

If $n = 0$, $T(n) = 0$, and $2^0 = 1 > 0 = n$.

Induction step: assume that for all n less than some integer $m > 0$ the formula is correct.

If $m = 2l$, then $T(m) = T(2l) = T(l) + 1$, and $l < m$. Let $T(l) = k$. Then by the assumption of induction, k is the smallest non-negative integer such that $2^k > l$. Then $2^{k+1} > 2l = m$. Also, $m = 2l \geq 2^k$, because otherwise $2l < 2^k$, and so $l < 2^{k-1}$, but this contradicts the fact that $T(l) = k$, because by assumption of induction $T(l)$ if $l < 2^{k-1}$, then $T(l) \leq k - 1$. Thus $T(m) = T(l) + 1 = k + 1$, and $2^k \leq m < 2^{k+1}$, so the formula is also correct for m .

If $m = 2l + 1$, then $T(m) = T(2l + 1) = T(l) + 1$, and $l < m$. Let $T(l) = k$. Again, by the assumption of induction, k is the smallest non-negative integer such that $2^k > l$. Then $2^k \geq l + 1$, and $2^{k+1} \geq 2(l + 1) = 2l + 2 > 2l + 1 = m$. Also, $m = 2l + 1 \geq 2^k$, because otherwise $2l < 2l + 1 < 2^k$, and so $l < 2^{k-1}$. Thus $T(m) = T(l) + 1 = k + 1$, and $2^k \leq m < 2^{k+1}$, so the formula is also correct for m .

9. What is the value of the sum $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}$?

☒ 0.99

☐ 0.9

☐ 1

☐ 0.5

☒ **Correct**

Correct!

First, let us generalize the problem: what is the sum of $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n}$?

To come up with the answer, note that $\frac{1}{k \cdot (k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Then

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n} &= \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-1} - \frac{1}{n}. \end{aligned}$$

All the summands in the middle cancel out, and the sum becomes $\frac{1}{1} - \frac{1}{n} = 1 - \frac{1}{n}$. For $n = 100$, $1 - \frac{1}{n} = 1 - \frac{1}{100} = 1 - 0.01 = 0.99$.

We can also prove the formula

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n} = 1 - \frac{1}{n}$$

by induction.

Induction base: $n = 2$.

$$\frac{1}{1 \cdot 2} = \frac{1}{2} = 1 - \frac{1}{2}.$$

Induction step: $n \rightarrow n + 1$.

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1) \cdot n} + \frac{1}{n \cdot (n+1)} &\stackrel{!}{=} \\ &\stackrel{!}{=} 1 - \frac{1}{n} + \frac{1}{n \cdot (n+1)} = 1 - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}. \end{aligned}$$

We used the assumption of the induction for n in the equality marked by "!".