The First Project Report

Xu Shu
201818000206008
University of Chinese Academy of Sciences

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1 Problem Description

Consider the following Riemann Problem

$$\begin{cases} u_t + ax_u = 0, \quad x \in [\alpha, \beta],\ t > 0, \\ u_0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases} \\ u(\alpha, t) = 1, \quad u(\beta, t) = 0. \end{cases}$$

Parameters a, α and β are all fixed in this problem where $a > 0$ and $\alpha < 0 < \beta$.

The objective of the project is to solve this problem in four different numerical schemes and compare the numerical outcome with the exact solution respectively at time $T = 0.5$.

2 Implementation and Comparison

2.1 The Exact Solution

Taking as initial condition

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases}$$

the characteristic line issuing from the point $(x_0, 0)$ is given by

$$x(t) = x_0 + at.$$

Thus its solution is given by

$$u(x,t) = u_0(x - at) = \begin{cases} 1, & x - at \leq 0, \\ 0, & x - at > 0. \end{cases}$$

It is worth noting that since u_0 is discontinuous at 0, then such a discontinuity propagates along the characteristics issuing from 0, which is shown in Figure 1 below.

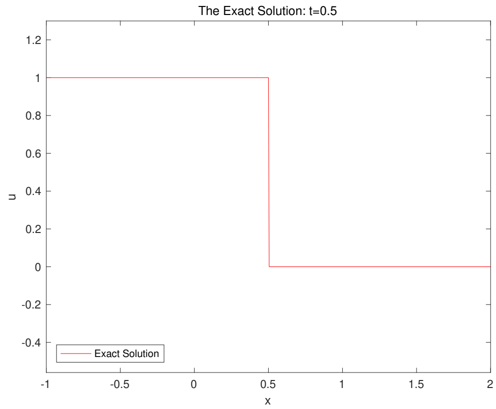

Figure 1: The Exact Solution, a=1

2.2 Upwind Scheme

The first-order upwind scheme is given by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

or

$$u_j^{n+1} = (1 - ar)u_j^n + aru_{j-1}^n$$

for $a > 0$ where $r = \frac{\Delta t}{\Delta x}$.

The upwind scheme is stable if the following Courant–Friedrichs–Lewy condition(CFL) is satisfied.

$$|\frac{a\Delta t}{\Delta x}| \leq 1.$$

A Taylor series analysis of the upwind scheme discussed above will show that it is first-order accurate in space and time.

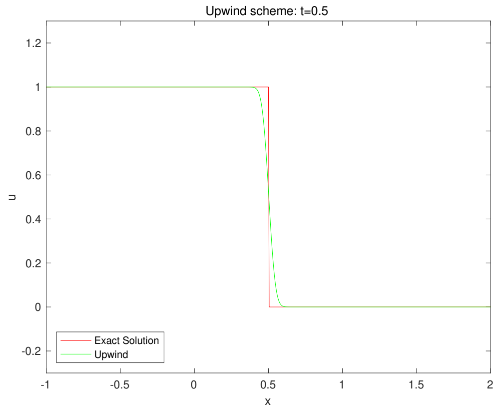

Figure 2: The Outcome of Upwind Scheme, a=1

As can be seen from the figure 2, the first-order upwind scheme has a good result, but there is a polishing effect near the discontinuity.

2.3 Lax-Friedrichs Scheme

The Lax–Friedrichs method for solving the above partial differential equation is given by:

$$\frac{u_i^{n+1} - \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\,\Delta x} = 0$$

Or, rewriting this as

$$u_i^{n+1} = \frac{1-ar}{2}u_{i+1}^n + \frac{1+ar}{2}u_{i-1}^n$$

where $r = \frac{\Delta t}{\Delta x}$.

This method is explicit and first order accurate in time and first order accurate in space. It is stable if and only if the following condition is satisfied:

$$|a\frac{\Delta t}{\Delta x}| \leq 1.$$

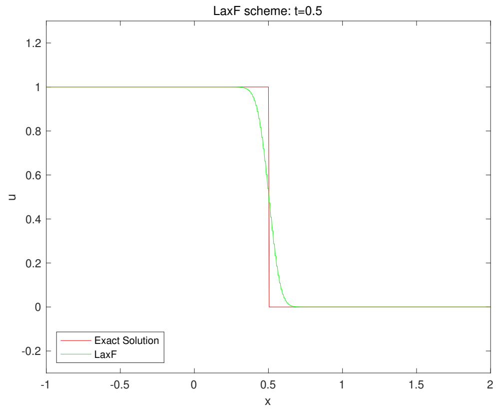

Figure 3: The Outcome of Lax–Friedrichs Scheme, a=1

It turns out that Lax–Friedrichs scheme has a stronger polishing effect than upwind scheme.(Figure 3)

2.4 Lax-Wendroff Scheme

The Lax–Wendroff method is given by:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x} a(u_{i+1}^n - u_{i-1}^n) + \frac{\Delta t^2}{2\Delta x^2} a^2 (u_{i+1}^n - 2u_i^n + u_{i-1}^n).$$

This method is explicit and second order accurate in time and second order accurate in space. It is stable if and only if the following condition is satisfied:

$$|a\frac{\Delta t}{\Delta x}| \leq 1.$$

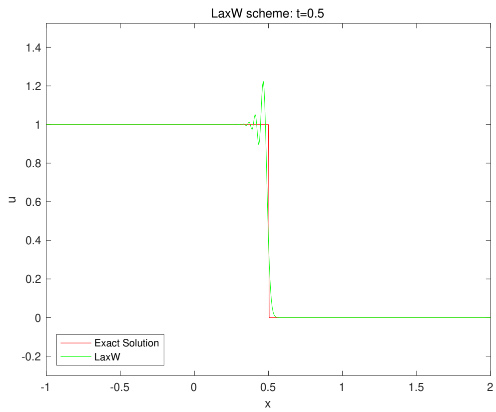

Figure 4: The Outcome of Lax–Wendroff Scheme

Unlike the previous first-order scheme, the result of Lax–Wendroff sheme oscillates to the left of the discontinuity. Using Taylor's expansion to the second order to represent the truncation error, it is possible to associate with Lax–Wendroff scheme an equivalent differential equation of the form

$$u_t + au_x = \mu u_{xx}$$

where the terms μu_{xx} represent dissipation. Group velocity $C_g(\xi) = a + 3\mu\xi^2$ can better characterize the behavior of the solution near the discontinuity. In this experiment, $\mu < 0$ and $C_g(\xi) < a$ explains why oscillations occur to the left of the discontinuity.

2.5 Beam-Warming Scheme

The Beam-Warming scheme is given by:

$$u_j^{n+1} = u_j^n - ar(u_j^n - u_{j-1}^n) - \frac{ar(1-ar)}{2}(u_j^n - 2u_{j-1}^n + u_{j-2}^n).$$

This scheme is second order accurate in time and second order accurate in space. It is stable if and only if the following condition is satisfied:

$$a > 0, \quad a\frac{\Delta t}{\Delta x} \le 2.$$

In this numerical experiment, we choose Lax-Wendroff method to calculate u_2^{n+1} in advance, guaranteeing Beam-Warming scheme's second order accuracy in both time and space.

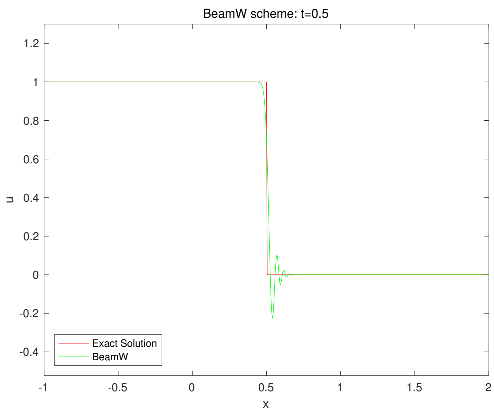

Figure 5: The Outcome of Beam-Warming Scheme

In contrary to Lax–Wendroff sheme, it oscillates to the right of the discontinuity for $\mu > 0$ and $C_g(\xi) > a$.

3 Actual Convergence Order

Let u be the exact solution and u_n be the numerical solution. Then the L^1 error can be calculated according to the following formula

$$E_n = ||u^n - u||_{L^1} \approx \sum_{1}^{N_x} |u_j^n - u(x_j, t^n)|\Delta x.$$

If a scheme is γ-order accurate, we have $E_n = C\Delta t^\gamma + o(\Delta t^\gamma)$ and $E_n \approx C\Delta t^\gamma$ when Δt is small enough. Adapting two different time step length Δt_{n_1} and Δt_{n_2} leads to

$$\gamma \approx \frac{log(E_{n_1}/E_{n_2})}{log(n_2/n_1)}.$$

Fixing $r = 0.5$ and partitioning $T = 0.5$ into $Nt = 100 * 2^{n-1}$ equal length intervals, the corresponding actual convergence orders of the four numerical schemes are listed respectively below.

n	Upwind	Lax-Friedrichs	Lax–Wendroff	Beam-Warming
1	/	/	/	/
2	0.4982	0.4974	0.5990	0.5990
3	0.4991	0.4987	0.5919	0.5919
4	0.4995	0.4993	0.5989	0.5989
5	0.4998	0.4997	0.5991	0.5991
6	0.4999	0.4998	0.5987	0.5987
7	0.4999	0.4999	0.6013	0.6013
8	0.5000	0.5000	0.6010	0.6010

Table 1: Actual Convergence Order γ

The result shows that the first order accurate schemes are actually only half order while the second order's are $\frac{2}{3}$ order, which is consistent with the theoretical analysis.