$$\max_{q(z_j)} \mathcal{L}(q(z)) = \int_z q(z) \log \frac{p(x,z)}{q(z)} \qquad \text{Assume } q(z) = \prod_{i=1}^{I} q_i(z_i)$$

$$\mathcal{L}(q(z)) = \mathcal{L}\left(\prod q(z_i)\right) = \int_z \prod q(z_i) \log \frac{p(x,z)}{\prod q(z_\ell)} dz$$

$$\log \hat{p}(x, z_j) = E_{\prod_{i \neq j} q(z_i)}[\log p(x,z)]$$

$$= \int_{z_j} q(z_j) \left[\int_{z_{-j}} \prod_{i \neq j} q(z_i) \log p(x,z) dz_{-j} \right] dz_j - \int_z \prod q(z_i) \sum_\ell \log q(z_\ell) dz$$

$$= \int_{z_j} q(z_j) \log \hat{p}(x, z_j) dz_j - \int_{z_j} q(z_j) \log q(z_j) dz_j$$

$$= \int_{z_j} q(z_j) \log \frac{\hat{p}(x, z_j)}{q(z_j)} dz_j = -KL(q(z_j) \| \hat{p}(x, z_j))$$

$$q(z_j) = \hat{p}(x, z_j) \quad \max \mathcal{L} \Rightarrow \quad \log q(z_j) = \log \hat{p}(x, z_j)$$

So $\log q(z_j) = E_{\prod_{i \neq j} q(z_i)}[\log p(x,z)]$

$$\int_z \prod q(z_i) \sum_\ell \log q(z_\ell)\, dz \stackrel{+}{=} \int_{z_j} q(z_j) \log q(z_j)\, dz_j$$

$$\text{LHS} = \sum_\ell \int_z \log q(z_\ell) \prod_i q(z_i)\, dz$$

$$= \sum_\ell \int_{z_\ell} q(z_\ell) \log q(z_\ell) \cancel{\left[\int_{z_{-\ell}} \prod_{i \neq \ell} q(z_i)\, dz_{-\ell} \right]} dz_\ell$$

$$\stackrel{+}{=} \int_{z_j} q(z_j) \log q(z_j)\, dz_j$$