25.01.23

Ch1.

1.1 Mixed state. (probabilistic state).

ex. $\frac{1}{\sqrt{2}}\left(|00\rangle_{A,B} + |11\rangle_{AB}\right) \Rightarrow$ Alice has Mixed State.

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|+\rangle$ are different if you measure them in $(|+\rangle, |-\rangle)$.

- A mixed quantum state is a clean way of describing the State.

$$\rho \begin{cases} |e_1\rangle & \text{w.p } p_1 . \\ \vdots & \\ |e_t\rangle & \text{w.p. } p_t . \end{cases}$$

- We will write this

$$\rho = \sum_i p_i |e_i\rangle\langle e_i|$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}.$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|+\rangle\langle +| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad |-\rangle\langle -| = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Definition: A mixed state on qubits is a matrix $\rho = \sum_i p_i |e_i\rangle\langle e_i|$ where each $|e_i\rangle$ is a n-qubit pure state, each $p_i \geq 0$ and $\sum_i p_i = 1$.

- Properties of quantum states
  - ρ is Hermision $\rho = \rho^{*} = \rho^{-T}$
  - $Tr(\rho) = 1$
  - Because ρ is Hermition, it is diagonalizable with real-valued eigenvalues.

This means we can write $\rho = \sum_i \lambda_i |f_i\rangle\langle f_i|$ with $\{|f_i\rangle\}$ orthonormal basis and $\lambda_i \geq 0$.

$$\rho_1 = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \Rightarrow \begin{cases} |0\rangle \text{ w.p } 3/4 \\ |1\rangle \text{ w.p } 1/4. \end{cases}$$

$$\Rightarrow \begin{cases} 1/2 & \text{to get } |+\rangle \\ 1/2 & \text{to get } |-\rangle \end{cases}$$

$$\rho_2 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -| = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \Rightarrow \begin{cases} |0\rangle \text{ w.p. } 1/2 \\ |+\rangle \text{ wp } 1/4 \\ |-\rangle \text{ wp } 1/4. \end{cases}$$

$$\Rightarrow \begin{cases} 1/2 & \text{to get } |+\rangle \\ 1/2 & \text{to get } |-\rangle \end{cases}$$

1.2 Applying quantum operations on mixed state.

If we start from $|e_i\rangle$ and apply U, we obtain $|f_i\rangle = U|e_i\rangle$.

$$|f_i\rangle\langle f_i| = U|e_i\rangle\langle e_i|U^\dagger = U(|e_i\rangle\langle e_i|)U^\dagger.$$

$$\rho = \sum_i p_i |e_i\rangle\langle e_i|.$$

$$\rho \xrightarrow{U} \sum_i p_i |f_i\rangle\langle f_i| = \sum_i p_i U|e_i\rangle\langle e_i|U^\dagger$$

$$= U\left(\sum_i p_i |e_i\rangle\langle e_i|\right)U^\dagger = U\rho U^\dagger.$$

Projective measurements

$B = \{|b_1\rangle, \ldots, |b_n\rangle\}$.

we start from

$$\Pr[\text{output } k \mid |e_i\rangle] = |\langle e_1|b_1\rangle|^2.$$

$$\rho \xrightarrow{\substack{\text{measurement}\\ \text{basis } B}} \text{output "}k\text{" w.p. } \sum_i p_i |\langle e_i|b_k\rangle|^2$$

$$= \sum_i p_i \langle b_k|e_i\rangle\langle e_i|k\rangle$$

$$= \langle b_k|\left(\sum_i p_i |e_i\rangle\langle e_i|\right)|b_k\rangle$$

$$= \langle b_k|\rho|b_k\rangle.$$

ex.

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |1+\rangle\right)_{AB} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$= \sqrt{\frac{3}{4}}\underbrace{\left(\sqrt{\frac{3}{2}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle\right)}_{|\phi\rangle}|0\rangle + \frac{1}{2}|11\rangle$$

$$\rho_A = \frac{3}{4}|\phi\rangle\langle\phi| + \frac{1}{4}|1\rangle\langle 1| = \begin{pmatrix} 1/2 & \sqrt{2}/4 \\ \sqrt{2}/4 & 1/2 \end{pmatrix}.$$

Definition: For a (possibly mixed) state

ρ_{AB} we define.

$$\mathrm{Tr}_B(\rho_{AB}) = \sum_j (I_A \otimes \langle j|)\, \rho_{AB}\, (I_A \otimes |j\rangle).$$

Special cases.

$|\psi\rangle = \sum_i \alpha_i |e_i\rangle|f_i\rangle$ where $\{|e_i\rangle\}$ forms an orthonormal basis.

$$\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \sum_i |\alpha_i|^2 |f_i\rangle\langle f_i|$$

but, $\rho_A \neq \sum_i |\alpha_i|^2 |e_i\rangle\langle e_i|$.

$$|\psi\rangle_{AB} = \sum_i \alpha_i |e_i\rangle\langle f_i|$$ where the $\{|f_i\rangle\}$ form a orthonormal basis.

$$\rho_A = Tr_B(\rho_{AB}) = \sum_i |\alpha_i|^2 |e_i\rangle\langle e_i|.$$

1.3 Generalized measurements

Definition : A POVM is an measurement of matricies $\{M_i\}$, s.t. $\sum_i M_i M_i^\dagger = I$

Measuring a state ρ with this POVM gives outcome i w.p. $P_i = tr(\rho M_i M_i^\dagger)$.

and the resulting state is

$$\boxed{\rho_i = \frac{M_i \rho M_i^\dagger}{tr(M_i \rho M_i^\dagger)}}$$

Measurement in an orthonormal basis, $\{|b_1\rangle, \ldots, |b_n\rangle\}$.

Take $M_i = |b_i\rangle \rightarrow M_i M_i^\dagger = |b_i\rangle\langle b_i|$

$$- P_i = tr(\rho |b_i\rangle\langle b_i|) = tr(\langle b_i|\rho|b_i\rangle)$$
$$= \langle b_i|\rho|b_i\rangle.$$

- Resulting state is $\dfrac{|b_i\rangle \rho \langle b_i|}{tr(|b_i\rangle \rho \langle b_i|)} = |b_i\rangle\langle b_i|$

- Sometimes, the POVM is charactrized by.

$$F_i = M_i M_i^{+} \qquad \sum_i F_i = Id.$$

$$tr(\rho M_i M_i^{+}) = tr(\rho F_i).$$