$$S = \sum_{i \in \mathbb{N}} S_i x^i$$

$$\exists T \rightarrow ST = 1 \text{ iff } S_0 \neq 0$$

Power series inversion

$$\begin{bmatrix} S_0 & 0 & \cdots & & 0 \\ S_1 & S_0 & \cdots & & 0 \\ S_2 & S_1 & S_0 & 0, \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_n & & \cdots & & S_0 \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$t_0 = S_0^{-1}$$
$$t_1 = (-S_1 t_0) S_0^{-1}$$
$$t_2 = -(S_1 t_1 + S_2 t_0) S_0^{-1}$$
$$\vdots$$
$$t_n = -S_0^{-1} \sum_{k=0}^{n} S_k t_{i-k}$$

Addition : $\sum_{i=0}^{n} (i-1) = \frac{n(n-1)}{2}$

Multiplicaton : $\sum_{i=0}^{n} (i+1) + 1 = \frac{(n+1)(n+2)}{2} + 1$

$$= \frac{1}{2}(n^2 + 3n + 2) + 1$$

$$= \frac{1}{2} M(n^2)$$

$$\Downarrow$$

$$\frac{1}{2} M(n^2) + O(n^2)$$

Example IV.6.

$S = 3 + 2x^2 + x^3 + x^7 \in \mathbb{F}_5[[x]]$

$n = 8$,

$S = 3 + 2x^2 + x^3 + O(x^4)$ $O(x^{\lceil n/2 \rceil})$

$n = 4$.

$S = 3 + O(x^2)$

$n = 1$

$T = 3^{-1} + O(x) = 2 + O(x)$ $3^{-1} \bmod 5$.

return....

$U = T + (1 - TS)T + O(x^2)$ at $O(x^2)$, $n = 4$. $S = 3 + O(x^2)$

$= 2 + (1 - 2 \cdot 3)2 + O(x^2)$

$= 2 + (\underbrace{-5 \bmod 5}_{\to 0})2 + O(x^2)$

$= 2 + O(x^2)$

$V = U + (1 - US)U + O(x^4)$

$= 2 + (1 - 2 \cdot (3 + 2x^2 + x^3))2 + O(x^4)$

$= 2 + (1 - 6 - 4x^2 - 2x^3) \cdot 2 + O(x^4)$

$-5 - 4x^2 - 2x^3 \bmod 5$.

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$$V = 2 + (x^2 + 3x^3)\cdot 2 + O(x^4)$$

$$= 2 + 2x^2 + 6x^3 + O(x^4) \quad \text{mod } 5$$

$$= 2 + 2x^2 + x^3 + O(x^4)$$

$$W = V + (1 - VS)\cdot V + O(x^8)$$

* you don't consider any term higher than degree 8

$$= (2 + 2x^2 + x^3) + \left(1 - (2 + 2x^2 + x^3)(3 + 2x^2 + x^3 + x^7)\right)(2 + 2x^2 + x^3)$$
$$+ O(x^8)$$

$$\cancel{6} + \cancel{4x^2} + \cancel{2x^3} + 2x^7 + \cancel{6x^2} + \cancel{4x^4} + 2x^5 + \cancel{3x^3} + \cancel{2x^5} + x^6$$

$$\cancel{6 + 4x^2 + 2x^3 + 2x^4 + 6x^4 + 2x^5 + 3x^3 + 2x^5 + x^6}$$

$$6 + 10x^2 + 5x^3 + 4x^4 + 4x^5 + x^6 + 2x^7$$

$$= 2 + 2x^2 + x^3 + (\cancel{1 - 6} - \cancel{10x^2} - \cancel{5x^3} - 4x^4 - 4x^5 - x^6 - 2x^7)$$
$$(2 + 2x^2 + x^3) + O(x^8) \quad \text{mod } 5$$

$$= 2 + 2x^2 + x^3 + (\cancel{x^2} + x^4 + x^5 + 4x^6 + 3x^7)(2 + 2x^2 + x^3)$$

$$= 2 + 2x^2 + x^3 + (2x^4 + 2x^5 + 8x^6 + 6x^7 + 2x^6 + 2x^7 + x^7) + O(x^8) \quad \text{mod } 5$$

$$= 2 + 2x^2 + x^3 + 2x^4 + 2x^5 + 4x^7 + O(x^8) \checkmark$$

- With Newton iteration, pay attention to $O(x^{\lceil n/2 \rceil})$ (ignore all terms with degree higher than $\lceil n/2 \rceil$) and $\mathbb{F}_n[(x)]$ (which means, computation is on mod n)

Question IV.7

mod 7. $O(x^8)$

$S = \sum_{i \in \mathbb{N}} (i+2)x^i \in \mathbb{F}_7[[x]]$. at precision 8.

mod 7

mod 7 0 x^6 $2x^7$

$S = \underbrace{2}_{i=0} + \underbrace{3x}_{i=1} + \underbrace{4x^2}_{i=2} + \underbrace{5x^3}_{i=3} + \underbrace{6x^4}_{i=4} + \underbrace{\cancel{7x^5}}_{i=5} + \underbrace{\cancel{8x^6}}_{i=6} + \underbrace{\cancel{9x^7}}_{i=7} + \underbrace{O(x^8)}_{i=8 \ldots, i=9}$

$n=1$. $\cancel{O(x^{\lceil 1/2 \rceil}) = O(x)}$

EEA $(2,7)$

7		1	0
2	3	0	1
1	2	1	-3
0			

$S_0^{-1} = 2^{-1} \bmod 7 = -3 \bmod 7 = \underline{\underline{4}}$

$n=8$: $O(x^4)$.

$S = 2 + 3x + 4x^2 + 5x^3 + O(x^4)$

$n=4$: $O(x^2)$

$S = 2 + 3x + O(x^2)$

$n=2$: $O(x)$

$S = 2 + O(x)$

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at n=1, return $T = 2^{-1} + O(x) = 4 + O(x)$

$$U_1 = T + (1 - TS)\cdot T + O(x^2)$$

$$= 4 + (1 - 4\cdot \overset{S+O(x^2)}{(2+3x)})\cdot 4 + O(x^2)$$

$$= 4 + (\overset{\cancel{-7}\ 0\ \text{mod } 7}{\cancel{1-8}}\ \ 12x)\cdot 4 + O(x^2)$$

$$= 4 + (0 - \overset{2x}{\cancel{5x}})\cdot 4 + O(x^2) = 4 + 8x + O(x^2) = 4 + x + O(x^2)$$

$$U_2 = U_1 + (1 - U_1\cdot \overset{S+O(x^4)}{S})\, U_1 + O(x^4)$$

$$= 4 + x + (1 - (4+x)(2+3x+4x^2+5x^3))\overset{(4+x)}{} + O(x^4)$$

ignore term with degree $\leq n$.

$$= 4 + x + (1 - (8 + 12x + 16x^2 + 20x^3 + 2x + 3x^2 + 4x^3))\overset{(4+x)}{} + O(x^4)$$

$$= 4 + x + (1 - (\overset{1}{\cancel{8}} + \cancel{14x} + \overset{5}{\cancel{19}}x^2 + \overset{3}{\cancel{24}}x^3))\overset{(4+x)}{} + O(x^4)$$

$$= 4 + x + (1 - 1 - 5x^2 - 3x^3)\overset{(4+x)}{} + O(x^4)$$

$$= 4 + x + (2x^2 + 4x^3)(4+x)$$

$$= 4 + x + (\overset{1}{\cancel{8}}x^2 + 2x^3 + \overset{2}{\cancel{16}}x^3) + O(x^4)$$

$$= 4 + x + x^2 + 4x^3 + O(x^4)$$

$$U_3 = U_2 + (1 - U_2 S)U_2 + O(x^8)$$

$$= (4+x+x^2+4x^3) + \left(\left(1 - (4+x+x^2+4x^3)(2+3x+4x^2+5x^3+6x^4 + x^6+2x^7)\right)\right) = V$$

$$(4+x+x^2+4x^3) + O(x^8)$$

$$V = 1 - (\cancel{8}^{1} + \cancel{12}^{5}x + \cancel{16}^{2}x^2 + \cancel{20}^{6}x^3 + \cancel{24}^{3}x^4 + 4x^6 + \cancel{8}^{1}x^7$$
$$+ 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + x^7$$
$$+ 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6$$
$$+ \cancel{8}^{1}x^3 + \cancel{12}^{5}x^4 + \cancel{16}^{2}x^5 + \cancel{20}^{6}x^6 + \cancel{24}^{3}x^7)$$

6 5 2 3 5 4 5 6 4 1 3 1 1 3 1 6 6 4

$$= 1 - (1 + \cancel{7x} + \cancel{7x^2} + \cancel{14x^3} + \cancel{17}^{3}x^4 + \cancel{13}^{6}x^5 + \cancel{16}^{2}x^6 + 5x^7)$$

$$= \cancel{1 - 1} - 3x^4 - 6x^5 - 2x^6 - 5x^7$$

$$V = 4x^4 + x^5 + 5x^6 + 2x^7$$

$$V \cdot U_2 = (4x^4 + x^5 + 5x^6 + 2x^7)(4+x+x^2+4x^3)$$

$$= \cancel{16}^{2}x^4 + 4x^5 + 4x^6 + \cancel{16}^{2}x^7 + 4x^5 + x^6 + x^7 + \cancel{20}^{6}x^6 + 5x^7$$

$$+ \cancel{8}^{1}x^7 = 2x^4 + 8x^5 + 11x^6 + 9x^7$$

4 4 2 1 5 1 6

$$VU_2 = 2x^4 + x^5 + 4x^6 + 2x^7$$

$$U_3 = U_2 + VU_2 + O(x^8)$$

$$\boxed{= 4 + x + x^2 + 4x^3 + 2x^4 + x^5 + 4x^6 + 2x^7}$$

$$= \ddot{S}^{-1} + O(x^8)$$