QINTRO - TD7

Ex. 5

1. Alice and Bob share $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

- Alice perform $R_{\theta_x}^{A}$, Bob perform $R_{\theta_y}^{B}$ on their qubits, where

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

(a) $(R_\theta^A \otimes R_\theta^B)|\phi\rangle$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(R_\theta^A \otimes R_\theta^B)(|0\rangle\otimes|0\rangle - |1\rangle\otimes|1\rangle)$$

$$= \frac{1}{\sqrt{2}}\left[(R_\theta^A \otimes R_\theta^B)|0\rangle\otimes|0\rangle - (R_\theta^A\otimes R_\theta^B)|1\rangle\otimes|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left(R_\theta^A|0\rangle \otimes R_\theta^B|0\rangle - R_\theta^A|1\rangle \otimes R_\theta^B|1\rangle\right.$$

$$R_\theta|0\rangle = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$

$$R_\theta|1\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\Big((\cos\theta^A|0\rangle_A + \sin\theta^A|1\rangle_A)\otimes(\cos\theta^B|0\rangle_B + \sin\theta^B|1\rangle)$$

$$- (-\sin\theta^A|0\rangle_A + \cos\theta^A|1\rangle_A)\otimes(-\sin\theta^B|0\rangle_B + \cos\theta^B|1\rangle_B)\Big)$$

$$= \frac{1}{\sqrt{2}}\Big((\cos\theta^A|0\rangle_A \cos\theta^B|0\rangle_B + \cos\theta^A|0\rangle_A\sin\theta^B|1\rangle + \sin\theta^A|1\rangle_A\cos\theta^B|0\rangle_B$$

$$+ \cos\theta^A|1\rangle_A\cos\theta^B|1\rangle_B) - (-\sin\theta^A\sin\theta^B|00\rangle_{AB} + \sin\theta^A\cos\theta^B|01\rangle_{AB}$$

$$+ \cos\theta^A\sin\theta^B|10\rangle_{AB} - \cos\theta^A\cos\theta^B|11\rangle_{AB})\Big)$$

$= \frac{1}{\sqrt{2}}\Big(\cos(\theta^A+\theta^B)(|00\rangle - |11\rangle)$

$\quad + \sin(\theta^A+\theta^B)(|01\rangle + |10\rangle)\Big)$

$\cos\theta^A \cos\theta^B - \sin\theta^A \sin\theta^B$
$= \cos(\theta^A+\theta^B)$

$\cos\theta^A \sin\theta^B + \sin\theta^A \cos\theta^B$
$= \sin(\theta^A+\theta^B)$

(b) Probability of winning the game?

Measuring $|\Phi_1\rangle$ in the computational basis, we obtain

$|00\rangle$ w.p. $\frac{1}{2}\cos^2(\theta_A+\theta_B)$, $(a=0, b=0)$

$|11\rangle$ w.p. $\frac{1}{2}\cos^2(\theta_A+\theta_B)$ $(a=b=1)$

$|01\rangle$ w.p $\frac{1}{2}\sin^2(\theta_A+\theta_B)$ $(a=0, b=1)$

$|10\rangle$ w.p. $\frac{1}{2}\sin^2(\theta_A+\theta_B)$ $(a=1, b=0)$

If inputs are

$x=0, y=0$ $\rightarrow$ win if get $|00\rangle$ or $|11\rangle$

So probability of winning is

& $x=0, y=1$

& $x=1, y=0$

$$\frac{1}{2}\left(\cos^2(\theta_A^x+\theta_B^y) + \cos^2(\theta_A^x+\theta_B^y)\right)$$
$$= \cos^2(\theta_A^x+\theta_B^y)$$

Alice and Bob win if $a \oplus b = 0$ i.e. $a=b$

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

Alice and Bob win if $a \oplus B = 1$ i.e. $a \neq b$.

If $x=y=1$. $\rightarrow$ win with prob. $\sin^2(\theta_A^x+\theta_B^y)$

$$\text{Probab. of winning} = \frac{1}{4}\Big(\cos^2(\underbrace{\theta_A^0+\theta_B^0}_{-\pi/8}) + \cos^2(\underbrace{\theta_A^0+\theta_B^1}_{\pi/8})$$
$$+ \cos^2(\underbrace{\theta_A^1+\theta_B^0}_{\pi/8}) + \sin^2(\theta_A^1+\theta_B^1)\Big)$$

$\frac{6\pi}{16} = \frac{3\pi}{8}$

$\theta_0^A = \theta_0^B = -\frac{\pi}{16}$ and $\theta_1^A = \theta_1^B = \frac{3\pi}{16}$

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expectation value

$$E(f) = \sum_{x \in X} P(x) \times f(x)$$

2.

x	y	$x \wedge y$	$(-1)^{x \wedge y}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	-1

First three rows: Alice and Bob wins if $ab = 1$ ie. $a = b$

Last row: Alice and Bob wins if $ab = -1$ ie. $a \neq b$

(a) $[ab] = \sum_{a,b \in \{-1,1\}} ab \underbrace{\langle \psi | A_a^x \otimes B_b^y | \psi \rangle}_{\text{prob. of outputting } a,b}$ $\longleftrightarrow$ $E[ab|xy]$

$= 1 \times P(ab=1|x,y) + (-1) \times P(ab=-1|x$

$$= \langle \psi | A_1^x \otimes B_1^y | \psi \rangle - \langle \psi | A_1^x \otimes B_{-1}^y | \psi \rangle - \langle \psi | A_{-1}^x \otimes B_1^y | \psi \rangle$$
$$+ \langle \psi | A_{-1}^x \otimes B_{-1}^y | \psi \rangle$$
$$= \langle \psi | (A_1^x - A_{-1}^x) \otimes (B_1^y - B_{-1}^y) | \psi \rangle = \langle \psi | A^x \otimes B^y | \psi \rangle$$

$$\begin{cases} A^x = A_1^x - A_{-1}^x \\ B^y = B_1^y - B_{-1}^y \end{cases}$$

(b) $P(ab=1|x,y) + P(ab=-1|xy) = 1.$

$2P[ab=s|x,y] = E[ab|xy] + P(ab=-1|xy) + P(ab=1|x,y)$

$$= 1 + s \langle \psi | A^x \otimes B^y | \psi \rangle$$

$$P[ab=s|x,y] = \frac{1}{2} + \frac{s}{2} \langle \psi | A^x \otimes B^y | \psi \rangle$$

(c) $C = A^0 \otimes B^0 + A^0 \otimes B^1 + A^1 \otimes B^0 - A^1 \otimes B^1$

4 different possible cases to win.

- $ab = 1$, and $x = y = 0$

$\frac{1}{2} + \frac{1}{2}\langle\psi|A^0 \times B^0|\psi\rangle \to \alpha$

- $ab = 1$, $x = 0, y = 1$

$\frac{1}{2} + \frac{1}{2}\langle\psi|A^0 \otimes B^1|\psi\rangle \to \beta$

- $ab = 1$, $x = 1$, $y = 0$

$\frac{1}{2} + \frac{1}{2}\langle\psi|A^1 \otimes B^0|\psi\rangle \to \lambda$

- $ab = -1$, $x = 1$, $y = 1$

$\frac{1}{2} + \frac{(-1)}{2}\langle\psi|A^1 \otimes B^1|\psi\rangle \to \gamma$

Sum all probability, than we get

$$\frac{1}{4}(\alpha + \beta + \lambda + \gamma)$$

$$= \frac{1}{4}\left(2 + \frac{1}{2}\langle\psi|C|\psi\rangle\right) = \frac{1}{2} + \frac{1}{8}\langle\psi|C|\psi\rangle$$

(d) $C = \underbrace{A^0 \otimes B^0} + \underbrace{A^0 \otimes B^1} + \underbrace{A^1 \otimes B^0} - \underbrace{A^1 \otimes B^1}$

$C^2 = 4I + (A^0A^1 - A^1A^0) \otimes (B^1B^0 - B^0B^1)$

$(A^x)^2 = (B^y)^2 = I \quad x, y \in \{0, 1\}$

$I \otimes A^0B^1 \otimes B^0A^1 \otimes B^0B^1$

$$\begin{aligned}
C^2 = & \Big[\underbrace{(A^0 \otimes B^0)(A^0 \otimes B^0)}_{I} + (A^0 \otimes B^0)(A^0 \otimes B^1) + (A^0 \otimes B^0)(A^1 \otimes B^0) - (A^0 \otimes B^0)(A^1 \otimes B^1)\Big] \\
& + \Big[(A^0 \otimes B^1)(A^0 \otimes B^0) + \underbrace{(A^0 \otimes B^1)(A^0 \otimes B^1)}_{I} + (A^0 \otimes B^1)(A^1 \otimes B^0) - (A^0 \otimes B^1)(A^1 \otimes B^1)\Big] \\
& + \Big[(A^1 \otimes B^0)(A^0 \otimes B^0) + (A^1 \otimes B^0)(A^0 \otimes B^1) + \underbrace{(A^1 \otimes B^0)(A^1 \otimes B^0)}_{I} - (A^1 \otimes B^0)(A^1 \otimes B^1)\Big] \\
& + \Big[-(A^1 \otimes B^1)(A^0 \otimes B^0) - (A^1 \otimes B^1)(A^0 \otimes B^1) - (A^1 \otimes B^1)(A^1 \otimes B^0) + \underbrace{(A^1 \otimes B^1)(A^1 \otimes B^1)}_{I}\Big]
\end{aligned}$$

Use $(M\otimes N)(M'\otimes N') = MM'\otimes NN'$

$$C^2 = \left[I + \cancel{B^0B^1} + \cancel{A^0A^1} - (A^0A^1 \otimes B^0B^1)\right]$$
$$+ \left[\cancel{B^1B^0} + I + (A^0A^1\otimes B^1B^0) - \cancel{A^0A^1}\right]$$
$$+ \left[\cancel{A^1A^0} + (A^1A^0\otimes B^0B^1) + I - \cancel{B^0B^1}\right]$$
$$+ \left[-(A^1A^0 \otimes B^1B^0) - \cancel{A^1A^0} - \cancel{B^1B^0} + I\right]$$

$$= 4I - (A^0A^1\otimes B^0B^1) + (A^0A^1\otimes B^1B^0) + (A^1A^0\otimes B^0B^1) - (A^1A^0\otimes B^1B^0)$$

$$= 4I + (A^1A^0 - A^0A^1)\otimes(B^1B^0 - B^0B^1)$$

e) $\langle\psi|C^2|\psi\rangle = \langle\psi|\, 4I + (A^{01} - A^{10})\otimes(B^{10} - B^{01})\,|\psi\rangle$

$$= \overset{=4}{\langle\psi|4I|\psi\rangle} + \overset{\leq 1}{\langle\psi|A^{01}\otimes B^{01}|\psi\rangle} -$$
$$\overset{\leq 1}{\langle\psi|A^0A^1\otimes B^1B^0|\psi\rangle} - \overset{\leq 1}{\langle\psi|A^1A^0\otimes B^{01}|\psi\rangle}$$
$$+ \overset{\leq 1}{\langle\psi|A^1A^0\otimes B^1B^0|\psi\rangle} \leq 4+4 = 8$$

f) $\frac{1}{2} + \frac{\sqrt{8}}{8} = \cos^2\left(\frac{\pi}{8}\right)$ $\qquad \langle\psi|C|\psi\rangle \le \sqrt{8}$

From (c), probability of winning $P = \frac{1}{2} + \frac{\langle\psi|C|\psi\rangle}{8}$

$\langle\psi|C|\psi\rangle \le \sqrt{8} \implies P \le \frac{1}{2} + \frac{\sqrt{8}}{8}$

RECAP

goal of ex. 5 : Prove that the best quantum strategy wins the CHSH game with probability $\cos^2\left(\frac{\pi}{8}\right)$

Part 1 : We exhibit such a quantum strategy to show that there exists a quantum strategy winning with prob. $\cos^2\left(\frac{\pi}{8}\right)$

Part 2 : We show that the winning probability of any quantum strategy is $\le \cos^2\left(\frac{\pi}{8}\right)$

①

(a), (b), (c) $\implies$ winning prob. of general quantum strategy is $\frac{1}{2} + \frac{1}{8}\langle\psi|C|\psi\rangle$

$\to$ eq(3) ②

(d), (e) $\implies$ Find upper bound of $\langle\psi|C^2|\psi\rangle$

(f) $\implies$ Deduce upper bound of $\langle\psi|C|\psi\rangle$

$\hookrightarrow$ eq(4) ③

3 Missing Steps.

① A^x and B^y are unitary with ± 1 eigenvalues

$A^x = A^x_1 - A^x_{-1}$ $\qquad$ in a certain basis, $A^x_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ $\qquad A^x_{-1} = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

$A^x = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$

② eq(3): $|\langle\psi|A^{x_0}A^{x_1} \otimes B^{y_0}B^{y_1}|\psi\rangle| \leq 1$

$$|\langle\psi|A^x|\psi\rangle| \leq \sum_i |x_i|^2 \underbrace{|\langle e_i|A^x|e_i\rangle|}_{\pm 1} = \sum_i |x_i|^2 = 1$$

$$|\psi\rangle = \sum_i x_i |e_i\rangle$$

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

③ $\max \langle\phi|C^2|\phi\rangle \leq 8$

Show $\Rightarrow \langle\phi|C|\phi\rangle \leq \sqrt{8}$

$x \geq 0 \iff$ all eigenvalues are ≥ 0

$\iff \forall|\phi\rangle,\ \langle\phi|x|\phi\rangle \geq 0$

$\langle\lambda|x\lambda\rangle = \lambda\langle\lambda|\lambda\rangle = \lambda$

show $\max\langle\phi|C^2|\phi\rangle$ = largest eigenvalue of $C^2 = \lambda_{max}$

1) Show that for many $|\phi\rangle$, $\langle\phi|C^2|\phi\rangle \leq \lambda_{max}$

$$C^2 = \sum_i \lambda_i |e_i\rangle\langle e_i|$$

$$\langle\phi|C^2|\phi\rangle = \sum_i \lambda_i |x_i|^2 \leq \sum_i \lambda_{max} |x_i| = \lambda_{max}$$

1

2) Take $|\phi\rangle = |\lambda_{max}\rangle \to \langle\phi|\lambda_{max}|\phi\rangle = \lambda_{max}$

$\langle\phi|C^2|\phi\rangle = \lambda_{max}$

$\Rightarrow \lambda_{max} \leq 8$

largest eigenvalue of C is $\sqrt{\lambda_{max}} \leq \sqrt{8} = \langle\phi|C|\phi\rangle$