SORBONNE UNIVERSITÉ

Fondement de l'algorithmique algébrique - FLAG

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Notes

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Contents

I. Introduction

I.1. Ring and Field

Rough definition : Field is a set F with two operations $(+, \times)$ with "usual" properties (e.g. $a(b+c) = ab+ac$, $a(bc) = (ab)c$ etc...). And every element has an inverse. If one of the elements does not have an inverse, it is a ring.

- So, elements in the field can be inverted except 0.

- Every element $a \in F$ has an opposite for $+$, these act on b s.t. $a + b = 0$. This is true for all rings and fields.

- Therefore, invertibility is about "$\times$"

Remark : Ring $\leftarrow$ Field

Examples

- $\mathbb{R}$ the set of real numbers (field). $\mathbb{Q}$ rational (field). $\mathbb{C}$ complex (field).

- $\mathbb{Z}$ integers is not a field, thus a ring. Why?

This is because 2 is not invertible. For instance, there is no $x \in \mathbb{Z}$ s.t. $2x = 1$.

- $\mathbb{Z}/p\mathbb{Z}$ is a field. In general, $a \in \{1, ..., p-1\}$, there are u, v s.t. $au + pv = 1$, so $au = 1 \, mod \, p$.

- $\mathbb{Z}/4\mathbb{Z}$ is not a field, why? $\rightarrow$ because 2 is not invertible.

I.2. Complexity

In this course, we manipulate algebraic objects. For instance, polynomials (ring), matrices (field), and more. Complexity measures the efficiency of algorithms. Algebraic complexity is counting the number of operations in the base ring and field.

The complexity here does not take into account of ...

- all memory-related operations

- the size of the elements in the base ring or field. (e.g. : addition of polynomials of degree $\leq d$ in $\mathbb{Q}[x]$, algebraic complexity is $O(d)$ and ignore the size of the rational coefficients.

*You should be familiar with the complexity from MODEL.

I.3. Matrices

For a field $\mathbb{K}$ or denote by $\mathbb{K}^{m \times n}$ the set of $m \times n$ matrices over $\mathbb{K}$.

Recall : Not a ring because .... ??

It is represented as a two-dimensional array with rows and columns. $A = (a_{i\,j}) \in \mathbb{K}^{m \times n}$ where $0 \leq i \leq m$ and $0 \leq j \leq n$.

Addition

Complexity is $O(m\,n)$, operations in $\mathbb{K}$.

Multiplication

Multiplication of $A \in \mathbb{K}^{m \times n}$ by $B \in \mathbb{K}^{n \times p}$

- Naive algorithm : $O(m\,n\,p)$

- Strassen's algorithm (recall form MODEL) has complexity $O(n^{log_2(7)})$ where $m = n = p$

- As an alternative, there is a lot of work on matrix multiplication algorithms. The best performance today is $O(n^{2.37})$ but for the moment it is impractical.

I.4. Polynomials

For a field $\mathbb{K}$ or a ring $\mathbb{R}$, the sets $\mathbb{K}[x]$, $\mathbb{R}[x]$ are those of polynomials in one variable x.

$\mathbb{K}[x] = \{P = P_0 + P_1 x + P_2 x^2 + ... + P_d x^d\}$ in $\mathbb{K}$. (same for $\mathbb{R}$)

They are represented as a one-dimensional array $[P_0, P_1, P_2, ..., P_d]$.

Addition

of two polynomials in $\mathbb{R}[x]$ of degree $\leq d$ has a complexity $O(d)$ operations in $\mathbb{R}$.

Multiplication :

- Naive algorithm : $O(d^2)$ operations in $\mathbb{R}$ - Karatsuba (recall from MODEL) : $O(d^{log_2(3)}) \approx O(d^{1.584})$ operations in $\mathbb{R}$

- For alternative FFT (again, recall from MODEL) : $O(d\,log(d)\,log\,log(d))$

II. Euclid's division and GCD

- $A|B$ means A divides B. Meaning there is no reminder (and A and B are polynomials here)

II.1. Algebraic structure

Definition : $\mathbb{K}$ field

$f \in \mathbb{K}[x] \Rightarrow f = a_0 + a_1 x + ... + a_n x^n$ with $a_i \in \mathbb{K}$

$n = deg(F)$

- Roots : $\alpha \in \mathbb{K} s.t f(\alpha) = 0$

Lemma : α is a root of $f \iff (x - \alpha)|f$, in other word $f \in \mathbb{K}[x]$ and $\alpha \in \mathbb{K} s.t\ f(\alpha) = 0$, the $(X - a)|f$

Lemma : f has degree n, then f has at most n roots.

Definition (Algebraic closure) : $\overline{\mathbb{K}}$ is the algebraic closure of K if :

- $\mathbb{K} \subset \overline{\mathbb{K}}$

- $\forall f \in \overline{\mathbb{K}}\ [x], f$ has exactly $degf$ roots in $\overline{\mathbb{K}}$.

Example :

$\mathbb{C}$ is algebraic closed.

$\mathbb{R} \subset \mathbb{C}$ and $\mathbb{Q} \subset \mathbb{C}$ are algebraic closure.

But, $\mathbb{R}$ is not $\rightarrow x^2 + 1$ has 0 roots in $\mathbb{R}$.

Definition (Recalling a definition of ring) :

Let R a set equipped with two binary operation s.t $(R, +, \times)$. It is said to be a ring if the followings hold $\rightarrow$

- $R, +)$ commutative group with neutral 0_R (what is 0_R means ???) - $\times$ has a neutral element 1_R and is associative

$a \times b \times c = (a \times b) \times c = a \times (b \times c)$

- $\times$ is distributive with respect to $+$.

$a(b + c) = ab + ac$

Example :

$\mathbb{Z}, \mathbb{K}[x], \mathbb{R}, \mathbb{C}$, but $\mathbb{N}$ is not a ring because $-1 \in \mathbb{N}$.

Proposition :

$f = a_0 + ... + a_n x^n$ with $a_i \in \mathbb{R}$, $\mathbb{R}[x]$

if R is a ring so is $R[x]$

Definition (Monic Polynomial): Monic polynomial is a polynomial with a leading coefficient equal to 1.

Example

- Monic polynomial : $x + 1$, $x^2 - 7x + 99$, $x^{1000} + x^{99} - 10000$

- Polynomial that are not monic : $5x^{99} + 4$, $8x^3 + x^2 - 7x + 0$

Definition :

Let R be a ring and $f \in R[x]$ be monic. Also, $s, t \in R[x]$.

- We say that s and t are equivalent modulo $f \iff s \equiv t \ mod\, f$, if f divides $s - t$.

- The set of classes of equivalences is denoted by $R[x]/\langle f \rangle$ and is called quotient ring.

Example :

$x^2 + x + 1 \equiv 1 \ mod \ x + 1$, since $x^2 + x + 1 = x(x + 1) + 1$

II.2. Division algorithm for polynomials in $\mathbb{K}[x]$

Definition (Euclidean domain) :

$\mathbb{R}$ is a Euclidean domain if there exists a Euclidean division (and R is an integral domain).

Algorithm 1: PolynomialDivisionAlgorithm

Input: Two polynomials $A = a_m X^m + \cdots + a_0$ and $B = b_n X^n + \cdot + b_0$ in $\mathbb{K}[X]$.
Output: Two polynomials $Q = q_{m-n} X^{m-n} + \cdots + q_0$ and $R = r_p X^p + \cdots + r_0$ in $\mathbb{K}[X]$ such that $A = BQ + R$ and $p = \deg R < \deg B = n$.

$R := A$, $Q = 0$, $b = \mathrm{lc}(B)$
While $\deg R \geq \deg B$ **do**
  $a := \mathrm{lc}(R)$ → it is the leading coefficient of $f = a_0 + ... + a_n x^n$
  $Q := Q + \frac{a}{b} X^{\deg R - \deg B}$
  $R := R - \frac{a}{b} X^{\deg R - \deg B} B$
Return (Q, R)

Proposition:

On input A and B in $\mathbb{K}[x]$ with degree m and n, with $m > n$. Polynomial division algorithm perform $O(n(m - n))$ arithmetic operation in $\mathbb{K}$.

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$R := A, Q = 0, b = \mathrm{lc}(B)$
While $\deg R \geq \deg B$ **do** ← $m-n$ iterations
  $a := \mathrm{lc}(R)$
  $Q := Q + \frac{a}{b} X^{\deg R - \deg B}$
  $R := R - \frac{a}{b} X^{\deg R - \deg B} B$
  ← n operations at each iteration.
Return (Q, R)

Remark:

(1) If $\mathbb{K}$ is just a ring, the algorithm works if and only if B is monic.

(2) $A \equiv R \ mod \ B \quad \rightarrow$ Euclidean division allows to perform operations in $\mathbb{K}[x]/(B)$.

$A_1 + A_2 \equiv R_1 + R_2 \ mod \ B$

$A_1 \times A_2 \equiv R_1 \times R_2 \ mod \ B$

II.3. Euclidean algorithm

Definition:

R Euclidean domain and $a, b \in \mathbb{R}$, g is a gcd of a and b : $g = gcd(a, b)$

if : $g \in R$

$g|a$

$g|b$

any common divisor of $a$and$b$ divides g.

Proposition :

In R Euclidean, such a gcd always exist. **Remark** : g may not be unique. **Proposition** :

If $a = bq + r$ with $h(r) < h(b)$ then, $gcd(a, b) = gcd(b, r)$.

Algorithm 2: EuclidAlgorithm

Input: Two elements a and b in a Euclidean domain $\mathcal{R}$ with a height function h.
Output: A gcd of a and b in $\mathcal{R}$.

$r_0 := a, r_1 := b, i := 1$
While $r_i \neq 0$ **do**
  $r_{i+1} := \mathrm{rem}(r_{i-1}, r_i)$
  $i := i + 1$
Return r_{i-1}

Complexity : $O(deg(a) \times deg(b))$

Proposition: $a, b \in R$

If $g = gcd(a, b)$ then, $\exists (u, v) \in R^2$ s.t $au + bv = g$ and $h(ug) < h(b), h(vg) < h(a)$

- u and v are cofactors.

EEA

Algorithm 3: ExtendedEuclideanAlgorithm

Input: Two elements a and b in a Euclidean domain $\mathcal{R}$ with a height function h.
Output: A gcd of a and b in $\mathcal{R}$ together with the corresponding cofactors.
$r_0 := a, u_0 := 1, v_0 := 0.$
$r_1 := b, u_1 := 0, v_1 := 1, i := 1$
While $r_i \neq 0$ **do**
  $(q_i, r_{i+1}) :=$ PolynomialDivisionAlgorithm(r_{i-1}, r_i)
  $u_{i+1} = u_{i-1} - q_i u_i, v_{i+1} = v_{i-1} - q_i v_i$
  $i := i + 1$
Return $r_{i-1}, u_{i-1}, v_{i-1}$

Complexity : $O(deg(a) \times deg(b))$

Application of Extended Euclidean Algorithm: Modulo inversion

If a and b are coprimers, then $\exists (u, v) / au + bv = 1$.

So, $au \equiv 1 \ mod \ b \ bv \equiv 1 \ mod \ a$

- u is a inverse of $a \ mod \ b$

- v is a inverse of $b \ mod \ a$

If $a \in \mathbb{Z}$ and $au + bv = 1$, then $a^{-1} \equiv u \ mod \ n$

$\bar{a} = a + kn, K \in \mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$, $\bar{a}^{-1} = \bar{u} = u + kn, k \in \mathbb{Z}$

- If n is prime number then for all $a \in \mathbb{Z}$, $gcd(a, n) = 1$. So for all $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$, $\bar{a}^{-1}$ exists.

  Proposition : $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is prime.

Definition :

$P \in \mathbb{K}[x]$ is irreducible if for any $\mathbb{Q}, \mathbb{R} \in \mathbb{K}[x]$ s.t $P = QR$, then either $\mathbb{Q} \in \mathbb{K}$ or $\mathbb{R} \in \mathbb{K}$.

Proposition : If P is irreducible then $\forall \mathbb{Q} \in \mathbb{K}[x], \quad gcd(P, Q) = 1$

Theorem : $\mathbb{K}[x]/(P)$ is a field if and only if P is irreducible

: we can computer the inverse with Extended Euclidean Algorithm.

III. Finite field

Definition: A finite field is a field with a finite number of elements.

III.1. Integer $\mathbb{Z}$

- for $n \in \mathbb{Z}/0 \approx 0, ..., n-1$ with add/ multiplication modulo n

- $a \in \mathbb{Z}/n\mathbb{Z}$ is invertible if and only if $gcd(a, n) = 1$

- n is prime $\iff$ $\mathbb{Z}/n\mathbb{Z}$ is a field

- computing a^{-1} : run EEA to obtain $1 = au + nv$

Theorem (Bezout's relation) :

Let $R = \mathbb{Z} or R = \mathbb{K}[x]$. If a and b in R, there exist u and v in R s.t $au + bv = gcd(a, b)$

Theorem:

Let $R = \mathbb{Z} or R = \mathbb{K}[x]$. If a and b are coprime, then a invertible modulo b and b invertible modulo a. Thus, $\mathbb{Z}/n\mathbb{Z}$ is a field, if and only if n is a prime.

III.2. Polynomial $\mathbb{K}[x]$

(where $\mathbb{K}$ is a field) - for $f \in \mathbb{K}[x] \setminus \{0\}$, $\mathbb{K}[x]/\langle f\rangle \approx \{P(X) \in \mathbb{K}[x]/deg(p) < deg(f)\}$ with add/multiplication mod f.

- $P \in \mathbb{K}[x]/\langle f\rangle$: P is invertible if and only if $gcd(p, f) = 1$

-f is invertible $\iff$ $\mathbb{K}[x]/\langle f\rangle$ field.

- computing P^{-1}: run EEA to obtain $1 = pu + fv$

- $\mathbb{K}[x]/\langle f\rangle$ is a field, for irreducible polynomial f.

Proof:

Suppose f irreducible, let $P \in \mathbb{K}[x]/\langle f\rangle \setminus \{0\}$. To show that P is invertible which means $gcd(P, f) = 1$. The gcd of P and f divides both P and f. But f has only $\mathbb{K}\{0\}$ and $\propto f, \propto \in \mathbb{K} \setminus \{0\}$ as divides. Since $deg(P) < deg(f)$, P cannot be divisible by f, so $gcd(P, f) = 1$.

III.3. Building a finite field

- If $K = \mathbb{Z}/p\mathbb{Z}$ and $deg(f) = d$ then $\mathbb{Z}/p\mathbb{Z}[x]/\langle f\rangle$ is a finite field of cardinality p^d. This is because $\mathbb{Z}/p\mathbb{Z}[x]/\langle f\rangle = \{a_0 + a_1x + ... + a_{d-1}x^{d-1}, (a_0, a_1....., a_{d-1}) \in (\mathbb{Z}/p\mathbb{Z})^d\}$ and cardinality of $(\mathbb{Z}/p\mathbb{Z}))^d$ is p^d

Theorem

A finite field must have p elements for same prime p and $d \in \{1, 2, ...\}$. If $d = 1$ then finite field is $\mathbb{Z}/p\mathbb{Z}$. If $d > 1$, then $\mathbb{Z}/p\mathbb{Z}$ is not a field.

- $\mathbb{F}_q$ for $q = p^d$ is the notation for a finite field of cardinality q

III.6.