QINTRO - TD9

Ex.1

1. $M = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$

$$M = \alpha X + \beta Y$$

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha - i\beta \\ \alpha + i\beta & 0 \end{pmatrix}$$ thus, $\begin{cases} x = \alpha - i\beta \\ y = \alpha + i\beta \end{cases}$

$\Rightarrow \begin{cases} \alpha = \dfrac{x+y}{2} \\ \beta = i\dfrac{x-y}{2} \end{cases}$

2. M : any 2×2 complex matrix.

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

$$M = \alpha I + \beta X + \gamma Y + \delta Z$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}}_{\text{express in terms of } I \text{ and } Z} + \underbrace{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}}_{\text{express in terms of } X \text{ and } Y \text{ using (1)}}$$

$$= \alpha I + \delta Z + \begin{pmatrix} 0 & \beta - i\gamma \\ \beta + i\gamma & 0 \end{pmatrix}$$

M

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha+\delta & 0 \\ 0 & \alpha-\delta \end{pmatrix} + \begin{pmatrix} 0 & \beta - i\gamma \\ \beta + i\gamma & 0 \end{pmatrix}$$

$\begin{cases} \alpha + \delta = a \\ \alpha - \delta = d \end{cases}$ $\quad \alpha = \dfrac{a+d}{2}$, $\delta = \dfrac{a-d}{2}$

$$M = \frac{a+d}{2} I + \frac{b+c}{2} X + \frac{ic-b}{2} Y + \frac{a-d}{2} Z$$

3. $$XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = iY$$

$$\mathrm{tr}(XZ) = 0$$

$$XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iZ$$

$$\mathrm{tr}(XY) = 0$$

$$YZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = iX$$

$$\mathrm{tr}(YZ) = 0$$

$X^2 = Id$ $\quad Y^2 = Id$ $\quad Z^2 = Id$

$\mathrm{tr}(X^2) = 2$ $\quad \mathrm{tr}(Y^2) = 2$ $\quad \mathrm{tr}(Z^2) = 2$

Ex.1

4) $U = \alpha I + \beta X + \gamma Y + \delta Z$

$U^\dagger = \alpha^* I + \beta^* X + \gamma^* Y + \delta^* Z$

$$tr(UU^\dagger) = tr\left((\alpha I + \beta I + \gamma Y + \delta Z)(\alpha^* I + \beta^* X + \gamma^* Y + \delta^* Z)\right)$$

$$= tr\left(\alpha I \alpha^* I + \beta X \beta^* X + \gamma Y \gamma^* Y + \delta Z \delta^* Z\right)$$

$$= 2\left(\alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + \delta\delta^*\right)$$

$$= 2\left(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2\right)$$

$UU^\dagger = I$ and $tr(UU^\dagger) = 2$

So, $2(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2) = 2$

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$$

Ex. 2

1. $H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}(X + Z)$

2. H-error => phase flip.

$|\tilde{0}\rangle = |000\rangle$

$|\tilde{1}\rangle = |111\rangle$

$|\tilde{\tilde{0}}\rangle = |\tilde{+}\tilde{+}\tilde{+}\rangle$ where $|\tilde{+}\rangle = \frac{|\tilde{0}\rangle + |\tilde{1}\rangle}{\sqrt{2}}$

$|\tilde{\tilde{1}}\rangle = |\tilde{-}\tilde{-}\tilde{-}\rangle$ $\quad |\tilde{-}\rangle = \frac{|\tilde{0}\rangle + |\tilde{1}\rangle}{\sqrt{2}}$

A state $\alpha|0\rangle + \beta|1\rangle$ is mapped to
$|\Omega\rangle = \alpha|\tilde{\tilde{0}}\rangle + \beta|\tilde{\tilde{1}}\rangle$ using Shor's code

-> Compute $H_1|\Omega\rangle$, error

$|\tilde{\tilde{0}}\rangle = |\tilde{+}\tilde{+}\tilde{+}\rangle = (|000\rangle + |111\rangle)^{\otimes 3}$

$|\tilde{\tilde{1}}\rangle = |\tilde{-}\tilde{-}\tilde{-}\rangle = (|000\rangle - |111\rangle)^{\otimes 3}$

$|\Omega\rangle = \alpha\left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right)^{\otimes 3} + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|000\rangle - |111\rangle\right)^{\otimes 3}$

$$H_1|\Omega\rangle = \alpha\left(\left(\frac{|+00\rangle}{\sqrt{2}} + \frac{|-11\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|000\rangle}{\sqrt{2}} + \frac{|111\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|000\rangle}{\sqrt{2}} + \frac{|111\rangle}{\sqrt{2}}\right)\right)$$
$$+ \beta\left(\left(\frac{|+00\rangle}{\sqrt{2}} + \frac{|-11\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|000\rangle}{\sqrt{2}} + \frac{|111\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|000\rangle}{\sqrt{2}} + \frac{|111\rangle}{\sqrt{2}}\right)\right)$$

Ex.2

2. $X_1|\Omega\rangle = \alpha\left(\frac{1}{\sqrt{2}}(|100\rangle + |011\rangle) + \left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right)^{\otimes 2}\right)$

$+\beta\left(\frac{1}{\sqrt{2}}(|100\rangle + |011\rangle) + \left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right)^{\otimes 2}\right)$

After the process for bit-flip detection and correction

$\alpha\left(\frac{1}{\sqrt{2}}(|100\rangle + |011\rangle)|1\rangle + \left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)|0\rangle\right)^{\otimes 2}\right)$

...

Correct by applying X_1, we get

$\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$

For $Z_1|\Omega\rangle$.

$\alpha\left(\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) \otimes \left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right)^{\otimes 2}\right)^{\otimes 2}$

$+\beta\cdot\left(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \otimes \left(\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)\right)^{\otimes 2}\right)^{\otimes 2}$.

After correction of the bit-flip, the state is

$$\frac{1}{\sqrt{2}}\left(|\Omega\rangle + Z|\Omega\rangle\right)$$

Express $|\Omega\rangle$ in the $\{|\hat{+}\rangle, |\hat{-}\rangle\}$ basis to apply the phase detection and correction.

$$|\Omega\rangle = \alpha|\hat{+}\hat{+}\hat{+}\rangle + \beta|\hat{-}\hat{-}\hat{-}\rangle$$

$$Z_1|\Omega\rangle = \alpha|\hat{-}\hat{+}\hat{+}\rangle + \beta|\hat{+}\hat{-}\hat{-}\rangle$$

- No phase shift in $|\Omega\rangle \rightarrow$ syndrome $|0\rangle$
  $\Rightarrow$ after detection get $|\Omega\rangle|0\rangle$.

For $Z_1|\Omega\rangle$. $\rightarrow$ syndrome 1

$\Rightarrow$ phase flip on 1st register.

$\Rightarrow$ correction : $\alpha|\hat{+}\hat{+}\hat{+}\rangle + \beta|\hat{-}\hat{-}\hat{-}\rangle$