FLAG - TD

- Multiplication of two polynomials requires n^2 multiplications and $(n-1)$ additions

- karatsuba

$$f = f_0 + x f_1$$

$$g = g_0 + x g_1$$

$$f \cdot g = f_0 g_0 + (f_0 g_1 + g_1 f_0) x + f_1 g_1 x^2 \qquad n = 2$$

$$f_0 g_1 + g_0 f_1 = (f_0 + f_1)(g_0 + g_1) - f_0 g_0 - f_1 g_1$$

3 multiplication , 4 additions

It is under the assumption that the cost of multiplication is more expensive than addition. Karatsuba reduce the number of multiplication in cost of increasing number of additions.

```
Karatsuba(F, G)  # size = n.
  If n = 1 :
     return F·G
  Else :
    F = F0 + F1 x^⌈n/2⌉
    G = G0 + G1 x^⌈n/2⌉
    M0 = Karatsuba(F0, G0)
    M1 = Karatsuba(F0 + F1). (G0 + G1)
    M2 = Karatsuba (F1 , G1)
    M = M0 + (M1 - M0 - M2) x^(n/2) + M2 x^n
    return M.
```

Ex

$F = 2 + x + 2x^2 + 3x^3$

$G = 1 + 4x + 3x^2 + x^3$

$$\boxed{\mathbb{Z}/7\mathbb{Z}}$$

$F = \underbrace{(2+x)}_{F_0} + \underbrace{(2+3x)}_{f_1 \to \deg \leq n = 2} x^2$

$n = 4$

$m = 2$

$G = \underbrace{(1+4x)}_{G_0} + \underbrace{(3+x)x^2}_{G_1}$

$(F_0 + F_1)(G_0 + G_1) - H_0 - H_2$

$H = F \cdot G = H_0 + H_1 x^2 + H_2 x^4$

$F \cdot G = \overset{H_0}{(2+x)(1+4x)} + \overset{H_1}{\left[(2+x)(3+x) + (2+3x)(1+4x)\right]} x^2 + \overset{H_2}{\left[(2+3x)(3+x)\right]} x^4$

$-H_0 - H_2$

$H_0 = 4x^2 + 2x + 2$

$H_2 = 3x^2 + 4x + 6$

$H_1 = 2 + x + 6x^2 - 4x^2 - 2x - 2 - 3x^2 - 4x - 6$

$= 6x^2 + 2x + 1$?

x^2: $-1 \bmod 7 = 6$

x: $-5 \bmod 7 = 2$

$-6 \bmod 7 = 1$

$H_0 = 4x^2 + 2x + 2$

$H_1 x^2 = x^2 + 2x^3 + 6x^4$

$H_2 x^4 = 6x^4 + 4x^5 + 3x^6$.

$H_0 + H_1 x^2 + H_2 x^4$

$= 2 + 2x + 5x^2 + 2x^3 + 5x^4 + 4x^5 + 3x^6$

x^4: $12 \bmod 7 = 5$

Master theorem

$m = 3$ (number of recursive call)

$C(n) \leq m\, C(n/p) + T(n)$

$p = q = 2$

$T(n) = 4n$

$q < m$

$k(n) \leq 3k(n/2) + 4n$

$C(n) = k(n) = O\left(n^{\log_p (m/q)} T(n)\right) = O\left(n^{\log_2 (3/2)}\, 4n\right)$

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- Polynomial division

Input : f. degree n

g. degree m where $m < n$

Output : q; r where $f = q \cdot g + r$

$deg(r) < deg(g)$

Example

$f = 4x^3 + 2x^2 + 5x + 1$

$g = x^2 - 1$

$f/g = ?$

$$
\begin{array}{r l}
 & 4x + 2 \\
x^2 - 1 \,\big)\!\! & 4x^3 + 2x^2 + 5x + 1 \\
 & -(4x^3 - 4x) \longrightarrow q \\ \hline
 & 2x^2 + 9x + 1 \\
 & -(2x^2 + 0x - 2) \\ \hline
 & 9x + 3 \rightarrow r
\end{array}
$$

$$
\begin{aligned}
(q \cdot g) + r &= (x^2 - 1)(4x + 2) + (9x + 3) \\
&= 4x^3 + 2x^2 - 4x - 2 + 9x + 3 \\
&= 4x^3 + 2x^2 + 5x + 1 \checkmark
\end{aligned}
$$

Polynomial division

Input : F, G Output : Q, R

lead(x) returns leading term of x.

$R := F$; $Q := 0$

while($deg(R) \geq deg(G)$) — $O(n)$

$t = \dfrac{lead(f)}{lead(g)} X^{(\underset{m}{deg(R)} - \underset{n}{deg(G)})}$ $O(m-n)$

$Q := Q + t$

$R := R - t \cdot G$

$O(n(m-n))$