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$A, B \in k[x]$

$n = \text{degree}(A)$, $m = \text{degree}(B)$

easier than compute gcd

$d = \deg(\gcd(A,B)) = m + n - \text{rank}(\text{Syl}(A,B))$

I have to recall how to compute rank of matrix

$uA + vB = G = \gcd(A,B)$ — linear system with quasi-Toeplitz matrix

$\exists (u, v)$, $\deg(u) < m - d$

$\deg(v) < n - d$

rational num.

If we're working with $\mathbb{Q}$, you can say leading coeff of gcd, G is 1.

ex $G = 5a + x$

Ex

$A = x^3 + 2x^2 + x + 4$

$B = 3x^2 + 2x + 1$

$A \cdot B = (x^3 + 2x^2 + x + 4)(3x^2 + 2x + 1)$

$= 3x^5 + 2x^4 + x^3 + 6x^4 + 4x^3 + 2x^2 + 3x^3 + 2x^2 + x$
$+ 12x^2 + 8x + 4$

$= 3x^5 + 8x^4 + 8x^3 + 16x^2 + 9x + 4$

$A \cdot B \rightarrow T_A \cdot b = v$

$\deg(A \cdot B) = 6$

$$\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ \vdots \\ v_5 \end{bmatrix}$$

~~ex~~ Pr. 8.5 $\quad uA + vB = G = gcd(A,B)$

example $A = x^4 - x^3 - 7x^2 + 2x + 3 \qquad n = 4$

$B = x^3 - 4x^2 + 2x + 3 \qquad m = 3$

$d = deg(gcd(A,B)) = 1$

$deg(u) < 2$
$deg(v) < 3$ $\Rightarrow$ $u = u_0 + u_1 x$
$v = v_0 + v_1 x + v_2 x^2$ $\qquad G = g_0 + x$

$Au + Bv \longmapsto T_A \cdot u + T_B \cdot v$

$deg(Au) = 5$.

$$\begin{bmatrix} 3 & 0 \\ 2 & 3 \\ -7 & 2 \\ -1 & -7 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ -4 & 2 & 3 \\ 1 & -4 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$T_A u + T_B \cdot v - g_0 = 0$ (1st $\qquad$ b/c $G = g_0 + x$

$$\left[\begin{array}{cc|ccc|c} 3 & 0 & 3 & 0 & 0 & -1 \\ 2 & 3 & 2 & 3 & 0 & 0 \\ -7 & 2 & -4 & 2 & 3 & 0 \\ -1 & -7 & 1 & -4 & 2 & 0 \\ 1 & -1 & 0 & 1 & -4 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \times \begin{bmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \\ v_2 \\ g_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

continue 8.5

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$$\Phi : A \longmapsto A - ZAZ^T$$

→ shift a matrix one to right then one to down.

$$Z = \begin{bmatrix} 0 & \cdots & & 0 \\ 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$\Phi(T) = \left[\begin{array}{cc|cccc|c} 3 & 0 & b_0 & 0 & \cdots & 0 & c_0 \\ 2 & 0 & b_1 & & & \vdots & c_1 \\ -7 & 0 & \vdots & & & \vdots & \vdots \\ -1 & 0 & \vdots & & & \vdots & \vdots \\ 1 & 0 & \vdots & & & \vdots & \vdots \\ 0 & 0 & b_5 & 0 & \cdots & 0 & c_5 \end{array}\right] \to \operatorname{rank}(\Phi(T)) = 3$$

polynomial problems ⟷ matrix problems

you can model these problems one another, so you can use faster algorithm

Vander matr ⟺ poly evalu

~~Tho~~ Toep. matrix ⟺ poly multi

Vander. matrix ⟷ poly. inter

Toep matrix ⟷ linear reccurence

bivariate polynomial.

8.6

$$P(x,y) = \sum_{j=0}^{2} \sum_{i=0}^{n-1} P_{ij} x^i y^j \in \mathbb{K}[x,y]$$

$$= P_{00}, + P_{10} x + \dots\dots P_{n-1,0} x^{n-1}$$

$$+ P_{01} + P_{02} y + P_{11} xy + \dots\dots + P_{n-1,1} x^{n-1} y$$

$$+ P_{02} y^2 + \dots\dots\dots \quad P_{n-1,2} x^{n-1} y^2$$

→ Give matrix $W_{x,y}$ corresponding to multipoint evaluation of $P(x,y)$

Multi point ⟵⟶ Vandermonde matrix

Given $P(x)$, and $x = (x_0 \dots x_n)$

Compute y : ⟶ $P(x_0) = y$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & & \vdots \\ 1 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & & x_n^n \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

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Continue 8.6

$$P(x,y) = \sum_{j=0}^{2} \sum_{i=0}^{n-1} P_{ij}\, x^i y^j$$

3n coefficients P_{ij}

(x_i, y_j) $0 \le i \le n-1$ & $0 \le j \le 2$

↳ 3n points

recall that

$$P(x,y) = P_{00} + P_{10}x + \dots\dots + P_{n-1,0}x^{n-1}$$
$$+ P_{01}y + P_{02}y + P_{11}xy + \dots + P_{n-1,1}x^{n-1}y$$
$$+ P_{02}y^2 + P_{12}xy^2 + \dots\dots P_{n-1,2}x^{n-1}y^2$$

$$\Downarrow$$

V_x

$$
\begin{array}{c}
(x_0,y_0)\\ (x_1,y_0)\\ \vdots \\ (x_{n-1},y_0)\\ (x_0,y_1)\\ \vdots\\ (x_{n-1},y_1)\\ (x_0,y_2)\\ \vdots \\ (x_{n-1},y_2)
\end{array}
\left[\begin{array}{c|c|c}
\begin{matrix}1 & x_0 & \cdots & x_0^{n-1}\\ 1 & x_1 & \cdots & x_1^{n-1}\\ \vdots & \vdots & & \vdots \\ 1 & x_{n-1} & \cdots & x_{n-1}^{n-1}\end{matrix} & y_0 V_x & y_0^2 V_x \\ \hline
V_x & y_1 V_x & y_1^2 V_x \\ \hline
V_x & y_2 V_x & y_2^2 V_x
\end{array}\right]
\left[\begin{array}{c} P_{00}\\ \vdots \\ P_{n-1,0}\\ \hline P_{01}\\ \vdots\\ P_{n-1,1}\\ \hline P_{02}\\ \vdots \\ P_{n-1,2}\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{matrix}P\\ \vdots\\ P\end{matrix}}\right\} n \\ \vphantom{\begin{matrix}P\\ \vdots\\ P\end{matrix}} \\ \vphantom{\begin{matrix}P\\ \vdots\\ P\end{matrix}}
\end{array}
$$