ANUM - TD2 Feb. 15. 23

Exercise 7

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle = \frac{1}{2} x^T A x - b^T x$$

A is symmetric matrix of $n \times n$.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\cancel{\nabla f(x) = \begin{pmatrix} \frac{df}{dx_1} \\ \vdots \\ \frac{df}{dx_n} \end{pmatrix}} =$$ complicated to compute derivative of $f(x)$ herer. buddy.

O, not zero.

$$\boxed{f(x+h) = f(x) + \nabla f(x)^T h + O(\|h\|)} \qquad h \in \mathbb{R}^n.$$

$$f(x+h) = \frac{1}{2}(x+h)^T A (x+h) - b^T (x+h)$$

$$= \frac{1}{2} x^T A x + \frac{1}{2} h^T A x + \frac{1}{2} x^T A h + \frac{1}{2} h^T A h - b^T x - b^T h$$

$$= \underbrace{\left(\frac{1}{2} x^T A x - b^T x\right)}_{f(x)} + \underbrace{x^T A h - b^T h}_{(Ax - b)^T h} + \underbrace{\frac{1}{2} h^T A h}_{O(\|h\|)}$$

$$\underbrace{(Ax-b)}_{\nabla f(x)}{}^T h$$

$\hookrightarrow \nabla f(x) = Ax - b.$

2) $\nabla f(x+h) = \nabla f(x) + H_f h + O(\|h\|)$

$$\nabla f(x+h) = A(x+h) - b = \underbrace{Ax - b}_{\nabla f(x)} + Ah$$

So $H_f(x) = A$.

3) let $f(x) = x^T A x - bx$.

The only critical point of A is a solution of

$\nabla f(x) = 0 = Ax - b$

let $x_0 = A^{-1}b$

Moreover, the $H_g(x_0)$ is SPD so x_0 is a local minimum clearly also global because $\forall x$ $H_f(x)$ is SPD.

Exercise 8

$f : \mathbb{R}^n \to \mathbb{R}, \quad n \geq 2$

$$f(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n} x_k^2 + \left(\sum_{k=1}^{n} x_k\right)^2 - \sum_{k=1}^{n} x_k$$

1.) f is polynomial in $x_1 \dots x_n$, so it is C^∞.
$n \geq 2$

$$\nabla f(x_1, \dots, x_n) = \begin{pmatrix} \frac{df}{dx_1}(x_1 \dots x_n) \\ \vdots \\ \frac{df}{dx_n}(x_1 \dots x_n) \end{pmatrix} = \begin{pmatrix} 2x_1 + 2\left(\sum_{k=1}^{n} x_k\right) - 1 \\ 2x_2 + 2\left(\sum_{k=1}^{n} x_k\right) - 1 \\ \vdots \\ 2x_n + 2\left(\sum_{k=1}^{n} x_k\right) - 1 \end{pmatrix}$$

$(0, 0)$

$\Rightarrow 2. + 2.$

$$H_f(x) = \left(\frac{df}{dx_i \, dx_j}\right) = \begin{pmatrix} 4 & 2 & \cdots & 2 \\ 2 & \ddots & (2) & \vdots \\ \vdots & (2) & \ddots & 4 \\ 2 & \cdots & & \end{pmatrix}$$

2) $\bar{x}$ is a critical point $\nabla f(\bar{x}) = 0$

$$\Leftrightarrow \begin{cases} 2\bar{x}_1 = 1 - 2\left(\sum_{k=1}^{n} \bar{x}_k\right) \\ 2\bar{x}_2 = 1 - 2\left(\sum_{k=1}^{n} \bar{x}_k\right) \\ \vdots \\ 2\bar{x}_n = 1 - 2\left(\sum_{k=1}^{n} \bar{x}_k\right) \end{cases}$$

So, $\bar{x}_1 = \bar{x}_2 = \dots = \bar{x}_n$.

and $2\bar{x}_1 = 1 - 2\left(\sum_{k=1}^{n} \bar{x}_k\right) = 1 - 2n\bar{x}_1 = \frac{1}{2(n+1)}$

$2\bar{x}_2 = 1 - 2\left(\sum_{k=1}^{n} \bar{x}_k\right) = 1 - 2n\bar{x}_2 = \frac{1}{2(n+1)}$

$\vdots$

$\bar{x}_n = \frac{1}{2(n+1)}$

So, the critical point is $x = \frac{1}{2(n+1)}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

Ex. 8

3) a) Let, $I = Id$, and $J = \begin{pmatrix} 1 & - & 1 \\ 1 & 1 & 1 \\ 1 & - & 1 \end{pmatrix}$

So, $2(I_n + J_n) = 2\begin{pmatrix} 2 & 1 & - & 1 \\ 1 & \ddots & (1) & 1 \\ (1) & (1) & \ddots & 1 \\ & - & & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 & - & 2 \\ 2 & \ddots & 2 & 1 \\ 1 & 2 & \ddots & \\ 2 & - & & 4 \end{pmatrix} = H_f(x)$

b). $J_n = \begin{pmatrix} 1 & - & 1 \\ | & (1) & | \\ 1 & - & 1 \end{pmatrix}$

$Im(J_n) = \text{space}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ so $rk(J_n) = 1$.

With rank theorem. $\dim(\ker(J_n)) = n - rk(J_n)$
$= n - 1 \geq 1$

So, $\ker(J_n)$ is non-empty so there exists $x \neq 0$ such that $J_n x = 0$ which means the 0 is eigenvalue of J_n.

$(J_n x = \lambda x)$

c) $J_n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ so n is also eigenvalue of J_n.

- $E_n(J_n)$ eigenspace of eigenvalue n.
- $E_0(J_n)$ eigenspace of eigenvalue 0.

then, $\dim E_n \geq 1$ and $\dim(E_n) + \dim(E_0) \leq n$

So $\dim(E_n) = 1$.

As a consequence, the only eigenvalues of J_n are 0 and n.

d) $$H_f(\bar{x}) = \begin{pmatrix} 4 & 2 & \cdots & 2 \\ 2 & \ddots & 2 & \vdots \\ \vdots & 2 & \ddots & 2 \\ 2 & \cdots & 2 & 4 \end{pmatrix} = 2(I_n + J_n)$$

so the eigenvalues of $H_f(\bar{x})$ are 2 and $2(n+1)$.

so, $H_f(x)$ is symetric positive-definite so $\bar{x}$ is a local minimum.

Moreover, as $H_f(x)$ is always SDP, it means that f is strickly convex as $\bar{x}$ is indeed a global minimum.