Feb. 01. 23 QINTRO - TO 1

Ex. 1

- norm of density matrix is one

$|\psi_{AB}\rangle$

$\rho_A = tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$

$\rho_B = tr_A(|\psi_{AB}\rangle\langle\psi_{AB}|)$ orthonormal basis

$|\phi\rangle_{AB} = \sum_{a,b} \lambda_{a,b} |\varepsilon_a\rangle|e_b\rangle$

$tr_A(|\phi_{AB}\rangle\langle\phi_{AB}|) = \sum_{a,b} |\lambda_{a,b}|^2 |e_b\rangle\langle e_b|_B$

$|\phi'_{AB}\rangle = \sum_{a,b} \mu_{ab} |e_a\rangle|\varepsilon_b\rangle$ orthonormal basis

$tr_B(|\phi'_{AB}\rangle\langle\phi'_{AB}|) = \sum_{a,b} |\mu_{ab}|^2 |e_a\rangle\langle e_b|$

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1. $|\psi_{AB}\rangle = \sqrt{\frac{1}{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle$

$\rho_A = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1| = Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$

$tr_A(|\psi_{AB}\rangle\langle\psi_{AB}|) = \rho_B = \rho_A$

2. $|\psi_{AB}\rangle = \sqrt{\frac{3}{5}}|0\rangle|0\rangle + \sqrt{\frac{2}{5}}|1\rangle|0\rangle = \overbrace{\left(\sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle\right)}^{|\phi\rangle}|0\rangle$

$\rho_B = Tr_A(|\psi_{AB}\rangle\langle\psi_{AB}|) = \frac{3}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 0| = |0\rangle\langle 0|$

$\rho_A = Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = |\phi\rangle\langle\phi|$

3. $|\psi_{AB}\rangle = \frac{1}{\sqrt{3}}\left(|01\rangle + |12\rangle + |22\rangle\right)$

$= \frac{1}{\sqrt{3}}\left(|01\rangle + (|1\rangle + |2\rangle)|2\rangle\right) = \frac{1}{\sqrt{3}}|01\rangle + \left(\frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle\right)|2\rangle$

$= \frac{1}{\sqrt{3}}|01\rangle + \sqrt{\frac{2}{3}}\underbrace{\left(\sqrt{\frac{1}{2}}|1\rangle + \sqrt{\frac{1}{2}}|2\rangle\right)}_{|+\rangle_{12}}|2\rangle$

$\rho_B = tr_A\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right)$

$= \frac{1}{3}|1\rangle + \frac{1}{3}|2\rangle + \frac{1}{3}|2\rangle = \frac{1}{3}|1\rangle + \frac{2}{3}|2\rangle$

$\rho_A = tr_B\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right)$

$= \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|+_{12}\rangle\langle +_{12}|$

4. $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |1+\rangle\right)$

$\rho_B = tr_A\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$

$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + \frac{|1\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

$= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes |0\rangle + \frac{1}{2}|1\rangle|1\rangle$

$= \frac{\sqrt{3}}{2}\underbrace{\left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)}_{|\phi\rangle}|0\rangle + \frac{1}{2}|1\rangle|1\rangle$

$\rho_A = tr_B\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right) = \frac{3}{4}|\phi\rangle\langle\phi| + \frac{1}{4}|1\rangle\langle 1|$

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Ex. 2

1. $|\psi_0\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $|\psi_1\rangle = \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}$

$\langle\psi_0| = (\alpha \ \beta)$ $\langle\psi_1| = (\beta \ -\alpha)$

$$\frac{1}{2}\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|\right) = \frac{1}{2}\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}(\alpha\ \beta) + \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}(\beta\ -\alpha)\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix} + \begin{pmatrix} \beta^2 & -\alpha\beta \\ -\alpha\beta & \alpha^2 \end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} \overset{1}{\cancel{\alpha^2+\beta^2}} & 0 \\ 0 & \overset{1}{\cancel{\beta^2+\alpha^2}} \end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$$

2. $\rho = a|0\rangle\langle 0| + (1-a)|1\rangle\langle 1|$

$\sum_i p_i |e_i\rangle\langle e_i|$

$\rho = a|0\rangle\langle 0| + a|1\rangle\langle 1| - a|1\rangle\langle 1| + (1-a)|1\rangle\langle 1|$

$= a|0\rangle\langle 0| + a|1\rangle\langle 1| - (2a-1)|1\rangle\langle 1|$

for any 2 state $|\psi_0\rangle, |\psi_1\rangle$

$\rho = a\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|\right) + \overset{\geq 0 \text{ if } a<\frac{1}{2}}{(1-2a)}|1\rangle\langle 1|$ (if $a < \frac{1}{2}$)

Else

$$\rho = a|0\rangle\langle 0| + (1-a)|1\rangle\langle 1| + (1-a)|0\rangle\langle 0| - (1-a)|0\rangle\langle 0|$$

$$= (2a-1)|0\rangle\langle 0| + \underbrace{(1-a)}_{\geq 0}\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|\right)$$

Ex. 3

$|\varphi_0\rangle = |0\rangle$

$|\varphi_1\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle \qquad \theta \in [0, \frac{\pi}{2}]$

① $|0\rangle \to |\varphi_0\rangle$

$|1\rangle \to |\varphi_1\rangle$

$b = 0$ measures the state sent is $|\varphi_0\rangle = |0\rangle$

He measures $|0\rangle$ with probability of 1 so $P_0 = 1$.

$b = 1$ measures the state sent is $|\varphi_1\rangle$

He gets $|1\rangle$ and then they get a result $|\varphi_1\rangle$ with probability $|\langle 1|\varphi_1\rangle|^2 = \sin^2\theta$

$$\frac{1}{2}(1 + \sin^2\theta)$$

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2. $|\psi_0\rangle$ s.t. $B = \{|\psi_0\rangle, |\psi_1\rangle\}$ althonormal basis

if the output state after measurement is

$|\psi_0\rangle \longrightarrow |\phi_0\rangle$

$|\phi_1\rangle \longrightarrow |\phi_1\rangle$

$|\psi_0\rangle \perp |\phi_1\rangle$

$|\psi_0\rangle = (\sin\theta|0\rangle - \cos\theta|1\rangle) e^{i\theta}$ ← you don't need this

$b=0 \quad |\langle\psi_0|\phi_0\rangle|^2 = \sin^2\theta$

$b=1 \quad |\langle\phi_1|\phi_1\rangle|^2 = \underline{1}$

$\frac{1}{2}(1 + \sin^2\theta)$

$b=0$, probability of success in $|\psi_0\rangle$ is $1 - |\langle\phi_0|\phi_1\rangle|^2$

$= 1 - \cos^2\theta$

$= \underline{\sin^2\theta}$