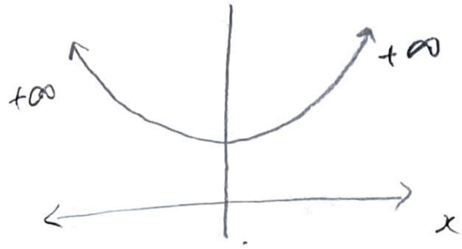
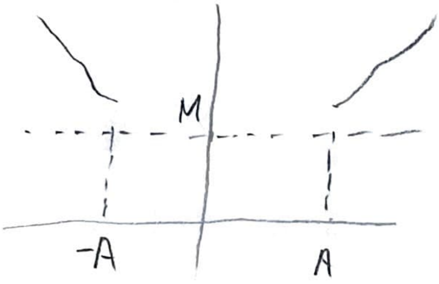
ANUM - TD2 Feb. 08. 23

Ex. 1

$f : \mathbb{R}^n \to \mathbb{R}$, coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$



(1) means that $\forall M > 0, \exists A > 0$ s.t $\left(\|x\| \geq A \Rightarrow f(x) \geq M \right)$



let us define $M = f(0) + 1$.

let $\bar{B}(0,A) = \{x \in \mathbb{R}^n \text{ st. } \|x\| \leq A\}$

As $\bar{B}(0,A)$ is closed, bounded and as f is continuous on $\bar{B}(0,A)$, f admits a local min. denoted $x^* \in \bar{B}(0,A)$

So, for all $x \in \bar{B}(0,A)$, $f(x^*) \leq f(x)$

Moreover, as $0 \in \bar{B}(0,A)$, $f(x^*) \leq f(0)$ (2)

For $x \notin \bar{B}(0,A)$, $\|x\| \geq A \Rightarrow f(x) \geq M = f(0) + 1$

so $f(x) \geq f(0) + 1 \geq f(x^*)$ because of (2).

As a cosequence the all $x \in \mathbb{R}^n$, $f(x^*) \leq f(x)$.

so, x^* is a global minimum

Closed if $\geq$, $\leq$
if it's $>$, $<$, not closed.

Ex. 2.

$f : \mathbb{R}^n \to \mathbb{R}$

$$\nabla f(x) = \begin{pmatrix} \frac{df}{dx_n}(x) \\ \vdots \\ \frac{df}{dx_n}(x) \end{pmatrix}$$

Let, $t \in \mathbb{R}^+$ and $d \in \mathbb{R}^n$

$$f(a+td) = f(a) + \nabla f(a)^T (td) + \overbrace{|t| \, \|d\| \, P(t)}^{o(td)}$$

with $\underset{t \to 0^+}{P(t) \to 0}$

$$\text{So, } \nabla f(a)^T d = \frac{\overbrace{f(a+td) - f(a)}^{\geq 0}}{t} - \frac{|t|}{t} \|d\| \, P(t)$$

For $t \geq 0$ small, $f(a+td) \geq f(a)$

$$\nabla f(a)^T d \geq -\frac{|t|}{t} \|d\| \, p(t)$$

So for $t \to 0^+$ then $\nabla f(a)^T d \geq 0$.

For $d = -\nabla f(a) \Rightarrow \nabla f(a)^T d \geq 0 \Rightarrow \|\nabla f(a)\|^2 \leq 0$

So, $\nabla f(x)$

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Ex 4

2) $f(x,y) = 3x^2 + 2y^2 + 2xy + x + y + 10$

f is $\mathcal{C}^\infty$

so, $\nabla f(x,y) = \begin{pmatrix} \frac{df}{dx}(x,y) \\ \frac{df}{dy}(x,y) \end{pmatrix} = \begin{pmatrix} 6x + 2y + 1 \\ 4y + 2x + 1 \end{pmatrix}$

· Compute of the critical points

$$f(x,y) = 0 \Leftrightarrow \begin{pmatrix} 6x + 2y + 1 = 0 \\ 4y + 2x + 1 = 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 10y + 2 = 0 \\ 10x + 1 = 0 \end{pmatrix}$$

so, $(x,y) = \begin{pmatrix} -1/10 \\ -1/5 \end{pmatrix}$

$$H_f(x,y) = \begin{pmatrix} \frac{d^2f}{d^2x}(x,y) & \frac{d^2f}{dx\,dy}(x,y) \\ \frac{df}{dy\,dx}(x,y) & \frac{d^2f}{dy^2}(x,y) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$$

$\lambda(H_f) = \det(xI - H_f)$

$$= \begin{vmatrix} x-6 & -2 \\ -2 & x-4 \end{vmatrix} = (x-6)(x-4) - 4$$

$$= x^2 - 10x + 20$$

$\Delta = 10^2 - 4 \times 20 = 20$

$$x_1 = \frac{10 + \sqrt{20}}{2} = 5 + \sqrt{5} > 0$$

$$x_2 = \frac{10 - \sqrt{20}}{2} = 5 - \sqrt{5} > 0$$

H_f is positive semidefinite so the point $(x, y) = \left(-\frac{1}{10}, -\frac{1}{5}\right)$ is a local min.

As f is convex, the point is for a global minimum.

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Ex3

function is convex if tangent is smaller thant the function

$\forall y.$

$f(x) \geq f(y) + f'(y)(x-y)$

$f(x) \geq f(y) + \nabla f(y)^T (x-y)$

$\langle \nabla f(y), x-y \rangle$

let f convex via a local minimum is a.

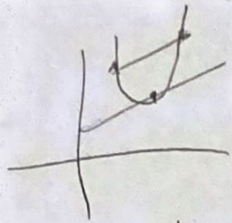
$\forall x, \quad f(x) \geq f(a) + \langle \underset{\overset{\Downarrow}{0}}{\nabla f(a)}, x-a \rangle$

$\Rightarrow \forall x \in \mathbb{R}^2 \quad f(x) \geq f(a) \Rightarrow$ a is a global minimum

let us assume that thee exists 2 distincts global minimum $x_1 \neq x_2$, $m = f(x_1) = f(x_2)$

$f\left(\frac{x_1+x_2}{2}\right) < \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2)$

$< m$ so the is contradiction.



f' is non decreasing $\longleftrightarrow$ $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$

$f'' \geq 0$ $\longleftrightarrow$ $Hf(x)$ is positive semi-definite

$\Updownarrow$

all eigenvalues ≥ 0