Past exam

Ex 1.

1. If $a \approx b$, $\det(A) \simeq 0$ so matric is singular. Therefore, matrix is mal-condition. Solution of this system can have rounding errors.

2. By solving the linear system, we get

$$x = \frac{-a}{b^2 - a^2} \quad \text{and} \quad y = \frac{b}{b^2 - a^2}$$

As $a \approx b$, we will have very bad cancellation error from calculating $b^2 - a^2$.
On the other hand, as $x + y = \frac{1}{a+b}$, $z = x + y$ we can calculate it numerically with precision.

Ex 2

1. For $n=3$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

$$= \begin{pmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{31} l_{11} & l_{31} l_{21} + l_{32} l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$$

2. The MATLAB function will be the following algorithm

for $i = 1:n$

$$l_{ii} = \sqrt{a_{ii} - \sum_{j=1}^{i-1} l_{ij}^2}$$

for $j = i+1 : n$

$$l_{ji} = \left(a_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik}\right) / l_{ii}$$

end

end

3. By applying algorithm we get $A = LL^T$ with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Ex.3

1. $f_1(x,y) = xy - y + x + 1$

f_1 is from class C^∞ on $\mathbb{R}^2$ and $\nabla f_1(x,y) = \begin{pmatrix} y+1 \\ x-1 \end{pmatrix}$.

Therefore, the only critical point which is a solution $\nabla f_1(x^*, y^*) = 0$ and $(x^*, y^*) = (1, -1)$

The Hessian matrix $Hf_1(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ does not depend on point (x,y).

Eigenvalue of $Hf_1(x,y)$ are 1 and -1 so $(1,-1)$ is saddle point

2. $f_2(x,y) = \frac{x^4}{4} - 2x^2 + y^2 - 4y$

f_2 is from class C^∞ on $\mathbb{R}^2$ and $\nabla f_2(x,y) = \begin{pmatrix} x^3 - 4x \\ 2y - 4 \end{pmatrix}$

The solution of the equation $\nabla f_2(x,y) = 0$ are $(0,2)$, $(2,2)$ and $(-2,2)$. There is 3 critical points.

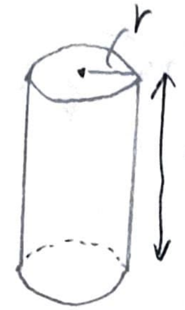
The Hessian matrix is given by $Hf_2(x,y) = \begin{pmatrix} 3x^2 - 4 & 0 \\ 0 & 2 \end{pmatrix}$

Eigenvalue of $Hf_2(x,y)$ are $d_1 = 3x^2 - 4$ and $d_2 = 2$

Therefore, $Hf_2(0,2)$ has a value of -4 and 2, So it is saddle point.

The matrix $Hf_2(2,2) = Hf_2(-2,2)$ have value of 8 and 2 so, $(2,2)$ and $(-2,2)$ are local minimums,

Ex4



1. $V(h,r) = \pi r^2 h$,

$S(h,r) = 2\pi r h + 2\pi r^2$

$= 2\pi r(h+r)$

2. We want to solve the problem

$$\begin{cases} \min S(h,r) \text{ one the constraints} \\ h>0,\ r>0,\ V(h,r) = V_0 \end{cases}$$

3. Using the fact that $V_0 = \pi r^2 h$, we can rewrite the problem as

$$\left\{ \min\left(2\pi r^2 + \frac{2V_0}{r}\right) \text{ under the constraint } r>0. \right.$$

Ex4

4. Let $f(r) = 2\pi r^2 + \frac{2V_0}{r}$ on $\mathbb{R}^+$,

$f'(r) = 4\pi r - \frac{V_0}{r^2}$

$f''(r) = 4\pi + \frac{4V_0}{r^3}$.

The only critical point r_0 verifying

$f'(r_0) = 0$ and $r_0 = \sqrt[3]{V_0/(2\pi)}$

Moreover, as second derivative f'' is always strickly positive, the function is strickly convex and the local minimum is global.

So, $r_0 = \sqrt[3]{V_0/(2\pi)}$ and $h_0 = V_0/(\pi r_0^2)$

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