Feb 01.23 ANUM - TD

Σ and V are square matrix

Ex.1 $A \in \mathbb{R}_{m \geq n}$, $A = U\Sigma V^T$

1. $\|Ax - b\|_2^2 = \|U\Sigma V^T x - b\|^2$

As A is full rank, Σ is also.

The col of U are a free family so that we can compute it $[U, \tilde{U}]$ such that $[U, \tilde{U}]$ is a square althogonal matrix.

$$\|U\Sigma V^T x - b\|_2^2 = \left\| \begin{bmatrix} U^T \\ \tilde{U}^T \end{bmatrix} (U\Sigma V^T x - b) \right\|^2$$

this is not zero in general. but it is if $A \in \mathbb{R}_{m=n}$

$$= \left\| \begin{bmatrix} \Sigma V^T x - U^T b \\ -\tilde{U}^T b \end{bmatrix} \right\|_2^2$$

constant, in this case

$$= \|\Sigma V^T x - U^T b\|^2 + \|\tilde{U}^T b\|^2$$

To minimize quantity, we need to have

$$\Sigma V^T x = U^T b \iff \boxed{x = V\Sigma^{-1} U^T b}$$

M is max of d_i if. $|d_i| \le M$

2. if $\begin{pmatrix} d_1 & & (0) \\ & \ddots & \\ (0) & & d_n \end{pmatrix}$, $\|D\|_2 = \max_{i=1\ldots n} |d_i|$

$$\sum_{i=1}^{n} \left(|d_i||x_i|^2\right) \le \sum_{i=1}^{n} \left(m|x_i|^2\right) \le m\left(\sum |x_i|^2\right)$$

$$\|Dx\|_2^2 = \sum_{i=1}^{n} |d_i|^2 |x_i|^2 \le \left(\max_{i=1\ldots n} |d_i|^2\right) \|x\|_2^2$$

so. $\dfrac{\|Dx\|}{\|x\|} \le \max_{i=1\ldots n} |d_i|$. for all $x \neq 0$

As a consequence $\|D\|_2 = \sup_{x \neq 0} \dfrac{\|Dx\|}{\|x\|} \le \max_{i=1\ldots n} |d_i|$

Let us denote j such $\max_{i=1\ldots n} |d_i| = |d_j|$

$$\frac{\|De_j\|_2}{\|e_j\|_2} = \frac{|d_j|}{1} = |d_j|$$

$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$

So we can conclude that $\|D\|_2 = \max_{i=1\ldots n} |d_i|$

As $A = U\Sigma V^T$, $\|A\|_2 = \|U\Sigma V^T\|_2 = \|\Sigma\|_2 = \left\|\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}\right\|_2 = \sigma_1$

$\|A^{-1}\|_2 = \|U\Sigma^{-1}V^T\|_2 = \|\Sigma^{-1}\|_2$

when you rotate a matrix, norm doesn't change.

$= \left\|\begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n \end{bmatrix}\right\|_2 = \dfrac{1}{\sigma_n}$

$k(A) = \|A\|_2 \|A^{-1}\|_2 = \dfrac{\sigma_1}{\sigma_n}$

condition number.

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3. $\|A - A_k\|_2 = \left\| \sum_{i=k+1}^{n} \sigma_i u_i v_i^T \right\|_2$

$$= \left\| U \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & \sigma_{k+1} & \\ 0 & & & \ddots \ \sigma_n \end{pmatrix} V^T \right\|_2$$

$$\underbrace{\dim(F+G)}_{\leq n} = \underbrace{\dim(F) + \mathrm{kre}(G)}_{> n} - \underbrace{\dim(F \cap G)}_{\geq 1}$$

$$= \left\| \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & \sigma_{k+1} & \\ 0 & & & \ddots \ \sigma_n \end{bmatrix} \right\|_2 = \sigma_{k+1}$$

Let B a matrix of rank k and show that $\|A - B\| \geq \sigma_{k+1}$

rank

r

$rk(B) = k$ so $\dim \ker(B) = n - k$

$rk(\mathrm{span}(v_1 \dots v_{k+1})) = k+1$

As $\dim \ker(B) + rk(\mathrm{span}(v_1 - v_{k+1})) \Rightarrow n+1 > n$

it means $\ker(B) \cap \mathrm{span}(v_1 - v_{k+1}) \neq \{0\}$

let $h \in \ker(B) \cap \mathrm{span}(v_1 - v_{k+1})$ st. $h \neq 0$, $\|h\|_2 = 1$

$\|A - B\|_2^2 \geq \|(A-B)h\|^2 = \|Ah\|^2$

$= \|U \Sigma V^T h\|_2^2 = \|\Sigma V^T h\|_2^2$

$= \sum_{i=1}^{n} \sigma_i^2 (v_i^T h)^2 = \sum_{i=1}^{k+1} \sigma_i^2 (v_i^T h)^2$

$\geq \sigma_{k+1}^2 \sum_{i=1}^{k+1} (v_i^T h)^2$

$\|A\| = \max_{n \neq 0} \frac{\|An\|}{\|n\|}$

$\|A\| \geq \frac{\|An\|}{\|n\|}$

$\|A - B\| = \frac{\|(A-B)h\|}{\cancel{\|h\|} = 1}$

$$\sigma_{k+1}^2 \sum_{i=1}^{k+1} (V_i^T h)^2 = \sigma_{h+1}^2 |V^T h|_2^2$$

$$= \sigma_{h+1}^2 \|h\|_2^2 = \sigma_{h+1}^2$$

Ex. 2

1. Show that $|Ax| \leq |A||x|$

$$(Ax)_i = \sum_{j=i}^{n} a_{ij} x_j$$

so. $$|(Ax)_i| \leq \sum_{j<i}^{n} |a_{ij}||x_j| = (|A||x|)_i$$

and so $|Ax| \leq |A||x|$

2. Show that, $A \geq 0 \Leftrightarrow (x \geq 0 \Rightarrow Ax \geq 0)$

"$\Rightarrow$" if. $A \geq 0$ and $x \geq 0$ $\Rightarrow Ax \geq 0$

"$\Leftarrow$" let $x = e_i = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - i \geq 0$

$Ae_i \geq 0 \Rightarrow$ i th col of A is non-negative.

for all $i \Rightarrow A \geq 0$

3. spectral radius $\rho(A) = \max(|\lambda|, \lambda \text{ eigenvalue of } A)$

let λ s.t $|\lambda| = \rho(A)$ and v an associate eigenvector

We have $\rho(A)|v| = |\lambda||v| = |\lambda v| = |Av| \leq |A||v| \leq A|v|$

Let us denote C, the domain of $\mathbb{R}$ defined by

$x \in \mathbb{R}^n$ satisfying $\begin{cases} \sum_{i=1}^{n} x_i = 1 \\ x \geq 0 \\ Ax \geq \rho(A)x \end{cases}$

C is non-empty as it $\frac{|v|}{\|v\|_1}$

C is also convex and closed and bounded $(0 \leq x_i \leq 1)$

So, C is a compact set.

2 cases

a) There exist $x \in C$ s.t. $Ax = 0$

then as $Ax \geq \rho(A)x \Rightarrow \rho(A) = 0$

$Ax = 0 \Rightarrow 0$ is a eigenvalue with associated eigenvector $x \geq 0$.

b) For all $x \in C$, $Ax \neq 0$

$f : C \to \mathbb{R}^x \quad x \mapsto \frac{1}{\|Ax\|} Ax$

we have $f(x) \geq 0$, $\|f(x)\|_1 = 1$. and

$Af(x) = \frac{1}{\|Ax\|_1} AAx \geq \frac{1}{\|Ax\|_1} \rho(A) Ax = \rho(A) f(x)$ which means that $f(x) \in C$

So $f(C) \subset C$.

By, Brouwer theorem, there exists

y s.t. $f(y) = y$ which means that

$$\frac{Ay}{\|Ay\|_1} = y \Rightarrow Ay = \|Ay\|_1 \, y$$

y is a non-negative eigenvalue associated to the eigenvector $n = \|Ay\|_1$.

But as $y \in C$ $\quad ny = f(y) \geq \rho(A) y$

and so $n \geq \rho(A) \Rightarrow n = \rho(A)$