Ex. 2

2) output 0 :

$\mathrm{tr}(F_0 |\phi\rangle\langle\phi|) = \frac{1}{2} |\langle 0|\phi\rangle|^2$, resulting state is 0.

Output 1:

$\mathrm{tr}(F_1 |\phi\rangle\langle\phi|) = \frac{1}{2} \mathrm{tr}(|+\rangle\langle+|\phi\rangle\langle\phi|) = \frac{1}{2}\langle+|\phi\rangle \, \mathrm{tr}(|+\rangle\langle\phi|)$

$= \frac{1}{2}|\langle+|\phi\rangle|^2$

$\Rightarrow$ state $|+\rangle$

Output 2

$\mathrm{tr}(F_2 |\phi\rangle\langle\phi|) = \frac{1}{2}|\langle-|\phi\rangle|^2$ state $|-\rangle$

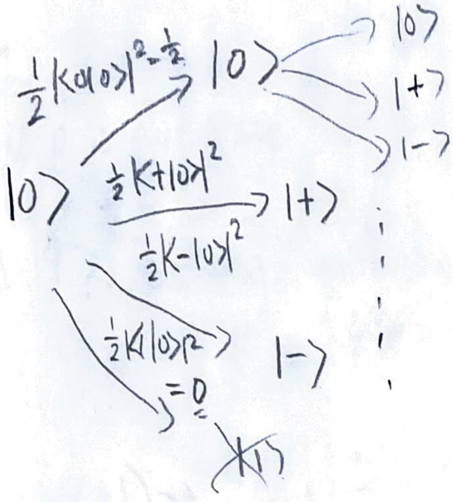
Output 3

$\mathrm{tr}(F_3 |\phi\rangle\langle\phi|) = \frac{1}{2}|\langle 1|\phi\rangle|^2$ State $|1\rangle$

3) From previous question, it is probability of outcome 3 when input $|\phi\rangle = |0\rangle$

$= \frac{1}{2}|\langle 0|1\rangle|^2 = 0.$ it happens with probability 0.

4) Same idea as question 3.



Ex. 1

If ρ is pure state.

$\exists |\phi\rangle$ s.t. $\rho = |\phi\rangle\langle\phi|$

$$\rho^2 = |\phi\rangle\overbrace{\langle\phi|\phi\rangle}^{1}\langle\phi| = |\phi\rangle\langle\phi| = \rho$$

$$Tr(\rho^2) = 1$$

$\langle e_i | e_j \rangle = \delta_{ij}$

$$\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|$$

$$\rho^2 = \sum_i \lambda_i^2 |e_i\rangle\langle e_j|$$

$$tr(\rho^2) = \sum_i \lambda_i^2$$

Assume ρ is not pure

we know $tr(\rho) = \sum_i \underbrace{\lambda_i}_{\geq 0} = 1$

There exist at least one λ_i that is not 0,

let this to be λ_1.

$$0 < \lambda_1 < 1 \Rightarrow \lambda_1^2 < \lambda_1$$

$$tr(\rho^2) = \sum_i \lambda_i^2 = \lambda_1^2 + \sum_i \lambda_i^2 < \lambda_1 + \sum_i \lambda_i$$

$$= \sum_i \lambda_i = tr(\rho)$$

$$\Rightarrow tr(\rho^2) < 1$$

Ex. 3

1. $p_i = tr(F_i \rho)$, $q_i = tr(F_i \sigma)$

p_i = probability of outcomes : with state ρ

q_i = " σ

2.

$\rho - \sigma = \sum_i d_i |e_i\rangle\langle e_i|$

$|p_i - q_i| \leq tr(F_i |\rho - \sigma|)$

$p_i - q_i = Tr(F_i \rho) - Tr(F_i \sigma)$

$= Tr(F_i(\rho - \sigma)) = Tr(F_i(Q - S))$

$= \underbrace{Tr(F_i Q)}_{a} - \underbrace{Tr(F_i S)}_{b.}$

$|a - b| \leq |a| + |b|$

$|p_i - q_i| \leq |Tr(F_i Q)| + |Tr(F_i S)|$

$= tr(F_i(Q + S)) = tr(F_i |\rho - \sigma|)$

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3.

$$\sum_i |p_i - q_i| \leq \sum_i \operatorname{Tr}(F_i |\rho - \sigma|)$$

$$= \operatorname{tr}(\underbrace{\sum_i F_i}_{Id.} |\rho - \sigma|)$$

$$= \operatorname{tr}(|\rho - \sigma|)$$

$$\Delta(p, q) = \frac{1}{2}\sum_i |p_i - q_i| \leq \frac{1}{2} \operatorname{tr}(|\rho - \sigma|)$$

$$= \frac{1}{2}\sum_i |\lambda_i| = \Delta(\rho, \sigma)$$

Ex. 4

Both uses a certain POVM. F_0, F_1
If outcome is i, Bob guesses that he had ρ_i. Probability of success $= \frac{1}{2}(\operatorname{Tr}(F_0 \rho_0) + \operatorname{tr}(F_1 \rho_1))$

$$= \frac{1}{2}(p_0 + q_1)$$

goal is to show that max. success prob. $\leq \Delta(\rho_0, \rho_1)$ where the optimization is done on the POVM

To show that it is true for the max., we will show that it is the case for any POVM.

$$P_{success} = \frac{1}{2}(p_0 + q_1)$$

Result from ex.3 can thus be written as

$$\Delta(P_0, P_1) \geq \frac{1}{2}\left(|P_0-(1-q_1)| + |1-P_0-q_1|\right)$$

$$= \frac{1}{2}\left(|P_0+q_1-1| + |1-(P_0+q_1)|\right)$$

$$= \frac{1}{2}\left(|2P_{success}-1| + |1-2P_{success}|\right)$$

$$= \left(|1-2P_{success}|\right)$$

$$|1-2P_{success}| \leq \Delta(P_0, P_1)$$

If $1-2P_{success} \leq 0 \Rightarrow 2P_{success}-1 \leq \Delta(P_0, P_1)$

$$P_{success} \leq \frac{1}{2} + \frac{\Delta(P_0, P_1)}{2}.$$

if $1-2P_{sucess} > 0$, $P_{success} < \frac{1}{2}$

then there is a both strategy for Bob choosing at random.