Important

ANUM - TD6 08.03.23

Ex. 1

(1.) ~~$\nabla f(x) = \left(\frac{\partial f}{\partial x}\right) = f'(x)$~~

$$\boxed{f(x) = \nabla f(x)^T h + o(h)}$$

$$f(x) = \|Ax-b\|^2 = (Ax-b)^T(Ax-b) = x^TA^TAx - x^TA^Tb - b^TAx - b^Tb$$

$$f(x+h) = (A(x+h)-b)^T(A(x+h)-b)$$

$$= (Ax+Ah-b)^T(Ax+Ah-b)$$

$$= x^TA^TAx + x^TA^TAh - A^Tx^Tb + h^TA^TAx + h^TAAh$$
$$- h^TA^Tb - b^TAx - b^Tb$$

$$= f(x) + x^TA^TAh + h^TA^TAx - h^TA^Tb - b^TAh + h^TA^TAh$$

$$= f(x) + x^TA^TAh + (A^TAx)^Th - (A^Tb)^Th$$

$\boxed{x^Th = h^Tx}$ $$- b^TAh + h^TA^TAh$$

$$= f(x) + (A^TAx + A^TAx - A^Tb - A^Tb)h + h^TA^TAh$$

$$= f(x) + \underbrace{(2A^TAx - 2A^Tb)^T}_{\nabla f(x)} h + h^TA^TAh$$

So, $\nabla f(x) = 2(A^TAx - A^Tb) = \boxed{2A^T(Ax-b)}$

For the Hessiarm, Taylor formula of order 2

$$f(x+h) = f(x) + \nabla f(x)^T h + \frac{1}{2} h^T Hf(x) h + O(\|h\|^2)$$

$$Hf(x) = 2A^T A$$

(2) Let x^* a minimum of $f(x)$ on $\mathbb{R}^n$.
then a necessary condition is that

$$\nabla f(x^*) = 0 \iff 2A^T(Ax^* - b) = 0$$
$$\iff A^T A x^* = A^T b$$

(3) f is continuous and $\lim_{\|x\| \to +\infty} f(x) = +\infty$ (f is coercive)
So the minimization problem has a solution.
This solution must satisfy $\nabla f(x^*) = 0$

So $x^* = (A^T A)^{-1} A^T b$.

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Ex.2

1. $g(x) = \frac{1}{x} - a$

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}}$$

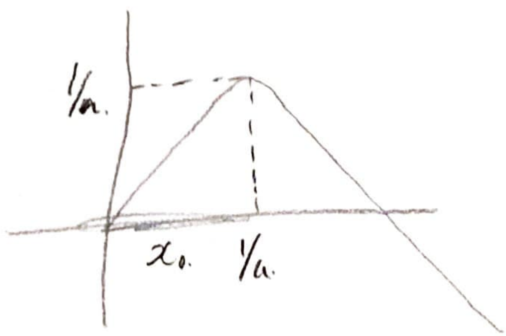
$$= x_n + x_n - x_n^2 a$$

$$\boxed{= x_n(2 - a x_n)}$$

2. $x_{n+1} = g(x_n)$ with $g(x) = x(2-ax)$

$g'(x) = 2 - 2ax$

x	0		$1/a$		$+\infty$
$g'(x)$		$+$		$-$	
$g(x)$	0	$\nearrow$	$1/a$	$\searrow$	$-\infty$



$g :]0, 1/a[\to]0, 1/a[$

for $x \in]0, 1/a[$

$$g(x) - x = x - ax^2 = x(1-ax) \geq 0 \quad \text{for } 0 < x < 1/a$$

So, $x_{n+1} = g(x) \geq x_n$.

So, (x_n) is increasing and bounded by $1/a$, this means that (x_n) is convergent.

As g is continus if $l = \lim\limits_{n \to +\infty} x_n$.

then $g(l) = l \iff l(2-al) = l$

$$\iff l - al^2 = 0$$

$$\iff l(1-al) = 0$$

$$\iff l = 1/a \quad \text{as } l \neq 0$$

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Ex 2.

polynomial of degree n
is equal to Taylor expansion of order n.

3. $g(x) = x(2-ax)$

Taylor expansion

$$g(x) = g(1/a) + g'(1/a)\left(x - \frac{1}{2}\right) + \frac{1}{2} g''(1/a)(x - 1/a)^2$$

$$g(x) = 1/a + \frac{1}{2}(-2a)\left(n - \frac{1}{a}\right)^2$$

$$g(x_n) = \frac{1}{a} - a(x_n - 1/a)^2$$

means it's quadratic.

$$|x_{n+1} - 1/a| \leq |a|\, |x_n - 1/a|^2$$