Proof of Dependency

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For the two proofs of **QP → gr-co** as well as **SCT and AE → ↑DB, ↑AB**, I received significant assistance from my co-author, Ryuta Arisaka.

QP → gr-co

Proof. QP $\rightarrow$ gr-co is directly derived from the following lemma that, with QP in place, every member of the grounded extension of AF ranks higher than every argument that is not in the grounded extension and that every argument attacked by a member of the grounded extension ranks lower than every other arguments.

As necessary preparation for that, we introduce a level of every member of AF ' s grounded extension, and thus of AF. A member a of AF ' s grounded extension is leveled as follows:

- $a = 1$: $\{\}$ defends a.
- $a = k + 1$: There exists a subset A of the grounded extension satisfying the following four conditions:
  1. The levels of all $a' \in A$ are k or lower.
  2. There exists $a' \in A$ such that the level of a' is k.
  3. A defends a.
  4. There is no smaller subset of the grounded extension satisfying 1, 2, and 3.

The level of AF is then defined as the maximum level among the arguments in the grounded extension. Note that if $gr(AF) = \{\{\}\}$, the level of AF is 0.

We prove the lemma using mathematical induction based on the level of AF.

For $k = 0$, that is, when $gr(AF) = \{\{\}\}$, there are no conditions for gr-co to hold, so it trivially holds.

For $k = 1$, that is, when AF ' s grounded extension consists only of arguments that are not attacked, the arguments in the grounded extension are not attacked and thus are ranked higher than any arguments not in the grounded extension. By QP, arguments attacked by arguments in the grounded extension are ranked lower than all other arguments.

Assume that this lemma holds for all $k \leq n$**.** For any AF with $k = n + 1$, AF can then be constructed by a simple extension of some graph AF' with $k = n$. (Specifically, adding arguments of level $k = n + 1$ that are only attacked by arguments attacked from AF' ' s grounded extension, adding arguments that are attacked by these nodes, and adding some edges, results in a graph AF where AF' necessarily exists.)

In this AF', the arguments can be divided into three sets:

1. Arguments in AF' ' s grounded extension,
2. Arguments not attacked by AF' ' s grounded extension but not in AF' ' s grounded extension,
3. Arguments attacked by AF' ' s grounded extension.

Since AF' has $k = n$ and the induction hypothesis holds, we have $1 \succ 2 \succ 3$. The set of arguments included in set 2, if only attacked by arguments in set 3, are defended by the grounded extension, thus becoming part of set 1. Hence, they are attacked either set 1 or set 2, and if they are attacked by set 1, they would be included in set 3, so therefore, arguments in set 2 are being attacked by arguments in set 2 (and indeed, the arguments in set 2 actually form loop structures).

For the graph AF'' where the arguments of level $k = n + 1$ are re-added to AF', for any argument a of level $k = n + 1$ in AF'' ' s grounded extension, a is defended by arguments of level n or lower in AF'' ' s grounded extension. Hence, attackers of a belong only to set 3. By QP, arguments a attacked only by set 3 are ranked higher than arguments in set 3 attacked by set 1, and those attacked by set 2. Therefore, in AF'', we have $1 + a \succ 2 \succ 3$. Returning all arguments that are attacked (including by a) and some edges to form the complete graph AF. (If an edge is drawn from a to an argument in set 2, then in the AF, that argument in set 2 will simply be included in set 3, and it will not affect the subsequent arguments.) By QP, all such arguments b are ranked lower than set 2 of AF, not attacked by 1. Therefore, in AF:

$$1 + a \succ 2 \succ 3 + b$$

where a belongs to AF ' s grounded extension and b is attacked by AF ' s grounded extension. Hence, for AF with $k = n + 1$, arguments in AF ' s grounded extension $\succ$ arguments not attacked by AF ' s grounded extension but not in AF's grounded extension $\succ$ arguments attacked by AF's grounded extension holds, so the lemma also holds for AF with $k = n + 1$.

Thus, by mathematical induction, QP $\rightarrow$ gr-co is proven. □

CT → MDP

Proof. For a, b where $|R_1^-(a)| = |R_1^-(b)|$, $R_2^+(a) \neq \emptyset$, and $R_2^+(b) = \emptyset$. There exists $c \in R_1^-(a)$ such that $R_1^-(c) \neq \emptyset$, and if CT is satisfied, then $\forall d \in R_1^-(b)$, $d \succeq c$ and $R_1^-(b) \succeq_G R_1^-(a)$ hold, leading to $a \succeq b$. Therefore, if CT is satisfied, MDP is also satisfied. □

gr-co → AvsFD

Proof. For $a, b \in A$ where $|BR^-(a)| = 0$, $|R_1^-(b)| = 1$, and $|R_2^+(b)| = 0$, and $AF = \langle A, R \rangle$ is acyclic. a only having defense edges implies it must be included in the grounded extension, and since b only receives attack from one argument in the grounded extension, b cannot be in the grounded extension. Therefore, if gr-co is satisfied, $a \succ b$, then AvsFD is satisfied. □

adm-co → AvsFD

Proof. For $a, b \in A$ where $|BR^-(a)| = 0$, $|R_1^-(b)| = 1$, and $|R_2^+(b)| = 0$, and $AF = \langle A, R \rangle$ is acyclic. Since all complete extensions include the grounded extension, a is always included in complete extensions, and since complete extensions are chosen from admissible extensions, a must also be in admissible extensions. Also, The argument attacks b is in admissible extensions, so

b cannot be in admissible extensions. Therefore, if adm-co is satisfied, $a \succ b$, then AvsFD is also satisfied. □

st-co → AvsFD

Proof. For $a, b \in A$ where $|BR^{-}(a)| = 0$, $|R_1^{-}(b)| = 1$, and $|R_2^{+}(b)| = 0$, and $AF = \langle A, R \rangle$ is acyclic. Since stable extensions equal preferred extensions because AF is acyclic, and since adm-co = pr-co, if st-co is satisfied, $a \succ b$, then AvsFD is also satisfied. □

comp-weak → adm-weak

Proof. Assuming adm-weak holds, meaning a weakly adm-supports b, then since complete extensions are a subset of admissible extensions and both contain the same arguments, a also weakly co-support b. In other words, the condition of adm-weak is included in the condition of comp-weak. Therefore, if comp-weak is satisfied, then adm-weak is also satisfied. □

comp-weak → gr-weak

Proof. Assuming gr-weak holds, meaning a weakly gr-supports b, then since complete extensions always include grounded extensions, a also weakly co-support b. In other words, the condition of gr-weak is included in the condition of comp-weak. Therefore, if comp-weak is satisfied, then gr-weak is also satisfied. □

gr-weak → NaE

Proof. For a, b where $R_1(a) = \emptyset$ and $R_1(b) = \emptyset$. Since both are not attacked, they must be included in the grounded extension. As the grounded extension is unique, a and b mutually weakly gr-support each other. Therefore, if gr-weak is satisfied, $a \simeq b$, then NaE is also satisfied. □

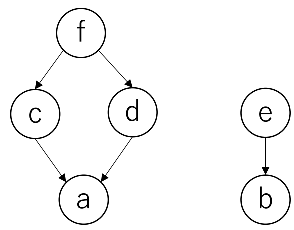
pr-weak → comp-weak

Proof. Assuming comp-weak holds, meaning a weakly supports b, then since preferred extensions are a subset of complete extensions and both contain the same arguments, a also weakly pr-support b. In other words, the condition of comp-weak is included in the condition of pr-weak. Therefore, if pr-weak is satisfied, then comp-weak is also satisfied. □

gr-co and gr-weak → MVP

Proof. For a not attacked and b attacked. If b is included in the grounded extension, then $a \simeq b$ by gr-weak. If b is not included in the grounded extension, then $a \succ b$ by gr-co. Therefore, if both gr-co and gr-weak are satisfied, $a \succeq b$ holds, so MVP is also satisfied. □

MCP and gr-weak → COM21



Proof. Consider an AF as depicted in the figure. For a, b satisfying the conditions of COM21, where $|R_1^-(a)| = 2$, $R_1^-(a) \subseteq C^1(AF)$, $|R_1^-(b)| = 1$, and $R_1^-(b) \subseteq C^0(AF)$.

All defenders of a and attackers of b are arguments not attacked, hence included in the grounded extension. Also, since a only has defense edges, it is included in the grounded extension. Therefore, for gr-weak to hold, $a \simeq e \simeq f$. Additionally, for MCP to hold, $f \succeq b \succeq a$, considering the number of attackers.

Thus, for MCP and gr-weak to hold, $a \simeq b$ must be satisfied. Consequently, if MCP and gr-weak are satisfied, COM21 is also satisfied. □

SCT and AE → ↑DB, ↑AB

Proof. They can be simultaneously demonstrated by mathematical induction, on the maximum length n of the path $P(x,a)$, in which x is the defense (attack) root of a and the argument added to a branch. (If traversing the same edge twice is allowed, the maximum length can become infinite. Therefore, we constrain our analysis to paths where each edge is traversed at most once. This constraint does not affect the satisfaction of $\uparrow$DB and $\uparrow$AB, and it ensures that all possible AF are considered. Consequently, this restriction does not pose any issues for the proof.)

For $n = 1$, since x is not attacked and $\gamma(x)$ is attacked, for the arguments z where zRa are not on $P(x,a)$, AE implies $z \simeq \gamma(z)$, thus by SCT, $\gamma(a) \succ a$.

For $n = 2$, if xRa and $x \in BR^{-}(a)$, there is no condition, so $\uparrow$DB naturally holds. If not, let b be the arguments attacking a on $P(x,a)$. Then, by the result for $n = 1$, $\gamma(b) \succ b$. For the arguments z where zRa is not on $P(x,a)$, AE implies $z \simeq \gamma(z)$. Therefore, by SCT, $a \succ \gamma(a)$.

Assume SCT and AE $\rightarrow\uparrow$DB, $\uparrow$AB holds for all cases up to $n = 2k$. For n = 2k + 1, if $x \in BR^{+}(a)$, there is no condition, so $\uparrow$AB naturally holds. If not, for the arguments y where yRa on $P(x,a)$, by the assumption, $y \succ \gamma(y)$ holds. For the arguments z where zRa is not on $P(x,a)$, AE implies $z \simeq \gamma(z)$. Therefore, by SCT, $\gamma(a) \succ a$, and $\uparrow$AB holds.

Assume SCT and AE $\rightarrow\uparrow$DB, $\uparrow$AB holds for all cases up to $n = 2k+1$. For n = 2k + 2, if $x \in BR^{-}(a)$, there is no condition, so $\uparrow$DB naturally holds. If not, for the arguments y where yRa on $P(x,a)$, by the assumption, $\gamma(y) \succ y$ holds. For the arguments z where zRa is not on $P(x,a)$, AE implies $z \simeq \gamma(z)$. Therefore, by SCT, $a \succ \gamma(a)$, and $\uparrow$DB holds.

In conclusion, by mathematical induction, SCT and AE $\rightarrow\uparrow$DB,AB holds. □

adm-co and comp-weak → MVP

Proof. For a not attacked and b attacked. If only a is included in admissible extensions, then according to adm-co, $a \succ b$. If both a and b are included in admissible extensions, according to adm-co = comp-co, there exists a complete extension that includes b, and since a is not attacked and included in all complete extensions, a comp-weakly supports b. Therefore, by comp-weak, $a \succeq b$. Therefore, if adm-co and comp-weak are satisfied, either $a \succ b$ or $a \succeq b$ holds, then MVP is also satisfied. □

adm-co and comp-strong → gr-co

Proof. For a included in the grounded extension and b not included. If only a is included in complete extensions, according to adm-co = comp-co, $a \succ b$. If both a and b are included in complete extensions, the grounded extension contains a but not b, and since all complete extensions must include the grounded extension, and grounded extension itself is a complete extension, so a comp-strongly supports b. Thus, by comp-strong, $a \succ b$. Therefore, if adm-co and comp-strong are satisfied, $a \succ b$, then gr-co is also satisfied. □