MIT
Applied Mathematics

Globally Synchronizing Graphs – Afonso Bandeira

Abstract:

In the 1600s, Christiaan Hyugens realized that two pendulum clocks (an invention of his!) placed in the same wooden table eventually fall into synchrony. Since then, synchronization of coupled oscillators has been an important subject of study in classical mechanics and nonlinear dynamics. The Kuramoto model, proposed in the 1970s, has become a prototypical model used for rigorous mathematical analysis in this field. A realization of this model consists of a collection of identical oscillators with interactions given by a network, which we identify respectively with vertices and edges of a graph.

In this talk we discuss which graphs are globally synchronizing, meaning that all but a measure-zero set of initial conditions converge into the fully synchronized state. We show that large expansion of the underlying graph is a sufficient condition (but far from necessary) and solve a conjecture of Ling, Xu and Bandeira stating that Erdos-Renyi random graphs are globally synchronizing above their connectivity threshold.

Time permitting, we will discuss connections with studying the non-convex landscape of the Burer-Monteiro algorithm for Community Detection in the Stochastic Block Model. Joint work with Pedro Abdalla (ETHZ), Martin Kassabov (Cornell), Victor Souza (Cambridge), Steven H. Strogatz (Cornell), Alex Townsend (Cornell).

§1 Introduction

- Coupled oscillators converge to synchronization. Question: How?
- Oscillator: $\Theta_i : [0, \infty)$ function that
  - $\Theta(t)$ = phase at time t
  - Have natural frequency: $\frac{d\theta}{dt} = \omega$
- Take graph structure $G = (v, e)$ vertices and edges
- Describe dynamics (via ordinary differential equations)
- Coupling strength? $\frac{d\theta_i}{dt} = \omega_i - \sum A_{ij} \sin(\theta_i \theta_j)$
- Oscillators have same frequency, they are homogenous
- WLOG (w loss of generality)

§2 Kuramoto

Model: $\frac{d\theta}{dt} = -\nabla E(\theta)$

where: E - energy function

Definition: G is globally synchronizing if the only stable fixed points are on the synchronous states $\theta_i = c \, \forall i$ (minimum energy)

- Global rotation? - Synchronize at local min of $E(\theta)$

- This is akin to gradient descent... here $E(\theta) = \frac{1}{2} \sum_{i,j \in V} A_{ij} (1 - \cos(\theta_i, \theta_j))$

§3 Examples of Globally Synchronyzing Graphs

- Cycles (powers of cycles)
- Graphs (varying across connectedness)
- Twisted state
- Gradient — Heshin... maybe hetian
- Stability — Regularity of graphs

§4 Conditions for Global Synchronization

Random Graphs

Erdös-Renyo Graphs: $G(n,p)$ of n nodes where every edge appears ind wit $P = p$

- If $p = 1$, G is globally synchronizing and K_n (fully connected?)
- If $p < \frac{(1-\varepsilon)}{n} \log(n)$, G is disconnected and not GS whp

Theorem: If $p > \frac{(1-\varepsilon)}{n} \log(n)$, G is GS whp

~ If G is a good expander, then G is GS.
(good expander means the results of G closely approximate that of a fully-connected graph. density is homogenous enough?)
G is d-regular if $\|A - \frac{d}{n}\| \leq \lambda$, all nodes have d neighbors

~ If $\frac{\lambda}{d} < 0.0816$ then G is GS.