Portfolio Selection
Harry Markowitz (1952)

§1 Introduction

- Portfolio selection can be broken down into two phases
  1) Predictions of asset returns
  2) Selection of assets
- Assertion: investors should attempt to maximize discounted expected return and minimize expected variance (risk)

§2 On Diversification

- The assertion above without considering variance is equivalent to asserting that an investor should place all their investments in the single security with the highest expected return. This fails to incorporate diversification, which is 'observed and sensible'

Analytically:
$$R = \sum_{t=1}^{\infty} \sum_{i=1}^{N} d_{it} r_{it} X$$
$$= \sum_{i=1}^{N} X \sum_{t=1}^{\infty} d_{it} r_{it}$$
$$R_i = \sum_{t=1}^{\infty} d_{it} r_{it}$$

where: d_{it} = discount rate for asset i at time t

r_{it} = return of i at t

X_i = weight of asset i in portfolio

R_i = discounted expected return of i

Thus, $R = \sum X_i R_i$

Assuming $\sum X_i = 1$, R is a weighted avg

If any R_{a_n} for $a = 1 \dots k$,

then max R is any portfolio where $\sum_{a=1}^{k} X_{a_n} = 1$

Thus, in no case is a diversified portfolio optimal

§3 The E-V Rule

- Investors face a trade-off between expected returns and variance.
- Efficient portfolios are those that maximize expected returns at a specified variance or minimize variance at a specified expected return.
- It is an 'exception' to see a diversified portfolio that simultaneously maximizes expected return and minimizes variance.

§4 Mathematical Statistics

- Random Variable: Y, can take on $y_1, y_2, \dots, y_N$
- Expected Value: $E[Y] = P_1 y_1 + P_2 y_2 + \dots + P_N y_N$
- Variance: $V = P_1(y_1 - E)^2 + P_2(y_2 - E)^2 + \dots + P_N(y_N - E)^2$
- Standard Deviation: $\sigma = \sqrt{V}$
- Coeficient of Variation: σ / E
- Random Variable: Linear Combination of Random Variables (weighted sum)
  $$R = \alpha_1 R_1 + \alpha_2 R_2 + \dots + \alpha_N R_N$$
  - Expected Value: $E[R] = \alpha_1 E[R_1] + \alpha_2 E[R_2] + \dots + \alpha_N E[R_N]$
  - Variance: 1) Covariance: $\sigma_{ij} = E[(R_i - E[R_i])(R_j - E[R_j])]$
    2) Correlation: $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$
    3) Variance: $V(R) = \sum_{i=1}^{N} \alpha_i V(R_i) + 2 \sum_{i=1}^{N} \sum_{i>1}^{N} \alpha_i \alpha_j \sigma_{ij}$
    $= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \sigma_{ij}$

§5 Efficient Frontier