The Inverse Trigonometric Functions

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WHAT'S COVERED

In this lesson, you will learn about the inverse trigonometric functions and how they are evaluated.

1. The Inverse Trigonometric Functions
2. Evaluating the Inverse Sine, Cosine, and Tangent Functions for Known Ratios
3. Evaluating the Inverse Cosecant, Secant, and Cotangent Functions for Known Ratios

1. The Inverse Trigonometric Functions

Recall the six basic trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$, and $\cot x$.

For each of them, the input is some angle and the output is a real number.

The **inverse trigonometric functions** do just the reverse. The input is the real number, while the output is the angle that produces the ratio.

For example, we define the inverse sine function as $y = \sin^{-1}x$, which means $x = \sin y$. Looking at the equation $x = \sin y$, it's clear that x must be between -1 and 1 (inclusive) since the sine function only returns ratios between -1 and 1.

The six inverse trigonometric functions, with their domains and ranges, are summarized in the table below.

Function	Domain	Range
$y = \sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1}x$	All real numbers	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \sec^{-1}x$	$(-\infty, -1]U[1, \infty)$	$\left[0, \frac{\pi}{2}\right)U\left(\frac{\pi}{2}, \pi\right]$
$y = \csc^{-1}x$	$(-\infty, -1]U[1, \infty)$	$\left[-\frac{\pi}{2}, 0\right)U\left(0, \frac{\pi}{2}\right]$
$y = \cot^{-1}x$	All real numbers	$(0, \pi)$

Note: when you use your calculator to evaluate an inverse trigonometric function, it will return the correct value.

HINT

The inverse trigonometric functions often go by other names. For example, $\sin^{-1}x$ can also be written as $\arcsin x$. This is sometimes more convenient since the "-1" in $\sin^{-1}x$ is often mistaken for an exponent of -1. Naturally, the other trigonometric functions follow suit. For example, $\tan^{-1}x$ is also known as $\arctan x$, etc.

TERM TO KNOW

Inverse Trigonometric Functions

A function that receives a real number as its input and returns an angle as its output.

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2. Evaluating the Inverse Sine, Cosine, and Tangent Functions for Known Ratios

Recall that $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$. Then, we can say $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$ or $\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}$.

When evaluating inverse trigonometric functions, we need to keep the range in mind.

EXAMPLE $\sin^{-1}(1)=\frac{\pi}{2}$ since $\frac{\pi}{2}$ is inside the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\left(\frac{\pi}{2}\right)=1$.

$\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$ since $\frac{2\pi}{3}$ is inside the interval $[0,\pi]$ and $\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$.

$\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$ since $\frac{\pi}{3}$ is inside the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\tan\frac{\pi}{3}=\sqrt{3}$.

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3. Evaluating the Inverse Cosecant, Secant, and Cotangent Functions for Known Ratios

Most calculators do not have dedicated buttons for $\csc^{-1}x$, $\sec^{-1}x$, or $\cot^{-1}x$; as you might suspect, these are related to their corresponding reciprocal functions.

For example, let's say we wish to find $\csc^{-1}(2)$.

This means that we want to find y so that $\csc y=2$.

Since $\csc y$ and $\sin y$ are reciprocals, this is equivalent to writing $\sin y=\frac{1}{2}$, which means $y=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$.

Through all this, something to notice is that $\csc^{-1}(2) = \sin^{-1}\left(\frac{1}{2}\right)$. This leads to some important identities.

FORMULA

Evaluating Inverse Cosecant

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Secant

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Cotangent

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

EXAMPLE Find $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$.

$$\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right) = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Note: $\frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{2\cdot 3} = \frac{\sqrt{3}}{2}$

EXAMPLE Find $\cot^{-1}(-1)$.

$$\cot^{-1}(-1) = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

EXAMPLE Find $\csc^{-1}\left(\frac{1}{2}\right)$.

$$\csc^{-1}\left(\frac{1}{2}\right) = \sin^{-1}(2)$$

Since $\sin^{-1}(2)$ is undefined, $\csc^{-1}\left(\frac{1}{2}\right)$ is undefined as well.

TRY IT

Consider the following inverse trigonometric function:

Inverse Trigonometric Function	Exact Value
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$	?
$\sec^{-1}(\sqrt{2})$	?
$\tan^{-1}(-\sqrt{3})$	?

Find the exact value of each inverse trigonometric function. +

Inverse Trigonometric Function	Exact Value
$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$	$\frac{\pi}{4}$
$\sec^{-1}(\sqrt{2})$	$\frac{\pi}{4}$
$\tan^{-1}(-\sqrt{3})$	$-\frac{\pi}{3}$

SUMMARY

In this lesson, you learned that **the inverse trigonometric functions** provide a way to express the angle as a function of the trigonometric ratio. You also learned how to **evaluate the inverse trigonometric functions for known ratios**, noting that while not all of these inverse functions are available on most calculators, there are identities that can be used to relate to other more common functions.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 7 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Inverse Trigonometric Functions
A function that receives a real number as its input and returns an angle as its output.

FORMULAS TO KNOW

Evaluating Inverse Cosecant

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Cotangent

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$$

Evaluating Inverse Secant

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$$

