Areas, Integrals, and Antiderivatives

by Sophia

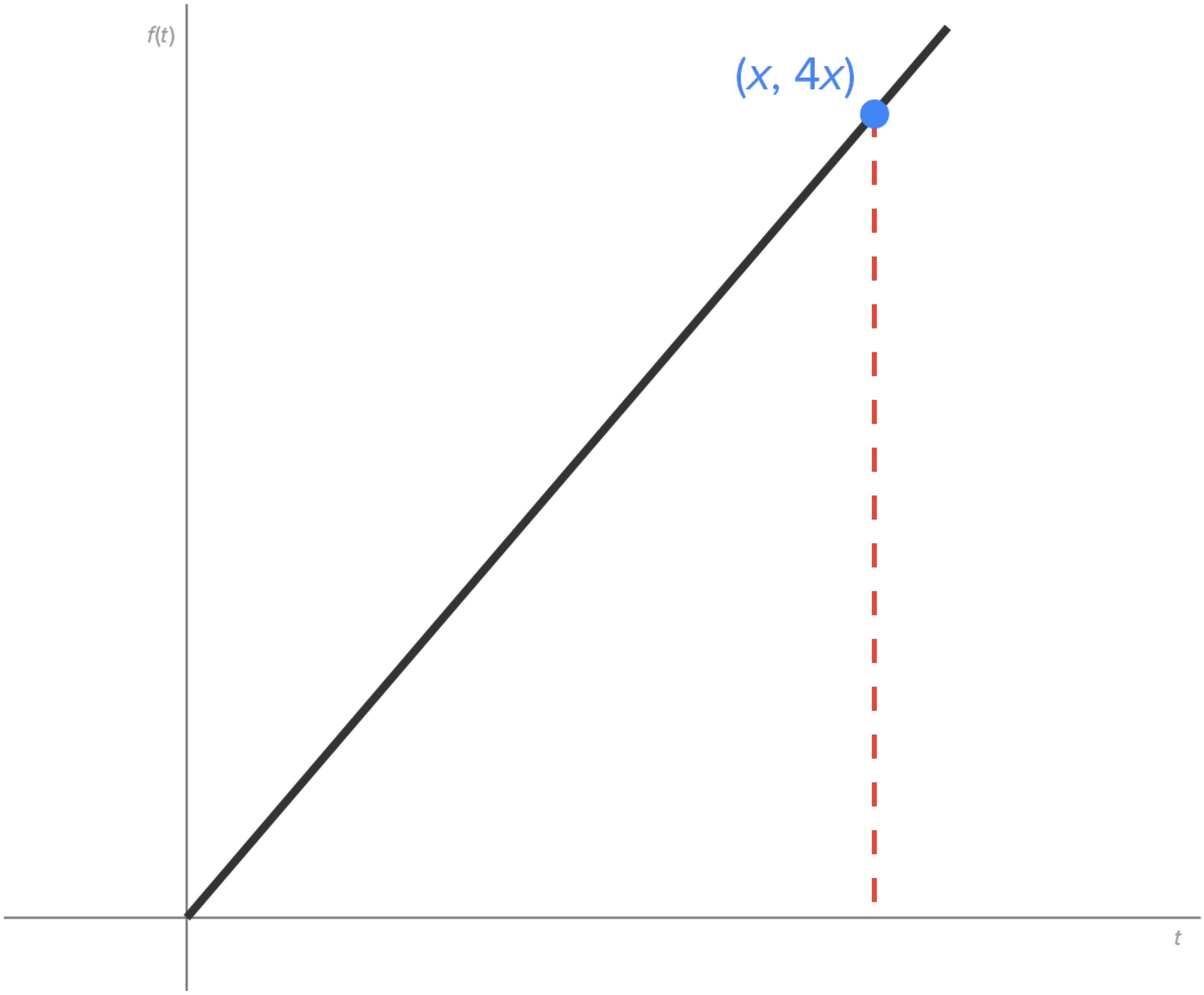
WHAT'S COVERED

In this lesson, you will connect the idea of area, definite integrals, and antiderivatives. Specifically, this lesson will cover:

1. The Area Function, $A(x)$ and Its Relationship to $f(x)$
2. Finding Basic Antiderivatives
3. The First and Second Fundamental Theorems of Calculus
   a. The First Fundamental Theorem of Calculus
   b. The Second Fundamental Theorem of Calculus
4. Using Antiderivatives to Calculate Area

1. The Area Function, $A(x)$ and Its Relationship to $f(x)$

Consider the area of the region bounded by the t-axis (horizontal axis), the function $f(t)=4t$, and the vertical line $t=x$. The graph is shown in the figure below.



As the value of x changes, the area of the region changes, meaning that the area depends on x, meaning the area is a function of x.

Let $A(x) =$ the area of the region, which is a triangle.

Since $A(x)$ defines the area of a region between $f(t)$ and the t-axis, we can define $A(x)$ as a definite integral:

$$A(x) = \int_0^x 4t\,dt$$

Assuming that $x > 0$, the area of the region is $A(x) = \frac{1}{2}(x)(4x) = 2x^2$.

TRY IT

Consider the region bounded by $f(t) = 3$ and the t-axis between $t = 0$ and $t = x$.

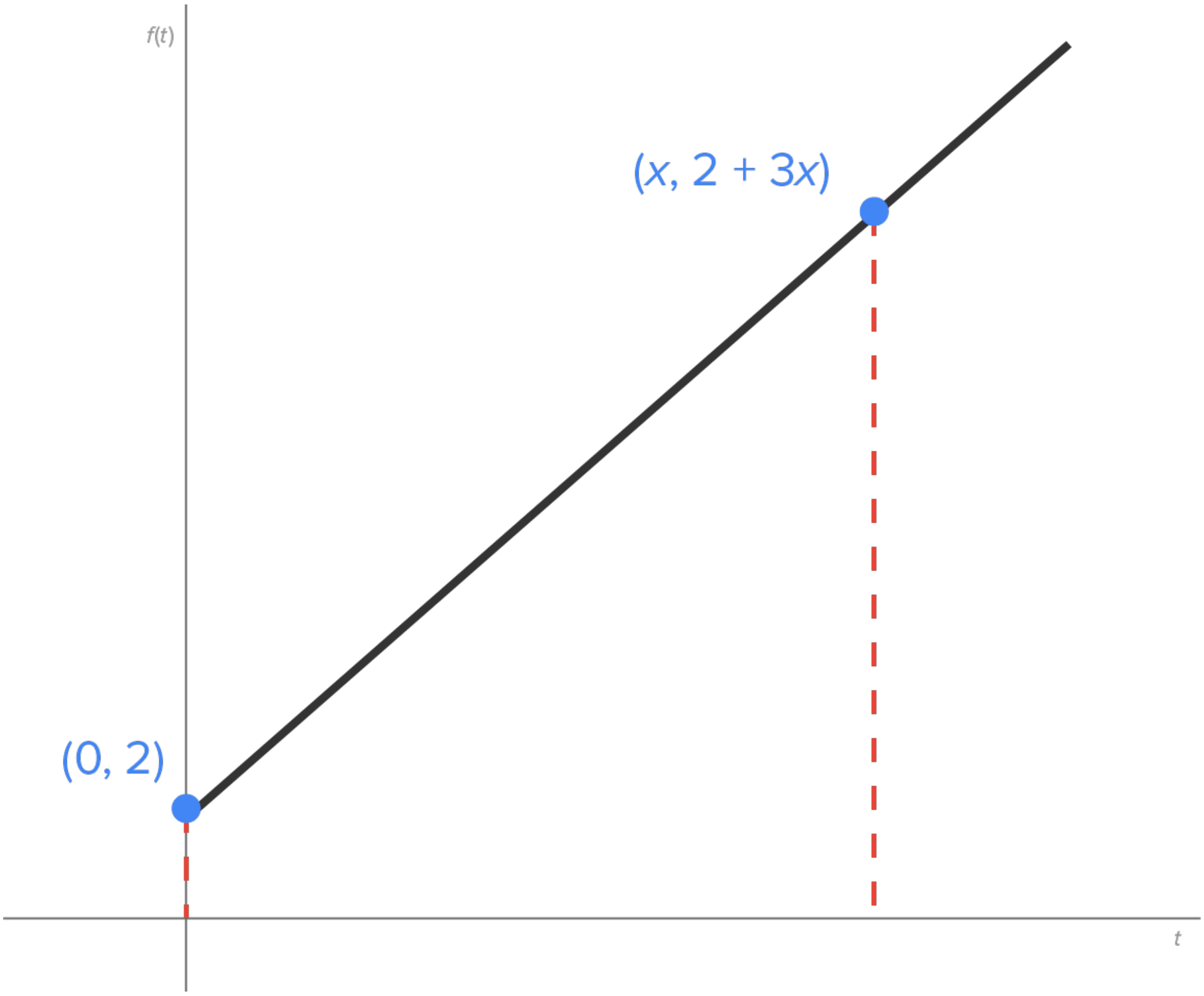
Write the area function $A(x)$ as both a definite integral and as a function of x. +

$$A(x) = \int_0^x 3\,dt = 3x$$

BIG IDEA

If x is one of the limits of integration, it is important that another variable be used inside the integral sign. This is because we can't have variables serving two roles at once (upper limit of integration and the variable inside the integral).

EXAMPLE Consider the region bounded by the t-axis and the line $f(t) = 2 + 3t$ between $t = 0$ and $t = x$.



As x increases, the shape remains a trapezoid.

As a definite integral, the area is:

$A(x) = \int_0^x (2 + 3t)dt$

Using the trapezoid area formula, we have:

$\frac{1}{2}(x)(2 + (2 + 3x))$	Use the trapezoid area formula.
$\frac{1}{2}x(4 + 3x)$	Simplify parentheses.
$2x + \frac{3}{2}x^2$	Distribute.

Thus, $A(x)=2x+\frac{3}{2}x^2$.

You might notice that there is a relationship between the area function $A(x)$ and the associated curve $y=f(x)$. We're going to explore this in this next segment.

Consider the last three examples. Here is a summary of the area functions with their associated curves, as well as the derivatives of each area function.

Regions	Area Function, $A(x)$	"Height" Function, $f(x)$	$A'(x)$
Region bounded by the t-axis (horizontal axis), the function $f(t)=4t$, and the vertical line $t=x$	$A(x)=2x^2$	$f(x)=4x$	$A'(x)=4x$
Region bounded by $f(t)=3$ and the t-axis between $t=0$ and $t=x$	$A(x)=3x$	$f(x)=3$	$A'(x)=3$
Region bounded by the t-axis and the line $f(t)=2+3t$ between $t=0$ and $t=x$	$A(x)=2x+\frac{3}{2}x^2$	$f(x)=2+3x$	$A'(x)=2+3x$

Note that in each situation, $A'(x)=f(x)$. It turns out that this is always the case, which is a very useful idea in finding areas of regions that use any choice of $f(x)$. First, we need to learn a bit about antiderivatives.

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2. Finding Basic Antiderivatives

We call $F(x)$ an **antiderivative** of $f(x)$ if $F'(x)=f(x)$. That is, $F(x)$ is the function whose derivative is $f(x)$. For instance, an antiderivative of $f(x)=3x^2$ is $F(x)=x^3$ since $D[x^3]=3x^2$. In fact, we could also say that $F(x)=x^3+4$ is an antiderivative of $f(x)=3x^2$ since $D[x^3+4]=3x^2$.

As it turns out, any function of the form $F(x)=x^3+C$ (where C is constant) is an antiderivative of $f(x)=3x^2$ since $D[x^3+C]=3x^2$.

EXAMPLE Find three antiderivatives of $f(x)=\cos x$.

Since $D[\sin x]=\cos x$, it follows that $F(x)=\sin x$ is an antiderivative of $f(x)=\cos x$. To find others, all you would need to do is add a constant.

Two more antiderivatives are $F(x)=\sin x+2$ and $F(x)=\sin x-3$.

In summary, any function of the form $F(x)=\sin x+C$ is an antiderivative of $f(x)=\cos x$.

TRY IT

Consider the function $f(x)=6x^2$.

Write three antiderivatives of this function. +

Any function of the form $F(x)=2x^3+C$ is an antiderivative of $f(x)=6x^2$. Here are three examples:

- $F(x)=2x^3$
- $F(x)=2x^3+2$
- $F(x)=2x^3-4$

TERM TO KNOW

Antiderivative
$F(x)$ is an antiderivative of $f(x)$ if $F'(x)=f(x)$.

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3. The First and Second Fundamental Theorems of Calculus

3a. The First Fundamental Theorem of Calculus

Consider the area function $A(x)=\int_0^x f(t)dt$.

By substituting $x=1$ and $x=2$, we have $A(1)=\int_0^1 f(t)dt$ and $A(2)=\int_0^2 f(t)dt$.

By properties of integrals, $\int_0^2 f(t)dt=\int_0^1 f(t)dt+\int_1^2 f(t)dt$.

Replacing the first two integrals by their values, we have $A(2)=A(1)+\int_1^2 f(t)dt$.

Finally, let's write the definite integral to one side: $A(2)-A(1)=\int_1^2 f(t)dt$

Remember that $A'(x)=f(x)$, meaning that $A(x)$ is an antiderivative of $f(x)$. Therefore, we could write $A(x)=F(x)$.

Thus, we can rewrite as $\int_1^2 f(t)dt=F(2)-F(1)$. This is generalized in **the first fundamental theorem of calculus**, as shown below:

Let $F(x)$ be an antiderivative of $f(x)$, meaning that $F'(x)=f(x)$. Then, $\int_a^b f(x)dx=F(b)-F(a)$, which means we evaluate the antiderivative at the endpoints, then subtract.

To show that we are substituting *a* and *b* into $F(x)$, we use the following notation: $F(x)\Big|_a^b$

Then, it follows that $F(x)\Big|_a^b = F(b) - F(a)$.

EXAMPLE Evaluate $\int_0^2 3x^2 dx$.

Since any function of the form $F(x) = x^3 + C$ is an antiderivative of $f(x) = 3x^2$, we have the following:

$\int_0^2 3x^2 dx$	Start with the original expression.
$= (x^3 + C)\Big|_0^2$	Apply the first fundamental theorem of calculus with $F(x) = x^3 + C$.
$= (2^3 + C) - (0^3 + C)$	Substitute $x = 2$ and $x = 0$ into $F(x)$, then subtract.
$= 8 + C - C$	Evaluate operations in the parentheses.
$= 8$	Simplify.

Thus, $\int_0^2 3x^2 dx = 8$.

BIG IDEA

Notice that the "$+C$" dropped out when evaluating the definite integral. Intuitively, this will always happen. Thus, when evaluating a definite integral, select $C = 0$. Some students even say "You don't need the *C*."

TERM TO KNOW

The First Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of $f(x)$, meaning that $F'(x) = f(x)$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$, which means we evaluate the antiderivative at the endpoints, then subtract.

3b. The Second Fundamental Theorem of Calculus

Recall that $A'(x) = f(x)$. If we replace $A(x)$ with $F(x)$ to correspond with $f(x)$, we have another important theorem in calculus, **the second fundamental theorem of calculus**:

Let $f(x)$ be a continuous function on the closed interval $[a, b]$ with $a \le x \le b$.

Let $F(x) = \int_a^x f(t)dt$. Then, $F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

EXAMPLE Let $F(x) = \int_2^x \sqrt{t+1}\, dt$.

Since $f(t) = \sqrt{t+1}$ is continuous on $[-1, \infty)$, which includes 2, then $F'(x) = \sqrt{x+1}$.

TRY IT

Let $F(x)=\int_{\sqrt{\pi}}^{x}\sin(t^2)dt$.

Find $F'(x)$. +

$F'(x)=\sin(x^2)$

Suppose x is replaced by u, where u is a function of x.

That is, $F(x)=\int_a^u f(t)dt$. Then, by the chain rule, $F'(x)=f(u)\cdot\dfrac{du}{dx}$.

EXAMPLE Let $F(x)=\int_1^{x^3} e^t dt$.

Then, $F'(x)=e^{x^3}\cdot 3x^2=3x^2e^{x^3}$.

TRY IT

Let $F(x)=\int_0^{\sin x}\ln t\,dt$.

Find $F'(x)$. +

$F'(x)=\cos x\cdot\ln(\sin x)$

TERM TO KNOW

The Second Fundamental Theorem of Calculus

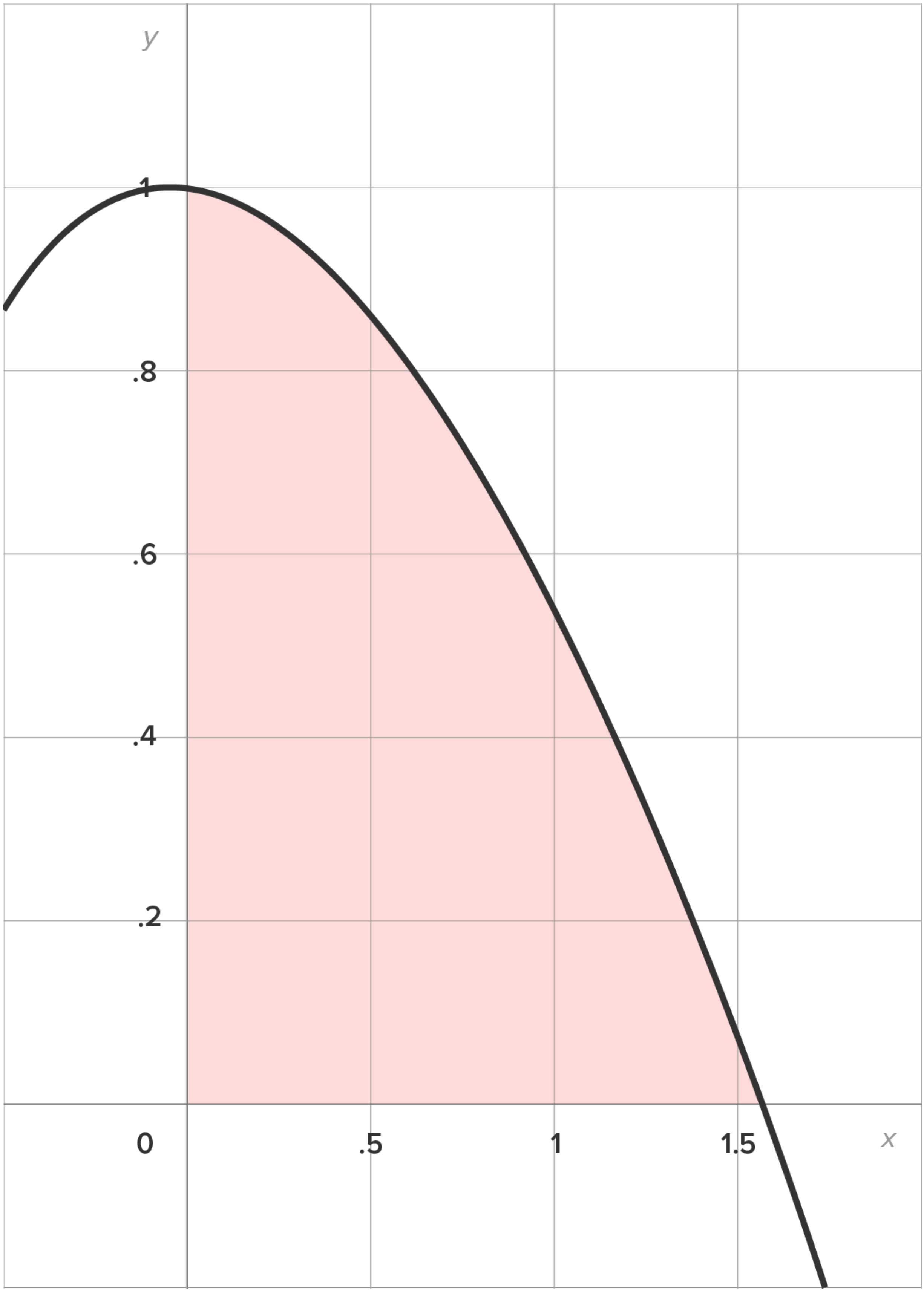
Let $f(x)$ be a continuous function on the closed interval $[a,b]$ with $a\le x\le b$.
Let $F(x)=\int_a^x f(t)dt$. Then, $F'(x)=\dfrac{d}{dx}\int_a^x f(t)dt=f(x)$.

4. Using Antiderivatives to Calculate Area

As a result of the fundamental theorem of calculus, we have a new way to compute areas with definite integrals. Instead of relying on a sketch of the region, we can use antiderivatives to compute areas.

EXAMPLE Find the area between the graph of $f(x)=\cos x$ and the x-axis between $x=0$ and $x=\dfrac{\pi}{2}$.

The region is shown in the figure:



We know the area is given by the definite integral $\int_0^{\pi/2} \cos x dx$.

Earlier, we saw that sin x is an antiderivative of cos x. Therefore, the area is as follows:

$\int_0^{\pi/2} \cos x dx$	Start with the original expression.
$= \sin x \Big\|_0^{\pi/2}$	Use the fundamental theorem of calculus with $F(x) = \sin x$. Remember, we do not need to write "+C," meaning we are choosing $C = 0$.
$= \sin\frac{\pi}{2} - \sin 0$	Substitute $x = \frac{\pi}{2}$ and $x = 0$ into $F(x)$, then subtract.
$= 1$	Simplify. Recall $\sin\frac{\pi}{2} = 1$ and $\sin 0 = 0$.

Thus, the area of the region is 1 square unit.

TRY IT

Consider the graphs of $f(x) = e^x$ and the x-axis between $x = 0$ and $x = 2$.

Calculate the area between these graphs. +

$e^2 - 1$ square units

SUMMARY

In this lesson, you learned that with a **link between the area function** $A(x)$ **and a region with** $f(x)$ as the upper boundary, **basic antiderivatives** can be used to **calculate areas** and compute definite integrals by using **the first and second fundamental theorems of calculus**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Antiderivative

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

The First Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of $f(x)$, meaning that $F'(x) = f(x)$.

Then, $\int_a^b f(x)dx = F(b) - F(a)$, which means we evaluate the antiderivative at the endpoints, then subtract.

The Second Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function on the closed interval $[a, b]$ with $a \le x \le b$.

Let $F(x) = \int_a^x f(t)dt$. Then, $F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.