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Differentiability

by Sophia

WHAT'S COVERED

In this lesson, you will investigate the differentiability of a function by using analytical techniques, which include a determination of continuity. Specifically, this lesson will cover:

1. Defining Differentiability
2. Determining Differentiability at $x = a$ Analytically
   a. Continuous but Not Differentiable
   b. Differentiable for All Real Numbers
   c. Not Continuous

1. Defining Differentiability

Differentiability is an important concept in calculus since it pertains to the "smoothness" of a curve. A function $y = f(x)$ is said to be **differentiable** at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

TERM TO KNOW

Differentiable
A function $y = f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.

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2. Determining Differentiability at $x = a$ Analytically

The following statements are equivalent:

- If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.
- If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

How to interpret these statements:

- If $f(x)$ is not continuous at $x = a$, then it is never differentiable at $x = a$.

- If $f(x)$ is differentiable at $x = a$, then it is always continuous at $x = a$.

Note: This means that if $f(x)$ is continuous at $x = a$, $f(x)$ may or may not be differentiable at $x = a$.

HINT

Recall that the definition of continuity of $f(x)$ at $x = a$ is $\lim_{x \to a} f(x) = f(a)$.

2a. Continuous but Not Differentiable

Here is an example of a function that is continuous but not differentiable at a point.

EXAMPLE Determine if $f(x) = \sqrt[3]{x}$ is differentiable at $x = 0$. First, check for continuity at $x = 0$:

$$f(0) = \sqrt[3]{0} = 0$$

$$\lim_{x \to 0} \sqrt[3]{x} = 0$$

Since $\sqrt[3]{x}$ is defined for positive and negative real numbers, there is no need to use one-sided limits. Since the limit and $f(0)$ are equal, $f(x)$ is continuous at $x = 0$.

Now, let's check the derivative. Note that $f(x) = \sqrt[3]{x} = x^{1/3}$. Then, $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$.

Since $f'(0)$ is undefined (0 in the denominator), $f(x)$ is not differentiable at $x = 0$.

Note: In this case, this means that the slope of the tangent line is undefined, and that the tangent line is vertical.

2b. Differentiable for All Real Numbers

Here is an example of a function that is differentiable for all real numbers.

EXAMPLE Show that $f(x) = x^3$ is differentiable for all real numbers.

Check continuity: Since $f(x)$ is a polynomial function, it is continuous for all real numbers (this was established in Challenge 2.3).

Check the derivative: $f'(x) = 3x^2$, which is defined for all real numbers.

Thus, $f(x) = x^3$ is differentiable for all real numbers.

2c. Not Continuous

Here is an example of a function that is not continuous at a point, which means that it is also not differentiable at the point.

EXAMPLE Consider the function $f(x) = \frac{2}{x-1}$.

Since $f(x)$ is not continuous at $x = 1$, it is also not differentiable at $x = 1$.

TRY IT

Consider the following functions and x-values.

Function	Given x-value	Differentiable (Yes or No)?
$f(x)=\cos x$	$x=0$	?
$g(x)=\sqrt{x}$	$x=4$	?
$h(x)=\frac{x}{2x-1}$	$x=\frac{1}{2}$	?

Determine if each function is differentiable at the given x-value in the table. +

Function	Given x-value	Differentiable (Yes or No)?
$f(x)=\cos x$	$x=0$	Yes
$g(x)=\sqrt{x}$	$x=4$	Yes
$h(x)=\frac{x}{2x-1}$	$x=\frac{1}{2}$	No (not continuous, therefore not differentiable at $x=\frac{1}{2}$)

SUMMARY

In this lesson, you explored the first of two ways to **define differentiability**, noting that if a function is to be differentiable at $x=a$, it must be continuous at $x=a$ and $f'(a)$ needs to be defined. You learned how to **determine differentiability at $x = a$ analytically**, exploring examples of functions that are **continuous but not differentiable**, **differentiable for all real numbers**, and also **not continuous** and therefore not differentiable at the points of discontinuity.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Differentiable

A function $y=f(x)$ is said to be differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)$ is defined.