Linear Approximation

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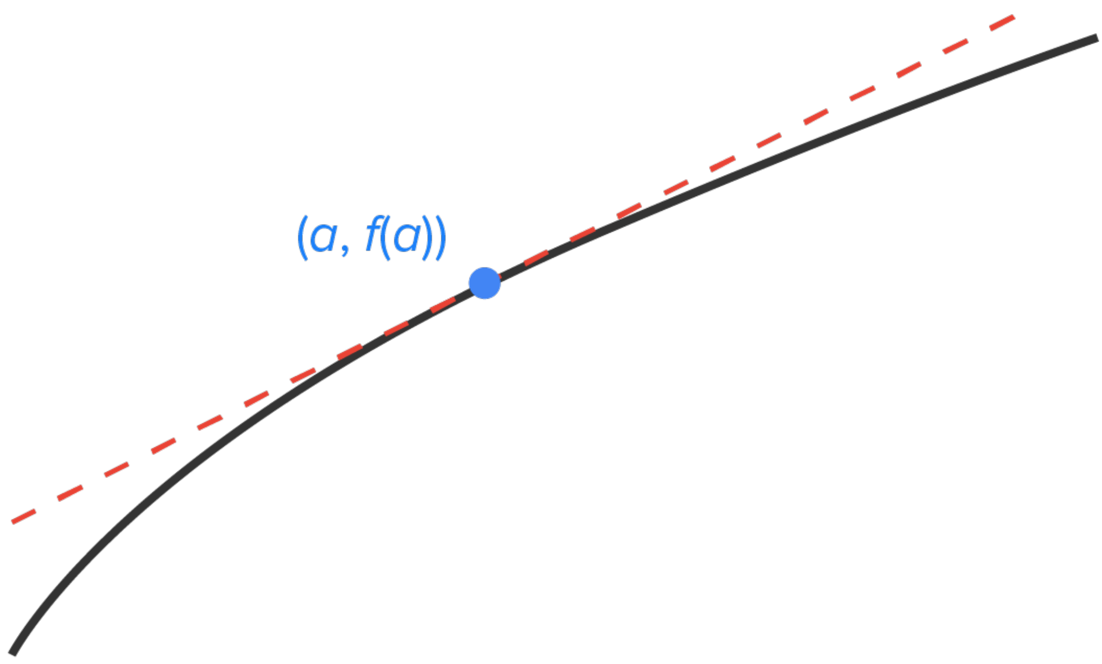
WHAT'S COVERED

In this lesson, you will see how the derivative can be used to approximate the values of various types of functions that would be difficult to compute otherwise, such as power, root, logarithmic, exponential, and trigonometric functions. For example, a well-placed tangent line for the function $f(x)=\sqrt{x}$ can be used to approximate $\sqrt{17}$. Specifically, this lesson will cover:

1. The Linear Approximation of $f(x)$ at $x=a$
2. Approximating Values of $f(x)$

1. The Linear Approximation of $f(x)$ at $x = a$



Consider the graph above.
The solid curve is the graph of $y=f(x)$.
The dashed line is the tangent line at $x=a$.

Notice how the tangent line and the graph of $y=f(x)$ are very close to each other near $x=a$. This means that the expression for the tangent line can be used to approximate the value of $y=f(x)$ near $x=a$. As a result, the tangent line at $x=a$ is called the **linear approximation of *f*(*x*) at $x=a$**.

In the context of finding a linear approximation, the equation for the tangent line is written as $L(x)=f(a)+f'(a)(x-a)$. Note that the only change is using $L(x)$ to represent the output instead of *y*.

> EXAMPLE Find the linear approximation of $f(x)=\sqrt{x}$ when $x=16$. Remember that this is really the equation of the tangent line at $x=16$.
>
> Find the derivative.
>
> | | |
> |---|---|
> | $f(x)=\sqrt{x}=x^{1/2}$ | Rewrite as a power of *x*. |
> | $f'(x)=\frac{1}{2}x^{-1/2}$ | Find the derivative. |
> | $f'(x)=\frac{1}{2x^{1/2}}=\frac{1}{2\sqrt{x}}$ | Rewrite with a positive exponent, then the radical. |
> | $f'(16)=\frac{1}{2\sqrt{16}}=\frac{1}{8}$ | Substitute 16 in for *x* and simplify. |
>
> Now, form the linear approximation:
>
> | | |
> |---|---|
> | $L(x)=f(a)+f'(a)(x-a)$ | Use the equation for the tangent line. |
> | $L(x)=f(16)+f'(16)(x-16)$ | $a=16$ |
> | $L(x)=4+\frac{1}{8}(x-16)$ | $f(16)=4$ and $f'(16)=\frac{1}{8}$ |
>
> The linear approximation at $x=16$ is $L(x)=4+\frac{1}{8}(x-16)$. This means that $L(x)$ approximates the value of $f(x)$ near $x=16$.
>
> Note: Traditionally, we would simplify $L(x)$, but in this context, we are using $L(x)$ to approximate values of $f(x)$ near $x=16$. Thus, the "$x-16$" serves as a reminder that the linear approximation is based on the tangent line at $x=16$.

> EXAMPLE Find the linear approximation to $f(x)=e^x$ at $x=0$.
>
> First, find the derivative: $f(x)=e^x$, $f'(x)=e^x$
>
> Now, use the linear approximation:
>
> | | |
> |---|---|
> | $L(x)=f(a)+f'(a)(x-a)$ | Use the equation for the tangent line. |
> | $L(x)=f(0)+f'(0)(x-0)$ | $a=0$ |

$L(x) = 1 + 1(x - 0)$ $f(0) = 1$ and $f'(0) = 1$

$L(x) = 1 + x$ Simplify.

This means that $L(x) = 1 + x$ approximates the value of $f(x) = e^x$ near $x = 0$.

WATCH

Here is a video example finding the linear approximation of $f(x) = \sqrt[3]{x}$ when $x = -64$.

TERM TO KNOW

Linear Approximation of *f*(*x*) at $x = a$
The tangent line to the graph of $f(x)$ at $x = a$.

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2. Approximating Values of *f*(*x*)

The point of the linear approximation is to approximate the value of $f(x)$ for values close to $x = a$. Thus, in the previous example, $L(x) = 4 + \frac{1}{8}(x - 16)$ will be used to approximate values of $f(x) = \sqrt{x}$ for *x* near 16.

EXAMPLE Considering the last example, use the linear approximation to approximate $f(17) = \sqrt{17}$.

The linear approximation we found in the last example gives $L(17) = 4 + \frac{1}{8}(17 - 16) = 4.125$.

From a calculator, $\sqrt{17} \approx 4.123$. The linear approximation for $\sqrt{17}$ is very close.

TRY IT

Consider the function $f(x) = \sqrt[3]{x}$.

Find the linear approximation of this function when *x* = 8. +

$L(x) = \frac{1}{12}(x - 8) + 2$

Suppose you want to use this above linear approximation to approximate $\sqrt[3]{9}$.

Approximate to three decimal places. +

$\sqrt[3]{9} \approx L(9) = 2.083$

SUMMARY

In this lesson, you learned that the tangent line to a function at $x = a$ can be used to find the **the linear approximation of $f(x)$ at *x* = *a*.** Then, you were able to apply this knowledge to explore a few examples **approximating values of** $f(x)$ at *x* values near *x* = *a*. Later in this challenge, we will discuss the errors involved in using the linear approximation and how using other values of *x* affect the approximation.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Linear Approximation of f(x) at x = a
The tangent line to the graph of $f(x)$ at $x = a$.

