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Derivative of y = a^x

by Sophia

WHAT'S COVERED

In this lesson, you will find derivatives of exponential functions with any base. For example, $f(x)=2^x$, $g(x)=3^{-x^2}$, and $A(t)=\left(\frac{1}{2}\right)^{t/300}$. Sometimes it is more convenient to model situations with bases other than *e*, so it is important that we learn about the derivatives of $y=a^x$ and $y=a^u$. Specifically, this lesson will cover:

1. Derivatives of $y=a^x$ and Combinations of Functions With $y=a^x$
2. Derivatives of $y=a^u$ and Combinations of Functions With $y=a^u$, Where *u* Is a Function of *x*

1. Derivatives of $y = a^x$ and Combinations of Functions With $y = a^x$

WATCH

Please view this video to see how we arrive at the derivative formula for $f(x)=a^x$, where $a>0$.

Video Transcription

[MUSIC PLAYING] Hello. And in this video, we're going to take a little bit of a different track. We're going to derive a derivative formula based on facts we already know. So here, we have f of x equals a to the x. So a could be any base. It doesn't have to be e. But we're going to use the fact that we know the derivative of e to the x to find the derivative of a to the x.

So the first thing to realize is that e to the natural log of a is equal to a. That's one of the logarithm properties discussed way back in unit 1. And we're going to replace a with e to the natural log of a. So we have e to the natural log of a raised to the x. And then by properties of exponents, we know that that's equal to e to the natural log of a times x. So all three of these things are equivalent, and that's going to be important later on in the problem.

So we know that the derivative of e to the u is e to the u times the derivative of u, And we're going to use that fact to find the derivative of this expression. So we have f prime of x is equal to-- now, remember, this natural log of a is basically just a constant. So we're going to use that when we find our derivative. So that means the derivative of e to the natural log of a times x is e to the natural log of a

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times x-- e to the something-- times the derivative of the something, OK?

So that's equal to-- and I'm going to leave this the way it is-- e to the natural log of a times x times-- now, remember, this is a constant. So that means we're basically taking the derivative of a constant times x, and that is just the constant. And now to finish this off, remember that e to the natural order of a times x, which is right here, is equivalent to a to the x. All three of those things in the first three lines or equivalent. So I'm going to rewrite that as a to the x, because, one, it's simpler, and it's related to what was written as the primary function, which was a to the x. So that means that the derivative is a to the x times the natural log of a.

[MUSIC PLAYING]

So, we can say the derivative of a^x can be expressed with the following formula:

FORMULA

The Derivative of a^x

$D[a^x] = a^x \cdot \ln a$

For instance, this means that $D[3^x] = 3^x \cdot \ln 3$ and $D\left[\left(\frac{1}{2}\right)^x\right] = \left(\frac{1}{2}\right)^x \cdot \ln\left(\frac{1}{2}\right)$.

Let's look at a few examples where $f(x) = a^x$ is combined with other functions.

EXAMPLE Consider the function $f(x) = x \cdot 10^x$. Find its derivative.

$f(x) = x \cdot 10^x$	Start with the original function.
$f'(x) = D[x] \cdot 10^x + x \cdot D[10^x]$	Use the product rule.
$f'(x) = (1) \cdot 10^x + x \cdot (10^x \cdot \ln 10)$	$D[x] = 1$, $D[10^x] = 10^x \cdot \ln 10$
$f'(x) = 10^x + x \cdot 10^x \cdot \ln 10$	Remove extra grouping symbols.

Thus, $f'(x) = 10^x + x10^x \ln 10$.

This could also be rewritten by factoring out 10^x: $f'(x) = 10^x(1 + x\ln 10)$

2. Derivatives of $y = a^u$ and Combinations of Functions With $y = a^u$, Where u Is a Function of x

As a result of the chain rule, we have the following derivative formula:

FORMULA

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The Derivative of a^u, Where u Is a Function of x

$D[a^u]=(a^u\cdot\ln a)\cdot u'$

↪ EXAMPLE Consider the function $f(x)=3^{-x^2}$. Find its derivative.

$f(x)=3^{-x^2}$	Start with the original function.
$f'(x)=\left(3^{-x^2}\cdot\ln 3\right)\cdot(-2x)$	$D[3^u]=(3^u\cdot\ln 3)\cdot u'$ Here, $u=-x^2$.
$f'(x)=-2x3^{-x^2}\ln 3$	Write "-2x" in front and remove unnecessary grouping symbols.

Thus, $f'(x)=-2x3^{-x^2}\ln 3$.

TRY IT

Consider the function $f(x)=\sqrt{5^x+2}$.

Find its derivative. +

$f'(x)=\frac{1}{2}(5^x+2)^{-1/2}(5^x\cdot\ln 5)$ or $f'(x)=\frac{5^x\ln 5}{2\sqrt{5^x+2}}$

↪ EXAMPLE A drug has a half-life of 6 hours, which means that after 6 hours in the bloodstream, half of the original amount remains. When 40mg of this drug is introduced into the bloodstream, the amount remaining after t hours is $A(t)=40\left(\frac{1}{2}\right)^{t/6}$.

At what rate is the amount of drug in the bloodstream changing after 8 hours?

In this problem, we want to find $A'(8)$. So, let's first find $A'(t)$.

$A(t)=40\left(\frac{1}{2}\right)^{t/6}$	Start with the original function.
$A'(t)=40\left[\left(\frac{1}{2}\right)^{t/6}\cdot\ln\left(\frac{1}{2}\right)\right]\cdot\frac{1}{6}$	$D[a^u]=a^u\cdot\ln a\cdot u'$ $u=\frac{t}{6}=\frac{1}{6}t,\ u'=\frac{1}{6}$
$A'(t)=\frac{20}{3}\cdot\left(\frac{1}{2}\right)^{t/6}\cdot\ln\left(\frac{1}{2}\right)$	$40\left(\frac{1}{6}\right)=\frac{40}{6}=\frac{20}{3}$ Remove extra symbols.

Then, $A'(8)=\frac{20}{3}\cdot\left(\frac{1}{2}\right)^{8/6}\cdot\ln\left(\frac{1}{2}\right)\approx-1.83384$. This means that the amount of drug in the bloodstream is decreasing at a rate of about 1.83 mg/hr.

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SUMMARY

In this lesson, you explored finding the **derivatives of** $y = a^x$ **and combinations of functions with** $y = a^x$. You also learned how to find the **derivatives of the general exponential function** $y = a^u$**, where** ***u*** **is a function of** ***x*** **(and related combinations of functions)**, which allows you to explore even more functions and applications. Remember that the derivative rule for $y = a^u$ is very similar to that of $y = e^u$ where *u* is a function of *x*, but with an extra factor of $\ln a$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

The Derivative of a^x

$$D[a^x] = a^x \cdot \ln a$$

The Derivative of a^u, Where u Is a Function of x

$$D[a^u] = (a^u \cdot \ln a) \cdot u'$$

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