The Algorithm for Newton's Method

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WHAT'S COVERED

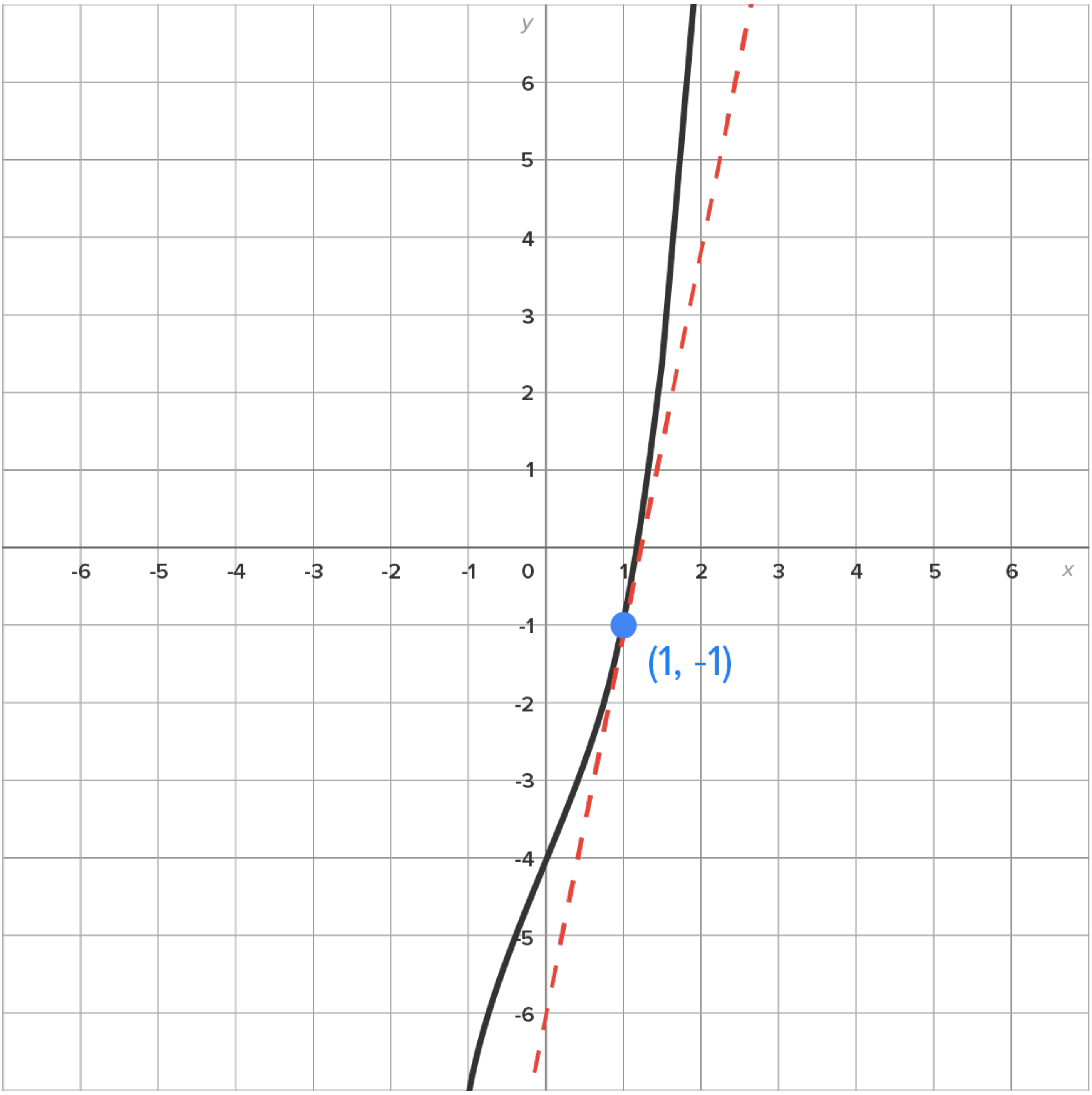
As you have seen in this challenge, the tangent line to a function at $x = a$ provides a good estimate to $f(x)$ near $x = a$. Another way to use the tangent line is to find its x-intercept to approximate the x-intercept of $f(x)$. In this lesson, you will learn Newton's method, which uses successive tangent lines to approximate an x-intercept. Most graphing utilities use Newton's method to locate x-intercepts and points of intersections of graphs. Specifically, this lesson will cover:

1. The Idea Behind Newton's Method
2. Applying Newton's Method
   a. Newton's Method: The Algorithm
   b. Approximating x-Intercepts with Newton's Method

1. The Idea Behind Newton's Method

The goal of Newton's method is to use tangent lines to approximate an x-intercept of the graph of $y = f(x)$. In other words, the goal is to solve the equation $f(x) = 0$.

Consider the function $f(x) = x^3 + 2x - 4$.



Now, consider the picture shown above, which has two graphs:

- The solid curve is the graph of $f(x)$.
- The dashed line is the tangent line at $x = 1$ (this corresponds to our "guess").

To start the process for Newton's method, we're going to "guess" $x = 1$ as the x-intercept.
Notice that the x-intercept of $f(x)$ is very close to the x-intercept of the tangent line. The advantage of using the tangent line is that it is much easier to solve a linear equation than it is a cubic equation.

First step: Find the equation of the tangent line at $x = 1$.

Given $f(x) = x^3 + 2x - 4$, the derivative is $f'(x) = 3x^2 + 2$. Then, the slope of the tangent line is $f'(1) = 3(1)^2 + 2 = 5$.

Then, the equation of the tangent line is:

$$y = f(1) + f'(1)(x-1)$$
$$y = -1 + 5(x-1)$$
$$y = -1 + 5x - 5$$
$$y = 5x - 6$$

Then, the x-intercept of the tangent line is found by letting $y = 0$ and solving for *x*:

$$0 = 5x - 6$$
$$6 = 5x$$
$$\frac{6}{5} = x \text{ (or 1.2 in decimal form)}$$

Thus, our approximation for the x-intercept is (1.2, 0).
So, where would we go from here?

We now have a new “guess” for the x-intercept of the graph of $f(x)$. To continue with this process, find the equation of the tangent line to $f(x)$ at $x = 1.2$, then find its x-intercept. We’ll formalize this process and then complete this problem in the next part of this challenge.

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2. Applying Newton’s Method

Consider a function $y = f(x)$ and let x_0 be the first guess for its x-intercept.

Write the equation of the tangent line at $x = x_0$: $y = f(x_0) + f'(x_0)(x - x_0)$.

Find the x-intercept of the tangent line, which means $y = 0$:

$0 = f(x_0) + f'(x_0)(x - x_0)$	Replace *y* with 0.
$-f(x_0) = f'(x_0)(x - x_0)$	Subtract $f(x_0)$ from both sides.
$-\dfrac{f(x_0)}{f'(x_0)} = x - x_0$	Divide both sides by $f'(x_0)$.
$x_0 - \dfrac{f(x_0)}{f'(x_0)} = x$	Add x_0 to both sides.

Now, this x-intercept is the next guess for the intercept, which under normal conditions, is a closer estimate than x_0. Since this process will continue, let’s call the x-intercept of the tangent line x_1. Then, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

Now, suppose we want to continue this process:

- Find the equation of the tangent line at $x = x_1$.
- Find the x-intercept of the tangent line and call it x_2. Then, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

If we continue this process, we get a sequence of estimates x_0, x_1, x_2, ... for the estimates of the x-intercept that get closer to some number (which would be the actual x-intercept). Performing these iterations is what is known as Newton’s method.

2a. Newton's Method: The Algorithm

Suppose the goal is to find an approximation to an x-intercept of a function $y=f(x)$, which is equivalent to finding a solution to $f(x)=0$. Starting with an initial guess at $x=x_0$, the sequence of guesses x_1, x_2, x_3, ... is generated by the formula $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

The process stops when one of two things occurs:

- Two consecutive x-values are "close enough" together.
- The x-values are jumping around to the point where they aren't getting closer to a common number.

FORMULA

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

2b. Approximating x-Intercepts with Newton's Method

↪ EXAMPLE Let's pick back up with the function $f(x)=x^3+2x-4$. When we left off, we had $x_0=1$ and $x_1=1.2$. Let's perform two more iterations of Newton's Method to get a better approximation of the x-intercept. To use Newton's method, it is best to organize the information into a table:

Note: $f(x)=x^3+2x-4$ and $f'(x)=3x^2+2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
0	1	-1	5	1.2
1	1.2	0.128	6.32	1.179746835
2	1.179746835	0.001468379	6.175407787	1.179509057
3	1.179509057	0.0000002	6.173724847	1.179509025

The last two estimates are identical to 6 decimal places, so we conclude that the x-intercept to six decimal places of $f(x)$ is (1.179509, 0). This also means that the equation $x^3+2x-4=0$ has the solution $x\approx 1.179509$.

WATCH

Use Newton's method to find the approximate solution to $x-\cos x=0$.

Video Transcription

Hello, and welcome to today's video on how to complete three iterations of Newton's method to find an approximate solution to the equation x minus the cosine of x is equal to 0. Now, Newton's method is a method to approximate the x-coordinates of the x-intercepts of a graph.

And it's an iterative method, meaning that you run a value through the process, and you take the value you get out after running it through the process and run it through the process again, getting hopefully a more refined or closer approximation than the previous one. And you can keep doing that over and over to get closer and closer approximations to your original equation or x-coordinate of your x-intercept.

There are some situations where it actually does not get us closer to a solution, but for all of the problems that we will do in this course, Newton's method will work. So how do we do this? We will take our initial guess and put it in our table. And then what we'll do after that is run that initial guess through the function, run the initial guess through the derivative of the function.

And then our Newton's method process is, we take that initial guess and subtract off of it the function evaluated at that initial guess divided by the derivative evaluated at that initial guess. And we'll get a value out that will be our first iteration. Then we will bring that down into the next row of the table and do the process all over again.

So when we're starting this and we have our equation that we want to use Newton's method on, that x minus cosine x is equal to 0, because we want to apply the process of Newton's method, we need to make sure that our equation is set up so that everything is on one side equal to 0 on the other side.

So it's nice that this is already set up in that manner. When we have it that way, then our function we take is just the expression that's on the side opposite of 0. So we have our x minus cosine x. Now, I've drawn the graph of f of x equal x minus cosine x just to the left of what I just wrote. And if you notice, the curve is crossing the x-axis really close to an x value of 1.

So our initial guess is 1. So you want to get that from, actually, just drawing the graph of the function and seeing the integer x value that is really close to where the graph crosses. Now we want to find f of our x value, our first guess. And f of 1 is equal to 1 minus the cosine of 1.

Make sure you set your calculator in radian mode, and when you calculate 1 minus the cosine of 1, you'll get 0.4596976941. And that's what we'll put in our table right underneath the f of x sub n. So we have our 0.4596976941.

Next, we want f prime of x sub n. Well, f of x is x minus cosine x. So f prime of x is equal to-- well, the derivative of x is 1, minus the derivative of cosine x is negative sine x. So the derivative f prime of x is equal to 1 plus sine x. For our f prime of 1, we get 1 plus the sine of 1. And f prime of 1 then is about equal to, approximately equal to, 1.841470985.

And we don't want to round these at all. We want to use this with all the digits in place, or we cause the problem not to be able to get the accuracy that we can get from Newton's method. So that's going to go into our table under the f prime of x sub n. So we've got our 1.841470985.

And now for our x sub 1, we are going to get that by taking x sub 0, which is 1, minus f of x sub 0, which is the 0.4596976941, divided by f prime of 1, so divided by 1.841470985. And when you calculate that, you'll have 0.7503638678.

And that's what we will put down underneath the x sub n and across from the 1, meaning that it's x sub 1. Now we're going to go through the process again. So we are going to take f of this 0.7503638678. And once we run it through the function, we'll get our value out of 0.0189230738.

When we run it through the derivative, so remember, we're running that 0.7503638678 through the derivative as well, and you'll get 1.68190495291. And then we will take our 0.7503638678 minus the quotient of 0.0189230738 and 1.68190495291. And you'll have the next iteration approximation of the 0.7391128909.

So now, I'm going to just go ahead and fill out this table. And then you'll see what we have in our box under the x sub n across from the 3 will be our value for our third iteration. So here, I've filled out the rest of the table, and our final approximation after three iterations is reporting that entire decimal expression there.

So we have our x sub 3 is approximately equal to the 0.7390851334. And that's Newton's method to approximate the solution of x minus cosine x equals 0 and completing three iterations.

SUMMARY

In this lesson, you learned **the idea behind Newton's method**, which is to use tangent lines to approximate an x-intercept of the graph of $y=f(x)$. Newton's method is a very straightforward approximation method designed to solve equations of the form $f(x)=0$ (equivalent to finding the x-intercepts of the graph of $y=f(x)$). You learned how to **apply Newton's method** using its **algorithm**, by starting with an initial guess at $x=x_0$, then generating a sequence of guesses $x_1, x_2, x_3, \ldots$ to arrive at a close **approximation of the x-intercept**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Newton's Method

To find the next estimate for an x-intercept, use the formula $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$.