The Area Between Two Curves That Do Not Intertwine

by Sophia

WHAT'S COVERED

In this lesson, you will use the fundamental theorem of calculus to find the areas of regions bounded by two curves (for now, the regions will not intertwine). Specifically, this lesson will cover:

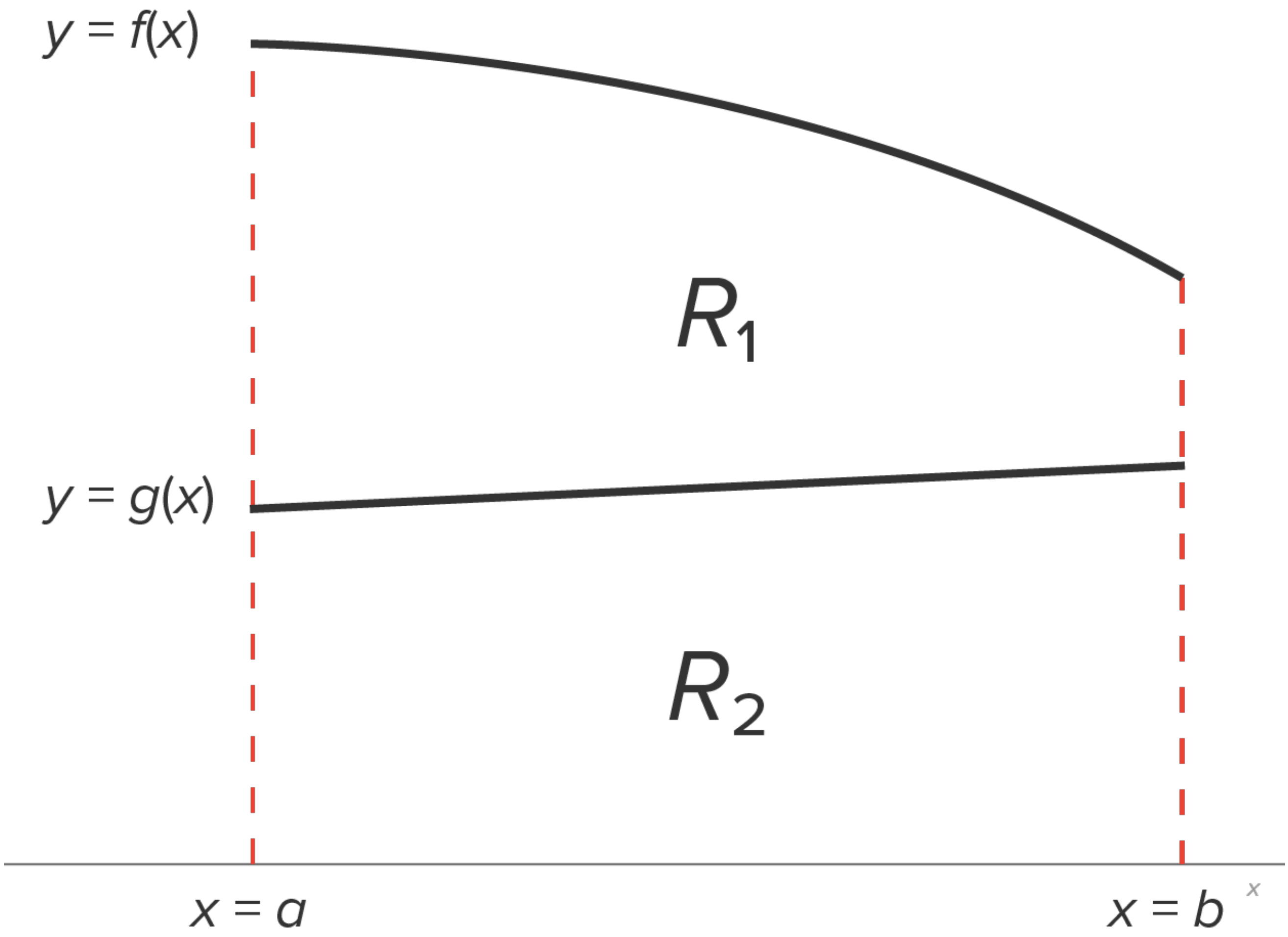
1. The Idea Behind the Area Between Two Curves That Do Not Intertwine
2. Finding the Area Between Two Curves That Do Not Intertwine

1. The Idea Behind the Area Between Two Curves That Do Not Intertwine

In addition to the regions we have worked with up to now, we also need to consider regions that do not have an axis as an edge.

Consider the areas in the figure below where:

- R_1 is the area of the region between the graphs of $y=f(x)$ and $y=g(x)$ on the interval $[a, b]$.
- R_2 is the area of the region between the x-axis and $y=g(x)$ on the interval $[a, b]$.
- R_1+R_2 is the area of the region between the x-axis and the graph of $y=f(x)$ on $[a, b]$.



From the picture, we know that $\int_a^b f(x)dx = R_1 + R_2$. We also see that $R_2 = \int_a^b g(x)dx$.

Replacing R_2 with $\int_a^b g(x)dx$ into the first equation, we have $\int_a^b f(x)dx = R_1 + \int_a^b g(x)dx$.

Solving for R_1, we have $R_1 = \int_a^b f(x)dx - \int_a^b g(x)dx$.

By properties of definite integrals, we know this is equal to $R_1 = \int_a^b [f(x) - g(x)]dx$.

Thus, assuming that the graph of $y = f(x)$ is above the graph of $y = g(x)$ on $[a, b]$, we have the following formula to find the area of the region between the graphs.

FORMULA

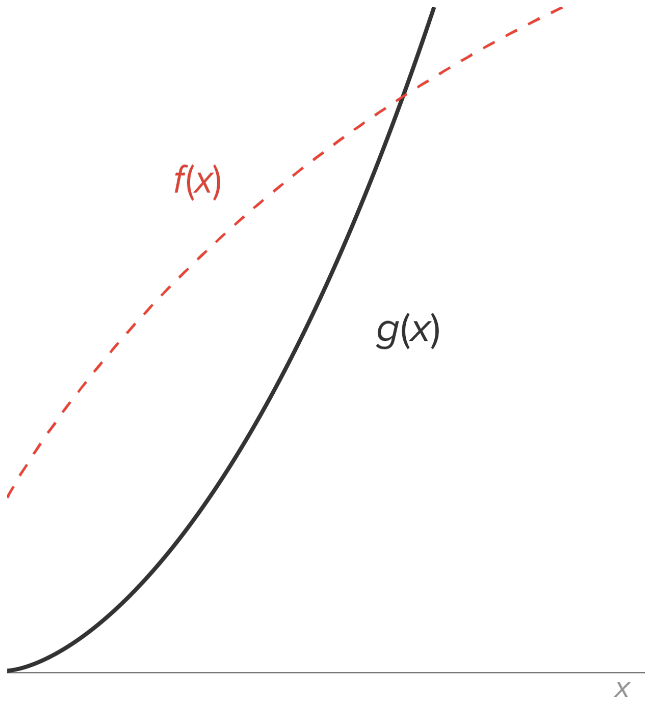
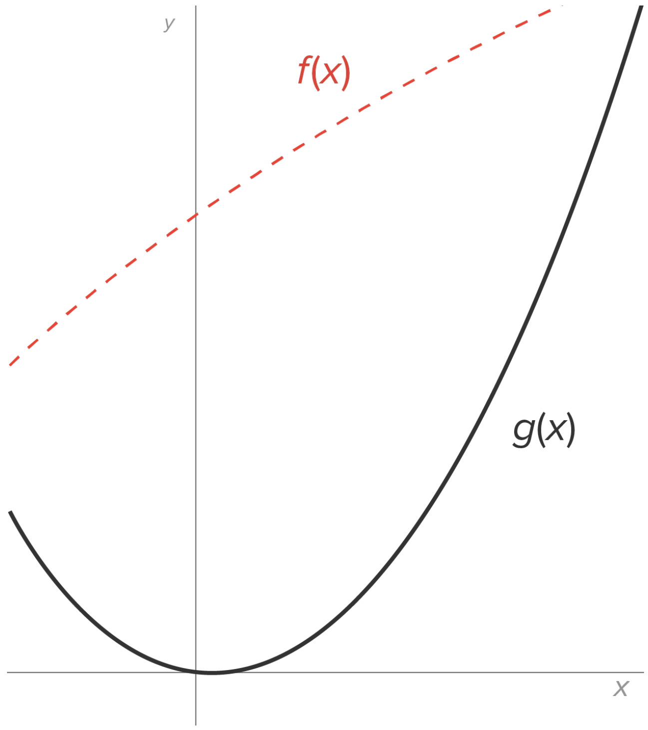
Area Between Two Curves, $y = f(x)$ and $y = g(x)$, Assuming $f(x) \geq g(x)$ on $[a, b]$

$$Area = \int_a^b [f(x) - g(x)]dx$$

We'll use this idea to find the areas of some regions bounded between two curves.

2. Finding the Area Between Two Curves That Do Not Intertwine

In order to find the area of this type of region, first verify that the graphs do not intertwine.

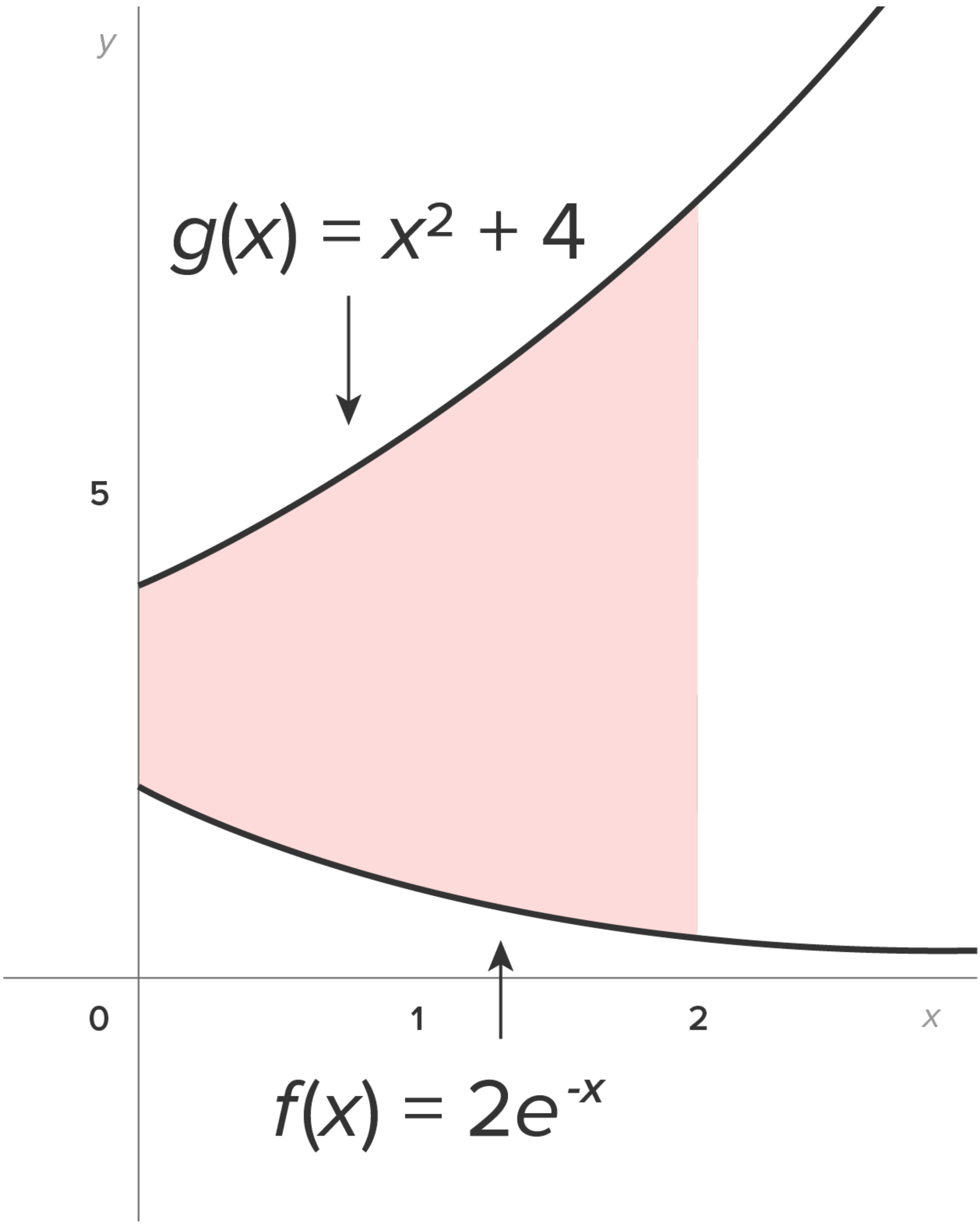
When $f(x)$ and $g(x)$ Intertwine	When $f(x)$ and $g(x)$ Do NOT Intertwine
	

HINT

If the graphs meet at either endpoint, they are not considered intertwining. We will discuss intertwining graphs in the next tutorial.

EXAMPLE Find the exact area between the graphs of $f(x)=2e^{-x}$ and $g(x)=x^2+4$ between $x=0$ and $x=2$.

The graph of the two curves on $[0, 2]$ is shown in the figure below.



The figure shows that $g(x) = x^2 + 4$ is higher than $f(x) = 2e^{-x}$ on the entire interval. Then, the area of the region is $\int_0^2 (x^2 + 4 - 2e^{-x})dx$.

Now we evaluate the integral:

$\int_0^2 (x^2 + 4 - 2e^{-x})dx$	Start with the original expression.
	Apply the fundamental theorem of calculus.

$$= \left(\frac{1}{3}x^3+4x+2e^{-x}\right)\Big|_0^2$$ Note: $\int 2e^{-x}dx = \frac{2}{-1}e^{-x}+C = -2e^{-x}+C$

$$=\left[\frac{1}{3}(2)^3+4(2)+2e^{-2}\right]-\left[\frac{1}{3}(0)^3+4(0)+2e^{-0}\right]$$ Substitute the upper and lower endpoints.

$$=\frac{8}{3}+8+2e^{-2}-2e^0$$ Evaluate the brackets.

$$=\frac{8}{3}+8+\frac{2}{e^2}-2$$ Note: $e^0=1$

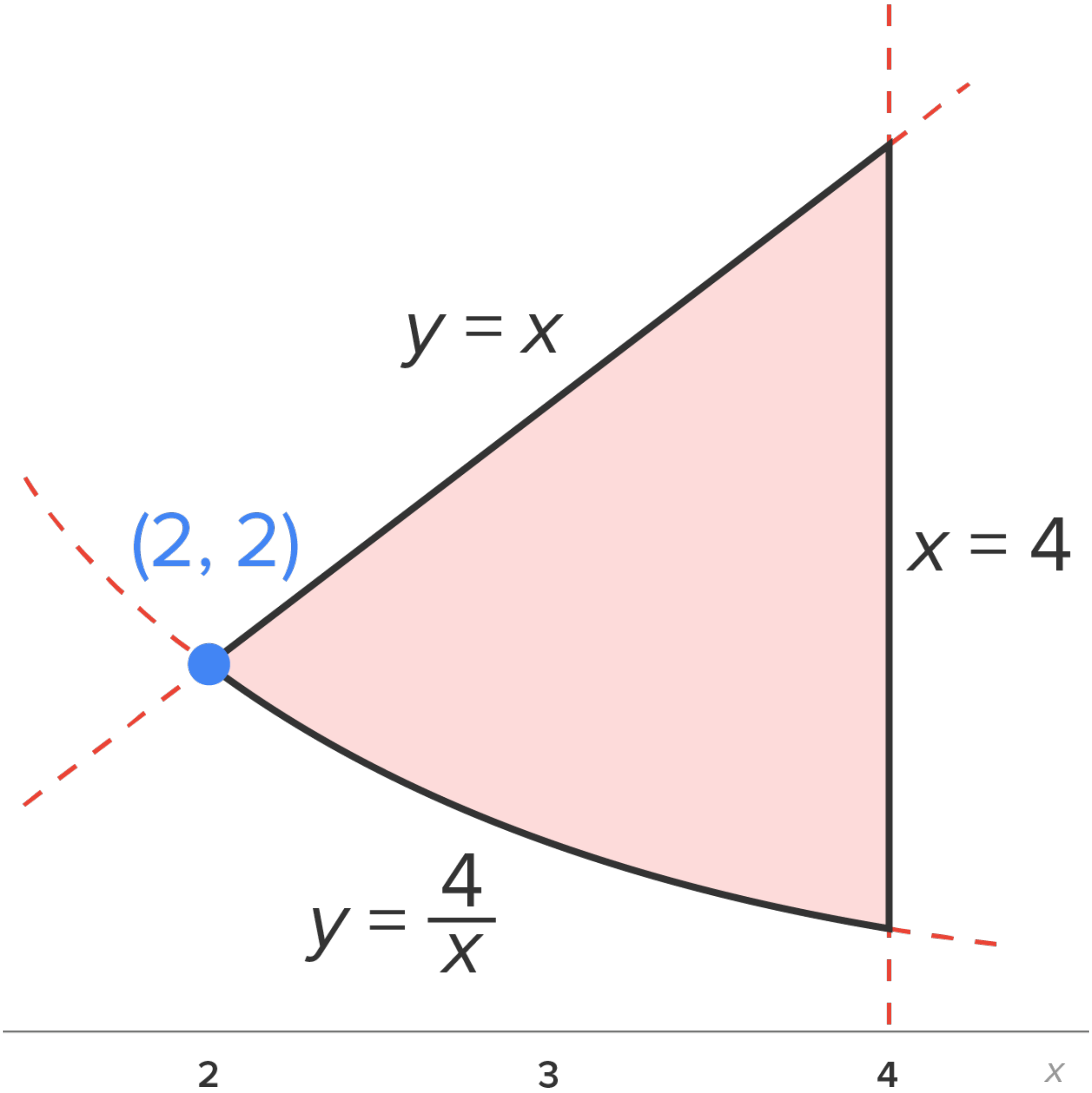
$$=\frac{26}{3}+\frac{2}{e^2}$$ Simplify.

Then, the area of the region is $\frac{26}{3}+\frac{2}{e^2}$ units^2.

Here is an example where the interval is not given.

EXAMPLE Find the exact area of the region in the first quadrant that is bounded by the graphs of $y=x$, $y=\frac{4}{x}$, and $x=4$.

The graph of the region in the first quadrant is shown in the figure.



To find the intersection point, set $x=\frac{4}{x}$ and solve:

$$x=\frac{4}{x}$$

$$x^2=4$$

$$x=\pm 2$$

Since the region is in the first quadrant, only $x=2$ is considered.

Also, since the graph of $y=x$ is above the graph of $y=\frac{4}{x}$ on the interval $[2, 4]$, the definite integral that gives the area is $\int_2^4\left(x-\frac{4}{x}\right)dx$.

Now evaluate the definite integral.

$\int_2^4 \left(x - \frac{4}{x}\right)dx$	Start with the original expression.
$= \left(\frac{1}{2}x^2 - 4\ln\lvert x\rvert\right)\Big\rvert_2^4$	Apply the fundamental theorem of calculus.
$= \left[\frac{1}{2}(4)^2 - 4\ln\lvert 4\rvert\right] - \left[\frac{1}{2}(2)^2 - 4\ln\lvert 2\rvert\right]$	Substitute the upper and lower endpoints.
$= [8 - 4\ln\lvert 4\rvert] - [2 - 4\ln\lvert 2\rvert]$	Evaluate the parentheses.
$= 6 - 4\ln 4 + 4\ln 2$	Simplify.
$= 6 - 4(\ln 4 - \ln 2)$	Factor.
$= 6 - 4\ln\left(\frac{4}{2}\right)$	Use the property $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.
$= 6 - 4\ln 2$	Simplify.

Thus, the area of the region is equal to $6 - 4\ln 2$ units^2.

WATCH

Check out this video to learn how to find the exact area between $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

TRY IT

Consider the region bounded by the graphs of $y = 4$ and $y = \cos x$ on the interval $[0, 2\pi]$.

Find the exact area of the region. +

8π units^2

TRY IT

Consider the region bounded by the graphs of $y = 4x$ and $y = x^2 + 3$.

Find the exact area of the region. +

$\frac{4}{3}$ units^2

WATCH

There are situations where it is advantageous to use horizontal subrectangles instead of vertical ones. Check out this video to learn how to find the exact area between $x = y^2 - 4$ and $x = -2y - 1$.

To summarize the previous video, we have the following formula to calculate the area between two curves using horizontal subrectangles:

FORMULA

Area Between Two Curves, $x = h(y)$ and $x = k(y)$, Assuming $h(y) \geq k(y)$ on $[c, d]$ (Horizontal Subrectangles)

$$Area = \int_c^d [h(y) - k(y)]dy$$

SUMMARY

In this lesson, you learned how to apply the fundamental theorem of calculus to compute the area of a region that does not have the x-axis as a boundary, **finding the area between two curves that do not intertwine**. Note that if the graphs meet at either endpoint, they are not considered intertwining. In the next tutorial, we will apply what we've learned in this tutorial to tackle areas where the boundary curves do intertwine.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Area Between Two Curves, x = h(y) and x = k(y), Assuming h(y) ≥ k(y) on [c, d] (Horizontal Subrectangles)

$$Area = \int_c^d [h(y) - k(y)]dy$$

Area Between Two Curves, y = f(x) and y = g(x), Assuming f(x) ≥ g(x) on [a, b]

$$Area = \int_a^b [f(x) - g(x)]dx$$