Which Functions Are Continuous?

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☰ WHAT'S COVERED

In this lesson, you will explore how continuity can be applied to combinations of functions. Specifically, this lesson will cover:

1. Functions That Are Continuous for All Real Numbers
2. Rational Functions
3. Radical Functions
4. Combinations of Functions

1. Functions That Are Continuous for All Real Numbers

To gain an understanding of continuity, let's recall some non-piecewise functions that contain no breaks or holes in their graphs (i.e., no points where the function is undefined). In other words, these functions are continuous for every real number.

- Polynomial functions
- Sine and cosine functions
- Absolute value functions

Moreover, given that $f(x)$ and $g(x)$ are continuous at $x = a$, it follows that these functions are also continuous for $x = a$.

Function	Operations
$f \pm g$	Sum/difference
$f \cdot g$	Product
$\dfrac{f}{g}$	Quotient, provided $g(a) \neq 0$
$[f(x)]^n, n = 0, 1, 2, \ldots$	Raise $f(x)$ to a whole number power
$f(g(x))$	Composition of f and g

It follows that any of these combinations of polynomial, sine, cosine, and absolute value functions are continuous for all real numbers (provided that we do not create a denominator that could be zero).

The following functions are continuous for all real numbers.

Functions	Combinations
$f(x) = -2x^3 + 12x^2 + 5x - 8$	Polynomial
$g(x) = \sin^2 x - 2\cos x$	Powers of sine and cosine
$h(x) = x^3 \sin 2x$	Product of polynomial and sine, with composition
$j(x) = \dfrac{2x}{x^2 + 4}$	Quotient of two functions, but no value of x will make the denominator equal to 0

2. Rational Functions

Let's explore the quotient of two functions a bit more. We have to be careful though, since a function is undefined when its denominator is equal to 0.

A **rational function** has the form $f(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.

↪ EXAMPLE $f(x) = \dfrac{2x}{x-1}$ is continuous for every real number except where $x - 1 = 0$, which means $x = 1$. The intervals over which $f(x)$ is continuous are $(-\infty, 1) \cup (1, \infty)$.

↪ EXAMPLE $g(x) = \dfrac{2x^2 - 3}{x^2 - 5x + 6}$ is continuous for every real number except where $x^2 - 5x + 6 = 0$.

Writing in factored form, we have $(x - 2)(x - 3) = 0$, which has solutions $x = 2$ and $x = 3$. Thus, $g(x)$ is continuous for all real numbers except 2 and 3.

Using interval notation, $g(x)$ is continuous on the intervals $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

📄 TERM TO KNOW

Rational Function

A function in the form $f(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.

3. Radical Functions

Recall that the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$ when n is even (even root), and $(-\infty, \infty)$ when n is odd (odd root).

It follows that $f(x) = \sqrt[n]{x}$ is continuous on its domain, which depends on the type of root (even or odd).

⮕ EXAMPLE $f(x) = \sqrt[3]{x-1}$ is continuous for all real numbers.

⮕ EXAMPLE $g(x) = \sqrt{x-4}$ is continuous when $x-4 \geq 0$, which means the interval $[4, \infty)$.

4. Combinations of Functions

Consider the two functions $f(x)$ and $g(x)$. Given that we know where $f(x)$ and $g(x)$ are continuous, the following can be said for some combinations of $f(x)$ and $g(x)$:

- The functions $f(x) + g(x)$, $f(x) - g(x)$, and $f(x) \cdot g(x)$ are all continuous on the interval(s) over which $f(x)$ and $g(x)$ are both continuous.
- The function $\dfrac{f(x)}{g(x)}$ is continuous on the interval(s) over which $f(x)$ and $g(x)$ are both continuous, with the added condition that $g(x) \neq 0$.

> ⮕ EXAMPLE Consider the function $f(x) = x^3\sqrt{x-4}$, which is a product of $y = x^3$ and $y = \sqrt{x-4}$.
> - $y = x^3$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
> - $y = \sqrt{x-4}$ is continuous where $x-4 \geq 0$, which is the interval $[4, \infty)$.
>
> Therefore, $f(x) = x^3\sqrt{x-4}$ is continuous on the interval $[4, \infty)$.

> ⮕ EXAMPLE Consider the function $f(x) = \dfrac{\cos x}{x-5}$, which is the quotient of $y = \cos x$ and $y = x-5$.
> - $y = \cos x$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
> - $y = x-5$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
> - Since the denominator is $x-5$, $f(x)$ is not continuous when $x-5 = 0$, which means $x = 5$.
>
> Therefore, $f(x)$ is continuous for all real numbers except $x = 5$, which means $(-\infty, 5) \cup (5, \infty)$ in interval notation.

> ⮕ EXAMPLE Consider the function $f(x) = \dfrac{2x}{\sqrt{x-3}}$, which is the quotient of $y = 2x$ and $y = \sqrt{x-3}$.
> - $y = 2x$ is continuous for all real numbers, or the interval $(-\infty, \infty)$.
> - $y = \sqrt{x-3}$ is continuous where $x-3 \geq 0$, which is the interval $[3, \infty)$.
> - Since the denominator is $\sqrt{x-3}$, $f(x)$ is also not continuous when $x-3 = 0$, which means $x = 3$.
>
> Therefore, $f(x)$ is continuous when $x > 3$, which means the interval $(3, \infty)$.

> ☑ TRY IT
>
> Consider the function $f(x) = 3x + \sqrt{2x-5}$.
>
> **Determine the interval(s) over which this function is continuous.**   +
>
> $\left[\dfrac{5}{2}, \infty\right)$

The following video goes through the examples determining the intervals of continuity for the functions $f(x) = 4x^3\sqrt{x+15}$, $f(x) = \dfrac{x^2-16}{x+4}$, and $f(x) = \dfrac{9\sin x}{\sqrt{x-8}}$.

✎ TRY IT

Consider the following table:

Function	Continuous Interval
$f(x) = 3x\cos(x^2)$	?
$g(x) = \dfrac{x^3-8}{x-2}$	?
$h(x) = \sqrt[5]{x^3 - 3x^2 + 10x - 4}$	?

Determine the intervals over which each function is continuous.   **+**

Function	Continuous Interval
$f(x) = 3x\cos(x^2)$	All real numbers or $(-\infty, \infty)$
$g(x) = \dfrac{x^3-8}{x-2}$	All real numbers except 2 or $(-\infty, 2)\cup(2, \infty)$
$h(x) = \sqrt[5]{x^3 - 3x^2 + 10x - 4}$	All real numbers or $(-\infty, \infty)$

☑ SUMMARY

In this lesson, you learned about some non-piecewise **functions that are continuous for all real numbers**, including polynomial functions, sine and cosine functions, and absolute value functions. You explored the quotient of two functions, noting that you have to be careful since a function is undefined when its denominator is equal to 0. You learned about **rational functions**, which have the form $f(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials; a rational function is continuous at all real numbers except for those where $D(x) = 0$. You also learned about **radical functions**, understanding that $f(x) = \sqrt[n]{x}$ is continuous on its domain, which depends on the type of root (even or odd). Lastly, you learned that for **combinations of functions**, take special care with radical and rational functions. If $f(x)$ has no values where it is undefined, then $f(x)$ is continuous for all real numbers.

📄 TERMS TO KNOW

Rational Function

A function in the form $f(x) = \dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials. A rational function is continuous at all real numbers except for those where $D(x) = 0$.