Apply L'Hopital's Rule to the Indeterminate Forms "∞ - ∞" and "∞ * 0"

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WHAT'S COVERED

In this lesson, you will learn strategies to use when evaluating limits that have other indeterminate forms. Specifically, this lesson will cover:

1. The Indeterminate Form $\infty - \infty$
2. The Indeterminate Form $\infty \cdot 0$

1. The Indeterminate Form $\infty - \infty$

The form $\infty - \infty$ occurs when there is a difference between two expressions that are both tending toward ∞ as $x \to a$.

↳ EXAMPLE Evaluate the following limit: $\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right)$

Since $\frac{1}{x} \to \infty$ and $\frac{1}{x^2} \to \infty$ as $x \to 0^+$, we have a limit of the form $\infty - \infty$. One strategy is to write it as a single fraction, since this is a more familiar scenario.

Since $\frac{1}{x} - \frac{1}{x^2} = \frac{x}{x^2} - \frac{1}{x^2} = \frac{x-1}{x^2}$, we have the following:

$\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right)$	Start with the limit that needs to be evaluated.
$= \lim_{x \to 0^+}\left(\frac{x-1}{x^2}\right)$	Replace the expression with a single fraction.
$= \frac{\text{close to } -1}{\text{small positive number}}$	As x approaches 0 from the right, $x - 1$ approaches -1 and x^2 is a small positive number.
$= -\infty$	A negative number divided by a small positive number is a large negative number.

Thus, $\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right) = -\infty$.

BIG IDEA

Many might think that a limit of the form $\infty - \infty$ should be 0 since you are “subtracting something from itself.” As we can see, this is not the case. Once we see $\infty - \infty$ produce another value, we will see why it is an indeterminate form.

WATCH

In this video, we'll evaluate $\lim_{x \to \infty}\left(\sqrt{x^2+9x} - x\right)$.

Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. We're going to do in this video is look at another indeterminate form, infinity minus infinity, and it's indeterminate because it does not always yield the same value. There are times when it ends up being 0. There's times when it ends up being infinity. There's times when it ends up being negative infinity. It could end up being one. There's just a lot of options here.

So the question is, how do we analytically go through this to figure out what the limit is? One of the strategies we like to use with radicals is to employ the conjugate because that helps us to rewrite the expression. I'm going to multiply this by the square root of x squared plus 9x, plus x, over the square root of x squared plus 9x, plus x. So let's see what we have here. And we'll put these in parentheses here as well. So we have the limit as x approaches infinity of-- Well, let's just see here.

So the denominator is now the square root of x squared plus 9x, plus x. Now the numerator, when these two get multiplied together the radicals just go away. So we have x squared plus 9x. The whole purpose of multiplying by the conjugate is to eliminate the outers and the inners and you notice we get x times the square root and then minus x times the square root. So they do go away. Then the last times the last is minus x squared.

What we notice is this has simplified quite considerably. So you have limit as x approaches infinity of 9x over square root, x squared plus 9x, plus x. Now you might remember before in an earlier challenge that we evaluated a limit just like this. The process is we divide by the highest power of x in the denominator. Now, what that means is I'm going to divide by square root of x squared, which remember is x. It just so happens that they're the same power in the denominator so we're going to use each one as it's relevant.

So I'm going to divide 9x by x. I'm going to divide the square root by the square root of x squared, and I'm going to divide x by x. Remember, x squared under the square root and x are the same because x is positive. We're going to positive infinity. This is the limit, x approaches infinity of 9 over-- Now that's a more complicated square root now. This is x squared plus 9 over x squared, plus 1. Now we'll simplify under the radical a little bit more. So that's equals the limit as x approaches infinity, 9 over.

Now, if I divide through by x squared, that is the square root of 1 plus 9 over x squared plus 1. And now if we take the limit as x approaches infinity, this term right here goes to 0. And now we're down to 9 over square root of 1 plus 1, which is 9/2. So maybe not an intuitive result, but that limit is 9/2. So infinity minus infinity could be 9/2. Could be 1, could be 4, it could be 0. It's indeterminate, which means we don't always know it's value. So there's our limit.

[MUSIC PLAYING]

Considering the results from these last two examples, it is clear now why $\infty - \infty$ is an indeterminate form. In one case, the result was $-\infty$, and in another case, the result was $\frac{9}{2}$.

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2. The Indeterminate Form ∞ · 0

The indeterminate form $\infty \cdot 0$ is handled in one of two ways.

Loosely speaking, we can say that a limit of the form $\frac{1}{\infty}$ will approach 0 and a limit of the form $\frac{1}{0}$ will approach $\pm\infty$.

That said, we can treat "0" and "∞" as reciprocals as far as limits are concerned.

This means that the indeterminate form $\infty \cdot 0$ could be rewritten as either $\frac{0}{0}$ or $\frac{\infty}{\infty}$, whichever is more convenient.

↪ EXAMPLE Evaluate the following limit: $\lim\limits_{x \to \infty} x^2 e^{-2x}$

If we look at each factor separately, we see that $x^2 \to \infty$ and $e^{-2x} \to 0$ as $x \to \infty$. Thus, this limit has the form $\infty \cdot 0$.

To rewrite, consider the fact that $e^{-2x} = \frac{1}{e^{2x}}$, which means $\lim\limits_{x \to \infty} x^2 e^{-2x} = \lim\limits_{x \to \infty} x^2 \frac{1}{e^{2x}} = \lim\limits_{x \to \infty} \frac{x^2}{e^{2x}}$, which now has the form $\frac{\infty}{\infty}$.

To evaluate, use L'Hopital's rule.

$\lim\limits_{x \to \infty} \frac{x^2}{e^{2x}}$	Start with the limit that needs to be evaluated.
$= \lim\limits_{x \to \infty} \frac{2x}{2e^{2x}}$	Since x^2 and e^{2x} are differentiable and the limit has the form $\frac{\infty}{\infty}$, L'Hopital's rule is used. $D[x^2] = 2x$, $D[e^{2x}] = 2e^{2x}$
$= \lim\limits_{x \to \infty} \frac{x}{e^{2x}}$	Remove the common factor of 2.
$= \lim\limits_{x \to \infty} \frac{1}{2e^{2x}}$	Since $\lim\limits_{x \to \infty} \frac{x}{e^{2x}}$ has the form $\frac{\infty}{\infty}$, continue to use L'Hopital's rule.

$$D[x]=1,\ D\left[e^{2x}\right]=2e^{2x}$$

$=0$ Since the denominator grows very large as $x \to \infty$, the limit is 0.

Thus, $\lim\limits_{x \to \infty} x^2 e^{-2x} = 0$.

BIG IDEA

If $\lim\limits_{x \to a} f(x) \cdot g(x)$ has the form $\infty \cdot 0$, write $\lim\limits_{x \to a} \dfrac{f(x)}{\left(\dfrac{1}{g(x)}\right)}$ or $\lim\limits_{x \to a} \dfrac{g(x)}{\left(\dfrac{1}{f(x)}\right)}$, then use L'Hopital's rule.

WATCH

In this video, we'll evaluate $\lim\limits_{x \to 0^+} x^3 \cdot \ln x$.

Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. What we're going to do in this video is evaluate a limit that has yet another indeterminate form. In this function we have x to the 1/3 times the natural log of x. As x gets close to 0, x to the third gets close to 0, and the natural log of x drives down to negative infinity. So this qualifies as a limit of the form 0 to infinity.

Remember our strategy for this kind of limit we rewrite one of them as a reciprocal so that it's a rational function, that we're able to use l'hopital's rule because 0 times infinity can either be written as 0 divided by 1 over infinity or infinity over 1 over 0. We consider 1 over infinity to be basically zero and we consider 1 over 0 to be basically infinity. So basically that just means flip one of them. So which one do we want to flip? Do we want to flip the x to the third or do we want to flip the natural log of x?

Thinking about derivatives, because l'hopital's rule involves a derivative, it's much easier to take the reciprocal of x to the third than it is to take the reciprocal of natural log of x. So we're going to write the natural log of x divided by 1 over x to the third, which remember, thinking ahead about derivatives is the limit as x approaches 0 from the right, natural log of x over x to the negative 3. So just to double check, by rewriting that what have we done? Natural log of x still drives off to negative infinity, but 1 over x to the third you get 1 over a very tiny positive number, which ends up being a large positive number. We have infinity over infinity and we're ready to use l'hopital's rule.

So to use l'hopital's rule we apply the derivative to the numerator and the denominator. So this is the limit as x approaches 0 from the right, 1 over x over negative 3x to the negative 4. It would seem that we're at a crossroads here, but remember, now we have powers of x. We can simplify this very nicely. So this is equal to the limit as x approaches 0 from the right, we have 1 over x over negative 3 over x to the fourth, which is the limit as x approaches 0 from the right. 1 over x times the reciprocal of that fraction, which turns out to be the limit as x approaches 0 from the right of x to the third over negative 3.

We can now use direct substitution for this limit. We get 0 to the third over negative 3, which is just 0. So the limit of that original function is 0. You can easily check that by looking at a graph. The graph would make it look like it's passing through 0,0. It's just getting so close to that point that it just looks like it. And there we have it.

[MUSIC PLAYING]

WATCH

In this video, we'll evaluate $\lim\limits_{x \to \infty} x \cdot \sin\left(\frac{1}{x}\right)$.

Video Transcription

[MUSIC PLAYING] Hi there. What we're going to do is look at another limit that involves an indeterminate form. And we're going to try to rearrange this particular one and try to use L'Hospital's rule to evaluate the limit. So let's see what we come up with here.

So we have the limit as x approaches infinity of x times sin of 1/x. So it's a product of two things. So as x goes to infinity, the first term goes to infinity-- the first factor goes to infinity. The sine, however-- it's sine of 1 over [INAUDIBLE], which basically is sine of 0, which is 0. So this is the limit of the form infinity times 0.

Now, remember our strategy. The strategy is we write it as a fraction where we take the reciprocal of the factors. So it's either the reciprocal of x in the denominator, or it's the reciprocal of sin of 1/x in the denominator. I think it's going to be the reciprocal of x in the denominator. So we have the limit as x approaches infinity of sin of 1/x divided by 1/x.

And just to double check, this is going to 0. And 1/x is going to 0. So we're good to go in using L'Hospital's rule. So L'Hospital's rule says, take the derivative of the numerator. Take the derivative of the denominator. And just a little fun fact here, if we want the derivative of 1/x-- which, remember, is the derivative of x to the negative 1-- that is negative 1x to the negative 2, which is negative 1 over x squared.

That is one of those derivatives that we just happen to see often enough, it might be worth memorizing going from 1/x directly to negative 1 over x squared. So when we apply L'Hospital's rule here, the derivative of sin of 1/x is the cosine of the same of the something times the derivative of the inside.

And then the derivative of the denominator is negative 1 over x squared. And that's actually really good news, because the same thing showed up in the numerator and denominator, those essentially cross out, which means we have the limit as x approaches infinity of cosine 1/x.

But as x goes to infinity, this goes to 0. So this ends up being cosine of 0, which is 1. So that original limit is 1. And that we get by L'Hospital's rule.

[MUSIC PLAYING]

SUMMARY

In this lesson, you learned that with the addition of new indeterminate forms, more strategies need to be used. Specifically, you learned that for **the indeterminate form** $\infty - \infty$, combining the fractions or rationalizing are the most common strategies; for **the indeterminate form** $\infty \cdot 0$, rewriting the expression using reciprocals then using L'Hopital's rule is the main strategy.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.