Indefinite Integrals of Functions Requiring Rewriting Before Applying Rules

by Sophia

WHAT'S COVERED

In this lesson, you will use algebraic manipulation so that the power rule can be used to find the antiderivative. Specifically, this lesson will cover:

1. Radicals and Powers in the Denominator of the Integrand
2. Performing Multiplication and/or Division First
   a. Performing Multiplication First
   b. Performing Division First

1. Radicals and Powers in the Denominator of the Integrand

Let's remind ourselves of the following:

- When you want to take the *derivative* of $f(x)=\sqrt[n]{x}$, you first rewrite $f(x)$ as $x^{1/n}$.
- When you want to take the *derivative* of $g(x)=\frac{1}{x^n}$, you first rewrite $g(x)$ as x^{-n}.

These rewrites are also used when finding antiderivatives!

BIG IDEA

When the integrand of $\int f(x)dx$ contains a radical and/or a power in a monomial denominator, rewrite these expressions in exponential form so that the power rule may be used.

EXAMPLE Find the indefinite integral: $\int \sqrt[3]{t}\,dt$

We need to rewrite the expression as a power, then find the antiderivative.

$$\int \sqrt[3]{t}\,dt$$ Start with the original expression.

$$= \int t^{1/3}dt$$ Rewrite $\sqrt[3]{t}$ as $t^{1/3}$.

$$= \frac{1}{\left(\frac{4}{3}\right)}t^{4/3}+C$$ Use the power rule with $n=\frac{1}{3}$: $\frac{1}{3}+1=\frac{4}{3}$

$$= \frac{3}{4}t^{4/3}+C$$ Simplify.

Thus, $\int \sqrt[3]{t}\,dt = \frac{3}{4}t^{4/3}+C$.

EXAMPLE Find the indefinite integral: $\int \frac{6}{x^3}dx$

We need to rewrite the expression as a power, then find the antiderivative.

$$\int \frac{6}{x^3}dx$$ Start with the original expression.

$$= \int 6x^{-3}dx$$ Rewrite $\frac{1}{x^3}$ as x^{-3}.

$$= 6\left(\frac{x^{-2}}{-2}\right)+C$$ Use the power rule with $n=-3$: $-3+1=-2$

$$= -3x^{-2}+C$$ Simplify.

$$= \frac{-3}{x^2}+C$$ You can also write the final answer using positive exponents.

Thus, $\int \frac{6}{x^3}dx = \frac{-3}{x^2}+C$.

TRY IT

Consider $\int (12\sqrt[3]{t} + 12\sqrt[4]{t})dt$.

Find the indefinite integral. +

$$9t^{4/3}+\frac{48}{5}t^{5/4}+C$$

EXAMPLE Find the indefinite integral: $\int \left(\frac{3}{\sqrt{x}}-\frac{4}{x^2}\right)dx$

We need to rewrite each expression as a power, then find the antiderivative.

$\int\left(\frac{3}{\sqrt{x}}-\frac{4}{x^2}\right)dx$	Start with the original expression.
$=\int(3x^{-1/2}-4x^{-2})dx$	Rewrite each expression as a power of *x*. Note: $\frac{3}{\sqrt{x}}=\frac{3}{x^{1/2}}=3x^{-1/2}$
$=3\int x^{-1/2}dx-4\int x^{-2}dx$	Use the sum/difference rules and the constant multiple rules.
$=3\left(\frac{x^{1/2}}{\left(\frac{1}{2}\right)}\right)-4\left(\frac{x^{-1}}{-1}\right)+C$	Apply the power rule to both terms $(n=-\frac{1}{2}$ and $n=-2)$.
$=6x^{1/2}+4x^{-1}+C$	Simplify.
$=6\sqrt{x}+\frac{4}{x}+C$	Rewrite in more “common” form.

Thus, $\int\left(\frac{3}{\sqrt{x}}-\frac{4}{x^2}\right)dx=6\sqrt{x}+\frac{4}{x}+C$.

TRY IT

Consider $\int\left(\frac{12}{\sqrt[3]{x}}-\frac{8}{x^5}\right)dx$.

Find the indefinite integral. +

$18x^{2/3}+\frac{2}{x^4}+C$

2. Performing Multiplication and/or Division First

Now, we'll look at examples where algebraic manipulation is required before using the power rule. This could include multiplication and division.

2a. Performing Multiplication First

EXAMPLE Consider the indefinite integral $\int(2x-3)(x+5)dx$.

The integrand $(2x-3)(x+5)$ is not a power of *x*, nor is it a sum or difference of powers of *x*.

If we perform the multiplication, the integrand becomes $(2x-3)(x+5)=2x^2+7x-15$, which is a polynomial, which we learned to antidifferentiate in the last tutorial.

This means we now have $\int(2x-3)(x+5)dx=\int(2x^2+7x-15)dx$.

$$\int(2x-3)(x+5)dx$$ Start with the original expression.

$$=\int(2x^2+7x-15)dx$$ Multiply to form the expanded expression.

$$=2\int x^2dx+7\int xdx-\int 15dx$$ Use the sum/difference properties followed by the constant multiple rule.

$$=2\left(\frac{x^3}{3}\right)+7\left(\frac{x^2}{2}\right)-15x+C$$ Apply the power rule.

$$=\frac{2}{3}x^3+\frac{7}{2}x^2-15x+C$$ Simplify.

Thus, $\int(2x-3)(x+5)dx=\frac{2}{3}x^3+\frac{7}{2}x^2-15x+C$.

TRY IT

Consider $\int 3x^2\left(2x+\frac{2}{x^2}\right)dx$.

Find the indefinite integral. +

$\frac{3}{2}x^4+6x+C$

EXAMPLE Find the definite integral: $\int(4x^2+3)^2dx$

Here, we can write $(4x^2+3)^2=(4x^2+3)(4x^2+3)=16x^4+24x^2+9$, which is a polynomial. Now, use the power rule and properties to find the indefinite integral.

$$\int(4x^2+3)^2dx$$ Start with the original expression.

$$=\int(16x^4+24x^2+9)dx$$ Multiply out the square of the binomial.

$$=16\int x^4dx+24\int x^2dx+\int 9dx$$ Use the sum/difference properties followed by the constant multiple rule.

$$=16\left(\frac{x^5}{5}\right)+24\left(\frac{x^3}{3}\right)+9x+C$$ Apply the power rule.

$$=\frac{16}{5}x^5+8x^3+9x+C$$ Simplify.

Thus, $\int(4x^2+3)^2dx=\frac{16}{5}x^5+8x^3+9x+C$.

HINT

When the integrand is a quantity raised to a power, there are only certain instances in which performing the multiplication is desirable.

When the power is 2, this is fairly straightforward. For instance, $(2x^3-7x)^2=4x^6-28x^4+49x^2$.

If the power is 3, this could take some time, but is still reasonable. For instance,
$(5x^2+4)^3=125x^6+300x^4+240x^2+64$.

For higher whole-number powers, the process is much longer and time-consuming. Consider examples like $(2x+1)^4$ or $(x^2-4)^{10}$. These can be done, but there may be faster methods presented later.

2b. Performing Division First

If the integrand has a certain form, division can be used to rewrite so that the power rule can be used.

EXAMPLE Perform the division in the expression $\frac{3x^2+14x}{x}$.

This expression isn't a power of *x*, nor a sum/difference of powers of *x*. Since the denominator is a single term, we can divide each term in the numerator by the denominator (and some simplification should occur).

$$\frac{3x^2+14x}{x}=\frac{3x^2}{x}+\frac{14x}{x}=3x+14$$

This expression is a polynomial, which means it can be integrated.

EXAMPLE Perform the division in the expression $\frac{6x^2-20x^5}{4x^3}$.

This expression isn't a power of *x*, nor a sum/difference of powers of *x*. Once again, since the denominator is a single term, we can divide each term in the numerator by the denominator (and some simplification should occur).

$$\frac{6x^2-20x^5}{4x^3}=\frac{6x^2}{4x^3}-\frac{20x^5}{4x^3}=\frac{3}{2x}-5x^2$$

This can be integrated.

EXAMPLE Find the indefinite integral: $\int\frac{3x^2+14x}{x}dx$

As we saw in the previous example, performing the division gives a sum of powers of *x*.

$\int\frac{3x^2+14x}{x}dx$	Start with the original expression.

$$= \int (3x + 14)dx$$ Perform the division, as seen in the previous example.

$$= 3\int x dx + \int 14 dx$$ Use the sum property and the constant multiple rule.

$$= 3\left(\frac{x^2}{2}\right) + 14x + C$$ Apply the power rule.

$$= \frac{3}{2}x^2 + 14x + C$$ Simplify.

Thus, $\int \frac{3x^2 + 14x}{x} dx = \frac{3}{2}x^2 + 14x + C$.

Now, let's look at the other expression we simplified in the first example.

EXAMPLE Find the definite integral: $\int \frac{6x^2 - 20x^5}{4x^3} dx$

As we saw in the previous example, performing the division gives a difference of powers of *x*.

$$\int \frac{6x^2 - 20x^5}{4x^3} dx$$ Start with the original expression.

$$= \int \left(\frac{3}{2x} - 5x^2\right) dx$$ Perform the division, as seen in the previous example.

$$= \frac{3}{2}\int \frac{1}{x} dx - 5\int x^2 dx$$ Use the difference property followed by the constant multiple rule.

$$= \frac{3}{2}\ln|x| - 5\left(\frac{x^3}{3}\right) + C$$ Apply the natural logarithm rule and power rule.

$$= \frac{3}{2}\ln|x| - \frac{5}{3}x^3 + C$$ Simplify.

Thus, $\int \frac{6x^2 - 20x^5}{4x^3} dx = \frac{3}{2}\ln|x| - \frac{5}{3}x^3 + C$.

TRY IT

Consider $\int \frac{6x^4 + 1}{3x^3} dx$.

Find the indefinite integral. +

$$x^2 - \frac{1}{6x^2} + C$$

SUMMARY

In this lesson, you learned that when there are **radicals and powers in the denominator of the**

integrand of $\int f(x)dx$, it is sometimes necessary to rewrite these expressions in exponential form so that the power rule may be used. You also explored examples where algebraic manipulation is required before using the power rule, including **performing multiplication first** and **performing division first**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.