Inflection Points

by Sophia

WHAT'S COVERED

In this tutorial, we will find inflection points that occur when a curve changes concavity. Inflection points are useful in modeling an epidemic. Specifically, this lesson will cover:

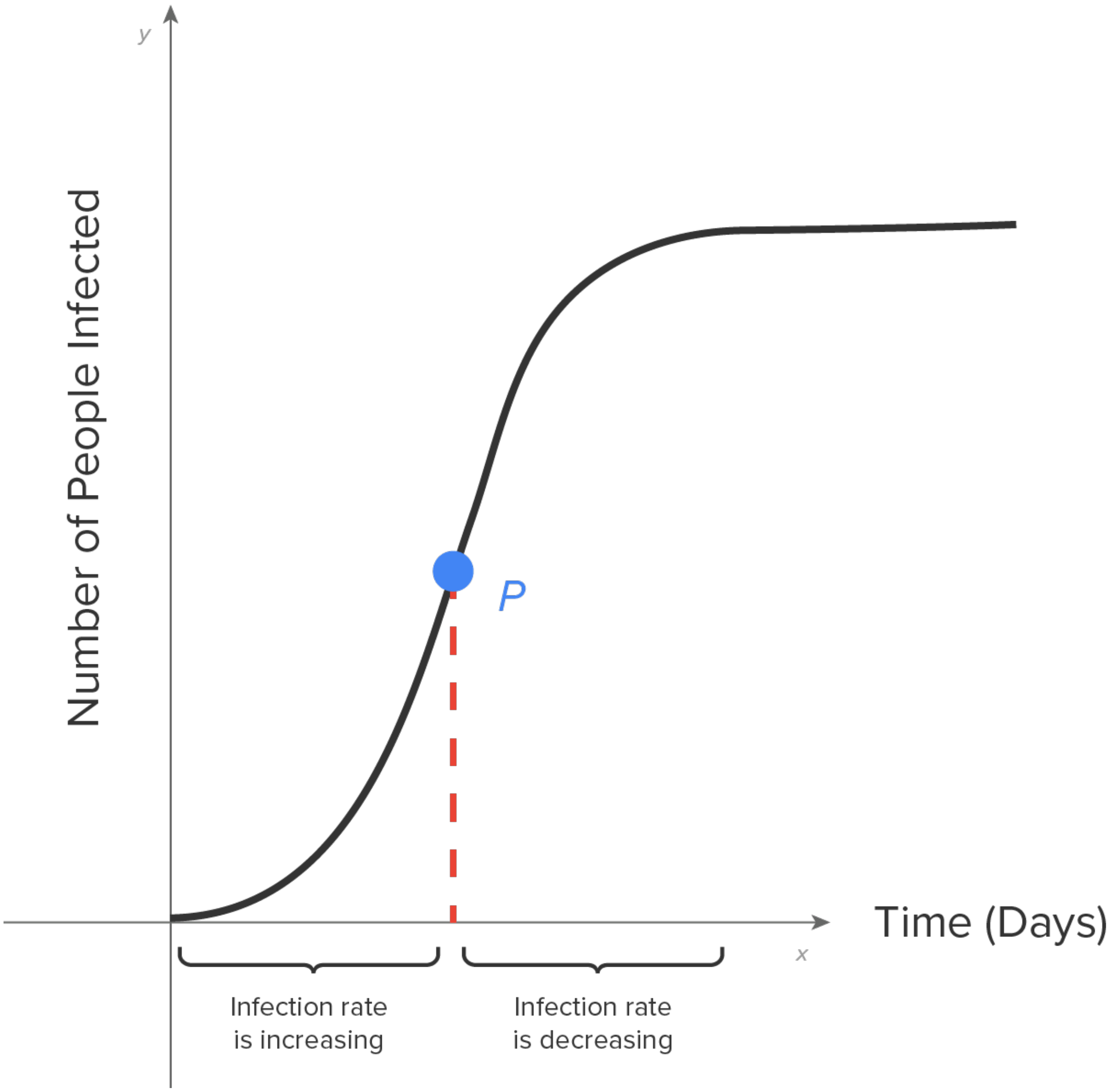
1. Defining the Point of Inflection
2. Determining the Inflection Points of a Function

1. Defining the Point of Inflection

In the last tutorial, we learned that $f''(x)$ gives information about the concavity of $f(x)$.

When a curve has a point where it transitions between being concave up and concave down, and the tangent line exists, the point is called an **inflection point** (or **point of inflection**).

Consider the graph shown below, which represents the number of people who have become infected with a disease.



As you can see, the number of cases is increasing over the entire domain.

To the left of point *P*, the slopes of the tangent lines are increasing. This means that the rate of infection is increasing.

To the right of point *P*, the slopes of the tangent lines are decreasing. This means that the rate of infection is decreasing.

The inflection point is the transition point between these two events. In terms of disease control, this point is important since it represents the point at which the disease is beginning to get under control.

In the last tutorial, you learned that the graph of $f(x)$ is concave down when $f''(x) < 0$ and the graph of $f(x)$ is concave up when $f''(x) > 0$.

BIG IDEA

As long as $f(x)$ is continuous at $x = c$, the graph of $f(x)$ could have a point of inflection when $f''(c) = 0$ or $f''(c)$ is undefined. To verify this, make a sign graph of $f''(x)$.

TERM TO KNOW

Inflection Point (Point of Inflection)

A point on a curve at which concavity changes.

2. Determining the Inflection Points of a Function

EXAMPLE Consider the function $f(x) = -2x^3 + 18x^2 + 30x - 40$. Find any points of inflection.

$f(x) = -2x^3 + 18x^2 + 30x - 40$	Start with the original function; the domain is all real numbers.
$f'(x) = -6x^2 + 36x + 30$	Take the first derivative.
$f''(x) = -12x + 36$	Take the second derivative.
$-12x + 36 = 0$	Any inflection points could occur when $f''(x) = 0$. (Note: $f''(x)$ is never undefined.)
$-12x = -36$	Subtract 36 from both sides.
$x = 3$	Divide both sides by -12.

Therefore, there could be a point of inflection when $x = 3$.

Now, select one number (called a *test value*) inside the intervals $(-\infty, 3)$ and $(3, \infty)$ to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 3)$	$(3, \infty)$
Test Value	0	4
Value of $f''(x) = -12x + 36$	36	-12
Behavior of $f(x)$	Concave up	Concave down

Therefore, $f(x)$ is concave up on the interval $(-\infty, 3)$ and concave down on the interval $(3, \infty)$. Thus, a point of inflection occurs when $x = 3$.

On the graph of $f(x)$, the inflection point is located at $(3, f(3)) = (3, 158)$.

Let's take a look at a different function.

EXAMPLE Consider the function $f(x) = x^4 - 2x$. Find any points of inflection.

$f(x) = x^4 - 2x$	Start with the original function; the domain is all real numbers.
$f'(x) = 4x^3 - 2$	Take the first derivative.
$f''(x) = 12x^2$	Take the second derivative.
	Any inflection points could occur when $f''(x) = 0$. (Note: $f''(x)$ is never

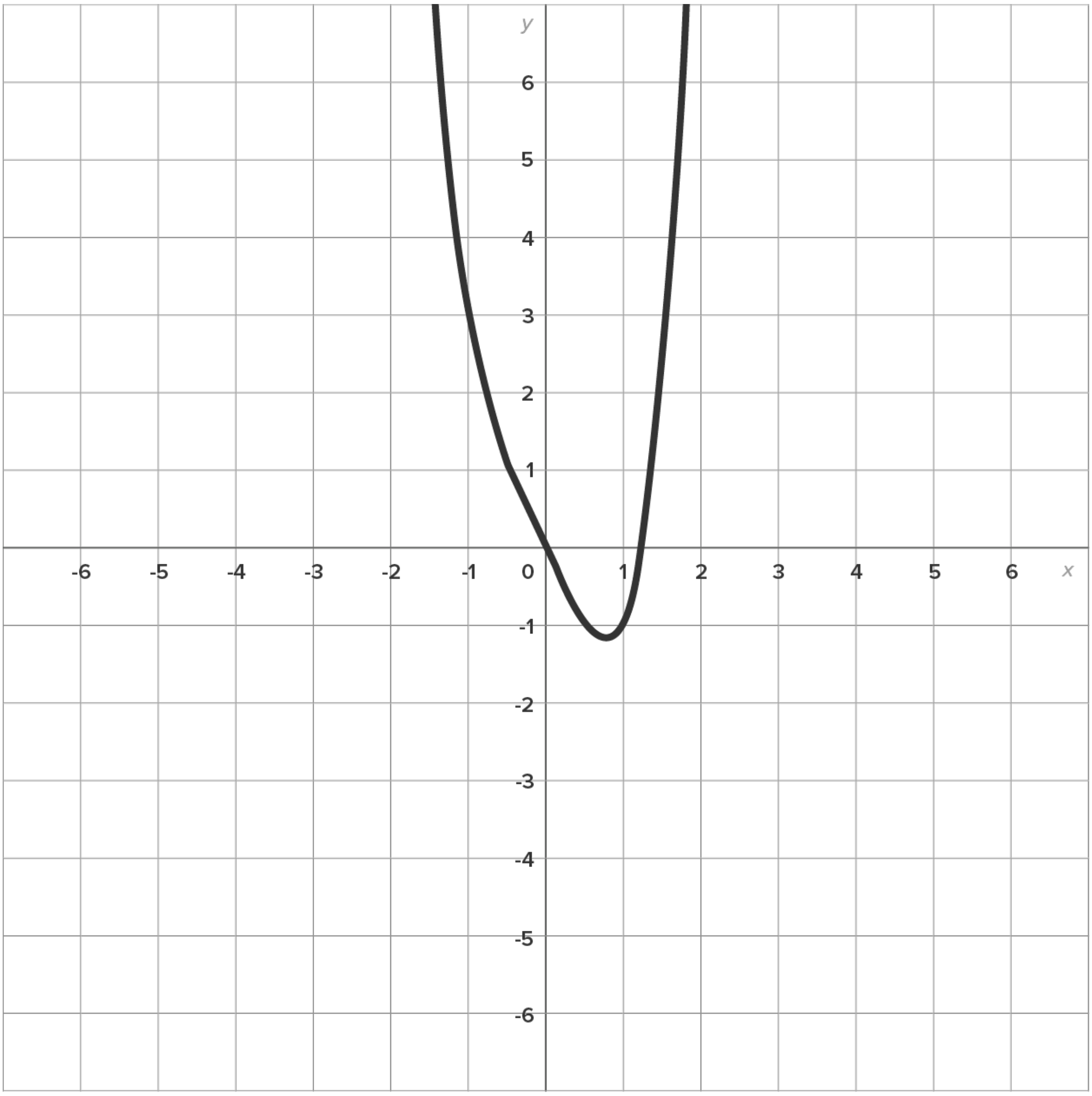
$12x^2=0$	undefined.)
$x^2=0$	Divide both sides by 12.
$x=0$	Take the square root of both sides.

Therefore, an inflection point possibly occurs when $x=0$.

Now, select one number (called a *test value*) inside the intervals $(-\infty, 0)$ and $(0, \infty)$ to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	-1	1
Value of $f''(x)=12x^2$	12	12
Behavior of $f(x)$	Concave up	Concave up

Since the concavity does not change at $x=0$, there is no inflection point when $x=0$. Since there were no other possible points of inflection, $f(x)=x^4-2x$ has no points of inflection. As you can see, the graph is always concave up.



WATCH

In this video, we'll analyze the function $f(x) = 2x - 3x^{2/3}$ for points of inflection.

SUMMARY

In this lesson, you learned that **the point of inflection is defined** as the point on a curve where it transitions between being concave up and concave down (and the tangent line exists). This point is useful in modeling an epidemic, for example, since it can represent the point at which the disease is beginning to get under control. You also learned that to **determine the inflection points of a function**, first find all values in the domain of $f(x)$ where $f''(x)$ is either 0 or undefined, then use a graph of the signs of $f''(x)$ to determine the x-values where inflection points occur.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Inflection Point (Point of Inflection)

A point on a curve at which concavity changes.