Rolle's Theorem

by Sophia

WHAT'S COVERED

In this lesson, you will learn about Rolle's theorem, a seemingly simple yet powerful theorem whose consequences are used in Unit 5 (Antiderivatives). Specifically, this lesson will cover:

1. Introduction to Rolle's Theorem
2. Applying Rolle's Theorem

1. Introduction to Rolle's Theorem

Let's say we have a function that passes through the points (1, 6) and (5, 6).

TRY IT

Take a piece of paper and draw the points (1, 6) and (5, 6). Connect the two points with a curve that is continuous and differentiable (something other than a horizontal line between them). This means that the graph has no break and no sharp turn.

What do you notice about your curve? Does your curve contain at least one horizontal tangent? +

Hopefully you have at least one horizontal tangent. As it turns out, under certain circumstances, this will always happen.

BIG IDEA

Rolle's theorem:

Let $f(x)$ be continuous on the closed interval $[a, b]$ with $f(a) = f(b)$, and differentiable on the open interval (a, b).

Then, there is at least one value of *c* between *a* and *b* for which $f'(c) = 0$.

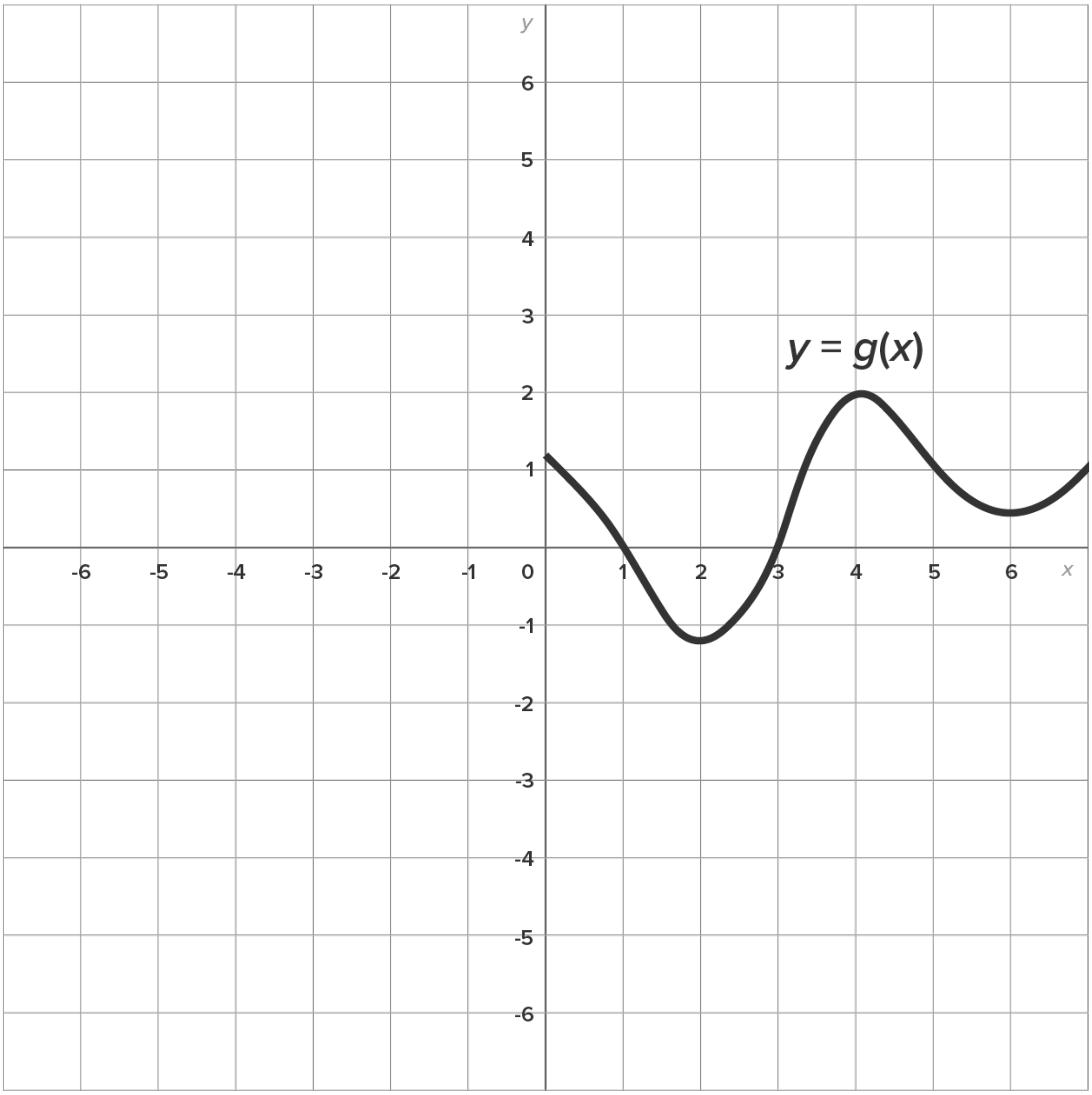
---

2. Applying Rolle's Theorem

Now, let's look at a few examples of how Rolle's theorem can be applied.

$\hookrightarrow$ EXAMPLE Here is the graph of some function $y = g(x)$, where $g(0) = g(7)$.

Since $g(x)$ is continuous and differentiable, it follows by Rolle's theorem that there is at least one value of c between 0 and 7 where $f'(c) = 0$.



In the graph, we can see there are three x-values where a horizontal tangent line occurs: $x = 2$, $x = 4$, and $x = 6$. Therefore, the guaranteed values of c are 2, 4, and 6.

$\hookrightarrow$ EXAMPLE Consider the function $f(x) = 3x + \frac{3}{x}$ on the interval $\left[\frac{1}{2}, 2\right]$.

First, check requirements for Rolle's theorem.

$f(x)$ is continuous on any interval not including 0, and therefore is continuous on $\left[\frac{1}{2}, 2\right]$.

$f(x)$ is differentiable everywhere except where $x = 0$, so $f(x)$ is certainly differentiable on $\left(\frac{1}{2}, 2\right)$.

$f\left(\frac{1}{2}\right)=3\left(\frac{1}{2}\right)+\frac{3}{\left(\frac{1}{2}\right)}=\frac{15}{2}$ and $f(2)=3(2)+\frac{3}{2}=\frac{15}{2}$. Therefore, $f(a)=f(b)$.

Thus, the conditions of Rolle's theorem have been met and there is at least one value of *c* between $\frac{1}{2}$ and 2 such that $f'(c)=0$.

To find all values of *c*, take the derivative, then set equal to 0, then solve.

$f(x)=3x+\frac{3}{x}$	Start with the original function.
$f(x)=3x+3x^{-1}$	Rewrite to use the power rule.
$f'(x)=3-3x^{-2}$	Take the derivative.
$f'(x)=3-\frac{3}{x^2}$	Rewrite with positive exponents.
$3-\frac{3}{x^2}=0$	Set equal to 0.
$3=\frac{3}{x^2}$	Add $\frac{3}{x^2}$ to both sides.
$3x^2=3$	Multiply both sides by x^2.
$x^2=1$	Divide both sides by 3.
$x=\pm 1$	Take the square root of both sides.

Since we want all values on the interval $\left(\frac{1}{2}, 2\right)$, the value guaranteed by Rolle's theorem is $c=1$. (In other words, since $c=-1$ is not on the interval $\left(\frac{1}{2}, 2\right)$, it is not considered.)

WATCH

In this video, we will find all values of *c* guaranteed by Rolle's theorem for $f(x)=20\sqrt{x}-2x$ on the interval $[16, 36]$.

Video Transcription

Hi there, and welcome back. What we're going to look at, here is an example to illustrate Rolle's theorem, which remember Rolle's theorem says that if you have a continuous function on a closed interval, and the value of the function at each endpoint is the same, then we're guaranteed a horizontal tangent somewhere in the interval. So that's what we're going to check here.

Our function here, as we see, 20 square root of x minus 2x. It is continuous on the interval 16 to 36. There are no breaks. We can see that with a square root function. And it is differentiable on that interval as well. So the thing we have to verify is that f of 16 and f of 36 are the same.

So f of 16 is 20 square root of 16 minus 2 times 16. Square root of 16 is 4, times 20 is 80. So that is 80 minus 32, which is 48. And f of 36 is 20 square root 36 minus 2 times 36, which is 120 minus 72. 120 minus 72, which is also 48. So those two function values are equal.

So that means that somewhere in the middle of the interval, we are guaranteed a horizontal tangent. So what we do is we take the derivative of f And we set it equal to 0 to find where that horizontal tangent is. And we know that location should be somewhere between 16 and 36.

So let's find out. So f of x, I'm going to rewrite as 20x to the 1/2 minus 2x. And then I'm going to take the derivative. So 20 times 1/2 is 10x, to the-- take one away from the power. That's x to the negative 1/2, minus 2 equals 0.

Now, before I get to solving, I'm going to rewrite x to the negative 1/2 in a more familiar form, which means I'm going to write this as 10 over x to the 1/2 minus 2 equals 0, which means 10 over square root of x minus 2 equals 0. So then to solve this, I'm going to add 2 to both sides. And then multiply both sides by the square root of x. And then divide both sides by 2, just to isolate the square root of x to one side. And then finally, we're going to square both sides.

And as we can see, this is the value of c, even though we're calling it x. This is the value of c that is guaranteed by Rolle's theorem. The horizontal tangent occurs at 25. The endpoints were 16 and 36. 25 is right between them. And there we have Rolle's theorem.

SUMMARY

In this lesson, you learned that when a function is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then **Rolle’s theorem** guarantees that there is a value of *c* between *a* and *b* such that $f'(c) = 0$, which means that there is a guaranteed horizontal tangent line at *c*. Then, you examined a few examples involving the**application of Rolle's theorem**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.