Properties of the Definite Integral

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WHAT'S COVERED

In this lesson, you will learn some useful properties of definite integrals. They are mostly quite intuitive and have been hinted at in previous tutorials, but this is where we will establish these properties officially. Specifically, this lesson will cover:

1. Properties of the Definite Integral
2. Properties of Definite Integrals of Combinations of Functions
3. Comparison Properties
   a. Comparing Two Functions
   b. Bounds on the Value of a Definite Integral

1. Properties of the Definite Integral

FORMULA

Property	Integral Formula	In Words
1	**Definite Integral When Lower and Upper Bounds Are Equal** $\int_a^a f(x)dx = 0$	When the limits of integration are equal, the value of the definite integral is 0.
2	**Definite Integral When Upper and Lower Bounds Are Interchanged** $\int_b^a f(x)dx = -\int_a^b f(x)dx$	When the order of the limits of integration are interchanged, the values of the definite integrals are opposites.
3	**Definite Integral of a Constant Function** $\int_a^b kdx = k(b-a)$	The definite integral of a constant is equal to the constant multiplied by the width of the interval $(b-a)$.
	Definite Integral of a Constant	

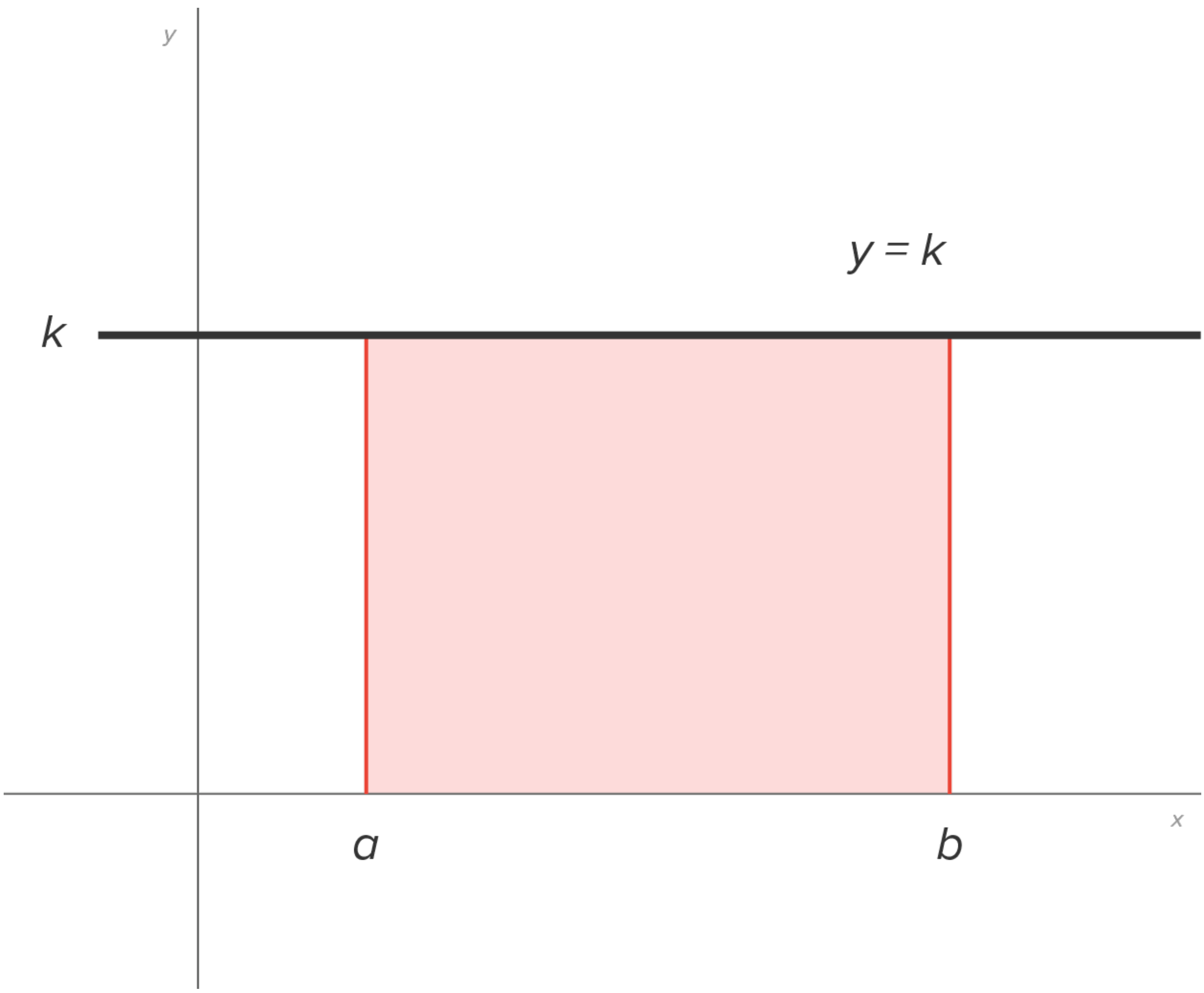
4	**Multiple of a Function** $\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$	The constant *k* can be moved outside, and the definite integral of $f(x)$ is multiplied by *k*.
5	**Definite Integral Over a Partition of an Interval, with a ≤ b ≤ c** $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$	Adding areas Note: $a \le b \le c$

To understand some of these properties, we'll look at Riemann sums and areas.

Property 1: If the lower and upper limits of integration are equal, then the width of the interval is 0, which means there is no accumulated area.

Property 2: As we have seen, $\int_a^b f(x)dx$ accumulates area from $x = a$ to $x = b$. Then, moving in the opposite direction from $x = b$ to $x = a$, whatever was added would be subtracted, and vice versa. Thus, the definite integrals have opposite signs.

Property 3: Consider the graph of $y = k$ on the interval $[a, b]$.



Assume $k > 0$, as in the picture. The region formed by $y = k$, $x = a$, and $x = b$ is a rectangle with height *k* and width $b - a$. Then, the area is $k(b - a)$.

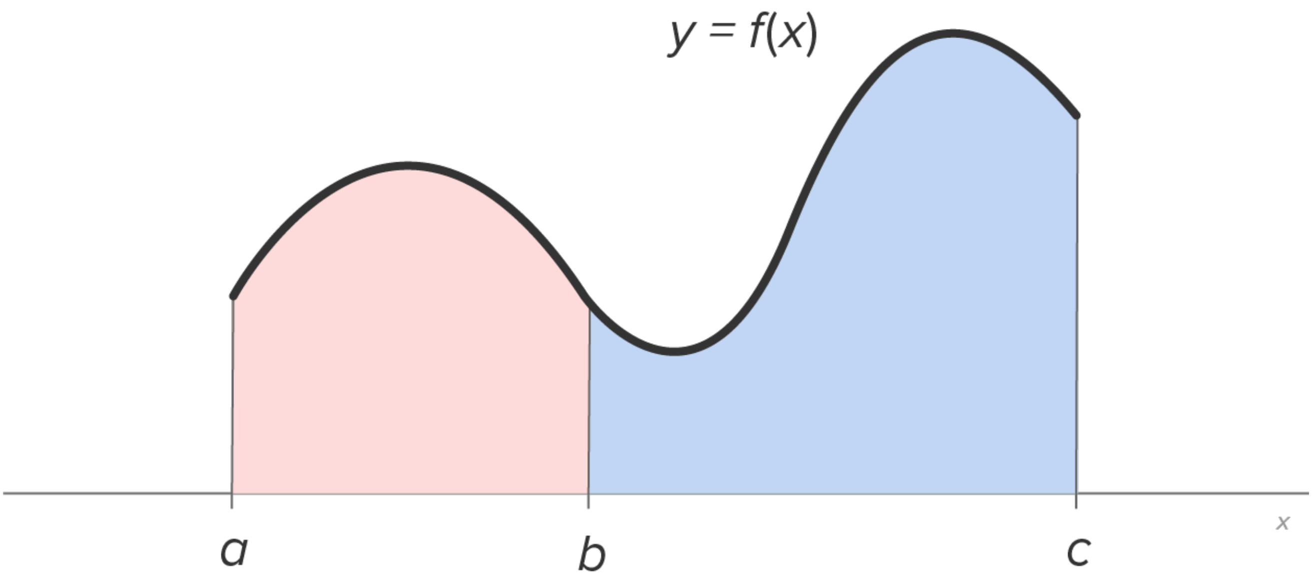
- When $k > 0$, the definite integral is equal to the area.
- When $k < 0$, $k(b - a) < 0$, which is also true since then the rectangle is below the x-axis.

Property 4: Consider the Riemann sums of each function, where the same partition is used:

Riemann Sum for $y = f(x)$	Riemann Sum for $y = k \cdot f(x)$
$\sum_{i=1}^{n} f(c_i) \Delta x$	$\sum_{i=1}^{n} k \cdot f(c_i) \Delta x$
Height of each rectangle: $f(c_i)$	Height of each rectangle: $k \cdot f(c_i)$
Area of each rectangle: $f(c_i) \cdot \Delta x$	Area of each rectangle: $k \cdot f(c_i) \cdot \Delta x$

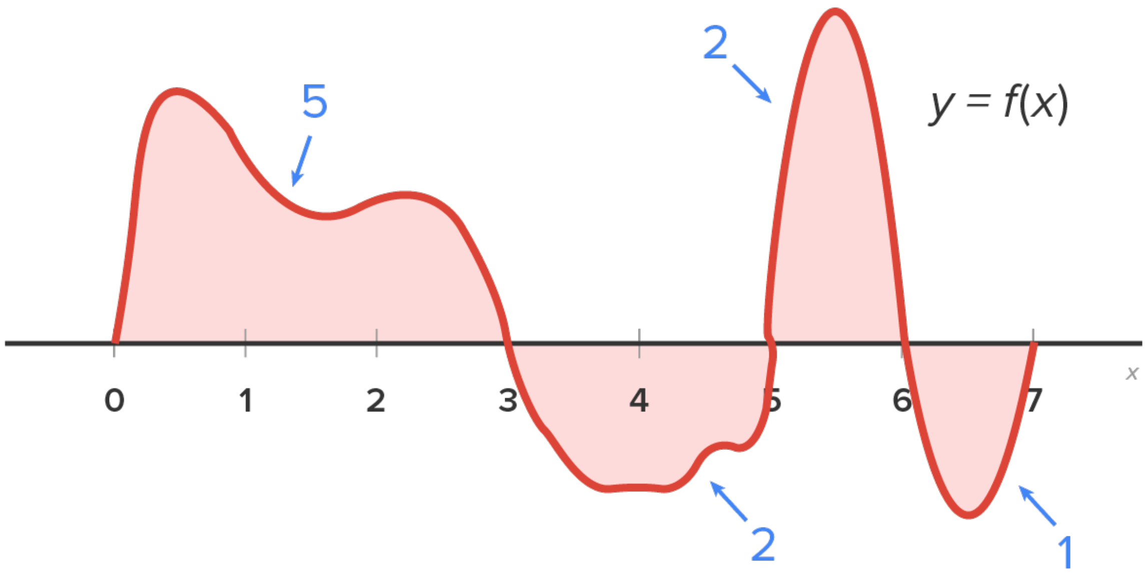
In the Riemann sum, all terms have a common factor of *k*, meaning it can be factored outside the sum, $k \cdot \sum_{i=1}^{n} f(c_i) \Delta x$. Since this is the original Riemann sum multiplied by *k*, this justified the integral version of this property.

Property 5: Consider the graph in the figure:



By adding areas, we see that the area on $[a, c]$ is the sum of the areas on $[a, b]$ and $[b, c]$.

↱ EXAMPLE The graph in the figure shows a function $f(x)$ and areas between $f(x)$ and the x-axis.



Find the definite integrals of f for each of the following:

a. Find the definite integral: $\int_3^5 f(x)dx$

$\int_3^5 f(x)dx$	Evaluate this definite integral.
$= -2$	The area of the region is 2, but is below the x-axis.

Conclusion: $\int_3^5 f(x)dx = -2$

b. Find the definite integral: $\int_0^5 f(x)dx$

$\int_0^5 f(x)dx$	Evaluate this definite integral.
$= \int_0^3 f(x)dx + \int_3^5 f(x)dx$	Use the following property: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ Note, "3" was chosen since at $x = 3$, the first region ends and the second one begins.
$= 5 + (-2)$	Substitute values from the graph: $\int_0^3 f(x)dx = 5$, $\int_3^5 f(x)dx = -2$
$= 3$	Simplify.

Conclusion: $\int_0^5 f(x)dx = 3$

c. Find the definite integral: $\int_5^3 f(x)dx$

$\int_5^3 f(x)dx$	Evaluate this definite integral.
$= -\int_3^5 f(x)dx$	Use the following property: $\int_b^a f(x)dx = -\int_a^b f(x)dx$
	Since the limits of integration are in reverse order, this property is appropriate to use.
$= -(-2) = 2$	Substitute values from the graph: $\int_3^5 f(x)dx = -2$
$= 2$	Simplify.

Conclusion: $\int_5^3 f(x)dx = 2$

d. Find the definite integral: $\int_0^5 4f(x)dx$

$\int_0^5 4f(x)dx$	Evaluate this definite integral.
$= 4\int_0^5 f(x)dx$	Use the following property: $\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$
$= 4(3)$	$\int_0^5 f(x)dx$ was evaluated in part b.
$= 12$	Simplify.

Conclusion: $\int_0^5 4f(x)dx = 12$

TRY IT

Consider the following definite integrals in relation to the same graph as in the previous example.

a. $\int_4^4 [f(x)]^3 dx$

b. $\int_5^7 -3f(x)dx$

c. $\int_3^6 f(x)dx$

Evaluate each definite integral.

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a. $\int_4^4 [f(x)]^3 dx = 0$

b. $\int_5^7 -3f(x)dx = -3$

c. $\int_3^6 f(x)dx = 0$

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2. Properties of Definite Integrals of Combinations of Functions

FORMULA

Formula	In Words
Definite Integral of a Sum of Two Functions $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$	The definite integral of a sum of two functions is the sum of the definite integrals of the functions.
Definite Integral of a Difference of Two Functions $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$	The definite integral of a difference of two functions is the difference of the definite integrals of the functions.

These properties follow directly from Riemann sums (using properties of summations).

For the sum property:

$$\int_a^b (f(x)+g(x))dx$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}(f(c_k)+g(c_k))\Delta x\right]$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}(f(c_k)\Delta x+g(c_k)\Delta x)\right]$$

By the property of summations, this is written as $\lim_{n\to\infty}\left[\sum_{k=1}^{n} f(c_k)\Delta x + \sum_{k=1}^{n} g(c_k)\Delta x\right]$, which by the limit property is equal to $\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x + \lim_{n\to\infty}\sum_{k=1}^{n} g(c_k)\Delta x$, which is equal to $\int_a^b f(x)dx + \int_a^b g(x)dx$.

A very similar sequence of steps can be followed for the difference between *f* and *g*.

↪ EXAMPLE Given $\int_2^6 f(x)dx = 10$ and $\int_2^6 g(x)dx = 4$, find each of the following:

a. Find the definite integral: $\int_2^6 [f(x)+g(x)]dx$

$\int_2^6 [f(x)+g(x)]dx$	Evaluate this definite integral.
$=\int_2^6 f(x)dx+\int_2^6 g(x)dx$	Use the definite integral of a sum of two functions property.
$=10+4$	Substitute values.
$=14$	Simplify.

Conclusion: $\int_2^6 [f(x)+g(x)]dx=14$

b. Find the definite integral: $\int_2^6 [3+f(x)]dx$

$\int_2^6 [3+f(x)]dx$	Evaluate this definite integral.
$=\int_2^6 3dx+\int_2^6 f(x)dx$	Use the property: $\int_a^b [f(x)+g(x)]dx=\int_a^b f(x)dx+\int_a^b g(x)dx$
$=3(6-2)+10$	For the first integral, $\int_a^b kdx=k(b-a)$.
	For the second integral, the value is given: 10
$=22$	Simplify.

Conclusion: $\int_2^6 [3+f(x)]dx=22$

c. Find the definite integral: $\int_2^6 [3f(x)-g(x)]dx$

$\int_2^6 [3f(x)-g(x)]dx$	Evaluate this definite integral.
$=\int_2^6 3f(x)dx-\int_2^6 g(x)dx$	Use the property: $\int_a^b [f(x)-g(x)]dx=\int_a^b f(x)dx-\int_a^b g(x)dx$
$=3\int_2^6 f(x)dx-\int_2^6 g(x)dx$	Use the property: $\int_a^b k\cdot f(x)dx=k\int_a^b f(x)dx$

$= 3(10) - 4$ Substitute given values of integrals.

$= 26$ Simplify.

Conclusion: $\int_2^6 [3f(x) - g(x)]dx = 26$

WATCH

In this video, given $\int_1^4 f(x)dx = 12$, we'll find the value of $\int_1^4 (f(x) + x)dx$.

Video Transcription

[MUSIC PLAYING] Hello there. Welcome back. What we're going to do is take a look at a known definite integral-- the integral from 1 to 4 of f of x dx. And its value is 12. And we're going to use it to evaluate the integral from 1 to 4 of f of x plus x dx. And that's going to be by using the various properties of definite integrals that we've learned.

So the first property, remember, is that when you have a sum as your integrand, we can break the definite integral into pieces. So this is the integral from 1 to 4 of f of x dx plus the integral from 1 to 4 of just x dx. Now, we already know that the value of this first definite integral is 12. And because it's a known formula-- a known shape, I should say-- we know what the definite integral of f of x equals x is.

So I just sketch a graph of that. And we have our axes here. y equals x is the diagonal line, making a 45-degree angle with the x-axis. And we have x equals 1 here and x equals 4 here. Now since the region is above the x-axis on the interval 1 to 4, we know that the definite integral from 1 to 4 is the area of said region, which is in the shape of a trapezoid.

So the height of this side here is 1. The height of this side here is 4. Those are your parallel bases. And then what we call the height of the trapezoid-- the side that's perpendicular to the parallel bases-- is 3. So that means that this area is 1/2 times 3 times 1 plus 4, which looks to be 15/2. So that means that this second definite integral-- I'm going to have to go to the next line here-- has value 15 over 2, since, again, it's the area of the region because the region is above the x-axis. So that means this is equal to 12 plus 15/2, which is 7.5. So I can write this as either 19.5 or 39/2. and that is using properties to evaluate a definite integral.

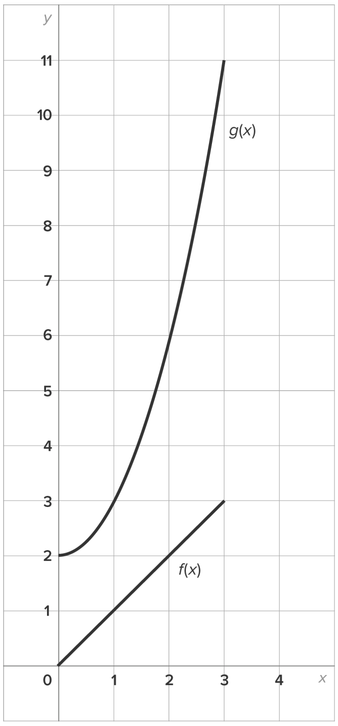
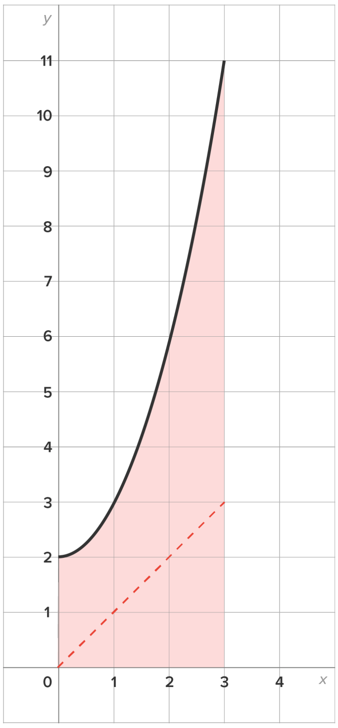
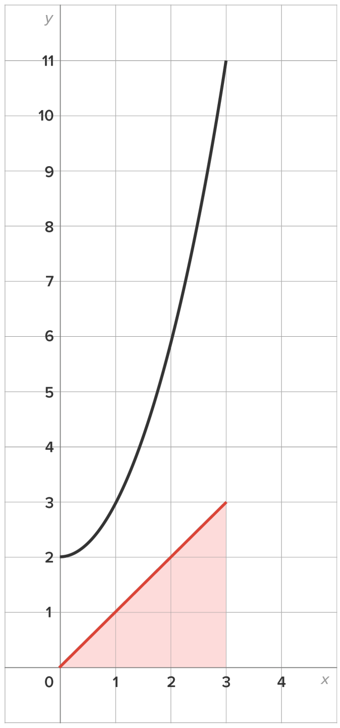
[MUSIC PLAYING]

3. Comparison Properties

3a. Comparing Two Functions

If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

To visualize this, consider these graphs. Clearly, the area between the graph of $g(x)$ and the x-axis is greater than the area between the graph of $f(x)$ and the x-axis.

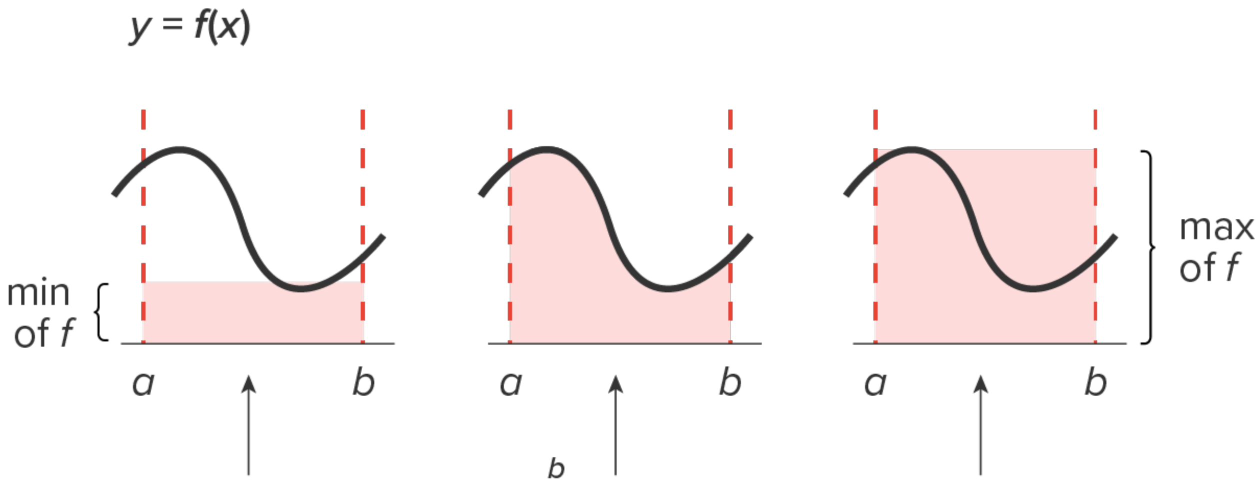
The graphs of $f(x)$ and $g(x)$ together on the same axes.	The region bounded by the graphs of $g(x)$ and the x-axis. Area $= \int_0^3 g(x)dx$	The region bounded by the graphs of $f(x)$ and the x-axis. Area $= \int_0^3 f(x)dx$
		

3b. Bounds on the Value of a Definite Integral

Let m = the minimum value of $f(x)$ on $[a, b]$.
Let M = the maximum value of $f(x)$ on $[a, b]$.

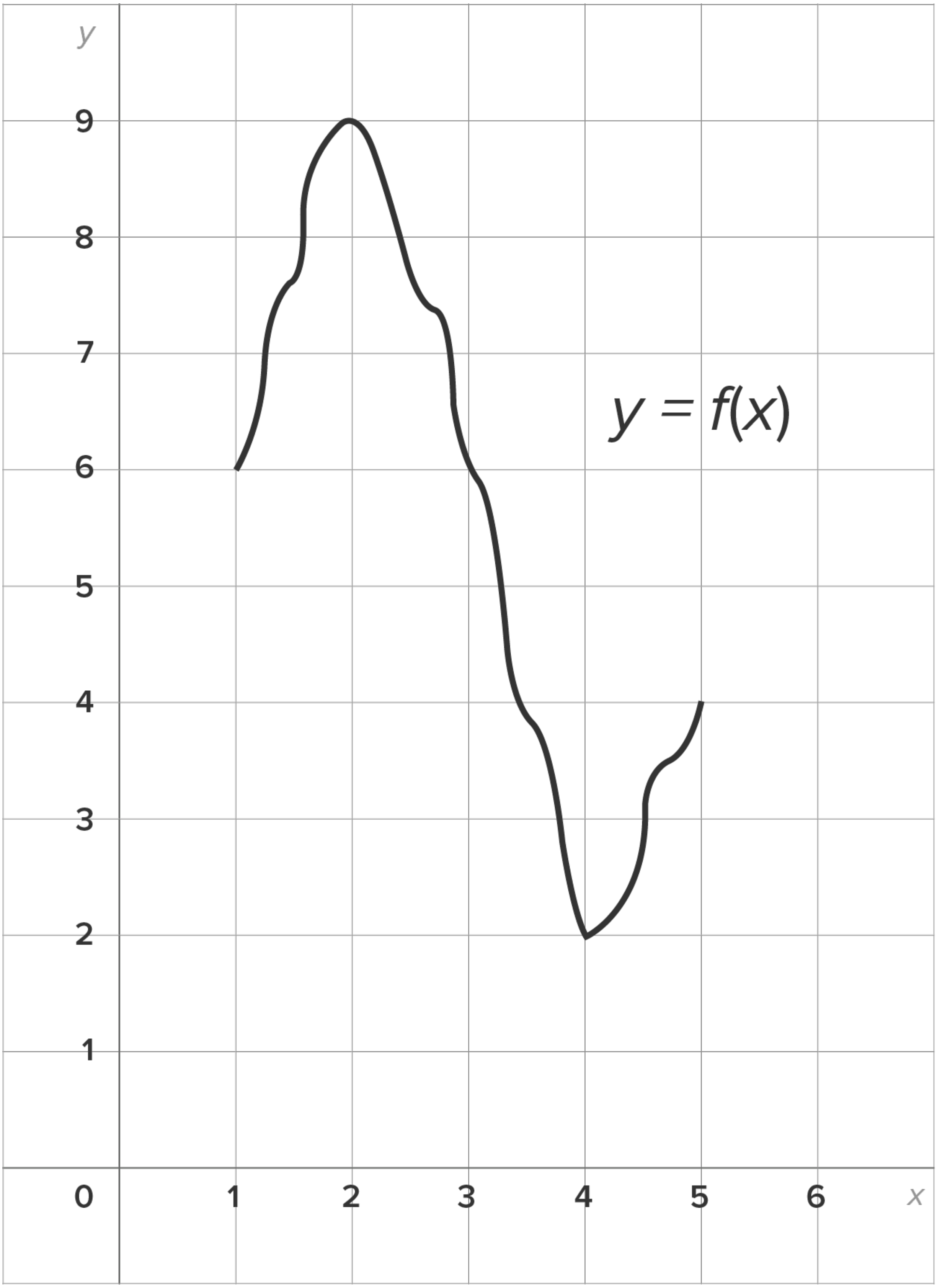
Given that $m \le f(x) \le M$ on the interval $[a, b]$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$.

Below is the graphical justification:



$$\{\text{ min of } f\,\}\,(b - a) \;\leq\; \int_a^b f(x)\,dx \;\leq\; \{\text{ max of } f\,\}\,(b - a)$$

→ EXAMPLE Use the graph to determine the upper and lower bounds of the value of $\int_1^5 f(x)dx$.



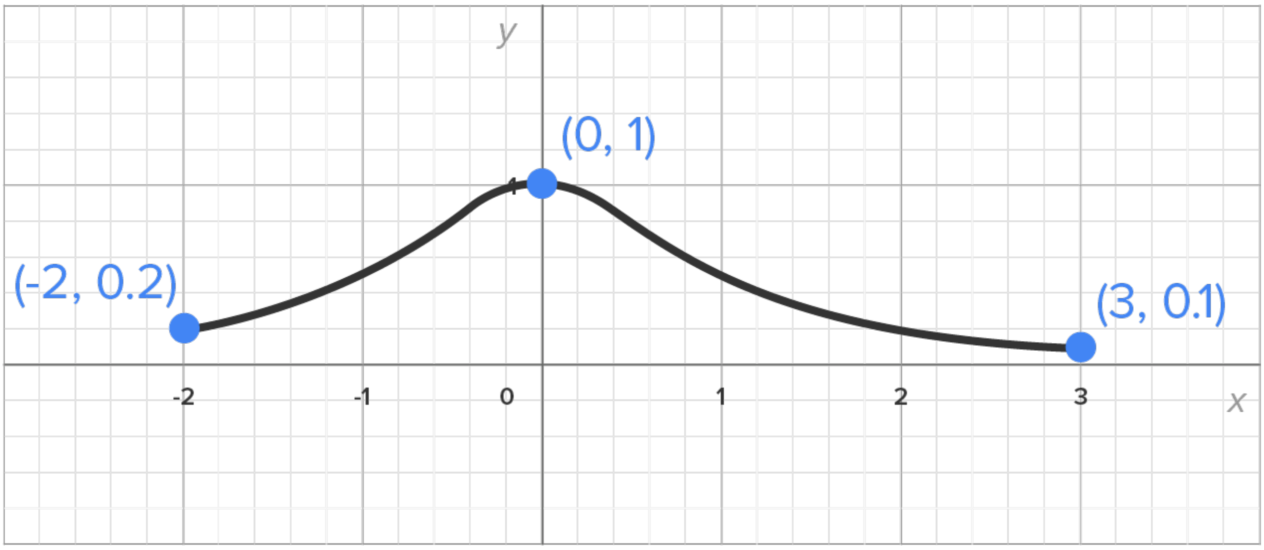
The minimum value of $f(x)$ on $[1, 5]$ is 2; the maximum value is 9.

Thus, the minimum value for the definite integral is $2(5-1)=8$ and the maximum value is $9(5-1)=36$.

That is, $8 \le \int_1^5 f(x)dx \le 36$.

TRY IT

Consider the graph below in relation to the integral $\int_{-2}^{3} f(x)dx$.



Use the graph to determine the upper and lower bounds of the value of this integral. +

The minimum value is 0.5 and the maximum value is 5.

SUMMARY

In this lesson, you examined some useful **properties of the definite integral** as well as **properties of definite integrals of combinations of functions**. As a result of properties of areas, you are now able to find definite integrals of sums, differences, and constant multiples of functions, as well as utilize **comparison properties** to **compare two functions** and put **bounds on the value of a definite integral**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Definite Integral Over a Partition of an Interval, with a ≤ b ≤ c

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Definite Integral When Lower and Upper Bounds Are Equal

$$\int_a^a f(x)dx = 0$$

Definite Integral When Upper and Lower Bounds Are Interchanged

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Definite Integral of a Constant Function

$$\int_a^b k dx = k(b-a)$$

Definite Integral of a Constant Multiple of a Function

$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$$

Definite Integral of a Difference of Two Functions

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Definite Integral of a Sum of Two Functions

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$