Implicit Differentiation

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WHAT'S COVERED

In this lesson, you will apply techniques of derivatives when the equation defines y implicitly. So far, we know how to find derivatives when y is explicitly a function of x, meaning $y=f(x)$. The equation $x^2+2y^2=22$ is an example of an equation where y is defined implicitly, meaning y is not isolated to one side. The equation still defines a curve, so it makes sense to discuss the derivative and slopes of tangent lines, etc. Specifically, this lesson will cover:

1. Implicit Differentiation
2. Slopes and Equations of Tangent Lines

1. Implicit Differentiation

If y is some function of x, we know that the derivative of y is $\frac{dy}{dx}$.

Then, by the chain rule, we know the following:

$$\frac{d}{dx}[y^2]=2yD[y]=2y\frac{dy}{dx}$$

$$\frac{d}{dx}[\sin y]=\cos yD[y]=\cos y\frac{dy}{dx}$$

$$\frac{d}{dx}[\ln y]=\frac{1}{y}D[y]=\frac{1}{y}\frac{dy}{dx}$$

Now, consider the equation $x^2+2y^2=22$, where y is some function of x. If we take the derivative of both sides of the equation with respect to x, we get:

$\frac{d}{dx}[x^2]+\frac{d}{dx}[2y^2]=\frac{d}{dx}[22]$	Use the sum/difference rules.
$2x+4y\frac{dy}{dx}=0$	$D[x^2]=2x$, $D[2y^2]=4y\frac{dy}{dx}$, $D[22]=0$

At this point, notice that $\frac{dy}{dx}$ is a quantity in the equation. In order to get an expression for $\frac{dy}{dx}$, we solve for it as if it were a variable.

$2x+4y\frac{dy}{dx}=0$	Start where we left off.
$4y\frac{dy}{dx}=-2x$	Subtract $2x$ from both sides.
$\frac{dy}{dx}=-\frac{2x}{4y}$	Divide both sides by $4y$.
$\frac{dy}{dx}=-\frac{x}{2y}$	Simplify the fraction to its lowest terms.

This means that $\frac{dy}{dx}=-\frac{x}{2y}$. Note that the expression is written in terms of both x and y. This is very common with implicit differentiation.

STEP BY STEP

To find $\frac{dy}{dx}$ implicitly, perform these steps to the equation.

1. Differentiate both sides with respect to x.
2. Collect all terms with $\frac{dy}{dx}$ to one side.
3. Solve for $\frac{dy}{dx}$.

EXAMPLE Now, let's look at another example. Given $2x^2+3xy+4y^2=100$, compute $\frac{dy}{dx}$.

$2x^2+3xy+4y^2=100$	Start with the original relation.
$\frac{d}{dx}[2x^2]+\frac{d}{dx}[3xy]+\frac{d}{dx}[4y^2]=\frac{d}{dx}[100]$	Apply the derivative to each term (use the sum/difference rule).
$4x+3(y)+3x\frac{dy}{dx}+8y\frac{dy}{dx}=0$	$D[2x^2]=4x$ $D[3xy]=D[3x\cdot y]=(3)y+3x\frac{dy}{dx}$ (product rule) $D[4y^2]=8y\frac{dy}{dx}$
$3x\frac{dy}{dx}+8y\frac{dy}{dx}=-4x-3y$	Subtract $4x$ and $3y$ from both sides.
$(3x+8y)\frac{dy}{dx}=-4x-3y$	Factor out $\frac{dy}{dx}$.
$\frac{dy}{dx}=\frac{-4x-3y}{3x+8y}$	Divide both sides by $3x+8y$.

Thus, $\frac{dy}{dx}=\frac{-4x-3y}{3x+8y}$.

TRY IT

Consider the equation $10x^2y^2+4x^3-3y^5=11$.

Find the derivative implicitly. +

$$\frac{dy}{dx} = \frac{-20xy^2 - 12x^2}{20x^2y - 15y^4}$$

WATCH

Here is a video in which we find $\frac{dy}{dx}$ of $\cos(xy) = -\frac{1}{2} + e^y$.

Video Transcription

Hi, there. Welcome to the video and how to use implicit differentiation to find dy dx when I have a mathematical relation where the variables are mixed together and one's not sorted out on one side of the equal sign?

Well, we can still find the derivative without having to go through and trying to solve it by this implicit differentiation. Here, our mathematical relation is the cosine of xy is equal to negative 1/2 plus e to the y.

Now when we are using implicit differentiation, it's really important to look at what they're asking you to find. When we are asked to find the derivative, we want to note that what's in our numerator, dy, that's telling me that y is the dependent variable. And the denominator tells me the variable x is the independent variable. So it's very important to pay attention to that so that you know the roles that they play.

Now implicit differentiation-- we need to be careful as we go through this to use our derivative rules that we've already learned. And when we are differentiating the independent variable-- here the x-- that we will just go ahead with our rules. We don't need to do any extra factor there. But when we are differentiating and the process involves doing that on the dependent variable, we don't know what the dependent variable is equal to in terms of the independent variable, so we need to use the chain rule, which is to also multiply by the notation of the dy dx.

So let's get started and see how that plays out. So on the left-hand side of the equal sign, I have the cosine of the angle xy. Now remember, the derivative of cosine of an angle is negative sine the angle times the derivative of the angle. Now the derivative of xy is the derivative of a product. So I'm going to need to apply the product rule when I'm differentiating that inside product. So remember, that's the derivative of the first factor-- the derivative of x is 1-- and it's the x variable times the second factor unchanged plus the first factor times the derivative of the second factor.

Well, the second factor here is the letter y. The derivative of y is 1, but it's the dependent variable. So by the chain rule, I also have to multiply by dy dx. And this is equal to. On the right-hand side, the first term is just a constant, just a number. The derivative of a number is 0, and then plus the derivative of e to the y. Well, the derivative of e to the power is e to the power times the derivative of the power. And the derivative of y is 1, but it's the dependent variable, so I have to multiply by dy dx.

Now let's just rewrite that so it's a little bit more manageable to look at. So I have negative sine of xy times y plus x dy dx is equal to e to the dy dx. Next, what we need to do is to remove our parentheses that we can and sort all the terms with the dy dx on one side of the equation and all the terms without the dy dx on the other side. So I have negative sine xy times y. That's a negative y sine of xy. And then negative sine of xy times x dy dx-- well, negative times positive is a negative. x sine of xy times dy dx is equal to e to the y dy dx.

Now when I look at these terms, the first term does not have a dy dx in it. The second term does have a dy dx. And then I have my equal sign, and then on the right-hand side, I also have a dy dx. I want all the dy dx terms on the same side of the equal sign. So we are going to add this x sine of xy dy dx to both sides.

So I have negative y sine of xy is equal to e to the y dy dx plus x sine of xy dy dx. And have there been any terms on the right that didn't have the dy dx, I would have moved those over to the left.

Now what we've done by this is we've forced a common factor of dy dx in each of the terms on the right-hand side. So we are going to factor the did x out of the terms on the right-hand side. And we're going to factor it out the back. So we have e to the y plus x sine of xy and then dy dx. And then to solve for dy dx, divide both sides by that, we'll just multiply to it. So we're going to divide both sides by the e to the y plus x sine of xy.

And so our final result is that dy dx is equal to negative y sine of xy divided by e to the y plus x sine of xy. And that is how you use implicit differentiation to find dy dx for cosine of xy is equal to negative 1/2 plus e to the y.

Once we know the derivative, it is possible to find the slope of the tangent line, then the equation of the tangent line.

Since the implicit derivatives use the notation $\frac{dy}{dx}$ for the derivative, we need a way to show that we are evaluating the derivative at a point.

BIG IDEA

The notation $\left.\frac{dy}{dx}\right|_{(a,\, b)}$ means to evaluate $\frac{dy}{dx}$ when $x = a$ and $y = b$.

Now, we are ready to find slopes of tangent lines with implicit functions.

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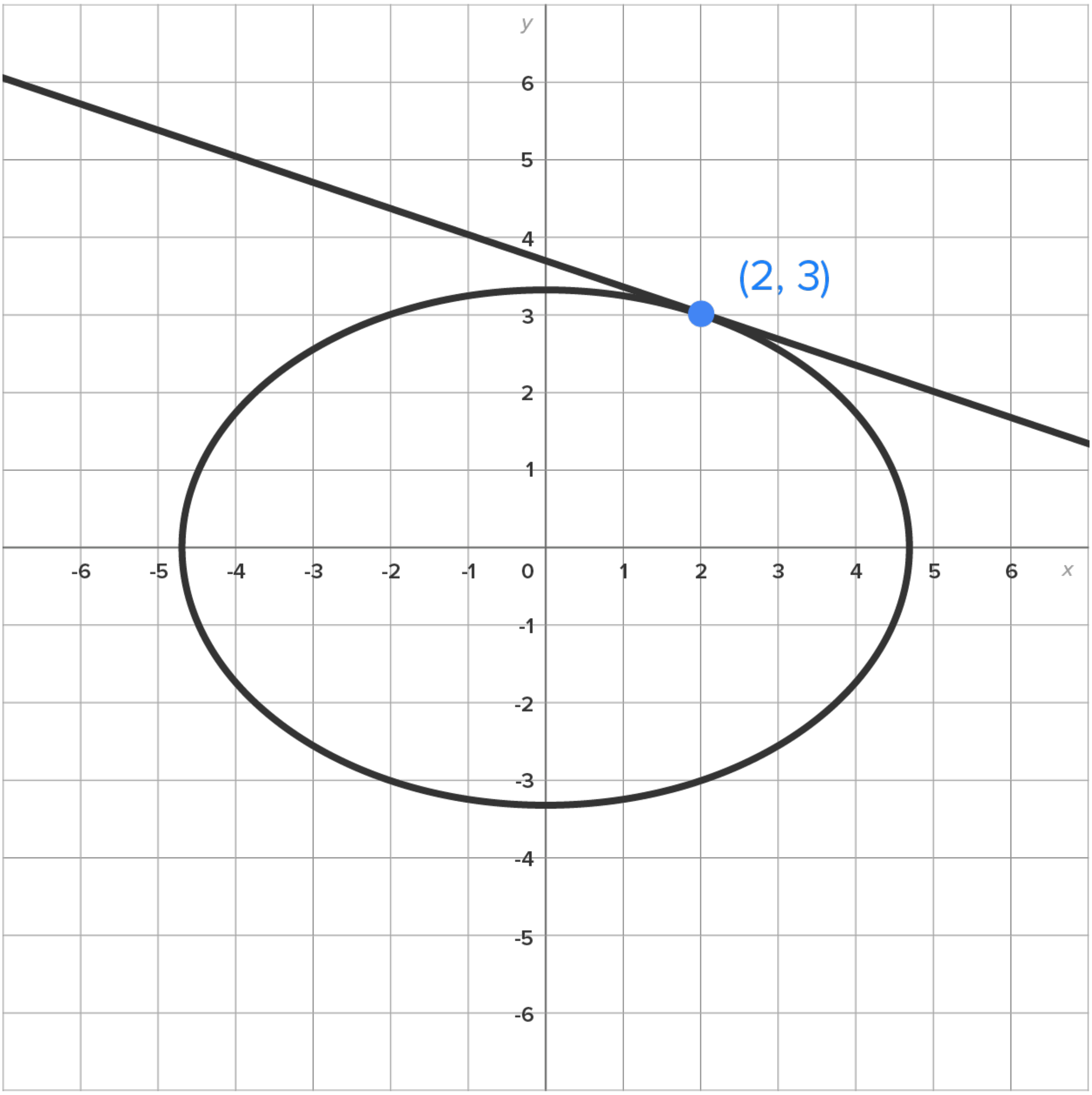
2. Slopes and Equations of Tangent Lines

Earlier in this challenge, we computed $\frac{dy}{dx}$ for the curve $x^2 + 2y^2 = 22$.

Shown in the graph below is the curve (the ellipse), and its tangent line at the point (2, 3).



The derivative formula we calculated earlier is $\frac{dy}{dx} = -\frac{x}{2y}$.

Then, the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(2,\ 3)} = -\frac{2}{2(3)} = -\frac{1}{3}$.

To write the equation of the tangent line, we normally need $f(a)$ and $f'(a)$. Since y is defined implicitly, we do not have the "f" notation. That being the case, we'll make use of the point-slope form of a line.

Now, let's find the equation of the tangent line.

$y - y_1 = m(x - x_1)$	Use the point-slope form.
$y - 3 = -\frac{1}{3}(x - 2)$	The line passes through (2, 3) and has slope $-\frac{1}{3}$.
$y - 3 = -\frac{1}{3}x + \frac{2}{3}$	Distribute $-\frac{1}{3}$.

$$y = -\frac{1}{3}x + \frac{11}{3}$$ Add 3 to both sides.

The equation of the tangent line is $y = -\frac{1}{3}x + \frac{11}{3}$.

TRY IT

Consider the curve $2x^2 + 3xy + 4y^2 = 100$ with $\frac{dy}{dx} = \frac{-4x - 3y}{3x + 8y}$.

Write the equation of the line tangent to this curve at the point (0, 5). +

$$y = -\frac{3}{8}x + 5$$

WATCH

Watch this video to see an example of writing an equation of a tangent line to $x^2 + 2xy + 4y^2 = 12$ at the point (2, 1).

Video Transcription

[BRIGHT MUSIC] Welcome to the video on how to write the equation of a tangent line to a mathematical relation at a given point. So for this problem, we are asked to write the equation of the line tangent to x squared plus 2xy plus 4y squared equal 12 at the point 2,1. Now recall, to write the equation of a line, what we need is first, the slope, second, a point that the line goes through, and then we plug that information into our point slope form, y minus y sub 1 is equal to m times x minus x sub 1.

Let's look at the first thing, the slope. We'll recall that the slope of a tangent to a curve is given by the value of the derivative there. So the derivative is the equation that you put your values in, and you get your slope of your tangent out. So what we need to do, then, is find the derivative of our mathematic relation. Now here, the x's and the y's are intermixed in the equation. And so the relation is implied between the x's and y's, meaning that when we do our derivative, we can use our implicit differentiation.

Recall implicit differentiation. We're going to follow all of our derivative rules, and when we're differentiating the independent variable, we will just do the derivative rule. When we are taking the derivative of the dependent variable, once we do the derivative rule, we need to multiply by dy, dx for the chain rule. So let's get started. Here, I'm going to differentiate term by term.

The derivative of x squared is 2x, and that's the independent variable I differentiated. So we just do the rule and move on. In the next term, I have, actually, a multiplication of 2x, which is an expression with the independent variable, and then y, which is my dependent variable. When I'm differentiating a product of variable expressions by the derivative rules, I have to use the product rule.

So we're going to take the derivative of the first factor-- and the derivative of 2x is 2-- times-- keep the second factor y alone-- and then plus-- this time, we're going to leave the 2x factor the way it is, and we are going to multiply that by the derivative of that second factor, y. But y is the dependent variable. So once I apply the derivative rule, I'm going to also multiply by dy, dx.

Well, the derivative rule for y is 1. But it's the y letter, so I multiply by dy, dx. And then plus-- now my last term, this 4y squared. That is a term that involves the dependent variable, y. So again, we apply the derivative rule. 4 times my variable base to the number 2 power is the general power rule, so I have 8y.

But it was the y letter that we were working the derivative on, so once I do that by the chain rule, I need to multiply by dy, dx. And that's equal to. And the derivative of 12 on the right-hand side-- well, 12 is a number. It's a constant. And the derivative of constant term is 0.

OK. So this gives us our implicit differentiation. Next, what we want to do is solve for dy, dx. So I'm going to write this in a little bit friendlier manner, by within the multiplications, write that with whatever multiplications I can do out. And then we want all the terms with dy, dx on one side of the equal sign. So my equal sign is there, and my 2x dy, dx plus my 8y dy, dx both have dy, dxx in them.

And any term that doesn't have the dy, dx, we are going to move over to the other side. So we are going to subtract a 2x and subtract a 2y from both sides. And we will get 2x dy, dx plus 8y dy, dx is equal to negative 2x minus 2y. Factor out the dy, dx. I have 2x plus 8y.

Quantity times dy, dx is equal to negative 2x minus 2y. And then divide both sides by what's ever multiplied to your dy, dx. So dy, dx is equal to negative 2x minus 2y over 2x plus 8y.

Now, that's the formula that gives me the slope of that mathematical relation at any point on the curve. I specifically want to know the slope at the .21, so we are going to take that derivative dy, dx, and we are going to evaluate it-- so we do this vertical line to show we're going to evaluate it-- at the ordered pair 2,1. And remember, in the ordered pair, your first coordinate is your x-coordinate, and your second coordinate is your y-coordinate. So we're going to come over to our formula. Wherever we see an x, we're going to take it out and put in 2. Wherever we see a y, we're going to take it out and put in 1.

So I have negative 2 times 2 minus 2 times 1 over 2 times 2 plus 8 times 1. So that gives me negative 4 minus 2, which is negative 6, over 4 plus 8, which is 12. And so my dy, dx evaluated at 2,1 is negative one-half. And that is my slope of my line.

Now, the point is actually given to me with both coordinates. So I have my point is 2,1. So now, remember that's your x sub 1, y sub 1. And so our equation is y minus the y-coordinate of the point 1 is equal to negative one-half times x minus the x-coordinate of the point 2. Removing the parentheses, I get y minus 1 equals negative one-half x plus 1. Adding 1 to both sides, I get y is equal to negative one half x plus 2. And that is the equation of the tangent line to x squared plus 2xy plus 4y squared equals 12 at the point 2,1.

SUMMARY

In this lesson, you learned that through **implicit differentiation**, it is possible to find the derivative of a mathematical relation that is not explicitly solved for y. You also learned that in an equation where y is defined implicitly, when asked to write the equation of a tangent line, you will be given a point on the curve; therefore, you can use the **point-slope form to write the equation of the tangent line**.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.