Continuous Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn what it means for a function to be continuous, including how limits are used in relation to continuity. Specifically, this lesson will cover:

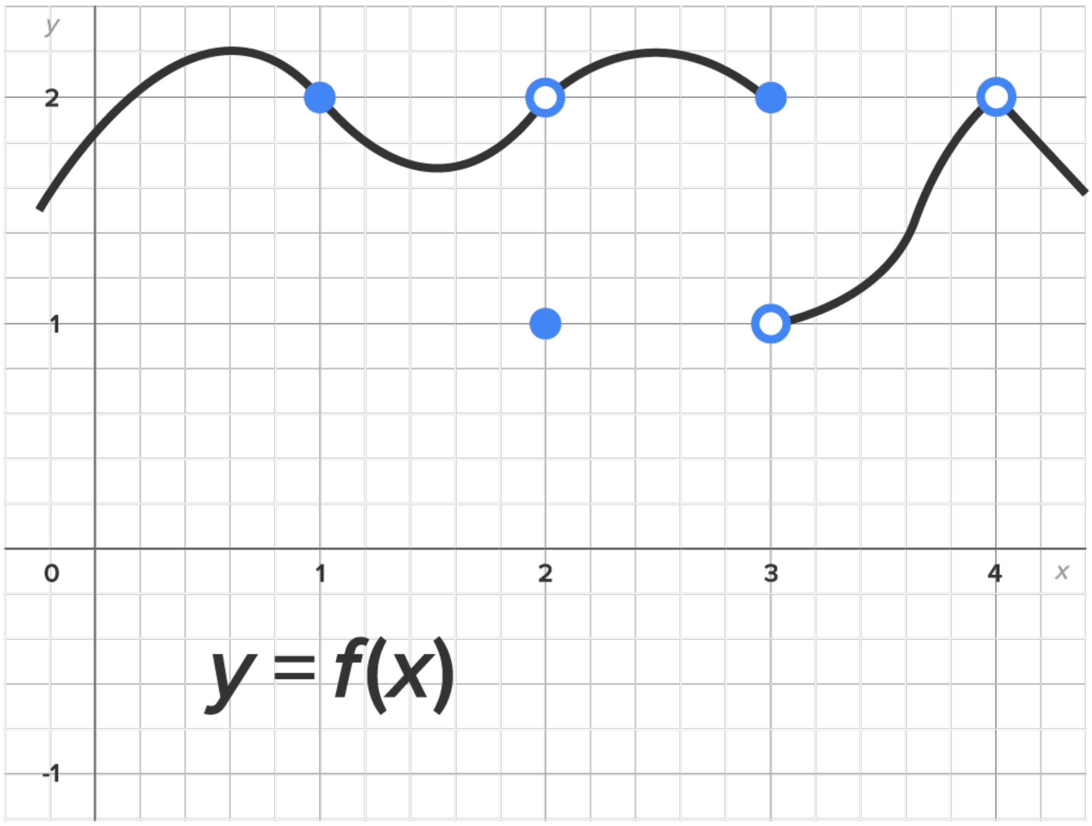
1. The Definition of Continuity
2. Determining if a Function Is Continuous at $x = a$
3. Determining Intervals Over Which a Function Is Continuous

1. The Definition of Continuity

A function is called **continuous** at a point where there is no break in the graph *at that point*.

That is, $\lim_{x \to a} f(x) = f(a)$.

Consider the graph of $y = f(x)$ shown below. We will examine the continuity of $f(x)$ when $x = 1, 2, 3$, and 4.



Given Point	Continuity of $f(x)$ at the Given Point
$x = 1$	The graph of $f(x)$ is continuous when $x = 1$ since there are no breaks in the graph at that point. Looking just before $x = 1$, the graph passes through the point $(1, f(1))$ and continues to "flow" afterwards.
$x = 2$	The graph of $f(x)$ is NOT continuous when $x = 2$. There is a hole in the graph when $x = 2$, meaning there is a break in the graph.
$x = 3$	The graph is NOT continuous when $x = 3$. There is a break in the graph.
$x = 4$	The graph is NOT continuous when $x = 4$. There is a hole in the graph.

Now, considering these 4 points, let's examine the limits at these points and the values of $f(x)$ at these points as well as whether or not the function is continuous at these points:

x-value	$\lim_{x \to a} f(x)$	$f(a)$	Continuous?
$x = 1$	$\lim_{x \to 1} f(x) = 2$	$f(1) = 2$	Yes
$x = 2$	$\lim_{x \to 2} f(x) = 2$	$f(2) = 1$	No
$x = 3$	$\lim_{x \to 3} f(x)$ does not exist (the left-hand and right-hand limits are not equal).	$f(3) = 2$	No
$x = 4$	$\lim_{x \to 4} f(x) = 2$	$f(4)$ is not defined.	No

From this table, we can conclude the following:

- A function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$. That is, $\lim_{x \to a} f(x)$ exists and is equal to the value of $f(a)$.
- A function $f(x)$ is not continuous at $x = a$ if any of the following occur:
  - $\lim_{x \to a} f(x)$ does not exist.
  - $f(a)$ is undefined.
  - $\lim_{x \to a} f(x)$ exists, but is not equal to $f(a)$.

TERM TO KNOW

Continuous Function

A function that has no breaks in the graph. That is, $\lim_{x \to a} f(x) = f(a)$.

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2. Determining if a Function Is Continuous at $x = a$

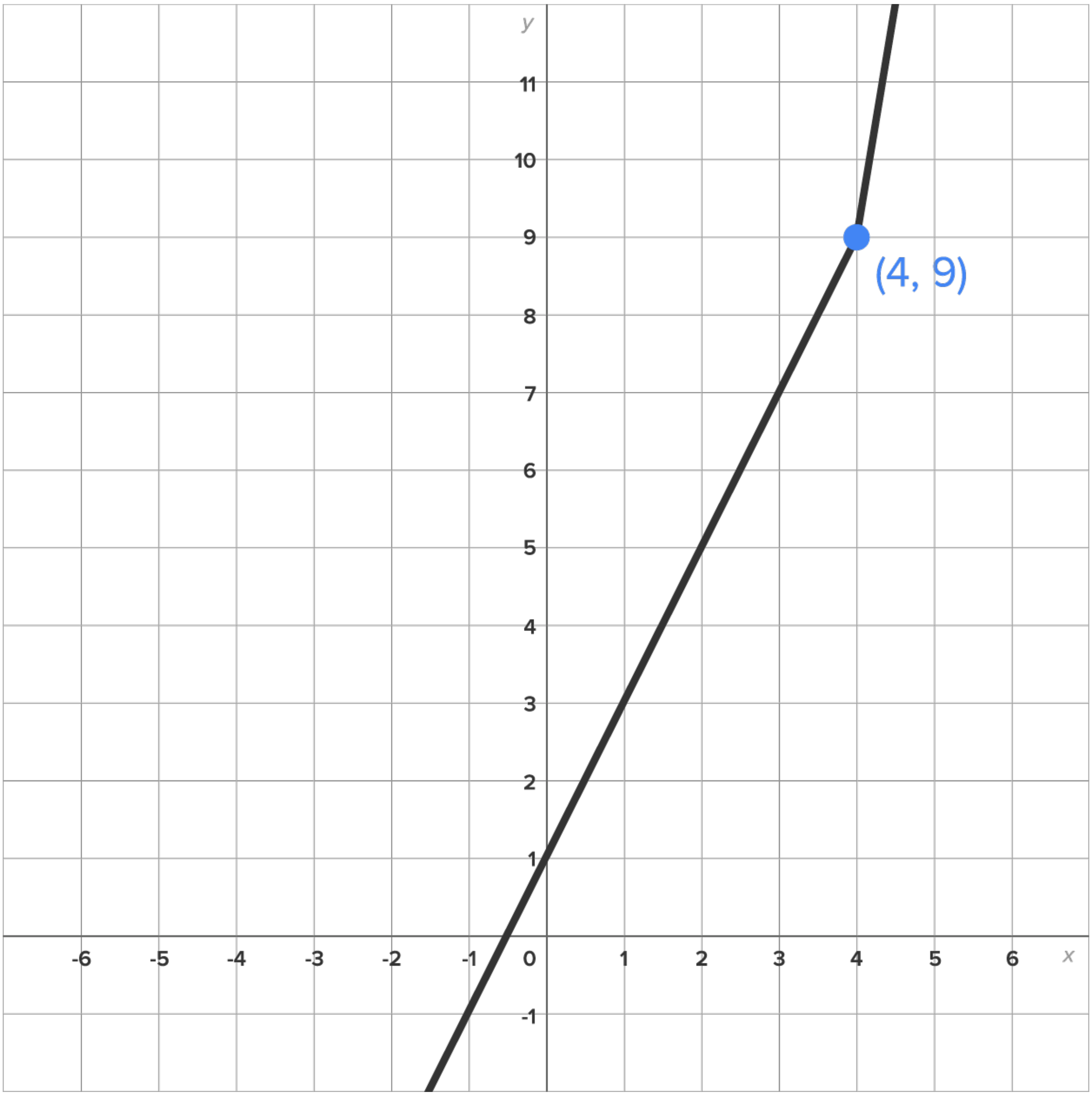
To determine if a function is continuous at $x = a$, we need to compare the values of $\lim_{x \to a} f(x)$ and $f(a)$. While computing $f(a)$ is straightforward, computing $\lim_{x \to a} f(x)$ requires more care, and sometimes requires one-sided limits.

EXAMPLE Consider the function $f(x) = \begin{cases} 2x+1 & \text{if } x < 4 \\ (x-1)^2 & \text{if } x \geq 4 \end{cases}$. Determine if $f(x)$ is continuous at $x = 4$.

First, check to see if $\lim_{x \to 4} f(x)$ exists. Since $f(x)$ changes definition when $x = 4$, we need to consider the one-sided limits:

- Left-sided limit: $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (2x+1) = 2(4)+1 = 9$
- Right-sided limit: $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} (x-1)^2 = (4-1)^2 = 9$
- Conclusion: $\lim_{x \to 4} f(x) = 9$, which means it exists and is equal to 9.

From looking at the function definition, $f(4) = (4-1)^2 = 9$. Thus, the limit and the function value are the same; therefore the function is continuous at $x = 4$. Here is the graph of $f(x)$ to help visualize this:

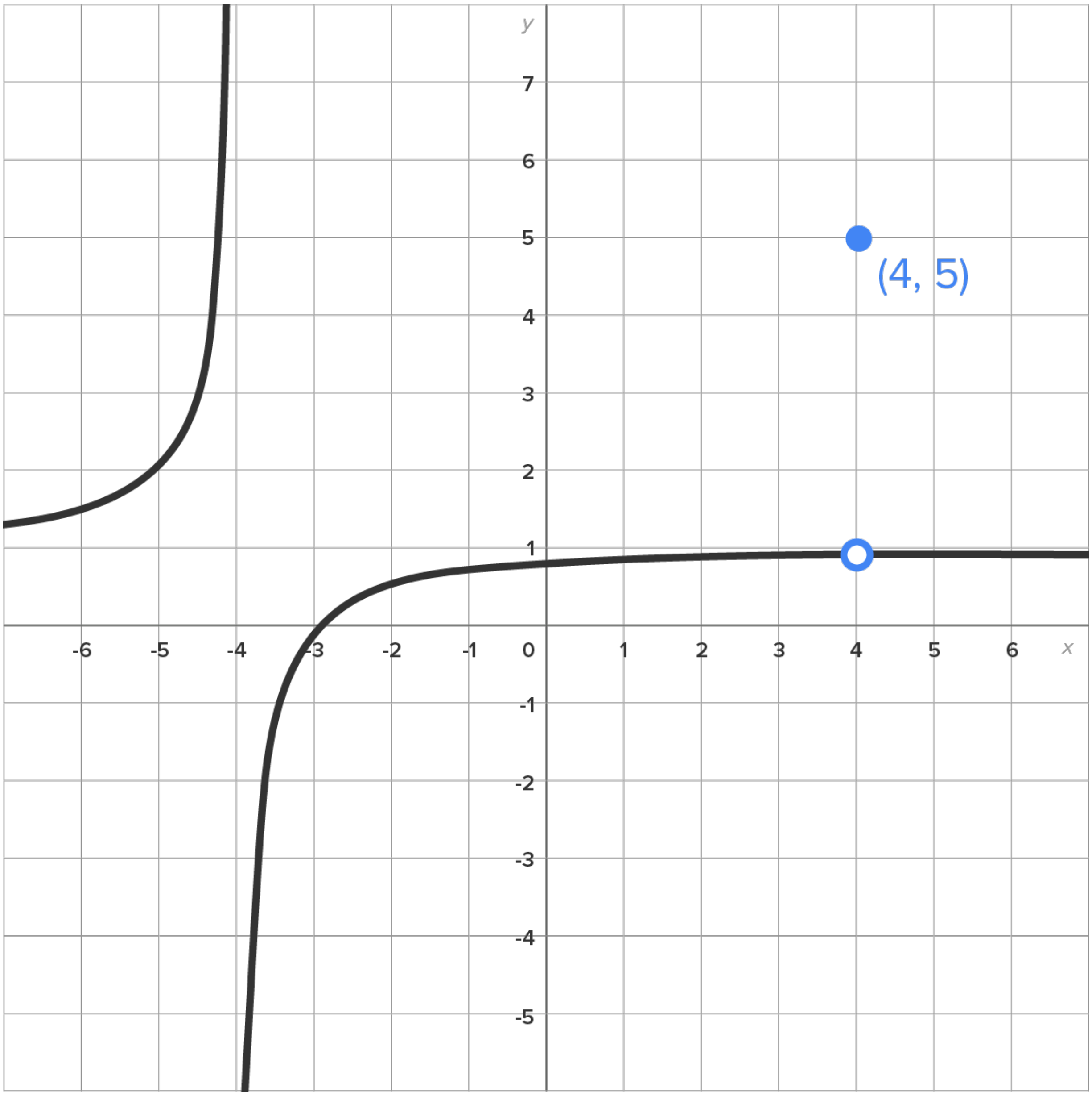


↪ EXAMPLE Consider the function $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x^2 - 16} & \text{if } x \neq 4 \\ 5 & \text{if } x = 4 \end{cases}$. Determine if $f(x)$ is continuous at $x = 4$.

First, evaluate $\lim\limits_{x \to 4} f(x)$. Since $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x^2 - 16} & \text{if } x \neq 4 \\ 5 & \text{if } x = 4 \end{cases}$ is defined on both sides of $x = 4$, there is no need to compute one-sided limits.

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - x - 12}{x^2 - 16} = \lim_{x \to 4} \frac{(x-4)(x+3)}{(x+4)(x-4)} = \lim_{x \to 4} \frac{(x+3)}{(x+4)} = \frac{7}{8}$$

However, $f(4) = 5$. Since the limit and the function value are different, this function is not continuous at $x = 4$. Here is a graph to help visualize this:



TRY IT

Consider the function: $f(x) = \begin{cases} 3x+4 & \text{if } x<1 \\ \sqrt{x+8} & \text{if } x \geq 1 \end{cases}$

Determine if $f(x)$ is continuous when $x=1$.

+

The function is not continuous. $\lim_{x \to 1} f(x)$ does not exist.

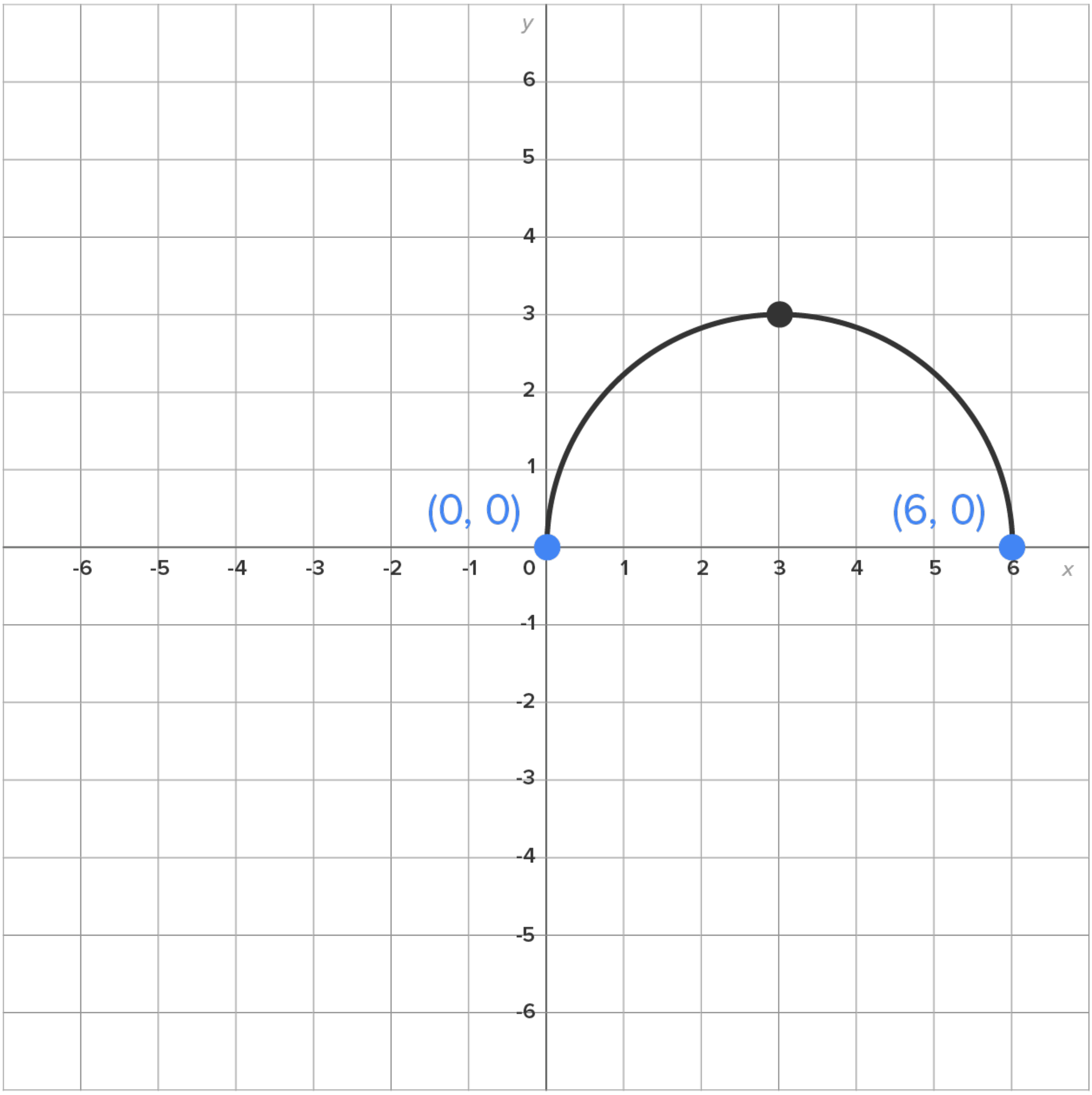
3. Determining Intervals Over Which a Function Is Continuous

For a function to be continuous on an interval of values, it has to be continuous at every point contained in the interval.

↪ EXAMPLE $f(x) = x^2 - 4x + 5$ is continuous at every real number. Thus, we say that $f(x)$ is continuous on the interval $(-\infty, \infty)$.

↪ EXAMPLE $f(x) = \frac{2}{x-1}$ is continuous at every value except $x = 1$. We can say that $f(x)$ is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$. This can also be written as $(-\infty, 1) \cup (1, \infty)$.

It is also possible to define continuity at an endpoint. For example, consider $f(x) = \sqrt{6x - x^2}$, whose graph is shown below. Note that the domain of this function is $[0, 6]$.



This means that defining continuity at $x = 0$ and $x = 6$ takes a bit more care.

Consider the endpoint $x = 0$. It can only be approached from the right. Looking at the graph, observe that $\lim_{x \to 0^+} f(x) = 0$ and $f(0) = 0$.

Consider the endpoint $x = 6$. It can only be approached from the left. Looking at the graph, observe that

$\lim_{x \to 6^-} f(x) = 0$ and $f(6) = 0$.

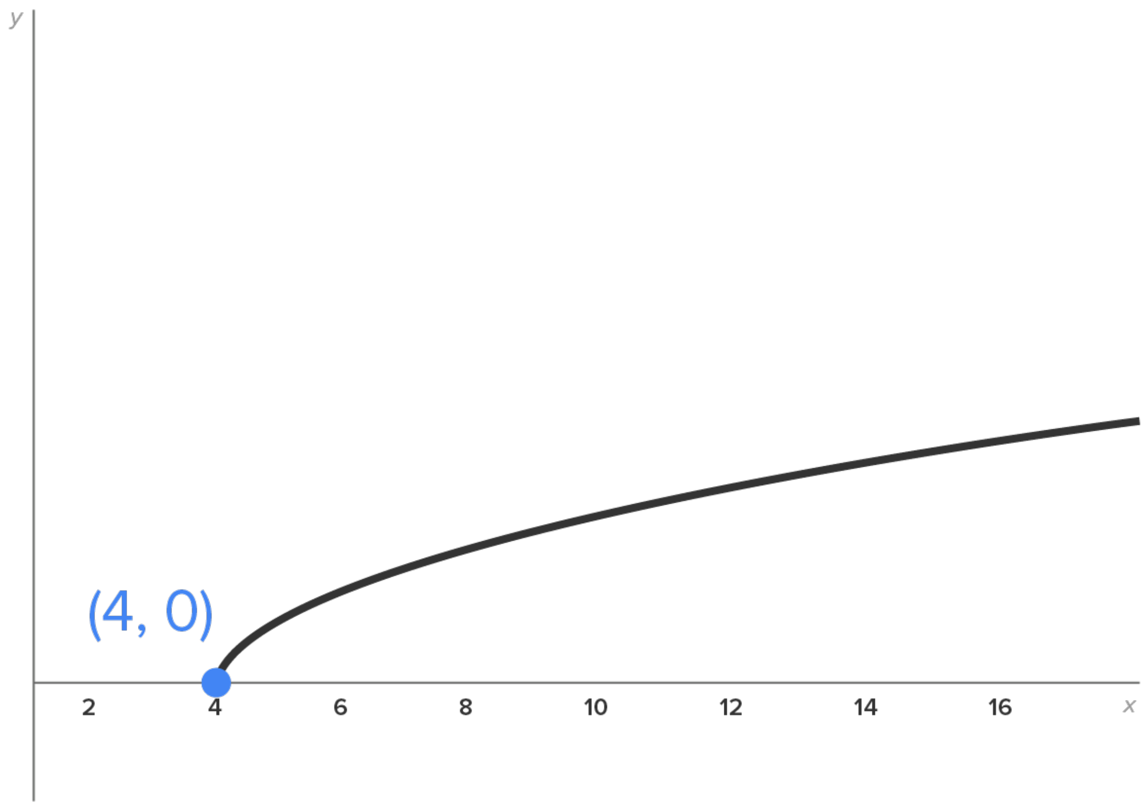
BIG IDEA

A function is **continuous from the left** at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

A function is **continuous from the right** at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

Thus, in the previous problem, we can say that $f(x)$ is continuous from the left at $x = 6$ and continuous from the right at $x = 0$. This enables us to say that $f(x)$ is continuous for all values on the interval $[0, 6]$.

→ EXAMPLE Determine the interval(s) over which $f(x) = \sqrt{x-4}$ is continuous. The graph is shown below.



Note that the domain of $f(x)$ is $[4, \infty)$. It follows that $f(x)$ is continuous on the interval $[4, \infty)$, noting that it is continuous from the right at $x = 4$.

TRY IT

Consider the following table:

Function	Continuous Interval
$f(x) = 3x - x^4$	?
$g(x) = \dfrac{x}{x+4}$	?

$h(x)=\sqrt{2x-1}$?

Determine the interval(s) over which each function is continuous. +

Function	Continuous Interval
$f(x)=3x-x^4$	$(-\infty, \infty)$
$g(x)=\frac{x}{x+4}$	$(-\infty, -4)\cup(-4, \infty)$
$h(x)=\sqrt{2x-1}$	$\left[\frac{1}{2}, \infty\right)$

TERMS TO KNOW

Continuous From the Left

A function is continuous from the left at $x=a$ if $\lim\limits_{x\to a^-} f(x)=f(a)$.

Continuous From the Right

A function is continuous from the right at $x=a$ if $\lim\limits_{x\to a^+} f(x)=f(a)$.

SUMMARY

In this lesson, you learned **the definition of continuity**, understanding that when given a graph, continuity is determined by locations where the graph has no breaks, jumps, or holes. A continuous function has no breaks in the graph; that is, $\lim\limits_{x\to a} f(x)=f(a)$. You learned that you can use limits to **determine if a function is continuous at** $x=a$ (a specific point) by comparing the values of $\lim\limits_{x\to a} f(x)$ and $f(a)$. It's important to note that while computing $f(a)$ is straightforward, computing $\lim\limits_{x\to a} f(x)$ requires more care, and sometimes requires one-sided limits. Lastly, you learned that by examining the domain of a function, you can use it to **determine the intervals over which a function is continuous**, noting that the function has to be continuous at every point contained in the interval in order to say the function is continuous on the interval.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 1 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Continuous From the Left

A function is continuous from the left at $x=a$ if $\lim\limits_{x\to a^-} f(x)=f(a)$.

Continuous From the Right

A function is continuous from the left at $x=a$ if $\lim\limits_{x\to a^+} f(x)=f(a)$.

Continuous Function

A function that has no breaks in the graph. That is, $\lim_{x \to a} f(x) = f(a)$.