The Average Value of a Continuous Function on a Closed Interval

by Sophia

WHAT'S COVERED

In this lesson, you will learn about the average value of a continuous function over an interval $[a, b]$. Recall that the average of a set of numbers is the sum of the numbers, divided by the number of numbers. This takes on a different meaning for continuous functions. Specifically, this lesson will cover:

1. The Idea Behind Average Value
2. Computing the Average Value of a Continuous Function

1. The Idea Behind Average Value

When finding the average of a set of numbers, you add up all the numbers, then divide by how many numbers there are.

EXAMPLE Given the numbers 81, 85, 89, and 71, the average of these four numbers is $\frac{81+85+89+71}{4}=81.5$.

Now consider a function $y=f(x)$ on some interval $[a, b]$. Break up the interval $[a, b]$ into n equal subintervals. Then, select a value of x from each subinterval. Call these values $x_1, x_2, \ldots, x_n$.

Then, the average of these values is $\frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}=\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{1}{n}\right]$.

The summation resembles a Riemann sum, but the Δx term is missing inside the summation. Recall that $\Delta x=\frac{b-a}{n}$.

We can multiply the summation by $\frac{b-a}{b-a}$ as follows:

$$\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{b-a}{b-a}\,\frac{1}{n}\right]$$
$$=\sum_{k=1}^{n}\left[f(x_k)\cdot\frac{b-a}{n}\,\frac{1}{b-a}\right]$$

We replace $\frac{b-a}{n}$ with Δx:

$$= \sum_{k=1}^{n}\left[f(x_k)\cdot\Delta x\cdot\frac{1}{b-a}\right]$$

Since $\frac{1}{b-a}$ is a constant, it can be factored out and written in front of the summation:

$$\frac{1}{b-a}\sum_{k=1}^{n}\left[f(x_k)\cdot\Delta x\right]$$

Recall that the summation $\sum_{k=1}^{n}\left[f(x_k)\cdot\Delta x\right]$ approaches the value of $\int_a^b f(x)dx$ as $n\to\infty$ as long as $f(x)$ is integrable on $[a, b]$. Since we are assuming $f(x)$ is continuous on $[a, b]$, $f(x)$ is also integrable on $[a, b]$. Note that the summation for the average value is the Riemann sum for $f(x)$ but multiplied by $\frac{1}{b-a}$.

This leads to an integral formula to find the average value of a continuous function $f(x)$ on an interval $[a, b]$.

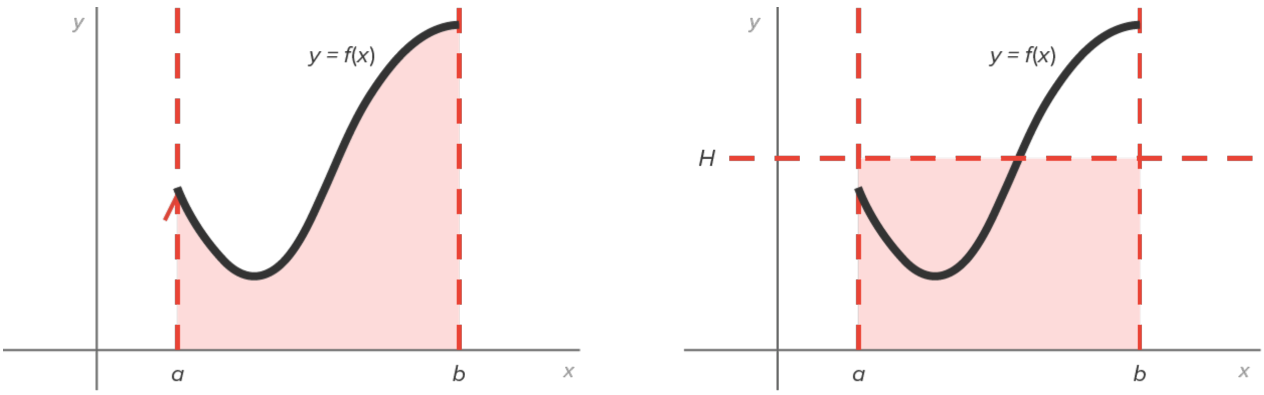
FORMULA

Average Value of a Function

If $f(x)$ is continuous on the closed interval $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

BIG IDEA

For a geometric interpretation of average value, let $H=$ the average value of a nonnegative function $f(x)$ on $[a, b]$. The figure below shows an illustration of this.



(1) (2)

- The graph in (1) is the region bounded by the graph of $f(x)$ and the x-axis on $[a, b]$.
- The graph in (2) is the rectangle with an area equal to $\int_a^b f(x)dx$. Note that the base is $b-a$, and its height is H, where H is the average value of $f(x)$ on $[a, b]$.

The area of the rectangle with height H and width $b-a$ is equal to the area of the region bounded by the graph of $f(x)$ and the x-axis on $[a, b]$.

---

2. Computing the Average Value of a Continuous Function

Now that we have a formula for average value, let's compute and interpret average values.

EXAMPLE Find the average value of $f(x)=\sin x$ on the interval $[0, \pi]$.

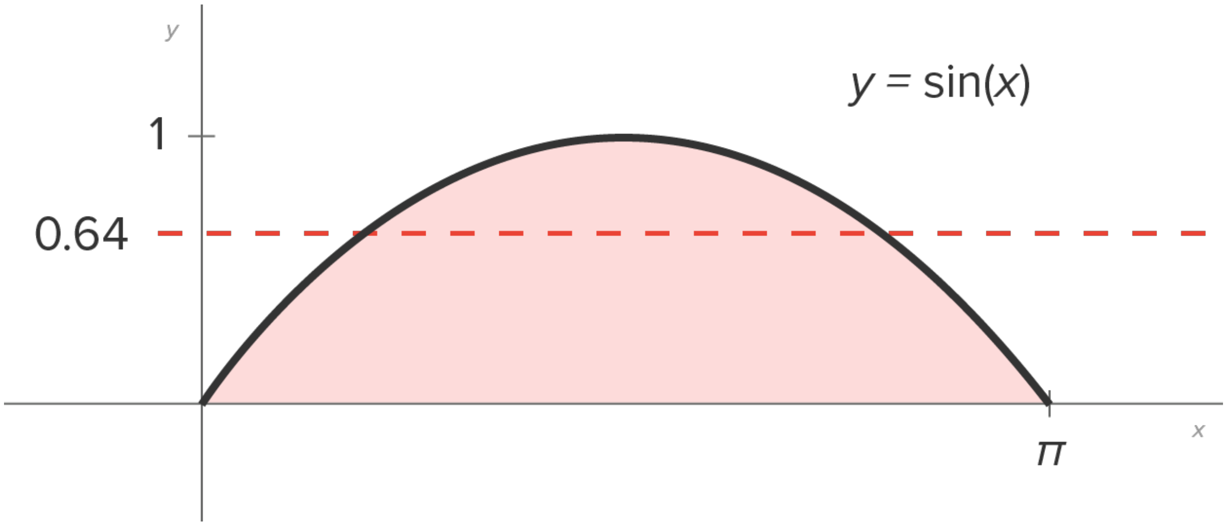
From the formula, this is equal to $\frac{1}{\pi-0}\int_0^\pi \sin x\,dx = \frac{1}{\pi}\int_0^\pi \sin x\,dx$.

Now, we evaluate the definite integral:

$\frac{1}{\pi}\int_0^\pi \sin x\,dx$	Start with the original expression.
$=\frac{1}{\pi}(-\cos x)\Big\vert_0^\pi$	Apply the fundamental theorem of calculus.
$=\frac{1}{\pi}(-\cos\pi)-\frac{1}{\pi}(-\cos 0)$	Substitute the upper and lower endpoints.
$=\frac{1}{\pi}+\frac{1}{\pi}$	Evaluate the parentheses.
$=\frac{2}{\pi}$	Simplify.

The average value of $f(x)=\sin x$ on the interval $[0, \pi]$ is equal to $\frac{2}{\pi}$.

To see the geometric interpretation, here is the graph of the region bounded by $f(x)=\sin x$ and the x-axis on the interval $[0, \pi]$ and the rectangle whose height is the average value and whose width is π.
Note: $\frac{2}{\pi}\approx 0.64$



WATCH

Find the average value of $f(x)=\frac{15x}{x^2+1}$ on the interval $[0, 5]$.

TRY IT

Consider the following table.

Function on a Given Interval	Average Value
$f(x)=x^2+2$ on the interval $[0, 4]$	?
$f(x)=\frac{4}{x^2}$ on the interval $[1, 2]$	?

Find the average value of each function on the given interval. +

Function on a Given Interval	Average Value
$f(x)=x^2+2$ on the interval $[0, 4]$	$\frac{22}{3}$
$f(x)=\frac{4}{x^2}$ on the interval $[1, 2]$	2

EXAMPLE During a 9-hour workday, the production rate at time t hours is $r(t)=5+\sqrt{t}$ cars per hour. What is the average hourly production rate?

We seek the average value of $r(t)$ over the interval $[0, 9]$.

$\text{Average Value}=\frac{1}{9-0}\int_0^9 (5+t^{1/2})dt$	Start with the original expression. Rewrite $\sqrt{t}=t^{1/2}$ to be able to use the power rule.
$=\frac{1}{9}\left(5t+\frac{2}{3}t^{3/2}\right)\Big\vert_0^9$	Apply the fundamental theorem of calculus.

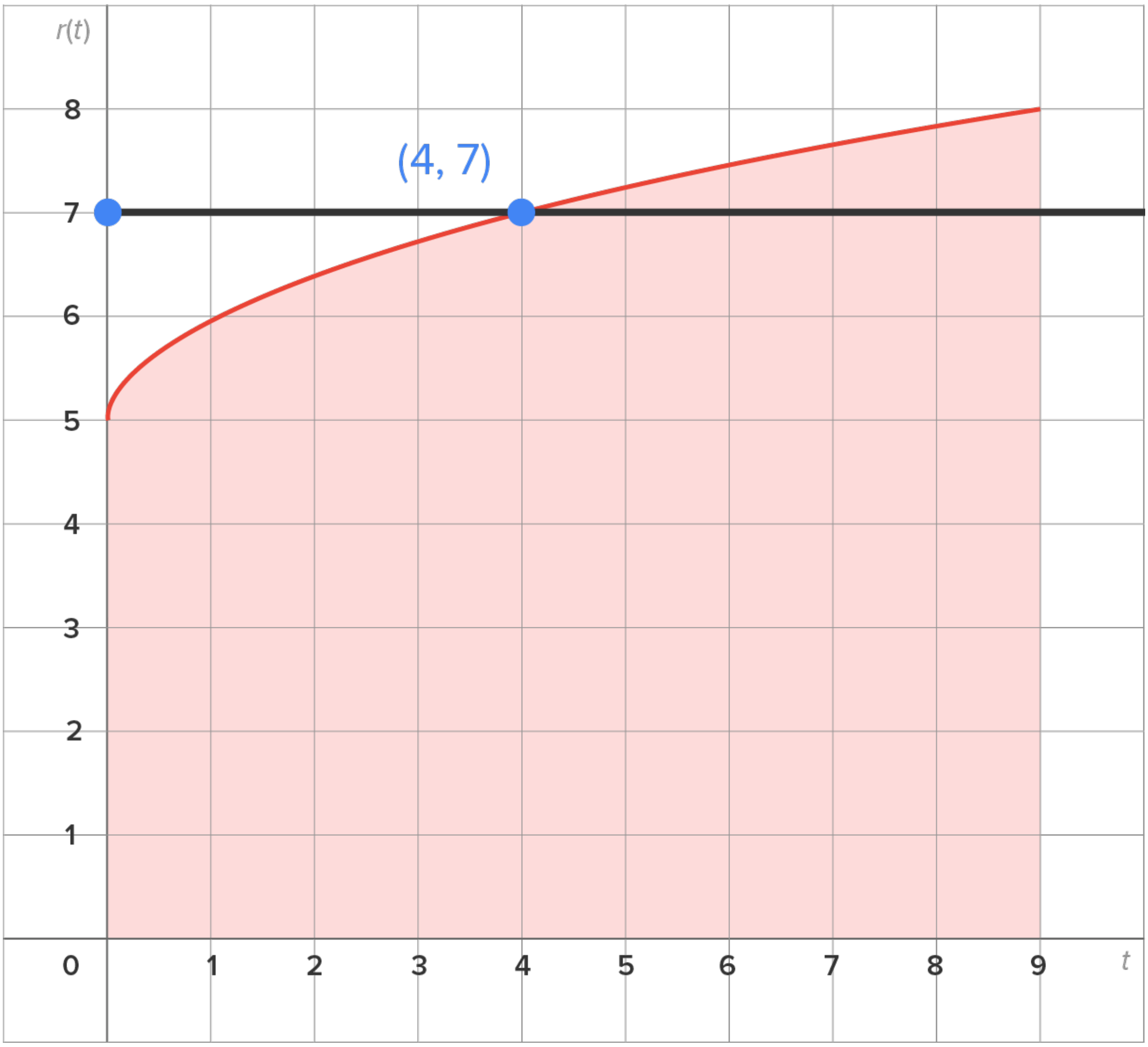
$$= \frac{1}{9}\left[5(9) + \frac{2}{3}(9)^{3/2}\right] - \frac{1}{9}\left[5(0) + \frac{2}{3}(0)^{3/2}\right]$$ Substitute the upper and lower endpoints.

$$= \frac{1}{9}(45 + 18) - \frac{1}{9}(0)$$ Evaluate.

$$= 7$$ Simplify.

The average rate of production is 7 cars per hour.

Shown in the figure is the region between $r(t) = 5 + \sqrt{t}$ and the t-axis, as well as the horizontal line $r(t) = 7$. Note that the area between $r(t)$ and the t-axis is equal to the area of the rectangle with the same base (9) and height 7 (the average value).



SUMMARY

In this lesson, you began by understanding **the idea behind average value**, following the path from the formula to find the average of a set of numbers to an integral formula to find the average value of a continuous function $f(x)$ on an interval $[a, b]$. You also learned how the fundamental theorem of calculus can be used to **compute the average value of a continuous function** $f(x)$ on an interval $[a, b]$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Average Value of a Function

If $f(x)$ is continuous on the closed interval $[a, b]$, then the average value of $f(x)$ on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

