The Chain Rule

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WHAT'S COVERED

In this lesson, you will learn how to find derivatives of general composite functions by using the chain rule. In a previous tutorial, you learned how to find derivatives of $y=(f(x))^n$, which is a composite function, but this doesn't cover all composite functions. For example, we are still unable to find the derivative of functions such as $f(x)=\sin(3x)$ and $y=\tan(x^2)$. In this tutorial, we'll learn the techniques necessary to find derivatives of said functions. Specifically, this lesson will cover:

1. Motivation for the Chain Rule
   a. Composite Functions
   b. Examining Rates of Change
2. Applying the Chain Rule
   a. Basic Functions
   b. Applying the Chain Rule Twice
   c. Combining the Chain Rule With Other Rules

1. Motivation for the Chain Rule

1a. Composite Functions

Recall that a composite function has the form $y=f(g(x))$. We call $g(x)$ the "inner function" since it is plugged into the function *f*.

To find derivatives of composite functions, it will be helpful to first identify the inner function. Given $y=f(g(x))$, let $u=g(x)$. Then $y=f(u)$, a less complicated function. Here are a few examples:

Function	Inner Function	Written in Terms of *u*
$y=\sqrt{2x^2+5x}$	$u=2x^2+5x$	$y=\sqrt{u}$
$y=\sin(3x)$	$u=3x$	$y=\sin u$
$y=\dfrac{3}{(2+\sin x)^4}$	$u=2+\sin x$	$y=\dfrac{3}{u^4}$

1b. Examining Rates of Change

The chain rule is a derivative technique that uses several rates of change in one problem. Here is a real-life consideration:

A factory can produce 30 units of a certain item per hour at a cost of $20 per item. What is the cost per hour of producing the items?

If you think the answer is $600, great! That is correct. But let's look at this more closely so that we can understand the rates of change.

Let x = the number of hours, u = the number of units produced, and C, the cost.

We can translate the given information into rates of change:

- The factory can produce 30 units per hour: Slope = $\frac{\Delta u}{\Delta x} = 30$
- The units cost $20 each to produce: Slope = $\frac{\Delta C}{\Delta u} = 20$

Then, the cost per hour is $\frac{\Delta C}{\Delta x}$, which can be written as $\frac{\Delta C}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = (20)(30) = \$600$ per hour.

Generally speaking, let's say that C is a function of u, and u is a function of x. Then, C is also a function of x, and $\frac{\Delta C}{\Delta x} = \frac{\Delta C}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$.

While not a proof, this idea can be extended to derivatives (instantaneous rates of change). That is, given C is a function of u and u is a function of x, $\frac{dC}{dx} = \frac{dC}{du} \cdot \frac{du}{dx}$. This derivative rule applies to composite functions and is called the chain rule.

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2. Applying the Chain Rule

2a. Basic Functions

The chain rule can be expressed with the following formula:

FORMULA

Chain Rule

Suppose $y = f(u)$, a composite function, where u is a function of x.

Then, $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$.

Using "prime" notation, we can write $y' = f'(u) \cdot u'$.

Using "D" notation, we can write $D[y] = f'(u) \cdot D[u]$.

BIG IDEA

With the chain rule in mind, we can write the derivative of each basic function.

- $D[u^n] = n \cdot u^{n-1} \cdot D[u]$
- $D[\sin u] = \cos u \cdot D[u]$
- $D[\cos u] = -\sin u \cdot D[u]$

- $D[\tan u] = \sec^2 u \cdot D[u]$
- $D[\cot u] = -\csc^2 u \cdot D[u]$
- $D[\sec u] = \sec u \tan u \cdot D[u]$
- $D[\csc u] = -\csc u \cot u \cdot D[u]$

HINT

$D[u^n] = n \cdot u^{n-1} \cdot D[u]$ is the power rule from **The General Power Rule for Functions**.

EXAMPLE Consider the function $y = \sin(3x)$. Find its derivative.

$y = \sin u$	Let $u = 3x$, the inner function.
$\frac{dy}{dx} = \cos u \cdot D[u]$	Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\frac{dy}{dx} = \cos u \cdot 3$	$D[u] = D[3x] = 3$
$\frac{dy}{dx} = 3\cos 3x$	Rewrite "3" in front and replace u with $3x$.

Thus, $\frac{dy}{dx} = 3\cos 3x$.

In challenge 3.2, you took derivatives of functions of the form $y = [f(x)]^n$, which is a specific form of the chain rule ($y = u^n$ where $u = f(x)$). Here is a reminder of a power function that requires the chain rule.

EXAMPLE Consider the function $y = \sqrt[3]{4x^2 + \sin x}$. Find its derivative.

$y = \sqrt[3]{u} = u^{1/3}$	Let $u = 4x^2 + \sin x$, the inner function.
$\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \cdot D[u]$	Apply the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\frac{dy}{dx} = \frac{1}{3} u^{-2/3} \cdot (8x + \cos x)$	$D[u] = D\left[4x^2 + \sin x\right] = 8x + \cos x$
$\frac{dy}{dx} = \frac{1}{3}(4x^2 + \sin x)^{-2/3}(8x + \cos x)$	Replace u with $4x^2 + \sin x$.
$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{(4x^2 + \sin x)^{2/3}}(8x + \cos x)$	Rewrite with positive exponents.
$\frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}}$	Combine fractions.

Thus, $\frac{dy}{dx} = \frac{8x + \cos x}{3(4x^2 + \sin x)^{2/3}}$.

EXAMPLE Consider the function $y = \tan(x^2 + 1)$. Find its derivative.

$y = \tan u$	Let $u = x^2 + 1$, the inner function.

$\frac{dy}{dx} = \sec^2 u \cdot D[u]$ Apply the chain rule: $\frac{dy}{dx} = f'(u) \cdot D[u]$

$\frac{dy}{dx} = (\sec^2 u) \cdot 2x$ $D[u] = D[x^2 + 1] = 2x$ (parentheses added to show separation)

$\frac{dy}{dx} = 2x \cdot \sec^2(x^2 + 1)$ Rewrite "2x" in front and replace u with $x^2 + 1$.

Thus, $\frac{dy}{dx} = 2x \cdot \sec^2(x^2 + 1)$.

TRY IT

Consider the function $y = \cos(2x^3)$.

Find its derivative. +

$\frac{dy}{dx} = -6x^2 \sin(2x^3)$

WATCH

Here is a video in which we find the derivative of $y = \sec(3x + 1)$.

2b. Applying the Chain Rule Twice

When the inner function is itself a composite function, the chain rule is applied more than once. Rather than assign a letter to each inside function, we'll present another way to organize this.

EXAMPLE Consider the function $y = \sin^2 3x$. Find its derivative.

$y = (\sin(3x))^2$ Rewrite as a quantity squared.

$\frac{dy}{dx} = 2(\sin(3x)) \cdot D[\sin(3x)]$ Apply the chain rule: $D[u^2] = 2u \cdot u'$

$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot D[3x]$ Apply the chain rule again: $D[\sin u] = \cos u \cdot u'$

$\frac{dy}{dx} = 2(\sin 3x) \cdot \cos 3x \cdot 3$ $D[3x] = 3$

$\frac{dy}{dx} = 6 \sin 3x \cos 3x$ $2 \cdot 3 = 6$; omit unnecessary parentheses.

Thus, $\frac{dy}{dx} = 6 \sin 3x \cos 3x$.

TRY IT

Consider the function $y = \tan^4 2x$.

Find its derivative. +

$$\frac{dy}{dx} = 8\tan^3 2x \sec^2 2x$$

2c. Combining the Chain Rule With Other Rules

Sometimes sums, differences, products, and quotients contain composite functions. The key is to approach the derivative carefully.

EXAMPLE Consider the function $y = x\sin 4x$. Find its derivative.

$y = x\sin 4x$	Start with the given function.
$\frac{dy}{dx} = D[x]\cdot\sin 4x + x\cdot D[\sin 4x]$	Apply the product rule.
$\frac{dy}{dx} = 1\cdot\sin 4x + x\cdot\cos 4x\cdot D[4x]$	$D[x] = 1$, $D[\sin u] = \cos u\cdot D[u]$
$\frac{dy}{dx} = 1\cdot\sin 4x + x\cdot\cos 4x\cdot(4)$	$D[4x] = 4$
$\frac{dy}{dx} = \sin 4x + 4x\cos 4x$	Omit unnecessary grouping symbols.

Thus, $\frac{dy}{dx} = \sin 4x + 4x\cos 4x$.

SUMMARY

In this lesson, you began by understanding the **motivation for the chain rule**, a derivative technique used to compute derivatives of **composite functions**, namely that it expands your ability to find **rates of change**. You learned how to **apply the chain rule**, exploring the formula and the derivative of each **basic function**. You also learned that when the inner function is itself a composite function, you need to **apply the chain rule twice**. Lastly, you examined an example of finding a derivative by **combining the chain rule with other rules**, such as the product rule.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Chain Rule

Suppose $y = f(u)$, a composite function, where u is a function of x.

Then, $\frac{dy}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$.

Using “prime” notation, we can write $\frac{dy}{dx} = f'(u)\cdot u'$.

Using “D” notation, we can write $\frac{dy}{dx} = f'(u) \cdot D[u]$.