Derivatives of Natural Logarithmic Functions

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WHAT'S COVERED

In this lesson, you will learn how to differentiate logarithmic functions. Recall that a logarithmic function is the inverse of an exponential function. Thus, in any situation in which the rate of change of an exponential function is desired, it makes sense to also discuss the rates of change of logarithmic functions. Specifically, this lesson will cover:

1. The Derivative of $f(x) = \ln x$ and Functions Involving $\ln x$
2. The Derivative of $f(u) = \ln u$ and Functions Involving $\ln u$, Where u Is a Function of x
3. Using Properties of Logarithms Before Differentiating

1. The Derivative of $f(x) = \ln x$ and Functions Involving $\ln x$

WATCH

Please view this video to see how to derive a formula for the derivative of $f(x) = \ln x$.

Video Transcription

[MUSIC PLAYING] Hello and welcome back. What we're going to do with this video is another derivation of a derivative rule. We're going to focus on the function natural log of x. And, again, using known properties, we'll be able to come up with a formula for the derivative.

So one other known fact, we used it in the last video as well, is that e to the natural log of x is equal to x. So those are two functions that are equivalent to each other. So we can deduce then that the derivative of each function is the same.

So you're probably wondering at this point, how does that relate to what we're trying to figure out here? Well, let's keep taking a look here. So the derivative of e to the something, I'm going to write that off to the side. The derivative of e to the u is, remember, itself times the derivative of the inside.

So the derivative of e to the natural log of x is e to the natural log of x times the derivative of the natural log of x. Now, that is the thing we're trying to find. We're trying to figure out what the derivative of natural log of x is. So how are we going to get there? Stay tuned.

On the right-hand side, the derivative of x is just 1. Now, remember from our relationship up here, e to the natural log of x is equivalent or is equal to x. So I'm going to replace e to the natural log of x with x, because that is much simpler.

So we have x times the derivative of natural log of x equals 1. And I'm gonna divide both sides by x, because that allows me to solve for the derivative expression. And this is why the derivative of the natural log of x is 1 over x, a very simple derivative formula. So we're going to use this to differentiate functions dealing with natural logarithms.

So, we can say the derivative of the natural log function can be expressed with the following formula:

FORMULA

Derivative of the Natural Logarithmic Function

$$D[\ln x] = \frac{1}{x}$$

With this new derivative rule, let's compute a few derivatives.

EXAMPLE Consider the function $f(x) = x^2 \ln x$.

$f(x) = x^2 \ln x$	Start with the original function.
$f'(x) = D[x^2] \cdot \ln x + x^2 \cdot D[\ln x]$	Use the product rule.
$f'(x) = 2x \cdot \ln x + x^2 \cdot \frac{1}{x}$	$D[x^2] = 2x$, $D[\ln x] = \frac{1}{x}$
$f'(x) = 2x \ln x + x$	Simplify $x^2 \cdot \frac{1}{x} = x$ and remove extra symbols.

Thus, $f'(x) = 2x \ln x + x$.

TRY IT

Consider the function $f(x) = \frac{\ln x}{x}$.

Find its derivative. +

$$f'(x) = \frac{1 - \ln x}{x^2}$$

HINT

Similar to trigonometric functions, powers of natural logarithmic functions are sometimes written with the power after the "ln". For example, $\ln^4 x$ means $(\ln x)^4$.

EXAMPLE Consider the function $f(x)=\ln^3 x$. Find its derivative.

$f(x)=\ln^3 x=(\ln x)^3$	Start with the original function. Rewrite in a more recognizable form.
$f'(x)=3(\ln x)^2\cdot\frac{1}{x}$	$D[u^3]=3u^2\cdot u'$ (Apply the chain rule.)
$f'(x)=\frac{3(\ln x)^2}{x}$	Combine as a single fraction.

Thus, $f'(x)=\frac{3(\ln x)^2}{x}$. It is also acceptable to write $f'(x)=\frac{3\ln^2 x}{x}$.

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2. The Derivative of $f(u)=\ln u$ and Functions Involving $\ln u$, Where u Is a Function of x

In step with the chain rule, and the fact that $D[\ln x]=\frac{1}{x}$, we have the following rule for the derivative of $\ln u$:

FORMULA

Derivative of $\ln u$, Where u Is a Function of x

$$D[\ln u]=\frac{1}{u}\cdot u'$$

EXAMPLE Consider the function $f(x)=\ln(x^2+1)$. Find its derivative.

$f(x)=\ln(x^2+1)$	Start with the original function.
$f'(x)=\frac{1}{x^2+1}\cdot 2x$	$D[\ln u]=\frac{1}{u}\cdot u'$
$f'(x)=\frac{2x}{x^2+1}$	Rewrite as a single fraction.

Thus, $f'(x)=\frac{2x}{x^2+1}$.

HINT

The rule $D[\ln u]=\frac{1}{u}\cdot u'$ can also be written as $D[\ln u]=\frac{u'}{u}$.

EXAMPLE Consider the function $f(x)=\ln(\cos x)$. Find its derivative.

$f(x)=\ln(\cos x)$	Start with the original function.
$f'(x)=\frac{-\sin x}{\cos x}$	$D[\ln u]=\frac{u'}{u}$

$f'(x) = -\tan x$ Use this trigonometric identity: $\frac{\sin x}{\cos x} = \tan x$

Thus, $f'(x) = -\tan x$.

TRY IT

Consider the function $f(x) = \ln(2 + \sin x)$.

Find its derivative. +

$$f'(x) = \frac{\cos x}{2 + \sin x}$$

WATCH

The video below illustrates how to find the derivative of $f(x) = x \cdot \ln(x^3 + 1)$, which requires a combination of the product and chain rules.

Video Transcription

[MUSIC PLAYING] Hello, and welcome back to more derivatives with natural algorithms. In this video, we're going to find the derivative of a function that combines a few rules. As you can see here, we have x times the natural log of x to the third plus 1. So that's going to involve a product rule. So just to remember what the products rule is, the derivative of-- I guess we'll call it u times v-- is the derivative of the first times the second plus the first times the derivative of the second.

So what we have here is our first and our second. So the derivative is the derivative of x, which is 1-- so I'm just going to actually write that as the derivative of x-- times the natural log of x to the third plus 1 plus the first times the derivative of the natural log of x to the third plus 1.

So now looking at each derivative, the derivative of x-- that's a nice, simple one-- that's 1. So you have 1 times natural log of x to the third plus 1. And then we have x times-- now, the derivative of natural log of something-- I'm going to write that up here. Derivative of natural log of u, is 1 over u times u prime or u prime over u, as we have used a couple of times.

So derivative of natural log of x to the third plus 1. So we know it's something over x to the third plus 1. And the derivative of x to the third plus 1 is 3x squared. So there's the derivative of that. And now, we'll just clean things up a bit here. f prime of x is equal to the natural log of x to the third plus 1 plus-- now, remember, to pull these two expressions here together, this is really over 1.

So we have x to the third-- I'm sorry-- 3x to the third over x to the third plus 1. And that is the derivative of our function. So combining the product rule and the chain rule.

3. Using Properties of Logarithms Before Differentiating

↱ EXAMPLE Consider the function $f(x)=\ln(x\cdot e^{-2x})$. Find the derivative of this function.

$f'(x)=\frac{1}{x\cdot e^{-2x}}\cdot D[x\cdot e^{-2x}]$	$D[\ln u]=\frac{1}{u}\cdot u'$
$f'(x)=\frac{1}{x\cdot e^{-2x}}\cdot[D[x]\cdot e^{-2x}+x\cdot D[e^{-2x}]]$	Use the product rule.
$f'(x)=\frac{1}{x\cdot e^{-2x}}\cdot[1\cdot e^{-2x}+x\cdot e^{-2x}(-2)]$	$D[x]=1$, $D[e^u]=e^u\cdot u'$
$f'(x)=\frac{1}{x\cdot e^{-2x}}\cdot[e^{-2x}-2x\cdot e^{-2x}]$	Simplify and remove unnecessary symbols.
$f'(x)=\frac{e^{-2x}}{x\cdot e^{-2x}}-\frac{2xe^{-2x}}{x\cdot e^{-2x}}$	Distribute.
$f'(x)=\frac{1}{x}-2$	Remove the common factors.

Thus, $f'(x)=\frac{1}{x}-2$.

This process was quite cumbersome. However, if we use the properties of logarithms that we reviewed in Unit 1, this can be made simpler.

FORMULA

Product Property
$\ln(ab)=\ln a+\ln b$

Quotient Property
$\ln\left(\frac{a}{b}\right)=\ln a-\ln b$

Power Property
$\ln(a^b)=b\cdot\ln a$

↱ EXAMPLE Consider the function $f(x)=\ln(x\cdot e^{-2x})$. Find its derivative by first using logarithm properties.

Since, $\ln(x\cdot e^{-2x})$ is the logarithm of a product, use properties of logarithms to rewrite:

$f(x)=\ln(x\cdot e^{-2x})$	Start with the original function.
$f(x)=\ln x+\ln(e^{-2x})$	$\ln(ab)=\ln a+\ln b$
$f(x)=\ln x+(-2x)\ln e$	$\ln(a^b)=b\cdot\ln a$
$f(x)=\ln x-2x$	$\ln e=1,\ -2x(1)=-2x$

So, in expanded (and simpler) form, $f(x)=\ln x-2x$.

Then, $f'(x) = D[\ln x] - D[2x] = \frac{1}{x} - 2$.

HINT

To find the derivative of $\ln u$, where u is a product, quotient, or power (or any combination of them), use logarithm properties before finding the derivative. This results in simpler derivatives.

WATCH

In this video, we'll use properties of logarithms to find the derivative of $f(x) = \ln\left(\frac{x}{\sqrt{2x+1}}\right)$.

Video Transcription

Hello. I hope all is well in your world. What we're going to do is, in this video, look at the derivative of a natural log function where properties can be used to find the derivative. And this is a prime example, no pun intended, of a function where you don't want to really deal with the function as it's written. Because remember, the derivative of natural log-- well, let's review that.

The derivative of the natural log of u is 1 over u times the derivative of u. I really don't want to deal with the derivative of x divided by the square root of 2x plus 1 because that's going to involve a quotient rule. That's quite complicated. There's the quotient rule. There's a chain rule. There's a whole lot going on there.

So one thing we can do if our goal is just to find a derivative expression is to rewrite the logarithm using properties. So the first thing to remember is that we have the natural logarithm of a quotient. So remember that is the natural log of the numerator minus the natural log of the denominator.

So already that's looking better because we can handle the natural log of x no problem. Natural log of the square root of 2x plus 1, that might be a little challenging. But remember, square roots can be written as powers. So that's really the natural log of 2x plus 1 to the 1/2. And remember another logarithm property is that the power, the 1/2 power, can be brought down in front of the natural logarithm.

And that is as much as this simplify. So remember, this is all just y. We have not taken a derivative yet. So now our goal is to find the derivative of this function. But now, so we have dy dx equals the derivative of the natural log of x, which is 1 over x, minus-- now that's 1/2 times a natural log. So that means the constant multiple rule is in effect here. It's going to be 1/2 times whatever the derivative of natural log of 2x plus 1 is.

And according to our formula up here, the derivative of natural log of something is 1 over the something times the derivative of the something. So this is 1 over the something, 1 over 2x plus 1, times the derivative of 2x plus 1, which is 2. So this 2 comes from the derivative of 2x plus 1. Now all we have to do is simplify, and then we are done.

So dy dx is equal 2. Also the 1 over x is out there on its own. If we look here, we have a 1/2 times 1 over 2x plus 1 times a 2. I know that 1/2 times 2 is 1, so those cancel. And we're left with 1 over x minus 1 over 2x plus 1. And that is a perfectly simplified expression for the derivative.

And there we have it. Using logarithm properties makes derivatives much simpler, at least when we have a product, quotient, or power inside of the natural algorithm.

SUMMARY

In this lesson, you learned how to find **the derivative of a natural logarithmic function** (represented by $f(x) = \ln x$) and, given the chain rule, **the derivative of** $f(u) = \ln u$. These are the latest additions to our library of derivatives, and you've seen through examples and videos the different way the natural logarithmic function can be combined with other functions. You also learned that since logarithms have special properties, it is more advantageous to **use properties of logarithms before differentiating** functions that involve products, quotients, and powers.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

FORMULAS TO KNOW

Derivative of lnu, Where u Is a Function of x

$$D[\ln u] = \frac{1}{u} \cdot u'$$

Derivative of the Natural Logarithmic Function

$$D[\ln x] = \frac{1}{x}$$

Power Property

$$\ln(a^b) = b \cdot \ln a$$

Product Property

$$\ln(ab) = \ln a + \ln b$$

Quotient Property

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$