Related Rates Problems Using Proportional Reasoning and Trigonometry

by Sophia

WHAT'S COVERED

In this lesson, you will continue to explore related rates problems that involve proportional reason (such as similar triangles) and trigonometry. For example, we may want to determine how an angle of inclination changes for a camera that is following a rocket that is taking off. Specifically, this lesson will cover:

1. Related Rates Problems Involving Proportions
2. Related Rates Problems Involving Trigonometry

1. Related Rates Problems Involving Proportions

We will now look at problems where we need to use proportional reasoning (similar triangles, etc.) to get a relationship between the variables.

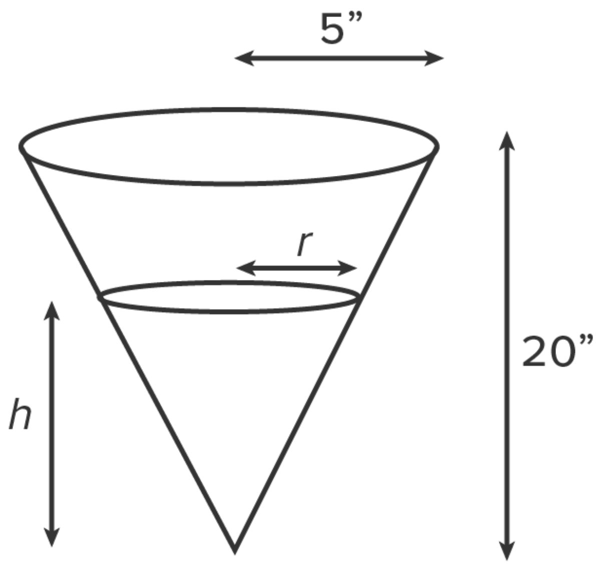
→ EXAMPLE Water is filling a cone-shaped container at a rate of $30\pi \text{ in}^3/\text{min}$. The container is 10" wide at the top and 20" deep. At what rate is the height of the water changing when the water is 10 inches deep?

This information means we are given $\frac{dV}{dt} = 30\pi$, and we want $\frac{dh}{dt}$ when $h = 10$.

From the geometry formulas, we know that $V = \frac{\pi}{3}r^2h$, where:

- r = the radius of the water in the cone
- h = the height of the water in the cone

Notice that there isn't any information given about the radius of the conical shape. However, we do have the information we need to solve this problem.



Since the cone is 10” wide across at the top, its radius is 5”.

Notice that the water in the cone forms a smaller version of the conical container, which means that the height and radius are in proportion to each other. In the full cone, the height is 4 times the radius. To figure out the relationship, we know that $\frac{h}{r}=\frac{20}{5}$.

Now we have to decide whether to solve for *h* or *r*.

Since there is no information given about the radius in this problem, we want to replace *r* in the formula. This means we want to solve for *r*. Solving $\frac{h}{r}=\frac{20}{5}$ gives $5h=20r$, or $r=\frac{1}{4}h$.

Now, we can write a volume formula in terms of only *h*.

$$V=\frac{\pi}{3}r^2h=\frac{\pi}{3}\left(\frac{h}{4}\right)^2h=\frac{\pi}{3}\cdot\frac{h^2}{16}\cdot h=\frac{\pi}{48}h^3$$
$$V=\frac{\pi}{48}h^3$$

Since rates are involved, find the derivative with respect to time:

$V=\frac{\pi}{48}h^3$	Start with the original equation.
$\frac{dV}{dt}=\frac{\pi}{48}(3h^2)\cdot\frac{dh}{dt}$	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.
$\frac{dV}{dt}=\frac{\pi}{16}h^2\cdot\frac{dh}{dt}$	Simplify.
$30\pi=\frac{\pi}{16}(10)^2\cdot\frac{dh}{dt}$	$\frac{dV}{dt}=30\pi$, $h=10$, $\frac{dh}{dt}=?$

$$30\pi = \frac{25\pi}{4} \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = 30\pi \cdot \frac{4}{25\pi} = \frac{24}{5}\ \text{in/min}$$

In conclusion, the height is increasing at a rate of 4.8 inches per minute when the water is 10 inches deep.

WATCH

This is an example of related rates of distance from a flagpole.

Video Transcription

Welcome. In this video, we're going to look at an example of a related rate question. Here, a woman walks away from a flagpole at the rate of 4 feet per second. If the flagpole is 25 feet tall, at what rate is she walking away from the top of the flagpole when she is 15 feet from its base.

So what I've done is drawn a picture to give this scenario, and we need to put some variables in here. We are assuming that it's on level ground and that the flagpole is set perpendicular to the ground, and so that gives us the right triangle. The flagpole is 25 feet tall. So I've done the vertical part of our right triangle as 25 feet.

And so the ground is the horizontal component of the right triangle, and she is walking away from the flagpole. So her distance from the flagpole we're going to call x. And we're measuring this distance that she's away from the ground to the top of the flagpole. So from her feet up is the distance that we're talking about.

Now, we also need a variable for the hypotenuse. We'll just call this s. So when we look at the information, it's important, with the related rates, to identify the information that's given and identify the information that you want to find, and also notice any sort of geometrical or other sorts of formulas that might help you.

So it says that she is walking away from the flagpole at the rate of 4 feet per second. And so, since that's her distance from the flagpole and it's a rate-- it's the change of distance with respect to the change in time, we have that dx dt-- the change in her X distance away from the flagpole with respect to the change in time-- is that 4 feet per second.

Now, we have that the flagpole is 25 feet tall, and it says, at what rate is she walking away from the top of the flagpole? So it's asking me for the rate that she's walking away from the top of the flagpole. So that is from the top of the flagpole down to her feet. So that is the rate of change of s.

So ds dt is what we are looking for. And we're looking for that when she is 15 feet from the base. So that's when x is equal to 15 feet.

So now, what geometrical formula can we use here? Well, it's a right triangle. And when we are talking about the relationships between the sides of the right triangle, that's Pythagorean theorem. So we have legs squared-- 25 squared plus x squared, the other leg squared, is equal to the hypotenuse, s squared.

Now, 25 squared is 625, plus x squared is equal 2x squared. And we can go through and find our derivative now with implicit differentiation, and we are looking at it where x and s are both dependent variables on the independent variable of time t. And we know that because of the way our derivatives are set up. The dx dt-- x is dependent on t, and the dt s is dependent on t.

So the derivative of the constant term, 625, is 0, plus the derivative of x squared is 2x for the derivative rule, but x is dependent on t. So I have to multiply by dx dt by the chain rule-- is equal to the derivative of s squared is 2s but s is dependent on dt so by the chain rule, I have to multiply by ds dt.

Now, when we look at what we are given and what we want to know, I have a value for x. It's 15. I have a value for dx dt. That's that 4 feet per second. I don't yet have a value that I've calculated for s, and I want to know ds dt.

So at first glance, it looks like maybe we don't have enough information because we don't have our s, but we have to remember that when x is 15, we have that right triangle where I have a 25-foot vertical on my right triangle. When x is 15, the horizontal leg of the right triangle is 15. So I can use the Pythagorean theorem-- 25 squared-- plus 15 squared is equal to s squared-- to find the value of s at that time.

So I have 625 plus 225 is equal to s squared. That gives me 850 is equal to s squared. So s is equal to the square root of 850 when x is 15. And your flagpole stays at its 25 feet, which-- as a simplified radical, s is equal to 5 times the square root of 34.

So now we have all but one thing missing. So we can plug in our values and solve for that. So I have 2 times-- take out the x and put in my 15 feet-- times-- take out the dx dt and put in 4 feet per second-- is equal to 2 times s. Take out the s, and put in 5 square roots of 34 feet, and then times that ds dt.

So simplifying the left-hand side, I have 2 times my 15 times-- feet times my 4. That gives me a value of 120 square feet per second, just bringing our units along, and we will see how they work out as we go through the rest of the calculation. And then I have 2 times 5 times the square root of 34 feet times ds dt. So that's 10 times the square root of 34 feet times ds dt.

And then solving for ds dt, we want to divide both sides by the 10 times the square root of 34 feet. So I have 1 over 10 times the square root of 34 feet times my 120 feet squared over seconds is equal to the ds dt, and my feet-- no unit in the denominator. We'll remove one of those in the numerator. And then, simplifying the values numerically, I have 12 over the square root of 34 feet per second is what we get for our ds dt. And so the units work out to be what they should be because it's in the change in distance with respect to the change in time.

Now, there's a couple other things we can do with this exact answer. First thing we can do is rationalize. And if you rationalize and simplify it, you get ds dt is equal to 6 times the square root of 34 over 17. And if you approximate that, it's approximately 2.058 feet per second. And that is the rate that she is walking away from the top of the flagpole when she is 15 feet from its base.

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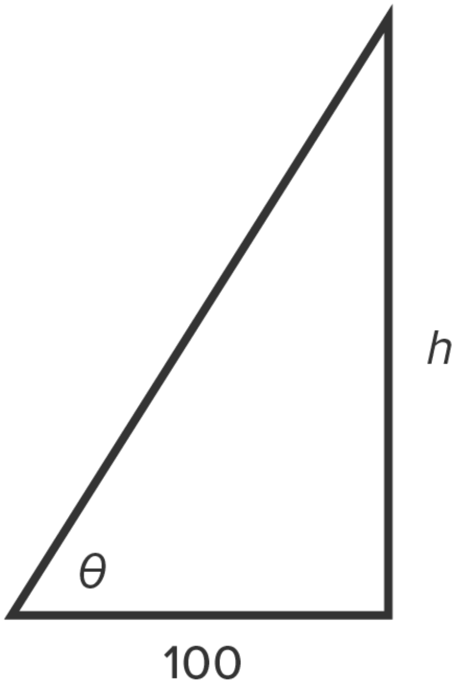
2. Related Rates Problems Involving

Trigonometry

Related rates can also help answer questions about angles of inclination.

A camera is on ground level 100 feet from a rocket's launchpad and inclines upward as a rocket takes off vertically at a rate of 250 ft/s. At what rate is the angle of inclination changing when the rocket is 1000 feet off the ground?



Notice that the base is always 100 feet. This is not changing. Since the height of the rocket changes, the vertical side is a variable. Since the angle is changing, it is labeled with a variable (θ) as well.

To relate all the relevant quantities, we need to use a trigonometric function. Since the opposite and adjacent sides to angle θ are labeled, tangent is the best choice.

The equation is $\tan\theta = \frac{h}{100}$. Solving for θ, we have $\theta = \tan^{-1}\left(\frac{h}{100}\right)$.

We were given $\frac{dh}{dt} = 250$ and we want to find $\frac{d\theta}{dt}$ when $h = 1000$ feet.

Now, we take the derivative:

$\theta = \tan^{-1}\left(\frac{h}{100}\right)$	Start with the original equation.
	Take the derivative of both sides, remembering that each variable is being differentiated implicitly.

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{h}{100}\right)^2} \cdot \frac{d}{dt}\left[\frac{h}{100}\right]$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{h}{100}\right)^2} \cdot \frac{1}{100} \cdot \frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{1000}{100}\right)^2} \cdot \frac{1}{100} \cdot 250$$ Substitute quantities.

$$\frac{d\theta}{dt} = \frac{1}{101} \cdot \frac{1}{100} \cdot 250 \approx 0.02475 \text{ radians/sec}$$ Simplify.

The angle is increasing at a rate of 0.02475 radians per second. For reference, this is about 1.42°/second.

SUMMARY

In this lesson, you learned how to solve **related rates problems involving proportions and trigonometry** (which can help answer questions about angles of inclination). In these problems, equations were derived using mathematical facts more so than a standard formula. Once the equation is determined, the procedure for finding the related rates is exactly the same: take the derivative, substitute what it is known, then solve for the unknown.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 2 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

