Limits with Variable Bases and Exponents

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WHAT'S COVERED

In this lesson, you will learn strategies to evaluate indeterminate forms that have both variable bases and exponents. Specifically, this lesson will cover:

1. The Strategy for Evaluating Limits With Variable Bases and Exponents
2. Evaluating Limits With Variable Bases and Exponents

1. The Strategy for Evaluating Limits With Variable Bases and Exponents

Consider a function that has the form $y = f(x)^{g(x)}$. Of all the possible behaviors of $f(x)$ and $g(x)$ that could occur in a limit, there are three situations that lead to indeterminate forms.

Form	Explanation
0^0	The base and exponent both approach 0.
∞^0	The base grows without bound and at the same time, the exponent approaches 0.
1^∞	When the base approaches 1 and at the same time, the exponent increases without bound.

Since L'Hopital's rule can only be applied to limits with indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, limits with the indeterminate forms 0^0, ∞^0, or 1^∞ will need to be manipulated in order to use L'Hopital's rule.

To see how to start, consider the identity $a = e^{\ln a}$, which is valid as long as $a > 0$.

Replacing a with $f(x)^{g(x)}$, we can write $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}}$.

By the property of logarithms, we know that $\ln(f(x)^{g(x)}) = g(x) \cdot \ln f(x)$, which allows us to write $f(x)^{g(x)} = e^{g(x) \cdot \ln f(x)}$.

This also means that $\lim\limits_{x \to a} f(x)^{g(x)} = \lim\limits_{x \to a} e^{g(x) \cdot \ln f(x)}$.

The limit on the right-hand side suggests that we can focus on the exponent $g(x) \cdot \ln f(x)$, which is a product, something that we have already handled using L'Hopital's rule.

BIG IDEA

If $\lim_{x \to a} g(x) \cdot \ln f(x) = L$, then the limit we seek is $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x) \cdot \ln f(x)} = e^L$.

To summarize, these steps will help to evaluate limits with indeterminate forms 0^0, ∞^0, or 1^∞.

STEP BY STEP

To evaluate a limit with an indeterminate form 0^0, 1^∞, or ∞^0:

1. Let $y = f(x)^{g(x)}$. Then, $\ln y = g(x) \cdot \ln f(x)$.
2. Find $\lim_{x \to a} \ln y$.
3. Assuming that $\lim_{x \to a} \ln y = L$, we know $\lim_{x \to a} y = e^L$, where $y = f(x)^{g(x)}$.

Let's see how this methodology is applied to specific examples.

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2. Evaluating Limits With Variable Bases and Exponents

Now that we have a strategy, let's evaluate a few limits that have one of these indeterminate forms.

EXAMPLE Evaluate the following limit: $\lim_{x \to 0^+} x^x$

Note that this is a limit of the form 0^0, which will use our new strategy:

1. Take the natural logarithm of x^x: $\ln x^x = x \ln x$
2. Now find the limit:

$\lim_{x \to 0^+} x^x$	Start with the limit that needs to be evaluated.
$\lim_{x \to 0^+} x \ln x$	Evaluate the limit of the natural logarithm of the function. This has the form $0 \cdot (-\infty)$, which is another indeterminate form.
$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}$	The strategy here is to rewrite as either $\frac{x}{\left(\frac{1}{\ln x}\right)}$ or $\frac{\ln x}{\left(\frac{1}{x}\right)}$. The latter is preferable.
$= \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$	The limit has the form $\frac{\infty}{\infty}$ and both numerator and denominator are differentiable, so L'Hopital's rule can be used. $D\left[\frac{1}{x}\right] = D[x^{-1}] = -x^{-2} = \frac{-1}{x^2}$, $D[\ln x] = \frac{1}{x}$
$= \lim_{x \to 0^+} (-x)$	

Simplify $\dfrac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \frac{1}{x} \cdot \frac{x^2}{-1} = -x$.

$= 0$ Use direct substitution.

3. Then, the limit of the original function is $e^0 = 1$.

Thus, $\lim_{x \to 0^+} x^x = 1$.

WATCH

In this video, we will evaluate the limit $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$.

Video Transcription

[MUSIC PLAYING] Hi there-- good to see you again. What we're going to do in this video is take the limit of this function, which is going to end up being yet another one of our indeterminate forms. So remember, the goal is to rearrange-- rewrite-- the expression so that L'Hopital's rule can possibly be used. So let's look at the strategy for this one. So as x goes to infinity, we have a variable base and a variable exponent. So this is actually a limit of the form 1 to the infinity, which, again, doesn't tell us anything just by being 1 to the infinity. We have to analyze this a little more.

So remember that the strategy when you have a variable base and a variable exponent is, we take the natural log of 1 plus 2 over x to the x and then bring the power down. And that's the whole purpose of taking the natural log-- is to bring the power down. So then what we want is the limit as x approaches infinity of the logarithm of the expression, because, remember, we find the limit of the logarithm. We go through the paces. We find our limit. And then the original limit is e to whatever we get.

So, in this case, we have a infinity. And that is approaching-- let's see, natural log of 1 plus 0 is natural log of 1, which is 0. So we have an infinity times 0, which means, rewrite using a reciprocal. And here again, we're either going to take the reciprocal of x, which seems easier, versus the reciprocal of the natural log. I'm going to take the reciprocal of x.

So this is going to be the limit as x approaches infinity of the natural log of 1 plus 2 over x divided by 1 over x. And just to double-check, this still goes to 0. But now 1 over x also goes to 0. We are good to use L'Hopital's rule. So this is equal to the limit as x approaches infinity. Now here's where we have to be careful. Remember that the derivative of 1 over x is negative 1 over x squared. That showed up in a previous video.

So when we take the derivative of these expressions, we'll just keep that in mind so I don't have to keep writing as the negative power and doing that. You can if you want to. Derivative of natural log of something is 1 over the something times the derivative of the something. So the derivative of 1 plus 2 over x is negative 2 over x squared. And then the derivative of the denominator is negative 1 over x squared.

OK, so that actually majorly helps us here. So this is equal to the limit as x approaches infinity. Now let's just see what happens here. We can multiply by x squared over x squared. And they would just go away. We have a negative over a negative, which makes those positive. So we have 2 over 1 plus 2 over x. And as x goes to infinity, this 2 over x just goes to 0.

So now we have the limit of 2 over 1, which we know is just 2. Now, that's not the final answer. So remember the original limit. It was the limit as x approaches infinity 1 plus 2 over x to the x. Since this is the limit of the natural log, that means we have to counteract that by making it a base e. So that limit is e squared. And there we have it.

[MUSIC PLAYING]

TRY IT

Consider the following limit: $\lim_{x \to \infty} x^{3/x}$

Evaluate the limit. +

$\lim_{x \to \infty} x^{3/x} = 1$

SUMMARY

In this lesson, you learned **the strategy for evaluating limits with variable bases and exponents**. For instance, when evaluating $\lim_{x \to a} f(x)^{g(x)}$ and the limit results in one of the indeterminate forms (0^0, 1^∞, and ∞^0), the limit will need to be manipulated using logarithms in order to use L'Hopital's rule.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

