Exponential and Logarithmic Functions

by Sophia

WHAT'S COVERED

In this lesson, you will review the basics of exponential and logarithmic functions and their properties. Specifically, this lesson will cover:

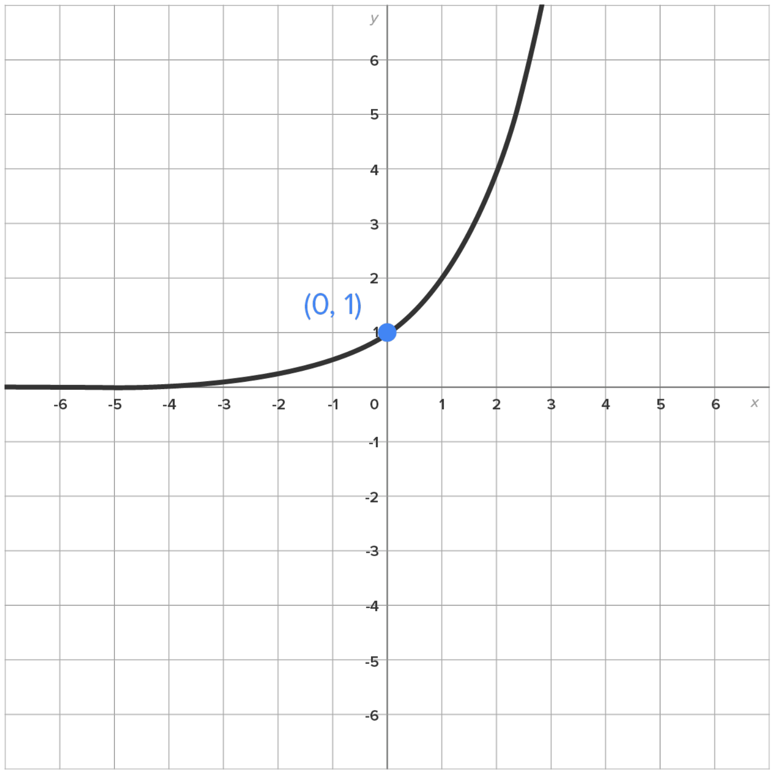
1. Exponential Functions
2. Logarithmic Functions
   a. Evaluating Logarithms
   b. Graphs of Logarithmic Functions
   c. Properties of Logarithms
   d. Expanding Logarithmic Expressions
   e. Condensing a Logarithmic Expression Into a Single Logarithm

1. Exponential Functions

Consider the function $f(x) = 2^x$ with some input-output pairs:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	0.0625	0.125	0.25	0.5	1	2	4	8	16

This leads us to the graph on the right:

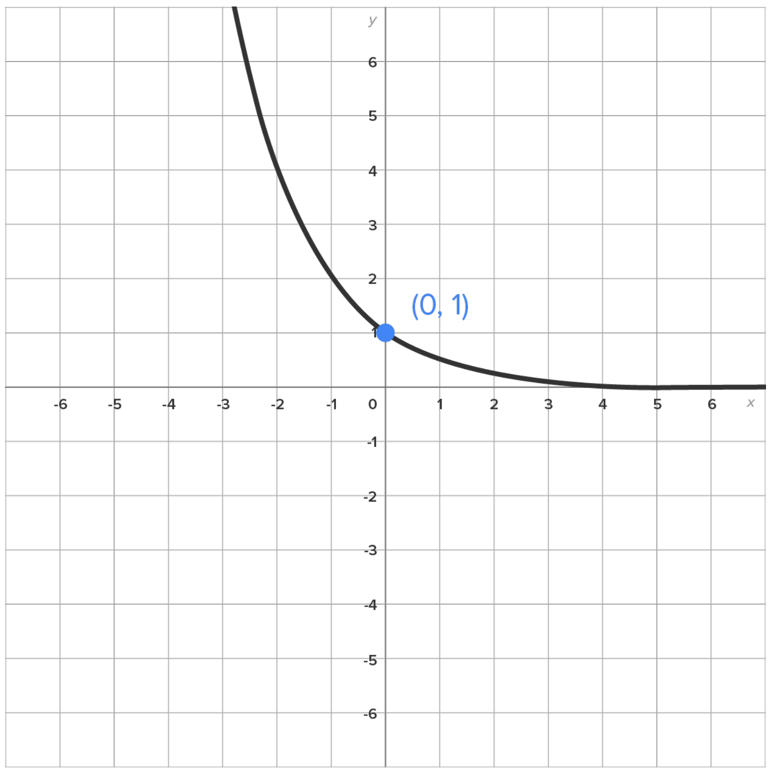


- The portion of the graph to the right of the y-axis increases sharply.
- The portion of the graph to the left of the y-axis decreases gradually toward $y = 0$, but it never quite gets there.
- This is because there is no value of x for which $2^x = 0$.

Let's now look at the graph of $f(x) = (0.5)^x$ with some input-output pairs.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = (0.5)^x$	16	8	4	2	1	0.5	0.25	0.125	0.0625

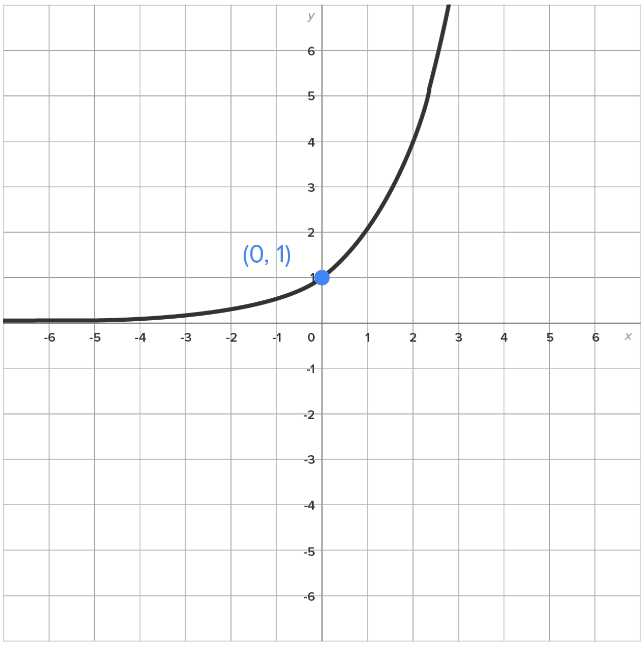
This leads us to the graph on the right:



- The portion of the graph to the left of the y-axis increases sharply.
- The portion of the graph to the right of the y-axis decreases gradually toward $y = 0$, but it never quite gets there.
- This is because there is no value of x for which $(0.5)^x = 0$.

In general, define the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

$f(x) = a^x$, *where* $a > 1$	$f(x) = a^x$, *where* $0 < a < 1$



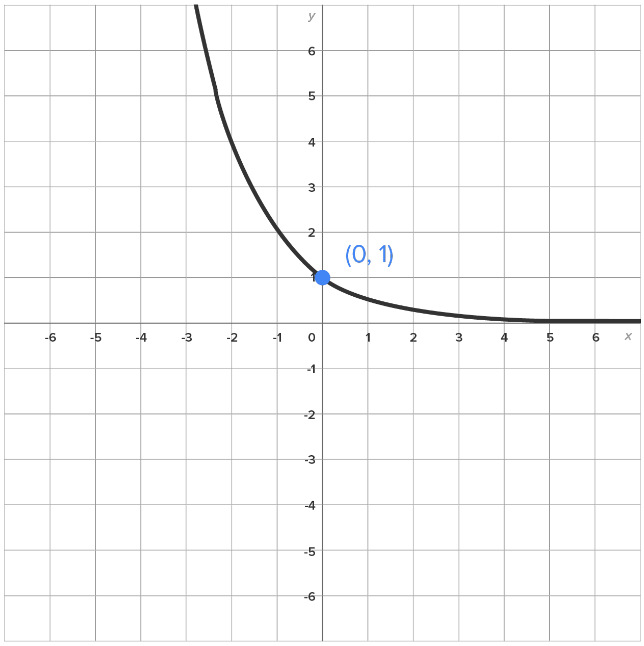
The graph is increasing at every point.

The domain is $(-\infty, \infty)$.

The range is $(0, \infty)$.

The graph contains the point (0, 1).

There is a horizontal asymptote at $y = 0$.



The graph is decreasing at every point.

The domain is $(-\infty, \infty)$.

The range is $(0, \infty)$.

The graph contains the point (0, 1).

There is a horizontal asymptote at $y = 0$.

HINT

Exponential functions can only be defined for $a > 0$ and $a \neq 1$ for the following reasons:

- If $a < 0$, there would be infinite values that are undefined due to fractional exponents. This would not be a useful function.
- If $a = 0$, the function is undefined when $x \leq 0$ and equal to 0 when $x > 0$, which is not an exponential function.
- If $a = 1$, then $f(x) = 1$ for all values of *x*, which is simply a horizontal line, which is not an exponential function.

A commonly used base is the number e, which is called the natural base, where $e \approx 2.718281828$... (this pattern does not repeat). Since $e > 1$, its graph is the increasing exponential graph as seen above.

2. Logarithmic Functions

2a. Evaluating Logarithms

Recall that the input of an exponential function is the exponent. The output of the exponential function is called the **power**, the result of raising a number to an exponent.

With a **logarithmic function**, the input is the power and the output is the exponent. In other words, a logarithm is the exponent *y* needed to complete the equation $a^y = x$ for given values of *a* and *x*.

That said, to find the value of *y*, we can write $f(x) = \log_a x$ (logarithm with "base *a*" of *x*).

FORMULA

Logarithm Definition

$y = \log_a x$ if $a^y = x$ where $a > 0$ and $a \neq 1$

EXAMPLE

Find the value of $\log_2 8$.

$y = \log_2 8$	Start with the original logarithmic function.
$2^y = 8$	Rewrite in exponential form.
$8 = 2^3$	Write 8 as a power of 2.
$2^y = 2^3$	Equate the exponential expressions.
$y = 3$	Solve for *y*.

Thus, $\log_2 8 = 3$.

EXAMPLE

Find the value of $\log_{10} 0.01$.

$y = \log_{10} 0.01$	Start with the original logarithmic function.
$10^y = 0.01$	Rewrite in exponential form.
$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$	Write 0.01 as a power of 10.
$10^y = 10^{-2}$	Equate the two expressions.
$y = -2$	Solve for *y*.

Thus, $\log_{10} 0.01 = -2$.

HINT

There are two special logarithms that will be handy to know:

- $\log_a 1 = 0$ (We know this because $a^0 = 1$ for any value of *a*.)
- $\log_a a = 1$ (We know this because $a^1 = a$ for any value of *a*.)

Notation Used for Logarithms of Special Bases	
Base 10	$\log_{10} x$ is written $\log x$. No base written means the base is 10.
Base *e*	$\log_e x$ is written $\ln x$, which means the natural logarithm of *x*. You may remember that e is called the natural base, where $e \approx 2.718281828...$ (this pattern does not repeat).

TERMS TO KNOW

A Power

The result of raising a number to an exponent. For example, $2^5 = 32$, and we say that 32 is the 5th power of 2.

Logarithmic Function

$f(x) = \log_a x$ uses the power as its input and returns the exponent required to produce that power when the base is a.

2b. Graphs of Logarithmic Functions

Earlier, we graphed the function $y = 2^x$ by using the following table.

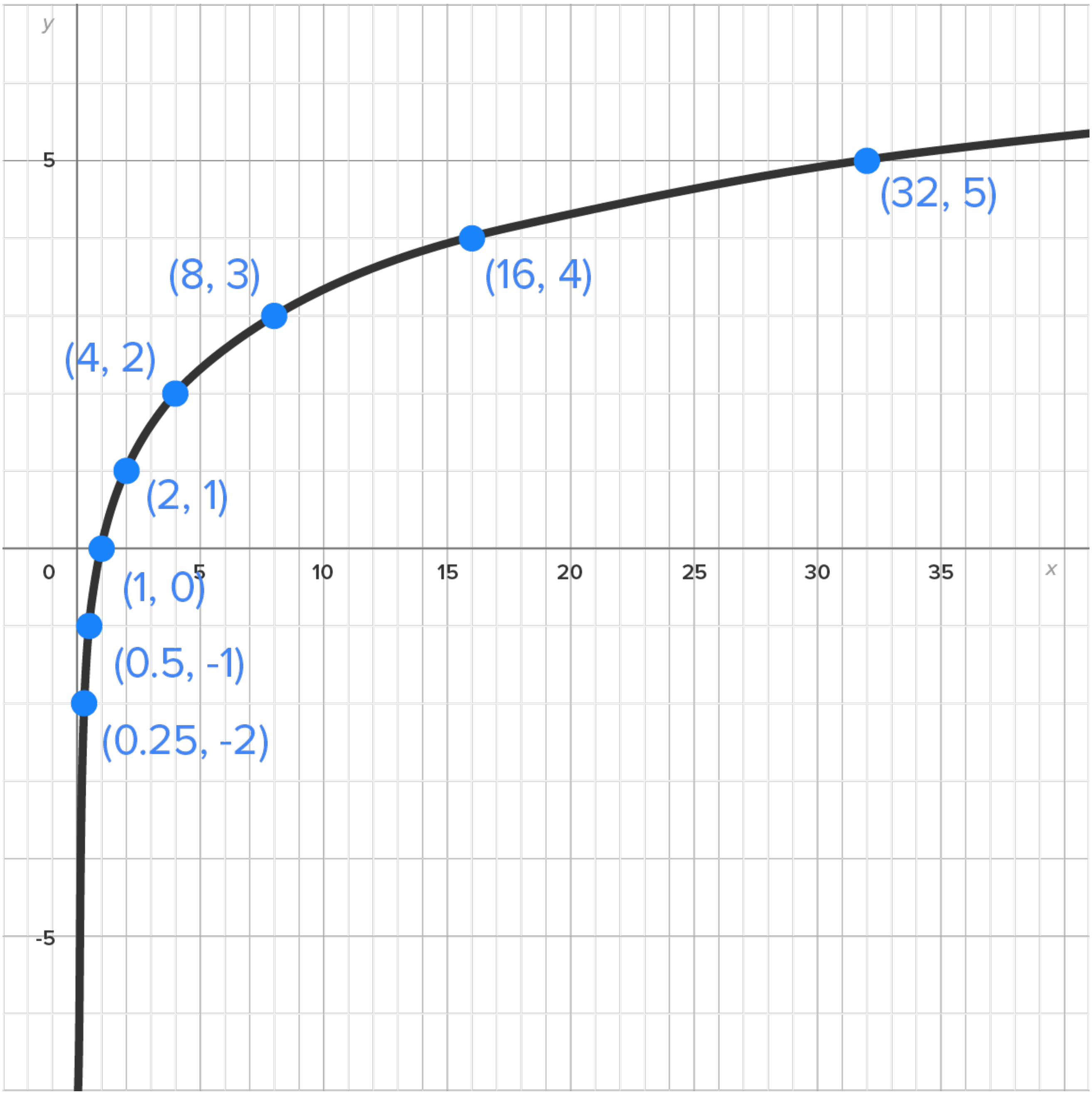
x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	0.0625	0.125	0.25	0.5	1	2	4	8	16

The logarithmic function $y = \log_2 x$ would interchange these values:

x	0.0625	0.125	0.25	0.5	1	2	4	8	16
$y = \log_2 x$	-4	-3	-2	-1	0	1	2	3	4

For example, $\log_2 16 = 4$ since $2^4 = 16$ and $\log_2 1 = 0$ since $2^0 = 1$.

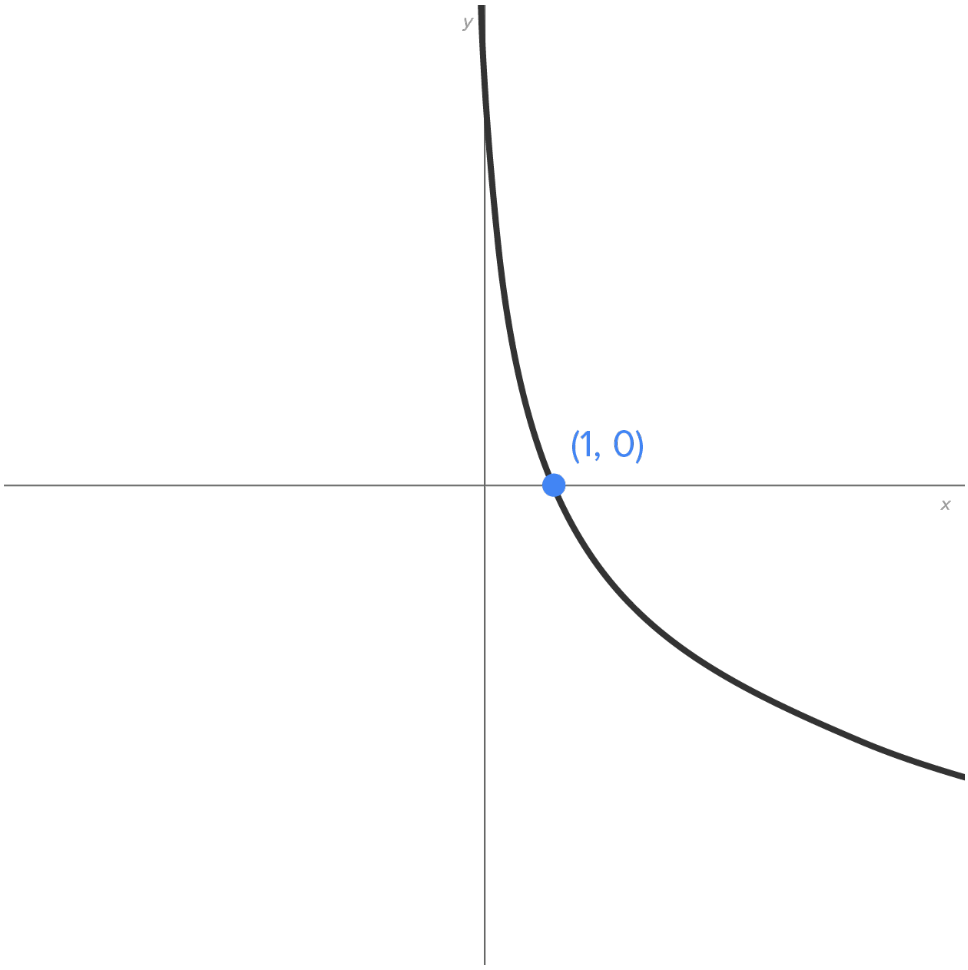
Here is the graph of $y = \log_2 x$ based on these points:



The graph has a vertical asymptote at $x = 0$.

In general, this is what the graph of $y = \log_a x$ looks like when $a > 1$.

When $0 < a < 1$, the graph has this general shape:



Properties of the graph of $y = \log_a x$:

- The domain is $x > 0$.
- The range is all real numbers.
- There is a vertical asymptote at $x = 0$.
- If $a > 1$, the graph is increasing, and if $0 < a < 1$, the graph is decreasing.

2c. Properties of Logarithms

You may recall the following properties of exponents:

$a^x \cdot a^y = a^{x+y}$	Multiply Exponential Expressions, Add Exponents
$\dfrac{a^x}{a^y} = a^{x-y}$	Divide Exponential Expressions, Subtract Exponents
$(a^x)^y = a^{xy}$	Raise an Exponential Expression to a Power, Multiply the Exponents

Now, remember that a logarithm is an exponent. Thus, the logarithm properties tell us what happens to the exponents when expressions are multiplied, divided, and raised to a power.

FORMULA

Product Property
$\log_a(xy) = \log_a x + \log_a y$

Quotient Property

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Power Property

$$\log_a(x^y) = y \cdot \log_a x$$

These properties are used to rewrite logarithmic expressions in two ways:

- Expand a single logarithm as a sum, difference, or multiple of logarithms.
- Write an expanded logarithmic expression as a single logarithm.

2d. Expanding Logarithmic Expressions

There is a process that you can follow to expand logarithmic expressions:

1. Apply product/quotient property first to "break up" the expression into a sum/difference.
2. Apply power property where relevant.

EXAMPLE Use logarithm properties to expand the expression $\ln\left(\frac{2x}{y}\right)$.

$\ln\left(\frac{2x}{y}\right)$	Start with the original expression.
$\ln(2x) - \ln y$	$\frac{2x}{y}$ is a quotient; apply the quotient property.
$\ln 2 + \ln x - \ln y$	$2x$ is a product; apply the product property.

The expanded form of $\ln\left(\frac{2x}{y}\right)$ is $\ln 2 + \ln x - \ln y$.

EXAMPLE Use logarithm properties to expand the expression $\log(x^2y^4)$.

$\log(x^2y^4)$	Start with the original expression.
$\log(x^2) + \log(y^4)$	x^2y^4 is a product; apply the product property.
$2\log x + 4\log y$	Apply the power property on each logarithm.

The expanded form of $\log(x^2y^4)$ is $2\log x + 4\log y$.

TRY IT

Consider the expression $\log_4\left(\frac{2x}{y^3}\right)$.

Use logarithm properties to expand this expression. +

$\log_4\left(\frac{2x}{y^3}\right)$	Start with the original expression.
$\log_4(2x) - \log_4(y^3)$	$\frac{2x}{y^3}$ is a quotient; apply the quotient property.

$\log_4 2 + \log_4 x - \log_4(y^3)$ $2x$ is a product; apply the product property.

$\log_4 2 + \log_4 x - 3\log_4 y$ Apply the power property.

The expanded form of $\log_4\left(\frac{2x}{y^3}\right)$ is $\log_4 2 + \log_4 x - 3\log_4 y$.

2e. Condensing a Logarithmic Expression Into a Single Logarithm

To condense a logarithmic expression into a single logarithm, apply the properties as we did when expanding an expression, but in reverse. This means:

1. Reverse the power property first for any expressions: $y \cdot \log_a x = \log_a(x^y)$
2. Reverse the sum/difference properties: $\log_a x + \log_a y = \log_a(xy)$ or $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$

EXAMPLE Use logarithm properties to write $3\log_4 x + \log_4 5 - 2\log_4 z$ as a single logarithm.

$3\log_4 x + \log_4 5 - 2\log_4 z$	Start with the original expression.
$\log_4 x^3 + \log_4 5 - \log_4 z^2$	Reverse the power property.
$\log_4(5x^3) - \log_4 z^2$	Reverse the product property.
$\log_4\left(\frac{5x^3}{z^2}\right)$	Reverse the quotient property.

The condensed form of $3\log_4 x + \log_4 5 - 2\log_4 z$ is $\log_4\left(\frac{5x^3}{z^2}\right)$.

TRY IT

Consider the expression $2\ln x - 3\ln y + 4\ln(z+1)$.

Write this expression as a single logarithm. +

$\ln\left[\frac{x^2(z+1)^4}{y^3}\right]$

SUMMARY

In this lesson, to add to the library of functions, you explored **exponential functions** and **logarithmic functions** and their **properties**. You learned how to **evaluate logarithms** by rewriting logarithmic functions in exponential form and also explored **graphs of logarithmic functions**. You also learned how to use properties of logarithms to **expand logarithmic expressions**. Lastly, you learned that to **condense a logarithmic expression into a single logarithm**, you need to apply the properties as you did when expanding an expression, but in reverse.

TERMS TO KNOW

A Power

The result of raising a number to an exponent. For example, $2^5 = 32$, and we say that 32 is the 5th power of 2.

Logarithmic Function

$f(x) = \log_a x$ uses the power as its input and returns the exponent required to produce that power when the base is *a*.

FORMULAS TO KNOW

Logarithm Definition

$y = \log_a x$ if $a^y = x$ where $a > 0$ and $a \neq 1$.

Power Property

$\log_a(x^y) = y \cdot \log_a x$

Product Property

$\log_a(xy) = \log_a x + \log_a y$

Quotient Property

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$