Mean Value Theorem for Derivatives

by Sophia

WHAT'S COVERED

In this lesson, you will learn another theorem called the mean value theorem, whose consequences are used in Unit 5 (Antiderivatives). Specifically, this lesson will cover:

1. The Mean Value Theorem for Derivatives and a Real-Life Connection
2. Applying the Mean Value Theorem for Derivatives

1. The Mean Value Theorem and a Real-Life Connection

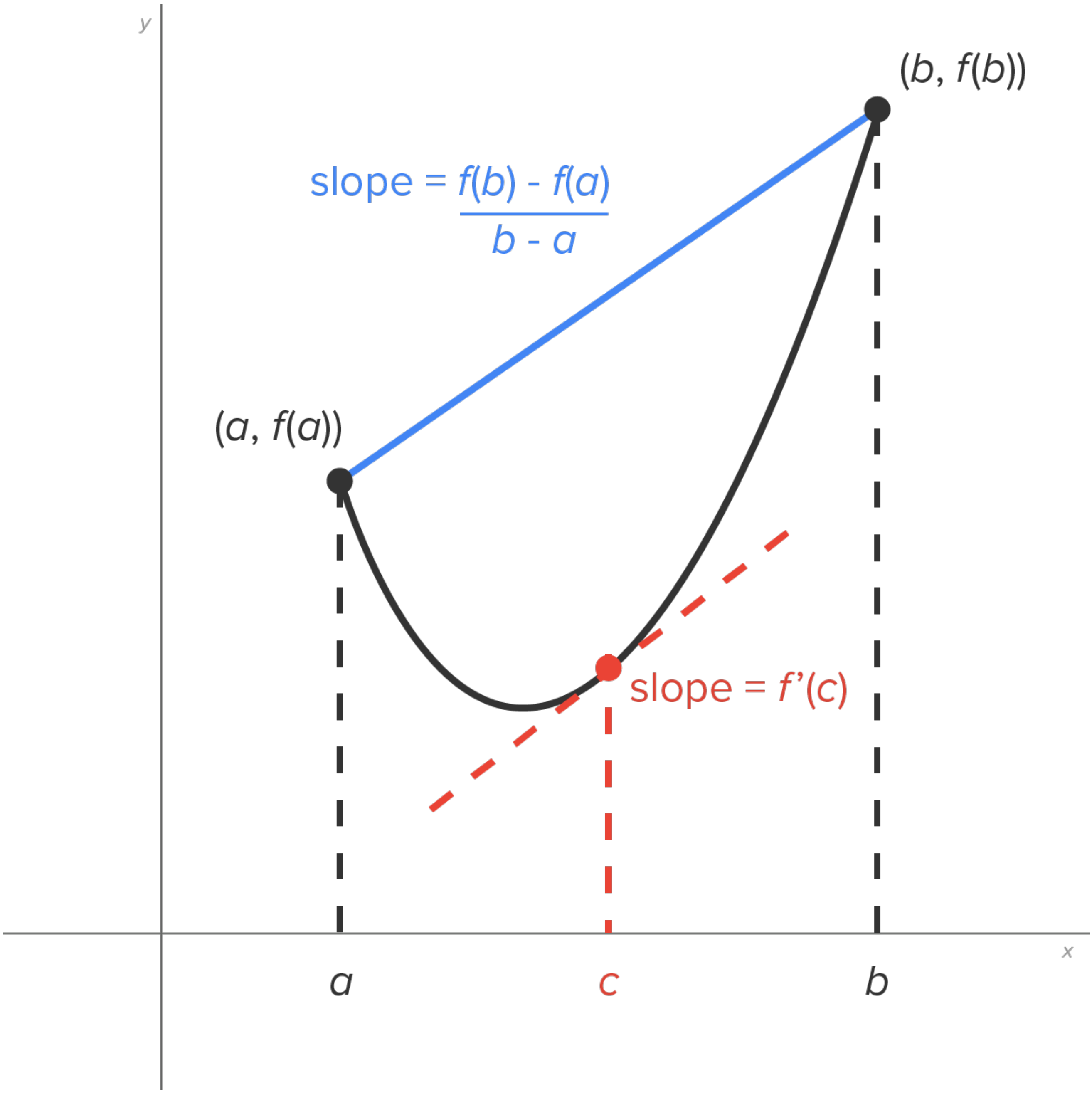
The distance between two toll booths on a major highway is 2 miles, and it took you 1.5 minutes to get from one toll booth to the other. Assume the speed limit on the highway is 65mph. Seeing no police cars on the route, you are surprised that you are being pulled over, and it turns out it is for speeding. What happened?

Let's look at this situation: It took 1.5 minutes to travel 2 miles. On average, your velocity was:

$$\frac{2 \text{ miles}}{1.5 \text{ minutes}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = \frac{120 \text{ miles}}{1.5 \text{ hours}} = \frac{80 \text{ miles}}{\text{hour}}$$

It stands to reason that at some point, you had to be traveling at a speed of 80 mph. Therefore, we are saying that there is a single point where the instantaneous velocity between the toll booths is equal to the average velocity traveled between the toll booths. This is the idea behind the **mean value theorem for derivatives**.

That is to say, there is a value of c where the instantaneous rate of change in f is equal to the average rate of change of f over the interval $[a, b]$.



As you can see, the tangent line at c and the slope of the line between the endpoints are identical.

TERM TO KNOW

Mean Value Theorem for Derivatives
Let $f(x)$ be continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b). Then, there is at least one value of c between a and b for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

2. Applying the Mean Value Theorem for Derivatives

Now that we have defined the mean value theorem for derivatives, let's see how it applies to some functions.

➦ EXAMPLE Consider the function $f(x) = x^2 - 3x$ on the interval $[1, 3]$. Note that $f(x)$ is continuous on $[1, 3]$ and differentiable on the interval $(1, 3)$.

Then, we should be able to find a value of *c* between 1 and 3 where $f'(c) = \frac{f(3) - f(1)}{3 - 1}$.

First we'll compute $\frac{f(3) - f(1)}{3 - 1}$.

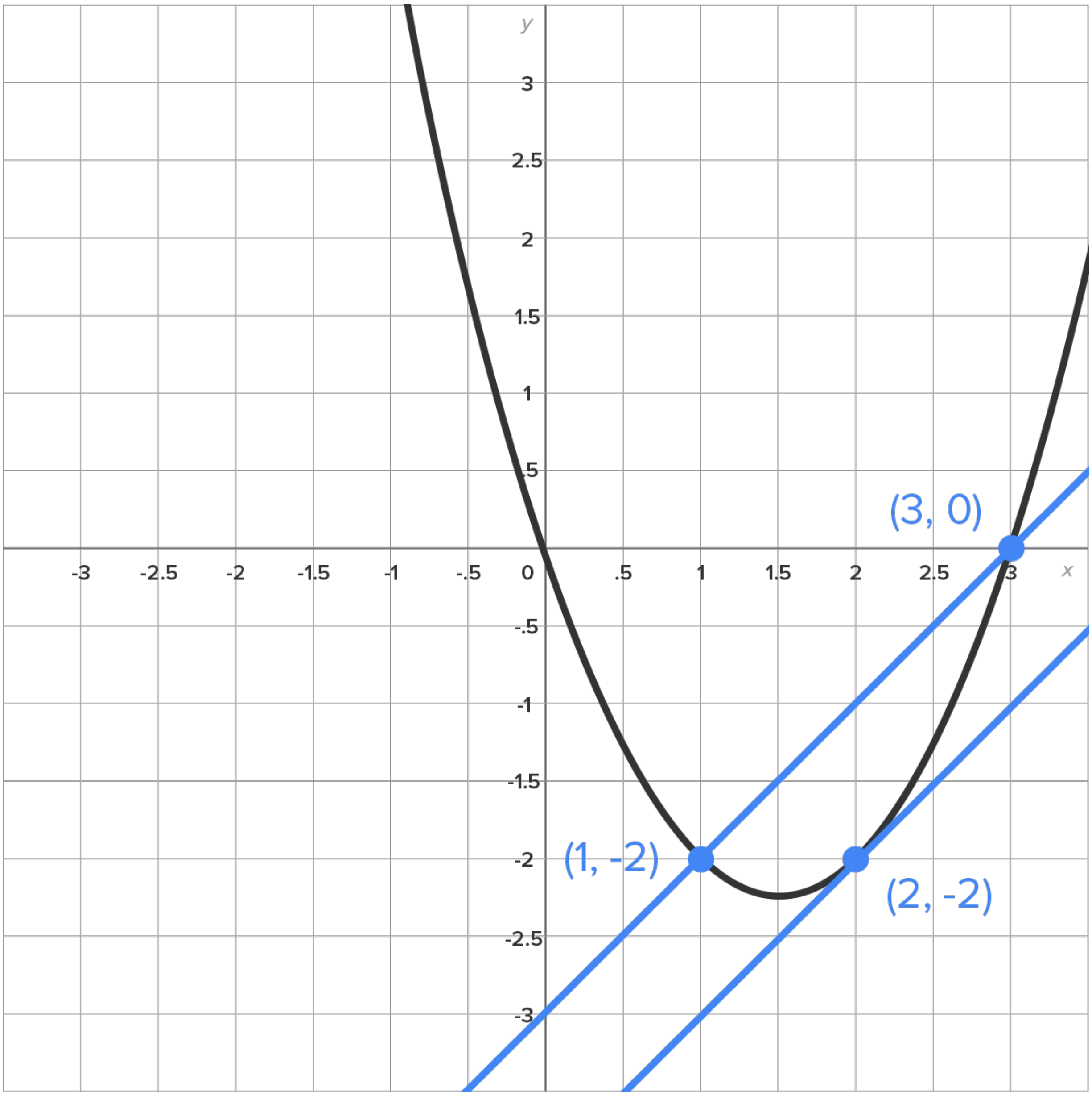
- We have $f(1) = -2$ and $f(3) = 0$.
- Then, $\frac{f(3) - f(1)}{3 - 1} = \frac{0 - (-2)}{3 - 1} = 1$.

Now, we want to find the value of *c* guaranteed by the mean value theorem.

$f(x) = x^2 - 3x$	Start with the original function.
$f'(x) = 2x - 3$	Take the derivative.
$2x - 3 = 1$	Set the derivative equal to $\frac{f(3) - f(1)}{3 - 1}$.
$2x = 4$	Add 3 to both sides.
$x = 2$	Divide both sides by 2.

Since $x = 2$ is clearly inside the interval $(1, 3)$, $c = 2$.

Conclusion: The value guaranteed by the mean value theorem for derivatives is $c = 2$. The graph of this situation is shown here.



The curve is $f(x) = x^2 - 3x$.

Note how the slope of the tangent line at $x = 2$ is equal to the slope of the secant line between $x = 1$ and $x = 3$.

WATCH

In this video we will find the value(s) of c guaranteed by the mean value theorem for derivatives for the function $f(x) = x^3 - 2x$ on the interval $[1, 4]$.

SUMMARY

In this lesson, you learned that **the mean value theorem for derivatives** guarantees that for a continuous function on a closed interval $[a, b]$, the derivative (slope of the tangent line) at some point on the interval is equal to the average rate of change (slope of the secant line) over the entire interval. You examined a **real-life connection** by seeing how the MVT can be used to catch drivers who drive too fast, followed by several examples **applying the mean value theorem for derivatives** to some functions. This idea will be utilized in Unit 5 when we start to explore antiderivatives.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Mean Value Theorem for Derivatives

Let $f(x)$ be continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b).

Then, there is at least one value of *c* between *a* and *b* for which $f'(c) = \frac{f(b) - f(a)}{b - a}$.

