f'' and Extreme Values of f

by Sophia

WHAT'S COVERED

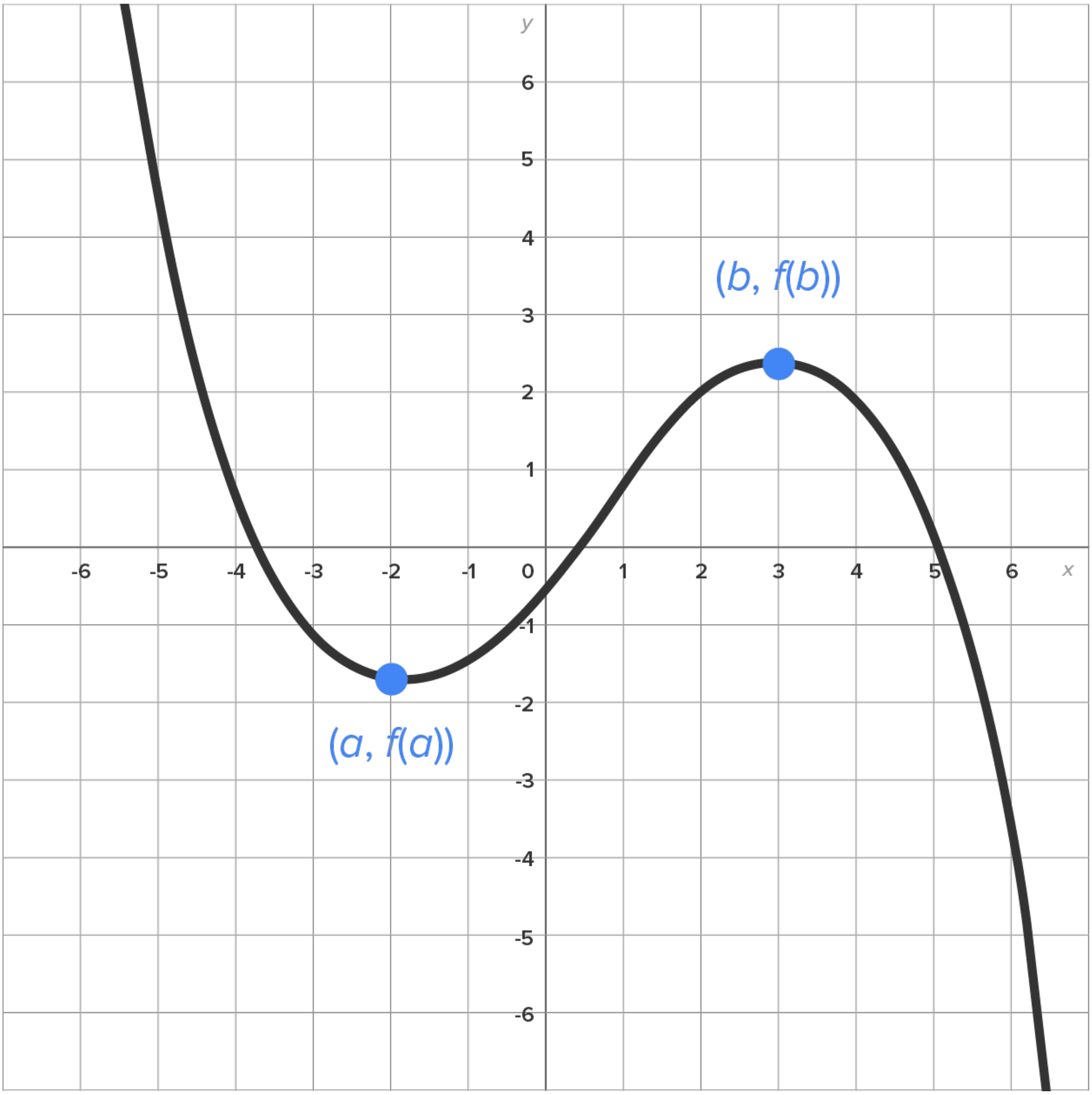
In this lesson, you will use the second derivative to determine if a maximum or minimum occurs at a critical number. Specifically, this lesson will cover:

1. The Second Derivative Test
2. Locating Maximum/Minimum With the Second Derivative Test

1. The Second Derivative Test

When a function has several critical numbers, and when the second derivative is relatively easy to get, the second derivative test is much more efficient to use than the first derivative test when locating maximum and minimum values. However, there are conditions under which the second derivative doesn't give enough information, and we need to use information from the first derivative to determine maximum and minimum points of a function.

Consider the graph shown below:



Observe the following at the local extreme points:

The point $(a, f(a))$ is a local minimum.

- Since there is a horizontal tangent at $x = a$, we know $f'(a) = 0$.
- Since the graph is concave up at $x = a$, we know that $f''(a) > 0$.

The point $(b, f(b))$ is a local maximum.

- Since there is a horizontal tangent at $x = b$, we know $f'(b) = 0$.
- Since the graph is concave down at $x = b$, we know that $f''(b) < 0$.

Based on this graph, there is a connection between the concavity of $f(x)$ when there is a horizontal tangent line and the type of local extrema at that point. Formally stated, this is called the **second derivative test**.

HINT

Only critical numbers where $f'(c)=0$ are considered for the second derivative test. If $f'(c)$ is undefined, then $f''(c)$ is also undefined, which means we cannot determine if $f''(c)$ is positive, negative, or zero.

That said, if $f(x)$ has critical numbers where $f'(c)$ is undefined, then the first derivative test will need to be used to determine if any local extrema occur at $x=c$.

TERM TO KNOW

Second Derivative Test

Suppose $f'(c)=0$, which means $f(x)$ has a horizontal tangent at $x=c$.

- If $f''(c)<0$, this means $f(x)$ is concave down around *c*, which means there is a local maximum at *c*.
- If $f''(c)>0$, this means $f(x)$ is concave up around *c*, which means there is a local minimum at *c*.
- If $f''(c)=0$, the test is inconclusive, and the first derivative test needs to be used to determine the behavior at *c*.

2. Locating Maximum/Minimum With the Second Derivative Test

Let's look at a few examples of how the second derivative test is implemented.

EXAMPLE Determine the local maximum and minimum values of $f(x)=-2x^3+6x^2+15$.

First, find the values of *c* for which $f'(c)=0$.

$f(x)=-2x^3+6x^2+15$	Start with the original function; the domain is all real numbers.
$f'(x)=-6x^2+12x$	Take the derivative.
$-6x^2+12x=0$ $-6x(x-2)=0$ $-6x=0,\ x-2=0$ $x=0,\ x=2$	Set $f'(x)=0$ and solve.

The critical numbers are $x=0$ and $x=2$.

Now, take the second derivative using $f'(x)=-6x^2+12x$ and substitute $x=0$ and $x=2$.

$f'(x)=-6x^2+12x$	Take the first derivative.
$f''(x)=-12x+12$	Take the second derivative.
$f''(0)=-12(0)+12=12$	Substitute the critical number, $x=0$. Since $f''(0)$ is positive, $f(0)$ is a local minimum.

$f''(2) = -12(2) + 12 = -12$	Substitute the critical number, $x = 2$. Since $f''(2)$ is negative, $f(2)$ is a local maximum.

Therefore, the local minimum value is $f(0) = 15$ and the local maximum value is $f(2) = 23$. On the graph, the local minimum is located at (0, 15) and the local maximum is located at (2, 23).

Let's look at another example.

↪ EXAMPLE Determine the local maximum and minimum values of $f(x) = 5x + \frac{20}{x}$.

First, find the values of c for which $f'(c) = 0$.

$f(x) = 5x + \frac{20}{x} = 5x + 20x^{-1}$	Start with the original function; rewrite to use the power rule. The domain is $(-\infty, 0) \cup (0, \infty)$.
$f'(x) = 5 - 20x^{-2}$	Take the derivative.
$5 - 20x^{-2} = 0$ $5 - \frac{20}{x^2} = 0$ $5x^2 - 20 = 0$ $5(x^2 - 4) = 0$ $5(x+2)(x-2) = 0$ $x = 2, x = -2$	Set $f'(x) = 0$ and solve.

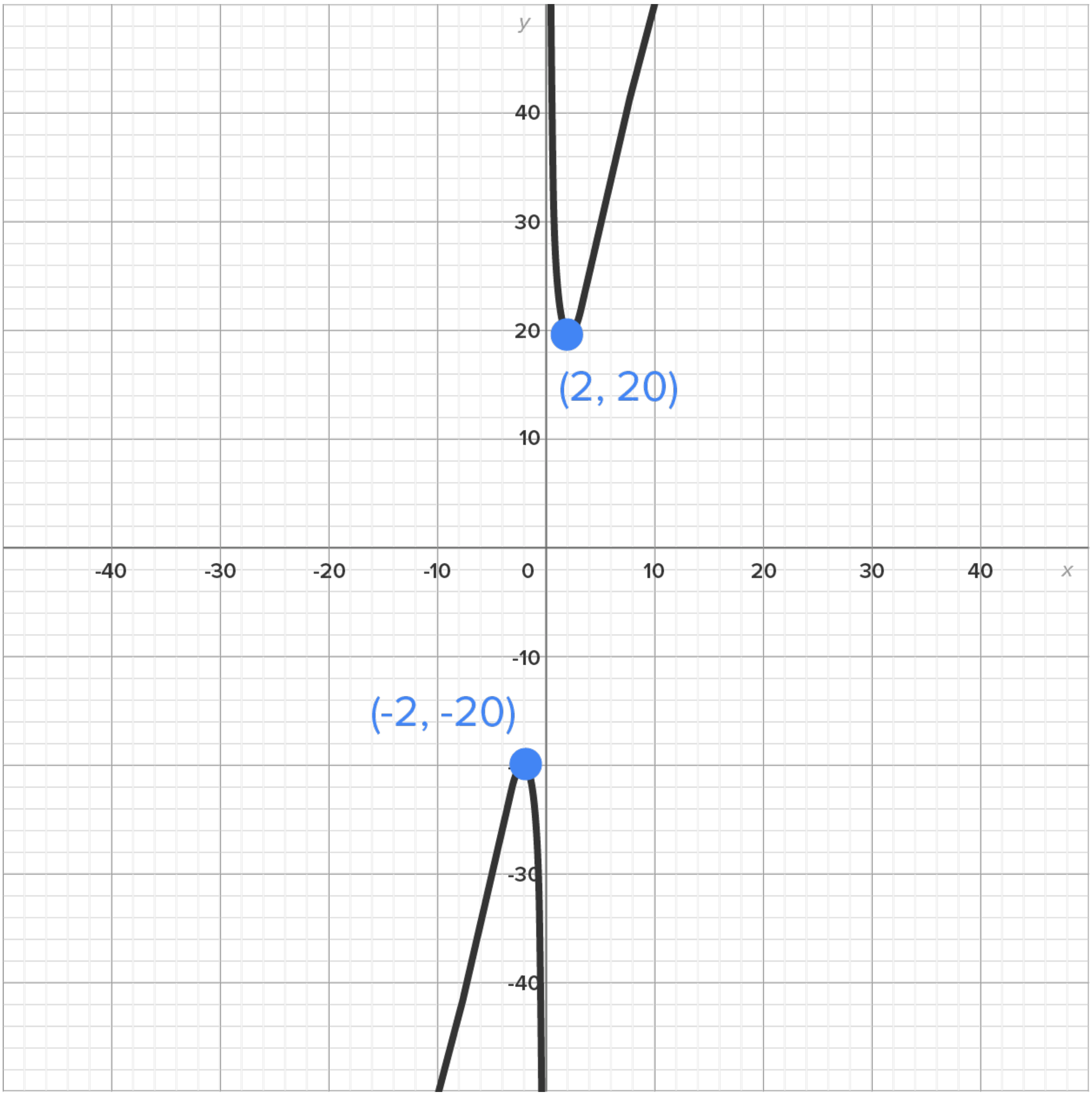
The critical numbers are $x = -2$ and $x = 2$.

Now, take the second derivative using $f'(x) = 5 - 20x^{-2}$, then substitute $x = -2$ and $x = 2$.

$f'(x) = 5 - 20x^{-2}$	Take the first derivative.
$f''(x) = 40x^{-3} = \frac{40}{x^3}$	Take the second derivative.
$f''(-2) = \frac{40}{(-2)^3} = -5$	Substitute the critical number, $x = -2$. Since $f''(-2)$ is negative, $f(-2)$ is a local maximum.
$f''(2) = \frac{40}{2^3} = 5$	Substitute the critical number, $x = 2$. Since $f''(2)$ is positive, $f(2)$ is a local minimum.

Therefore, the local maximum value is $f(-2) = -20$ and the local minimum value is $f(2) = 20$. On the graph, the local maximum is located at (-2, -20) and the local minimum is located at (2, 20).

Here is a graph of the function:



Let's now look at an example where the second derivative test cannot be used.

↪ EXAMPLE Determine the local maximum and minimum values of $f(x) = x^3 - 6x^2 + 12x + 10$.

First, find the values of c for which $f'(c) = 0$.

$f(x) = x^3 - 6x^2 + 12x + 10$	Start with the original function; the domain is all real numbers.
$f'(x) = 3x^2 - 12x + 12$	Take the derivative.
$3x^2 - 12x + 12 = 0$	Set $f'(x) = 0$ and solve.
$3(x^2 - 4x + 4) = 0$	
$3(x-2)(x-2) = 0$	
$x = 2$	

There is only one critical number, $x = 2$.

Now, take the second derivative using $f'(x) = 3x^2 - 12x + 12$, then substitute $x = 2$.

$f'(x) = 3x^2 - 12x + 12$	Take the first derivative.
$f''(x) = 6x - 12$	Take the second derivative.
$f''(2) = 6(2) - 12 = 0$	Substitute the critical number, $x = 2$.

This means that the second derivative cannot be used to determine if $f(x)$ attains a local maximum or minimum at $x = 2$.

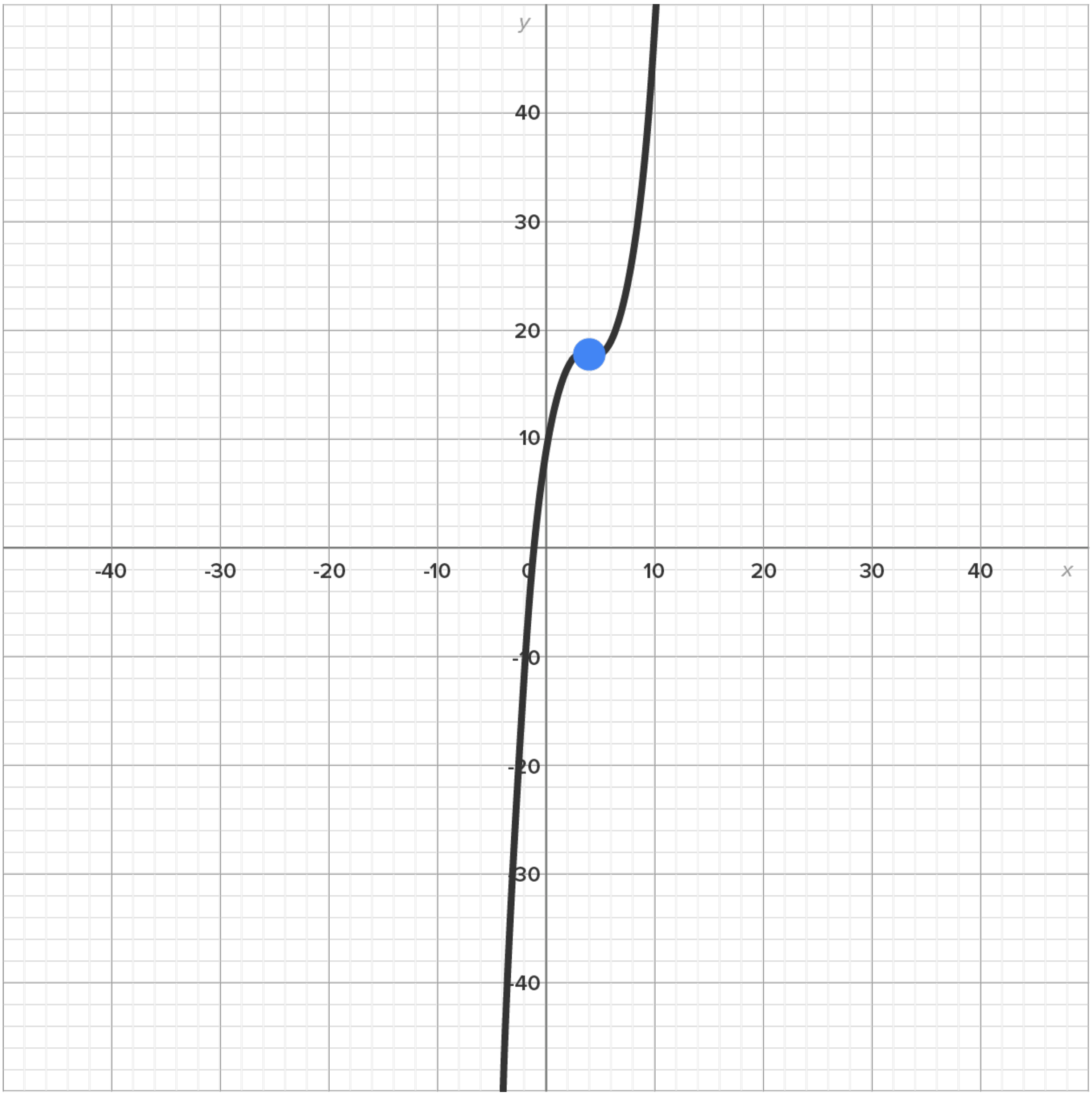
To determine the behavior of $f(x)$ at $x = 2$, we now turn back to the first derivative, $f'(x) = 3x^2 - 12x + 12$.

Evaluate on either side by finding $f'(1)$ and $f'(3)$:

- $f'(1) = 3(1)^2 - 12(1) + 12 = 3$ (increasing)
- $f'(3) = 3(3)^2 - 12(3) + 12 = 3$ (increasing)

Since $f(x)$ is increasing on both sides of $x = 2$, this means that there is no minimum or maximum when $x = 2$.

The graph of $f(x)$ is shown in the figure:



WATCH

In this video, we'll use the second derivative test to locate local maximum and minimum values of the function $f(x) = xe^{-0.5x}$.

Video Transcription

Well, hello there. Welcome back. We're going to continue our quest of maxes and mins and derivatives by looking at a function and using the second derivative test to determine if there's a minimum or a maximum at a value where there's a horizontal tangent, a.k.a., a critical number, but not where it's undefined.

So let's just look over to the right at the criteria for using the second derivative test. Let's say that f prime of c is equal to 0. If the second derivative happens to be positive at that value, we have a local minimum at c. And if the second derivative is negative at that value, then we have a local maximum at c.

So we need to first find the critical numbers where the first derivative is equal to 0, and then use those in the second derivative. So here, we have our function, and it does require a product rule to find the derivative. So I'm going to say x is the first and e to the negative 0.5x is the second.

So the derivative, as we know with the product rule, is the derivative of the first, which is 1, times the second, which is e to the negative 0.5x, plus the first x, times the derivative of the second. Now, the derivative of an exponential is e to the negative 0.5 x times negative 0.5 It's itself times the derivative of the exponent, the inside.

So I'm going to clean things up, and all that means is I'm not going to write the 1. And then I'm going to say, minus 0.5xe to the negative 0.5x. And actually, what I'm going to do is, before I say it's simplified, I am going to factor out e to the negative 0.5x. And that's going to leave you with 1 minus 0.5x. So this is my first derivative. So I want to know where that's equal to 0. So I set it equal to 0.

So if we need a point of reference here, this is finding critical numbers. And remember that since it's in factored form, we know that each factor could be 0. Now remember, an exponential never takes on a value of 0, because there is no exponent that creates 0. So this is no solution.

And on the right, if we add 0.5x to both sides and divide by 0.5, we have x equals 2. So x equals 2 is the critical number we're going to pay attention to. Now we move to the second derivative. So using the equation in the box, we're just going to use the products rule once again.

So if double prime is-- so the derivative of the first, which is e to the negative 0.5x, the derivative of that is e to the negative 0.5x times negative 0.5, and then times the second, which is 1 minus 0.5x, plus the first e to the negative 0.5x times the derivative of the second, which is 0 minus 0.5. The derivative of 1 is 0. The derivative of 0.5x is 0.5.

Now, at that point, we could simplify, but remember, we are just going to be substituting a value in to see what the second derivative is. So I'm going to leave it as is and just substitute 2. And we have a lot of 0.5s and 2s here. So let's just remember. Let's just look at this here. So 0.5 times 2 is 1.

So this is e to the negative 1 times negative 0.5 times-- well, this is 1. That makes it look like a fraction, though. There we go. So that's 1 minus 1. So that's 0. So that whole term drops out. And then we have minus 0.5e to the negative 1. So this whole term goes away, and we have -0.5 times 1 over e.

Now remember, e is a positive number, so this whole thing is negative. Therefore, there is a local maximum at x equals 2 by the second derivative test, what is the value of f at that point? Well, remember, the maximum value would then be f of 2 equals-- now remember, f of x was xe to the negative 0.5x. So that'd be 2e to the negative 0.5 times 2, which is 2 times e to the negative 1, which is 2 over 8.

So the local maximum point occurs at 2, comma 2 over e. And I have a graph here just to confirm our results. Here is the graph of y equals xe to the -0.5 x. And as you can see, the maximum point turns out to be a global max at x equals 2. And 0.736 is the decimal approximation of 1 over e. So there we have it. Another way to find mins and maxes is to use a second derivative test.

SUMMARY

In this lesson, you learned that under certain conditions, the second derivative can be used to determine if $f(x)$ has a local maximum or minimum at a critical value *c* for which $f'(c)=0$, known as **the second derivative test**. Next, you explored several examples **locating the maximum and minimum values with the second derivative test**; however, when those conditions aren't met, the first derivative test is used to determine the locations of local maximum and minimum values.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

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