Changing the Variable: u-Substitution with Trigonometric Functions

by Sophia

WHAT'S COVERED

In this lesson, you will continue using substitutions to find antiderivatives, but now we will focus on the trigonometric functions. Specifically, this lesson will cover:

1. The Inner Function Is Trigonometric
2. The Outer Function Is Trigonometric

1. The Inner Function Is Trigonometric

Since we have already been through u-substitution, there really is nothing too different about these indefinite integrals. The idea is the same, it's just that we have to use rules for trigonometric functions now.

Let's take a look at an example.

EXAMPLE Find the indefinite integral: $\int \sin^2 x \cos x dx$

First, note that the integral can be written $\int (\sin x)^2 \cos x dx$. At this point, notice that $\sin x$ is the "inner" function since it is raised to a power. Notice also that its derivative, $\cos x$, is also in the integral. This means that u-substitution should work!

$\int \sin^2 x \cos x dx$	Start with the original expression.
$= \int u^2 du$	First, make the substitution: $u = \sin x$ Write the differential: $du = \cos x dx$ Replace $\sin x$ with u and $\cos x dx$ with du.
$= \frac{1}{3}u^3 + C$	Find the antiderivative with respect to u.
$= \frac{1}{3}\sin^3 x + C$	Back-substitute $u = \sin x$.

Thus, $\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x + C$.

TRY IT

Consider $\int \frac{\sin x}{\sqrt{\cos x}} dx$.

Find the indefinite integral. +

$-2\sqrt{\cos x} + C$

We can also use substitution to find antiderivatives of certain trigonometric functions.

WATCH

In this video, we'll find $\int \tan x dx$.

Video Transcription

[MUSIC PLAYING] Hello, there, and welcome back. What we're going to do in this video is find an antiderivative of tangent of x. As you see here we have the integral of tangent of xdx. So the first thing we'll do since it's not quite clear what the antiderivative of tangent is, we're going to rewrite tangent as sine x over cosine x. While it might be becoming a little bit more clear we're still going to do one more rewrite. I'm going to rewrite this as 1 over cosine x times sine xdx.

Now, if you look at the function the way it's written now the cosine x is actually part of a bigger function. We have 1 over something. So here it makes sense to do a u substitution with cosine, but before doing so we need to make sure that its derivative is inside the integral in some form as well. Remember that the derivative of cosine is negative sine. So we essentially have the sine of x, but we're going to have to adjust for a negative. So u substitution seems to be a good strategy to use here.

So we're going to let u equal cosine of x because then we get 1 over u. Then du is negative sine of xdx, the differential form. Since we don't have a negative inside of the integral sign we're going to get sine of xdx by itself by multiplying both sides by negative 1. So you have negative du equals sine of xdx. Now to make our substitution we have the integral of 1 over u times negative du. Since that negative is really a constant we're going to write it on the outside of the integral sign. It's really a negative 1 getting multiplied.

Now, remember that the antiderivative of 1 over u is the natural log of the absolute value of u plus a constant. Now we back substitute, u is equal to cosine of x. So we have the natural log of the absolute value of cosine of x plus c-- negative natural log of absolute value of cosine of x plus c. Sorry, I misspoke there.

Now, that is a perfectly fine form for the antiderivative, but we can take it one step further. Remember that the negative out front of the natural log is really a negative 1 multiplied, so by property of logarithms we can write the natural log of the absolute value of cosine x to the negative 1 plus c. Remember negative 1 means reciprocal, and remember that the reciprocal of cosine is secant. So we can also write this as the natural log of secant x plus c.

So that means we have two forms for the antiderivative. I'm just going to write the other one here right next to it. Either one of these is an acceptable form for the antiderivative of tangent of x. So there we have it, natural log of absolute value of secant x, that one appeases the crowd that doesn't like negatives in expressions. And we have the negative natural log of absolute value of cosine x plus c, that one appeases the crowd that likes to write trig functions in terms of sines and cosine. So there's something for everyone here.

[MUSIC PLAYING]

TRY IT

Consider $\int \cot x\,dx$.

Find the indefinite integral. +

$\ln|\sin x| + C$

TRY IT

Consider $\int (2+\tan x)^4 \sec^2 x\,dx$.

Find the indefinite integral. +

$\frac{1}{5}(2+\tan x)^5 + C$

2. The Outer Function Is Trigonometric

EXAMPLE Find the indefinite integral: $\int \cos(4x)dx$

$\int \cos(4x)dx$	Start with the original expression.
$= \int \cos u \cdot \frac{1}{4}du$	Make the substitution: $u = 4x$ Find the differential: $du = 4dx$ Solve for *dx*: $dx = \frac{1}{4}du$ Replace $4x$ with *u* and *dx* with $\frac{1}{4}du$.
$= \frac{1}{4}\int \cos u\,du$	Move the constant $\frac{1}{4}$ outside the integral sign.

$= \frac{1}{4}\sin u + C$ Use the antiderivative rule for $\cos x$.

$= \frac{1}{4}\sin(4x) + C$ Back-substitute $u = 4x$.

Thus, $\int \cos(4x)dx = \frac{1}{4}\sin(4x) + C$.

↱ EXAMPLE Find the indefinite integral: $\int x\sec(3x^2)\tan(3x^2)dx$

$\int x\sec(3x^2)\tan(3x^2)dx$ Start with the original expression.

$= \int \sec u \tan u \cdot \frac{1}{6}du$ Make the substitution: $u = 3x^2$

Find the differential: $du = 6xdx$

Solve for *xdx*: $xdx = \frac{1}{6}du$

Replace $3x^2$ with *u* and *xdx* with $\frac{1}{6}du$.

$= \frac{1}{6}\int \sec u \tan u\, du$ Move the constant $\frac{1}{6}$ outside the integral sign.

$= \frac{1}{6}\sec u + C$ Use the antiderivative rule for $\sec x \tan x$.

$= \frac{1}{6}\sec(3x^2) + C$ Back-substitute $u = 3x^2$.

Thus, $\int x\sec(3x^2)\tan(3x^2)dx = \frac{1}{6}\sec(3x^2) + C$.

TRY IT

Consider $\int \frac{\sin(4\ln x)}{x}dx$.

Find the indefinite integral. +

$\frac{-1}{4}\cos(4\ln x) + C$

THINK ABOUT IT

Consider the antiderivative $\int \cos(x^2)dx$.

A first instinct is to let $u = x^2$, then $du = 2xdx$, which means $xdx = \frac{1}{2}du$. Here is the problem: the original integral only has a "*dx*" term, not an "*x dx*" term. From the substitution, we could also write $x = \sqrt{u}$. Let's see where that takes us.

$x dx = \frac{1}{2} du$ becomes $\sqrt{u}\, dx = \frac{1}{2} du$, so $dx = \frac{1}{2\sqrt{u}} du$.

Making all the substitutions, our integral becomes $\int \frac{\cos u}{\sqrt{u}} du$, which is much more complicated. As it turns out, there is no substitution that would solve $\int \cos(x^2) dx$ because there is no antiderivative for $f(x) = \cos(x^2)$. There are several other functions that do not have antiderivatives.

So, when making substitutions, be careful that all the variables are covered in your substitution. If they aren't, you either should check your work, try another substitution, or it is possible that the antiderivative doesn't exist.

SUMMARY

In this lesson, you learned how to apply your knowledge of u-substitution to find indefinite integrals when **the inner function is trigonometric** and **the outer function is trigonometric**. With the addition of trigonometric functions, your abilities to find antiderivatives expands even further. As you saw in one example, however, there are several functions that do not have antiderivatives.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.