Applied Maximum and Minimum Problems

by Sophia

WHAT'S COVERED

In this lesson, you will apply your knowledge of derivatives to real-world maximization and minimization problems (which collectively are called *optimization problems*). Specifically, this lesson will cover:

1. Strategy for Solving Optimization Problems
2. Solving Applied Optimization Problems

1. Strategy for Solving Optimization Problems

An **optimization problem** is a problem in which the maximum or minimum value is sought, whichever is relevant.

STEP BY STEP

To solve an optimization problem:

1. Identify the function to be optimized. This is called the primary equation.
   a. If the goal is to maximize the area, the primary equation expresses the area as a function.
   b. If the goal is to minimize the amount of material used, then the primary equation gives the total amount of material used.
2. If your primary equation has more than one variable (for example: $A = xy$), you will need to form a secondary equation based on other information that is given in the problem.
3. If applicable, use the secondary equation in Step 2 to write the primary equation in Step 1 in terms of one independent variable. Also, state the domain of the function.
4. Find critical numbers.
5. Keeping the requirements in mind, use one of methods covered in this challenge to determine where the extreme points are located:
   a. Extreme Value Theorem
   b. First Derivative Test
   c. Second Derivative Test

Now that we have a game plan, let's solve a few optimization problems.

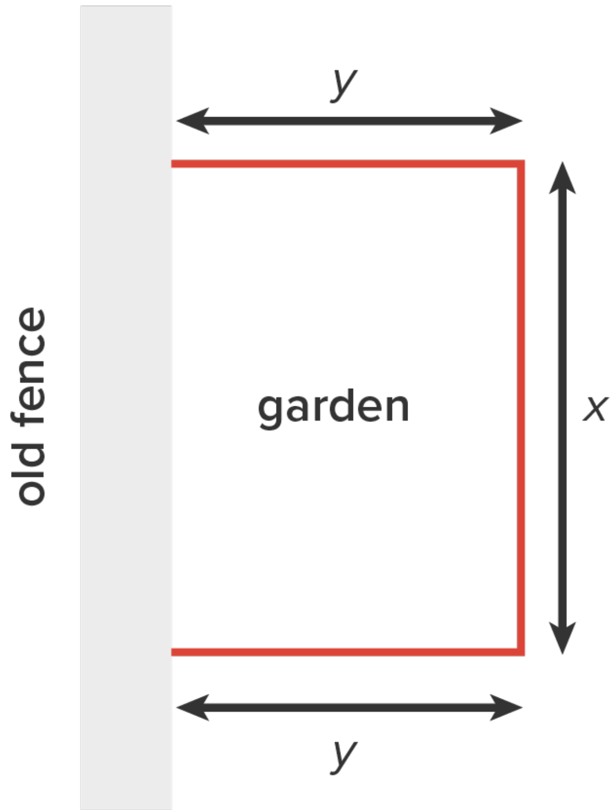
TERM TO KNOW

Optimization Problem

A problem in which the maximum or minimum value is sought, whichever is relevant.

2. Solving Applied Optimization Problems

EXAMPLE A garden is to be constructed against an old fence, as shown in the figure. Trim is to be placed around the garden on the three remaining sides but is not needed on the fence side. If 24 feet of trim is to be used, what is the largest area that can be enclosed?



As the figure suggests, let x = the side parallel to the fence and let y = the length of the other two sides.

We want to maximize the area of the garden, which means our primary equation is $A = xy$, but this equation has too many variables for us to use calculus just yet. Thus, there should be a secondary equation we can use from information in the problem.

We also know there is 24 feet of trim available, which means $y + y + x = 24$, or $2y + x = 24$. This is the secondary equation.

Since it is easier to solve for x, the equation can be written $x = 24 - 2y$.

Now, the area equation can be written $A = xy = (24-2y)y = 24y-2y^2$, which leads us to:

The function to optimize (maximize) is $A(y) = 24y-2y^2$.

The next thing we should look at is the domain of the function. Since y is a side of the rectangle, it must be nonnegative and can be no more than 12 since the total amount of fencing is 24 feet, and there are two sides with length y.

Thus, the domain is $0 \le y \le 12$.

To determine the maximum value, we first take the derivative (with respect to y) and find critical points. Since $A(y)$ is continuous on the closed interval $[0, 12]$, we can apply the extreme value theorem, which means evaluating $A(y)$ at its endpoints and at any critical numbers.

First, find the derivative and critical numbers:

$A(y) = 24y-2y^2$	Start with the original function.
$A'(y) = 24-4y$	Take the derivative.
$24-4y=0$ $24=4y$ $6=y$	Set $A'(y)=0$, then solve.

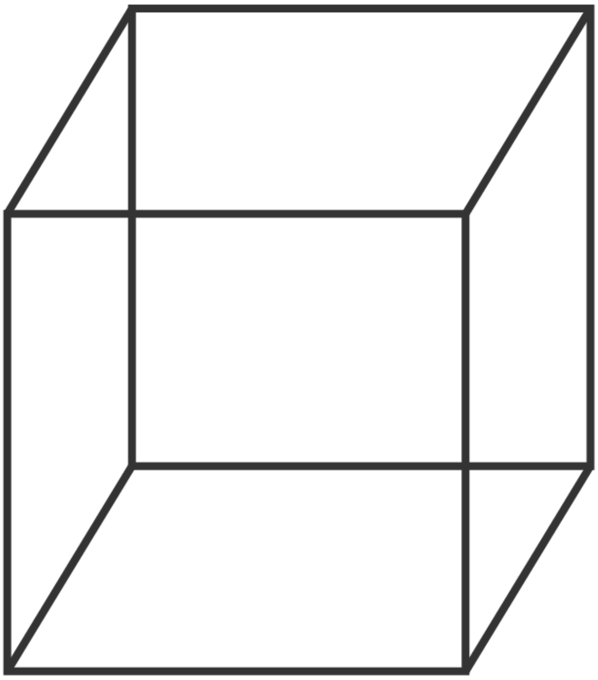
The critical number is $y=6$, which is inside the interval $[0, 12]$. Now, evaluate $A(y)$ at each endpoint and at $y=6$.

y	0	6	12
$A(y) = 24y-2y^2$	0	72	0

Thus, the maximum area is 72 ft^2, which occurs when $y=6$.

Let's now look at an example where we minimize the surface area of a rectangular box with known volume.

→ EXAMPLE A rectangular box with a square base and no lid has volume 500 in^3. What is the least amount of material that could be used to construct such a box? (In other words, what is the minimum surface area?)



- Let x = the length of the base.
- Let h = the height of the box.

Primary Equation:
The surface area is the sum of the areas of all sides. Since this box has no lid, we do not count the area of the top.

The base has area x^2 and each of the four sides have area xh. This means that the surface area is $S = x^2 + 4xh$. This is the primary (optimization) equation.

Secondary Equation:
Since volume is (length)(width)(height), this translates to volume $= x \cdot x \cdot h = x^2 h$.

We are given that the volume is 500 in^3, so this is written $x^2 h = 500$. This is the secondary equation that will be used to write the primary equation in terms of one variable.

To do so, it is easiest to replace h with an expression in terms of x. Using the volume (secondary) equation, rewrite $x^2 h = 500$ as $h = \dfrac{500}{x^2}$.

Substituting into the surface area (primary) equation gives $S = x^2 + 4xh = x^2 + 4x\left(\dfrac{500}{x^2}\right) = x^2 + \dfrac{2000}{x}$,

which leads us to:

The function we want to optimize (minimize) is $S(x)=x^2+\frac{2000}{x}$.

Now, let's discuss the domain of this function. Looking at the equation $h=\frac{500}{x^2}$, the value of *h* will be positive (and therefore valid) for any positive value of *x*. Therefore, the domain is $(0, \infty)$.

Since this is an open interval (no endpoints), we will only focus on finding the minimum value by investigating the function at the critical values in the interval, not the endpoints.

Now, we find the derivative and all critical numbers.

$S(x)=x^2+\frac{2000}{x}=x^2+2000x^{-1}$	Start with the original function; rewrite so the power rule can be used.
$S'(x)=2x-2000x^{-2}$	Take the derivative.
$2x-2000x^{-2}=0$	Set $S'(x)=0$, then solve.
$2x-\frac{2000}{x^2}=0$	Rewrite with positive exponents.
$2x=\frac{2000}{x^2}$	Add $\frac{2000}{x^2}$ to both sides.
$2x^3=2000$	Multiply both sides by x^2.
$x^3=1000$	Divide both sides by 2.
$x=10$	Take the cube root of both sides.

Thus, there is a critical number at $x=10$. To determine if it is a minimum, use the second derivative test.

$S'(x)=2x-2000x^{-2}$	Take the first derivative.
$S''(x)=2+4000x^{-3}$	Take the second derivative.
$S''(10)=2+4000(10)^{-3}=6$	Evaluate $S''(10)$.

Since $S''(10)$ is positive, there is a minimum when $x=10$.

Thus, the minimum amount of material to build the box is $S(10)=10^2+\frac{2000}{10}=300\text{ in}^2$.

WATCH

In this video, we will determine the optimal route to lay cable across two terrains (underground and underwater, with water being more expensive).

Video Transcription

Hi there. In this video, what we're going to do is take all the knowledge we learned about derivatives and finding maximum and minimum points, and we're going to solve an optimization problem. And this particular problem, if you follow along the red, we're trying to find the optimal path to go from point A to point B by selecting a point along the shoreline so that we minimize the cost of laying the cable.

And what we know is that every mile on land costs $5,000, and every mile going underwater costs $8,000. And what we have is that the point A is 2 miles from the shoreline. And we're talking about a distance of 8 feet or 8 miles along the shoreline. So we're trying to find the optimal path. So let's just take a moment and just consider some easy paths just to see.

So one path might be, knowing that the water is the most expensive, one choice might be, well, why don't I go directly across the water and then directly down the shoreline. And while that might appear to be cost effective, the problem is, it's a longer distance than any other distance we could ever find. you know, we're going the whole way across and then the whole way down. That's a total of 10 miles. So it could be the optimal path. It's just not clear.

The other possible path is if we go directly from A to B. And while that is the shortest distance, the problem is it's also the most expensive when it comes to materials. $8,000 per mile, and we're using all of the distance underwater. So that can really drive the cost up.

So what we do is we pick a point along the shoreline for the cable to stop, and then we go the rest of the way down the shoreline. Maybe that'll end up with an optimal solution. We'll see. And of course, the calculus will help us figure out where on the shoreline that should be.

So looking at the two distances, we need distances in terms of x. Now, this straight-line distance here, the horizontal distance, since the entire horizontal distance is 8 and we're saying x is the distance we're going from the left, that means this distance here is 8 minus x. That one wasn't too bad.

The other one, the slanted distance, however, notice that kind of makes a right triangle. Well, it does make a right triangle where one side is x, the other side is 2, and the distance we want is the hypotenuse of that triangle. That, we have to use Pythagorean theorem to get. And just setting it up, we 2 squared plus x squared is equal to-- I'm going to call this side AC, because it's going from point A to point C.

And if we solve for AC, we get the square root of x squared plus 4. So this side here is the square root of x squared plus 4. So when we formulate our cost function, we have to keep those two distances in mind. One distance is the square root of x squared plus 4, and that's going to cost us $8,000 per mile. So the total cost of that piece is $8,000 times the square root of x squared plus 4.

The other piece is going to cost us $5,000 per mile, and we're going 8 minus x miles. There's our cost function. So now, here's an added bonus. We have a nice domain that we can find here too. We know that because x is the location along the shoreline, I know that x cannot be any more than 8. And I know it can't be any less than 0.

So that means we are finding the optimal value of a function on a closed interval. That means we can apply the extreme value theorem, which basically says if you have a continuous function on a closed interval, there is a minimum and there is a maximum guaranteed. So we're going to utilize that instead of using the second derivative test this time.

OK, so ready to find the derivative and find the critical numbers? And then off we go. So first thing you need to do before taking the derivative is to rewrite the square root, because we know we deal better when it's a 1/2 power. So that means C of x is 8,000. x squared plus 4 raised to the 1/2 plus-- and I'm going to go ahead and multiply the 5,000 through. So you have 40,000 minus 5,000x, OK.

So now we're ready to take the derivative. So C prime is-- OK. So 8,000 times something with the power rule. So it's 8,000 times the 1/2, which is 4,000, times the something to the negative 1/2 times the derivative of the inside. The derivative of 40,000 is 0, and the derivative of minus 5,000x is minus 5,000.

And we're going to clean up the derivative first before we do anything else with it. So this is 8,000x times x squared plus 4 to the negative 1/2 minus 5,000. And I'm going to rewrite the negative 1/2 power as dividing by the square root. Remember, that would be over the 1/2 power, and then a 1/2 power mean square root. So you have 8,000x over square root X squared plus 4 minus 5,000.

Now, remember that critical numbers occur when the derivative is either equal to 0 or is undefined. And you might be thinking, this is undefined because there's a denominator here. But remember, that denominator is x squared plus 4. There is no value of x that makes that 0. X squared is non-negative, and adding 4 just means my denominator is always at least 4, never 0.

So I'm going to set the derivative equal to 0 in hopes of finding a critical number. So what we'll do, we'll isolate the x terms to one side. So I'm going to add 5,000 to both sides. So you have 8,000x over the square root equals 5,000. And I'm going to go ahead and multiply both sides by the square root. So that means we have 8,000x equals 5,000 square root x squared plus 4.

And right away, I notice both sides have a common factor of 1,000. So I'm going to divide both sides by 1,000 right away. That's going to go a long way in making our computations much easier, see, if 8x equals 5 squared root x squared plus 4. And now to solve for x, this is an old trick you might remember from your algebra experience. We're going to square both sides.

And after squaring both sides, let's see what we end up with here. So this is 64x squared equals 5 squared is 25. And the square root squared is just what's under the radical. So we have that. Going to distribute the 25. And then subtract 25x squared from both sides. That means we have 39x squared equals 100, which means x squared is equal to 100 over 39. Kind of a weird number there.

So that means x is equal to-- now normally, it's plus and minus. But we know in this problem, we're only dealing with positives. X is a distance. So it's the square root of 100 over 39, which oddly enough, is just very close to 1.6. So that number is on our interval. So it does need to be checked.

So we substitute 1.6 into the cost function. We substitute 0 and 8, because those were the endpoints. And here is the analysis. So when x equals 0, meaning we're going straight across the river and then 8 miles down, the total cost is $56,000. At 1.6, the total cost is $52,490.

And at 8, then that would be corresponding to going directly across the river from point A to point B, the slanted distance. It turns out that is the most costly. It's almost $66,000. So the minimum cost occurs when x is equal to 1.6, and a minimum cost of $52,490.

↳ EXAMPLE A 30-inch rod is to be cut into at most two pieces. One piece will be made into a square, and the other piece will be made into an equilateral triangle. How long should each piece be in order to maximize the combined area, and what is the maximum area?

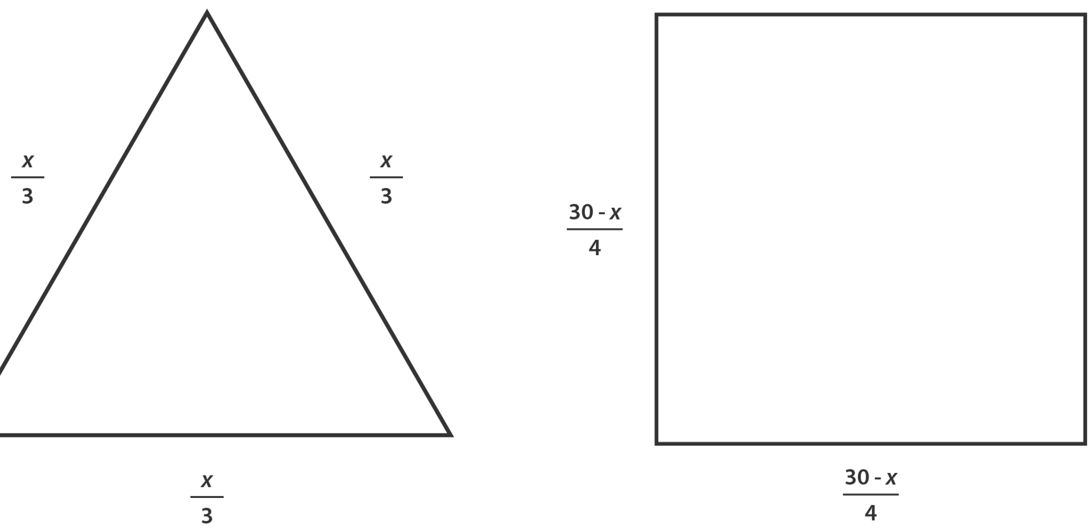
First, start by drawing a picture. Here is the rod being divided into two pieces:

- Let x = the length of the piece that will be used for the triangle.
- Then, $30-x=$ the length of the piece that will be used for the square.

It is clear that x must be between 0 and 30 inches, therefore the domain is $[0, 30]$, a closed interval.

Now, let's form the expressions for the lengths of the sides.

- Triangle: Since the piece of the rod has length x, each of its 3 sides has length $\frac{x}{3}$.
- Square: Since the piece of the rod has length $30-x$, each of its 4 sides has length $\frac{30-x}{4}$.



Forming the area function:

The area of an equilateral triangle with a side of length s is $A=\frac{\sqrt{3}}{4}s^2$. Then, the area of our equilateral triangle is $A_T=\frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2=\frac{\sqrt{3}}{36}x^2$.

The area of a square with sides of length s is $A = s^2$. Then, the area of the square is

$$A_S = \left(\frac{30-x}{4}\right)^2 = \frac{900-60x+x^2}{16} = \frac{225}{4} - \frac{15}{4}x + \frac{1}{16}x^2.$$

Since we want to find the maximum combined area, the optimization function is $A_T + A_S$, which leads us to:

$A(x) = \frac{\sqrt{3}}{36}x^2 + \frac{225}{4} - \frac{15}{4}x + \frac{1}{16}x^2$ on the interval $[0, 30]$.

Now, take the derivative and find all critical numbers on the interval $[0, 30]$. Since $A(x)$ is continuous on the closed interval $[0, 30]$, the extreme value theorem can be used to determine the minimum and maximum values of $A(x)$.

$A(x) = \frac{\sqrt{3}}{36}x^2 + \frac{1}{16}x^2 + \frac{225}{4} - \frac{15}{4}x$	Start with the original function (place like terms next to each other).
$A'(x) = \frac{\sqrt{3}}{36}(2x) + \frac{1}{16}(2x) + 0 - \frac{15}{4}$	Take the derivative.
$A'(x) = \frac{\sqrt{3}}{18}x + \frac{1}{8}x - \frac{15}{4}$	Simplify.
$\frac{\sqrt{3}}{18}x + \frac{1}{8}x - \frac{15}{4} = 0$	Set $A'(x) = 0$. There is no possibility for $A'(x)$ to be undefined since it is a linear function.
$72\left(\frac{\sqrt{3}}{18}x + \frac{1}{8}x - \frac{15}{4}\right) = 72(0)$ $4\sqrt{3}x + 9x - 270 = 0$	Multiply both sides by the LCD (72) to clear the fractions.
$4\sqrt{3}x + 9x = 270$ $(4\sqrt{3} + 9)x = 270$ $x = \frac{270}{4\sqrt{3}+9} \approx 16.95$	Solve for *x*. Since the exact value is complicated, use the approximate value.

Now, make a table to compare the values of $A(x)$ at the critical number as well as the endpoints.

x	0	30	16.95
***A*(x)**	$\frac{225}{4} = 56.25$	$25\sqrt{3} \approx 43.30$	24.47 (approx.)

The maximum area occurs when $x = 0$ (which means the entire 30 inches will be used to make the square and none of it will be used to make the triangle).

Thus, the maximum possible area is 56.25 in^2.

SUMMARY

In this lesson, you learned about the **strategy for solving optimization problems**, which are problems in which the maximum or minimum value is sought (whichever is relevant). As you learned by examining several real-world examples, **solving applied optimization problems** can be particularly challenging since you have to come up with the function on your own. This takes practice, and drawing pictures or making tables is often helpful.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Optimization Problem

A problem in which the maximum or minimum value is sought, whichever is relevant.