Derivatives of Inverse Trigonometric Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn and use rules to differentiate the inverse trigonometric functions. Specifically, this lesson will cover:

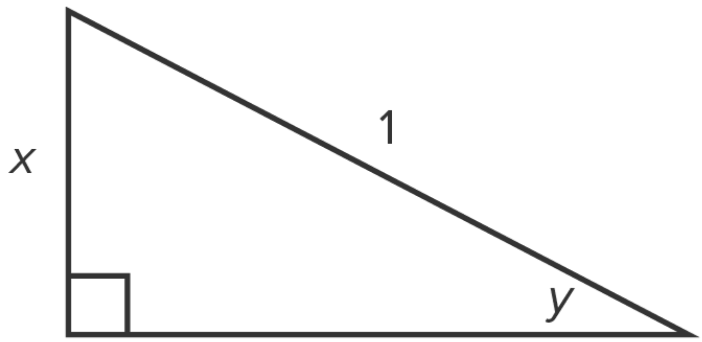
1. Derivatives of the Inverse Trigonometric Functions
2. Derivatives of Functions That Involve Inverse Trigonometric Functions

1. Derivatives of the Inverse Trigonometric Functions

Consider the function $y = \sin^{-1}x$, which is also written $x = \sin y$. To find $\frac{dy}{dx}$, we will use the equation $x = \sin y$ and find the derivative implicitly.

$x = \sin y$	Start with the original equation.
$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$	Set up the derivative on each side.
$1 = \cos y \frac{dy}{dx}$	Take the derivative of each side.
$\frac{dy}{dx} = \frac{1}{\cos y}$	Solve for $\frac{dy}{dx}$.

At this point, it would appear that we are done, but the goal is to get an expression in terms of *x* alone, instead of a function of *y*.



To do so, let's use a right triangle with angle y. Since $x = \sin y$, this means the side opposite y is x and the hypotenuse is 1.

By using the Pythagorean theorem, the length of the adjacent side is $\sqrt{1-x^2}$.

Then, $\cos y = \frac{adjacent}{hypotenuse} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.

Thus, $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$.

In summary, $D[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$.

Through similar reasoning, the derivatives of all six inverse trigonometric functions are shown below. Note that each formula has the basic version (with x as the variable) and the chain rule version (with u as the variable, where u represents a function of x.)

FORMULA

Derivative of the Inverse Sine Function

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\sin^{-1} u] = \frac{u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Cosine Function

$$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\cos^{-1} u] = \frac{-u'}{\sqrt{1-u^2}}$$

Derivative of the Inverse Tangent Function

$$\frac{d}{dx}[\tan^{-1}x] = \frac{1}{1+x^2}$$
$$\frac{d}{dx}[\tan^{-1}u] = \frac{u'}{1+u^2}$$

Derivative of the Inverse Cotangent Function

$$\frac{d}{dx}[\cot^{-1}x] = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}[\cot^{-1}u] = \frac{-u'}{1+u^2}$$

Derivative of the Inverse Secant Function

$$\frac{d}{dx}[\sec^{-1}x] = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}[\sec^{-1}u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

Derivative of the Inverse Cosecant Function

$$\frac{d}{dx}[\csc^{-1}x] = \frac{-1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}[\csc^{-1}u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

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2. Derivatives of Functions That Involve Inverse Trigonometric Functions

With our new derivative rules, we can now find derivatives of functions that contain inverse trigonometric functions.

EXAMPLE Find the derivative of $y = \tan^{-1}(2x)$.

$y = \tan^{-1}(2x)$	Start with the original equation.
$\frac{dy}{dx} = \frac{2}{1+(2x)^2}$	$\frac{dy}{dx} = \frac{u'}{1+u^2}$, $u = 2x$, $u' = 2$
$\frac{dy}{dx} = \frac{2}{1+4x^2}$	Simplify.

Thus, $\frac{dy}{dx} = \frac{2}{1+4x^2}$.

EXAMPLE Consider the function $f(x) = x^2 \cdot \sin^{-1}x$. Find its derivative.

$f(x) = x^2 \sin^{-1}x$	Start with the original equation.
$f'(x) = 2x\sin^{-1}x + x^2\frac{1}{\sqrt{1-x^2}}$	Use the product rule with x^2 and $\sin^{-1}x$.

$$f'(x) = 2x\sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}}$$ Simplify.

Thus, $f'(x) = 2x\sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}}$.

TRY IT

Consider the function $f(x) = \cos^{-1}(x^3)$.

Find the derivative. +

$$f'(x) = \frac{-3x^2}{\sqrt{1-x^6}}$$

WATCH

Find the derivative of $y = 2x^3\arctan(5x^2+3)$.

Naturally, we can apply what we know about inverse trigonometric functions to applications such as finding the slope of the tangent line.

EXAMPLE Compute the slope of the line tangent to the function $y = \sec^{-1}(x^2+1)$ when $x = -1$.

First, find the derivative of $y = \sec^{-1}(x^2+1)$.

$y = \sec^{-1}(x^2+1)$	Start with the original equation.
$\frac{dy}{dx} = \frac{2x}{\lvert x^2+1\rvert\sqrt{(x^2+1)^2-1}}$	$\frac{dy}{dx} = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$, $u = x^2+1$, $u' = 2x$
$\frac{dy}{dx} = \frac{2x}{\lvert x^2+1\rvert\sqrt{x^4+2x^2}}$	Simplify $(x^2+1)^2-1 = x^4+2x^2+1-1 = x^4+2x^2$.
$m_{\tan} = -\frac{\sqrt{3}}{3}$	Substitute -1 for x to get $-\frac{1}{\sqrt{3}}$, then rationalize the denominator.

Thus, the slope of the tangent line is $-\frac{1}{\sqrt{3}}$, which after rationalizing the denominator, is $-\frac{\sqrt{3}}{3}$.

SUMMARY

In this lesson, you learned that by knowing the **derivative rules for the inverse trigonometric functions**, you can now find **derivatives of functions that involve inverse trigonometric functions**, thus expanding on the types of functions you are able to analyze for slope and rates of change, etc.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 7 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

Derivative of the Inverse Cosecant Function

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