Extreme Value Theorem - Endpoint Extremes

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WHAT'S COVERED

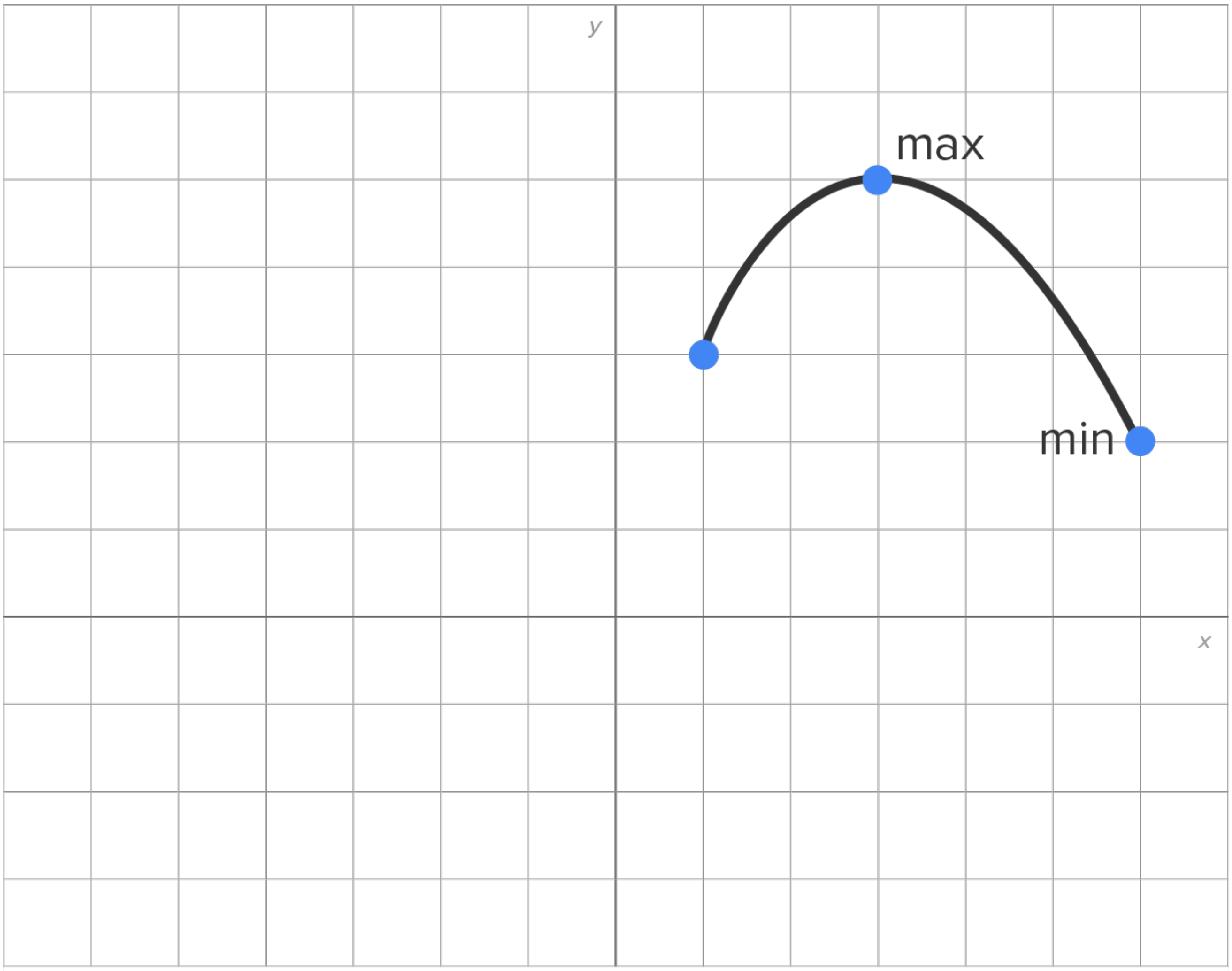
In this lesson, you will use critical numbers and endpoint analysis to determine the maximum and minimum values of a continuous function on some closed interval. Specifically, this lesson will cover:

1. The Extreme Value Theorem
2. Finding Extreme Values of a Continuous Function on a Closed Interval

1. The Extreme Value Theorem

If a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is guaranteed to have global maximum and global minimum values on the interval $[a, b]$. This is known as the **extreme value theorem**.

Here is an illustration of the extreme value theorem:

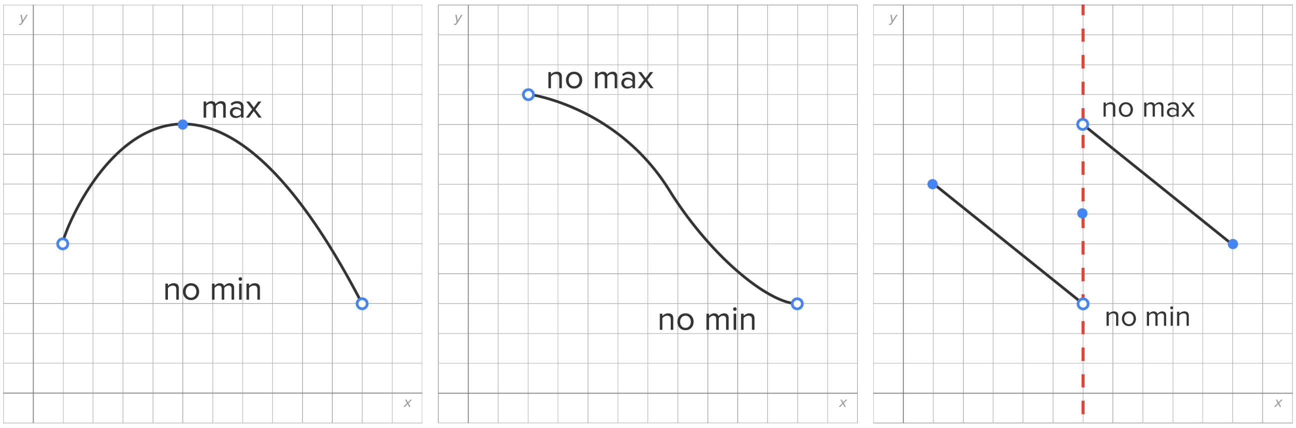


f continuous closed interval

- The function is continuous on the interval $[a, b]$.
- The maximum point occurs inside the interval.
- The minimum occurs at an endpoint.

The following are examples of situations in which one of the criteria is violated.

f Continuous Open Interval	*f* Continuous Open Interval	*f* Not Continuous Closed Interval



TERM TO KNOW

Extreme Value Theorem

If $f(x)$ is a continuous function on some closed interval $[a, b]$, then $f(x)$ has global maximum and global minimum values on the interval $[a, b]$.

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2. Finding Extreme Values of a Continuous Function on a Closed Interval

As a result of the theorem, here is what we need to do in order to find the global minimum and maximum values of $f(x)$ on a closed interval $[a, b]$.

1. Find all critical numbers of $f(x)$ that are in the interval $[a, b]$.
2. Evaluate $f(x)$ at each endpoint and each critical number. The largest value of f is the global maximum and the smallest value of f is the global minimum.

EXAMPLE Find the global maximum and minimum points of the function $f(x)=x^3-6x^2+5$ on the interval $[-1, 3]$.

First, find the critical numbers.

$f(x)=x^3-6x^2+5$	Start with the original function.
$f'(x)=3x^2-12x$	Take the derivative.
$3x^2-12x=0$ $3x(x-4)=0$ $x=0, x=4$	Since $f'(x)$ is a polynomial, it is never undefined. Set $f'(x)=0$ and solve for x.

Therefore, the critical numbers are $x=0$ and $x=4$.

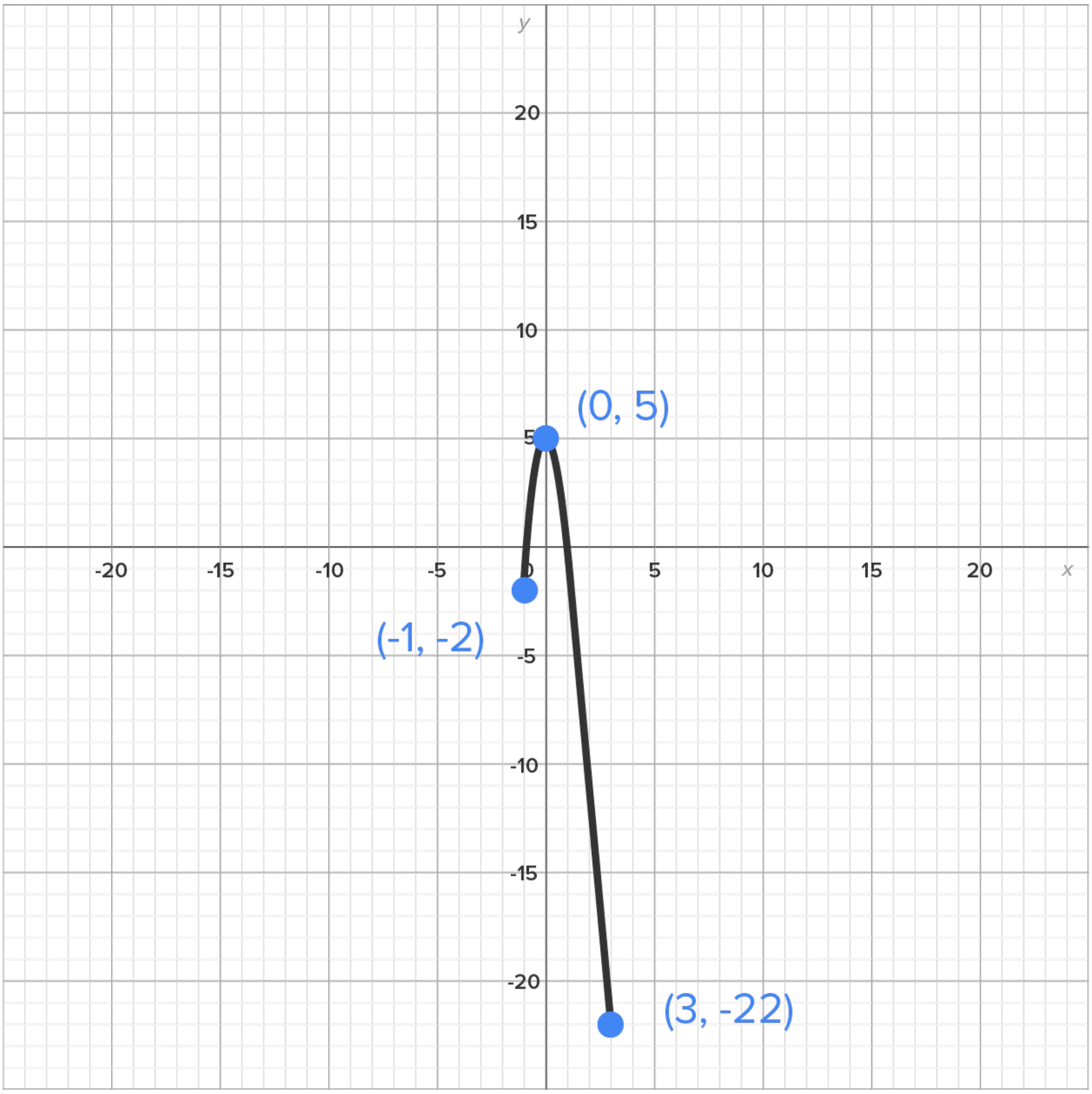
However, since only the closed interval $[-1, 3]$ is considered, the critical value $x=4$ is not used.

Now, evaluate $f(x)$ at the endpoints, $x=-1$ and $x=3$, and the remaining critical number, $x=0$.

x	$f(x)$	Result
-1	$(-1)^3-6(-1)^2+5=-2$	Neither a Global Maximum or Global Minimum
0	$(0)^3-6(0)^2+5=5$	Global Maximum
3	$3^3-6(3)^2+5=-22$	Global Minimum

In conclusion, the global maximum occurs at the point (0, 5) and the global minimum occurs at the point (3, -22). In order words, the global maximum value is 5 and occurs when $x=0$; and the global minimum value is -22 and occurs when $x=3$.

The graph of the function on $[-1, 3]$ is shown below, which confirms the results.



WATCH

In this video, we'll find the global minimum and maximum values of $f(x)=10\sqrt{x}-x$ on the interval $[16, 64]$.

Video Transcription

Hello there. Welcome back. What we're going to look at in this video is the extreme value theorem, meaning that if you have a continuous function on a closed interval, there is a guaranteed absolute maximum value, or as we sometimes call global maximum value and a global minimum value.

And looking at our function here, our function is definitely continuous. There are no breaks in the graph. So therefore, we can use the theorem. So the way mins and maxes work, we definitely have to check the endpoints. So we have x equals 16 and x equals 64 as candidates for where maximum or minimum could occur.

But we also need to check the inside of the interval by finding the critical number. And those are values, remember, where the derivative is either 0 or undefined. And more importantly, those are values of x where we could see a minimum or a maximum. So it comes down to taking the derivative, setting it equal to 0, finding any critical numbers that are on the interval, and then comparing those values against the ones at the endpoints.

So I am going to start by making a table, since we are going to be substituting several values of x, and we already know that 16 is going to get substituted and we already know that 64 is going to get substituted. So let's see what else we can find here. So I'm going to first write the function in a form that's easier to find the derivative. And that is using exponents instead of radical symbols. And then the derivative is-- well, let's see.

1/2 times 10, is 5x to the negative 1/2 minus 1. And just looking at that, we have x to a negative power, which means 1 over x to a positive power. I'm going to write it that way so we can more clearly see what's happening. We see that this is undefined when x equals 0. But that value is of no concern to us, because that's outside of the interval that we're interested in. So we can ignore the fact that this is not defined when x equals 0.

So I'm going to focus on where the derivative is equal to 0. So then I'm going to add 1 to both sides. And I'm going to multiply both sides by x to the 1/2 power. Now, remember, x to the 1/2 is really the square root. So if it helps you to solve that, convert that to a square root. And then I'm going to square both sides and we're going to get 25 equals x. And that, remember, is what we call a critical number.

More importantly, it's inside of our interval, so that means there definitely could be a min or a max happening when x equals 25. So we'll substitute that there, or we'll place that there. Now we're ready to substitute all of our values. We're going to plug in 16, 64 and 25, and see what the biggest and smallest value of f are.

So plugging in 16, we get 10 square root of 16 minus 16. 10 times 4 is 40, minus 16 is 24. If we substitute 64, we have 10 root 64 minus 64. 10 times 8 is 80, minus 64 is 16. And finally, substituting 25, 10 root 25 minus 25.

Root 25 is 5 times 10 is 50, minus 25 is 25. So look at that. We have our maximum value here and our minimum value here. So that means that the global minimum value is 16, which occurs when x equals 64. So we can say at the point 64, comma 16. And the global maximum value is 25, which occurs at the point 25, 25. And there, we have the extreme value theorem.

SUMMARY

In this lesson, you learned that when $f(x)$ is continuous on a closed interval, **the extreme value theorem** guarantees a global minimum value and a global maximum value at some location within the closed interval. Then, you applied this theorem to **find extreme values of a continuous function on a closed interval**, by first finding all critical numbers of $f(x)$ that are in the interval $[a, b]$, then evaluating $f(x)$ at each endpoint and each critical number. This concept is going to be very useful once we use derivatives to solve optimization problems.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Extreme Value Theorem

If $f(x)$ is a continuous function on some closed interval $[a, b]$, then $f(x)$ has global maximum and global minimum values on the interval $[a, b]$.

