Solving y' = f(x)

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WHAT'S COVERED

In this lesson, you will solve differential equations of the form $y' = f(x)$. Specifically, this lesson will cover:

1. What Is a Differential Equation?
2. Solving a Differential Equation
   a. Finding the General Solution
   b. Finding a Particular Solution Based on Initial Conditions
3. Applications
   a. Finding Height Given Velocity and Initial Position
   b. Finding Velocity and Position Given Acceleration

1. What Is a Differential Equation?

A **differential equation** is an equation that contains derivatives of some function y. The solution of the differential equation is the function y that satisfies the equation.

Examples of differential equations include:

- $y' = 2x + 5$
- $y'' + 4y' + 4y = 5\sin t$

In this course, we will solve differential equations of the form $y' = f(x)$.

TERM TO KNOW

Differential Equation
An equation that contains derivatives of some function y.

2. Solving a Differential Equation

2a. Finding the General Solution

Consider the differential equation $y' = f(x)$. The solution can be written $y = \int f(x)dx$. This tells us to take the antiderivative of $f(x)$ to solve this type of differential equation.

Now, recall that we often use $F(x)$ to represent the antiderivative of $f(x)$. That is, $F(x) = \int f(x)dx$.

This all considered, the function $y = F(x) + C$ is called the **general solution** of the differential equation $y' = f(x)$. The general solution is actually a family of solutions since the equation represents a set of functions that differ only by the value of C, the arbitrary constant.

> ↪ EXAMPLE Consider the differential equation $y' = 2x$, which we know has solution $y = x^2 + C$.
>
> This means that $y = x^2$, $y = x^2 - 3$, and $y = x^2 + 5$ are solutions to the differential equation, to name a few.
>
> In fact, if you were to graph each solution, they only differ by a vertical shift.
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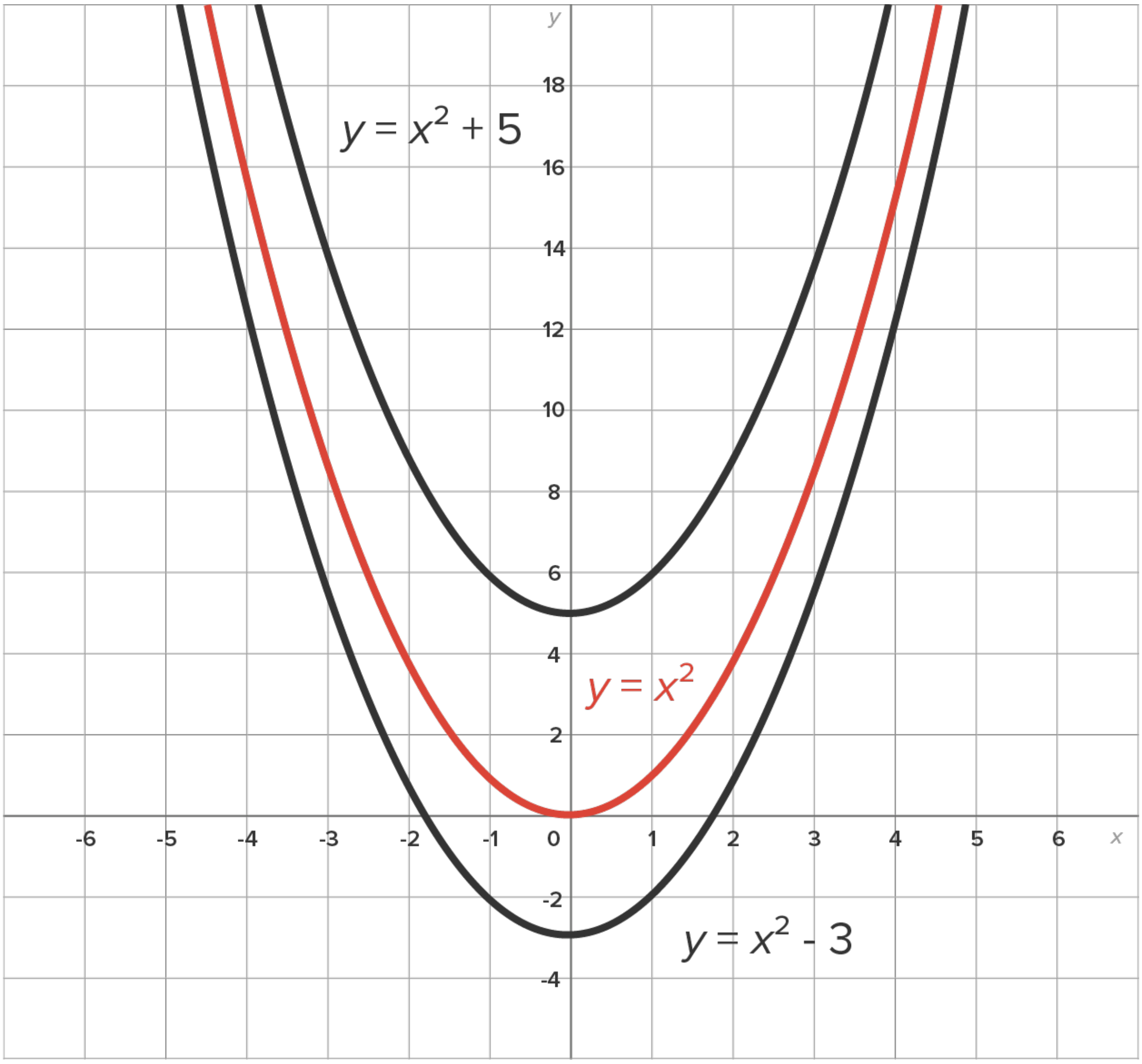
> 
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BIG IDEA

To solve a differential equation of the form $y' = f(x)$, find the antiderivative of $f(x)$, which is called

$F(x)=\int f(x)dx$. Then, the general solution is $y=F(x)+C$.

↪ EXAMPLE Solve $y'=\cos(2x)$.

This means that y is the antiderivative of $\cos(2x)$, written $y=\int \cos(2x)dx$.

Let's go through the antiderivative process.

$\int \cos(2x)dx$	Start with the original expression.
$=\int \cos u \cdot \frac{1}{2}du$	Let $u=2x$. Then, the differential is $du=2dx$. Then, $dx=\frac{1}{2}du$.
$=\frac{1}{2}\int \cos u du$	Move the constant $\frac{1}{2}$ outside the integral sign.
$=\frac{1}{2}\sin u+C$	Use the antiderivative rule for $\cos u$.
$=\frac{1}{2}\sin(2x)+C$	Back-substitute $u=2x$.

Thus, the solution to the differential equation $y'=\cos(2x)$ is $y=\frac{1}{2}\sin(2x)+C$.

TRY IT

Consider the differential equation $y'=x^2-4x+7$.

Find the solution. +

$y=\frac{1}{3}x^3-2x^2+7x+C$

TRY IT

Consider the differential equation $y'=\cos(3x)+3e^x-9x^2$.

Find the solution. +

$y=\frac{1}{3}\sin(3x)+3e^x-3x^3+C$

↪ EXAMPLE Solve $y'=\frac{3}{2x+1}$.

This means $y = \int \frac{3}{2x+1}dx$.

Again, let's go through the antiderivative process.

$y = \int \frac{3}{2x+1}dx$	Start with the original expression.
$= \int \frac{1}{u} \cdot \frac{3}{2}du$	Let $u = 2x+1$. Then, the differential is $du = 2dx$. Then, $dx = \frac{1}{2}du$.
$= \frac{3}{2}\int \frac{1}{u}du$	Move the constant $\frac{3}{2}$ outside the integral sign.
$= \frac{3}{2}\ln\lvert u\rvert + C$	Use the antiderivative rule for $\frac{1}{u}$.
$= \frac{3}{2}\ln\lvert 2x+1\rvert + C$	Back-substitute $u = 2x+1$.

Thus, the solution to the differential equation $y' = \frac{3}{2x+1}$ is $y = \frac{3}{2}\ln\lvert 2x+1\rvert + C$.

TERM TO KNOW

General Solution

The general solution of a differential equation is a function of the form $y = F(x) + C$ that satisfies a differential equation regardless of the value of C.

2b. Finding a Particular Solution Based on Initial Conditions

When obtaining the solution to the differential equation, sometimes we are given a value of y when x is some number in the domain. This is called an **initial condition**.

For example, if the graph is to pass through the point $(1, 5)$, then the initial condition is written $y(1) = 5$.

→ EXAMPLE Solve $y' = 2x$, given that the solution passes through the point $(3, 20)$.

We know $\int 2x dx = x^2 + C$. Thus, the solution to the differential equation is $y = x^2 + C$. Use this equation to find the solution that passes through $(3, 20)$.

$y = x^2 + C$	Use this equation to find the solution.
$20 = 3^2 + C$	Replace x with 3 and y with 20.
$11 = C$	Simplify.

Substituting this answer back into $y = x^2 + C$, the solution to the differential equation is $y = x^2 + 11$.

BIG IDEA

The initial condition is used to find the value of C, the constant of integration.

A **particular solution** is the solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

↪ EXAMPLE Solve $y' = e^{-3x} + x + \sin x$, given that the solution passes through the point $(0, 4)$.

First, find the family of solutions.

$y' = e^{-3x} + x + \sin x$	Start with the original expression.
$y = \int (e^{-3x} + x + \sin x)dx$	If $y' = f(x)$, then $y = \int f(x)dx$.
$y = \int e^{-3x}dx + \int xdx + \int \sin xdx$	Use the sum of antiderivatives property.
$y = \frac{-1}{3}e^{-3x} + \frac{1}{2}x^2 - \cos x + C$	Apply antiderivative formulas.
$4 = \frac{-1}{3}e^{(3 \cdot 0)} + \frac{1}{2}(0)^2 - \cos 0 + C$	Apply the initial condition and replace x with 0 and y with 4.
$C = \frac{16}{3}$	Solve for C.

Thus, the particular solution is $y = \frac{-1}{3}e^{-3x} + \frac{1}{2}x^2 - \cos x + \frac{16}{3}$.

TRY IT

Consider the differential equation $y' = 2\cos(4x) - 12\sin(6x)$ with $y(0) = -1$.

Solve the differential equation. +

$$y = \frac{1}{2}\sin(4x) + 2\cos(6x) - 3$$

WATCH

In this video, we'll solve a differential equation: $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+16}}$, $y(3) = 1$

Video Transcription

[MUSIC PLAYING] Well, hello there. Thank you for joining me today. What we have here is a differential equation, dy dx is equal to x divided by the square root of x squared plus 16, where we know that the y value when x equals 3 is equal to 1. So the goal is to solve this differential equation, which means we're trying to find the function y that satisfies all these conditions. So if dy dx is equal to this expression, that means that y is equal to the antiderivative of this expression.

This is what's going to get us the indefinite answer, the general answer, the family of curves. Then we apply the initial condition to find the value of c. So that's our framework here. The first thing to think about here is, remember that whenever we see a square root in calculus we always write it as a one half power. Since this one is in a denominator, it's really a negative one half power.

That's going to be where I start here. Then it looks like we have a complicated expression to the negative one half power. I'm going to call that u equals x squared plus 16 and that means that differential form is 2x dx. Since I don't have the 2, I'm going to isolate x dx. So that means this piece. This means that x is equal to one half du, just divide both sides by 2. OK.

So looking at that now, x squared plus 16 is u. So u to the negative one half, and the x dx, that all becomes one half du. So I'm going to put the one half here and the du here. OK. Remember that constants can come outside, so I'm going to put the one half outside, u to the negative 1/2 du. Then we use our power rule.

So this is one half times, going up 1 power is one half. I divide by a half right away, plus c. A couple of ways to handle this, I could either bring the one half up as a reciprocal, make it a 2, and then the 2 and the one half would cancel each other out or if I just multiply my fraction straight across that's a one. So either way, I just end up with u to the one half plus c, which is-- Well, we could write that as the square root of u plus c just to make it more familiar, which is the square root of x squared plus 16 plus c. Remember, that's all y.

So the family of curves that solves the functional part of the differential equation is y equals square root of x squared plus 16 plus c. So c could be any constant at this point. We want to know the particular one where 3,1 is a point on the curve. So that means when x equals 3, y is equal to 1. So we substitute 1 for y, we substitute 3 for x, and we see what happens.

So 1 is equal to square root. 3 squared is 9, plus 16, plus c. So you have 1 equals 9 plus 16 is 25 and the square root of 25 is 5, plus c. So that means that c is negative 4. So that we're almost at the final answer. Taking this, this is where we started from, the final answer is y equals square root, x squared plus 16 minus 4. I will share this with you, a lot of times there is confusion about where the square root starts and stops so it is often customary to write the constant first, followed by the square root. That way you know that 16 is the last thing intended to be under the square root, otherwise somebody might think that the minus 4 is also under the square root.

So that's just a little tidbit of information there for you, but there's our solution to our differential equation. The function that satisfies both the condition about y of 3 equals 1 and dy dx equals, as we saw up here, x divided by square root x squared plus 16.

[MUSIC PLAYING]

TERMS TO KNOW

Initial Condition

From a differential equation, a point that the solution's graph passes through.

Particular Solution

The solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.

3. Applications

3a. Finding Height Given Velocity and Initial Position

↱ EXAMPLE Suppose the velocity of an object that is falling from a height of 400 feet is given by $v(t)=120-120e^{-0.4t}$, where t is measured in seconds, and $v(t)$ is measured in meters per second.

What is the object's height above the ground after t seconds, given that the object's initial position was 400 feet above the ground?

We know that the height, $h(t)$, is the antiderivative of the velocity $v(t)$. Let's start there.

$v(t)=120-120e^{-0.4t}$	Start with the original expression.
$h(t)=\int(120-120e^{-0.4t})dt$	We can say $h(t)=\int v(t)dt$.
$h(t)=120t+300e^{-0.4t}+C$	Integrate the right-hand side.
$400=120(0)+300e^{(-0.4\cdot 0)}+C$	We were also told that the initial position was 400 feet away in the positive direction. This means $h(0)=400$.
$400=300+C$	Simplify.
$100=C$	Solve for C.

Thus, the object's height function is $h(t)=120t+300e^{-0.4t}+100$.

3b. Finding Velocity and Position Given Acceleration

If $s(t)$ represents a function's position at time t (usually measured in seconds), recall the following:

- If $v(t)$ represents its velocity at time t, then $v(t)=s'(t)$.
- If $a(t)$ represents its acceleration at time t, then $a(t)=v'(t)=s''(t)$.
- It follows that $v(t)=\int a(t)dt$ and $s(t)=\int v(t)dt$.

↱ EXAMPLE A tennis ball is dropped (no starting velocity) from a height of 400 feet at time $t=0$, where t is measured in seconds. This means that $v(0)=0$ and $s(0)=400$.

If the tennis ball's acceleration (due to gravity) is -32 ft/s^2, find its velocity and position as functions of time.

Since we are given acceleration, we find the velocity function, $v(t)$, first.

$v(t)=\int a(t)dt$	Given the acceleration, we can find the velocity first.
$v(t)=\int(-32)dt$	Plug in the known value of acceleration, -32.

$v(t) = -32t + C$ Evaluate.

$0 = -32(0) + C$ Knowing that $v(0) = 0$, substitute to find C.

$0 = C$ Simplify.

Thus, $v(t) = -32t$.

Now repeat the process to find the position function, $s(t)$.

$s(t) = \int v(t)dt$ Given the velocity, we can find the position.

$s(t) = \int (-32t)dt$ Plug in the known velocity function, $v(t) = -32t$.

$s(t) = -32\left(\frac{1}{2}\right)t^2 + C$ Evaluate.

$s(t) = -16t^2 + C$ Simplify.

$400 = -16(0)^2 + C$ Knowing that $s(0) = 400$, substitute to find C.

$400 = C$ Simplify.

Then, $s(t) = -16t^2 + 400$.

TRY IT

At time $t = 0$, a tomato is launched with an upward velocity of 20 feet per second from a height of 300 feet, where t is measured in seconds. This means that $v(0) = 20$ and $s(0) = 200$. Assume the tomato's acceleration due to gravity is -32 ft/s^2.

Find the velocity of the tomato as a function of time. +

$v(t) = -32t + 20$

Find the position of the tomato as a function of time. +

$s(t) = -16t^2 + 20t + 200$

DID YOU KNOW

In the metric system, the basic unit of distance is meters. The acceleration due to gravity is approximately -9.8 m/s^2. Therefore, we can find $v(t)$ and $s(t)$ for a moving object when the distance is measured in meters rather than feet.

BIG IDEA

If an object moves with acceleration a (constant) with initial velocity v_0 and has initial position s_0, we know the following:

- Its velocity after *t* seconds is $v(t) = at + v_0$.
- Its position at time *t* is $s(t) = \frac{1}{2}at^2 + v_0t + s_0$.

SUMMARY

In this lesson, you began by defining **a differential equation**, which is an equation that contains derivatives of some function *y*. You learned that when **solving a differential equation** of the form $y' = f(x)$, there are two ways to express the solution. When there is no initial condition, **finding the general solution** of the differential equation has the form $y = F(x) + C$, where *C* is any constant. This produces a family of solutions that are vertical shifts of one another. When **finding a particular solution based on an initial condition** that is given, you can find the specific curve from the same family of solutions that satisfied the initial condition.

Extending this idea to other **applications**, you are able to **find the height** of an object in motion **given velocity and initial position** as well as **find the velocity and position** of an object in motion, **given its acceleration** at any time *t*.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 6 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Differential Equation
An equation that contains derivatives of some function *y*.

General Solution
The general solution of a differential equation is a function of the form $y = F(x) + C$ that satisfies a differential equation regardless of the value of *C*.

Initial Condition
From a differential equation, a point that the solution's graph passes through.

Particular Solution
The solution to a differential equation that doesn't contain an arbitrary constant. The particular solution satisfies the differential equation as well as the initial condition.