Greatest Integer Functions

by Sophia

WHAT'S COVERED

In this lesson, you will learn how to graph the greatest integer function. Specifically, this lesson will cover:

1. The Greatest Integer Function
   a. Motivation: Why Do We Need This?
   b. The Basic Greatest Integer Function
   c. The Graph of the Basic Greatest Integer Function
2. Compositions That Involve the Greatest Integer Function

1. The Greatest Integer Function

1a. Motivation: Why Do We Need This?

You have 60 pieces of candy to give a group of x people, and you want to distribute them evenly.

- If there are 15 people, then you would give each person 60/15 = 4 pieces of candy.
- If there are 12 people, then you would give each person 60/12 = 5 pieces of candy.

What happens when you can't divide evenly?
If there are 13 people in the room, then each person would get 60/13 = 4.615... pieces of candy. Since this is impossible and you want to be fair to each person, you would give each person 4 pieces of candy, with some left over.

So, what would be a mathematical rule for this situation?

Let $p(x)$ = the number of pieces of candy each person receives when there are x people.

Then, the value of $p(x)$ is found as follows:

- If $\frac{60}{x}$ is a whole number, then use $\frac{60}{x}$.
- If $\frac{60}{x}$ is not a whole number, then use the next whole number less than $\frac{60}{x}$.

1b. The Basic Greatest Integer Function

This leads us to a need to define another function, called the **greatest integer function**, which is denoted $\lfloor x \rfloor$. The piecewise definition of $\lfloor x \rfloor$ is as follows:

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{the closest integer less than } x & \text{if } x \text{ is NOT an integer} \end{cases}$$

HINT

There are other commonly used notations for the greatest integer function:

$INT(x)$ $[x]$ $floor(x)$

(In computer science, this function is often called the floor function, since it usually produces a lower value).

Here are a few function values for $f(x) = \lfloor x \rfloor$:

Function Statement	Reasoning
$f(3.4) = 3$	Since 3.4 is not an integer, the function returns the greatest integer below 3.4, which is 3.
$f(5) = 5$	Since 5 is an integer, the function returns 5.
$f(-2.1) = -3$	Since -2.1 is not an integer, the function returns the greatest integer that is below -2.1, which is -3.
$f(10.8) = 10$	Since 10.8 is not an integer, the function returns the greatest integer that is below 10.8, which is 10.

TRY IT

Consider the greatest integer function $f(x) = \lfloor x \rfloor$.

What does the function return when $x = 2.9$? +

$f(2.9) = 2$

What does the function return when $x = -4$? +

$f(-4) = -4$

What does the function return when $x = 0.8$? +

$f(0.8) = 0$

FORMULA

The Piecewise Greatest Integer Function

$$\lfloor x \rfloor = \begin{cases} x & \textit{if } x \textit{ is an integer} \\ \textit{the closest integer less than } x & \textit{if } x \textit{ is NOT an integer} \end{cases}$$

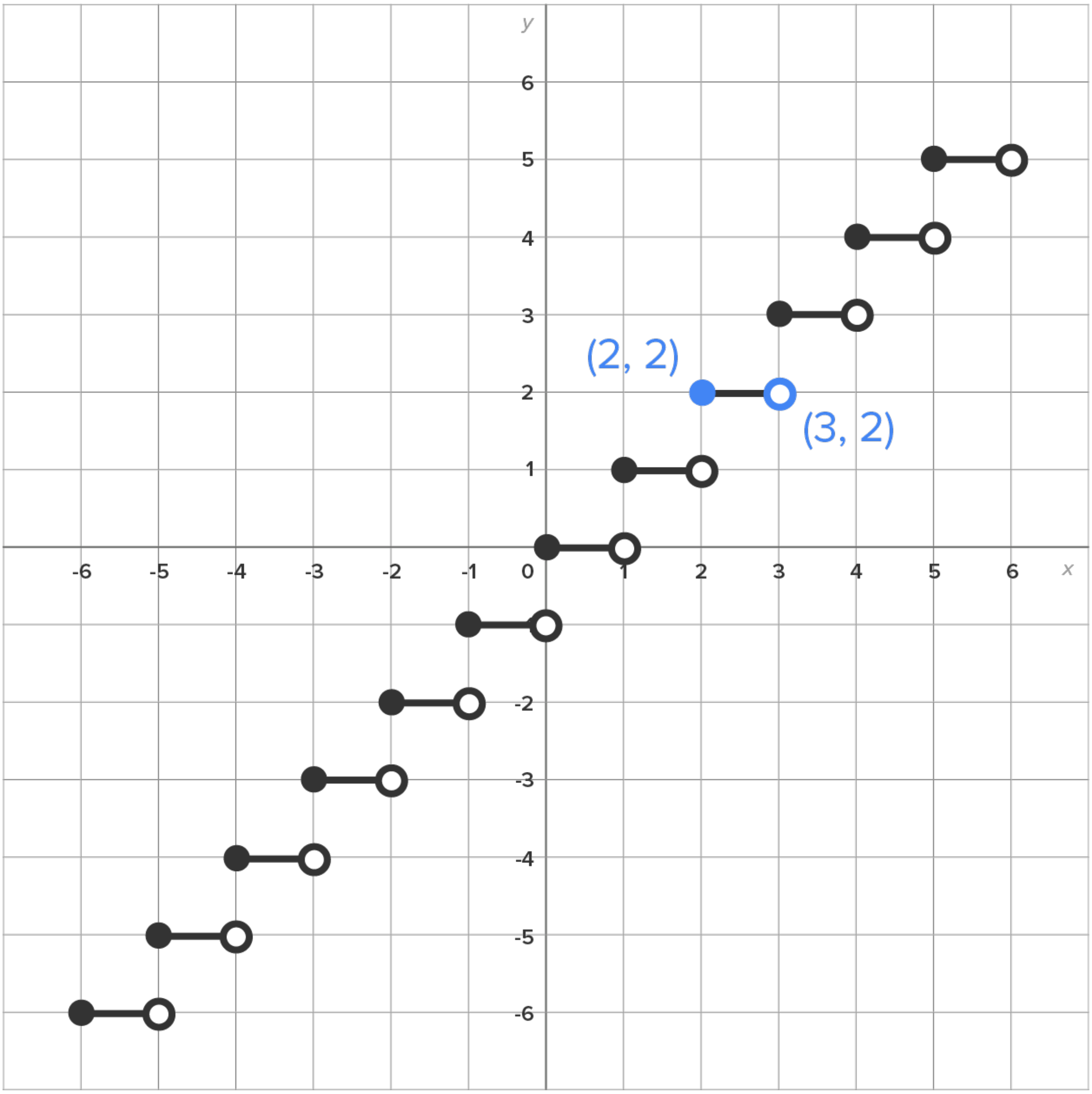
TERM TO KNOW

Greatest Integer Function

Returns the greatest integer that is less than or equal to the input value.

1c. The Graph of the Basic Greatest Integer Function

The graph of the basic greatest integer function is shown below.



Note, the "stair step" pattern continues indefinitely in both directions.

To understand how this graph works, consider the two points that are labelled (2, 2) and (3, 2). If $2 \le x < 3$, the greatest integer function returns the value of 2. We can see this by using some input-output pairs:

x	2	2.2	2.5	2.8	2.9	2.99	2.999	2.9999
$\lfloor x \rfloor$	2	2	2	2	2	2	2	2
(x, y)	(2, 2)	(2.2, 2)	(2.5, 2)	(2.8, 2)	(2.9, 2)	(2.99, 2)	(2.999, 2)	(2.9999, 2)

As soon as the value of x jumps to exactly 3, then the greatest integer function returns a value of 3, which means the graph moves to the next "stair step."

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2. Compositions That Involve the Greatest Integer Function

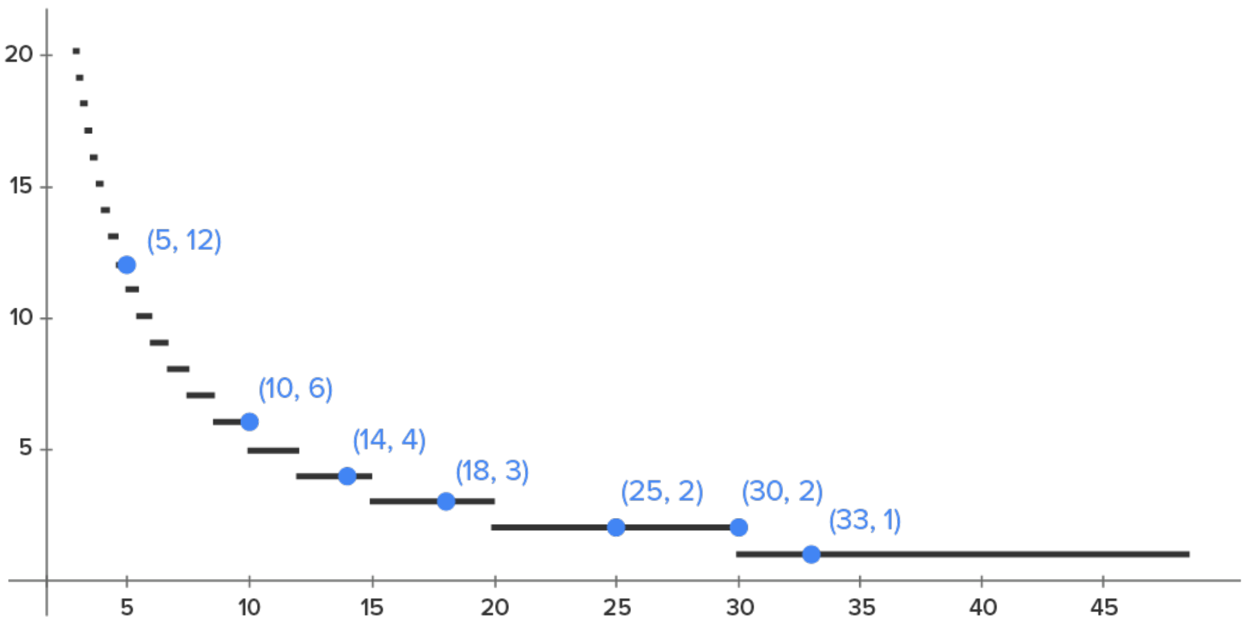
Let's now return to the situation where you were handing out candy to a group of x people.

To find the number of pieces, we calculate $\frac{60}{x}$, then round down if necessary. This is the essence of a greatest integer function!

Using the greatest integer function, the number of pieces received by each person is:

$$p(x) = \left\lfloor \frac{60}{x} \right\rfloor$$

Here is the graph of $p(x)$, with selected points labelled:



Note, the piece of the graph where $y = 1$ would extend out to the point (60, 1) until falling to 0 for $x > 60$. Why is this? If you have more than 60 people in your group and you only have 60 pieces of candy to give out, there is no way to give the same amount of candy to each of them (so each person would receive no candy). On the bright side, that means you get to keep it all.

SUMMARY

In this lesson, you were introduced to **the greatest integer function**, which returns the greatest integer that is less than or equal to the input value. You investigated a real-life situation in which finding a composition involving a greatest integer function was useful, in order to understand the

motivation behind why we need this function (everyone wants their fair share of candy!). In learning about **the basic greatest integer function**, you explored what **the graph of the basic greatest integer function** looks like, noting its "stair step" pattern which continues indefinitely in both directions. Lastly, you circled back to the original real-life situation in order to apply your knowledge to **compositions that involve the greatest integer function.**

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 0 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Greatest Integer Function

Returns the greatest integer that is less than or equal to the input value.

FORMULAS TO KNOW

The Piecewise Greatest Integer Function

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \text{ is an integer} \\ \text{the closest integer less than } x & \text{if } x \text{ is NOT an integer} \end{cases}$$