Concavity

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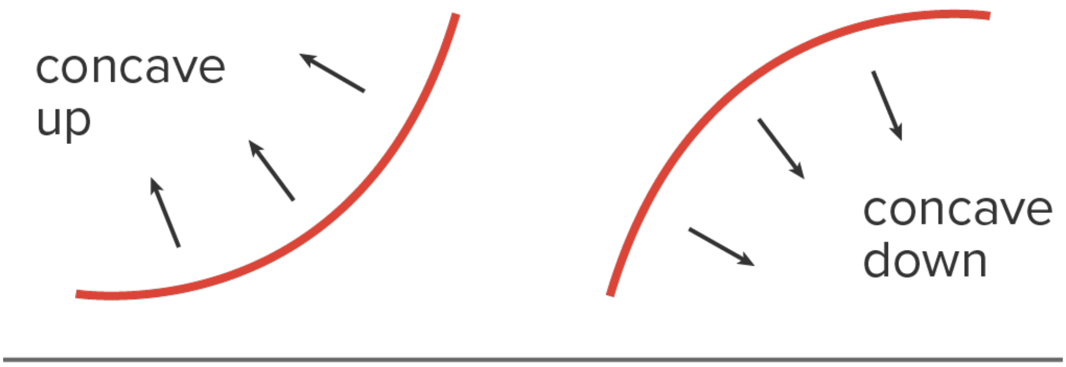
WHAT'S COVERED

In this lesson, you will learn how to use the second derivative to determine the direction that the graph of a function opens, also known as its concavity. Specifically, this lesson will cover:

1. Defining Concavity
2. Determining Where a Function Is Concave Up/Concave Down

1. Defining Concavity

Concavity refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward. A graph is **concave up** on an interval if it opens upward on that interval. A graph is **concave down** on an interval if it opens downward on that interval.



WATCH

This video shows how concavity relates to how slopes of tangent lines change.

Video Transcription

[MUSIC PLAYING] OK, so what we have in this picture is the graph of a function, f of x. And what we're going to do to try to motivate concavity and how it comes about is to analyze the slopes of tangent lines. So as you notice at this point here on the left, we have a slope, and we're about an x equals about negative 3.5 or so. The slope of the tangent line there is a positive 43, so pretty steep, OK? So we're going to do is analyze what happens to the slopes as we move along the curve, as x increases.

So looking at that, if I move along a little bit, you notice that the slopes are decreasing as we move along the curve here. So that's causing the curve to bend in a certain way, OK? And we keep moving to the right, and we hit the horizontal tangent. And the slopes keep decreasing. Now we're into the negative slopes.

And up until we hit this point, p-- now keep your eye on the values. The slope there is negative 14.25, negative 14 and almost 3/4, and then we hit this point p, and then our slope is negative 15. Now, as we move through p, going to the other side, notice the slopes are going another way, which is causing the curve to bend in a different way. If the slopes are increasing, it's causing the curve to bend upward. Because as you move to the right, the curve is starting to go further and further up more steeply.

So how does this relate to the shape of the curve? Well, let's look at this. So coming back to this part, when the slopes were decreasing, the curve was bending in a downward fashion. We call that concave down. Now let's think about how that relates to derivatives. So the slope, which is f prime, is decreasing. So remember that when a function is decreasing, its derivative is negative. So if f prime is decreasing, its derivative is negative. But the derivative of f prime is f double prime, OK? So this is where the second derivative comes into play.

When the second derivative is negative, that means the curve is concave down, because that comes from the slopes decreasing, OK? If we go to the other side of point p here, this is where the slopes are increasing, causing the curve to bend upward, just like you see here. And that's where f prime is increasing, which means its derivative, f double prime, is positive. So that is where the link between the second derivative and concavity comes from, as you're going to read about in the tutorial.

[MUSIC PLAYING]

BIG IDEA

Based on the video, we make the following observations:

- If $f''(x) > 0$ on an interval, then the graph of $f(x)$ is concave up on the same interval.
- If $f''(x) < 0$ on an interval, then the graph of $f(x)$ is concave down on the same interval.

HINT

Remember that a function can change between positive and negative when it is either equal to 0 or when it is undefined. Therefore, to determine where the graph of the function is concave up or concave down, find all values where $f''(x) = 0$ or $f''(x)$ is undefined. Then, make a sign graph similar to what you did for the first derivative test.

TERMS TO KNOW

Concavity

Refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward.

Concave Up
When a graph opens upward on an interval.

Concave Down
When a graph opens downward on an interval.

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2. Determining Where a Function Is Concave Up/Concave Down

↪ EXAMPLE Determine the interval(s) over which the graph of $f(x) = x^3 - 3x^2 + 5$ is concave up or concave down. Since concavity is determined from the second derivative, we start there.

$f(x) = x^3 - 3x^2 + 5$	Start with the original function.
$f'(x) = 3x^2 - 6x$	Take the first derivative.
$f''(x) = 6x - 6$	Take the second derivative.

Since $f''(x)$ is never undefined, we set it to 0 and solve:

$6x - 6 = 0$	The second derivative is set to 0.
$6x = 6$	Add 6 to both sides.
$x = 1$	Divide both sides by 6.

Thus, $f(x)$ could be changing concavity when $x = 1$. This means that at any x-value on the interval $(-\infty, 1)$, the concavity is the same. The same can be said for the interval $(1, \infty)$.

Now, select one number (called a test value) inside each interval to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 1)$	$(1, \infty)$
Test Value	0	2
Value of $f''(x) = 6x - 6$	-6	6
Behavior of $f(x)$	Concave down	Concave up

Therefore, the graph of $f(x)$ is concave down on the interval $(-\infty, 1)$ and concave up on the interval $(1, \infty)$.

↪ EXAMPLE Determine the interval(s) over which the graph of $f(x) = 5x^2 - 18x^{5/3}$ is concave up or concave down. Note that the domain of $f(x)$ is all real numbers.

Since concavity is determined from the second derivative, we start there.

$f(x)=5x^2-18x^{5/3}$	Start with the original function.
$f'(x)=10x-18\cdot\frac{5}{3}x^{2/3}$ $=10x-30x^{2/3}$	Take the first derivative.
$f''(x)=10-30\left(\frac{2}{3}\right)x^{-1/3}$ $=10-20x^{-1/3}$ $=10-\frac{20}{x^{1/3}}$	Take the second derivative.

Note that $f''(x)$ is undefined when $x=0$.

To find other possible transition points, set $f''(x)=0$ and solve:

$10-\frac{20}{x^{1/3}}=0$	The second derivative is set to 0.
$10x^{1/3}-20=0$	Multiply everything by $x^{1/3}$.
$10x^{1/3}=20$	Add 20 to both sides.
$x^{1/3}=2$	Divide both sides by 10.
$x=8$	Cube both sides.

Thus, $f(x)$ could be changing concavity when $x=0$ or $x=8$. This means that at any x-value on the interval $(-\infty, 0)$, the concavity is the same. The same can be said for the intervals $(0, 8)$ and $(8, \infty)$.

Now, select one number (called a test value) inside each interval to determine the sign of $f''(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, 8)$	$(8, \infty)$
Test Value	-1	1	27
Value of $f''(x)=10-\frac{20}{x^{1/3}}$	30	-10	$\frac{10}{3}$
Behavior of $f(x)$	Concave up	Concave down	Concave up

Thus, the graph of $f(x)$ is concave up on $(-\infty, 0)\cup(8, \infty)$ and concave down on the interval $(0, 8)$.

WATCH

In this video, we'll determine the intervals over which the function $f(x)=\ln(x^2+1)$ is concave up or concave down.

Video Transcription

[MUSIC PLAYING] Hi there. Welcome back. What we're going to take a look at in this example is given the function f of x equals the natural log of x squared plus 1, we're going to determine, analytically, where the function is concave up and where the function is concave down.

So remember that concavity-- that information is found from the second derivative. So we're going to first take the second derivative and then do our analysis that way.

So first thing we'll do is take the first derivative. Derivative of the natural log function is 1 over the something, and then times the derivative of the inside, and that gives us 2x over x squared plus 1. And now the second derivative is going to be the derivative of 2x over x squared plus 1, which requires the quotient rule.

So what I'm going to do is, knowing that we need the derivative of each piece, I'm going to write the derivative as 2 here, and I'm going to write the derivative as 2x there. And then the second derivative is-- well, let's see. It's low d high, which means the denominator times the derivative of the numerator, so x squared plus 1 times 2 minus high d low so that is 2x as the numerator times 2x as the derivative of the denominator, and then all over low squared.

So naturally what we will do is simplify the numerator. So we have distributing the two, 2x squared plus 2 minus 4x squared, all over x squared plus 1 squared, which is 2 minus 2x squared over x squared plus 1 squared. And that looks simplified enough for now.

So what we first have to do is figure out where all the possible points of transition are and with any expression that is where it's either equal to 0 or where it's undefined. The good news is, with this one, there is no place where the derivative of the second derivative is undefined because x squared plus 1 can never be 0, which certainly means that the square can never be 0. So that means we don't have to worry about undefined here.

So we're just going to set this equal to 0. And there's two ways to look at this. If you have a fraction that's equal to 0, the only way for a fraction to be equal to 0 is if its numerator is 0. Or we could just go back to basics and multiply both sides by the denominator, which is x squared plus 1 the quantity squared. Either way, we end up with the equation 2 minus 2x squared equals 0.

So we'll solve for x. I'm going to add 2x squared to both sides. I'm going to divide both sides by 2 and I'm going to take the square root of both sides and get plus or minus 1. So those are the possible places where the graph is transitioning between being concave up and concave down.

So we are going to make a sine graph of f double prime, very similar to what we did in f prime in the last video to determine increasing and decreasing-- very similar concept here. So I know that I have a possible transition at negative 1 and another possible transition at 1. And what we're going to do is the same thing. We're going to pick some test numbers. We're going to plug them into the second derivative and try to tell what exactly is happening.

Now remember, the first thing is we divided the number line into three intervals. So let's just mark those down. This is negative infinity to negative 1. This is -1 to 1. And this is 1 to infinity. So for my test numbers, I'm going to pick negative 2 for this interval just because it's close to negative 1. You could pick anything you want to. On the interval -1 to 1, I'm going to pick 0. And on the interval 1 to infinity, I'm going to pick 2.

So we're going to plug all of those into the second derivative to see what's happening. So f double prime at negative 2 is-- and remember, we're going to use the last version of it before we started manipulating it, so we're going to plug it into that. So you have 2 minus 2 times negative 2 squared-- lots of 2's-- divided by negative 2 squared plus 1 squared.

Well, the numerator is, let's see-- negative 2 squared is 4. 2 times 4 is 8. 2 minus 8 is negative 6. The denominator is 4 plus 1 squared, which is 25, which all that tells me is I have a negative value, which means we are concave down.

So I'm just going to draw a picture that looks something like this-- just a concave down. At 0-- and we have 2 minus 2 times 0 squared all over 0 squared plus 1 squared. Now after you work with all the 0's there, the numerator is 2 and the denominator is 1, which is 2, which means we're concave up.

So we are definitely transitioning between concavity types there. And then plugging in 2-- we have 2 minus 2 times 2 squared all divided by 2 squared plus 1 squared. And turns out, that's the same as if we plugged in negative 2. That's still negative 6 over 25. So that result doesn't change. That means we are concave down.

So as far as writing the intervals, it's concave up on the interval. The only interval we have there is -1 to 1. And it's concave down on the interval-- well, let's see. We have negative infinity to negative 1. We also have 1 to infinity. And again, those are both intervals that describe the same set of values. So we put a union in between them-- same set of values that are-- the answer to our problem, I should say.

So there are the intervals where the function is concave up and concave down. And just to kind confirm the results, I do have a graph of the function y equals natural log x squared plus 1. And as you can see, at those points right there, where x is 1 and negative 1, we do indeed have inflection points. You notice it's concave up in the middle, and it switches to concave down as you go to the extremes.

SUMMARY

In this lesson, you learned that **concavity is defined** as the direction in which a graph opens, noting that a graph is concave up if it opens upward on an interval and concave down if it opens downward on an interval. You also learned that you can **determine where a function is concave up/concave down** by using the second derivative of the function $f(x)$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Concave Down
When a graph opens downward on an interval.

Concave Up

When a graph opens upward on an interval.

Concavity

Refers to the direction in which a graph opens. A graph is concave up if it opens upward and concave down if it opens downward.