Some Applications of "Area"

by Sophia

WHAT'S COVERED

In this lesson, you will use the connection between area and velocity graphs in order to find the distance traveled on an interval. This will be the key to understanding the connection between differential calculus and integral calculus. Specifically, this lesson will cover:

1. Applications of Area
2. Calculating Distance Using Area

1. Applications of Area

Let's first look at an example where an object is moving with constant velocity.

EXAMPLE An object is moving at $v(t) = 40$ miles per hour, where *t* is the number of hours since the object started to move.

If we wanted to find the distance traveled between $t = 1$ and $t = 3$ (time measured in hours), we use the distance formula, $\text{distance} = \text{rate} \cdot \text{time}$.

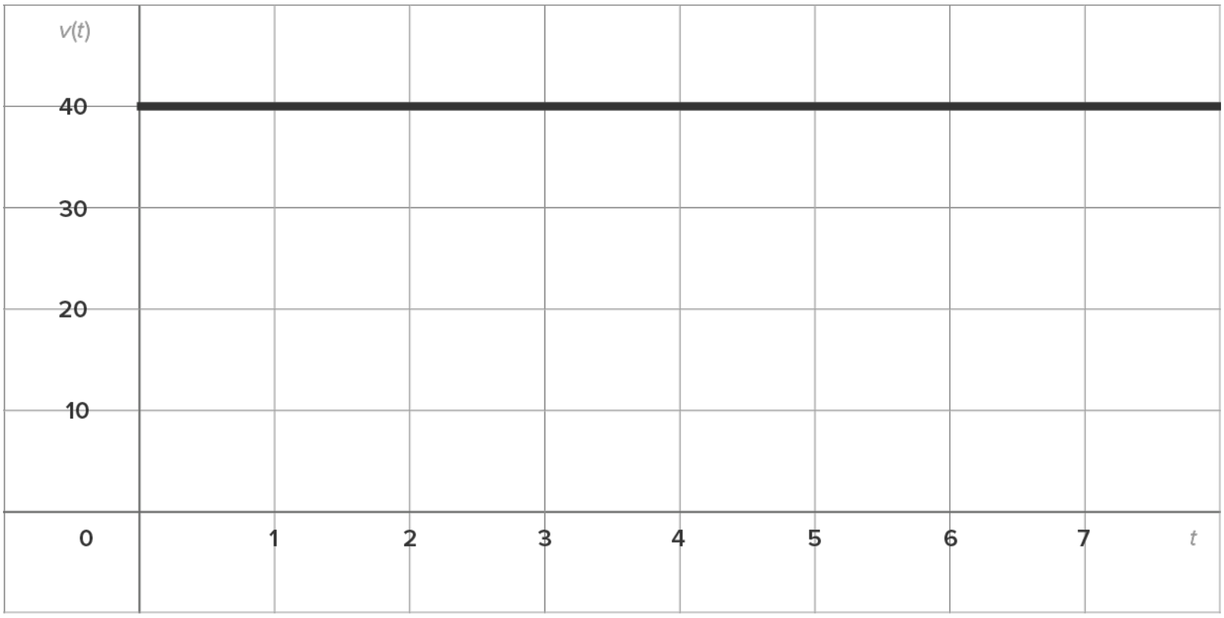
The time elapsed is 2 hours, and the distance is $\frac{40 \text{ miles}}{1 \text{ hour}} \cdot 2 \text{ hours} = 80 \text{ miles}$. Note how the "hours" are canceled out.

The distance on the interval $[2, 7]$ is found in a similar way. The time elapsed is 5 hours, and the distance is $\frac{40 \text{ miles}}{1 \text{ hour}} \cdot 5 \text{ hours} = 200 \text{ miles}$.

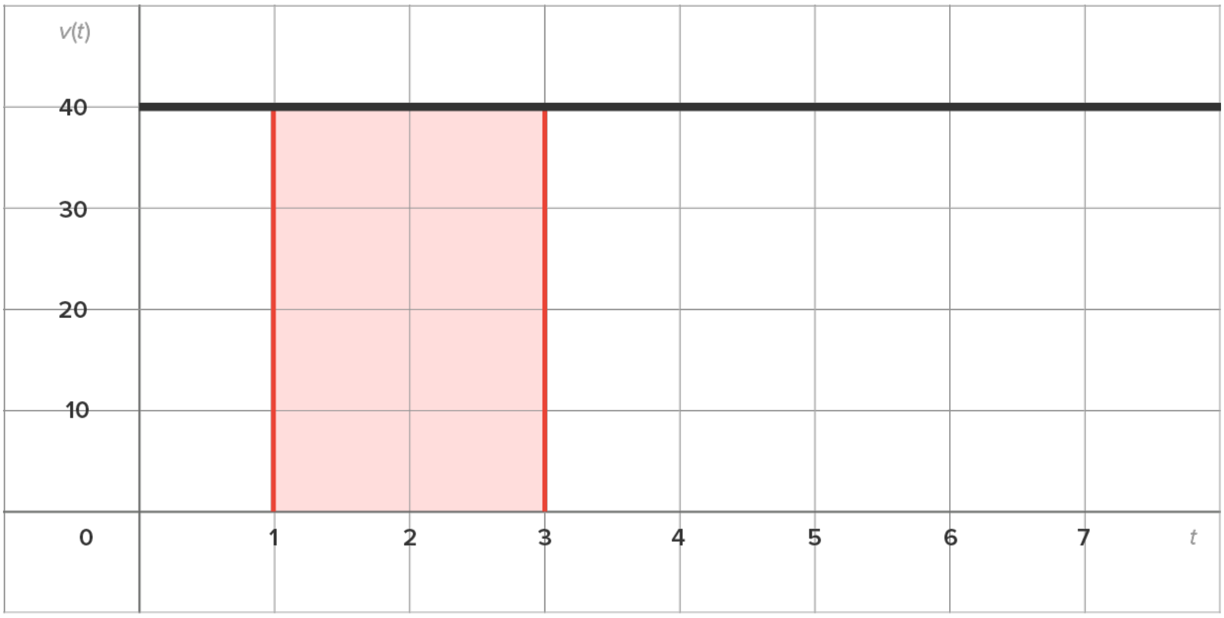
Let's now look at the previous situation, but graphically.

EXAMPLE An object is moving at $v(t) = 40$ miles per hour, where *t* is the number of hours since the object started to move.

Here is a graph of the velocity of the object:



To find the distance traveled between $t = 1$ and $t = 3$, now consider this graph:

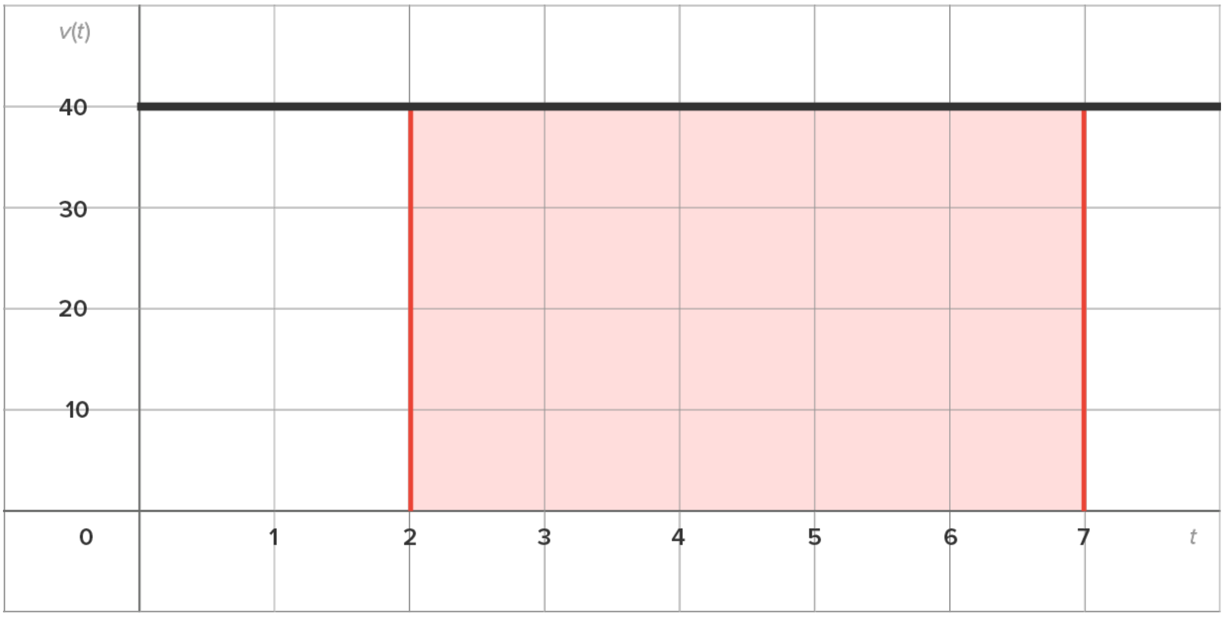


Notice that the area of the region is $2(40) = 80$. Since t is measured in hours and $v(t)$ is measured in miles per hour, we have the same calculation as in the previous example.

$$\text{Area} = (2\text{ hours}) \cdot (40\text{ miles/hour}) = 80\text{ miles}$$

Thus, the area between the velocity graph and the t-axis is the distance traveled.

On the interval $[2, 7]$, we see the same thing:



$\text{Area} = (5 \text{ hours}) \cdot (40 \text{ miles/hour}) = 200 \text{ miles}$, which is the same as the answer in the previous exercise.

BIG IDEA

If given $v(t)$, the velocity after t units of time (seconds, minutes, hours), the distance traveled between $t = a$ and $t = b$ is the area between the graph of $v(t)$ and the t-axis on the interval $[a, b]$.

The area concept can be extended to other applications as well. If $c(x) =$ the cost per unit and $x =$ the number of units, the area under $c(x)$ gives the total cost.

If $f(x)$ measures the force (pounds) at a distance of x (feet), then the area under the graph of $f(x)$ has units "pound-feet," which is a measure of work done.

In summary, the area's units are the product of the units of the horizontal axis and the units of the vertical axis. Therefore, the product has to make sense for the area to be useful.

EXAMPLE If both the horizontal and vertical axes were measured in dollars, then the area would be dollars^2, which makes no sense.

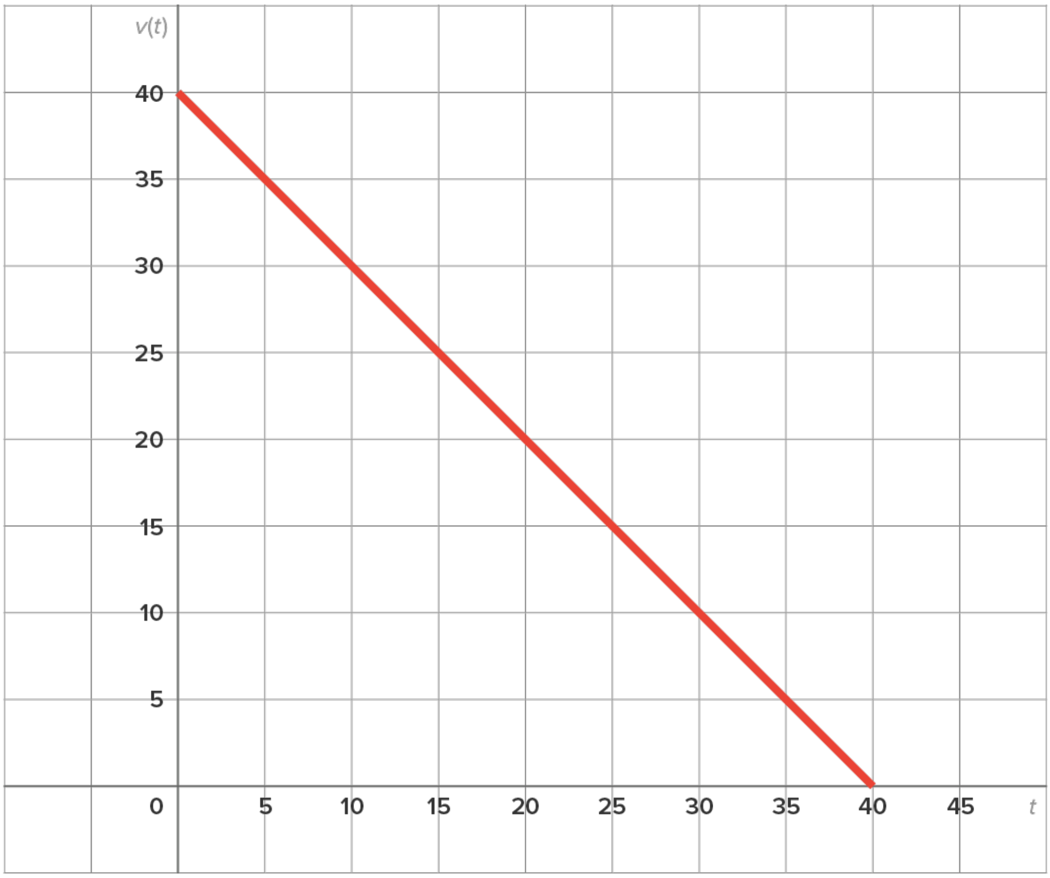
2. Calculating Distance Using Area

Now, let's explore the relationship between distance and velocity by looking at some varied situations.

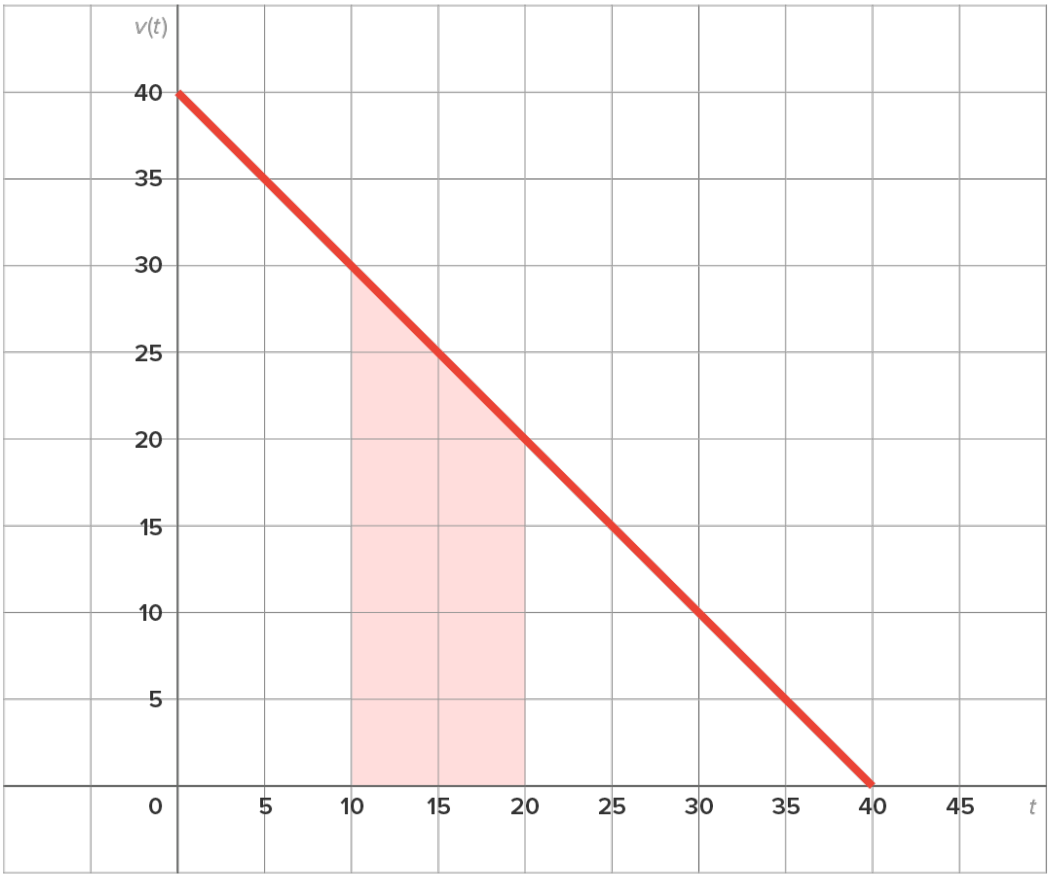
EXAMPLE Suppose the velocity of some object is $v(t) = 40 - t$, where $0 \le t \le 40$.

Here, t is the time in seconds and $v(t)$ is the velocity in meters per second.

The graph of $v(t)$ is shown below:

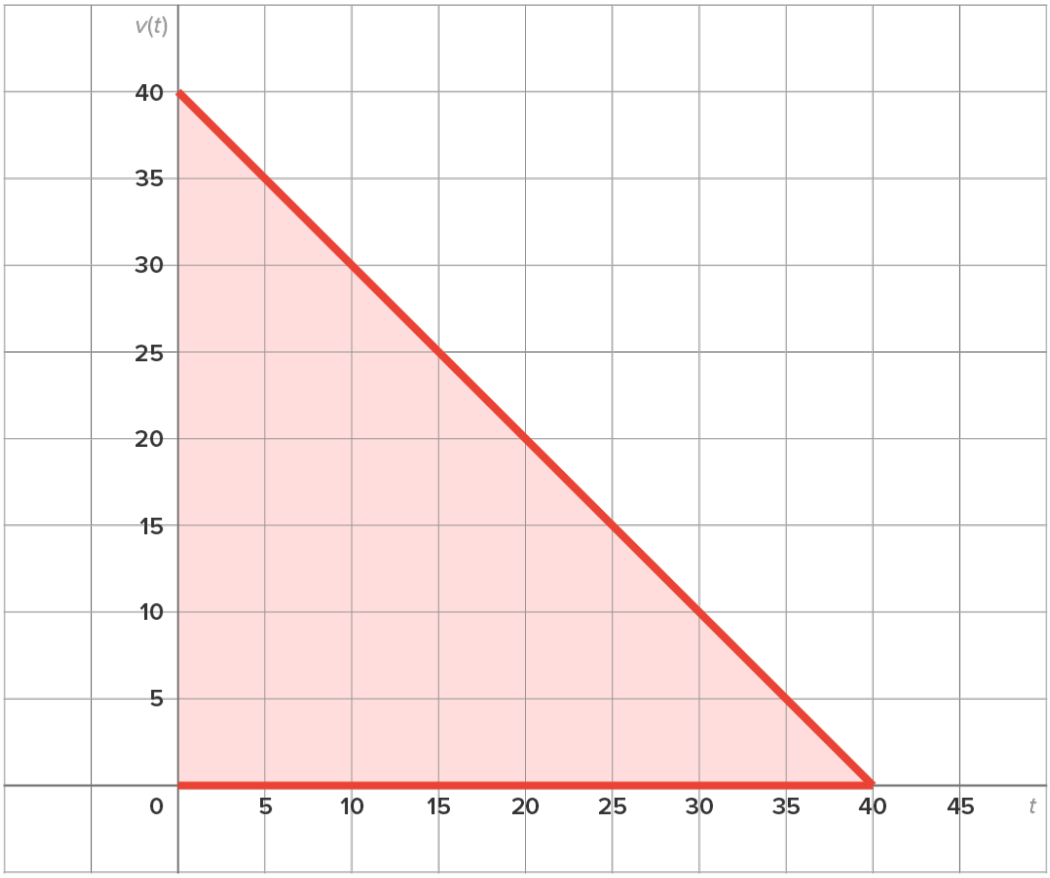


If we want to know the total distance traveled between 10 and 20 seconds, we want the area between $v(t)$ and the t-axis, between $t = 10$ and $t = 20$, as shown below:



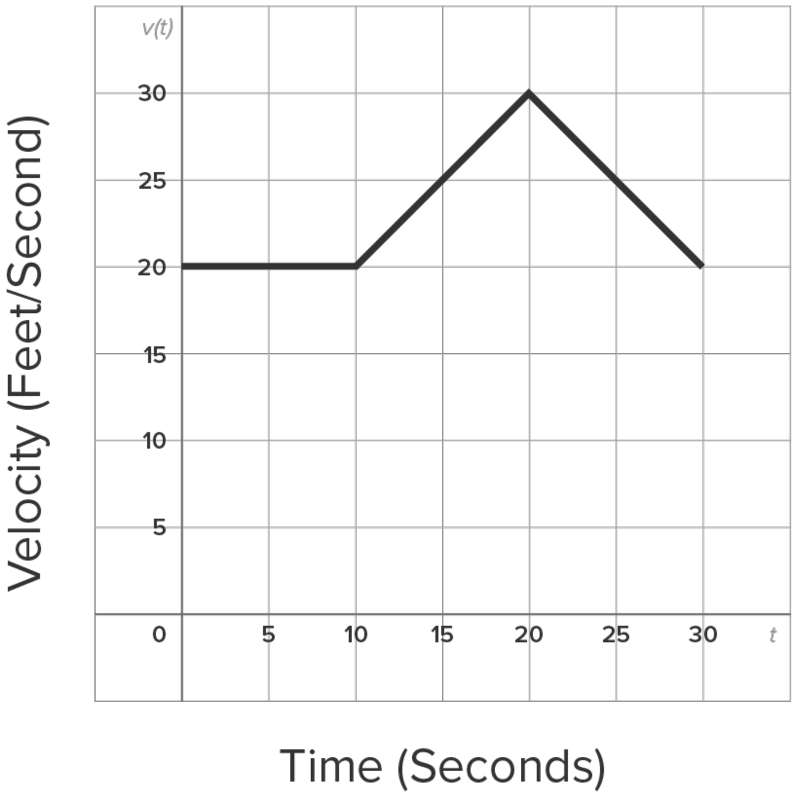
The region is a trapezoid. Note that $v(10) = 30$ and $v(20) = 20$. The parallel bases are 30 and 20 and the height (along the t-axis) is 10. The area is $\frac{1}{2}(10)(30+20)=250,$ which means that the object traveled 250 meters in that time.

Now, if we want to know the total distance traveled between 0 and 40 seconds, we need the area between $v(t)$ and the t-axis, between $t=0$ and $t=40,$ as shown below:

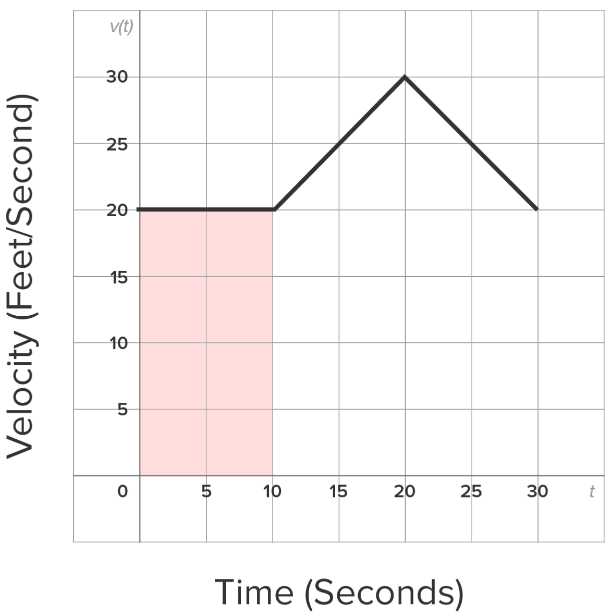


The region is a triangle. Note that $v(0)=40$ and $v(40)=0.$ The area is $\frac{1}{2}(40)(40)=800,$ which means the object traveled 800 meters in that time.

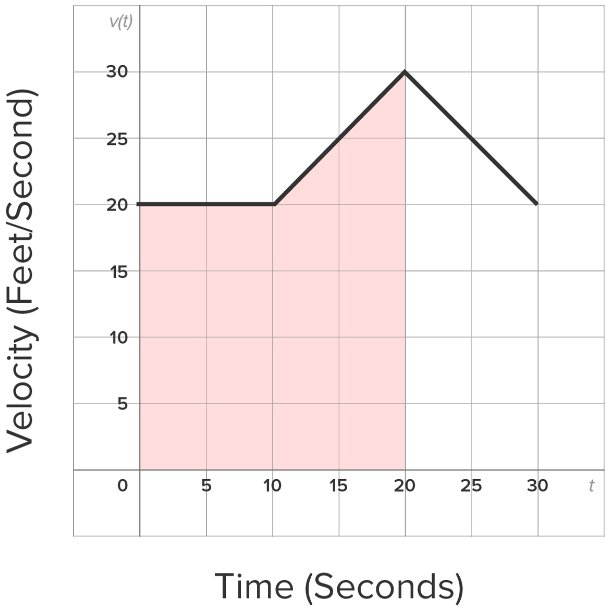
EXAMPLE The graph shows the velocity of a bicycle over time.



Let's see how the distance traveled changes starting at $t=0$ and ending at $t=10$, 20, and 30.

Time	Graph	Description
$t=10$		The region is a rectangle with base 10 and height 20. Thus, the area is $(10)(20)=200$. The distance traveled on $[0, 10]$ is 200 feet.

$t = 20$



Since we already know the area on $[0, 10]$ from above, let's find the area on $[10, 20]$.

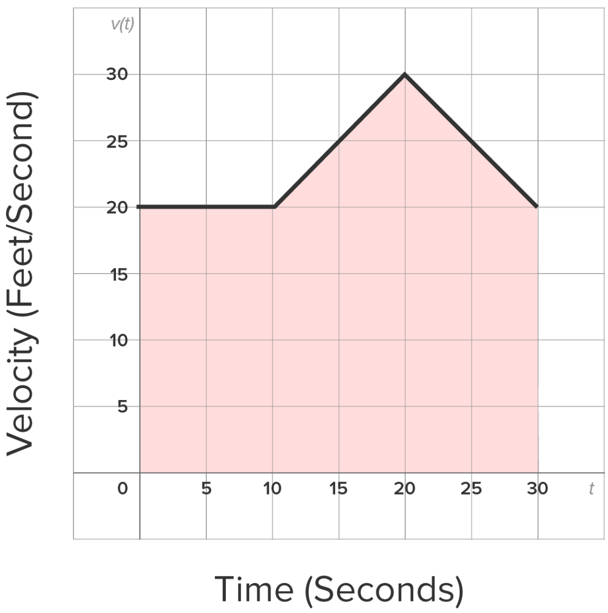
The region is a trapezoid (or triangle on top of a rectangle).

The rectangle has area $(20)(10) = 200$. The triangle has area $\frac{1}{2}(10)(10) = 50$.

The distance traveled on $[10, 20]$ is 250 feet.

Thus, the distance traveled on $[0, 20]$ is $200 + 250 = 450$ feet.

$t = 30$



Since we already know the area on $[0, 20]$ from previous work, let's find the area on $[20, 30]$.

The region is a trapezoid (or triangle on top of a rectangle).

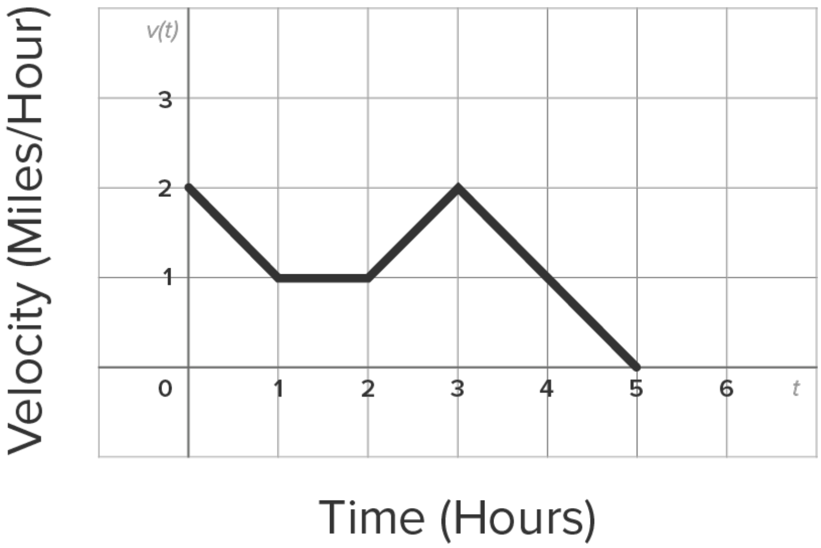
The rectangle has area $(20)(10) = 200$. The triangle has area $\frac{1}{2}(10)(10) = 50$.

The distance traveled on $[20, 30]$ is 250 feet.

Thus, the distance traveled on $[0, 30]$ is $450 + 250 = 700$ feet.

TRY IT

Tom was hiking from some starting point and his velocity is shown in the graph below. Assume the horizontal axis shows the time in hours and the vertical axis shows the velocity in miles per hour.



Find Tom's total distance traveled on the interval [0, 1]. +

1.5 miles

Find Tom's total distance traveled on the interval [0, 2]. +

2.5 miles

Find Tom's total distance traveled on the interval [0, 3]. +

4 miles

Find Tom's total distance traveled on the interval [0, 4]. +

5.5 miles

Find Tom's total distance traveled on the interval [0, 5]. +

6 miles

SUMMARY

In this lesson, you learned about some **applications of area**, notably when units of the product of two quantities are useful, the area between the graph of a function and the x-axis can be used to interpret the result. More specifically, you learned about **calculating distance using area**, understanding that the area between the velocity graph and the t-axis, between $t = a$ and $t = b$, is interpreted as the total distance traveled on that interval.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

