Basic Derivative Rules

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WHAT'S COVERED

In this lesson, you will learn more derivative rules for specific types of functions. Specifically, this lesson will cover:

1. Derivatives of x^n
2. Derivatives of $\sin x$ and $\cos x$
3. Derivatives of Absolute Value Functions
4. Finding the Slope of a Tangent Line

1. Derivatives of x^n

So far, here is what we know about the derivative of $f(x) = x^n$:

Value of n	$f(\mathbf{x})$	$f'(\mathbf{x})$
$n = 1$	$f(x) = x$	$f'(x) = 1$ (Derivative of linear function)
$n = 2$	$f(x) = x^2$	$f'(x) = 2x$ (Derived in last challenge)

Now, let's look at other values of n.

If $n = 3$, then $f(x) = x^3$.

Also, $f(x+h) = (x+h)^3 = x^3 + 3hx^2 + 3h^2x + h^3$.

Then, evaluate the limit:

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$	Apply the limit definition of a derivative.
$= \lim\limits_{h \to 0} \dfrac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}$	Replace $f(x+h)$ and $f(x)$ with their expressions.
$= \lim\limits_{h \to 0} \dfrac{3hx^2 + 3h^2x + h^3}{h}$	Simplify the numerator.

$$= \lim_{h \to 0} \left(\frac{3hx^2}{h} + \frac{3h^2x}{h} + \frac{h^3}{h} \right)$$ Divide each term by h.

$$= \lim_{h \to 0} (3x^2 + 3hx + h^2)$$ Remove the common factor of h in each fraction.

$$= 3x^2$$ Substitute 0 for h.

Thus, when $f(x) = x^3$, its derivative is $f'(x) = 3x^2$.

Let’s look at one more power:

If $n = 4$, then $f(x) = x^4$.

Also, $f(x+h) = (x+h)^4 = x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4$.

Then, evaluate the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ Apply the limit definition of a derivative.

$$= \lim_{h \to 0} \frac{x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4 - x^4}{h}$$ Replace $f(x+h)$ and $f(x)$ with their expressions.

$$= \lim_{h \to 0} \frac{4hx^3 + 6h^2x^2 + 4h^3x + h^4}{h}$$ Simplify the numerator.

$$= \lim_{h \to 0} \left(\frac{4hx^3}{h} + \frac{6h^2x^2}{h} + \frac{4h^3x}{h} + \frac{h^4}{h} \right)$$ Divide each term by h.

$$= \lim_{h \to 0} (4x^3 + 6hx^2 + 4h^2x + h^3)$$ Remove the common factor of h in each fraction.

$$= 4x^3$$ Substitute 0 for h.

Thus, when $f(x) = x^4$, its derivative is $f'(x) = 4x^3$.

Now, let’s put the derivatives we’ve seen together:

Value of n	$f(x)$	$f'(x)$
$n = 1$	$f(x) = x$	$f'(x) = 1$
$n = 2$	$f(x) = x^2$	$f'(x) = 2x$
$n = 3$	$f(x) = x^3$	$f'(x) = 3x^2$
$n = 4$	$f(x) = x^4$	$f'(x) = 4x^3$

In these functions, it appears that the original exponent becomes the coefficient, while the new exponent is 1 less than the original exponent.

FORMULA

Power Rule

$$\frac{d}{dx}[x^n] = n \cdot x^{n-1}$$

This is also written in other ways:

- $D[x^n] = n \cdot x^{n-1}$
- If $y = x^n$, then $y' = n \cdot x^{n-1}$ or $\frac{dy}{dx} = n \cdot x^{n-1}$

HINT

Recall the other functions that can be written with exponents:

- Radical functions: $f(x) = \sqrt[n]{x} = x^{1/n}$
- Reciprocal functions: $f(x) = \frac{1}{x^n} = x^{-n}$

EXAMPLE Find the derivative of $f(x) = x^7$.

Apply the power rule: $f'(x) = 7x^{7-1} = 7x^6$

EXAMPLE Find the derivative of $g(x) = \frac{1}{x^3}$.

First, rewrite as $g(x) = x^{-3}$.

Now apply the power rule: $g'(x) = -3x^{-3-1} = -3x^{-4}$

Since there is a negative exponent in the answer, this is not considered to be in simplest form. Using properties of exponents, write $g'(x) = \frac{-3}{x^4}$.

EXAMPLE Find the derivative of $h(x) = \sqrt{x}$.

First, rewrite as $h(x) = x^{1/2}$.

Now apply the power rule: $h'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$

Since there is a negative exponent in the answer, this is not considered to be in simplest form. Using properties of exponents, write $h'(x) = \frac{1}{2x^{1/2}}$. This could also be written as $h'(x) = \frac{1}{2\sqrt{x}}$.

TRY IT

Consider the functions $f(x) = x^{14}$, $g(x) = \frac{1}{x}$, and $h(x) = \frac{1}{\sqrt[3]{x}}$.

Find the derivative of *f*. +

$f'(x) = 14x^{13}$

Find the derivative of *g*. +

$g'(x) = -\frac{1}{x^2}$

Find the derivative of *h*. +

$h'(x) = -\frac{1}{3x^{4/3}}$

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2. Derivatives of sin *x* and cos *x*

In order to use the limit definition, let's keep the following identities and limits in mind:

- $\sin(x+h) = \sin x \cos h + \sin h \cos x$
- $\cos(x+h) = \cos x \cos h - \sin x \sin h$
- $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$
- $\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = 0$

For the derivative of $f(x) = \sin x$, we set up the limit definition as usual:

$f'(x) = \lim\limits_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$	Apply the limit definition of a derivative.
$= \lim\limits_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$	Replace $\sin(x+h)$ with $\sin x \cos h + \cos x \sin h$.
$= \lim\limits_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$	Factor sin*h*.
$= \lim\limits_{h \to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right)$	Write each part over *h*.
$= \lim\limits_{h \to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right)$	Group "*h*" terms together.
$= \sin x(0) + \cos x(1)$	$\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = 0$, $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$
$= 0 + \cos x$	Simplify.
$= \cos x$	Simplify.

Thus, if $f(x) = \sin x$, $f'(x) = \cos x$.

For the derivative of $f(x) = \cos x$, we again set up the limit definition as follows:

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$ Apply the limit definition of a derivative.

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$ Replace $\cos(x+h)$ with $\cos x \cos h - \sin x \sin h$.

$$= \lim_{h \to 0} \frac{\cos x \cos h - \cos x - \sin x \sin h}{h}$$ Group $\cos x$ and $\sin x$ terms.

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - (\sin x)\sin h}{h}$$ Factor $\cos x$.

$$= \lim_{h \to 0} \left(\frac{\cos x(\cos h - 1)}{h} - \frac{(\sin x)\sin h}{h} \right)$$ Write each part over h.

$$= \lim_{h \to 0} \left(\cos x \left(\frac{\cos h - 1}{h} \right) - \sin x \left(\frac{\sin h}{h} \right) \right)$$ Group "h" terms together.

$$= \cos x(0) - \sin x(1)$$ $\lim_{h \to 0} \frac{\sin h}{h} = 1$, $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

$$= 0 - \sin x$$ Simplify.

$$= -\sin x$$ Simplify.

Thus, if $f(x) = \cos x$, $f'(x) = -\sin x$.

FORMULA

Derivative of Sine

$$\frac{d}{dx}[\sin x] = \cos x$$

Derivative of Cosine

$$\frac{d}{dx}[\cos x] = -\sin x$$

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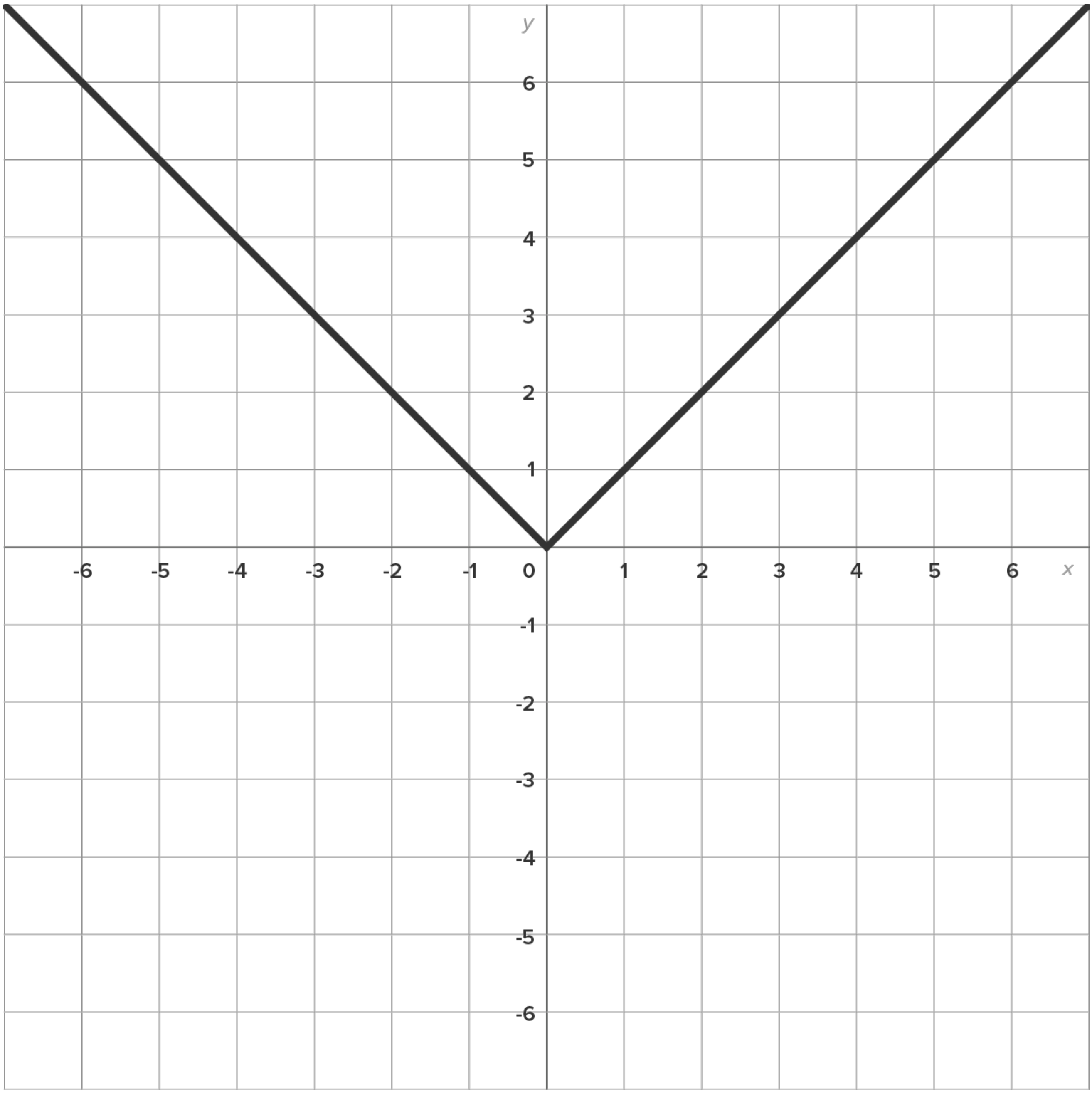
3. Derivatives of Absolute Value Functions

Recall the piecewise definition of $|x|$ and its graph:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$



- When $x < 0$, the graph is the line $y = -x$, which has slope -1.
- When $x > 0$, the graph is the line $y = x$, which has slope 1.
- When $x = 0$, the slope changes abruptly from -1 to 1, suggesting that there is no derivative when $x = 0$.

We can investigate this more closely using the limit definition of derivative for $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

Now, let's examine the expression $\frac{|h|}{h}$ when $h < 0$ and when $h > 0$.

- When $h < 0$, $|h| = -h$, therefore $\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$.
- When $h > 0$, $|h| = h$, therefore $\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1$.

Thus, $\lim\limits_{h \to 0} \frac{|h|}{h}$ does not exist. Since this limit is $f'(0)$, we also say that $f'(0)$ does not exist.

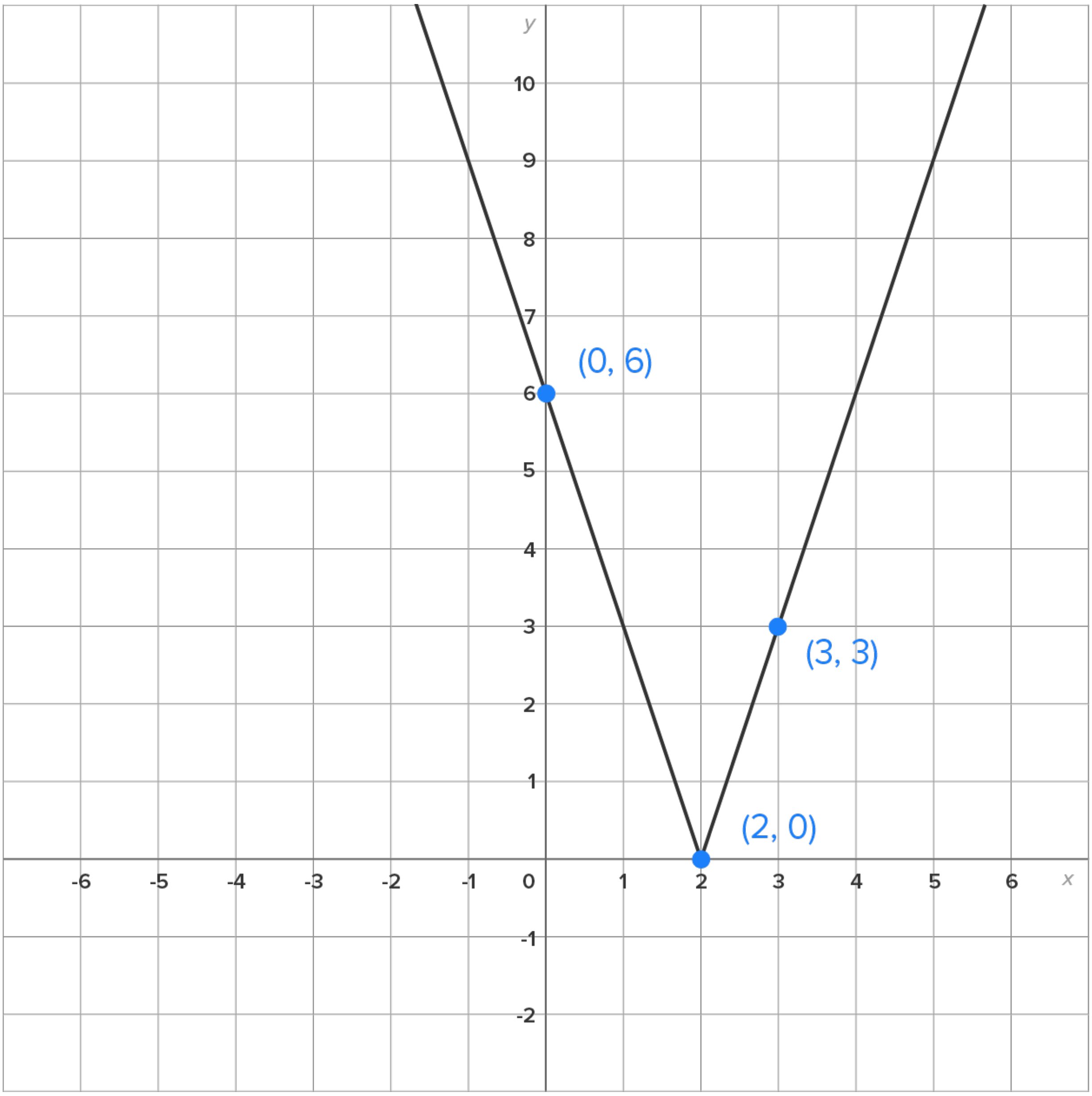
This means that the derivative of $f(x) = |x|$ is as follows:

$$D[|x|] \begin{cases} -1 & if\ x < 0 \\ 1 & if\ x > 0 \\ \text{undefined} & if\ x = 0 \end{cases}$$

This idea can be applied to any absolute value function. We tend to analyze absolute value functions graphically rather than by using formulas.

↱ EXAMPLE Find the derivative of $f(x) = 3|x - 2|$ graphically.

The graph of $f(x)$ is shown below.



When $x < 2$, the slope of the graph is -3.
When $x > 2$, the slope of the graph is 3.

When $x=2$, the graph has a corner point and therefore the derivative is undefined there.

Therefore,

$$f'(x)=\begin{cases} -3 & \text{if } x<2 \\ 3 & \text{if } x>2 \\ \text{undefined} & \text{if } x=2 \end{cases}$$

4. Finding the Slope of a Tangent Line

Now that we have some “shortcut” rules for finding derivatives, finding the slope of a tangent line is now a much easier process.

EXAMPLE Find the slope of the tangent line to the graph of $f(x)=\frac{1}{x}$ when $x=3$ and $x=6$.

First, we need to find $f'(x)$. To do so, we need to rewrite $f(x)=\frac{1}{x}=x^{-1}$.

Now apply the power rule: $f'(x)=-1x^{-2}=\frac{-1}{x^2}$

The slope of the tangent line when $x=3$ is $f'(3)=\frac{-1}{3^2}=-\frac{1}{9}$.

The slope of the tangent line when $x=6$ is $f'(6)=\frac{-1}{6^2}=-\frac{1}{36}$.

SUMMARY

In this lesson, you learned that the limit definition of derivative is useful in establishing “shortcut” rules for finding **derivatives of x^n, sin*x*, cos*x*, and absolute value functions**. Using these rules enables us to solve problems involving derivatives and rates of change much more quickly and succinctly, such as **finding the slope of a tangent line**.

FORMULAS TO KNOW

Derivative of Cosine

$$\frac{d}{dx}[\cos x]=-\sin x$$

Derivative of Sine

$$\frac{d}{dx}[\sin x]=\cos x$$

Power Rule

$$\frac{d}{dx}\left[x^{n}\right]=n\cdot x^{n-1}$$

