Sophia

Sigma Notation

by Sophia

WHAT'S COVERED

In this lesson, you will use sigma notation to evaluate sums, numbers that are a result of some formula. This is a key idea in integral calculus, since we will be adding up several areas. Specifically, this lesson will cover:

1. Introduction to Sigma Notation
2. Using Sigma Notation
3. Special Sums and Summation Properties
   a. Adding Powers of Positive Integers
   b. Summation Properties

1. Introduction to Sigma Notation

When there are many numbers to add together, sigma notation (Σ) is used to represent this sum (sometimes called a **summation**). Here are some examples of how sums can be written using sigma notation.

EXAMPLE

Sum	Sigma Notation
$1^2+2^2+3^2+4^2+5^2$	$\sum_{k=1}^{5} k^2$
$2^0+2^1+2^2+2^3$	$\sum_{k=0}^{3} 2^k$
$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$	$\sum_{k=3}^{8} \frac{1}{k}$

BIG IDEA

In general, sigma notation is written as:

$$\sum_{k=1}^{n} a_k$$

- The symbol Σ (capital sigma) is used to represent a sum.
- "k" is called the index of summation or the counter.
- The starting value (in this instance) is 1 and the stopping (ending) value is n.
- The notation a_k represents the "formula" for the sequence of numbers being added. In the three problems in the previous example, these were k^2, 2^k, and $\frac{1}{k}$. This is also called the **summand**, in other words, the expression being used to determine the numbers that are added in a sum.

When using many rectangles to represent an area under a curve, sigma notation will be used to represent that sum cohesively. Before discussing the area application, let's see how to use sigma notation.

TERMS TO KNOW

Summation

An expression that implies that several numbers are being added together. These are often written using sigma notation.

Summand

The expression being used to determine the numbers that are added in a sum.

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2. Using Sigma Notation

We saw earlier how a summation translates to sigma notation; now let's use sigma notation to find sums.

EXAMPLE Evaluate the sum $\sum_{k=1}^{5} k^2$.

To evaluate, substitute the values of k from 1 to 5, then add the results.

$$\sum_{k=1}^{5} k^2 = 1^2+2^2+3^2+4^2+5^2 = 1+4+9+16+25 = 55$$

TRY IT

Consider the sum $\sum_{k=0}^{4} 3^k$.

Evaluate the sum. +

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EXAMPLE Write out the sum $\sum_{k=1}^{4} f(x_k)$.

Substituting the values of k, we have $f(x_1)+f(x_2)+f(x_3)+f(x_4)$.

Now, let's get some practice writing sigma notation.

EXAMPLE Write the sigma notation for 2 + 4 + 6 + ... + 80. Assume the starting value is $k = 1$.

In this example, $k = 1$ corresponds to “2”, $k = 2$ corresponds to “4”, and $k = 3$ corresponds to “6.” Deduce that the formula for the summand is 2*k*. Then, since the last value is 80, the ending value for the sum is $k = 40$.

The sigma notation is $\sum_{k=1}^{40} 2k$.

TRY IT

Suppose you have 9 + 16 + 25 + ... + 144.

Express this sum using sigma notation. Use a convenient starting value. +

$$\sum_{k=3}^{12} k^2$$

3. Special Sums and Summation Properties

3a. Adding Powers of Positive Integers

Consider the sum $\sum_{k=1}^{40} 2 = 2+2+2+...+2$ (there are 40 terms; each of them are 2s). We can see that the sum is $40(2) = 80$, but there is a “shortcut” we can take.

FORMULA

The Summation of a Constant

If C is a constant, $\sum_{k=1}^{n} C = C \cdot n$

HINT

In this formula, there are n terms, where each term has the value of C. Also note that this formula only works when the starting value is 1.

Here are summation formulas for powers of consecutive positive integers.

FORMULA

Summations of Powers of Consecutive Numbers

$$\sum_{k=1}^{n} k = 1+2+\ldots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = 1^4+2^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

HINT

These formulas only work when the starting value is 1. Note also that the value of these sums depends on the ending value.

EXAMPLE Use a formula to evaluate the sum $\sum_{k=1}^{10} k^2$.

We can use the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. To use the formula, note that $k=1$ and $n=10$. Then, the sum is equal to $\frac{10(10+1)(2(10)+1)}{6} = 385$.

Now, you try one.

TRY IT

Consider the sum $\sum_{k=1}^{6} k^3$.

Use a formula to evaluate this sum. +

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3b. Summation Properties

Consider the summation $\sum_{k=1}^{3} 2k^3 = 2(1)^3+2(2)^3+2(3)^3$. Notice that each term has a factor of 2, which means this can be written $2(1^3+2^3+3^3)$. Then, the result can be rewritten using sigma notation: $2\cdot\sum_{k=1}^{3} k^3$

This leads to the following property of summations:

FORMULA

Summation of a Constant Multiple

$$\sum_{k=1}^{n} C\cdot a_k = C\cdot\sum_{k=1}^{n} a_k$$

HINT

In other words, you can move the constant multiple outside of the summation.

Now, consider the summation $\sum_{k=1}^{3}(k^3+k^2)=(1^3+1^2)+(2^3+2^2)+(3^3+3^2)$. Rearrange the terms so that the common powers are together: $(1^3+2^3+3^3)+(1^2+2^2+3^2)$. Then, the result can be written using two summations: $\sum_{k=1}^{3}k^3+\sum_{k=1}^{3}k^2$

This leads to two more summation formulas:

FORMULA

Summation of a Sum or Difference

$$\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$

$$\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$$

BIG IDEA

For the summation of a sum, you can break into a sum of two simpler summations. For the summation of a difference, you can break into a difference of two simpler summations.

EXAMPLE Evaluate the summation $\sum_{k=1}^{8}(3k^2+4k-2)$.

By the sum and difference formulas, this is equivalent to $\sum_{k=1}^{8}3k^2+\sum_{k=1}^{8}4k-\sum_{k=1}^{8}2$.

By the constant multiple formulas, this is equivalent to $3\sum_{k=1}^{8}k^2+4\sum_{k=1}^{8}k-\sum_{k=1}^{8}2$.

Now, use the summation formulas:

$$\sum_{k=1}^{8}k^2=\frac{8(8+1)(2\cdot 8+1)}{6}=204$$

$$\sum_{k=1}^{8}k=\frac{8(8+1)}{2}=36$$

$$\sum_{k=1}^{8}2=8(2)=16$$

When you substitute these values, the value of the sum is $3(204)+4(36)-16=740$.

TRY IT

Consider the summation $\sum_{k=1}^{10}(k^4-2k)$.

Evaluate this summation. +

25,223

SUMMARY

In this lesson, you learned that when a sum contains many terms, **sigma notation** is a convenient way to express the sum, as long as the terms follow some pattern. Next, you practiced **using sigma notation** to find sums. You also learned about some **special sums and summation properties** that enable you to find the sum quickly without having to add a large set of numbers together, such as when **adding powers of positive integers**. In the next section, we'll see why sigma notation is useful when calculating area.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Summand

The expression being used to determine the numbers that are added in a sum.

Summation

An expression that implies that several numbers are being added together. These are often written using sigma notation.

FORMULAS TO KNOW

Summation of a Constant Multiple

$$\sum_{k=1}^{n} C \cdot a_k = C \cdot \sum_{k=1}^{n} a_k$$

Summation of a Sum or Difference

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

Summations of Powers of Consecutive Numbers

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

The Summation of a Constant

If C is a constant, $\sum_{k=1}^{n} C = C \cdot n$