Second Shape Theorem

by Sophia

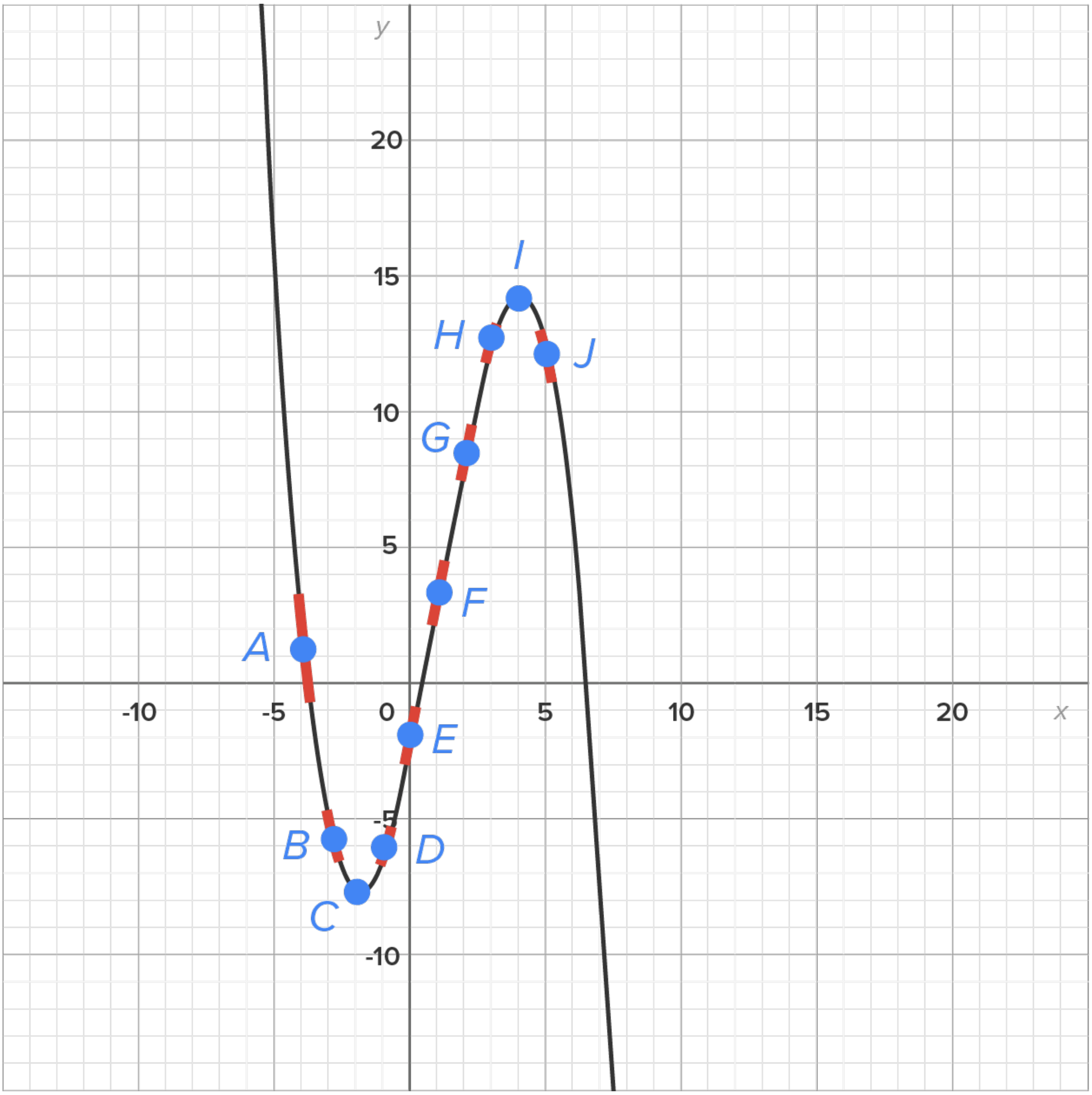
WHAT'S COVERED

In this lesson, you will sketch the shape of a function $f(x)$ given values of its derivative, $f'(x)$. Specifically, this lesson will cover:

1. Using Values of $f'(x)$ to Sketch the Shape of the Graph of $f(x)$
2. Using $f'(x)$ to Detect Local Maximum and Minimum Values

1. Using Values of $f'(x)$ to Sketch the Shape of the Graph of $f(x)$

Consider the graph below of a function $y = f(x)$, which is the same as the graph in the last tutorial.



Notice that the graph of $f(x)$ is increasing at every point where its tangent line has a positive slope, and the graph of $f(x)$ is decreasing at every point where its tangent line has a negative slope. This means we can extend the ideas of increasing and decreasing from points to intervals.

BIG IDEA

If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that same interval.
If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that same interval.

Based on this fact, we can use the derivative to determine entire intervals over which a function $f(x)$ is increasing or decreasing.

WATCH

In the following video, we'll find all intervals over which $f(x) = -x^3 + 3x^2 + 24x + 10$ is increasing or decreasing. One of the main objectives of this video is to show how the information gets organized, so viewing this video is important!

Video Transcription

Hi there. I hope you're ready to do some math, because we're going to look at a function analytically to determine where it's increasing and decreasing. And remember that that information is obtained by looking at the first derivative. We know that a positive slope means that the graph is increasing, and we know that a negative slope means that the graph is decreasing.

So what we're going to do to do the analytic piece is we first want to figure out where the graph is transitioning. And that is as simple as figuring out where the derivative is 0, because if it's 0, I know that it's either positive or negative on one side and positive or negative on the other side. So it's easier to find those points first and then compare.

So here's what we do. We take our first derivative. And this is just the power rule-- negative 3x squared plus 6x plus 24. And we know that this transition point could occur either when the derivative is equal to 0 or undefined. Now we have a polynomial function here, which we know is never undefined. So we are going to stick with setting it equal to 0.

So we're going to solve, and that means I'm going to factor out a negative 3, like so. And I noticed that factor is quite nicely. So I'm going to factor that into x minus 4 and x plus 2. And by setting each factor equal to 0, I know I can ignore the negative 3 because negative 3 is never 0. This one means that x equals 4, and this one means that x equals negative 2. So these are our points where the graph could be transitioning between increasing and decreasing. We call those the critical numbers.

So what do we do with this? And the major point of this video is to see how to organize this information to get the most out of it. So we're going to make a sign graph, kind of a number line, and what we notice is this number line gets broken into three pieces. We have our critical number at negative 2. We have our other critical number at 4, which divides the number line into three intervals. We have negative infinity to negative 2, we have negative 2 up to 4, and we have 4 to infinity.

Now what is the what's the business with these intervals? Picking the first interval, negative infinity to negative 2, this tells me that whatever value of x I pick on in that interval. If I plug it in to f prime, I'm going to get the same sign-- sign. It's either going to always be positive or always be negative because the only way you could change from negative to positive or positive to negative is if you pass through 0 if it's a continuous function, which we know we have here. So we use test numbers for each interval to tell what the whole interval is behaving like.

So I'm going to select three values of x because there's three intervals. And there's no science behind what values you pick, but we're going to try to keep it simple for ourselves. So since the critical number is negative 2, I'm going to select negative 3. For the interval negative 2 to 4, I'm going to pick 0 because that's probably the easiest number to use. And on 4 to infinity, I'm going to select the number 5, just because it's close to 4.

And what we're going to do is substitute these into the first derivative to see if they're positive or negative. And that will, in turn, tell us whether or not the function is-- or whether the function is increasing or decreasing. So f prime of negative 3 is negative 3 times negative 3 squared. I'm basically using this right here, using the derivative before we've manipulated it-- plus 6 times negative 3 plus 24. And if you crunch that out, you get negative 21, which tells me I have a negative slope, which tells you that my function is decreasing.

So I tend to write it like this here-- just write a decreasing line just to give it a little bit of visual appeal. So plugging in 0-- now one thing we notice is that any of the terms are going to just drop out because 0 to any power is 0. But it's nice to show it out. So we have 24, which means graph is increasing on that interval.

And last but not least, at 5, we have negative 3 times 5 squared plus times 5 plus 24. And that is negative 21 again, so we're back to decreasing.

OK, so looking at that sign graph, there's actually a lot of information there, and it looks like we have a minimum point somewhere and a maximum point somewhere, but that's for further on down the line. We can write out the intervals over which this function is increasing and decreasing. So we could say this function is increasing on the interval, while the only interval where it's increasing is the one in the middle, negative 2 to 4 and it's basically decreasing everywhere else.

But remember, there's a certain way we write this. Decreasing on the interval now, we have negative infinity to negative 2. And we also have the interval 4 to infinity. Since those two intervals describe a set of values that have something in common-- in this case, where the function is decreasing-- we put a U in between them to represent that there's a union of those sets.

And that's our answer. So analytically, we were able to determine where the function was increasing and decreasing. And just so you have another visual to go along with this, I have the graph of the function right here. And as you can see, it is decreasing up to x equals negative 2, increasing between negative 2 and 4, and then decreasing again after x equals 4. So that more or less confirms our results.

Now that you see how to organize the information, let's look at more examples.

↱ EXAMPLE Let $f(x)=\frac{x}{x^2+1}$. Determine the intervals over which $f(x)$ is increasing or decreasing. Since this information comes from the first derivative, we start there. Finding the derivative of this function requires the quotient rule.

$f(x)=\frac{x}{x^2+1}$	Start with the original function; the domain is all real numbers.
$f'(x)=\frac{(1)(x^2+1)-x(2x)}{(x^2+1)^2}$	Apply the quotient rule.
$f'(x)=\frac{x^2+1-2x^2}{(x^2+1)^2}=\frac{1-x^2}{(x^2+1)^2}$	Simplify.

$f'(x)$ can only change its sign when $f'(x)=0$ or is undefined. It is never undefined since the expression in the denominator is never zero.

Now, set $f'(x)=0$ and solve.

$$\frac{1-x^2}{(x^2+1)^2}=0$$ Set $f'(x)=0$. Since $f'(x)$ is never undefined, this case is not considered.

$1-x^2=0$ Multiply both sides by $(x^2+1)^2$.

$1=x^2$ Solve for *x*.

$x=\pm 1$

This means that the real number line is broken into three intervals:

$(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$

Now, select one number (called a test value) inside each interval to determine the sign of $f'(x)$ on that interval:

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test Value	-2	0	2
Value of $f'(x)=\frac{1-x^2}{(x^2+1)^2}$	$\frac{-3}{25}$	1	$\frac{-3}{25}$
Behavior of $f(x)$	Decreasing	Increasing	Decreasing

Thus, $f(x)$ is increasing on the interval $(-1, 1)$ and decreasing on $(-\infty, -1)\cup(1, \infty)$.

Let's look at an exponential function now.

→ EXAMPLE Let $f(x)=e^{-x^2}$. Determine the intervals over which $f(x)$ is increasing or decreasing.

Since this information comes from the first derivative, we start there. This derivative requires the chain rule.

$f(x)=e^{-x^2}$ Start with the original function.

$f'(x)=-2x\cdot e^{-x^2}$ Apply the chain rule.

$f'(x)$ can only change its sign when $f'(x)=0$ or is undefined. It is never undefined since there are no domain restrictions.

Now, set $f'(x)=0$ and solve.

$-2x\cdot e^{-x^2}=0$ The derivative is set to 0.

$-2x=0$ or $e^{-x^2}=0$ Set each factor equal to 0.

$x=0$ $-2x=0$ implies $x=0$.

$e^{-x^2}=0$ has no solution.

This means that the real number line is broken into two intervals:

$(-\infty, 0)$, $(0, \infty)$

Now, select one number (called a test value) inside each interval to determine the sign of $f'(x)$ on that interval:

Interval	$(-\infty, 0)$	$(0, \infty)$
Test Value	-1	1
Value of $f'(x) = -2xe^{-x^2}$	$2e^{-1}$	$-2e^{-1}$
Behavior of $f(x)$	Increasing	Decreasing

Thus, $f(x)$ is increasing on the interval $(-\infty, 0)$ and decreasing on $(0, \infty)$.

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2. Using $f'(x)$ to Detect Local Maximum and Minimum Values

Consider the function $f(x) = \frac{x}{x^2+1}$. In part 1, we obtained the following sign graph for $f'(x)$. Notice that the bottom row has been added to show the direction the graph is moving when increasing or decreasing.

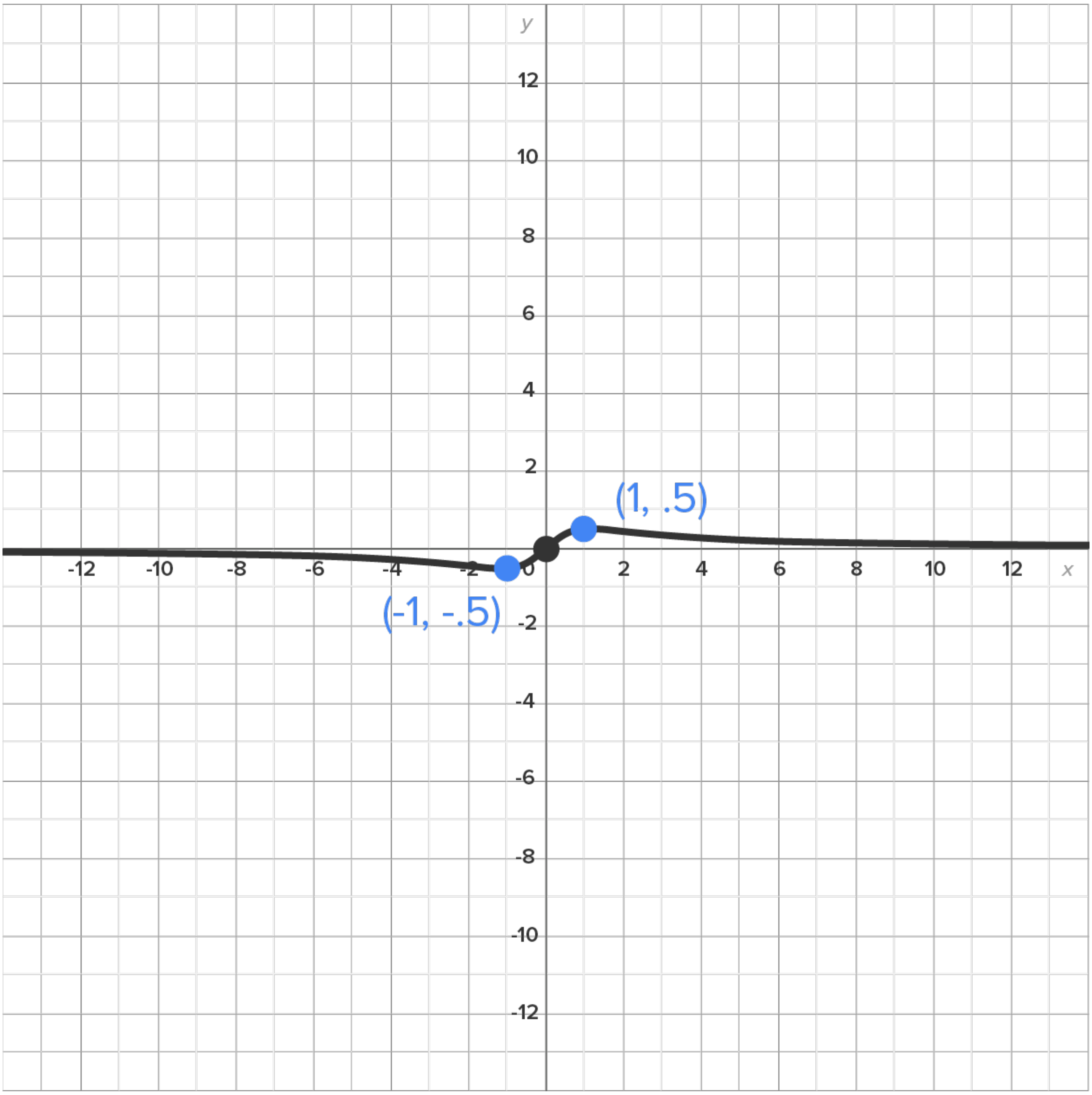
Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test Value	-2	0	2
Value of $f'(x) = \frac{1-x^2}{(x^2+1)^2}$	$\frac{-3}{25}$	1	$\frac{-3}{25}$
Behavior of $f(x)$	Decreasing	Increasing	Decreasing
Direction	$\searrow$	$\nearrow$	$\searrow$

This means that when $x = -1$, the graph is transitioning from decreasing to increasing; and when $x = 1$, the graph is transitioning from increasing to decreasing. Since $f(x)$ is continuous at $x = -1$ and $x = 1$, this means that there is a local minimum at $x = -1$ and a local maximum at $x = 1$.

To find the coordinates of these points on the graph, substitute each value into $f(x)$:

- Local minimum: $(-1, f(-1))$, or $\left(-1, \frac{-1}{2}\right)$
- Local maximum: $(1, f(1))$, or $\left(1, \frac{1}{2}\right)$

The graph of $f(x)$ is shown here for reference:



Given a function $f(x)$, the first derivative, $f'(x)$, can be used to locate maximum and minimum values. To do so, use the same sign graph you used to determine where the function is increasing or decreasing. Then, observe the critical numbers where $f(x)$ transitions between increasing and decreasing. When $f(x)$ is continuous at these critical numbers, there is either a local minimum or local maximum point.

Performing this analysis to locate local minimum and maximum points is called the**first derivative test**.

> ↪ EXAMPLE Use the first derivative test to determine all local minimum and maximum points of the function $f(x) = x^4 - 12x^3$.
>
> First, find all critical numbers:
>
> | | |
> |---|---|
> | $f(x) = x^4 - 12x^3$ | Start with the original function; the domain is all real numbers. |
> | | Take the derivative. |

$f'(x) = 4x^3 - 36x^2$

$4x^3 - 36x^2 = 0$ Set $f'(x) = 0$. Since $f'(x)$ is a polynomial, there is no value of x for which $f'(x)$ is undefined, so this case is not considered.

$4x^2(x - 9) = 0$ Solve by factoring.

$4x^2 = 0,\ x - 9 = 0$

$x = 0,\ x = 9$

Thus, the critical numbers are $x = 0$ and $x = 9$. Now we move to the first derivative test, which means making a sign graph, determining the intervals of increase and decrease, then observing which critical numbers produce a local maximum or local minimum.

Interval	$(-\infty, 0)$	$(0, 9)$	$(9, \infty)$
Test Value	-1	1	10
Value of $f'(x) = 4x^3 - 36x^2$	-40	-32	400
Behavior of $f(x)$	Decreasing	Decreasing	Increasing
Direction	$\searrow$	$\searrow$	$\nearrow$

At $x = 0$, $f(x)$ does not change direction; it is decreasing on both sides. There is no local extreme value at $x = 0$.

At $x = 9$, the $f(x)$ transitions from decreasing to increasing, indicating that there is a local minimum value when $x = 9$.

Thus, there is a local minimum at $(9, f(9))$, or $(9, -2187)$.

BIG IDEA

There is no guarantee that a local extreme value occurs at a critical number. This is why we perform the first derivative test.

TRY IT

Consider the function $f(x) = xe^{-x}$.

Use the first derivative test to determine the local extrema of the function. +

Local maximum at $\left(1, \frac{1}{e}\right)$.

TERM TO KNOW

First Derivative Test

Used to identify possible local maximum and minimum points.

SUMMARY

In this lesson, you learned that the derivative, $f'(x)$, can be used to provide information about where $f(x)$ is increasing or decreasing. In other words, you can**use values of $f'(x)$ to sketch the shape of the graph of $f(x)$**. You also learned that you can apply these ideas further,**using $f'(x)$ to detect local maximum and minimum values** by performing an analysis called the first derivative test.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

First Derivative Test

Used to identify possible local maximum and minimum points.