Horizontal and Vertical Asymptotes

by Sophia

WHAT'S COVERED

In this lesson, you will connect limits with horizontal and vertical asymptotes. Specifically, this lesson will cover:

1. Horizontal Asymptotes and Limits
2. Vertical Asymptotes and Limits

1. Horizontal Asymptotes and Limits

The graph of $f(x)$ has a **horizontal asymptote** $y=c$ if either $\lim_{x\to\infty} f(x)=c$ or $\lim_{x\to-\infty} f(x)=c$.

EXAMPLE Find all horizontal asymptotes of the graph of $f(x)=\frac{7x+1}{8x+3}$.

To find horizontal asymptotes, evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

$\lim_{x\to\infty}\frac{7x+1}{8x+3}$	Start with the required limit to evaluate.
$=\lim_{x\to\infty}\frac{\frac{7x}{x}+\frac{1}{x}}{\frac{8x}{x}+\frac{3}{x}}$	Divide each term by the highest power of x in the denominator, which is x.
$=\lim_{x\to\infty}\frac{7+\frac{1}{x}}{8+\frac{3}{x}}$	Simplify.
$=\frac{\lim_{x\to\infty}\left(7+\frac{1}{x}\right)}{\lim_{x\to\infty}\left(8+\frac{3}{x}\right)}$	$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$
$=\frac{7}{8}$	$\lim_{x\to\infty}\left(7+\frac{1}{x}\right)=\lim_{x\to\infty} 7+\lim_{x\to\infty}\frac{1}{x}=7+0=7$ $\lim_{x\to\infty}\left(8+\frac{3}{x}\right)=\lim_{x\to\infty} 8+\lim_{x\to\infty}\frac{3}{x}=8+0=8$

Thus, the graph of $f(x)$ has a horizontal asymptote at $y=\frac{7}{8}$.

Now, check $\lim_{x \to -\infty} \frac{7x+1}{8x+3}$.

$$\lim_{x \to -\infty} \frac{7x+1}{8x+3}$$ Start with the required limit to evaluate.

$$= \lim_{x \to -\infty} \frac{\frac{7x}{x}+\frac{1}{x}}{\frac{8x}{x}+\frac{3}{x}}$$ Divide each term by the highest power of x in the denominator, which is x.

$$= \lim_{x \to -\infty} \frac{7+\frac{1}{x}}{8+\frac{3}{x}}$$ Simplify.

$$= \frac{\lim_{x \to -\infty}\left(7+\frac{1}{x}\right)}{\lim_{x \to -\infty}\left(8+\frac{3}{x}\right)}$$ $$\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to -\infty} f(x)}{\lim_{x \to -\infty} g(x)}$$

$$= \frac{7}{8}$$ $$\lim_{x \to -\infty}\left(7+\frac{1}{x}\right) = \lim_{x \to -\infty} 7 + \lim_{x \to -\infty} \frac{1}{x} = 7+0=7$$ $$\lim_{x \to -\infty}\left(8+\frac{3}{x}\right) = \lim_{x \to -\infty} 8 + \lim_{x \to -\infty} \frac{3}{x} = 8+0=8$$

This produces the same result as the other limit, so there is no additional horizontal asymptote.

We can conclude that the graph of $f(x)$ has one horizontal asymptote at $y=\frac{7}{8}$. If you graph this function, you would see that the graph approaches the horizontal line $y=\frac{7}{8}$ as $x \to \pm\infty$.

TRY IT

Consider the function $f(x)=\frac{3x}{x^2+3}$.

Find all horizontal asymptotes of the graph of the function. +

$f(x)$ has one horizontal asymptote, and its equation is $y=0$.

If $f(x)=\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are both polynomials, the following are results of limits:

- If $N(x)$ and $D(x)$ have the same degree, then the horizontal asymptote is $y=\frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
- If the degree of $N(x)$ is less than the degree of $D(x)$, then the horizontal asymptote is $y=0$.
- If the degree of $N(x)$ is more than the degree of $D(x)$, then there is no horizontal asymptote, and this case is discussed in the next tutorial.

↪ EXAMPLE The function $f(x)=\frac{2x^2+4x+1}{3x^2+5x}$ has the horizontal asymptote $y=\frac{2}{3}$ since the degrees are the same.

↪ EXAMPLE The function $f(x)=\frac{3x}{x^2+3}$ has the horizontal asymptote $y=0$, as you saw in the Try It above, since the degree of the numerator is less than the degree of the denominator.

WATCH

In this video, we'll use limits to find the equations of all horizontal asymptotes of the function $f(x)=\frac{|x|}{2x+3}$.

Video Transcription

Hello there. Good to see you. What we're going to do in this video is considering the function f of x equals the absolute value of x divided by 2x plus 3. We're going to find the equations of the horizontal asymptotes. And we're used to functions having just one horizontal asymptote. We'll see if that's the case with this one here too.

So remember that a horizontal asymptote results from the limit as x approaches infinity of the function. And as well, the limit as x approaches negative infinity of the function. Now, remember that the rational functions we've dealt with in the past usually have the same horizontal asymptote on both sides.

And this isn't exactly a rational function because of the absolute value in the numerator. But one thing that's going to help us to evaluate these limits is, remember that the absolute value of x is equal to x if x is greater than or equal to 0, negative x if x is less than 0, which basically means the absolute value of x is the number underneath the bars if you have a non-negative number. And it's the opposite of the number underneath the bars if x is negative.

So let's look at each of these limits separately. So as x approaches infinity, we know that that means that x is positive. So this becomes the limit as x approaches infinity of x over 2x plus 3. Now that's looking more familiar. We go through, and we divide by the highest power in the denominator, which is x. So we do limit as x approaches infinity. We divide every term by x. So you have x over x over 2x over x plus 3 over x. And then we simplify.

So you have 1 over 2 plus 3 over x. Now, ordinarily, I would just write the limit of the numerator over the limit of the denominator. But in this case, we can see, here's what's going to happen. As x approaches infinity, this term is going to go to 0 and this limit is 1/2.

So that means that one of the horizontal asymptotes is y equals 1/2. Now, let's see if the other limit gives us the same thing. So as x approaches negative infinity, the absolute value of x is going to be negative x, because we're saying that x is less than 0 for sure. So this is the limit as x approaches negative infinity of negative x over 2x plus 3.

So we go through the same motions here. We have the limit as x approaches negative infinity. I'm going to divide everything by x again. So negative x over x over 2x over x plus 3 over x. And there's some simplification that can happen here. We have the limit as x approaches negative infinity. Negative 1 over 2 plus 3 over x.

And a very similar thing happens here as what happened in the other limit. We have negative 1 to the numerator for sure. We have 2 in the denominator for sure. 3 over x will tend to 0, and this limit is negative 1/2. So that means that another horizontal asymptote is y equals 2 negative 1/2. And if you were to graph the function, you would notice that to the right, the graph levels off to y equals 1/2. And to the extreme left, the graph levels off to y equals negative 1/2. Kind of cool.

TERM TO KNOW

Horizontal Asymptote

A horizontal line in the form $y=c$ for the graph of $f(x)$ if either $\lim_{x\to\infty} f(x)=c$ or $\lim_{x\to-\infty} f(x)=c$.

2. Vertical Asymptotes and Limits

The graph of $f(x)$ has a **vertical asymptote** $x=a$ if either $\lim_{x\to a^-} f(x)=\pm\infty$ or $\lim_{x\to a^+} f(x)=\pm\infty$.

The "$\pm\infty$" in the above definition means that the limit could be either $-\infty$ or ∞ for there to be a vertical asymptote when $x=a$.

HINT

If $f(x)$ is a rational function, the only values of x where a vertical asymptote could occur are those values where the denominator is equal to 0.

EXAMPLE Determine the vertical asymptotes of $f(x)=\frac{x^2-2x}{x^2-4x}$.

First, find all values of x for which the denominator is 0:

$$x^2-4x=0$$
$$x(x-4)=0$$
$$x=0,\ x=4$$

Thus, the possible vertical asymptotes are $x=0$ and $x=4$. To determine which are vertical asymptotes, we need to evaluate a one-sided limit for each x-value. For this example, we'll choose right-sided limits.

Is $x=0$ a vertical asymptote?

$\lim_{x\to 0^+} \frac{x^2-2x}{x^2-4x}$	Start with the required limit to evaluate.
$\lim_{x\to 0^+} \frac{x(x-2)}{x(x-4)} = \lim_{x\to 0^+} \frac{x-2}{x-4}$	Factor, then remove the common factor.

$$= \frac{0-2}{0-4} = \frac{1}{2}$$ Direct substitution works!

Since the limit is not $\pm\infty$, there is no vertical asymptote at $x = 0$. (Note: The left-sided limit would produce the same result.)

Is $x = 4$ a vertical asymptote?

$$\lim_{x \to 4^+} \frac{x^2 - 2x}{x^2 - 4x}$$ Start with the required limit to evaluate.

$$\lim_{x \to 4^+} \frac{x(x-2)}{x(x-4)} = \lim_{x \to 4^+} \frac{x-2}{x-4}$$ Factor, then remove the common factor.

$$= \infty$$ As *x* approaches 4 from the right, *x* - 2 is around 2, and *x* - 4 is a small positive number.

$\dfrac{\textit{around 2}}{\textit{small positive number}}$ = a large number, so the limit is ∞.

We can conclude that there is a vertical asymptote at $x = 4$, but not at $x = 0$. If you were to graph the function, this would confirm this result.

TRY IT

Consider the function $f(x) = \dfrac{2x+1}{x^2 - 6x + 5}$.

Find the equations of all vertical asymptotes of the function. +

The vertical asymptotes are $x = 1$ and $x = 5$.

TERM TO KNOW

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$.

SUMMARY

In this lesson, you learned that **the horizontal and vertical asymptotes of a function are related to limits** of a function where infinity is involved. Specifically, a function $f(x)$ has a horizontal asymptote at $y = c$ if $\lim\limits_{x \to \infty} f(x) = c$ or $\lim\limits_{x \to -\infty} f(x) = c$, and a function $f(x)$ has a vertical asymptote at $x = a$ if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 3 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.

TERMS TO KNOW

Horizontal Asymptote

A horizontal line in the form $y = c$ for the graph of $f(x)$ if either $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

Vertical Asymptote

A vertical line in the form $x = a$ for the graph of $f(x)$ if either $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.