Changing the Variable: u-substitution with Power Rule

by Sophia

WHAT'S COVERED

In this lesson, you will explore indefinite integrals of composite functions. Specifically, this lesson will cover:

1. Introduction to u-Substitution/Review of Chain Rule
2. Using u-Substitution With the Power Rule

1. Introduction to u-Substitution/Review of Chain Rule

One way to view u-substitution is "undoing the chain rule." Consider the function $f(x) = (x^2+1)^9$, which is a composite function.

Taking the derivative, we get $f'(x) = 9(x^2+1)^8 \cdot 2x = 18x(x^2+1)^8$.

It follows that $f(x) = (x^2+1)^9 + C$ is the antiderivative of $f'(x) = 18x(x^2+1)^8$.

In other words, $\int 18x(x^2+1)^8 dx = (x^2+1)^9 + C$.

The big question: How do we get the antiderivative without having to guess? The answer to this question lies in the very way we find the derivative.

Let's look again at $f(x) = (x^2+1)^9$.

When we first learned the chain rule, we let $u = x^2+1$. Then, $f(u) = u^9$, and $f'(u) = 9u^8 \cdot \frac{du}{dx}$.

As you can see, the derivative is less complicated when the "u" is used. You can focus on the "inner" function. While you may have gotten used to not using the "u" idea in derivatives, it is very useful for antiderivatives. Let's walk through an example.

↪ EXAMPLE Find the indefinite integral: $\int \sqrt{4x^2+3} \cdot 8x dx$

In the spirit of the chain rule, let $u=4x^2+3$.

Since a chain rule derivative also contains $\frac{du}{dx}$, we'll find that as well: $\frac{du}{dx}=8x$

Notice that the integrand is written in differential form (a function multiplied by *dx*).

To align with that, we'll write $\frac{du}{dx}=8x$ in differential form: $du=8xdx$

The goal is to get an integral that has *u* as its only variable. With these substitutions, $4x^2+3$ gets replaced by *u* and $8xdx$ is replaced by *du*.

This means the integral can now be written as $\int \sqrt{u}\, du$, which is much simpler, and it is clearer what to do:

$\int \sqrt{u}\, du$	Start with the original expression.
$=\int u^{1/2} du$	Rewrite as a power so that the power rule can be used.
$=\frac{u^{3/2}}{\left(\frac{3}{2}\right)}+C$	Apply the power rule with $n=\frac{1}{2}$.
$=\frac{2}{3}u^{3/2}+C$	Simplify.
$=\frac{2}{3}(4x^2+3)^{3/2}+C$	Replace *u* with $4x^2+3$.

The last step is the key step. The original function was in terms of *x*, which means that the final answer should also be in terms of *x*. The *u*-substitution was more or less used to help us to get organized.

Thus, $\int \sqrt{4x^2+3} \cdot 8x dx = \frac{2}{3}(4x^2+3)^{3/2}+C$.

This is the essence with *u*-substitution. When you identify an antiderivative that requires *u*-substitution, here is what you do:

STEP BY STEP

1. Identify the "inner" function, which is the function within another function. For now, it will be the function that is raised to a power (or under a radical). If we call the inside function $g(x)$, then let $u=g(x)$.
2. Find the differential $du=g'(x)dx$. This will only work if $g'(x)$ or some multiple of $g'(x)$ is in the integrand before the substitutions are made.
3. Substitute *u* and *du* into the integral so that the integral has *u* as its only variable.

4. Find the antiderivative with respect to u (and don't forget $+C$!).
5. Replace $u=g(x)$ in the antiderivative. This is called back-substitution.

---

2. Using *u*-Substitution With the Power Rule

Now that we have a process, let's look at some examples.

↪ EXAMPLE Find the indefinite integral: $\int 2x(x^2+3)^7 dx$

$\int 2x(x^2+3)^7 dx$	Start with the original expression.
$=\int u^7 du$	Make the substitution: $u=x^2+3$ Find the differential: $du=2xdx$ Replace x^2+3 with u and $2xdx$ with du.
$=\frac{1}{8}u^8+C$	Apply the power rule with $n=7$.
$=\frac{1}{8}(x^2+3)^8+C$	Back-substitute $u=x^2+3$.

Thus, $\int 2x(x^2+3)^7 dx=\frac{1}{8}(x^2+3)^8+C$.

The next example will illustrate what happens when *du* is not exactly in the integral, but is a constant multiple.

WATCH

In this video, we will find $\int x^2(4x^3+5)^6 dx$.

Video Transcription

[MUSIC PLAYING] Hi, there, and welcome back. What we're going to do in this video is find an antiderivative that requires the use of u substitution. In other words, reversing the chain rule. So we have the antiderivative of x squared times the quantity, 4x to the third plus 5 raised to the sixth power. What signals us to use u substitution for this is that we have a function within a function. In this case, it's a quantity that's being raised to a power. That's what's telling us that it's a composite function.

So we are going to let u equal 4x to the 1/3 plus 5. Then the differential form is du equals 12x squared dx. Now, notice that we do not have a 12x squared dx in the integral. We just have x squared dx. That turns out to be the most important part. We can work with constants, it's just the variable piece does have to be there. Since x squared dx is in the integral, I am going to go ahead and isolate-- Somehow my x got cut off there. There we go. I'm going to isolate x squared dx to one side.

So this means that 1/12 du is equal to x squared dx. So then u equals 4x to the 1/3 plus 5 is going to replace this piece, and x squared dx becomes 1/12 du. We're going to have a whole brand new integral with u as the variable instead of x.

We have the integral of u to the sixth and x squared dx is 1/12 d u. That integral sign is a little bit too far away from my taste. How about that? So now we know that the 1/12 can go outside of the integral by the constant multiple rule. So we have 1/12 integral of u to the sixth du. Now this is where the power rule is going to be used. So we have 1/12 times 1/7 u to the seventh plus c, which is 1 over 84 u to the seventh plus c.

Then u is 4x to the 1/3 plus 5. We need to remember to write our answer in terms of the original variable. So we have 1 over 84, 4x to the 1/3 plus 5 to the seventh plus c, which of course, can also be written as 4x to the 1/3 plus 5 to the seventh, all over 84, plus c. So either one of those two forms is acceptable for a final answer. There is an integral using u substitution.

[MUSIC PLAYING]

→ EXAMPLE Find the indefinite integral: $\int \sqrt{8x+1}\,dx$

$\int \sqrt{8x+1}\,dx$	Start with the original expression.
$= \int \frac{1}{8}\sqrt{u}\,du$	First, make the substitution: $u = 8x+1$ Write the differential: $du = 8dx$ Solve for *dx*: $dx = \frac{1}{8}du$ Replace $8x+1$ with *u* and *dx* with $\frac{1}{8}du$.
$= \frac{1}{8}\int \sqrt{u}\,du$	Move the constant $\frac{1}{8}$ outside the integral sign.
$= \frac{1}{8}\int u^{1/2}du$	Rewrite as a power.
$= \frac{1}{8}\frac{u^{3/2}}{\left(\frac{3}{2}\right)}+C$	Use the power rule with $n = \frac{1}{2}$.
$= \frac{1}{12}u^{3/2}+C$	Simplify.
$= \frac{1}{12}(8x+1)^{3/2}+C$	Back-substitute $u = 8x+1$.

Thus, $\int \sqrt{8x+1}\,dx = \frac{1}{12}(8x+1)^{3/2}+C$.

In the next example, we'll look at a power in the denominator.

→ EXAMPLE Find the indefinite integral: $\int \frac{2x+1}{(x^2+x+4)^5}dx$

$\int \frac{2x+1}{(x^2+x+4)^5}dx$	Start with the original expression.

$= \int \frac{1}{u^5} du$	First, make the substitution: $u = x^2 + x + 4$ Write the differential: $du = (2x+1)dx$ Replace $x^2 + x + 4$ with u and $(2x+1)dx$ with du. Note: dx is multiplied by the expression, which means it is multiplied by the numerator.
$= \int u^{-5} du$	Rewrite as a negative power so that the power rule can be used.
$= \frac{1}{-4} u^{-4} + C$	Apply the power rule with $n = -5$.
$= \frac{-1}{4} u^{-4} + C$	Simplify.
$= \frac{-1}{4}(x^2 + x + 4)^{-4} + C$	Back-substitute $u = x^2 + x + 4$.
$= \frac{-1}{4(x^2 + x + 4)^4} + C$	Write in terms of positive exponents if desired or directed.

Thus, $\int \frac{2x+1}{(x^2 + x + 4)^5} dx = \frac{-1}{4(x^2 + x + 4)^4} + C$.

TRY IT

Consider $\int \frac{4x}{(x^2+8)^{3/4}} dx$.

Find the indefinite integral. +

$8(x^2+8)^{1/4} + C$

Finally, here is an example where there doesn't appear to be an inner function.

EXAMPLE Find the indefinite integral: $\int \frac{4x}{8x^2+1} dx$

At first glance, it looks like we could manipulate the integrand, but since the denominator has more than one term, this is not possible. Taking a closer look, notice that the substitution $u = 8x^2 + 1$ has differential $du = 16x dx$, which is a constant multiple of $4x$ (which is in the integral). This is the direction we'll go.

$\int \frac{4x}{8x^2+1} dx$	Start with the original expression.
$= \int \frac{1}{u} \cdot \frac{1}{4} du$	First, make the substitution: $u = 8x^2 + 1$ Write the differential: $du = 16x dx$ Solve for $x \cdot dx$: $x dx = \frac{1}{16} du$. Replace $8x^2 + 1$ with u and dx with $\frac{1}{16} du$. Note: dx is multiplied by the expression, which means it is multiplied by

the numerator.

Note: $4xdx = 4\left(\frac{1}{16}du\right) = \frac{1}{4}du$

We isolate *xdx* since the goal is to rewrite the integrand in terms of *u* and *du*.

$= \frac{1}{4}\int \frac{1}{u}du$ Move the constant $\frac{1}{4}$ outside the integral sign.

$= \frac{1}{4}\ln|u| + C$ Apply the natural logarithm rule.

$= \frac{1}{4}\ln|8x^2+1| + C$ Back-substitute $u = 8x^2+1$.

$= \frac{1}{4}\ln(8x^2+1) + C$ It is worth mentioning that since $8x^2+1$ is positive for all real numbers, there is no need to use absolute value. That said, it is not incorrect to use absolute value, but it is not necessary in this case.

Thus, $\int \frac{4x}{8x^2+1}dx = \frac{1}{4}\ln(8x^2+1) + C$.

The substitution method isn't exclusively used for reversing the chain rule. It can also be used to rewrite expressions that could not otherwise be manipulated for antidifferentiation.

EXAMPLE Find the indefinite integral: $\int x(x+3)^{15}dx$

Since we certainly don't want to multiply $(x+3)^{15}$ out, we'll try a substitution.

$\int x(x+3)^{15}dx$ Start with the original expression.

$= \int (u-3)\cdot u^{15}du$ Make the substitution: $u = x+3$
Use the differential form: $du = dx$
At this point, there is no replacement for the "*x*" in front. Since $x+3$ will be replaced with *u*, we need a replacement for *x*.
From the substitution $u = x+3$, we can solve for *x* to obtain $x = u-3$.
$x \to u-3,\ x+3 \to u,\ dx \to du$

$= \int (u^{16} - 3u^{15})du$ Distribute u^{15}.

$= \frac{1}{17}u^{17} - \frac{3}{16}u^{16} + C$ Apple the power rule and combine constants.

$= \frac{1}{17}(x+3)^{17} - \frac{3}{16}(x+3)^{16} + C$ Back-substitute $u = x+3$.

Thus, $\int x(x+3)^{15}dx = \frac{1}{17}(x+3)^{17} - \frac{3}{16}(x+3)^{16} + C$.

TRY IT

Consider $\int x(x-9)^{10}dx$.

Find the indefinite integral. +

$$\frac{1}{12}(x-9)^{12}+\frac{9}{11}(x-9)^{11}+C$$

WATCH

In this video, we'll find $\int \frac{(2+\ln x)^3}{x}dx$.

Video Transcription

[MUSIC PLAYING] Hi there. Welcome back. What we're going to do in this video is look at another example of an antiderivative that needs to be found using u-substitution. So, looking at this antiderivative, we have 2 plus the natural log of x quantity raised to the third divided by x. And this is what we're trying to find the antiderivative of. So our convention for u-substitution is we let u equal the quantity that's being raised to the power of the function that's within the function.

So you have 2 plus the natural log of x. Oh, and that's it. So then the differential is 0 plus 1 over x dx. Now, it might not be apparent right away that is in the integral. So I'm going to rewrite this integral another way. I'm going to write 2 plus natural log of x raised to the third times 1 over x dx. And now it's very clear where everything belongs. This right here is going to be du. And this right here is going to be u to the third. So that means our integral successfully converts to an integral with only u as the variable.

So we have u to the third du. And now the power rule can be used. So this means this is 1/4 u to the fourth plus c. And then we're back substituting because, again, we want our answer to be in terms of the original variable, which is 1/4 2 plus natural log x to the fourth plus c. And that is our antiderivative. Remember, to check an antiderivative answer, you can find its derivative. And as long as you do everything correctly, you will end up with the integrand. So there's our antiderivative-- 1/4 quantity 2 plus natural log x raised to the fourth plus an arbitrary constant.

[MUSIC PLAYING]

TRY IT

Consider $\int \frac{\ln x}{x}dx$.

Find the indefinite integral. +

$$\frac{1}{2}(\ln x)^2+C$$

SUMMARY

In this lesson, you learned about the ***u*-substitution** method, which is primarily used to reverse **the chain rule**. When the *u* is used, the derivative is less complicated, allowing you to focus on the “inner” function. You also practiced **using *u*-substitution with the power rule**. As you learned, *u*-substitution isn't exclusively used for reversing the chain rule; it can also be used to “rearrange” the expression so that it can be manipulated for antidifferentiation.

SOURCE: THIS WORK IS ADAPTED FROM CHAPTER 4 OF *CONTEMPORARY CALCULUS* BY DALE HOFFMAN.