AN INTERNAL LOGIC OF VIRTUAL DOUBLE CATEGORIES

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ABSTRACT. We present a type theory called fibrational virtual double type theory (FVDblTT) designed specifically for formal category theory, which is a succinct reformulation of New and Licata's Virtual Equipment Type Theory (VETT). FVDblTT formalizes reasoning on isomorphisms that are commonly employed in category theory. Virtual double categories are one of the most successful frameworks for developing formal category theory, and FVDblTT has them as a theoretical foundation. We validate its worth as an internal language of virtual double categories by providing a syntax-semantics duality between virtual double categories and specifications in FVDblTT as a biadjunction.

1. **Introduction**

Category theory is the art of studying mathematical objects through how they interact with each other. It captures how mathematical objects behave not from their internal structures but from (homo)morphisms between them and describes their properties systematically with universal properties or other concepts. This abundant expressiveness of category theory led to the development of categorical semantics, the way to take models of type theories or programming languages in categorical structures. Good examples include Lawvere theories in categories with finite products [Law63], simply typed lambda calculus in cartesian closed categories [LS86], extensional Martin-Löf type theory in locally cartesian closed categories [See84], and homotopy type theory in ∞-groupoids [HS98, Str14]. Thus, it has been discovered that there are dualities between syntactical datasets and categorical structures [Jac99, CD14], endorsing the principle that type theory corresponds to category theory.

Since categories themselves are mathematical objects, categorical analysis of categories is crucial. This methodology is called *formal category theory* [Gra74] by category theorists; see Subsection 1.1 for further details. In short, it is a way to develop category theory in a general categorical structure, such as a 2-category, by imagining it as the totality of categories. Here, the adjective "formal" does not imply that the theory is written in a "formal language," but having a formal language for formal category theory is still engaging. New and Licata's Virtual Equipment Type Theory (VETT) [NL23] is pioneering in this direction. It is designed to facilitate the use of category theory in formal proofs by diminishing tedious but straightforward proofs, such as checking naturality or functoriality.

This paper aims to give an alternative type theory called *fibrational virtual double type theory* (FVDblTT) for the same purpose, which is more succinct and manageable, and to demonstrate its value from the viewpoint of 2-category theory by constructing a syntax-semantics biadjunction. In this sense, FVDblTT is an internal language for corresponding categorical structures: *fibrational virtual double categories*. Each fibrational virtual double category can be seen as a totality of categories of some kind, such as enriched categories or internal categories, and the type theory treats them as if they were mere categories. Arguing category theories is then divided into two parts: one is a common argument independent of different category theories, which occasionally falls into *abstract nonsense*, and the other is a specific discussion particular to a certain category theory. What we can do with FVDblTT is to deal with massive proofs belonging to the former part in the formal language and make people focus on the latter part. Proving something in FVDblTT implies proving corresponding statements in those various category theories that fibrational virtual double categories can model. A comparison of FVDblTT with VETT requires a detailed discussion, which is given in Section 6.

FVDblTT handles the four fundamental components of category theory: categories, functors, natural transformations, and *profunctors*.[1] A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$, written as $P(-,\bullet)\colon \mathcal{C} \rightarrow\!\!\!\mapsto \mathcal{D}$, is a functor from $\mathcal{C}^{\mathsf{op}} \times \mathcal{D}$ to the category of sets $\mathcal{S}\mathrm{et}$, which is an enhancement of a presheaf on $\mathcal{C}$ and a copresheaf on $\mathcal{D}$. It has been recognized that profunctors or their counterparts cannot be avoided in formal category theory [SW78, Woo82]. A naive framework to address them is double categories, which are compositional structures with two kinds of arrows, vertical and horizontal, and cells between them, which model functors, profunctors, and natural transformations. However, the composition of profunctors is not always well-defined since it is defined by a certain kind of colimits called coends in $\mathcal{S}\mathrm{et}$, which do not always exist because of the size of the categories. One way to overcome this difficulty is to consider *virtual cells* as a generalization of cells in double categories and forget about the composition of profunctors. A *virtual cell* (Figure 1) in the virtual double category of

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[1] Profunctors are also called *distributors* or *bimodules*.

profunctors $\mathbb{P}\mathrm{ROF}$ is defined as a natural family of functions $\mu_{C_0,\dots,C_n}$ from $P_1(C_0,C_1)\times\cdots\times P_n(C_{n-1},C_n)$ to $Q(F(C_0),G(C_n))$ for $C_0,\dots,C_n$. What we want to apprehend by composing profunctors is now absorbed into the virtual cells since, if we suppose the top boundary of the cell μ has a composite, then the above family of functions coincides with the unzipped data of the same cell but with its top boundary replaced by the composite.

A structure defined on the basis of these virtual cells is called a *virtual double category* (VDC), and it is a widely accepted foundation of formal category theory. While its name first appeared in the work of Cruttwell and Shulman [CS10], the idea of virtual double categories has been studied in various forms in the past under different names such as ***fc**-multicategories* [Lei02, Lei04], and *lax double categories* [DPP06]. For these years, virtual double categories have gained the status of a guidepost for working out new category theories, especially in the ∞-categorical setting [GH15, RV17, Rui24].

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "P_1", "|" marking] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "|" marking] & \cdots \arrow[r, "P_n", "|" marking] & \mathcal{C}_n \arrow[d, "G"] \\
\mathcal{D}_0 \arrow[rrr, "Q"', "|" marking] \arrow[urrr, phantom, "\mu"] & & & \mathcal{D}_1
\end{tikzcd}$$

FIGURE 1. A general virtual cell in $\mathbb{P}\mathrm{ROF}$.

$$\text{Type}\quad \mathsf{I}\ \mathsf{type}\ ,\qquad \text{Term}\quad \mathit{\Gamma}\vdash \mathsf{s} : \mathsf{I}\ ,$$
$$\text{Protype}\quad \mathit{\Gamma}\fatsemi\mathit{\Delta}\vdash \alpha\ \mathsf{protype}\ ,$$
$$\text{Proterm}\quad \mathit{\Gamma}_0\fatsemi\dots\fatsemi\mathit{\Gamma}_n \mid \mathsf{a}_1:\alpha_1\fatsemi\dots\fatsemi\mathsf{a}_n:\alpha_n\vdash\mu:\beta\ ,$$
$$\text{Protype Isomorphism}\quad \mathit{\Gamma}\fatsemi\mathit{\Delta}\vdash\mathit{\Upsilon}:\alpha\cong\beta\ .$$

FIGURE 2. Judgments of FVDblTT. ($\mathit{\Gamma},\mathit{\Delta},\dots$ are contexts like $x_1 : I_1,\dots,x_n : I_n$.)

Central judgments of FVDblTT are shown in Figure 2. Models of the type theory are taken in *fibrational* virtual double categories[2]. Fibrationality relates to the possibility of substitution of tight arrows (functors) into a loose arrow (profunctor) as $P(F-,G\bullet)$, which is indispensable to performing category theory. Let us illustrate how the basic components of the type theory are interpreted using $\mathbb{P}\mathrm{ROF}$. Types, terms, and ***pro**types* are interpreted as categories, functors, and **pro**functors. *Proterms* are interpreted as virtual cells as Figure 1 but are limited to those with the functors on both sides being identities. This limitation renders the linearized presentation of virtual cells in the type theory without losing the expressive power owing to fibrationality. Derivation rules for these constituents of the type theory are given in line with the axioms of virtual double categories, which are naturally interpreted in $\mathbb{P}\mathrm{ROF}$. Table 1 is an example of the interpretation of the prosubstitution in proterms within $\mathbb{P}\mathrm{ROF}$.

Prosubstitution into a protype

$$\dfrac{\begin{array}{c} x_0:I_0\fatsemi x_1:I_1 \mid a_1:\alpha_1\vdash\mu\{a_1\}:\beta_1 \\ x_1:I_1\fatsemi x_2:I_2 \mid a_2:\alpha_2\vdash\nu\{a_2\}:\beta_2 \\ x_0:I_0\fatsemi x_1:I_1\fatsemi x_2:I_2 \mid b_1:\beta_1\fatsemi b_2:\beta_2\vdash\lambda\{\beta_1\fatsemi\beta_2\}:\gamma \end{array}}{x_0:I_0\fatsemi x_1:I_1\fatsemi x_2:I_2 \mid a_1:\alpha_1\fatsemi a_2:\alpha_2\vdash\lambda\{\mu\{a_1\}\fatsemi\nu\{a_2\}\}:\gamma}$$

Category Theory – the interpretation in $\mathbb{P}\mathrm{ROF}$

$$\dfrac{\begin{array}{ll} \mu_{C_0,C_1}: P_1(C_0,C_1)\longrightarrow Q_1(C_0,C_1) & (\text{natural in } C_0,C_1) \\ \nu_{C_1,C_2}: P_2(C_1,C_2)\longrightarrow Q_2(C_1,C_2) & (\text{natural in } C_1,C_2) \\ \lambda_{C_0,C_1,C_2}: Q_1(C_0,C_1)\times Q_2(C_1,C_2)\longrightarrow R(C_0,C_2) & \left(\begin{array}{c}\text{natural in } C_0,C_2\\ \text{and dinatural in } C_1\end{array}\right) \end{array}}{P_1(C_0,C_1)\times P_2(C_1,C_2)\xrightarrow{\mu\times\nu}Q_1(C_0,C_1)\times Q_2(C_1,C_2)\xrightarrow{\lambda}R(C_0,C_2)\quad \left(\begin{array}{c}\text{natural in } C_0,C_2\\ \text{and dinatural in } C_1\end{array}\right)}$$

Predicate Logic – the interpretation in $\mathbb{R}\mathrm{el}$

$$\dfrac{\begin{array}{rl} \varphi_1(x_0,x_1)\Rightarrow\psi_1(x_0,x_1) & (\forall x_0,\forall x_1) \\ \varphi_2(x_1,x_2)\Rightarrow\psi_2(x_1,x_2) & (\forall x_1,\forall x_2) \\ \psi_1(x_0,x_1),\psi_2(x_1,x_2)\Rightarrow\chi(x_0,x_2) & (\forall x_0,\forall x_1,\forall x_2) \end{array}}{\varphi_1(x_0,x_1),\varphi_2(x_1,x_2)\Rightarrow\chi(x_0,x_2)\quad(\forall x_0,\forall x_1,\forall x_2)}\ \textit{MODUS PONENS}$$

TABLE 1. An example of interpretation of prosubstitution

The feature of FVDblTT compared with VETT is protype isomorphisms. They model up-to-isomorphism reasoning that is ubiquitous in category theory. One often proves two things are isomorphic by constructing some pieces of mutual inverses and then combining them to form the intended isomorphism. We bring this custom into the type theory as protype isomorphisms, which are interpreted as isomorphisms between profunctors, *i.e.*, an invertible natural transformation between profunctors. Moreover, they capture isomorphisms between functors as well since isomorphisms between functors $F,G\colon\mathcal{C}\to\mathcal{D}$ correspond to natural isomorphisms between $\mathcal{D}(F-,\bullet),\mathcal{D}(G-,\bullet)\colon\mathcal{C}\twoheadrightarrow\mathcal{D}$ according to the Yoneda lemma.

A byproduct of providing the internal language of fibrational virtual double categories is its aspect as an all-encompassing language for predicate logic. The double category of sets, functions, relations as objects, tight

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[2] The difference between virtual equipments and fibrational virtual double categories lies in that the former requires every object to have a unit while the latter does not.

arrows, and loose arrows would also serve as the stage of semantics of FVDblTT. In this approach, protypes symbolize relations (two-sided **pro**positions) but with a direction, and proterms symbolize Horn formulas, as exemplified in Table 1. In other words, category theory based on categories, functors, and profunctors can be perceived as *generalized logic*. The unity of category theory and logic dates back to the work of Lawvere [Law73], in which he proposed that the theories of categories or metric spaces are generalized logic, with the truth value sets being some closed monoidal categories. How we interpret the type theory in these different virtual double categories is summarized in Table 2. The last two rows of the table show some of the additional constructors for FVDblTT, which we will discuss in Section 4.

Items in FVDblTT	**Formal category theory**	**Predicate logic**
Types I	categories $\mathcal{C}$	sets A
Terms $x : I \vdash s : J$	functors $T \colon \mathcal{C} \to \mathcal{D}$	functions $f \colon A \to B$
Protypes $\alpha(x \fatsemi y)$	profunctors $P \colon \mathcal{C} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{D}$	formulas $\varphi(x, y)$ $(x \in A, y \in B)$
Proterms $a : \alpha(x \fatsemi y) \fatsemi b : \beta(y \fatsemi z)$ $\vdash \mu : \gamma(x \fatsemi z)$	natural transformations $\mu_{x,y,z} \colon P(x, y) \times Q(y, z) \to S(x, z)$	Horn sequences (and its proofs) $\varphi(x, y), \psi(y, z) \Rightarrow \chi(x, z)$
Protype Isomorphisms $\Upsilon : \alpha \cong \beta$	natural isomorphisms $\iota_{x,y} \colon P(x, y) \cong Q(x, y)$	equivalence of formulas $\varphi(x, y) \equiv \psi(x, y)$
Product types $I \times J$	product categories $\mathcal{C} \times \mathcal{D}$	product sets $A \times B$
Product protypes $\alpha \wedge \beta$	product profunctors $P(x, y) \times Q(x, y)$	conjunctions $\varphi(x, y) \wedge \psi(x, y)$
Substitution in terms $t(s_1, \ldots, s_n)$	composition of functors $T \circ \langle S_1, \ldots, S_n \rangle$	composition of functions $f \circ \langle s_1, \ldots, s_n \rangle$
Substitution in protypes $\alpha(s \fatsemi t)$	restriction of profunctors $P(S(x), T(y))$	instantiated formulas $\varphi(s(x), t(y))$
Substitution in proterms $\mu(s_0 \fatsemi s_1 \fatsemi s_2)$	instantiated natural transformations $\mu_{S_0(x'), S_1(y'), S_2(z')}$	instantiated Horn sequences $\varphi(s_0(x'), s_1(y')), \psi(s_1(y'), s_2(z'))$ $\Rightarrow \chi(s_0(x'), s_2(z'))$
path protype $\nrightarrow$	hom profunctor $\mathcal{C}(-, -)$	equality relation $=_A$
composition protype $\odot$	composition of profunctors by coend	composition of relations by $\exists$

Table 2. Interpretation of FVDblTT in $\mathbb{PROF}$ and $\mathbb{R}\mathrm{el}$

The paramount principle of categorical logic is the syntax-semantics duality [Uem23]. The main contribution of this paper is to establish the duality of the type theory FVDblTT and virtual double categories. Stating that a type theory is an internal language for a categorical structure should always be accompanied by a precise statement that there is a biadjunction between the 2-category of the specifications (or theories) of the type theory and the 2-category of the categorical structures. We present the biadjunction in our setting in Section 5, one between the 2-category consisting of specifications for this type theory and that of fibrational virtual double categories, and show the soundness and completeness of the type theory for the virtual double categories. In the proof, we pay special care to protype isomorphisms, whose behavior is peculiar to FVDblTT, by bridging the two 2-categories with another 2-category.

1.1. **Formal category theory** Formal category theory aims to provide an adequate framework in which we can develop category theory itself. One benefit to doing so is that one can argue, in parallel, multiple category theories, including enriched category theory [Kel05], internal category theory [Joh02], and enriched indexed category theory [Shu13]. Therefore, results in formal theory reduce to individual category theories, and one can focus on particularities therein, just as category theory offers abstract perspectives to individual mathematics. A comprehensive exposition of formal category theory is given in [LHLL17].

The earliest attempt was to perform category theory in an arbitrary 2-category by pretending that it is a 2-category of categories [Gra74]. However, more than mere 2-categories are needed to capture the big picture of category theory. The core difficulty is that this approach does not embody the notion of presheaves, or “set-valued functors” inside a 2-category. Subsequently, many solutions have emerged to address this problem, such as *Yoneda structures* [SW78] and *proarrow equipments* [Woo82, Woo85]. An expedient approach uses double categories, virtual double categories, or *augmented virtual double categories* [Shu08, Kou20]. The idea is to encapsulate functors and profunctors in a single setting in order to express categorical concepts in terms of internal behaviors of virtual double categories. General theories in (augmented) virtual double categories have recently been developed, the successful examples of which are the Yoneda structures and total categories in augmented virtual double categories by Koudenburg [Kou20, Kou24] and the theory of relative monads in virtual equipments by Arkor and McDermott [AM24]. In this paper, we cherish the philosophy of formal category theory and provide a type theory for it, respecting categorical behaviors of fibrational virtual double categories.

1.2. **Outline** Section 2 summarizes the terminology and notation used in this paper. Section 3 introduces the syntax and the equational theory of FVDblTT and its semantics in virtual double categories. Section 4 explains the type theory's possible extensions with additional constructors and how they work in the semantics with examples. In Section 5, we present the main result of this paper, the biadjunction between the 2-category of virtual double categories and the 2-category of FVDblTT specifications. This result directly implies the soundness and completeness of the type theory.

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2. Preliminaries on Virtual Double Categories

In this section, we briefly recall the definition of a virtual double category and introduce the notion of a cartesian fibrational virtual double category. It is this notion that we will construct the internal logic of.

First, we recall the definition of a virtual double category. The definitions are based on the paper [CS10].

Definition 2.1 ([CS10, Definition 2.1])**.** A ***virtual double category*** (VDC) $\mathbb{D}$ is a structure consisting of the following data.

- A category $\mathbb{D}_{\mathbf{t}}$. Its objects are simply called ***objects***, and its arrows are called ***tight arrows***, which are depicted vertically in this paper.
- A class of ***loose arrows*** $\mathbb{D}(I,J)_0$ for each pair of objects $I,J\in\mathbb{D}_{\mathbf{t}}$. These arrows are depicted horizontally with slashes as $\alpha\colon I\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} J$.
- A class of ***(virtual) cells***

$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "s"'] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, "t"] \\
J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] \arrow[urrr, phantom, "\mu"] & & & J_1
\end{tikzcd} \tag{2.1}$$

for each dataset consisting of $n\geq 0$, objects $I_0,\dots,I_n,J_0,J_1\in\mathbb{D}_{\mathbf{t}}$, tight arrows $s\colon I_0\to J_0$ and $t\colon I_n\to J_1$, and loose arrows $\alpha_1,\dots,\alpha_n,\beta$. We will write the finite sequence of loose arrows as $\overline{\alpha}=\alpha_1;\dots;\alpha_n$. When s and t are identities, we call the cell a ***globular cell*** and let $\mu\colon\overline{\alpha}\Rightarrow\beta$ denote the cell. The class of globular cells $\overline{\alpha}\Rightarrow\beta$ would be denoted by $\mathbb{D}(\overline{I})(\overline{\alpha},\beta)$ in which $\overline{I}=I_0;\dots;I_n$.
- A composition operation on cells that assigns to each dataset of cells

$$\begin{tikzcd}
I_{0,0} \arrow[r, dashed, "\overline{\alpha}_1", "\shortmid"{marking}] \arrow[d, "s_0"'] \arrow[dr, phantom, "\mu_1"] & I_{1,0} \arrow[r, dashed, "\overline{\alpha}_2", "\shortmid"{marking}] \arrow[d, "s_1"'] \arrow[dr, phantom, "\mu_2"] & I_{2,0} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "s_2"'] & \cdots \arrow[r, dashed, "\overline{\alpha}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\mu_n"] & I_{n,m_n} \arrow[d, "s_n"] \\
J_0 \arrow[r, "\beta_1", "\shortmid"{marking}] \arrow[d, "t_0"'] & J_1 \arrow[r, "\beta_2", "\shortmid"{marking}] & J_2 \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & \cdots \arrow[r, "\beta_n", "\shortmid"{marking}] & J_n \arrow[d, "t_1"] \\
K_0 \arrow[rrrr, "\gamma"', "\shortmid"{marking}] & & {} & & K_1
\end{tikzcd}$$

a cell

$$\begin{tikzcd}
I_{0,0} \arrow[r, dashed, "\overline{\alpha}_1", "\shortmid"{marking}] \arrow[d, "s_0"'] & I_{1,0} \arrow[r, dashed, "\overline{\alpha}_2", "\shortmid"{marking}] & I_{2,0} \arrow[r, dashed, "\shortmid"{marking}] & \cdots \arrow[r, dashed, "\overline{\alpha}_n", "\shortmid"{marking}] & I_{n,m_n} \arrow[d, "s_n"] \\
J_0 \arrow[d, "t_0"'] & & \nu\{\mu_1;\dots;\mu_n\} & & J_n \arrow[d, "t_1"] \\
K_0 \arrow[rrrr, "\gamma"', "\shortmid"{marking}] & & & & K_1
\end{tikzcd},$$

where the dashed line represents finite sequences of loose arrows for which associativity axioms hold. We will write the finite sequence of cells as $\overline{\mu}=\mu_1;\dots;\mu_n$.
- An identity cell for each loose arrow $\alpha\colon I\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} J$

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "\mathtt{id}_I"'] \arrow[dr, phantom, "\mathtt{id}_\alpha"] & J \arrow[d, "\mathtt{id}_J"] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] & J
\end{tikzcd},$$

for which identity laws axioms hold. (Henceforth, we will just write $=$ for the identity tight arrows.)

■

We say two object I, J in a virtual double category are isomorphic if they are isomorphic in the underlying tight category $\mathbb{D}_{\mathtt{t}}$, and write $I \cong J$. For any objects I, J in a virtual double category, we write $\mathbb{D}(I, J)$ for the category whose objects are loose arrows $\alpha\colon I \nrightarrow J$ and whose arrows are cells $\mu\colon \alpha \Rightarrow \beta$. A cell is called an ***isomorphism cell*** if it is invertible in this category. More generally, we say two loose arrows α, β are isomorphic if there exist two cells

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, phantom, "\mu"] & J \arrow[d, "t"] & & K \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, "s'"'] \arrow[dr, phantom, "\nu"] & L \arrow[d, "t'"] \\
K \arrow[r, "\beta"', "\shortmid"{marking}] & L & \text{and} & I \arrow[r, "\alpha"', "\shortmid"{marking}] & J
\end{tikzcd}$$

such that $\mu\{\nu\} = \mathtt{id}_\beta$ and $\nu\{\mu\} = \mathtt{id}_\alpha$, and call the cells μ and ν ***isomorphism cells***. It is always the case that $I \cong K$ and $J \cong L$ through the tight arrows s, t, s', t'.

Example 2.2. A double category is a virtual double category where every sequence of loose arrows is composable. In this case, a cell (2.1) is a cell whose top loose arrow is the composite of the loose arrows $\alpha_1, \dots, \alpha_n$.

Definition 2.3 ([CS10, Definition 3.1])**.** A ***virtual double functor*** $F\colon \mathbb{D} \to \mathbb{E}$ between virtual double categories $\mathbb{D}$ and $\mathbb{E}$ consists of the following data and conditions:

- A functor $F_{\mathtt{t}}\colon \mathbb{D}_{\mathtt{t}} \to \mathbb{E}_{\mathtt{t}}$.
- A family of functions $F_1\colon \mathbb{D}(I, J)_0 \to \mathbb{E}(F_{\mathtt{t}}(I), F_{\mathtt{t}}(J))_0$ for each pair of objects I, J of $\mathbb{D}$.
- A family of functions sending each cell μ of $\mathbb{D}$ on the left below to a cell $F_1(\mu)$ of $\mathbb{E}$ on the right below:

$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "s_0"'] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, "s_1"] & & F_{\mathtt{t}}(I_0) \arrow[r, "F_1(\alpha_1)", "\shortmid"{marking}] \arrow[d, "F_{\mathtt{t}}(s_0)"'] & F_{\mathtt{t}}(I_1) \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "F_1(\alpha_n)", "\shortmid"{marking}] & F_{\mathtt{t}}(I_n) \arrow[d, "F_{\mathtt{t}}(s_1)"] \\
 & \mu & & & \mapsto & & F_1(\mu) & & \\
J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] & & & J_1 & & F_{\mathtt{t}}(J_0) \arrow[rrr, "F_1(\beta)"', "\shortmid"{marking}] & & & F_{\mathtt{t}}(J_1)
\end{tikzcd} . \tag{2.2}$$

- The identity cells are preserved.
- Composition of cells is preserved.

As usual, we will often omit the subscripts of the functor and functions $F_{\mathtt{t}}$ and F_1.

A ***vertical transformation*** $\theta\colon F \to G$ between virtual double functors $F, G\colon \mathbb{D} \to \mathbb{E}$ consists of the following data and conditions:

- A natural transformation $\theta_0\colon F_{\mathtt{t}} \to G_{\mathtt{t}}$.
- A cell $\theta_{1,\alpha}$ for each loose arrow $\alpha\colon I \nrightarrow J$ of $\mathbb{D}$:

$$\begin{tikzcd}
FI \arrow[r, "F\alpha", "\shortmid"{marking}] \arrow[d, "\theta_{0,I}"'] \arrow[dr, phantom, "\theta_{1,\alpha}"] & FJ \arrow[d, "\theta_{0,J}"] \\
GI \arrow[r, "G\alpha"', "\shortmid"{marking}] & GJ
\end{tikzcd}$$

- The naturality condition for cells:

$$\begin{tikzcd}
FI_0 \arrow[r, dashed, "F\overline{\alpha}", "\shortmid"{marking}] \arrow[d, "Fs_0"'] \arrow[dr, phantom, "F\mu"] & FI_n \arrow[d, "Fs_n"] & & FI_0 \arrow[r, dashed, "F\overline{\alpha}", "\shortmid"{marking}] \arrow[d, "\theta_{I_0}"'] \arrow[dr, phantom, "\theta_{\overline{\alpha}}"] & FI_n \arrow[d, "\theta_{I_n}"] \\
FJ_0 \arrow[r, "F\beta"', "\shortmid"{marking}] \arrow[d, "\theta_{J_0}"'] \arrow[dr, phantom, "\theta_\beta"] & FJ_1 \arrow[d, "\theta_{J_1}"] & = & GI_0 \arrow[r, dashed, "G\overline{\alpha}"', "\shortmid"{marking}] \arrow[d, "Gs_0"'] \arrow[dr, phantom, "G\mu"] & GI_n \arrow[d, "Gs_n"] \\
GJ_0 \arrow[r, "G\beta"', "\shortmid"{marking}] & GJ_1 & & GJ_0 \arrow[r, "G\beta"', "\shortmid"{marking}] & GJ_1
\end{tikzcd} .$$

VDbl is the 2-category of virtual double categories, virtual double functors, and vertical transformations. ■

Definition 2.4 ([CS10, Definition 7.1])**.** Let $\mathbb{D}$ be a virtual double category. A ***restriction*** of a loose arrow $\alpha\colon I \nrightarrow J$ along a pair of tight arrows $s\colon I' \to I$ and $t\colon J' \to J$ is the loose arrow $\alpha[s \,⨟\, t]\colon I' \nrightarrow J'$ equipped with a cell

$$\begin{tikzcd}
I' \arrow[r, "{\alpha[s ⨟ t]}", "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, phantom, "\mathtt{rest}"] & J' \arrow[d, "t"] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] & J
\end{tikzcd}$$

with the following universal property: any cell μ of the form on the left below factors uniquely through the cell $\mathtt{rest}$ as on the right below.

$$\begin{tikzcd}
K \arrow[r, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "u"'] & L \arrow[d, "v"] & & K \arrow[r, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\widehat{\mu}"] & L \arrow[d, "v"] \\
I' \arrow[d, "s"'] & J' \arrow[d, "t"] & = & I' \arrow[r, "{\alpha[s ⨟ t]}"', "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, phantom, "\mathtt{rest}"] & J' \arrow[d, "t"] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[uu, phantom, "\mu" near start, shift left=3ex] & J & & I \arrow[r, "\alpha"', "\shortmid"{marking}] & J
\end{tikzcd}$$

If the restrictions exist for all triples (α, s, t), then we say that $\mathbb{D}$ is a ***fibrational virtual double category (*FVDC*)***.

A ***fibrational virtual double functor*** $F\colon \mathbb{D} \to \mathbb{E}$ between fibrational virtual double categories $\mathbb{D}$ and $\mathbb{E}$ is a virtual double functor that preserves restrictions. **FibVDbl** is the 2-category of fibrational virtual double categories, fibrational virtual double functors, and vertical transformations. ■

Lemma 2.5. A virtual double functor $F\colon \mathbb{D} \to \mathbb{E}$ is an equivalence in **VDbl** if and only if

(i) the functor $F_{\mathbf{t}}\colon \mathbb{D}_{\mathbf{t}} \to \mathbb{E}_{\mathbf{t}}$ for F is an equivalence of categories,
(ii) for any loose arrow $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ in $\mathbb{E}$, there exists a loose arrow $\beta\colon I' \relbar\joinrel\mapstochar\joinrel\rightarrow J'$ in $\mathbb{D}$ and an isomorphism cell μ as below:

$$\begin{tikzcd} FI' \arrow[r, "F\beta", "\shortmid"{marking}] \arrow[d, "\wr"'] \arrow[dr, phantom, "\mu\ \wr"] & FJ' \arrow[d, "\wr"] \\ I \arrow[r, "\alpha"', "\shortmid"{marking}] & J \end{tikzcd} \quad , \quad \text{and}$$

(iii) for any quadruple $(s, t, \overline{\alpha}, \beta)$, the function F on the cells (2.2) is a bijection.

A fibrational virtual double functor $F\colon \mathbb{D} \to \mathbb{E}$ is an equivalence in **FibVDbl** if and only if (i), (ii), and the special case of (iii) where s and t are identities are satisfied.

Proof. If we are given an inverse G of F, then $G_{\mathbf{t}}$ is the inverse of $F_{\mathbf{t}}$, and the isomorphism $FG \Rightarrow \mathrm{Id}$ gives the isomorphism cells μ above. The inverse of functions F in (2.2) is given by sending a cell ν on the right to $G_1(\nu)$ and composing with the isomorphism cells obtained from the isomorphism $GF \Rightarrow \mathrm{Id}$.

Conversely, given the conditions, we can construct an inverse G of F. The vertical part of G is given by an inverse of $F_{\mathbf{t}}$. Then, for each loose arrow $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ in $\mathbb{E}$, we can show that a loose arrow $\beta\colon GI \relbar\joinrel\mapstochar\joinrel\rightarrow GJ$ in $\mathbb{D}$ is isomorphic to α by the second condition. The bijection in (iii) determines how to send a cell in $\mathbb{E}$ to a cell in $\mathbb{D}$. The functoriality of G follows from the one-to-one correspondence between cells in $\mathbb{D}$ and $\mathbb{E}$ in (iii).

For the fibrational case, we only need to check that (iii), in general, follows from the cases where s and t are identities. However, it is straightforward by the universal property of the restrictions. It follows that the inverse is fibrational from the fact that any equivalence preserves restrictions. □

Next, we introduce the notion of a cartesian fibrational virtual double category.

Definition 2.6. A ***cartesian object*** in a 2-category $\mathbf{B}$ with finite products $\mathbb{1}, \otimes$ is an object x of $\mathbf{B}$ such that the canonical 1-cells $!\colon x \to \mathbb{1}$ and $\varDelta\colon x \to x \otimes x$ have right adjoints $1\colon \mathbb{1} \to x$ and $\times\colon x \otimes x \to x$, respectively. A ***cartesian 1-cell*** (or ***cartesian arrow***) in $\mathbf{B}$ is a 1-cell $f\colon x \to y$ between cartesian objects x and y of $\mathbf{B}$ such that the canonical 2-cells obtained by the mate construction $\times \circ (f \otimes f) \Rightarrow f \circ \times$ and $f \circ 1 \Rightarrow 1$ are invertible.

For a 2-category $\mathbf{B}$ with finite products, we write $\mathbf{B_{cart}}$ for the 2-category of cartesian objects, cartesian 1-cells, and arbitrary 2-cells in $\mathbf{B}$. ■

Lemma 2.7. Let $\mathbf{B}$ be a 2-category with finite products. A 1-cell $f\colon x \to y$ in $\mathbf{B_{cart}}$ is an equivalence in $\mathbf{B_{cart}}$ if and only if the underlying 1-cell of f is an equivalence in $\mathbf{B}$.

Proof. The only if part is clear since we have the forgetful 2-functor $\mathbf{B_{cart}} \to \mathbf{B}$. For the if part, take the right adjoint g of the underlying 1-cell of f as its inverse. Taking the right adjoint of both sides of the isomorphism 2-cells $! \circ f \cong !$ and $(f \otimes f) \circ \varDelta \cong \varDelta \cong f$, we obtain the isomorphism 2-cells $1 \fatsemi g \cong 1$ and $\times \circ (g \otimes g) \cong g \circ \times$. This shows that g gives a cartesian morphism from y to x, and g is indeed the inverse of f in $\mathbf{B_{cart}}$. □

Proposition 2.8. An FVDC $\mathbb{D}$ is cartesian if and only if the following conditions are satisfied:

(i) $\mathbb{D}_{\mathbf{t}}$ has finite products;
(ii) $\mathbb{D}$ locally has finite products, that is, for each $I, J \in \mathbb{D}_{\mathbf{t}}$,
  (a) for any loose arrows $\alpha, \beta\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ in $\mathbb{D}$, there exists a loose arrow $\alpha \wedge \beta\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ and cells $\pi_0\colon \alpha \wedge \beta \Rightarrow \alpha$, $\pi_1\colon \alpha \wedge \beta \Rightarrow \beta$ such that for any finite sequence of loose arrows $\overline{\gamma}$ where $\gamma_i\colon I_{i-1} \relbar\joinrel\mapstochar\joinrel\rightarrow I_i$ for $1 \leq i \leq n$ where $I_0 = I$ and $I_n = J$, the function

$$\mathbb{D}(\overline{I})(\overline{\gamma}, \alpha \wedge \beta) \to \mathbb{D}(\overline{I})(\overline{\gamma}, \alpha) \times \mathbb{D}(\overline{I})(\overline{\gamma}, \beta) \quad ; \quad \mu \mapsto (\pi_0 \circ \mu, \pi_1 \circ \mu)$$

  is a bijection, and
  (b) there exists a loose arrow $\top\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ such that $\mathbb{D}(\overline{I})(\overline{\gamma}, \top)_0$ is a singleton for any finite sequence of loose arrows $\overline{\gamma}$;
(iii) the local finite products are preserved by restrictions.

A morphism between cartesian FVDCs is a cartesian morphism if and only if the underlying tight functor preserves finite products and the morphism preserves local finite products.

Proof sketch. The proof is similar to that of [Ale18, Prop 4.12]. First, suppose that $\mathbb{D}$ is cartesian. Let $\Delta_I\colon I \to I \times I$ be the diagonal of I and $!_I\colon I \to 1$ be the unique arrow to the terminal object. If $\mathbb{D}$ is cartesian, then $\alpha \wedge \beta$ and $\top$ in $\mathbb{D}(I, J)$ are given by $(\alpha \times \beta)[\Delta_I \fatsemi \Delta_J]$ and $\mathtt{Id}_1(!_I, !_J)$, which brings the finite products in $\mathbb{D}(I, J)$. The local finite products are preserved by restrictions since, by the universal property of the restrictions, we have

$$(\alpha \times \beta)[\Delta_I \fatsemi \Delta_J][s \fatsemi t] \cong (\alpha \times \beta)[(s \times s) \fatsemi (t \times t)][\Delta_{I'} \fatsemi \Delta_{J'}] \cong (\alpha[s \fatsemi t] \times \beta[s \fatsemi t])[\Delta_{I'} \fatsemi \Delta_{J'}],$$

and similarly for $\top$. Conversely, if $\mathbb{D}$ locally has finite products, then assigning

$$\alpha \times \beta \coloneqq \alpha[\pi_I \fatsemi \pi_J] \wedge \beta[\pi_K \fatsemi \pi_L] \colon I \times K \mathrel{\mapstochar\rightarrow} J \times L$$

to each pair $\alpha\colon I \mathrel{\mapstochar\rightarrow} J$, $\beta\colon K \mathrel{\mapstochar\rightarrow} L$ and a cell $\mu \times \nu$ naturally obtained from the universal property of the restrictions induces the functor $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ right adjoint to the diagonal functor, and the functor $1\colon \mathbb{1} \to \mathbb{D}$ obtained by the terminal object in $\mathbb{D}_{\mathbf{t}}$ is the right adjoint of $!$. The second statement follows from the construction of the equivalence above. ☐

Remark 2.9. The third condition in Proposition 2.8 is necessary for FVDC but not for equipments as in [Ale18] since the latter has extensions.

Example 2.10. One of the motivations for our type theory is to formalize category theory in formal language. The following examples of virtual double categories will provide a multitude of category theories that can be formalized in our type theory.

(i) The double category $\mathbb{P}\mathrm{rof}$ consists of small categories, functors, and profunctors as objects, tight arrows, and loose arrows, respectively. When we consider not necessarily small categories, however, we do not have a composition of profunctors in general. Nevertheless, we can still define a virtual double category $\mathbb{P}\mathrm{ROF}$ of categories, functors, and profunctors. It is a cartesian fibrational virtual double category (CFVDC).

(ii) Similarly, we can define CFVDCs $\mathcal{V}$-$\mathbb{P}\mathrm{rof}$ and $\mathcal{V}$-$\mathbb{P}\mathrm{ROF}$ of $\mathcal{V}$-enriched categories, functors, and profunctors, where $\mathcal{V}$ is a symmetric monoidal category.

(iii) We can also define virtual double categories $\mathbb{P}\mathrm{rof}(\mathcal{S})$ of internal categories, functors, and profunctors in categories $\mathcal{S}$ with finite limits.

Example 2.11. Predicate logic deals with functions and relations between sets. We can see these two as tight and loose arrows, respectively, although we limit ourselves to the case where relations are binary.

(i) We have a double category called $\mathbb{R}\mathrm{el}$ consisting of sets, functions, and relations [Lam22], where there is at most one cell for each frame and a cell (2.1) exists whenever

$$\alpha_1(x_0, x_1) \wedge \cdots \wedge \alpha_n(x_{n-1}, x_n) \Rightarrow \beta(s_0(x_0), s_1(x_n)) \quad (\forall x_i \in I_i, \quad 0 \leq i \leq n).$$

This can be generalized to a regular category and further to a category with finite limits equipped with a stable factorization system as in [HN23].

(ii) Given a category $\mathcal{S}$ with finite limits, a span in $\mathcal{S}$ is a pair of arrows $f\colon Z \to X$ and $g\colon Z \to Y$. We can define a double category called $\mathbb{S}\mathrm{pan}(\mathcal{S})$ consisting of objects, arrows, and spans in $\mathcal{S}$ [Ale18]. It can be seen as a proof-relevant version of $\mathbb{R}\mathrm{el}$.

3. Fibrational Virtual Double Type Theory

This section will present a type theory that serves as an internal language for a CFVDC, which we call ***fibrational virtual double type theory*** (FVDblTT). We will first introduce the type theory FVDblTT in Subsection 3.1. Then, we will present how to interpret the type theory in a CFVDC in Subsection 3.2.

3.1. Syntax

In this subsection, we will present the syntax and equational theory of FVDblTT and explain some advantages of the type theory.

Typing Judgments

Type	I type
Context	Γ ctx
Term	$\Gamma \vdash s : I$
Term Substitution	$\Gamma \vdash S \,/\, \Delta$
Protype	$\Gamma \fatsemi \Delta \vdash \alpha$ protype
Procontext	$\Gamma_0 \fatsemi \ldots \fatsemi \Gamma_n \mid A$ proctx
Proterm	$\Gamma_0 \fatsemi \ldots \fatsemi \Gamma_n \mid A \vdash \mu : \beta$

Equality Judgments

Term	$\Gamma \vdash s \equiv t : I$
Protype	$\Gamma \fatsemi \Delta \vdash \alpha \equiv \beta$
Proterm	$\Gamma_0 \fatsemi \ldots \fatsemi \Gamma_n \mid A \vdash \mu \equiv \nu$

Isomorphism Judgments

Protype	$\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta$

FIGURE 3. The three kinds of judgments in FVDblTT

Three kinds of judgments. In FVDblTT, judgments are divided into three kinds, as in Figure 3. The first kind of judgments is the typing judgments for all the syntactic entities in the type theory, which regulates the well-formedness of the entities. Types, contexts, terms, and term substitutions are the same as those in the ordinary algebraic theory as in [Cro94, Jac99]. This fragment of the type theory serves as the theory of categories and functors. As usual, substitution of terms for variables in terms is defined by induction on the structure of terms.

Protypes and proterms are particular to this type theory and encode the loose arrows and cells in an FVDC. The prefix ***pro-*** stands for "pro"positions and "pro"functors. A protype α depends on two contexts, Γ and Δ, which will be interpreted as the source and the target of a loose arrow representing the protype. We call the pair (Γ, Δ) the ***two-sided context*** of the protype. In the type theory, we distinguish semicolons $\fatsemi$ from the ordinary concatenating symbol commas , by restricting using the former to concatenate items in the horizontal direction in a diagram in a VDC. Since the source and the target of a loose arrow can not be exchanged in any sense in a general VDC, we need to respect the order when we use the semicolons. Accordingly, a procontext $\alpha_1 \fatsemi \ldots \fatsemi \alpha_n$, which is a finite sequence of protypes, is only well-typed so that the second (target) context of a protype is the first (source) context of the subsequent protype, and hence a procontext depends on a sequence of contexts. As a particular case, we have the empty procontext $\cdot$ depending on a single context Γ. Another item is proterms $A \vdash \mu : \beta$, which are to be interpreted as globular cells in a VDC whose domains and codomains are the interpretation of A and β, respectively.

The second kind of judgments is the equality judgments for terms and proterms. We incorporate the ordinary algebraic theory of terms with the equality judgments for terms, and we also have the equality judgments for proterms to capture the equality of cells in a VDC. The rules for equality judgments, or the equational theory of the type theory, are based on the usual axioms of reflexivity, symmetry, transitivity, and congruence. The rules for equality judgments of terms are standard, and the rules for those of protyopes are given so that the type theory is sound and complete for the CFVDCs. The details of the rules will be given in Appendix B.1.

Type

$$\frac{I\ \mathsf{type} \qquad J\ \mathsf{type}}{I \times J\ \mathsf{type}}\ \times \qquad \frac{}{\mathbb{1}\ \mathsf{type}}\ \mathbb{1}$$

Context

$$\frac{}{\cdot\ \mathsf{ctx}}\ \textsc{EmpCtx} \qquad \frac{\Gamma\ \mathsf{ctx} \qquad I\ \mathsf{type}}{\Gamma, x : I\ \mathsf{ctx}}\ \textsc{ConsCtx}$$

Term

$$\frac{}{\Gamma, x : I, \Delta \vdash x : I}\ \textsc{Var} \qquad \frac{I\ \mathsf{type} \qquad J\ \mathsf{type} \qquad \Gamma \vdash s : I \qquad \Gamma \vdash t : J}{\Gamma \vdash \langle s, t \rangle : I \times J}\ \text{I-}\times$$

$$\frac{\Gamma \vdash t : I \times J}{\Gamma \vdash \mathsf{pr}_0(t) : I}\ \text{E-}\times\text{-L} \qquad \frac{\Gamma \vdash t : I \times J}{\Gamma \vdash \mathsf{pr}_1(t) : J}\ \text{E-}\times\text{-R} \qquad \frac{}{\Gamma \vdash \langle\ \rangle : \mathbb{1}}\ \text{I-}\mathbb{1}$$

Term Substitution

$$\frac{}{\Gamma \cdot \vdash / \cdot}\ \textsc{EmpSubs} \qquad \frac{\Gamma \vdash S \,/\, \Delta \qquad \Gamma \vdash s : I}{\Gamma \vdash S, s \,/\, \Delta, x : I}\ \textsc{ConsSubs}$$

Protype

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha\ \mathsf{protype} \qquad \Gamma \fatsemi \Delta \vdash \beta\ \mathsf{protype}}{\Gamma \fatsemi \Delta \vdash \alpha \wedge \beta\ \mathsf{protype}}\ \textsc{ProProd}$$

$$\frac{}{\Gamma \fatsemi \Delta \vdash \top\ \mathsf{protype}}\ \textsc{ProTer}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \mid \alpha\ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \vdash \alpha[S/\Gamma \fatsemi T/\Delta]\ \mathsf{protype}}\ \textsc{ProtpRes}$$

Procontext

$$\frac{}{\Gamma \mid \cdot\ \mathsf{proctx}}\ \textsc{EmpProctx}$$

$$\frac{\Gamma_0 \fatsemi \ldots \fatsemi \Gamma_n \mid A\ \mathsf{proctx} \qquad \Gamma_n \fatsemi \Delta \vdash \alpha\ \mathsf{protype}}{\Gamma_0 \fatsemi \ldots \fatsemi \Gamma_n \fatsemi \Delta \mid A, a : \alpha\ \mathsf{proctx}}\ \textsc{ConsPro}$$

Proterm

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha\ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash a : \alpha}\ \textsc{ProVar} \qquad \frac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mathsf{tr}_{\Upsilon}(a) : \beta}\ \textsc{Transport}$$

$$\frac{\overline{\Gamma_i} \mid a_{i,1} : \alpha_{i,1} \fatsemi \ldots \fatsemi a_{i,n_i} : \alpha_{i,n_i} \vdash \mu_i : \beta_i\ (i = 1, \ldots, m) \qquad \widetilde{\Gamma} \mid b_1 : \beta_1 \fatsemi \ldots \fatsemi b_n : \beta_n \vdash \nu : \gamma}{\overline{\Gamma} \mid a_{1,1} : \alpha_{1,1} \fatsemi \ldots \fatsemi a_{m,n_m} : \alpha_{m,n_m} \vdash \nu\{\mu_1/b_1 : \beta_1 \fatsemi \ldots \fatsemi \mu_m/b_m : \beta_m\} : \gamma}\ \textsc{Prosubstitute}$$

$$\frac{\overline{\Gamma} \mid A \vdash \mu : \alpha \qquad \overline{\Gamma} \mid A \vdash \nu : \beta}{\overline{\Gamma} \mid A \vdash \langle \mu, \nu \rangle : \alpha \wedge \beta}\ \text{I-}\wedge \qquad \frac{\Gamma \fatsemi \Delta \vdash \alpha\ \mathsf{protype} \qquad \Gamma \fatsemi \Delta \mid \beta\ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \wedge \beta \vdash \pi_0 : \alpha}\ \text{E-}\wedge\text{-L}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha\ \mathsf{protype} \qquad \Gamma \fatsemi \Delta \mid \beta\ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \wedge \beta \vdash \pi_1 : \beta}\ \text{E-}\wedge\text{-R} \qquad \frac{}{\overline{\Gamma} \mid A \vdash \langle\ \rangle : \top}\ \text{I-}\top$$

$$\frac{\Gamma_i \vdash S_i \,/\, \Delta_i\ (i = 0, \ldots, n) \qquad \Delta_0 \fatsemi \ldots \fatsemi \Delta_n \mid a_1 : \alpha_1 \fatsemi \ldots \fatsemi a_n : \alpha_n \vdash \mu : \beta}{\begin{array}{l}\Gamma_1 \fatsemi \ldots \fatsemi \Gamma_n \mid c_1 : \alpha_1[S_0/\Delta_0 \fatsemi S_1/\Delta_1] \fatsemi \ldots \fatsemi c_n : \alpha_n[S_{n-1}/\Delta_{n-1} \fatsemi S_n/\Delta_n] \\ \quad \vdash \mu[S_0/\Delta_0 \fatsemi \ldots \fatsemi S_n/\Delta_n] : \beta[S_0/\Delta_0 \fatsemi S_n/\Delta_n]\end{array}}\ \textsc{ProtmRest}$$

FIGURE 4. The rules for typing judgments

Protype isomorphisms. The third kind of judgments is what we call isomorphism judgments for protypes, which are the central component of this type theory. They are understood half as the typing judgments for protype isomorphisms and half as the equality judgments for protypes up to isomorphism. Protype isomorphisms are considered as codes for the two proterms mutually inverse to each other so that proterms can track what they actually represent in the type theory. On the other hand, they allow us to reason about the equality of protypes up to isomorphism handily. In category theory, one often proves that two objects, functors, or profunctors are isomorphic by exhibiting a sequence of those isomorphisms between them that one has already constructed or known to exist. With protype isomorphisms, we can do the same in the type theory without showing proterms in both directions explicitly every time but still keeping track of the proterms that represent the isomorphisms. We do not have equality judgments for protype isomorphisms since one can identify or distinguish them by the proterms they represent using the equality judgments for proterms.

The rules for isomorphism judgments for protypes are given in Appendix B.1. They are divided into three kinds. The first kind is for the groupoid structure of protypes, for instance, the introduction rule of the inverse of a protype isomorphism. Next, the second kind is for compatibility conditions of restriction and other operations that must be satisfied in a CFVDC, such as the introduction rule of the isomorphism between $\top$ and $\top$ after substitution. Finally, the third kind constructs protype isomorphisms from pairs of mutually inverse proterms.

The transformation of protype isomorphisms into proterms is achieved by the conversion rule. The resulting proterms are written as $\mathsf{tr}_{\Upsilon}(a)$ formally, but we will often write them as $\Upsilon\{a\}$ by abuse of notation. Corresponding to some of the rules for protype isomorphisms, such as inverses and composition of protype isomorphisms, we have rules for proterm equalities that guarantee that the protype isomorphisms are those that are expected to be.

Fibrationality and cartesianness. We restrict our interest to fibrational virtual double categories and an internal language for them. This is because the fibrationality condition enables us to describe the data of cells in a VDC more conveniently. A cell in a VDC comes equipped with two tight arrows, one finite sequence of loose arrows, and another loose arrow, which is unwieldy to bring into the syntax of the type theory. Owing to the fibrationality condition, under which every cell corresponds to a globular cell in a VDC, judgments for proterms achieve a linear presentation of the data of cells in a VDC.

While the fibrationality facilitates the presentation of the type theory, it does not confine what we are trying to model with it. Rather, it reflects what people are genuinely interested in when they work with the VDCs of profunctors and relations. The fibrationality condition indicates the possibility of substituting terms (tight arrows) for a variable in loose arrows, which is fundamental to both the VDCs of profunctors and relations. For a profunctor $Q : \mathcal{A} \nrightarrow \mathcal{B}$ and two functors, $F : \mathcal{C} \to \mathcal{A}$ and $G : \mathcal{D} \to \mathcal{B}$, the instantiated profunctor $Q(F, G)$ is defined as $Q(F, G)(C, D) = Q(F(C), G(D))$. A cell of the form on the left below in the VDC $\mathbb{P}\mathrm{ROF}$ of profunctors is a family of functions on the right below that is natural in C and D and dinatural in E, which shows the primacy of instantiated profunctors in $\mathbb{P}\mathrm{ROF}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "P_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathcal{E} \arrow[r, "P_2", "\shortmid"{marking}] \arrow[d, phantom, "\mu"] & \mathcal{D} \arrow[d, "G"] & & \\
\mathcal{A} \arrow[rr, "Q"', "\shortmid"{marking}] & & \mathcal{B} & &
\end{tikzcd}
\quad \leftrightsquigarrow \quad P_1(C, E) \times P_2(E, D) \xrightarrow{\mu_{C,E,D}} Q(F(C), G(D))$$

The substitution in relations is similarly a common practice in predicate logic. To sum up, the fibrationality condition is a natural condition for VDCs of these kinds and should be worth considering whatever category theory or predicate logic we are working with.

The cartesianness assumption is natural for dealing with tuples of terms or proterms in the type theory. The type theory has the product type $I \times J$ and the protype $\alpha \wedge \beta$, interpreted as the product of two objects and the local product of two loose arrows in a VDC. When we establish our type theory on the bedrock of the 2-category of CFVDCs, one benefit of assuming cartesianness is that cartesianness in 2-categories is not just a structure but a property-like one, which makes it easier to legitimate the rules on compatibility with additional constructors as we will introduce in Appendix B.1. It might be preferable to have an internal language for FVDCs that is not necessarily cartesian but with a monoidal structure and its type-theoretic counterpart, which does not seem to be a difficult task to achieve but is beyond the scope of this paper.

Explicit substitution. Substitution is vital to type theory. In FVDblTT, we have two kinds of variables: ones for terms into variables and ones for proterms into provariables. For the term substitution, we have a term $t[S/\Gamma]$, a protype $\alpha[S/\Gamma \fatsemi T/\Delta]$, and a proterm $\mu[S_0/\Gamma_0 \fatsemi \ldots \fatsemi S_n/\Gamma_n]$. For the proterm substitution, we have a instantiated proterm $\mu\{\mu_1/b_1 : \beta_1, \ldots, \mu_n/b_n : \beta_n\}$.

The prevailing approach to substitution in type theory is to define it inductively concerning the structure of terms. In FVDblTT, however, we adopt an alternative approach called explicit substitution, as in [ACCL91]. Explicit substitution is a method of introducing substitution as a syntactic construct, and it is often used when they encounter the problem that the syntactic equality is too strict for the intended semantics. For instance, Curien's work uses explicit substitution to resolve the problem of the semantics of dependent type theory in locally cartesian closed categories [Cur93]. While the usual substitution is defined in the meta-theory, explicit substitution enables us to define it in the syntax of the type theory.

We adopt explicit substitution for protypes and proterms in order to accommodate the difference between two isomorphic protypes that are not strictly equal. The meta-theoretic substitution would ignore the difference between a term with a substitution and the result of the substitution, unintendedly forcing us to interpret the two isomorphic protypes, say $\alpha[S/\Gamma, T/\Delta][S'/\Gamma', T'/\Delta']$ and $\alpha\,[S[S'/\Gamma] \fatsemi T[T'/\Delta]/\Delta]$, as the same. On the other hand, in a FVDC, restrictions are not even chosen but are just given by universal properties. Therefore, the two protypes above, for instance, should be distinguished in the type theory but interpreted as canonically isomorphic loose arrows in a VDC, for which we add a protype isomorphism between these two protypes. This is the reason why we adopt explicit substitution in FVDblTT.

3.2. **Semantics** As previously mentioned, the semantics of FVDblTT are to be given in a CFVDC. The elements in the type theory are to be interpreted as the following elements in a VDC.

- $I \ \mathsf{type}$ is to be interpreted as an object $\llbracket I \rrbracket$ of $\mathbb{D}$.
- $\mathit{\Gamma} \ \mathsf{ctx}$ is to be interpreted as the product of the objects $\llbracket \mathit{\Gamma} \rrbracket = \prod_i \llbracket I_i \rrbracket$ in $\mathbb{D}$ where $\mathit{\Gamma} = I_0, \dots, I_{n-1}$.
- $\mathit{\Gamma} \vdash t : I$ is to be interpreted as a tight arrow $\llbracket t \rrbracket : \llbracket \mathit{\Gamma} \rrbracket \to \llbracket I \rrbracket$ in $\mathbb{D}$.
- $\mathit{\Gamma} \vdash S \,/\, \mathit{\Delta}$ is to be interpreted as a tight arrow $\llbracket S \rrbracket : \llbracket \mathit{\Gamma} \rrbracket \to \llbracket \mathit{\Delta} \rrbracket$ in $\mathbb{D}$.
- $\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha \ \mathsf{protype}$ is to be interpreted as a loose arrow $\llbracket \alpha \rrbracket : \llbracket \mathit{\Gamma} \rrbracket \looparrowright \llbracket \mathit{\Delta} \rrbracket$ in $\mathbb{D}$.
- $\mathit{\Gamma}_0 \fatsemi \dots \fatsemi \mathit{\Gamma}_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \ \mathsf{proctx}$ is to be interpreted as a path of loose arrows

$$\begin{tikzcd}
\llbracket \mathit{\Gamma}_0 \rrbracket \arrow[r, "\llbracket \alpha_1 \rrbracket", "\shortmid"{marking}] & \llbracket \mathit{\Gamma}_1 \rrbracket \arrow[r, "\shortmid"{marking}] & \dots \arrow[r, "\llbracket \alpha_n \rrbracket", "\shortmid"{marking}] & \llbracket \mathit{\Gamma}_n \rrbracket
\end{tikzcd} \quad \text{in } \mathbb{D}.$$

- $\overline{\mathit{\Gamma}} \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu : \beta$ is to be interpreted as a cell

$$\begin{tikzcd}
\llbracket \mathit{\Gamma}_0 \rrbracket \arrow[r, "\llbracket \alpha_1 \rrbracket", "\shortmid"{marking}] \arrow[d, equal] & \dots & \dots \arrow[r, "\llbracket \alpha_n \rrbracket", "\shortmid"{marking}] & \llbracket \mathit{\Gamma}_n \rrbracket \arrow[d, equal] \\
\llbracket \mathit{\Gamma}_0 \rrbracket \arrow[rrr, "\llbracket \beta \rrbracket"', "\shortmid"{marking}] \arrow[rrru, phantom, "\llbracket \mu \rrbracket"] & & & \llbracket \mathit{\Gamma}_n \rrbracket
\end{tikzcd} \quad \text{in } \mathbb{D}.$$

- $\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta$ is to be interpreted as an isomorphism of loose arrows $\llbracket \Upsilon \rrbracket \colon \llbracket \alpha \rrbracket \Rightarrow \llbracket \beta \rrbracket \colon \llbracket \mathit{\Gamma} \rrbracket \looparrowright \llbracket \mathit{\Delta} \rrbracket$ in $\mathbb{D}$.

Let us turn to the inductive definition of the interpretation of the terms and protypes in a CFVDC.

Definition 3.1. The interpretation of the terms, protypes, protype isomorphisms, and proterms in a CFVDC $\mathbb{D}$ is defined inductively as follows:

- Product types $\times, \mathbb{1}$ are interpreted as the product and terminal object of $\mathbb{D}$, respectively. Pairing, projections, and the unit are interpreted in an obvious way.
- Product protypes $\wedge, \top$ in context $\mathit{\Gamma} \fatsemi \mathit{\Delta}$ are interpreted as the product and terminal loose arrow from $\llbracket \mathit{\Gamma} \rrbracket$ to $\llbracket \mathit{\Delta} \rrbracket$, respectively. Pairing, projections, and the unit are interpreted in an obvious way.
- The interpretation of the term $t[S/\mathit{\Delta}]$ is the composite of the tight arrow $\llbracket t \rrbracket : \llbracket \mathit{\Delta} \rrbracket \to \llbracket I \rrbracket$ with $\llbracket S \rrbracket : \llbracket \mathit{\Gamma} \rrbracket \to \llbracket \mathit{\Delta} \rrbracket$.
- The interpretation of the protype $\alpha[S/\mathit{\Gamma} \fatsemi T/\mathit{\Delta}]$ is the restriction of the loose arrow $\llbracket \alpha \rrbracket : \llbracket \mathit{\Gamma} \rrbracket \looparrowright \llbracket \mathit{\Delta} \rrbracket$ along the tight arrow $\llbracket S \rrbracket : \llbracket \mathit{\Gamma}' \rrbracket \to \llbracket \mathit{\Gamma} \rrbracket$ and $\llbracket T \rrbracket : \llbracket \mathit{\Delta}' \rrbracket \to \llbracket \mathit{\Delta} \rrbracket$.

$$\begin{tikzcd}
\llbracket \mathit{\Gamma}' \rrbracket \arrow[r, "\llbracket \alpha[S/\mathit{\Gamma} \fatsemi T/\mathit{\Delta}] \rrbracket", "\shortmid"{marking}] \arrow[d, "\llbracket S \rrbracket"'] & \llbracket \mathit{\Delta}' \rrbracket \arrow[d, "\llbracket T \rrbracket"] \\
\llbracket \mathit{\Gamma} \rrbracket \arrow[r, "\llbracket \alpha \rrbracket"', "\shortmid"{marking}] \arrow[ru, phantom, "\mathtt{rest}"] & \llbracket \mathit{\Delta} \rrbracket
\end{tikzcd}$$

- The interpretations of the protype isomorphisms $\mathsf{id}_\alpha, \Upsilon^{-1}, \Upsilon \circ \Omega$ are the identity cell on $\llbracket \alpha \rrbracket$, the inverse of the cell $\llbracket \Upsilon \rrbracket$, and the composite of the cells $\llbracket \Omega \rrbracket$ and $\llbracket \Upsilon \rrbracket$, respectively.
- The interpretations of the protype isomorphisms $\mathsf{rest\text{-}iter}, \mathsf{rest\text{-}ide}, \mathsf{repl}$ are all canonical cells determined by the universal properties of restriction.
- The interpretations of the protype isomorphisms $\mathsf{rest}_\wedge, \mathsf{rest}_\top$ are the canonical cells determined by the universal properties of the restriction, the universal properties of restriction, and finite products of loose arrows. Note that the interpretation surely defines isomorphism cells by Lemma 2.7.
- The interpretation of the protype isomorphism $(\!|\mu, \nu|\!)$ is the cell $\llbracket \mu \rrbracket$.
- The interpretation of term substitution in proterms obtained by the following rule

$$\frac{\mathit{\Gamma}_i \vdash S_i \,/\, \mathit{\Delta}_i \ (i = 0, \dots, n) \qquad \mathit{\Delta}_0 \fatsemi \dots \fatsemi \mathit{\Delta}_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu : \beta}{\begin{array}{c} \mathit{\Gamma}_1 \fatsemi \dots \fatsemi \mathit{\Gamma}_n \mid c_1 : \alpha_1[S_0/\mathit{\Delta}_0 \fatsemi S_1/\mathit{\Delta}_1] \fatsemi \dots \fatsemi c_n : \alpha_n[S_{n-1}/\mathit{\Delta}_{n-1} \fatsemi S_n/\mathit{\Delta}_n] \\ \vdash \mu[S_0/\mathit{\Delta}_0 \fatsemi \dots \fatsemi S_n/\mathit{\Delta}_n] : \beta[S_0/\mathit{\Delta}_0 \fatsemi S_n/\mathit{\Delta}_n] \end{array}}$$

  is the unique cell $\llbracket \mu[S_0/\mathit{\Delta}_0 \fatsemi \dots \fatsemi S_n/\mathit{\Delta}_n] \rrbracket$ that makes the following composites equal.

$$\begin{tikzcd}
\llbracket \mathit{\Gamma}_0 \rrbracket \arrow[r, "\llbracket \alpha_1[S_0/\mathit{\Delta}_0 \fatsemi S_1/\mathit{\Delta}_1] \rrbracket", "\shortmid"{marking}] \arrow[d, "\llbracket S_0 \rrbracket"] & \llbracket \mathit{\Gamma}_1 \rrbracket \arrow[d, "\llbracket S_1 \rrbracket"] & \cdots & \llbracket \mathit{\Gamma}_{n-1} \rrbracket \arrow[r, "\llbracket \alpha_n[S_{n-1}/\mathit{\Delta}_{n-1} \fatsemi S_n/\mathit{\Delta}_n] \rrbracket", "\shortmid"{marking}] \arrow[d, "\llbracket S_{n-1} \rrbracket"] & \llbracket \mathit{\Gamma}_n \rrbracket \arrow[d, "\llbracket S_n \rrbracket"] \\
\llbracket \mathit{\Delta}_0 \rrbracket \arrow[r, "\llbracket \alpha_1 \rrbracket"', "\shortmid"{marking}] \arrow[ru, phantom, "\mathtt{rest}"] \arrow[d, equal] & \llbracket \mathit{\Delta}_1 \rrbracket & \cdots & \llbracket \mathit{\Delta}_{n-1} \rrbracket \arrow[r, "\llbracket \alpha_n \rrbracket"', "\shortmid"{marking}] \arrow[ru, phantom, "\mathtt{rest}"] & \llbracket \mathit{\Delta}_n \rrbracket \arrow[d, equal] \\
\llbracket \mathit{\Delta}_0 \rrbracket \arrow[rrrr, "\llbracket \beta \rrbracket"', "\shortmid"{marking}] & & \llbracket \mu \rrbracket & & \llbracket \mathit{\Delta}_n \rrbracket
\end{tikzcd}$$

$$\begin{tikzcd}
\llbracket\Gamma_0\rrbracket \arrow[r, "\shortmid"{marking}, "\llbracket\alpha_1[S_0/\Delta_0 \fatsemi S_1/\Delta_1]\rrbracket"] \arrow[d, equal] & \llbracket\Gamma_1\rrbracket & \cdots \arrow[d, phantom, "\llbracket\mu[S_0/\Delta_0 \fatsemi \dots \fatsemi S_n/\Delta_n]\rrbracket"] & \llbracket\Gamma_{n-1}\rrbracket \arrow[r, "\shortmid"{marking}, "\llbracket\alpha_n[S_{n-1}/\Delta_{n-1} \fatsemi S_n/\Delta_n]\rrbracket"] & \llbracket\Gamma_n\rrbracket \arrow[d, equal] \\
\llbracket\Gamma_0\rrbracket \arrow[rrrr, "\shortmid"{marking}, "\llbracket\beta[S_0/\Delta_0 \fatsemi S_n/\Delta_n]\rrbracket"'] \arrow[d, "\llbracket S_0\rrbracket"] & & \arrow[d, phantom, "\texttt{rest}"] & & \llbracket\Gamma_n\rrbracket \arrow[d, "\llbracket S_n\rrbracket"] \\
\llbracket\Delta_0\rrbracket \arrow[rrrr, "\shortmid"{marking}, "\llbracket\beta\rrbracket"'] & & & & \llbracket\Delta_n\rrbracket
\end{tikzcd}$$

- The interpretation of the prosubstitution in proterms obtained by the following rule

$$\frac{\overline{\Gamma_i} \mid a_{i,1} : \alpha_{i,1} \fatsemi \dots \fatsemi a_{i,n_i} : \alpha_{i,n_i} \vdash \mu_i : \beta_i \ (i = 1, \dots, m) \qquad \widetilde{\Gamma} \mid b_1 : \beta_1 \fatsemi \dots \fatsemi b_n : \beta_n \vdash \nu : \gamma}{\overline{\Gamma} \mid a_{1,1} : \alpha_{1,1} \fatsemi \dots \fatsemi a_{m,n_m} : \alpha_{m,n_m} \vdash \nu\{\mu_1/b : \beta_1 \fatsemi \dots \fatsemi \mu_m/b_m : \beta_m\} : \gamma}$$

  is the following composite of cells.

$$\begin{tikzcd}
\llbracket\Gamma_{0,0}\rrbracket \arrow[r, "\shortmid"{marking}, "\llbracket\alpha_{1,1}\rrbracket"] \arrow[d, equal] & \cdots \arrow[d, phantom, "\llbracket\mu_1\rrbracket"] \arrow[r, "\shortmid"{marking}, "\llbracket\alpha_{1,n_1}\rrbracket"] & \llbracket\Gamma_{1,0}\rrbracket \arrow[r, "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[r, "\shortmid"{marking}] & \llbracket\Gamma_{m,0}\rrbracket \arrow[r, "\shortmid"{marking}, "\llbracket\alpha_{m,1}\rrbracket"] \arrow[d, equal] & \cdots \arrow[d, phantom, "\llbracket\mu_m\rrbracket"] \arrow[r, "\shortmid"{marking}, "\llbracket\alpha_{m,n_m}\rrbracket"] & \llbracket\Gamma_{m,n_m}\rrbracket \arrow[d, equal] \\
\llbracket\Gamma_{0,0}\rrbracket \arrow[rr, "\shortmid"{marking}, "\llbracket\beta_1\rrbracket"'] \arrow[d, equal] & & \llbracket\Gamma_{1,0}\rrbracket \arrow[r, "\shortmid"{marking}] & \cdots \arrow[d, phantom, "\llbracket\nu\rrbracket"] \arrow[r, "\shortmid"{marking}] & \llbracket\Gamma_{m,0}\rrbracket \arrow[rr, "\shortmid"{marking}, "\llbracket\beta_m\rrbracket"'] & & \llbracket\Gamma_{m,n_m}\rrbracket \arrow[d, equal] \\
\llbracket\Gamma_{0,0}\rrbracket \arrow[rrrrrr, "\shortmid"{marking}, "\llbracket\gamma\rrbracket"'] & & & & & & \llbracket\Gamma_{m,n_m}\rrbracket
\end{tikzcd} .$$

- The interpretation of the proterm $a : \alpha \vdash \mathsf{tr}_{\Upsilon}(a)$ is the globular isomorphism cell $\llbracket\Upsilon\rrbracket\colon \llbracket\alpha\rrbracket \Rightarrow \llbracket\beta\rrbracket$ itself.

■

Taking semantics in VDCs listed in Examples 2.10 and 2.11 justifies how FVDblTT expresses formal category theory and predicate logic in Table 2.

4. Protype and type constructors for FVDblTT

4.1. Common structures in VDCs and the corresponding constructors

In this section, we will specify the type and protype constructors that can be added to FVDblTT. The virtual double categories of relations and those of profunctors have many structures in common. We would like to introduce the inductive types and protypes corresponding to the common structures in these kinds of virtual double categories. We first list the additional types will introduce for the type theory.

Structures	Formal category theory	Predicate logic	Constructors in FVDblTT
Units [CS10]	hom-profuntors $\mathcal{C}(-, \bullet)$	equality $=$	path $\nrightarrow$
Composition [CS10]	composition via coends $\int$	composition via $\exists$	composition $\odot$
Extension [RV22]	profunctor extension $\triangleright$	contraction via $\forall$	extension $\triangleright$
Tabulators [GP99]	two-sided Grothendieck construction	comprehension $\{\text{-}\}$	tabulator $\{\!\vert\text{-}\vert\!\}$

Table 3. The common structures and the corresponding constructors

The constructors we will add to FVDblTT are $\nrightarrow$, $\odot$, $\triangleright$, $\triangleleft$, and $\{\!\vert\text{-}\vert\!\}$. Even though we can add the constructors for the loose adjunctions and the companions and conjoints independently of the other constructors, we would take the approach of defining them in terms of $\nrightarrow$ and $\odot$ in this paper.

Path protype $\nrightarrow$ for the units. (Appendix B.3) The path protype is the protype that represents the units in a VDC. In a double category, the units are just the identity loose morphisms, but in a VDC, the units are formulated via a universal property. The definition of units is due to [CS10].

Definition 4.1 (Units [CS10, Definition 5.1])**.** A ***unit*** of an object I in a VDC is a loose arrow $U_I\colon I \nrightarrow I$ equipped with a cell $\eta_I\colon \cdot \Rightarrow U_I$ with the following universal property. Given any cell ν on the left below where $\overline{\alpha} = \left(I_0 \overset{\alpha_1}{\nrightarrow} \cdots \overset{\alpha_n}{\nrightarrow} I\right)$ and $\overline{\alpha}' = \left(I \overset{\alpha'_1}{\nrightarrow} \cdots \overset{\alpha'_{n'}}{\nrightarrow} I'_{n'}\right)$ are arbitrary sequences of loose arrows, it uniquely factors through the sequence of the identity cells with η_I as on the right below.

$$\begin{tikzcd}
I_0 \arrow[r, dashed, "\shortmid"{marking}, "\overline{\alpha}"] \arrow[d, "f"'] & I \arrow[d, phantom, "\nu"] \arrow[r, dashed, "\shortmid"{marking}, "\overline{\alpha}'"] & I'_{n'} \arrow[d, "f'"] \\
J \arrow[rr, "\shortmid"{marking}, "\beta"'] & & J'
\end{tikzcd}
=
\begin{tikzcd}
I_0 \arrow[r, dashed, "\shortmid"{marking}, "\overline{\alpha}"] \arrow[d, equal] & & I \arrow[r, dashed, "\shortmid"{marking}, "\overline{\alpha}'"] \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_I"] & & I'_{n'} \arrow[d, equal] \\
I_0 \arrow[r, dashed, "\shortmid"{marking}, "\overline{\alpha}"'] \arrow[d, "f"'] & I \arrow[rr, "\shortmid"{marking}, "U_I"'] & \arrow[d, phantom, "\widetilde{\nu}"] & I \arrow[r, dashed, "\shortmid"{marking}, "\overline{\alpha}'"'] & I'_{n'} \arrow[d, "f'"] \\
J \arrow[rrrr, "\shortmid"{marking}, "\beta"'] & & & & J'
\end{tikzcd}$$

■

The formation rule for the ***path protype*** is on the left below, and it comes equipped with the introduction rule on the right below:

$$\frac{I\ \mathsf{type}}{x : I \fatsemi y : I \vdash x \nrightarrow_I y\ \mathsf{protype}}\ . \qquad\qquad \frac{I\ \mathsf{type}}{x : I \mid\ \ \vdash \mathsf{refl}_I(x) : x \nrightarrow_I x}$$

The proterm refl corresponds to the unit η_I in the definition of the units. To let the path protype encode the units in the VDCs, we need to add elimination and computation rules as in Appendix B.3. The path protypes behave as inductive (pro)types, and their inductions look very similar to path induction in homotopy type theory, but with the difference that the path protype is directed.

The semantics of the path protypes $\nrightarrow$ are given by the units in any VDC with units, with the proterm constructor refl_I interpreted as the cell $\eta_{[\![I]\!]}$. For instance, in the VDCs $\mathbb{PROF}$ and $\mathbb{R}\mathrm{el}$, the interpretations of the path protypes are given as the hom profunctors and the equality relations, respectively. These follow from the fact that the identity loose morphisms in a double category serve as the units when we see it as a VDC.

In order to make the path protypes behave well with the product types in FVDblTT, we need to add the compatibility rules between the path protypes and the product types as in Appendix B.3. For instance, when we consider the hom-profunctor on a product category $\mathcal{C} \times \mathcal{D}$, we expect its components to be isomorphic to the product $\mathcal{C}(C, C') \times \mathcal{D}(D, D')$. Correspondingly, we would like to add the following rule, which does not follow from other rules *a priori*:

$$\frac{I\ \mathsf{type} \qquad J\ \mathsf{type}}{x : I, y : J \fatsemi x' : I, y' : J \vdash \mathsf{exc}_{\nrightarrow,\wedge} : \langle x, y\rangle \nrightarrow_{I\times J} \langle x', y'\rangle \cong x \nrightarrow_I x' \wedge y \nrightarrow_J y'}\ .$$

Appendix B.3 will give the whole set of rules for the compatibility between the path protypes and the product types. The rules we introduce are justified by the fact that with them, the syntactic VDCs we will give in Subsection 5.3 become cartesian objects in the 2-category of CFVDCs with units. See Proposition A.2 for a detailed explanation from the 2-categorical perspective.

Composition protype $\odot$ for the composites. (Appendix B.3) The composition protype is the protype that represents the composition of paths of loose arrows just of length 2 in the virtual double categories. Here we follow the definition of the composition of paths of loose arrows in [CS10].

Definition 4.2 (Composites [CS10, Definition 5.2])**.** A ***composite*** of a given sequence of loose arrows $\overline{\alpha} = \left(I_0 \overset{\alpha_1}{\nrightarrow} I_1 \nrightarrow \cdots \overset{\alpha_m}{\nrightarrow} I_m\right)$ in a virtual double category is a loose arrow $\odot\overline{\alpha}$ from I_0 to I_m equipped with a cell

$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "|" marking] \arrow[d, equal] & I_1 \arrow[r, "|" marking] & \cdots \arrow[r, "\alpha_m", "|" marking] & I_m \arrow[d, equal] \\
I_0 \arrow[rrr, "\odot\overline{\alpha}"', "|" marking] \arrow[rrru, phantom, "\epsilon_{\overline{\alpha}}"] & & & I_m
\end{tikzcd}$$

with the following universal property. Given any cell ν on the left below where $\overline{\beta}, \overline{\beta}'$ are arbitrary sequences of loose arrows, it uniquely factors through the sequence of the identity cells with $\mu_{\overline{\alpha}}$ as on the right below.

$$\begin{tikzcd}
J_0 \arrow[r, dashed, "\overline{\beta}", "|" marking] \arrow[d, "f"'] & I_0 \arrow[r, dashed, "\overline{\alpha}", "|" marking] & I_m \arrow[r, dashed, "\overline{\beta}'", "|" marking] & J'_{n'} \arrow[d, "f'"] \\
K \arrow[rrr, "\gamma"', "|" marking] \arrow[rrru, phantom, "\nu"] & & & K'
\end{tikzcd}
=
\begin{tikzcd}
J_0 \arrow[r, dashed, "\overline{\beta}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\|"] & I_0 \arrow[r, dashed, "\overline{\alpha}", "|" marking] \arrow[d, equal] & I_m \arrow[r, dashed, "\overline{\beta}'", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\|"] & J'_{n'} \arrow[d, equal] \\
J_0 \arrow[r, dashed, "\overline{\beta}"', "|" marking] \arrow[d, "f"'] & I_0 \arrow[r, "\odot\overline{\alpha}"', "|" marking] \arrow[u, phantom, ""] \arrow[ru, phantom, "\epsilon_{\overline{\alpha}}"] & I_m \arrow[r, dashed, "\overline{\beta}'"', "|" marking] & J'_{n'} \arrow[d, "f'"] \\
K \arrow[rrr, "\gamma"', "|" marking] \arrow[rrru, phantom, "\widetilde{\nu}"] & & & K'
\end{tikzcd}$$

■

The units are the special cases of the composition of paths of length 0, and the composition of paths longer than 2 can be realized by the iterated use of the composition of paths of length 2. In order to gain access to the composition of paths of positive length in the type theory, we introduce the ***composition protype*** $\odot$ to FVDblTT. The formation rule for the composition protype is the following:

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad x : J \fatsemi y : K \vdash \beta(x \fatsemi y)\ \mathsf{protype}}{w : I \fatsemi y : K \vdash \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y)\ \mathsf{protype}}$$

This comes equipped with the introduction rule:

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad x : J \fatsemi y : K \vdash \beta(x \fatsemi y)\ \mathsf{protype}}{w : I \fatsemi x : J \fatsemi y : K \mid a : \alpha(w \fatsemi x) \fatsemi b : \beta(x \fatsemi y) \vdash a \odot b : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y)}$$

For the detailed rules of the composition protype, see Appendix B.3. Plus, we need the compatibility rules for the composition protype and the product types as we did for the path protype, see Appendix B.3.

If we load the path protype $\nrightarrow$ and the composite protype $\odot$ to FVDblTT, procontexts can be equivalently expressed by a single protype. In this sense, such a type theory can be seen as an internal language of double categories. This is supported by the fact that a VDC is equivalent to one arising from a double category if and only if it has composites of all paths of loose arrows, including units [CS10, Theorem 5.2].

The semantics of the composition protypes $\odot$ is given by the composites in VDCs if they have ones of sequences of length 2 in an appropriate way. For example, in the VDC $\mathbb{P}\mathrm{rof}$, the composite of paths of length 2 is the composite of profunctors, given by the coend $\int$. In the VDC $\mathbb{R}\mathrm{el}$, the composites of paths of length 2 are the composites of relations, given by the existential quantification $\exists$.

$$\llbracket \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \rrbracket = \int^{X \in \llbracket J \rrbracket} \llbracket \alpha \rrbracket(-, X) \times \llbracket \beta \rrbracket(X, \bullet) \colon \llbracket I \rrbracket \to\!\!\!\!\!\shortmid\;\; \llbracket K \rrbracket \text{ in } \mathbb{P}\mathrm{rof}$$
$$\llbracket \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \rrbracket = \{\, (w, y) \mid \exists x \in \llbracket J \rrbracket . \llbracket \alpha \rrbracket(w, x) \wedge \llbracket \beta \rrbracket(x, y) \,\} : \llbracket I \rrbracket \to\!\!\!\!\!\shortmid\;\; \llbracket K \rrbracket \text{ in } \mathbb{R}\mathrm{el}$$

Filler protype $\triangleright, \triangleleft$ for the closed structure. (Appendix B.3) Having obtained the ability to express a particular kind of coends in formal category theory, and existential quantification in predicate logic, we would like to introduce the protypes for ends and universal quantification in the type theory. First of all, we recall the definition of the right extension and the right lift [RV22, AM24] in a VDC, which are straightforward generalizations of the right extension and the right lift in a bicategory.

Definition 4.3. A ***right extension*** of a loose arrow $\beta \colon I \to\!\!\!\!\!\shortmid\;\; K$ along a loose arrow $\alpha \colon I \to\!\!\!\!\!\shortmid\;\; J$ is a loose arrow $\alpha \triangleright \beta \colon J \to\!\!\!\!\!\shortmid\;\; K$ equipped with a cell

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha\triangleright\beta", "\shortmid"{marking}] \arrow[d, phantom, "\varpi_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

with the following universal property. Given any cell ν on the left below where $\overline{\gamma}$ is an arbitrary sequence of loose arrows, it uniquely factors through the cell $\varpi_{\alpha,\beta}$ as on the right below.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\nu}"] & K \arrow[d, equal] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha\triangleright\beta"', "\shortmid"{marking}] \arrow[d, phantom, "\varpi_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

A ***right lift*** of a protype $\beta \colon I \to\!\!\!\!\!\shortmid\;\; K$ along a protype $\alpha \colon J \to\!\!\!\!\!\shortmid\;\; K$ is a protype $\beta \triangleleft \alpha \colon I \to\!\!\!\!\!\shortmid\;\; J$ equipped with a cell

$$\begin{tikzcd}
I \arrow[r, "\beta\triangleleft\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, phantom, "\varpi'_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

with the following universal property. Given any cell ν on the left below where $\overline{\gamma}$ is an arbitrary sequence of loose arrows, it uniquely factors through the cell $\varpi'_{\alpha,\beta}$ as on the right below.

$$\begin{tikzcd}
I \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\nu}"] & J \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] & K \arrow[d, equal] \\
I \arrow[r, "\beta\triangleleft\alpha"', "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[d, phantom, "\varpi'_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

$\blacksquare$

With this notion, one can handle the concept of weighted limits and colimits internally in virtual double categories. We now introduce the ***filler protypes*** $\triangleright$ and $\triangleleft$ to FVDblTT to express the right extension and the right lift in the type theory. The formation rule for the ***right extension protype*** is the following:

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x) \ \mathsf{protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y) \ \mathsf{protype}}{x : J \fatsemi y : K \vdash \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \ \mathsf{protype}}$$

The constructor for the right extension protype is given in the elimination rule since the orientation of the universal property of the right extension is opposite to that of the composition protype and the path protype.

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x) \ \mathsf{protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y) \ \mathsf{protype}}{w : I \fatsemi x : J \fatsemi y : K \mid a : \alpha(w \fatsemi x) \fatsemi e : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \vdash a \blacktriangleright e : \beta(w \fatsemi y)}$$

The semantics of the right extension protype $\triangleright$ is given by the right extension in VDCs. The constructor $\blacktriangleright$ is interpreted using the cell $\varpi_{\llbracket \alpha \rrbracket, \llbracket \beta \rrbracket}$ above. To illustrate the semantics of the right extension protype, we give the interpretations of the right extension protype in the VDCs $\mathbb{P}\mathrm{rof}$ and $\mathbb{R}\mathrm{el}$.

$$\llbracket \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \rrbracket = \int_{W \in \llbracket I \rrbracket} [\llbracket \alpha \rrbracket(W,-), \llbracket \beta \rrbracket(W,\bullet)] : \llbracket J \rrbracket \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} \llbracket K \rrbracket \text{ in } \mathbb{P}\mathrm{rof}$$

$$\llbracket \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \rrbracket = \{\, (x,y) \mid \forall w \in \llbracket I \rrbracket . \, (\llbracket \alpha \rrbracket(w,x) \Rightarrow \llbracket \beta \rrbracket(w,y)) \,\} : \llbracket J \rrbracket \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} \llbracket K \rrbracket \text{ in } \mathbb{R}\mathrm{el}$$

Here, $[X, Y]$ is the function set from X to Y.
Comprehension type $\{\!|\text{-}|\!\}$ for the tabulators. (Appendix B.3) The last one is not a protype but a type constructor. A relation $R \colon A \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} B$ in a general category can be seen as a subobject of the product $A \times B$, and its legs to A and B give a triangle cell

$$\begin{tikzcd} & X_R \arrow[ld, "\pi_1"'] \arrow[rd, "\pi_2"] \arrow[d, phantom, "\tau_R"] & \\ A \arrow[rr, "R"', "\shortmid" marking] & & B \end{tikzcd} \qquad \text{where} \quad X_R = \{\, (a,b) \in A \times B \mid R(a,b) \,\}.$$

This triangle cell is a universal triangle cell whose base is R. In a general virtual double category, such a universal object is called a ***tabulator*** of a loose arrow $A \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} B$.

Definition 4.4 (Tabulators[GP99])**.** A ***(1-dimensional) tabulator*** of a loose arrow $\alpha \colon I \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} J$ is an object $\{\!|\alpha|\!\}$ equipped with a pair of tight arrows $\ell_\alpha \colon \{\alpha\} \to I$ and $r_\alpha \colon \{\alpha\} \to J$ and a cell

$$\begin{tikzcd} & \{\alpha\} \arrow[ld, "\ell_\alpha"'] \arrow[rd, "r_\alpha"] \arrow[d, phantom, "\tau_\alpha"] & \\ I \arrow[rr, "\alpha"', "\shortmid" marking] & & J \end{tikzcd}$$

such that, for any cell ν on the left below, there exists a unique tight arrow $t_\nu \colon X \to \{\!|\alpha|\!\}$ that makes the following two cells equal.

$$\begin{tikzcd} & X \arrow[ld, "h"'] \arrow[rd, "k"] \arrow[d, phantom, "\nu"] & \\ I \arrow[rr, "\alpha"', "\shortmid" marking] & & J \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[d, "t_\nu"] \arrow[ldd, bend right, "h"'] \arrow[rdd, bend left, "k"] & \\ \circlearrowleft & \{\alpha\} \arrow[ld, "\ell_\alpha"'] \arrow[rd, "r_\alpha"] \arrow[d, phantom, "\tau_\alpha"] & \circlearrowleft \\ I \arrow[rr, "\alpha"', "\shortmid" marking] & & J \end{tikzcd}$$

Henceforth, we call a dataset (X, h, k, ν) a ***cone*** over α with the apex X. ■

Corresponding to the tabulators in the virtual double categories, we introduce the ***comprehension type*** $\{\!|\text{-}|\!\}$ to FVDblTT. The formation rule for the comprehension type is the following:

$$\frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype}}{\{\!|\alpha|\!\} \ \mathsf{type}}$$

This comes equipped with the constructor

$$\frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype}}{w : \{\!|\alpha|\!\} \vdash l(w) : I \qquad w : \{\!|\alpha|\!\} \vdash r(w) : J \qquad w : \{\!|\alpha|\!\} \mid\!\vdash \mathsf{tab}_{\{\!|\alpha|\!\}}(w) : \alpha[l(w)/x \fatsemi r(w)/y]}$$

The comprehension type $\{\!|\text{-}|\!\}$ is interpreted as the tabulators in the VDCs. In the VDC $\mathbb{P}\mathrm{rof}$, the tabulator of a profunctor $P \colon \mathcal{C} \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathcal{D}$ is given by ***two-sided Grothendieck construction***, which results in ***a two-sided discrete fibration*** from $\mathcal{C}$ to $\mathcal{D}$. A frequently used example of this construction is the comma category for a pair of functors $F \colon \mathcal{C} \to \mathcal{E}$ and $G \colon \mathcal{D} \to \mathcal{E}$ as the tabulator of the profunctor $\mathcal{E}(F(-), G(-))$, see [LR20] for more details. The VDC $\mathbb{R}\mathrm{el}$ has the tabulators if we ground the double category to an axiomatic system of set theory with the comprehension axiom, as the tabulator of a relation $R \colon A \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} B$ is given by the set of all the pairs (a, b) such that $R(a, b)$ holds.

In the presence of the unit protype $\nrightarrow$, we should add some rules concerning the compatibility between the comprehension type and the path protype. This is because, in many examples of double categories, the tabulators have not only the universal property as in Definition 4.4 but also respect the units, although the original universal property of the tabulators is enough to detect the tabulators in a double category. This issue is thoroughly discussed in [GP99]. Here, we give a slightly generalized version of the tabulators in virtual double categories with units.

Definition 4.5 (2-dimensional universal property of tabulators)**.** In a virtual double category with units, an ***unital tabulator*** $\{\alpha\}$ of a loose arrow $\alpha \colon I \mathrel{\relbar\joinrel\mapstochar\joinrel\rightarrow} J$ is a tabulator of α in the sense of Definition 4.4, which also satisfies the following universal property. Suppose we are given any pair of cones (X, h, k, ν) and (X', h', k', ν')

over α and a pair of cells ς, ϑ such that the following equality holds.

$$\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\varsigma"] & X' \arrow[d, "h'"'] \arrow[dr, "k'"] \arrow[dr, phantom, "\nu'"', near end] & & & & X \arrow[dl, "h"'] \arrow[dl, phantom, "\nu", near end] \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, "k"] \arrow[dr, phantom, "\vartheta"] & X' \arrow[d, "k'"] \\
I \arrow[r, "U_I"', "\shortmid"{marking}] \arrow[d, equal] & I \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[d, phantom, "\wr\|"] & J \arrow[d, equal] & = & I \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "U_J"', "\shortmid"{marking}] \arrow[d, phantom, "\wr\|"] & J \arrow[d, equal] \\
I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J & & I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J
\end{tikzcd}$$

Then, there exists a unique cell ϱ for which the following equalities hold.

$$\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, "t_\nu"'] \arrow[dr, phantom, "\varrho"] & X' \arrow[d, "t_{\nu'}"] \\
\{\alpha\} \arrow[r, "U_{\{\alpha\}}"', "\shortmid"{marking}] \arrow[d, "\ell_\alpha"'] \arrow[dr, phantom, "U_{\ell_\alpha}"] & \{\alpha\} \arrow[d, "\ell_\alpha"] \\
I \arrow[r, "U_I"', "\shortmid"{marking}] & I
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\varsigma"] & X' \arrow[d, "h'"] \\
I \arrow[r, "U_I"', "\shortmid"{marking}] & I
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, "t_\nu"'] \arrow[dr, phantom, "\varrho"] & X' \arrow[d, "t_{\nu'}"] \\
\{\alpha\} \arrow[r, "U_{\{\alpha\}}"', "\shortmid"{marking}] \arrow[d, "r_\alpha"'] \arrow[dr, phantom, "U_{r_\alpha}"] & \{\alpha\} \arrow[d, "r_\alpha"] \\
J \arrow[r, "U_J"', "\shortmid"{marking}] & J
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, "k"'] \arrow[dr, phantom, "\vartheta"] & X' \arrow[d, "k'"] \\
J \arrow[r, "U_J"', "\shortmid"{marking}] & J
\end{tikzcd}$$

$\blacksquare$

This universal property determines what the unit on the apex of the tabulator should be. Appendix B.3 will present the corresponding rules for the comprehension type $\{\!|{-}|\!\}$ in FVDblTT with the unit protype $\nrightarrow$.

Predicate logic. When we work with the type theory FVDblTT for the purpose of reasoning about predicate logic, we consider types, terms, protypes, and proterms to represent sets, functions, predicates (or propositions), and proofs, respectively. However, the type theory FVDblTT, as it is, treats the protypes in a context $\mathit{\Gamma} \fatsemi \mathit{\Delta}$ and those in a context $\mathit{\Delta} \fatsemi \mathit{\Gamma}$ as different things. In this sense, the type theory FVDblTT as predicated logic has directionality. If one wants to develop a logic without a direction, one can simply add the following rules to the type theory.

$$\frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha \ \mathsf{protype}}{\mathit{\Delta} \fatsemi \mathit{\Gamma} \vdash \alpha^\circ \ \mathsf{protype}} \qquad\qquad \frac{\mathit{\Gamma}_0 \fatsemi \cdots \fatsemi \mathit{\Gamma}_m \mid a_1 : \alpha_1 \cdots a_n : \alpha_n \vdash \mu : \beta}{\mathit{\Gamma}_m \fatsemi \cdots \fatsemi \mathit{\Gamma}_0 \mid a_n : \alpha_n^\circ \cdots a_1 : \alpha_1^\circ \vdash \mu^\circ : \beta^\circ}$$

These rules are the counterparts of the structure of involution in VDCs.

If one also wants to make the type theory FVDblTT proof irrelevant, one can reformulate protype isomorphism judgment as equality judgments of protypes and add the rule stating that all the proterms are equal. It is the counterpart of the flatness [GP99] or local preorderedness [HN23] of VDCs.

4.2. **Examples of calculus** This section exemplifies how one can reason about category theory and logic formally in the type theory FVDblTT.

Example 4.6 (Ninja (co)Yoneda Lemma)**.** One of the most fundamental results in category theory is the Yoneda Lemma, and it has a variety of presentations in the literature. Here we present one called the Ninja Yoneda Lemma [Lor21, Proposition 2.2.1]: *given a category $\mathcal{C}$ and a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, we have the canonical isomorphism*

$$F \cong \int_{X \in \mathcal{C}} [\mathcal{C}(X, -), FX].$$

This follows from the categorical fact that $\mathbb{P}\mathrm{rof}$ is an FVDC with the structures listed above. Indeed, in the type theory FVDblTT with the path protype $\nrightarrow$ and the filler protype $\triangleright$, one can deduce the following:

$$y : I \fatsemi \cdot \vdash \mathsf{Yoneda} : (x \nrightarrow_I y) \triangleright_{x:I} \alpha(x) \cong \alpha(y)$$

Similarly, we have

$$y : I \fatsemi \cdot \vdash \mathsf{CoYoneda} : (y \nrightarrow_I x) \odot_{x:I} \alpha(x) \cong \alpha(y)$$

which expresses the coYoneda Lemma:

$$\int^{X \in \mathcal{C}} \mathcal{C}(-, X) \times FX \cong F.$$

In short, all the theorems in category theory that can be proven using this type theory fall into corollaries of the theorem that $\mathbb{P}\mathrm{rof}$ is a CFVDC with the structures corresponding to the constructors. Other examples include the unit laws and the associativity of the composition of profunctors or the iteration of extensions and lifts of profunctors.

Turning to the aspect of predicate logic, we can interpret the protype isomorphisms as the following logical equivalences.

$$\begin{aligned}
\varphi(y) &\equiv \forall x \in I.\, (x = y) \Rightarrow \varphi(x) \\
\varphi(y) &\equiv \exists x \in I.\, (x = y) \wedge \varphi(x)
\end{aligned}$$

Example 4.7 (Isomorphism of functors)**.** A natural transformation $\xi\colon F \to G$ between two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is given by a family of arrows $\xi_X\colon FX \to GX$ satisfying some naturality conditions. In the type theory FVDblTT with the path protype $\nrightarrow$, this natural transformation can be represented by a proterm $x : I \mid\!\vdash \xi(x) : f(x) \nrightarrow_I g(x)$. Here, the naturality condition automatically holds because we describe it as a proterm. The isomorphism of functors can be expressed using this notion, but an alternative way is to use the protype isomorphism.

Lemma 4.8. Given two terms, $f(x)$ and $g(x)$, in the same context, the following are equivalent.

(i) There are proterms $\xi(x) : f(x) \nrightarrow_I g(x)$ and $\eta(x) : g(x) \nrightarrow_I f(x)$ such that $\xi(x) \boxdot \eta(x) \equiv \mathsf{refl}_{f(x)}$ and $\eta(x) \boxdot \xi(x) \equiv \mathsf{refl}_{g(x)}$.
(ii) There is a protype isomorphism $Z : y \nrightarrow_J f(x) \cong y \nrightarrow_I g(x)$.

Here, $\boxdot$ is a tailored constructor defined as follows.

$$y : J \fatsemi y' : J \fatsemi y'' : J \mid\ a : y \nrightarrow_J y' \fatsemi b : y' \nrightarrow_J y'' \vdash a \boxdot b :\equiv \mathsf{ind}_{\nrightarrow_J}(a) : y \nrightarrow_J y''.$$

Proof. First, suppose (i) holds. We define a proterm ζ by the following:

$$\dfrac{\dfrac{\begin{array}{c} x : I \mid\!\vdash \xi : f(x) \nrightarrow_J g(x) \\ y : J \fatsemi y' : J \fatsemi y'' : J \mid a : y \nrightarrow_J y' \fatsemi b : y' \nrightarrow_J y'' \vdash a \boxdot b : y \nrightarrow_J y'' \end{array}}{y : J \fatsemi x : I \fatsemi x' : I \mid a : y \nrightarrow_J f(x) \fatsemi b : f(x) \nrightarrow_J g(x) \vdash a \boxdot b[y/y \fatsemi f(x)/y' \fatsemi g(x)/y''] : y \nrightarrow_J g(x)}}{y : J \fatsemi x : I \mid a : y \nrightarrow_J f(x) \vdash \zeta(a) : y \nrightarrow_J g(x)}$$

Therefore, we have $\zeta(a)$, and in the same way, we can define a proterm $b : y \nrightarrow_J g(x) \vdash \zeta'(b) : y \nrightarrow_J f(x)$, which is the inverse of ζ by simple reasoning.

Next, suppose (ii) holds. Let $a : y \nrightarrow_J f(x) \vdash \zeta(a) : y \nrightarrow_J g(x)$ be the proterm witnessing the isomorphism. By substituting $f(x)$ for y and the refl for a, we obtain a proterm $\xi(x) : f(x) \nrightarrow_J g(x)$. In the same way, we can define a proterm $\eta(x) : g(x) \nrightarrow_J f(x)$, for which the two desired equalities hold. □

We therefore use the equalities $y \nrightarrow_J f(x)$ and $y \nrightarrow_J g(x)$ when f and g are already proven to be isomorphic.

Example 4.9 (Adjunction)**.** In a virtual double category, the ***companion*** and ***conjoint*** of a tight arrow $f\colon A \to B$ is defined as the loose arrows $f_*\colon A \nrightarrow B$ and $f^*\colon B \nrightarrow A$ equipped with cells satisfying some equations of cells [GP04, CS10]. In a virtual equipment, it is known that the companion and conjoint of a tight arrow $f\colon A \to B$ are the restrictions of the units on B along the pairs of tight arrows $(f, \mathtt{id}_B)$ and $(\mathtt{id}_B, f)$, respectively. These notions are the formalization of the representable profunctors in the virtual double categories. Therefore, the companions and conjoints of a term $t(x)$ in the type theory FVDblTT should be defined as $t(x) \nrightarrow_I y$ and $y \nrightarrow_I t(x)$, respectively.

The ***adjunction*** between two functors is described in terms of representable profunctors, which motivates the following definition of the adjunction in the type theory FVDblTT. Remember a functor $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G\colon \mathcal{D} \to \mathcal{C}$ if there is a natural isomorphism between the hom-sets

$$\mathcal{D}(F-, \bullet) \cong \mathcal{C}(-, G\bullet).$$

In the type theory FVDblTT, a term $t(x)$ is announced to be a left adjoint to a term $u(y)$ if the following equality holds:

$$x : I \fatsemi y : J \vdash t(x) \nrightarrow_J y \equiv x \nrightarrow_I u(y).$$

Example 4.10 (Kan extension)**.** In [Kel05], the (pointwise) left Kan extension $\mathrm{Lan}_G F$ of a functor $F\colon \mathcal{C} \to \mathcal{D}$ along a functor $G\colon \mathcal{C} \to \mathcal{E}$ is defined as a functor $H\colon \mathcal{D} \to \mathcal{E}$ equipped with a natural transformation

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & \arrow[d, phantom, "\overset{\mu}{\Downarrow}"] & \mathcal{D} \\
& \mathcal{E} \arrow[ur, "H"'] &
\end{tikzcd}$$

with the following canonical natural transformation being an isomorphism:

$$\mathcal{D}(HE, D) \overset{\cong}{\to} \widehat{\mathcal{C}}\left(\mathcal{E}(G-, E), \mathcal{D}(F-, D)\right) \quad \text{naturally in } D \in \mathcal{D}, E \in \mathcal{E}.$$

A protype isomorphism corresponding to this isomorphism is given by the following.

$$z : K \fatsemi y : J \vdash \mathsf{LeftKan} : h(z) \nrightarrow_J y \cong (g(x) \nrightarrow_K z) \triangleright_{x:I} (f(x) \nrightarrow_J y)$$

We will demonstrate how proofs in category theory can be done in the type theory FVDblTT.

Proposition 4.11 ([Kel05, Theorem 4.47])**.** $\mathrm{Lan}_{G'}\mathrm{Lan}_G F \cong \mathrm{Lan}_{G'\circ G}F$ hold for any functors $F\colon \mathcal{C}\to\mathcal{D}$, $G\colon\mathcal{C}\to\mathcal{E}$, and $G'\colon\mathcal{E}\to\mathcal{F}$ if the Kan extensions exist.

$$\begin{tikzcd}
\mathcal{C} \arrow[rrr, "F"] \arrow[dr, "G"'] \arrow[dr, phantom, shift left=4, "\Downarrow"] & & & \mathcal{D} \\
& \mathcal{E} \arrow[urr, "\mathrm{Lan}_G F"] \arrow[dr, "G'"'] \arrow[dr, phantom, shift left=4, "\Downarrow"] & & \\
& & \mathcal{E}' \arrow[uur, "\mathrm{Lan}_{G'}\mathrm{Lan}_G F\cong\mathrm{Lan}_{G'\circ G}F"'] &
\end{tikzcd}$$

Proof. We associate $F, G, G', \mathrm{Lan}_G F, \mathrm{Lan}_{G'}\mathrm{Lan}_G F, \mathrm{Lan}_{G'\circ G}F$ with the terms $f(x), g(x), g'(z), h(z), h'(z')$, and $h''(z')$. We will have the desired protype isomorphism judgment by composing the protype isomorphisms in the following order.

$$\begin{aligned}
&z' : K' \fatsemi y : J \mid h'(z') \nrightarrow_J y \\
&\qquad\cong \left(g'(z) \nrightarrow_{K'} z'\right) \triangleright_{z:K} \left(h(z) \nrightarrow_J y\right) && (\mathsf{LeftKan}) \\
&\qquad\cong \left(g'(z) \nrightarrow_{K'} z'\right) \triangleright_{z:K} \left(\left(g(x) \nrightarrow_K z\right) \triangleright_{x:I} \left(f(x) \nrightarrow_J y\right)\right) && \left(\left(g'(z) \nrightarrow_{K'} z'\right) \triangleright_{z:K} \mathsf{LeftKan}\right) \\
&\qquad\cong \left(\left(g(x) \nrightarrow_K z\right) \odot_{z:K} \left(g'(z) \nrightarrow_{K'} z'\right)\right) \triangleright_{x:I} \left(f(x) \nrightarrow_J y\right) && (\mathsf{Fubini}) \\
&\qquad\cong \left(g'(g(x)) \nrightarrow_{K'} z'\right) \triangleright_{x:I} \left(f(x) \nrightarrow_J y\right) && \left(\mathsf{CoYoneda} \triangleright_{x:I} \left(f(x) \nrightarrow_J y\right)\right) \\
&\qquad\cong h''(z') \nrightarrow_{K'} y && (\mathsf{LeftKan}^{-1})
\end{aligned}$$

Here, the protype isomorphism Fubini is given as $\llparenthesis \mathsf{Fubini}_1, \mathsf{Fubini}_2 \rrparenthesis$, where Fubini_1 and Fubini_2 are the proterms derived as follows.

$$\begin{array}{c}
x_0 : I_0 \fatsemi x_1 : I_1 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid a : \alpha \fatsemi b : \beta \fatsemi c : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma) \vdash a \blacktriangleright (b \blacktriangleright c) : \gamma \\
\hline
x_0 : I_0 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid d : \alpha \odot_{x_1:I_1} \beta \fatsemi c : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma) \vdash \_\_ : \gamma \\
\hline
x_2 : I_2 \fatsemi x_3 : I_3 \mid c : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma) \vdash \mathsf{Fubini}_1 : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma
\end{array}$$

$$\begin{array}{c}
x_0 : I_0 \fatsemi x_1 : I_1 \fatsemi x_2 : I_2 \mid a : \alpha \fatsemi b : \beta \vdash a \odot b : \alpha \odot_{x_1:I_1} \beta \\
x_0 : I_0 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid d : \alpha \odot_{x_1:I_1} : \beta \fatsemi e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash d \blacktriangleright e : \gamma \\
\hline
x_0 : I_0 \fatsemi x_1 : I_1 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid a : \alpha \fatsemi b : \beta \fatsemi e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash \_\_ : \gamma \\
\hline
x_1 : I_1 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid b : \beta \fatsemi e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash \_\_ : \alpha \triangleright_{x_0:I_0} \gamma \\
\hline
x_2 : I_2 \fatsemi x_3 : I_3 \mid e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash \mathsf{Fubini}_2 : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma)
\end{array}$$

□

5. A syntax-semantics adjunction for FVDblTT

Stating that a type theory is the internal language of a categorical structure always comes with the notion of a syntax-semantics adjunction. We set out to construct the term model of FVDblTT by following the standard procedure of categorical logic.

5.1. **Syntactic presentation of virtual double categories** In order to let the type theory be an internal language for FVDCs, we proceed to define the notion of a signature and a specification for the type theory. Signatures assign the base type symbols, function symbols, protype symbols, and cell symbols of the type theory, and specifications assign the equality that the type theory should satisfy.

Definition 5.1. A ***signature*** Σ for FVDblTT is a quadruple $(T_\Sigma, F_\Sigma, P_\Sigma, C_\Sigma)$ where

- T_Σ is a class of ***type symbols***,
- $F_\Sigma(\sigma,\tau)$ is a family of classes of ***function symbols*** for any $\sigma,\tau \in T_\Sigma$,
- $P_\Sigma(\sigma \fatsemi \tau)$ is a family of classes of ***protype symbols*** for any $\sigma,\tau \in T_\Sigma$,
- $C_\Sigma(\rho_1 \fatsemi \dots \fatsemi \rho_n \mid \omega)$ is a family of classes of ***cell symbols*** for any $\sigma_0,\dots,\sigma_n \in T_\Sigma$, $\rho_i \in P_\Sigma(\sigma_{i-1} \fatsemi \sigma_i)$ for $i = 1,\dots,n$, and $\omega \in P_\Sigma(\sigma_0 \fatsemi \sigma_n)$ where $n \geq 0$.

For simplicity, in the last item, we omit the dependency of the class of cells on σ_i's. Henceforth, $f\colon \sigma \to \tau$ denotes a function symbol $f \in F_\Sigma(\sigma,\tau)$, $\rho\colon \sigma \nrightarrow \tau$ denotes a protype symbol $\rho \in P_\Sigma(\sigma \fatsemi \tau)$, and $\kappa\colon \rho_1 \fatsemi \dots \fatsemi \rho_n \Rightarrow \omega$ denotes a cell symbol $\kappa \in C_\Sigma(\rho_1 \fatsemi \dots \fatsemi \rho_n \mid \omega)$.

A ***morphism of signatures*** $\Phi\colon \Sigma \to \Sigma'$ is a family of functions sending the symbols of each kind in Σ to symbols of the same kind in Σ' so that a symbol dependent on another kind of symbol is sent to a symbol dependent on the image of the former symbol. For instance, $\rho\colon \sigma \nrightarrow \tau$ is sent to a protype symbol of the form $\Phi(\rho)\colon \Phi(\sigma) \nrightarrow \Phi(\tau)$ where the assignment of type symbols has already been determined. ■

A typical example of a signature is the signature defined by a CFVDC $\mathbb{D}$.

Definition 5.2. The ***associated signature*** of a CFVDC $\mathbb{D}$ is the signature $\Sigma_{\mathbb{D}}$ defined by

- $T_{\mathbb{D}}$ is the set of objects of $\mathbb{D}$, where we write $\ulcorner I \urcorner$ for $I \in \mathbb{D}$ as a type symbol,
- $F_{\mathbb{D}}(\ulcorner I \urcorner, \ulcorner J \urcorner)$ is the set of tight arrows $I \to J$ in $\mathbb{D}$, where we write $\ulcorner f \urcorner$ for $f \in F_{\mathbb{D}}(\ulcorner I \urcorner, \ulcorner J \urcorner)$ as a function symbol,
- $P_{\mathbb{D}}(\ulcorner I \urcorner \fatsemi \ulcorner J \urcorner)$ is the set of loose arrows $\alpha \colon I \nrightarrow J$ in $\mathbb{D}$, where we write $\ulcorner \alpha \urcorner$ for $\alpha \in P_{\mathbb{D}}(\ulcorner I \urcorner \fatsemi \ulcorner J \urcorner)$ as a protype symbol,
- $C_{\mathbb{D}}(\ulcorner \alpha_1 \urcorner \fatsemi \dots \fatsemi \ulcorner \alpha_n \urcorner \mid \ulcorner \beta \urcorner)$ is the set of cells $\mu \colon \alpha_1; \dots; \alpha_n \Rightarrow \beta$ in $\mathbb{D}$, where we write $\ulcorner \mu \urcorner$ for $\mu \in C_{\mathbb{D}}(\ulcorner \alpha_1 \urcorner \fatsemi \dots \fatsemi \ulcorner \alpha_n \urcorner \mid \ulcorner \beta \urcorner)$ as a cell symbol.

$\blacksquare$

Now, we turn to the definition of a specification. We use the word "***multi-class***" to mean a class X with multiplicities $(\mathrm{M}_x)_{x \in \mathrm{X}}$, where M_x is a class. One can think of a multi-class as a (class-large) family of classes.

Definition 5.3. A ***specification*** E for a signature Σ is a pair $(\mathrm{E}^{\mathsf{pty}}, \mathrm{E}^{\mathsf{tm}}, \mathrm{E}^{\mathsf{ptm}})$ where

- $\mathrm{E}^{\mathsf{pty}} = \left(\mathrm{E}^{\mathsf{pty}}_{\rho,\omega}\right)_{\rho,\omega}$ is a multi-class of pairs of protype symbols (ρ, ω) of the same two-sided arity, say $\sigma \nrightarrow \tau$, whose elements are called ***protype isomorphism symbols***.
- E^{tm} is a class of pair of terms of the same type that are well-formed in Σ and $\mathrm{E}^{\mathsf{pty}}$,
- $\mathrm{E}^{\mathsf{ptm}}$ is a class of proterm equality judgments that are well-formed in $\Sigma, \mathrm{E}^{\mathsf{pty}}$ and E^{tm}.

When we say (Σ, E) is a specification, we mean that Σ is a signature and E is a specification for Σ.

A ***morphism of specifications*** $\Phi \colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$ is a morphism of signatures $\Phi \colon \Sigma \to \Sigma'$ by which every judgment in E is translated to a judgment that is derivable from E', together with a multi-class function $\mathrm{E}^{\mathsf{pty}} \to \mathrm{E}'^{\mathsf{pty}}$ with the index function being the translation by Φ, that is, a family of functions $\mathrm{E}^{\mathsf{pty}}_{\rho,\omega} \to \mathrm{E}'^{\mathsf{pty}}_{\rho^{\Phi},\omega^{\Phi}}$ for each pair of protype symbols (ρ, ω). $\blacksquare$

Note that we say a judgment is well-formed in (Σ, E) if it is derivable from the basic derivation rules and the following introduction rules:

$$\frac{\sigma \in T_{\Sigma}}{\sigma \ \mathsf{type}}\ \textsc{BaseType} \qquad \frac{f \in F_{\Sigma}(\sigma, \tau) \qquad \Gamma \vdash t : \sigma}{\Gamma \vdash f(t) : \tau}\ \textsc{Function} \qquad \frac{\rho \in P_{\Sigma}(\sigma, \tau)}{x : \sigma \fatsemi y : \tau \vdash \rho(x, y) \ \mathsf{protype}}\ \textsc{BasePro}$$

$$\frac{\kappa \in C_{\Sigma}(\rho_1 \fatsemi \dots \fatsemi \rho_n \mid \omega)}{\Gamma \mid a_1 : \rho_1 \fatsemi \dots \fatsemi a_n : \rho_n \vdash \kappa\{a_1 \fatsemi \dots \fatsemi a_n\} : \omega \quad (\Gamma = x_0 : \sigma_0, \dots, x_n : \sigma_n)}\ \textsc{BaseCell} \qquad \frac{m \in \mathrm{E}^{\mathsf{pty}}_{\rho,\omega}}{x : \sigma \fatsemi y : \tau \vdash \Lambda_m : \rho \cong \omega}\ \textsc{BaseProIso}$$

Λ_m is a mere symbol for readability. We treat terms like $f(s)$ simply as an abbreviation for $f(x)[s/x]$ and similarly for proterms and cells, as mentioned in Subsection 3.1.

Definition 5.4. For a signature Σ and a CFVDC $\mathbb{D}$, a ***Σ-structure*** $\mathcal{M}$ in $\mathbb{D}$ is a morphism of signatures $\Sigma \to \Sigma_{\mathbb{D}}$. The identity morphism on $\Sigma_{\mathbb{D}}$ can be deemed a $\Sigma_{\mathbb{D}}$-structure in $\mathbb{D}$, which we call the ***canonical ($\Sigma_{\mathbb{D}}$-)structure*** in $\mathbb{D}$. $\blacksquare$

A Σ-structure in $\mathbb{D}$ defines what the type symbols, function symbols, protype symbols, and cell symbols in Σ are interpreted as in $\mathbb{D}$, and with the interpretations for the constructors of the type theory, we can interpret types, terms, protypes, and proterms in Σ as objects, tight arrows, loose arrows, and globular cells in $\mathbb{D}$, respectively. We write $\mathcal{M}(s)$ for the interpretation of a symbol s in $\mathbb{D}$ under $\mathcal{M}$ and similarly for the other items in the type theory.

Definition 5.5. Let $\Phi \colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$ be a morphism of signatures, and J be a judgment in the type theory based on Σ. We write J^{Φ} for the judgment in (Σ, E) defined by replacing each symbol in J with its image under Φ. J^{Φ} is called the ***translation*** of J via Φ. In particular, we write $J^{\mathcal{M}}$ for the judgment in the type theory based on $\Sigma_{\mathbb{D}}$ for a Σ-structure $\mathcal{M}$ in $\mathbb{D}$. $\blacksquare$

Definition 5.6 (Validity of equality judgments)**.** We define the validity of equality judgments in a CFVDC as follows.

- A term equality judgment $t \equiv t'$ is ***valid*** in a Σ-structure $\mathcal{M}$ in a CFVDC $\mathbb{D}$ if $\mathcal{M}(t)$ and $\mathcal{M}(t')$ are equal as tight arrows in $\mathbb{D}$.
- A proterm equality judgment $\mu \equiv \mu'$ is ***valid*** in a Σ-structure $\mathcal{M}$ in a CFVDC $\mathbb{D}$ if $\mathcal{M}(\mu)$ and $\mathcal{M}(\mu')$ are equal as loose arrows in $\mathbb{D}$.

$\blacksquare$

With the definition of validity, one can canonically associate a specification $\mathrm{E}_{\mathbb{D}}$ to a CFVDC $\mathbb{D}$, which exhaustively contains the information of $\mathbb{D}$.

Definition 5.7. The ***associated specification*** $\mathrm{Sp}(\mathbb{D})$ of a CFVDC $\mathbb{D}$ is the specification $(\Sigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$ with $\Sigma_{\mathbb{D}}$ as above, $\mathrm{E}^{\mathsf{tm}}_{\mathbb{D}}$ (resp. $\mathrm{E}^{\mathsf{ptm}}_{\mathbb{D}}$) the set of all the valid equality judgments for terms (resp. proterms) in the canonical structure in $\mathbb{D}$, and $\mathrm{E}^{\mathsf{pty}}_{\mathbb{D}}$ the multi-class whose class indexed by (α, β) is the class of isomorphism cells from $\mathcal{M}(\alpha)$ to $\mathcal{M}(\beta)$ in $\mathbb{D}$. $\blacksquare$

5.2. **An overview of the proof** We will provide an overview of our proof of the syntax-semantics biadjunction for FVDblTT. The full proof is given in Subsection 5.3. First of all, we present the statement of the main theorem.

Theorem 5.8. When we define the 2-cells of specifications in an appropriate way, we have a biadjunction between the 2-category of CFVDCs and the 2-category of specifications in FVDblTT as follows:

$$\begin{tikzcd}
\mathbf{Speci} \arrow[r, shift left=1.5, "\$"] \arrow[r, phantom, "\perp"] & \mathbf{FibVDbl_{cart}} \arrow[l, shift left=1.5, "\mathrm{Sp}"]
\end{tikzcd} .$$

Here, Sp takes a VDC to its associated specification, and we will call $\$(\Sigma,\mathrm{E})$ the ***syntactic virtual double category*** of the specification (Σ,E). Moreover, the components of the counit of this biadjunction are equivalences.

The proof of this theorem is done in the following steps:

(i) We provide a more scarce version of FVDblTT, which we call crude FVDblTT. This type theory lacks protype isomorphisms, and accordingly, we will arrange additional pairs of proterms as backups for the protype isomorphism constructors.
(ii) We construct a biadjunction between the 2-category of CFVDCs and the 2-category of a crude version of the specifications whose components of the counit are equivalences, which is much more straightforward than the direct construction of the expected biadjunction.
(iii) We construct a relative biadjunction between the 2-category of crude specifications and that of specifications in FVDblTT, which requires translations between the two type theories.
(iv) We show that the composite of the biadjunctions constructed in the previous two steps brings us the expected biadjunction.

5.3. **The syntactic virtual double category of a specification** This section provides a detailed proof of the main result of this paper, Theorem 5.8.

Definition 5.9. The type theory FVDblTT$^\diamondsuit$ has the same syntax as FVDblTT except that judgments are limited to typing judgments for all entities and equality judgments for terms and proterms, but not for protype isomorphisms. Accordingly, we drop the rules for the protype isomorphisms. Instead, we add the rules for the typing judgments and the equality judgments for protype isomorphisms as in Figure 5, consisting of the proterm constructors in both directions for the protype isomorphism constructors $\mathsf{rest\text{-}ide}$, $\mathsf{rest\text{-}iter}$, $\mathsf{rest}_\wedge$, $\mathsf{rest}_\top$, repl and the equality judgments showing that they are mutual inverses to each other. Their interpretations in a CFVDC are defined in the same way as the protype isomorphism constructors in Subsection 3.2. ■

Since the type theory FVDblTT$^\diamondsuit$ lacks protype isomorphisms, we need to redefine the notion of a specification for it accordingly.

Definition 5.10. A ***crude specification*** $\mathrm{E}^\diamondsuit$ for a signature Σ is a pair $(\mathrm{E}^\diamondsuit_{\mathsf{tm}}, \mathrm{E}^\diamondsuit_{\mathsf{ptm}})$ where $\mathrm{E}^\diamondsuit_{\mathsf{tm}}$ is a class of term equality judgments, and $\mathrm{E}^\diamondsuit_{\mathsf{ptm}}$ is a class of proterm equality judgments. A ***morphism of crude specifications*** $(\Sigma,\mathrm{E}) \to (\Sigma',\mathrm{E}')$ is similarly defined as in Definition 5.3 but without the data for protype isomorphisms.

For a CFVDC $\mathbb{D}$, the ***associated crude specification*** $(\Sigma_{\mathbb{D}}, \mathrm{E}^\diamondsuit_{\mathbb{D}})$ is the crude specification for $\Sigma_{\mathbb{D}}$ with $\mathrm{E}^\diamondsuit_{\mathbb{D}}$ as the equality judgments part of the associated specification of $\mathbb{D}$. ■

In Subsection 3.2, we have seen how judgments in the type theory are interpreted in a CFVDC. Following the philosophy of functorial semantics, the interpretation in a CFVDC should be captured by functors from what should be called the syntactic VDC of a specification to the CFVDC. We will first define the syntactic VDC of a crude specification and then use it to define the syntactic VDC of a specification.

Definition 5.11. For a crude specification (Σ,E), the ***syntactic virtual double category*** (or classifying virtual double category) $\$^\diamondsuit(\Sigma,\mathrm{E})$ is the virtual double category whose

- objects are contexts Γ ctx for Σ,
- tight arrows $\Gamma \to \Delta = (y_1 : J_1, \dots, y_n : J_n)$ are equivalence classes of sequences of terms $\Gamma \vdash t_1 : J_1, \dots, t_n : J_n$ (or substitutions) modulo equality judgments derivable from (Σ,E),
- loose arrows $\Gamma \mathrel{\mathrlap{\to}{\,\mid}} \Delta$ are protypes $\Gamma \mathbin{⨾} \Delta \vdash \alpha$ $\mathsf{protype}$ derived from (Σ,E),
- cells of form

$$\begin{tikzcd}
\Gamma_0 \arrow[r, "\alpha_1", "|" marking] \arrow[d, "s_0"'] & \cdots & \cdots \arrow[r, "\alpha_n", "|" marking] & \Gamma_n \arrow[d, "s_1"] \\
\Delta_0 \arrow[rrr, "\beta"', "|" marking] & \mu & & \Delta_1
\end{tikzcd} \tag{5.1}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest\text{-}ide}^{\rightarrow}\{a\} : \alpha} \qquad \frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mathsf{rest\text{-}ide}^{\leftarrow}\{a\} : \alpha[\Gamma/\Gamma \fatsemi \Delta/\Delta]}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha[\Gamma/\Gamma \fatsemi \Delta/\Delta] \vdash \mathsf{rest\text{-}iter}^{\leftarrow}\{\mathsf{rest\text{-}iter}^{\rightarrow}\{a\}\} \equiv a : \alpha[\Gamma/\Gamma \fatsemi \Delta/\Delta]} \qquad \frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mathsf{rest\text{-}iter}^{\rightarrow}\{\mathsf{rest\text{-}iter}^{\leftarrow}\{a\}\} \equiv a : \alpha}$$

$$\frac{\Gamma'' \vdash S' \,/\, \Gamma' \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta'' \vdash T' \,/\, \Delta' \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma'' \fatsemi \Delta'' \mid a : (\alpha[S/\Gamma \fatsemi T/\Delta])\,[S'/\Gamma' \fatsemi T'/\Delta'] \vdash \mathsf{rest\text{-}iter}^{\rightarrow}\{a\} : \alpha[S[S'/\Gamma']/\Gamma \fatsemi T[T'/\Delta']/\Delta]}$$

$$\frac{\Gamma'' \vdash S' \,/\, \Gamma' \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta'' \vdash T' \,/\, \Delta' \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma'' \fatsemi \Delta'' \mid a : \alpha[S[S'/\Gamma']/\Gamma \fatsemi T[T'/\Delta']/\Delta] \vdash \mathsf{rest\text{-}iter}^{\leftarrow}\{a\} : (\alpha[S/\Gamma \fatsemi T/\Delta])\,[S'/\Gamma' \fatsemi T'/\Delta']}$$

$$\frac{\Gamma'' \vdash S' \,/\, \Gamma' \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta'' \vdash T' \,/\, \Delta' \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma'' \fatsemi \Delta'' \mid a : (\alpha[S/\Gamma \fatsemi T/\Delta])\,[S'/\Gamma' \fatsemi T'/\Delta'] \vdash \mathsf{rest\text{-}iter}^{\leftarrow}\{\mathsf{rest\text{-}iter}^{\rightarrow}\{a\}\} \equiv a : (\alpha[S/\Gamma \fatsemi T/\Delta])\,[S'/\Gamma' \fatsemi T'/\Delta']}$$

$$\frac{\Gamma'' \vdash S' \,/\, \Gamma' \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta'' \vdash T' \,/\, \Delta' \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma'' \fatsemi \Delta'' \mid a : [S[S'/\Gamma']/\Gamma \fatsemi T[T'/\Delta']/\Delta] \vdash \mathsf{rest\text{-}iter}^{\rightarrow}\{\mathsf{rest\text{-}iter}^{\leftarrow}\{a\}\} \equiv a : [S[S'/\Gamma']/\Gamma \fatsemi T[T'/\Delta']/\Delta]}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S/\Gamma \fatsemi T/\Delta] \wedge \beta[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest}^{\leftarrow}_{\wedge}\{a\} : (\alpha \wedge \beta)[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : (\alpha \wedge \beta)[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest}^{\leftarrow}_{\wedge}\{\langle \pi_0[S/\Gamma \fatsemi T/\Delta], \pi_1[S/\Gamma \fatsemi T/\Delta]\rangle\} \equiv a : (\alpha \wedge \beta)[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S/\Gamma \fatsemi T/\Delta] \wedge \beta[S/\Gamma \fatsemi T/\Delta] \vdash \langle \pi_0[S/\Gamma \fatsemi T/\Delta], \pi_1[S/\Gamma \fatsemi T/\Delta]\rangle\,\{\mathsf{rest}^{\leftarrow}_{\wedge}\{a\}/a\} \equiv a : (\alpha \wedge \beta)[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \top \vdash \mathsf{rest}^{\leftarrow}_{\top} : \top[S/\Gamma \fatsemi T/\Delta]} \qquad \frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \top[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest}^{\leftarrow}_{\top}\{\langle\ \rangle\} \equiv a : \top[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S_0/\Gamma \fatsemi T_0/\Delta] \vdash \mathsf{repl}^{\rightarrow}_{S_0,S_1 \fatsemi T_0,T_1}\{a\} : \alpha[S_1/\Gamma \fatsemi T_1/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S_0/\Gamma \fatsemi T_0/\Delta] \vdash \mathsf{repl}^{\rightarrow}_{S_1,S_0 \fatsemi T_1,T_0}\{\mathsf{repl}^{\rightarrow}_{S_0,S_1 \fatsemi T_0,T_1}\{a\}\} \equiv a : \alpha[S_0/\Gamma \fatsemi T_0/\Delta]}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{repl}^{\rightarrow}_{S,S \fatsemi T,T}\{a\} \equiv a : \alpha[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S_0/\Gamma \fatsemi T_0/\Delta] \vdash \mathsf{repl}^{\rightarrow}_{S_1,S_0 \fatsemi T_1,T_0}\left\{\mathsf{repl}^{\rightarrow}_{S_0,S_1 \fatsemi T_0,T_1}\{a\}\right\} \equiv a : \alpha[S_0/\Gamma \fatsemi T_0/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Gamma' \vdash S_1 \,/\, \Gamma \equiv S_2 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Delta' \vdash T_1 \,/\, \Delta \equiv T_2 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha\,[S_0/\Gamma \fatsemi T_0/\Delta] \vdash \mathsf{repl}^{\rightarrow}_{S_1,S_2 \fatsemi T_1,T_2}\{\mathsf{repl}^{\rightarrow}_{S_0,S_1 \fatsemi T_0,T_1}\{a\}\} \equiv \mathsf{repl}^{\rightarrow}_{S_0,S_2 \fatsemi T_0,T_2}\{a\} : \alpha\,[S_2/\Gamma \fatsemi T_2/\Delta]}$$

FIGURE 5. FVDblTT$^{\diamondsuit}$: The rules for the proterm constructors and the equality judgments (excluding the counterparts for the last nine rules in Appendix B.1, which can be directly obtained by replacing rest-iter, rest-ide, repl with rest-iter $^{\rightarrow}$, rest-ide $^{\rightarrow}$, repl $^{\rightarrow}$ in those rules).

are equivalence classes of proterms

$$\overline{\Gamma} \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu : \beta[S_0/\Delta_0 \fatsemi S_n/\Delta_n]$$

modulo equality judgments derivable from (Σ, E). It makes no difference which representatives we choose for the equivalence classes of terms S_i's when we fix them as replacing S_i's with another family of terms S_i''s, which the transformation by repl allows, gives the bijection between the proterms for each equivalence class.

■

Although tight arrows and cells are equivalence classes of terms and proterms, respectively, we will often abuse the notation and write t and μ for the representatives of the equivalence classes.

Proposition 5.12. The syntactic VDC $\$^{\diamondsuit}(\Sigma, \mathrm{E})$ for a crude specification (Σ, E) is a cartesian fibrational virtual double category.

Proof sketch. The verification of the objects and the tight arrows making a category with finite products is the same as the classical result [Jac99, Theorem 3.3.4]. The derivation of the proterm a in the context $a : \alpha$ gives the identity cell on α, and the substitution and the prosubstitution give the composition of cells. We illustrate this with a small example.[3] Suppose we have cells as follows:

$$\begin{tikzcd}
\Gamma_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "S_0"'] & \Gamma_1 \arrow[r, "\alpha_2", "\shortmid"{marking}] & \Gamma_2 \arrow[d, "S_1"] \arrow[dr, bend left, "S_2"] & \\
\Delta_0 \arrow[rr, "\beta_1"', "\shortmid"{marking}] \arrow[d, "T_0"'] & \mu_1 & \Delta_1 \arrow[r, "\beta_2"', "\shortmid"{marking}] & \Delta_2 \arrow[d, "T_1"] \\
\Theta_0 \arrow[rrr, "\gamma"', "\shortmid"{marking}] & & \nu & \Theta_1
\end{tikzcd} \mu_2 \quad ,$$

meaning that we have proterms

$$\Gamma_0 \fatsemi \Gamma_1 \fatsemi \Gamma_2 \mid a_1 : \alpha_1 \fatsemi a_2 : \alpha_2 \vdash \mu_1 : \beta_1[S_0/\Delta_0 \fatsemi S_1/\Delta_1],$$
$$\Gamma_2 \mid \ \vdash \mu_2 : \beta_2[S_1/\Delta_1 \fatsemi S_2/\Delta_2],$$
$$\Delta_0 \fatsemi \Delta_1 \fatsemi \Delta_2 \mid b_1 : \beta_1 \fatsemi b_2 : \beta_2 \vdash \nu : \gamma[T_0/\Theta_0 \fatsemi T_1/\Theta_1].$$

From the third proterm, we can derive the proterm

$$\begin{aligned}
&\Gamma_0 \fatsemi \Gamma_1 \fatsemi \Gamma_2 \mid b_1 : \beta_1[S_0/\Delta_0 \fatsemi S_1/\Delta_1] \fatsemi b_2 : \beta_2[S_1/\Delta_1 \fatsemi S_2/\Delta_2] \\
&\qquad \vdash \mathsf{rest\text{-}iter}^{\rightarrow} \{\nu[S_0/\Delta_0 \fatsemi S_1/\Delta_1 \fatsemi S_2/\Delta_2]\} : \gamma\,[T_0[S_0/\Delta_0]/\Theta_1 \fatsemi T_1[S_2/\Delta_2]/\Theta_1]\,.
\end{aligned}$$

By the prosubstitution rule, we then have the proterm

$$\begin{aligned}
&\Gamma_0 \fatsemi \Gamma_1 \fatsemi \Gamma_2 \mid a_1 : \alpha_1 \fatsemi a_2 : \alpha_2 \\
&\quad \vdash \Big(\mathsf{rest\text{-}iter}^{\rightarrow} \{\nu[S_0/\Delta_0 \fatsemi S_1/\Delta_1 \fatsemi S_2/\Delta_2]\}\Big) \{\mu_1/b_1 \fatsemi \mu_2/b_2\} : \gamma\,[T_0[S_0/\Delta_0]/\Theta_1 \fatsemi T_1[S_2/\Delta_2]/\Theta_1]\,,
\end{aligned}$$

which we define as the composite of the cells. The computation rules for proterms make this composition satisfy the axioms of VDCs.

A restriction is given by the substitution in protypes. It is straightforward to check that the canonical cell

$$\begin{tikzcd}
\Gamma_0 \arrow[r, "{\alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]}", "\shortmid"{marking}] \arrow[d, "S_0"'] & \Gamma_1 \arrow[d, "S_1"] \\
\Delta_0 \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[ur, phantom, "\mathtt{rest}"] & \Delta_1
\end{tikzcd} \quad \text{given by} \quad \Gamma_0 \fatsemi \Gamma_1 \mid a : \alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1] \vdash a : \alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]$$

exhibits $\alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]$ as a restriction of a loose arrow α along S_0 and S_1 as tight arrows. By Proposition 2.8, it is enough to show that $\mathbb{S}^{\diamondsuit}(\Sigma, \mathrm{E})$ has loose arrows $\alpha \wedge \beta$ for any α, β and $\top$ for any pair of objects satisfying the conditions we have shown therein. However, protype constructors $\wedge$ and $\top$ that we have defined in the type theory achieve this condition by the computation rules for proterms. □

The following lemma is easy to check.

Lemma 5.13. For any morphism of specifications $\Phi\colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$, the translation $(-)^{\Phi}$ defines a morphism $\mathbb{S}^{\diamondsuit}(\Phi)\colon \mathbb{S}^{\diamondsuit}(\Sigma, \mathrm{E}) \to \mathbb{S}^{\diamondsuit}(\Sigma', \mathrm{E}')$, and this is functorial in Φ.

Following this lemma, we define the 2-cells of crude specifications.

Definition 5.14. A ***2-cell of crude specifications*** $\Phi \to \Psi\colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$ is a transformation of VDCs from $\mathbb{S}^{\diamondsuit}(\Phi)$ to $\mathbb{S}^{\diamondsuit}(\Psi)$. We write **$\diamondsuit$-Speci** for the 2-category of crude specifications, morphisms of crude specifications, and 2-cells of crude specifications. ■

Corollary 5.15. The assignment $(\Sigma, \mathrm{E}) \mapsto \mathbb{S}^{\diamondsuit}(\Sigma, \mathrm{E})$ defines a 2-functor $\mathbb{S}^{\diamondsuit}\colon$ **$\diamondsuit$-Speci** $\to$ $\mathbf{FibVDbl}_{\mathbf{cart}}$.

Proposition 5.16. The assignment that sends a CFVDC $\mathbb{D}$ to the associated crude specification $(\Sigma_{\mathbb{D}}, \mathrm{E}^{\diamondsuit}_{\mathbb{D}})$ extends to a 2-functor $\mathrm{Sp}^{\diamondsuit}\colon \mathbf{FibVDbl}_{\mathbf{cart}} \to$ **$\diamondsuit$-Speci** which is a right biadjoint to $\mathbb{S}^{\diamondsuit}$. The components of the counit $\epsilon_{\mathbb{D}}\colon \mathbb{S}^{\diamondsuit}(\mathrm{Sp}^{\diamondsuit}(\mathbb{D})) \to \mathbb{D}$ are equivalences.

Proof. The assignment $\mathbb{D} \mapsto (\Sigma_{\mathbb{D}}, \mathrm{E}^{\diamondsuit}_{\mathbb{D}})$ extends to a 1-functor $\mathrm{Sp}^{\diamondsuit}$ by the definition of the associated crude specification.

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[3] See (ii) in Appendix B.2 for the detailed proof.

Before arguing that $\mathrm{Sp}^{\diamondsuit}$ is a 2-functor, we construct a virtual double functor $\varepsilon_{\mathbb{D}} \colon \mathbb{S}^{\diamondsuit}(\mathrm{Sp}^{\diamondsuit}(\mathbb{D})) \to \mathbb{D}$ that will turn out to be an equivalence. We have the canonical $\Sigma_{\mathbb{D}}$-structure in $\mathbb{D}$. In the way we showed in Subsection 3.2, we can interpret all the items in $\mathrm{Sp}^{\diamondsuit}(\mathbb{D})$ in $\mathbb{D}$. Now, we are to show that this defines a virtual double functor from $\mathbb{S}^{\diamondsuit}(\Sigma_{\mathbb{D}}, \mathrm{E}^{\diamondsuit}_{\mathbb{D}})$ to $\mathbb{D}$.[4] The way we assign the data of VDCs is straightforward using Definition 3.1 except for the cells. A cell of $\mathbb{S}^{\diamondsuit}(\Sigma_{\mathbb{D}}, \mathrm{E}^{\diamondsuit}_{\mathbb{D}})$ of the form (5.1) is interpreted as the composite of the cartesian cell on the left and the cell $[\![\mu]\!]$ on the right, which is inductively defined in Definition 3.1.

$$\begin{tikzcd}
{[\![\Gamma_0]\!]} \arrow[r, "{[\![\beta[S_0/\Delta_0 \fatsemi S_1/\Delta_1]]\!]}", "\shortmid"{marking}] \arrow[d, "{[\![S_0]\!]}"'] \arrow[dr, phantom, "\texttt{rest}"] & {[\![\Gamma_1]\!]} \arrow[d, "{[\![S_1]\!]}"] \\
{[\![\Delta_0]\!]} \arrow[r, "{[\![\beta]\!]}"', "\shortmid"{marking}] & {[\![\Delta_1]\!]}
\end{tikzcd}
,
\begin{tikzcd}
{[\![\Gamma_0]\!]} \arrow[r, "{[\![\alpha_1]\!]}", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[d, phantom, "{[\![\mu]\!]}"] & \cdots \arrow[r, "{[\![\alpha_n]\!]}", "\shortmid"{marking}] & {[\![\Gamma_n]\!]} \arrow[d, equal] \\
{[\![\Gamma_0]\!]} \arrow[rrr, "{[\![\beta[S_0/\Delta_0 \fatsemi S_1/\Delta_1]]\!]}"', "\shortmid"{marking}] & & & {[\![\Gamma_n]\!]}
\end{tikzcd}$$

The assignments are independent of the choice of terms and proterms since the equivalence relation is up to the equality belonging to $\mathrm{E}^{\diamondsuit}_{\mathbb{D}}$, that is, the equality in $\mathbb{D}$. Proving that this defines a morphism in $\mathbf{FibVDbl}_{\mathbf{cart}}$ is a routine verification: the translation of the β-rule and the η-rule to the universal properties. We consult Lemmas 2.5 and 2.7 to see that this is an equivalence. We only confirm the limited case of (iii) in Lemma 2.5; the remaining part is similar. Up to isomorphism, loose arrows in the syntactic VDC are of the form $\boldsymbol{\alpha}(x \fatsemi y)$ for a protype symbol $\boldsymbol{\alpha}$ in the signature $\Sigma_{\mathbb{D}}$. This allows us to reduce the proof to the case where the cell is framed by arrows of the above form. Given a cell in $\mathbb{D}$ on the left below, we can find a cell in $\mathbb{S}^{\diamondsuit}(\Sigma_{\mathbb{D}}, \mathrm{E}^{\diamondsuit}_{\mathbb{D}})$ on the right below that is mapped to the given cell by $\varepsilon_{\mathbb{D}}$.

$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[d, phantom, "\mu"] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, equal] \\
J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] & & & J_1
\end{tikzcd}
\longleftarrow\!\shortmid
\begin{tikzcd}
x_0 : \ulcorner I_0 \urcorner \arrow[r, "\ulcorner \alpha_1 \urcorner(x_0 \fatsemi x_1)", "\shortmid"{marking}] \arrow[d, equal] & \cdots \arrow[d, phantom, "\ulcorner \mu \urcorner\{a_1 \fatsemi \ldots \fatsemi a_n\}"] & \cdots \arrow[r, "\ulcorner \alpha_n \urcorner(x_{n-1} \fatsemi x_n)", "\shortmid"{marking}] & x_n : \ulcorner I_n \urcorner \arrow[d, equal] \\
x_0 : \ulcorner I_0 \urcorner \arrow[rrr, "\ulcorner \beta \urcorner(x_0 \fatsemi x_n)"', "\shortmid"{marking}] & & & x_n : \ulcorner I_n \urcorner
\end{tikzcd}$$

If another cell in $\mathbb{S}^{\diamondsuit}(\Sigma_{\mathbb{D}}, \mathrm{E}^{\diamondsuit}_{\mathbb{D}})$ framed by the arrows above is mapped to the same cell in $\mathbb{D}$ by $\varepsilon_{\mathbb{D}}$, then the equality of the two proterms representing the cells belongs to $\mathrm{E}^{\diamondsuit}_{\mathbb{D}}$ by definition. Therefore, $\varepsilon_{\mathbb{D}}$ satisfies (iii) in Lemma 2.5.

Using the equivalence $\varepsilon_{\mathbb{D}}$, we know that there is the only way to define how the 2-functor $\mathrm{Sp}^{\diamondsuit}$ should send 2-cells in such a way that $\varepsilon_{\mathbb{D}}$ is a 2-natural transformation.

Note that $\mathbb{S}^{\diamondsuit}(-)$ is locally fully faithful. In order to see that $\varepsilon_{\mathbb{D}}$ gives a counit of the biadjunction, it is enough to show that any morphism $F \colon \mathbb{S}^{\diamondsuit}(\Sigma, \mathrm{E}) \to \mathbb{D}$ in $\mathbf{FibVDbl}_{\mathbf{cart}}$ is presented, up to isomorphism, by the composite of the image of a morphism $\widetilde{F} \colon (\Sigma, \mathrm{E}) \to \mathrm{Sp}^{\diamondsuit}(\mathbb{D})$ by $\mathbb{S}^{\diamondsuit}$ and $\varepsilon_{\mathbb{D}}$. We can pick such a morphism of signatures $\widetilde{F}$ by taking the image of each symbol in Σ as some item in the syntactic VDC $\mathbb{S}^{\diamondsuit}(\Sigma, \mathrm{E})$. The equality judgments in E are translated to the valid equality judgments in $\Sigma_{\mathbb{D}}$, meaning that the morphism of signatures indeed defines a morphism of specifications $\widetilde{F} \colon (\Sigma, \mathrm{E}) \to \mathrm{Sp}^{\diamondsuit}(\mathbb{D})$. The morphism $\varepsilon_{\mathbb{D}} \circ \mathbb{S}^{\diamondsuit}(\widetilde{F})$ only differs from F by the inductive assignments of items in $\mathrm{Sp}^{\diamondsuit}(\mathbb{D})$ using the universal properties in $\mathbb{D}$; hence, they are isomorphic. □

One can translate specifications into crude specifications without losing information.

Definition 5.17. A ***crude translation*** of a proterm judgment $\Delta \fatsemi \Gamma \mid A \vdash \mu : \beta$ in FVDblTT is a proterm judgment $\Delta \fatsemi \Gamma \mid A \vdash \mu^{\diamondsuit} : \beta$ in $\mathrm{FVDblTT}^{\diamondsuit}$ of the same context defined inductively as follows:

- a judgment derived by the rules except for TRANSPORT is translated to the judgment derived by the same rules from the judgments already translated from the premises,
- the translation of a judgment derived by TRANSPORT is defined by induction on the derivation of protype isomorphism judgments as in Figure 6.

A ***de-crude translation*** of a proterm judgment $\Delta \fatsemi \Gamma \mid A \vdash \mu : \beta$ in $\mathrm{FVDblTT}^{\diamondsuit}$ is a proterm judgment $\Delta \fatsemi \Gamma \mid A \vdash \mu^{\clubsuit} : \beta$ in FVDblTT of the same context defined by replacing the constructor added in Definition 5.9 with the corresponding constructor in FVDblTT, for example, $\mathsf{rest\text{-}iter}^{\rightarrow}\{a\}$ with $\mathsf{rest\text{-}iter}\{a\}$ and $\mathsf{rest\text{-}iter}^{\leftarrow}\{a\}$ with $\mathsf{rest\text{-}iter}^{-1}\{a\}$. ■

The rules for $\mathrm{FVDblTT}^{\diamondsuit}$ and the (de-)crude translations are formulated so that the following lemma holds.

Lemma 5.18. For a proterm judgment $\Delta \fatsemi \Gamma \mid A \vdash \mu : \beta$ derived from a specification (Σ, E) without the use of TRANSPORT in FVDblTT, $(\mu^{\diamondsuit})^{\clubsuit} \equiv \mu$ is derivable in FVDblTT. Conversely, for a proterm judgment $\Delta \fatsemi \Gamma \mid A \vdash \mu : \beta$ derived from Σ in $\mathrm{FVDblTT}^{\diamondsuit}$, $(\mu^{\clubsuit})^{\diamondsuit} \equiv \mu$ is derivable in $\mathrm{FVDblTT}^{\diamondsuit}$.

Definition 5.19. A ***crude encoding*** $\mathrm{CD}(\Sigma, \mathrm{E})$ of a specification (Σ, E) is a crude specification defined by

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[4] See (i) in Appendix B.2.

- the signature consists of data in Σ plus additional cell symbols $\varphi_m \colon \rho \Rightarrow \omega$ and $\psi_m \colon \omega \Rightarrow \rho$ for each element $m \in \mathrm{E}^{\mathsf{pty}}_{\rho,\omega}$,
- the equality judgments consist of the crude translations of equality judgments in E plus the equality judgments

  $$x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \psi_m\{\varphi_m\{a\}\} \equiv a : \rho \quad \text{and} \quad x : \sigma \fatsemi y : \tau \mid b : \omega \vdash \varphi_m\{\psi_m\{b\}\} \equiv b : \omega \tag{5.2}$$

  for each $m \in \mathrm{E}^{\mathsf{pty}}_{\rho,\omega}$.

■

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mathsf{id}_\alpha\{a\}^{\Diamond} \doteq a : \alpha} \qquad \frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mathsf{id}_\alpha^{-1}\{a\}^{\Diamond} \doteq a : \alpha} \qquad \frac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta \qquad \Gamma \fatsemi \Delta \vdash \Omega : \beta \cong \gamma}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash (\Omega \circ \Upsilon)\{a\}^{\Diamond} \doteq \Omega\{b\}^{\Diamond} \left\{ \Upsilon\{a\}^{\Diamond} \middle/ b \right\} : \gamma}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha[\Gamma/\Gamma \fatsemi \Delta/\Delta] \vdash \mathsf{rest\text{-}ide}\{a\}^{\Diamond} \doteq \mathsf{rest\text{-}ide}^{\rightarrow}\{a\} : \alpha} \qquad \frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mathsf{rest\text{-}ide}^{-1}\{a\}^{\Diamond} \doteq \mathsf{rest\text{-}ide}^{\leftarrow}\{a\} : \alpha[\Gamma/\Gamma \fatsemi \Delta/\Delta]}$$

$$\frac{\Gamma'' \vdash S' \,/\, \Gamma' \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta'' \vdash T' \,/\, \Delta' \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma'' \fatsemi \Delta'' \mid a : (\alpha[S/\Gamma \fatsemi T/\Delta])\,[S'/\Gamma' \fatsemi T'/\Delta'] \vdash \mathsf{rest\text{-}iter}\{a\}^{\Diamond} \doteq \mathsf{rest\text{-}iter}^{\rightarrow}\{a\} : \alpha[S[S'/\Gamma']/\Gamma \fatsemi T[T'/\Delta']/\Delta]}$$

$$\frac{\Gamma'' \vdash S' \,/\, \Gamma' \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta'' \vdash T' \,/\, \Delta' \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma'' \fatsemi \Delta'' \mid a : \alpha\left[S[S'/\Gamma']/\Gamma \fatsemi T[T'/\Delta']/\Delta\right] \vdash \mathsf{rest\text{-}iter}^{-1}\{a\}^{\Diamond} \doteq \mathsf{rest\text{-}iter}^{\leftarrow}\{a\} : (\alpha[S/\Gamma \fatsemi T/\Delta])\,[S'/\Gamma' \fatsemi T'/\Delta']}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \qquad \Gamma \fatsemi \Delta \vdash \beta \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : (\alpha \wedge \beta)\,[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest}_{\wedge}\{a\}^{\Diamond} \doteq \langle \pi_0[S/\Gamma \fatsemi T/\Delta], \pi_1[S/\Gamma \fatsemi T/\Delta] \rangle : \alpha[S/\Gamma \fatsemi T/\Delta] \wedge \beta[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \qquad \Gamma \fatsemi \Delta \vdash \beta \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S/\Gamma \fatsemi T/\Delta] \wedge \beta[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest}_{\wedge}^{-1}\{a\}^{\Diamond} \doteq \overset{\leftarrow}{\mathsf{rest}}_{\wedge}\{a\} : (\alpha \wedge \beta)\,[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \top[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest}_{\top}\{a\}^{\Diamond} \doteq \langle\rangle : \top} \qquad \frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \top \vdash \mathsf{rest}_{\top}^{-1}\{a\}^{\Diamond} \doteq \overset{\leftarrow}{\mathsf{rest}}_{\top}\{a\} : \top[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S_0/\Gamma \fatsemi T_0/\Delta] \vdash \mathsf{repl}_{S_0,S_1 \fatsemi T_0,T_1}\{a\}^{\Diamond} \doteq \overset{\rightarrow}{\mathsf{repl}}_{S_0,S_1 \fatsemi T_0,T_1}\{a\} : \alpha[S_1/\Gamma \fatsemi T_1/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S_1/\Gamma \fatsemi T_1/\Delta] \vdash \mathsf{repl}^{-1}_{S_0,S_1 \fatsemi T_0,T_1}\{a\}^{\Diamond} \doteq \overset{\rightarrow}{\mathsf{repl}}_{S_1,S_0 \fatsemi T_1,T_0}\{a\} : \alpha[S_0/\Gamma \fatsemi T_0/\Delta]}$$

$$\frac{\begin{array}{cc} \Gamma \fatsemi \Delta \mid a : \alpha \vdash \mu\{a\} : \beta & \Gamma \fatsemi \Delta \mid b : \beta \vdash \mu\{\nu\{b\}\} \equiv b : \beta \\ \Gamma \fatsemi \Delta \mid b : \beta \vdash \nu\{b\} : \alpha & \Gamma \fatsemi \Delta \mid a : \alpha \vdash \nu\{\mu\{a\}\} \equiv a : \alpha \end{array}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \llparenthesis \mu, \nu \rrparenthesis\{a\}^{\Diamond} \doteq \mu\{a\} : \beta} \qquad \frac{\begin{array}{cc} \Gamma \fatsemi \Delta \mid a : \alpha \vdash \mu\{a\} : \beta & \Gamma \fatsemi \Delta \mid b : \beta \vdash \mu\{\nu\{b\}\} \equiv b : \beta \\ \Gamma \fatsemi \Delta \mid b : \beta \vdash \nu\{b\} : \alpha & \Gamma \fatsemi \Delta \mid a : \alpha \vdash \nu\{\mu\{a\}\} \equiv a : \alpha \end{array}}{\Gamma \fatsemi \Delta \mid b : \beta \vdash \llparenthesis \mu, \nu \rrparenthesis^{-1}\{a\}^{\Diamond} \doteq \nu\{b\} : \alpha}$$

$$\frac{m \in \mathrm{E}^{\mathsf{pty}}_{\rho,\omega}}{x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \Lambda_m\{a\}^{\Diamond} \doteq \varphi_m\{a\} : \omega} \qquad \frac{m \in \mathrm{E}^{\mathsf{pty}}_{\rho,\omega}}{x : \sigma \fatsemi y : \tau \mid b : \omega \vdash \Lambda_m^{-1}\{b\}^{\Diamond} \doteq \psi_m\{b\} : \rho}$$

FIGURE 6. Translations of transported proterms

We define the 2-category **Speci** of specifications so that the above assignment is a 2-functor.

Lemma 5.20. The assignment $(\Sigma, \mathrm{E}) \mapsto \mathrm{CD}(\Sigma, \mathrm{E})$ induces a (1-)functor from the category of specifications and morphisms of specifications to the category of crude specifications and morphisms of crude specifications.

Proof sketch. For a morphism of specifications $\Phi \colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$, the assignment $\mathrm{CD}(\Phi)$ sends the cell symbols φ_m and ψ_m to $\varphi_{\Phi(m)}$ and $\psi_{\Phi(m)}$. The equality judgments (5.2) are translated into the equality judgments of the same form and hence derivable from $\mathrm{CD}(\Sigma', \mathrm{E}')$.[5] □

Definition 5.21. A ***2-cell of specifications*** $\Phi \to \Psi \colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$ is a 2-cell $\mathrm{CD}(\Phi) \to \mathrm{CD}(\Psi)$ in $\Diamond$-**Speci**, hence a transformation of VDCs from $\$^{\Diamond}(\mathrm{CD}(\Phi))$ to $\$^{\Diamond}(\mathrm{CD}(\Psi))$. We write **Speci** for the 2-category of specifications, morphisms of specifications, and 2-cells of specifications. ■

Corollary 5.22. The assignment $(\Sigma, \mathrm{E}) \mapsto \$^{\Diamond}(\mathrm{CD}(\Sigma, \mathrm{E}))$ defines a 2-functor $\mathrm{CD} \colon \mathbf{Speci} \to \mathbf{FibVDbl_{cart}}$, which is locally fully faithful.

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[5] See (iv) in Appendix B.2.

This 2-functor does not have a right biadjoint globally but a partial one.

Definition 5.23. A crude specification $(\mathit{\Sigma}, \mathrm{E})$ is ***unary-cell-saturated*** if, for any proterm judgment $x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \vartheta : \omega$ derivable from E where $\sigma, \tau, \rho, \omega$ belongs to the signature $\mathit{\Sigma}$, there is a cell symbol $\kappa_{\vartheta} \colon \rho \Rightarrow \omega$ in $\mathit{\Sigma}$ such that the equality judgment

$$x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \kappa_{\vartheta}\{a\} \equiv \vartheta : \omega$$

is derivable from E. ■

A crude specification being saturated means that the symbols in the signature constitute a virtual double category that is equivalent to the syntactic VDC of the specification. Let $\Diamond$-$\mathbf{Speci}_{\mathsf{sat}}$ be the full subcategory of $\Diamond$-**Speci** whose objects are unary-cell-saturated crude specifications. Note that the associated crude specification $(\mathit{\Sigma}_{\mathbb{D}}, \mathrm{E}^{\Diamond}_{\mathbb{D}})$ of a CFVDC $\mathbb{D}$ is unary-cell-saturated. Note that the choice of the cell symbols κ_{ϑ} may not be unique but does not affect anything in the presence of E.

Proposition 5.24. The 2-functor $\mathrm{CD} \colon \mathbf{Speci} \to \Diamond$-**Speci** has a relative right biadjoint $\mathrm{DCD} \colon \Diamond$-$\mathbf{Speci}_{\mathsf{sat}} \to \mathbf{Speci}$ over the inclusion $J \colon \Diamond$-$\mathbf{Speci}_{\mathsf{sat}} \hookrightarrow \Diamond$-**Speci**.

$$\begin{tikzcd}
\mathbf{Speci} \arrow[r, "\mathrm{CD}"] & \Diamond\text{-}\mathbf{Speci} \\
& \Diamond\text{-}\mathbf{Speci}_{\mathsf{sat}} \arrow[u, hook, "J"'] \arrow[ul, "\mathrm{DCD}"] \arrow[ul, phantom, shift left=3, "\overset{\upsilon}{\Rightarrow}"]
\end{tikzcd} .$$

The components of the counit $\upsilon_{(P,\mathrm{D})} \colon \mathrm{CD}(\mathrm{DCD}(P, \mathrm{D})) \to (P, \mathrm{D})$ are sent to the equivalence by $\mathbb{S}^{\Diamond}$.

Here, a relative right biadjoint means that there is a natural equivalence

$$\Diamond\text{-}\mathbf{Speci}(\mathrm{CD}(-), J(*)) \simeq \mathbf{Speci}(-, \mathrm{DCD}(*))$$

induced by the 2-cell above. See [FGHW18] on relative biadjunctions, where the notion is called a relative pseudo-adjunction.

Proof. The outline of the proof is as follows. We first construct a specification $\mathrm{DCD}(P, \mathrm{D})$ and a morphism of specifications $\upsilon_{(P,\mathrm{D})} \colon \mathrm{CD}(\mathrm{DCD}(P, \mathrm{D})) \to (P, \mathrm{D})$ from a unary-cell-saturated crude specification (P, D). Then, we show that for each $(\mathit{\Sigma}, \mathrm{E})$ in **Speci**, the functor $\mathbf{Speci}\,((\mathit{\Sigma}, \mathrm{E}), \mathrm{DCD}(P, \mathrm{D})) \to \Diamond\text{-}\mathbf{Speci}\,(\mathrm{CD}(\mathit{\Sigma}, \mathrm{E}), (P, \mathrm{D}))$ defined by $\mathit{\Psi} \mapsto \upsilon_{(P,\mathrm{D})} \circ \mathrm{CD}(\mathit{\Psi})$ is an equivalence. We divided the proof into two parts: the essential surjectivity and the full faithfulness. We precede with the former, and then before the latter, we show the last statement in the proposition to facilitate the proof.

For a unary-cell-saturated crude specification (P, D), a specification $\mathrm{DCD}(P, \mathrm{D})$ consists of the same signature P, the multi-class $\mathrm{D}^{\cong}$ defined from D by setting $\mathrm{D}^{\cong}_{\rho,\omega}$ to be the class of the pairs (ϑ, ς) of proterms

$$x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \vartheta\{a\} : \omega \quad \text{and} \quad x : \sigma \fatsemi y : \tau \mid b : \omega \vdash \varsigma\{b\} : \rho$$

for which D derives the equality judgments

$$x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \varsigma\{\vartheta\{a\}\} \equiv a : \rho \quad \text{and} \quad x : \sigma \fatsemi y : \tau \mid b : \omega \vdash \vartheta\{\varsigma\{b\}\} \equiv b : \omega,$$

and the classes of term and proterm equality judgments consisting of the de-crude translations of the equality judgments in D^{tm} and $\mathrm{D}^{\mathsf{ptm}}$ plus the equality judgments

$$x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \mathit{\Lambda}_{(\vartheta,\varsigma)}\{a\} \equiv \vartheta\{a\}^{\clubsuit} : \omega \quad \text{and} \quad x : \sigma \fatsemi y : \tau \mid b : \omega \vdash \mathit{\Lambda}^{-1}_{(\vartheta,\varsigma)}\{b\} \equiv \varsigma\{b\}^{\clubsuit} : \rho \tag{5.3}$$

for each protype isomorphism symbol (ϑ, ς) in $\mathrm{D}^{\cong}_{\rho,\omega}$. Then we will have a morphism of specifications $\upsilon_{(P,\mathrm{D})}$ that sends the new cell symbols $\varphi_{(\vartheta,\varsigma)}$ and $\psi_{(\vartheta,\varsigma)}$ to the cell symbols κ_{ϑ} and κ_{ς} in the definition of the unary-cell-saturated crude specification (P, D) for any pair (ϑ, ς) in $\mathrm{D}^{\cong}$. It follows that $\upsilon_{(P,\mathrm{D})}$ defines a morphism of specifications since the equality judgments in $\mathrm{CD}(\mathrm{DCD}(P, \mathrm{D}))$ are either in D or those of the form (5.2) for the pairs in $\mathrm{D}^{\cong}$, which are translated derivable from D.

We first prove that this $\upsilon_{(P,\mathrm{D})}$ leads to the essential surjectivity. Given a morphism of crude specifications $\mathit{\Phi} \colon \mathrm{CD}(\mathit{\Sigma}, \mathrm{E}) \to (P, \mathrm{D})$, we can restrict it to a morphism of signatures $\mathit{\Phi}|_{\mathit{\Sigma}} \colon \mathit{\Sigma} \to P$. We now prove that it defines a morphism of specifications $\widehat{\mathit{\Phi}} \colon (\mathit{\Sigma}, \mathrm{E}) \to \mathrm{DCD}(P, \mathrm{D})$. By assumption, for each element $m \in \mathrm{E}^{\mathsf{pty}}_{\rho,\omega}$ in E, the cell symbols φ_m and ψ_m in $\mathrm{CD}(\mathit{\Sigma}, \mathrm{E})$ are sent to some cell symbols χ_m and λ_m in P. Since $\mathrm{CD}(\mathit{\Sigma}, \mathrm{E})$ can derive the equality judgments (5.2), D derives the equality judgments

$$x : \sigma^{\mathit{\Phi}} \fatsemi y : \tau^{\mathit{\Phi}} \mid a : \rho^{\mathit{\Phi}} \vdash \lambda_m\{\chi_m\{a\}\} \equiv a : \rho^{\mathit{\Phi}} \quad \text{and} \quad x : \sigma^{\mathit{\Phi}} \fatsemi y : \tau^{\mathit{\Phi}} \mid b : \omega^{\mathit{\Phi}} \vdash \chi_m\{\lambda_m\{b\}\} \equiv b : \omega^{\mathit{\Phi}}.$$

We define the function $\widehat{\mathit{\Phi}} \colon \mathrm{E}^{\mathsf{pty}}_{\rho,\omega} \to \mathrm{D}^{\cong}_{\rho^{\mathit{\Phi}},\omega^{\mathit{\Phi}}}$ to send m to the pair (χ_m, λ_m). Crude translations of the equality judgments in E are translated by $\mathrm{CD}(\mathit{\Phi})$ to the equality judgments that are derivable from D in $\mathrm{FVDblTT}^{\Diamond}$ by assumption. Hence, in FVDblTT, we can simulate the derivation of the equality judgments in E translated by Φ through the de-crude translations of the equality judgments in D in FVDblTT.[6] Therefore, $\widehat{\Phi}$ is a morphism of specifications, and it is straightforward to check that $\upsilon_{(P,\mathrm{D})} \circ \mathrm{CD}(\widehat{\Phi})$ is equal to Φ.

$$\begin{tikzcd}
\mathrm{CD}(\Sigma,\mathrm{E}) \arrow[d, "\mathrm{CD}(\Phi|_{\Sigma})"'] \arrow[rd, "\Phi"] \arrow[rd, phantom, "/\!/" description] & \\
\mathrm{CD}(\mathrm{DCD}(P,\mathrm{D})) \arrow[r, "\upsilon_{(P,\mathrm{D})}"'] & (P,\mathrm{D})
\end{tikzcd} \quad \text{in } \mathbf{Speci}^{\diamondsuit}.$$

To see that $\$^{\diamondsuit}(\upsilon_{(P,\mathrm{D})})$ is an equivalence, we confer Lemma 2.5. The equivalence on the tight part is apparent since $\upsilon_{(P,\mathrm{D})}$ does not change anything on types and terms. Next, for each loose arrow in $\$^{\diamondsuit}(\mathrm{CD}(\mathrm{DCD}(P,\mathrm{D})))$, we can find a corresponding loose arrow in $\$^{\diamondsuit}(\mathrm{CD}(\mathrm{DCD}(P,\mathrm{D})))$ by taking the protype with precisely the same presentation. Finally, when fixing a frame, the function on globular cells defined by $\upsilon_{(P,\mathrm{D})}$ is to send proterm judgments with the additional cell symbols $\varphi_{(\vartheta,\varsigma)}$ and $\psi_{(\vartheta,\varsigma)}$ to the proterm judgments without them by replacing the cell symbols with κ_{ϑ} and κ_{ς}. The surjectiveness is checked similarly to the above argument. We can also see the injectiveness up to derivable equality by induction on the construction of the proterms using the fact that from the crude translations of the equality judgments (5.3), the equalities $\varphi_{(\vartheta,\varsigma)}(a) \equiv \vartheta(a)$ and $\psi_{(\vartheta,\varsigma)}(b) \equiv \varsigma(b)$ are derivable from $\mathrm{CD}(\mathrm{DCD}(P,\mathrm{D}))$.

The full faithfulness follows from the fact that the 2-cells in **Speci** and $\diamondsuit$-**Speci** are defined as the transformations of their syntactic VDCs, and $\$^{\diamondsuit}(\upsilon_{(P,\mathrm{D})})$ is an equivalence. ☐

We obtain the main theorem by combining the (relative) biadjunctions Propositions 5.16 and 5.24,

Theorem 5.25 (Restatement of Theorem 5.8). Let $\$ \colon \mathbf{Speci} \to \mathbf{FibVDbl}_{\mathbf{cart}}$ be the composite $\$^{\diamondsuit} \circ \mathrm{CD}$. The composite $\mathrm{DCD} \circ \mathrm{Sp}^{\diamondsuit}$ assigns the associated specification in the sense of Definition 5.7 to a CFVDC, and this 2-functor Sp and \$ form a biadjunction

$$\begin{tikzcd}
\mathbf{Speci} \arrow[r, shift left, "\$"] \arrow[r, phantom, "\perp"] & \mathbf{FibVDbl}_{\mathbf{cart}} \arrow[l, shift left, "\mathrm{Sp}"]
\end{tikzcd}, \quad \text{i.e.,} \quad
\begin{tikzcd}
\mathbf{Speci} \arrow[r, "\mathrm{CD}"] & \mathbf{Speci}^{\diamondsuit} \arrow[r, "\$^{\diamondsuit}(-)"] \arrow[d, phantom, "\perp"] & \mathbf{FibVDbl}_{\mathbf{cart}} \arrow[ld, "\mathrm{Sp}^{\diamondsuit}"] \\
& \mathbf{Speci}^{\diamondsuit}_{\mathsf{sat}} \arrow[lu, "\mathrm{DCD}"] &
\end{tikzcd}.$$

The right adjoint Sp is a local equivalence, that is, $\mathrm{Sp} \colon \mathbf{FibVDbl}_{\mathbf{cart}}(\mathbb{D}, \mathbb{D}') \to \mathbf{Speci}(\mathrm{Sp}(\mathbb{D}), \mathrm{Sp}(\mathbb{D}'))$ is an equivalence for any pair $\mathbb{D}, \mathbb{D}'$.

Proof. Through Propositions 5.16 and 5.24, the expected biadjunction follows from the general theory of relative biadjunctions. Explicitly, for a specification S and a CFVDC $\mathbb{D}$,

$$\begin{aligned}
\mathbf{FibVDbl}_{\mathbf{cart}}\left(\$^{\diamondsuit}\left(\mathrm{CD}(\mathrm{S})\right), \mathbb{D}\right) &\simeq \mathbf{Speci}^{\diamondsuit}\left(\mathrm{CD}(\mathrm{S}), \mathrm{Sp}^{\diamondsuit}(\mathbb{D})\right) && (\textit{by Proposition } 5.16) \\
&\simeq \mathbf{Speci}\left(\mathrm{S}, \mathrm{DCD}(\mathrm{Sp}^{\diamondsuit}(\mathbb{D}))\right) && (\textit{by Proposition } 5.24)
\end{aligned}$$

The local equivalence of Sp follows from the fact that the counit of the biadjunction is pointwise an equivalence because of Propositions 5.16 and 5.24. ☐

The unit of a syntax-semantics adjunction is called a term model in the context of type theory.

Definition 5.26. A ***term model*** of a specification (Σ, E) in a CFVDC is a canonical Σ-structure in the syntactic VDC $\$(\Sigma, \mathrm{E})$, defined by regarding symbols σ, $f \colon \sigma \to \tau$, $\rho \colon \sigma \nrightarrow \tau$, and $\kappa \colon \rho_1 \fatsemi \dots \fatsemi \rho_n \Rightarrow \omega$ as an object $x : \sigma$, a tight arrow $f(x)$, a loose arrow $\rho(x \fatsemi y)$, and a globular cell $\kappa\llparenthesis a_1 \fatsemi \dots \fatsemi a_n \rrparenthesis$ in $\$(\Sigma, \mathrm{E})$, respectively. ■

When we define the 2-category of models of a specification (Σ, E) to be the comma 2-category $(\Sigma, \mathrm{E}) \downarrow \mathrm{Sp}(-)$, the theorem above implies that the term model is bi-initial in this 2-category.

From the biadjunction, we can deduce the soundness and completeness theorems for the type theory.

Corollary 5.27 (Soundness and Completeness). Let (Σ, E) be a specification.

(i) Let $\mathcal{M}$ be its model in a CFVDC $\mathbb{D}$, that is, a morphism $\mathcal{M} \colon (\Sigma, \mathrm{E}) \to \mathrm{Sp}(\mathbb{D})$. An equality judgment J is valid in $\mathcal{M}$ whenever J is derivable from (Σ, E). If a protype isomorphism judgment $\Upsilon : \alpha \cong \beta$ is derivable from (Σ, E), then $\mathcal{M}(\alpha) \cong \mathcal{M}(\beta)$ holds.

---

[6] See (iii) in Appendix B.2.

(ii) If an equality judgment J is valid in every model of (Σ, E) in a CFVDC $\mathbb{D}$, then J is derivable from (Σ, E).

Proof. If J is derivable from (Σ, E), then the term model $\mathbb{S}(\Sigma, \mathrm{E})$ validates J. The Σ-structure $\mathcal{M}$ is isomorphic to the term model composed with the morphism $\mathbb{S}(\Sigma, \mathrm{E}) \to \mathbb{D}$ induced by $\mathcal{M}$, and hence $\mathcal{M}$ validates J. Take the term model of (Σ, E), and we will have the expected result. ☐

We end this section by discussing the extension of the results to the enhanced version of the type theory. In Section 4, we have listed the protype constructors $\nrightarrow, \odot, \triangleleft, \triangleright$ and the type constructor $\{\!|\,|\!\}$. One can make the syntactic VDCs and the biadjunctions work for each enhanced version of the type theory by modifying the specifications and other constructions accordingly. We briefly describe how to do this. Let $\mathfrak{T}$ be a subset of the set $\{\nrightarrow, \odot, \triangleleft, \triangleright, \{\!|\,|\!\}\}$. We write $\mathrm{FVDblTT}_{\mathfrak{T}}$ for the type theory with the constructors in $\mathfrak{T}$ and $\mathbf{FibVDbl}_{\mathfrak{T},\mathbf{cart}}$ for the 2-category of cartesian objects in $\mathrm{FVDblTT}_{\mathfrak{T}}$, the 2-category of FVDCs with the structures corresponding to the constructors in $\mathfrak{T}$. We define the 2-category $\mathbf{Speci}_{\mathfrak{T}}$ and $\Diamond$-$\mathbf{Speci}_{\mathfrak{T}}$ of specifications and crude specifications for $\mathrm{FVDblTT}_{\mathfrak{T}}$ and $\mathrm{FVDblTT}^{\Diamond}_{\mathfrak{T}}$ in the same way as $\mathbf{Speci}$, the only difference being that the equality judgments and the isomorphism judgments can include the constructors in $\mathfrak{T}$. Rules for the constructors in $\mathfrak{T}$ are added to the original ones to admit a biadjunction between $\Diamond$-$\mathbf{Speci}_{\mathfrak{T}}$ and $\mathbf{FibVDbl}_{\mathfrak{T},\mathbf{cart}}$, see Appendices A and B.3. The relative biadjunction between $\mathbf{Speci}_{\mathfrak{T}}$ and $\Diamond$-$\mathbf{Speci}_{\mathfrak{T}}$ is also constructed in the same way as Proposition 5.24, and the biadjunction between $\mathbf{Speci}_{\mathfrak{T}}$ and $\mathbf{FibVDbl}_{\mathfrak{T},\mathbf{cart}}$ is obtained as in Theorem 5.25.

6. Related and Future Work

In this section, we will discuss how FVDblTT, in comparison with New and Licata's "Virtual Equipment Type Theory" (VETT) [NL23], stands out with its practicality and 2-categorical accuracy in the semantics. It is protype isomorphism judgments that achieve the practicality, and the 2-categorical accuracy is what we have developed in Section 5.

The most significant difference resides in protype isomorphism judgments in FVDblTT. The equality of sets in VETT does not incorporate isomorphisms into the equational theory. VETT proves isomorphisms of sets (profunctors) by constructing mutual inverses meta theoretically but cannot directly reason about them inside the type theory. FVDblTT directly handles protype isomorphisms inside the type theory and shows what can be done in a VDC as it is, which simulates reasoning on isomorphisms in category theory. The difference comes partly from the presentation of profunctors. Profunctors in VETT look like $\phi : \lambda\alpha; \beta.R(\alpha \mathbin{⨾} \beta)$, and "$\lambda$" represents quantification that respects the naturality of ϕ. Although this kind of presentation suits how one argues about profunctors in ordinary category theory, it does not reflect how one formally develops category theory in a VDC. In short, VETT is a formal language for category theory that can treat a wide range of category theories, while FVDblTT is a formal language for formal category theory.

The semantics of VETT are given in a *split fibrational virtual equipment*, where all the restrictions are chosen in a compatible way. If one considers a complete set of connectives given in this paper such as composition or extensions, the stage of the semantics requires many choices and compatibility conditions, which are sometimes complicated to handle. On the other hand, the semantics of FVDblTT is given in a CFVDC with no restrictions chosen. We naively interpret the entities of the type theory, which is justified by allowing the interpretation to be up to equivalence in the 2-category of VDCs. In practice, this enables us to take any VDC with additional (non-split) structures that we need to interpret the type theory without worrying about how they are chosen. On the theoretical side, a syntax-semantics adjunction is, in general, established concerning 2-dimensional structures since the interpretation of the type theory is given only up to isomorphism. The author does not know why the initiality of the syntactic model in VETT does not require 2-dimensional care in [NL23]. In the context of Martin-Löf type theory, for example, similar problems called the coherence problems have been studied in [Cur93, Hof95, CD14]. This paper is intended to give a 2-categorically accurate semantics of a type theory for formal category theory. In particular, we pay special attention to how two constructors in the type theory should be interrelated to each other from the perspective of 2-categories of VDCs with additional structures, as discussed in Appendix A, providing persuasive evidence that the type theory is a beneficial language both as an internal language for VDCs and as a language for formal category theory in VDCs.

It is fair to mention what cannot be done in FVDblTT now. The type theory lacks the meta-level kinds as in VETT, meaning that we cannot deal with categories or functors using polymorphism in the type theory. On the other hand, VETT has different type-theoretic entities corresponding to the hierarchy of abstractness. It has *categories*, *sets*, and meta-level entities called *types*, all with equational theory. The distinction between categories in VETT and types in FVDblTT is that the former has the equational theory as elements of a meta-level type "Cat" while the latter does not. When we consider the semantics of these type theories, it makes a significant difference that VETT requires VDCs indexed by a category with families (VETT judgmental models) as a model, while FVDblTT merely requires a single VDC.

	FVDblTT in this paper	**VETT in [NL23]**
Type-theoretic entities	types, terms, protypes, proterms	categories, sets, types
Isomorphic reasoning	✓	×
Semantics	cartesian fibrational VDCs	indexed split-fibrational virtual equipments
Syntax-semantics duality	biadjunction	initiality (without 2-dimensional care)
Polymorphism	×	✓

TABLE 4. Comparison between FVDblTT and VETT

There are several directions for future work. First, we would like to extend the type theory FVDblTT to include *augmented* virtual double categories [Kou20, Kou24]. The latter conceptualizes the notion of a Kan extension and a Yoneda embedding inside this framework, and develops a formal category theory more flexibly than the original virtual double categories. It would be interesting to investigate how category theory can be formalized in the extended type theory. Second, the dependent version of the type theory FVDblTT should be developed. There are several studies on directed type theory [LH11, Nor19, ANv23], and those are all based on dependent types. One of the primary objectives of those studies is to obtain a substantial type theory for higher categories as Martin-Löf type theory is for higher groupoids. The dependent version of the type theory FVDblTT might offer another candidate for this purpose. Finally, we are interested in the relationship between the type theory FVDblTT and other type theories or calculi for relations. In particular, we are interested in the connection to diagrammatic calculi for relations such as the one in [BPS17, BDGHS24], or more directly, the string diagrams for double categories [Mye18]. They may be understood as a string diagrammatic presentation of the type theory FVDblTT. We hope to explore these connections in future work.

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Appendix A. Cartesianness of Structured Virtual Double Categories

We provide rationale for the rules in appendix B.3, by unpacking the cartesianness of virtual double categories with structures.

First, we provide a general lemma on the cartesianness in the 2-category of some structured objects in a 2-category.

Lemma A.1. Let $\mathbf{B}, \mathbf{B}'$ be 2-categories with finite products $(\mathbb{1}, \otimes)$, and $|-|\colon \mathbf{B}' \to \mathbf{B}$ be a 2-functor preserving finite products and locally full-inclusion, *i.e.*, injective on 1-cells and bijective on 2-cells. For an object x of $\mathbf{B}'$ to be cartesian, it is necessary and sufficient that $|x|$ is cartesian in $\mathbf{B}$ and that the 1-cells $1\colon \mathbb{1} \to |x|$ and $\times\colon |x| \otimes |x| \to |x|$ right adjoint to the canonical 1-cells are essentially in the image of $|-|$.

Moreover, for a 1-cell $f\colon x \to y$ of $\mathbf{B}'$ where x and y are cartesian in $\mathbf{B}'$, f is cartesian in $\mathbf{B}'$ if and only if $|f|$ is cartesian in $\mathbf{B}$.

Proof. The necessity of the first condition follows from the fact that any 2-functor preserves adjunctions, that right adjoints are unique up to isomorphism, and that $|-|$ preserves finite products. Since $|-|$ is locally fully faithful, it also reflects units, counits, and the triangle identities with respect to the adjunctions, and hence the sufficiency of the first condition follows.

The necessity of the second condition is again immediate from the fact that $|-|$ preserves finite products. The sufficiency follws from the fact that $|-|$ is locally fully faithful, in particular, reflects isomorphisms. ☐

Proposition A.2. Let **FibUVDbl** be the locally-full sub-2-category of **FibVDbl** spanned by the FVDCs with units and functors preserving units. Then, a FVDC $\mathbb{D}$ with units is cartesian in **FibUVDbl** if and only if

(i) $\mathbb{D}$ is a cartesian FVDC,
(ii) $U_1 \cong \top_{1,1}$ in $\mathbb{D}(1,1)$ canonically, and
(iii) for any $I, J \in \mathbb{D}$, $U_{I,J} \cong U_I \times U_J$ canonically in $\mathbb{D}(I \times J, I \times J)$.

Proof. By Lemma A.1, $\mathbb{D}$ is cartesian as a unital FVDC if and only if it is cartesian as a FVDC and the 1-cells $1\colon \mathbb{1} \to \mathbb{D}$ and $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ are in **FibUVDbl**. The first condition is equivalent to *(ii)* since it sends the only loose arrow in $\mathbb{1}$, which is the unit loose arrow, to $\top_{1,1}$. The second condition is equivalent to *(iii)* since the unit loose arrow of (I, J) in $\mathbb{D}(I \times J, I \times J)$ is (δ_I, δ_J), which is sent to $\delta_I \times \delta_J$ in $\mathbb{D}(I \times J, I \times J)$. ☐

The key idea is that in the virtual double categories $\mathbb{D} \times \mathbb{D}$ and $\mathbb{1}$, the unit loose arrows are given pointwise by the unit loose arrows of $\mathbb{D}$. We can discuss the cartesianness of some classes of FVDCs in parallel with the above proposition.

Proposition A.3. Let **FibCVDbl** be the locally-full sub-2-category of **FibVDbl** spanned by the FVDCs with composites of sequences of loose arrows of positive length and functors preserving those composites. A VDC $\mathbb{D}$ in **FibCVDbl** is cartesian in this 2-category if and only if

(i) $\mathbb{D}$ is a cartesian FVDC,
(ii) $\top_{1,1} \odot \cdots \odot \top_{1,1} \cong \top_{1,1}$ canonically in $\mathbb{D}(1,1)$, and
(iii) for any paths of positive length

$$\begin{tikzcd} I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] & I_1 \arrow[r, "\shortmid"{marking}] & \dots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n & \text{and} & J_0 \arrow[r, "\beta_1", "\shortmid"{marking}] & J_1 \arrow[r, "\shortmid"{marking}] & \dots \arrow[r, "\beta_n", "\shortmid"{marking}] & J_n \end{tikzcd}$$

in $\mathbb{D}$, we have

$$(\alpha_1 \odot \cdots \odot \alpha_n) \times (\beta_1 \odot \cdots \odot \beta_n) \cong (\alpha_1 \times \beta_1) \odot \cdots \odot (\alpha_n \times \beta_n)$$

canonically in $\mathbb{D}(I_0 \times J_0, I_n \times J_n)$.

Proposition A.4. Let **FibREVDbl** be the locally-full sub-2-category of **FibVDbl** spanned by the FVDCs with right extensions and functors preserving right extensions. A VDC $\mathbb{D}$ in **FibREVDbl** is cartesian in this 2-category if and only if

(i) $\mathbb{D}$ is a cartesian FVDC,
(ii) $\top_{1,1} \triangleright \top_{1,1} \cong \top_{1,1}$ canonically in $\mathbb{D}(1,1)$, and
(iii) for any quadruples of loose arrows

$$\begin{tikzcd} I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[rr, bend right, "\alpha_2"', "\shortmid"{marking}] & I_1 & I_2 & \text{and} & J_0 \arrow[r, "\beta_1", "\shortmid"{marking}] \arrow[rr, bend right, "\beta_2"', "\shortmid"{marking}] & J_1 & J_2 \end{tikzcd}$$

in $\mathbb{D}$, we have

$$(\alpha_1 \triangleright \alpha_2) \times (\beta_1 \triangleright \beta_2) \cong (\alpha_1 \times \beta_1) \triangleright (\alpha_2 \times \beta_2)$$

canonically in $\mathbb{D}(I_1 \times J_1, I_2 \times J_2)$.

Appendix B. Details on type theory

B.1. The rules for equational and isomorphism theory

In this section, we explicitly provide the rules for the equational theory of terms and proterms. To begin with, we introduce some notations. Basically, we use overlines $\overline{\bullet}$ to denote horizontal concatenation of items, for example, $\nu\{\overline{\mu}/\overline{b}\}$ means $\nu\{\mu_1/b_1 : \beta_1 \fatsemi \dots \fatsemi \mu_n/b_n : \beta_n\}$. Other notations are given in Figure 7.

In addition, although the presentation of replacing a variable with a term does not make sense since we follow the convention of explicit substitution, we still use it just as an abbreviation for the sake of readability. For example, the items on the left-hand side in the following are abbreviations for the items on the right-hand side.

$$t(s_0, \dots, s_n) \doteq t[s_0/x_0, \dots, s_n/x_n]$$

Abbreviation	**Given Data**	**Meaning**
$\overline{\Delta}$	$\Delta = (\Delta_0, \dots, \Delta_n)$	$\Delta_0 \fatsemi \dots \fatsemi \Delta_n$
$\overline{\Gamma} = \overline{\Gamma_{\underline{i}, \underline{j}}}$	$\Gamma = (\Gamma_{i,j})_{(i,j) \in P}$ (The index set P is the set of pairs (i,j) such that $0 \leq i \leq n$ and $0 \leq j \leq m_i - 1$ or $(i,j) = (n, m_n)$ for a given $(m_i)_{i=0}^{n}$.)	$\Gamma_{0,0} \fatsemi \Gamma_{0,1} \fatsemi \dots \fatsemi \Gamma_{0,n_0-1} \fatsemi \Gamma_{1,0} \fatsemi \dots \fatsemi \Gamma_{m,n_m}$
$\overline{\Gamma_i} = \overline{\Gamma_{i, \underline{j}}}$		$\Gamma_{i,0} \fatsemi \Gamma_{i,1} \fatsemi \dots \fatsemi \Gamma_{i,n_i}$
$\widetilde{\Gamma} = \widetilde{\Gamma_{\underline{i}, \tilde{j}}}$		$\Gamma_{0,0} \fatsemi \Gamma_{1,0} \fatsemi \dots \fatsemi \Gamma_{m,0} \fatsemi \Gamma_{m,n_m}$
$\overline{A_{\underline{i}}}$	$A_i = \alpha_{i,0} \fatsemi \dots \fatsemi \alpha_{i,n_i}$ for $1 \leq i \leq m$ $\alpha_{i,n_i} = \alpha_{i+1,0}$ for $1 \leq i \leq m-1$	$\alpha_{1,0} \fatsemi \dots \fatsemi \alpha_{1,n_1-1} \fatsemi \alpha_{2,0} \fatsemi \dots \fatsemi \alpha_{m,n_m}$

FIGURE 7. Abbreviations of contexts

$$\alpha(s_0, \dots, s_n \fatsemi t_0, \dots, t_m) \doteq \alpha[s_0/x_0, \dots, s_n/x_n \fatsemi t_0/y_0, \dots, t_m/y_m]$$
$$\nu(\overrightarrow{s_0} \fatsemi \dots \fatsemi \overrightarrow{s_n}) \doteq \nu[\overrightarrow{s_0}/\overrightarrow{x_0} \fatsemi \dots \fatsemi \overrightarrow{s_n}/\overrightarrow{x_n}]$$
$$\nu\{\mu_1 \fatsemi \dots \fatsemi \mu_n\} \doteq \nu\{\mu_1/b_1, \dots, \mu_n/b_n\}$$

The rules for protype isomorphisms are given as follows.

$$\dfrac{\Gamma \fatsemi \Delta \vdash \alpha \text{ protype}}{\Gamma \fatsemi \Delta \vdash \mathsf{id}_\alpha : \alpha \cong \alpha} \qquad \dfrac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta}{\Gamma \fatsemi \Delta \vdash \Upsilon^{-1} : \beta \cong \alpha} \qquad \dfrac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta \qquad \Gamma \fatsemi \Delta \vdash \Omega : \beta \cong \gamma}{\Gamma \fatsemi \Delta \vdash \Omega \circ \Upsilon : \alpha \cong \gamma}$$

$$\dfrac{\Gamma \fatsemi \Delta \vdash \alpha \text{ protype}}{\Gamma \fatsemi \Delta \vdash \mathsf{rest\text{-}ide} : \alpha[\Gamma/\Gamma \fatsemi \Delta/\Delta] \cong \alpha} \qquad \dfrac{\Gamma'' \vdash S' \,/\, \Gamma' \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta'' \vdash T' \,/\, \Delta' \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \text{ protype}}{\Gamma'' \fatsemi \Delta'' \vdash \mathsf{rest\text{-}iter} : (\alpha[S/\Gamma \fatsemi T/\Delta])\,[S'/\Gamma' \fatsemi T'/\Delta'] \cong \alpha\left[S[S'/\Gamma']/\Gamma \fatsemi T[T'/\Delta']/\Delta\right]}$$

$$\dfrac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \text{ protype} \qquad \Gamma \fatsemi \Delta \vdash \beta \text{ protype}}{\Gamma' \fatsemi \Delta' \vdash \mathsf{rest}_\wedge : (\alpha \wedge \beta)\,[S/\Gamma \fatsemi T/\Delta] \cong \alpha[S/\Gamma \fatsemi T/\Delta] \wedge \beta[S/\Gamma \fatsemi T/\Delta]} \qquad \dfrac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \vdash \mathsf{rest}_\top : \top[S/\Gamma \fatsemi T/\Delta] \cong \top}$$

$$\dfrac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \text{ protype}}{\Gamma' \fatsemi \Delta' \vdash \mathsf{repl}_{S_0, S_1 \fatsemi T_0, T_1} : \alpha[S_0/\Gamma \fatsemi T_0/\Delta] \cong \alpha[S_1/\Gamma \fatsemi T_1/\Delta]}$$

$$\dfrac{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mu\{a\} : \beta \qquad \Gamma \fatsemi \Delta \mid b : \beta \vdash \nu\{b\} : \alpha \qquad \Gamma \fatsemi \Delta \mid b : \beta \vdash \mu\{\nu\{b\}\} \equiv b : \beta \qquad \Gamma \fatsemi \Delta \mid a : \alpha \vdash \nu\{\mu\{a\}\} \equiv a : \alpha}{\Gamma \fatsemi \Delta \vdash (\!|\mu, \nu|\!) : \alpha \cong \beta}$$

The rules for the equational theory of proterms are given as follows. It is worth noting that there are some rules for the conversion $\mathsf{tr}_\Upsilon\{a\}$ to guarantee that the introduced protype isomorphisms behave as expected. For example, we have the rule $(\!|\mu, \nu|\!)\{a\} \equiv \mu\{a\}$ for the proterm μ and ν that are mutually inverse to each other, and the rule $\Upsilon^{-1}\{\Upsilon\{a\}\} \equiv a$ for a protype isomorphism Υ. From these rules, one can derive that the inverse of $(\!|\mu, \nu|\!)$ also has the expected behavior: $(\!|\mu, \nu|\!)^{-1}\{b\} \equiv (\!|\nu, \mu|\!)^{-1}\left\{\mu\left\{\nu\{b\}\right\}\right\} \equiv (\!|\nu, \mu|\!)^{-1}\left\{(\!|\nu, \mu|\!)\left\{\nu\{b\}\right\}\right\} \equiv \nu\{b\}$.

Equational theory of proterms

$$\dfrac{\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu : \beta}{\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash b\{\mu\} \equiv \mu : \beta} \qquad \dfrac{\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu : \beta}{\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu\{a_1 \fatsemi \dots \fatsemi a_n\} \equiv \mu : \beta}$$

$$\dfrac{\overline{\Gamma} \mid A \vdash \mu : \alpha \qquad \overline{\Gamma} \mid A \vdash \nu : \beta}{\overline{\Gamma} \mid A \vdash \pi_0\{\langle \mu, \nu \rangle\} \equiv \mu : \alpha} \qquad \dfrac{\overline{\Gamma} \mid A \vdash \mu : \alpha \qquad \overline{\Gamma} \mid A \vdash \nu : \beta}{\overline{\Gamma} \mid A \vdash \pi_1\{\langle \mu, \nu \rangle\} \equiv \nu : \beta} \qquad \dfrac{\overline{\Gamma} \mid A \vdash \mu : \alpha \wedge \beta}{\overline{\Gamma} \mid A \vdash \langle \pi_0\{\mu\}, \pi_1\{\mu\} \rangle \equiv \mu : \alpha \wedge \beta} \qquad \dfrac{\overline{\Gamma} \mid A \vdash \mu : \top}{\overline{\Gamma} \mid A \vdash \mu \equiv \langle\ \rangle : \top}$$

$$\dfrac{\overline{\Gamma_i} \mid A_i \vdash \mu_i : \beta_i \quad (i = 1, \dots, n) \qquad \widetilde{\Gamma} \mid \beta_1, \dots, \beta_n \vdash \nu_j : \gamma_j \quad (j = 1, 2)}{\overline{\Gamma} \mid \overline{A_{\underline{i}}} \vdash \langle \nu_1, \nu_2 \rangle\{\overline{\mu}/\overline{b}\} \equiv \langle \nu_1\{\overline{\mu}/\overline{b}\}, \nu_2\{\overline{\mu}/\overline{b}\} \rangle : \gamma_1 \wedge \gamma_2}$$

$$\dfrac{\overline{\Gamma_i} \mid A_i \vdash \mu_i : \beta_i \quad (i = 1, \dots, n) \qquad \widetilde{\Gamma} \mid \beta_1, \dots, \beta_n \vdash \nu : \gamma_1 \wedge \gamma_2}{\overline{\Gamma} \mid \overline{A_{\underline{i}}} \vdash \pi_0\{\nu\{\overline{\mu}/\overline{b}\}\} \equiv \pi_0\{\nu\}\{\overline{\mu}/\overline{b}\} : \gamma_1} \qquad \dfrac{\overline{\Gamma_i} \mid A_i \vdash \mu_i : \beta_i \quad (i = 1, \dots, n) \qquad \widetilde{\Gamma} \mid \beta_1, \dots, \beta_n \vdash \nu : \gamma_1 \wedge \gamma_2}{\overline{\Gamma} \mid \overline{A_{\underline{i}}} \vdash \pi_1\{\nu\{\overline{\mu}/\overline{b}\}\} \equiv \pi_1\{\nu\}\{\overline{\mu}/\overline{b}\} : \gamma_2}$$

$$\dfrac{\begin{array}{c} \overline{\Gamma_{i,j,\underline{k}}} \mid A_{i,j} \vdash \mu_{i,j} : \beta_{i,j} \quad (i = 1, \dots, m,\ j = 1, \dots, n_i) \\ \widetilde{\Gamma_{i,\underline{j},\tilde{k}}} \mid b_{i,1} : \beta_{i,1} \fatsemi \dots b_{i,n_i} : \beta_{i,n_i} \vdash \nu_i : \gamma_i \quad (i = 1, \dots, m) \qquad \widetilde{\Gamma_{\underline{i},\tilde{j},\tilde{k}}} \mid c_1 : \gamma_1 \fatsemi \dots \fatsemi c_m : \gamma_m \vdash \lambda : \delta \end{array}}{\overline{\Gamma} \mid \overline{A} \vdash (\lambda\{\overline{\nu}/\overline{c}\})\,\{\overline{\mu}/\overline{b} : \overline{\beta}\} \equiv \lambda\left\{\overline{\nu_{\underline{i}}\{\overline{\mu_{i,\underline{j}}}/\overline{b_{i,\underline{j}}}\}} \Big/ \overline{c}\right\} : \delta}$$

$$\dfrac{\Gamma_i \vdash S_i \,/\, \Delta_i \ (i = 1, \dots, n) \qquad \overline{\Delta} \mid A \vdash \mu : \beta \qquad \overline{\Delta} \mid A \vdash \nu : \gamma}{\overline{\Gamma} \mid A[\overline{S}/\overline{\Delta}] \vdash \langle \mu, \nu \rangle[S_0/\Delta_0 \fatsemi \dots \fatsemi S_n/\Delta_n] \equiv \left\langle \mu[\overline{S}/\overline{\Delta}], \nu[\overline{S}/\overline{\Delta}] \right\rangle : (\beta \wedge \gamma)\,[S_0/\Delta_0 \fatsemi S_n/\Delta_n]}$$

$$\dfrac{\Gamma_i \vdash S_i \,/\, \Delta_i \ (i = 1, \dots, n) \qquad \overline{\Delta} \mid A \vdash \mu : \beta \wedge \gamma}{\overline{\Gamma} \mid A[\overline{S}/\overline{\Delta}] \vdash \pi_0\{\mu[\overline{S}/\overline{\Delta}]\} \equiv \pi_0\{\mu\}[\overline{S}/\overline{\Delta}] : \beta[S_0/\Delta_0 \fatsemi S_n/\Delta_n]} \qquad \dfrac{\Gamma_i \vdash S_i \,/\, \Delta_i \ (i = 1, \dots, n) \qquad \overline{\Delta} \mid A \vdash \mu : \beta \wedge \gamma}{\overline{\Gamma} \mid A[\overline{S}/\overline{\Delta}] \vdash \pi_1\{\mu[\overline{S}/\overline{\Delta}]\} \equiv \pi_1\{\mu\}[\overline{S}/\overline{\Delta}] : \gamma[S_0/\Delta_0 \fatsemi S_n/\Delta_n]}$$

$$\frac{\Gamma_i \vdash S_i \,/\, \Delta_i \ (i = 1, \dots, n) \qquad \Delta_i \vdash T_i \,/\, \Theta_i \ (i = 1, \dots, n) \qquad \overline{\Theta} \mid A \vdash \mu : \beta}{\overline{\Gamma} \mid A[\overline{T}/\overline{\Theta}][\overline{S}/\overline{\Delta}] \vdash \left(\mu[\overline{T}/\overline{\Theta}]\right)[\overline{S}/\overline{\Delta}] \equiv \mu\left[\overline{T}[\overline{S}/\overline{\Delta}]/\overline{\Theta}\right] : \beta[T_0/\Theta_0 \fatsemi T_n/\Theta_n][S_0/\Delta_0 \fatsemi S_n/\Delta_n]}$$

$$\frac{\Gamma_{i,j} \vdash S_{i,j} \,/\, \Delta_{i,j} \ (i = 1, \dots, m,\ j = 1, \dots, n_i) \qquad \overline{\Delta_{i,\underline{j}}} \mid A_i \vdash \mu_i : \beta_i \ (i = 1, \dots, m) \qquad \widetilde{\Delta_{\underline{i},\tilde{j}}} \mid b_1 : \beta_1 \fatsemi \dots \fatsemi b_n : \beta_n \vdash \nu : \gamma}{\overline{\Gamma} \mid \overline{A}[\overline{S}/\overline{\Delta}] \vdash \nu\{\overline{\mu}/\overline{b}\}[\overline{S}/\overline{\Delta}] \equiv \nu\left[\widetilde{S_{\underline{i},\tilde{j}}}\middle/\widetilde{\Delta_{\underline{i},\tilde{j}}}\right]\left\{\overline{\mu_{\underline{i}}[\overline{S_{i,\underline{j}}}/\overline{\Delta_{i,\underline{j}}}]}\middle/\overline{b_{\underline{i}}}\right\} : \gamma[S_{0,0}/\Delta_{0,0} \fatsemi S_{m,n_m}/\Delta_{m,n_m}]}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mathsf{id}_\alpha\{a\} \equiv a : \alpha} \qquad \frac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \Upsilon^{-1}\{\Upsilon\{a\}\} \equiv a : \alpha} \qquad \frac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta}{\Gamma \fatsemi \Delta \mid b : \beta \vdash \Upsilon\{\Upsilon^{-1}\{a\}\} \equiv a : \alpha}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta \qquad \Gamma \fatsemi \Delta \vdash \Omega : \beta \cong \gamma}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash (\Omega \circ \Upsilon)\{a\} \equiv \Omega\{\Upsilon\{a\}\} : \gamma} \qquad \frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{repl}_{S,S \fatsemi T,T}\{a\} \equiv a : \alpha[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha\,[S_0/\Gamma \fatsemi T_0/\Delta] \vdash \mathsf{repl}_{S_0,S_1 \fatsemi T_0,T_1}\{a\} \equiv \mathsf{repl}^{-1}_{S_1,S_0 \fatsemi T_1,T_0}\{a\} : \alpha\,[S_1/\Gamma \fatsemi T_1/\Delta]}$$

$$\frac{\Gamma' \vdash S_0 \,/\, \Gamma \equiv S_1 \,/\, \Gamma \qquad \Gamma' \vdash S_1 \,/\, \Gamma \equiv S_2 \,/\, \Gamma \qquad \Delta' \vdash T_0 \,/\, \Delta \equiv T_1 \,/\, \Delta \qquad \Delta' \vdash T_1 \,/\, \Delta \equiv T_2 \,/\, \Delta \qquad \Gamma' \fatsemi \Delta' \vdash \alpha \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : \alpha\,[S_0/\Gamma \fatsemi T_0/\Delta] \vdash \mathsf{repl}_{S_1,S_2 \fatsemi T_1,T_2}\{\mathsf{repl}_{S_0,S_1 \fatsemi T_0,T_1}\{a\}\} \equiv \mathsf{repl}_{S_0,S_2 \fatsemi T_0,T_2}\{a\} : \alpha\,[S_2/\Gamma \fatsemi T_2/\Delta]}$$

$$\frac{\Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta \qquad \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \qquad \Gamma \fatsemi \Delta \vdash \beta \ \mathsf{protype}}{\Gamma' \fatsemi \Delta' \mid a : (\alpha \wedge \beta)[S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest}_\wedge\{a\} \equiv \langle \pi_0[S/\Gamma \fatsemi T/\Delta], \pi_1[S/\Gamma \fatsemi T/\Delta]\rangle : \alpha[S/\Gamma \fatsemi T/\Delta] \wedge \beta[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \mu\{a\} : \beta \qquad \Gamma \fatsemi \Delta \mid b : \beta \vdash \nu\{b\} : \alpha \qquad \Gamma \fatsemi \Delta \mid b : \beta \vdash \mu\{\nu\{b\}\} \equiv b : \beta \qquad \Gamma \fatsemi \Delta \mid a : \alpha \vdash \nu\{\mu\{a\}\} \equiv a : \alpha}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \llparenthesis \mu, \nu \rrparenthesis\{a\} \equiv \mu\{a\} : \beta}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \alpha[\Gamma/\Gamma \fatsemi \Delta/\Delta][S/\Gamma \fatsemi T/\Delta] \vdash \mathsf{rest\text{-}ide}\{a\}[S/\Gamma \fatsemi T/\Delta] \equiv \mathsf{rest\text{-}iter}\{a\} : \alpha[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \qquad \Gamma' \vdash S \,/\, \Gamma \qquad \Delta' \vdash T \,/\, \Delta}{\Gamma' \fatsemi \Delta' \mid a : \alpha[S/\Gamma \fatsemi T/\Delta][\Gamma/\Gamma \fatsemi \Delta/\Delta] \vdash \mathsf{rest\text{-}ide}\{a\} \equiv \mathsf{rest\text{-}iter}\{a\} : \alpha[S/\Gamma \fatsemi T/\Delta]}$$

$$\frac{\Gamma_{i+1} \vdash S_i \,/\, \Gamma_i \ (i = 0,1,2) \qquad \Delta_{i+1} \vdash T_i \,/\, \Delta_i \ (i = 0,1,2) \qquad \Gamma_0 \fatsemi \Delta_0 \vdash \alpha \ \mathsf{protype}}{\begin{array}{l} \Gamma_3 \fatsemi \Delta_3 \mid a : \alpha[S_0/\Gamma_0 \fatsemi T_0/\Delta_0][S_1/\Gamma_1 \fatsemi T_1/\Delta_1][S_1/\Gamma_2 \fatsemi T_2/\Delta_2] \\ \quad \vdash \mathsf{rest\text{-}iter}\,\{\mathsf{rest\text{-}iter}\{a\}[S_2/\Gamma_2 \fatsemi T_2/\Delta_2]\} \equiv \mathsf{rest\text{-}iter}\,\{\mathsf{rest\text{-}iter}\{a\}\} : \alpha\,[\,S_0[S_1/\Gamma_1][S_2/\Gamma_2]/\,\Gamma_0 \fatsemi\ T_0[T_1/\Delta_1][T_2/\Delta_2]/\,\Delta_0] \end{array}}$$

$$\frac{\Gamma_{i+1} \vdash S_i \,/\, \Gamma_i \ (i = 0,1) \qquad \Delta_{i+1} \vdash T_i \,/\, \Delta_i \ (i = 0,1) \qquad \Gamma_0 \fatsemi \Delta_0 \vdash \alpha \ \mathsf{protype} \qquad \Gamma_0 \fatsemi \Delta_0 \vdash \beta \ \mathsf{protype}}{\begin{array}{l} \Gamma_2 \fatsemi \Delta_2 \mid a : (\alpha \wedge \beta)\,[S_0/\Gamma_0 \fatsemi T_0/\Delta_0][S_1/\Gamma_1 \fatsemi T_1/\Delta_1] \\ \quad \vdash \mathsf{rest}_\wedge\,\{\mathsf{rest\text{-}iter}\{a\}\} \equiv \langle \mathsf{rest\text{-}iter}\{\pi_0\{a\}\}, \mathsf{rest\text{-}iter}\{\pi_1\{a\}\}\rangle\,\{\mathsf{rest}_\wedge\,\{\mathsf{rest}_\wedge\{a\}[S_1/\Gamma_1 \fatsemi T_1/\Delta_1]\}\,/a\} \\ \quad : \alpha\,[\,S_0[S_1/\Gamma_1]/\,\Gamma_0 \fatsemi\ T_0[T_1/\Delta_1]/\,\Delta_0] \wedge \beta\,[\,S_1/\,\Gamma_1 \fatsemi\ T_1/\,\Delta_1] \end{array}}$$

$$\frac{\Gamma_1 \vdash S_0/\Gamma_0 \equiv S'_0/\Gamma_0 \qquad \Delta_1 \vdash T_0/\Delta_0 \equiv T'_0/\Delta_0 \qquad \Gamma_2 \vdash S_1 \,/\, \Gamma_1 \qquad \Delta_2 \vdash T_1 \,/\, \Delta_1}{\begin{array}{l} \Gamma_2 \fatsemi \Delta_2 \mid a : \alpha[S_0/\Gamma_0 \fatsemi T_0/\Delta_0][S_1/\Gamma_1 \fatsemi T_1/\Delta_1] \\ \quad \vdash \mathsf{repl}_{S_0[S_1],S'_0[S_1] \fatsemi T_0[T_1],T'_0[T_1]}\{\mathsf{rest\text{-}iter}\{a\}\} \equiv \mathsf{rest\text{-}iter}\left\{\mathsf{repl}_{S_0,S'_0 \fatsemi T_0,T'_0}\{a\}[S_1/\Gamma_1 \fatsemi T_1/\Delta_1]\right\} : \alpha\left[\,S'_0[S_1/\Gamma_1]\middle/\,\Gamma_0 \fatsemi\ T'_0[T_1/\Delta_1]\middle/\,\Delta_0\right] \end{array}}$$

$$\frac{\Gamma_1 \vdash S_0/\Gamma_0 \qquad \Delta_1 \vdash T_0/\Delta_0 \qquad \Gamma_2 \vdash S_1 \,/\, \Gamma_1 \equiv S'_1/\Gamma_1 \qquad \Delta_2 \vdash T_1 \,/\, \Delta_1 \equiv T'_1/\Delta_1}{\begin{array}{l} \Gamma_2 \fatsemi \Delta_2 \mid a : \alpha[S_0/\Gamma_0 \fatsemi T_0/\Delta_0][S_1/\Gamma_1 \fatsemi T_1/\Delta_1] \\ \quad \vdash \mathsf{repl}_{S_0[S_1],S_0[S'_1] \fatsemi T_0[T_1],T_0[T'_1]}\{\mathsf{rest\text{-}iter}\{a\}\} \equiv \mathsf{rest\text{-}iter}\left\{\mathsf{repl}_{S_1,S'_1 \fatsemi T_1,T'_1}\{a\}\right\} : \alpha\left[\,S_0[S'_1]\middle/\,\Gamma_0 \fatsemi\ T_0[T'_1]\middle/\,\Delta_0\right] \end{array}}$$

$$\frac{\overline{\Gamma} \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_m : \alpha_m \vdash \mu : \beta}{\overline{\Gamma'} \mid a_1 : \alpha_1[\Gamma_0/\Gamma_0 \fatsemi \Gamma_1/\Gamma_1] \fatsemi \dots \fatsemi a_m : \alpha_m[\Gamma_{m-1}/\Gamma_{m-1} \fatsemi \Gamma_m/\Gamma_m] \vdash \mathsf{rest\text{-}ide}\left\{\mu[\overline{\Gamma}/\overline{\Gamma}]\right\} \equiv \mu\{\mathsf{rest\text{-}ide}\{a_1\} \fatsemi \dots \fatsemi \mathsf{rest\text{-}ide}\{a_m\}\} : \beta}$$

$$\frac{\overline{\Gamma} \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_m : \alpha_m \vdash \mu : \beta \qquad \Gamma'_i \vdash S_i \,/\, \Gamma_i \ (i = 0, \dots, m) \qquad \Gamma''_i \vdash S'_i \,/\, \Gamma'_i \ (i = 0, \dots, m)}{\begin{array}{l} \overline{\Gamma''} \mid a_1 : \alpha_1[S_0/\Gamma_0 \fatsemi S_1/\Gamma_1][S'_0/\Gamma'_0 \fatsemi S'_1/\Gamma'_1] \fatsemi \dots \fatsemi a_m : \alpha_m[S_{m-1}/\Gamma_{m-1} \fatsemi S_m/\Gamma_m][S'_{m-1}/\Gamma'_{m-1} \fatsemi S'_m/\Gamma'_m] \\ \quad \vdash \mathsf{rest\text{-}iter}\left\{\mu[\overline{S}/\overline{\Gamma}][\overline{S}'/\overline{\Gamma}']\right\} \equiv \left(\mu\left[\overline{S_i[S'_i/\Gamma'_i]/\Gamma_i}\right]\right)\{\mathsf{rest\text{-}iter}\{a_1\} \fatsemi \dots \fatsemi \mathsf{rest\text{-}iter}\{a_m\}\} : \beta\left[S_0[S'_0/\Gamma'_0] \fatsemi S_1[S'_1/\Gamma'_1]\right] \end{array}}$$

$$\frac{\overline{\Gamma} \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_m : \alpha_m \vdash \mu : \beta \qquad \Gamma'_i \vdash S_i \,/\, \Gamma_i \equiv T_i \,/\, \Gamma_i \ (i = 0, \dots, m)}{\begin{array}{l} \overline{\Gamma'} \mid a_1 : \alpha_1[S_0/\Gamma_0 \fatsemi S_1/\Gamma_1] \fatsemi \dots \fatsemi a_m : \alpha_m[S_{m-1}/\Gamma_{m-1} \fatsemi S_m/\Gamma_m] \\ \quad \vdash \mathsf{repl}_{S_0,T_0 \fatsemi S_m,T_m}\left\{\mu[\overline{S}/\overline{\Gamma}]\right\} \equiv \mu\left[\overline{T}/\overline{\Gamma}\right]\left\{\mathsf{repl}_{S_0,T_0 \fatsemi S_1,T_1}\{a_1\} \fatsemi \dots \fatsemi \mathsf{repl}_{S_{m-1},T_{m-1} \fatsemi S_m,T_m}\{a_m\}\right\} : \beta\,[T_0 \fatsemi T_m] \end{array}}$$

B.2. **Justification of the derivation rules** Justification of the derivation rules of an internal language has two parts: to verify that each step of derivation is valid (soundness) and to ensure that they are sufficient to derive all they should (completeness), both with respect to the semantics. In this paper, we have achived these two parts implicitly along the way, but not thoroughly in the main text. This is because theese sorts of verification always require massive effort of induction, and it is not always easy to present them in a readable manner. For FVDblTT and its semantics in CFVDCs, all we need to justify the rules is the following:

(i) The interpretation of typing judgments in FVDblTT$^\diamondsuit$ is well-defined, and two (pro)terms that are derivably equal in FVDblTT$^\diamondsuit$ are interpreted as equal in CFVDCs.

(ii) When two (pro)terms are derivably equal in FVDblTT, so are their crude translations in FVDblTT$^\diamondsuit$. For a derivable protype isomorphism judgment in FVDblTT, there is a corresponding pair of mutually inverse proterms in FVDblTT$^\diamondsuit$, and they are interpreted as the same loose isomorphism in CFVDCs.
(iii) The syntactic VDC $\mathbb{S}^\diamondsuit(\Sigma, \mathrm{E})$ (Definition 5.10) of a crude specification is indeed a CFVDC.
(iv) When two (pro)terms are derivably equal in FVDblTT$^\diamondsuit$, so are their decrude translations in FVDblTT.

Since FVDblTT$^\diamondsuit$ connects FVDblTT and CFVDCs, the two parts of the justification are split into the four points above. We have footnoted where these ingredients are used implicitly in the main text. We do not provide the full proof, but instead illustrate the idea of the proof with instances.

(ii) and (iv) are straightforward to verify, because the translations and the rules are defined in a way that they satisfy these properties.

What is critical to verify (i) concerns the interpretation of proterms and their equalities. Once you define the interpretation of protypes, at which point you choose restrictions of loose arrows and other data using some universal properties, two derivably equal proterms involving protypes whose interpretations require these choices are interpreted as equal in CFVDCs when you fix the choices. For example, consider the following rule:

$$\frac{\Gamma_{i,j} \vdash S_{i,j} \,/\, \Delta_{i,j}\ (i = 1, \dots, m,\ j = 1, \dots, n_i) \qquad \overline{\Delta_{i,\underline{j}}} \mid A_i \vdash \mu_i : \beta_i\ (i = 1, \dots, m) \qquad \widetilde{\Delta_{\underline{i},j}} \mid b_1 : \beta_1 \fatsemi \dots \fatsemi b_n : \beta_n \vdash \nu : \gamma}{\overline{\Gamma} \mid \overline{A}[\overline{S}/\overline{\Delta}] \vdash \nu\{\overline{\mu}/\overline{b}\}[\overline{S}/\overline{\Delta}] \equiv \nu \left[\overline{\widetilde{S}}/\overline{\widetilde{\Delta}}\right] \left\{ \overline{\mu_{\underline{i}}[\overline{S_{i,\underline{j}}}/\overline{\Delta_{i,\underline{j}}}]} \Big/ \overline{b_{\underline{i}}} \right\} : \gamma[S_{0,0}/\Delta_{0,0} \fatsemi S_{m,n_m}/\Delta_{m,n_m}]}$$

The interpretations of the proterms are shown to be equal using the universal properties of restriction and the following equalities (here we omit the symbol $[\![\cdot]\!]$ for simplicity and the names of the cells are written in red):

$$\begin{tikzcd}
\Gamma_{0,0} \arrow[r, dashed, "\shortmid"{marking}, "\overline{A}[\overline{S}/\overline{\Delta}]"] \arrow[d, equal] \arrow[rd, phantom, "\nu\{\overline{\mu}/\overline{b}\}[\overline{S}/\overline{\Delta}]"] & \Gamma_{m,n_m} \arrow[d, equal] \\
\Gamma_{0,0} \arrow[r, dashed, "\shortmid"{marking}, "\gamma[S_{0,0}/\Delta_{0,0} \fatsemi S_{m,n_m}/\Delta_{m,n_m}]"'] \arrow[d, "S_{0,0}"'] \arrow[rd, phantom, "\mathsf{rest}"] & \Gamma_{m,n_m} \arrow[d, "S_{m,n_m}"] \\
\Delta_{0,0} \arrow[r, "\shortmid"{marking}, "\gamma"'] & \Delta_{m,n_m}
\end{tikzcd}$$

$$\overset{(1)}{=}$$

$$\begin{tikzcd}
\Gamma_{0,0} \arrow[r, "\shortmid"{marking}, "{\alpha_{1,1}[S_{0,1}/\Delta_{0,1} \fatsemi S_{1,0}/\Delta_{1,0}]}"] \arrow[d, "S_{0,0}"'] \arrow[rd, phantom, "\mathsf{rest}"] & \Gamma_{0,1} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "S_{0,1}"'] \arrow[rd, phantom, "\overline{\mathsf{rest}}"] & \Gamma_{1,0} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "S_{1,0}"'] \arrow[rd, phantom, "\overline{\mathsf{rest}}"] & \Gamma_{m,n_m} \arrow[d, "S_{m,n_m}"] \\
\Delta_{0,0} \arrow[r, "\shortmid"{marking}, "\alpha_{1,1}"'] \arrow[d, equal] \arrow[rrd, phantom, "\mu_1"] & \Delta_{0,1} \arrow[r, dashed, "\shortmid"{marking}] & \Delta_{1,0} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\cdots"] & \Delta_{m,n_m} \arrow[d, equal] \\
\Delta_{0,0} \arrow[rr, "\shortmid"{marking}, "\beta_1"'] \arrow[d, equal] \arrow[rrrd, phantom, "\nu"] & & \Delta_{1,0} \arrow[r, dashed, "\shortmid"{marking}] & \Delta_{m,n_m} \arrow[d, equal] \\
\Delta_{0,0} \arrow[rrr, "\shortmid"{marking}, "\gamma"'] & & & \Delta_{m,n_m}
\end{tikzcd}$$

$$\overset{(2)}{=}$$

$$\begin{tikzcd}
\Gamma_{0,0} \arrow[r, "\shortmid"{marking}, "{\alpha_{1,1}[S_{0,1}/\Delta_{0,1} \fatsemi S_{1,0}/\Delta_{1,0}]}"] \arrow[d, equal] \arrow[rrd, phantom, "{\mu_1[\overline{S_{1,\underline{j}}}/\overline{\Delta_{1,\underline{j}}}]}"] & \Gamma_{0,1} \arrow[r, dashed, "\shortmid"{marking}] & \Gamma_{1,0} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\cdots"] & \Gamma_{m,n_m} \arrow[d, equal] \\
\Gamma_{0,0} \arrow[rr, "\shortmid"{marking}, "{\beta_1[S_{0,0}/\Delta_{0,0} \fatsemi S_{1,0}/\Delta_{1,0}]}"'] \arrow[d, "S_{0,0}"'] \arrow[rrd, phantom, "\mathsf{rest}"] & & \Gamma_{1,0} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "S_{1,0}"'] \arrow[rd, phantom, "\overline{\mathsf{rest}}"] & \Gamma_{m,n_m} \arrow[d, "S_{m,n_m}"] \\
\Delta_{0,0} \arrow[rr, "\shortmid"{marking}, "\beta_1"'] \arrow[d, equal] \arrow[rrrd, phantom, "\nu"] & & \Delta_{1,0} \arrow[r, dashed, "\shortmid"{marking}] & \Delta_{m,n_m} \arrow[d, equal] \\
\Delta_{0,0} \arrow[rrr, "\shortmid"{marking}, "\gamma"'] & & & \Delta_{m,n_m}
\end{tikzcd}$$

$$\overset{(3)}{=}$$

$$\begin{tikzcd}
\Gamma_{0,0} \arrow[r, "\shortmid"{marking}, "{\alpha_{1,1}[S_{0,1}/\Delta_{0,1} \fatsemi S_{1,0}/\Delta_{1,0}]}"] \arrow[d, equal] \arrow[rrd, phantom, "{\mu_1[\overline{S_{1,\underline{j}}}/\overline{\Delta_{1,\underline{j}}}]}"] & \Gamma_{0,1} \arrow[r, dashed, "\shortmid"{marking}] & \Gamma_{1,0} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, equal] \arrow[rd, phantom, "\cdots"] & \Gamma_{m,n_m} \arrow[d, equal] \\
\Gamma_{0,0} \arrow[rr, "\shortmid"{marking}] \arrow[d, equal] \arrow[rrrd, phantom, "{\nu[\overline{\widetilde{S}}/\overline{\widetilde{\Delta}}]}"] & & \Gamma_{1,0} \arrow[r, dashed, "\shortmid"{marking}] & \Gamma_{m,n_m} \arrow[d, equal] \\
\Gamma_{0,0} \arrow[rrr, dashed, "\shortmid"{marking}, "{\gamma[S_{0,0}/\Delta_{0,0} \fatsemi S_{m,n_m}/\Delta_{m,n_m}]}"'] \arrow[d, "S_{0,0}"'] \arrow[rrrd, phantom, "\mathsf{rest}"] & & & \Gamma_{m,n_m} \arrow[d, "S_{m,n_m}"] \\
\Delta_{0,0} \arrow[rrr, "\shortmid"{marking}, "\gamma"'] & & & \Delta_{m,n_m}
\end{tikzcd}$$

$$\overset{(4)}{=}$$

$$\begin{tikzcd}
\Gamma_{0,0} \arrow[r, dashed, "\shortmid"{marking}, "\overline{A}[\overline{S}/\overline{\Delta}]"] \arrow[d, equal] \arrow[rd, phantom, "{\nu \left[\overline{\widetilde{S}}/\overline{\widetilde{\Delta}}\right] \left\{ \overline{\mu_{\underline{i}}[\overline{S_{i,\underline{j}}}/\overline{\Delta_{i,\underline{j}}}]} \Big/ \overline{b_{\underline{i}}} \right\}}"] & \Gamma_{m,n_m} \arrow[d, equal] \\
\Gamma_{0,0} \arrow[r, dashed, "\shortmid"{marking}, "{\gamma[S_{0,0}/\Delta_{0,0} \fatsemi S_{m,n_m}/\Delta_{m,n_m}]}"'] \arrow[d, "S_{0,0}"'] \arrow[rd, phantom, "\mathsf{rest}"] & \Gamma_{m,n_m} \arrow[d, "S_{m,n_m}"] \\
\Delta_{0,0} \arrow[r, "\shortmid"{marking}, "\gamma"'] & \Delta_{m,n_m}
\end{tikzcd}$$

$\overline{\mathsf{rest}}$ represents the horizontal concatenation of restrictions cells. The definition of the interpretation of substitutions $[\overline{-}/\overline{-}]$ are used in the equalities (1), (2), and (3), and the definition of the interpretation of prosubstitutions $\{-/-\}$ is used in the equalities (1) and (4). Because of the universal properties of restriction, we can show that the interpretations of the two proterms are equal.

(iii) is presented as Proposition 5.12 in the main text, and is the most nontrivial part of the four. We give a more detailed explanation of the proof of (iii) here. Let us recall the definition of the syntactic VDC $\$^{\diamondsuit}(\Sigma, \mathrm{E})$ (Definition 5.10). In this virtual double category, a cell

$$\begin{tikzcd}
\Gamma_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "S_0"'] & \cdots & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & \Gamma_n \arrow[d, "S_1"] \\
\Delta_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] & \mu & & \Delta_1
\end{tikzcd}$$

is an equivalence class of proterms

$$\overline{\Gamma} \mid a_1 : \alpha_1 \fatsemi \ldots \fatsemi a_n : \alpha_n \vdash \mu : \beta[S_0/\Delta_0 \fatsemi S_n/\Delta_n]$$

modulo equality judgments derivable from (Σ, E). The identity cell on α is given by the equivalence class of the provariable α as a proterm. A general composition of cells

$$\begin{tikzcd}
\Gamma_{0,0} \arrow[r, dashed, "\overline{\alpha_1}", "\shortmid"{marking}] \arrow[d, "S_0"'] & \Gamma_{1,0} \arrow[r, dashed, "\overline{\alpha_2}", "\shortmid"{marking}] \arrow[d, "S_1"'] & \Gamma_{2,0} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "S_2"'] & \cdots \arrow[r, dashed, "\overline{\alpha_m}", "\shortmid"{marking}] & \Gamma_{m,0} \arrow[d, "S_n"] \\
\Delta_0 \arrow[r, "\beta_1"', "\shortmid"{marking}] \arrow[d, "T_0"'] & \Delta_1 \arrow[r, "\beta_2"', "\shortmid"{marking}] & \Delta_2 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\beta_m"', "\shortmid"{marking}] & \Delta_m \arrow[d, "T_1"] \\
\Theta_0 \arrow[rrrr, "\gamma"', "\shortmid"{marking}] & & & & \Theta_1
\end{tikzcd}$$

with μ_1, μ_2, …, μ_m in the upper squares and ν in the lower one, is given as follows: first, the cells μ_i's and ν are

$$\overline{\Gamma_{i-1,\underline{j}}} \mid a_{i,1} : \alpha_{i,1} \fatsemi \ldots \fatsemi a_{i,n_i} : \alpha_{i,n_i} \vdash \mu_i : \beta_i[S_{i-1}/\Delta_{i-1} \fatsemi S_i/\Delta_i], \qquad (i = 1, \ldots, m)$$
$$\overline{\Delta_{\underline{i}}} \mid b_1 : \beta_1 \fatsemi \ldots \fatsemi b_m : \beta_m \vdash \nu : \gamma[T_0/\Theta_0 \fatsemi T_1/\Theta_1].$$

From the second proterm, we can derive the proterm

$$\widetilde{\Gamma_{\underset{\sim}{\underline{i},j}}} \mid b_1 : \beta_1[S_0/\Delta_0 \fatsemi S_1/\Delta_1] \fatsemi \ldots \fatsemi b_m : \beta_m[S_0/\Delta_0 \fatsemi S_1/\Delta_1] \vdash \mathsf{rest\text{-}iter}^{\rightarrow}\left\{\nu\left[\overline{S}/\overline{\Delta}\right]\right\} : \gamma\left[T_0[S_0/\Delta_0] \fatsemi T_1[S_1/\Delta_1]\right].$$

The proof of (iii) is to show that this gives a well-defined CFVDC. Once we have shown it is a VDC, the rest of the proof is routine as explained in the main text. The laws for the identities follow from the first two axioms in Appendix B.1. The hardest part of the proof is to show that the composition of cells is associative. The proof is as follows: the associativity of the composition of cells is presented as

$$(\lambda\{\nu_1 \fatsemi \ldots \fatsemi \nu_m\})\{\mu_{1,1} \fatsemi \ldots \fatsemi \mu_{1,n_1} \fatsemi \ldots \fatsemi \mu_{m,1} \fatsemi \ldots \fatsemi \mu_{m,n_m}\} = \lambda\left\{\nu_1\{\mu_{1,1} \fatsemi \ldots \fatsemi \mu_{1,n_1}\} \fatsemi \ldots \fatsemi \nu_m\{\mu_{m,1} \fatsemi \ldots \fatsemi \mu_{m,n_m}\}\right\}$$

for any cells $\lambda, \nu_1, \ldots, \nu_m, \mu_{i,j}$ which are appropriately composable. For the syntactic VDC, this amounts to showing that, for any proterms $\lambda, \nu_1, \ldots, \nu_m, \mu_{i,j}$ as follows:

$$\overline{\Theta} \mid \overline{c_{\underline{i}}} : \overline{\gamma_{\underline{i}}} \vdash \lambda : \delta[U_0/\Phi_0 \fatsemi U_1/\Phi_1],$$
$$\overline{\Delta_{i-1,\underline{j}}} \mid \overline{b_{i,\underline{j}}} : \overline{\beta_{i,\underline{j}}} \vdash \nu_i : \gamma_i[T_{i-1}/\Theta_{i-1} \fatsemi T_i/\Theta_i], \qquad (i = 1, \ldots, m)$$
$$\overline{\Gamma_{i-1,j-1,\underline{k}}} \mid \overline{a_{i,j,\underline{k}}} : \overline{\alpha_{i,j,\underline{k}}} \vdash \mu_{i,j} : \beta_{i,j}[S_{i-1,j-1}/\Delta_{i-1,j-1} \fatsemi S_{i-1,j}/\Delta_{i-1,j}], \qquad (i = 1, \ldots, m,\ j = 1, \ldots, n_i)$$

the following equality is derivable:

$$\begin{aligned}
&\left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\left(\left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\right\}\right)\left\{\overline{\nu_{\underline{i}}}/\overline{c_{\underline{i}}}\right\}\right)\left[\overline{S_{\underline{i,j}}}/\overline{\Delta_{\underline{i,j}}}\right]\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\lambda\left[\overline{T_{\underline{i}}[\overline{S_{\underline{i},\underline{j}}}/\overline{\Delta_{\underline{i},\underline{j}}}]}\Big/\overline{\Theta_{\underline{i}}}\right]\right\}\right)\left\{\overline{\left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\nu_i\left[\overline{S_{\underline{i}-1,\underline{j}}}/\overline{\Delta_{\underline{i}-1,\underline{j}}}\right]\right\}\right)\left\{\overline{\mu_{\underline{i},\underline{j}}}/\overline{b_{\underline{i},\underline{j}}}\right\}}\Big/\overline{c_{\underline{i}}}\right\}
\end{aligned}$$

. This is proved as follows: $(\widetilde{\widetilde{S}}/\widetilde{\widetilde{\Delta}} = S_{0,0}/\Delta_{0,0} \fatsemi S_{m,n_m}/\Delta_{m,n_m})$

$$\begin{aligned}
(\mathrm{LHS}) &\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\left\{\overline{\nu_{\underline{i}}}/\overline{c_{\underline{i}}}\right\}\right\}\right)\left[\overline{S_{\underline{i,j}}}/\overline{\Delta_{\underline{i,j}}}\right]\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\left(\mathsf{rest\text{-}iter}^{\rightarrow}\{d\}\left\{\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\left\{\overline{\nu_{\underline{i}}}/\overline{c_{\underline{i}}}\right\}\Big/d\right\}\right)\left[\overline{S_{\underline{i,j}}}/\overline{\Delta_{\underline{i,j}}}\right]\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\left(\mathsf{rest\text{-}iter}^{\rightarrow}\{d\}\left[\widetilde{\widetilde{S}}/\widetilde{\widetilde{\Delta}}\right]\right)\left\{\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\left\{\overline{\nu_{\underline{i}}}/\overline{c_{\underline{i}}}\right\}\left[\overline{S_{\underline{i,j}}}/\overline{\Delta_{\underline{i,j}}}\right]\Big/d\right\}\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\mathsf{rest\text{-}iter}^{\rightarrow}\{d\}\left[\widetilde{\widetilde{S}}/\widetilde{\widetilde{\Delta}}\right]\right\}\right)\left\{\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\left\{\overline{\nu_{\underline{i}}}/\overline{c_{\underline{i}}}\right\}\left[\overline{S_{\underline{i,j}}}/\overline{\Delta_{\underline{i,j}}}\right]\Big/d\right\}\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\left\{\overline{\nu_{\underline{i}}}/\overline{c_{\underline{i}}}\right\}\left[\overline{S_{\underline{i,j}}}/\overline{\Delta_{\underline{i,j}}}\right]\right\}\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\left(\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\left[\widetilde{S_{\underline{i},\tilde{j}}}/\widetilde{\Delta_{\underline{i},\tilde{j}}}\right]\right)\left\{\overline{\nu_{\underline{i}}}\left[\overline{S_{\underline{i}-1,\underline{j}}}/\overline{\Delta_{\underline{i}-1,\underline{j}}}\right]\Big/\overline{c_{\underline{i}}}\right\}\right\}\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\lambda\left[\overline{T_{\underline{i}}}/\overline{\Theta_{\underline{i}}}\right]\left[\widetilde{S_{\underline{i},\tilde{j}}}/\widetilde{\Delta_{\underline{i},\tilde{j}}}\right]\right\}\right)\left\{\overline{\nu_{\underline{i}}}\left[\overline{S_{\underline{i}-1,\underline{j}}}/\overline{\Delta_{\underline{i}-1,\underline{j}}}\right]\Big/\overline{c_{\underline{i}}}\right\}\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\} \\
&\equiv \left(\mathsf{rest\text{-}iter}^{\rightarrow}\left\{\left(\lambda\left[\overline{T_{\underline{i}}[\overline{S_{\underline{i},\underline{j}}}/\overline{\Delta_{\underline{i},\underline{j}}}]}\Big/\overline{\Theta_{\underline{i}}}\right]\left\{\overline{\mathsf{rest\text{-}iter}^{\rightarrow}}/\overline{c_{\underline{i}}}\right\}\right)\left\{\overline{\nu_{\underline{i}}}\left[\overline{S_{\underline{i}-1,\underline{j}}}/\overline{\Delta_{\underline{i}-1,\underline{j}}}\right]\Big/\overline{c_{\underline{i}}}\right\}\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\}
\end{aligned}$$

$$\equiv \left(\mathsf{rest\text{-}iter}^{\to}\left\{\lambda\left[\overline{T_{\underline{i}}[\overline{S_{\underline{i},\underline{j}}}/\overline{\Delta_{\underline{i},\underline{j}}}]}\Big/\overline{\Theta_{\underline{i}}}\right]\left\{\overline{\mathsf{rest\text{-}iter}^{\to}\left\{\nu_{\underline{i}}\left[\overline{S_{\underline{i}-1,\underline{j}}}/\overline{\Delta_{\underline{i}-1,\underline{j}}}\right]\right\}}\Big/\overline{c_{\underline{i}}}\right\}\right\}\right)\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\}$$

$$\equiv \left(\mathsf{rest\text{-}iter}^{\to}\left\{\lambda\left[\overline{T_{\underline{i}}[\overline{S_{\underline{i},\underline{j}}}/\overline{\Delta_{\underline{i},\underline{j}}}]}\Big/\overline{\Theta_{\underline{i}}}\right]\right\}\right)\left\{\overline{\mathsf{rest\text{-}iter}^{\to}\left\{\nu_{\underline{i}}\left[\overline{S_{\underline{i}-1,\underline{j}}}/\overline{\Delta_{\underline{i}-1,\underline{j}}}\right]\right\}}\Big/\overline{c_{\underline{i}}}\right\}\left\{\overline{\mu_{\underline{i,j}}}/\overline{b_{\underline{i,j}}}\right\}$$

$$\equiv \text{(RHS)}.$$

B.3. **The derivation rules for the additional constructors** In Section 4, we explain some additional constructors of FVDblTT that are meaningful both in the contexts of formal category theory and predicate logic. In this section, we provide all the derivation rules of the constructs.

Unit protype.

$$\frac{I\ \mathsf{type}}{x : I \fatsemi y : I \vdash x \nrightarrow_I y\ \mathsf{protype}}\ \nrightarrow\text{-}\textsc{Form} \qquad \frac{I\ \mathsf{type}}{x : I \mid\ \ \vdash \mathsf{refl}_I(x) : x \nrightarrow_I x}\ \nrightarrow\text{-}\textsc{Intro}$$

$$\frac{w_0 : J_0 \fatsemi z_m : K_m \vdash \gamma(w_0 \fatsemi z_m)\ \mathsf{protype} \qquad \overline{w} : \overline{J} \fatsemi x : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi \overline{B}(x \fatsemi \overline{z}) \vdash \mu : \gamma(w_0 \fatsemi z_m)}{\overline{w} : \overline{J} \fatsemi x : I \fatsemi y : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi p : x \nrightarrow_I y \fatsemi \overline{B}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\nrightarrow_I}\{\mu\} : \gamma(w_0 \fatsemi z_m)}\ \nrightarrow\text{-}\textsc{Elim}$$

$$\frac{\overline{w} : \overline{J} \fatsemi x : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi \overline{B}(x \fatsemi \overline{z}) \vdash \mu : \gamma(w_0 \fatsemi z_m)}{\overline{w} : \overline{J} \fatsemi x : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi \overline{B}(x \fatsemi \overline{z}) \vdash \left(\mathsf{ind}_{\nrightarrow_I}\{\mu\}\right)[x/y]\{\mathsf{refl}_I(x)/p\} \equiv \mu : \gamma(w_0 \fatsemi z_m)}\ \nrightarrow\text{-}\textsc{Comp}\beta$$

$$\frac{\overline{w} : \overline{J} \fatsemi x : I \fatsemi y : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi p : x \nrightarrow_I y \fatsemi \overline{B}(y \fatsemi \overline{z}) \vdash \nu : \gamma(w_0 \fatsemi z_m)}{\overline{w} : \overline{J} \fatsemi x : I \fatsemi y : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi p : x \nrightarrow_I y \fatsemi \overline{B}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\nrightarrow_I}\{\nu[x/y]\{\mathsf{refl}_I(x)/p\}\} \equiv \nu : \gamma(w_0 \fatsemi z_m)}\ \nrightarrow\text{-}\textsc{Comp}\eta$$

Unit protype meets product type.

$$\frac{}{\cdot \fatsemi \cdot \vdash \mathsf{exc}_{\nrightarrow,\top} : \langle\rangle \nrightarrow_{\mathbf{1}} \langle\rangle \cong \top}\ \nrightarrow\text{-}\top \qquad \frac{I\ \mathsf{type} \qquad J\ \mathsf{type}}{x : I, y : J \fatsemi x' : I, y' : J \vdash \mathsf{exc}_{\nrightarrow,\wedge} : \langle x, y\rangle \nrightarrow_{I\times J} \langle x', y'\rangle \cong x \nrightarrow_I x' \wedge y \nrightarrow_J y'}\ \nrightarrow\text{-}\wedge$$

$$\frac{I\ \mathsf{type} \qquad J\ \mathsf{type}}{x : I, y : J \fatsemi x' : I, y' : J \mid a : \langle x, y\rangle \nrightarrow_{I\times J} \langle x', y'\rangle \vdash \mathsf{exc}_{\nrightarrow,\wedge}\{a\} \equiv \mathsf{ind}_{\nrightarrow_{I\times J}}\{\langle \mathsf{refl}_I(x), \mathsf{refl}_J(y)\rangle\} : x \nrightarrow_I x' \wedge y \nrightarrow_J y'}$$

$$\text{where} \quad \frac{x : I, y : J \mid \langle \mathsf{refl}_I(x), \mathsf{refl}_J(y)\rangle : x \nrightarrow_I x' \wedge y \nrightarrow_J y'}{x : I, y : J \fatsemi x' : I, y' : J \mid a : \langle x, y\rangle \nrightarrow_{I\times J} \langle x', y'\rangle \vdash \mathsf{ind}_{\nrightarrow_{I\times J}}\{\langle \mathsf{refl}_I(x), \mathsf{refl}_J(y)\rangle\} : x \nrightarrow_I x' \wedge y \nrightarrow_J y'}$$

Composition protype.

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad x : J \fatsemi y : K \vdash \beta(x \fatsemi y)\ \mathsf{protype}}{w : I \fatsemi y : K \vdash \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y)\ \mathsf{protype}}\ \odot\text{-}\textsc{Form}$$

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad x : J \fatsemi y : K \vdash \beta(x \fatsemi y)\ \mathsf{protype}}{w : I \fatsemi x : J \fatsemi y : K \mid a : \alpha(w \fatsemi x) \fatsemi b : \beta(x \fatsemi y) \vdash a \odot b : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y)}\ \odot\text{-}\textsc{Intro}$$

$$\frac{\overline{v} : \overline{H} \fatsemi w : I \fatsemi x : J \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi a : \alpha(w \fatsemi x) \fatsemi b : \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mu : \gamma(v_0 \fatsemi z_m)}{\overline{v} : \overline{H} \fatsemi w : I \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi p : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\odot_{\alpha,\beta}}\{\mu\} : \gamma(v_0 \fatsemi z_m)}\ \odot\text{-}\textsc{Elim}$$

$$\frac{\overline{v} : \overline{H} \fatsemi w : I \fatsemi x : J \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi \alpha(w \fatsemi x) \fatsemi \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mu : \gamma(v_0 \fatsemi z_m)}{\begin{array}{r}\overline{v} : \overline{H} \fatsemi w : I \fatsemi x : J \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi a : \alpha(w \fatsemi x) \fatsemi b : \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \\ \vdash \left(\mathsf{ind}_{\odot_{\alpha,\beta}}\{\mu\}\right)\{a \odot b/p\} \equiv \mu : \gamma(v_0 \fatsemi z_m)\end{array}}\ \odot\text{-}\textsc{Comp}\beta$$

$$\frac{\overline{v} : \overline{H} \fatsemi w : I \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi p : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \nu : \gamma(v_0 \fatsemi z_m)}{\overline{v} : \overline{H} \fatsemi w : I \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi p : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\odot_{\alpha,\beta}}\{(\nu\{a \odot b/p\})\} \equiv \nu : \gamma(v_0 \fatsemi z_m)}\ \odot\text{-}\textsc{Comp}\eta$$

Composition protype meets product type.

$$\frac{}{\cdot \fatsemi \cdot \vdash \mathsf{exc}_{\odot,\top} : \top \odot_{\langle\rangle:\cdot} \top \cong \top}\ \odot\text{-}\top$$

$$\frac{\begin{array}{c}x : I \fatsemi y : J \vdash \alpha(x \fatsemi y)\ \mathsf{protype} \\ y : J \fatsemi z : K \vdash \beta(y \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v)\ \mathsf{protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype}\end{array}}{\begin{array}{r}x : I, u : L \fatsemi z : K, w : N \vdash \mathsf{exc}_{\odot,\wedge} : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \odot_{\langle y,v\rangle:J\times M} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \\ \cong (\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}\ \odot\text{-}\wedge$$

$$\frac{x : I \fatsemi y : J \vdash \alpha(x \fatsemi y)\ \mathsf{protype} \qquad y : J \fatsemi z : K \vdash \beta(y \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v)\ \mathsf{protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype}}{\begin{array}{l}x : I, u : L \fatsemi z : K, w : N \mid e : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \odot_{\langle y,v\rangle:J\times M} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{exc}_{\odot,\wedge}\{e\} \equiv \mathsf{ind}_{\odot_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \odot \pi_0\{b\}, \pi_1\{a\} \odot \pi_1\{b\}\rangle\} : (\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}$$

$$\text{where} \quad \frac{\begin{array}{l}x : I \fatsemi u : L \fatsemi y : J \fatsemi v : M \fatsemi z : K \fatsemi w : N \mid a : \alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v) \fatsemi b : \beta(y \fatsemi z) \wedge \delta(v \fatsemi w) \\ \quad \vdash \langle \pi_0\{a\} \odot \pi_0\{b\}, \pi_1\{a\} \odot \pi_1\{b\}\rangle : (\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}{\begin{array}{l}x : I \fatsemi u : L \fatsemi z : K \fatsemi w : N \mid e : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \odot_{\langle y,v\rangle:J\times M} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{ind}_{\odot_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \odot \pi_0\{b\}, \pi_1\{a\} \odot \pi_1\{b\}\rangle\} : (\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}$$

Filler protype.

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y)\ \mathsf{protype}}{x : J \fatsemi y : K \vdash \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y)\ \mathsf{protype}}\ \triangleright\text{-}\textsc{Form}$$

$$\frac{w : I \fatsemi x : J \fatsemi \overline{y} : \overline{L} \mid a : \alpha(w \fatsemi x) \fatsemi \overline{C}(x \fatsemi \overline{y}) \vdash \mu : \beta(w \fatsemi y_m)}{x : J \fatsemi \overline{y} : \overline{L} \mid \overline{C}(x \fatsemi \overline{y}) \vdash \mathsf{ind}_{\triangleright_{\alpha,\beta}}\{\mu\} : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y_m)}\ \triangleright\text{-}\textsc{Intro}$$

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y)\ \mathsf{protype}}{w : I \fatsemi x : J \fatsemi y : K \mid a : \alpha(w \fatsemi x) \fatsemi e : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \vdash a \blacktriangleright e : \beta(w \fatsemi y)}\ \triangleright\text{-}\textsc{Elim}$$

$$\frac{w : I \fatsemi x : J \fatsemi \overline{y} : \overline{L} \mid a : \alpha(w \fatsemi x) \fatsemi \overline{C}(x \fatsemi \overline{y}) \vdash \mu : \beta(w \fatsemi y_m)}{w : I \fatsemi x : J \fatsemi \overline{y} : \overline{L} \mid a : \alpha(w \fatsemi x) \fatsemi \overline{C}(x \fatsemi \overline{y}) \vdash a \blacktriangleright \left(\mathsf{ind}_{\triangleright_{\alpha,\beta}}\{\mu\}\right) \equiv \mu : \beta(w \fatsemi y_m)}\ \triangleright\text{-}\textsc{Comp}\beta$$

$$\frac{x : J \fatsemi \overline{y} : \overline{L} \mid \overline{C}(x \fatsemi \overline{y}) \vdash \nu : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y_m)}{x : J \fatsemi \overline{y} : \overline{L} \mid \overline{C}(x \fatsemi \overline{y}) \vdash \mathsf{ind}_{\triangleright_{\alpha,\beta}}\{a \blacktriangleright \nu\} \equiv \nu : \beta(w \fatsemi y_m)}\ \triangleright\text{-}\textsc{Comp}\eta$$

$$\frac{y : J \fatsemi z : K \vdash \alpha(y \fatsemi z)\ \mathsf{protype} \qquad x : I \fatsemi z : K \vdash \beta(x \fatsemi z)\ \mathsf{protype}}{x : I \fatsemi y : J \vdash \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z)\ \mathsf{protype}}\ \triangleleft\text{-}\textsc{Form}$$

$$\frac{\overline{x} : \overline{J} \fatsemi y : J \fatsemi z : K \mid \overline{C}(\overline{x} \fatsemi y) \fatsemi a : \alpha(y \fatsemi z) \vdash \mu : \beta(x \fatsemi z)}{\overline{x} : \overline{J} \fatsemi y : J \mid \overline{C}(\overline{x} \fatsemi y) \vdash \mathsf{ind}_{\triangleleft_{\alpha,\beta}}\{\mu\} : \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z)}\ \triangleleft\text{-}\textsc{Intro}$$

$$\frac{x : I \fatsemi y : J \vdash \beta(x \fatsemi z)\ \mathsf{protype} \qquad y : J \fatsemi z : K \vdash \alpha(y \fatsemi z)\ \mathsf{protype}}{x : I \fatsemi y : J \fatsemi z : K \mid a : \beta(x \fatsemi z) \fatsemi e : \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z) \vdash a \blacktriangleleft e : \alpha(y \fatsemi z)}\ \triangleleft\text{-}\textsc{Elim}$$

$$\frac{x : I \fatsemi y : J \fatsemi \overline{z} : \overline{L} \mid a : \beta(x \fatsemi z) \fatsemi \overline{C}(x \fatsemi \overline{z}) \vdash \mu : \alpha(y \fatsemi z_m)}{x : I \fatsemi y : J \fatsemi \overline{z} : \overline{L} \mid a : \beta(x \fatsemi z) \fatsemi \overline{C}(x \fatsemi \overline{z}) \vdash a \blacktriangleleft \left(\mathsf{ind}_{\triangleleft_{\alpha,\beta}}\{\mu\}\right) \equiv \mu : \alpha(y \fatsemi z_m)}\ \triangleleft\text{-}\textsc{Comp}\beta$$

$$\frac{y : J \fatsemi \overline{z} : \overline{L} \mid \overline{C}(y \fatsemi \overline{z}) \vdash \nu : \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z)}{y : J \fatsemi \overline{z} : \overline{L} \mid \overline{C}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\triangleleft_{\alpha,\beta}}\{a \blacktriangleleft \nu\} \equiv \nu : \alpha(y \fatsemi z_m)}\ \triangleleft\text{-}\textsc{Comp}\eta$$

Filler protype meets product type.

$$\frac{}{\cdot \fatsemi \cdot \mid \mathsf{exc}_{\triangleright,\top} : \top \triangleright_{.} \top \cong \top}\ \triangleright\text{-}\top$$

$$\frac{\begin{array}{c} x : I \fatsemi y : J \vdash \alpha(x \fatsemi y)\ \mathsf{protype} \\ x : I \fatsemi z : K \vdash \beta(x \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v)\ \mathsf{protype} \qquad u : L \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype} \end{array}}{\begin{array}{l} y : J, v : M \fatsemi z : K, w : N \vdash \mathsf{exc}_{\triangleright,\wedge} : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \cong (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \triangleright_{x:I,u:L} (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}\ \triangleright\text{-}\wedge$$

$$\frac{\begin{array}{c} x : I \fatsemi y : J \vdash \alpha(x \fatsemi y)\ \mathsf{protype} \\ x : I \fatsemi z : K \vdash \beta(x \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v)\ \mathsf{protype} \qquad u : L \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype} \end{array}}{\begin{array}{l} y : J, v : M \fatsemi z : K, w : N \mid e : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{exc}_{\triangleright,\wedge}\{e\} \equiv \mathsf{ind}_{\triangleright_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \blacktriangleright \pi_0(e), \pi_1\{a\} \blacktriangleright \pi_1(e)\rangle\} : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \triangleright_{x:I,u:L} (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}\ \triangleright\text{-}\wedge\text{-}\textsc{canon}$$

where
$$\frac{\begin{array}{l} x : I, u : L, y : J, v : M, z : K, w : N \mid a : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \fatsemi e : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \quad \vdash \langle \pi_0\{a\} \blacktriangleright \pi_0(e), \pi_1\{a\} \blacktriangleright \pi_1(e)\rangle : (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}{\begin{array}{l} y : J, v : M \fatsemi z : K, w : N \mid e : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{ind}_{\triangleright_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \blacktriangleright \pi_0(e), \pi_1\{a\} \blacktriangleright \pi_1(e)\rangle\} : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \triangleright_{x:I,u:L} (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}$$

$$\frac{}{\cdot \fatsemi \cdot \vdash \mathsf{exc}_{\triangleleft,\top} : \top \triangleleft_{.} \top \equiv \top}\ \triangleleft\text{-}\top$$

$$\frac{\begin{array}{c} x : I \fatsemi z : K \vdash \alpha(x \fatsemi z)\ \mathsf{protype} \\ y : J \fatsemi z : K \vdash \beta(y \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi w : N \vdash \gamma(u \fatsemi w)\ \mathsf{protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype} \end{array}}{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M \vdash \mathsf{exc}_{\triangleleft,\wedge} : (\alpha(x \fatsemi z) \triangleleft_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \triangleleft_{w:N} \delta(v \fatsemi w)) \\ \cong (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \triangleleft_{z:K,w:N} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}\ \triangleleft\text{-}\wedge$$

$$\frac{\begin{array}{c} x : I \fatsemi z : K \vdash \alpha(x \fatsemi z)\ \mathsf{protype} \\ y : J \fatsemi z : K \vdash \beta(y \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi w : N \vdash \gamma(u \fatsemi w)\ \mathsf{protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype} \end{array}}{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M \mid e : (\alpha(x \fatsemi z) \triangleleft_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \triangleleft_{w:N} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{exc}_{\triangleleft,\wedge}\{e\} \equiv \mathsf{ind}_{\triangleleft_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \blacktriangleleft \pi_0(e), \pi_1\{a\} \blacktriangleleft \pi_1(e)\rangle\} : (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \triangleleft_{z:K,w:N} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}\ \triangleleft\text{-}\wedge\text{-}\textsc{canon}$$

where
$$\frac{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M, z : K, w : N \mid a : (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \fatsemi e : (\alpha(x \fatsemi z) \triangleleft_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \triangleleft_{w:N} \delta(v \fatsemi w)) \\ \quad \vdash \langle \pi_0\{a\} \blacktriangleleft \pi_0(e), \pi_1\{a\} \blacktriangleleft \pi_1(e)\rangle : (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M \mid e : (\alpha(x \fatsemi z) \triangleleft_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \triangleleft_{w:N} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{ind}_{\triangleleft_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \blacktriangleleft \pi_0(e), \pi_1\{a\} \blacktriangleleft \pi_1(e)\rangle\} : (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \triangleleft_{z:K,w:N} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}$$

Comprehension type.

$$\frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype}}{\{\!|\alpha|\!\} \ \mathsf{type}}\ \{\!|\,|\!\}\text{-}\textsc{Form} \qquad \frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype}}{w : \{\!|\alpha|\!\} \vdash l(w) : I}\ \{\!|\,|\!\}\text{-}\textsc{Elim-}\ell \qquad \frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype}}{w : \{\!|\alpha|\!\} \vdash r(w) : J}\ \{\!|\,|\!\}\text{-}\textsc{Elim-}r$$

$$\frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype}}{w : \{\!|\alpha|\!\} \mid\!\vdash \mathsf{tab}_{\{\!|\alpha|\!\}}\{w\} : \alpha[l(w)/x \fatsemi r(w)/y]}\ \{\!|\,|\!\}\text{-}\textsc{Elim-cell}$$

$$\frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype} \qquad \Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash \mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu) : \{\!|\alpha|\!\}}\ \{\!|\,|\!\}\text{-}\textsc{Intro}$$

$$\frac{\Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash l(\mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu)) \equiv s : I}\ \{\!|\,|\!\}\text{-}\textsc{Comp-}\ell \qquad \frac{\Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash r(\mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu)) \equiv t : J}\ \{\!|\,|\!\}\text{-}\textsc{Comp-}r$$

$$\frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype} \qquad \Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash \mathsf{tab}_{\{\!|\alpha|\!\}}\{\mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu)\} \equiv \nu : \alpha[s/x \fatsemi t/y]}\ \{\!|\,|\!\}\text{-}\textsc{Comp-}\beta$$

$$\frac{x : I \fatsemi y : J \vdash \alpha \ \mathsf{protype}}{w : \{\!|\alpha|\!\} \vdash \mathsf{ind}_{\{\!|\,|\!\}}(l(w), r(w), \mathsf{tab}_{\{\!|\alpha|\!\}}\{w\} \ \equiv w : \{\!|\alpha|\!\}}\ \{\!|\,|\!\}\text{-}\textsc{Comp-}\eta$$

Comprehension type meets unit protype.

$$\frac{\begin{array}{c} \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_0 \vdash t_0 : J \qquad \Gamma_m \vdash t_1 : J \qquad x : I, y : J \vdash \alpha(x, y) \ \mathsf{protype} \\ \Gamma_0 \mid\!\vdash \mu_0 : \alpha(s_0 \fatsemi t_0) \qquad \Gamma_m \mid\!\vdash \mu_1 : \alpha(s_1 \fatsemi t_1) \qquad \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \overline{\Gamma} \mid B \vdash j : t_0 \nrightarrow_J t_1 \qquad \overline{\Gamma} \mid B \vdash i \boxdot \mu_1 \equiv \mu_0 \boxdot j \end{array}}{\overline{\Gamma} \mid B \vdash \mathsf{ind}_{\{\!|\,|\!\}}(i, j, \mu_0, \mu_1) : \mathsf{ind}_{\{\!|\,|\!\}}(s_0, t_0, \mu_0) \nrightarrow_{\{\!|\alpha|\!\}} \mathsf{ind}_{\{\!|\,|\!\}}(s_1, t_1, \mu_1)}\ \{\!|\,|\!\}\text{-}\textsc{Elim}$$

where

$$\frac{\begin{array}{c} \dfrac{\dfrac{x : I \fatsemi y : J \mid a : \alpha(x \fatsemi y) \vdash a : \alpha(x' \fatsemi y)}{x : I \fatsemi x' : I \fatsemi y : J \mid p : x \nrightarrow_I x' \fatsemi a : \alpha(x \fatsemi y) \vdash \mathsf{ind}_{\nrightarrow}\{a\} : \alpha(x \fatsemi y)} \qquad \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_m \vdash t_1 : J}{\overline{\Gamma} \mid p : s_0 \nrightarrow_I s_1 \fatsemi a : \alpha(s_1 \fatsemi t_1) \vdash \mathsf{ind}_{\nrightarrow}\{a\}[s_1/x' \fatsemi t_1/y] : \alpha(s_0 \fatsemi t_1)} \\ \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \Gamma_m \mid\!\vdash \mu_1 : \alpha(s_1 \fatsemi t_1) \end{array}}{\overline{\Gamma} \mid B \vdash i \boxdot \mu_1 :\equiv \mathsf{ind}_{\nrightarrow}\{a\}[s_1/x' \fatsemi t_1/y]\{i/p : s_0 \nrightarrow_I s_1 \fatsemi \mu_1/a : \alpha(s_1 \fatsemi t_1)\} : \alpha(s_0 \fatsemi t_1)}$$

and similarly for $\mu_0 \boxdot j$.

$$\frac{\begin{array}{c} \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_0 \vdash t_0 : J \qquad \Gamma_m \vdash t_1 : J \qquad x : I, y : J \vdash \alpha(x, y) \ \mathsf{protype} \\ \Gamma_0 \mid\!\vdash \mu_0 : \alpha[s_0/x \fatsemi t_0/y] \qquad \Gamma_m \mid\!\vdash \mu_1 : \alpha[s_1/x \fatsemi t_1/y] \qquad \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \overline{\Gamma} \mid B \vdash j : t_0 \nrightarrow_J t_1 \qquad \overline{\Gamma} \mid B \vdash i \boxdot \mu_1 \equiv \mu_0 \boxdot j \end{array}}{\overline{\Gamma} \mid B \vdash \mathsf{app}_l(\mathsf{ind}_{\{\!|\,|\!\}}(i, j, \mu_0, \mu_1)) \equiv i : s_0 \nrightarrow_I s_1}\ \{\!|\,|\!\}\text{-}\textsc{Comp}$$

$$\frac{\begin{array}{c} \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_0 \vdash t_0 : J \qquad \Gamma_m \vdash t_1 : J \qquad x : I, y : J \vdash \alpha(x, y) \ \mathsf{protype} \\ \Gamma_0 \mid\!\vdash \mu_0 : \alpha[s_0/x \fatsemi t_0/y] \qquad \Gamma_m \mid\!\vdash \mu_1 : \alpha[s_1/x \fatsemi t_1/y] \qquad \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \overline{\Gamma} \mid B \vdash j : t_0 \nrightarrow_J t_1 \qquad \overline{\Gamma} \mid B \vdash i \boxdot \mu_1 \equiv \mu_0 \boxdot j \end{array}}{\overline{\Gamma} \mid B \vdash \mathsf{app}_r(\mathsf{ind}_{\{\!|\,|\!\}}(i, j, \mu_0, \mu_1)) \equiv j : t_0 \nrightarrow_J t_1}\ \{\!|\,|\!\}\text{-}\textsc{Comp}$$

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