Double categories of relations relative to factorisation systems

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Introduction

Objects in $\mathcal{C}$ + Morphisms in $\mathcal{C}$ + Relations in $\mathcal{C}$ $R \rightarrowtail A \times B$; Spans in $\mathcal{C}$ $X \xrightarrow[\langle f, g\rangle]{} A \times B$ + ...

$\rightsquigarrow$ Double categories $\mathbb{R}\mathrm{el}(\mathcal{C})$, $\mathbb{S}\mathrm{pan}(\mathcal{C})$.

What is a common generalization?

How can we characterize them?

Structure

1. Double categories
2. How should double categories of relations be?
3. The correspondence between DCRs and SOFSs

   — Cauchy double categories of relations
4. Conclusions and future work

This talk is based on

Double categories of relations relative to factorisation systems, ArXiv 2310. 19428

1. Double categories
2. How should double categories of relations be?
3. The correspondence between DCRs and SOFSs
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Double categories

A **double category** $\mathbb{D}$ consists of the following data:

- objects $A, B, C, \dots$
- vertical arrows $\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd}, \dots$
- horizontal arrows $A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B, \dots$
- cells $\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}, \dots$

with compositions of vertical arrows / horizontal arrows / cells such that $\cdots$.

Examples

$\mathbb{R}\mathrm{el}(\mathbf{Set})$

objects : sets , vertical arrows : functions

horizontal arrows : (binary) relations

composition of horizontal arrows :

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \;:=\; A \xrightarrow[\quad]{\{(a,c) \mid \exists b \in B \;\; aRb \wedge bSc\}} C$$

cells : In this case, at most one cell can exist for each frame.

$$\begin{tikzcd} A \arrow[r, "R", "|"' marking] \arrow[d, "f"'] \arrow[dr, phantom, "\wedge\mathrm{I}"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "|" marking] & D \end{tikzcd} \quad \Longleftrightarrow \quad \begin{array}{l} \forall a \in A \;\; \forall c \in C \\ \quad aRc \Longrightarrow f(a) \; S \; g(c) \end{array}$$

Examples

$\mathrm{Span}(\mathcal{C})$ for a finitely complete category $\mathcal{C}$

objects : objects in $\mathcal{C}$, **vertical arrows** : arrows in $\mathcal{C}$

horizontal arrows : spans

$$\frac{A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B}{\overline{A \xleftarrow{l_R} R \xrightarrow{r_R} B \text{ in } \mathcal{C}}}$$

composition of horizontal arrows :

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] & B \arrow[r, "S", "|" marking] & C \end{tikzcd} := \begin{tikzcd} & & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45] & & \\ A & R \arrow[l, "l_R"'] \arrow[dr, "r_R"'] & & S \arrow[dl, "l_S"] \arrow[r, "r_S"] & C \\ & & B & & \end{tikzcd}$$

cells :

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "|" marking] & D \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & R \arrow[l, "l_R"'] \arrow[r, "r_R"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "g"] \\ B & S \arrow[l, "l_S"] \arrow[r, "r_S"'] & D \end{tikzcd} \text{ in } \mathcal{C}$$

Some Definitions

$\mathbb{D}$: a double category

- **the vertical category $\mathcal{V}(\mathbb{D})$** is a category consisting of objects and vertical arrows in $\mathbb{D}$.
- **the horizontal bicategory $\mathcal{H}(\mathbb{D})$** is a bicategory whose 0-cells are objects, 1-cells are horizontal arrows, and 2-cells are cells with the identity vertical arrows in $\mathbb{D}$.

$\mathcal{V}(\mathbb{D})$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \arrow[d, "g"'] \\ C \end{tikzcd}$$

$\mathcal{H}(\mathbb{D})$

$$\begin{tikzcd} A \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} := \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equal] \\ A \arrow[r, "S"', "\shortmid"{marking}] & B \end{tikzcd}$$

Historical Remarks

Rel Span

1984 Carboni, Kasangian, and Street defined bicategories of spans and bicategories of relations on regular categories.

1987 Carboni and Walters characterized bicategories of relations on regular categories

2010 Lack, Walters, and Wood characterized bicategories of spans.

2018 Aleiferi characterized? double categories of spans ↓ DC

2022 Lambert characterized double categories of relations on regular categories

Historical Remarks continued & Motivations

Kelly defined bicategories of relations relative to stable proper factorization systems (E, M).

└ e.g., (Surj, Inj) in Set

M-relation $A \overset{R}{\longrightarrow\!\!\!\!\!\!+\;} B$ $\|$ $R \longrightarrow A \times B \in M$

Motivation

To characterize double categories of relations relative to stable orthogonal factorization systems.

This treatment includes DCs of spans / relations.

Some Definitions

Definition

A stable orthogonal factorization system (SOFS) on a category $\mathcal{C}$ is a pair of classes of morphisms (E, M) such that:

(i) E and M are closed under composition and contain iso's.

(ii) E and M are orthogonal:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "E \ni"'] & \bullet \arrow[d, "\in M"] \\ \bullet \arrow[r] \arrow[ur, dashed, "\exists!"] & \bullet \end{tikzcd}$$

(iii) Every morphism in $\mathcal{C}$ is factored as $\bullet \xrightarrow[\in E]{} \bullet \xrightarrow[\in M]{} \bullet$.

(iv) E is stable under pullback.

It is proper if $M \subseteq \mathrm{Mono}$, $E \subseteq \mathrm{Epi}$

$\mathrm{Rel}_{(E,M)}(\mathcal{C})$ is the double category whose vertical arrows are arrows in $\mathcal{C}$ and horizontal arrows are M-relations.

Why double categories ?

A. They have potentials of rich structures and enable us to describe behaviours of relations effectively!

Bicategories

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B$$

- ○ have compositions
- × have no 'functions'

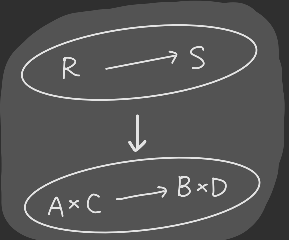
Fibered Categories

- × have no compositions
- ○ have 'functions' on the base

Double Categories

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\wedge\!\mid"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$$

- ○ have compositions
- ○ have 'functions'



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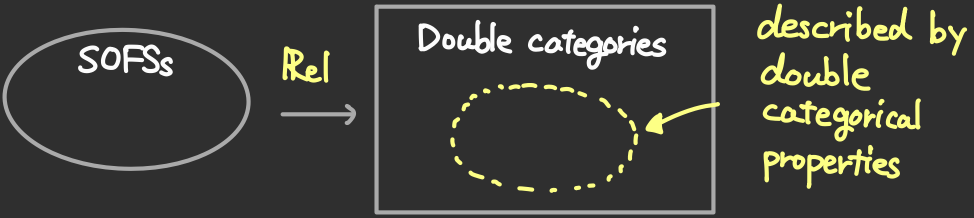
Characterisation Theorem

Theorem [HN.]

For a double category $\mathbb{D}$, the following are equivalent:

(i) $\mathbb{D} \simeq \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ for some SOFS (E,M) on some finitely complete category $\mathcal{C}$.

(ii) $\mathbb{D}$ is a cartesian equipment, has Beck-Chevalley pullbacks, and admits an M-comprehension scheme for some M.



Cartesian double categories

Since $\mathcal{M}$-relations $A \nrightarrow B$ are defined as $\mathcal{M}$-subobjects of $A \times B$, we assume double categories to be **cartesian**.

A double category $\mathbb{D}$ is **cartesian** iff (if it is an equipment)

- $\mathcal{V}(\mathbb{D})$ has finite products
- $\mathcal{H}(\mathbb{D})$ has local finite products, i.e., $\mathcal{H}(\mathbb{D})(A, B)$ has finite products for any $A, B \in \mathbb{D}$
- these products 'respect' horizontal composition

More precisely, it is a cartesian object in the 2-cat $\mathcal{D}\mathrm{bl}\mathcal{C}\mathrm{at}$.

Functions $\longrightarrow$ Relations

In $\mathbb{R}\mathrm{el}(\mathrm{Set})$, every function $f : A \to B$ defines binary relations called the graphs of f :

$$A \xrightarrow{f_*} B := \{(a, b) \mid f(a) = b\}.$$

$$B \xrightarrow{f^*} A := \{(b, a) \mid b = f(a)\}.$$

> **Remark** the identity horizontal arrow in $\mathbb{R}\mathrm{el}(\mathrm{Set})$ is defined by the equality relation $=$.

We have two cells

$$\begin{tikzcd}
A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\rotatebox{90}{$\geq$}"] & B \arrow[d, equal] \\
B \arrow[r, equal, "\shortmid"{marking}] & B
\end{tikzcd}
,\quad
\begin{tikzcd}
A \arrow[r, equal, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\rotatebox{90}{$\geq$}"] & A \arrow[d, "f"] \\
A \arrow[r, "f_*"', "\shortmid"{marking}] & B
\end{tikzcd}
.$$

Functions $\longrightarrow$ Relations

f_* is the largest relation among those R's with
$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[dr, "f"'] & B \arrow[d, equal] \arrow[dl, phantom, "\leq" near start] \\ & B \end{tikzcd},$$

and also the smallest one among those S's with
$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\leq"] & \\ A \arrow[rr, "S"', "\shortmid"{marking}] & {} & B \end{tikzcd}.$$

A double category $\mathbb{D}$ is called an **equipment** iff

for $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$, there exist two universal cells
$$\begin{tikzcd} A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[dr, "f"'] & B \arrow[d, equal] \arrow[dl, phantom, "\alpha" near start] \\ & B \end{tikzcd}, \quad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\beta"] & \\ A \arrow[rr, "f_*"', "\shortmid"{marking}] & {} & B \end{tikzcd}$$

s.t.
$$\begin{tikzcd} C \arrow[rr, "R", "\shortmid"{marking}] \arrow[d, "s"'] & {} \arrow[d, phantom, "\xi"] & D \arrow[d, "t"] \\ A \arrow[dr, "f"'] & {} & B \arrow[dl, equal] \\ & B & \end{tikzcd} = \begin{tikzcd} C \arrow[rr, "R", "\shortmid"{marking}] \arrow[d, "s"'] & {} \arrow[d, phantom, "\exists!"] & D \arrow[d, "t"] \\ A \arrow[rr, "f_*", "\shortmid"{marking}] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\alpha"] & B \arrow[dl, equal] \\ & B & \end{tikzcd}$$
and $\cdots$, and there exist

$f^* : B \looparrowright A \quad \cdots$.

Restrictions / extensions

In an equipment, for
$$
\begin{tikzcd}
A \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "\shortmid" marking, "R"'] & D
\end{tikzcd}
,
\qquad
\begin{tikzcd}
A \arrow[r, "\shortmid" marking, "f_*"] \arrow[dr, "f"'] \arrow[r, phantom, shift right=4, "\alpha"] & B \arrow[r, "\shortmid" marking, "R"] \arrow[d, equal] \arrow[dr, phantom, "="] & D \arrow[r, "\shortmid" marking, "g^*"] \arrow[d, equal] & C \arrow[dl, "g"] \arrow[l, phantom, shift right=4, "\gamma"] \\
& B \arrow[r, "\shortmid" marking, "R"'] & D &
\end{tikzcd}
$$
is the universal cell,

$$
\begin{tikzcd}
E \arrow[r, "\shortmid" marking, "S"] \arrow[d, "s"'] & F \arrow[d, "t"] \\
A \arrow[d, "f"'] \arrow[r, phantom, "\xi"] & C \arrow[d, "g"] \\
B \arrow[r, "\shortmid" marking, "R"'] & D
\end{tikzcd}
=
\begin{tikzcd}
E \arrow[r, "\shortmid" marking, "S"] \arrow[d, "s"'] \arrow[dr, phantom, "\exists!"] & F \arrow[d, "t"] \\
A \arrow[d, "f"'] \arrow[r, "\shortmid" marking, "f_*Rg^*"'] & C \arrow[d, "g"] \\
B \arrow[r, "\shortmid" marking, "R"'] & D
\end{tikzcd}
$$

This is called **restriction** of R along f and g.

A cell with such a universal property is called **cartesian**.

Example In $\mathbb{R}\mathrm{el}(\mathrm{Set})$,
$$
\begin{tikzcd}
A \arrow[r, "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{cart}"] & C \arrow[d, "g"] \\
B \arrow[r, "\shortmid" marking, "R"'] & D
\end{tikzcd}
$$
$\{(a,c) \mid f(a)\, R\, g(c)\}$

Restrictions / extensions

Dually, for $\begin{tikzcd} A \arrow[r, "Q", "|" marking] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B & D \end{tikzcd}$, an **extension** $\begin{tikzcd} A \arrow[r, "Q", "|" marking] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "f^* Q g_*"', "|" marking] & D \end{tikzcd}$ is the universal cell for downward composition. Such a cell is called **opcartesian**.

Example In $\mathbb{R}\mathrm{el}(\mathrm{Set})$,

$$\begin{tikzcd} A \arrow[r, "Q", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\text{opcart}"] & C \arrow[d, "g"] \\ B \arrow[r, "|" marking] & D \end{tikzcd}$$

$\{(f(a), g(c)) \mid a \, Q \, c\}$

In particular,

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[rr, "|" marking] \arrow[rr, phantom, bend left, "\text{opcart}"] & & D \end{tikzcd}$$

$\{(f(a), g(a)) \mid a \in A\}$

Relations $\longrightarrow$ Functions

In Set, a relation $A \overset{R}{\nrightarrow} B$ is expressed by two projections from $|R| = \{(a,b) \mid aRb\}$: $A \xleftarrow{\pi_1} |R| \xrightarrow{\pi_2} B$.

These two morphisms have the following property :

$$\begin{array}{c}\{(f(x),g(x))\}\\ \cap \\ R\end{array} \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & \\ A \arrow[rr, "R"', "\shortmid"{marking}] & {} & B \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[d, "\exists!"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & {|R|} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & \\ A \arrow[rr, "R"', "\shortmid"{marking}] & {} & B \end{tikzcd}$$

Moreover, $A \overset{R}{\nrightarrow} B$ is recoverable from π_1 and π_2 :

$$\begin{tikzcd} & {|R|} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & \\ A \arrow[rr, "R"', "\shortmid"{marking}] & {} & B \end{tikzcd}$$ is opcartesian.

$R = \{(\pi_1(x), \pi_2(x))\}$

Relations $\longrightarrow$ Functions

A **tabulator** of a horizontal arrow $A \xrightarrow{R} B$ is an object $|R|$

with
$$\begin{tikzcd} & {|R|} \arrow[dl, "\ell"'] \arrow[dr, "r"] \arrow[d, phantom, "\tau"] & \\ A \arrow[rr, "R"', "|" marking] & {} & B \end{tikzcd}$$
satisfying the following universal property:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ A \arrow[rr, "R"', "|" marking] & {} & B \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[d, "\exists!"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & {|R|} \arrow[dl, "\ell"'] \arrow[dr, "r"] \arrow[d, phantom, "\tau"] & \\ A \arrow[rr, "R"', "|" marking] & {} & B \end{tikzcd} .$$

The tabulator is called **strong** if τ is opcartesian.

Plus, a relation $A \xrightarrow{R} B$ is a subset R of $A \times B$.

In general, we expect a relation $A \xrightarrow{R} B$ to be a "**sub**" of $A \times B$.

Relations $\longrightarrow$ Functions

A class M of morphisms in a category is called a **stable system** if it contains all iso's, is closed under composition, and stable under pullback.

For a stable system M, an **M-relation** $R : A \to\!\!\!\!\!+\; B$ is a morphism $R \longrightarrow A \times B$ in M.

In $\mathbb{R}\mathrm{el}(\mathbb{S}\mathrm{et})$,

$$\begin{tikzcd} & {|R|} \arrow[dl, "\ell"'] \arrow[dr, "r"] \arrow[d, phantom, "\tau"] & \\ A \arrow[rr, "R"', "\shortmid" marking] & & B \end{tikzcd} : \text{tabulator} \implies \begin{tikzcd} {|R|} \arrow[d, "\langle \ell, r \rangle"] \\ A \times B \end{tikzcd} \in \mathrm{Mono}.$$

For a horizontal arrow R, its tabulator

$$\begin{tikzcd} & {|R|} \arrow[dl, "\ell"'] \arrow[dr, "r"] \arrow[d, phantom, "\tau"] & \\ A \arrow[rr, "R"', "\shortmid" marking] & & B \end{tikzcd}$$

is called an **M-tabulator** if $\langle \ell, r \rangle \in M$.

M-comprehension scheme

If $\mathbb{D}$ has M-tabulators for every horizontal arrow for a stable system M in $\mathbb{V}(\mathbb{D})$,

$$\mathcal{H}(\mathbb{D})(A,B) \underset{\text{tab}}{\overset{\text{ext}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} M/_{A\times B} \overset{\text{full}}{\subset} \mathbb{V}(\mathbb{D})/_{A\times B}$$

$$\begin{tikzcd} A \arrow[r, "f^* g_*", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\tilde{\alpha}"] & B \arrow[d, equal] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ A \arrow[rr, "R"', "\shortmid"{marking}] & {} & B \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} & X \arrow[d, "\tilde{\tilde{\alpha}}"'] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & {|R|} \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

$$\boxed{\langle f, g\rangle \in M}$$

$\mathbb{D}$ is said to admit an M-comprehension scheme if the adjoints are equivalences.

Beck-Chevalley pullbacks

$\mathbb{D}$ has **Beck-Chevalley pullbacks** if $\mathbf{V}(\mathbb{D})$ has all pullbacks

and every
$$\begin{tikzcd}
& P \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[d, phantom, "\vee"] & \\
A \arrow[dr, "f"'] & {} & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$
is factored as
$$\begin{tikzcd}
& P \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[d, phantom, "\text{opcart}"] & \\
A \arrow[rr, "|" marking] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\text{cart}"] & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$
.

Example $\mathbb{R}\mathrm{el}(\mathbf{Set})$ has Beck-Chevalley pullbacks :

$$\begin{tikzcd}
& {|P|} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{opcart}"] & \\
A \arrow[rr, "|" marking] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\text{cart}"] & B \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

$$\{(a,b) \mid f(a) = g(b)\} =: P$$

Characterisation Theorem

Theorem [HN.]
For a double category $\mathbb{D}$, the following are equivalent:

(i) $\mathbb{D} \simeq \mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$ for some SOFS (E,M) on some finitely complete category $\mathcal{C}$.

(ii) $\mathbb{D}$ is a cartesian equipment, has Beck-Chevalley pullbacks, and admits an M-comprehension scheme for some M.

Remark Another equivalent condition is given in the paper without "the variable M", and purely double categorically.

SOFS (E, M)	M-relations	$\mathbb{R}\mathrm{el}_{(E,M)}$
(Regepi, Mono) in a regular category	(usual) relations	$\mathbb{R}\mathrm{el}(\mathcal{C})$ [Lam21]
(Iso, Mor) in a finitely complete category	spans	$\mathbb{S}\mathrm{pan}(\mathcal{C})$
(Epi, Regmono) in a quasi-topos	strong relations	

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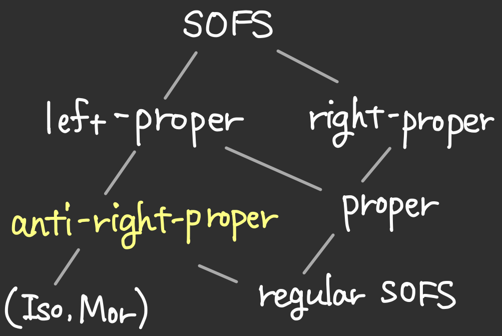
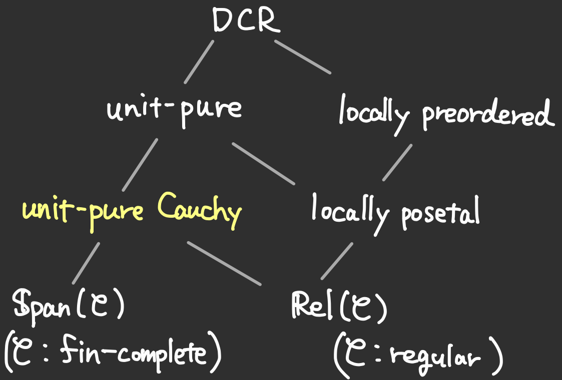
Correspondence of SOFSs and DCRs

Properties of SOFSs are translated to those of DCRs

The correspondence of SOFSs and DCRs (TABLE1 in HN.)

SOFSs	DCRs
	

Cauchy equipments

$\mathcal{C}$ is **Cauchy complete** (idempotent complete) iff for any adjunction of profunctors
$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left, "P", "|"{marking}] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "Q", "|"{marking}] \end{tikzcd}$$
from a small category, there exists a functor $F : \mathcal{D} \to \mathcal{C}$ with $P \cong \mathcal{C}(F-,-)$.

(cf. $\mathbb{P}\mathrm{rof}$: DC of categories, functors, and profunctors)

In an equipment $\mathbb{D}$,
$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left, "f_*", "|"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^*", "|"{marking}] \end{tikzcd} \text{ in } \mathcal{H}(\mathbb{D})$$

unit
$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[d, "f"] \arrow[dr, equal] & \\ A \arrow[r, "f_*"', "|"{marking}] \arrow[ur, phantom, "\text{opc.}"] & B \arrow[r, "f^*"', "|"{marking}] \arrow[ur, phantom, "\text{opc}"] & A \end{tikzcd}$$

counit
$$\begin{tikzcd} B \arrow[r, "f^*", "|"{marking}] \arrow[dr, equal] & A \arrow[r, "f_*", "|"{marking}] \arrow[d, "f"] \arrow[dl, phantom, "\text{cart.}"] \arrow[dr, phantom, "\text{cart.}"] & B \arrow[dl, equal] \\ & B & \end{tikzcd}$$

This kind of adjunctions is called **representable**.

Cauchy equipments

An equipment $\mathbb{D}$ is called Cauchy if any adjunction in $\mathcal{H}(\mathbb{D})$ is representable. (Paré '21)

Example

$\mathbb{P}\mathrm{rof}_{cc}$: double categories of small Cauchy complete categories, functors, and profunctors

Q. How does this condition behave in a DCR?

What is Cauchy DCR?

If we think of horizontal arrows as binary predicates

$$
\begin{tikzcd} A \arrow[r, shift left, "P", "\shortmid"'] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left, "Q", "\shortmid"'] \end{tikzcd}
\rightsquigarrow
\begin{cases}
(\text{unit}) & \forall a : A \ \exists b : B \quad P(a,b) \wedge Q(b,a) \\
(\text{counit}) & \forall b, b' : B, \ \forall a : A \quad Q(b,a) \wedge P(a,b') \to b = b'
\end{cases}
$$

$$\Rightarrow \quad \forall a : A \ \exists ! \, b : B \quad P(a,b)$$

Cauchy condition behaves as the unique choice principle :

$$\forall a : A \ \exists ! \, b : B \quad P(a,b) \Longrightarrow \text{"}\exists\text{"} \, f : A \longrightarrow B \quad P = f_* .$$

Classical results

Proposition [Kelly '91]
For a proper SOFS (E, M), a left adjoint M-relation is of the form $A \xleftarrow{e} X \xrightarrow{f} B$ where $e \in E \cap \mathrm{Mono}$.

In particular, for a regular category $\mathcal{C}$, $\mathbf{Rel}(\mathcal{C})$ is Cauchy.

Proposition [Carboni, Kasangian, Street '84 (in terms of DC)]
$\mathbf{Span}(\mathcal{C})$ is Cauchy.

Cauchy unit-pure DCR

A double category is called unit-pure if

a cell of the form
$$\begin{tikzcd} A \arrow[r, equal, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "g"] \\ B \arrow[r, equal, "|" marking] & B \end{tikzcd}$$
must be
$$\begin{tikzcd} A \arrow[r, equal, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\ddagger"] & A \arrow[d, "f"] \\ B \arrow[r, equal, "|" marking] & B \end{tikzcd}$$
.

Theorem [HN.]

In a unit-pure DCR $\mathbb{R}\mathrm{el}_{(E,M)}(\mathcal{C})$, a horizontal left adjoint is of the form
$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$
where $e \in E \cap \mathrm{Mono}$.

Cauchy unit-pure DCR

We have

$$\boxed{\text{Cauchy unit-pure DCRs} = \text{DCRs with } \mathrm{Mono} \subseteq M}$$

because a unit-pure DCR is Cauchy iff

$E \cap \mathrm{Mono} = \mathrm{Iso} \iff \mathrm{Mono} \subseteq M$.

There is also a "Cauchization" 2-functor

$$\begin{tikzcd} \mathcal{C}\mathrm{auchyUnitpure}\mathcal{DCR} \arrow[r, shift right, hook] \arrow[r, phantom, "\perp"] & \mathrm{Unitpure}\mathcal{DCR} \arrow[l, shift right, "\mathbb{C}\mathrm{au}(-)"'] \end{tikzcd}$$

1. Double categories
2. How should double categories of relations be?
3. The correspondence between DCRs and SOFSs

   — Cauchy double categories of relations
4. Conclusions and future work

Conclusions

- We defined double categories of relations and characterized them using comprehension schemes which involve some double-categorical universal properties.
- Cauchy DCRs are those admitting "unique choice" and correspond to SOFSs (E, M) with Mono CM.
- Other significant classes of SOFSs correspond to those of DCRs.

Future Work

Extending the correspondences

to non-stable OFSs, AWFSs, etc.

Developing logic in double categories

double categories

bicategories

fibered categories (hyperdoctrines)

horizontal composition ⟷ existensial quantifier

horizontal identity ⟷ equality

Thank you!

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References are

[Ale18], [CKS84], [Kel91], [Kle70], [Lam22], [LWW10], [Par21], [Shu08],

and others in the reference list of

ArXiv 2310.19428.