A Formal Theory of Anticolimits

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Introduction

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[rr, squiggly, "\exists ?", "\text{Taking colimit}"'] & & A \arrow[r] \arrow[d] & C \arrow[d] & & & C \arrow[d] \arrow[lll, squiggly, "\exists ?"', "\text{Taking anticolimit}"] \\
B & & & B \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & D & & B \arrow[r] & D
\end{tikzcd}$$

The central question is:

"How can we know if a cocone is a colimit?"

Tataru, Vicary. "The theory and applications of anticolimits" (2024)

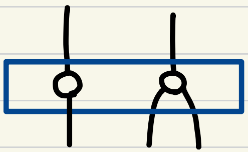
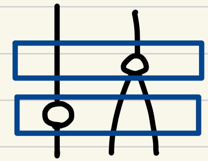
So many examples are out there!

For categories, 2-categories, additive categories, ...

Goal : A conceptual understanding of anticolimits.

Original work

- Tataru and Vicary introduced anticolimits in the study of homotopy.io.

 is a "colimit" of 

- Their problem is like:

Given
$$\begin{tikzcd} A_0 \arrow[dr, bend right, "\varphi_0"'] & A_1 \arrow[d, "\varphi_1"] & \cdots & A_n \arrow[dll, bend left, "\varphi_n"] \\ & Z & & \end{tikzcd}$$
in a category $\mathcal{C}$,

find a diagram
$$\begin{tikzcd} & X_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & & \cdots & & X_n \arrow[dl, "l_n"'] \arrow[dr, "r_n"] & \\ A_0 & & A_1 & \cdots & & & A_n \end{tikzcd}$$
whose colimit is this.

- J : poset, $X : \max J \longrightarrow \mathcal{C}$, $c \in \mathcal{C}$, $\kappa : X \Rightarrow \Delta c$

An anticolimit of κ is an extension $\widetilde{X} : J \to \mathcal{C}$ of X such that κ induces a colimit cocone of κ.

A recipe for anticolimits

J : poset, $X : \max \mathcal{J} \longrightarrow \mathcal{C}$, $c \in \mathcal{C}$, $\kappa : X \Rightarrow \Delta_c$

<u>Def</u> $\Pi_{\mathcal{J}}(\kappa) : \mathcal{J} \longrightarrow \mathcal{C}$ is defined (if possible) as follows:

- $j \in \mathcal{J}$ is mapped to a multiple pullback of $(X_i \xrightarrow{\kappa_i} c)_{i \geq j}$.
- $j \to j'$ in $\mathcal{J}$ is mapped to the canonical arrow in $\mathcal{C}$.

$$\begin{tikzcd}
 & & X_0 \arrow[dr, "\kappa_0"] & \\
 & X_0 \times_c X_1 \arrow[ur] \arrow[r] & X_1 \arrow[r, "\kappa_1"'] & c \\
X_0 \times_c X_1 \times_c X_2 \arrow[ur] \arrow[rr] & & X_2 \arrow[ur, "\kappa_2"'] &
\end{tikzcd}$$

<u>Theorem</u> [TV24] If $\Pi_{\mathcal{J}}(\kappa)$ exists and κ has an anticolimit, then $\Pi_{\mathcal{J}}(\kappa)$ is an anticolimit of κ.

When $\mathcal{C}$ has enough limits, whether κ has an anticolimit can be checked just by looking at $\Pi_{\mathcal{J}}(\kappa)$.

Further Examples

① Regular epi

Def $f : A \to B$ in a category $\mathcal{C}$ is a <u>regular epimorphism</u> if it is a coequalizer of some arrows.

$$X \underset{h}{\overset{k}{\rightrightarrows}} A \xrightarrow{f} B$$

Prop If $\mathcal{C}$ has pullbacks,
f is a regular epimorphism iff it is a coequalizer of its kernel pair.

$$\begin{tikzcd} A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$\in$ an effective epimorphism

② Subcanonicity of sites

Def A site $(\mathcal{C}, T)$ is <u>subcanonical</u> if every representable presheaf is a sheaf w.r.t. T.

Prop TFAE when $\mathcal{C}$: complete.

(i) $(\mathcal{C}, T)$ is sub nical.

(ii) For any T-coverin $A_i \xrightarrow{\kappa_i} C)_i$,
C is a colimit of
$$\{\kappa_i\} \xhookrightarrow{f.f.} \mathcal{C}/C \xrightarrow{\mathrm{dom}} \mathcal{C}$$

(iii) For any T-covering $(A_i \xrightarrow{\kappa_i} C)_i$,
C is a colimit of $\left(\begin{tikzcd}[column sep=small] & A_i \times_C A_j \arrow[dl] \arrow[dr] & \\ A_i & & A_j \end{tikzcd}\right)_{i,j}$.

Further Examples

③ Normal epi in Ab-cats

Def $f: A \to B$ in an Ab-cat. $\mathcal{C}$

is a normal epimorphism if

it is a cokernel of some arrow.

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Prop In a finitely complete Ab-cat,

an arrow is a normal epimorphism

if it is a cokernel of its kernel.

④ Localization in 2-categories

Def An 1-cell is localization if

it is a coinverter of some 2-cell.

Prop A functor $F: \mathcal{C} \to \mathcal{D}$ is

a localization iff

$$\begin{tikzcd} & \mathcal{C}[W^{-1}] \arrow[d, "\cong"] \\ \mathcal{C} \arrow[ur] \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\circlearrowright"] & \mathcal{D} \end{tikzcd}$$

with $W := \{ f \mid Ff : \text{iso} \}$.

⑤ Effective tabulator in double cats

(Strong)

Prop In a flat double cat $\mathbb{D}$

Cells ≤ 1 for each frame

with tabulators,

$A \overset{p}{\nrightarrow} B$ is presented as

$$\begin{tikzcd} & \exists\, C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, "p"', "|" marking] \arrow[rr, phantom, shift left=4, "\text{opc.}"] & & B \end{tikzcd}$$

if it has an effective tabulator

$$\begin{tikzcd} & \{p\} \arrow[dl, "l_p"'] \arrow[dr, "r_p"] & \\ A \arrow[rr, "p"', "|" marking] \arrow[rr, phantom, shift left=4, "\text{opc.}"] & & B \end{tikzcd}$$

Interlude: Formal Category Theory

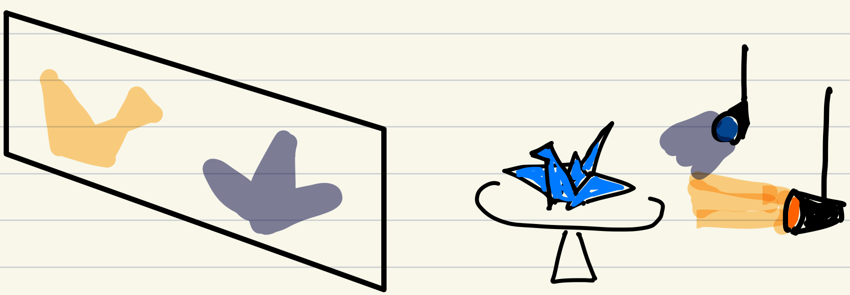
Formal Category Theory = Category Theory of Category Theories

Conceptual treatment from a abstract viewpoint

- $\mathcal{V}$-enriched category theory
- $\mathcal{S}$-internal category theory
- $\mathcal{S}$-fibred category theory

It studies how one can develop category theory inside ~~2-categories~~ (Virtual) double cats by imagining it as the ~~2-category~~ of categories.

Mathematical Phenomena	Categorical Treatment of ...	Categories
Categorical Phenomena	Formal theory of ...	VDCs (or other structures)

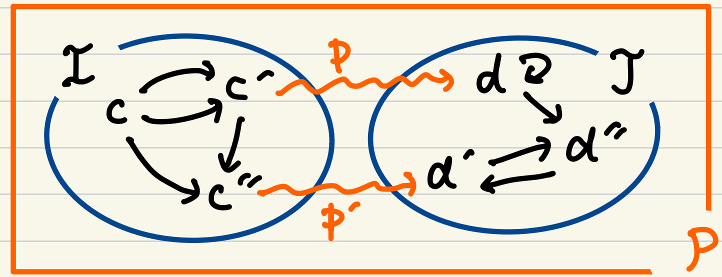


Goal : A formal theory of anticolimits.

Profunctors and Virtual equipments

Def A profunctor $P : \mathcal{I} \nrightarrow \mathcal{J}$ is a functor $P : \mathcal{I}^{op} \times \mathcal{J} \to \mathbf{Set}$.

⚠ Contravariant on its domain.



Prop The following correspond bijectively.

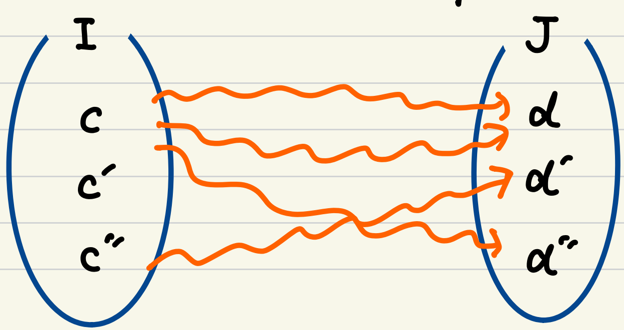
(i) Profunctos $P : \mathcal{I} \nrightarrow \mathcal{J}$

(ii) Pairs of embeddings $\mathcal{I} \overset{i}{\hookrightarrow} P \overset{j}{\hookleftarrow} \mathcal{J}$

s.t. $\mathrm{ob}\,\mathcal{I} \sqcup \mathrm{ob}\,\mathcal{J} \xrightarrow[\cong]{\langle i, j \rangle} \mathrm{ob}\,P$ and $\mathcal{E}(j(d), i(c)) = \emptyset$

Ex • For a category $\mathcal{C}$, we have the hom-profunctor $\mathcal{C}(-, \bullet) : \mathcal{C} \nrightarrow \mathcal{C}$.

• For two sets I, J seen as discrete categories, a profunctor $I \nrightarrow J$ is a bipartite graph (or span.)



Profunctors and Virtual equipments (continued)

A natural trans. $\begin{tikzcd} \mathcal{C} \arrow[d, bend right, "F"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] \\ \mathcal{D} \end{tikzcd}$ is a natural family $(\alpha_c \in \mathcal{D}(Fc, Gc))_{c \in \mathcal{C}}$

Naturality only involves the structure of the hom-profunctor $\mathcal{D}(-,\bullet)$.

generalize

A natural trans. $\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "\alpha"] & \\ \mathcal{D} \arrow[rr, "P"', "\shortmid"{marking}] & {} & \mathcal{D}' \end{tikzcd}$ is a natural family $\underline{\underline{(\alpha_c \in P(Fc, Gc))_{c \in \mathcal{C}}}}$

$(\alpha_{c,c'} : \mathcal{C}(c,c') \to P(Fc, Gc'))_{c,c'}$

$\rightsquigarrow$ $\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{C}(-{,}\bullet)", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{D} \arrow[r, "P"', "\shortmid"{marking}] & \mathcal{D} \end{tikzcd}$

generalize

A natural trans. $\begin{tikzcd} \mathcal{C}_0 \arrow[r, "Q_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "Q_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \mathcal{C}_n \arrow[d, "G"] \\ \mathcal{D} \arrow[rrr, "P"', "\shortmid"{marking}] & & & \mathcal{D}' \end{tikzcd}$ is a natural family

$$\left(Q_0(c_0, c_1) \times \cdots \times Q_n(c_{n-1}, c_n) \xrightarrow{\alpha} P(Fc_0, Gc_n)\right)_{c_0, \dots, c_n}$$

Profunctors and Virtual equipments (continued)

A natural trans.
$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "Q_1", "|"'] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "|"] & \cdots \arrow[r, "Q_n", "|"'] & \mathcal{C}_n \arrow[d, "G"] \\ \mathcal{D} \arrow[rrr, "P"', "|"] \arrow[rrru, phantom, "\alpha"] & & & \mathcal{D}' \end{tikzcd}$$
is a natural family

$$\left(Q_0(c_0, c_1) \times \cdots \times Q_n(c_{n-1}, c_n) \xrightarrow{\alpha} P(Fc_0, Gc_n)\right)_{c_0, \dots, c_n}$$

These natural transformations can be composed like
$$\begin{tikzcd} \cdot \arrow[r, "|"] \arrow[d] & \cdots \arrow[r, "|"] & \cdot \arrow[d] & \cdots & \cdot \arrow[r, "|"] \arrow[d] & \cdots \arrow[r, "|"] & \cdot \arrow[d] \\ \cdot \arrow[rr, "|"] \arrow[d] \arrow[rru, phantom, "\alpha_1"] & & \cdot & \cdots & \cdot \arrow[rr, "|"] \arrow[rru, phantom, "\alpha_n"] & & \cdot \arrow[d] \\ \cdot \arrow[rrrrrr, "|"] \arrow[rrrrrru, phantom, "\beta"] & & & & & & \cdot \end{tikzcd}$$
,

and constitute a virtual double category $\mathbb{PROF}$.

Def A virtual double category $\mathbb{X}$ consists of

- a category $\mathbb{X}_t$ of objects and tight arrows. $\quad \begin{tikzcd} \cdot \arrow[d, "f"] \\ \cdot \end{tikzcd}$
- a family of classes of loose arrows $(\mathbb{X}(x, x'))_{x, x' \in \mathbb{X}}$ $\quad \cdot \xrightarrow{P} \cdot$
- a family of classes of cells for each frame
$$\begin{tikzcd} \cdot \arrow[r, "p_1", "|"'] \arrow[d, "f"'] & \cdots \arrow[r, "p_n", "|"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[rr, "q"', "|"] & & \cdot \end{tikzcd}$$
.
- Data of composition and identities.

Profunctors and Virtual equipments (continued)

Def A restriction of

$$\begin{tikzcd} I \arrow[d, "f"'] & J \arrow[d, "g"] \\ I' \arrow[r, "p"', "\shortmid" marking] & J' \end{tikzcd} \quad \text{is a cell} \quad \begin{tikzcd} I \arrow[r, "{p[f;g]}", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{rest}"] & J \arrow[d, "g"] \\ I' \arrow[r, "p"', "\shortmid" marking] & J' \end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd} K \arrow[r, "\shortmid" marking] \arrow[d, "k"'] & \cdots \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\forall\ \alpha"] & L \arrow[d, "h"] \\ I \arrow[d, "f"'] & & J \arrow[d, "g"] \\ I' \arrow[rr, "p"', "\shortmid" marking] & & J' \end{tikzcd} = \begin{tikzcd} K \arrow[r, "\shortmid" marking] \arrow[d, "k"'] & \cdots \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\exists!\ \tilde{\alpha}"] & L \arrow[d, "h"] \\ I \arrow[rr, "{p[f;g]}", "\shortmid" marking] \arrow[d, "f"'] & \arrow[d, phantom, "\mathrm{rest}"] & J \arrow[d, "g"] \\ I' \arrow[rr, "p"', "\shortmid" marking] & & J' \end{tikzcd} .$$

Def A (loose) unit on I is

a loose arrow $U_I : I \nrightarrow I$

together with a cell

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_I"] & \\ I \arrow[rr, "\shortmid" marking] & {} & I \end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd} K_n \arrow[r, "\shortmid" marking] \arrow[d, "f"'] & \cdots \arrow[r, "\shortmid" marking] & K_1 \arrow[r, "\shortmid" marking] & I \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\forall\ \alpha"] & L_1 \arrow[r, "\shortmid" marking] & \cdots \arrow[r, "\shortmid" marking] & L_m \arrow[d, "g"] \\ N \arrow[rrrrrr, "p"', "\shortmid" marking] & & & & & & M \end{tikzcd}$$

$$\|$$

$$\begin{tikzcd} K_n \arrow[r, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}"] & \cdots \arrow[r, "\shortmid" marking] & K_1 \arrow[r, "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}"] & I \arrow[dl, equal] \arrow[d, phantom, "\eta_I"] \arrow[dr, equal] \arrow[r, "\shortmid" marking] & L_1 \arrow[r, "\shortmid" marking] \arrow[d, equal] & \cdots \arrow[r, "\shortmid" marking] & L_m \arrow[d, equal] \\ K_n \arrow[r, "\shortmid" marking] \arrow[d, "f"'] & \cdots \arrow[r, "\shortmid" marking] & K_1 \arrow[r, "\shortmid" marking] & I \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\exists!\ \tilde{\alpha}"] & I \arrow[r, "\shortmid" marking] & L_1 \arrow[r, "\shortmid" marking] & \cdots \arrow[r, "\shortmid" marking] & L_m \arrow[d, "g"] \\ N \arrow[rrrrrrr, "p"', "\shortmid" marking] & & & & & & & M \end{tikzcd}$$

Def A virtual equipment is

a virtual double categories with restrictions and units on every object.

Colimits via profunctors

Lem The following are in 1-to-1 correspondence.

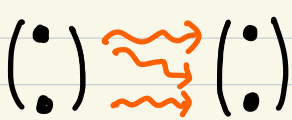
(i) A natural transformation

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "P", "\shortmid"{marking}] \arrow[dr, "F"'] & \arrow[d, phantom, "\alpha"] & \mathcal{J} \arrow[dl, "G"] \\ & \mathcal{C} & \end{tikzcd} \quad \left(= \begin{tikzcd} \mathcal{I} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathcal{J} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "\mathcal{C}(-,\bullet)"', "\shortmid"{marking}] & \mathcal{C} \end{tikzcd} \right)$$

(ii) A functor $A : \mathcal{P} \to \mathcal{C}$ with

$$\begin{tikzcd} \mathcal{I} \arrow[r, hook, "i"] \arrow[dr, "F"'] & \mathcal{P} \arrow[d, "A"] & \mathcal{J} \arrow[l, hook', "j"'] \arrow[dl, "G"] \\ & \mathcal{C} & \end{tikzcd}$$

Example P :



The data above amounts to

$$\begin{tikzcd} F_1 \arrow[r, "f"] \arrow[dr, "g"] & G_1 \\ F_2 \arrow[r, "h"'] & G_2 \end{tikzcd} \quad \text{in } \mathcal{C}$$

Colimits can be captured with triangle cells of this form:

Lem Let $F : \mathcal{I} \to \mathcal{C}$, $P : \underline{\mathcal{I} \not\to 1}$
$\mathcal{I}^{op} \to \mathrm{Set}$

The weighted colimit $\mathrm{colim}^P F$ shows the following universal property

$$\begin{tikzcd} \mathcal{I} \arrow[r, "P", "\shortmid"{marking}] \arrow[dr, "F"'] \arrow[r, phantom, shift right=4, "\forall\alpha"] & 1 \arrow[dl, bend left, "\forall c"] \\ & \mathcal{C} \end{tikzcd} = \begin{tikzcd} \mathcal{I} \arrow[r, "P", "\shortmid"{marking}] \arrow[dr, "F"'] \arrow[r, phantom, shift right=4, "\lambda"] & 1 \arrow[d, "\mathrm{colim}^P F"'] \arrow[dl, bend left, "c"] \\ & \mathcal{C} \end{tikzcd} \quad \exists!\ \tilde{\alpha}$$

$$\begin{tikzcd} F_i \arrow[dr, "\alpha_p"] \arrow[dd, "Fu"'] & \\ & c \\ F_{i'} \arrow[ur, "\alpha_{p'}"'] & \end{tikzcd} = \begin{tikzcd} F_i \arrow[dr, "\lambda_p"] \arrow[dd, "Fu"'] & & \\ & \mathrm{colim}^P F \arrow[r, "\exists!\,\tilde{\alpha}"] & c \\ F_{i'} \arrow[ur, "\lambda_{p'}"'] & & \end{tikzcd}$$

⚠ This is not the "correct" notion of limit.

Anticolimits via profunctors: balances

We take the example of regular epimorphisms as a model case,

and will solve the inverse problem of $X \overset{f}{\underset{g}{\rightrightarrows}} A \xmapsto{\text{colimit}} A \xrightarrow{e} Z$

Fixed data

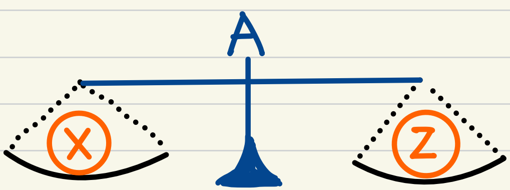
Def [Street. '80]

A gamut is a cell of the form

$$\begin{tikzcd} & \mathcal{I} \arrow[dr, "N", "|" marking] & \\ \mathcal{J} \arrow[ur, "M", "|" marking] \arrow[rr, bend right, "P"', "|" marking] \arrow[rr, phantom, "\mu"] & & \mathcal{K} \end{tikzcd}$$

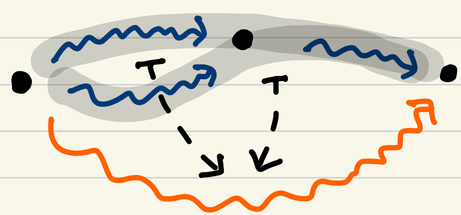
Def A balance on $\mathcal{C}$ consists of

a gamut μ and $\begin{tikzcd} \mathcal{I} \arrow[d, "A"] \\ \mathcal{C} \end{tikzcd}$ (fulcrum)



Ex

$$\mu_{coeq} := \begin{tikzcd} (\bullet) \arrow[r, "2", "|" marking] \arrow[rr, bend right, "1"', "|" marking] \arrow[rr, phantom, "\Downarrow"] & (\bullet) \arrow[r, "1", "|" marking] & (\bullet) \end{tikzcd}$$



Ex A fulcrum for μ_{coeq} is an object $A \in \mathcal{C}$.

Anticolimits via profunctors: diagrams and bicones

Def (μ, A) : balance

A <u>left (right) diagram</u> of (μ, A) is a pair (X, ξ) (resp. (Z, ζ)) where

$$
\begin{tikzcd}
\mathcal{J} \arrow[r, "M", "|" marking] \arrow[dr, "X"'] & \mathcal{I} \arrow[d, "A"] \arrow[dl, phantom, "\xi"] \\
& \mathcal{C}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{I} \arrow[r, "N", "|" marking] \arrow[d, "A"'] & \mathcal{K} \arrow[dl, "Z"] \arrow[dl, phantom, "\zeta"'] \\
\mathcal{C} &
\end{tikzcd}
.
$$

Ex For (μ_{coeq}, A),

$$\left(\begin{array}{l}\text{Left} \\ \text{diagrams}\end{array}\right) = \left(\begin{array}{l}\text{Parallel arrows} \\ X \overset{f}{\underset{g}{\rightrightarrows}} A \quad \text{into } A\end{array}\right)$$

$$\left(\begin{array}{l}\text{Right} \\ \text{diagrams}\end{array}\right) = \left(\begin{array}{l}\text{Arrows} \\ A \xrightarrow{e} Z \quad \text{from } A\end{array}\right)$$

The left/right diagrams constitute categories $\mathbf{LD}(\mu, A)$ and $\mathbf{RD}(\mu, A)$.

$(X, \xi) \xrightarrow{\sigma} (X', \xi')$ in $\mathbf{LD}(\mu, A)$ $\quad\Big\|\quad$

$$
\begin{tikzcd}
& \mathcal{J} \arrow[dl, bend right, "X"'] \arrow[dr, bend left, "X'"] & \\
\mathcal{C} \arrow[rr, phantom, "\sigma"] & & \mathcal{C}
\end{tikzcd}
$$

$X \overset{\sigma}{\Rightarrow} X' : \mathcal{J} \to \mathcal{C}$ s.t.

$$
\begin{tikzcd}
\mathcal{J} \arrow[r, "M", "|" marking] \arrow[dr, bend right=60, "X"'] \arrow[dr, "X'"] \arrow[dr, phantom, bend right=30, "\sigma"] & \mathcal{I} \arrow[dl, "A"] \arrow[dl, phantom, bend left=20, "\xi'"] \\
& \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{J} \arrow[r, "M", "|" marking] \arrow[dr, "X"'] & \mathcal{I} \arrow[dl, "A"] \arrow[dl, phantom, "\xi"'] \\
& \mathcal{C}
\end{tikzcd}
$$

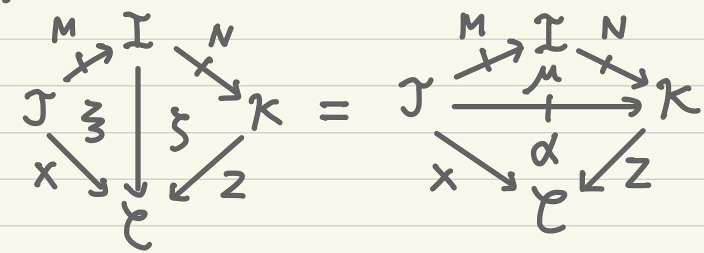
Def A <u>bicone</u> of (μ, A) is a triple of

- a left diagram $\xi = (X, \xi)$
- a right diagram $\zeta = (Z, \zeta)$, and
- a cell α such that ...

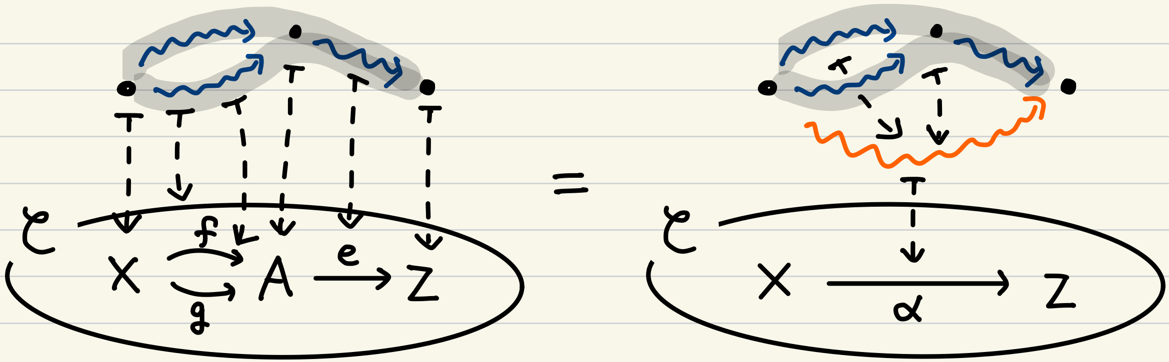
$$
\begin{tikzcd}
& \mathcal{I} \arrow[dd, "A"] \arrow[dr, "N", "|" marking] & \\
\mathcal{J} \arrow[ur, "M", "|" marking] \arrow[dr, "X"'] \arrow[r, phantom, "\xi"] & {} \arrow[r, phantom, "\zeta"] & \mathcal{K} \arrow[dl, "Z"] \\
& \mathcal{C} &
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{I} \arrow[dr, "N", "|" marking] & \\
\mathcal{J} \arrow[ur, "M", "|" marking] \arrow[rr, "\mu", "|" marking] \arrow[dr, "X"'] & \alpha & \mathcal{K} \arrow[dl, "Z"] \\
& \mathcal{C} &
\end{tikzcd}
$$

Def A bicone of (μ, A) is a triple of

- a left diagram (X, ξ)
- a right diagram (Z, ζ), and
- a cell α such that ...



Ex For (μ_{coeq}, A),



α is unique if it exists.

Prop $\mathrm{Bicone} : \mathbf{LD}(\mu, A)^{op} \times \mathbf{RD}(\mu, A) \to \mathrm{Set}$ is a functor,

$$(X, \xi) \quad , \quad (Z, \zeta) \longmapsto \left\{ \text{bicones over } \begin{matrix} (X,\xi) \\ (Z,\zeta) \end{matrix} \right\}$$

which gives a profunctor $\mathrm{Bicone} : \mathbf{LD}(\mu, A) \nrightarrow \mathbf{RD}(\mu, A)$.

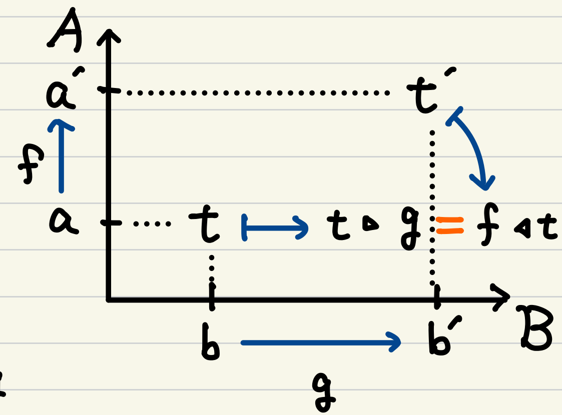
General results on profunctors (1)

Lem (Two-sided Grothendieck construction)

For a profunctor $T : A \nrightarrow B$,

the category $\mathcal{T}$ defined by

- objects : $(a \in A,\ b \in B,\ t \in T(a,b))$
- arrows $\begin{array}{c}(a,b,t)\\ \downarrow \\ (a',b',t')\end{array} : \left(\begin{array}{cc} a & b \\ f\downarrow & ,\ \downarrow g \\ a' & b' \end{array}\right)$ s.t. $f \triangleleft t' = t \triangleright g$



induces a two-sided discrete fibration with the obvious forgetful functors.

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "L"', "\text{opf}"] \arrow[dr, "R", "\text{fib}"'] & \\ A & & B \end{tikzcd}$$

For BiCone, we write $\mathbf{Bico}(\mu, A)$ for this category.

Notation
- ${}_a\mathcal{T}$: the fiber of $\mathcal{T} \xrightarrow{L} A$ at $a \in A$.
- $\mathcal{T}_b$: the fiber of $\mathcal{T} \xrightarrow{R} B$ at $b \in B$.

For a left diagram ξ, objects of ${}_{\xi}\mathbf{Bico}(\mu, A)$ are called anticones.
(resp. right 〃 ζ) (resp. $\mathbf{Bico}(\mu, A)_{\zeta}$) (resp. anticocones.)

Anticolimits via profunctors: limit / colimit bicone

Def A **colimit bicone** of $\xi \in \mathbf{LD}(\mu, A)$ is an initial objects in ${}_{\xi}\mathbf{Bico}(\mu, A)$.

A **limit bicone** of $\zeta \in \mathbf{RD}(\mu, A)$ is a terminal objects in $\mathbf{Bico}(\mu, A)_{\zeta}$

$\begin{pmatrix}\text{Left}\\ \text{diagrams}\end{pmatrix} = \begin{pmatrix}\text{Parallel arrows}\\ X \overset{f}{\underset{g}{\rightrightarrows}} A \quad \text{into } A\end{pmatrix}$

$\begin{pmatrix}\text{Right}\\ \text{diagrams}\end{pmatrix} = \begin{pmatrix}\text{Arrows}\\ A \xrightarrow{e} Z \quad \text{from } A\end{pmatrix}$

for (μ_{coeq}, A),

Ex The case of (μ_{coeq}, A).

- For $\xi := \left(X \overset{f}{\underset{g}{\rightrightarrows}} A\right)$, ${}_{\xi}\mathbf{Bico} \cong \left\{ \begin{array}{c} A \\ \downarrow k \\ Y \end{array} \;\middle|\; X \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{k} Y \text{ : cofork} \right\} \overset{\text{f.f.}}{\hookrightarrow} A/\mathcal{C}$.

A colimit bicone is a **coequalizer**.

- For $\zeta := (A \xrightarrow{e} Z)$, $\mathbf{Bico}_{\zeta} \cong \left\{ A \xleftarrow{p} W \xrightarrow{q} A \;\middle|\; W \overset{p}{\underset{q}{\rightrightarrows}} A \xrightarrow{e} Z \text{ : cofork} \right\} \overset{\text{f.f.}}{\hookrightarrow} {}_{A}\mathbf{Span}(\mathcal{C})_{A}$.

A limit bicone is a **kernel pair**.

General results on profunctors (2)

Lem For a two-sided discrete fibration $\begin{tikzcd} & T \arrow[dl, "L"'] \arrow[dr, "R"] & \\ A & & B \end{tikzcd}$,

(i) L has a left adjoint iff ${}_aT$ has an initial for all $a \in A$.

(ii) R has a right adjoint iff T_b has a terminal for all $b \in B$.

The values of these adjoints are given by the initials/terminals.

The resulting adjoints are fully-faithful.

Assuming the existence of limit/colimit bicones, we obtain:

$$\boxed{\begin{tikzcd} \mathbf{LD}(\mu, A) \arrow[r, shift left=1.5, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & \mathbf{Bico}(\mu, A) \arrow[l, shift left=1.5, "L"] \arrow[r, shift left=1.5, "R"] \arrow[r, phantom, "\perp"] & \mathbf{RD}(\mu, A) \arrow[l, shift left=1.5, "\mathrm{Lim}"] \end{tikzcd}.}$$

Ex For (μ_{coeq}, A), this adjunction is:

$$\begin{tikzcd} {}_A\mathrm{Span}(\mathcal{C})_A \arrow[r, shift left=1.5, "\mathrm{coeq.}"] \arrow[r, phantom, "\perp"] & A/\mathcal{C} \arrow[l, shift left=1.5, "\mathrm{kerpair.}"] \end{tikzcd}.$$

Remark Most of the results can be developed without assuming (co)limits; we could use relative adjoints on both sides.

General results on profunctors (3)

$$\boxed{\begin{tikzcd} LD(\mu, A) \arrow[r, shift left, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & \mathrm{Bico}(\mu, A) \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & RD(\mu, A) \arrow[l, shift left, "\mathrm{Lim}"] \end{tikzcd}.}$$

Prop If $T : A \to\!\!\!\!\!\!\!\!\to B$ is a propositional profunctor and induces

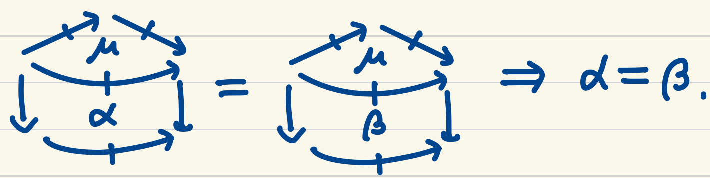
$:\Leftrightarrow \# T(a,b) \leqslant 1.$

the adjunction $\begin{tikzcd} A \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \Upsilon \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & B \arrow[l, shift left] \end{tikzcd}$, then this is idempotent.

Ex If the gamut μ is epimorphic w.r.t. vertical composition,

Bicone is propositional.

μ_{coeq} is of this kind.



Remark Idempotency of adjunctions are described in various ways.

$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "G"] \end{tikzcd}$ is idempotent $\Longleftrightarrow$ $F\eta$: iso $\Longleftrightarrow$ $\mathrm{Fix}(FG) = \mathrm{Im}(F) \subseteq B$

$\Longleftrightarrow \cdots$ $\qquad := \{ b \mid \varepsilon_b : \mathrm{iso} \}$

Main result : anticolimit and effectiveness

Thm Let (μ, A) be a balance.

$$LD(\mu,A) \underset{L}{\overset{\mathrm{Colim}}{\rightleftarrows}} \mathrm{Bico}(\mu,A) \underset{\mathrm{Lim}}{\overset{R}{\rightleftarrows}} RD(\mu,A).$$

If Bicone of this is propositional, then the following are equivalent for $\zeta \in RD(\mu,A)$.

(i) $\zeta \cong R \circ \mathrm{Colim}(\xi)$ for some $\xi \in LD(\mu,A)$.

$:\Longleftrightarrow$ ζ has an anticolimit.

(ii) $\zeta \cong R \circ \mathrm{Colim} \circ L \circ \mathrm{Lim}(\zeta)$ (canonically)

ζ is "the colimit of the limit" of itself.

Ex For (μ_{coeq}, A),

(i) e : regular epi

(ii) e : coeq. of its kernel pair

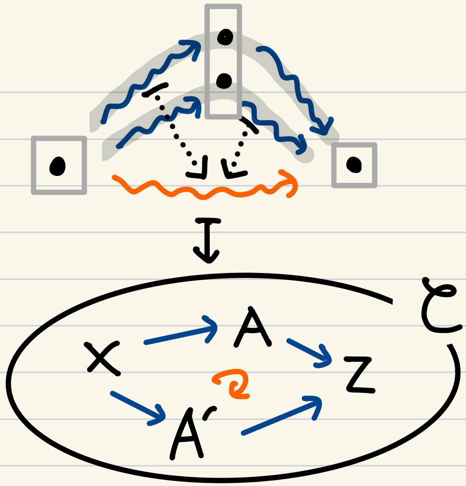
Cor The adjunction reduces to an equivalence :

$$\left\{ \xi \in LD \,\middle|\, \begin{array}{l} \xi \text{ has} \\ \text{an antilimit} \end{array} \right\} \simeq \left\{ \zeta \in RD \,\middle|\, \begin{array}{l} \zeta \text{ has} \\ \text{an anticolimit} \end{array} \right\}.$$

Captured Examples (1)

① Set-enriched case

(i) $\mu_{pbpo} :=$
$$\begin{tikzcd} & 2 \arrow[dr, "1" description] \arrow[d, phantom, "\Downarrow !"] & \\ 1 \arrow[ur, "1" description] \arrow[rr, "1"'] & & 1 \end{tikzcd}$$



The resulting adj. is
$$\begin{array}{c} {}_A\mathrm{Span}(\mathcal{C})_{A'} \\ \text{p.b.} \uparrow \vdash \downarrow \text{p.o.} \\ {}_A\mathrm{Cospan}(\mathcal{C})_{A'} \end{array}$$

(ii) S : set

$\mu_{mk,S} :=$
$$\begin{tikzcd} & S \arrow[dr, "1" description] \arrow[d, phantom, "\Downarrow !"] & \\ S\times S \arrow[ur, "M" description] \arrow[rr, "1"'] & & 1 \end{tikzcd}$$
where M only has $(s,s') \overset{l_{s,s'}}{\rightsquigarrow} s$, $(s,s') \overset{r_{s,s'}}{\rightsquigarrow} s'$ $(\forall s,s')$

$(\zeta_s : A_s \to Z)_{s\in S}$ has an anticolimit iff it is a colimit of
$$\left(\begin{tikzcd}[row sep=small] & A_s \\ A_s \times_Z A_{s'} \arrow[ur] \arrow[dr] & \\ & A_{s'} \end{tikzcd} \right)_{s,s'}.$$

② Ab-enriched case

$\mu_{ck} :=$
$$\begin{tikzcd} & \Delta 1 \arrow[dr, "\mathbb{Z}" description] \arrow[d, phantom, "\Downarrow !"] & \\ \Delta 1 \arrow[ur, "\mathbb{Z}" description] \arrow[rr, "0"'] & & \Delta 1 \end{tikzcd}.$$

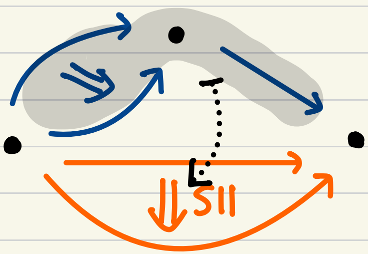
The resulting adjunction is
$$\begin{tikzcd} \mathcal{C}/A \arrow[r, shift left, "\mathrm{Coker}"] \arrow[r, phantom, "\perp"] & A/\mathcal{C} \arrow[l, shift left, "\mathrm{Ker}"] \end{tikzcd}.$$

Similar for $(\mathrm{Set}_*, \wedge)$-cats.

Captured Examples (2)

③ Cat-enriched case

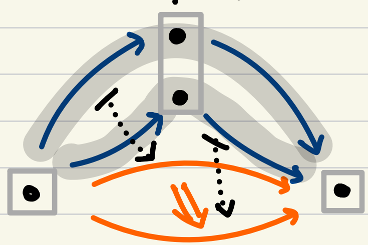
(i)



$$\left\{ \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & A \end{tikzcd} \right\} \begin{tikzcd} {} \arrow[r, bend left, "\text{“Coinvt”}"] \arrow[r, phantom, "\perp"] & {} \arrow[l, bend left] \end{tikzcd} A/\mathcal{K}_0$$

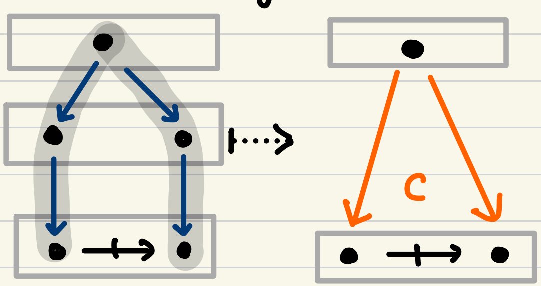
$$\begin{tikzcd} \mathrm{Lim}(\xi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z^{(\bullet\cong\bullet)} \arrow[d] & & A \arrow[d, "\xi"] \\ A^{(\bullet\to\bullet)} \arrow[r, "\xi^{\bullet\to\bullet}"'] & Z^{(\bullet\to\bullet)} & & Z \arrow[ll, mapsto] \end{tikzcd}$$

(ii) (⚠ Non-propositional)



$${}_A\mathrm{Span}(\mathcal{K})_B \begin{tikzcd} {} \arrow[r, bend left, "\text{Cocomma}"] \arrow[r, phantom, "\perp"] & {} \arrow[l, bend left, "\text{Comma}"] \end{tikzcd} {}_A\mathrm{Cospan}(\mathcal{K})_B$$

④ Double categories



$${}_A\mathrm{Span}(\mathbb{D}_0)_B \begin{tikzcd} {} \arrow[r, bend left, "\text{Ext}"] \arrow[r, phantom, "\perp"] & {} \arrow[l, bend left, "\text{Tab}"] \end{tikzcd} \left\{ \begin{tikzcd} A \arrow[d] & B \arrow[d] \\ \bullet \arrow[r, "\shortmid" marking] & \bullet \end{tikzcd} \text{ in } \mathbb{D} \right\}$$

$$\perp \qquad \mathbb{D}(A, B)$$

Future application

(i) Formal theory of homology ?

For a gamut of the form

$$\begin{tikzcd} & \mathcal{I} \arrow[dr, "M", "|"] \arrow[d, phantom, "\mu"] & \\ \mathcal{I} \arrow[ur, "M", "|"] \arrow[rr, "P"', "|"] & {} & \mathcal{I} \end{tikzcd}$$

(vertically epimorphic)

we can define chain complex as

$$\begin{tikzcd} \cdots \arrow[r, "|"] & \mathcal{I} \arrow[r, "M", "|"'] \arrow[drr, bend right, "c_2"'] \arrow[r, phantom, shift right=4, "\varphi_{21}"] & \mathcal{I} \arrow[r, "M", "|"'] \arrow[dr, "c_1"] \arrow[r, phantom, shift right=4, "\varphi_{10}"] & \mathcal{I} \arrow[r, "M", "|"'] \arrow[d, "c_0"] \arrow[r, phantom, shift right=4, "\varphi_{0,-1}"] & \mathcal{I} \arrow[r, "|"] \arrow[dl, bend left, "c_{-1}"] & \cdots \\ & & & \mathcal{C} & & \end{tikzcd}$$

s.t. the composites of two adjacent φ's factor through μ.

Ex

$$\begin{tikzcd} & \Delta 1 \arrow[dr, "\mathbb{Z}", "|"] \arrow[d, phantom, "\Downarrow !"] & \\ \Delta 1 \arrow[ur, "\mathbb{Z}", "|"] \arrow[rr, "0"', "|"] & {} & \Delta 1 \end{tikzcd}$$

in Ab-Prof

leads to a usual one.

(ii) Formal theory of regularity ?

Bourke and Garner developed theory of 2-dimensional regularity based on kernel-quotient systems.

Def [BG14] A kernel-quotient system is an f.f. embedding $J : \underset{(0 \to 1)}{\overset{}{2}} \hookrightarrow F$

$2 = (0 \to 1)$

Let $K := F \setminus \{J1\} \overset{\text{f.f.}}{\hookrightarrow} F$

With this, they construct an adjunction

$$\begin{tikzcd} {[K, \mathcal{C}]} \arrow[r, shift left=2, "\text{quot}"] \arrow[r, phantom, "\perp"] & {[2, \mathcal{C}]} \arrow[l, shift left=2, "\text{ker}"] \end{tikzcd}$$

We can recover this by taking μ as

$$\begin{tikzcd} & 1 \arrow[dr, "I", "|"] \arrow[d, phantom, "\Downarrow (J0 \to J1)_*"] & \\ F \setminus \mathrm{Im} J \arrow[ur, "{F(-, J0)}", "|"] \arrow[rr, "{F(-, J1)}"', "|"] & {} & 1 \end{tikzcd}$$

Final slide

Summary

- Anticolimit is the inverse problem of colimits. Solutions are occasionally given "effectively".
- We provided a general theory of anticolimits in virtual equipment.
- We constructed the (possibly relative) adjoint of limit and colimit from a data of diagram shape (= gamut).

$$\mathrm{LD}(\mu, A) \underset{L}{\overset{\mathrm{Colim}}{\rightleftarrows}} \mathrm{Bico}(\mu, A) \underset{\mathrm{Lim}}{\overset{R}{\rightleftarrows}} \mathrm{RD}(\mu, A).$$

- A sufficient condition for this adjunction to be idempotent is that the gamut μ is vertically epimorphic.

What I couldn't include

- Pointwiseness of limits / colimits
- The cases of relative adjoints

What I want to look into

- Preservation / Reflection / Stability
- Relation to other formal theories

References

[TV24] Callin Tataru and Jamie Vicary. The theory and applications of anticolimits. 2024. preprint on arXiv.

[BG14] John Bourke and Richard Garner. Two-dimensional regularity and exactness. 2014. Journal of Pure and Applied Algebra.

[St80] Ross Street. Fibrations in bicategories. 1980, Cahier de Topologie et Géométrie Différentielle.

[nLab+] nLab page. Generalized kernels

Thank you!

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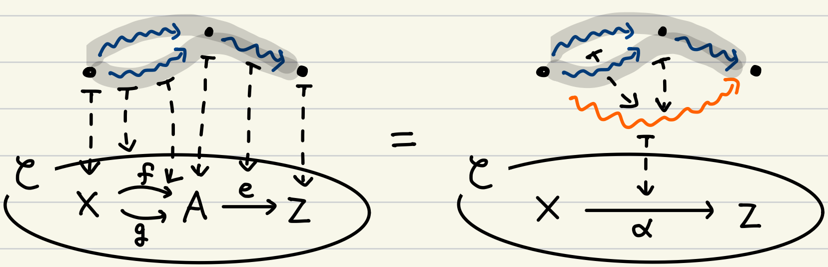
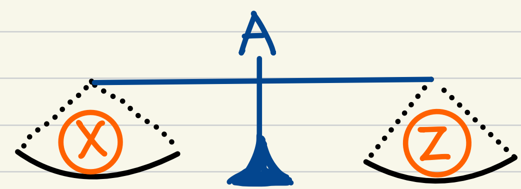
Please let me know if you hit upon any example of anticolimit!





$$\boxed{\begin{tikzcd} \mathrm{LD}(\mu, A) \arrow[r, shift left, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & \mathrm{Bico}(\mu, A) \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & \mathrm{RD}(\mu, A). \arrow[l, shift left, "\mathrm{Lim}"] \end{tikzcd}}$$