

From Fibrations

to Virtual Double Categories

Categorical Logic Meets Double Categories – Side A

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Outline

This talk is based on my recent thesis:

Logical Aspects of
Virtual Double Categories
Chapter 2 :
Categorical Logic meets
Double Categories

Today's slides



Fibrations (Lawvere.'70 , Jacobs.'99)

- ✓ terms(functions)
- ✗ binary relations

Today's focus

Virtual Double Categories

- ✓ terms(functions)
- ✓ binary relations

Bicategories (Carboni, Walters.'87)
(Freyd, Scedrov.'90)

- ✗ terms(functions)
- ✓ binary relations

1. Fibrations and Logic

2. Virtual Double Categories
of Relations

3. Comparing Fibrations
and Virtual Double Categories

4. Conclusion

1. Fibrations and Logic

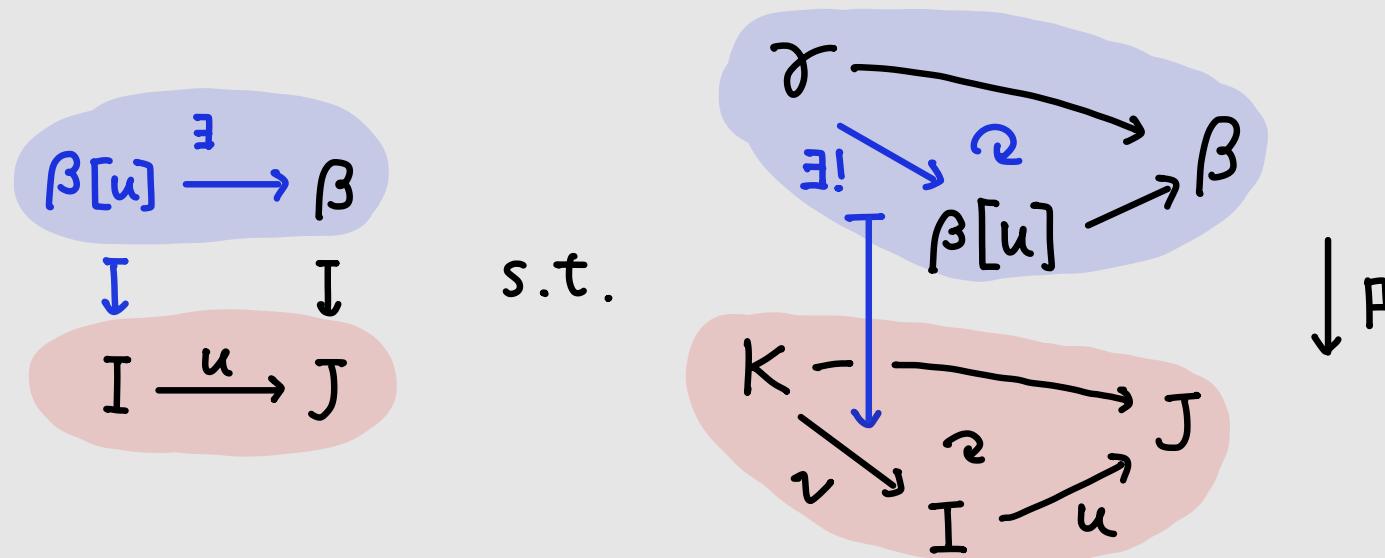
2. Virtual Double Categories
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Taking semantics in fibrations

A functor $\downarrow_P^{\mathcal{E}}$ is a (Grothendieck) fibration



Idea

B ... the category of contexts Γ

and sequences of terms $\Gamma \vdash S : \Delta$

$x_1 : R, x_2 : R, x_3 : M$

$$\begin{array}{c} \downarrow \\ (x_2 \cdot x_3, x_1 - x_2) \\ \downarrow \\ y_1 : M, y_2 : R \end{array}$$

\mathcal{E}_Γ (the fiber category)

... the category of formulas $\Gamma \vdash \psi$

and proofs of implication $\Gamma \vdash p : \psi \Rightarrow \gamma$

Example

(E, M) : a stable factorization system on \mathcal{C}

$\rightsquigarrow M$ (M : the full subcategory of \mathcal{C}^\rightarrow spanned by $m \in M$)
 \downarrow
 \mathcal{C} is a fibration.

$$\begin{array}{c} \exists y : R (y + x = y) \\ \downarrow \\ x = 0 \end{array}$$

$\mathcal{E}_{x : R}$

Taking semantics in fibrations

Given a language \mathcal{L} , what structures do we need on a fibration to interpret formulas in \mathcal{L} ?

- a sort $\sigma \in \mathcal{L}$ \rightsquigarrow an object $[\sigma]$ in \mathbf{B}
- a context $x_1 : \sigma_1, \dots, x_n : \sigma_n$
 \rightsquigarrow the product $[\sigma_1] \times \dots \times [\sigma_n]$
- a function symbol $f : \sigma_1, \dots, \sigma_n \rightarrow \tau$
 \rightsquigarrow an arrow $[\mathbf{f}] : [\sigma_1] \times \dots \times [\sigma_n] \rightarrow [\tau]$ in \mathbf{B}
- the interpretation of terms
is inductively defined
e.g. $[\mathbf{f}(g(x), y)] = [\mathbf{f}] \circ ([\mathbf{g}] \times \text{id})$

- a relation symbol $R : \sigma_1, \dots, \sigma_n$

\rightsquigarrow an object $[\mathbf{R}]$ in the fiber of $[\sigma_1] \times \dots \times [\sigma_n]$

- substitution $\varphi[t_1/x_1, \dots, t_n/x_n]$

$$\begin{array}{ccc} [\varphi[t_1/x_1, \dots, t_n/x_n]] & \longrightarrow & [\varphi] \\ \downarrow & & \downarrow \\ \langle [\mathbf{t}_1], \dots, [\mathbf{t}_n] \rangle & & \\ [[\tau_1]] \times \dots \times [[\tau_m]] & \longrightarrow & [[\sigma_1]] \times \dots \times [[\sigma_n]] \end{array}$$

using the cartesian (prone) lifting.

- the conjunction $\varphi \wedge \gamma \rightsquigarrow$ fiberwise product $[\varphi] \wedge [\gamma]$

- the existential quantifier $\exists z : \sigma. \varphi(\vec{x}, z)$

\rightsquigarrow the image of $[\varphi(\vec{x}, z)] \in \Sigma_{[[\Gamma]] \times [\sigma]}$
by the left adjoint

$$\begin{array}{ccc} \Sigma_{[[\Gamma]] \times [\sigma]} & \xrightarrow{\exists \pi} & \Sigma_{[[\Gamma]]} \\ \xleftarrow{(\dashv) [\pi]} & & \end{array}$$

$(\pi : [[\Gamma]] \times [\sigma] \rightarrow [[\Gamma]])$

Taking semantics in fibrations

A sufficient structure for regular logic
is an **elementary existential doctrine** by Lawvere.

Def A fibration P is **cartesian** if

- the base category B has finite products,
- the fiber categories E_I have finite products,
and the reindexings $(-)[u]$ preserve them.

Def A fibration P is **elementary existential**

- if
- it is cartesian,
 - all reindexing functors $(-)[u]$ for \xrightarrow{u} ,

of the forms

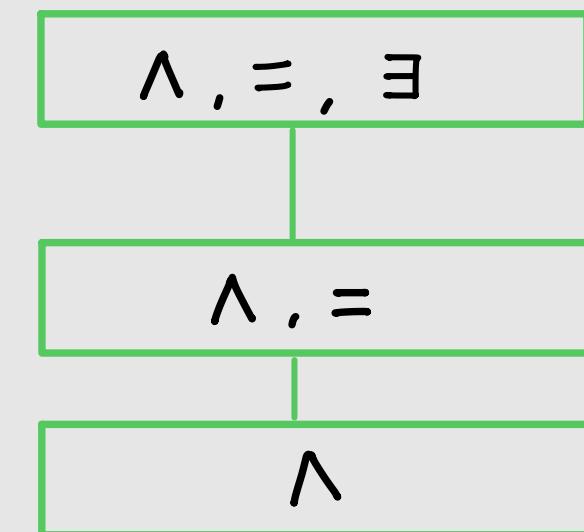
$$\begin{cases} I \times J \xrightarrow{\pi_2} J \\ I \times J \xrightarrow{\Delta \times \text{id}} I \times I \times J \end{cases}$$

have a left adjoint $\exists u$, and

- it satisfies some axioms.

(the Beck-Chevalley condition)
(the Frobenius reciprocity)

We have hierarchies of
logical systems and classes of fibrations.



**elementary
existential fibrations**

elementary fibrations

cartesian fibrations

We will add a new column of hierarchy:
the hierarchy of virtual double categories.

Constructing a new fibration

⚠ Fibers are pre-ordered in this slide.

Quotient completion (Maietti, Rosolini, '13)

The quotient completion is an analogue of the exact completion in the context of doctrines.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\text{Des}(\mathbb{P})} & \\ \mathbb{P} \downarrow & \rightsquigarrow & \downarrow \mathbb{Q}(\mathbb{P}) \\ B & & Q(\mathbb{P}) \end{array}$$

Objects in $Q(\mathbb{P})$ are \mathbb{P} -equivalence relations, that is, pairs of $I \in B$ and $\rho \in \Sigma_{I \times I}$ s.t.

- $\delta_I \leq \rho$ in $\Sigma_{I \times I}$ where $\delta_I = \exists_{\Delta_I}(\tau)$
- $\rho \leq \rho[\langle \pi_2, \pi_1 \rangle]$ in $\Sigma_{I \times I}$
- $\rho[\langle \pi_1, \pi_2 \rangle] \wedge \rho[\langle \pi_2, \pi_3 \rangle] \leq \rho[\langle \pi_1, \pi_3 \rangle]$ in $\Sigma_{I \times I \times I}$

Function comprehension (Pavlović '96)

\mathbb{P} is function-comprehensive

if for any $\rho \in \Sigma_{I \times J}$ with

- $\rho[\langle \pi_1, \pi_2 \rangle] \wedge \rho[\langle \pi_1, \pi_3 \rangle] \leq \delta[\langle \pi_2, \pi_3 \rangle]$
- $\tau \leq \exists_{\pi_1}(\rho)$,

there is $u: I \rightarrow J$ in B s.t. $\rho \cong \delta[u \times \text{id}_J]$.

= "Single-valued total relations come from a function."

The paper mentions function comprehensive completion.

Motivation

cumbersome constructions

fibrations with \star



relational structures with \star'

clearer?

fibrations with \heartsuit



relational structures with \heartsuit'

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Virtual double categories of relations

The double category Rel :

objects ... sets

tight arrows $I \xrightarrow{u} J$... functions

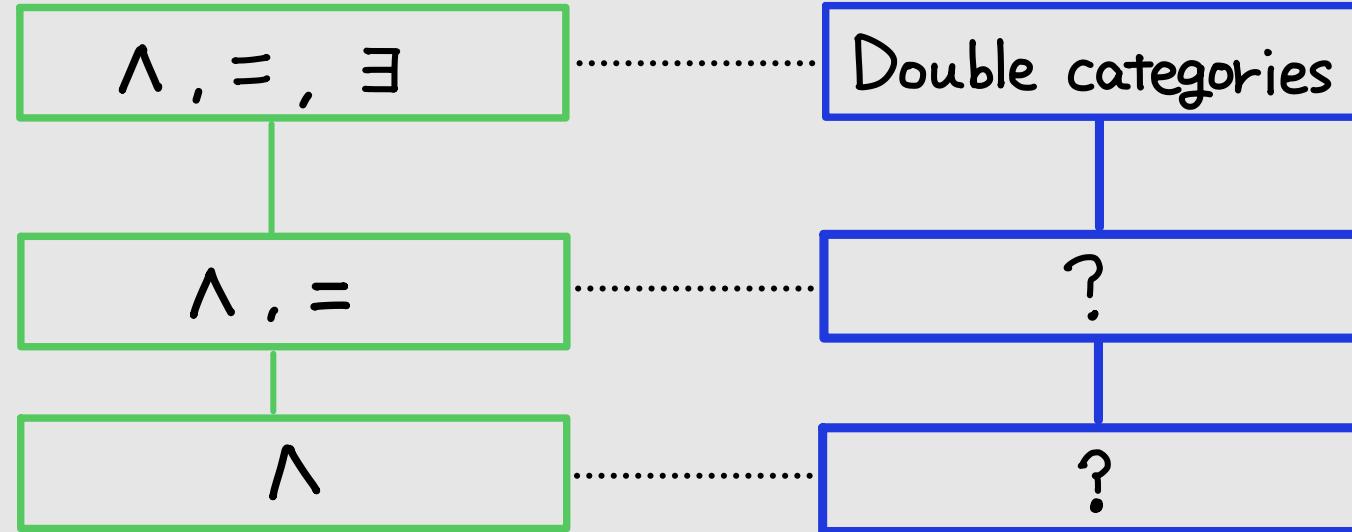
loose arrows $I \xrightarrow{\alpha} K$... binary relations

cells $\begin{array}{ccc} I & \xrightarrow{\alpha} & K \\ u \downarrow & \nwarrow & \downarrow v \\ J & \xrightarrow{\beta} & L \end{array}$... (proofs of) implication
 $\alpha(x,y) \Rightarrow \beta(u(x),v(y))$

The loosewise composition is given by

$$I \xrightarrow{Id_I} I \dots \{(x,x') \mid x=x'\}$$

$$I \xrightarrow{\alpha} J \xrightarrow{\beta} K \dots \{(x,z) \mid \exists y \in J. \alpha(x,y) \wedge \beta(y,z)\}$$



Def A virtual double category \mathbb{D} consists of

- objects I, J, \dots
- tight arrows $I \xrightarrow{u}, \dots$
- loose arrows $I \xrightarrow{\alpha} K, \dots$
- (virtual) cells $\begin{array}{ccc} I_0 & \xrightarrow{\alpha_1} & I_1 & \xrightarrow{\alpha_n} & I_n \\ u \downarrow & & & & \downarrow v \\ J_0 & \xrightarrow{\beta} & J_1 & & \end{array}$ for $n \geq 0$

with compositions of tight arrows and cells,
satisfying some laws.

Virtual double categories of relations

Example (A DC is a VDC)

In Rel , a virtual cell

$$\begin{array}{ccccc} I_0 & \xrightarrow{\alpha} & I_1 & \xrightarrow{\beta} & I_2 \\ u \downarrow & & \cap_1 & & \downarrow v \\ J_0 & \xrightarrow[\gamma]{} & J_1 & & J_2 \end{array}$$

exists if and only if

$$(\exists y \in I_2. \alpha(x, y) \wedge \beta(y, z)) \Rightarrow \gamma(u(x), v(z)), \quad (\forall x, z)$$

but this is equivalent to

$$\alpha(x, y) \wedge \beta(y, z) \Rightarrow \gamma(u(x), v(z)) \quad (\forall x, y, z)$$

By adopting this presentation, weaker systems

than regular logic are modeled in a VDC

with "substitution" and "finite products".

Def A restriction of $\begin{array}{ccc} I & & K \\ \downarrow u & & \downarrow v \\ J & \xrightarrow[\alpha]{} & L \end{array}$ 13 / 20

is a loose arrow $I \xrightarrow[\alpha[u; v]]{} K$, equipped with a cell

$$\begin{array}{ccc} I & \xrightarrow{\alpha[u; v]} & K \\ \downarrow u & p & \downarrow v \\ J & \xrightarrow[\alpha]{} & L \end{array}$$

with the following universal property:

$$\begin{array}{ccc} M_0 \xrightarrow{\dots} M_n & & M_0 \xrightarrow{\dots} M_n \\ s \downarrow \text{v} & & s \downarrow \text{v} \\ I & \xrightarrow{v} & K \\ \downarrow u & & \downarrow v \\ J & \xrightarrow[\alpha]{} & L \end{array} = \begin{array}{ccc} M_0 \xrightarrow{\dots} M_n & & M_0 \xrightarrow{\dots} M_n \\ s \downarrow \text{v} & & s \downarrow \text{v} \\ I & \xrightarrow{v} & K \\ \downarrow u & p & \downarrow v \\ J & \xrightarrow[\alpha]{} & L \end{array}$$

Def A VDC \mathbb{D} is cartesian fibrational if

- every $\begin{array}{ccc} \vdots & \vdots & \vdots \\ \downarrow & \rightarrow & \downarrow \\ \vdots & \rightarrow & \vdots \end{array}$ in \mathbb{D} has a restriction,
- the tight category \mathbb{D}_t has finite products,
- the loose hom-categories $\mathbb{D}(I, J)$ have finite products preserved by $(-)^{[u; v]}$'s.

Def A restriction of $u \downarrow_{J \xrightarrow{\alpha} L}$

is a loose arrow $I \xrightarrow{\alpha[u; v]} K$, equipped with a cell

$I \xrightarrow{\alpha[u; v]} K$ with the following universal property:

$$\begin{array}{ccc} M_0 \xrightarrow{\quad} \cdots \xrightarrow{\quad} M_n & M_0 \xrightarrow{\quad} \cdots \xrightarrow{\quad} M_n \\ s \downarrow \textcolor{red}{\forall} & s \downarrow \exists! \textcolor{red}{\tilde{v}} & \\ I & K & = \\ u \downarrow & \downarrow v & \\ J \xrightarrow{\alpha} L & & \end{array}$$

Def A VDC \mathbb{D} is cartesian fibrational if

- every $\downarrow \xrightarrow{\cdot}$ in \mathbb{D} has a restriction,
- the tight category \mathbb{D}_t has finite products,
- the loose hom-categories $\mathbb{D}(I, J)$ have finite products preserved by $(-)[u; v]$'s.

Now, we construct

the VDC of relations relative to \mathbb{P} .

Def (Shulman '08 [Fr](#), Lawler '15 [Matr](#), Nasu '25)

A VDC $\mathbb{Bil}(\mathbb{P})$ for a cartesian fibration $\mathcal{E} \downarrow_{\mathbb{P}} \mathcal{B}$ is defined as follows:

- its tight category $\mathbb{Bil}(\mathbb{P})_0 : \mathcal{B}$
- its loose arrows $I \xrightarrow{\alpha} J : \alpha \in \mathcal{E}_{I \times J}$

• cells $I_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} I_n$
 $u \downarrow \quad \quad \quad \downarrow v$
 $J_0 \xrightarrow{\beta} J_1$

$$\alpha_1[\langle \pi_0, \pi_1 \rangle] \wedge \cdots \wedge \alpha_n[\langle \pi_{n-1}, \pi_n \rangle] \rightarrow \beta[(u \times v) \circ \langle p_{\pi_0}, p_{\pi_n} \rangle]$$

in $\mathcal{E}_{I_0 \times \cdots \times I_n}$

proofs of $\alpha_1(x_0, x_1) \wedge \cdots \wedge \alpha_n(x_{n-1}, x_n)$
 $\Rightarrow \beta(u(x_0), v(x_n))$

Prop $\mathbb{Bil}(\mathbb{P})$ is a cartesian fibrational VDC.

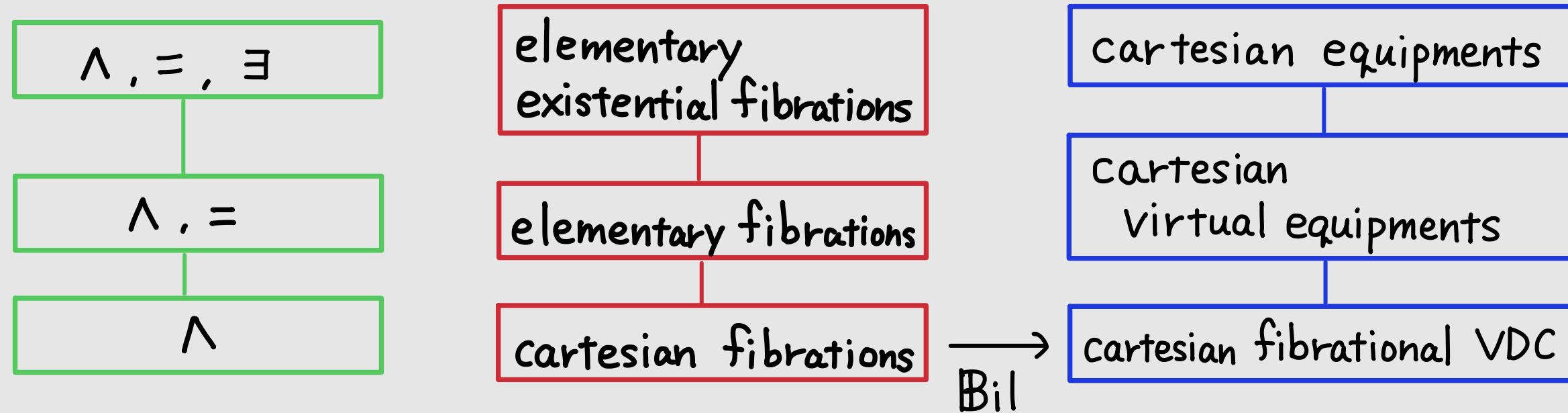
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Comparing fibrations and VDCs



Equipments and virtual equipments can be / are formulated as VDCs with a certain kind of composition in terms of universal properties.

Def A composite of $I_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} I_n$ is a loose arrow together with the following universal property :

$$J \xrightarrow{\bar{\beta}} I_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} I_n \xrightarrow{\bar{\gamma}} K$$

$\forall \quad \exists$

$u \downarrow \qquad \qquad \nu \downarrow$

$L \xrightarrow{f} M$

$I_0 \xrightarrow{\odot\alpha} I_n$ together with

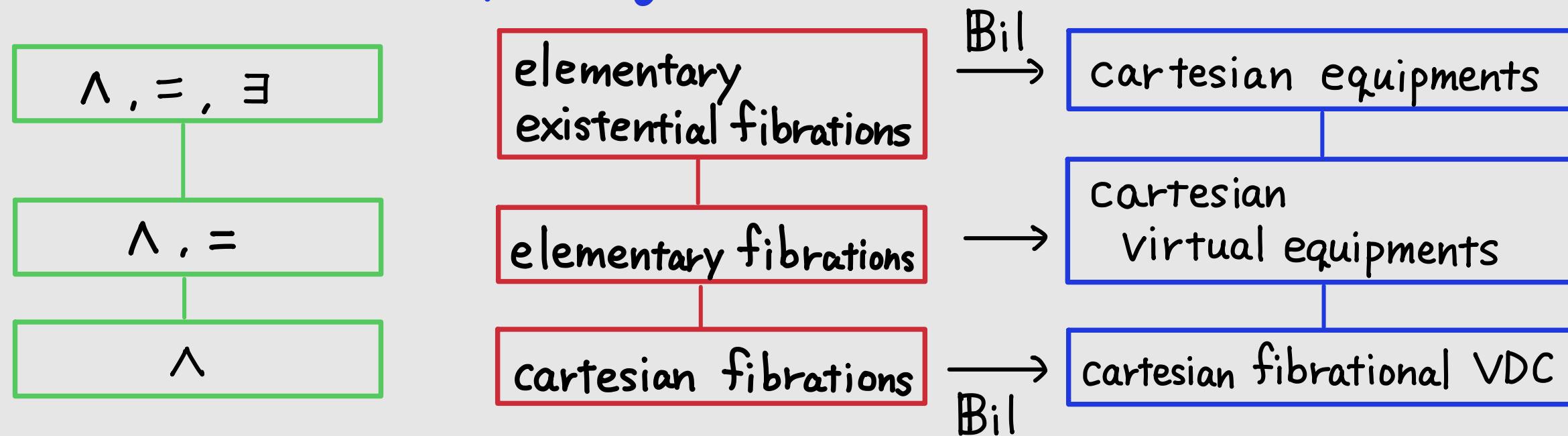
$$\begin{array}{c} J \xrightarrow{\bar{\beta}} I_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} I_n \xrightarrow{\bar{\gamma}} K \\ \parallel \qquad \parallel \qquad \parallel \qquad \parallel \end{array}$$

$\exists! \quad \widetilde{\nu}$

$L \xrightarrow{f} M$

$$\begin{array}{c} I_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} I_n \\ \parallel \qquad \parallel \\ I_0 \xrightarrow{\odot\alpha} I_n \end{array}$$

Comparing fibrations and VDCs



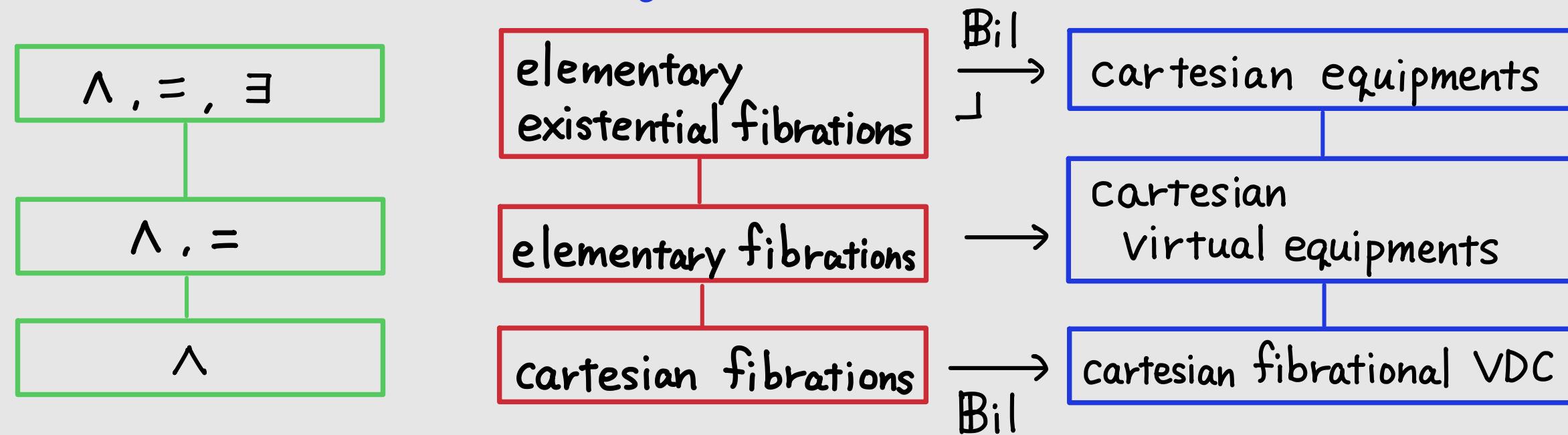
Prop

When P is $\{ \text{elementary} \}$, $\{ \text{elementary existential} \}$, $Bil(P)$ is $\{ \text{a cartesian virtual equipment.} \}$
 $\{ \text{a cartesian equipment.} \}$

More precisely, we have 2-functors

$$\begin{array}{ccc}
 \text{EEFib} & \rightarrow & \text{CartEq}' \\
 \downarrow & & \downarrow \\
 \text{EleFib} & \rightarrow & \text{CartVEq} \\
 \downarrow & & \downarrow \\
 \text{CartFib} & \xrightarrow{\text{Bil}} & \text{CartFVDC}
 \end{array}$$

Comparing fibrations and VDCs



Thm For a cartesian fibration P ,

P is elementary existential $\iff \text{Bil}(P)$ is a cartesian equipment

Moreover, we have a 2-pullback

$$\begin{array}{ccc} \text{EEFib} & \xrightarrow{\quad} & \text{CartEq}' \\ \downarrow & \lrcorner & \downarrow \\ \text{CartFib} & \xrightarrow[\text{Bil}]{} & \text{CartFVDC} \end{array}$$

Prop $\text{Bil} : \text{EEFib} \rightarrow \text{CartEq}'$ is locally an equivalence.

The essential image of this is characterized by Frobenius axiom.
(Walters, Wood '08)
(Nasu '25 2.3.2)

Comparing fibrations and VDCs

Example

From the fibration $\begin{array}{c} M \\ \downarrow \\ \mathcal{E} \end{array}$ for a stable

factorization system (E, M) , we obtain

$$\text{Bil}\left(\begin{array}{c} M \\ \downarrow \\ \mathcal{E} \end{array}\right) = \frac{\text{Rel}_{(E,M)}(\mathcal{E})}{\perp}$$

the double category of relations
relative to (E, M) . (Hoshino, Nasu. '23)

Application (Function comprehension)

For an elementary existential fibration $P: \mathcal{E} \rightarrow B$,
a total and single-valued relation $\lambda \in \mathcal{E}_{I \times J}$

is equivalent to a loose adjunction

$$\begin{array}{ccc} I & \xrightleftharpoons[\perp]{\lambda} & J \\ \perp & & \perp \\ P & & P \end{array} \quad \text{in } \text{Bil}(P).$$

The function comprehension property is
translated as Cauchyness of double categories:

$$\begin{array}{ccc} A & \begin{array}{c} \xrightleftharpoons[\perp]{\lambda} \\ \perp \\ P \end{array} & \exists u \\ I & \xrightleftharpoons[\perp]{\lambda+P} & \downarrow J \\ \perp & & \perp \\ P & & P \end{array} \quad (\lambda+P) \cong (u_* \dashv u^*)$$

The function comprehensive completion
is achieved as follows:

$$\begin{array}{ccccc} \text{EEFib} & \xrightleftharpoons[\perp]{\dashv} & \text{EEFib}_{\text{funccomp}} \\ \downarrow & & \uparrow \\ \text{Bil} & & & & \uparrow \\ \text{CartEq} & \xrightleftharpoons[\perp]{\dashv} & & & \text{CartEq}_{\text{Cauchy}} \end{array}$$

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Conclusion

- introduced cartesian fibrational virtual double categories as a relation-oriented framework for predicate logic.
- presented the Bil -construction which produces a cartesian fibrational VDC from a cartesian fibration.

$$\begin{array}{ccc} \text{EEFib} & \xrightarrow{\text{Bil}} & \text{CartEq}' \\ \downarrow & \lrcorner & \downarrow \\ \text{CartFib} & \rightarrow & \text{CartFVDC} \end{array}$$

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Others in my thesis

- Recovering a characterization theorem in (Hoshino, Nasu. '23).
- Connection to bicategories.
- Self-duality on Frobenius equipments.
- (• An internal logic of fibrational VDCs.)

Future Work

- develop theories of exact completion and other kinds of completion in terms of virtual double categories.
- ?

Reference

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Thank you!

my homepage :

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my thesis :

arXiv : 2501.17869

