Logical Aspects of Virtual Double Categories

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Preface

Abstract

This thesis deals with two main topics: virtual double categories as semantics environments for predicate logic, and a syntactic presentation of virtual double categories as a type theory. One significant principle of categorical logic is bringing together the semantics and the syntax of logical systems in a common categorical framework. This thesis is intended to propose a double-categorical method for categorical logic in line with this principle. On the semantic side, we investigate virtual double categories as a model of predicate logic, and illustrate that this framework subsumes the existing frameworks properly. On the syntactic side, we develop a type theory called FVDblTT that is designed as an internal language for virtual double categories.

Structure of this Thesis

The thesis is divided into three chapters. The first chapter is devoted to the preliminaries necessary to understand the main content of this thesis. The second chapter deals with the first theme, the virtual double categories as a model of predicate logic. The third chapter studies the type theory that is designed as an internal language for virtual double categories. The material in the last chapter and necessary background in the first chapter has already been made public as a preprint [**Nas24**]. Some parts of the first and second chapters, mostly the definitions and theorems on double categories, are based on the author's joint work [**HN23**] with Keisuke Hoshino.

Each chapter has its abstract at the beginning, and the last two chapters have their own introduction sections Sections 2.1 and 3.1, which can be read independently of the other chapters.

Summary of Contributions

The major contributions of this thesis are as follows:

Chapter 2: Categorical Logic Meets Virtual Double Categories

- We construct a 2-functor $\mathbb{B}\mathrm{il}$ from the 2-category $\mathbf{Fib}_{\mathbf{cart}}$ of cartesian fibrations to the 2-category $\mathbf{FVDbl}_{\mathbf{cart}}$ of cartesian fibrational virtual double categories. (Proposition 2.3.4)
- We characterize the 2-category $\mathbf{Fib}_{\times\wedge=\exists}$ of elementary existential fibrations as the pullback of the 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{Fib}_{\mathbf{cart}} \to \mathbf{FVDbl}_{\mathbf{cart}}$ along the forgetful 2-functor from $\mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}}$. (Theorem 2.3.17)
- We also characterize the image of the 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{Fib}_{\times\wedge=\exists} \to \mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}}$ as the sub-2-category of $\mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}}$ consisting of Frobenius cartesian equipments. (Corollary 2.3.37)
- We prove that the loose bicategory of a cartesian equipment is a cartesian bicategory. (Theorem 2.4.8)
- We revisit some existing results in the literature from the perspective of the $\mathbb{B}\mathrm{il}$-construction. (Corollary 2.5.8 and Remark 2.5.21)

Chapter 3: Type Theory for Virtual Double Categories

- We developed a type theory called FVDblTT and established a biadjunction between the 2-category of cartesian fibrational virtual double categories and the 2-category of specifications for this type theory, whose counit is a pointwise equivalence.

Notations

$\mathcal{C}, \mathcal{D}, \mathcal{E}$	(1-)categories
$\mathcal{C}^{\mathsf{op}}$	the opposite category of $\mathcal{C}$
$\mathbf{1}$	the terminal category, or the terminal 2-category
$\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$	a fibration
$\mathcal{E}_I$	the fiber of $\mathcal{E}$ over $I \in \mathcal{B}$
$\alpha[f]$	the reindexing of $\alpha \in \mathcal{E}_I$ along $f\colon J \to I$
$\mathbf{K}, \mathbf{L}, \mathbf{M}$	2-categories and bicategories
$\mathbf{K}^{\mathsf{op}}$	the 1-cell opposite bicategory of $\mathbf{K}$
$\mathbf{K}^{\mathsf{co}}$	the 2-cell opposite bicategory of $\mathbf{K}$
$\mathbb{D}, \mathbb{E}$	double categories
$\mathtt{id}_I, \mathtt{id}_J$	the identity arrows in a category
δ_I, δ_J	the identity (resp. unit) loose arrows in a (virtual) double category, the identity 1-cells in a bicategory, or the objects in a fiber $\mathcal{E}_{I\times I}$ that represent equality
Id	the identity (1-, 2-, double) functor
$1, \times$	the finite products in (1-,double) categories
$\top, \wedge$	the finite products in fiber categories or loose hom-categories in double categories

On Projections of Products. In this thesis, we will write $\langle f_0, \ldots, f_{n-1}\rangle\colon A \to B_0 \times \cdots \times B_{n-1}$ for the arrow induced by $f_i\colon A \to B_i$ for $i = 0, \ldots, n-1$. In the case where f_i's are all projections, we will adopt a more suggestive notation: we will write $\langle i_0, \ldots, i_{n-1}\rangle\colon A_0 \times \cdots \times A_{m-1} \to A_{i_0} \times \cdots \times A_{i_{n-1}}$ for the arrow whose j-th component is the i_j-th projection for $j = 0, \ldots, n-1$. For example, we will write $\langle 0, 0\rangle\colon A \to A \times A$ for the diagonal arrow, $\langle 0\rangle\colon A \times B \to A$ for the first projection, and $\langle 1\rangle\colon A \times B \to B$ for the second projection. This notation facilitates calculation of the composition of arrows given by the projections. For example,

$$A \times C \times C \xleftarrow[\langle 0,2,4\rangle]{} A \times A \times C \times B \times C \xleftarrow[\langle \underline{0},0,\underline{2},1,\underline{2}\rangle]{} A \times B \times C \;=\; A \times C \times C \xleftarrow[\langle 0,2,2\rangle]{} A \times B \times C \;.$$

Accordingly, we have

$$\alpha[\langle 0,2,4\rangle][\langle 0,0,2,1,2\rangle] \cong \alpha[\langle 0,2,2\rangle] \quad \text{in} \quad \mathcal{E}_{A\times B\times C}$$

for a fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ and an object $\alpha \in \mathcal{E}_{A\times C\times C}$. On the other hand, we will write ! for the unique arrow to the terminal object, not $\langle\rangle$.

Introducing Terminology. The first and second chapters of this thesis include a brief introduction to the basic notions of fibrations, double categories, and virtual double categories. We introduce the basic terminology and a few new terms that we use throughout the thesis, and those terms are written in ***boldface and italics***. We also mention some terminology that appears in the literature but that we do not use again in the main body of the thesis, and those terms are written in *italics but not boldface*.

Contents

Chapter 1

Preliminaries on 2-dimensional Structures

This chapter is devoted to the preliminaries on cartesian objects, double categories, and virtual double categories. Cartesian objects are a generalization of the notion of categories with finite products. This concept is convenient when we make a statement on finite products for general categorical structures. Double categories are a generalization of categories. They have two kinds of arrows called tight and loose arrows, which can be composed with arrows of the same kind, and also cells that fill those arrows. Virtual double categories are a further generalization of double categories, in which loose arrows are not equipped with composition. This chapter is intended to provide the reader with the necessary background to understand the main content of this thesis. Part of this chapter is based on the author's joint work with Keisuke Hoshino [HN23].

1.1. Cartesian Objects in 2-categories

Definition 1.1.1 ([CKW91, §5.1])**.** A ***cartesian object*** in a 2-category $\mathbf{K}$ with strict (2-dimensional) finite products $\mathbb{1}, \otimes$ is an object x of $\mathbf{K}$ such that the canonical 1-cells $!\colon x \to \mathbb{1}$ and $\Delta\colon x \to x \otimes x$ have right adjoints $1\colon \mathbb{1} \to x$ and $\times\colon x \otimes x \to x$, respectively. A ***cartesian 1-cell*** (or ***cartesian arrow***) in $\mathbf{K}$ is a 1-cell $f\colon x \to y$ between cartesian objects x and y of $\mathbf{K}$ such that the canonical 2-cells obtained by the mate construction $\times \circ (f \otimes f) \Rightarrow f \circ \times$ and $f \circ 1 \Rightarrow 1$ are invertible.

For a 2-category $\mathbf{K}$ with strict finite products, we write $\mathbf{K}_{\mathbf{cart}}$ for the 2-category of cartesian objects, cartesian 1-cells, and arbitrary 2-cells in $\mathbf{K}$. ⌟

Remark 1.1.2. By strict finite products, we mean the most strict notion of finite products, that is, the terminal object $\mathbb{1}$ and the binary product $\otimes$ come with the isomorphisms in the 2-category of categories:

$$\mathbf{K}(x, \mathbb{1}) \cong \mathbf{1}$$

$$\mathbf{K}(x, y \otimes z) \cong \mathbf{K}(x, y) \times \mathbf{K}(x, z)$$

2-naturally in x, y, z. We call this kind of limits ***strict (2-)limits***. ⌟

Example 1.1.3. In the 2-category **Cat** of categories, functors, and natural transformations, the cartesian objects are the categories with finite products, where the right adjoints 1 and $\times$ are the functors of the terminal object and the binary product, respectively. ⌟

Lemma 1.1.4. Let $\mathbf{K}$ be a 2-category with strict finite products. A 1-cell $f\colon x \to y$ in $\mathbf{K}_{\mathbf{cart}}$ is an equivalence in $\mathbf{K}_{\mathbf{cart}}$ if and only if the underlying 1-cell of f is an equivalence in $\mathbf{K}$. ⌟

Proof. The only if part is clear since we have the forgetful 2-functor $\mathbf{K}_{\mathbf{cart}} \to \mathbf{K}$. For the if part, take the right adjoint g of the underlying 1-cell of f as its inverse. Taking the right adjoint of both sides of the isomorphism 2-cells $! \circ f \cong !$ and $(f \otimes f) \circ \Delta \cong \Delta \circ f$, we obtain the isomorphism 2-cells $g \circ 1 \cong 1$ and $\times \circ (g \otimes g) \cong g \circ \times$. This shows that g gives a cartesian morphism from y to x, and g is indeed the inverse of f in $\mathbf{K}_{\mathbf{cart}}$. □

Lemma 1.1.5. Let $\mathbf{K}, \mathbf{K}'$ be 2-categories with strict finite products $(\mathbb{1}, \otimes)$, and $|{-}|\colon \mathbf{K}' \to \mathbf{K}$ be a 2-functor preserving strict finite products and locally full-inclusion, *i.e.*, injective on 1-cells and bijective on 2-cells. For an object x of $\mathbf{K}'$ to be cartesian, it is necessary and sufficient that $|x|$ is cartesian in $\mathbf{K}$ and that the 1-cells $1\colon \mathbb{1} \to |x|$ and $\times\colon |x| \otimes |x| \to |x|$ right adjoint to the canonical 1-cells are essentially in the image of $|{-}|$.

Moreover, for a 1-cell $f\colon x \to y$ of $\mathbf{K}'$ where x and y are cartesian in $\mathbf{K}'$, f is cartesian in $\mathbf{K}'$ if and only if $|f|$ is cartesian in $\mathbf{K}$. ⌟

Proof. The necessity of the first condition follows from the fact that any 2-functor preserves adjunctions, that right adjoints are unique up to isomorphism, and that $|{-}|$ preserves finite products.

Since $|-|$ is locally fully faithful, it also reflects units, counits, and the triangle identities with respect to the adjunctions, and hence the sufficiency of the first condition follows.

The necessity of the second condition is again immediate from the fact that $|-|$ preserves finite products. The sufficiency follws from the fact that $|-|$ is locally fully faithful, in particular, reflects isomorphisms. ☐

Lemma 1.1.6. Let $\mathbf{K}$, $\mathbf{L}$, and $\mathbf{M}$ be a 2-category with strict finite products $(1, \otimes)$, and $T\colon \mathbf{K} \to \mathbf{M}$ and $S\colon \mathbf{L} \to \mathbf{M}$ be 2-functors preserving the finite products strictly, and locally isofibrations. Then, the canonical 2-functor

$$(\mathbf{K} \times_{\mathbf{M}} \mathbf{L})_{\mathbf{cart}} \to \mathbf{K}_{\mathbf{cart}} \times_{\mathbf{M}_{\mathbf{cart}}} \mathbf{L}_{\mathbf{cart}}$$

is a 2-equivalence, where $- \times_{\mathbf{M}} -$ denotes the strict pullback of 2-categories, that is, the 2-category of pairs (k, l) of 0-cells $k \in \mathbf{K}$ and $l \in \mathbf{L}$ with $T(k) = S(l)$ in $\mathbf{M}$. $\lrcorner$

Proof. finite products in $\mathbf{K} \times_{\mathbf{M}} \mathbf{L}$ are given by pointwise finite products in $\mathbf{K}$ and $\mathbf{L}$, namely, $(k, l) \otimes (k', l') := (k \otimes k', l \otimes l')$, and $1 := (1, 1)$. In addition, a 1-cell $(f, g)\colon (k, l) \to (k', l')$ has a right adjoint if and only if f and g have right adjoints in $\mathbf{K}$ and $\mathbf{L}$, respectively. Here, we use the assumption that T and S are locally isofibrations. From this, we see that the canonical 2-functor is essentially surjective, and locally fully faithful by the fact that natural isomorphisms in the pullback are pointwise. ☐

1.2. Double Categories

Broadly speaking, as far as the author is aware, the use of double categories has two aspects: the first aspect is as a framework for two distinguished kinds of arrows that are equivalent in their workings, and the second is as a framework for a category with a different composable structure that supports the original category. In the first aspect, double categories are usually given in a strict setting, where the associativity and unit laws are strict for the two kinds of arrows. In the second aspect, double categories are usually given in a weak setting, where the associativity and unit laws for the second kind of arrows are relaxed to isomorphisms. In this thesis, we will focus on the second aspect of double categories, and hence what we call a double category is a pseudo, or equivalently weak, double category. In the following, we will introduce the basic terminology and concepts that we use throughout the thesis. For a comprehensive introduction to double categories, we refer the reader to [Gra20].

By a *(pseudo-)double category* $\mathbb{D}$, we mean a pseudo-category in the 2-category $\mathbf{CAT}$ of locally small categories. In other words, a double category consists of two (locally small) categories $\mathbb{D}_0$, $\mathbb{D}_1$ and functors

$$\begin{tikzcd} \mathbb{D}_1 {}_{\mathsf{tgt}}\times_{\mathsf{src}} \mathbb{D}_1 \arrow[r, "\odot"] & \mathbb{D}_1 \arrow[r, shift left=3, "\mathsf{src}"] \arrow[r, shift right=3, "\mathsf{tgt}"'] & \mathbb{D}_0 \arrow[l, "\delta" description] \end{tikzcd} .$$

These data come equipped with natural isomorphisms that stand for the associativity law and the unit laws.

Objects and arrows of $\mathbb{D}_0$ are called ***objects*** and ***tight arrows*** of the double category $\mathbb{D}$. We use the notation $g \circ f$ for the composition of $I \xrightarrow{f} J \xrightarrow{g} K$ in $\mathbb{D}_0$, or ocasionally $f ; g$ in the diagrammatic order. An object α of $\mathbb{D}_1$ whose values of src and tgt are I and J, respectively, is called a ***loose arrow***[1] from I to J, and written as $\alpha\colon I \nrightarrow J$. We use the notation $\alpha \odot \beta$, or simply $\alpha\beta$, for the composite of $\alpha\colon I \nrightarrow J$ and $\beta\colon J \nrightarrow K$ in $\mathbb{D}_1$, and δ_I for the identity loose arrow on I. An arrow $\varphi\colon \alpha \to \beta$ in $\mathbb{D}_1$ is called a ***double cell*** (or merely a ***cell***) in the double category $\mathbb{D}$. This cell is drawn as below, where $\mathsf{src}(\varphi) = f$ and $\mathsf{tgt}(\varphi) = g$.

(1.2.1)
$$\begin{tikzcd} I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & J \arrow[d, "g"] \\ K \arrow[r, "\beta"', "\shortmid"{marking}] & L \end{tikzcd}$$

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[1] The term "tight" and "loose" are used not to confuse with the terms "vertical" and "horizontal" because there is no consensus on the terminology and notation on which class of arrows should be called "vertical" or "horizontal". The difference cannot be dismissed since only one class of arrows requires strict associativity and unit laws.

Interchanging the roles of src and tgt in a double category $\mathbb{D}$, we obtain another double category. We call it the ***loosewise opposite*** of $\mathbb{D}$ and write it as $\mathbb{D}^{\mathsf{lop}}$. Sending the data of $\mathbb{D}$ by the 2-functor $(-)^{\mathsf{op}}\colon \mathbf{CAT}^{\mathsf{co}} \to \mathbf{CAT}$, we get another double category. We call it the ***tightwise opposite*** of $\mathbb{D}$ and write it as $\mathbb{D}^{\mathsf{top}}$.

Since the category $\mathbb{D}_0$ is a category consisting of objects and tight arrows, we call it the ***tight category*** of the double category $\mathbb{D}$. If we consider the cells with the top and bottom loose arrows being identities and call them ***tight cells***, then we obtain a 2-category of objects, tight arrows, and tight cells. We write this 2-category as $\mathbf{T}(\mathbb{D})$. On the other hand, we can consider a bicategory consisting of objects, loose arrows, and ***globular cells***, meaning cells whose source and target are identities. We call this bicategory the ***loose bicategory*** of the double category $\mathbb{D}$, and write it as $\mathbf{L}(\mathbb{D})$. For later use, we summarize how the various notions of opposites are related:

$$(\mathbb{D}^{\mathsf{top}})_0 = (\mathbb{D}_0)^{\mathsf{op}}, \qquad (\mathbb{D}^{\mathsf{lop}})_0 = \mathbb{D}_0,$$

$$\mathbf{T}(\mathbb{D}^{\mathsf{top}}) = (\mathbf{T}(\mathbb{D}))^{\mathsf{op}}, \qquad \mathbf{T}(\mathbb{D}^{\mathsf{lop}}) = \mathbf{T}(\mathbb{D})^{\mathsf{co}}, \qquad \mathbf{L}(\mathbb{D}^{\mathsf{top}}) = \mathbf{L}(\mathbb{D})^{\mathsf{co}}, \qquad \mathbf{L}(\mathbb{D}^{\mathsf{lop}}) = \mathbf{L}(\mathbb{D})^{\mathsf{op}}.$$

By abuse of notation, we write $\mathbb{D}(I,J)$ for the hom-category of the loose bicategory $\mathbf{L}(\mathbb{D})$ for objects I and J of $\mathbb{D}_0$.

Remark 1.2.1. Strictly speaking, the composite $\alpha\beta\gamma$ does not make unique sense in a pseudo-double category, but rather we have $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ equipped with the canonical isomorphism between them. Still, the composite $\alpha\beta\gamma$ is determined up to the canonical isomorphisms, and we will use this notation in this thesis. This is supported by the strictification theorem [**GP99**, §7.5] saying that any pseudo-double category is equivalent to a strict double category. One may define a pseudo-double category in an unbiased way in which the n-ary compositions for general n are primitively defined. We will not use this notion in this thesis explicitly, but it is more similar to an equivalent formulation of double categories in terms of virtual double categories introduced in Section 1.4. ⌟

Remark 1.2.2. In this thesis, we often use diagrammatic presentations as in (1.2.1). We often use the convention that the identity arrows are contracted to vertices of objects. They are also drawn as arrows with double lines like $=$ and $\mathrel{=\mkern-11mu\shortmid\mkern5mu}$. By an alignment of arrows, we mean the composite of them. The following examplifies these conventions.

$$\begin{tikzcd} & I \arrow[dr, "f"] & \\ I \arrow[ur, equal] \arrow[rr, "\alpha"', "\shortmid"{marking}] & \varphi & J \end{tikzcd} \quad := \quad \begin{tikzcd} I \arrow[r, "\delta_I", "\shortmid"{marking}] \arrow[d, "\mathtt{id}_I"'] \arrow[dr, phantom, "\varphi"] & I \arrow[d, "f"] \\ I \arrow[r, "\alpha"', "\shortmid"{marking}] & J \end{tikzcd}, \qquad \begin{tikzcd} I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[dr, "g"'] & J \arrow[r, equal, "\shortmid"{marking}] \arrow[d, phantom, "\varphi"] & J \arrow[dl, "h"] \\ & K & \end{tikzcd} \quad := \quad \begin{tikzcd} I \arrow[r, "\alpha\odot\delta_J(\cong\alpha)", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\varphi"] & J \arrow[d, "h"] \\ K \arrow[r, "\delta_K"', "\shortmid"{marking}] & K \end{tikzcd}.$$

⌟

For double categories $\mathbb{D}$ and $\mathbb{E}$, a ***double functor*** $F\colon \mathbb{D} \to \mathbb{E}$ is an internal functor between the double categories as internal pseudo-categories in $\mathbf{CAT}$. It consists of two functors $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1\colon \mathbb{D}_1 \to \mathbb{E}_1$ such that $\mathsf{src} \circ F_1 = F_0 \circ \mathsf{src}$ and $\mathsf{tgt} \circ F_1 = F_0 \circ \mathsf{tgt}$, together with natural isomorphisms

$$\begin{tikzcd} \mathbb{D}_1 {}_{\mathsf{tgt}}\!\times_{\mathsf{src}} \mathbb{D}_1 \arrow[r, "\odot"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\cong"] & \mathbb{D}_1 \arrow[d, "F_1"] \arrow[dr, phantom, "\cong"] & \mathbb{D}_0 \arrow[l, "\delta"'] \arrow[d, "F_0"] \\ \mathbb{E}_1 {}_{\mathsf{tgt}}\!\times_{\mathsf{src}} \mathbb{E}_1 \arrow[r, "\odot"'] & \mathbb{E}_1 & \mathbb{E}_0 \arrow[l, "\delta"] \end{tikzcd}$$

that are compatible with the isomorphism cells for the associativity and unit laws of $\mathbb{D}$ and $\mathbb{E}$. Unpacking this definition, a double functor F consists of the following data:

- a functor $F_0\colon \mathbb{D}_0 \to \mathbb{E}_0$,
- a function that sends a loose arrow $\alpha\colon I \mathrel{\rightarrow\mkern-11mu\shortmid\mkern5mu} J$ to a loose arrow $F_1(\alpha)\colon F_0(I) \mathrel{\rightarrow\mkern-11mu\shortmid\mkern5mu} F_0(J)$,
- a function that sends a cell φ as in (1.2.1) to a cell $F_1(\varphi)$ framed by the images of the tight arrows and the loose arrows under F.
- invertible globular cells for all objects I and for all composable pairs (α,β) of loose arrows as follows

$$\begin{tikzcd} F_0(I) \arrow[r, "\delta_{F_0(I)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{\delta;I}\ \wr\|"] & F_0(I) \arrow[d, equal] \\ F_0(I) \arrow[r, "F_1(\delta_I)"', "\shortmid"{marking}] & F_0(I) \end{tikzcd} \qquad \begin{tikzcd} F_0(I) \arrow[r, "F_1(\alpha)", "\shortmid"{marking}] \arrow[d, equal] & F_0(J) \arrow[r, "F_1(\beta)", "\shortmid"{marking}] \arrow[d, phantom, "F_{\odot;\alpha,\beta}\ \wr\|"] & F_0(K) \arrow[d, equal] \\ F_0(I) \arrow[rr, "F_1(\alpha\odot\beta)"', "\shortmid"{marking}] & & F_0(K) \end{tikzcd}$$

such that the coherence conditions for the associativity and unit laws are satisfied.

A ***(double) tightwise transformation*** $\Gamma\colon F \Rightarrow G$ between double functors $F, G\colon \mathbb{D} \to \mathbb{E}$ is also defined in terms of internal pseudo-categories in **CAT**. It consists of natural transformations $\Gamma_0\colon F_0 \Rightarrow G_0$ and $\Gamma_1\colon F_1 \Rightarrow G_1$ that are compatible with the loose compostion. More concretely, it consists of the following data:

- a natural transformation $\Gamma_0\colon F_0 \Rightarrow G_0$,
- a family of cells $(\Gamma_{1,\alpha})$ in $\mathbb{E}$ indexed by the loose arrows $\alpha\colon I \nrightarrow J$ of $\mathbb{D}$, which are framed by $\Gamma_{0,I}$ and $\Gamma_{0,J}$, together with the images of α under F and G, and satisfy the following naturality condition and the coherence conditions for loosewise composition.

$$\begin{tikzcd}
F_0(I) \arrow[r, "F_1(\alpha)", "\shortmid"{marking}] \arrow[d, "\Gamma_{0,I}"'] \arrow[dr, phantom, "\Gamma_{1,\alpha}"] & F_0(J) \arrow[d, "\Gamma_{0,J}"] & & F_0(I) \arrow[r, "F_1(\alpha)", "\shortmid"{marking}] \arrow[d, "F_1(f)"'] \arrow[dr, phantom, "F_{1,\varphi}"] & F_0(J) \arrow[d, "F_1(g)"] \\
G_0(I) \arrow[r, "G_1(\alpha)"', "\shortmid"{marking}] \arrow[d, "G_1(f)"'] \arrow[dr, phantom, "G_{1,\varphi}"] & G_0(J) \arrow[d, "G_1(g)"] & = & F_0(K) \arrow[r, "F_1(\beta)"', "\shortmid"{marking}] \arrow[d, "\Gamma_{0,K}"'] \arrow[dr, phantom, "\Gamma_{1,\beta}"] & F_0(L) \arrow[d, "\Gamma_{0,L}"] \\
G_0(K) \arrow[r, "G_1(\beta)"', "\shortmid"{marking}] & G_0(L) & & G_0(K) \arrow[r, "G_1(\beta)"', "\shortmid"{marking}] & G_0(L)
\end{tikzcd} \quad \text{for} \quad \varphi \quad \text{as in (1.2.1)}$$

$$\begin{tikzcd}
F_0(I) \arrow[r, "\delta_{F_0(I)}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "F_{\delta;I}"] & F_0(I) \arrow[d, equal] & & F_0(I) \arrow[r, "\delta_{F_0(I)}", "\shortmid"{marking}] \arrow[d, "\Gamma_{0,I}"'] \arrow[dr, phantom, "\delta_{\Gamma_{0;I}}"] & F_0(I) \arrow[d, "\Gamma_{0,I}"] \\
F_0(I) \arrow[r, "F_1(\delta_I)"', "\shortmid"{marking}] \arrow[d, "\Gamma_{0,I}"'] \arrow[dr, phantom, "\Gamma_{1,\delta_I}"] & F_0(I) \arrow[d, "\Gamma_{0,I}"] & = & G_0(I) \arrow[r, "\delta_{G_0(I)}"', "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "G_{\delta;I}"] & G_0(I) \arrow[d, equal] \\
G_0(I) \arrow[r, "G_1(\delta_I)"', "\shortmid"{marking}] & G_0(I) & & G_0(I) \arrow[r, "G_1(\delta_I)"', "\shortmid"{marking}] & G_0(I)
\end{tikzcd}$$

$$\begin{tikzcd}
F_0(I) \arrow[r, "F_1(\alpha)", "\shortmid"{marking}] \arrow[d, equal] & F_0(J) \arrow[r, "F_1(\beta)", "\shortmid"{marking}] \arrow[d, phantom, "F_{\odot;\alpha,\beta}"] & F_0(K) \arrow[d, equal] & & F_0(I) \arrow[r, "F_1(\alpha)", "\shortmid"{marking}] \arrow[d, "\Gamma_{0,I}"'] \arrow[dr, phantom, "\Gamma_{1,\alpha}"] & F_0(J) \arrow[r, "F_1(\beta)", "\shortmid"{marking}] \arrow[d, "\Gamma_{0,J}"] \arrow[dr, phantom, "\Gamma_{1,\beta}"] & F_0(K) \arrow[d, "\Gamma_{0,K}"] \\
F_0(I) \arrow[rr, "F_1(\alpha\odot\beta)"', "\shortmid"{marking}] \arrow[d, "\Gamma_{0,I}"'] & {} \arrow[d, phantom, "\Gamma_{1,\alpha\odot\beta}"] & F_0(K) \arrow[d, "\Gamma_{0,K}"] & = & G_0(I) \arrow[r, "G_1(\alpha)"', "\shortmid"{marking}] \arrow[d, equal] & G_0(J) \arrow[r, "G_1(\beta)"', "\shortmid"{marking}] \arrow[d, phantom, "G_{\odot;\alpha,\beta}"] & G_0(K) \arrow[d, equal] \\
G_0(I) \arrow[rr, "G_1(\alpha\odot\beta)"', "\shortmid"{marking}] & {} & G_0(K) & & G_0(I) \arrow[rr, "G_1(\alpha\odot\beta)"', "\shortmid"{marking}] & {} & G_0(K)
\end{tikzcd}$$

In [**Gra20**, §3.8], the author distinguishes the notion of tightwise transformations from a weaker notion for which the naturality condition on objects is relaxed to isomorphisms in double categories. We will not use this weaker notion in this thesis. We write **Dbl** for the 2-category of double categories, double functors, and tightwise transformations. We will reformulate these concepts in terms of virtual double categories in the next section.

Example 1.2.3. We give some basic examples of double categories.

(i) The double category $\mathbb{R}\mathrm{el}(\mathcal{S}\mathrm{et})$ of relations between sets is defined as follows. Its tight category $\mathbb{R}\mathrm{el}(\mathcal{S}\mathrm{et})_0$ is the category $\mathcal{S}\mathrm{et}$ of sets and functions. The loose arrows are binary relations between sets, i.e., a loose arrow $\alpha\colon A \nrightarrow B$ is a subset of $A \times B$. A cell of the form (1.2.1) exists if and only if for any $a \in A$ and $b \in B$ such that $(a, b) \in \alpha$, we have $(f(a), g(b)) \in \beta$. There is at most one cell framed by a pair of tight arrows and a pair of loose arrows. The composite $\alpha \odot \beta$ of relations $\alpha\colon A \nrightarrow B$ and $\beta\colon B \nrightarrow C$ is defined as the following relation. For $a \in A$ and $c \in C$, we have $(a, c) \in \alpha \odot \beta$ if and only if there exists $b \in B$ such that $(a, b) \in \alpha$ and $(b, c) \in \beta$. The identity loose arrow δ_A is defined by the diagonal $\{ (a, a) \mid a \in A \}$. This construction is generalized to relations in a regular category [**Lam22**].

(ii) For a category $\mathcal{C}$ with pullbacks, we can form the double category of spans in $\mathcal{C}$, whose tight category is $\mathcal{C}$, whose loose arrows are spans in $\mathcal{C}$, that is, a pair of arrows with the same source, and whose cells are arrows from the vertex of the top loose arrow to the vertex of the bottom loose arrow.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & J \arrow[d, "g"] \\
K \arrow[r, "\beta"', "\shortmid"{marking}] & L
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & {|\alpha|} \arrow[dll, "\ell_\alpha"'] \arrow[drr, "r_\alpha"] \arrow[d, "\varphi"] & & \\
I \arrow[d, "f"'] & \circlearrowright & {|\beta|} \arrow[dll, "\ell_\beta"] \arrow[drr, "r_\beta"'] & \circlearrowright & J \arrow[d, "g"] \\
K & & & & L
\end{tikzcd}$$

The identity loose arrow δ_I is the span $(\mathtt{id}_I, \mathtt{id}_I)$, and the composition of spans is defined by the pullback in $\mathcal{C}$. The examples (i) and (ii) are generalized to the double category of relations relative to a stable factorization system [**HN23**].

(iii) The double category $\mathbb{P}\mathrm{rof}$ of profunctors has its tight category $\mathbb{P}\mathrm{rof}_0$ as the category $\mathcal{C}\mathrm{at}$ of small categories and functors, and its loose arrows from $\mathcal{C}$ to $\mathcal{D}$ are profunctors $\alpha\colon \mathcal{C} \nrightarrow \mathcal{D}$, namely, $\mathcal{S}\mathrm{et}$-valued functors $\alpha\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathcal{S}\mathrm{et}$. The cells are natural transformations between profunctors with respect to the source and target functors.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\varphi"] & \mathcal{D} \arrow[d, "g"] \\
\mathcal{E} \arrow[r, "\beta"', "\shortmid"{marking}] & \mathcal{F}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{D} \arrow[d, "f \times g"'] \arrow[dr, "\alpha"] & \\
\mathcal{E}^{\mathsf{op}} \times \mathcal{F} \arrow[r, "\beta"'] \arrow[ur, phantom, "\Downarrow \varphi"] & \mathcal{S}\mathrm{et}
\end{tikzcd}$$

The identity loose arrow $\delta_{\mathcal{C}}$ is the hom-profunctor $\mathcal{C}(-,-)$, and the composition of profunctors is defined by contraction in terms of coends.

$$(\alpha \odot \beta)(c,e) = \int^{d \in \mathcal{D}} \alpha(c,d) \times \beta(d,e) \quad \text{for} \quad c \in \mathcal{C}, e \in \mathcal{E}. \qquad (\alpha\colon \mathcal{C} \nrightarrow \mathcal{D}, \beta\colon \mathcal{D} \nrightarrow \mathcal{E})$$

⌟

We move on to illustrate several structures on double categories. Before we define the fibrational structure on double categories, let us review the notion of a (Grothendieck) fibration in category theory.

Definition 1.2.4. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a functor between categories. In the context of this thesis, $\mathcal{B}$ is called the ***base category*** and $\mathcal{E}$ is called the ***total category***. An object $\alpha \in \mathcal{E}$ (resp. an arrow $\varphi\colon \alpha \to \beta$ in $\mathcal{E}$) is called an ***object over*** $I \in \mathcal{B}$ (resp. an ***arrow over*** $f\colon I \to J$ in $\mathcal{B}$) if $\mathfrak{p}(I) = \alpha$ (resp. $\mathfrak{p}(f) = \varphi$). An arrow $\varphi\colon \alpha \to \beta$ is called $\mathfrak{p}$-***prone***[2] (or just ***prone***) over $f\colon I \to J$ if it is an arrow over f and for any arrow $\varphi'\colon \alpha' \to \beta$ in $\mathcal{E}$ such that $\mathfrak{p}(\varphi')$ factors through f as $\mathfrak{p}(\varphi') = f \circ g$, there exists a unique arrow $\psi\colon \varphi' \to \varphi$ over g such that $\varphi' = \varphi \circ \psi$.

$$\begin{tikzcd}
\mathcal{E} \arrow[dd, "\mathfrak{p}"'] & \alpha' \arrow[dr, dashed, "\exists!\ \psi"'] \arrow[drr, bend left, "\varphi'"] & & \\
& & \alpha \arrow[r, "\varphi"'] \arrow[u, phantom, "\circlearrowleft"] & \beta \\
\mathcal{B} & \mathfrak{p}(\alpha') \arrow[dr, "g"'] \arrow[drr, bend left, "\mathfrak{p}(\varphi')"] & & \\
& & I \arrow[r, "f"'] \arrow[u, phantom, "\circlearrowleft"] & J
\end{tikzcd}$$

Let $\mathcal{E}_I$ denote the subcategory of $\mathcal{E}$ consisting of objects over I and arrows over $\mathtt{id}_I$, which is called the ***fiber*** of $\mathcal{E}$ over I.

The functor $\mathfrak{p}$ is called a ***fibration*** if for any arrow $f\colon I \to J$ in $\mathcal{B}$ and any object $\beta \in \mathcal{E}_J$, there exists a prone arrow $\varphi\colon \alpha \to \beta$ over f. We call this arrow φ a ***prone lift*** of f to ψ, and write its domain φ as $\psi[f]$[3] The assignment $\psi \mapsto \psi[f]$ defines a functor $(-)[f]\colon \mathcal{E}_J \to \mathcal{E}_I$, which is called the ***base change*** or the ***reindexing*** along f.

A $\mathfrak{p}^{\mathsf{op}}$-prone arrow where $\mathfrak{p}^{\mathsf{op}}\colon \mathcal{E}^{\mathsf{op}} \to \mathcal{B}^{\mathsf{op}}$ is the opposite of $\mathfrak{p}$ is called a $\mathfrak{p}$-***supine*** arrow. An ***opfibration*** is a functor admitting supine lifts for every arrow in the base category. A ***bifibration*** is a functor that is both a fibration and an opfibration.

Proposition 1.2.5 ([**Shu08**, Theorem 4.1])**.** Let $\mathbb{D}$ be a double category, $f\colon I \to J$ be a tight arrow in $\mathbb{D}$, and $\alpha\colon I \nrightarrow J$ and $\beta\colon J \nrightarrow I$ be loose arrows. Then, the (structural) 2-out-of-3 condition holds for the following three data; i.e., given any two of the three pieces of data, the other is uniquely determined under a suitable ternary relation.

---

[2] Prone arrows are commonly called *cartesian* arrows in the literature. The term “prone” is borrowed from [**Tay99, Joh02a**]. The term “cartesian” is avoided in this thesis because “the word has been rather overworked by category-theorists, and deserves a rest” as Johnstone says [**Joh02a**, B 1.3, p.266].

[3] It is common to write $\psi[f]$ as $f^*\psi$ in the literature.

(i) Companion. A pair (φ, ψ) satisfying the following.

(1.2.2)
$$\begin{tikzcd} & I \arrow[r, "\alpha", "|"{marking}] \arrow[d, "f"] & J \\ I \arrow[ur, equals, "\psi"'] \arrow[r, "\alpha"', "|"{marking}] & J \arrow[ur, equals, "\varphi"] & \end{tikzcd} = \begin{tikzcd} I \arrow[r, "\alpha", "|"{marking}] \arrow[d, equals] \arrow[dr, phantom, "||"] & J \arrow[d, equals] \\ I \arrow[r, "\alpha"', "|"{marking}] & J \end{tikzcd} , \quad \begin{tikzcd} & I \arrow[dr, "f"] \arrow[d, phantom, "\psi"] & \\ I \arrow[ur, equals] \arrow[rr, "\alpha" description, "|"{marking}] \arrow[dr, "f"'] & {} & J \\ & J \arrow[ur, equals] \arrow[u, phantom, "\varphi"] & \end{tikzcd} = \begin{tikzcd} I \arrow[r, equals, "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\delta_f"] & I \arrow[d, "f"] \\ J \arrow[r, equals, "|"{marking}] & J \end{tikzcd}$$

If f and α come equipped with these structures, we say that α is a ***companion*** of f.

(ii) Conjoint. A pair (χ, υ) satisfying the following.

(1.2.3)
$$\begin{tikzcd} J \arrow[r, "\beta", "|"{marking}] \arrow[dr, equals, "\chi"] & I \arrow[d, "f"] \arrow[dr, equals, "\upsilon"'] & \\ & J \arrow[r, "\beta"', "|"{marking}] & I \end{tikzcd} = \begin{tikzcd} J \arrow[r, "\beta", "|"{marking}] \arrow[d, equals] \arrow[dr, phantom, "||"] & I \arrow[d, equals] \\ J \arrow[r, "\beta"', "|"{marking}] & I \end{tikzcd} , \quad \begin{tikzcd} & I \arrow[dl, "f"'] \arrow[dr, equals] \arrow[d, phantom, "\upsilon"] & \\ J \arrow[rr, "\beta" description, "|"{marking}] \arrow[dr, equals] & {} & I \arrow[dl, "f"] \\ & J \arrow[u, phantom, "\chi"] & \end{tikzcd} = \begin{tikzcd} I \arrow[r, equals, "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\delta_f"] & I \arrow[d, "f"] \\ J \arrow[r, equals, "|"{marking}] & J \end{tikzcd}$$

If f and β come equipped with these structures, we say that β is a ***conjoint*** of f.

(iii) Adjoint in $\mathbf{L}(\mathbb{D})$. A pair (η, ε) satisfying the following.

(1.2.4)
$$\begin{tikzcd} & J \arrow[r, "\beta", "|"{marking}] \arrow[d, phantom, "||"] & I \arrow[dr, equals] \arrow[d, phantom, "\eta"] & \\ J \arrow[ur, equals] \arrow[r, "\beta", "|"{marking}] \arrow[dr, equals] & I \arrow[ur, equals] \arrow[r, "\alpha", "|"{marking}] \arrow[d, phantom, "\varepsilon"] & J \arrow[r, "\beta"', "|"{marking}] \arrow[d, phantom, "||"] & I \\ & J \arrow[ur, equals] \arrow[r, "\beta"', "|"{marking}] & I \arrow[ur, equals] & \end{tikzcd} = \begin{tikzcd} J \arrow[r, "\beta", "|"{marking}] \arrow[d, equals] \arrow[dr, phantom, "||"] & I \arrow[d, equals] \\ J \arrow[r, "\beta"', "|"{marking}] & I \end{tikzcd} , \quad \begin{tikzcd} & I \arrow[r, "\alpha", "|"{marking}] \arrow[dr, equals] \arrow[d, phantom, "\eta"] & J \arrow[dr, equals] \arrow[d, phantom, "||"] & \\ I \arrow[ur, equals] \arrow[r, "\alpha"', "|"{marking}] \arrow[dr, equals] & J \arrow[r, "\beta", "|"{marking}] \arrow[dr, equals] \arrow[d, phantom, "||"] & I \arrow[r, "\alpha", "|"{marking}] \arrow[d, phantom, "\varepsilon"] & J \\ & I \arrow[r, "\alpha"', "|"{marking}] & J \arrow[ur, equals] & \end{tikzcd} = \begin{tikzcd} I \arrow[r, "\alpha", "|"{marking}] \arrow[d, equals] \arrow[dr, phantom, "||"] & J \arrow[d, equals] \\ I \arrow[r, "\alpha"', "|"{marking}] & J \end{tikzcd}$$

In particular, a tight arrow with companion and conjoint produces an adjoint in $\mathbf{L}(\mathbb{D})$. We call such an adjoint a ***representable*** adjoint. ⌟

Definition 1.2.6. A double category $\mathbb{D}$ is an ***equipment*** (or a ***fibrational double category***[4]) if the functor $\langle \mathsf{src}, \mathsf{tgt} \rangle \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a fibration. ⌟

Equipments are also known as 'framed bicategories' [**Shu08**] and 'fibrant double categories' [**Ale18**].

Remark 1.2.7. A double category $\mathbb{D}$ is an equipment if and only if $\langle \mathsf{src}, \mathsf{tgt} \rangle$ is an opfibration, hence a bifibration. Also, being an equipment is equivalent to the condition that for every tight arrow $f \colon I \to J$, there are loose arrows $\alpha \colon I \nrightarrow J$ and $\beta \colon J \nrightarrow I$, equipped with two (hence all) of the data listed in Proposition 1.2.5; see [**Shu08**, Theorem 4.1]. Under this correspondence, φ in (1.2.2) is the prone lifting of $(f \colon I \to J, \mathtt{id} \colon J \to J)$, and likewise for other cells. The companion and conjoint of $f \colon I \to J$ are written as f_* and f^*.

By a ***prone (resp. supine) cell***, we mean a prone (resp. supine) morphism of the bifibration $\langle \mathsf{src}, \mathsf{tgt} \rangle$. From a loose arrow $\alpha \colon J \nrightarrow K$ and tight arrows $f \colon H \to J$ and $g \colon I \to K$, the prone lift of (f, g) to α in the bifibration gives the prone cell as the cell on the left below.

$$\begin{tikzcd} H \arrow[r, "{\alpha[f \fatsemi g]}", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathtt{prn}"] & I \arrow[d, "g"] \\ J \arrow[r, "\alpha"', "|"{marking}] & K \end{tikzcd} , \quad \begin{tikzcd} H \arrow[r, "\beta", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathtt{spn}"] & I \arrow[d, "g"] \\ J \arrow[r, "{\mathit{opRest}(\beta; f, g)}"', "|"{marking}] & K \end{tikzcd} , \quad \begin{tikzcd} H \arrow[rr, "{\delta_K[f \fatsemi g]}", "|"{marking}] \arrow[dr, "f"'] & \mathtt{prn} & I \arrow[dl, "g"] \\ & K & \end{tikzcd} , \quad \begin{tikzcd} & I \arrow[dl, "f"'] \arrow[dr, "g"] & \\ J \arrow[rr, "{\mathit{opRest}(I; f, g)}"', "|"{marking}] & \mathtt{spn} & K \end{tikzcd}$$

Here the prone cell is unique up to invertible globular cell, so we just write `prn` for the prone cell and call the loose arrow $\alpha[f \fatsemi g]$ the ***restriction*** of α along f and g. Note that the tight composition of two prone cells is prone, and the tight composition of two supine cells is supine. Taking the loose dual, the supine cell is unique up to invertible globular cell, so we just write `spn` for the supine cell and call

---

[4] In the virtual setting, virtual equipments and fibrational virtual double categories are different concepts, but the difference disappears in double categories. Therefore, we use the terms interchangeably in this thesis depending on which framework we consider as its generalization.

the loose arrow $\mathit{opRest}(\beta; f, g)$ the ***oprestriction*** of β along f and g. In particular, as presented in the right half of the above diagrams, the restriction of δ_K through f and g is written as $K(f, g)$, and the oprestriction of δ_I through f and g is written as $\mathit{opRest}(I; f, g)$ for brevity.

The restriction $\alpha(f, g)$ and the oprestriction $\mathit{opRest}(\beta; f, g)$ are realized as $f_* p g^*$ and $f^* q g_*$, respectively, using the companion and conjoint, and the prone cell and the supine cell are realized as below.

$$\begin{tikzcd}
H \arrow[r, "f_*", "|" marking] \arrow[rd, "f"'] \arrow[rd, phantom, "\varphi", near start, shift left=2ex] & J \arrow[r, "\alpha", "|" marking] \arrow[d, equal] \arrow[rd, phantom, "="] & K \arrow[r, "g^*", "|" marking] \arrow[d, equal] & I \arrow[ld, "g"] \arrow[ld, phantom, "\chi", near start, shift right=2ex] \\
& J \arrow[r, "\alpha"', "|" marking] & K &
\end{tikzcd}
,\qquad
\begin{tikzcd}
& H \arrow[r, "\beta", "|" marking] \arrow[ld, "f"'] \arrow[d, equal] \arrow[rd, phantom, "="] & I \arrow[d, equal] \arrow[rd, "g"] & \\
J \arrow[r, "f^*"', "|" marking] \arrow[ru, phantom, "\upsilon", near start, shift left=2ex] & H \arrow[r, "\beta"', "|" marking] & I \arrow[r, "g_*"', "|" marking] \arrow[ru, phantom, "\psi", near start, shift left=2ex] & K
\end{tikzcd}$$

Put it another way, if we are given a prone cell φ and χ as above, then the above cell is the restriction of α through f and g. Since the φ and χ are prone cells and the ψ and υ are supine cells, we just write `prn` and `spn` for them as well. For a comprehensive treatment on equipments, see [**Shu08**, §4]. ⌟

Remark 1.2.8. By the general theory of fibrations, it is known that isomorphisms in $\mathbb{D}_1$, which we will call ***tightwise isomorphisms*** from now on, are prone and supine cells at the same time. In addition, prone and supine cells are closed under tightwise composition. ⌟

Example 1.2.9. The examples in Example 1.2.3 are all equipments.

(i) In the double category $\mathbb{R}\mathrm{el}(\mathcal{S}\mathrm{et})$ of relations between sets, the companion and conjoint of a function $f\colon I \to J$ are its graphs $\{(i, f(i)) \mid i \in I\}$ and $\{(f(i), i) \mid i \in I\}$ as relations.
(ii) In the double category of spans in a category with pullbacks, the companion and conjoint of an arrow $f\colon I \to J$ are $(\mathtt{id}_I, f)$ and $(f, \mathtt{id}_J)$, respectively.
(iii) In the double category of profunctors, the companion and conjoint of a functor $F\colon \mathcal{C} \to \mathcal{D}$ are the representable profunctors $\mathcal{D}(F(-), -)$ and $\mathcal{D}(-, F(-))$, respectively.

⌟

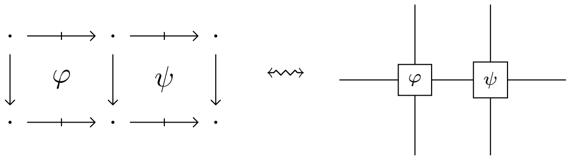
Remark 1.2.10. If one already knows that a double category is an equipment, then checking a cell is prone or supine becomes a simpler task. In an equipment, a cell τ is prone (resp. supine) if and only if it shows the universal property of the prone (resp. supine) cell only for the cells with the same tight arrows f and g and the same loose arrow β at the bottom (resp. the same loose arrow α at the top).

$$\tau \text{ is prone} \quad\iff\quad
\begin{tikzcd}
I \arrow[r, "\gamma", "|" marking] \arrow[d, "f"'] \arrow[rd, phantom, "\varphi"] & J \arrow[d, "g"] \\
K \arrow[r, "\beta"', "|" marking] & L
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "\gamma", "|" marking] \arrow[d, equal] \arrow[rd, phantom, "\exists!\ \widehat{\varphi}"] & J \arrow[d, equal] \\
I \arrow[r, "\alpha", "|" marking] \arrow[d, "f"'] \arrow[rd, phantom, "\tau"] & J \arrow[d, "g"] \\
K \arrow[r, "\beta"', "|" marking] & L
\end{tikzcd}$$

This follows from the corresponding fact in the context of bifibrations. ⌟

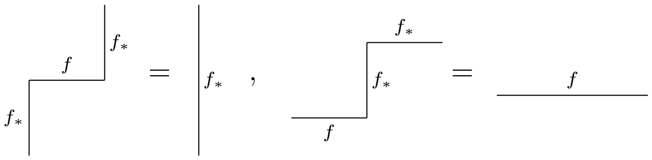
Remark 1.2.11 (String diagrams in equipments)**.** String diagrams are known to be a useful tool in reasoning about monoidal categories and bicategories as they offer visualized intuition for the composition of cells. They are naturally extended to double categories as well. The paper [**Mye18**] introduces string diagrams in double categories, and discusses soundness of the diagrammatic calculus.

In string diagrams for double categories, objects are drawn as regions, tight arrows are drawn as horizontal lines, loose arrows are drawn as vertical lines, and cells are drawn as vertices. Composition of cells is represented by concatenation of vertices along the lines as shown in the following diagram.

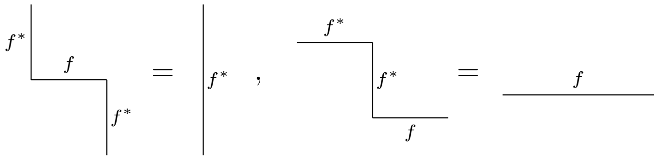


For the companion and conjoint of a tight arrow $f\colon I \to J$, we do not explicitly depict the vertices for the cells in (1.2.2) and (1.2.3). Instead, we express those cells with zigzag lines, and the equations in Proposition 1.2.5 are represented as follows.

(1.2.5)



(1.2.6)



The unit and counit of the adjunction $f_* \dashv f^*$ are represented with the zigzag strings like $\sqcap$ and $\sqcup$, and they satisfy the triangle identities by the above equations. ⌟

In the joint work [**HN23**], the author and Hoshino made the following small observation, which turns out to be a convenient and powerful tool in reasoning about double categories.

Lemma 1.2.12 (Sandwich Lemma, [**HN23**, Lemma 2.1.8]**).** Let $\mathbb{D}$ be an equipment. Given a sequence of loosewise composable cells

(1.2.7)
$$\begin{tikzcd}
\cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\texttt{prn}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\texttt{spn}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\texttt{prn}"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot
\end{tikzcd}$$

with the supine cell sandwiched between two prone cells, the composition of these cells is prone. The same thing holds when swapping the roles of 'prone' and 'supine'. ⌟

Proof. By Remark 1.2.7, we can rewrite the diagram (1.2.7) as follows, in which the names of the cells correspond to that in Proposition 1.2.5.

$$\begin{tikzcd}
\cdot \arrow[r, "f_*", "\shortmid"{marking}] \arrow[dr, "f"', "\varphi"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dl, "\chi"', "\upsilon"] \arrow[dr, phantom, "\shortparallel"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, "\psi"', "\varphi"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & \cdot \arrow[r, "k^*", "\shortmid"{marking}] \arrow[d, equal] & \cdot \arrow[dl, "\chi"', "k"] \\
& \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot &
\end{tikzcd}$$

Because of the equalities described in Proposition 1.2.5, the middle sequence of square cells are all identities. Again by Proposition 1.2.5, this implies that the composition of the cells in the diagram is prone. Considering the same statement for the tightwise opposite of $\mathbb{D}$, we obtain the dual. ⌟

Definition 1.2.13. Let $\mathbb{D}$ be a double category. We say a tight arrow $f\colon I \to J$ is an ***inclusion*** if the loose identity cell on f is prone. We say a tight arrow $f\colon I \to J$ is a ***cover*** if the loose identity cell on f is supine.

$$f \text{ is an inclusion} \iff \begin{tikzcd}
I \arrow[r, "\delta_I", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\texttt{prn}"] & I \arrow[d, "f"] \\
J \arrow[r, "\delta_J"', "\shortmid"{marking}] & J
\end{tikzcd}, \quad f \text{ is a cover} \iff \begin{tikzcd}
I \arrow[r, "\delta_I", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\texttt{spn}"] & I \arrow[d, "f"] \\
J \arrow[r, "\delta_J"', "\shortmid"{marking}] & J
\end{tikzcd}$$

⌟

Remark 1.2.14. In other words, $f\colon I \to J$ is an inclusion if the restriction $J(f, f)$ is isomorphic to the loose identity δ_A, and $f\colon I \to J$ is a cover if the oprestriction $\mathit{opRest}(I; f, f)$ is isomorphic to the loose identity δ_B. With inclusions and covers, we gain a better command of the diagrammatic calculation of prone and supine cells via the sandwich lemma Lemma 1.2.12. For example, the following cells are all prone, where $\hookrightarrow$ and $\twoheadrightarrow$ denote an inclusion and a cover, respectively.

$$\begin{tikzcd}
\cdot \arrow[r, "\shortmid"{marking}] \arrow[d, hook] \arrow[dr, phantom, "\texttt{spn}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\texttt{prn}"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot
\end{tikzcd}, \quad \begin{tikzcd}
\cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\texttt{prn}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, two heads] \arrow[dr, phantom, "\texttt{prn}"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot
\end{tikzcd}, \quad \begin{tikzcd}
& \cdot \arrow[dl] \arrow[dr] \arrow[d, phantom, "\texttt{prn}"] & \\
\cdot \arrow[r, "\shortmid"{marking}] \arrow[d, hook] \arrow[dr, phantom, "\texttt{spn}"] & \cdot \arrow[r, "\shortmid"{marking}] \arrow[d, hook] \arrow[dr, phantom, "\texttt{spn}"] & \cdot \arrow[d, hook] \\
\cdot \arrow[r, "\shortmid"{marking}] & \cdot \arrow[r, "\shortmid"{marking}] & \cdot
\end{tikzcd}$$

⌟

Example 1.2.15. Let us consider again the examples in Example 1.2.9.

(i) A restriction of a relation $\beta\colon J \relbar\joinrel\mapstochar\joinrel\rightarrow L$ along a pair of functions $f\colon I \to J$ and $g\colon K \to L$ is the relation $\beta[f \fatsemi g]$ defined by
$$(i,k) \in \beta[f \fatsemi g] \quad \iff \quad (f(i), g(k)) \in \beta.$$
Thus, inclusions in $\mathbb{R}\mathrm{el}(\mathcal{S}\mathrm{et})$ are precisely the monomorphisms. This is the same for the double category of relations in any regular category.

(ii) In the double category $\mathbb{S}\mathrm{pan}(\mathcal{C})$ of spans in a category $\mathcal{C}$ with pullbacks, an oprestriction of a span $(p,q)\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow K$ along a pair of arrows $f\colon I \to J$ and $g\colon K \to L$ is the span $(f \circ p, g \circ q)\colon J \relbar\joinrel\mapstochar\joinrel\rightarrow L$. Thus, covers in this double category are limited to the isomorphisms.

(iii) In the double category of profunctors, a restriction of a profunctor $\alpha\colon \mathcal{C} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{D}$ along a pair of functors $F\colon \mathcal{I} \to \mathcal{C}$ and $G\colon \mathcal{J} \to \mathcal{D}$ is the profunctor $\alpha(F-, G-)$. In this double category, inclusions are the fully faithful functors, and covers are the absolutely dense functors.

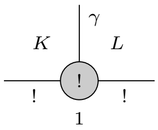
More details on inclusions and covers can be found in [**HN23**]. ⌟

Remark 1.2.16. Since the condition for a double category to be an equipment is characterized by the existence of cells satisfying the equations in Proposition 1.2.5, a double functor between equipments preserves all the structures of equipments. In particular, a double functor preserves prone and supine cells as they are presented as composites of the identity cells and the cells satisfying the equations in Proposition 1.2.5. We write **Eqp** for the sub 2-category of **Dbl** spanned by all equipments. ⌟

The 2-category **Dbl** of double categories has strict finite products by naive pointwise construction, and the sub 2-category **Eqp** of equipments is closed under the formation of products. Following Definition 1.1.1, by ***cartesian double categories***, we mean cartesian objects in **Dbl**. In the same way, we define ***cartesian equipments*** as cartesian objects in **Eqp**. Since it is a full sub-2-category of **Dbl**, an equipment is cartesian as a double category if and only if it is cartesian as an equipment.

A comprehensive account of cartesian double categories and cartesian equipments can be found in [**Ale18**]. Here, we present a brief review of the argument. The right adjoints $1\colon \mathbb{1} \to \mathbb{D}$ and $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ of the double functors $!\colon \mathbb{D} \to \mathbb{1}$ and $\Delta\colon \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ have the following universal properties. The detailed discussion is given in [**Ale18**]. For a terminal object 1 in $\mathbb{D}$, it has the universal property that for any object K in $\mathbb{D}$, there is a unique tight arrow $!\colon K \to 1$, and for any loose arrow $\gamma\colon K \to L$ in $\mathbb{D}$, there is a unique cell $!$ whose bottom face is δ_1. Note that δ_1 does not appear in the diagram because it is a loose identity.

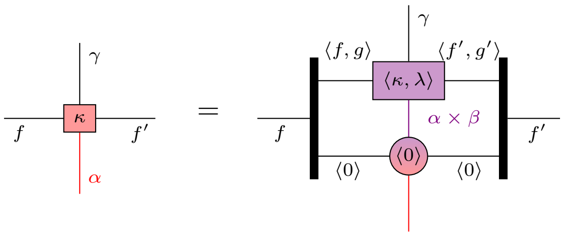
$$\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "!"'] \arrow[dr, phantom, "!"] & L \arrow[d, "!"] \\
1 \arrow[r, "\delta_1"', "\shortmid"{marking}] & 1
\end{tikzcd}$$

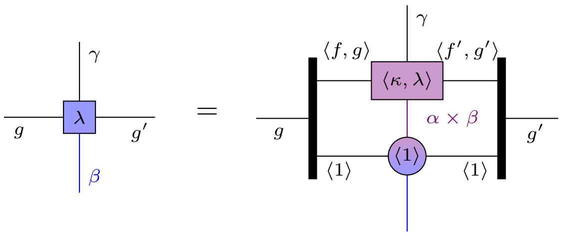


For binary products $I \times J$ in $\mathbb{D}$, they have the universal property that for any object K in $\mathbb{D}$ and any pair of tight arrows $f\colon K \to I$ and $g\colon K \to J$, there is a unique tight arrow $\langle f, g\rangle\colon K \to I \times J$ such that $\langle 0\rangle \circ \langle f, g\rangle = f$ and $\langle 1\rangle \circ \langle f, g\rangle = g$. For binary products of loose arrows $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow I'$ and $\beta\colon J \relbar\joinrel\mapstochar\joinrel\rightarrow J'$ in $\mathbb{D}$, they have the universal property that for any pair of cells κ and λ as below, there is a unique cell $\langle \alpha, \beta\rangle$ that makes the following two equations hold.

$$\forall \left(
\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\kappa"] & K' \arrow[d, "f'"] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] & I'
\end{tikzcd}
,
\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\lambda"] & K' \arrow[d, "g'"] \\
J \arrow[r, "\beta"', "\shortmid"{marking}] & J'
\end{tikzcd}
\right)
\exists !
\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "{\langle f,g\rangle}"'] \arrow[dr, phantom, "{\langle \kappa, \lambda\rangle}"] & K' \arrow[d, "{\langle f',g'\rangle}"] \\
I \times J \arrow[r, "\alpha\times\beta"', "\shortmid"{marking}] & I' \times J'
\end{tikzcd}$$

$$\text{s.t.}
\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\kappa"] & K' \arrow[d, "f'"] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] & I'
\end{tikzcd}
=
\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "{\langle f,g\rangle}"'] \arrow[dr, phantom, "{\langle \kappa, \lambda\rangle}"] & K' \arrow[d, "{\langle f',g'\rangle}"] \\
I \times J \arrow[r, "\alpha\times\beta"', "\shortmid"{marking}] \arrow[d, "\langle 0\rangle"'] \arrow[dr, phantom, "\langle 0\rangle"] & I' \times J' \arrow[d, "\langle 0\rangle"] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] & I'
\end{tikzcd}
\text{and}
\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\lambda"] & K' \arrow[d, "g'"] \\
J \arrow[r, "\beta"', "\shortmid"{marking}] & J'
\end{tikzcd}
=
\begin{tikzcd}
K \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, "{\langle f,g\rangle}"'] \arrow[dr, phantom, "{\langle \kappa, \lambda\rangle}"] & K' \arrow[d, "{\langle f',g'\rangle}"] \\
I \times J \arrow[r, "\alpha\times\beta"', "\shortmid"{marking}] \arrow[d, "\langle 1\rangle"'] \arrow[dr, phantom, "\langle 1\rangle"] & I' \times J' \arrow[d, "\langle 1\rangle"] \\
J \arrow[r, "\beta"', "\shortmid"{marking}] & J'
\end{tikzcd}
.$$





Here, the black squares represent the identity, which is only explicitly drawn in string diagrams because a composable sequence and its composite are depicted differently in string diagrams.

Obvious from the above universal properties, the tight category $\mathbb{D}_0$ of a cartesian double category $\mathbb{D}$ is a cartesian category. In addition, a cartesian double category induces the finite-product structure on the loose hom-category $\mathbb{D}(I, J)$ for a pair of objects I and J as follows. The terminal object and the binary product of loose arrows $\alpha, \beta \colon I \nrightarrow J$ in $\mathbb{D}$ are defined by

$$\top_{I,J} := !_* !^* \quad \text{and} \quad \alpha \wedge \beta := \langle 0, 0\rangle_* (\alpha \times \beta) \langle 0, 0\rangle^*,$$

where !'s are the unique tight arrows to the terminal object 1 and $\langle 0, 0\rangle$'s are the diagonal arrows.

However, the finite products on the tight category $\mathbb{D}_0$ and the loose hom-categories $\mathbb{D}(I, J)$ for every pair of objects I and J do not necessarily induce a cartesian structure on the double category $\mathbb{D}$. From these data, we can define a potential product of two loose arrows $\alpha \colon I \nrightarrow J$ and $\beta \colon K \nrightarrow L$ as

$$\alpha \times \beta := (\langle 0\rangle_* \alpha \langle 0\rangle^*) \wedge (\langle 1\rangle_* \beta \langle 1\rangle^*)$$

and a potential terminal object as the terminal object of the tight category $\mathbb{D}_0$. However, these data do not constitute the desired double functors $\times$ and 1 but only *lax double functors* in general. We do not give the precise definition of lax double functors here, because the concept can be defined as virtual double functors when we regard double categories as virtual double categories; see Definition 1.3.4. In light of this, we can state the following proposition.

Proposition 1.2.17 ([Ale18, Corollary 4.3.3]). An equipment $\mathbb{D}$ is cartesian if and only if

(i) $\mathbb{D}_0$ is a cartesian category,
(ii) $\mathbf{L}(\mathbb{D})$ locally has finite products, that is, for every pair of objects I and J in $\mathbb{D}_0$, $\mathbf{L}(\mathbb{D})(I, J)$ is a cartesian category,
(iii) the lax double functors $\times \colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and $1 \colon \mathbb{1} \to \mathbb{D}$ induced by the above data are actually double functors.

⌟

Example 1.2.18. The double categories $\mathbb{R}\mathrm{el}(\mathcal{S}\mathrm{et})$ of relations in $\mathcal{S}\mathrm{et}$, $\mathbb{S}\mathrm{pan}(\mathcal{C})$ of spans in a category $\mathcal{C}$ with pullbacks, and $\mathbb{P}\mathrm{rof}(\mathcal{C})$ of profunctors in a category $\mathcal{C}$ are all cartesian equipments. ⌟

Remark 1.2.19. In [Pat24], products are formulated in an unbiased way using the family construction. The paper also meticulously discusses the gradation of possible definitions of products in double categories. ⌟

A *Beck-Chevalley pullback square* in a double category plays a fundamental role in [HN23], where it serves as a double categorical version of the Beck-Chevalley condition in a bicategory [WW08, 2.4].

Definition 1.2.20 ([HN23, Definition 3.1.1]). Let $\mathbb{D}$ be a double category. A ***diamond cell*** in $\mathbb{D}$ is a quadruple of vertical arrows together with a vertical cell α of the form on the left below. A diamond cell is called an ***identity diamond cell*** if the vertical cell is the identity cell. We say a diamond cell satisfies the ***Beck-Chevalley condition*** if there exists a horizontal arrow $\alpha \colon B \nrightarrow C$ and α factors as an opcartesian cell followed by a cartesian cell as shown in the right below:

(1.2.8)
$$\begin{tikzcd}
& I \arrow[ddl, "g"'] \arrow[ddr, "f"] & & & & I \arrow[ddl, "g"'] \arrow[ddr, "f"] & \\
& & & & & \mathtt{spn} & \\
J \arrow[ddr, "h"'] & \alpha & K \arrow[ddl, "k"] & = & J \arrow[rr, "\alpha" description, "\shortmid" marking] \arrow[ddr, "h"'] & & K \arrow[ddl, "k"] \\
& & & & & \mathtt{prn} & \\
& L & & & & L &
\end{tikzcd}$$

Although this condition is defined for a diamond cell, we often abuse the terminology and say that a cell α satisfies the Beck-Chevalley condition when the quadruple of vertical arrows is evidently recognised from the context. ⌟

Definition 1.2.21 ([HN23, Definition 3.1.2])**.** A ***Beck-Chevalley pullback square*** in $\mathbb{D}$ is a pullback square in $\mathbb{D}_0$ as presented on the left below for which the two identity diamond cells placed in both directions as in the diagrams in the middle and right below satisfy the Beck-Chevalley condition.

$$\begin{tikzcd}
P \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "f"] \\
J \arrow[r, "g"'] & K
\end{tikzcd}
,\quad
\begin{tikzcd}
& P \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, phantom, "="] & \\
I \arrow[dr, "f"'] & & J \arrow[dl, "g"] \\
& K &
\end{tikzcd}
,\quad
\begin{tikzcd}
& P \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, phantom, "="] & \\
J \arrow[dr, "g"'] & & I \arrow[dl, "f"] \\
& K &
\end{tikzcd}$$

We say a double category $\mathbb{D}$ ***has the Beck-Chevalley pullbacks*** if the vertical category $\mathbb{D}_0$ has pullbacks and their pullback squares are all Beck-Chevalley pullback squares. ⌟

Lemma 1.2.22. Let $\mathbb{D}$ be a cartesian equipment.

(i) The pullback of an identity arrow along any arrow gives a Beck-Chevalley pullback square in $\mathbb{D}$.
(ii) Beck-Chavalley pullback squares in $\mathbb{D}$ are closed under finite products.
(iii) Beck-Chevalley pullback squares in $\mathbb{D}$ are closed under pasting.

$$\text{(i)}\quad
\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "\mathtt{id}_I"'] \arrow[dr, phantom, "\text{(BC)}"] & K \arrow[d, "\mathtt{id}_K"] \\
I \arrow[r, "f"'] & K
\end{tikzcd}
\qquad
\text{(ii)}\quad
\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{(BC)}"] & K \arrow[d, "h"] \\
J \arrow[r, "k"'] & L
\end{tikzcd}
,\;
\begin{tikzcd}
I' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\text{(BC)}"] & K' \arrow[d, "h'"] \\
J' \arrow[r, "k'"'] & L'
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
I \times I' \arrow[r, "f \times f'"] \arrow[d, "g \times g'"'] \arrow[dr, phantom, "\text{(BC)}"] & K \times K' \arrow[d, "h \times h'"] \\
J \times J' \arrow[r, "k \times k'"'] & L \times L'
\end{tikzcd}$$

$$\text{(iii)}\quad
\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{(BC)}"] & K \arrow[d, "n"] \\
J \arrow[r, "k"'] & L
\end{tikzcd}
,\;
\begin{tikzcd}
K \arrow[r, "h"] \arrow[d, "n"'] \arrow[dr, phantom, "\text{(BC)}"] & M \arrow[d, "l"] \\
L \arrow[r, "m"'] & N
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "g"'] \arrow[drr, phantom, "\text{(BC)}"] & K \arrow[r, "h"] & M \arrow[d, "l"] \\
J \arrow[r, "k"'] & L \arrow[r, "m"'] & N
\end{tikzcd}$$

⌟

Proof.

(i) The two identity diamond cells for this pullback square are given by the companion and the conjoint of the arrow f.
(ii) Since every double functor preserves prone and supine cells, the Beck-Chevalley condition on diamond cells is preserved under the product functor $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$.
(iii) In the following diagram, the big top triangle is a supine cell and the big bottom triangle is a prone cell because of the sandwich lemma Lemma 1.2.12.

$$\begin{tikzcd}
& & I \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "\mathtt{spn}"] & & \\
& I \arrow[rr, "\shortmid"{marking}] \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[d, phantom, "\mathtt{spn}"] & {} \arrow[d, phantom, "\mathtt{prn}"] & K \arrow[dl, equal] \arrow[dr, "h"] \arrow[d, phantom, "\mathtt{spn}"] & \\
J \arrow[rr, "\shortmid"{marking}] \arrow[dr, "h"'] & {} \arrow[d, phantom, "\mathtt{prn}"] & K \arrow[rr, "\shortmid"{marking}] \arrow[dl, "n"'] \arrow[dr, "h"] \arrow[d, phantom, "\mathtt{spn}"] & {} \arrow[d, phantom, "\mathtt{prn}"] & M \arrow[dl, equal] \\
& L \arrow[rr, "\shortmid"{marking}] \arrow[dr, "m"'] & {} \arrow[d, phantom, "\mathtt{prn}"] & M \arrow[dl, "l"] & \\
& & N & &
\end{tikzcd}$$

⌟

Lemma 1.2.23. Let $\mathbb{D}$ be a cartesian equipment. Suppose we have a pullback square and a loose arrow as follows.

$$\begin{tikzcd}
& I \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
J \arrow[dr, "f"'] & & K \arrow[r, "\alpha", "\shortmid"{marking}] & M \\
& L \arrow[from=ur, "g"] & &
\end{tikzcd}$$

Then the canonical cell σ on the right below is an isomorphism if the pullback square is a Beck-Chevalley pullback square.

$$\begin{tikzcd}
I \arrow[r, "t_*\alpha", "\shortmid"{marking}] \arrow[d, "t"'] \arrow[rd, phantom, "\mathtt{prn}"] & M \arrow[d, equal] \\
K \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[rd, phantom, "\mathtt{spn}"] & M \arrow[d, equal] \\
L \arrow[r, "g^*\alpha"', "\shortmid"{marking}] & M
\end{tikzcd}
= =
\begin{tikzcd}
I \arrow[r, "t_*\alpha", "\shortmid"{marking}] \arrow[d, "s"'] & M \arrow[d, equal] \\
J \arrow[r, bend left, "s^*t_*\alpha", "\shortmid"{marking}] \arrow[r, bend right, "f_*g^*\alpha"', "\shortmid"{marking}] \arrow[r, phantom, "\sigma"] \arrow[d, "f"'] & M \arrow[d, equal] \\
L \arrow[r, "g^*\alpha"', "\shortmid"{marking}] & M
\end{tikzcd}$$

The upper square of the right-hand side is labelled $\mathtt{spn}$ and the lower square $\mathtt{prn}$.

⌟

Proof. Applying the sandwich lemma Lemma 1.2.12 to the diagram below, we obtain the desired result.

$$\begin{tikzcd}
& I \arrow[ld, "s"'] \arrow[rd, "t"] \arrow[rr, "t_*\alpha", "\shortmid"{marking}] \arrow[d, phantom, "\mathtt{spn}"] & & M \arrow[d, equal] \\
J \arrow[rr, "\shortmid"{marking}] \arrow[rd, "f"'] & \arrow[d, phantom, "\mathtt{prn}"] & K \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[ld, "g"] & M \arrow[d, equal] \\
& L \arrow[rr, "g^*\alpha"', "\shortmid"{marking}] & & M
\end{tikzcd}$$

⌟

Lemma 1.2.24. Let $\mathbb{D}$ be a cartesian equipment. Suppose we have the following data in a double category $\mathbb{D}$.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "f"'] & K \\
J \arrow[r, "\beta"', "\shortmid"{marking}] & K
\end{tikzcd}$$

Then the canonical cell σ on the right below is an isomorphism if the pullback of $\langle 0,0\rangle$ and $f\times\mathtt{id}_K$, which is always given by the span f and $\langle\mathtt{id},f\rangle$, is a Beck-Chevalley pullback square.

$$\begin{tikzcd}
I \arrow[r, "\alpha\wedge f_*\beta", "\shortmid"{marking}] \arrow[d, "\langle\mathtt{id},f\rangle"'] \arrow[rd, phantom, "\mathtt{prn}"] & K \arrow[d, "\langle 0,0\rangle"] \\
I\times J \arrow[r, "\alpha\times\beta", "\shortmid"{marking}] \arrow[d, "f\times\mathtt{id}"'] \arrow[rd, phantom, "\mathtt{spn}"] & K\times K \arrow[d, equal] \\
J\times J \arrow[r, "f^*\alpha\times\beta"', "\shortmid"{marking}] & K\times K
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "\alpha\wedge f_*\beta", "\shortmid"{marking}] \arrow[d, "f"'] & K \arrow[d, equal] \\
J \arrow[r, bend left, "f^*(\alpha\wedge f_*\beta)", "\shortmid"{marking}] \arrow[r, bend right, "f^*\alpha\wedge\beta"', "\shortmid"{marking}] \arrow[r, phantom, "\sigma"] \arrow[d, "\langle 0,0\rangle"'] & K \arrow[d, "\langle 0,0\rangle"] \\
J\times J \arrow[r, "f^*\alpha\times\beta"', "\shortmid"{marking}] & K\times K
\end{tikzcd}$$

The upper square of the right-hand side is labelled $\mathtt{spn}$ and the lower square $\mathtt{prn}$.

⌟

Proof. By assumption, the pullback of $\langle 0,0\rangle$ and $f\times\mathtt{id}_K$ is a Beck-Chevalley pullback square. Thus, we have the following diagram and the sandwich lemma Lemma 1.2.12 gives the desired result.

$$\begin{tikzcd}
& I \arrow[ld, "f"'] \arrow[rd, "\langle\mathtt{id},f\rangle"] \arrow[rrr, "\alpha\wedge f_*\beta", "\shortmid"{marking}] \arrow[d, phantom, "\mathtt{spn}"] & & \mathtt{prn} & K \arrow[ld, "\langle 0,0\rangle"'] \arrow[rd, equal] \arrow[d, phantom, "\mathtt{spn}"] & \\
J \arrow[rr, "\shortmid"{marking}] \arrow[rd, "\langle 0,0\rangle"'] & \mathtt{prn} & I\times J \arrow[r, "\alpha\times\beta", "\shortmid"{marking}] \arrow[ld, "f\times\mathtt{id}"'] & K\times K \arrow[rr, "\langle 0,0\rangle^*", "\shortmid"{marking}] \arrow[rd, equal] \arrow[d, phantom, "\mathtt{spn}"] & \mathtt{prn} & K \arrow[ld, "\langle 0,0\rangle"] \\
& J\times J \arrow[rrr, "f_*\alpha\times\beta"', "\shortmid"{marking}] & & & K\times K &
\end{tikzcd}$$

⌟

Finally, we introduce the notion of local preorderedness in a double category.

Definition 1.2.25 ([HN23, Definition 4.1.7])**.** Let $\mathbb{D}$ be a double category. We say that $\mathbb{D}$ is ***locally preordered*** if there exists at most one cell framed by every square consisting of two tight arrows and two loose arrows

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "f"'] & J \arrow[d, "g"] \\
K \arrow[r, "\beta"', "\shortmid"{marking}] & L
\end{tikzcd}$$

in $\mathbb{D}$. A cell in a locally preordered double category is depicted simply as a symbol $_{|\wedge}$. ⌟

This condition is called *flat* in [**GP99**].

Remark 1.2.26. An equipment $\mathbb{D}$ is locally preordered if and only if the loose bicategory $\mathbf{L}(\mathbb{D})$ is locally preordered. For a locally preordered equipment $\mathbb{D}$, we obtain an equivalent equipment $\mathbb{D}'$ with the loose bicategory $\mathbf{L}(\mathbb{D}')$ being a locally posetal bicategory. Therefore, loosewise local posetality is not a stable property under equivalence of equipments.

It should be noted that *locally posetal* double categories in [**HN23**] are defined by requiring that the tight 2-category to be locally posetal, which is a stronger condition than the local preorderedness. ⌟

1.3. Fibrational Virtual Double Categories

This section is devoted to the basic concepts of fibrational virtual double categories. Virtual double categories were first introduced by Burroni in [**Bur71**] under the name of *multicatégories* as an example of T-categories. Since then, this concept has turned up in several papers under different names, such as **fc**-*multicategories* in [**Lei04**], or *lax double categories* in [**DPP06**]. The most common name "virtual double categories" was introduced by Cruttwell, Shulman ([**CS10**]).

Definition 1.3.1 ([**CS10**, Definition 2.1])**.** A ***virtual double category*** (VDC) $\mathbb{X}$ is a structure consisting of the following data.

- A category $\mathbb{X}_{\mathrm{t}}$. Its objects are simply called ***objects***, and its arrows are called ***tight arrows***, which are depicted vertically in this paper.
- A class of ***loose arrows*** $\mathbb{X}(I, J)_0$ for each pair of objects $I, J \in \mathbb{X}_{\mathrm{t}}$. These arrows are depicted horizontally with slashes as $\alpha\colon I \to\!\!\!\!\!\shortmid\;\, J$.
- A class of ***(virtual) cells***

(1.3.1)
$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "s"'] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, "t"] \\
J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] \arrow[rrr, phantom, "\mu", shift left=4ex] & & & J_1
\end{tikzcd}$$

  for each dataset consisting of $n \geq 0$, objects $I_0, \ldots, I_n, J_0, J_1 \in \mathbb{X}_{\mathrm{t}}$, tight arrows $s\colon I_0 \to J_0$ and $t\colon I_n \to J_1$, and loose arrows $\alpha_1, \ldots, \alpha_n, \beta$. To specify the number n of loose arrows, we call the cell an ***n-ary cell***. We will write the finite sequence of loose arrows as $\overline{\alpha} = \alpha_1; \ldots; \alpha_n$. When s and t are identities, we call the cell a ***globular cell*** and let $\mu\colon \overline{\alpha} \Rightarrow \beta$ denote the cell. The class of globular cells $\overline{\alpha} \Rightarrow \beta$ would be denoted by $\mathbb{X}(\overline{I})(\overline{\alpha}, \beta)$ in which $\overline{I} = I_0; \ldots; I_n$.
- A composition operation on cells that assigns to each dataset of cells

$$\begin{tikzcd}
I_{1,0} \arrow[r, dashed, "\overline{\alpha}_1", "\shortmid"{marking}] \arrow[d, "s_0"'] \arrow[dr, phantom, "\mu_1"] & I_{1,m_1} \arrow[r, dashed, "\overline{\alpha}_2", "\shortmid"{marking}] \arrow[d, "s_1"'] \arrow[dr, phantom, "\mu_2"] & I_{2,m_2} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "s_2"'] & \cdots \arrow[r, dashed, "\overline{\alpha}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\mu_n"] & I_{n,m_n} \arrow[d, "s_n"] \\
J_0 \arrow[r, "\beta_1"', "\shortmid"{marking}] \arrow[d, "t_0"'] & J_1 \arrow[r, "\beta_2"', "\shortmid"{marking}] & J_2 \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & \cdots \arrow[r, "\beta_n"', "\shortmid"{marking}] & J_n \arrow[d, "t_1"] \\
K_0 \arrow[rrrr, "\gamma"', "\shortmid"{marking}] & & {} & & K_1
\end{tikzcd}$$

  a cell

$$\begin{tikzcd}
I_{1,0} \arrow[r, dashed, "\overline{\alpha}_1", "\shortmid"{marking}] \arrow[d, "s_0"'] & I_{1,m_1} \arrow[r, dashed, "\overline{\alpha}_2", "\shortmid"{marking}] & I_{2,m_2} \arrow[r, dashed, "\shortmid"{marking}] & \cdots \arrow[r, dashed, "\overline{\alpha}_n", "\shortmid"{marking}] & I_{n,m_n} \arrow[d, "s_n"] \\
J_0 \arrow[d, "t_0"'] & & \nu\{\mu_1 \fatsemi \cdots \fatsemi \mu_n\} & & J_n \arrow[d, "t_1"] \\
K_0 \arrow[rrrr, "\gamma"', "\shortmid"{marking}] & & & & K_1
\end{tikzcd},$$

  where the dashed line represents finite sequences of loose arrows for which associativity axioms hold. We will write the finite sequence of cells as $\overline{\mu} = \mu_1; \ldots; \mu_n$.
- An identity cell for each loose arrow $\alpha\colon I \to\!\!\!\!\!\shortmid\;\, J$

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "\mathtt{id}_I"'] \arrow[dr, phantom, "\mathtt{id}_\alpha"] & J \arrow[d, "\mathtt{id}_J"] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] & J
\end{tikzcd},$$

  for which the identity axioms hold. (Henceforth, we will just write $=$ for the identity tight arrows.)

⌟

We say two object I, J in a virtual double category are isomorphic if they are isomorphic in the underlying tight category $\mathbb{X}_{\mathrm{t}}$, and write $I \cong J$. For any objects I, J in a virtual double category, we write $\mathbb{X}(I, J)$ for the category whose objects are loose arrows $\alpha\colon I \to J$ and whose arrows are cells $\mu\colon \alpha \Rightarrow \beta$. A cell is called an ***(tightwise) isomorphism cell*** if it is invertible in this category. More generally, we say two loose arrows α, β are isomorphic if there exist two cells

$$\begin{tikzcd} I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, phantom, "\mu"] & J \arrow[d, "t"] \\ K \arrow[r, "\beta"', "\shortmid"{marking}] & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, "s'"'] \arrow[dr, phantom, "\nu"] & L \arrow[d, "t'"] \\ I \arrow[r, "\alpha"', "\shortmid"{marking}] & J \end{tikzcd}$$

such that $\mu\{\nu\} = \mathtt{id}_\beta$ and $\nu\{\mu\} = \mathtt{id}_\alpha$, and call the cells μ and ν ***isomorphism cells***. It is always the case that $I \cong K$ and $J \cong L$ through the tight arrows s, t, s', t'.

Remark 1.3.2. As already mentioned, we will often use dashed horizontal arrows to represent sequences of loose arrows. Correspondingly, we will use the expression on the left below to represent a sequence of the identity cells on the right below:

$$\begin{tikzcd} I_0 \arrow[r, dashed, "\overline{\alpha}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & I_n \arrow[d, equal] \\ I_0 \arrow[r, dashed, "\overline{\alpha}"', "\shortmid"{marking}] & I_n \end{tikzcd} \quad := \quad \begin{tikzcd} I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "\mathtt{id}_{I_0}"'] \arrow[dr, phantom, "\mathtt{id}_{\alpha_1}"] & I_1 \arrow[r, "\shortmid"{marking}] \arrow[d, "\mathtt{id}_{I_1}"] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] \arrow[dr, phantom, "\mathtt{id}_{\alpha_n}"] & I_n \arrow[d, "\mathtt{id}_{I_n}"] \\ I_0 \arrow[r, "\alpha_1"', "\shortmid"{marking}] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n"', "\shortmid"{marking}] & I_n \end{tikzcd}$$

We also note that a cell whose top sequence of loose arrows is the empty sequence is depicted as a triangle:

$$\begin{tikzcd} & I \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[d, phantom, "\mu"] & \\ J_0 \arrow[rr, "\beta"', "\shortmid"{marking}] & {} & J_1 \end{tikzcd} .$$

⌟

Example 1.3.3. A double category can be seen as a virtual double category in the following way. A cell (1.3.1) is defined as a cell

$$\begin{tikzcd} I_0 \arrow[r, "\alpha_1 \odot \cdots \odot \alpha_n", "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, phantom, "\mu"] & I_n \arrow[d, "t"] \\ J_0 \arrow[r, "\beta"', "\shortmid"{marking}] & J_1 \end{tikzcd}$$

where $\odot$ is the horizontal composition of loose arrows in the double category. The composition of cells is given by first composing cells horizontally on each row and then composing vertically. ⌟

Definition 1.3.4 ([**CS10**, Definition 3.1])**.** A ***virtual double functor*** $F\colon \mathbb{X} \to \mathbb{Y}$ between virtual double categories $\mathbb{X}$ and $\mathbb{Y}$ consists of the following data and conditions:

- A functor $F_{\mathrm{t}}\colon \mathbb{X}_{\mathrm{t}} \to \mathbb{Y}_{\mathrm{t}}$.
- A family of functions $F_1\colon \mathbb{X}(I, J)_0 \to \mathbb{Y}(F_{\mathrm{t}}(I), F_{\mathrm{t}}(J))_0$ for each pair of objects I, J of $\mathbb{X}$.
- A family of functions sending each cell μ of $\mathbb{X}$ on the left below to a cell $F_1(\mu)$ of $\mathbb{Y}$ on the right below:

(1.3.2)

$$\begin{tikzcd} I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "s_0"'] & I_1 \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "\mu"] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, "s_1"] \\ J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] & & & J_1 \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F_{\mathrm{t}}(I_0) \arrow[r, "F_1(\alpha_1)", "\shortmid"{marking}] \arrow[d, "F_{\mathrm{t}}(s_0)"'] & F_{\mathrm{t}}(I_1) \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "F_1(\mu)"] & \cdots \arrow[r, "F_1(\alpha_n)", "\shortmid"{marking}] & F_{\mathrm{t}}(I_n) \arrow[d, "F_{\mathrm{t}}(s_1)"] \\ F_{\mathrm{t}}(J_0) \arrow[rrr, "F_1(\beta)"', "\shortmid"{marking}] & & & F_{\mathrm{t}}(J_1) \end{tikzcd} .$$

- The identity cells are preserved.
- Composition of cells is preserved.

As usual, we will often omit the subscripts of the functor and functions F_{t} and F_1.

A ***tightwise transformation*** $\theta\colon F \to G$ between virtual double functors $F, G\colon \mathbb{X} \to \mathbb{Y}$ consists of the following data and conditions:

- A natural transformation $\theta_0\colon F_{\mathrm{t}} \to G_{\mathrm{t}}$.

- A cell $\theta_{1,\alpha}$ for each loose arrow $\alpha\colon I \to\mkern-14mu\mapsto J$ of $\mathbb{X}$:

$$\begin{tikzcd}
FI \arrow[rr, "F\alpha", "\shortmid"{marking}] \arrow[d, "\theta_{0,I}"'] & & FJ \arrow[d, "\theta_{0,J}"] \\
{} & \theta_{1,\alpha} & {} \\[-2em]
GI \arrow[rr, "G\alpha"', "\shortmid"{marking}] & & GJ
\end{tikzcd}$$

- The naturality condition for cells:

$$\begin{tikzcd}
FI_0 \arrow[rr, dashed, "F\overline{\alpha}", "\shortmid"{marking}] \arrow[d, "Fs_0"'] & F\mu & FI_n \arrow[d, "Fs_n"] & & FI_0 \arrow[rr, dashed, "F\overline{\alpha}", "\shortmid"{marking}] \arrow[d, "\theta_{I_0}"'] & \theta_{\overline{\alpha}} & FI_n \arrow[d, "\theta_{I_n}"] \\
FJ_0 \arrow[rr, "F\beta"', "\shortmid"{marking}] \arrow[d, "\theta_{J_0}"'] & \theta_\beta & FJ_1 \arrow[d, "\theta_{J_1}"] & = & GI_0 \arrow[rr, dashed, "G\overline{\alpha}"', "\shortmid"{marking}] \arrow[d, "Gs_0"'] & G\mu & GI_n \arrow[d, "Gs_n"] \\
GJ_0 \arrow[rr, "G\beta"', "\shortmid"{marking}] & & GJ_1 & & GJ_0 \arrow[rr, "G\beta"', "\shortmid"{marking}] & & GJ_1
\end{tikzcd} .$$

VDbl is the 2-category of virtual double categories, virtual double functors, and tightwise transformations. ⌟

Definition 1.3.5 ([**CS10**, Definition 7.1])**.** Let $\mathbb{X}$ be a virtual double category. A ***restriction*** of a loose arrow $\alpha\colon I \to\mkern-14mu\mapsto J$ along a pair of tight arrows $s\colon I' \to I$ and $t\colon J' \to J$ is the loose arrow $\alpha[s \fatsemi t]\colon I' \to\mkern-14mu\mapsto J'$ equipped with a cell

$$\begin{tikzcd}
I' \arrow[rr, "{\alpha[s \fatsemi t]}", "\shortmid"{marking}] \arrow[d, "s"'] & \texttt{rest} & J' \arrow[d, "t"] \\
I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J
\end{tikzcd}$$

with the following universal property: any cell μ of the form on the left below factors uniquely through the cell `rest` as on the right below.

(1.3.3)
$$\begin{tikzcd}
K \arrow[rr, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "u"'] & & L \arrow[d, "v"] & & K \arrow[rr, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "u"'] & \widehat{\mu} & L \arrow[d, "v"] \\
I' \arrow[d, "s"'] & \mu & J' \arrow[d, "t"] & = & I' \arrow[rr, "{\alpha[s \fatsemi t]}"', "\shortmid"{marking}] \arrow[d, "s"'] & \texttt{rest} & J' \arrow[d, "t"] \\
I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J & & I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J
\end{tikzcd}$$

In this case, we call the cell `rest` a ***restricting cell***. If the restrictions exist for all triples (α, s, t), then we say that $\mathbb{X}$ is a ***fibrational virtual double category (*FVDC*)***[5]

A ***fibrational virtual double functor*** $F\colon \mathbb{X} \to \mathbb{Y}$ between fibrational virtual double categories $\mathbb{X}$ and $\mathbb{Y}$ is a virtual double functor that preserves restrictions. **FVDbl** is the 2-category of fibrational virtual double categories, fibrational virtual double functors, and tightwise transformations. ⌟

Example 1.3.6. An equipment is fibrational as a virtual double category. The converse also holds, as we will see in Remark 2.3.20. ⌟

Definition 1.3.7. Similarly to Definition 1.2.25, we define a ***local preordered virtual double category*** as one in which there exists at most one cell for each frame. ⌟

Our focus is on fibrational virtual double categories since most of the examples of virtual double categories that we are interested in are fibrational.

Lemma 1.3.8. A virtual double functor $F\colon \mathbb{X} \to \mathbb{Y}$ is an equivalence in **VDbl** if and only if

(i) the functor $F_{\mathrm{t}}\colon \mathbb{X}_{\mathrm{t}} \to \mathbb{Y}_{\mathrm{t}}$ for F is an equivalence of categories,

(ii) for any loose arrow $\alpha\colon I \to\mkern-14mu\mapsto J$ in $\mathbb{Y}$, there exists a loose arrow $\beta\colon I' \to\mkern-14mu\mapsto J'$ in $\mathbb{X}$ and an isomorphism cell μ as below:

$$\begin{tikzcd}
FI' \arrow[rr, "F\beta", "\shortmid"{marking}] \arrow[d, "\wr"'] & \mu\ \wr & FJ' \arrow[d, "\wr"] \\
I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J
\end{tikzcd} , \quad \text{and}$$

(iii) for any quadruple $(s, t, \overline{\alpha}, \beta)$, the function F on the cells (1.3.2) is a bijection.

---

[5] The term "fibrational" is not standard in the literature. If we follow the terminology of [**Ale18**], we should call it a *fibrant virtual double category*, but we prefer to use the term because it has nothing to do with any model structure, at least *a priori*.

A fibrational virtual double functor $F\colon \mathbb{X} \to \mathbb{Y}$ is an equivalence in **FVDbl** if and only if (i), (ii), and the special case of (iii) where s and t are identities are satisfied. ⌟

Proof. If we are given an inverse G of F, then G_{t} is the inverse of F_{t}, and the isomorphism $FG \Rightarrow \mathrm{Id}$ gives the isomorphism cells μ above. The inverse of functions F in (1.3.2) is given by sending a cell ν on the right to $G_1(\nu)$ and composing with the isomorphism cells obtained from the isomorphism $GF \Rightarrow \mathrm{Id}$.

Conversely, given the conditions, we can construct an inverse G of F. The tight part of G is given by an inverse of F_{t}. Then, for each loose arrow $\alpha\colon I \mathrel{\mid}\joinrel\rightarrow J$ in $\mathbb{Y}$, we can show that a loose arrow $\beta\colon GI \mathrel{\mid}\joinrel\rightarrow GJ$ in $\mathbb{X}$ is isomorphic to α by the second condition. The bijection in (iii) determines how to send a cell in $\mathbb{Y}$ to a cell in $\mathbb{X}$. The functoriality of G follows from the one-to-one correspondence between cells in $\mathbb{X}$ and $\mathbb{Y}$ in (iii).

To show the last statement, we need to show that the general case of (iii) follows from its special case where s and t are identities under the fibrational condition, which is straightforward by the universal property of the restrictions. It follows that the inverse is fibrational from the fact that any equivalence preserves restrictions. □

Next, we explicitly describe the notion of cartesian fibrational virtual double category (CFVDC), although it is already defined because we have the 2-category of fibrational virtual double categories **FibVDbl**, which has strict finite products.

Proposition 1.3.9. An FVDC $\mathbb{X}$ is cartesian if and only if the following conditions are satisfied:

(i) $\mathbb{X}_{\mathrm{t}}$ has finite products;

(ii) $\mathbb{X}$ locally has finite products, that is, for each $I, J \in \mathbb{X}_{\mathrm{t}}$,

  (a) for any loose arrows $\alpha, \beta\colon I \mathrel{\mid}\joinrel\rightarrow J$ in $\mathbb{X}$, there exists a loose arrow $\alpha \wedge \beta\colon I \mathrel{\mid}\joinrel\rightarrow J$ and globular cells $\pi_0\colon \alpha \wedge \beta \Rightarrow \alpha$, $\pi_1\colon \alpha \wedge \beta \Rightarrow \beta$ such that for any finite sequence of loose arrows $\overline{\gamma}$ where $\gamma_i\colon I_{i-1} \mathrel{\mid}\joinrel\rightarrow I_i$ for $1 \le i \le n$ where $I_0 = I$ and $I_n = J$, the function

$$\mathbb{X}(\overline{I})(\overline{\gamma}, \alpha \wedge \beta) \to \mathbb{X}(\overline{I})(\overline{\gamma}, \alpha) \times \mathbb{X}(\overline{I})(\overline{\gamma}, \beta) \quad ; \quad \mu \mapsto (\pi_0 \circ \mu, \pi_1 \circ \mu)$$

  is a bijection, and

  (b) there exists a loose arrow $\top\colon I \mathrel{\mid}\joinrel\rightarrow J$ such that $\mathbb{X}(\overline{I})(\overline{\gamma}, \top)_0$ is a singleton for any finite sequence of loose arrows $\overline{\gamma}$;

(iii) the local finite products are preserved by restrictions.

A morphism between cartesian FVDCs is a cartesian morphism if and only if the underlying tight functor preserves finite products and the morphism preserves local finite products. ⌟

Proof sketch. The proof is similar to that of [**Ale18**, Prop 4.12]. First, suppose that $\mathbb{X}$ is cartesian. Let $\Delta_I\colon I \to I \times I$ be the diagonal of I and $!_I\colon I \to 1$ be the unique arrow to the terminal object. If $\mathbb{X}$ is cartesian, then $\alpha \wedge \beta$ and $\top$ in $\mathbb{X}(I, J)$ are given by $(\alpha \times \beta)[\Delta_I \mathbin{⨟} \Delta_J]$ and $\delta_1(!_I, !_J)$, which brings the finite products in $\mathbb{X}(I, J)$. The local finite products are preserved by restrictions since, by the universal property of the restrictions, we have

$$(\alpha \times \beta)[\Delta_I \mathbin{⨟} \Delta_J][s \mathbin{⨟} t] \cong (\alpha \times \beta)[(s \times s) \mathbin{⨟} (t \times t)][\Delta_{I'} \mathbin{⨟} \Delta_{J'}] \cong (\alpha[s \mathbin{⨟} t] \times \beta[s \mathbin{⨟} t])[\Delta_{I'} \mathbin{⨟} \Delta_{J'}],$$

and similarly for $\top$. Conversely, if $\mathbb{X}$ locally has finite products, then assigning

$$\alpha \times \beta \coloneqq \alpha[\pi_I \mathbin{⨟} \pi_J] \wedge \beta[\pi_K \mathbin{⨟} \pi_L]\colon I \times K \mathrel{\mid}\joinrel\rightarrow J \times L$$

to each pair $\alpha\colon I \mathrel{\mid}\joinrel\rightarrow J$, $\beta\colon K \mathrel{\mid}\joinrel\rightarrow L$ and a cell $\mu \times \nu$ naturally obtained from the universal property of the restrictions induces the functor $\times\colon \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ right adjoint to the diagonal functor, and the functor $1\colon \mathbb{1} \to \mathbb{X}$ obtained by the terminal object in $\mathbb{X}_{\mathrm{t}}$ is the right adjoint of !. The second statement follows from the construction of the equivalence above. □

Remark 1.3.10. The third condition in Prop 1.3.9 is necessary for FVDC but not for equipments as in [**Ale18**] since the latter has oprestrictions of loose arrows. ⌟

Example 1.3.11. We give several examples from the context of predicate logic.

(i) In Example 1.2.3, we defined a double category $\mathbb{R}\mathrm{el}(\mathcal{S}\mathrm{et})$ of sets, functions, and relations, and mentioned that it is generalizable to $\mathbb{R}\mathrm{el}(\mathcal{B})$ for a regular category $\mathcal{B}$. We can drop the regularity condition and define a virtual double category $\mathbb{R}\mathrm{el}(\mathcal{B})$ of objects, arrows, and internal relations, meaning subobjects of the product of two objects in a category with finite limits $\mathcal{B}$. This is possible since without the regularity condition, since we can interpret Horn sentences internally in $\mathcal{B}$ when it has finite limits. More concretely, it is a local preordered virtual double category where a cell exists precisely when the corresponding Horn sentence is valid. For instance, nullary and binary cells are respectively defined as follows:

$$\begin{tikzcd}
 & I \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & \\
J \arrow[rr, "\alpha"', "\shortmid"{marking}] & {} & K
\end{tikzcd}
\iff
\begin{tikzcd}
I \arrow[r, dashed, "\exists"] \arrow[d, "{\langle 0,0\rangle}"'] \arrow[dr, phantom, "\circlearrowleft"] & \alpha \arrow[d, hook] \\
I \times I \arrow[r, "s\times t"'] & J \times K
\end{tikzcd},$$

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "s"'] & J \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & K \arrow[d, "t"] \\
L \arrow[rr, "\gamma"', "\shortmid"{marking}] & & M
\end{tikzcd}
\iff
\begin{tikzcd}
(\alpha \times K) \cap (I \times \beta) \arrow[r, dashed, "\exists"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \gamma \arrow[d, hook] \\
I \times J \times K \arrow[r, "{(s\times t)\circ\langle 0,2\rangle}"'] & L \times M
\end{tikzcd}$$

A restriction of a relation along a pair of functions is given by the pullback of the relation along the product of the functions. This double category is a CFVDC, and we will see that this is an instance of what we study in Chapter 2.

(ii) For a monoidal category $\mathcal{V}$, we can define a fibrational virtual double category $\mathcal{V}$-$\mathbb{M}$at as follows. Its tight category is $\mathcal{S}$et, and the loose arrows $I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ are matrices $(A_{i,j})_{i\in I, j\in J}$ of objects in $\mathcal{V}$. A cell of the form on the left below, for instance, is a family of morphisms in $\mathcal{V}$ on the right below:

$$\begin{tikzcd}
I \arrow[r, "{(A_{i,j})_{i,j}}", "\shortmid"{marking}] \arrow[d, "s"'] & J \arrow[r, "{\left(B_{j,k}\right)_{j,k}}", "\shortmid"{marking}] \arrow[d, phantom, "\mu"] & K \arrow[d, "t"] \\
L \arrow[rr, "{\left(C_{l,m}\right)_{l,m}}"', "\shortmid"{marking}] & & M
\end{tikzcd}
\quad\Bigg\|\quad
\left(\mu_{i,j,k} \colon A_{i,j} \otimes B_{j,k} \to C_{s(i),t(k)}\right)_{i,j,k}$$

Defining general cells and composition of cells involves the monoidal structure of $\mathcal{V}$. A restriction of a matrix along a pair of functions $s \colon I' \to I$ and $t \colon J' \to J$ is given by the matrix $\left(A_{s(i),t(j)}\right)_{i\in I', j\in J'}$. It is a CFVDC if $\mathcal{V}$ is cartesian monoidal.

⌟

Example 1.3.12. One of the motivations for the type theory in Chapter 3 is to formalize category theory in formal language. The following examples of virtual double categories will provide a multitude of category theories that can be formalized in our type theory.

(i) The double category $\mathbb{P}$rof in Example 1.2.3 is a CFVDC. When we consider not necessarily small categories, however, we do not have a composition of profunctors in general. Nevertheless, we can still define a virtual double category $\mathbb{P}$ROF of categories, functors, and profunctors. This is possible because even without colimits, we can define virtual cells with extranatural transformations. Namely, a cell on the left below is defined as a family of arrows (di)natural in $i_0, \dots, i_n$:

$$\begin{tikzcd}
\mathcal{I}_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathcal{I}_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] \arrow[d, phantom, "\mu"] & \mathcal{I}_n \arrow[d, "G"] \\
\mathcal{J}_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] & & & \mathcal{J}_1
\end{tikzcd}
\quad\Bigg\|\quad
\left(\mu_{i_0,\dots,i_n} \colon\ \alpha_1(i_0, i_1) \times \cdots \times \alpha_n(i_{n-1}, i_n) \to \beta(F(i_0), G(i_n))_{i_0,\dots,i_n}\right)$$

It is a CFVDC.

(ii) Similarly, we can define the FVDCs $\mathcal{V}$-$\mathbb{P}$rof and $\mathcal{V}$-$\mathbb{P}$ROF of $\mathcal{V}$-enriched categories, functors, and profunctors, without any assumption on the monoidal category $\mathcal{V}$. They are CFVDCs if $\mathcal{V}$ is cartesian monoidal.

(iii) We can also define virtual double categories $\mathbb{P}\text{rof}(\mathcal{S})$ of internal categories, functors, and profunctors in categories $\mathcal{S}$ with finite limits. This is a CFVDC.

⌟

For later use, we define restrictions of cells along a sequence of tight arrows.

Definition 1.3.13. Let $\mathbb{X}$ be an FVDC. Given a globular cell μ as in (1.3.1) with s and t identities and a sequence of tight arrows $f_i \colon K_i \to I_i$ for $0 \leq i \leq n$, we define the ***restriction*** of μ along the sequence $\overline{f} = f_0 \fatsemi \dots \fatsemi f_n$ as the globular cell $\mu[\overline{f}]$ in the diagram below defined as the unique cell that makes the following equation hold.

$$\begin{tikzcd}
K_0 \arrow[r, "{\alpha_1[f_0 \fatsemi f_1]}", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\mathtt{rest}"] & K_1 \arrow[d, "f_1"'] & \cdots & K_{n-1} \arrow[r, "{\alpha_n[f_{n-1} \fatsemi f_n]}", "\shortmid"{marking}] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\mathtt{rest}"] & K_n \arrow[d, "f_n"] \\
I_0 \arrow[r, "\alpha_1"', "\shortmid"{marking}] \arrow[d, equal] & I_1 & \cdots \arrow[d, phantom, "\mu"] & I_{n-1} \arrow[r, "\alpha_n"', "\shortmid"{marking}] & I_n \arrow[d, equal] \\
I_0 \arrow[rrrr, "\beta"', "\shortmid"{marking}] & & & & I_n
\end{tikzcd}
=
\begin{tikzcd}
K_0 \arrow[r, "{\alpha_1[f_0 \fatsemi f_1]}", "\shortmid"{marking}] \arrow[d, equal] & K_1 & \cdots \arrow[d, phantom, "{\mu[\overline{f}]}"] & K_{n-1} \arrow[r, "{\alpha_n[f_{n-1} \fatsemi f_n]}", "\shortmid"{marking}] & K_n \arrow[d, equal] \\
K_0 \arrow[rrrr, "{\beta[f_0 \fatsemi f_n]}"', "\shortmid"{marking}] \arrow[d, "f_0"'] & & \arrow[d, phantom, "\mathtt{rest}"] & & K_n \arrow[d, "f_n"] \\
I_0 \arrow[rrrr, "\beta"', "\shortmid"{marking}] & & & & I_n
\end{tikzcd}$$

⌟

1.4. Composition in Virtual Double Categories

In a virtual double category, composition of loose arrows is no longer a built-in operation, but rather a structure on a virtual double category defined by a universal property. In this chapter, we outline the definition of composition in a virtual double category and summarize basic results mostly from [**DPP06**, **CS10**].

Definition 1.4.1 ([**DPP06**, Definition 2.7],[**CS10**, Definition 5.2])**.** A ***composite*** of a given sequence of loose arrows $\overline{\alpha} = \left(I_0 \overset{\alpha_1}{\nrightarrow} I_1 \nrightarrow \cdots \overset{\alpha_m}{\nrightarrow} I_m\right)$ in a virtual double category is a loose arrow $\odot\overline{\alpha}$ from I_0 to I_m equipped with a cell

$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, equal] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_m", "\shortmid"{marking}] \arrow[d, phantom, "\varkappa_{\overline{\alpha}}"] & I_m \arrow[d, equal] \\
I_0 \arrow[rrr, "\odot\overline{\alpha}"', "\shortmid"{marking}] & & & I_m
\end{tikzcd}$$

with the following universal property: given any cell ν on the left below where $\overline{\beta}, \overline{\beta}'$ are arbitrary sequences of loose arrows, it uniquely factors through the sequence of the identity cells with $\mu_{\overline{\alpha}}$ as on the right below.

(1.4.1)

$$\begin{tikzcd}
J_0 \arrow[r, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "f"'] & I_0 \arrow[r, dashed, "\overline{\alpha}", "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & I_m \arrow[r, dashed, "\overline{\beta}'", "\shortmid"{marking}] & J'_{n'} \arrow[d, "f'"] \\
K \arrow[rrr, "\gamma"', "\shortmid"{marking}] & & & K'
\end{tikzcd}
=
\begin{tikzcd}
J_0 \arrow[r, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & I_0 \arrow[r, dashed, "\overline{\alpha}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\varkappa_{\overline{\alpha}}"] & I_m \arrow[r, dashed, "\overline{\beta}'", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\parallel"] & J'_{n'} \arrow[d, equal] \\
J_0 \arrow[r, dashed, "\overline{\beta}"', "\shortmid"{marking}] \arrow[d, "f"'] & I_0 \arrow[r, "\odot\overline{\alpha}"', "\shortmid"{marking}] \arrow[dr, phantom, "\widetilde{\nu}"] & I_m \arrow[r, dashed, "\overline{\beta}'"', "\shortmid"{marking}] & J'_{n'} \arrow[d, "f'"] \\
K \arrow[rrr, "\gamma"', "\shortmid"{marking}] & & & K'
\end{tikzcd}$$

We call the cell $\varkappa_{\overline{\alpha}}$ the ***composing cell*** of $\overline{\alpha}$. In particular, a composite of the empty sequence of loose arrows on I is called a ***unit*** on I and denoted by δ_I.

A virtual double functor is said to ***preserve the composite*** $\odot\overline{\alpha}$ if it sends the composing cell of $\overline{\alpha}$ to the cell that exhibits the image of $\odot\overline{\alpha}$ as the composite of the images of $\overline{\alpha}$. It is said to ***preserve composition*** if it preserves all composites. ⌟

In [**DPP06**], a composite of a sequence of loose arrows is defined as another virtual double category called the ***path double category***, and they say the composite is *strongly representable* if it comes with a loose arrow in the original virtual double category that satisfies the universal property of the composite in our definition. In [**CS10**], an adjective *opcartesian* for a cell is used to indicate what we call a composing cell. There is a weaker notion of composites which has the universal property only for the case where $\overline{\beta}$ and $\overline{\beta}'$ above are empty sequences. This is called *representable* composites in [**DPP06**], and the cells that satisfy this property are called *weakly opcartesian* in [**CS10**]. The weaker notion is not so useful in practice, because it does lead to the associativity of composition. See [**DPP06**, 2.9] and [**CS10**, Remark 5.8] for more details.

For the purpose of this thesis, we will separately discuss composability of sequences of loose arrows of positive length and those of length zero.

Definition 1.4.2. A virtual double category is called ***unital***[6] if it has composites of sequences of length zero, or equivalently, if it has units on all objects. In particular, a ***virtual equipment*** is a fibrational virtual double category that is also unital.

---

[6] For consistency, this should be called *zero-length composable*, but we respect the decent name *unital* in the literature.

A virtual double category is called ***positive-length composable*** (or ***PL-composable***) if it has composites of any sequence of loose arrows of positive length.

A virtual double category is called ***composable*** if it is both unital and positive-length composable.

We let $\mathbf{VDbl}_{\nrightarrow}$ (resp. $\mathbf{VDbl}_{\odot}$, $\mathbf{VDbl}_{\nrightarrow\odot}$) denote the locally full sub-2-categories of $\mathbf{VDbl}$ spanned by the unital (resp. positive-length composable, composable) virtual double categories and the virtual double functors preserving the composites assumed to exist. We also let $\mathbf{FVDbl}_{\nrightarrow}$ (resp. $\mathbf{FVDbl}_{\odot}$, $\mathbf{FVDbl}_{\nrightarrow\odot}$) denote the locally full sub-2-categories of $\mathbf{FVDbl}$ spanned by the unital (resp. positive-length composable, composable) virtual double categories and the virtual double functors preserving the composites. ⌟

The following proposition is a fundamental result in this context. The proof can be found in [**Her00**], which is mentioned in [**DPP06**].

Proposition 1.4.3. A composable virtual double category is presented as a double category seen as a virtual double category in the way described in Example 1.3.3. More precisely, the 2-category $\mathbf{VDbl}_{\nrightarrow\odot}$ of composable virtual double categories is biequivalent[7] to the 2-category $\mathbf{Dbl}$ of double categories. ⌟

A better proof should be given in higher generality using generalized multicategories. On the other hand, the explicit proof may provide us a better intuition when we go back and forth between the two perspectives.

Sketch of proof. Let $\mathbb{X}$ be a virtual double category with composites of any finite sequence of loose arrows. The data of a double category is almost ready in $\mathbb{X}$: the unit loose arrows and the composites of loose arrows are those defined in terms of VDCs, the cells of unary input give the cells of the double category, and the vertical composition of cells is already given.

The only thing left is to define the horizontal composition of cells and check the associativity and unitality of the loose composition. The composite of the two cells μ and ν in $\mathbb{X}$ on the left below is defined as the cell $\mu \odot \nu$ on the right below that satisfies the equation. Note that this equation uniquely determines $\mu \odot \nu$.

$$\begin{tikzcd}
I_0 \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "s_0"'] \arrow[dr, phantom, "\mu"] & I_1 \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, "s_1"'] \arrow[dr, phantom, "\nu"] & I_2 \arrow[d, "s_2"] \\
I'_0 \arrow[r, "\alpha'"', "\shortmid"{marking}] & I'_1 \arrow[r, "\beta'"', "\shortmid"{marking}] & I'_2
\end{tikzcd}$$

$$\begin{tikzcd}
& I_1 \arrow[dr, "\beta", "\shortmid"{marking}] \arrow[dd, "s_1"'] & \\
I_0 \arrow[ur, "\alpha", "\shortmid"{marking}] \arrow[dd, "s_0"'] \arrow[r, phantom, "\mu"] & & I_2 \arrow[dd, "s_2"] \arrow[l, phantom, "\nu"] \\
& I'_1 \arrow[dr, "\beta'", "\shortmid"{marking}] \arrow[d, phantom, "\varkappa_{\alpha';\beta'}"] & \\
I'_0 \arrow[ur, "\alpha'", "\shortmid"{marking}] \arrow[rr, bend right, "\alpha'\odot\beta'"', "\shortmid"{marking}] & & I'_2
\end{tikzcd}
=
\begin{tikzcd}
& I_1 \arrow[dr, "\beta", "\shortmid"{marking}] \arrow[d, phantom, "\varkappa_{\alpha;\beta}"] & \\
I_0 \arrow[ur, "\alpha", "\shortmid"{marking}] \arrow[rr, bend right, "\alpha\odot\beta"', "\shortmid"{marking}] \arrow[d, "s_0"'] & \mu\odot\nu & I_2 \arrow[d, "s_2"] \\
I'_0 \arrow[rr, bend right, "\alpha'\odot\beta'"', "\shortmid"{marking}] & & I'_2
\end{tikzcd}$$

We present an instance of the isomorphism cells in $\mathbb{X}$ that witness the unitality of the loose composition.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\parallel"] & J \arrow[d, equals] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] & J
\end{tikzcd}
=
\begin{tikzcd}
& I \arrow[dr, "\alpha", "\shortmid"{marking}] \arrow[dd, equals] & \\
\eta_I & \parallel & J \arrow[dd, equals] \\
& I \arrow[dr, "\alpha", "\shortmid"{marking}] & \\
I \arrow[uuur, equals, bend left] \arrow[ur, "\delta_I", "\shortmid"{marking}] \arrow[rr, bend right, "\delta_I\odot\alpha"', "\shortmid"{marking}] \arrow[r, phantom, "\varkappa_{\delta_I;\alpha}"] \arrow[d, equals] & \texttt{unit}\ \wr\| & J \arrow[d, equals] \\
I \arrow[rr, bend right, "\alpha"', "\shortmid"{marking}] & & J
\end{tikzcd}$$

The cell `unit` is uniquely determined by the iterated use of the universal properties of the composites, and its inverse is given by the upper part of the diagram on the right above.

There is a canonical 2-functor from $\mathbf{Dbl}$ to $\mathbf{VDbl}_{\nrightarrow\odot}$ that assigns to each double category the same thing seen as a virtual double category. It is indeed composable by the loose composition in the double category. What we have proved is that this 2-functor is surjective up to equivalence. It is also a local equivalence. □

---

[7] It is not a 2-equivalence because composability in virtual double categories does not choose composition of loose arrows while the composition in double categories is a built-in structure.

Remark 1.4.4. It was shown in [**CS10**, Theorem 7.24] that a unital virtual double functor between virtual equipments automatically preserves restrictions. Therefore, the 2-category $\mathbf{FVDbl}_{\nrightarrow}$ is a full sub-2-category of $\mathbf{VDbl}_{\nrightarrow}$, and the 2-category $\mathbf{FVDbl}_{\nrightarrow\odot}$ is a full sub-2-category of $\mathbf{VDbl}_{\nrightarrow\odot}$.

Since restrictions in composable virtual double categories are the same thing as in the corresponding double categories, we obtain a biequivalence between the 2-categories $\mathbf{FVDbl}_{\nrightarrow\odot}$ and the 2-category **Eqp** of equipments from the biequivalence between $\mathbf{VDbl}_{\nrightarrow\odot}$ and **Dbl**. $\lrcorner$

Remark 1.4.5. Analogously to Remark 1.2.10, we can check the composability condition for a cell by a simpler condition owing to restrictions. In an FVDC $\mathbb{X}$, a globular cell $\varkappa$ is a composing cell if and only if it has the universal property that for any cells ν in (1.4.1) with f and f' being identities, there is a unique cell $\widetilde{\nu}$ that makes the equation hold. $\lrcorner$

Once again, the 2-categories $(\mathbf{F})\mathbf{VDbl}_{\nrightarrow}$, $(\mathbf{F})\mathbf{VDbl}_{\odot}$ have strict finite products, and we can discuss cartesianness in these 2-categories in the sense of Definition 1.1.1. We now unravel the cartesianness of virtual double categories with those structures in the following.

Proposition 1.4.6. An FVDC $\mathbb{X}$ with units is cartesian in $\mathbf{FVDbl}_{\nrightarrow}$ if and only if

(i) $\mathbb{X}$ is a cartesian FVDC,
(ii) $\delta_1 \cong \top_{1,1}$ in $\mathbb{X}(1,1)$ canonically, and
(iii) for any $I, J \in \mathbb{X}$, $\delta_{I,J} \cong \delta_I \times \delta_J$ canonically in $\mathbb{X}(I \times J, I \times J)$.

$\lrcorner$

Proof. By Lem 1.1.5, $\mathbb{X}$ is cartesian as a unital FVDC if and only if it is cartesian as an FVDC and the 1-cells $1\colon \mathbb{1} \to \mathbb{X}$ and $\times\colon \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are in $\mathbf{FVDbl}_{\nrightarrow}$. The first condition is equivalent to *(ii)* since it sends the only loose arrow in $\mathbb{1}$, which is the unit loose arrow, to $\top_{1,1}$. The second condition is equivalent to *(iii)* since the unit loose arrow of (I, J) in $\mathbb{X}(I \times J, I \times J)$ is (δ_I, δ_J), which is sent to $\delta_I \times \delta_J$ in $\mathbb{X}(I \times J, I \times J)$. $\square$

The key idea is that in the virtual double categories $\mathbb{X} \times \mathbb{X}$ and $\mathbb{1}$, the unit loose arrows are given pointwise by the unit loose arrows of $\mathbb{X}$. We can discuss the cartesianness of some classes of FVDCs in parallel with the above proposition.

Proposition 1.4.7. Let $\mathbf{FVDbl}_{\odot}$ be the locally-full sub-2-category of **FVDbl** spanned by the FVDCs with composites of sequences of loose arrows of positive length and functors preserving those composites. A VDC $\mathbb{X}$ in $\mathbf{FVDbl}_{\odot}$ is cartesian in this 2-category if and only if

(i) $\mathbb{X}$ is a cartesian FVDC,
(ii) $\top_{1,1} \odot \top_{1,1} \cong \top_{1,1}$ canonically in $\mathbb{X}(1,1)$, and
(iii) for any paths of positive length

$$I_0 \xrightarrow{\alpha_1} I_1 \xrightarrow{} \dots \xrightarrow{\alpha_n} I_n \quad \text{and} \quad J_0 \xrightarrow{\beta_1} J_1 \xrightarrow{} \dots \xrightarrow{\beta_n} J_n$$

in $\mathbb{X}$, we have

$$(\alpha_1 \odot \cdots \odot \alpha_n) \times (\beta_1 \odot \cdots \odot \beta_n) \cong (\alpha_1 \times \beta_1) \odot \cdots \odot (\alpha_n \times \beta_n)$$

canonically in $\mathbb{X}(I_0 \times J_0, I_n \times J_n)$.

$\lrcorner$

Example 1.4.8. Let us check whether the virtual double categories we have seen in the previous section are composable. We start with Example 1.3.11.

(i) $\mathbb{R}\mathrm{el}(\mathcal{B})$ is unital, as its unit on I is given as the equality relation on I, namely, $\langle 0, 0\rangle\colon I \to I \times I$. It is positive-length composable if and only if $\mathcal{B}$ is regular. This follows from our main result Theorem 2.3.14.
(ii) $\mathcal{V}$-Mat is unital if $\mathcal{V}$ has an initial object 0 that is preserved by the tensor product. In this case, the unit on I is given by $(\delta_{i,j})_{i,j\in I}$ where $\delta_{i,j}$ is the monoidal unit of $\mathcal{V}$ when $i = j$ and the initial object otherwise. It is positive-length composable if and only if $\mathcal{V}$ has small coproducts that are preserved by the tensor product. In this case, the composing cell of a sequence of matrices is

given as follows.

$$I \xrightarrow{(A_{i,j})_{i,j}} J \xrightarrow{(B_{j,k})_{j,k}} K \ ; \qquad (A_{i,j})_{i,j} \odot (B_{j,k})_{j,k} = \left(\sum_{j \in J} A_{i,j} \otimes B_{j,k} \right)_{i \in I, k \in K} .$$

See [**CS10**, Example 5.3].

See Example 2.3.26 for another explanation for the compositability for theses virtual double categories. ⌟

Example 1.4.9. We proceed to Example 1.3.12

(i) $\mathbb{P}$rof is composable as its composition is given by coends in $\mathcal{S}$et. $\mathbb{PROF}$ does not have composites of sequences of positive length in general, and a unit on a category $\mathcal{I}$ exists if and only if $\mathcal{I}$ is locally small.

(ii) $\mathcal{V}$-$\mathbb{P}$rof is composable if $\mathcal{V}$ has small colimits that are preserved by the tensor product. See [**CS10**, Example 5.6].

(iii) $\mathbb{P}$rof$(\mathcal{S})$ is composable if $\mathcal{S}$ has coequalizers that are preserved by pullbacks, in particular, if $\mathcal{S}$ is a regular category.

Since these virtual double categories arise through the $\mathbb{M}$od-construction [**CS10**, Definition 2.8] these virtual double categories are unital by a general result [**CS10**, Proposition 5.5]. ⌟

Chapter 2

Categorical Logic Meets Double Categories

This chapter is aimed at proposing a double-categorical approach to categorical logic. The study of categorical logic has put emphasis on doctrines, fibrations, and occasionally bicategories as semantic environments for logical systems. In this chapter, we take an alternative approach using virtual double categories for this purpose. To this end, we will contrast virtual double categories with other categorical structures intended for categorical logic. The main result of this chapter concerning the comparison with fibrations is the construction of a 2-functor from the 2-category of cartesian fibrations to the 2-category of cartesian fibrational virtual double categories, and characterizing the elementary existential fibrations, which are known as a semantic counterpart of regular logic, as the fibrations that induce a cartesian equipment by this construction. We also prove that the loose bicategory of a cartesian equipment is a cartesian bicategory in the sense of Carboni, Kelly, Walters, and Wood.

The key idea behind this chapter is to incorporate both fibrations and bicategories into a single framework of virtual double categories. The prototypical example of this framework is the double category of sets, functions, and relations, which is a cartesian equipment in the sense of Aleiferi. However, the composition of relations relies on the nature of the category of sets that admits interpretaion of regular logic. If we consider a weaker logical system without the existential quantifier or the equality, a weaker structure than a double category is naturally taken into account. It is virtual double categories that is a suitable structure for this purpose. The main contribution of this chapter is to associate the condition for a cartesian fibrational virtual double category to be a cartesian equipment with the interpretability of regular logic in terms of fibrations.

Outline. Section 2.1 provides an introduction to this chapter. Section 2.2 introduces the background on fibrations and doctrines, mainly focusing on elementary existential fibrations.

Section 2.3 is the main part of this chapter. Subsection 2.3.1 presents the construction $\mathbb{B}\mathrm{il}$ from cartesian fibrations to cartesian fibrational virtual double categories and characterizes the elementary existential fibrations as cartesian fibrations that induce a cartesian equipment. Subsection 2.3.2 further characterizes the regular fibrations as the fibrations that induce a cartesian equipment with Beck-Chevalley pullbacks. Subsection 2.3.3 determines the image of the construction $\mathbb{B}\mathrm{il}$ by the Frobenius property on cartesian equipments.

Section 2.4 compares the double-categorical approach with other approaches to categorical logic, including regular categories and categories with stable factorization systems (Subsection 2.4.1), bicategorical approach (Subsection 2.4.2), and relational doctrines (Subsection 2.4.3). Section 2.5 translates the properties of fibrations and doctrines into the language of virtual double categories, including (predicate) comprehension (Subsection 2.5.1), function extensionality (Subsection 2.5.2), and the unique choice principle (or function comprehension) (Subsection 2.5.3).

One of the primary contributions of this chapter is to provide a comprehensive comparison of the double-categorical approach with other approaches to categorical logic, which is summarized in Figure 1.

2.1. Introduction

Categorical semantics provide a means to interpret logical systems and type theories in categorical structures. The origin of this idea dates back to Lawvere's seminal work on the functorial semantics of algebraic theories [**Law63**]. Given an algebraic theory, one can think of its models in a category with finite products by interpreting function symbols and terms as morphisms in the category and the equations as equalities of morphisms. When one proceeds to interpret first-order logic, the interpretation of logical predicates becomes more involved, as those predicates include quantifiers and logical connectives. A naive way is to interpret predicates with free variables as subobjects of the product of the objects where the variables range over. Moreover, one must enhance the category with additional structures to interpret quantifiers, connectives, or other operators like modalities. For instance, to interpret regular logic, a fragment of first-order logic constituted by the equality $=$, the existential

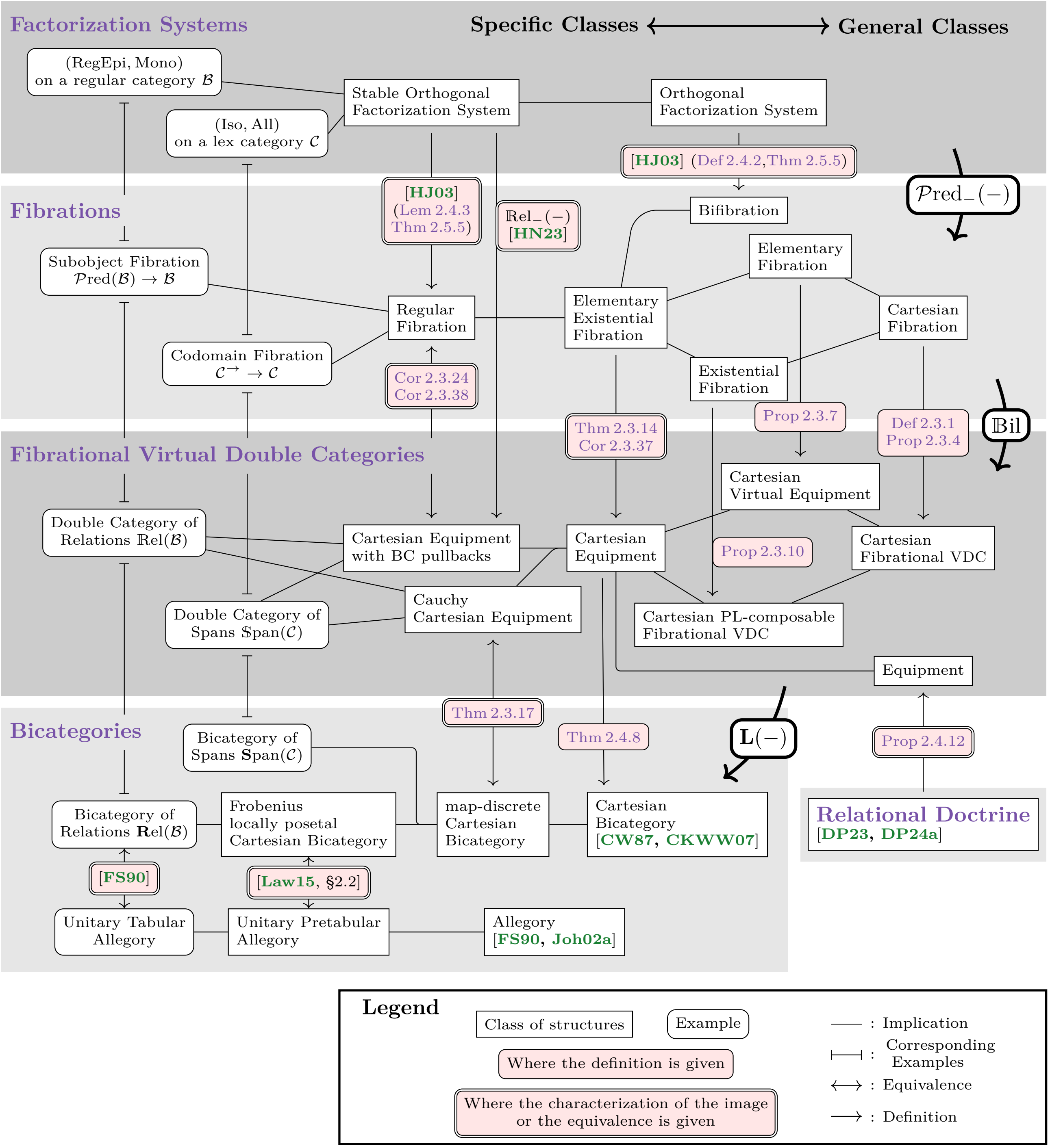


FIGURE 1. The relationship among the structures

quantifier ∃, and the conjunction ∧, the category in question should be a regular category. The existential quantifier is then interpreted using the factorization system consisting of regular epimorphisms and monomorphisms.

Various classes of categories and their corresponding logical systems have been studied, from categories with finite products to regular categories and beyond. A shared interest in these studies is the possibility of completing a category into a model of a given logical system. Exact completion is one of the most well-known examples of this direction and is known to have many applications, such as realizability theory [Men00] and constructive mathematics [MR13b]. In general, the basic idea is to freely add new operations to the category to make it a model of the logical system of interest and possibly to formulate it as a 2-dimensional universal property. However, the presentation of those completions sometimes gets clumsy, revealing the invisible constraints of relying solely on categories. The study of more flexible structures has thus been motivated. Let us review two of them: fibrations (or doctrines) and bicategories.

Lawvere [**Law69**] initiated the approach via doctrines, which can be seen as a special kind of fibrations. A *hyperdoctrine* is a category $\mathcal{C}$ equipped with a contravariant pseudofunctor $\mathfrak{P} : \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ to the category of posets, often with appropriate properties. The key idea is to unleash the interpretation of logical predicates from subobjects of a fixed category to elements of the indexed posets. The category $\mathcal{C}$ is then considered a category of contexts and terms, while predicates in a "context" I are interpreted as elements in $\mathfrak{P}(I)$. The order-preserving map $\mathfrak{P}(f) : \mathfrak{P}(J) \to \mathfrak{P}(I)$ for a "term" $f : I \to J$ represents the substitution of the term, which is a fundamental concept in predicate logic. The existential quantifier is then interpreted as the left adjoint of the substitution map, a groundbreaking idea by Lawvere in the aforementioned work. It should be noted that a fibration is a different formulation of the same notion, with the generalization of the target of $\mathfrak{P}$ to the category of categories.

The bicategorical approach was developed by Freyd and Scedrov [**FS90**]. They introduced a calculus based exclusively on binary relations and not with terms. Its roots can be traced back to 19th-century work by Peirce and Schroeder and its subsequent development known as Tarski's relational calculus [**Tar41**]. The categorical structure that models this calculus was named an *allegory*, a special kind of bicategory, and they found a close connection between allegories and regular categories. In the meantime, the study of bicategories of relations has been developed by many people, resulting in the notion of *cartesian bicategory* [**CKS84, CW87, CKWW07**]. In both approaches, the prototypical example is the bicategory of sets, relations between sets, and inclusions of relations as 0-cells, 1-cells, and 2-cells, respectively. Relations are composed in this bicategory by the existential quantifier as follows:

$$\begin{aligned} R\colon A \to\!\!\!\!\!\to B,\ S\colon B \to\!\!\!\!\!\to C \quad &\mapsto \quad R \odot S\colon A \to\!\!\!\!\!\to C, \\ (R \odot S)(a,c) \quad &\Leftrightarrow \quad \exists b \in B.\ R(a,b) \wedge S(b,c). \end{aligned}$$

In addition, the identity relations are defined as the diagonal relations $\delta_A = \{ (a,a') \mid a = a' \}$.

The two approaches have their advantages and disadvantages. A significant benefit of the bicategorical approach is the expressive power of compositionality of relations. For instance, the exact completion of a regular category refers to its internal equivalence relations. The condition of an endorelation being an equivalence relation is cumbersome to express in a regular category as it is. On the other hand, once we construct the bicategory of the internal relations in that category, we can express it simply as a monad with symmetry therein. In fact, the exact completion can be described through allegories in [**FS90, Joh02a**]. However, not having functions as a primitive notion can sometimes be a disadvantage. Although we could regard internal functional relations as functions and interpret terms using them, they do not capture how we reason about terms in mathematics, particularly their operational nature. This disadvantage is critical when we want to interpret logical systems on the foundation in which the principle of unique choice fails to hold. Again, we need to unleash the interpretation of functions from functional relations to a more flexible structure.

Meanwhile, the fibrational approach has a more transparent connection to the logical systems, as indicated by its completeness theorem for first-order logic and its fragments [**Jac99**]. However, the fibrational approach is not as flexible as the bicategorical approach regarding relations. In addition, the interpretation of the equality and the existential quantifier in fibrations involves the finite products in the base category, while the bicategorical approach can handle them with their built-in composition. Relatedly, there is insufficient category-theoretic justification for the conditions for fibrations to interpret those logical systems, such as the Beck-Chevalley condition and the Frobenius reciprocity; they are somewhat *ad hoc* conditions primarily designed to make the interpretation sound. Thus, the two approaches are complementary to each other, and there should be some framework that preserves the advantages of both.

Here, we propose that double categories are an adequate framework to achieve this goal. The core insight is that *relations are convenient tools, but functions are still indispensable.* Since both functions and relations involve composition, a double category is a natural setting to study them simultaneously. The double category of sets, functions, and relations is an archetypal example of this idea. Despite its intriguing properties, the double category of relations has not been studied as much as the bicategory of relations until recently. The paper [**Lam22**] was the first step to filling this gap, followed by [**HN23**]. Dagnino and Pasquali independently developed a similar idea but with a different framework in [**DP23, DP24a**]; see Subsection 2.4.3 for more details.

The reader may wonder why we go further to virtual double categories. The reason is that the composition of relations relies on the equality and the existential quantifier, which are unavailable in a weaker logical system than regular logic. In other words, the composition of relations is not as primitive as that of functions. Nevertheless, even in the absence of the existential quantifier, we can still define when the following inclusion holds:

$$\begin{tikzcd} & B \arrow[dr, bend left, "S", "|" marking] & \\ A \arrow[ur, bend left, "R", "|" marking] \arrow[rr, bend right, "T"', "|" marking] & \rotatebox{-90}{$\subseteq$} & C \end{tikzcd}
\qquad
\begin{aligned}
&\iff \forall a \in A.\forall b \in B.\forall c \in C.\ (R(a,b) \wedge S(b,c)) \Rightarrow T(a,c) \\
&\iff \forall a \in A.\forall c \in C.\ (\exists b \in B.\ R(a,b) \wedge S(b,c)) \Rightarrow T(a,c).
\end{aligned}$$

Indeed, with the existential quantifier $\exists$, the inclusion of relations can equivalently be expressed as the inculsion between the single relations using $\exists$, as shown in the second line. This observation encourages us to begin with virtual double categories, a structure that does not assume units and composites of loose arrows (arrows of the second kind) to be defined. According to this idea, it is conceptually natural to speculate that the virtual double category of relations based on a logical system becomes a double category precisely when the equality and the existential quantifier are available. This is the main motivation for this paper, and we substantiate this idea in the following way:

Theorem (Theorem 2.3.14). For a cartesian fibration $\mathfrak{p}$ to be elementary existential, it is necessary and sufficient that the VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a cartesian equipment. ⌟

Here, an elementary existential fibration is known to be a fibration that can interpret regular logic, and a cartesian equipment is a double category that can interpret substitution and has a double-categorical finite-product structure. When we regard cartesian fibrations as a logical system, based on the completeness theorem, this theorem states that a virtual double category being a cartesian equipment is necessary and sufficient to interpret regular logic. We also characterize the cartesian equipments that arise this way as Frobenius cartesian equipments in Corollary 2.3.37.

As a result, we can disassociate the equality and the existential quantifier from the finite-product structure in the base category using double categories, overcoming the limitation of fibrations mentioned earlier. This observation is now clearly formulated in terms of 2-categorical structures: elementary existential fibrations and cartesian bicategories are impossible to formulate as cartesian objects in any 2-category, while cartesian equipments are cartesian objects in the 2-category of equipments. Therefore, the Beck-Chevalley condition and the Frobenius reciprocity can be understood as the conditions to make the induced double categories cartesian.

This paper explores how double categories of relations are related to fibrations, bicategories, and other structures, including the abovementioned theorem. The overall picture of the known structures and the proposed framework is depicted in Figure 1. For instance, we prove that the loose bicategory of a cartesian equipment is a cartesian bicategory in Theorem 2.4.8. This suggests that the double categorical approach is a legitimate generalization of the bicategorical one. We also observe that some properties of fibrations are nicely captured with double categories. For instance, comprehension in a fibration, which transforms a predicate $\alpha(x)$ into a new context $x : \{\alpha\}$, is expressed as a double-categorical limit called a *tabulator*.

This study is a step toward understanding the capabilities of (virtual) double categories in place of the existing structures in the context of categorical logic. The seed of this idea can be found in a conference talk by Paré [**Par09**], suggesting the possibility to "put logic in the realm of double categories", as he wrote in his slides. Subsequently, a study based on the same motivation as ours was conducted in [**Law15**], but our approach is more bottom-up, starting from virtual double categories. Our future work includes the study of the exact completion, the tripos-to-topos construction, and other logical completion procedures in the context of virtual double categories. For instance, the existential completion [**Tro20**] is similar to the path construction in virtual double categories [**DPP06**] in concept once we swallow the idea that composition in double categories is the counterpart of the existential quantifier. We expect some connection between the two, although the details are yet to be explored. Quotient completions should also be studied, as they could be expressed elegantly as a quotient of a loose symmetric monoid in a double category, as suggested in [**DP24b**]. To this end, we hope to develop logical aspects of double categories further beyond regular logic.

2.2. Background on Fibrations

In this section, we provide an overview of the background on fibrations in order to clarify the terminology and the notation used in this thesis. We assume that the reader is familiar with basic fibered category theory, which can be found in [**Jac99**, **Joh02a**, **Pit00**]. The definition is already presented in Definition 1.2.4.

Example 2.2.1. We give some examples of fibrations.

(i) The codomain functor $\mathcal{B}^{\to} \to \mathcal{B}$ is a fibration if and only if $\mathcal{B}$ has pullbacks, which we call the ***codomain fibration*** over $\mathcal{B}$.

(ii) Let $\mathcal{S}\mathrm{ub}(\mathcal{B})$ be the category of a pair (I, m) of an object in $\mathcal{B}$ and its subobject m, that is an isomorphism class of monomorphisms into I. Then, the canonical functor $\mathcal{S}\mathrm{ub}(\mathcal{B}) \to \mathcal{B}$ is a fibration if and only if $\mathcal{B}$ has pullbacks of monomorphisms, which is called the ***subobject fibration*** over $\mathcal{B}$.

(iii) For a category $\mathcal{B}$, let $\mathcal{F}am(\mathcal{B})$ be the small coproduct cocompletion of $\mathcal{B}$, that is, the category whose objects are pairs $(I, (b_i)_i)$ where I is a set and $(b_i)_i$ is a family of objects in $\mathcal{B}$ indexed by I, and whose arrows from $(I, (b_i)_i)$ to $(J, (c_j)_j)$ are pairs $(u, (f_i)_i)$ where $u\colon I \to J$ is a function and $f_i\colon b_i \to c_{u(i)}$ for each $i \in I$. Then the forget functor $\mathcal{F}am(\mathcal{B}) \to \mathcal{S}\mathrm{et}$ is a fibration, which is called the ***family fibration***.

⌟

We skip the definitions of fibered functors with fixed and unfixed base categories, and natural transformations between them, but we write **Fib** for the 2-category of fibrations and $\mathbf{Fib}_{\mathcal{B}}$ for the 2-category of fibrations over $\mathcal{B}$. See [**Her93**, **Her94**, **Her99**] for the details.

A fibration is said to be ***cloven*** if the prone lifts are chosen, and further said to be ***split*** if the chosen prone lifts are strictly functorial: that is, $\beta[\mathtt{id}_I] = \beta$ and $\beta[g][f] = \beta[g \circ f]$ for any arrows $f\colon I \to J$ and $g\colon J \to K$ in $\mathcal{B}$. By the axiom of choice, any fibration admits an equivalent cloven fibration. Giving a cloven fibration whose base category is $\mathcal{B}$ is equivalent to giving a pseudofunctor $\mathcal{B}^{\mathsf{op}} \to \mathbf{CAT}$ where **CAT** is the 2-category of categories, functors, and natural transformations. Such a pseudofunctor is called an ***indexed category*** over $\mathcal{B}$ [**JP78**]. For a cloven fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$, the indexed category is formed by the assignment $I \mapsto \mathcal{E}_I$ and the base change functors $(-)[f]\colon \mathcal{E}_J \to \mathcal{E}_I$. The opposite construction from an indexed category to a cloven fibration is called the ***Grothendieck construction*** [**Gro71**].

An indexed category whose values are posets is sometimes called a ***doctrine*** [**Law70**, **KR77**] as a rudimentary version of a *hyperdoctrine* [**Law69**]. We will use the term ***doctrine*** in this thesis to refer to a fibration with each fiber being a poset, which is automatically split.

Proposition 2.2.2 ([Her94, Corollary 3.7], [Her99, 4.1]). Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a fibration where $\mathcal{B}$ is a category with finite products. Then, the following conditions are equivalent:

(i) $\mathfrak{p}$ is a cartesian object in **Fib**.
(ii) $\mathfrak{p}$ is a cartesian object in $\mathbf{Fib}_{\mathcal{B}}$.
(iii) $\mathcal{E}$ has finite products and the functor $\mathfrak{p}$ preserves them.
(iv) For any object $I \in \mathcal{B}$, the fiber $\mathcal{E}_I$ has finite products, and for any arrow $f\colon I \to J$ in $\mathcal{B}$, the base change functor $(-)[f]\colon \mathcal{E}_J \to \mathcal{E}_I$ preserves finite products.

⌟

Definition 2.2.3. A ***cartesian fibration***[1] is a fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ that satisfies the equivalent conditions in the proposition above. ⌟

The finite-product structure is the very minimum requirement for a fibration since it is necessary for the interpretation of sequences of types (contexts) and conjunctions of predicates in logic. A cartesian fibration is simply called a *fibration with finite products* in many references, and the corresponding class of doctrines is called by the term ***primary doctrine*** [**MR13b**] (or *prop-category* [**Pit00**], which is less common nowadays).

---

[1] We have not found a standard adjective for a fibration with finite products. This term is seemingly avoided in the literature to prevent confusion with cartesian lifts, or to distinguish it from a cartesian fibrations between quasicategories, but we use it in this thesis for convenience and integrity.

Remark 2.2.4 (Internal Language of Fibrations)**.** It is sometimes useful to have the formal language to describe what is going on in a categorical structure. The internal language of fibrations with preordered fibers is given in [**Jac99**, Section 4.3] as first-order predicate logic, and that of general fibrations is a proof-relevant version of the former, which appears in [**Pav96**]. We present here a language in the style we will use in the following sections, but our use is restricted to regular logic. This language is compatible with the type theory we will present in Chapter 3.

The language, or the type theory, has the algebraic type theory as its base. It also has another kind called the ***proposition*** depending on a context $\mathit{\Gamma} = x_1 \colon I_1, \dots, x_n \colon I_n$. Given propositions $\alpha_1, \dots, \alpha_n$ and β in the same context $\mathit{\Gamma}$, there is another syntactic entity called ***proof*** of a Horn clause $\alpha_1, \dots, \alpha_n \vdash \beta$. Here, we would prefer the following judgment declaration:

$$\begin{aligned} &\vdash I \ \mathsf{type} \\ \mathit{\Gamma} &\vdash t : I \\ \mathit{\Gamma} &\vdash \alpha \ \mathsf{prop} \\ \mathit{\Gamma} \mid a_1 : \alpha_1, \dots, a_n : \alpha_n &\vdash \mu : \beta. \end{aligned}$$

The variables $a_1, \dots, a_n$ will serve as ***proof variables***. If we make the variable dependency explicit, terms, propositions, and proofs are given by the following grammar:

$$t(x_1, \dots, x_n), \quad \alpha(x_1, \dots, x_n), \quad \mu(x_1, \dots, x_n)\{a_1, \dots, a_n\}.$$

Here $\{-\}$ denotes the proof variable dependency, but we will omit it, or even drop the proof variables from the notation and write a proof as

$$\mathit{\Gamma} \vdash \alpha_1, \dots, \alpha_n \vdash^{\mu} \beta.$$

We do not go into the details of the rules of the type theory because we will present in Chapter 3 a bilateral extension of this type theory called FVDblTT, which is a type theory for fibrational virtual double categories. We will only use the language to make the statements in the following sections more accessible to the reader. ⌟

If one is interested in the interpretation of other logical connectives and quantifiers, then the fibration should have more structure. In this thesis, we focus on the interpretation of equality and existential quantification, so we need a fibration with more structure related to the left adjoints of certain reindexing functors. It is widely known that the left adjoints of reindexing functors should satisfy some conditions so that the interpretation of added logical entities behave coherent with the existing ones.

We use the notion $\sum_f$ for the left adjoint of the reindexing functor $(-)[f] \colon \mathcal{E}_J \to \mathcal{E}_I$ along $f \colon I \to J$.

Definition 2.2.5. Let $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration.

(i) A ***functorial choice of pullback squares*** is a functor $\mathit{\Phi} \colon \mathcal{C} \to \mathcal{P}\mathrm{b}(\mathcal{B})$ into the wide-subcategory $\mathcal{P}\mathrm{b}(\mathcal{B}) \subseteq \mathcal{B}^{\to}$ of $\mathcal{B}^{\to}$ whose arrows are pullback squares in $\mathcal{B}$. For each object $c \in \mathcal{C}$, we write $\mathit{\Phi}_c \colon D_c \to C_c$ for the value of $\mathit{\Phi}$ at c.

For a cartesian fibration $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$ and a functorial choice of pullback squares $\mathit{\Phi} \colon \mathcal{C} \to \mathcal{P}\mathrm{b}(\mathcal{B})$, we say that $\mathfrak{p}$ ***has*** $\mathit{\Phi}$***-coproducts*** if for any object $c \in \mathcal{C}$, the functor $(-)[\mathit{\Phi}_c] \colon \mathcal{E}_{C_c} \to \mathcal{E}_{D_c}$ has a left adjoint $\sum_{\mathit{\Phi}_c}$.

$$\begin{tikzcd} \mathcal{E}_{D_c} \arrow[r, bend left, "\sum_{\mathit{\Phi}_c}"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{C_c} \arrow[l, bend left, "(-)[\mathit{\Phi}_c]"] \end{tikzcd}$$

(ii) We say that $\mathfrak{p}$ satisfies ***the Beck-Chevalley condition (BC condition) for a pullback square with direction*** $(g, h) \colon f \to f'$ in $\mathcal{B}^{\to}$ as in

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & I' \arrow[d, "f'"] \\ J \arrow[r, "g"'] & J' \end{tikzcd} \tag{2.2.1}$$

with f and f' admitting the left adjoint $\sum_f$ and $\sum_{f'}$ if, the following canonical natural transformation is an isomorphism:

$$\begin{tikzcd}
\mathcal{E}_I \arrow[d, "\sum_{f'}"'] \arrow[dr, phantom, "\Searrow"] & \mathcal{E}_{I'} \arrow[l, "h^*"'] \arrow[d, "\sum_f"] \\
\mathcal{E}_J & \mathcal{E}_{J'} \arrow[l, "g^*"]
\end{tikzcd} .$$

For a functorial choice of pullback squares $\Phi \colon \mathcal{C} \to \mathcal{P}\mathrm{b}(\mathcal{B})$ and a cartesian fibration $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$ with Φ-coproducts, we say that $\mathfrak{p}$ satisfies the ***Beck-Chevalley condition for*** Φ if if it satisfies the Beck-Chevalley condition for any pullback square with direction Φ_t for every $t \in \mathcal{C}$.

$$\Phi_t = \begin{tikzcd}
D_c \arrow[r, "D_t"] \arrow[d, "\Phi_c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D_{c'} \arrow[d, "\Phi_d"] \\
C_c \arrow[r, "C_t"'] & C_{c'}
\end{tikzcd}$$

(iii) We say that $\mathfrak{p}$ satisfies the ***Frobenius reciprocity*** for an arrow $f \colon I \to J$ that admits the left adjoint $\sum_f$ of the reindexing along f if the following canonical natural transformation is an isomorphism:

$$\begin{tikzcd}
\mathcal{E}_I \arrow[d, "\sum_f"'] & \mathcal{E}_I \times \mathcal{E}_I \arrow[l, "\wedge"'] \arrow[dr, Rightarrow, shorten=10pt] & \mathcal{E}_I \times \mathcal{E}_J \arrow[l, "\mathtt{id}\times f^*"'] \arrow[d, "\sum_f \times \mathtt{id}"] \\
\mathcal{E}_J & & \mathcal{E}_J \times \mathcal{E}_J \arrow[ll, "\wedge"]
\end{tikzcd} .$$

For a functorial choice of pullback squares $\Phi \colon \mathcal{C} \to \mathcal{P}\mathrm{b}(\mathcal{B})$ and a cartesian fibration $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$ with Φ-coproducts, we say that $\mathfrak{p}$ satisfies the ***Frobenius reciprocity for*** Φ if it satisfies the Frobenius reciprocity for the arrows Φ_c whenever $c \in \mathcal{C}$.

⌟

Although the Beck-Chevalley condition depends on the direction of the pullback square, we often omit the direction when it is clear from the context or when we consider the condition for both directions simultaneously.

The definition is mostly based on [**Jac99**, Section 1.9], but we have introduced the notion of a functorial choice of pullback squares. A standard method to choose the arrows along which the left (or right) adjoints of reindexing functors are defined is to take a subclass of arrows in the base category, as in a *display map category* ([**Tay83**, §4.3.2], [**Jac99**, Definition 10.4.1]). We prefer the functorial presentation because it can specify the form of the pullback squares for which the Beck-Chevalley condition should hold.

Definition 2.2.6 ([**EPR21**, Definition 2.5], [**EPR22**, Definition 4.1])**.** Let $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration. We define a functor $\Phi_= \colon \mathrm{ob}\,\mathcal{B} \times \mathcal{B} \to \mathcal{P}\mathrm{b}(\mathcal{B})$ that assigns to each pair (I, J) the arrow $\langle 0, 0, 1\rangle \colon I \times J \to I \times I \times J$, and assigns to each arrow $f \colon (I, J) \to (I, J')$, which is simply an arrow $f \colon J \to J'$ in $\mathcal{B}$, the pullback square on the left below.

$$\begin{tikzcd}
I \times J \arrow[r, "\mathtt{id}\times f"] \arrow[d, "\langle 0,0,1\rangle"'] \arrow[dr, phantom, very near start, "\lrcorner"] & I \times J' \arrow[d, "\langle 0,0,1\rangle"] \\
I \times I \times J \arrow[r, "\mathtt{id}\times\mathtt{id}\times f"'] & I \times I \times J
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}_{I\times J} \arrow[d, "\sum_{\langle 0,0,1\rangle}"'] \arrow[dr, Rightarrow, shorten=10pt] & \mathcal{E}_{I\times J'} \arrow[l, "(\mathtt{id}\times f)^*"'] \arrow[d, "\sum_{\langle 0,0,1\rangle}"] \\
\mathcal{E}_{I\times I\times J} & \mathcal{E}_{I\times I\times J'} \arrow[l, "(\mathtt{id}\times\mathtt{id}\times f)^*"]
\end{tikzcd}$$

We say that $\mathfrak{p}$ is an ***elementary fibration*** if it has $\Phi_=$-coproducts and satisfies the Beck-Chevalley condition and the Frobenius reciprocity for $\Phi_=$. Here, the Beck-Chevalley condition for $\Phi_=$ is the condition that the canonical natural transformation on the right above is an isomorphism. ⌟

In [**Jac99**, Section 3.4], a fibration with its base category having finite products is said to have *(simple) equality*[2] if it has $\Phi_=$-coproducts and satisfies the Beck-Chevalley condition for $\Phi_=$, and it is said to have *equality with the Frobenius property* if it further satisfies the Frobenius reciprocity for $\Phi_=$.

---

[2] Equality defined in [**Pit00**, Definition 5.6.1] does not require the Beck-Chevalley condition, but it is discussed in the following paragraphs.

An elementary fibration which is a doctrine is called an ***elementary doctrine*** ([**Law70**], [**MR13a**, Definition 2.1]) and one with preordered fibers is called an *Eq-fibration* [**Jac99**, Definition 3.5.1].

Lemma 2.2.7. Let $\mathfrak{p}\colon \mathcal{E}\to\mathcal{B}$ be an elementary fibration. Then, the Frobenius reciprocity for $\mathit{\Phi}_=$ induces the following isomorphisms:

$$\sum_{\langle 0,0,1\rangle}(\kappa_1\wedge\kappa_2)\overset{\iota}{\cong}\left(\sum_{\langle 0,0,1\rangle}\kappa_1\right)\wedge\kappa_2[\langle 0,2\rangle]\cong\left(\sum_{\langle 0,0,1\rangle}\kappa_1\right)\wedge\kappa_2[\langle 1,2\rangle]$$

for $\kappa_1,\kappa_2\in\mathcal{E}_{I\times J}$. Moreover, the above isomorphism ι makes the following diagram commute:

$$\begin{tikzcd}
\kappa_1\wedge\kappa_2 \arrow[r, "\eta_{\kappa_1\wedge\kappa_2}"] \arrow[d, "\eta_{\kappa_1}\wedge\mathtt{id}"'] & \left(\sum_{\langle 0,0,1\rangle}(\kappa_1\wedge\kappa_2)\right)[\langle 0,0,1\rangle] \arrow[d, "\iota[\langle 0,0,1\rangle]", "\wr"'] \\
\left(\sum_{\langle 0,0,1\rangle}\kappa_1\right)[\langle 0,0,1\rangle]\wedge\kappa_2 \arrow[r, "\cong"'] & \left(\left(\sum_{\langle 0,0,1\rangle}\kappa_1\right)\wedge\kappa_2[\langle 0\rangle]\right)[\langle 0,0,1\rangle]
\end{tikzcd},$$

where η is the unit of the adjunction $\sum_{\langle 0,0,1\rangle}\dashv(-)[\langle 0,0,1\rangle]$ and the bottom horizontal isomorphism is given by the preservation of finite products by base change functors. The corresponding statement holds if we replace $\langle 0,2\rangle$ with $\langle 1,2\rangle$. ⌟

Proof. The isomorphisms are obtained by pre-composing the base change functors $(-)[\langle 0,2\rangle]$ and $(-)[\langle 1,2\rangle]$ to the Frobenius reciprocity for $\mathit{\Phi}_=$, and the commutativity of the diagram is a direct consequence of the definition of the natural transformation for the Frobenius reciprocity.

$$\begin{tikzcd}
\mathcal{E}_{I\times J}\times\mathcal{E}_{I\times J} \arrow[r, "\sum_{\langle 0,0,1\rangle}\times\mathtt{id}", "\eta\times\mathtt{id}\ \Nearrow"'] \arrow[dr, equal] & \mathcal{E}_{I\times I\times J}\times\mathcal{E}_{I\times J} \arrow[r, "\mathtt{id}\times((-)[\langle 0,2\rangle])"] \arrow[d, "(-)[\langle 0,0,1\rangle]\times\mathtt{id}"] \arrow[dr, phantom, "="] & \mathcal{E}_{I\times J}\times\mathcal{E}_{I\times I\times J} \arrow[r, "\wedge"] \arrow[d, "(-)[\langle 0,0,1\rangle]\times\mathtt{id}"] \arrow[dr, phantom, "\cong"] & \mathcal{E}_{I\times I\times J} \arrow[d, "(-)[\langle 0,0,1\rangle]"] \\
 & \mathcal{E}_{I\times J}\times\mathcal{E}_{I\times J} \arrow[r, "\mathtt{id}\times((-)[\langle 0,2\rangle])"'] & \mathcal{E}_{I\times J}\times\mathcal{E}_{I\times I\times J} \arrow[r, "\wedge\circ(\mathtt{id}\times(-)[\langle 0,0,1\rangle])"'] & \mathcal{E}_{I\times J}
\end{tikzcd}$$

$$=\quad\begin{tikzcd}
\mathcal{E}_{I\times J}\times\mathcal{E}_{I\times J} \arrow[r, "\mathtt{id}\times((-)[\langle 0,2\rangle])"] & \mathcal{E}_{I\times J}\times\mathcal{E}_{I\times I\times J} \arrow[r, "\sum_{\langle 0,0,1\rangle}\times\mathtt{id}", "\eta\times\mathtt{id}\ \Nearrow"'] \arrow[dr, equal] & \mathcal{E}_{I\times I\times J}\times\mathcal{E}_{I\times I\times J} \arrow[r, "\wedge"] \arrow[d, "(-)[\langle 0,0,1\rangle]\times\mathtt{id}"] \arrow[dr, phantom, "\cong"] & \mathcal{E}_{I\times I\times J} \arrow[d, "(-)[\langle 0,0,1\rangle]"] \\
 & & \mathcal{E}_{I\times J}\times\mathcal{E}_{I\times I\times J} \arrow[r, "\wedge\circ(\mathtt{id}\times(-)[\langle 0,0,1\rangle])"'] & \mathcal{E}_{I\times J}
\end{tikzcd}$$

$$=\quad\begin{tikzcd}
\mathcal{E}_{I\times J}\times\mathcal{E}_{I\times J} \arrow[r, "\mathtt{id}\times((-)[\langle 0,2\rangle])"] & \mathcal{E}_{I\times J}\times\mathcal{E}_{I\times I\times J} \arrow[r, "\wedge\circ\left(\sum_{\langle 0,0,1\rangle}\times\mathtt{id}\right)"] \arrow[d, "\wedge\circ(\mathtt{id}\times(-)[\langle 0,0,1\rangle])"'] \arrow[dr, phantom, "\cong\ (\mathrm{FR})"] & \mathcal{E}_{I\times I\times J} \arrow[d, equal] \\
 & \mathcal{E}_{I\times J} \arrow[r, "\sum_{\langle 0,0,1\rangle}", "\eta\ \Nearrow"'] \arrow[dr, equal] & \mathcal{E}_{I\times I\times J} \arrow[d, "(-)[\langle 0,0,1\rangle]"] \\
 & & \mathcal{E}_{I\times J}
\end{tikzcd}$$

□

Corollary 2.2.8. Let $\mathfrak{p}\colon\mathcal{E}\to\mathcal{B}$ be an elementary fibration. Then, it satisfies the Beck-Chevalley condition for the following pullback square in $\mathcal{B}$ in both directions:

$$\begin{tikzcd}
I\times J \arrow[r, "\langle 0,0,1\rangle"] \arrow[d, "\langle 0,0,1\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & I\times I\times J \arrow[d, "\langle 0,0,1,2\rangle"] \\
I\times I\times J \arrow[r, "\langle 0,1,1,2\rangle"'] & I\times I\times I\times J
\end{tikzcd}.$$

⌟

Proof. We only prove it in one direction as the other direction is similar. Let $\alpha \in \mathcal{E}_{I\times I\times J}$. The canonical arrow that we need to show to be an isomorphism is at the top of the following diagram:

$$\begin{tikzcd}
\sum_{\langle 0,0,1\rangle}\left(\alpha[\langle 0,0,1\rangle]\right) \arrow[r] \arrow[dd, "\cong"'] & \left(\sum_{\langle 0,0,1,2\rangle}\alpha\right)[\langle 0,1,1,2\rangle] \arrow[d, "\cong"', "{\sigma[\langle 0,1,1,2\rangle]}"] \\
 & \left(\left(\sum_{\langle 0,0\rangle}\top\right)[\langle 0,1\rangle]\wedge\alpha[\langle 1,2,3\rangle]\right)[\langle 0,1,1,2\rangle] \arrow[d, "\cong"'] \\
\sum_{\langle 0,0,1\rangle}\left(\top\wedge\alpha[\langle 0,0,1\rangle]\right) \arrow[r, "\cong", "\iota"'] & \left(\sum_{\langle 0,0,1\rangle}\top\right)\wedge\alpha[\langle 0,0,1\rangle][\langle 1,2\rangle]
\end{tikzcd} ,$$

where ι is the isomorphism in Lemma 2.2.7, but for the second isomorphism. The arrow σ is the canonical isomorphsim following from the fact that the left adjoint of the reindexing functor $(-)[\langle 0,1,1,2\rangle]$ is realized by the functor $\left(\sum_{\langle 0,0\rangle}\top\right)[\langle 0,1\rangle]\wedge(-)[\langle 1,2,3\rangle]$, as shown in [**EPR22**]. Therefore, once the commutativity of the diagram is established, the Beck-Chevalley condition for the pullback square in the statement follows. However, the commutativity of the diagram is equivalent to that of the following diagram:

$$\begin{tikzcd}
\alpha[\langle 0,0,1\rangle] \arrow[r, "{\eta_\alpha[0,0,1]}"] \arrow[dr, "{\eta'_\alpha[0,0,1]}"'] \arrow[dd, "\cong"'] & \left(\sum_{\langle 0,0,1,2\rangle}\alpha\right)[\langle 0,1,1,2\rangle][\langle 0,0,1\rangle] \arrow[d, "\cong"', "{\sigma[\langle 0,1,1,2\rangle][\langle 0,0,1\rangle]}"] \\
 & \left(\left(\sum_{\langle 0,0\rangle}\top\right)[\langle 0,1\rangle]\wedge\alpha[\langle 1,2,3\rangle]\right)[\langle 0,1,1,2\rangle][\langle 0,0,1\rangle] \arrow[d, "\cong"'] \\
\top\wedge\alpha[\langle 0,0,1\rangle] \arrow[r, "{\eta_\top\wedge\mathtt{id}_{\alpha[\langle 0,0,1\rangle]}}"] & \left(\sum_{\langle 0,0,1\rangle}\top\right)[\langle 0,0,1\rangle]\wedge\alpha[\langle 0,0,1\rangle]
\end{tikzcd} ,$$

where η'_α is the unit of the adjunction. The traingle is commutative by the uniqueness of the left adjoints up to isomorphism, and the square is commutative because the unit η'_α is isomorphic to $\eta_\top\wedge\mathtt{id}_{\alpha[\langle 0,0,1\rangle]}$. Therefore, the Beck-Chevalley condition for the pullback square in the statement holds. ☐

Definition 2.2.9. Let $\mathfrak{p}\colon\mathcal{E}\to\mathcal{B}$ be a cartesian fibration. We define a functor $\varPhi_\exists\colon \mathrm{ob}\,\mathcal{B}\times\mathcal{B}\to\mathcal{P}\mathrm{b}(\mathcal{B})$ that assigns to each pair (I,J) the arrow $\langle 1\rangle\colon I\times J\to J$, and assigns to each arrow $f\colon(I,J)\to(I,J')$ (an arrow $f\colon J\to J'$ in $\mathcal{B}$) the pullback square on the left below.

$$\begin{tikzcd}
I\times J \arrow[r, "\mathtt{id}\times f"] \arrow[d, "\langle 1\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & I\times J' \arrow[d, "\langle 1\rangle"] \\
J \arrow[r, "f"'] & J'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}_{I\times J} \arrow[d, "\sum_{\langle 1\rangle}"'] \arrow[dr, Rightarrow] & \mathcal{E}_{I\times J'} \arrow[l, "(\mathtt{id}\times f)^*"'] \arrow[d, "\sum_{\langle 1\rangle}"] \\
\mathcal{E}_J & \mathcal{E}_{J'} \arrow[l, "f^*"]
\end{tikzcd}$$

We say that $\mathfrak{p}$ is an ***existential fibration*** if it has $\varPhi_\exists$-coproducts and satisfies the Beck-Chevalley condition and the Frobenius reciprocity for $\varPhi_\exists$. ⌟

The left adjoints to the reindexing functors along the product projections satisfying the Beck-Chevalley condition are called ***simple coproducts*** in [**Jac99**, 1.9.1]. We borrow the adjective "existential" from its doctrine counterpart called ***existential doctrine*** [**MR13b**, Definition 2.11] ([**Tro20**, Definition 3.3]).

Definition 2.2.10. A cartesian fibration $\mathfrak{p}\colon\mathcal{E}\to\mathcal{B}$ is called an ***elementary existential fibration*** if it is an elementary fibration and an existential fibration. ⌟

The doctrine counterpart for this was introduced by Lawvere in [**Law70**] as an ***elementary existential doctrine*** (*eed* for short). An elementary existential fibration with preordered fibers is called a *regular fibration* in [**Jac99**, Definition 4.2.1].

We now introduce a slightly different notion of a fibration.

Definition 2.2.11. A ***regular fibration*** is a cartesian fibration $\mathfrak{p}\colon\mathcal{E}\to\mathcal{B}$ such that the base category $\mathcal{B}$ has finite limits and the fibration $\mathfrak{p}$ has $\mathrm{Id}_{\mathcal{P}\mathrm{b}(\mathcal{B})}$-coproducts and satisfies the Beck-Chevalley condition and the Frobenius reciprocity for $\mathrm{Id}_{\mathcal{P}\mathrm{b}(\mathcal{B})}$. ⌟

Remark 2.2.12. A regular fibration is obviously an elementary existential fibrations. The converse is not true in general, but the difference is more subtle than it seems. A well-known result (see [**Jac99**, Examples 4.3.7]) states that if a fibration $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$ is an elementary existential fibration, then any base change functor $(-)[f]$ has a left adjoint $\sum_f$. One can elegantly prove this by using the internal language of fibtations. What is missing in an elementary existential fibration, even when the base category has finite limits, is the Beck-Chevalley condition for all pullback squares in $\mathcal{B}$. The Frobenius reciprocity for $\mathrm{Id}_{\mathcal{P}\mathrm{b}(\mathcal{B})}$ is actually a consequence of the Beck-Chevalley condition for all pullback squares in $\mathcal{B}$. This is because the left adjoint $\sum_f$ of the reindexing functor $(-)[f]$ is acheived by a combination of the left adjoints of the reindexing functors along the product projections and diagonal arrows together with the fiberwise finite products, and the Frobenius reciprocity for $(-)[f] \dashv \sum_f$ follows from the Frobenius reciprocity for these special cases. The situation is summarized by the following reasoning in the internal language of fibrations:

$$\begin{aligned}\sum_f (\alpha(x) \wedge \beta(f(x))) &\equiv \exists x : I.\, (y = f(x) \wedge (\alpha(x) \wedge \beta(f(x))))\\ &\equiv \exists x : I.\, (y = f(x) \wedge \alpha(x) \wedge \beta(y))\\ &\equiv (\exists x : I.\, (y = f(x) \wedge \alpha(x))) \wedge \beta(y)\\ &\equiv \left(\sum_f \alpha(x)\right) \wedge \beta(y).\end{aligned}$$

Here, the Frobenius reciprocity for $\Phi_\exists$ is used in the third equivalence, and that for $\Phi_=$ is used implicitly in the second equivalence. This statement is proved more rigorously but elegantly using double categories as we will see in Corollary 2.3.22. ⌟

Example 2.2.13. Let us see whether the examples in Example 2.2.1 belong to the classes of fibrations we have defined.

(i) The codomain fibration $\mathcal{B}^{\to} \to \mathcal{B}$ for a category $\mathcal{B}$ with pullbacks is a regular fibration. The Beck-Chevalley conditions for all pullback squares in $\mathcal{B}$ are satisfied by virtue of *the pullback lemma*.

(ii) The subobject fibration $\mathcal{S}\mathrm{ub}(\mathcal{B}) \to \mathcal{B}$ for a category $\mathcal{B}$ with finite limits is a cartesian fibration. We also have the following:

Proposition 2.2.14. Let $\mathcal{B}$ be a category with finite limits and $\mathcal{S}\mathrm{ub}(\mathcal{B}) \to \mathcal{B}$ be its subobject fibration.

(a) This fibration is an elementary fibration.
(b) The following are equivalent:
    (1) $\mathcal{B}$ is a regular category,
    (2) the subobject fibration over $\mathcal{B}$ is an elementary existential fibration, and
    (3) the subobject fibration over $\mathcal{B}$ is a regular fibration. ⌟

For the proof, see [**Jac99**, Examples 3.4.4,Theorem 4.4.4].

(iii) The family fibration $\mathcal{F}am(\mathcal{B}) \to \mathcal{S}\mathrm{et}$ is a cartesian fibration if $\mathcal{B}$ has finite products. We have the following:

Proposition 2.2.15. Let $\mathcal{B}$ be a category with finite products and $\mathcal{F}am(\mathcal{B}) \to \mathcal{S}\mathrm{et}$ be its family fibration.

(a) This fibration is an elementary fibration if $\mathcal{B}$ has distributive initial objects.
(b) The following are equivalent:
    (1) $\mathcal{B}$ has distributive small coproducts,
    (2) the family fibration over $\mathcal{B}$ is an elementary existential fibration, and
    (3) the family fibration over $\mathcal{B}$ is a regular fibration. ⌟

Proof. First, we show that (1) implies (3). The proof for the existence of the left adjoint of reindexing functors and the Frobenius reciprocity is given in [**Jac99**, Example 3.4.4 (iii)]. The left adjoint of the reindexing functor $(-)[f]$ for a function $f\colon I \to J$ is given by

$$\sum_f ((\alpha_i)_{i\in I}) = \left(\sum_{i\in f^{-1}(j)} \alpha_i\right)_{j\in J},$$

and the Beck-Chevalley condition for all pullback squares in $\mathcal{S}$et follows directly from this presentation. Evidently, (3) implies (2). To show that (2) implies (1), the coproduct of a family of objects $(\alpha_i)_{i\in I}$ is achieved by the left adjoint of the reindexing functor for the function $!_I\colon I \to 1$. The Frobenius reciprocity for this reindexing functor guarantees the distributivity of the coproducts. ⌟

⌟

Remark 2.2.16. The terms *regular fibration* and *elementary existential fibration* may be used in a different sense in the literature, sometimes interchangeably. Our terminology is based on our desire to consider the term *elementary existential fibration* as a conjunction of the elementary and existential fibrations, and the term *regular fibration* as a fibration with sufficiently similar properties that the subobject fibration of a regular category has. ⌟

The 2-category of the forementioned classes of doctrines is given in the style of indexed categories in [**MR13b, MR13a**]. Here, we give the 2-category of the corresponding classes of fibrations.

Definition 2.2.17. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ and $\mathfrak{p}'\colon \mathcal{E}' \to \mathcal{B}'$ be fibrations, and let $\mathfrak{f} = (\mathfrak{f}_0, \mathfrak{f}_1)$ be a morphism of fibrations from $\mathfrak{p}$ to $\mathfrak{p}'$; that is, a pair $(\mathfrak{f}_0, \mathfrak{f}_1)$ where $\mathfrak{f}_0\colon \mathcal{B} \to \mathcal{B}'$ is a functor and $\mathfrak{f}_1\colon \mathcal{E} \to \mathcal{E}'$ is a functor over $\mathfrak{f}_0$ that preserves prone arrows.

(i) In the case where $\mathfrak{p}, \mathfrak{p}'$ are cartesian fibrations, we say that $(\mathfrak{f}_0, \mathfrak{f}_1)$ is a ***morphism of cartesian fibrations*** if $\mathfrak{f}_0$ and $\mathfrak{f}_1$ preserve finite products. This is equivalent to saying that $\mathfrak{f}_0$ preserves finite products and for any object $I \in \mathcal{B}$, $\mathfrak{f}_{1,I}\colon \mathcal{E}_I \to \mathcal{E}'_{\mathfrak{f}_0(I)}$ preserves finite products.

(ii) Suppose that we have functorial choices of pullback squares $\varPhi\colon \mathcal{C} \to \mathcal{P}\mathrm{b}(\mathcal{B})$ and $\varPhi'\colon \mathcal{C}' \to \mathcal{P}\mathrm{b}(\mathcal{B}')$, that $\mathfrak{f}_0$ preserves the pullback squares that arise in the image of $\varPhi$, and that we have a functor $T\colon \mathcal{C} \to \mathcal{C}'$ such that $\varPhi' \circ T$ is naturally isomorphic to $\mathcal{P}\mathrm{b}(\mathfrak{f}_0) \circ \varPhi$, where $\mathcal{P}\mathrm{b}(\mathfrak{f}_0) \circ \varPhi$ makes sense by the second condition. Furthermore, $\mathfrak{p}$ and $\mathfrak{p}'$ have $\varPhi$-coproducts and $\varPhi'$-coproducts, respectively. We say that $(\mathfrak{f}_0, \mathfrak{f}_1)$ ***sends $\varPhi$-coproducts to $\varPhi'$-coproducts*** if for any object $c \in \mathcal{C}$, the following canonical natural transformation is an isomorphism[3]:

$$\begin{tikzcd}
\mathcal{E}_{D_c} \arrow[r, "F_{D_c}"] \arrow[d, "\sum_{\varPhi_c}"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{E}'_{D'_{Tc}} \arrow[d, "\sum_{\varPhi'_{Tc}}"] \\
\mathcal{E}_{C_c} \arrow[r, "F_{C_c}"'] & \mathcal{E}'_{C'_{Tc}}
\end{tikzcd}.$$

(iii) In the case where $\mathfrak{p}, \mathfrak{p}'$ are elementary fibrations, we say that $(\mathfrak{f}_0, \mathfrak{f}_1)$ is a ***morphism of elementary fibrations*** if it is a morphism of cartesian fibrations and sends $\varPhi_=$-coproducts in $\mathfrak{p}$ to $\varPhi_=$-coproducts in $\mathfrak{p}'$. Note that when $\mathfrak{f}_0$ preserves finite products, the pullback squares that arise in the image of $\varPhi_=$ are preserved by $\mathfrak{f}_0$.

(iv) In the case where $\mathfrak{p}, \mathfrak{p}'$ are existential fibrations, we say that $(\mathfrak{f}_0, \mathfrak{f}_1)$ is a ***morphism of existential fibrations*** if it is a morphism of cartesian fibrations and sends $\varPhi_\exists$-coproducts in $\mathfrak{p}$ to $\varPhi_\exists$-coproducts in $\mathfrak{p}'$. The same note as above applies.

(v) In the case where $\mathfrak{p}, \mathfrak{p}'$ are elementary existential fibrations, we say that $(\mathfrak{f}_0, \mathfrak{f}_1)$ is a ***morphism of elementary existential fibrations*** if it is a morphism of elementary fibrations and existential fibrations.

⌟

From now on, we will omit the indices 0 and 1 in the notation of morphisms of fibrations.

Definition 2.2.18. We define the 2-category $\mathbf{Fib_{cart}}$ (resp. $\mathbf{Fib}_{\times\wedge=}$, $\mathbf{Fib}_{\times\wedge\exists}$, $\mathbf{Fib}_{\times\wedge=\exists}$) of cartesian fibrations (resp. elementary fibrations, existential fibrations, elementary existential fibrations) as follows:

---

[3] We identify KDc with D'_{Tc} and KCc with C'_{Tc} by the natural isomorphism $\mathcal{P}\mathrm{b}(\mathfrak{f}_0)\circ\varPhi \cong \varPhi'\circ T$ since this identification does not affect the condition.

(i) The objects are cartesian fibrations (resp. elementary fibrations, existential fibrations, elementary existential fibrations).
(ii) The morphisms are morphisms of cartesian fibrations (resp. elementary fibrations, existential fibrations, elementary existential fibrations).
(iii) The 2-cells are natural transformations between morphisms of fibrations.

⌟

Note that how we define the 2-category of cartesian fibrations gives exactly the same 2-category as the 2-category of cartesian objects (c.f. Definition 1.1.1) in **Fib** because of Proposition 2.2.2.

Let **BiFib** be the 2-category of bifibrations, fibered functors preserving the left adjoints of reindexing functors, and arbitrary fibered natural transformations.

Lemma 2.2.19. An elementary existential fibration is a bifibration. This gives rise to a fully faithful 2-functor $\mathbf{Fib}_{\times\wedge=\exists} \to \mathbf{BiFib}$ that sends an elementary existential fibration to its associated bifibration.

⌟

Proof. The first statement is a classical result as explained in Remark 2.2.12. Since the left adjoint of $(-)[f]\colon \mathcal{E}_J \to \mathcal{E}_I$ is acheived by

$$\mathcal{E}_I \xrightarrow{\left\langle \delta_I, \sum_{\langle 0\rangle} \right\rangle} \mathcal{E}_{I\times I} \times \mathcal{E}_{I\times J} \xrightarrow{(-)[\mathtt{id}\times f]\times \mathrm{Id}} \mathcal{E}_{I\times J} \times \mathcal{E}_{I\times J} \xrightarrow{\wedge} \mathcal{E}_{I\times J} \ ,$$

all of which are preserved by a morphism of elementary existential fibrations, the 2-functor is well-defined. This also implies that the condition for a fibered functor to be a 1-cell in **BiFib** and $\mathbf{Fib}_{\times\wedge=\exists}$ are the same, that is, the preservation of the left adjoints of reindexing functors for all arrows in the base category, and the 2-functor is fully faithful. ☐

2.3. From Fibrations to Virtual Double Categories

2.3.1. The bilateral virtual double category of a cartesian fibration.

In this section, we show how to construct a virtual double category from a cartesian fibration, and figure out when the resulting virtual double category is a cartesian equipment.

We start with the definition of a virtual double category from a cartesian fibration. When we see objects in fibers of a fibration as predicates, the loose arrows in the resulting virtual double category are the binary relations described by these predicates. Since these relations respect two different contexts as their domain and codomain, we would rather call them ***bilateral relations***, and the resulting virtual double category the ***bilateral virtual double category*** of the fibration. This terminology is suggested by Hoshino.

Definition 2.3.1. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration. Then the following data form a VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$:

- The tight part of $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is the category $\mathcal{B}$.
- The loose arrows from I to J in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ are objects α in $\mathcal{E}$ over $I \times J$.
- The cells of the form

(2.3.1)
$$\begin{tikzcd}
I_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "s_0"'] \arrow[drrr, phantom, "\xi"] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & I_n \arrow[d, "s_1"] \\
J_0 \arrow[rrr, "\beta"', "\shortmid"{marking}] & & & J_n
\end{tikzcd}$$

  in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ are arrows $\xi\colon \alpha_1[\langle 0,1\rangle] \wedge \cdots \wedge \alpha_n[\langle n-1,n\rangle] \to \beta[(s_0 \times s_1) \circ \langle 0,n\rangle]$ in $\mathcal{E}_{I_0\times\cdots\times I_n}$, where $\langle i,j\rangle$ denotes the pairing of the i-th and j-th projections $I_0 \times \cdots \times I_n \to I_i \times I_j$. This is equivalent to the data of an arrow $\xi\colon \bigwedge_{1\leq i\leq n} \alpha_i[\langle i-1,i\rangle] \to \beta[s_0 \times s_1]$ over the projection $I_0 \times \cdots \times I_n \to I_0 \times I_n$.
- The composite of the following cells

$$\begin{tikzcd}
I_{1,0} \arrow[r, dashed, "\overline{\alpha}_1", "\shortmid"{marking}] \arrow[d, "s_0"'] \arrow[dr, phantom, "\xi_1"] & I_{1,m_1} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "s_1"'] & \cdots \arrow[r, dashed, "\overline{\alpha}_n", "\shortmid"{marking}] \arrow[dr, phantom, "\xi_n"] & I_{n,m_n} \arrow[d, "s_n"] \\
J_0 \arrow[r, "\beta_1"', "\shortmid"{marking}] \arrow[d, "t_0"'] \arrow[drrr, phantom, "\zeta"] & J_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\beta_n"', "\shortmid"{marking}] & J_n \arrow[d, "t_1"] \\
K_0 \arrow[rrr, "\gamma"', "\shortmid"{marking}] & & & K_1
\end{tikzcd}$$

in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ with the cells ξ_i and ζ given by the arrows

$$
\begin{aligned}
\xi_1 &\colon \bigwedge_{1\le i\le m_1} \alpha_1^i[\langle i-1,i\rangle] \to \beta_1[(s_0\times s_1)\circ\langle 0,m_1\rangle] && \text{in} \quad \mathcal{E}_{\prod_{0\le i\le m_1} I_{1,i}},\\
&\vdots\\
\xi_n &\colon \bigwedge_{1\le i\le m_n} \alpha_n^i[\langle i-1,i\rangle] \to \beta_n[(s_{n-1}\times s_n)\circ\langle 0,m_n\rangle] && \text{in} \quad \mathcal{E}_{\prod_{0\le i\le m_n} I_{n,i}}, \text{ and}\\
\zeta &\colon \bigwedge_{1\le j\le n} \beta_j[\langle j-1,j\rangle] \to \gamma[(t_0\times t_1)\circ\langle 0,n\rangle] && \text{in} \quad \mathcal{E}_{\prod_{0\le j\le n} J_j},\\
&(\text{where} \quad I_{j,0} := I_{j-1,m_{j-1}} \quad \text{for} \quad 1<j\le n).
\end{aligned}
$$

is the cell

$$
\begin{aligned}
&\bigwedge_{\substack{1\le j\le n\\ 1\le i\le m_j}} \alpha_j^i[\langle \textstyle\sum_{1\le k<j} m_k+i-1, \sum_{1\le k<j} m_k+i\rangle]\\
&\xrightarrow{\cong} \bigwedge_{1\le j\le n}\left(\bigwedge_{1\le i\le m_j} \alpha_j^i[\langle i-1,i\rangle]\right)[\langle \textstyle\sum_{1\le k<j} m_k,\ldots,\sum_{1\le k\le j} m_k\rangle]\\
&\xrightarrow{\bigwedge_{1\le j\le n}\xi_j[\langle\sum_{1\le k<j} m_k,\sum_{1\le k\le j} m_k\rangle]} \bigwedge_{1\le j\le n} \beta_j[(s_{j-1}\times s_j)\circ\langle 0,m_j\rangle][\langle \textstyle\sum_{1\le k<j} m_k,\ldots,\sum_{1\le k\le j} m_k\rangle]\\
&\xrightarrow{\cong} \bigwedge_{1\le j\le n} \beta_j[(s_{j-1}\times s_j)\circ\langle \textstyle\sum_{1\le k<j} m_k,\sum_{1\le k\le j} m_k\rangle]\\
&\xrightarrow{\cong} \left(\bigwedge_{1\le j\le n} \beta_j[\langle j-1,j\rangle]\right)[(s_0\times\cdots\times s_n)\circ\langle 0,m_1,\ldots,\textstyle\sum_{1\le j\le n} m_j\rangle]\\
&\xrightarrow{\zeta[(s_0\times\cdots\times s_n)\circ\langle 0,\ldots,\sum_{1\le j\le n} m_j\rangle]} \gamma[(t_0\times t_1)\circ\langle 0,n\rangle][(s_0\times\cdots\times s_n)\circ\langle 0,\ldots,\textstyle\sum_{1\le j\le n} m_j\rangle]\\
&\xrightarrow{\cong} \gamma[((t_0\circ s_0)\times(t_1\circ s_1))\circ\langle 0,\textstyle\sum_{1\le j\le n} m_j\rangle] \text{ in } \mathcal{E}_{\prod_{\substack{0\le j\le n\\ 0\le i\le m_j}} I_{j,i}}.
\end{aligned}
$$

- The identity cell for a loose arrow $\alpha\colon I \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} J$ is the canonical isomorphism $\alpha \to \alpha[\mathtt{id}_{I\times J}] = \alpha[(\mathtt{id}_I \times \mathtt{id}_J)]$ in $\mathcal{E}_{I\times J}$.

We write this virtual double category as $\mathbb{B}\mathrm{il}(\mathfrak{p})$. ⌟

It is easy but tedious to check that the data in Definition 2.3.1 form a virtual double category. One way to see this is to use the internal language of fibrations Remark 2.2.4. Loose arrows of $\mathbb{B}\mathrm{il}(\mathfrak{p})$ correspond to propositions $\alpha(x,y)$ in the context $x : I, y : J$, and its cells correspond to proofs ξ as follows:

$$x_0 : I_0,\ldots,x_n : I_n \mid \alpha_1(x_0,x_1),\ldots,\alpha_n(x_{n-1},x_n) \vdash^{\xi} \beta(f_0(x_0),f_1(x_n)).$$

Composition of cells in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ means constructing a new proof from given proofs. Suppose we have the following proofs:

$$
\begin{aligned}
x_{1,0} : I_{1,0},\ldots,x_{1,m_1} : I_{1,m_1} &\mid \alpha_1^1(x_{1,0},x_{1,1}),\ldots,\alpha_1^{m_1}(x_{1,m_1-1},x_{1,m_1})\\
&\vdash^{\xi_1} \beta_1(s_0(x_{1,0}),s_1(x_{1,m_1})),\\
&\vdots\\
x_{n-1,m_{n-1}} : I_{n-1,m_{n-1}},\ldots,x_{n,m_n} : I_{n,m_n} &\mid \alpha_n^1(x_{n-1,m_{n-1}},x_{n,1}),\ldots,\alpha_n^{m_n}(x_{n,m_n-1},x_{n,m_n})\\
&\vdash^{\xi_n} \beta_n(s_{n-1}(x_{n-1,m_{n-1}}),s_n(x_{n,m_n})),\\
y_0 : J_0,\ldots,y_n : J_n &\mid \beta_1(y_0,y_1),\ldots,\beta_n(y_{n-1},y_n)\\
&\vdash^{\zeta} \gamma(t_0(y_0),t_1(y_n)),
\end{aligned}
$$

then we can construct the following proof

$$x_{1,0} : I_{1,0}, x_{1,m_1} : I_{1,m_1}, \dots, x_{n,m_n} : I_{n,m_n} \mid \beta_1(s_0(x_{1,0}), s_1(x_{1,m_1})), \dots, \beta_n(s_{n-1}(x_{n-1,m_{n-1}}), s_n(x_{n,m_n}))$$
$$\vdash^{\zeta[\overline{s_{\underline{j}}(x_{\underline{j},m_j})}]} \gamma(t_0(s_0(x_{1,0})), t_1(s_n(x_{n,m_n}))),$$

by substituting $s_j(x_{j,m_j})$'s for y_j's in the proof of ζ. Subsequently, we can combine the proofs $\xi_1, \dots, \xi_n$ with $\zeta[\overline{s_{\underline{j}}(x_{\underline{j},m_j})}]$ to obtain a proof of

$$x_{1,0} : I_{1,0}, x_{1,1} : I_{1,1}, \dots, x_{n,m_n} : I_{n,m_n} \mid \alpha_1^1(x_{1,0}, x_{1,1}), \dots, \alpha_1^{m_1}(x_{1,m_1-1}, x_{1,m_1}), \dots, \alpha_n^{m_n}(x_{n,m_n-1}, x_{n,m_n})$$
$$\vdash \gamma(t_0(s_0(x_{1,0})), t_1(s_n(x_{n,m_n}))).$$

This corresponds to the composite of the corresponding cells in $\mathbb{B}\mathrm{il}(\mathfrak{p})$. The proof of the associativity of the composition in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is almost the same as the proof of Proposition 3.5.6, but without the bilaterality of the propositions and the proofs.

Remark 2.3.2. This construction $\mathbb{B}\mathrm{il}$ is a generalization of Shulman's $\mathbb{F}\mathrm{r}$-construction of framed bicategories from cartesian fibrations with additional structures [**Shu08**, Theorem 14.4]. The construction assumes these structures on the fibration so that the resulting entity is a framed bicategory, which in our terminology is an equipment; we will revisit this in Remark 2.3.13. On the other hand, the paper deals with more general fibrations than cartesian fibrations, which is called a *monoidal fibration* but with the monoidal structure on the base category cartesian. We could follow the same path and defined a virtual double category from a monoidal fibration with cartesian base, but we do not proceed in this direction in this thesis. ⌟

Proposition 2.3.3. For a cartesian fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$, $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a cartesian FVDC. ⌟

Proof. The restriction of a loose arrow $\alpha\colon I \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} J$, which is an object in $\mathcal{E}$ over $I \times J$, along a pair of tight arrows $f\colon I' \to I$ and $g\colon J' \to J$ is given by $\alpha[f \times g] \in \mathcal{E}_{I' \times J'}$, and the restricting cell `rest` is the identity arrow on $\alpha[f \times g]$. One can check this by seeing that a cell on the left-hand side of the equation corresponds to a morphism ξ in $\mathcal{E}_{\prod_{0 \le i \le n} K_i}$ on the right diagram below:

$$\begin{tikzcd}
K_0 \arrow[rr, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "h"'] & & K_n \arrow[d, "k"] & & K_0 \arrow[rr, dashed, "\overline{\beta}", "\shortmid"{marking}] \arrow[d, "h"'] & \widetilde{\xi} & K_n \arrow[d, "k"] \\
I' \arrow[d, "f"'] & \xi & J' \arrow[d, "g"] & = & I' \arrow[rr, "{\alpha[f\times g]}", "\shortmid"{marking}] \arrow[d, "f"'] & \texttt{rest} & J' \arrow[d, "g"] \\
I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J & & I \arrow[rr, "\alpha"', "\shortmid"{marking}] & & J
\end{tikzcd},
\qquad
\begin{tikzcd}
\bigwedge_{1 \le i \le n} \beta_i[\langle i-1, i\rangle] \arrow[r, "\widetilde{\xi}"] \arrow[dr, "\xi"'] & \alpha[f \times g][(h \times k) \circ \langle 0, n\rangle] \arrow[d, "\cong"] \\
& \alpha[(f \times g) \circ (h \times k) \circ \langle 0, n\rangle]
\end{tikzcd}
\quad \text{in } \mathcal{E}_{\prod_{0 \le i \le n} K_i}.$$

Here, post-composing the canonical isomorphism on the rightmost diagram represents post-composing the cell `rest` on the leftmost diagram. This shows that $\mathbb{B}\mathrm{il}(\mathfrak{p})$ has restrictions.

To show that $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is cartesian, let us recall Proposition 1.3.9, which provides an explicit description of the cartesian structure on an FVDC. The vertical part of $\mathbb{B}\mathrm{il}(\mathfrak{p})$ has finite products by definition, and for each pair of objects I and J in $\mathcal{B}$, the finite products in $\mathcal{E}_{I \times J}$ give the local finite products in $\mathbb{B}\mathrm{il}(\mathfrak{p})(I, J)$. Finally, these are preserved by restriction along tight arrows because it is given by base change in $\mathcal{E}$, which preserves finite products in a cartesian fibration. □

Proposition 2.3.4. The assignment of a CFVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ to a cartesian fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ extends to a 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{Fib}_{\mathbf{cart}} \to \mathbf{FVDbl}_{\mathbf{cart}}$. ⌟

Proof. A morphism of cartesian fibrations $\mathfrak{f}$ from $\mathfrak{p}$ to $\mathfrak{q}$ induces a morphism of CFVDCs $\mathbb{B}\mathrm{il}(\mathfrak{f})\colon \mathbb{B}\mathrm{il}(\mathfrak{p}) \to \mathbb{B}\mathrm{il}(\mathfrak{q})$. This is because the morphism $\mathfrak{f}$ preserves the structure of the fibration including the cartesian structure, and hence the structure of the CFVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$. The assignment of a CFVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ to a cartesian fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ is functorial accordingly.

A 2-cell $\theta\colon \mathfrak{f} \to \mathfrak{g}\colon \mathfrak{p} \to \mathfrak{q}$ induces a vertical 2-cell $\mathbb{B}\mathrm{il}(\theta)$. For an object I in $\mathcal{B}$, the tight arrow $\mathbb{B}\mathrm{il}(\theta)_I\colon \mathbb{B}\mathrm{il}(\mathfrak{f})_I \to \mathbb{B}\mathrm{il}(\mathfrak{g})_I$ is given by $\theta_I\colon \mathfrak{f}_I \to \mathfrak{g}_I$. For a loose arrow $\alpha\colon I \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} J$ in $\mathbb{B}\mathrm{il}(\mathfrak{p})$, the cell $\mathbb{B}\mathrm{il}(\theta)_\alpha$ on the left diagram below is given by the unique arrow $\widetilde{\theta}_\alpha$ that makes the right triangle diagram commute:

$$\begin{tikzcd}
\mathfrak{f}_I \arrow[r, "\mathfrak{f}_\alpha", "\shortmid"{marking}] \arrow[d, "\theta_I"'] \arrow[dr, phantom, "\theta_\alpha"] & \mathfrak{f}_J \arrow[d, "\theta_J"] \\
\mathfrak{g}_I \arrow[r, "\mathfrak{g}_\alpha"', "\shortmid"{marking}] & \mathfrak{g}_J
\end{tikzcd} ,
\qquad
\begin{tikzcd}
\mathfrak{f}_\alpha \arrow[d, "\widetilde{\theta}_\alpha"'] \arrow[dr, "\theta_\alpha"] & & \\
\mathfrak{g}_\alpha[\theta_I \times \theta_J] \arrow[r] & \mathfrak{g}_\alpha & \mathcal{F} \arrow[d, "\mathfrak{q}"] \\
\mathfrak{f}_I \times \mathfrak{f}_J \arrow[r, "\theta_I \times \theta_J"'] & \mathfrak{g}_I \times \mathfrak{g}_J & \mathcal{C}
\end{tikzcd}$$

The naturality conditions of $\mathbb{B}\mathrm{il}(\theta)$ follow from the naturality of θ. □

Example 2.3.5. Some examples of CFVDCs we have seen so far can be obtained through the construction $\mathbb{B}\mathrm{il}$. We mean the fibration itself by its domain category by abuse of notation. The resulting CFVDCs are shown in Table 1. ⌟

Cartesian fibration $\mathfrak{p}$	**CFVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$**
the codomain fibration $\mathcal{B}^{\to} \to \mathcal{B}$ ($\mathcal{B}$: a category with finite limits)	$\mathbb{S}\mathrm{pan}(\mathcal{B})$
the subobject fibration $\mathcal{S}\mathrm{ub}(\mathcal{B}) \to \mathcal{B}$ ($\mathcal{B}$: a category with finite limits)	$\mathbb{R}\mathrm{el}(\mathcal{B})$
the family fibration $\mathcal{F}am(\mathcal{B}) \to \mathcal{S}\mathrm{et}$ ($\mathcal{B}$: a cartesian category)	$\mathcal{B}$-$\mathbb{M}\mathrm{at}$ w.r.t. the cartesian monoidal structure

Table 1. Examples of the construction $\mathbb{B}\mathrm{il}$

Now we have the construction of a CFVDC from a cartesian fibration. The next step is to show how the properties of the fibration are reflected in the resulting CFVDC. The primary interest is in coproducts in fibrations, since they are the key ingredient to interpret regular logic in fibrations.

Lemma 2.3.6. For an elementary fibration $\mathfrak{p}$, the VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is unital. ⌟

Proof. We will prove that a unit on an object I in $\mathcal{B}$ is given as the object $\delta_I := \sum_{\langle 0,0\rangle} \top_I$ in $\mathcal{E}_{I\times I}$, where $\top_I$ is the terminal object in $\mathcal{E}_I$. Here, the unit cell η_I is the component of the unit η of the adjunction $\sum_{\langle 0,0\rangle} : \mathcal{E}_I \underset{\longleftarrow}{\overset{\perp}{\rightleftarrows}} \mathcal{E}_{I\times I} : (-)[\langle 0,0\rangle]$. at the object $\top_I$. The universal property of the unit cell η_I that we want to show is stated as follows: for any cell ν on the left below uniquely factors through the unit cell η_I as on the right below.

$$\begin{tikzcd}[column sep=small]
J_m \arrow[r, "\alpha_m", "\shortmid"{marking}] & J_{m-1} \arrow[r, dashed, "{\alpha_{m-1},\dots,\alpha_1}", "\shortmid"{marking}] & I \arrow[r, dashed, "{\beta_1,\dots,\beta_{n-1}}", "\shortmid"{marking}] & K_{n-1} \arrow[r, "\beta_n", "\shortmid"{marking}] & K_n \\
\| & & \nu & & \| \\
J_m \arrow[rrrr, "\gamma"', "\shortmid"{marking}] & & & & K_n
\end{tikzcd}
=
\begin{tikzcd}[column sep=tiny]
J_m \arrow[rr, "\alpha_m", "\shortmid"{marking}] & & J_{m-1} \arrow[rrr, dashed, "{\alpha_{m-1},\dots,\alpha_1}", "\shortmid"{marking}] & & & I \arrow[ddl, equal] \arrow[ddr, equal] \arrow[rrr, dashed, "{\beta_1,\dots,\beta_{n-1}}", "\shortmid"{marking}] & & & K_{n-1} \arrow[rr, "\beta_n", "\shortmid"{marking}] & & K_n \\
\| & \| & \| & \| & & \eta_I & & \| & \| & \| & \| \\
J_m \arrow[rr, "\alpha_m"', "\shortmid"{marking}] & & J_{m-1} \arrow[rr, dashed, "\shortmid"{marking}] & & I \arrow[rr, "\delta_I"', "\shortmid"{marking}] & & I \arrow[rr, dashed, "\shortmid"{marking}] & & K_{n-1} \arrow[rr, "\beta_n"', "\shortmid"{marking}] & & K_n \\
\| & & & & & \widetilde{\nu} & & & & & \| \\
J_m \arrow[rrrrrrrrrr, "\gamma"', "\shortmid"{marking}] & & & & & & & & & & K_n
\end{tikzcd} .$$

This amounts to saying that for any arrow ν as below, there is a unique arrow $\widetilde{\nu}$ for which $\widetilde{\nu}[\Delta]$ makes the following diagram commute:

$$\begin{tikzcd}[column sep=small]
\kappa := \bigwedge\limits_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \bigwedge\limits_{1\le j\le n} \beta_j[\langle m+j-1,m+j\rangle] \arrow[d, "\cong"'] \arrow[dr, "\nu"] \arrow[ddd, bend right=60, "\tau"'] & \\
\bigwedge\limits_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \top_I[\langle m\rangle] \wedge \bigwedge\limits_{1\le j\le n} \beta_j[\langle m+j-1,m+j\rangle] \arrow[d, "{\mathtt{id}\wedge\eta_{I,\top_I}[\langle m\rangle]\wedge\mathtt{id}}"'] \arrow[dr, phantom, "\circlearrowright"] & \gamma[\langle 0,m+n\rangle] \arrow[d, "\cong"] \\
\bigwedge\limits_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \delta_I[\langle 0,0\rangle][\langle m\rangle] \wedge \bigwedge\limits_{1\le j\le n} \beta_j[\langle m+j-1,m+j\rangle] \arrow[d, "\cong"'] & \gamma[\langle 0,m+n+1\rangle][\Delta] \\
\left(\bigwedge\limits_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \delta_I[\langle m,m+1\rangle] \wedge \bigwedge\limits_{1\le j\le n} \beta_j[\langle m+j,m+j+1\rangle]\right)[\Delta] \arrow[ur, dashed, "{\widetilde{\nu}[\Delta]}"'] &
\end{tikzcd}$$

Here, $J_0 := I$, $K_0 := I$, and $\Delta :=_{\langle 0,\dots,m-1,m,m,m+1,\dots,m+n\rangle}$. We now show that the composite τ of the arrows on the left column of the above diagram coincides with the component of the unit η' of the adjunction $\sum_{\mathit{\Delta}} \dashv (-)[\mathit{\Delta}]$ at the object κ. Let us consider the Beck-Chevalley condition for the following diagram:

(2.3.2)
$$\begin{tikzcd}
\mathcal{E}_{\prod_{1\le i\le m} J_i\times I\times\prod_{1\le j\le n} K_j} \arrow[r, bend left, "\sum_{\mathit{\Delta}}"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{\prod_{1\le i\le m} J_i\times I\times I\times\prod_{1\le j\le n} K_j} \arrow[l, bend left, "(-)[\mathit{\Delta}]"] \\
\mathcal{E}_I \arrow[u, "(-)[\langle m\rangle]"] \arrow[r, bend left, "\sum_{\langle 0,0\rangle}"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{I\times I} \arrow[u, "(-)[\langle m{,}m+1\rangle]"'] \arrow[l, bend left, "(-)[\langle 0{,}0\rangle]"]
\end{tikzcd}.$$

Looking at the components of the units at the terminal objects, we have the following commutative triangle:

$$\begin{tikzcd}
\top_I[\langle m\rangle] \arrow[r, "(\eta_{I,\top_I})[\langle m\rangle]"] \arrow[dr, "\eta'_{\top_I[\langle m\rangle]}"'] & (\delta_I[\langle 0,0\rangle])\,[\langle m\rangle] \arrow[d, "\cong"] \\
& (\sum_{\mathit{\Delta}}(\top_I[\langle m\rangle]))\,[\mathit{\Delta}]
\end{tikzcd}$$

in $\mathcal{E}_{\prod_{1\le i\le m} J_i\times I\times\prod_{1\le j\le n} K_j}$.

In addition, it follows from Lemma 2.2.7 that the following commutes in $\mathcal{E}_{\prod_{1\le i\le m} J_i\times I\times\prod_{1\le j\le n} K_j}$:

$$\begin{tikzcd}
\kappa \arrow[r, "\eta'_\kappa"] \arrow[d, "\cong"'] & (\sum_{\mathit{\Delta}} \kappa)\,[\mathit{\Delta}] \arrow[d, "\cong"] \\
\top_I[\langle m\rangle] \wedge \kappa \arrow[r, "\eta'_{\top_I[\langle m\rangle]}"] \arrow[d, "\eta'_{\top_I[\langle m\rangle]}\wedge\mathtt{id}_\kappa"'] & (\sum_{\mathit{\Delta}}(\top_I[\langle m\rangle] \wedge \kappa))\,[\mathit{\Delta}] \arrow[d, "\cong"] \\
(\sum_{\mathit{\Delta}}(\top_I[\langle m\rangle]))\,[\mathit{\Delta}] \wedge \kappa \arrow[r, "\cong"] & (\sum_{\mathit{\Delta}}(\top_I[\langle m\rangle]) \wedge \kappa[\mathtt{id}\times\langle 0\rangle\times\mathtt{id}])\,[\mathit{\Delta}]
\end{tikzcd}$$

With these diagrams, we can show that the composite τ is equal to the arrow η'_κ up to the isomorphisms that are given by the structures of $\mathfrak{p}$ as an elementary fibration. Note that the subexpression in the codomain of τ to which the base change $(-)[\mathit{\Delta}]$ is applied can be identified with the object $(\sum_{\mathit{\Delta}} \kappa)\,[\mathit{\Delta}]$ via the isomorphisms in the argument above; this can be summarized as the following sequence of isomorphisms:

$$\begin{aligned}
\sum_{\mathit{\Delta}} \kappa &\cong \sum_{\mathit{\Delta}} (\top_I[\langle m\rangle] \wedge \kappa) && \text{by the preservation of finite products by base change} \\
&\cong \sum_{\mathit{\Delta}} (\top_I[\langle m\rangle]) \wedge \kappa[\mathtt{id}\times\langle 0\rangle\times\mathtt{id}] && \text{by the implication of the Frobenius reciprocity Lemma 2.2.7} \\
&\cong \delta_I[\langle m{,}m+1\rangle] \wedge \kappa[\mathtt{id}\times\langle 0\rangle\times\mathtt{id}] && \text{by the Beck Chevalley condition (2.3.2)} \\
&\cong \bigwedge_{1\le i\le m} \alpha_i[\langle m-i{,}m-i+1\rangle] \wedge \delta_I[\langle m{,}m+1\rangle] \wedge \bigwedge_{1\le j\le n} \beta_j[\langle m+j{,}m+j+1\rangle].
\end{aligned}$$

□

Proposition 2.3.7. For an elementary fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$, the VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a cartesian unital FVDC, or equivalently, a cartesian virtual equipment. ⌟

Proof. We have shown that $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a unital FVDC in the previous lemma. Considering Proposition 2.3.7, it remains to show that units are compatible with the cartesian structure in the sense of *(ii)* and *(iii)* there. The second condition is easily satisfied because the diagonal $\langle 00\rangle$ on the terminal object 1 in $\mathcal{B}$ is an isomorphism. The third condition follows from the Beck-Chevalley condition and the Frobenius reciprocity. From the construction of the units in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ and the translation between the fiberwise and the total finite products Proposition 2.2.2, the canonical arrow in *(iii)* of Proposition 2.3.7 is given by the arrow

$$\delta_{I\times J} \xrightarrow{\langle\mu,\nu\rangle} \delta_I[\langle 0_I, 2_I\rangle] \wedge \delta_J[\langle 1_J, 3_J\rangle] \quad \text{in } \mathcal{E}_{I\times J\times I\times J},$$

where μ and ν are the arrows that correspond respectively to the following arrows in $\mathcal{E}_{I\times J}$:

$$\begin{aligned}\top_{I\times J} \;\cong\; \top_I[\langle 0_I\rangle] &\xrightarrow{\eta_I[\langle 0_I\rangle]} \delta_I[\langle 0_I, 0_I\rangle][\langle 0_I\rangle] \;\cong\; \delta_I[\langle 0_I, 2_I\rangle][\langle 0_I, 1_J, 0_I, 1_J\rangle],\\ \top_{I\times J} \;\cong\; \top_J[\langle 1_J\rangle] &\xrightarrow{\eta_J[\langle 0_J\rangle]} \delta_J[\langle 0_J, 0_J\rangle][\langle 1_J\rangle] \;\cong\; \delta_J[\langle 1_J, 3_J\rangle][\langle 0_I, 1_J, 0_I, 1_J\rangle].\end{aligned}$$

However, it is not hard to see that this coincides with the composite of the arrows as follows:

$$\begin{aligned}\delta_{I\times J} &\xrightarrow{\cong} \sum_{\langle 0_I,1_J,0_I,2_J\rangle}\sum_{\langle 0_I,1_J,1_J\rangle}\top_{I\times J} && \text{by the uniqueness of the left adjoint}\\ &\overset{\cong}{\mapsto} \sum_{\langle 0_I,1_J,0_I,2_J\rangle}\left(\top_{I\times J\times J}\wedge \sum_{\langle 0_I,1_J,1_J\rangle}\left(\top_{I\times I\times J}[\langle 0_I,0_I,1_J\rangle]\right)\right) &&\\ &\overset{\cong}{\mapsto} \sum_{\langle 0_I,1_J,0_I,2_J\rangle}\left(\top_{I\times J\times J}\wedge \left(\sum_{\langle 0_I,2_J,1_I,2_J\rangle}\top_{I\times I\times J}\right)[\langle 0_I,1_J,0_I,2_J\rangle]\right) && \text{by the Beck-Chevalley condition}\\ &\overset{\cong}{\mapsto} \sum_{\langle 0_I,1_J,0_I,2_J\rangle}\top_{I\times J\times J}\wedge \sum_{\langle 0_I,2_J,1_I,2_J\rangle}\top_{I\times I\times J} && \text{by the Frobenius reciprocity}\\ &\overset{\cong}{\mapsto} \delta_I[\langle 0_I, 2_I\rangle]\wedge\delta_J[\langle 1_J, 3_J\rangle] && \text{by the Beck-Chevalley condition.}\end{aligned}$$

This completes the proof. ☐

Corollary 2.3.8. The assignment of a cartesian unital FVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ to an elementary fibration $\mathfrak{p}\colon \mathcal{E}\to\mathcal{B}$ extends to a 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{Fib}_{\times\wedge\exists}\to\mathbf{FVDbl}_{\odot,\mathbf{cart}}$. ⌟

Proof. A morphism between elementary fibrations $\mathfrak{f}\colon\mathfrak{p}\to\mathfrak{q}$ preserves the left adjoints of base change functors, which define the units and the unit cells of the FVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$. Hence, the assignment of an FVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ to an elementary fibration $\mathfrak{p}\colon\mathcal{E}\to\mathcal{B}$ extends to a functor. Since $\mathbf{Fib}_{\times\wedge\exists}$ and $\mathbf{FVDbl}_{\odot,\mathbf{cart}}$ are both locally full sub-2-categories of $\mathbf{Fib}_{\mathbf{cart}}$ and $\mathbf{FVDbl}_{,\mathbf{cart}}$, respectively, the assignment extends to a 2-functor. ☐

Lemma 2.3.9. For an existential fibration $\mathfrak{p}\colon\mathcal{E}\to\mathcal{B}$, the VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ has composition of paths of loose arrows of positive length. ⌟

Proof. The proof is analogous to the proof of Lemma 2.3.6. It is enough to show that a composite of two arrows $\alpha_0\colon J_0\to I_0$ and $\beta_0\colon I_0\to K_0$ is given by the object $\sum_{\langle 0,2\rangle}(\alpha_0[\langle 0,1\rangle]\wedge\beta_0[\langle 1,2\rangle])$ in $\mathcal{E}_{J_0\times K_0}$. The cell $\varkappa_{\alpha_0;\beta_0}\colon\alpha_0;\beta_0\Rightarrow\alpha_0\odot\beta_0$ in the definition of composition of paths is, in this case, the unit component of the adjunction $\sum_{\langle 0,2\rangle}\dashv(-)[\langle 0,2\rangle]$ at the object $\alpha_0[\langle 0,1\rangle]\wedge\beta_0[\langle 1,2\rangle]$ in $\mathcal{E}_{J_0\times I_0\times K_0}$.

Let p be the prodcut projection $\prod_{0\le i\le m}J_i\times I_0\times\prod_{0\le j\le n}K_j\to\prod_{0\le i\le m}J_i\times\prod_{0\le j\le n}K_j$. The universal property of the cell $\varkappa_{\alpha_0;\beta_0}$ is that for any cell $\nu\colon\alpha_m;\ldots;\alpha_0;\beta_0;\ldots;\beta_n\Rightarrow\gamma$, there is a unique cell $\widetilde{\nu}\colon\alpha_m;\ldots;\alpha_0\odot\beta_0;\ldots;\beta_n\Rightarrow\gamma$ for which $\widetilde{\nu}[p]$ makes the composite of the following arrows equal to ν:

$$\theta := \bigwedge_{1\le i\le m}\alpha_i[\langle m{-}i,m{-}i{+}1\rangle]\wedge\alpha_0[\langle m,m{+}1\rangle]\wedge\beta_0[\langle m{+}1,m{+}2\rangle]\wedge\bigwedge_{1\le j\le n}\beta_j[\langle m{+}j{+}1,m{+}j{+}2\rangle]$$

$$\cong\downarrow$$

$$\bigwedge_{1\le i\le m}\alpha_i[\langle m{-}i,m{-}i{+}1\rangle]\wedge\left(\alpha_0[\langle 0,1\rangle]\wedge\beta_0[\langle 1,2\rangle]\right)[\langle m,m{+}1,m{+}2\rangle]\wedge\bigwedge_{1\le j\le n}\beta_j[\langle m{+}j{+}1,m{+}j{+}2\rangle]$$

$$\mathtt{id}\wedge\varkappa_{\alpha_0;\beta_0}[\langle m,m{+}1,m{+}2\rangle]\wedge\mathtt{id}\downarrow$$

$$\bigwedge_{1\le i\le m}\alpha_i[\langle m{-}i,m{-}i{+}1\rangle]\wedge\left(\sum_{\langle 0,2\rangle}\left(\alpha_0[\langle 0,1\rangle]\wedge\beta_0[\langle 1,2\rangle]\right)\right)[\langle 0,2\rangle][\langle m,m{+}1,m{+}2\rangle]\wedge\bigwedge_{1\le j\le n}\beta_j[\langle m{+}j{+}1,m{+}j{+}2\rangle]$$

$$\cong\downarrow$$

$$\begin{tikzcd}
\left(\bigwedge_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \left(\sum_{\langle 0,2\rangle} (\alpha_0[\langle 0,1\rangle] \wedge \beta_0[\langle 1,2\rangle])\right) [\langle m,m+1\rangle] \wedge \bigwedge_{1\le j\le n} \beta_j[\langle m+j,m+j+1\rangle]\right) [p] \arrow[d, dashed, "{\widetilde{\nu}[p]}"'] \\
\gamma[\langle 0,m+n+1\rangle][p] \arrow[d, "\cong"'] \\
\gamma[\langle 0,m+n+2\rangle]
\end{tikzcd}\;.$$

The following shows that the domain of the $\widetilde{\nu}$ is isomorphic to the image of θ under $\sum_p$:

$$\begin{aligned}
\sum_p \theta &\cong \sum_p \left(\bigwedge_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \bigwedge_{1\le j\le n} \beta_j[\langle m+j+1,m+j+2\rangle] \wedge (\alpha_0[\langle 0,1\rangle] \wedge \beta_0[\langle 1,2\rangle])\,[\langle m,m+1,m+2\rangle]\right) \\
&\cong \bigwedge_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \bigwedge_{1\le j\le n} \beta_j[\langle m+j,m+j+1\rangle] \wedge \sum_p ((\alpha_0[\langle 0,1\rangle] \wedge \beta_0[\langle 1,2\rangle])\,[\langle m,m+1,m+2\rangle]) \\
&\cong \bigwedge_{1\le i\le m} \alpha_i[\langle m-i,m-i+1\rangle] \wedge \bigwedge_{1\le j\le n} \beta_j[\langle m+j,m+j+1\rangle] \wedge \left(\sum_{\langle 0,2\rangle} \alpha_0[\langle 0,1\rangle] \wedge \beta_0[\langle 1,2\rangle]\right) [\langle m,m+1\rangle]
\end{aligned}$$

From the second to the third line, we used the Frobenius reciprocity, and from the third to the fourth line, we used the Beck-Chevalley condition. What remains is to show the image of this isomorphism under the base change $(-)[p]$ precomposed with the unit component of the adjunction $\sum_p \dashv (-)[p]$ at θ is indeed the same as the upper part of the diagram above, which is a straightforward calculation as in the proof of Lemma 2.3.6. $\square$

Proposition 2.3.10. For an existential fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$, the VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a cartesian positive-length composable FVDC. $\lrcorner$

Proof. The proof is analogous to the proof of Proposition 2.3.7 besides that we consult Proposition 2.3.10 instead. Again, we only give the canonical isomorphisms, but not the explicit calculations for verifying that the canonical isomorphisms indeed arise as the universal properties of composition of paths.

For (ii), we have

$$\top_1 \odot \top_1 = \sum_{\langle 0,2\rangle} (\top_1[\langle 0,1\rangle] \wedge \top_1[\langle 1,2\rangle]) \cong \top_1$$

since $\langle i,j\rangle$ is the identity on 1 for all i and j. For (iii), if we have $\alpha_i\colon I_{i-1} \relbar\joinrel\mapstochar\joinrel\rightarrow I_i$ and $\beta_i\colon J_{i-1} \relbar\joinrel\mapstochar\joinrel\rightarrow J_i$ for $i = 1,2$, then we obtain the following isomorphisms. Here, we do not write the projections by numbers but by the names of the objects for the sake of readability.

$$\begin{aligned}
&(\alpha_1 \times \beta_1) \odot (\alpha_2 \times \beta_2) \\
&= \sum_{\langle I_0,J_0,I_2,J_2\rangle} ((\alpha_1 \times \beta_1)[\langle I_0,J_0,I_1,J_1\rangle] \wedge (\alpha_2 \times \beta_2)[\langle I_1,J_1,I_2,J_2\rangle]) \\
&\cong \sum_{\langle I_0,J_0,I_2,J_2\rangle} (\alpha_1[\langle I_0,I_1\rangle] \wedge \alpha_2[\langle I_1,I_2\rangle] \wedge \beta_1[\langle J_0,J_1\rangle] \wedge \beta_2[\langle J_1,J_2\rangle]) \\
&\cong \sum_{\langle I_0,J_0,I_2,J_2\rangle} \sum_{\langle I_0,J_0,J_1,I_2,J_2\rangle} ((\alpha_1[\langle I_0,I_1\rangle] \wedge \alpha_2[\langle I_1,I_2\rangle]) \wedge (\beta_1[\langle J_0,J_1\rangle] \wedge \beta_2[\langle J_1,J_2\rangle])\,[\langle I_0,J_0,J_1,I_2,J_2\rangle]) \\
&\cong \sum_{\langle I_0,J_0,I_2,J_2\rangle} \left(\left(\sum_{\langle I_0,J_0,J_1,I_2,J_2\rangle} (\alpha_1[\langle I_0,I_1\rangle] \wedge \alpha_2[\langle I_1,I_2\rangle])[\langle I_0,J_0,I_1,I_2,J_2\rangle]\right) \wedge \beta_1[\langle J_0,J_1\rangle] \wedge \beta_2[\langle J_1,J_2\rangle]\right) \\
&\cong \sum_{\langle I_0,J_0,I_2,J_2\rangle} \left(\left(\sum_{\langle I_0,J_0,I_2,J_2\rangle} (\alpha_1[\langle I_0,I_1\rangle] \wedge \alpha_2[\langle I_1,I_2\rangle])\right) [\langle I_0,J_0,I_2,J_2\rangle] \wedge \beta_1[\langle J_0,J_1\rangle] \wedge \beta_2[\langle J_1,J_2\rangle]\right) \\
&\cong \left(\sum_{\langle I_0,J_0,I_2,J_2\rangle} (\alpha_1[\langle I_0,I_1\rangle] \wedge \alpha_2[\langle I_1,I_2\rangle])\right) \wedge \left(\sum_{\langle I_0,J_0,I_2,J_2\rangle} \beta_1[\langle J_0,J_1\rangle] \wedge \beta_2[\langle J_1,J_2\rangle]\right)
\end{aligned}$$

$$
\begin{aligned}
&\cong \left(\sum_{\langle I_0,I_2\rangle} (\alpha_1[\langle I_0,I_1\rangle] \wedge \alpha_2[\langle I_1,I_2\rangle]) \right) [\langle I_0,I_2\rangle] \wedge \left(\sum_{\langle J_0,J_2\rangle} \beta_1[\langle J_0,J_1\rangle] \wedge \beta_2[\langle J_1,J_2\rangle] \right) [\langle J_0,J_2\rangle] \\
&\cong (\alpha_1 \odot \alpha_2) \times (\beta_1 \odot \beta_2).
\end{aligned}
$$

□

Corollary 2.3.11. The assignment of a cartesian positive-length composable FVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ to an existential fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ extends to a 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{Fib}_{\times\wedge\exists} \to \mathbf{FVDbl}_{\odot,\mathbf{cart}}$. ⌟

Proof. The preservation of the left adjoints of base change functors along product projections leads to the preservation of composition of paths of positive length. Hence, the assignment of an FVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ to an existential fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ extends to a functor, and further to a 2-functor by the same argument as in the proof of the previous corollary. □

Corollary 2.3.12. The assignment of a cartesian composable FVDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ to an elementary existential fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ extends to a 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{Fib}_{\times\wedge=\exists} \to \mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}}$. ⌟

Remark 2.3.13. The paper [**Shu08**] gives two sufficient conditions for a monoidal fibration to induce a framed bicategory. Since our definition concentrates on cartesian fibrations, let us compare the two conditions with our results in this case, although there seem no obstacles to generalizing our results to monoidal fibrations and the corresponding comparison.

In both conditions, the given cartesian fibration is required to be a bifibration, which holds true for elementary existential fibrations (cf. discussion in Remark 2.2.12). The first condition in *loc. cit.* is that it is *strongly BC*, which means that the Beck-Chevalley condition holds for all pullback squares. By Corollary 2.3.22, this is a stronger assumption than that the fibration is elementary existential.

The second condition is that the fibration is *internally closed*, which means that the fibration is fiberwise cartesian closed and base changes preserve the internal homs, and is *weakly BC*, which means that the Beck-Chevalley condition holds for pullback squares with one of the legs being a projection of a binary product. Examining the proofs in [**Shu08**, §17], we see that the internal closure of the fibration is used to show that it satisfies what we call the Frobenius reciprocity for arbitrary arrows (cf. (16.7) and (16.8) in *loc. cit.*), and the Beck-Chevalley conditions they use are the ones with the concerning left adjoints are of the forms $\sum_{\langle 0,0,1\rangle}$ or $\sum_{\langle 0\rangle}$. ⌟

The main theorem for this chapter is the following.

Theorem 2.3.14. For a cartesian fibration $\mathfrak{p}$ to be elementary existential, it is necessary and sufficient that the VDC $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a cartesian composable FVDC. ⌟

To state the theorem more precisely in 2-categorical terms, we need to introduce the following definition.

Definition 2.3.15. A 2-functor $\mathbf{f}\colon \mathbf{K} \to \mathbf{L}$ is a local inclusion if for each pair of 0-cells k and k' in $\mathbf{K}$, the (1-)functor

$$\mathbf{f}_{k,k'}\colon \mathbf{K}(k,k') \to \mathbf{L}(\mathbf{f}(k),\mathbf{f}(k'))$$

that $\mathbf{f}$ induces is fully faithful and injective on objects. ⌟

Lemma 2.3.16. For a commutative square of 2-functors

$$\begin{tikzcd}
\mathbf{K}' \arrow[r, "\mathbf{f}'"] \arrow[d, "\mathbf{i}"'] & \mathbf{L}' \arrow[d, "\mathbf{j}"] \\
\mathbf{K} \arrow[r, "\mathbf{f}"'] & \mathbf{L}
\end{tikzcd}$$

where $\mathbf{i}$ and $\mathbf{j}$ are local inclusions and isofibrations, then it is a pullback if and only if

(i) a 0-cell k in $\mathbf{K}$ is (essentially) in the image of $\mathbf{i}$ precicely when $\mathbf{f}(k)$ is (essentially) in the image of $\mathbf{j}$, and

(ii) a 1-cell $t\colon \mathbf{i}(k'_0) \to \mathbf{i}(k'_1)$ in $\mathbf{K}'$, in which k'_0 and k'_1 are 0-cells in $\mathbf{K}'$, is in the image of $\mathbf{i}$ precicely when $\mathbf{f}'(t)\colon \mathbf{j}(\mathbf{f}'(k'_0)) \to \mathbf{j}(\mathbf{f}'(k'_1))$ is in the image of $\mathbf{j}$.

⌟

Proof. Since the 2-category of 2-categories allows a concrete description of strict pullbacks, what we need to show is that the induced 2-functor $\langle \mathbf{i}, \mathbf{f}' \rangle \colon \mathbf{K}' \to \mathbf{K} \times_{\mathbf{L}} \mathbf{L}'$ is an equivalence if and only if the two conditions hold. Here, we use the Whitehead theorem for 2-categories [**Kel05**, §1.11]: a 2-functor is an equivalence (in the 2-category of 2-categories) if and only if it is essentially surjective on 0-cells and locally isomorphic (not an equivalence).

The essential surjectivity on 0-cells for $\langle \mathbf{i}, \mathbf{f}' \rangle$ states that for each pair of 0-cells k and l' in $\mathbf{K}'$ and $\mathbf{L}'$ such that $\mathbf{f}'(k) = \mathbf{j}(l')$, there is a 0-cell k' in $\mathbf{K}$ such that $\langle \mathbf{i}, \mathbf{f}' \rangle(k') \cong (k, l')$. This is equivalent to the first condition in the lemma since $\mathbf{i}$ and $\mathbf{j}$ are isofibrations. For the second condition, note that we have

$$\begin{tikzcd}
\mathbf{K}'(k'_0, k'_1) \arrow[r, "\langle \mathbf{i},\mathbf{f}' \rangle"] \arrow[dr, "\mathbf{i}"'] & \mathbf{K}\left(\mathbf{i}(k'_0), \mathbf{i}(k'_1)\right) \times_{\mathbf{L}\left(\mathbf{j}(\mathbf{f}'(k'_0)), \mathbf{j}(\mathbf{f}'(k'_1))\right)} \mathbf{L}'\left(\mathbf{f}'(k'_0), \mathbf{f}'(k'_1)\right) \arrow[d, "\cong"] \arrow[dl, phantom, "\circlearrowright"] & , \\
 & \{\, t \colon \mathbf{i}(k'_0) \to \mathbf{i}(k'_1) \mid \mathbf{f}'(t) \text{ is in the image of } \mathbf{j} \,\} \underset{\text{full}}{\subseteq} \mathbf{K}\left(\mathbf{i}(k'_0), \mathbf{i}(k'_1)\right) &
\end{tikzcd}$$

since $\mathbf{j}$ induces the inclusion between the hom-categories of $\mathbf{L}'$ and $\mathbf{L}$. As $\mathbf{i}$ is also a local inclusion, the $\mathbf{i}$ in the diagram above is isomorphic if and only if it is bijective on objects (1-cells in the 2-categories), which is (ii) in the lemma. □

Theorem 2.3.17 (Restatement of Theorem 2.3.14 functorially). We have the following pullback square of 2-functors:

$$\begin{tikzcd}
\mathbf{Fib}_{\times\wedge=\exists} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}} \arrow[d] \\
\mathbf{Fib}_{\mathbf{cart}} \arrow[r, "\mathbb{Bil}"'] & \mathbf{FVDbl}_{,\mathbf{cart}}.
\end{tikzcd}$$

⌟

Proof. We have shown that the assignment of $\mathbb{Bil}(\mathfrak{p})$ in $\mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}}$ to an elementary existential fibration $\mathfrak{p}$ extends to a 2-functor $\mathbb{Bil} \colon \mathbf{Fib}_{\times\wedge=\exists} \to \mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}}$. Since $\mathbf{Fib}_{\times\wedge=\exists} \to \mathbf{Fib}_{\mathbf{cart}}$ and $\mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}} \to \mathbf{FVDbl}_{\mathbf{cart}}$ are local inclusions and isofibrations, we can apply Lemma 2.3.16 to show that the square is a pullback.

Therefore, what we need to show is

(i) for a cartesian fibration $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$, if $\mathbb{Bil}(\mathfrak{p})$ is a cartesian equipment[4], then $\mathfrak{p}$ is an elementary existential fibration.

(ii) for elementary existential fibrations $\mathfrak{p}$ and $\mathfrak{q}$ and $\mathfrak{f} \colon \mathfrak{p} \to \mathfrak{q}$ in $\mathbf{Fib}_{\times\wedge}$, if $\mathbb{Bil}(\mathfrak{f})$ is a morphism of cartesian equipments, then $\mathfrak{f}$ is a morphism of elementary existential fibrations.

The second statement is easy. If $\mathbb{Bil}(\mathfrak{f})$ is a morphism of cartesian equipments, it preserves any spine cells, which means every coproducts including $\Phi_{=}$- and $\Phi_{\exists}$-coproducts. We will show the first statement.

Since an equipment $\mathbb{Bil}(\mathfrak{p})$ gives a bifibration $\mathbb{Bil}(\mathfrak{p})_1 \to \mathcal{B} \times \mathcal{B}$, we can recover $\mathfrak{p}$ as a base change of this along the functor $\mathcal{B} \to \mathcal{B} \times \mathcal{B}$ that sends an object I to $(I, 1)$, and hence, we have that $\mathfrak{p}$ is a bifibration. Let us explain this in explicit terms. Recall that a cell ξ in $\mathbb{Bil}(\mathfrak{p})$ as in the left diagram in (2.3.3) corresponds to an arrow $\xi \colon \alpha \to \beta[f]$ in $\mathcal{E}_I$. Since $\mathbb{Bil}(\mathfrak{p})$ is a cartesian equipment, we know that there is a oprestriction $f^*\alpha$ of α along f and $\mathtt{id}_1$. Using this, we find that the arrow ξ corresponds to the arrow $\widetilde{\xi}$ in the right diagram in (2.3.3).

$$\begin{tikzcd}
I \arrow[r, "\alpha", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & 1 \arrow[d, equal] & \Bigg\| & J \arrow[r, "f^*\alpha", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\xi}"] & 1 \arrow[d, equal] \\
J \arrow[r, "\beta"', "|" marking] & 1 & & J \arrow[r, "\beta"', "|" marking] & 1
\end{tikzcd} \tag{2.3.3}$$

---

[4] Since the 2-functor $\mathbf{Eqp}_{\mathbf{cart}} \to \mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}}$ is essentially surjective on 0-cells, $\mathbb{Bil}(\mathfrak{p})$ being a cartesian equipment is equivalent to saying that it is in the image of the 2-functor $\mathbf{FVDbl}_{\nrightarrow\odot,\mathbf{cart}} \to \mathbf{FVDbl}_{,\mathbf{cart}}$. The only reason we phrase it in this way is to make the statement more readable.

This shows that the oprestriction functor $f^*(-)\colon \mathcal{E}_I \to \mathcal{E}_J$ gives a left adjoint to the base change functor $(-)[f]\colon \mathcal{E}_J \to \mathcal{E}_I$. Therefore, it remains to show that $\mathfrak{p}$ satisfies the Beck-Chevalley condition and the Frobenius reciprocity for the base change functors along arrows in $\Phi_\exists$ and $\Phi_=$.

For the Beck-Chevalley condition, note that the canonical arrow which we need to show is an isomorphism is the one in Lemma 1.2.23 for the cartesian equipment $\mathbb{B}\mathrm{il}(\mathfrak{p})$, where we take the pullback square in $\Phi_=$ and $\Phi_\exists$ and $M = 1$. Since those pullback squares are the products of the pullback squares in which one side is the identity arrow, we can apply Lemma 1.2.22 to show that these are Beck-Chevalley pullback squares, and then apply Lemma 1.2.23 to show that the canonical arrow is an isomorphism.

For the Frobenius reciprocity for $\Phi_\exists$, apply Lemma 1.2.24 to the case where f is the projection $\langle 1\rangle\colon I\times J \to J$ and K is the terminal object 1. Here, the assumption that the pullback of $\langle 0,0\rangle\colon J \to J\times J$ and $\langle 1\rangle \times \mathtt{id}_J = \langle 1,2\rangle\colon I\times J\times J \to J\times J$ is a Beck-Chevalley pullback square follows again from Lemma 1.2.22:

$$\begin{tikzcd}
I\times J \arrow[r, "{\langle 0,1,1\rangle}"] \arrow[d, "\langle 1\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & I\times J\times J \arrow[d, "{\langle 1,2\rangle}"] \\
J \arrow[r, "{\langle 0,0\rangle}"'] & J\times J
\end{tikzcd}
\cong
\begin{tikzcd}
I \arrow[r, "\mathtt{id}_I"] \arrow[d, "\langle\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, "\langle\rangle"] \\
1 \arrow[r, "\mathtt{id}_1"'] & 1
\end{tikzcd}
\times
\begin{tikzcd}
J \arrow[r, "{\langle 0,0\rangle}"] \arrow[d, "\mathtt{id}_J"'] \arrow[dr, phantom, "\lrcorner", very near start] & J\times J \arrow[d, "\mathtt{id}_{J\times J}"] \\
J \arrow[r, "{\langle 0,0\rangle}"'] & J\times J
\end{tikzcd}$$

The Frobenius reciprocity for $\Phi_=$ is not as straightforward as the other cases. Recall that we need to show that the canonical arrow from $\sum_{\langle 0,0,1\rangle}(\alpha\wedge\beta[\langle 0,0,1\rangle])$ to $(\sum_{\langle 0,0,1\rangle}\alpha)\wedge\beta$ is an isomorphism. The strategy we take is as follows.

(I) We reduce the problem to the case where α is the terminal and simultaneously present another equivalent form of the canonical arrow up to isomorphism.

(II) We show that the canonical arrow has a retraction.

(III) We show that the retraction in turn has a retraction, which implies that the original arrow is an isomorphism.

In the following, when we index the isomorphisms with the symbol $\cong$ without defining them explicitly, we mean that the isomorphisms are the canonical ones that are induced by the iterated base changes and the Beck-Chevalley condition.

(I) Reduction to a special case.

We have already shown that $\mathfrak{p}$ is a bifibration, and since the composites including the units in an FVDC are unique up to isomorphism, we can identify the unit on I with the object $\sum_{\langle 0,0\rangle}\top_I$ in $\mathcal{E}_{I\times I}$ and the composite of $\alpha\colon I \nrightarrow J$ and $\beta\colon J \nrightarrow K$ with the object $\sum_{\langle 0,2\rangle}(\alpha[\langle 0,1\rangle]\wedge\beta[\langle 1,2\rangle])$ in $\mathcal{E}_{I\times K}$, with the composing cells defined by the units of the adjunctions.

We postpone the proof of the following claim to the end of this proof.

Claim 2.3.18. The natural transformation $\sum_{\langle 0,0,1\rangle} \Rightarrow (\sum_{\langle 0,0,1\rangle}\top_{I\times K})\wedge(-)[\langle 1,2\rangle]$ defined by the universal property of the local binary product $\wedge$ from the natural transformations

$$\begin{tikzcd}
 & & \mathcal{E}_{I\times K} \arrow[rrd, "\sum_{\langle 0,0,1\rangle}"] & & \\
\mathcal{E}_{I\times K} \arrow[rru, equal] \arrow[rrd, "{(-)[\langle 1,2\rangle]}"'] & \cong & & \Searrow & \mathcal{E}_{I\times I\times K} \\
 & & \mathcal{E}_{I\times I\times K} \arrow[uu, "{(-)[\langle 0,0,1\rangle]}" description] \arrow[rru, equal] & &
\end{tikzcd}
,\quad
\begin{tikzcd}
 & & \mathcal{E}_{I\times K} \arrow[rrd, "\sum_{\langle 0,0,1\rangle}"] & & \\
\mathcal{E}_{I\times K} \arrow[rru, equal] \arrow[rrd, "!"'] & \Searrow & & \cong & \mathcal{E}_{I\times I\times K} \\
 & & \mathbf{1} \arrow[uu, "\top_{I\times K}" description] \arrow[rru, "\sum_{\langle 0,0.1\rangle}\top_{I\times K}"'] & &
\end{tikzcd}$$

is an isomorphism. Here, the 2-cells in the diagram are the unit and counit of the adjunctions. ⌟

Using this claim, we have the following commutative diagram for a pair of objects $\alpha \in \mathcal{E}_{I\times K}$ and $\beta \in \mathcal{E}_{I\times I\times K}$ with the vertical arrows being isomorphisms. The whole diagram is natural in α and β.

$$\begin{tikzcd}
\sum_{\langle 0,0,1\rangle}(\alpha \wedge \beta[\langle 0,0,1\rangle]) \arrow[r, "(i)"] \arrow[d, "\cong"', "\text{Claim 2.3.18}"'] \arrow[dr, phantom, "\circlearrowright"] & (\sum_{\langle 0,0,1\rangle} \alpha) \wedge \beta \arrow[d, "\cong", "\text{Claim 2.3.18}"] \\
(\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge (\alpha \wedge \beta[\langle 0,0,1\rangle])\,[\langle 1,2\rangle] \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \left(\sum_{\langle 0,0,1\rangle} \top_{I\times K} \wedge \alpha[\langle 1,2\rangle]\right) \wedge \beta \arrow[d, "\cong"] \\
(\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge \alpha[\langle 1,2\rangle] \wedge \beta[\langle 1,1,2\rangle] \arrow[r, "(ii)"'] & (\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge \alpha[\langle 1,2\rangle] \wedge \beta
\end{tikzcd}$$

The Frobenius reciprocity for $\Phi_=$ states that the canonical arrow (i) is an isomorphism, so it suffices to show that the canonical arrow (ii) is an isomorphism. By looking at how (i) are defined and the naturality on the unique arrow $!\colon \beta \to \top_{I\times K\times K}$ in $\mathcal{E}_{I\times I\times K}$, we can see that (ii) is compatible with the projections to $\sum_{\langle 0,0,1\rangle} \top_{I\times K}$ and $\alpha[\langle 1,2\rangle]$. Therefore, the proof reduces to showing that (ii) is an isomorphism in the case where α is the terminal object $\top_{I\times K}$. In this case, the above diagram that defines (ii) becomes the following diagram:

$$\begin{tikzcd}
\sum_{\langle 0,0,1\rangle}(\beta[\langle 0,0,1\rangle]) \arrow[r, "\langle\!\langle \sum_{\langle 0,0,1\rangle}!,\ \varepsilon_\beta \rangle\!\rangle"] \arrow[d, "\iota", "\cong"', "\text{Claim 2.3.18}"'] & (\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge \beta \\
(\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge \beta[\langle 1,1,2\rangle] \arrow[ur, "\zeta"'] &
\end{tikzcd} \tag{2.3.4}$$

By definition, the counterpart of (i) is the pairing of the image of the unique arrow $!\colon \beta \to \top_{I\times K\times K}$ by $\sum_{\langle 0,0,1\rangle}$ and the counit component ε_β of the adjunction $\sum_{\langle 0,0,1\rangle} \dashv (-)[\langle 0,0,1\rangle]$ at β. Here $\langle\!\langle -,- \rangle\!\rangle$ is the pairing of arrows in the fiber categories.

(II) Construction of a retraction of ζ.

Let ν be the arrow that corresponds to the following cell ν in the virtual double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$, where $\delta_I := \sum_{\langle 0,0\rangle} \top_I$ is the unit on I.

(2.3.5)

$$\begin{tikzcd}
\delta_I[\langle 0,1\rangle] \wedge \beta[\langle 1,2,3\rangle] \arrow[d, "\nu"'] \\
\beta[\langle 0,2,3\rangle]
\end{tikzcd} \quad \text{in } \mathcal{E}_{I\times I\times I\times K}$$

$$\begin{tikzcd}
 & I \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & I\times K \arrow[d, equal] \\
I \arrow[ur, equal, "\eta_I"'] \arrow[r, "\delta_I"', "\shortmid"{marking}] \arrow[d, equal] & I \arrow[r, "\beta"', "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & I\times K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & I\times K
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & I\times K \arrow[d, equal] \\
I \arrow[r, "\beta"', "\shortmid"{marking}] & I\times K
\end{tikzcd}$$

Then, let $\xi\colon (\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge \beta \to \beta[\langle 1,1,2\rangle]$ be the following composite

$$(\textstyle\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge \beta \xrightarrow{\cong} (\delta_I[\langle 0,1\rangle] \wedge \beta[\langle 1,2,3\rangle])\,[\langle 1,0,1,2\rangle] \xrightarrow{\nu[\langle 1,0,1,2\rangle]} \beta[\langle 0,2,3\rangle][\langle 1,0,1,2\rangle] \xrightarrow{\cong} \beta[\langle 1,1,2\rangle] \ .$$

The first isomorphism uses the Beck-Chevalley condition for $\Phi_=$. Our claim is that $\langle\!\langle \pi_0, \xi \rangle\!\rangle$ is the inverse of ζ, where π_0 is the 0-th projection.

We show that $\langle\!\langle \pi_0, \xi \rangle\!\rangle$ is a retraction of ζ. By (2.3.4), this is equivalent to showing that $\langle\!\langle \pi_0, \xi \rangle\!\rangle \circ \langle\!\langle \sum_{\langle 0,0,1\rangle}!,\ \varepsilon_\beta \rangle\!\rangle = \iota$. On the 0-th projection, the equation follows directly from the construction of ι in Claim 2.3.18. The 1-st projection of the equation is the commutativity of the following diagram:

$$\begin{tikzcd}
\sum_{\langle 0,0,1\rangle}(\beta[\langle 0,0,1\rangle]) \arrow[r, "\langle\!\langle \sum_{\langle 0,0,1\rangle}!,\ \varepsilon_\beta \rangle\!\rangle"] \arrow[dr, "\varepsilon"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & (\sum_{\langle 0,0,1\rangle} \top_{I\times K}) \wedge \beta \arrow[d, "\xi"] \\
 & \beta[\langle 1,1,2\rangle]
\end{tikzcd} .$$

The adjunction $\sum_{\langle 0,0,1\rangle} \dashv (-)[\langle 0,0,1\rangle]$ translates this to the commutativity of the diagram

$$\begin{tikzcd}
\beta[\langle 0,0,1\rangle] \arrow[r] \arrow[rr, bend left, "\langle\!\langle \eta\circ !,\ \mathtt{id}_{\beta[\langle 0,0,1\rangle]}\rangle\!\rangle"] \arrow[dr, "\cong"'] \arrow[drr, equal, bend right] & \left(\left(\sum_{\langle 0,0,1\rangle} \top_{I\times K}\right)\wedge\beta\right)[\langle 0,0,1\rangle] \arrow[r, "\cong"] \arrow[d, "\xi[\langle 0,0,1\rangle]"] & \delta_I[\langle 0,0\rangle]\wedge\beta[\langle 0,0,1\rangle] \arrow[d, "\nu[\langle 0,0,0,1\rangle]"] \\
 & \beta[\langle 1,1,2\rangle][\langle 0,0,1\rangle] \arrow[r, "\cong"'] & \beta[\langle 0,0,1\rangle]
\end{tikzcd},$$

which holds true by the definition of ν in (2.3.5).

(III) Construction of a retraction of $\langle\!\langle \pi_0, \xi\rangle\!\rangle$.

Considering the pairing with the 0-th projection and how the arrow ξ is defined, it is enough to show that there exists an arrow λ that makes the following triangle commute:

(2.3.6)
$$\begin{tikzcd}
(\delta_I[\langle 0,1\rangle]\wedge\beta[\langle 1,2,3\rangle])[\langle 1,0,1,2\rangle] \arrow[r, "\langle\!\langle \pi_0,\nu\rangle\!\rangle\,[\langle 1,0,1,2\rangle]"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & (\delta_I[\langle 0,1\rangle]\wedge\beta[\langle 0,2,3\rangle])\,[\langle 1,0,1,2\rangle] \arrow[d, "\cong"] \\
\delta_I[\langle 1,0\rangle]\wedge\beta \arrow[r, "\langle\!\langle \pi_0,\nu[\langle 1,0,1,2\rangle]\rangle\!\rangle"] \arrow[dr, "\pi_1"'] & \delta_I[\langle 1,0\rangle]\wedge\beta[\langle 1,1,2\rangle] \arrow[d, "\lambda"] \\
 & \beta
\end{tikzcd} \quad \text{in } \mathcal{E}_{I\times I\times K}.$$

Before the construction, we observe that we have the following cell μ and the corresponding arrow $\mu\colon \delta_I[\langle 0,1\rangle]\wedge\delta_I[\langle 1,2\rangle]\to\delta_I[\langle 0,2\rangle]$ in $\mathcal{E}_{I\times I\times I}$ satisfying the following commutative diagram:

$$\begin{tikzcd}
 & I \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & \\
I \arrow[r, "\delta_I"', "|" marking] \arrow[d, equal] & I \arrow[r, "\delta_I"', "|" marking] \arrow[d, phantom, "\mu"] & I \arrow[d, equal] \\
I \arrow[rr, "\delta_I"', "|" marking] & & I
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & I \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_I"] & \\
I \arrow[rr, "\delta_I"', "|" marking] & & I
\end{tikzcd}
\;,\qquad
\begin{tikzcd}
\top_I \arrow[d, "\langle\!\langle \eta_I,\eta_I\rangle\!\rangle"'] \arrow[dr, "\eta_I"] & \\
\delta_I[\langle 0,0\rangle]\wedge\delta_I[\langle 0,0\rangle] \arrow[d, "\cong"'] \arrow[r, phantom, "\circlearrowright"] & \delta_I[\langle 0,0\rangle] \arrow[d, "\cong"] \\
(\delta_I[\langle 0,1\rangle]\wedge\delta_I[\langle 1,2\rangle])\,[\langle 0,0,0\rangle] \arrow[r, "\mu[\langle 0,0,0\rangle]"'] & \delta_I[\langle 0,2\rangle][\langle 0,0,0\rangle]
\end{tikzcd}$$

Let $\sigma\colon \delta_I\to\delta_I[\langle 1,0\rangle]$ be the canonical isomorphism, which follows from the observation that $\delta_I[\langle 1,0\rangle]$ exhibits the same universal property as δ_I. From the natural isomorphism between $(-)[\langle 0,0,0\rangle]$ and $(-)[\langle 1,0,1\rangle][\langle 0,0\rangle]$ and from the above diagram, we obtain the following commutative diagrams in $\mathcal{E}_{I\times I}$:

(2.3.7)
$$\begin{tikzcd}
\delta_I \arrow[r, "!"] \arrow[d, "\langle\!\langle \sigma,\mathtt{id}_{\delta_I}\rangle\!\rangle"'] \arrow[dr, phantom, "\circlearrowright"] & \top_I \arrow[r, "\cong"] & \top_I[\langle 1\rangle] \arrow[d, "\eta_I[\langle 1\rangle]"] \\
\delta_I[\langle 1,0\rangle]\wedge\delta_I \arrow[r, "\mu[\langle 1,0,1\rangle]"'] & \delta_I[\langle 1,1\rangle] \arrow[r, "\cong"'] & \delta_I[\langle 0,0\rangle][\langle 1\rangle]
\end{tikzcd}
\quad \overset{(-)[\langle 10\rangle]}{\mapsto} \quad
\begin{tikzcd}
\delta_I[\langle 1,0\rangle] \arrow[r, "!"'] \arrow[d, "\langle\!\langle \sigma^{-1},\mathtt{id}_{\delta_I}\rangle\!\rangle"'] \arrow[dr, phantom, "\circlearrowright"] & \top_I \arrow[r, "\cong"] & \top_I[\langle 0\rangle] \arrow[d, "\eta_I[\langle 0\rangle]"] \\
\delta_I\wedge\delta_I[\langle 1,0\rangle] \arrow[r, "\mu[\langle 0,1,0\rangle]"'] & \delta_I[\langle 0,0\rangle] \arrow[r, "\cong"'] & \delta_I[\langle 0,0\rangle][\langle 0\rangle]
\end{tikzcd}$$

Note that the vertical arrow ! is also the counit component at $\top_I$. Now, we get back to the construction of λ. By the associativity of loose composition, or equivalently by the universal property, we have the following commutative diagram:

$$\begin{tikzcd}
\delta_I[\langle 0,1\rangle]\wedge\delta_I[\langle 1,2\rangle]\wedge\beta[\langle 2,3,4\rangle] \arrow[r, "\mu[\langle 0,1,2\rangle]\wedge\mathtt{id}_{\beta[\langle 2,3,4\rangle]}"] \arrow[d, "\mathtt{id}_{\delta_I[\langle 0,1\rangle]}\wedge\nu[\langle 1,2,3,4\rangle]"'] \arrow[dr, phantom, "\circlearrowright"] & \delta_I[\langle 0,2\rangle]\wedge\beta[\langle 2,3,4\rangle] \arrow[d, "\nu[\langle 0,2,3,4\rangle]"] \\
\delta_I[\langle 0,1\rangle]\wedge\beta[\langle 1,3,4\rangle] \arrow[r, "\nu[\langle 0,1,3,4\rangle]"'] & \beta[\langle 0,3,4\rangle]
\end{tikzcd} \quad \text{in} \quad \mathcal{E}_{I\times I\times I\times I\times K}.$$

Sending this diagram by $(-)[\langle 0,1,0,1,2\rangle]$, we obtain the following commutative diagram:

(2.3.8)
$$\begin{tikzcd}
\delta_I[\langle 0,1\rangle] \wedge \delta_I[\langle 1,0\rangle] \wedge \beta \arrow[r, "{\mu[\langle 0,1,0\rangle]\wedge \mathtt{id}_\beta}"] \arrow[d, "{\mathtt{id}_{\delta_I[\langle 0,1\rangle]}\wedge\nu[\langle 1,0,1,2\rangle]}"'] \arrow[dr, phantom, "\circlearrowright"] & \delta_I[\langle 0,0\rangle] \wedge \beta \arrow[d, "{\nu[\langle 0,0,1,2\rangle]}"] \\
\delta_I[\langle 0,1\rangle] \wedge \beta[\langle 1,1,2\rangle] \arrow[r, "{\nu[\langle 0,1,1,2\rangle]}"'] & \beta
\end{tikzcd} \quad \text{in} \quad \mathcal{E}_{I\times I\times K}.$$

These diagrams can be combined into the desired diagram (2.3.6):

$$\begin{tikzcd}[column sep=large]
\delta_I[\langle 1,0\rangle] \wedge \beta \arrow[rr, "\pi_1"] \arrow[d, "{\langle\!\langle \sigma^{-1}\circ\pi_0, \pi_0, \pi_1\rangle\!\rangle}"] \arrow[ddd, bend right=60, "{\langle\!\langle \pi_0, \nu[\langle 1,0,1,2\rangle]\rangle\!\rangle}"', ""{name=L}] & (2.3.7) & \beta \arrow[d, "{\eta_I[\langle 0\rangle]\wedge\mathtt{id}_\beta}"] \arrow[dd, bend left=60, equal, "(2.3.5)"] \\
\delta_I[\langle 0,1\rangle] \wedge \delta_I[\langle 1,0\rangle] \wedge \beta \arrow[rr, "{\mu[\langle 0,1,0\rangle]\wedge\mathtt{id}_\beta}"] \arrow[d, "{\mathtt{id}_{\delta_I[\langle 0,1\rangle]}\wedge\nu[\langle 1,0,1,2\rangle]}"] \arrow[l, phantom, "="] & (2.3.8) & \delta_I[\langle 0,0\rangle] \wedge \beta \arrow[d, "{\nu[\langle 0,0,1,2\rangle]}"] \\
\delta_I[\langle 0,1\rangle] \wedge \beta[\langle 1,1,2\rangle] \arrow[rr, "{\nu[\langle 0,1,1,2\rangle]}"'] \arrow[d, "{\sigma\wedge\mathtt{id}_{\beta[\langle 1,1,2\rangle]}}"] & \equiv & \beta \\
\delta_I[\langle 1,0\rangle] \wedge \beta[\langle 1,1,2\rangle] \arrow[urr, bend right, "\lambda"'] & &
\end{tikzcd}$$

Consequently, we have shown that $\langle\!\langle \pi_0, \xi\rangle\!\rangle$ has a retraction. $\square$

Proof of Claim 2.3.18. In a cartesian equipment, we have the following two canonical isomorphisms:

$$\begin{tikzcd}
I\times I \arrow[r, "{\delta_{I\times I}[\mathtt{id}\,⨾\,\langle 0,0\rangle]}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\wr\|"] & I \arrow[d, equal] \\
I\times I \arrow[r, "{\delta_I[\langle 0\rangle\,⨾\,\mathtt{id}]\wedge\delta_I[\langle 1\rangle\,⨾\,\mathtt{id}]}"', "\shortmid"{marking}] & I
\end{tikzcd}, \qquad
\begin{tikzcd}
I \arrow[r, "\delta_I", "\shortmid"{marking}] \arrow[d, equal] & I \arrow[r, "\gamma", "\shortmid"{marking}] \arrow[d, phantom, "\wr\|"] & I\times K \arrow[d, equal] \\
I \arrow[rr, "\gamma"', "\shortmid"{marking}] & & I\times K
\end{tikzcd} \qquad (\gamma \text{ is arbitrary.})$$

The first isomorphism derives from the cartesian condition for virtual double categories with units, and the second isomorphism is the unitality of the horizontal composition. These lead to the following isomorphisms in $\mathcal{E}_{I\times I\times K}$:

$$\begin{aligned}
\sum_{\langle 0,0,1\rangle}\alpha &\cong \sum_{\langle 0,2,3\rangle}(\delta_{I\times I}[\langle\langle 0,2\rangle,\langle 1,1\rangle\rangle]\wedge\alpha[\langle 1,3\rangle]) && \text{by the presentation of } \sum_{\langle 0,0,1\rangle} \text{ in (2.3.3)} \\
&\cong \sum_{\langle 0,2,3\rangle}(\delta_I[\langle 0,1\rangle]\wedge\delta_I[\langle 1,2\rangle]\wedge\alpha[\langle 1,3\rangle]) && \text{by the first isomorphism above} \\
&\cong \delta_I[\langle 0,1\rangle]\wedge\alpha[\langle 1,2\rangle] && \text{by the second isomorphism above} \\
&\cong \sum_{\langle 0,0,1\rangle}\top_{I\times K}\wedge\alpha[\langle 1,2\rangle] && \text{by the Beck-Chevalley condition for } \varPhi_{=}.
\end{aligned}$$

By tracing the isomorphisms, one can see that this isomorphism is the desired one. $\square$

Remark 2.3.19. What we have proved is that the fibration $\mathfrak{p}$ is elementary existential when $\mathbb{B}\mathrm{il}(\mathfrak{p})$ has units and binary loose composition that are compatible with the cartesian structure. It is worth noting that both the units and the binary loose composition are crucial for each of the properties of the fibration. Neither the existence of units nor the existence of binary loose composition alone induces the properties of fibrations in question.

The proof of $\mathfrak{p}$ being a bifibration heavily relies on the fact that a double category is fibrational if and only if it is a bifibration, which never holds for a virtual double category with only units or only binary loose composition. The proofs for the Beck-Chevalley condition and the Frobenius reciprocity rely on the sandwich lemma, which is applied to cells from Beck-Chevalley pullbacks. The full structure of double categories is hence required in the proof. It seems this mutual dependence that makes the proof work, but we do not have a counterexample to show that the mutual dependence is necessary. ⌟

Remark 2.3.20. Owing to Remark 2.3.20, we will regard $\mathbb{B}\mathrm{il}(\mathfrak{p})$ as an equipment when $\mathfrak{p}$ is an elementary existential fibration in the following sections. ⌟

2.3.2. Regular Fibrations and Cartesian Equipments with Beck-Chevalley Pullbacks. We turn our attention to restricting the class of elementary existential fibrations and cartesian equipments to those for which the 2-functor $\mathbb{B}\mathrm{il}$ falls into a biequivalence. To this end, we take a closer look at the Beck-Chevalley conditions for fibrations and the Beck-Chevalley pullbacks in cartesian equipments.

Sharing the same name, the Beck-Chevalley conditions for fibrations and the Beck-Chevalley pullbacks in cartesian equipments express the same idea in principle. However, one should be aware that they behave slightly differently in parctice. We start by recalling some consequences of Beck-Chevalley pullbacks in cartesian equipments. Applying Lemmas 1.2.23 and 1.2.24, we have the following corollaries.

Corollary 2.3.21. For an elementary existential fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$, if a pullback square in $\mathcal{B}$ is a Beck-Chevalley pullback square in $\mathbb{B}\mathrm{il}(\mathfrak{p})$, then $\mathfrak{p}$ satisfies the Beck-Chevalley condition for the pullback square in both directions. $\lrcorner$

Proof. In the statement of Lemma 1.2.23, take M to be the terminal object in the double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$. Then the canonical cell σ in Lemma 1.2.23 reduces to the component of the canonical natural transformation in the definition of the Beck-Chevalley condition at α. □

Corollary 2.3.22. For an elementary existential fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$, if the pullback of $f \times \mathtt{id}_J\colon I \times J \to J \times J$ along the diagonal $\langle 0, 0\rangle\colon J \to J \times J$ gives a Beck-Chevalley pullback square in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ for f in $\mathcal{B}$, then $\mathfrak{p}$ satisfies the Frobenius reciprocity for f.

In particular, an elementary existential fibration $\mathfrak{p}$ with the Beck-Chevalley condition for all pullback squares in $\mathcal{B}$ satisfies the Frobenius reciprocity for all arrows in $\mathcal{B}$. $\lrcorner$

Proof. In the statement of Lemma 1.2.24, take K to be the terminal object in the double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$. □

Our starting point is Corollary 2.3.21, which states that the Beck-Chevalley condition in $\mathfrak{p}$ for a certain pullback square follows from the condition that the square is a Beck-Chevalley pullback in $\mathbb{B}\mathrm{il}(\mathfrak{p})$. On the other hand, unwinding the condition of the Beck-Chevalley pullback in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ in terms of the original fibration $\mathfrak{p}$, we obtain a different condition.

Lemma 2.3.23. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be an elementary existential fibration. A pullback square in $\mathcal{B}$ as left below is a Beck-Chevalley pullback in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ if $\mathfrak{p}$ satisfies the Beck-Chevalley condition for the square as right below, when we see it as an arrow from $\langle f, g\rangle$ to $\langle 0, 0\rangle$ in $\mathcal{B}^{\to}$.

$$\begin{tikzcd}
I \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner" very near start] & J \arrow[d, "h"] \\
K \arrow[r, "k"'] & L
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
I \arrow[r] \arrow[d, "{\langle f,g\rangle}"'] \arrow[dr, phantom, "\lrcorner" very near start] & L \arrow[d, "{\langle 0,0\rangle}"] \\
J \times K \arrow[r, "h\times k"'] & L \times L
\end{tikzcd}$$

$\lrcorner$

Proof. The original square is a Beck-Chevalley pullback in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ when the following canonical arrow is an isomorphism:

$$\textstyle\sum_{f\times g}(\delta_I) \xrightarrow{\cong} \sum_{\langle f,g\rangle} \top_I \longrightarrow \delta_L[k \times h]\ .$$

Since $\delta_L = \sum_{\langle 0,0\rangle} \top_L$, the above isomorphism is equivalent to the component at $\top_L$ of the canonical transformation for the Beck-Chevalley condition for the second square in the lemma. □

Corollary 2.3.24. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be an elementary existential fibration with $\mathcal{B}$ having finite limits. Then, the following are equivalent:

(i) $\mathfrak{p}$ is a regular fibration, that is, the Beck-Chevalley condition holds for all pullback squares in $\mathcal{B}$ (see Corollary 2.3.22).
(ii) $\mathbb{B}\mathrm{il}(\mathfrak{p})$ has the Beck-Chevalley pullbacks.

$\lrcorner$

Proof. The implication (i)⇒(ii) follows from Lemma 2.3.23, and the implication (ii)⇒(i) follows from Corollary 2.3.21. □

Remark 2.3.25. The author could not find how the class of pullback squares in $\mathcal{B}$ for which the Beck-Chevalley condition holds in $\mathfrak{p}$ corresponds to the class of the Beck-Chevalley pullbacks in $\mathbb{B}\mathrm{il}(\mathfrak{p})$ in general. Another delicate point is that the latter is closed under taking products but the former is not in general. The gradation of the pullback squares in $\mathcal{B}$ with respect to these two conditions should be investigated further. Not much is known about the subtleties of the Beck-Chevalley conditions but [**See83, Law15**] give detailed discussions on the related topics. ⌟

Example 2.3.26. Combining the classical results on the examples of fibrations in Example 2.2.13 and the results in this section, we have the following characterizations of the CFVDCs, which are mentioned in Example 1.4.8.

(i) For a category $\mathcal{B}$ with finite limits, the CFVDC $\mathbb{S}\mathrm{pan}(\mathcal{B})$ is a cartesian equipment with Beck-Chevalley pullbacks since $\mathcal{B}^{\to} \to \mathcal{B}$ is a regular fibration.
(ii) For a category $\mathcal{B}$ with finite limits, the CFVDC $\mathbb{R}\mathrm{el}(\mathcal{B})$ is a cartesian virtual equipment. It is a cartesian equipment if and only if $\mathcal{B}$ is a regular category. In this case, all pullbacks in $\mathcal{B}$ are Beck-Chevalley pullbacks in $\mathbb{R}\mathrm{el}(\mathcal{B})$.
(iii) For a cartesian monoidal category $\mathcal{B}$, the CFVDC $\mathcal{B}$-$\mathbb{M}\mathrm{at}$ is a cartesian equipment if and only if $\mathcal{B}$ has distributive small coproducts. In this case, all pullbacks in $\mathcal{B}$ are Beck-Chevalley pullbacks in $\mathcal{B}$-$\mathbb{M}\mathrm{at}$.

⌟

2.3.3. Frobenius Axiom and Recovering Fibrations from Cartesian Equipments. We now turn our attention to the problem of determining how close the 2-functor $\mathbb{B}\mathrm{il}$ is to a biequivalence. First, we focus on the essential image of this 2-functor. To this end, we introduce the Frobenius axiom for cartesian equipments, and show that it is a characteristic property of cartesian equipments of the form $\mathbb{B}\mathrm{il}(\mathfrak{p})$.

One important type of pullbacks is the following.

$$\begin{tikzcd}
 & I \arrow[dl, "{\langle 0,0 \rangle}"'] \arrow[dr, "{\langle 0,0 \rangle}"] & \\
I \times I \arrow[dr, "{\langle 0,0,1 \rangle}"'] & & I \times I \arrow[dl, "{\langle 0,1,1 \rangle}"] \\
 & I \times I \times I &
\end{tikzcd} \tag{2.3.9}$$

Using the notation $\varDelta = \langle 0, 0\rangle$, the above square being a Beck-Chevalley pullback means the canonical isomorphism $\varDelta^*\varDelta \cong (\varDelta \times \delta_I)_*(\delta_I \times \varDelta)^*$ and $\varDelta^*\varDelta \cong (\delta_I \times \varDelta)_*(\varDelta \times \delta_I)^*$ hold. In the context of cartesian bicategories, this condition is known as the Frobenius axiom in [**WW08**][5].

Definition 2.3.27. Let $\mathbb{D}$ be a cartesian equipment. An object I in $\mathbb{D}$ is said to be ***Frobenius*** if the pullback square (2.3.9) is a Beck-Chevalley pullback. A cartesian equipment is said to be ***Frobenius*** if every object in it is Frobenius. ⌟

Proposition 2.3.28. Let $\mathfrak{p} \colon \mathcal{E} \to \mathcal{B}$ be an elementary existential fibration. Then, the bilateral cartesian equipment $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is Frobenius. ⌟

Proof. To see that $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is Frobenius, by Lemma 2.3.23, it suffices to show that $\mathfrak{p}$ satisfies the Beck-Chevalley condition for the following square: (we omit the product symbols × and the commas "," in $\langle - \rangle$ for the sake of readability)

$$\begin{tikzcd}
I \arrow[r, "\langle 000 \rangle"] \arrow[d, "\langle 0000 \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & III \arrow[d, "\langle 012012 \rangle"] \\
IIII \arrow[r, "\langle 011223 \rangle"'] & IIIIII
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & II \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & III \arrow[d, "{\langle 0,1,2,0 \rangle}"] \\
II \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & III \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & IIII \arrow[d, "{\langle 0,1,2,3,1 \rangle}"] \\
III \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & IIII \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & IIIII \arrow[d, "{\langle 0,1,2,3,4,2 \rangle}"] \\
IIII \arrow[r, "{\langle 0,1,1,2,3 \rangle}"'] & IIIII \arrow[r, "{\langle 0,1,2,3,3,4 \rangle}"'] & IIIIII
\end{tikzcd} .$$

---

[5] This is called *discreteness* in [**CW87**].

However, this can be decomposed into the six pullback squares as above, where the right bottom square belongs to $\Phi_{=}$, and the other five squares are of the form presented in Corollary 2.2.8. Since the Beck-Chevalley conditions are closed under pasting of pullback squares, the square above is a Beck-Chevalley pullback. ☐

Remark 2.3.29. For any pushout square

$$\begin{tikzcd} N_1 & N_2 \arrow[l, "f"'] \\ N_3 \arrow[u, "g"] \arrow[ur, phantom, "\lrcorner"] & N_4 \arrow[l, "k"] \arrow[u, "h"'] \end{tikzcd}$$

in the category of finite sets with $N_1 + N_4 = N_2 + N_3$ as natural numbers, the pullback square

$$\begin{tikzcd} I^{N_1} \arrow[r, "I^f"] \arrow[d, "I^g"'] \arrow[dr, phantom, "\lrcorner"] & I^{N_2} \arrow[d, "I^h"] \\ I^{N_3} \arrow[r, "I^k"'] & I^{N_4} \end{tikzcd}$$

can be decomposed as in the proof of Proposition 2.3.28. ⌟

What makes the Frobenius axiom interesting is that it leads to the self-dual structure on the objects in the cartesian equipment. This was observed in [**CW87, WW08**] in the context of cartesian bicategories, and in [**HN23**] in the context of equipments[6]. To discuss further results on the self-duality in Frobenius cartesian equipments, we give a brief summary of the results in the paper[7].

Proposition 2.3.30. Suppose that $\mathbb{D}$ is a Frobenius cartesian equipment. For any object I in $\mathbb{D}$, let $\iota_I \colon 1 \to I \times I$ and $\epsilon_I \colon I \times I \to 1$ be defined by the following oprestrictions:

$$\begin{tikzcd} & I \arrow[dl, "!"'] \arrow[dr, "{\langle 0,0 \rangle}"] \arrow[d, phantom, "\mathsf{spn}"] & \\ 1 \arrow[rr, "\iota_I"', "|" marking] & {} & I \times I \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "{\langle 0,0 \rangle}"'] \arrow[dr, "!"] \arrow[d, phantom, "\mathsf{spn}"] & \\ I \times I \arrow[rr, "\epsilon_I"', "|" marking] & {} & 1 \end{tikzcd}$$

Then, the following hold:

(i) They come equipped with isomorphsms $\zeta_I \colon \delta_I \Rightarrow (\iota_I \times \delta_I)(\delta_I \times \epsilon_I)$ and $\theta_I \colon \delta_I \Rightarrow (\delta_I \times \iota_I)(\epsilon_I \times \delta_I)$.

(ii) These data extend to the functors $\iota, \epsilon \colon \mathbb{D}_0 \to \mathbb{D}_1$ such that $\mathsf{src} \circ \iota = \mathsf{tgt} \circ \epsilon = 1 \circ !$ and $\mathsf{tgt} \circ \iota = \mathsf{src} \circ \epsilon = \times \circ \Delta$.

(iii) Objects in $\mathbb{D}$ are self-dual in the sense of [**Sta16**, Definition 4.11], thus, $\mathbf{L}(\mathbb{D})$ is a compact closed bicategory.

(iv) The dagger structure induced from the above self-dual structure extends to the whole of $\mathbb{D}$ as a indentity-on-tight-parts double functor $(-)^\dagger \colon \mathbb{D}^{\mathsf{lop}} \to \mathbb{D}$.

(v) For a tight arrow $f \colon I \to J$ in $\mathbb{D}$, $(f_*)^\dagger \cong f^*$ canonically, in particular, $(\delta_I)^\dagger \cong \delta_I$.

(vi) Let $f \colon I_0 \to I_1$, $g \colon J_0 \to J_1$ be tight arrows in $\mathbb{D}$, and $\Big(\alpha_i \colon I_i \to J_i, \beta_i \colon I_i \times J_i \to 1, \gamma_i \colon J_i \to I_i\Big)$ be triples of loose arrows in $\mathbb{D}$ for $i = 0, 1$ where $\alpha_i, \beta_i, \gamma_i$ correspond to each other using $\iota, \epsilon, \zeta, \theta$. Then, we also have the bijective correspondence between the cells below:

$$\begin{tikzcd} I_0 \arrow[r, "\alpha_0", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & J_0 \arrow[d, "g"] \\ I_1 \arrow[r, "\alpha_1"', "|" marking] & J_1 \end{tikzcd} \Bigg\| \begin{tikzcd} I_0 \times J_0 \arrow[r, "\beta_0", "|" marking] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\beta"] & 1 \arrow[d, equal] \\ I_1 \times J_1 \arrow[r, "\beta_1"', "|" marking] & 1 \end{tikzcd} \Bigg\| \begin{tikzcd} J_0 \arrow[r, "\gamma_0", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\gamma"] & I_0 \arrow[d, "f"] \\ J_1 \arrow[r, "\gamma_1"', "|" marking] & I_1 \end{tikzcd} .$$

---

[6] In [**HN23**], the authors assume discreteness, which is a stronger condition than the Frobenius axiom, to obtain the self-dual structure, but the Frobenius axiom suffices for most of the results in the paper except for Lemma 3.1.16.

[7] We do not define what the self-dual structure is in this paper, because the author is not confident at the moment that a definition we have is of the correct generality. A tentative definition is given [**HN23**, Definition 3.1.11].

This correspondence is functorial with respect to tightwise composition. In other words, when we define the category $\mathcal{LS}(\mathbb{D})$ by the pullback

$$\begin{tikzcd}
\mathcal{LS}(\mathbb{D}) \arrow[r] \arrow[d, "\langle \mathsf{src}', \mathsf{tgt}' \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[d, "\langle \mathsf{src}, \mathsf{tgt} \rangle"] \\
\mathbb{D}_0 \times \mathbb{D}_0 \arrow[r, "(I,J)\mapsto(I\times J,1)"'] & \mathbb{D}_0 \times \mathbb{D}_0
\end{tikzcd},$$

the above correspondence gives the fibered equivalence $\mathcal{LS}(\mathbb{D}) \simeq \mathbb{D}_1$ over $\mathbb{D}_0 \times \mathbb{D}_0$.

⌟

Proof. Most of these are presented in [**HN23**, Section 3.1] except for (iii). The definition of the self-dual structure in *loc.cit.* lacks the last condition in [**Sta16**, Definition 4.11] [8], which is the following axiom called "swallowtail equation":

$$\begin{tikzcd}
 & I \times I \arrow[dr, "\iota_I \times \delta_I \times \delta_I"] \arrow[drr, bend left, "\delta_{I\times I}"] & \Downarrow\ \theta_I \times \delta_I & & & & \\
1 \arrow[ur, "\iota_I"] \arrow[dr, "\iota_I"'] & \wr\| & I \times I \times I \times I \arrow[r, "\delta_I \times \epsilon_I \times \delta_I"] & I \times I & = & 1 \arrow[r, "\iota_I"] & I \times I\ . \\
 & I \times I \arrow[ur, "\delta_I \times \delta_I \times \iota_I"'] \arrow[urr, bend right, "\delta_{I\times I}"'] & \Downarrow\ \delta_I \times \zeta_I^{-1} & & & &
\end{tikzcd}$$

This axiom is satisfied because all the 2-cells in the diagram are induced from the supine cells in the cartesian equipment. More precisely, since we have the supine cell whose bottom face is ι_I, it suffices to show the equation composed with the supine cell, which is the following: (again, we omit the product symbols $\times$ and the commas "," for the sake of readability)

$$\begin{tikzcd}
 & & & I \arrow[ddll, equal] \arrow[dr, "\langle 00 \rangle"] & & & \\
 & & & & II \arrow[ddll, equal] \arrow[ddrr, equal] & & \\
 & I \arrow[dl, "!"'] \arrow[dr, "\langle 00 \rangle"] & & & \theta_I\delta_I & & \\
1 \arrow[rr, "\mathtt{spn}", "\iota_I"'] & & II \arrow[rr, "\iota_I\delta_I\delta_I"'] & & IIII \arrow[rr, "\delta_I\epsilon_I\delta_I"'] & & II
\end{tikzcd}
=
\begin{tikzcd}
 & & & & I \arrow[ddll, equal] \arrow[dr, "\langle 00 \rangle"] & & & \\
 & & & & & II \arrow[ddll, equal] \arrow[ddrr, equal] & & \\
 & & I \arrow[dl, "!"'] \arrow[dr, "\langle 00 \rangle"] & & & \delta_I\zeta_I & & \\
 & 1 \arrow[rr, "\mathtt{spn}", "\iota_I"'] \arrow[drr, "\iota_I"'] & & II \arrow[rr, "\delta_I\delta_I\iota_I"] \arrow[d, phantom, "\wr\|"] & & IIII \arrow[rr, "\delta_I\epsilon_I\delta_I"'] & & II \\
 & & & IIII \arrow[urr, "\iota_I\delta_I\delta_I"'] & & & &
\end{tikzcd}\ .$$

Note that the square in the diagram above is a Beck-Chevalley pullback. The left-hand side of the equation can be computed as follows:

$$(\mathrm{LHS}) =
\begin{tikzcd}
 & & & I \arrow[dl, "\langle 00 \rangle"'] \arrow[dr, "\langle 00 \rangle"] & & & \\
 & & II \arrow[dl, "\langle 1 \rangle"'] \arrow[dr, "\langle 011 \rangle"] & & II \arrow[dl, "\langle 001 \rangle"'] \arrow[dr, "\langle 001 \rangle"] & & \\
 & I \arrow[dl, "!"'] \arrow[dr, "\langle 00 \rangle"] & & III \arrow[dl, "\langle 12 \rangle"'] \arrow[dr, "\langle 0012 \rangle"] & & III \arrow[dl, "\langle 0112 \rangle"'] \arrow[dr, "\langle 02 \rangle"] & \\
1 \arrow[rr, "\mathtt{spn}", "\iota_I"'] & & II \arrow[rr, "\mathtt{spn}", "\iota_I\delta_I\delta_I"'] & & IIII \arrow[rr, "\mathtt{spn}", "\delta_I\epsilon_I\delta_I"'] & & II
\end{tikzcd}$$

$$=
\begin{tikzcd}
 & & I \arrow[dl, "\langle 00 \rangle"'] \arrow[ddrr, "\langle 000 \rangle"] & & & \\
 & II \arrow[ddl, "!"'] \arrow[ddrr, "\langle 0011 \rangle"] & & & & \\
 & \mathtt{spn} & & & III \arrow[dl, "\langle 0112 \rangle"'] \arrow[dr, "\langle 02 \rangle"] & \\
1 \arrow[r, "\iota_I"'] & II \arrow[rr, "\iota_I\delta_I\delta_I"'] & & IIII \arrow[rr, "\mathtt{spn}", "\delta_I\epsilon_I\delta_I"'] & & II
\end{tikzcd},$$

---

[8] This was pointed out by Zeinab Galal in a private communication with Keisuke Hoshino and the author.

$$
= \begin{tikzcd}
 & & & I \arrow[dl, swap, "\langle 00\rangle"] \arrow[dr, "\langle 00\rangle"] & & & \\
 & & II \arrow[dl, swap, "\langle 0\rangle"] \arrow[dr, "\langle 001\rangle"] & & II \arrow[dl, swap, "\langle 011\rangle"] \arrow[dr, "\langle 011\rangle"] & & \\
 & I \arrow[dl, swap, "!"] \arrow[dr, "\langle 00\rangle"] & & III \arrow[dl, swap, "\langle 01\rangle"] \arrow[dr, "\langle 0122\rangle"] & & III \arrow[dl, swap, "\langle 0112\rangle"] \arrow[dr, "\langle 02\rangle"] & \\
1 \arrow[rr, "|" marking, "\mathtt{spn}", "\iota_I"'] \arrow[drr, "|" marking, "\iota_I"'] & & II \arrow[rr, "|" marking, "\mathtt{spn}", "\delta_I\delta_I\iota_I"'] \arrow[d, phantom, "\wr\|"] & & IIII \arrow[rr, "|" marking, "\mathtt{spn}", "\delta_I\epsilon_I\delta_I"'] & & II \\
 & & II \arrow[urr, "|" marking, "\iota_I\delta_I\delta_I"'] & & & &
\end{tikzcd} = \text{(RHS)},
$$

where all pullback squares are Beck-Chevalley pullbacks by the trivial reasons Lemma 1.2.22 and the Frobenius axiom so that we can use the sandwich lemma Lemma 1.2.12 iteratively to deduce that all triangles pointing upwards are supine cells. □

The paper [**HN23**] does not explicitly mention how the loose composition relates to the compact closed structure, although the connection to the dagger structure is discussed in (iv) of Proposition 2.3.30.

Proposition 2.3.31. Suppose that $\mathbb{D}$ is a Frobenius cartesian equipment, and let ι_I and ϵ_I be as in Proposition 2.3.30. For loose arrows $\beta\colon I\times J \relbar\joinrel\mapstochar\joinrel\rightarrow 1$ and $\beta'\colon J\times K \relbar\joinrel\mapstochar\joinrel\rightarrow 1$, let $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ and $\alpha'\colon J \relbar\joinrel\mapstochar\joinrel\rightarrow K$ be the corresponding loose arrows induced by ι, ϵ. Then, $\alpha\circ\alpha'$ is given by the following composite:

$$
I \xrightarrow{\delta_I\times\iota_K} I\times K\times K \xrightarrow{(\delta_I\times\iota_J\times\delta_K)\times\delta_K} I\times J\times J\times K\times K \xrightarrow{(\beta\times\beta')\times\delta_K} K \ . \tag{2.3.10}
$$

This assignment is functorial with respect to the tightwise composition. ⌟

Proof. The first statement follows from the general properties of compact closed bicategories. The second statement is a straightforward calculation, as in the proof of [**HN23**, Proposition 3.1.15]. □

Corollary 2.3.32. Suppose the same setting as in Proposition 2.3.31. When we define the category $\mathcal{LS}(\mathbb{D})$ as in Proposition 2.3.30, define the composition functor $\odot'\colon \mathcal{LS}(\mathbb{D})\times_{\mathbb{D}_0}\mathcal{LS}(\mathbb{D})\to\mathcal{LS}(\mathbb{D})$ by

$$
\Big(\beta\colon I\times J \relbar\joinrel\mapstochar\joinrel\rightarrow 1, \beta'\colon J\times K \relbar\joinrel\mapstochar\joinrel\rightarrow 1\Big) \mapsto\ I\times K \xrightarrow{\delta_I\times\iota_J\times\delta_K} I\times J\times J\times K \xrightarrow{\beta\times\beta'} 1\ .
$$

Then, this data together with $\langle \mathsf{src}', \mathsf{tgt}'\rangle$ gives rise to a double category $\mathbb{LS}(\mathbb{D})$:

$$
\begin{tikzcd}
\mathcal{LS}(\mathbb{D})\times_{\mathbb{D}_0}\mathcal{LS}(\mathbb{D}) \arrow[r, "\odot'"] & \mathcal{LS}(\mathbb{D}) \arrow[r, shift left=2, "\mathsf{src}'"] \arrow[r, shift right=2, "\mathsf{tgt}'"'] & \mathbb{D}_0 \arrow[l, "\epsilon" description]
\end{tikzcd} .
$$

The fibered equivalence in (vi) of Proposition 2.3.30 is lifted to an equivalence $\mathbb{LS}(\mathbb{D})\simeq\mathbb{D}$ as double categories, i.e., an equivalence in the 2-category **Dbl**. ⌟

Therefore, with the Frobenius axiom, the compact closed structure behaves well with respect to both the tight and loose compositions.

Definition 2.3.33. Let $\mathbb{D}$ be a cartesian equipment. We define a ***unilateral*** fibration to be a fibration $\mathfrak{uni}(\mathbb{D})\colon \mathcal{U}\mathrm{ni}(\mathbb{D})\to\mathbb{D}_0$ defined by the pullback

$$
\begin{tikzcd}
\mathcal{U}\mathrm{ni}(\mathbb{D}) \arrow[r] \arrow[d, "\mathfrak{uni}(\mathbb{D})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[d, "\langle\mathsf{src},\mathsf{tgt}\rangle"] \\
\mathbb{D}_0 \arrow[r, "I\mapsto(I,1)"'] & \mathbb{D}_0\times\mathbb{D}_0
\end{tikzcd} .
$$

⌟

Although we have defined the unilateral fibration for arbitrary cartesian equipments, the resulting fibration loses the information of the original equipment when the Frobenius axiom is not assumed. For instance, from the equipment $\mathbb{P}\mathrm{rof}$ of profunctors, we obtain the fibration of the presheaves over the category of categories, which no longer remembers all profunctors, in particular, copreshaves. Note that, with the Frobenius axiom, the fibration $\mathfrak{uni}(\mathbb{D})$ is equivalent to the pullback of $\mathcal{LS}(\mathbb{D})$ over $\mathbb{D}_0\times\mathbb{D}_0$ along the diagonal functor by construction. The extraordinary symmetric nature that the

Frobenius axiom gives to the objects enables the equipment to be reconstructed from only one side of the loose arrows and cells.

Proposition 2.3.34. Let $\mathbb{D}$ be a Frobenius cartesian equipment. Then, the unilateral fibration $\mathfrak{uni}(\mathbb{D})$ is a cartesian fibration, and its bilateral virtual double category $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D}))$ is equivalent to $\mathbb{L}\$(\mathbb{D})$, hence to $\mathbb{D}$. In particular, $\mathfrak{uni}(\mathbb{D})$ is an elementary existential fibration. ⌟

Proof. The fibration $\langle \mathsf{src}, \mathsf{tgt} \rangle$ is a cartesian fibration since $\mathbb{D}$ is a cartesian equipment. Since base change along finite-product-preserving functors preserves cartesian fibrations, the unilateral fibration $\mathfrak{uni}(\mathbb{D})$ is a cartesian fibration. For the second statement, $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D}))$ has $\mathbb{D}_0$ as its tight part, and a loose arrow $\alpha \colon I \nrightarrow J$ in $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D}))$ is given by a loose arrow $\alpha \colon I \times J \nrightarrow 1$ in $\mathbb{D}$ by construction, which is a loose arrow of $I \nrightarrow J$ in $\mathbb{L}\$(\mathbb{D})$. In the same way, the unary cells in $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D}))$ are in bijection with those in $\mathbb{L}\$(\mathbb{D})$. However, we do not know that $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D}))$ is a double category, requiring to check the correspondence of the general n-ary globular cells. See Lemma 1.3.8 for the condition for the equivalence of fibrational virtual double categories. We will only show this in the case of $n = 2$, and the general case is similar.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, phantom, "\tau"] & K \arrow[d, equal] \\
I \arrow[rr, "\gamma"', "\shortmid"{marking}] & & I
\end{tikzcd}$$

In $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D}))$, the above cell is given by a cell

$$\begin{tikzcd}
I \times J \times K \arrow[r, "\widehat{\alpha}[\langle 0,1 \rangle \fatsemi \mathtt{id}] \wedge \widehat{\beta}[\langle 1,2 \rangle \fatsemi \mathtt{id}]", "\shortmid"{marking}] \arrow[d, "\langle 0,2 \rangle"'] \arrow[dr, phantom, "\tau"] & 1 \arrow[d, equal] \\
I \times K \arrow[r, "\widehat{\gamma}"', "\shortmid"{marking}] & 1
\end{tikzcd}$$

where $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ are the corresponding loose arrows in $\mathbb{D}$ by the compact closed structure. Here, the loose arrow at the top is isomorphic to the restriction of $\widehat{\alpha} \times \widehat{\beta}$ along the tight arrow $\langle 0, 1, 1, 2 \rangle$. In other words, the cell above is equivalently given by the cell

$$\begin{tikzcd}
& I \times J \times K \arrow[r, "\langle 0,1,1,2 \rangle_*", "\shortmid"{marking}] \arrow[d, phantom, shift left=10, "\tau"] & I \times J \times J \times K \arrow[dr, "\widehat{\alpha} \times \widehat{\beta}", "\shortmid"{marking}] & \\
I \times K \arrow[ur, "\langle 0,2 \rangle^*", "\shortmid"{marking}] \arrow[rrr, "\widehat{\gamma}"', "\shortmid"{marking}] & & & 1
\end{tikzcd} ,$$

but the composite on the top row is exactly the composite of $\widehat{\alpha}$ and $\widehat{\beta}$ in $\mathbb{L}\$(\mathbb{D})$. Therefore, the virtual cells in $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D}))$ are in bijection with those in $\mathbb{L}\$(\mathbb{D})$.

The last statement follows from Theorem 2.3.14. ☐

The equivalence $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D})) \simeq \mathbb{D}$ in Proposition 2.3.34 is natural in $\mathbb{D}$ in the following sense.

Lemma 2.3.35. The assignment of $\mathfrak{uni}(\mathbb{D})$ to $\mathbb{D}$ gives rise to a 2-functor $\mathfrak{uni} \colon \mathbf{Eqp}_{\mathbf{Frob}} \to \mathbf{Fib}_{\times \wedge = \exists}$. ⌟

Proof. We have the 2-functor $\mathbf{Eqp}_{\mathbf{Frob}} \to \mathbf{BiFib}$ that sends an equipment $\mathbb{D}$ to the associated bifibration $\langle \mathsf{src}, \mathsf{tgt} \rangle \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$, and since the base change preserves bifibrations, we have the 2-functor $\mathfrak{uni} \colon \mathbf{Eqp} \to \mathbf{BiFib}$[9]. On $\mathbf{Eqp}_{\mathbf{Frob}}$, this factors through the 2-functor $\mathbf{Fib}_{\times \wedge = \exists} \to \mathbf{BiFib}$ by Proposition 2.3.34 on the level of 0-cells, and also as a 2-functor because $\mathbf{Fib}_{\times \wedge = \exists} \to \mathbf{BiFib}$ is fully faithful by Lemma 2.3.35. ☐

Proposition 2.3.36. The equivalence $\mathbb{B}\mathrm{il}(\mathfrak{uni}(\mathbb{D})) \simeq \mathbb{D}$ is pseudo-natural in $\mathbb{D}$. ⌟

Sketch of proof. Cartesian double functors preserve loose composition, finite-product structure, and (op)restrictions, and in particular, ι_I and ϵ_I as in Proposition 2.3.30 are preserved. This ensures that the equivalence constructed above is pseudo-natural because the data of the equivalence is determined by those structures. ☐

---

[9] This should be better explained in terms of 2-fibrations [Her99].

On the other hand, we have the canonical isomorphism $\mathfrak{uni}(\mathbb{B}\mathrm{il}(\mathfrak{p})) \cong \mathfrak{p}$ for any elementary existential fibration $\mathfrak{p}$ by the construction, and it is 2-natural in $\mathfrak{p}$. Combining the arguments above, we have the following corollary.

Corollary 2.3.37. The 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{Fib}_{\times\wedge=\exists} \to \mathbf{Eqp}_{\mathbf{cart}}$ is locally an equivalence, and the essential image of $\mathbb{B}\mathrm{il}$ up to equivalence is $\mathbf{Eqp}_{\mathbf{Frob}}$. The inverse 2-functor is given by $\mathfrak{uni}\colon \mathbf{Eqp}_{\mathbf{Frob}} \to \mathbf{Fib}_{\times\wedge=\exists}$. ⌟

It is worth mentioning that the Frobenius axiom is an instance of the Beck-Chevalley pullback condition, so the biequivalence restricts to give the following corollary.

Corollary 2.3.38. The 2-functor $\mathbb{B}\mathrm{il}\colon \mathbf{RegFib} \to \mathbf{Eqp}_{\mathbf{BC}}$ is a biequivalence, with the 2-functor $\mathfrak{uni}\colon \mathbf{Eqp}_{\mathbf{BC}} \to \mathbf{RegFib}$ as its inverse. ⌟

PROOF. Corollary 2.3.24 ensures that the biequivalence restricts to the full sub-2-categories of regular fibrations and cartesian equipments with Beck-Chevalley pullbacks. ☐

Remark 2.3.39. A similar result to these corollaries is presented in [**Law15**, §4.2.2]. What is called *regular fibrations* in [**Law15**] lies between our *elementary existential fibrations* and *regular fibrations*, at least *a priori*, as they are assumed to satisfy the Beck-Chevalley condition for *product-absolute* pullbacks. The equivalence in that paper is therefore another restriction of the equivalence in Corollary 2.3.37. ⌟

2.4. Comparison with other approaches

Having observed the interaction between fibrations and fibrational virtual double categories, we now compare these with other approaches that capture regular logic and its fragments. We focus on how far we can interpret formulae in regular logic or more general logical systems in these frameworks.

2.4.1. Regular Categories and Factorization Systems.

Models of algebraic theories can be taken in any category with finite products. An equation $s(x) \equiv t(x)$ in an algebraic theory is satisfied in a category $\mathcal{C}$ if the interpretations of s and t in $\mathcal{C}$ are equal. However, if we want to consider to what extent the equation holds in $\mathcal{C}$, that is, to determine the "subset" where the equation holds, we need more structure on $\mathcal{C}$. The easiest way to do this is to consider a category with finite limits. Importantly, we have equalizers in such a category, which offer a way to describe predicate with equality. Once we assume this structure, we can interpret formulae in cartesian logic ([**Joh02b**, D1]), generalized algebraic theories ([**Car86**]), or partial Horn logic ([**PV07**]), all of which have the same expressive power in terms of the categories of models (locally finitely presentable categories).

In the same vein, the minimal structure we need to interpret formulae in regular logic is regular categories ([**BGO71**]). Since the existential quantifier is interpreted using regular epimorphisms (or covers) and should be preserved by substitution, regular epimorphisms are required to be stable under pullbacks in the definition of regular category.

The above line of thought is based on the view that predicates should be interpreted as subobjects in a category, but there is no reason to restrict ourselves to this view. One motivation one might jump to a more general setting is to consider proof-relevant semantics: not only do we want to know whether a formula is true in a model, but we also want to know how it is true. Moreover, if one wants to take semantics in a quasitopos, for example, one sometimes needs to restrict the interpretation of formulae to strong subobjects, not general subobjects, depending on what kind of semantics one wants to take ([**Mon86**]). These considerations push us to consider more general structures than regular categories, and this is where orthogonal factorization systems come in.

Definition 2.4.1 ([**FK72**])**.** An ***orthogonal factorization system*** on a category $\mathcal{C}$ consists of a pair (E, M) of classes of arrows in $\mathcal{C}$ that satisfies the following conditions:

(i) E and M are closed under composition and contain isomorphisms.

(ii) Every arrow $e\colon X \to Y$ in E is left orthogonal to every arrow $m\colon A \to B$ in M; that is, every commutative square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, two heads, "e"'] & A \arrow[d, tail, "m"] \\
Y \arrow[r] \arrow[ur, dotted, "f"] & B
\end{tikzcd}$$

has a unique diagonal filler $f\colon Y \to A$ that makes two triangles commute.

(iii) Every arrow f in $\mathcal{C}$ factors as $f = e \fatsemi m$ where e belongs to E and m belongs to M.

E and M are called ***the left class*** and ***the right class*** of the factorization system, respectively. For a factorization of f as $f = e \fatsemi m$, we say m is the M-***image*** of f if $e \in \mathrm{E}$ and $m \in \mathrm{M}$. An orthogonal factorization system (E, M) is called ***stable*** if E is stable under pullback. ⌟

We see the arrows in the right class as the "subobjects" of the category. In place of regular epimorphisms, with which we can interpret the existential quantifier in the case of regular categories, we can use the left class of the factorization system and perform the same interpretation. The stability condition on the left class is a generalization of the stability condition on regular epimorphisms.

Having discussed the more primitive frameworks for logic, we now turn to see how these frameworks can be seen as special cases of fibrations and virtual double categories. The connection between orthogonal factorization systems and fibrations is well-known and explicitly discussed in [**HJ03**].

Definition 2.4.2. Let $\mathcal{B}$ be a category and (E, M) be an orthogonal factorization system on $\mathcal{B}$. Suppose that $\mathcal{B}$ admits pullbacks of arrows in M along arbitrary arrows. The fibration $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B}) \to \mathcal{B}$ is the full subfibration of the codomain fibration $\mathcal{B}^{\rightarrow} \to \mathcal{B}$ spanned by the arrows in M. Explicitly,

- the total category $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B})$ is the full subcategory of $\mathcal{B}^{\rightarrow}$ spanned by the arrows in M;
- the functor $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B}) \to \mathcal{B}$ is the codomain functor.

By abuse of notation, we denote the fibration $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B}) \to \mathcal{B}$ by $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B})$. ⌟

Lemma 2.4.3 ([HJ03, Lemma 2.8]). If the orthogonal factorization system (E, M) on $\mathcal{B}$ is stable, then the fibration $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B}) \to \mathcal{B}$ is a regular fibration. ⌟

Remark 2.4.4. Without the stability condition, the fibration $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B}) \to \mathcal{B}$ is not elementary nor existential in general because we require some sort of stability for the Frobenius reciprocity to hold. We could not find a precise condition when the fibration is elementary or existential. ⌟

Applying the $\mathbb{B}\mathrm{il}$ construction to this type of fibration, we obtain a virtual double category. For a stable orthogonal factorization system (E, M) on a category $\mathcal{B}$, in particular, we obtain a cartesian equipment due to Lemma 2.4.3 and Thm 2.3.17. The resulting double category is the same as $\mathbb{R}\mathrm{el}_{(\mathrm{E},\mathrm{M})}(\mathcal{B})$ in the paper [**HN23**]. Therein, relations relative to the factorization system are studied as loose arrows in the double category $\mathbb{R}\mathrm{el}_{(\mathrm{E},\mathrm{M})}(\mathcal{B})$. Prior to this paper, the category or the bicategory of relations relative to a factorization system was studied in many papers, such as [**Kle70**, **Kaw73**, **Kel91**, **Pav95**, **Mil00**, **HNTY22**].

A regular category gives rise to a typical example of a stable orthogonal factorization system by taking the class of monomorphisms Mono as the right class and the class of regular epimorphisms RegEpi as the left class. The resulting fibration $\mathcal{P}\mathrm{red}_{\mathrm{Mono}}(\mathcal{B}) \to \mathcal{B}$ is equivalent to the subobject fibration $\mathcal{S}\mathrm{ub}(\mathcal{B}) \to \mathcal{B}$, and the virtual double category $\mathbb{R}\mathrm{el}_{(\mathrm{Mono},\mathrm{RegEpi})}(\mathcal{B})$ is equivalent to $\mathbb{R}\mathrm{el}(\mathcal{B})$. The difference between the two is whether we consider isomorphism classes of monomorphisms (subobjects) or not.

Another example is a trivial factorization system on a category with finite limits whose right class includes all arrows and whose left class only includes isomorphisms. This leads to the codomain fibration $\mathcal{B}^{\rightarrow} \to \mathcal{B}$ over $\mathcal{B}$ as $\mathcal{P}\mathrm{red}_{\mathrm{All}}(\mathcal{B}) \to \mathcal{B}$, which in turn gives rise to the cartesian equipment of spans $\mathbb{S}\mathrm{pan}(\mathcal{B})$.

2.4.2. Allegories and Cartesian Bicategories.

The framework of bicategories as models of totalities of relations has been popular and developed in categorical logic. The bicategory of sets, relations, and inclusion orders is a prototypical example. The earlier use of bicategories in this context is as a metamorphosed version of regular categories, and the two frameworks are interchangeably used for different purposes. With double categories, on the other hand, we can capture how the functions and relations in regular logic behave in a single structure, not in two separate structures. This is a significant advantage of double categories over bicategories.

There have been several attempts to obtain an axiomatic (or algebraic) presentation of the structures of relations. The prototypical example is the bicategory of sets, relations, and inclusion orders. They are broadly classified into two kinds of approaches, with or without involution.

The first kind pays attention to the involutional nature of relations and incorporates it as a structure on the bicategory. The original source of this idea seems to date back to the work of [**Suc75**], but the monumental work of [**FS90**] is widely recognized[10]. Section A.3 of [**Joh02a**] provides a comprehensive summary of the theory of allegories. A seemingly similar approach is taken in the theory of ordered categories with involution ([**CGR84**]). In the paper, the authors introduced the notion of a correspondence category, which has a similar but weaker structure than a tabular allegory. Ordered categories with involution were later adopted as a setting for extended diagrammatic chasing, such as the snake lemma, in [**Lam99**].

The other kind is represented by the theory of cartesian bicategories [**CKS84, CW87, CKWW07, LWW10**]. The series of studies has achieved characterizations of the bicategory of relations and spans without referring to the involutive structure of relations. A link between these two approaches was partially established in [**Law15**], where unitary pre-tabular allegories were proved to be equivalent to bicategories of relations in the sense of [**CW87**] ([**Law15**, Proposition 2.2.33, Theorem 2.2.34]), which is in other words locally posetal Frobenius cartesian bicategories.

While regular categories or categories with stable factorization systems may be seen as small devices to create an elementary existential fibration or a cartesian equipment, those bicategorical structures may be seen as what one can obtain from the fibrations or the equipments when one forgets how functions behave in regular logic. From a regular category or a category with a stable factorization system, one can obtain the bicategory of relations relative to this structure as in [**HNTY22**] or as the loose part of the double category $\mathbb{R}\mathrm{el}_{(\mathrm{E},\mathrm{M})}(\mathcal{B})$ as in [**HN23**]. We can go further and create a bicategory from an elementary existential fibration by taking the loose part of the double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$ as in Section 2.3. Although one can redefine what functions are in these bicategories by considering functional relations, this process is not a perfect restoration because the original functions do not necessarily coincide with the redefined ones. The result by [**BSSS21**] clearly exhibits this difference. They constructed an adjunction between the category of elementary existential doctrines and the category of cartesian bicategories, whose counit is an isomorphism but whose unit is not; the only way to make it an isomorphism is to restrict the doctrines to the ones whose base categories are "recoverable" from the fiber structure[11]. Therefore, the double categories we could obtain from bicategories do not range over all the structures we can obtain from the fibrations but only over double categories whose tight arrows are "functional relations". The same situation is observed within the context of allegories and stable factorization systems in [**HNTY22**]. We will see this point in Subsection 2.5.3.

From this perspective, it would be interesting to know what kind of double categories we can obtain allegories or cartesian bicategories from. In the following discussion, we will write the composition of 1-cells in a bicategory in a diagrammatic way, as for the loose composition in a double category, and use the same notation for other operations in loose parts of a double category.

First, we recall the definition of an allegory, originally given in [**FS90**], but we follow the presentation in [**Joh02a**].

Definition 2.4.5 ([**Joh02a**, Definition A.3.2])**.** An ***allegory*** is a locally posetal bicategory $\mathbf{A}$ equipped with an involutive structure $(-)^\circ \colon \mathbf{A}^{\mathsf{op}} \to \mathbf{A}$ such that

(i) hom-posets $\mathbf{A}(I,J)$ have binary products (intersections) for any pair I, J of 0-cells and
(ii) (***modular law***) for any triple $\alpha \colon I \looparrowright J$, $\beta \colon J \looparrowright K$, and $\gamma \colon I \looparrowright K$ of 1-cells,

$$\alpha\beta \wedge \gamma \leq (\alpha \wedge \gamma\beta^\circ)\beta$$

holds in $\mathbf{A}$.

An allegory is called ***unital*** if it has an object 1 such that

(i) the identity 1-cells on 1 is the terminal (largest) element in $\mathbf{A}(1,1)$, and

---

[10] According to [**Bun17**], the original content of [**FS90**] had been already presented in the 1970s, in the unpublished paper "On Canonizing Category Theory", or "On Functorializing Model Theory" by Freyd in 1974.

[11] Technically, the condition is that the doctrines satisfy the Axiom of Unique Choice and have comprehensive diagonals.

(ii) for any 0-cell I, there is a 1-cell $\phi\colon I \to 1$ with $\delta_I \leq \phi\phi^\circ$.

An allegory is called ***tabular*** if, for any 1-cell $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$, there is a pair of maps (i.e., left adjoints) $f\colon K \relbar\joinrel\mapstochar\joinrel\rightarrow I$ and $g\colon K \relbar\joinrel\mapstochar\joinrel\rightarrow J$ such that

$$\alpha = f^\circ g, \quad \text{and} \quad ff^\circ \wedge gg^\circ = \delta_K.$$

⌟

It is necessary for the double category to be locally posetal to be an allegory. The only observation we could make is that a locally posetal double category with Beck-Chevalley pullbacks and strong tabulators satisfies the modular law ([HN23, Proposition 3.1.10, Remark 3.1.10]), as well as the other conditions. Note that if f and g give a tabulator for α in a locally posetal double category, then f and g are jointly monic, and hence jointly an inclusion. This leads to the second equality in the definition of a tabular allegory. As already mentioned, [Law15] presented a characterization of unitary pre-tabular allegories in terms of cartesian bicategories, so we can expect more connections after the following link between cartesian bicategories and double categories.

Let us now turn to cartesian bicategories. We adopt the refined definition in [CKWW07], which is more general than the original definition in [CW87] in that it no longer requires the bicategory to be locally posetal.

Definition 2.4.6. For a bicategory $\mathbf{B}$, $\mathbf{Map}(B)$ is the locally full sub-bicategory of $\mathbf{B}$ spanned by the left adjoint 1-cells. We call it the ***map bicategory*** of $\mathbf{B}$[12]. By abuse of notation, we write $\mathbf{Map}(\mathbb{D})$ for $\mathbf{Map}(\mathbf{L}(\mathbb{D}))$ where $\mathbb{D}$ is a double category. ⌟

We write 1-cells as $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ as if they were loose arrows. For a map $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ in $\mathbf{Map}(B)$, we write $\alpha^*\colon J \relbar\joinrel\mapstochar\joinrel\rightarrow I$ for the right adjoint of α.

Definition 2.4.7. A bicategory $\mathbf{B}$ is a ***cartesian bicategory*** if

(i) each hom-category $\mathbf{B}(I, J)$ has finite products for any pair of objects I, J in $\mathbf{B}$,

(ii) $\mathbf{Map}(\mathbf{B})$ has finite products in the sense of bilimits, that is,
  (a) there is an object 1 such that for every object K in $\mathbf{B}$, $\mathbf{Map}(\mathbf{B})(K, 1)$ is equivalent to the terminal category,
  (b) for every pair of objects I, J in $\mathbf{B}$, there is an object $I \times J$ in $\mathbf{B}$ such that for every object K in $\mathbf{B}$, $\mathbf{Map}(\mathbf{B})(K, I \times J)$ is equivalent to the product category $\mathbf{Map}(\mathbf{B})(K, I) \times \mathbf{Map}(\mathbf{B})(K, J)$.

(iii) The lax functors $1\colon \mathbf{1} \to \mathbf{B}$ and $\times\colon \mathbf{B} \times \mathbf{B} \to \mathbf{B}$ induced by the terminal object and the product object in the method described in [CKWW07] are pseudo functors.

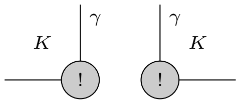
⌟

For cartesian bicategories, we found a quite surprising result.

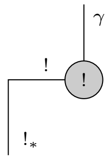
Theorem 2.4.8. The loose bicategory $\mathbf{L}(\mathbb{D})$ of a cartesian equipment $\mathbb{D}$ is a cartesian bicategory[13]. ⌟

The proof heavily uses the properties of companions and conjoints in an equipment, so we will not make a note every time we use them.

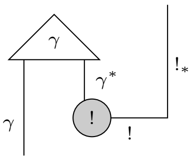
Proof. (i) is clear from the definition of a cartesian equipment. We prove (a) in (ii). Since we have an obvious object $!_*$ in $\mathbf{Map}(\mathbb{D})(K, 1)$ for any object K in $\mathbb{D}$, we can define a functor $\mathbf{1} \to \mathbf{Map}(\mathbb{D})(K, 1)$ by sending the unique object of $\mathbf{1}$ to $!_*$. There is a unique cell from $!_*$ to $!_*$ because of the universal property of 1, which implies that the functor is fully faithful. We prove that it is essentially surjective. Given a map $\gamma\colon K \relbar\joinrel\mapstochar\joinrel\rightarrow 1$ in $\mathbf{Map}(\mathbb{D})(K, 1)$, then we have the unique cells ! from γ and γ^* into δ_1, as shown on the left below. Define λ and ρ as follows.



$\lambda :=$



, $\rho :=$



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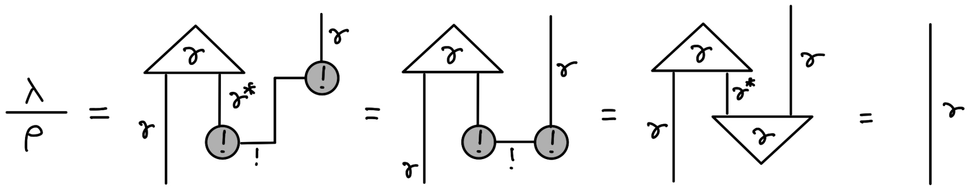
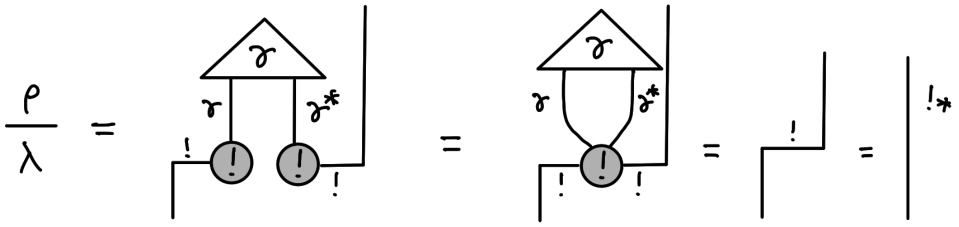
[12] This is because a left adjoint 1-cell is often called a *map*.

[13] In the paper [Lam22], the author gave a proof of this proposition in the local posetal case, but it has an error in that in the last line of the proof, the author applies the universal property of the binary product to loose arrows, which is not valid in general. Indeed, the bicategory of relations on sets has disjoint unions as the binary product, not the same as a category of functions. Our proof follows the basic idea of the proof in the paper, taking care of this point, and generalizes it to the non-locally posetal case.

Then we can deduce that they are mutually inverse by the following calculation.





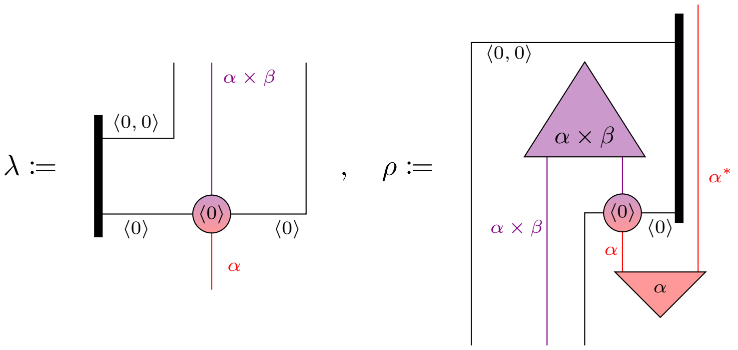
This implies that γ is isomorphic to $!_*$. Here, the triangle represent the unit of the adjunction $\gamma \vdash \gamma^*$.

Next, we prove (b) in (ii). For a triple of objects I, J, K in $\mathbb{D}$, we define two functors $\varPhi$ and $\varPsi$ as follows.

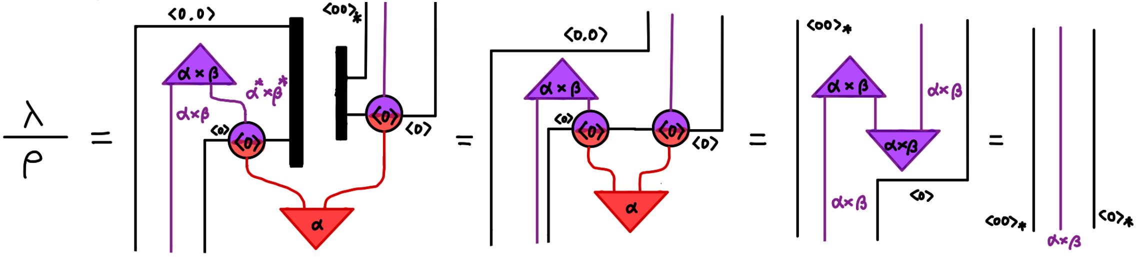
$$\mathbf{Map}(\mathbb{D})(K,I) \times \mathbf{Map}(\mathbb{D})(K,J) \underset{\varPsi}{\overset{\varPhi}{\rightleftarrows}} \mathbf{Map}(\mathbb{D})(K, I \times J) \ , \quad \begin{aligned} (\alpha,\beta) &\overset{\varPhi}{\mapsto} \langle 0,0\rangle_*(\alpha\times\beta), \\ \gamma &\overset{\varPsi}{\mapsto} (\langle 0\rangle_*(\gamma), \langle 1\rangle_*(\gamma)). \end{aligned}$$

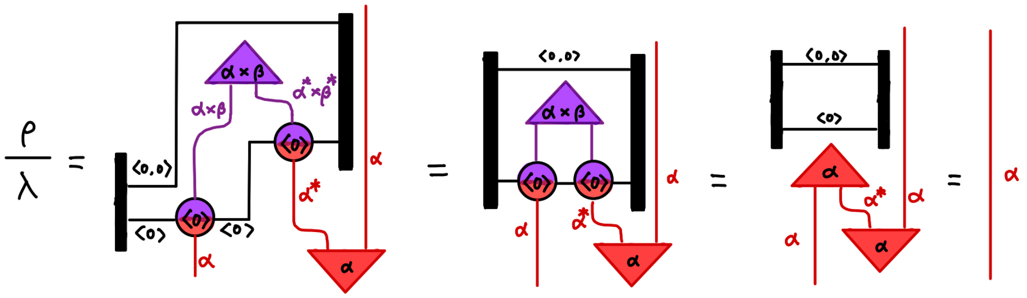
Note that composites of left adjoints are also left adjoints. We prove that $\varPhi$ and $\varPsi$ give an equivalence of categories. For one direction, we prove that $\varPsi \circ \varPhi$ is naturally isomorphic to the identity. We only show this only on the first component of the product.

Suppose we are given a pair of maps $\alpha\colon K \twoheadrightarrow I$ and $\beta\colon K \twoheadrightarrow J$ in $\mathbb{D}$. Let λ and ρ be defined as follows[14].



Then, λ and ρ are mutually inverse because of the following calculation.





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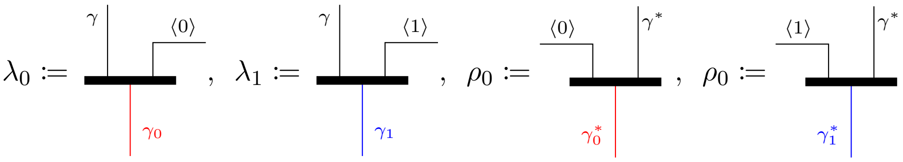
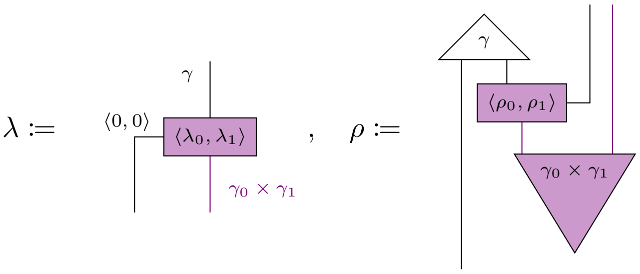
[14] Since we run out of letters, we reuse λ and ρ for different cells.

By this and the dual argument, we have that $\Psi(\Phi(\alpha, \beta))$ is isomorphic to (α, β).

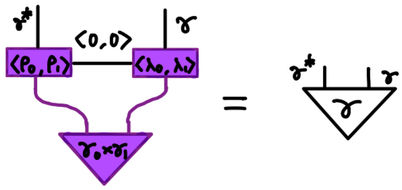
For the other direction, we prove that $\Phi \circ \Psi$ is naturally isomorphic to the identity. Suppose we are given a map $\gamma \colon K \twoheadrightarrow I \times J$ in $\mathbb{D}$. Let λ_i and ρ_i be defined as follows for $i = 0, 1$, where $\gamma_0 \coloneqq \gamma\langle 0\rangle_*$ and $\gamma_1 \coloneqq \gamma\langle 1\rangle_*$, and thus, γ_i^* can be taken as $\langle i\rangle^*\gamma^*$ for $i = 0, 1$.



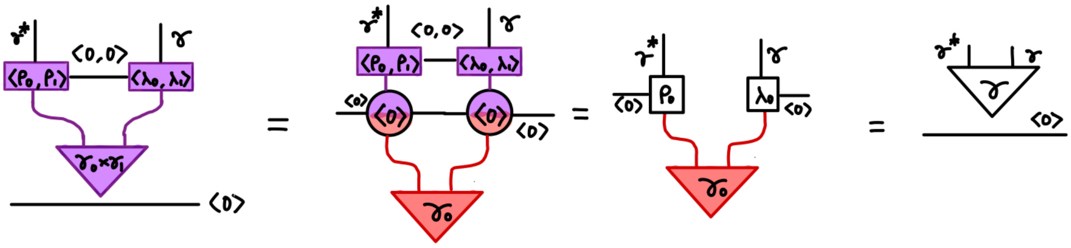
Using the universal property of the binary product in $\mathbb{D}$, we have the pairings $\langle\lambda_0, \lambda_1\rangle$ and $\langle\rho_0, \rho_1\rangle$. Then, we can define λ and ρ as follows.



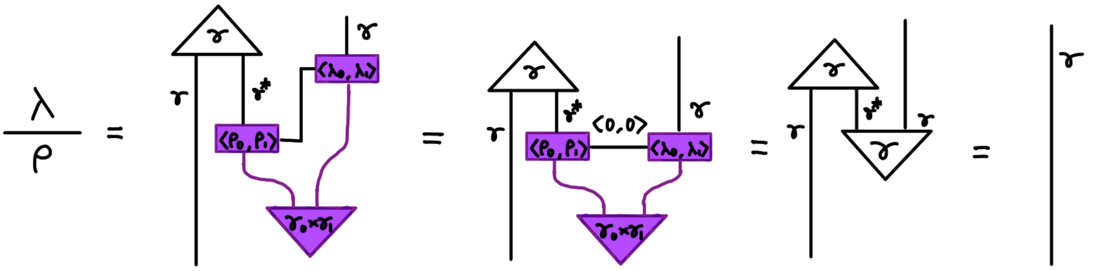
Before showing that λ and ρ are mutually inverse, we observe the following equality:



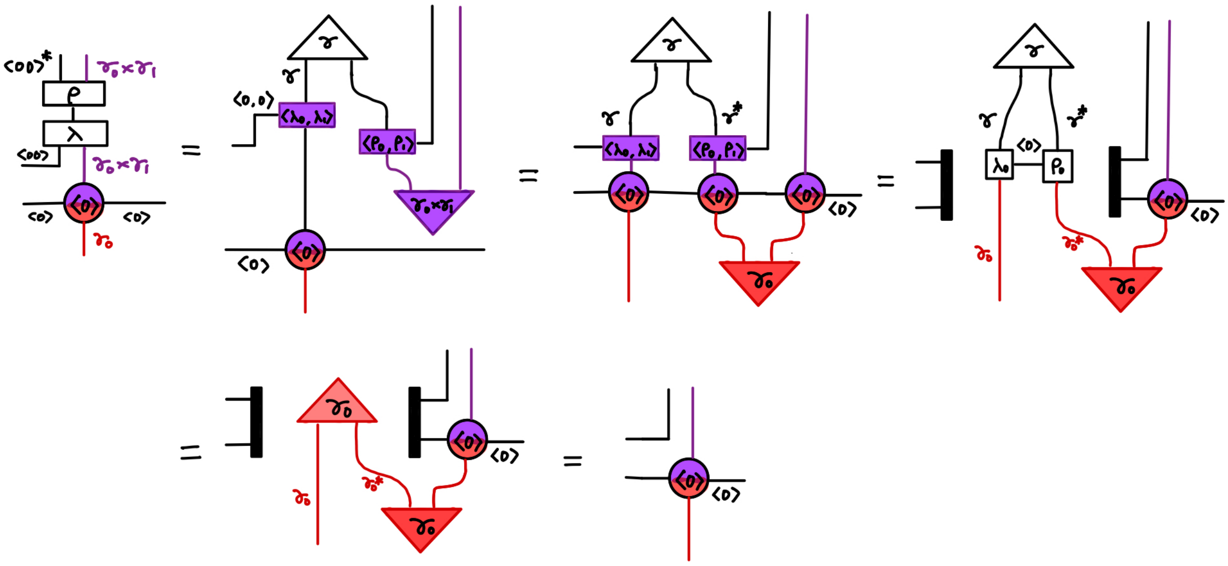
This follows from that the both sides postcomposed by the projections give the same result as shown below.



Then, with this equality, we can prove one direction of the mutual invertibility of λ and ρ.



Using the universal property of the binary product in $\mathbb{D}$ again, the other direction follows from the following equality and its dual.



This implies that $\varPhi(\varPsi(\gamma))$ is isomorphic to γ.

Finally, we prove (iii). The lax functors $1\colon \mathbf{1} \to \mathbf{L}(\mathbb{D})$ and $\times\colon \mathbf{L}(\mathbb{D}) \times \mathbf{L}(\mathbb{D}) \to \mathbf{L}(\mathbb{D})$ induced by the finite products in $\mathbf{Map}(\mathbb{D})$ are in fact the same as the loose parts of the double functors $1\colon \mathbb{1} \to \mathbb{D}$ and $\times\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$. This is because $\times\colon \mathbf{L}(\mathbb{D}) \times \mathbf{L}(\mathbb{D}) \to \mathbf{L}(\mathbb{D})$ induced by the finite products in $\mathbf{Map}(\mathbb{D})$ sends a pair of loose arrows (α, β) to $(\langle 0\rangle_* \alpha \langle 0\rangle^*) \wedge (\langle 1\rangle_* \beta \langle 1\rangle^*)$, which in a certain equipment is isomorphic to $\alpha \times \beta$. The laxity cells are also confirmed to be the same as the cells derived from the universal properties of the binary product in the cartesian equipment $\mathbb{D}$. Therefore, the lax functors 1 and $\times$ are pseudo since $\mathbb{D}$ is a cartesian equipment. □

2.4.3. Relational Doctrines. Dagnino and Pasquali introduced the notion of relational doctrines in [**DP23**, **DP24a**]. This is the closest notion to fibrational virtual double categories for regular logic and its fragments. The primary idea is to take a fibration over the product category $\mathcal{B} \times \mathcal{B}$ for a category $\mathcal{B}$ and regard the fibers over a pair (I, J) as the poset of binary predicates between I and J.

Definition 2.4.9 ([**DP23**, Definition 1])**.** A ***relational doctrine*** consists of

- a category $\mathcal{B}$;
- a functor $R\colon \mathcal{B}^{\mathsf{op}} \times \mathcal{B}^{\mathsf{op}} \to \mathcal{P}os$ where $\mathcal{P}os$ is the category of posets and monotone functions (for $s\colon I \to I'$, $t\colon J \to J'$, and $\alpha \in R(I', J')$, we write $\alpha[(s,t)]$ for $R(s,t)(\alpha)$);
- an element $\delta_I \in R(I, I)$ for each object I in $\mathcal{B}$ such that for any arrow $s\colon I \to J$, $\delta_I \leq \delta_J[(s,s)]$;
- a monotone function $- \odot_{I,J,K} -\colon R(I,J) \times R(J,K) \to R(I,K)$ for each triple of objects I, J, K in $\mathcal{B}$ such that for any arrows $s\colon I \to I'$, $t\colon J \to J'$, $u\colon K \to K'$, $\alpha \in R(I', J')$, and $\beta \in R(J', K')$, $\alpha[(s,t)] \odot \beta[(t,u)] \leq (\alpha \odot \beta)\,[(s,u)]$[15];
- a monotone function $(-)^{\dagger_{I,J}}\colon R(I,J) \to R(J,I)$ for each pair of objects I, J in $\mathcal{B}$ such that for any arrow $s\colon I \to I'$, $t\colon J \to J'$, and $\alpha \in R(I', J')$, $(\alpha[(s,t)])^\dagger \leq \alpha^\dagger[(t,s)]$[16];

satisfying the following equations for any I, J, K, L in $\mathcal{B}$, $\alpha \in R(I,J)$, $\beta \in R(J,K)$, and $\gamma \in R(K,L)$:

$$\alpha \odot (\beta \odot \gamma) = (\alpha \odot \beta) \odot \gamma, \qquad \delta_I \odot \alpha = \alpha, \qquad \alpha \odot \delta_J = \alpha,$$
$$(\alpha \odot \beta)^\dagger = \beta^\dagger \odot \alpha^\dagger, \qquad \delta_I^\dagger = \delta_I, \qquad (\alpha^\dagger)^\dagger = \alpha.$$

⌟

As pointed out in the conclusion of [**DP23**], relational doctrines can be naturally seen as double categories.

Proposition 2.4.10. A relational doctrine $(\mathcal{B}, R)$ bijectively corresponds to a locally posetal equipment $\mathbb{R}$ with a dagger structure, that is, a double functor $(-)^\dagger\colon \mathbb{R}^{\mathsf{lop}} \to \mathbb{R}$ that agrees with the identity on the tight part $\mathbb{R}_{\mathrm{t}}$ and $(-)^{\dagger\dagger} = \mathtt{id}$ as a double functor. ⌟

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[15] the subscripts such as I, J, K in $\odot_{I,J,K}$ are omitted when there is no confusion.

[16] the subscripts such as I, J in $\dagger_{I,J}$ are omitted when there is no confusion.

It should be noted that an involution structure on equipments is mentioned in [**Shu08**, §10].

Proof. The existence of a cell framed by the quadruple $s\colon I \to I'$, $t\colon J \to J'$, $\alpha \in R(I', J')$, and $\beta \in R(I, J)$ in a locally posetal equipment is equivalent to the order relation $\beta \leq \alpha[s \mathbin{⨟} t]$. This correspondence leads to the conclusion. Note that the local posetality of the equipment makes the restriction strictly functorial and composition strictly associative and unital. ☐

Let us give another perspective on relational doctrines. Recall that a strict double category is an internal category in the category of categories, which means that it is a monoid in the double category $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ of categories and spans of functors in the sense of [**CS10**].

Definition 2.4.11. Let $\mathbb{K}$ be a double category with a dagger structure $(-)^\dagger$. A ***symmetric monoid*** in $\mathbb{K}$ is a monoid $(I, \alpha\colon I \to\!\!\!\!\!\!\mapsto I, \eta\colon \delta_I \Rightarrow \alpha, \mu\colon \alpha \odot \alpha \Rightarrow \alpha)$ in $\mathbb{K}$ in the sense of [**CS10**] together with a globular cell $\sigma\colon \alpha \Rightarrow \alpha^\dagger$ such that the following equations hold:

$$\begin{tikzcd} & I \arrow[dr, "\alpha"] & \\ I \arrow[ur, "\alpha"] \arrow[rr, "\alpha"{description}] \arrow[rr, bend right=50, "\alpha^\dagger"'] \arrow[rr, phantom, shift left=3ex, "\mu"] \arrow[rr, phantom, shift right=3ex, "\sigma"] & & I \end{tikzcd} = \begin{tikzcd} & I \arrow[dr, bend left, "\alpha"] \arrow[dr, bend right, "\alpha^\dagger"'] \arrow[d, phantom, "\Uparrow"] & \\ I \arrow[ur, bend left, "\alpha"] \arrow[ur, bend right, "\alpha^\dagger"'] \arrow[rr, bend right=50, "\alpha^\dagger"'] \arrow[ur, phantom, "\sigma"] \arrow[rr, phantom, shift right=3ex, "\mu^\dagger"] & {} & I \arrow[ul, phantom, "\sigma"] \end{tikzcd}, \quad \begin{tikzcd} I \arrow[r, bend left=60, "\delta_I"] \arrow[r, "\alpha"{description}] \arrow[r, bend right=60, "\alpha^\dagger"'] \arrow[r, phantom, shift left=3ex, "\eta"] \arrow[r, phantom, shift right=3ex, "\sigma"] & I \end{tikzcd} = \begin{tikzcd} I \arrow[r, bend left, "\delta_I"] \arrow[r, bend right, "\alpha^\dagger"'] \arrow[r, phantom, "\eta^\dagger"] & I \end{tikzcd} \quad \begin{tikzcd} I \arrow[r, bend left=60, "\alpha"] \arrow[r, "\alpha^\dagger"{description}] \arrow[r, bend right=60, "\alpha"'] \arrow[r, phantom, shift left=3ex, "\sigma"] \arrow[r, phantom, shift right=3ex, "\sigma^\dagger"] & I \end{tikzcd} = \begin{tikzcd} I \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha"'] \arrow[r, phantom, "\|"] & I \end{tikzcd}$$

⌟

A double category with a loosewise dagger structure is, for instance, a symmetric monoid in $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$, where the dagger structure on $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ is defined by taking the opposite span. Similarly to the case of mere monoids, the symmetric monoids in a double category with a dagger structure form a double category with a dagger structure.

On the other hand, we can consider the double category whose objects are (small) categories, tight arrows are functors, and loose arrows from $\mathcal{B}$ to $\mathcal{B}'$ are "contravariant $\mathcal{P}os$-valued matrices", meaning that functors of the form $\mathcal{B}^{\mathsf{op}} \times \mathcal{B}'^{\mathsf{op}} \to \mathcal{P}os$. Cells in this double category are defined as oplax natural transformations:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "T"] \arrow[d, "F"'] \arrow[dr, phantom, "\tau"] & \mathcal{B}' \arrow[d, "F'"] \\ \mathcal{C} \arrow[r, "S"'] & \mathcal{C}' \end{tikzcd} \quad \Bigg\| \quad \begin{tikzcd} \mathcal{B}^{\mathsf{op}} \times \mathcal{B}'^{\mathsf{op}} \arrow[d, "F^{\mathsf{op}} \times F'^{\mathsf{op}}"'] \arrow[dr, "T"] \arrow[dr, phantom, shift right=3ex, "\tau \Downarrow_{oplax}"] & \\ \mathcal{C}^{\mathsf{op}} \times \mathcal{C}'^{\mathsf{op}} \arrow[r, "S"'] & \mathcal{P}os \end{tikzcd} \quad \Bigg\| \quad \begin{array}{c} \Big(\tau_{b,b'}: T(b, b') \to S(F(b), F'(b'))\Big)_{b,b'} \\ \text{with } \begin{tikzcd} T(b, b') \arrow[r, "\tau_{b,b'}"] \arrow[d, "{(-)[(s,s')]}"'] \arrow[dr, phantom, "\not\swarrow"] & S(F(b), F'(b')) \arrow[d, "{(-)[(F(s),F'(s'))]}"] \\ T(a, a') \arrow[r, "\tau_{a,a'}"'] & S(F(a), F'(a')) \end{tikzcd} \end{array}$$

The composition is defined as the composition of matrices using the finite products and coproducts in $\mathcal{P}os$[17]. Let us denote this double category by $\mathbb{M}$ for a moment. This double category naturally has a dagger structure, which is defined by taking the transpose of a matrix. Then, we observe the following.

Proposition 2.4.12. Symmetric monoids in the double category $\mathbb{M}$ are precisely the relational doctrines (on small categories). ⌟

There is a double functor $\mathbb{M} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$ that is bijective on tight part and sends a $\mathcal{P}os$-valued matrix to its Grothendieck construction. That is, a loose arrow $T\colon \mathcal{B} \times \mathcal{B}' \to \mathcal{P}os$ is sent to a span $\mathcal{B} \longleftarrow \mathcal{T} \to \mathcal{B}'$ in $\mathbb{S}\mathrm{pan}(\mathbf{Cat})$ where $\mathcal{T}$ is the category whose objects are the triples (I, J, α) with I in $\mathcal{B}$, J in $\mathcal{B}'$, and $\alpha \in T(I, J)$, and whose arrows $(I, J, \alpha) \to (I', J', \alpha')$ are the pairs (s, t) with $s\colon I \to I'$ in $\mathcal{B}$ and $t\colon J \to J'$ in $\mathcal{B}'$ such that $T(s, t)(\alpha) \leq \alpha'$. This double functor preserves the dagger structure, and therefore, it induces the double functor from the double category of relational doctrines to the double category of double categories with a dagger structure.

Note that the double functor $\mathbb{M} \to \mathbb{S}\mathrm{pan}(\mathbf{Cat})$ is loosewise fully faithful, and the essential image consists of the spans $(L\colon \mathcal{E} \to \mathcal{B}, R\colon \mathcal{E} \to \mathcal{B}')$ such that the pairing $\langle L, R\rangle\colon \mathcal{E} \to \mathcal{B} \times \mathcal{B}'$ is a split fibration. This observation suggests a potential generalization of relational doctrines to *relational fibrations*. Specifically, a relational fibration may be defined as a symmetric pseudo-monoid in the

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[17] We may encounter size issues when we consider non-small categories. One way to get through this is to define the notion of unital virtual double categories with a dagger structure and symmetric monoids in them. Since we do not go further in this direction in this paper, we do not elaborate on this point.

intercategory of spans of categories that are jointly fibrations over the product category. Although this generalization itself seems interesting, it would be rather convenient to use the language of double categories because the notion of relational fibrations should be equivalent to equipments with a dagger structure eventually.

2.5. Translation of Properties

2.5.1. Predicate Comprehension and Tabulators. In set theory, the comprehension axiom states that for any set I and any predicate $\alpha(x)$, there exists a subset $\{\alpha\}$ of I such that

$$\forall x : I.\ \alpha(x) \iff x \in \{\alpha\}$$

The following definition is a categorical reformulation of this axiom.

Definition 2.5.1. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a fibration with fiberwise terminal objects. For an object α in $\mathcal{E}_I$, a **predicate comprehension** of α is a terminal object in the comma category $\top_{-} \downarrow \alpha$, where $\top_{-}$ is the functor that sends an object I to the terminal object in $\mathcal{E}_I$. We say that $\mathfrak{p}$ has **predicate comprehension** if every object in $\mathcal{E}$ has a predicate comprehension, or equivalently, if the functor $\top_{-}$ has a right adjoint. ⌟

The statement that α has a predicate comprehension is unwound as follows. Let $\tau_\alpha\colon \top_{\{\alpha\}} \to \alpha$ be a terminal object in $\top_{-} \downarrow \alpha$, and let $c_\alpha\colon \{\alpha\} \to I$ be its image under $\mathfrak{p}$. Then, for any object J in $\mathcal{B}$ with an arrow $\varphi\colon \top_J \to \alpha$, there exists a unique arrow u such that $\tau_\alpha \circ \top_u = \varphi$. Using the internal language of fibrations, the data of τ_α and c_α can be encoded as a term $\mathsf{x} : \{\!|\boldsymbol{\alpha}|\!\} \vdash \mathsf{c}(\mathsf{x}) : \boldsymbol{I}$ and a proof of $\mathsf{x} : \{\!|\boldsymbol{\alpha}|\!\} \mid \top \vdash \boldsymbol{\alpha}(\mathsf{c}(\mathsf{x}))$. The statement above is then translated into the following. For any tuple of a type J, a term $\mathsf{y} : \mathsf{J} \vdash \mathsf{v}(\mathsf{y}) : \boldsymbol{I}$, and a proof of $\mathsf{y} : \mathsf{J} \mid \top \vdash \boldsymbol{\alpha}(\mathsf{v}(\mathsf{y}))$, there uniquely exists a term $\mathsf{y} : \mathsf{J} \vdash \mathsf{u}(\mathsf{y}) : \{\!|\boldsymbol{\alpha}|\!\}$ such that $\mathsf{v}(\mathsf{y}) = \mathsf{c}(\mathsf{u}(\mathsf{y}))$ and the proof of $\boldsymbol{\alpha}(\mathsf{c}(\mathsf{x}))$ for x replaced by $\mathsf{u}(\mathsf{y})$ is the same as the given proof. If one takes J to be the terminal type $\mathsf{1}$ and goes down to proof irrelevance, the above statement looks quite similar to the comprehension axiom in set theory.

The predicate comprehension is called the *subset (type)* in [**Jac99**, Definition 4.6.1], and just *comprehension* in [**MR13a**, §4]. It should be noted that this notion is a special case of what is called *comprehension structures* in several contexts, such as in [**MR20**], where a comprehension structure with section is defined as a section of the fibration $\mathfrak{p}$, not necessarily terminal, having a right adjoint.

Definition 2.5.2. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a fibration with fiberwise terminal objects and predicate comprehension. We say that $\mathfrak{p}$ has **full predicate comprehension** if the functor $c_{-}\colon \mathcal{E} \to \mathcal{B}^{\to}$ is fully faithful. Since this is shown to be a fibered functor, we can equivalently say that the functor $c_{-}\colon \mathcal{E}_I \to \mathcal{B}/I$ is fully faithful for every object I in $\mathcal{B}$. ⌟

This condition is required to ensure that the behaviors of predicates are completely determined by their comprehension. The definition is given in the references we have mentioned above with the adjective *full*. Note that, when $\mathfrak{p}$ is fiberwise preordered, the faithfulness of $\{-\}$ is automatically satisfied[18] since the counit components are necessarily epimorphisms.

In the paper [**HJ03**], full predicate comprehension plays a crucial role in characterization of factorization systems as special bifibrations. We briefly recall this result for a later discussion.

Definition 2.5.3 ([**HJ03**, Definition 2.12])**.** Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a bifibration with fiberwise terminal objects and full predicate comprehension. We say that $\mathfrak{p}$ has **strong products along subset projections**[19] if for any object I in $\mathcal{B}$, any object α in $\mathcal{E}_I$, and any object β in $\mathcal{E}_{\{\alpha\}}$, the supine lift of $c_\alpha\colon \{\alpha\} \to I$ to β induces an isomorphism $\{\beta\} \to \{\sum_{c_\alpha} \beta\}$.

$$\begin{tikzcd}
 & \beta \arrow[r, dashed, "\text{supine lift}"] \arrow[dd, mapsto, bend left] & \sum_{c_\alpha}\beta \arrow[dd, mapsto, bend left] \\
\{\beta\} \arrow[r] \arrow[dr, "c_\beta"'] & \{\sum_{c_\alpha}\beta\} \arrow[dr, "c_{\sum_{c_\alpha}\beta}"] & \\
 & \{\alpha\} \arrow[r, "c_\alpha"'] & I
\end{tikzcd}$$

⌟

---

[18] This seems why the term *full* is used in the definition.

[19] We leave the term as it is in the original paper.

Lemma 2.5.4. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a bifibration with fiberwise terminal objects and full predicate comprehension, and suppose that $\mathcal{B}$ has finite limits. We write $\mathrm{Prd}(\mathfrak{p})$ for the class of arrows in $\mathcal{B}$ that arise, up to isomorphism, as $c_\alpha\colon \{\alpha\} \to I$ for some $\alpha \in \mathcal{E}_I$. Then, the following are equivalent:

(i) $\mathfrak{p}$ has strong products along subset projections, and
(ii) $\mathrm{Prd}(\mathfrak{p})$ is closed under composition.

⌟

Proof. The implication (i)⇒(ii) is immediate from the definition of strong products along subset projections. Suppose we have a triple $\alpha, \gamma \in \mathcal{E}_I$ and $\beta \in \mathcal{E}_{\{\alpha\}}$ such that the diagram on the left below commutes and the top arrow is an isomorphism.

$$\begin{tikzcd}
 & \{\beta\} \arrow[r, "\cong"] \arrow[d, "c_\beta"'] & \{\gamma\} \arrow[d, "c_\gamma"] & & & & \\
\mathcal{B}^{\to} \ni & & & \arrow[l, mapsto, "c_-"'] & \beta \arrow[r, dashed, "\exists!\ \mu"] & \gamma & \in \mathcal{E} \\
 & \{\alpha\} \arrow[r, "c_\alpha"'] & I & & & &
\end{tikzcd}$$

By the fullness of predicate comprehension, this is the image of an arrow $\mu\colon \beta \to \gamma$ in $\mathcal{E}$ under the functor c_-. In the codomain fibration $\mathcal{B}^{\to} \to \mathcal{B}$, the arrow c_μ is supine since the top arrow is an isomorphism. Since a fully faithful fibered functor reflects supine arrows, the arrow μ is supine. Thus, (ii)⇒(i) holds. □

Theorem 2.5.5 ([HJ03, §3]). Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be a bifibration with fiberwise terminal objects, and suppose that $\mathcal{B}$ has all pullbacks. Then, the following are equivalent:

(i) $\mathfrak{p}$ has full predicate comprehension and strong products along subset projections.
(ii) $\mathfrak{p}$ is equivalent to $\mathcal{P}\mathrm{red}_{\mathrm{M}}(\mathcal{B})$ for some factorization system (E, M) on $\mathcal{B}$.

The factorization system (E, M) is uniquely determined by $\mathfrak{p}$. The bifibration is regular if and only if the factorization system is stable. ⌟

In the view that a double category is a generalization of an elementary existential fibration, a double categorical analog of predicate comprehension is the notion of tabulators.

Definition 2.5.6 (Tabulators [GP99])**.** A ***(1-dimensional) tabulator*** of a loose arrow $\alpha\colon I \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} J$ is an object $\{\alpha\}$ equipped with a pair of tight arrows $\ell_\alpha\colon \{\alpha\} \to I$ and $r_\alpha\colon \{\alpha\} \to J$ and a cell

$$\begin{tikzcd}
 & \{\alpha\} \arrow[dl, "\ell_\alpha"'] \arrow[dr, "r_\alpha"] \arrow[d, phantom, "\tau_\alpha"] & \\
I \arrow[rr, "\alpha"', "|" marking] & {} & J
\end{tikzcd}$$

such that, for any cell ν on the left below, there exists a unique tight arrow $t_\nu\colon X \to \{\alpha\}$ that makes the following two cells equal.

$$\begin{tikzcd}
 & L \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[d, phantom, "\nu"] & \\
I \arrow[rr, "\alpha"', "|" marking] & {} & J
\end{tikzcd}
=
\begin{tikzcd}
 & L \arrow[d, "t_\nu"] \arrow[ddl, bend right, "h"'] \arrow[ddr, bend left, "k"] & \\
\circlearrowleft & \{\alpha\} \arrow[dl, "\ell_\alpha"'] \arrow[dr, "r_\alpha"] \arrow[d, phantom, "\tau_\alpha"] & \circlearrowleft \\
I \arrow[rr, "\alpha"', "|" marking] & {} & J
\end{tikzcd}$$

Henceforth, we call the cell ν the ***tabulating cell*** of α. In other words, a tabulator is a terminal object in the comma category $\delta_- \downarrow \alpha$, where $\delta_-\colon \mathbb{D}_0 \to \mathbb{D}_1$ is the functor that sends an object I to the loose arrow δ_I, and α is seen as an object in $\mathbb{D}_1$. A tabulator is called ***effective***[20] if the tabulating cell is supine.

---

[20] In [HN23], the authors use the term *strong tabulator* to mean the same thing according to the definition in [Ale18]. We adopt the term *effective* because it is a fixed point of the adjunction between the category of spans between I and J and the category of loose arrows from I to J, as an effective epimorphism from I is a fixed point of the adjunction between the category of parallel pairs into I and the category of arrows from I.

In a double category $\mathbb{D}$ with a terminal object 1 in the tight category, tight arrow $f\colon I \to J$ is called a ***fibration*** if there exists a loose arrow $\alpha\colon I \to\!\!\!\!\!\shortmid\; 1$ and a tabulating cell

$$\begin{tikzcd} & I \arrow[dl, "f"'] \arrow[dr, "!"] \arrow[d, phantom, "\tau_f"] & \\ J \arrow[rr, "\alpha"', "\shortmid" marking] & {} & 1 \end{tikzcd} .$$

We write $\mathrm{Fib}(\mathbb{D})$ for the class of fibrations in $\mathbb{D}$. ⌟

Proposition 2.5.7. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be an elementary existential fibration. Then, the following are equivalent:

(i) $\mathfrak{p}$ has (full) predicate comprehension,
(ii) $\mathbb{B}\mathrm{il}(\mathfrak{p})$ has (effective) tabulators, and
(iii) $\mathbb{B}\mathrm{il}(\mathfrak{p})$ has left-sided (effective) tabulators.

Here, the additional conditions of fullness and effectiveness are satisfied simultaneously. Furthermore, $\mathrm{Prd}(\mathfrak{p})$ coincides with $\mathrm{Fib}(\mathbb{B}\mathrm{il}(\mathfrak{p}))$. ⌟

Proof. In the double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$, the comma category $\delta_{-} \downarrow \alpha$ for a loose arrow $\alpha\colon I \to\!\!\!\!\!\shortmid\; J$ is equivalent to the comma category $\top_{-} \downarrow \alpha$ where α is seen as an object in $\mathcal{E}_{I\times J} \subseteq \mathcal{E}$. Since the first condition is equivalent to the existence of a terminal object in $\top_{-} \downarrow \alpha$ for every I and $\alpha \in \mathcal{E}_I$, and the third condition is the statement for the cases where J is terminal, we obtain the equivalence of the three conditions. The last statement is immediate when one observes how the tabulator and the predicate comprehension are related by the above equivalence.

In an equipment $\mathbb{B}\mathrm{il}(\mathfrak{p})$, the operation of taking tabulators gives the right adjoints of the functors that sends each span to its oprestriction:

$$\begin{tikzcd}[column sep=huge] \mathbb{S}\mathrm{pan}(\mathcal{B})(I,J) \arrow[r, shift left=1.5, "(f,g)\mapsto f^*g_*"] \arrow[r, phantom, "\perp"] \arrow[d, "\cong"'] & \mathbb{B}\mathrm{il}(\mathfrak{p})(I,J) \arrow[l, shift left=1.5, "\langle \ell_\alpha, r_\alpha\rangle \mapsfrom \alpha"] \arrow[d, "\cong"] \\ \mathcal{B}/I\times J \arrow[r, shift left=1.5, "\langle f,g\rangle \mapsto \sum_{\langle f,g\rangle} \top"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{I\times J} \arrow[l, shift left=1.5, "c_\alpha \mapsfrom \alpha"] \end{tikzcd} .$$

The effectiveness of tabulators is equivalent to the counit components of the adjunction being isomorphisms, while the fullness of predicate comprehension is equivalent to the right adjoint c_{-} being fully faithful for every I and J. Therefore, these conditions are satisfied simultaneously. ☐

In [**HN23**], the authors provide a characterization of stable factorization systems in terms of double categories with additional structure. We now give another proof of this result using the above propositions.

Corollary 2.5.8 ([HN23, Theorem 3.3.20]). The following are equivalent for a double category $\mathbb{D}$:

(i) $\mathbb{D}$ is equivalent to $\mathbb{R}\mathrm{el}_{(\mathrm{E},\mathrm{M})}(\mathcal{B})$ for some category with finite limits $\mathcal{B}$ and a stable factorization system (E,M) on $\mathcal{B}$,
(ii) $\mathrm{Fib}(\mathbb{D})$ is closed under composition, and $\mathbb{D}$ is a cartesian equipment with Beck-Chevalley pullbacks and effective tabulators, and
(iii) $\mathrm{Fib}(\mathbb{D})$ is closed under composition, and $\mathbb{D}$ is a cartesian equipment with Beck-Chevalley pullbacks and left-sided effective tabulators.

⌟

Proof. By Corollary 2.3.38, the three conditions are subsumed by the condition that $\mathbb{D}$ is of the form $\mathbb{B}\mathrm{il}(\mathfrak{p})$ for some regular fibration $\mathfrak{p}$. Then, the equivalence follows from Proposition 2.5.7, Lem 2.5.4, and Thm 2.5.5. ☐

2.5.2. Function Extensionality and Unit-Pureness. The function extensionality is a principle that states that two functions are equal if they are equal at every point.

$$\forall f, g : I \to J.\ (\forall x : I.\ f(x) = g(x)) \Longrightarrow f = g$$

In the context of doctrines or fibrations, this principle is formulated by interpreting the equality of functions as the equality in the base category and the equality of elements as the predicate expressed in the fiber category. An elementary preordered fibration with this property, meaning that the inequality $\top_I \leq \delta_J[\langle f, g\rangle]$ implies $f = g$ for every $f, g\colon I \to J$ in the base category, is said to have *very strong equality* in [**Jac99**, Section 3.4]. This property is equivalent to the property that the diagonal arrow $\langle 0, 0\rangle\colon I \to I \times I$ is a predicate comprehension of δ_I for every object I in the base category (see [**MPR17**, Proposition 2.12]). From this observation, an elementary doctrine with this property is said to have *comprehensive diagonals* in [**MR13a**], and the combination of this property with the property of full predicate comprehension is called *m-variational* in [**MPR17**, Definition 2.16]. In the paper [**DP23**]introducing relational doctrines, the authors use the term *extensional* for the corresponding property to the very strong equality. We adopt the term ***comprehensive diagonals*** in this paper.

Note that in the context of allegories, extensionality does not make good sense since functions are defined as maps there and pointwise equality leads to the equality of the maps themselves (cf. [**Joh02a**, Proposition A 3.2.3]).

In the context of double categories, the corresponding property should be *unit-pureness.*

Definition 2.5.9 ([**Ale18**, Definition 4.3.7])**.** A double category (or a unital virtual double category) is called **unit-pure** if a cell of the form

$$\begin{tikzcd} I \arrow[r, "\delta_I", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mu"] & I \arrow[d, "g"] \\ J \arrow[r, "\delta_J"', "\shortmid"{marking}] & J \end{tikzcd}$$

is necessarily the identity cell δ_f with $f = g$. In other words, the functor δ_{-} is fully faithful. ⌟

Similarly to the case of fibrations, the unit-pureness of a double category is equivalent to the property that the span $(\mathtt{id}_I, \mathtt{id}_I)$ exhibits I as a tabulator of δ_I for every object I.

The following is an easy observation.

Proposition 2.5.10. Let $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$ be an elementary existential fibration. Then, the following are equivalent:

(i) $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is unit-pure,

(ii) for every parallel pair $f, g\colon I \to J$ in $\mathcal{B}$ and every arrow $\mu\colon \top_I \to \delta_J[\langle f, g\rangle]$ in $\mathcal{E}_I$, we have $f = g$ and μ is the composite of the following:

$$\top_I \xrightarrow{\cong} \top_J[\langle f\rangle] \xrightarrow{\eta_J[f]} \delta_J[\langle 0, 0\rangle][f] \xrightarrow{\cong} \delta_J[\langle f, f\rangle]$$

In particular, when $\mathfrak{p}$ is fiberwise preordered, the unit-pureness of $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is equivalent to the strong equality of $\mathfrak{p}$. ⌟

For the double category $\mathbb{R}\mathrm{el}_{(\mathrm{E},\mathrm{M})}(\mathcal{B})$ for a stable factorization system (E, M) on $\mathcal{B}$, the unit-pureness is equivalent to the property that the left class E is included in the class of epimorphisms in $\mathcal{B}$ [**HN23**, Theorem 4.1.6].

2.5.3. Unique Choice Principle and Cauchyness.

The unique choice principle, or functional comprehension, is a principle stating that for any predicate $\alpha(x, y)$, if it is total and single-valued in the sense that

$$\forall x : I.\exists y : J.\ \alpha(x, y)$$

$$\forall x : I.\forall y, y' : J.\ \ (\alpha(x, y) \wedge \alpha(x, y')) \implies y = y',$$

then there exists a function $f\colon I \to J$ such that

$$\forall x : I.\forall y : J.\,(\alpha(x, y) \iff f(x) = y)\,.$$

The totalness and single-valuedness of α are equivalently expressed as the following when $\beta(y, x) :\sim \alpha(x, y)$:

$$\forall x, x' : I.\forall y : J.\ \ (x = x' \implies \alpha(x, y) \wedge \beta(y, x'))$$

$$\forall x : I.\forall y, y' : J.\ \ (\beta(y, x) \wedge \alpha(x, y')) \implies y = y'.'$$

which serve as the unit and the counit of the adjunction $\alpha \dashv \beta$. The triangle identities are only meaningful when we respect the proofs of these implications, but they should be satisfied proof-theoretically, as explained in [**Pav95**]. This shows the importance of left adjoints (or maps) in a bicategory as an appropriate categorical counterpart of functional relations. If we follow this view, the unique choice principle is a statement that a left adjoint relation always arises from a function.

Definition 2.5.11 ([**Par21**, Definition 19])**.** A double category $\mathbb{D}$ is ***Cauchy*** if any adjoint $\alpha\colon I \underset{\perp}{\rightleftarrows} J : \beta$ in the bicategory $\mathbf{L}(\mathbb{D})$ is representable, namely, is of the form $f_*\colon I \underset{\perp}{\rightleftarrows} J : f^*$ for some tight arrow $f\colon I \to J$. ⌟

The name stems from the fact that a category is Cauchy-complete if and only if every left adjoint profunctor into it is representable. There have been several studies on this topic in various contexts, such as [**Pav95, Pav96**] in fibrations[21], and [**DP24a**] in relational doctrines. In this paper, we follow the terminology by Pavlović [**Pav96**] and call an elementary existential fibration with this property ***function comprehensive***. A double categorical account of this principle is given in [**HN23**, §4.2].

Together with the function extensionality, the unique choice principle guarantees that functions are in bijection with total and single-valued relations. Conceptually, this implies that the data of the functions are perfectly recoverable from the data of the relations, while the predicate comprehension implies the other way around.

The unique choice principle makes no sense in the context of bicategories, since there is no *a priori* notion of function therein: a function is defined as a map. Instead, a more appropriate way to proceed is to consider the condition when a cartesian bicategory or an allegory creates a Cauchy cartesian equipment. A previous study related to this is [**JW00**], where the authors study limits in the category of functional relations in the bicategory of relations relative to a stable factorization system. One problem surrounding our goal is that the composition of tight arrows in a double category is strictly associative and unital, and hence, when we create a double category from those bicategories, one needs to take the quotient of the left adjoints, which brings about a coherence issue. One way to avoid this is to consider doubly-pseudo double categories, or *double bicategories* introduced in [**Ver11**]. In Example 1.5.19 of this paper, the author constructs a double-bicategorical equivalent of an equipment from a bicategory, and characterizes cartesian bicategories in the sense of [**CKWW07**] as those induces a cartesian object in the bicategory-enriched category of (double-bicategorical) equipments and homomorphisms (p.152). In this paper, we take a different approach to this problem by assuming further conditions on bicategories which ensure the construction of a double category with the desired property.

Remark 2.5.12. A category is equivalent to a discrete category if and only if every object is subterminal, meaning that there is at most one arrow into it, and for every arrow $f\colon I \to J$, there exists an arrow $g\colon J \to I$ (which automatically becomes the inverse of f). ⌟

Definition 2.5.13. A bicategory is called ***map-discrete*** if the locally full sub-bicategory $\mathbf{Map}(\mathbf{B})$ of $\mathbf{B}$ is locally equivalent to discrete categories, namely, for every pair of objects I and J in $\mathbf{B}$, $\mathbf{Map}(\mathbf{B})(I, J)$ is equivalent to a discrete category. A double category is called ***map-discrete*** if the loose bicategory is map-discrete. ⌟

Definition 2.5.14. Let $\mathbf{B}$ be a map-discrete bicategory. We define the **double category of maps** $\mathbb{M}\mathrm{ap}(\mathbf{B})$ as follows:

- The objects are the same as the 0-cells of $\mathbf{B}$.
- The tight arrows are the isomorphism classes of the maps in $\mathbf{B}$. The composition of tight arrows is the composition as in $\mathbf{B}$.
- The loose arrows are the same as the 1-cells of $\mathbf{B}$. The composition of loose arrows is the composition as in $\mathbf{B}$.
- The cells of the form depicted on the left below are the 2-cells in $\mathbf{B}$.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "|" marking] \arrow[d, "{[f]}"'] & J \arrow[d, "{[g]}"] \\
K \arrow[r, "\beta"', "|" marking] \arrow[ur, phantom, "\mu"] & L
\end{tikzcd}
\qquad \Bigg\| \qquad
\begin{tikzcd}
& J \arrow[dr, "g", "|" marking] & \\
I \arrow[ur, "\alpha", "|" marking] \arrow[dr, "f"', "|" marking] & \mu & K \\
& L \arrow[ur, "\beta"', "|" marking] &
\end{tikzcd}$$

Note that the choice of representatives f and g is arbitrary.

---

[21] Note that what is called the unique choice in [**Jac99**, Definition 4.9.1] is different from the unique choice principle we are discussing here, as it mentions the existence of coproducts along product projections when a predicate is single-valued, not necessarily total. [**MPR17**, Proposition 5.3] seems to assume that these two conditions are equivalent, but the author of this paper is not sure about this.

- The composition of cells is defined by the composition in $\mathbf{B}$ as follows:

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "{[f]}"'] \arrow[rd, phantom, "\mu"] & J \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, "{[g]}"] \arrow[rd, phantom, "\nu"] & K \arrow[d, "{[h]}"] \\
I' \arrow[r, "\alpha'"', "\shortmid"{marking}] & J' \arrow[r, "\beta'"', "\shortmid"{marking}] & K'
\end{tikzcd}
\quad\Bigg\Vert\quad
\begin{tikzcd}
 & & K \arrow[rd, "h", "\shortmid"{marking}] & \\
 & J \arrow[ru, "\beta", "\shortmid"{marking}] \arrow[rd, "g"', "\shortmid"{marking}] \arrow[rr, phantom, "\nu \Downarrow"] & & K' \\
I \arrow[ru, "\alpha", "\shortmid"{marking}] \arrow[rd, "f"', "\shortmid"{marking}] \arrow[rr, phantom, "\mu \Downarrow"] & & J' \arrow[ru, "\beta'"', "\shortmid"{marking}] & \\
 & I' \arrow[ru, "\alpha'"', "\shortmid"{marking}] & &
\end{tikzcd}
,\qquad
\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "{[f]}"'] \arrow[rd, phantom, "\mu"] & J \arrow[d, "{[g]}"] \\
I' \arrow[r, "\alpha'"', "\shortmid"{marking}] \arrow[d, "{[f']}"'] \arrow[rd, phantom, "\mu'"] & J' \arrow[d, "{[g']}"] \\
I'' \arrow[r, "\alpha''"', "\shortmid"{marking}] & J''
\end{tikzcd}
\quad\Bigg\Vert\quad
\begin{tikzcd}
 & J \arrow[rd, "g", "\shortmid"{marking}] & & \\
I \arrow[ru, "\alpha", "\shortmid"{marking}] \arrow[rd, "f"', "\shortmid"{marking}] \arrow[rr, phantom, "\mu \Downarrow"] & & J' \arrow[rd, "g'", "\shortmid"{marking}] & \\
 & I' \arrow[ru, "\alpha'"', "\shortmid"{marking}] \arrow[rd, "f'"', "\shortmid"{marking}] \arrow[rr, phantom, "\mu' \Downarrow"] & & J'' \\
 & & I'' \arrow[ru, "\alpha''"', "\shortmid"{marking}] &
\end{tikzcd}
.$$

These compositions are well-defined independently of the choice of representatives by the map-discreteness of $\mathbf{B}$. In particular, the horizontal composition of cells requires the representative of the tight arrow in the middle by composing the unique isomorphsim, but the result is independent of the choice of the representative.

⌟

Lemma 2.5.15. The map double category $\mathbb{M}\mathrm{ap}(\mathbf{B})$ for a map-discrete bicategory $\mathbf{B}$ is unit-pure Cauchy equipment, and the loose bicategory $\mathbf{L}(\mathbb{M}\mathrm{ap}(\mathbf{B}))$ is equivalent to $\mathbf{B}$. ⌟

PROOF. Since the cells with the loose arrows at the top and the bottom being identities correspond to the 2-cells between the maps on the left and the right, the two maps are equal up to unique isomorphism and hence the cell is the identity. This shows the unit-pureness of $\mathbb{M}\mathrm{ap}(\mathbf{B})$. The companion and the conjoint for a map f, which has a right adjoint f^*, are given by the following cells:

(companion)
$$\begin{tikzcd}
I \arrow[rr, "f_*=f", "\shortmid"{marking}, "\mathtt{prn}"'] \arrow[rd, "f"'] & & J \arrow[ld, equal] \\
 & J &
\end{tikzcd}
\quad\Bigg\Vert\quad
\begin{tikzcd}
 & J \arrow[rd, equal, "\shortmid"{marking}] & \\
I \arrow[ru, "f", "\shortmid"{marking}] \arrow[rr, bend right, "f"', "\shortmid"{marking}] \arrow[rr, phantom, "\Vert"] & & J
\end{tikzcd}
,\qquad
\begin{tikzcd}
 & I \arrow[rd, "f"] & \\
I \arrow[ru, equal] \arrow[rr, "f_*=f"', "\shortmid"{marking}, "\mathtt{spn}"] & & J
\end{tikzcd}
\quad\Bigg\Vert\quad
\begin{tikzcd}
I \arrow[rr, bend left, "f", "\shortmid"{marking}] \arrow[rd, equal, "\shortmid"{marking}] \arrow[rr, phantom, "\Vert"] & & J \\
 & I \arrow[ru, "f"', "\shortmid"{marking}] &
\end{tikzcd}$$

(conjoint)
$$\begin{tikzcd}
J \arrow[rr, "f^*", "\shortmid"{marking}, "\mathtt{prn}"'] \arrow[rd, equal] & & I \arrow[ld, "f"] \\
 & J &
\end{tikzcd}
\quad\Bigg\Vert\quad
\begin{tikzcd}
 & I \arrow[rd, "f", "\shortmid"{marking}] & \\
J \arrow[ru, "f^*", "\shortmid"{marking}] \arrow[rr, equal, bend right, "\shortmid"{marking}] \arrow[rr, phantom, "\varepsilon_f"] & & J
\end{tikzcd}
,\qquad
\begin{tikzcd}
 & I \arrow[ld, "f"'] \arrow[rd, equal] & \\
J \arrow[rr, "f^*"', "\shortmid"{marking}, "\mathtt{spn}"] & & I
\end{tikzcd}
\quad\Bigg\Vert\quad
\begin{tikzcd}
I \arrow[rr, equal, bend left, "\shortmid"{marking}] \arrow[rd, "f"', "\shortmid"{marking}] \arrow[rr, phantom, "\eta_f"] & & I \\
 & J \arrow[ru, "f^*"', "\shortmid"{marking}] &
\end{tikzcd}$$

Thus, $\mathbb{M}\mathrm{ap}(\mathbf{B})$ is an equipment. The Cauchyness also follows immediately from the above construction. The equivalence of the loose bicategory of $\mathbb{M}\mathrm{ap}(\mathbf{B})$ and $\mathbf{B}$ is clear from the construction. □

Proposition 2.5.16. (i) For a map-discrete cartesian bicategory $\mathbf{B}$, the double category of maps $\mathbb{M}\mathrm{ap}(\mathbf{B})$ is a unit-pure Cauchy cartesian equipment, with $\mathbf{L}(\mathbb{M}\mathrm{ap}(\mathbf{B})) \simeq \mathbf{B}$.
(ii) A unit-pure Cauchy cartesian equipment $\mathbb{D}$ is map-discrete, with $\mathbb{M}\mathrm{ap}(\mathbf{L}(\mathbb{D})) \simeq \mathbb{D}$.
In this way, the map-discrete cartesian bicategories and the unit-pure Cauchy cartesian equipments are equivalent categories. ⌟

PROOF. (i) By Lemma 2.5.15, $\mathbb{M}\mathrm{ap}(\mathbf{B})$ is a unit-pure Cauchy equipment. The cartesianness of $\mathbb{M}\mathrm{ap}(\mathbf{B})$ is proven in the same way as in [**Ver11**, Example 1.5.19]. Recall the characterization of cartesian equipments Proposition 1.2.17. The category $\mathbb{M}\mathrm{ap}(\mathbf{B})_0$ is biequivalent to $\mathbf{Map}(\mathbf{B})$ as a bicategory, and since the latter has finite biproducts, so does the former. However, it is locally discrete by the assumption, hence it has strict finite products. The finite products in the loose hom-categories follow from the assumption, and the last condition in Proposition 1.2.17 follows from the corresponding condition in the definition of cartesian bicategories.

(ii) Since $\mathbb{D}$ is Cauchy, the category $\mathbf{Map}(\mathbb{D})(I, J)$ is equivalent to the category $\mathbf{T}(\mathbb{D})(I, J)$ for every pair of objects I and J. The unit-pureness implies that this is a discrete category. The equivalence of $\mathbb{M}\mathrm{ap}(\mathbf{L}(\mathbb{D}))$ and $\mathbb{D}$ follows from the above argument.

□

Although we have the 2-category of unit-pure Cauchy double categories, it seems unnecessarily complicated to consider the 2-category of cartesian bicategories. Therefore, we do not pursue the functorial aspect of this construction, but conceive the 2-category of those double categories as instead, we focus on a free construction of a Cauchy unit-pure cartesian equipment from a map-discrete cartesian equipment.

Definition 2.5.17 ([**HN23**, Definition 4.2.13])**.** Let $\mathbb{D}$ be an equipment. An equipment $\widehat{\mathbb{D}}$ is a ***Cauchisation*** of $\mathbb{D}$ if $\mathbf{L}(\widehat{\mathbb{D}}) = \mathbf{L}(\mathbb{D})$ holds and $\widehat{\mathbb{D}}$ is Cauchy and unit-pure, and denoted by $\mathbb{C}\mathrm{au}(\mathbb{D})$. ⌟

Lemma 2.5.18. Let $\mathbb{D}$ be a map-discrete cartesian equipment. Then, $\mathbb{M}\mathrm{ap}(\mathbf{L}(\mathbb{D}))$ is a Cauchisation of $\mathbb{D}$. ⌟

Proof. This is a direct consequence of Lemma 2.5.15. □

This simple definition of Cauchisation is enough to provide a universal property with respect to Cauchy unit-pure equipments.

Proposition 2.5.19 ([**HN23**, Proposition 4.2.14])**.** Let $\mathbb{D}$ be an equipment and $\widehat{\mathbb{D}}$ be a Cauchisation of $\mathbb{D}$. Then, we have a canonical double functor $C\colon \mathbb{D} \to \widehat{\mathbb{D}}$. Moreover, for any Cauchy unit-pure equipment $\mathbb{E}$ and a double functor $F\colon \mathbb{D} \to \mathbb{E}$, there exists a unique double functor $\widetilde{F}\colon \widehat{\mathbb{D}} \to \mathbb{E}$ such that $F = \widetilde{F} \circ C$. It also has the 2-dimensional universal property, that is, for any 2-cell $\varPsi\colon F \Rightarrow G$ in $\mathbb{D}$, there exists a unique 2-cell $\widetilde{\varPsi}\colon \widetilde{F} \Rightarrow \widetilde{G}$ in $\widehat{\mathbb{D}}$ such that $\varPsi = \widetilde{\varPsi} \circ C$. ⌟

Proof. We define C as the identity on the loose bicategory and send each tight arrow f to $(\mathtt{id}, f)$. Since a cell τ in $\mathbb{D}$ of the form below is in one-to-one correspondence with a globular cell $\widetilde{\tau}\colon \alpha g_* \Rightarrow f_*\beta$ in $\mathbf{L}(\mathbb{D})$, so C sends such a cell to the cell on the right below. Note that a pseudo-functor preserves companions.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\tau"] & J \arrow[d, "g"] \\
K \arrow[r, "\beta"', "\shortmid"{marking}] & L
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
A \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\|"] & B \arrow[d, equal] \arrow[dr, bend left, "Cg"] \arrow[dr, phantom, "\mathtt{spn}"] & \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "g_*"', "\shortmid"{marking}] \arrow[d, phantom, "\widetilde{\tau}"] & L \arrow[d, equal] \\
I \arrow[r, "f_*", "\shortmid"{marking}] \arrow[dr, bend right, "Cf"'] \arrow[dr, phantom, "\mathtt{prn}"] & K \arrow[r, "\beta", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\|"] & L \arrow[d, equal] \\
& K \arrow[r, "\beta"', "\shortmid"{marking}] & L
\end{tikzcd}$$

Thus defined C is easily shown to be a double functor.

In a unit-pure Cauchy equipment, a tight arrow is uniquely determined by its representative adjoint pair. Therefore, for any pseudo-functor $F\colon \mathbb{D} \to \mathbb{E}$, assignment of the image of a tight arrow in $\widehat{\mathbb{D}}$ is uniquely determined by the image of its representative adjoint pair. We can also reduce general cells to a combination of the pair of tight arrows and the corresponding globular cells in the loose bicategory, which implies that $\widetilde{F}$ is uniquely determined and also well-defined.

Since C is identity on the loose bicategory, the data of tightwise transformations $\varPsi\colon F \Rightarrow G$ and $\widetilde{\varPsi}\colon \widetilde{F} \Rightarrow \widetilde{G}$ are the same. We prove that the naturality for the tight arrows in $\widehat{\mathbb{D}}$ automatically follows from the naturality of F. Given a tight arrow $f\colon I \to J$ in $\widehat{\mathbb{D}}$, we have the cell on the left below in $\mathbb{E}$, which leads to the cell on the right below.

$$\begin{tikzcd}
FI \arrow[r, "Ff_*", "\shortmid"{marking}] \arrow[d, "\varPsi_I"'] \arrow[dr, phantom, "\varPsi_{f_*}"] & FJ \arrow[d, "\varPsi_J"] \\
GI \arrow[r, "Gf_*"', "\shortmid"{marking}] & GJ
\end{tikzcd}
\quad \Bigg\| \quad
\begin{tikzcd}
& FI \arrow[dl, "\varPsi_I"'] \arrow[dr, "Ff"] & \\
GI \arrow[dr, "Gf"'] & \widetilde{\varPsi}_{f_*} & FJ \arrow[dl, "\varPsi_J"] \\
& GJ &
\end{tikzcd}$$

By the unit-pureness of $\widehat{\mathbb{D}}$, we have $\varPsi_J \circ Ff = Gf \circ \varPsi_I$. The naturality for the additional cells in $\widehat{\mathbb{D}}$ is also shown in a similar way. □

Example 2.5.20.
(i) For a category $\mathcal{C}$ with finite limits and a stable factorization system (E, M) on $\mathcal{C}$ with $\mathrm{E} \subset \mathrm{Epi}$, the bicategory $\mathbb{R}\mathrm{el}_{(\mathrm{E},\mathrm{M})}(\mathcal{C})$ is map-discrete. This follows from the discussion in [**HN23**, Corollary 4.2.17].
(ii) More generally, for a regular fibration $\mathfrak{p}\colon \mathcal{E} \to \mathcal{B}$, $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is map-discrete [**Pav96**, Proposition 4.2, Theorem 4.3]. In Section 8 of the same paper, the author discusses the *function comprehension completion*, which is equivalent to $\mathfrak{uni}(\mathbb{C}\mathrm{au}(\mathbb{B}\mathrm{il}(\mathfrak{p})))$ in our notation.

⌟

Remark 2.5.21. The paper [**BSSS21**] provides an adjunction between the category of elementary existential doctrines and the category of Frobenius and locally-posetal cartesian bicategories[22]. This can be understood as the following composite of the adjunctions restricted to the subcategories spanned by the locally or fiberwise posetal structures:

$$\begin{tikzcd}
\mathbf{Fib}_{\times\wedge=\exists} \arrow[r, shift left, "\mathbb{B}\mathrm{il}"] \arrow[r, phantom, "\simeq"] & \mathbf{Eqp}_{\mathbf{Frob}} \arrow[l, shift left, "\mathfrak{uni}"] \arrow[r, shift left, "\mathbb{C}\mathrm{au}"] \arrow[r, phantom, "\perp"] \arrow[rr, shift left=4, "\mathbf{L}(-)"] & \mathbf{Eqp}_{\mathbf{Frob},\mathbf{Cauchy}} \arrow[l, hook', shift left] \arrow[r, shift left, "\mathbf{L}(-)"] \arrow[r, phantom, "\simeq"] & \mathbf{CartBi}_{\mathbf{Frob},\mathbf{MD}} \arrow[l, shift left, "\mathbb{M}\mathrm{ap}(-)"]
\end{tikzcd} .$$

We need some remarks to clarify the situation. First, we have not yet defined the 2-category **CartBi** in this paper, but it can be defined by importing the 2-categorical structure on $\mathbf{Eqp}_{\mathbf{cart}}$, and this is what we mean by the notation $\mathbf{CartBi}_{\mathbf{Frob}}$. The 2-category $\mathbf{CartBi}_{\mathbf{Frob},\mathbf{MD}}$ is the full sub-2-category of **CartBi** spanned by the Frobenius and map-discrete cartesian bicategories. Second, the 2-functor $\mathbb{C}\mathrm{au}$ is defined partially as the construction requires the map-discreteness of the input. However, it is defined on the image of $\mathbb{B}\mathrm{il}$ by (ii) of Example 2.5.20,. We also know that any locally-posetal Frobenius cartesian bicategory is map-discrete as in [**CW87**, Corollary 2.6], or by [**WW08**] and the fact that posetal groupoids are discrete. Therefore, we have the composite of the adjunctions as in the diagram above, which restricts to the adjunction between the categories of elementary existential doctrines and the Frobenius and locally-posetal cartesian bicategories in [**BSSS21**]. Moreover, its counit is pointwise an isomorphism by the above construction, and the image of the right adjoint is characterized by the unique choice principle for elementary existential doctrines, as shown in [**BSSS21**, Theorem 35]. ⌟

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[22] Note that in this paper, the term *cartesian bicategory* is used only for locally-posetal ones, following the terminology in [**CW87**]. However, it seems that they also assume the Frobenius law in the definition of cartesian bicategories, which is not compatible with the definition in [**CW87**] nor [**CKWW07, WW08**] (see Definition 2.4.7). The corresponding notion is rather called *'bicategories of relations'* in [**CW87**].

Chapter 3

Type Theory for Virtual Double Categories

An Internal Logic for Virtual Double Categories

We present a type theory called FVDblTT designed specifically for formal category theory, which is a succinct reformulation of New and Licata's Virtual Equipment Type Theory (VETT). FVDblTT formalizes reasoning on isomorphisms that are commonly employed in category theory. Virtual double categories are one of the most successful frameworks for developing formal category theory, and FVDblTT has them as a theoretical foundation. We validate its worth as an internal language of virtual double categories by providing a syntax-semantics duality between virtual double categories and specifications in FVDblTT as an adjunction.

Outline. Section 3.1 gives an introduction of this chapter. Section 3.2 introduces the syntax and the equational theory of FVDblTT and its semantics in virtual double categories. Section 3.3 explains the type theory's possible extensions with additional constructors and how they work in the semantics with examples. In Section 3.5, we present the main result of this chapter, the adjunction between the category of split cartesian fibrational virtual double categories and the category of specifications for FVDblTT.

3.1. Introduction

Variants of category theory have been developed over the decades, each with its own characteristics but sharing some basic concepts and principles. For instance, monoidal category theory [**Sel11**], enriched category theories over monoidal categories [**Kel05**], internal category theories in toposes [**Joh02a**], and fibered category theory [**Str23**] all have well-developed theories and significant applications. They often share several concepts, such as limits, representable functors, adjoints, and fundamental results like the Yoneda lemma, though there may be slight differences in their presentations.

Formal category theory [**Gra74**] is the abstract method that unifies these various category theories. As category theory offers us abstract results that can universally be applied to mathematical structures, formal category theory enables us to enjoy the universal results that hold for general category theories. A comprehensive exposition of this field is given in [**LHLL17**]. The earliest attempt was to perform category theory in an arbitrary 2-category by pretending it is a 2-category of categories [**Gra74**]. However, more than mere 2-categories are needed to capture the big picture of category theory. The core difficulty is that this approach does not embody the notion of presheaves, or "set-valued functors" inside a 2-category. Subsequently, many solutions have emerged to address this problem, such as *Yoneda structures* [**SW78**] and *proarrow equipments* [**Woo82**, **Woo85**].

A prominent approach to formal category theory is to use *virtual double categories* or *augmented virtual double categories* [**Shu08**, **Kou20**]. General theories in (augmented) virtual double categories have recently been developed, successful examples of which include the Yoneda structures and total categories in augmented virtual double categories by Koudenburg [**Kou20**, **Kou24**] and the theory of relative monads in virtual equipments by Arkor and McDermott [**AM24a**]. The advantage of this framework is that it is built up with necessary components of category theory as primitive structures. A virtual double category models the structure constituted by categories, functors, natural transformations, and *profunctors*, a common generalization of presheaves and copresheaves. This allows us to capture far broader classes of category theories since the virtual double category for a category can at least be defined even when essential components, like presheaves or natural transformations, do not behave as well as in the ordinary category theory.

In this paper, we provide a type theory called ***fibrational virtual double type theory*** (FVDblTT), which is designed specifically for formal category theory and serves as an internal language of virtual double categories. It aims to function as a formal language to reason about category theory that can be applied to various category theories, which may be used as the groundwork for computer-assisted proofs. Arguing category theories is often divided into two parts: one is a common argument independent of different category theories, which occasionally falls into *abstract nonsense*, and the other is a specific discussion particular to a certain category theory. What we can do with this type theory is to deal with massive proofs belonging to the former part in the formal language and make people focus on the latter part. Our attempt is not the first in this direction, as it follows New and Licata's Virtual Equipment Type Theory (VETT) [**NL23**]. However, we design FVDblTT based on the following desiderata that set apart from the previous work:

(i) It admits a syntax-semantics duality between the category of virtual double categories (with suitable structures) and the category of syntactic presentations of them.
(ii) It is built up from a plain type theory but allows enhancement compatible with existing and future results in formal category theory.
(iii) It allows reasoning with isomorphisms, a common practice in category theory.

In order to explain how FVDblTT achieves these goals, we overview its syntax and semantics.

3.1.1. Syntax and Semantics.

We start with reviewing virtual double categories. While its name first appeared in the work of Cruttwell and Shulman [**CS10**], the idea of virtual double categories has been studied in various forms in the past under different names such as *multicatégories* [**Bur71**], ***fc***-*multicategories* [**Lei02, Lei04**], and *lax double categories* [**DPP06**]. For these years, virtual double categories have gained the status of a guidepost for working out new category theories, especially in the ∞-categorical setting [**GH15, RV17, Rui24**].

A virtual double category has four kinds of data: *objects*, *tight arrows*, *loose arrows*, and *virtual cells*. The typical example is $\mathbb{P}\mathrm{rof}$, which has categories, functors, *profunctors*, and (generalized) natural transformations as these data. A profunctor from a category $\mathcal{I}$ to a category $\mathcal{J}$, written as $P(-,\bullet)\colon \mathcal{I} \nrightarrow \mathcal{J}$, is a functor from $\mathcal{I}^{\mathsf{op}} \times \mathcal{J}$ to the category of sets $\mathcal{S}\mathrm{et}$, which is a common generalization of a presheaf on $\mathcal{I}$ and a copresheaf on $\mathcal{J}$. One would expect these two kinds of arrows to have compositional structures, and indeed, two profunctors $P(-,\bullet)\colon \mathcal{I} \nrightarrow \mathcal{J}$ and $Q(-,\bullet)\colon \mathcal{J} \nrightarrow \mathcal{K}$ can be composed by a certain kind of colimits called coends in $\mathcal{S}\mathrm{et}$. However, the composition of profunctors may not always be defined within a general category theory, for instance, for an enriched category theory with the enriching base category lacking enough colimits. Virtual cells are introduced to liberate loose arrows from their composition and yet to keep seizing their compositional behaviors. As in Figure 1, a virtual cell has two tight arrows, one loose arrow, and one sequence of loose arrows as its underlying data, and in the case of $\mathbb{P}\mathrm{rof}$, virtual cells are natural families with multiple inputs. This pliability enables us to express category theoretic phenomena with a weaker assumption on the category theory one works with.

- A virtual cell in $\mathbb{P}\mathrm{rof}$:

$$\begin{tikzcd}
\mathcal{I}_0 \arrow[r, "\alpha_1", "|"{marking}] \arrow[d, "S"'] & \mathcal{I}_1 \arrow[r, "|"{marking}] & \cdots \arrow[r, "\alpha_n", "|"{marking}] & \mathcal{I}_n \arrow[d, "T"] \\
\mathcal{J}_0 \arrow[rrr, "\beta"', "|"{marking}] & \mu & & \mathcal{J}_1
\end{tikzcd}$$

- A family of functions natural in i_0, i_n and dinatural in $i_1, \ldots, i_{n-1}$:

$$\mu_{i_0,\ldots,i_n}\colon \alpha_1(i_0,i_1) \times \cdots \times \alpha_n(i_{n-1},i_n) \to \beta(S(i_0),T(i_n))$$

- An interpretation of the proterm

$$x_0 : I_0 \fatsemi \ldots \fatsemi x_n : I_n \mid a_1 : \alpha_1(x_0 \fatsemi x_1) \fatsemi \ldots \fatsemi a_n : \alpha_n(x_{n-1} \fatsemi x_n)$$
$$\vdash \mu : \beta(S(x_0), T(x_n)).$$

FIGURE 1. A virtual cell in $\mathbb{P}\mathrm{rof}$ and a proterm that corresponds to it.

Corresponding to these four kinds of entities, FVDblTT has four kinds of core judgments: *types*, *terms*, ***pro****types*, and *proterms*. In the semantics in the virtual double category $\mathbb{P}\mathrm{rof}$, types, terms, and protypes are interpreted as categories, functors, and **pro**functors, while proterms are interpreted as virtual cells with the functors on both sides being identities. We restrict the interpretation in this way in order to have the linearized presentation of virtual cells in the type theory. This enables us to bypass diagrammatic presentations of virtual cells, which often occupy considerable space in papers[1]. Nevertheless, it does not lose the expressive power because we assume the semantic stage to be a ***fibrational*** virtual double category.

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[1] This thesis is a good example of this.

$$\begin{array}{rl} \text{Type} & I \ \mathsf{type} \ , \\ \text{Term} & \Gamma \vdash s : I \ , \\ \text{Protype} & \Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype} \ , \\ \text{Proterm} & \Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu : \beta \ , \end{array}$$

$$(\Gamma, \Delta, \dots \ \text{are contexts like } x_1 : I_1, \dots, x_n : I_n.)$$

FIGURE 2. Judgments of FVDblTT.

Fibrationality is satisfied in most virtual double categories for our purposes and is conceptually a natural assumption since it represents the possibility of substituting functors S and T in a profunctor $\alpha(-, \bullet)$. For instance, a virtual cell in $\mathbb{P}\mathrm{rof}$ is defined as a natural family μ as in Figure 1, and it only refers to the instantiated profunctor $\beta(S(-), T(\bullet))$. Accordingly, we let the type theory describe a virtual cell as a proterm as in Figure 1. The fibrationality condition is defined in terms of universal property and assumed to hold in the semantics. We will further assume VDCs to have suitable finite products to interpret finite products in FVDblTT, which alleviates the complexity of syntactical presentation.

A byproduct of this type theory is its aspect as an all-encompassing language for predicate logic. The double category $\mathbb{R}\mathrm{el}$ of sets, functions, relations as objects, tight arrows, and loose arrows would also serve as the stage of the semantics of FVDblTT. In this approach, protypes symbolize relations (two-sided **pro**positions), and proterms symbolize Horn formulas. In other words, category theory based on categories, functors, and profunctors can be perceived as *generalized logic*. The unity of category theory and logic dates back to the work of Lawvere [**Law73**], in which he proposed that the theories of categories or metric spaces are generalized logic, with the truth value sets being some closed monoidal categories.

The interpretation of FVDblTT is summarized in Table 1.

Items in FVDblTT	**Formal category theory**	**Predicate logic**
Types I	categories $\mathcal{I}$	sets I
Terms $x : I \vdash s : J$	functors $S \colon \mathcal{I} \to \mathcal{J}$	functions $s \colon I \to J$
Protypes $\alpha(x \fatsemi y)$	profunctors $\alpha \colon \mathcal{I} \nrightarrow \mathcal{J}$	formulas $\alpha(x, y)$ $(x \in I, y \in J)$
Proterms $a : \alpha(x \fatsemi y) \fatsemi b : \beta(y \fatsemi z)$ $\vdash \mu : \gamma(x \fatsemi z)$	natural transformations $\mu_{x,y,z} \colon \alpha(x, y) \times \beta(y, z) \to \gamma(x, z)$	proofs of Horn clauses $\alpha(x, y), \beta(y, z) \Rightarrow \gamma(x, z)$
Product types $I \times J$	product categories $\mathcal{I} \times \mathcal{J}$	product sets $I \times J$
Product protypes $\alpha \wedge \beta$	product profunctors $\alpha(x, y) \times \beta(x, y)$	conjunctions $\alpha(x, y) \wedge \beta(x, y)$
path protype $\nrightarrow$	hom profunctor $\mathcal{I}(-, -)$	equality relation $=_I$
composition protype $\odot$	composition of profunctors by coend	composition of relations by $\exists$
Protype Isomorphisms $\Upsilon : \alpha \cong \beta$	natural isomorphisms $\Upsilon_{x,y} \colon \alpha(x, y) \cong \beta(x, y)$	equivalence of formulas $\alpha(x, y) \equiv \beta(x, y)$

TABLE 1. Interpretation of FVDblTT in $\mathbb{P}\mathrm{ROF}$ and $\mathbb{R}\mathrm{el}$ (All rows except the last three are included in the core of FVDblTT.)

3.1.2. Realizing the desiderata.

(i) Syntax-semantics duality for VDC. Categorical structures have been studied as the stages for semantics. Good examples include the Lawvere theories in categories with finite products [**Law63**], simply typed lambda calculus in cartesian closed categories [**LS86**], extensional Martin-Löf type theory in locally cartesian closed categories [**See84**], and homotopy type theory in ∞-groupoids [**HS98, Str14**]. Thus, it has been discovered that there are dualities between syntax and categorical structures [**Jac99, CD14**], endorsing the principle that type theory corresponds to category theory. It is worth noting that the above examples all started from the development of calculi, and the corresponding categorical structures were determined.

We will define specifications for FVDblTT and construct an adjunction between the category of virtual double categories with some structures and the category of those specifications whose counit is componentwise an equivalence, which justifies the type theory as an internal language and directly implies the soundness and completeness of the type theory. Here, we have proceeded in the reversed direction to the traditional developments: knowing that virtual double categories are the appropriate structures for formal category theory, we extract a calculus from it. This principle can be seen in [**ANv23**].

(ii) Additional constructors. Additional type and protype constructors are introduced to make FVDblTT expressive enough to describe sophisticated arguments in category theory. For example, the hom-profunctor $\mathcal{I}(-,\bullet)\colon \mathcal{I} \nrightarrow \mathcal{I}$ cannot be achieved in the core FVDblTT, and we introduce *path protype* $x : I \fatsemi y : I \vdash x \nrightarrow_I y : \mathsf{protype}$ as its counterpart. Just as a variable $x : I$ serves as an object variable in $\mathcal{I}$, a provariable $a : x \nrightarrow_I y$ serves as a morphism variable in $\mathcal{I}$. The introduction rule for this is similar to the path induction in homotopy type theory. Using this constructor, one can formalize, for instance, the fully-faithfulness of a functor (Figure 3), as it is defined merely through the bahevior on the hom-sets. In addition, we introduce *composition protype*, *filler protype*, and *comprehension type* in this paper, by which one can formalize a myriad of concepts in category theory, including (weighted) (co)limits, pointwise Kan extensions, and the Grothendieck construction of (co)presheaves, which is only possible with the protype constructors.

A term $x : I \vdash s(x) : J$ is fully faithfull if the following proterm has an inverse.

$$\cfrac{\cfrac{y : J \mid \vdash \mathsf{refl}_J : y \nrightarrow_J y}{x : I \mid \vdash \mathsf{refl}_J[s(x)/y] : s(x) \nrightarrow_I s(x)}}{x : I \fatsemi x' : I \mid a : x \nrightarrow_I x' \quad \vdash \mathsf{ind}_{\nrightarrow_I}\{\mathsf{refl}_J[s(x)/y]\} : s(x) \nrightarrow_J s(x')}$$

Having an inverse is formulated using protype isomorphisms: it comes with the following protype isomorphism

$$x : I \fatsemi x' : I \vdash \mathsf{FF} : x \nrightarrow_I x' \cong s(x) \nrightarrow_J s(x')$$

that satisfies the following equation (context is omitted).

$$\mathsf{FF}\{a\} \equiv \mathsf{ind}_{\nrightarrow_I}\{\mathsf{refl}_J[s(x)/y]\}$$

FIGURE 3. Fully faithfulness

A term $y : J \vdash l(y) : K$ is a pointwise left Kan extension of $x : I \vdash s(x) : K$ along $x : I \vdash t(x) : J$ if it comes with the following protype isomorphism.

$$y : J \fatsemi z : K \vdash \mathsf{Lan} : l(y) \nrightarrow_K z \cong (t(x) \nrightarrow_J y) \triangleright_{x:I} (s(x) \nrightarrow_K z)$$

FIGURE 4. Pointwise Kan extensions

A pointwise left Kan extension $l(y)$ of $s(x)$ along a fully faithful functor $t(x)$ admits an isomorphism $l(t(x)) \cong s(x)$.

Proof. (Contexts are omitted.)

$$\begin{aligned} l(t(x')) \nrightarrow_K z &\cong (t(x) \nrightarrow_J (t(x'))) \triangleright_{x:I} (s(x) \nrightarrow_K z) && (\mathsf{Lan}) \\ &\cong (x \nrightarrow_I x') \triangleright_{x:I} (s(x) \nrightarrow_K z) && (\mathsf{FF}^{-1} \triangleright_{x:I} (s(x) \nrightarrow_K z)) \\ &\cong s(x') \nrightarrow_K z && (\text{Yoneda Example 3.4.1}) \end{aligned}$$

FIGURE 5. Pointwise Kan extensions along fully faithful functors

(iii) Isomorphism reasoning. We will enhance our type theory with *protype isomorphisms*, a new kind of judgments for isomorphisms between protypes.

$$\text{Protype Isomorphism} \quad \Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta\ .$$

They serve as a convenient gadget for up-to-isomorphism reasoning that is ubiquitous in category theory. One often proves two things are isomorphic by constructing some pieces of mutual inverses and then combining them to form the intended isomorphism. We bring this custom into the type theory as protype isomorphisms, interpreted as isomorphisms between profunctors, *i.e.*, an invertible natural transformation between profunctors. For instance, pointwise Kan extensions are concisely defined using protype isomorphisms (Figure 4). Protype isomorphisms capture isomorphisms between functors as well since isomorphisms between functors $F, G\colon \mathcal{I} \to \mathcal{J}$ correspond to natural isomorphisms between $\mathcal{J}(F-,\bullet), \mathcal{J}(G-,\bullet)\colon \mathcal{I} \nrightarrow \mathcal{J}$ according to the Yoneda lemma. A formal proof of a well-known fact that a pointwise left Kan extension along a fully faithful functor admits an isomorphism to the original functor can be given by isomorphism reasoning (Figure 5). Although we do not present the proof that this isomorphism is achieved by the unit of the Kan extension here, we can formalize it within the type theory as a protype isomorphism introduces a proterm that witnesses the isomorphism by the following rule:

$$\frac{\Gamma \fatsemi \Delta \vdash \Upsilon : \alpha \cong \beta}{\Gamma \fatsemi \Delta \mid a : \alpha \vdash \Upsilon\{a\} : \beta}$$

3.1.3. Related Work.

The most closely related work to FVDblTT is VETT by New and Licata [**NL23**]. Along with the desiderata, we compare the differences between the two type theories. As for (i), their type theory is designed to have the adjunction between the category of hyperdoctrines of virtual equipments and that of its syntax, which originates from the polymorphic feature of VETT. The type theory has different type-theoretic entities corresponding to the hierarchy of abstractness. It has *categories*, *sets*, and meta-level entities called *types*, all with equational theory. The distinction between categories in VETT and types in FVDblTT is that the former has the equational theory as elements of a meta-level type “Cat,” while the latter does not. While this is advantageous when different layers of category theories are in question, it possibly obfuscates the overall type theory as a language for formal category theory. In contrast, FVDblTT has a single layer of category theory, namely one virtual double category, and the type theory is designed correspondingly to its components. It also gives rise to the syntax-semantics duality between the category of cartesian fibrational virtual double categories and the category of its syntax, which substantiates the type theory as an internal language of those virtual double categories.

As for (ii), VETT has more constructors for types and terms than FVDblTT in its core. On the other hand, we focus on minimal type theory to start with and introduce additional constructors as needed. This is because we aim to have a type theory that reflects results in formal category theory, which is still under development, and we do not want to go on ahead of the developments. For instance, when we introduce the path protype to FVDblTT, it seems plausible to have its compatibility with the default finite products in the type theory as in Subsection 3.3.2, which is supported by a category-theoretic observation in Section 1.4.

As for (iii), the capability of isomorphism reasoning is a novel feature of our type theory, which is not present in VETT. It facilitates reasoning in a category theory, as explained above.

There have been other attempts to obtain a formal language for category theory. A calculus for profunctors is presented in [**Lor21**] on the semantical level, which is followed by its type theoretic treatment in [**LLV24**]. Its usage is quite similar to that of FVDblTT, but they have different focuses. Although the calculus is similar to FVDblTT in that it deals with profunctors and some constructors for them, the semantics uses ordinary categories, functors, and profunctors, and general categorical structures as its semantic environment is not given, still less its syntax-semantics duality. On the other hand, the coend of an endoprofunctor $\alpha(-,\bullet)$, which cannot be handled it using FVDblTT at the moment, is in the scope of their calculus. It would be interesting to know the general categorical setting where the calculus can be interpreted and how it leads to the syntax-semantics duality in our style.

3.2. Fibrational Virtual Double Type Theory

3.2.1. Syntax. The syntax of FVDblTT is given by the following grammar.

Type	$I\ \mathsf{type}$
Context	$\Gamma\ \mathsf{ctx}$
Term	$\Gamma \vdash s : I$
Term Substitution	$\Gamma \vdash S \,/\, \Delta$
Protype	$\Gamma \fatsemi \Delta \vdash \alpha\ \mathsf{protype}$
Procontext	$\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid A\ \mathsf{proctx}$
Proterm	$\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid A \vdash \mu : \beta$
Term Equality	$\Gamma \vdash s \equiv t : I$
Protype Equality	$\Gamma \fatsemi \Delta \vdash \alpha \equiv \beta$
Proterm Equality	$\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid A \vdash \mu \equiv \nu : \beta$

Figure 6. Judgments in FVDblTT

Types, contexts, terms, and term substitutions are the same as those in the algebraic theory as in [**Cro94, Jac99**]. This fragment of the type theory serves as the theory of categories and functors. As usual, substitution of terms for variables in terms is defined by induction on the structure of terms.

Protypes and proterms are particular to this type theory and encode the loose arrows and cells in an CFVDC. The prefix ***pro-*** stands for “pro”positions and “pro”functors. A protype α depends on two contexts, Γ and Δ, which will be interpreted as the source and the target of a loose arrow representing the protype. We call the pair (Γ, Δ) the ***two-sided context*** of the protype and write $\Gamma \fatsemi \Delta$ for it. In the type theory, we distinguish semicolons “$\fatsemi$” from the ordinary concatenating symbol commas “,” by restricting using the former to concatenate items in the horizontal direction in a diagram in a VDC. Since the source and the target of a loose arrow can not be exchanged in any sense in a general VDC, we need to respect the order when we use the semicolons. Accordingly, a procontext $a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n$ with *provariables* a_i’s, which is formally defined as a finite sequence of protypes, is only well-typed so that the second (target) context of a protype is the first (source) context of the subsequent protype, and hence a procontext depends on a sequence of contexts. As a particular case, we have the empty procontext $\cdot$ depending on a single context $\mathit{\Gamma}$. Another item is proterms $A \vdash \mu : \beta$ where A is a procontext and β is a protype, which are interpreted as globular cells in a VDC whose domains and codomains are the interpretation of A and β, respectively.

The type theory also has the equality judgments for terms, protypes, and proterms. We incorporate the ordinary algebraic theory of terms with the equality judgments for terms, and we also have the equality judgments for proterms to capture the equality of cells in a VDC. The rules for equality judgments, or the equational theory of the type theory, are based on the basic axioms of reflexivity, symmetry, transitivity, and replacement with respect to the substitution we will define later. The equational theory for protypes is designed only to reflect the equational theory of terms by the replacement rule for substitution of terms, and we do not have any other rules for protypes except for the basic axioms. This is because, in formal category theory, we are mainly interested in isomorphisms of loose arrows, which we will incorporate in the type theory as the protype isomorphisms later.

$$\dfrac{I \ \mathsf{type} \qquad J\ \mathsf{type}}{I \times J\ \mathsf{type}} \qquad \dfrac{}{\mathsf{1}\ \mathsf{type}} \qquad \dfrac{}{\cdot\ \mathsf{ctx}} \qquad \dfrac{\mathit{\Gamma}\ \mathsf{ctx} \qquad I\ \mathsf{type}}{\mathit{\Gamma}, x : I\ \mathsf{ctx}} \qquad \dfrac{}{\mathit{\Gamma}, x : I, \mathit{\Delta} \vdash x : I}$$

$$\dfrac{I\ \mathsf{type} \qquad J\ \mathsf{type} \qquad \mathit{\Gamma} \vdash s : I \qquad \mathit{\Gamma} \vdash t : J}{\mathit{\Gamma} \vdash \langle s, t \rangle : I \times J} \qquad \dfrac{\mathit{\Gamma} \vdash t : I \times J}{\mathit{\Gamma} \vdash \mathsf{pr}_0(t) : I} \qquad \dfrac{\mathit{\Gamma} \vdash t : I \times J}{\mathit{\Gamma} \vdash \mathsf{pr}_1(t) : J} \qquad \dfrac{}{\mathit{\Gamma} \vdash \langle\ \rangle : \mathsf{1}} \qquad \dfrac{}{\mathit{\Gamma} \cdot \vdash /\cdot}$$

$$\dfrac{\mathit{\Gamma} \vdash S \,/\, \mathit{\Delta} \qquad \mathit{\Gamma} \vdash s : I}{\mathit{\Gamma} \vdash S, s \,/\, \mathit{\Delta}, x : I} \qquad \dfrac{\mathit{\Gamma} \vdash s : I \qquad \mathit{\Gamma} \vdash t : J}{\mathit{\Gamma} \vdash \mathsf{pr}_0(\langle s, t \rangle) \equiv s : I} \qquad \dfrac{\mathit{\Gamma} \vdash s : I \qquad \mathit{\Gamma} \vdash t : J}{\mathit{\Gamma} \vdash \mathsf{pr}_1(\langle s, t \rangle) \equiv t : J} \qquad \dfrac{\mathit{\Gamma} \vdash s : I \times J}{\mathit{\Gamma} \vdash \langle \mathsf{pr}_0(s), \mathsf{pr}_1(s) \rangle \equiv s : I \times J}$$

$$\dfrac{\mathit{\Gamma} \vdash s : \mathsf{1}}{\mathit{\Gamma} \vdash s \equiv \langle\ \rangle : \mathsf{1}}$$

FIGURE 7. The rules for types, contexts, and terms

$$\dfrac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha\ \mathsf{protype} \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \beta\ \mathsf{protype}}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha \wedge \beta\ \mathsf{protype}} \qquad \dfrac{}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \top\ \mathsf{protype}} \qquad \dfrac{}{\mathit{\Gamma} \mid \cdot\ \mathsf{proctx}}$$

$$\dfrac{\mathit{\Gamma}_0 \fatsemi \ldots \fatsemi \mathit{\Gamma}_n \mid A\ \mathsf{proctx} \qquad \mathit{\Gamma}_n \fatsemi \mathit{\Delta} \vdash \alpha\ \mathsf{protype}}{\mathit{\Gamma}_0 \fatsemi \ldots \fatsemi \mathit{\Gamma}_n \fatsemi \mathit{\Delta} \mid A, a : \alpha\ \mathsf{proctx}} \qquad \dfrac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha\ \mathsf{protype} \qquad \mathit{\Gamma}' \vdash S_0 \equiv S_1 \,/\, \mathit{\Gamma} \qquad \mathit{\Delta}' \vdash T_0 \equiv T_1 \,/\, \mathit{\Delta}}{\mathit{\Gamma}' \fatsemi \mathit{\Delta}' \vdash \alpha[S_0/\mathit{\Gamma} \fatsemi T_0/\mathit{\Delta}] \equiv \alpha[S_1/\mathit{\Gamma} \fatsemi T_1/\mathit{\Delta}]}$$

$$\dfrac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha\ \mathsf{protype}}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash a : \alpha} \qquad \dfrac{\overline{\mathit{\Gamma}_i} \mid a_{i,1} : \alpha_{i,1} \fatsemi \ldots \fatsemi a_{i,n_i} : \alpha_{i,n_i} \vdash \mu_i : \beta_i\ (i = 1, \ldots, m) \qquad \widetilde{\mathit{\Gamma}} \mid b_1 : \beta_1 \fatsemi \ldots \fatsemi b_n : \beta_n \vdash \nu : \gamma}{\overline{\mathit{\Gamma}} \mid a_{1,1} : \alpha_{1,1} \fatsemi \ldots \fatsemi a_{m,n_m} : \alpha_{m,n_m} \vdash \nu\{\mu_1/b_1 : \beta_1 \fatsemi \ldots \fatsemi \mu_m/b_m : \beta_m\} : \gamma}$$

$$\dfrac{\overline{\mathit{\Gamma}} \mid A \vdash \mu : \alpha \qquad \overline{\mathit{\Gamma}} \mid A \vdash \nu : \beta}{\overline{\mathit{\Gamma}} \mid A \vdash \langle \mu, \nu \rangle : \alpha \wedge \beta} \qquad \dfrac{\overline{\mathit{\Gamma}} \mid A \vdash \mu : \alpha \wedge \beta}{\overline{\mathit{\Gamma}} \mid A \vdash \pi_0\{\mu\} : \alpha} \qquad \dfrac{\overline{\mathit{\Gamma}} \mid A \vdash \mu : \alpha \wedge \beta}{\overline{\mathit{\Gamma}} \mid A \vdash \pi_1\{\mu\} : \beta} \qquad \dfrac{}{\overline{\mathit{\Gamma}} \mid A \vdash \langle\ \rangle : \top}$$

$$\dfrac{\overline{\mathit{\Gamma}} \mid A \vdash \mu : \alpha \qquad \overline{\mathit{\Gamma}} \mid A \vdash \nu : \beta}{\overline{\mathit{\Gamma}} \mid A \vdash \pi_0(\langle \mu, \nu \rangle) \equiv \mu : \alpha} \qquad \dfrac{\overline{\mathit{\Gamma}} \mid A \vdash \mu : \alpha \qquad \overline{\mathit{\Gamma}} \mid A \vdash \nu : \beta}{\overline{\mathit{\Gamma}} \mid A \vdash \pi_1(\langle \mu, \nu \rangle) \equiv \nu : \beta} \qquad \dfrac{\overline{\mathit{\Gamma}} \mid A \vdash \mu : \alpha \wedge \beta}{\overline{\mathit{\Gamma}} \mid A \vdash \langle \pi_0(\mu), \pi_1(\mu) \rangle \equiv \mu : \alpha \wedge \beta}$$

$$\dfrac{\overline{\mathit{\Gamma}} \mid A \vdash \mu : \top}{\overline{\mathit{\Gamma}} \mid A \vdash \mu \equiv \langle\ \rangle : \top}$$

$$\dfrac{\overline{\mathit{\Gamma}} \mid a_1 : \alpha_1 \fatsemi \ldots \fatsemi a_n : \alpha_n \vdash \mu : \beta \qquad \mathit{\Gamma}_0 \fatsemi \mathit{\Gamma}_1 \vdash \alpha_1 \equiv \alpha_1' \qquad \ldots \qquad \mathit{\Gamma}_{n-1} \fatsemi \mathit{\Gamma}_n \vdash \alpha_n \equiv \alpha_n' \qquad \mathit{\Gamma}_0 \fatsemi \mathit{\Gamma}_n \vdash \beta \equiv \beta'}{\mathit{\Gamma}_0 \fatsemi \mathit{\Gamma}_n \mid a_1 : \alpha_1' \fatsemi \ldots \fatsemi a_n : \alpha_n' \vdash \mu : \beta'}$$

FIGURE 8. The rules for protypes, procontexts, and proterms

Signatures. In algebraic theories, one often starts with a signature that specifies the sorts and operations of the theory. We present the signature for FVDblTT as follows.

Definition 3.2.1. A ***signature*** $\mathit{\Sigma}$ for FVDblTT is a quadruple $(T_{\mathit{\Sigma}}, F_{\mathit{\Sigma}}, P_{\mathit{\Sigma}}, C_{\mathit{\Sigma}})$ where

- $T_{\mathit{\Sigma}}$ is a class of ***category symbols***,
- $F_{\mathit{\Sigma}}(\sigma, \tau)$ is a family of classes of ***functor symbols*** for any $\sigma, \tau \in T_{\mathit{\Sigma}}$,
- $P_{\mathit{\Sigma}}(\sigma \fatsemi \tau)$ is a family of classes of ***profunctor symbols*** for any $\sigma, \tau \in T_{\mathit{\Sigma}}$,
- $C_{\mathit{\Sigma}}(\rho_1 \fatsemi \ldots \fatsemi \rho_n \mid \omega)$ is a family of classes of ***transformation symbols*** for any $\sigma_0, \ldots, \sigma_n \in T_{\mathit{\Sigma}}$, $\rho_i \in P_{\mathit{\Sigma}}(\sigma_{i-1} \fatsemi \sigma_i)$ for $i = 1, \ldots, n$, and $\omega \in P_{\mathit{\Sigma}}(\sigma_0 \fatsemi \sigma_n)$ where $n \geq 0$.

For simplicity, in the last item, we omit the dependency of the class of transformation symbols on σ_i's. Henceforth, $f\colon \sigma \to \tau$ denotes a functor symbol $f \in F_{\mathit{\Sigma}}(\sigma, \tau)$, $\rho\colon \sigma \nrightarrow \tau$ denotes a profunctor symbol $\rho \in P_{\mathit{\Sigma}}(\sigma \fatsemi \tau)$, and $\kappa\colon \rho_1 \fatsemi \ldots \fatsemi \rho_n \Rightarrow \omega$ denotes a transformation symbol $\kappa \in C_{\mathit{\Sigma}}(\rho_1 \fatsemi \ldots \fatsemi \rho_n \mid \omega)$.

A ***morphism of signatures*** $\Phi\colon \Sigma \to \Sigma'$ is a family of functions sending the symbols of each kind in Σ to symbols of the same kind in Σ' so that a symbol dependent on another kind of symbol is sent to a symbol dependent on the image of the former symbol. For instance, $\rho\colon \sigma \twoheadrightarrow \tau$ is sent to a profunctor symbol of the form $\Phi(\rho)\colon \Phi(\sigma) \twoheadrightarrow \Phi(\tau)$ where the assignment of category symbols has already been determined. ⌟

A typical example of a signature is the signature defined by a CFVDC $\mathbb{D}$.

Definition 3.2.2. The ***associated signature*** of a CFVDC $\mathbb{D}$ is the signature $\Sigma_{\mathbb{D}}$ defined by

- $T_{\mathbb{D}}$ is the set of objects of $\mathbb{D}$, where we write $\ulcorner I \urcorner$ for $I \in \mathbb{D}$ as a category symbol,
- $F_{\mathbb{D}}(\ulcorner I \urcorner, \ulcorner J \urcorner)$ is the set of tight arrows $I \to J$ in $\mathbb{D}$, where we write $\ulcorner f \urcorner$ for $f \in F_{\mathbb{D}}(\ulcorner I \urcorner, \ulcorner J \urcorner)$ as a functor symbol,
- $P_{\mathbb{D}}(\ulcorner I \urcorner \fatsemi \ulcorner J \urcorner)$ is the set of loose arrows $\alpha\colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ in $\mathbb{D}$, where we write $\ulcorner \alpha \urcorner$ for $\alpha \in P_{\mathbb{D}}(\ulcorner I \urcorner \fatsemi \ulcorner J \urcorner)$ as a profunctor symbol,
- $C_{\mathbb{D}}(\ulcorner \alpha_1 \urcorner \fatsemi \dots \fatsemi \ulcorner \alpha_n \urcorner \mid \ulcorner \beta \urcorner)$ is the set of cells $\mu\colon \alpha_1; \dots; \alpha_n \Rightarrow \beta$ in $\mathbb{D}$, where we write $\ulcorner \mu \urcorner$ for $\mu \in C_{\mathbb{D}}(\ulcorner \alpha_1 \urcorner \fatsemi \dots \fatsemi \ulcorner \alpha_n \urcorner \mid \ulcorner \beta \urcorner)$ as a transformation symbol.

⌟

A signature Σ is what we start derivations with in the type theory. In terms of formal category theory, it signifies what one regard as categories, functors, profunctors, and natural transformations. The rules for the signature are given as follows.

$$\frac{\sigma \in T_{\Sigma}}{\sigma \ \mathsf{type}} \qquad \frac{f \in F_{\Sigma}(\sigma, \tau) \qquad \Gamma \vdash s : \sigma}{\Gamma \vdash f(s) : \tau} \qquad \frac{\rho \in P_{\Sigma}(\sigma, \tau) \qquad \Gamma \vdash s : \sigma \qquad \Delta \vdash t : \tau}{\Gamma \fatsemi \Delta \vdash \rho(s \fatsemi t) : \mathsf{protype}}$$

$$\frac{\kappa \in C_{\Sigma}(\rho_1 \fatsemi \dots \fatsemi \rho_n \mid \omega) \qquad \Gamma_i \vdash s_i : \sigma_i \quad (i = 0, \dots, n) \qquad \Gamma_{i-1} \fatsemi \Gamma_i \mid A_i \vdash \mu_i : \rho_i(s_{i-1} \fatsemi s_i) \quad (i = 1, \dots, n)}{\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid A_1 \fatsemi \dots \fatsemi A_n \vdash \kappa(s_0 \fatsemi \dots \fatsemi s_n)\{\mu_1 \fatsemi \dots \fatsemi \mu_n\}}$$

FIGURE 9. The rules for the signature

Substitution. The substitution of terms for variables in terms, protypes, and proterms is defined inductively as follows.

$$\begin{aligned}
x_i[S/\Delta] &\equiv s_i \qquad (i = 1, \dots, n,\ S = (s_1, \dots, s_n)) \\
f(s_1, \dots, s_n)[S/\Delta] &\equiv f(s_1[S/\Delta], \dots, s_n[S/\Delta]) \\
\langle s, t \rangle[S/\Delta] &\equiv \langle s[S/\Delta], t[S/\Delta] \rangle \\
\mathsf{pr}_0(t)[S/\Delta] &\equiv \mathsf{pr}_0(t[S/\Delta]) \\
\mathsf{pr}_1(t)[S/\Delta] &\equiv \mathsf{pr}_1(t[S/\Delta]) \\
\langle\ \rangle[S/\Delta] &\equiv \langle\ \rangle \\
(\rho(s \fatsemi t))[S/\Delta] &\equiv \rho(s[S/\Delta] \fatsemi t[S/\Delta]) \\
(\alpha \wedge \beta)[S \,/\, \Delta \fatsemi T \,/\, \Theta] &\equiv \alpha[S/\Delta \fatsemi T/\Theta] \wedge \beta[S/\Delta \fatsemi T/\Theta] \\
\top[S \,/\, \Delta \fatsemi T \,/\, \Theta] &\equiv \top \\
a[S \,/\, \Delta \fatsemi T \,/\, \Theta] &\equiv a \\
\left(\kappa(\overline{s_{\underline{i}}})\{\overline{\mu_{\underline{i}}}\}\right) [\overline{S_{\underline{i},\underline{j}}}/\overline{\Delta_{\underline{i},\underline{j}}}] &\equiv \kappa\left(\overline{s_{\underline{i}}[S_{\underline{i},n_{\underline{i}}}/\Delta_{\underline{i},n_{\underline{i}}}]}\right) \{\overline{\mu_{\underline{i}}[\overline{S_{\underline{i},\underline{j}}}/\overline{\Delta_{\underline{i},\underline{j}}}]}\} \quad (S_{0,n_0} := S_{1,0}) \\
\langle \mu, \mu' \rangle[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}] &\equiv \langle \mu[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}], \mu'[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}] \rangle \\
\pi_i\{\mu\}[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}] &\equiv \pi_i\{\mu[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}]\} \\
\langle\ \rangle[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}] &\equiv \langle\ \rangle
\end{aligned}$$

Since the type theory has a different layer consisting of protypes and proterms, we need to define substitution for them as well, which we call ***prosubstitution*** and symbolize by ⁅ · ⁆ to distinguish it from the usual substitution. The prosubstitution is defined inductively as follows.

$$\begin{aligned}
a ⁅\mu/a⁆ &\equiv \mu \\
\left(\kappa(\overline{s_{\underline{i}}})\{\overline{\mu_{\underline{i}}}\}\right) ⁅\overline{\nu_{\underline{i},\underline{j}}}/\overline{b_{\underline{i},\underline{j}}}⁆ &\equiv \kappa(\overline{s_{\underline{i}}}) \left\{\overline{\mu_{\underline{i}} ⁅\overline{\nu_{i,\underline{j}}}/\overline{b_{i,\underline{j}}}⁆}\right\} \\
\langle \mu, \mu' \rangle ⁅\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}⁆ &\equiv \langle \mu ⁅\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}⁆ , \mu' ⁅\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}⁆ \rangle \\
\pi_i\{\mu\} ⁅\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}⁆ &\equiv \pi_i\{\mu ⁅\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}⁆ \} \\
\langle\ \rangle ⁅\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}⁆ &\equiv \langle\ \rangle
\end{aligned}$$

In the above, we use overline notation to denote the concatenation of terms, protypes, or proterms by $\fatsemi$, and we use underlined notation to specify the range of indices traversing the concatenation. For example, we write $\kappa(s_0 \fatsemi \dots \fatsemi s_n)\{\mu_1 \fatsemi \dots \fatsemi \mu_n\}$ as $\kappa(\overline{s_{\underline{i}}})\{\overline{\mu_{\underline{i}}}\}$. These are interpreted as sequences aligned in horizontal direction in a VDC. Note that a mere sequence of terms in a context, for instance, is not written with the overline notation.

Lemma 3.2.3 (Substitution lemmas). The following equations hold for substitution and prosubstitution.

(i) $\alpha\,[S/\Delta \fatsemi T/\Theta]\,[S'/\Delta' \fatsemi T'/\Theta'] \equiv \alpha\,[S\,[S'/\Delta']\,/\Delta \fatsemi T\,[T'/\Theta']\,/\Theta]$.

(ii) $\mu\,\left[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}\right]\,\left[\overline{S'_{\underline{i}}}/\overline{\Delta'_{\underline{i}}}\right] \equiv \mu\,\left[\overline{S_{\underline{i}}\,[S'_i/\Delta_i]}/\overline{\Delta_{\underline{i}}}\right]$.

(iii) $\mu\,\left[\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}\right]\,\left[\overline{\nu'_{\underline{i},\underline{j}}}/\overline{b'_{\underline{i},\underline{j}}}\right] \equiv \mu\,\left[\overline{\nu_{\underline{i}}\,\left[\overline{\nu'_{i,\underline{j}}}/\overline{b'_{i,\underline{j}}}\right]}\Big/\overline{b_{\underline{i}}}\right]$.

(iv) $\mu\,\left[\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}}\right]\,\left[\overline{S_{\underline{i},\underline{j}}}/\overline{\Delta_{\underline{i},\underline{j}}}\right] \equiv \left(\mu\,\left[\overline{S_{\underline{i},n_{\underline{i}}}}/\overline{\Delta_{\underline{i},n_{\underline{i}}}}\right]\right)\,\left[\overline{\nu_{\underline{i}}\,\left[\overline{S_{i,\underline{j}}}/\overline{\Delta_{i,\underline{j}}}\right]}\Big/\overline{b_{\underline{i}}}\right]$.

⌟

Proof. The proof is straightforward by induction on the structure of terms, protypes, and proterms. □

3.2.2. Semantics. As previously mentioned, the semantics of FVDblTT are taken in CFVDCs. The elements in the type theory are to be interpreted as the following elements in a CFVDC $\mathbb{D}$:

- I type and Γ ctx are to be interpreted as an object $[\![I]\!]$ and $[\![\Gamma]\!]$ in $\mathbb{D}$, respectively.
- $\Gamma \vdash t : I$ and $\Gamma \vdash S \,/\, \Delta$ are to be interpreted as tight arrows $[\![t]\!] \colon [\![\Gamma]\!] \to [\![I]\!]$ and $[\![S]\!] \colon [\![\Gamma]\!] \to [\![\Delta]\!]$ in $\mathbb{D}$, respectively.
- $\Gamma \fatsemi \Delta \vdash \alpha$ protype is to be interpreted as a loose arrow $[\![\alpha]\!] \colon [\![\Gamma]\!] \nrightarrow [\![\Delta]\!]$ in $\mathbb{D}$.
- $\Gamma_0 \fatsemi \dots \fatsemi \Gamma_n \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n$ proctx is to be interpreted as a path of loose arrows
$$[\![\Gamma_0]\!] \xrightarrow{[\![\alpha_1]\!]} [\![\Gamma_1]\!] \longrightarrow \dots \xrightarrow{[\![\alpha_n]\!]} [\![\Gamma_n]\!] \quad \text{in } \mathbb{D}.$$
- $\overline{\Gamma} \mid a_1 : \alpha_1 \fatsemi \dots \fatsemi a_n : \alpha_n \vdash \mu : \beta$ is to be interpreted as a globular cell $[\![\mu]\!] \colon \overline{[\![\alpha_{\underline{i}}]\!]} \Rightarrow [\![\beta]\!]$ in $\mathbb{D}$.

The semantics of FVDblTT consists of two parts: assignment of data in a CFVDC to the ingredients of a signature, and inductive definition of the interpretation.

Definition 3.2.4. For a signature Σ and a CFVDC $\mathbb{D}$, a ***Σ-structure*** $\mathcal{M}$ in $\mathbb{D}$ is a morphism of signatures $\Sigma \to \Sigma_{\mathbb{D}}$. The identity morphism on $\Sigma_{\mathbb{D}}$ can be deemed a $\Sigma_{\mathbb{D}}$-structure in $\mathbb{D}$, which we call the ***canonical ($\Sigma_{\mathbb{D}}$-)structure*** in $\mathbb{D}$. ⌟

Instead of writing $\mathcal{M}(\sigma)$ for the image of a category symbol σ under $\mathcal{M}$, we write $[\![\sigma]\!]_{\mathcal{M}}$, or simply $[\![\sigma]\!]$ when $\mathcal{M}$ is clear from the context.

Definition 3.2.5. Suppose we are given a Σ-structure $\mathcal{M}$ in a CFVDC $\mathbb{D}$. The interpretation of the terms, protypes, protype isomorphisms, and proterms for Σ in $\mathbb{D}$ is defined inductively as follows:

- The interpretation of the type σ is the object $[\![\sigma]\!]$ of $\mathbb{D}$.
- The interpretation of the context $\cdot$ is the terminal object of $\mathbb{D}$.
- The interpretation of the context $\Gamma, x : \sigma$ is the product $[\![\Gamma]\!] \times [\![\sigma]\!]$ of $[\![\Gamma]\!]$ and $[\![\sigma]\!]$.
- The interpretation of the term $\Gamma, x : \sigma, \Delta \vdash x : \sigma$ is the projection onto $[\![\sigma]\!]$.
- The interpretation of the term $f(t)$ is the composite $[\![f]\!] \circ [\![t]\!]$ of the tight arrows $[\![f]\!] \colon [\![\sigma]\!] \to [\![\tau]\!]$ and $[\![t]\!] \colon [\![\Gamma]\!] \to [\![\sigma]\!]$.
- Product types $\times, \mathtt{1}$ are interpreted as the product and terminal object of $\mathbb{D}$, respectively. Pairing, projections, and the unit are interpreted in an obvious way.
- The interpretation of the protype $\rho(s \fatsemi t)$ is the restriction of the loose arrow $[\![\rho]\!] : [\![\sigma]\!] \nrightarrow [\![\tau]\!]$ along the tight arrows $[\![s]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$ and $[\![t]\!] : [\![\Delta]\!] \to [\![\tau]\!]$.

$$\begin{tikzcd}
{[\![\Gamma]\!]} \arrow[r, "{[\![\rho(s \fatsemi t)]\!]}", "\shortmid"{marking}] \arrow[d, "{[\![s]\!]}"'] \arrow[dr, phantom, "\mathtt{rest}"] & {[\![\Delta]\!]} \arrow[d, "{[\![t]\!]}"] \\
{[\![\sigma]\!]} \arrow[r, "{[\![\rho]\!]}"', "\shortmid"{marking}] & {[\![\tau]\!]}
\end{tikzcd}$$

- Product protypes $\wedge, \top$ in context $\Gamma \fatsemi \Delta$ are interpreted as the product and terminal loose arrow from $\llbracket \Gamma \rrbracket$ to $\llbracket \Delta \rrbracket$, respectively. Pairing, projections, and the unit are interpreted in an obvious way.
- The interpretation of the proterm $a : \alpha \vdash a : \alpha$ is the identity cell on $\llbracket \alpha \rrbracket$.
- To define the interpretation of the proterm $\kappa(\overline{s_{\underline{i}}})\{\overline{\mu_{\underline{i}}}\}$, we first define a cell $\llbracket \kappa(\overline{s_{\underline{i}}}) \rrbracket$ as the restriction $\llbracket \kappa \rrbracket \left[\overline{\llbracket s_{\underline{i}} \rrbracket} \right] : \llbracket \rho_1(s_0 \fatsemi s_1) \rrbracket \fatsemi \dots \fatsemi \llbracket \rho_n(s_{n-1} \fatsemi s_n) \rrbracket \Rightarrow \llbracket \omega(s_0 \fatsemi s_n) \rrbracket$ in the sense of Definition 1.3.13. Then, the interpretation of the proterm $\kappa(\overline{s_{\underline{i}}})\{\overline{\mu_{\underline{i}}}\}$ is the composite $\llbracket \kappa(\overline{s_{\underline{i}}}) \rrbracket \{ \llbracket \mu_1 \rrbracket \fatsemi \dots \fatsemi \llbracket \mu_n \rrbracket \}$ of the cell $\llbracket \kappa(\overline{s_{\underline{i}}}) \rrbracket$ and the cells $\llbracket \mu_i \rrbracket \colon \llbracket A_i \rrbracket \Rightarrow \llbracket \rho_i(s_{i-1} \fatsemi s_i) \rrbracket$ for $i = 1, \dots, n$.

⌟

Taking semantics in the VDCs listed in Examples 1.3.11 and 1.3.12 justifies how FVDblTT expresses formal category theory and predicate logic.

We have naively used restrictions in the interpretation of protypes, but they are only defined up to isomorphism *a priori*. To make the definition precise, we need to consider strict functoriality in the following sense.

Definition 3.2.6. A CFVDC $\mathbb{D}$ is ***split*** if it comes with chosen finite products of its tight category, chosen restrictions $(-)[- \fatsemi -]$, and chosen terminals $\top$ and binary products $(-) \wedge (-)$ in the loose hom-categories. that satisfy the following equalities:

- $\alpha[\mathtt{id}_I \fatsemi \mathtt{id}_J] = \alpha$ for any $\alpha \colon I \nrightarrow J$.
- $\alpha[s \fatsemi t][s' \fatsemi t'] = \alpha[s \circ s' \fatsemi t \circ t']$ for any $\alpha \colon I \nrightarrow J$ and s, t, s', t'.
- $\top[s \fatsemi t] = \top$ for any s, t.
- $(\alpha \wedge \beta)[s \fatsemi t] = \alpha[s \fatsemi t] \wedge \beta[s \fatsemi t]$ for any $\alpha, \beta \colon I \nrightarrow J$ and s, t.

A ***morphism of split*** CFVDCs is a 1-cell in $\mathbf{FVDbl_{cart}}$ that preserves the chosen tightwise finite products, restrictions, terminals, and binary products on the nose. We will denote the category of split CFVDCs by $\mathbf{FVDbl^{split}_{cart}}$. ⌟

Note that in a split CFVDC, restrictions of globular cells along tight arrows in Definition 1.3.13 are uniquely determined by the chosen restrictions.

Lemma 3.2.7. Let $\mathbb{D}$ be a split CFVDC, and let $\mathcal{M}$ be a Σ-structure in $\mathbb{D}$. Suppose we choose the chosen restrictions in $\mathbb{D}$ in the definition of the interpretation.

(i) The interpretation of term substitutions is obtained by restrictions of loose arrows or globular cells along tight arrows. Explicitly, we have $\llbracket \alpha[S/\Gamma \fatsemi T/\Delta] \rrbracket = \llbracket \alpha \rrbracket [\llbracket S \rrbracket \fatsemi \llbracket T \rrbracket]$ and $\llbracket \mu[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}] \rrbracket = \llbracket \mu \rrbracket [\overline{\llbracket S_{\underline{i}} \rrbracket}]$ whenever the substitutions are well-typed.

(ii) The interpretation of proterm prosubstitutions is obtained by composition of globular cells. Explicitly, we have $\llbracket \mu \left[\overline{\nu_{\underline{i}}}/\overline{b_{\underline{i}}} \right] \rrbracket = \llbracket \mu \rrbracket \{ \overline{\llbracket \nu_{\underline{i}} \rrbracket} \}$ whenever the prosubstitutions are well-typed.

⌟

Proof. By induction on the structure of term substitutions and prosubstitutions. ☐

Assuming splitness for a CFVDC is too strong for practical purposes, but we can replace an arbitrary CFVDC by an equivalent split one.

Lemma 3.2.8. For any CFVDC $\mathbb{D}$, there exists a split CFVDC $\mathbb{D}^{\mathbf{split}}$ that is equivalent to $\mathbb{D}$ in the 2-category $\mathbf{FVDbl_{cart}}$. ⌟

Proof sketch. The proof is analogous to the proof for splitness of fibrational virtual double categories in [**AM24b**, Theorem A.1]. For a CFVDC $\mathbb{D}$, fix chosen terminals and binary products in each loose hom-category and chosen restrictions. We define a split CFVDC $\mathbb{D}^{\mathbf{split}}$ by taking the same objects and tight arrows as $\mathbb{D}$, but its loose arrows from I to J are finite tuples of triples $(f_i, g_i, \alpha_i)_i$ where $f_i \colon I \to K_i$ and $g_i \colon J \to L_i$ are tight arrows and $\alpha_i \colon K_i \nrightarrow L_i$ are loose arrows in $\mathbb{D}$. From a loose arrow $(f_i, g_i, \alpha_i)_i$, we can define its realization in $\mathbb{D}$ by taking $\bigwedge_i \alpha_i[f_i \fatsemi g_i]$. Then, we can define cells in $\mathbb{D}^{\mathbf{split}}$ framed by two tight arrows and loose arrows as those in $\mathbb{D}$ framed by the same tight arrows and the realization of the corresponding loose arrows. The associativity and unitality of cell composition in $\mathbb{D}^{\mathbf{split}}$ are inherited from those in $\mathbb{D}$. There is a virtual double functor $\mathbb{D}^{\mathbf{split}} \to \mathbb{D}$ that is the identity on the tight part and sends a loose arrow to its realization and a cell to itself. This is an equivalence of virtual double categories. To verify that $\mathbb{D}^{\mathbf{split}}$ admits the structure of a split CFVDC, we define the chosen restrictions, terminals, and binary products in $\mathbb{D}^{\mathbf{split}}$ as follows:

- The restriction of a loose arrow $(f_i, g_i, \alpha_i)_i$ along a pair of tight arrows (h, k) is the tuple $(f_i \circ h, g_i \circ k, \alpha_i)_i$.
- The terminal object in the loose hom-category from I to J is the empty tuple.
- The binary product of two loose arrows $(f_i, g_i, \alpha_i)_{i\in\mathrm{I}}$ and $(f'_j, g'_j, \alpha'_j)_{j\in\mathrm{J}}$ is the sum of the two tuples.

It is straightforward to verify that these chosen structures strictly satisfy the equalities in the definition of split CFVDCs. ☐

3.2.3. Protype isomorphisms. In category theory, one often proves that two objects, functors, or profunctors are isomorphic by exhibiting a sequence of those isomorphisms between them that one has already constructed or known to exist. Protype isomorphisms enable us to do the same in the type theory without showing proterms in both directions explicitly every time but still keeping track of the proterms that represent the isomorphisms. We introduce protype isomorphisms as additional typing judgments but they also serve partially as equality judgments for protypes up to isomorphism. Protype isomorphisms are also considered as codes for the two proterms mutually inverse to each other so that proterms can track what they actually represent in the type theory. They are also used to express isomorphisms between functors (terms) as we will see in Section 3.4. It should be noted that we do not have equality judgments for protype isomorphisms since one can identify or distinguish them by the proterms they represent using the equality judgments for proterms.

We call this extension of the type theory with protype isomorphisms $\mathrm{FVDblTT}^{\cong}$. The judgments for protype isomorphisms are presented as $\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta$ where α and β are protypes in the context $\mathit{\Gamma} \fatsemi \mathit{\Delta}$. The rules for protype isomorphisms are given as follows:

$$\frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha \text{ protype}}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \mathsf{id}_\alpha : \alpha \cong \alpha} \qquad \frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon^{-1} : \beta \cong \alpha} \qquad \frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Omega : \beta \cong \gamma}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Omega \circ \Upsilon : \alpha \cong \gamma}$$

$$\frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \mu\{a\} : \beta \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \mid b : \beta \vdash \nu\{b\} : \alpha \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \mid b : \beta \vdash \mu\{\nu\{b\}\} \equiv b : \beta \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \nu\{\mu\{a\}\} \equiv a : \alpha}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \llparenthesis \mu, \nu \rrparenthesis : \alpha \cong \beta}$$

$$\frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \Upsilon\{a\} : \beta} \qquad \frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \alpha \text{ protype}}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \mathsf{id}_\alpha\{a\} \equiv a : \alpha} \qquad \frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \Upsilon^{-1}\{\Upsilon\{a\}\} \equiv a : \alpha}$$

$$\frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid b : \beta \vdash \Upsilon\{\Upsilon^{-1}\{a\}\} \equiv a : \alpha} \qquad \frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Omega : \beta \cong \gamma}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash (\Omega \circ \Upsilon)\{a\} \equiv \Omega\{\Upsilon\{a\}\} : \gamma}$$

$$\frac{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \mu\{a\} : \beta \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \mid b : \beta \vdash \nu\{b\} : \alpha \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \mid b : \beta \vdash \mu\{\nu\{b\}\} \equiv b : \beta \qquad \mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \nu\{\mu\{a\}\} \equiv a : \alpha}{\mathit{\Gamma} \fatsemi \mathit{\Delta} \mid a : \alpha \vdash \llparenthesis \mu, \nu \rrparenthesis\{a\} \equiv \mu\{a\} : \beta}$$

If one has a pair of proterms μ and ν that are mutually inverse to each other, one can form a protype isomorphism $\llparenthesis \mu, \nu \rrparenthesis$. Conversely, protype isomorphisms are realized as proterms via the rule that introduces the proterm $\Upsilon\{a\}$ for a protype isomorphism Υ. We have the rule $\llparenthesis \mu, \nu \rrparenthesis\{a\} \equiv \mu\{a\}$, which is sufficient to derive that the inverse of $\llparenthesis \mu, \nu \rrparenthesis$ also has the expected behavior: $\llparenthesis \mu, \nu \rrparenthesis^{-1}\{b\} \equiv \llparenthesis \mu, \nu \rrparenthesis^{-1}\{\mu\{\nu\{b\}\}\} \equiv \llparenthesis \mu, \nu \rrparenthesis^{-1}\{\llparenthesis \mu, \nu \rrparenthesis\{\nu\{b\}\}\} \equiv \nu\{b\}$. The other rules are designed to ensure that protype isomorphisms behave as a groupoid as a whole.

The semantics of $\mathrm{FVDblTT}^{\cong}$ are also given in a CFVDC. A protype isomorphism judgment $\mathit{\Gamma} \fatsemi \mathit{\Delta} \vdash \Upsilon : \alpha \cong \beta$ is to be interpreted as an isomorphism of loose arrows $[\![\Upsilon]\!] \colon [\![\alpha]\!] \Rightarrow [\![\beta]\!] \colon [\![\mathit{\Gamma}]\!] \twoheadrightarrow [\![\mathit{\Delta}]\!]$ in $\mathbb{D}$. The interpretations of the protype isomorphisms $\mathsf{id}_\alpha, \Upsilon^{-1}, \Upsilon \circ \Omega$ are defined as the identity cell, the inverse cell, and the composite cell of the corresponding cells in $\mathbb{D}$, and the interpretation of the protype isomorphism $\llparenthesis \mu, \nu \rrparenthesis$ is the cell $[\![\mu]\!]$.

3.3. Protype and type constructors for FVDblTT

3.3.1. Further structures in VDCs and the corresponding constructors. In this section, we will specify the type and protype constructors that can be added to FVDblTT. The virtual double categories of relations and those of profunctors have many structures in common. We would like to introduce the inductive types and protypes corresponding to the common structures in these kinds of virtual double categories. We first list the additional types will introduce for the type theory.

Structures	**Formal category theory**	**Predicate logic**	**Constructors in FVDblTT**
Units [**CS10**]	hom-profuntors $\mathcal{C}(-,\bullet)$	equality $=$	path $\nrightarrow$
Composition [**CS10**]	composition via coends $\int$	composition via $\exists$	composition $\odot$
Extension [**RV22**]	profunctor extension $\triangleright$	contraction via $\forall$	extension $\triangleright$
Tabulators [**GP99**]	two-sided Grothendieck construction	comprehension $\{\text{-}\}$	tabulator $\{\!\vert\text{-}\vert\!\}$

Table 2. The common structures and the corresponding constructors

The constructors we will add to FVDblTT are $\nrightarrow$, $\odot$, $\triangleright$, $\triangleleft$, and $\{\!|\text{-}|\!\}$. Even though we can add the constructors for the loose adjunctions and the companions and conjoints independently of the other constructors, we would take the approach of defining them in terms of $\nrightarrow$ and $\odot$ in this paper.

Path protype $\nrightarrow$ for the units. The path protype is the protype that represents the units in a VDC. In a double category, the units are just the identity loose morphisms, but in a VDC, the units are formulated via a universal property Definitions 1.4.1 and 1.4.2.

The formation rule for the ***path protype*** is on the left below, and it comes equipped with the introduction rule on the right below:

$$\frac{I\ \mathsf{type}\qquad \Gamma\vdash s:I\qquad \Delta\vdash t:I}{\Gamma\fatsemi\Delta\vdash s\nrightarrow_I t\ \mathsf{protype}}\ \nrightarrow\text{-}\textsc{Form}\qquad\qquad \frac{I\ \mathsf{type}}{x:I\mid\ \ \vdash\mathsf{refl}_I(x):x\nrightarrow_I x}$$

The proterm refl corresponds to the unit η_I in the definition of the units. To let the path protype encode the units in the VDCs, we need to add elimination and computation rules as in Subsection 3.3.2. The path protypes behave as inductive (pro)types, and their inductions look very similar to path induction in homotopy type theory, but with the difference that the path protype is directed.

The semantics of the path protypes $\nrightarrow$ are given by the units in any VDC with units, with the proterm constructor refl_I interpreted as the cell $\eta_{[\![I]\!]}$. For instance, in the VDCs $\mathbb{PROF}$ and $\mathbb{R}\mathrm{el}$, the interpretations of the path protypes are given as the hom profunctors and the equality relations, respectively. These follow from the fact that the identity loose morphisms in a double category serve as the units when we see it as a VDC.

In order to make the path protypes behave well with the product types in FVDblTT, we need to add the compatibility rules between the path protypes and the product types as in Subsection 3.3.2. For instance, when we consider the hom-profunctor on a product category $\mathcal{C}\times\mathcal{D}$, we expect its components to be isomorphic to the product $\mathcal{C}(C,C')\times\mathcal{D}(D,D')$. Correspondingly, we would like to add the following rule, which does not follow from other rules *a priori*:

$$\frac{I\ \mathsf{type}\qquad J\ \mathsf{type}}{x:I,y:J\fatsemi x':I,y':J\vdash\mathsf{exc}_{\nrightarrow,\wedge}:\langle x,y\rangle\nrightarrow_{I\times J}\langle x',y'\rangle\cong x\nrightarrow_I x'\wedge y\nrightarrow_J y'}\ .$$

Subsection 3.3.2 will give the whole set of rules for the compatibility between the path protypes and the product types. The rules we introduce are justified by the fact that with them, the syntactic VDCs we will introduce in Section 3.5 become cartesian objects in the 2-category of FVDCs with units. See Proposition 2.3.7 for a detailed explanation from the 2-categorical perspective.

Composition protype $\odot$ for the composites. The composition protype is the protype that represents the composition of paths of loose arrows just of length 2 in virtual double categories Definition 1.4.1.

In order to gain access to the composition of paths of positive length in the type theory, we introduce the ***composition protype*** $\odot$ to FVDblTT. The formation rule for the composition protype is the following:

$$\frac{w:I\fatsemi x:J\vdash\alpha(w\fatsemi x)\ \mathsf{protype}\qquad x:J\fatsemi y:K\vdash\beta(x\fatsemi y)\ \mathsf{protype}}{w:I\fatsemi y:K\vdash\alpha(w\fatsemi x)\odot_{x:J}\beta(x\fatsemi y)\ \mathsf{protype}}$$

This comes equipped with the introduction rule:

$$\frac{w:I\fatsemi x:J\vdash\alpha(w\fatsemi x)\ \mathsf{protype}\qquad x:J\fatsemi y:K\vdash\beta(x\fatsemi y)\ \mathsf{protype}}{w:I\fatsemi x:J\fatsemi y:K\mid a:\alpha(w\fatsemi x)\fatsemi b:\beta(x\fatsemi y)\vdash a\odot b:\alpha(w\fatsemi x)\odot_{x:J}\beta(x\fatsemi y)}$$

For the detailed rules of the composition protype, see Subsection 3.3.2. Plus, we need the compatibility rules for the composition protype and the product types as we did for the path protype, see Subsection 3.3.2.

If we load the path protype $\twoheadrightarrow$ and the composite protype $\odot$ to FVDblTT, procontexts can be equivalently expressed by a single protype. In this sense, such a type theory can be seen as an internal language of double categories. This is supported by the fact that a VDC is equivalent to one arising from a double category if and only if it has composites of all paths of loose arrows, including units [**CS10**, Theorem 5.2].

The semantics of the composition protypes $\odot$ is given by the composites in VDCs if they have ones of sequences of length 2 in an appropriate way. For example, in the VDC $\mathbb{P}\mathrm{rof}$, the composite of paths of length 2 is the composite of profunctors, given by the coend $\int$. In the VDC $\mathbb{R}\mathrm{el}$, the composites of paths of length 2 are the composites of relations, given by the existential quantification $\exists$.

$$\llbracket \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \rrbracket = \int^{X \in \llbracket J \rrbracket} \llbracket \alpha \rrbracket(-, X) \times \llbracket \beta \rrbracket(X, \bullet) \colon \llbracket I \rrbracket \nrightarrow \llbracket K \rrbracket \text{ in } \mathbb{P}\mathrm{rof}$$

$$\llbracket \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \rrbracket = \{\, (w, y) \mid \exists x \in \llbracket J \rrbracket . \llbracket \alpha \rrbracket(w, x) \wedge \llbracket \beta \rrbracket(x, y) \,\} : \llbracket I \rrbracket \nrightarrow \llbracket K \rrbracket \text{ in } \mathbb{R}\mathrm{el}$$

Filler protype $\triangleright, \triangleleft$ for the closed structure. Having obtained the ability to express a particular kind of coends in formal category theory, and existential quantification in predicate logic, we would like to introduce the protypes for ends and universal quantification in the type theory. First of all, we recall the definition of the right extension and the right lift [**RV22, AM24a**] in a VDC, which are straightforward generalizations of the right extension and the right lift in a bicategory.

Definition 3.3.1. A ***right extension*** of a loose arrow $\beta \colon I \nrightarrow K$ along a loose arrow $\alpha \colon I \nrightarrow J$ is a loose arrow $\alpha \triangleright \beta \colon J \nrightarrow K$ equipped with a cell

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha\triangleright\beta", "\shortmid"{marking}] \arrow[d, phantom, "\varpi_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

with the following universal property. Given any cell ν on the left below where $\overline{\gamma}$ is an arbitrary sequence of loose arrows, it uniquely factors through the cell $\varpi_{\alpha,\beta}$ as on the right below.

$$\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\|"] & J \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\nu}"] & K \arrow[d, equal] \\
I \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha\triangleright\beta"', "\shortmid"{marking}] \arrow[d, phantom, "\varpi_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

A ***right lift*** of a protype $\beta \colon I \nrightarrow K$ along a protype $\alpha \colon J \nrightarrow K$ is a protype $\beta \triangleleft \alpha \colon I \nrightarrow J$ equipped with a cell

$$\begin{tikzcd}
I \arrow[r, "\beta\triangleleft\alpha", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, phantom, "\varpi'_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

with the following universal property. Given any cell ν on the left below where $\overline{\gamma}$ is an arbitrary sequence of loose arrows, it uniquely factors through the cell $\varpi'_{\alpha,\beta}$ as on the right below.

$$\begin{tikzcd}
I \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, dashed, "\overline{\gamma}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\widetilde{\nu}"] & J \arrow[r, "\alpha", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\|"] & K \arrow[d, equal] \\
I \arrow[r, "\beta\triangleleft\alpha"', "\shortmid"{marking}] \arrow[d, equal] & J \arrow[r, "\alpha"', "\shortmid"{marking}] \arrow[d, phantom, "\varpi'_{\alpha,\beta}"] & K \arrow[d, equal] \\
I \arrow[rr, "\beta"', "\shortmid"{marking}] & & K
\end{tikzcd}$$

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With this notion, one can handle the concept of weighted limits and colimits internally in virtual double categories. We now introduce the ***filler protypes*** $\triangleright$ and $\triangleleft$ to FVDblTT to express the right extension and the right lift in the type theory. The formation rule for the ***right extension protype*** is the following:

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x) \ \mathsf{protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y) \ \mathsf{protype}}{x : J \fatsemi y : K \vdash \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \ \mathsf{protype}}$$

The constructor for the right extension protype is given in the elimination rule since the orientation of the universal property of the right extension is opposite to that of the composition protype and the path protype.

$$\dfrac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x) \text{ protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y) \text{ protype}}{w : I \fatsemi x : J \fatsemi y : K \mid a : \alpha(w \fatsemi x) \fatsemi e : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \vdash a \blacktriangleright e : \beta(w \fatsemi y)}$$

The semantics of the right extension protype $\triangleright$ is given by the right extension in VDCs. The constructor $\blacktriangleright$ is interpreted using the cell $\varpi_{[\![\alpha]\!],[\![\beta]\!]}$ above. To illustrate the semantics of the right extension protype, we give the interpretations of the right extension protype in the VDCs $\mathbb{P}\mathrm{rof}$ and $\mathbb{R}\mathrm{el}$.

$$[\![\alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y)]\!] = \int_{W \in [\![I]\!]} [[\![\alpha]\!](W,-), [\![\beta]\!](W,\bullet)] : [\![J]\!] \relbar\joinrel\mapstochar\joinrel\rightarrow [\![K]\!] \text{ in } \mathbb{P}\mathrm{rof}$$
$$[\![\alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y)]\!] = \{ (x,y) \mid \forall w \in [\![I]\!].\, ([\![\alpha]\!](w,x) \Rightarrow [\![\beta]\!](w,y)) \} : [\![J]\!] \relbar\joinrel\mapstochar\joinrel\rightarrow [\![K]\!] \text{ in } \mathbb{R}\mathrm{el}$$

Here, $[X, Y]$ is the function set from X to Y.

Comprehension type $\{\!|\text{-}|\!\}$ for the tabulators. The last one is not a protype but a type constructor. First, we note that the definition of tabulators Definition 2.5.6 is directly generalizable to virtual double categories, where we interpret the triangle cells in the definition as cells with nullary inputs.

Corresponding to the tabulators in virtual double categories, we introduce the ***comprehension type*** $\{\!|\text{-}|\!\}$ to FVDblTT. The formation rule for the comprehension type is the following:

$$\dfrac{x : I \fatsemi y : J \vdash \alpha \text{ protype}}{\{\!|\alpha|\!\} \text{ type}}$$

This comes equipped with the constructor

$$\dfrac{x : I \fatsemi y : J \vdash \alpha \text{ protype}}{w : \{\!|\alpha|\!\} \vdash l(w) : I \qquad w : \{\!|\alpha|\!\} \vdash r(w) : J \qquad w : \{\!|\alpha|\!\} \mid\!\vdash \mathsf{tab}_{\{\!|\alpha|\!\}}(w) : \alpha[l(w)/x \fatsemi r(w)/y]}$$

The comprehension type $\{\!|\text{-}|\!\}$ is interpreted as the tabulators in the VDCs. In the VDC $\mathbb{P}\mathrm{rof}$, the tabulator of a profunctor $P \colon \mathcal{C} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathcal{D}$ is given by ***two-sided Grothendieck construction***, which results in ***a two-sided discrete fibration*** from $\mathcal{C}$ to $\mathcal{D}$. A frequently used example of this construction is the comma category for a pair of functors $F \colon \mathcal{C} \to \mathcal{E}$ and $G \colon \mathcal{D} \to \mathcal{E}$ as the tabulator of the profunctor $\mathcal{E}(F(-), G(-))$, see [**LR20**] for more details. The VDC $\mathbb{R}\mathrm{el}$ has the tabulators if we ground the double category to an axiomatic system of set theory with the comprehension axiom, as the tabulator of a relation $R \colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ is given by the set of all the pairs (a, b) such that $R(a, b)$ holds.

In the presence of the unit protype $\nrightarrow$, we should add some rules concerning the compatibility between the comprehension type and the path protype. This is because, in many examples of double categories, the tabulators have not only the universal property as in Definition 2.5.6 but also respect the units, although the original universal property of the tabulators is enough to detect the tabulators in a double category. This issue is thoroughly discussed in [**GP99**]. Here, we give a slightly generalized version of the tabulators in virtual double categories with units.

Definition 3.3.2 (2-dimensional universal property of tabulators)**.** In a virtual double category with units, a ***unital tabulator*** $\{\alpha\}$ of a loose arrow $\alpha \colon I \relbar\joinrel\mapstochar\joinrel\rightarrow J$ is a tabulator of α in the sense of Definition 2.5.6, which also satisfies the following universal property. Suppose we are given any pair of cones (X, h, k, ν) and (X', h', k', ν') over α and a pair of cells ς, ϑ such that the following equality holds.

$$\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}"] \arrow[d, "h"'] \arrow[dr, phantom, "\varsigma"] & X' \arrow[d, "h'"'] \arrow[dr, "k'"] \arrow[dr, phantom, "\nu'"' near end] & & & & X \arrow[r, dashed, "\overline{\gamma}"] \arrow[dl, "h"'] \arrow[d, "k"] \arrow[dr, phantom, "\vartheta"] \arrow[dl, phantom, "\nu" near end] & X' \arrow[d, "k'"] \\
I \arrow[r, "U_I"'] \arrow[d, equal] & I \arrow[r, "\alpha"'] \arrow[d, phantom, "\wr\|"] & J \arrow[d, equal] & = & I \arrow[r, "\alpha"'] \arrow[d, equal] & J \arrow[r, "U_J"'] \arrow[d, phantom, "\wr\|"] & J \arrow[d, equal] \\
I \arrow[rr, "\alpha"'] & & J & & I \arrow[rr, "\alpha"'] & & J
\end{tikzcd}$$

Then, there exists a unique cell ϱ for which the following equalities hold.

$$\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "|" marking] \arrow[d, "t_\nu"'] \arrow[dr, phantom, "\varrho"] & X' \arrow[d, "t_{\nu'}"] \\
\{\alpha\} \arrow[r, "U_{\{\alpha\}}"', "|" marking] \arrow[d, "\ell_\alpha"'] \arrow[dr, phantom, "U_{\ell_\alpha}"] & \{\alpha\} \arrow[d, "\ell_\alpha"] \\
I \arrow[r, "U_I"', "|" marking] & I
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\varsigma"] & X' \arrow[d, "h'"] \\
I \arrow[r, "U_I"', "|" marking] & I
\end{tikzcd}
,\qquad
\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "|" marking] \arrow[d, "t_\nu"'] \arrow[dr, phantom, "\varrho"] & X' \arrow[d, "t_{\nu'}"] \\
\{\alpha\} \arrow[r, "U_{\{\alpha\}}"', "|" marking] \arrow[d, "r_\alpha"'] \arrow[dr, phantom, "U_{r_\alpha}"] & \{\alpha\} \arrow[d, "r_\alpha"] \\
J \arrow[r, "U_J"', "|" marking] & J
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, dashed, "\overline{\gamma}", "|" marking] \arrow[d, "k"'] \arrow[dr, phantom, "\vartheta"] & X' \arrow[d, "k'"] \\
J \arrow[r, "U_J"', "|" marking] & J
\end{tikzcd}$$

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This universal property determines what the unit on the apex of the tabulator should be. Subsection 3.3.2 will present the corresponding rules for the comprehension type $\{\!|\text{-}|\!\}$ in FVDblTT with the unit protype $\nrightarrow$.

Remark 3.3.3 (Substitution into the additional constructor)**.** There are options how we define substitution for the additional constructors. For example, we may define the substitution for the composition protype as follows.

$$(\alpha \odot_{y:J} \beta)[s/x \fatsemi t/z] \coloneqq \alpha[s/x \fatsemi y/y] \odot_{y:J} \beta[y/y \fatsemi t/z]$$

This seems reasonable for our use in formal category theory, but this equality is not always satisfied in a general PL-composable FVDC unless it is actually a virtual equipment. Instead, we may extend the introduction rule for the composition protype so that the substituted composition protypes are directly introduced.

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x) \text{ protype} \qquad x : J \fatsemi y : K \vdash \beta(x \fatsemi y) \text{ protype} \qquad \Gamma \vdash s : I \qquad \Delta \vdash t : K}{\Gamma \fatsemi \Delta \vdash (\alpha \odot_{x:J} \beta)[s/w \fatsemi t/y] : \text{protype}}$$

Then, the substitution for the composition protype is obvious. Indeed we take the latter approach for the path protype.

Therefore, it depends on the purpose of the type theory how we define the substitution for the additional constructors, and we do not specify it in this paper because our main focus is the syntax-semantics duality for the very basic type theory. ⌟

Predicate logic.

When we work with the type theory FVDblTT for the purpose of reasoning about predicate logic, we consider types, terms, protypes, and proterms to represent sets, functions, predicates (or propositions), and proofs, respectively. However, the type theory FVDblTT, as it is, treats the protypes in a context $\Gamma \fatsemi \Delta$ and those in a context $\Delta \fatsemi \Gamma$ as different things. In this sense, the type theory FVDblTT as predicate logic has directionality. If one wants to develop a logic without a direction, one can simply add the following rules to the type theory.

$$\frac{\Gamma \fatsemi \Delta \vdash \alpha \text{ protype}}{\Delta \fatsemi \Gamma \vdash \alpha^\circ \text{ protype}} \qquad \frac{\Gamma_0 \fatsemi \cdots \fatsemi \Gamma_m \mid a_1 : \alpha_1 \cdots a_n : \alpha_n \vdash \mu : \beta}{\Gamma_m \fatsemi \cdots \fatsemi \Gamma_0 \mid a_n : \alpha_n^\circ \cdots a_1 : \alpha_1^\circ \vdash \mu^\circ : \beta^\circ} \qquad \frac{\Gamma_0 \fatsemi \cdots \fatsemi \Gamma_m \mid a_1 : \alpha_1 \cdots a_n : \alpha_n \vdash \mu : \beta}{\Gamma_0 \fatsemi \cdots \fatsemi \Gamma_m \mid a_1 : \alpha_1 \cdots a_n : \alpha_n \vdash \mu^{\circ\circ} \equiv \mu : \beta}$$

These rules are the counterparts of the structure of involution in VDCs.

This perspective is better understood with the $\mathbb{B}$il-construction in Chapter 2. This operation sending a cartesian fibration to a CFVDC corresponds to translating predicate logic with proofs as an internal logic of cartesian fibrations Remark 2.2.4 in terms of the type theory FVDblTT. More precisely, there is a comparison virtual double functor from the $\mathbb{B}$il of the syntactic cartesian fibration to the syntactic VDC in Section 3.5. This seems to be the 1-cell into the syntactic VDC from its "cofree Frobenius CFVDC" in the 2-category of CFVDCs, although we do not have a formal proof of this statement and leave it as a conjecture.

If one also wants to make the type theory FVDblTT proof irrelevant, one can reformulate protype isomorphism judgment as equality judgments of protypes and add the rule stating that all the proterms are equal. It is the counterpart of the flatness [**GP99**] or local preorderedness [**HN23**] of VDCs.

3.3.2. The derivation rules for the additional constructors. In Section 3.3, we explain some additional constructors of FVDblTT that are meaningful both in the contexts of formal category theory and predicate logic. In this section, we provide all the derivation rules of the constructs.

Unit protype.

$$\frac{I\ \mathsf{type} \qquad \Gamma \vdash s : I \qquad \Delta \vdash t : I}{\Gamma \fatsemi \Delta \vdash s \nrightarrow_I t\ \mathsf{protype}}\ {\nrightarrow}\text{-}\textsc{Form} \qquad\qquad \frac{I\ \mathsf{type}}{x : I \mid\ \ \vdash \mathsf{refl}_I(x) : x \nrightarrow_I x}\ {\nrightarrow}\text{-}\textsc{Intro}$$

$$\frac{w_0 : J_0 \fatsemi z_m : K_m \vdash \gamma(w_0 \fatsemi z_m)\ \mathsf{protype} \qquad \overline{w} : \overline{J} \fatsemi x : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi \overline{B}(x \fatsemi \overline{z}) \vdash \mu : \gamma(w_0 \fatsemi z_m)}{\overline{w} : \overline{J} \fatsemi x : I \fatsemi y : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi p : x \nrightarrow_I y \fatsemi \overline{B}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\nrightarrow_I}\{\mu\} : \gamma(w_0 \fatsemi z_m)}\ {\nrightarrow}\text{-}\textsc{Elim}$$

$$\frac{\overline{w} : \overline{J} \fatsemi x : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi \overline{B}(x \fatsemi \overline{z}) \vdash \mu : \gamma(w_0 \fatsemi z_m)}{\overline{w} : \overline{J} \fatsemi x : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi \overline{B}(x \fatsemi \overline{z}) \vdash (\mathsf{ind}_{\nrightarrow_I}\{\mu\})\,[x/y]\,⁅\mathsf{refl}_I(x)/p⁆ \equiv \mu : \gamma(w_0 \fatsemi z_m)}\ {\nrightarrow}\text{-}\textsc{Comp}\beta$$

$$\frac{\overline{w} : \overline{J} \fatsemi x : I \fatsemi y : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi p : x \nrightarrow_I y \fatsemi \overline{B}(y \fatsemi \overline{z}) \vdash \nu : \gamma(w_0 \fatsemi z_m)}{\overline{w} : \overline{J} \fatsemi x : I \fatsemi y : I \fatsemi \overline{z} : \overline{K} \mid \overline{A}(\overline{w} \fatsemi x) \fatsemi p : x \nrightarrow_I y \fatsemi \overline{B}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\nrightarrow_I}\{\nu[x/y]\,⁅\mathsf{refl}_I(x)/p⁆\} \equiv \nu : \gamma(w_0 \fatsemi z_m)}\ {\nrightarrow}\text{-}\textsc{Comp}\eta$$

Unit protype meets product type.

$$\frac{}{\cdot \fatsemi \cdot \vdash \mathsf{exc}_{\nrightarrow,\top} : \langle\rangle \nrightarrow_{\mathbf{1}} \langle\rangle \ \cong\ \top}\ {\nrightarrow}\text{-}\top \qquad \frac{I\ \mathsf{type} \qquad J\ \mathsf{type}}{x : I, y : J \fatsemi x' : I, y' : J \vdash \mathsf{exc}_{\nrightarrow,\wedge} : \langle x, y\rangle \nrightarrow_{I\times J} \langle x', y'\rangle \ \cong\ x \nrightarrow_I x' \wedge y \nrightarrow_J y'}\ {\nrightarrow}\text{-}\wedge$$

$$\frac{I\ \mathsf{type} \qquad J\ \mathsf{type}}{x : I, y : J \fatsemi x' : I, y' : J \mid a : \langle x, y\rangle \nrightarrow_{I\times J} \langle x', y'\rangle \vdash \mathsf{exc}_{\nrightarrow,\wedge}\{a\} \equiv \mathsf{ind}_{\nrightarrow_{I\times J}}\{\langle \mathsf{refl}_I(x), \mathsf{refl}_J(y)\rangle\} : x \nrightarrow_I x' \wedge y \nrightarrow_J y'}$$

$$\text{where}\quad \frac{x : I, y : J \mid \langle \mathsf{refl}_I(x), \mathsf{refl}_J(y)\rangle : x \nrightarrow_I x' \wedge y \nrightarrow_J y'}{x : I, y : J \fatsemi x' : I, y' : J \mid a : \langle x, y\rangle \nrightarrow_{I\times J} \langle x', y'\rangle \vdash \mathsf{ind}_{\nrightarrow_{I\times J}}\{\langle \mathsf{refl}_I(x), \mathsf{refl}_J(y)\rangle\} : x \nrightarrow_I x' \wedge y \nrightarrow_J y'}$$

Composition protype.

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad x : J \fatsemi y : K \vdash \beta(x \fatsemi y)\ \mathsf{protype}}{w : I \fatsemi y : K \vdash \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y)\ \mathsf{protype}}\ \odot\text{-}\textsc{Form}$$

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x)\ \mathsf{protype} \qquad x : J \fatsemi y : K \vdash \beta(x \fatsemi y)\ \mathsf{protype}}{w : I \fatsemi x : J \fatsemi y : K \mid a : \alpha(w \fatsemi x) \fatsemi b : \beta(x \fatsemi y) \vdash a \odot b : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y)}\ \odot\text{-}\textsc{Intro}$$

$$\frac{\overline{v} : \overline{H} \fatsemi w : I \fatsemi x : J \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi a : \alpha(w \fatsemi x) \fatsemi b : \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mu : \gamma(v_0 \fatsemi z_m)}{\overline{v} : \overline{H} \fatsemi w : I \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi p : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\odot_{\alpha,\beta}}\{\mu\} : \gamma(v_0 \fatsemi z_m)}\ \odot\text{-}\textsc{Elim}$$

$$\frac{\overline{v} : \overline{H} \fatsemi w : I \fatsemi x : J \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi \alpha(w \fatsemi x) \fatsemi \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mu : \gamma(v_0 \fatsemi z_m)}{\begin{array}{l}\overline{v} : \overline{H} \fatsemi w : I \fatsemi x : J \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi a : \alpha(w \fatsemi x) \fatsemi b : \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \vdash \left(\mathsf{ind}_{\odot_{\alpha,\beta}}\{\mu\}\right)⁅a \odot b/p⁆ \equiv \mu : \gamma(v_0 \fatsemi z_m)\end{array}}\ \odot\text{-}\textsc{Comp}\beta$$

$$\frac{\overline{v} : \overline{H} \fatsemi w : I \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi p : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \nu : \gamma(v_0 \fatsemi z_m)}{\overline{v} : \overline{H} \fatsemi w : I \fatsemi y : K \fatsemi \overline{z} : \overline{L} \mid \overline{C}(\overline{v} \fatsemi w) \fatsemi p : \alpha(w \fatsemi x) \odot_{x:J} \beta(x \fatsemi y) \fatsemi \overline{D}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\odot_{\alpha,\beta}}\{(\nu\,⁅a \odot b/p⁆)\} \equiv \nu : \gamma(v_0 \fatsemi z_m)}\ \odot\text{-}\textsc{Comp}\eta$$

Composition protype meets product type.

$$\frac{}{\cdot \fatsemi \cdot \vdash \mathsf{exc}_{\odot,\top} : \top \odot_{\langle\rangle:\cdot} \top \ \cong\ \top}\ \odot\text{-}\top$$

$$\frac{\begin{array}{c} x : I \fatsemi y : J \vdash \alpha(x \fatsemi y)\ \mathsf{protype} \\ y : J \fatsemi z : K \vdash \beta(y \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v)\ \mathsf{protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype}\end{array}}{\begin{array}{l} x : I, u : L \fatsemi z : K, w : N \vdash \mathsf{exc}_{\odot,\wedge} : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \odot_{\langle y,v\rangle : J\times M} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \cong\ \left(\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)\right) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}\ \odot\text{-}\wedge$$

$$\frac{\begin{array}{c} x : I \fatsemi y : J \vdash \alpha(x \fatsemi y)\ \mathsf{protype} \\ y : J \fatsemi z : K \vdash \beta(y \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v)\ \mathsf{protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype}\end{array}}{\begin{array}{l} x : I, u : L \fatsemi z : K, w : N \mid e : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \odot_{\langle y,v\rangle : J\times M} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{exc}_{\odot,\wedge}\{e\} \equiv \mathsf{ind}_{\odot_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \odot \pi_0\{b\}, \pi_1\{a\} \odot \pi_1\{b\}\rangle\} : \left(\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)\right) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}$$

$$\text{where}\quad \frac{\begin{array}{l} x : I \fatsemi u : L \fatsemi y : J \fatsemi v : M \fatsemi z : K \fatsemi w : N \mid a : \alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v) \fatsemi b : \beta(y \fatsemi z) \wedge \delta(v \fatsemi w) \\ \quad \vdash \langle \pi_0\{a\} \odot \pi_0\{b\}, \pi_1\{a\} \odot \pi_1\{b\}\rangle : \left(\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)\right) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}{\begin{array}{l} x : I \fatsemi u : L \fatsemi z : K \fatsemi w : N \mid e : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \odot_{\langle y,v\rangle : J\times M} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{ind}_{\odot_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \odot \pi_0\{b\}, \pi_1\{a\} \odot \pi_1\{b\}\rangle\} : \left(\alpha(x \fatsemi y) \odot_{y:J} \beta(y \fatsemi z)\right) \wedge (\gamma(u \fatsemi v) \odot_{v:M} \delta(v \fatsemi w))\end{array}}$$

Filler protype.

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x) \text{ protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y) \text{ protype}}{x : J \fatsemi y : K \vdash \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \text{ protype}} \ \triangleright\text{-}\textsc{Form}$$

$$\frac{w : I \fatsemi x : J \fatsemi \overline{y} : \overline{L} \mid a : \alpha(w \fatsemi x) \fatsemi \overline{C}(x \fatsemi \overline{y}) \vdash \mu : \beta(w \fatsemi y_m)}{x : J \fatsemi \overline{y} : \overline{L} \mid \overline{C}(x \fatsemi \overline{y}) \vdash \mathsf{ind}_{\triangleright_{\alpha,\beta}}\{\mu\} : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y_m)} \ \triangleright\text{-}\textsc{Intro}$$

$$\frac{w : I \fatsemi x : J \vdash \alpha(w \fatsemi x) \text{ protype} \qquad w : I \fatsemi y : K \vdash \beta(w \fatsemi y) \text{ protype}}{w : I \fatsemi x : J \fatsemi y : K \mid a : \alpha(w \fatsemi x) \fatsemi e : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y) \vdash a \blacktriangleright e : \beta(w \fatsemi y)} \ \triangleright\text{-}\textsc{Elim}$$

$$\frac{w : I \fatsemi x : J \fatsemi \overline{y} : \overline{L} \mid a : \alpha(w \fatsemi x) \fatsemi \overline{C}(x \fatsemi \overline{y}) \vdash \mu : \beta(w \fatsemi y_m)}{w : I \fatsemi x : J \fatsemi \overline{y} : \overline{L} \mid a : \alpha(w \fatsemi x) \fatsemi \overline{C}(x \fatsemi \overline{y}) \vdash a \blacktriangleright \left(\mathsf{ind}_{\triangleright_{\alpha,\beta}}\{\mu\}\right) \equiv \mu : \beta(w \fatsemi y_m)} \ \triangleright\text{-}\textsc{Comp}\beta$$

$$\frac{x : J \fatsemi \overline{y} : \overline{L} \mid \overline{C}(x \fatsemi \overline{y}) \vdash \nu : \alpha(w \fatsemi x) \triangleright_{w:I} \beta(w \fatsemi y_m)}{x : J \fatsemi \overline{y} : \overline{L} \mid \overline{C}(x \fatsemi \overline{y}) \vdash \mathsf{ind}_{\triangleright_{\alpha,\beta}}\{a \blacktriangleright \nu\} \equiv \nu : \beta(w \fatsemi y_m)} \ \triangleright\text{-}\textsc{Comp}\eta$$

$$\frac{y : J \fatsemi z : K \vdash \alpha(y \fatsemi z) \text{ protype} \qquad x : I \fatsemi z : K \vdash \beta(x \fatsemi z) \text{ protype}}{x : I \fatsemi y : J \vdash \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z) \text{ protype}} \ \triangleleft\text{-}\textsc{Form}$$

$$\frac{\overline{x} : \overline{J} \fatsemi y : J \fatsemi z : K \mid \overline{C}(\overline{x} \fatsemi y) \fatsemi a : \alpha(y \fatsemi z) \vdash \mu : \beta(x \fatsemi z)}{\overline{x} : \overline{J} \fatsemi y : J \mid \overline{C}(\overline{x} \fatsemi y) \vdash \mathsf{ind}_{\triangleleft_{\alpha,\beta}}\{\mu\} : \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z)} \ \triangleleft\text{-}\textsc{Intro}$$

$$\frac{x : I \fatsemi y : J \vdash \beta(x \fatsemi z) \text{ protype} \qquad y : J \fatsemi z : K \vdash \alpha(y \fatsemi z) \text{ protype}}{x : I \fatsemi y : J \fatsemi z : K \mid a : \beta(x \fatsemi z) \fatsemi e : \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z) \vdash a \blacktriangleleft e : \alpha(y \fatsemi z)} \ \triangleleft\text{-}\textsc{Elim}$$

$$\frac{x : I \fatsemi y : J \fatsemi \overline{z} : \overline{L} \mid a : \beta(x \fatsemi z) \fatsemi \overline{C}(x \fatsemi \overline{z}) \vdash \mu : \alpha(y \fatsemi z_m)}{x : I \fatsemi y : J \fatsemi \overline{z} : \overline{L} \mid a : \beta(x \fatsemi z) \fatsemi \overline{C}(x \fatsemi \overline{z}) \vdash a \blacktriangleleft \left(\mathsf{ind}_{\triangleleft_{\alpha,\beta}}\{\mu\}\right) \equiv \mu : \alpha(y \fatsemi z_m)} \ \triangleleft\text{-}\textsc{Comp}\beta$$

$$\frac{y : J \fatsemi \overline{z} : \overline{L} \mid \overline{C}(y \fatsemi \overline{z}) \vdash \nu : \beta(x \fatsemi z) \triangleleft_{z:K} \alpha(y \fatsemi z)}{y : J \fatsemi \overline{z} : \overline{L} \mid \overline{C}(y \fatsemi \overline{z}) \vdash \mathsf{ind}_{\triangleleft_{\alpha,\beta}}\{a \blacktriangleleft \nu\} \equiv \nu : \alpha(y \fatsemi z_m)} \ \triangleleft\text{-}\textsc{Comp}\eta$$

Filler protype meets product type.

$$\frac{}{\cdot \fatsemi \cdot \mid \mathsf{exc}_{\triangleright,\top} : \top \triangleright .\ \top \cong \top} \ \triangleright\text{-}\top$$

$$\frac{\begin{array}{c} x : I \fatsemi y : J \vdash \alpha(x \fatsemi y) \text{ protype} \\ x : I \fatsemi z : K \vdash \beta(x \fatsemi z) \text{ protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v) \text{ protype} \qquad u : L \fatsemi w : N \vdash \delta(v \fatsemi w) \text{ protype} \end{array}}{\begin{array}{l} y : J, v : M \fatsemi z : K, w : N \vdash \mathsf{exc}_{\triangleright,\wedge} : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \cong (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \triangleright_{x:I,u:L} (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}} \ \triangleright\text{-}\wedge$$

$$\frac{\begin{array}{c} x : I \fatsemi y : J \vdash \alpha(x \fatsemi y) \text{ protype} \\ x : I \fatsemi z : K \vdash \beta(x \fatsemi z) \text{ protype} \qquad u : L \fatsemi v : M \vdash \gamma(u \fatsemi v) \text{ protype} \qquad u : L \fatsemi w : N \vdash \delta(v \fatsemi w) \text{ protype} \end{array}}{\begin{array}{l} y : J, v : M \fatsemi z : K, w : N \mid e : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{exc}_{\triangleright,\wedge}\{e\} \equiv \mathsf{ind}_{\triangleright_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \blacktriangleright \pi_0(e), \pi_1\{a\} \blacktriangleright \pi_1(e)\rangle\} : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \triangleright_{x:I,u:L} (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}} \ \triangleright\text{-}\wedge\text{-}\textsc{canon}$$

where

$$\frac{\begin{array}{l} x : I, u : L, y : J, v : M, z : K, w : N \mid a : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \fatsemi e : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \quad \vdash \langle \pi_0\{a\} \blacktriangleright \pi_0(e), \pi_1\{a\} \blacktriangleright \pi_1(e)\rangle : (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}{\begin{array}{l} y : J, v : M \fatsemi z : K, w : N \mid e : (\alpha(x \fatsemi y) \triangleright_{x:I} \beta(x \fatsemi z)) \wedge (\gamma(u \fatsemi v) \triangleright_{u:L} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{ind}_{\triangleright_{\alpha\wedge\gamma,\beta\wedge\delta}}\{\langle \pi_0\{a\} \blacktriangleright \pi_0(e), \pi_1\{a\} \blacktriangleright \pi_1(e)\rangle\} : (\alpha(x \fatsemi y) \wedge \gamma(u \fatsemi v)) \triangleright_{x:I,u:L} (\beta(x \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}$$

$$\frac{}{\cdot \fatsemi \cdot \vdash \mathsf{exc}_{\triangleleft,\top} : \top \triangleleft .\ \top \equiv \top} \ \triangleleft\text{-}\top$$

$$\frac{\begin{array}{c} x : I \fatsemi z : K \vdash \alpha(x \fatsemi z) \text{ protype} \\ y : J \fatsemi z : K \vdash \beta(y \fatsemi z) \text{ protype} \qquad u : L \fatsemi w : N \vdash \gamma(u \fatsemi w) \text{ protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w) \text{ protype} \end{array}}{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M \vdash \mathsf{exc}_{\triangleleft,\wedge} : (\alpha(x \fatsemi z) \triangleleft_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \triangleleft_{w:N} \delta(v \fatsemi w)) \\ \cong (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \triangleleft_{z:K,w:N} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}} \ \triangleleft\text{-}\wedge$$

$$\frac{\begin{array}{c} x : I \fatsemi z : K \vdash \alpha(x \fatsemi z)\ \mathsf{protype} \\ y : J \fatsemi z : K \vdash \beta(y \fatsemi z)\ \mathsf{protype} \qquad u : L \fatsemi w : N \vdash \gamma(u \fatsemi w)\ \mathsf{protype} \qquad v : M \fatsemi w : N \vdash \delta(v \fatsemi w)\ \mathsf{protype} \end{array}}{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M \mid e : (\alpha(x \fatsemi z) \lhd_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \lhd_{w:N} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{exc}_{\lhd,\wedge}\{e\} \equiv \mathsf{ind}_{\lhd_{\alpha\wedge\gamma,\beta\wedge\delta}} \{\langle \pi_0\{a\} \blacktriangleleft \pi_0(e), \pi_1\{a\} \blacktriangleleft \pi_1(e)\rangle\} : (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \lhd_{z:K,w:N} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}} \ \lhd\text{-}\wedge\text{-CANON}$$

where
$$\frac{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M, z : K, w : N \mid a : (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \fatsemi e : (\alpha(x \fatsemi z) \lhd_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \lhd_{w:N} \delta(v \fatsemi w)) \\ \quad \vdash \langle \pi_0\{a\} \blacktriangleleft \pi_0(e), \pi_1\{a\} \blacktriangleleft \pi_1(e)\rangle : (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}{\begin{array}{l} x : I, u : L \fatsemi y : J, v : M \mid e : (\alpha(x \fatsemi z) \lhd_{z:K} \beta(y \fatsemi z)) \wedge (\gamma(u \fatsemi w) \lhd_{w:N} \delta(v \fatsemi w)) \\ \quad \vdash \mathsf{ind}_{\lhd_{\alpha\wedge\gamma,\beta\wedge\delta}} \{\langle \pi_0\{a\} \blacktriangleleft \pi_0(e), \pi_1\{a\} \blacktriangleleft \pi_1(e)\rangle\} : (\alpha(x \fatsemi z) \wedge \gamma(u \fatsemi w)) \lhd_{z:K,w:N} (\beta(y \fatsemi z) \wedge \delta(v \fatsemi w)) \end{array}}$$

Comprehension type.

$$\frac{x : I \fatsemi y : J \vdash \alpha\ \mathsf{protype}}{\{\!|\alpha|\!\}\ \mathsf{type}} \ \{\!|\,|\!\}\text{-FORM} \qquad \frac{x : I \fatsemi y : J \vdash \alpha\ \mathsf{protype}}{w : \{\!|\alpha|\!\} \vdash l(w) : I} \ \{\!|\,|\!\}\text{-ELIM-}\ell \qquad \frac{x : I \fatsemi y : J \vdash \alpha\ \mathsf{protype}}{w : \{\!|\alpha|\!\} \vdash r(w) : J} \ \{\!|\,|\!\}\text{-ELIM-}r$$

$$\frac{x : I \fatsemi y : J \vdash \alpha\ \mathsf{protype}}{w : \{\!|\alpha|\!\} \mid\!\vdash \mathsf{tab}_{\{\!|\alpha|\!\}}\{w\} : \alpha[l(w)/x \fatsemi r(w)/y]} \ \{\!|\,|\!\}\text{-ELIM-CELL}$$

$$\frac{x : I \fatsemi y : J \vdash \alpha\ \mathsf{protype} \qquad \Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash \mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu) : \{\!|\alpha|\!\}} \ \{\!|\,|\!\}\text{-INTRO}$$

$$\frac{\Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash l(\mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu)) \equiv s : I} \ \{\!|\,|\!\}\text{-COMP-}\ell \qquad \frac{\Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash r(\mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu)) \equiv t : J} \ \{\!|\,|\!\}\text{-COMP-}r$$

$$\frac{x : I \fatsemi y : J \vdash \alpha\ \mathsf{protype} \qquad \Gamma \vdash s : I \qquad \Gamma \vdash t : J \qquad \Gamma \mid\!\vdash \nu : \alpha[s/x \fatsemi t/y]}{\Gamma \vdash \mathsf{tab}_{\{\!|\alpha|\!\}}\{\mathsf{ind}_{\{\!|\,|\!\}}(s, t, \nu)\} \equiv \nu : \alpha[s/x \fatsemi t/y]} \ \{\!|\,|\!\}\text{-COMP-}\beta$$

$$\frac{x : I \fatsemi y : J \vdash \alpha\ \mathsf{protype}}{w : \{\!|\alpha|\!\} \vdash \mathsf{ind}_{\{\!|\,|\!\}}(l(w), r(w), \mathsf{tab}_{\{\!|\alpha|\!\}}\{w\} \ \equiv w : \{\!|\alpha|\!\}} \ \{\!|\,|\!\}\text{-COMP-}\eta$$

Comprehension type meets unit protype.

$$\frac{\begin{array}{c} \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_0 \vdash t_0 : J \qquad \Gamma_m \vdash t_1 : J \qquad x : I, y : J \vdash \alpha(x, y)\ \mathsf{protype} \\ \Gamma_0 \mid\!\vdash \mu_0 : \alpha(s_0 \fatsemi t_0) \qquad \Gamma_m \mid\!\vdash \mu_1 : \alpha(s_1 \fatsemi t_1) \qquad \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \overline{\Gamma} \mid B \vdash j : t_0 \nrightarrow_J t_1 \qquad \overline{\Gamma} \mid B \vdash i \boxdot \mu_1 \equiv \mu_0 \boxdot j \end{array}}{\overline{\Gamma} \mid B \vdash \mathsf{ind}_{\{\!|\,|\!\}}(i, j, \mu_0, \mu_1) : \mathsf{ind}_{\{\!|\,|\!\}}(s_0, t_0, \mu_0) \nrightarrow_{\{\!|\alpha|\!\}} \mathsf{ind}_{\{\!|\,|\!\}}(s_1, t_1, \mu_1)} \ \{\!|\,|\!\}\text{-ELIM}$$

where
$$\frac{\begin{array}{c} \dfrac{x : I \fatsemi y : J \mid a : \alpha(x \fatsemi y) \vdash a : \alpha(x' \fatsemi y)}{x : I \fatsemi x' : I \fatsemi y : J \mid p : x \nrightarrow_I x' \fatsemi a : \alpha(x \fatsemi y) \vdash \mathsf{ind}_{\nrightarrow}\{a\} : \alpha(x \fatsemi y)} \qquad \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_m \vdash t_1 : J \\ \hline \overline{\Gamma} \mid p : s_0 \nrightarrow_I s_1 \fatsemi a : \alpha(s_1 \fatsemi t_1) \vdash \mathsf{ind}_{\nrightarrow}\{a\}[s_1/x' \fatsemi t_1/y] : \alpha(s_0 \fatsemi t_1) \\ \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \Gamma_m \mid\!\vdash \mu_1 : \alpha(s_1 \fatsemi t_1) \end{array}}{\overline{\Gamma} \mid B \vdash i \boxdot \mu_1 :\equiv \mathsf{ind}_{\nrightarrow}\{a\}[s_1/x' \fatsemi t_1/y] \,\{\!\!\{ i/p : s_0 \nrightarrow_I s_1 \fatsemi \mu_1/a : \alpha(s_1 \fatsemi t_1) \}\!\!\} : \alpha(s_0 \fatsemi t_1)}$$

and similarly for $\mu_0 \boxdot j$.

$$\frac{\begin{array}{c} \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_0 \vdash t_0 : J \qquad \Gamma_m \vdash t_1 : J \qquad x : I, y : J \vdash \alpha(x, y)\ \mathsf{protype} \qquad \Gamma_0 \mid\!\vdash \mu_0 : \alpha[s_0/x \fatsemi t_0/y] \\ \Gamma_m \mid\!\vdash \mu_1 : \alpha[s_1/x \fatsemi t_1/y] \qquad \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \overline{\Gamma} \mid B \vdash j : t_0 \nrightarrow_J t_1 \qquad \overline{\Gamma} \mid B \vdash i \boxdot \mu_1 \equiv \mu_0 \boxdot j \end{array}}{\overline{\Gamma} \mid B \vdash \mathsf{app}_l(\mathsf{ind}_{\{\!|\,|\!\}}(i, j, \mu_0, \mu_1)) \equiv i : s_0 \nrightarrow_I s_1} \ \{\!|\,|\!\}\text{-COMP}$$

$$\frac{\begin{array}{c} \Gamma_0 \vdash s_0 : I \qquad \Gamma_m \vdash s_1 : I \qquad \Gamma_0 \vdash t_0 : J \qquad \Gamma_m \vdash t_1 : J \qquad x : I, y : J \vdash \alpha(x, y)\ \mathsf{protype} \qquad \Gamma_0 \mid\!\vdash \mu_0 : \alpha[s_0/x \fatsemi t_0/y] \\ \Gamma_m \mid\!\vdash \mu_1 : \alpha[s_1/x \fatsemi t_1/y] \qquad \overline{\Gamma} \mid B \vdash i : s_0 \nrightarrow_I s_1 \qquad \overline{\Gamma} \mid B \vdash j : t_0 \nrightarrow_J t_1 \qquad \overline{\Gamma} \mid B \vdash i \boxdot \mu_1 \equiv \mu_0 \boxdot j \end{array}}{\overline{\Gamma} \mid B \vdash \mathsf{app}_r(\mathsf{ind}_{\{\!|\,|\!\}}(i, j, \mu_0, \mu_1)) \equiv j : t_0 \nrightarrow_J t_1} \ \{\!|\,|\!\}\text{-COMP}$$

Concerning the filler protype, we have the following supporting observation.

Proposition 3.3.4. Let $\mathbf{FVDbl}_{\triangleright}$ be the locally-full sub-2-category of $\mathbf{FVDbl}$ spanned by the FVDCs with right extensions and functors preserving right extensions. A VDC $\mathbb{X}$ in $\mathbf{FVDbl}_{\triangleright}$ is cartesian in this 2-category if and only if

(i) $\mathbb{X}$ is a cartesian FVDC,
(ii) $\top_{1,1} \triangleright \top_{1,1} \cong \top_{1,1}$ canonically in $\mathbb{X}(1,1)$, and
(iii) for any quadruples of loose arrows

$$\begin{tikzcd} I_0 \arrow[r, "\alpha_1", "|"{marking}] \arrow[rr, bend right, "\alpha_2"', "|"{marking}] & I_1 & I_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} J_0 \arrow[r, "\beta_1", "|"{marking}] \arrow[rr, bend right, "\beta_2"', "|"{marking}] & J_1 & J_2 \end{tikzcd}$$

in $\mathbb{X}$, we have

$$(\alpha_1 \triangleright \alpha_2) \times (\beta_1 \triangleright \beta_2) \cong (\alpha_1 \times \beta_1) \triangleright (\alpha_2 \times \beta_2)$$

canonically in $\mathbb{X}(I_1 \times J_1, I_2 \times J_2)$.

⌟

3.4. Examples of calculus

This section exemplifies how one can reason about category theory and logic formally in the type theory FVDblTT.

Example 3.4.1 ((co)Yoneda Lemma)**.** One of the most fundamental results in category theory is the Yoneda Lemma, and it has a variety of presentations in the literature. Here we present one called the Yoneda Lemma [**Lor21**, Proposition 2.2.1]: *given a category $\mathcal{C}$ and a functor $F\colon \mathcal{C}^{op} \to \mathcal{S}\mathrm{et}$, we have the canonical isomorphism*

$$F \cong \int_{X\in\mathcal{C}} [\mathcal{C}(X,-), FX].$$

This follows from the categorical fact that $\mathbb{P}\mathrm{rof}$ is an FVDC with the structures listed above. Indeed, in the type theory FVDblTT with the path protype $\not\twoheadrightarrow$ and the filler protype $\triangleright$, one can deduce the following:

$$y : I \fatsemi \cdot \vdash \mathsf{Yoneda} : (x \not\twoheadrightarrow_I y) \triangleright_{x:I} \alpha(x) \ \overset{\sim}{\equiv}\ \alpha(y)$$

Similarly, we have

$$y : I \fatsemi \cdot \vdash \mathsf{CoYoneda} : (y \not\twoheadrightarrow_I x) \odot_{x:I} \alpha(x) \ \overset{\sim}{\equiv}\ \alpha(y)$$

which expresses the coYoneda Lemma:

$$\int^{X\in\mathcal{C}} \mathcal{C}(-,X) \times FX \cong F.$$

In short, all the theorems in category theory that can be proven using this type theory fall into corollaries of the theorem that $\mathbb{P}\mathrm{rof}$ is a CFVDC with the structures corresponding to the constructors. Other examples include the unit laws and the associativity of the composition of profunctors or the iteration of extensions and lifts of profunctors.

Turning to the aspect of predicate logic, we can interpret the protype isomorphisms as the following logical equivalences.

$$\begin{array}{rcl} \varphi(y) & \equiv & \forall x \in I.\,(x = y) \Rightarrow \varphi(x) \\ \varphi(y) & \equiv & \exists x \in I.\,(x = y) \wedge \varphi(x) \end{array}$$

⌟

Example 3.4.2 (Isomorphism of functors)**.** A natural transformation $\xi\colon F \to G$ between two functors $F, G\colon \mathcal{C} \to \mathcal{D}$ is given by a family of arrows $\xi_X\colon FX \to GX$ satisfying some naturality conditions. In the type theory FVDblTT with the path protype $\not\twoheadrightarrow$, this natural transformation can be represented by a proterm $x : I \mathrel{|}\!\vdash \xi(x) : f(x) \not\twoheadrightarrow_I g(x)$. Here, the naturality condition automatically holds because we describe it as a proterm. The isomorphism of functors can be expressed using this notion, but an alternative way is to use the protype isomorphism.

Lemma 3.4.3. Given two terms, $f(x)$ and $g(x)$, in the same context, the following are equivalent.

(i) There are proterms $\xi(x) : f(x) \not\twoheadrightarrow_I g(x)$ and $\eta(x) : g(x) \not\twoheadrightarrow_I f(x)$ such that $\xi(x) \boxdot \eta(x) \equiv \mathsf{refl}_{f(x)}$ and $\eta(x) \boxdot \xi(x) \equiv \mathsf{refl}_{g(x)}$.

(ii) There is a protype isomorphism $Z : y \not\twoheadrightarrow_J f(x) \ \overset{\sim}{\equiv}\ y \not\twoheadrightarrow_I g(x)$.

Here, $\boxdot$ is a tailored constructor defined as follows.

$$y : J \fatsemi y' : J \fatsemi y'' : J \mid\ a : y \not\twoheadrightarrow_J y' \fatsemi b : y' \not\twoheadrightarrow_J y'' \vdash a \boxdot b :\equiv \mathsf{ind}_{\not\twoheadrightarrow_J}(a) : y \not\twoheadrightarrow_J y''.$$

⌟

Proof. First, suppose (i) holds. We define a proterm ζ by the following:

$$\begin{array}{c} x : I \mathrel{|}\!\vdash \xi : f(x) \not\twoheadrightarrow_J g(x) \\ y : J \fatsemi y' : J \fatsemi y'' : J \mid a : y \not\twoheadrightarrow_J y' \fatsemi b : y' \not\twoheadrightarrow_J y'' \vdash a \boxdot b : y \not\twoheadrightarrow_J y'' \\ \hline y : J \fatsemi x : I \fatsemi x' : I \mid a : y \not\twoheadrightarrow_J f(x) \fatsemi b : f(x) \not\twoheadrightarrow_J g(x) \vdash a \boxdot b[y/y \fatsemi f(x)/y' \fatsemi g(x)/y''] : y \not\twoheadrightarrow_J g(x) \\ \hline y : J \fatsemi x : I \mid a : y \not\twoheadrightarrow_J f(x) \vdash \zeta(a) : y \not\twoheadrightarrow_J g(x) \end{array}$$

Therefore, we have $\zeta(a)$, and in the same way, we can define a proterm $b : y \nrightarrow_J g(x) \vdash \zeta'(b) : y \nrightarrow_J f(x)$, which is the inverse of ζ by simple reasoning.

Next, suppose (ii) holds. Let $a : y \nrightarrow_J f(x) \vdash \zeta(a) : y \nrightarrow_J g(x)$ be the proterm witnessing the isomorphism. By substituting $f(x)$ for y and the refl for a, we obtain a proterm $\xi(x) : f(x) \nrightarrow_J g(x)$. In the same way, we can define a proterm $\eta(x) : g(x) \nrightarrow_J f(x)$, for which the two desired equalities hold. ⌟

We therefore use the equalities $y \nrightarrow_J f(x)$ and $y \nrightarrow_J g(x)$ when f and g are already proven to be isomorphic. ⌟

Example 3.4.4 (Adjunction)**.** In a virtual double category, the ***companion*** and ***conjoint*** of a tight arrow $f\colon A \to B$ is defined as the loose arrows $f_*\colon A \nrightarrow B$ and $f^*\colon B \nrightarrow A$ equipped with cells satisfying some equations of cells [**GP04**, **CS10**]. In a virtual equipment, it is known that the companion and conjoint of a tight arrow $f\colon A \to B$ are the restrictions of the units on B along the pairs of tight arrows $(f, \mathtt{id}_B)$ and $(\mathtt{id}_B, f)$, respectively. These notions are the formalization of the representable profunctors in the virtual double categories. Therefore, the companions and conjoints of a term $t(x)$ in the type theory FVDblTT should be defined as $t(x) \nrightarrow_I y$ and $y \nrightarrow_I t(x)$, respectively.

The ***adjunction*** between two functors is described in terms of representable profunctors, which motivates the following definition of the adjunction in the type theory FVDblTT. Remember a functor $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G\colon \mathcal{D} \to \mathcal{C}$ if there is a natural isomorphism between the hom-sets

$$\mathcal{D}(F-, \bullet) \cong \mathcal{C}(-, G\bullet).$$

In the type theory FVDblTT, a term $t(x)$ is announced to be a left adjoint to a term $u(y)$ if the following equality holds:

$$x : I \fatsemi y : J \vdash t(x) \nrightarrow_J y \equiv x \nrightarrow_I u(y).$$

⌟

Example 3.4.5 (Kan extension)**.** In [**Kel05**], the (pointwise) left Kan extension $\mathrm{Lan}_G F$ of a functor $F\colon \mathcal{C} \to \mathcal{D}$ along a functor $G\colon \mathcal{C} \to \mathcal{E}$ is defined as a functor $H\colon \mathcal{D} \to \mathcal{E}$ equipped with a natural transformation

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & \arrow[d, phantom, "\mu\Downarrow"] & \mathcal{D} \\
& \mathcal{E} \arrow[ur, "H"'] &
\end{tikzcd}$$

with the following canonical natural transformation being an isomorphism:

$$\mathcal{D}(HE, D) \overset{\cong}{\to} \widehat{\mathcal{C}}\,(\mathcal{E}(G-, E), \mathcal{D}(F-, D)) \quad \text{naturally in } D \in \mathcal{D}, E \in \mathcal{E}.$$

A protype isomorphism corresponding to this isomorphism is given by the following.

$$z : K \fatsemi y : J \vdash \mathsf{LeftKan} : h(z) \nrightarrow_J y \cong (g(x) \nrightarrow_K z) \triangleright_{x:I} (f(x) \nrightarrow_J y)$$

We will demonstrate how proofs in category theory can be done in the type theory FVDblTT.

Proposition 3.4.6 ([Kel05, Theorem 4.47]). $\mathrm{Lan}_{G'}\mathrm{Lan}_G F \cong \mathrm{Lan}_{G'\circ G} F$ hold for any functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{C} \to \mathcal{E}$, and $G'\colon \mathcal{E} \to \mathcal{F}$ if the Kan extensions exist.

$$\begin{tikzcd}
\mathcal{C} \arrow[rrr, "F"] \arrow[dr, "G"'] & \Downarrow & & \mathcal{D} \\
& \mathcal{E} \arrow[urr, "\mathrm{Lan}_G F"] \arrow[dr, "G'"'] & \Downarrow & \\
& & \mathcal{E}' \arrow[uur, "\mathrm{Lan}_{G'}\mathrm{Lan}_G F \cong \mathrm{Lan}_{G'\circ G} F"'] &
\end{tikzcd}$$

⌟

Proof. We associate $F, G, G', \mathrm{Lan}_G F, \mathrm{Lan}_{G'}\mathrm{Lan}_G F, \mathrm{Lan}_{G'\circ G} F$ with the terms $f(x), g(x), g'(z), h(z), h'(z')$, and $h''(z')$. We will have the desired protype isomorphism judgment by composing the protype isomorphisms in the following order.

$$\begin{aligned}
&z' : K' \fatsemi y : J \mid h'(z') \nrightarrow_J y \\
&\qquad \cong (g'(z) \nrightarrow_{K'} z') \triangleright_{z:K} (h(z) \nrightarrow_J y) && (\mathsf{LeftKan}) \\
&\qquad \cong (g'(z) \nrightarrow_{K'} z') \triangleright_{z:K} ((g(x) \nrightarrow_K z) \triangleright_{x:I} (f(x) \nrightarrow_J y)) && (\,(g'(z) \nrightarrow_{K'} z') \triangleright_{z:K} \mathsf{LeftKan}\,)
\end{aligned}$$

$$
\begin{aligned}
&\cong ((g(x) \nrightarrow_{K} z) \odot_{z:K} (g'(z) \nrightarrow_{K'} z')) \triangleright_{x:I} (f(x) \nrightarrow_{J} y) && (\mathsf{Fubini})\\
&\cong (g'(g(x)) \nrightarrow_{K'} z') \triangleright_{x:I} (f(x) \nrightarrow_{J} y) && (\,\mathsf{CoYoneda} \triangleright_{x:I} (f(x) \nrightarrow_{J} y)\,)\\
&\cong h''(z') \nrightarrow_{K'} y && (\mathsf{LeftKan}^{-1})
\end{aligned}
$$

Here, the protype isomorphism Fubini is given as $\llparenthesis \mathsf{Fubini}_1, \mathsf{Fubini}_2 \rrparenthesis$, where Fubini_1 and Fubini_2 are the proterms derived as follows.

$$
\dfrac{\dfrac{x_0 : I_0 \fatsemi x_1 : I_1 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid a : \alpha \fatsemi b : \beta \fatsemi c : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma) \vdash a \blacktriangleright (b \blacktriangleright c) : \gamma}{x_0 : I_0 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid d : \alpha \odot_{x_1:I_1} \beta \fatsemi c : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma) \vdash \_ : \gamma}}{x_2 : I_2 \fatsemi x_3 : I_3 \mid c : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma) \vdash \mathsf{Fubini}_1 : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma}
$$

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c} x_0 : I_0 \fatsemi x_1 : I_1 \fatsemi x_2 : I_2 \mid a : \alpha \fatsemi b : \beta \vdash a \odot b : \alpha \odot_{x_1:I_1} \beta \\ x_0 : I_0 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid d : \alpha \odot_{x_1:I_1} : \beta \fatsemi e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash d \blacktriangleright e : \gamma \end{array}}{x_0 : I_0 \fatsemi x_1 : I_1 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid a : \alpha \fatsemi b : \beta \fatsemi e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash \_ : \gamma}}{x_1 : I_1 \fatsemi x_2 : I_2 \fatsemi x_3 : I_3 \mid b : \beta \fatsemi e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash \_ : \alpha \triangleright_{x_0:I_0} \gamma}}{x_2 : I_2 \fatsemi x_3 : I_3 \mid e : (\alpha \odot_{x_1:I_1} \beta) \triangleright_{x_0:I_0} \gamma \vdash \mathsf{Fubini}_2 : \beta \triangleright_{x_1:I_1} (\alpha \triangleright_{x_0:I_0} \gamma)}
$$

⌟

⌟

3.5. A syntax-semantics adjunction for FVDblTT

Stating that a type theory is the internal language of a categorical structure always comes with the notion of a syntax-semantics adjunction. We set out to construct the term model of FVDblTT by following the standard procedure of categorical logic.

3.5.1. Syntactic presentation of virtual double categories.

Now, we turn to the definition of a specification for a signature in the type theory.

Definition 3.5.1. Let $\Phi \colon \Sigma \to \Sigma'$ be a morphism of signatures, and J be a judgment in the type theory based on Σ. We write J^{Φ} for the judgment in (Σ, E) defined by replacing each symbol in J with its image under Φ. J^{Φ} is called the ***translation*** of J via Φ. ⌟

Definition 3.5.2. A ***specification*** E for a signature Σ is a pair $(\mathrm{E}^{\mathsf{tm}}, \mathrm{E}^{\mathsf{ptm}})$ where

- E^{tm} is a class of pair of terms of the same type that are well-formed in Σ and $\mathrm{E}^{\mathsf{pty}}$,
- $\mathrm{E}^{\mathsf{ptm}}$ is a class of proterm equality judgments that are well-formed in $\Sigma, \mathrm{E}^{\mathsf{pty}}$ and E^{tm}.

When we say (Σ, E) is a specification, we mean that Σ is a signature and E is a specification for Σ.

A ***morphism of specifications*** $\Phi \colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$ is a morphism of signatures $\Phi \colon \Sigma \to \Sigma'$ by which every judgment in E is translated to a judgment that is derivable from E'. ⌟

Definition 3.5.3 (Validity of equality judgments)**.** We define the validity of equality judgments in a CFVDC as follows.

- A term equality judgment $t \equiv t'$ is ***valid*** in a Σ-structure $\mathcal{M}$ in a CFVDC $\mathbb{D}$ if $[\![t]\!]_{\mathcal{M}}$ and $[\![t']\!]_{\mathcal{M}}$ are equal as tight arrows in $\mathbb{D}$.
- A proterm equality judgment $\mu \equiv \mu'$ is ***valid*** in a Σ-structure $\mathcal{M}$ in a CFVDC $\mathbb{D}$ if $[\![\mu]\!]_{\mathcal{M}}$ and $[\![\mu']\!]_{\mathcal{M}}$ are equal as cells in $\mathbb{D}$.

⌟

With the definition of validity, one can canonically associate a specification $\mathrm{E}_{\mathbb{D}}$ to a CFVDC $\mathbb{D}$, which exhaustively contains the information of $\mathbb{D}$.

Definition 3.5.4. The ***associated specification*** $\mathrm{Sp}(\mathbb{D})$ of a CFVDC $\mathbb{D}$ is the specification $(\Sigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$ with $\Sigma_{\mathbb{D}}$ as above, $\mathrm{E}_{\mathbb{D}}^{\mathsf{tm}}$ (resp. $\mathrm{E}_{\mathbb{D}}^{\mathsf{ptm}}$) the set of all the valid equality judgments for terms (resp. proterms) in the canonical structure in $\mathbb{D}$. ⌟

3.5.2. Constructing the adjunction. We will construct a biadjunction between the 2-category of virtual double categories and the 2-category of specifications in FVDblTT.

The first goal is to construct a 1-adjunction between the category of specifications and the category of split CFVDCs and morphisms between them.

Definition 3.5.5. For a specification (Σ, E), the ***syntactic virtual double category*** (or classifying virtual double category) $\mathbb{S}(\Sigma, \mathrm{E})$ is the virtual double category whose

- objects are contexts $\Gamma \ \mathsf{ctx}$ in Σ,
- tight arrows $\Gamma \to \Delta = (y_1 : J_1, \ldots, y_n : J_n)$ are equivalence classes of sequences of terms (or, term substitutions) $\Gamma \vdash s_1 : J_1, \ldots, s_n : J_n$ (or substitutions) modulo equality judgments derivable from (Σ, E),
- loose arrows $\Gamma \looparrowright \Delta$ are protypes $\Gamma \fatsemi \Delta \vdash \alpha \ \mathsf{protype}$ in Σ modulo equality judgments derivable from (Σ, E),
- cells of form

(3.5.1)
$$\begin{tikzcd}
\Gamma_0 \arrow[r, "\alpha_1", "\shortmid"{marking}] \arrow[d, "S_0"'] & \cdots & & \cdots \arrow[r, "\alpha_n", "\shortmid"{marking}] & \Gamma_n \arrow[d, "S_1"] \\
\Delta_0 \arrow[rrrr, "\beta"', "\shortmid"{marking}] & & \mu & & \Delta_1
\end{tikzcd}$$

are equivalence classes of proterms

$$\overline{\Gamma} \mid a_1 : \alpha_1 \fatsemi \ldots \fatsemi a_n : \alpha_n \vdash \mu : \beta[S_0/\Delta_0 \fatsemi S_n/\Delta_n]$$

modulo equality judgments derivable from (Σ, E). It makes no difference which representatives we choose for the equivalence classes of terms S_i's and protypes α_i's because of the replacement axioms, and the congruence problem does not arise because the equality judgments for protypes are limited to those coming from the equality judgments for terms by the replacement axiom.

⌟

Proposition 3.5.6. The syntactic VDC $\mathbb{S}(\Sigma, \mathrm{E})$ for a specification (Σ, E) has a structure of a split CFVDC. ⌟

Proof. The tight structure is given as usual in algebraic theories. The composite of the following cells

$$\begin{tikzcd}
\Gamma_{1,0} \arrow[r, dashed, "\overline{\alpha}_1", "\shortmid"{marking}] \arrow[d, "S_0"'] & \Gamma_{1,n_1} \arrow[r, dashed, "\shortmid"{marking}] \arrow[d, "S_1"'] & \cdots \arrow[r, dashed, "\overline{\alpha}_n", "\shortmid"{marking}] & \Gamma_{n,m_n} \arrow[d, "S_n"] \\
\Delta_0 \arrow[r, "\beta_1"', "\shortmid"{marking}] \arrow[d, "T_0"'] & \Delta_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "\beta_n"', "\shortmid"{marking}] & \Delta_n \arrow[d, "T_1"] \\
\Theta_0 \arrow[rrr, "\gamma"', "\shortmid"{marking}] & & & \Theta_1
\end{tikzcd}$$

with cells μ_1, μ_n in the upper row and ν in the lower row, is given as

$$\overline{\Gamma} \mid \overline{\alpha_1} \fatsemi \ldots \fatsemi \overline{\alpha_n} \vdash \nu[\overline{S_{\underline{i}}}/\overline{\Delta_{\underline{i}}}]\, \{\mu_1 \fatsemi \ldots \fatsemi \mu_n\} : \gamma[T_0/\Theta_0 \fatsemi T_1/\Theta_1][S_0/\Delta_0 \fatsemi S_n/\Delta_n].$$

The associativity and unit laws follow from Lemma 3.2.3.

The chosen restrictions are given by the term substitution into protypes. It is straightforward to check that the canonical cell

$$\begin{tikzcd}
\Gamma_0 \arrow[r, "{\alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]}", "\shortmid"{marking}] \arrow[d, "S_0"'] \arrow[dr, phantom, "\mathtt{rest}"] & \Gamma_1 \arrow[d, "S_1"] \\
\Delta_0 \arrow[r, "\alpha"', "\shortmid"{marking}] & \Delta_1
\end{tikzcd}
\quad \text{given by} \quad \Gamma_0 \fatsemi \Gamma_1 \mid a : \alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1] \vdash a : \alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]$$

exhibits $\alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]$ as a restriction of a loose arrow α along S_0 and S_1 as tight arrows. The chosen terminals and binary products are given by the constructors $\top$ and $\wedge$, whose universal properties can be confirmed by the computation rules for them. By Lemma 3.2.3, the choice gives a split CFVDC. □

The functoriality is easy to check.

Lemma 3.5.7. For any morphism of specifications $\Phi\colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$, the translation $(-)^{\Phi}$ by Φ defines a morphism $\mathbb{S}(\Phi)\colon \mathbb{S}(\Sigma, \mathrm{E}) \to \mathbb{S}(\Sigma', \mathrm{E}')$. This defines a (1-)functor $\mathbb{S}\colon \mathbf{Speci} \to \mathbf{FVDbl}^{\mathbf{split}}_{\mathbf{cart}}$.

⌟

Theorem 3.5.8. The assignment that sends a CFVDC $\mathbb{D}$ to the associated specification $(\Sigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$ extends to a functor $\mathrm{Sp}\colon \mathbf{FVDbl}^{\mathbf{split}}_{\mathbf{cart}} \to \mathbf{Speci}$ which is a right adjoint to $\mathbb{S}$. The counit components of the adjunction $\varepsilon_{\mathbb{D}}\colon \mathbb{S}(\mathrm{Sp}(\mathbb{D})) \to \mathbb{D}$ are an equivalence as a 1-cell in $\mathbf{FVDbl}_{\mathbf{cart}}$. ⌟

Proof. We construct a virtual double functor $\varepsilon_{\mathbb{D}}\colon \mathbb{S}(\mathrm{Sp}(\mathbb{D})) \to \mathbb{D}$. We have the canonical $\Sigma_{\mathbb{D}}$-structure in $\mathbb{D}$. In the way we showed in Subsection 3.2.2, we can interpret all the items in $\mathrm{Sp}(\mathbb{D})$ in $\mathbb{D}$. Now, we show that this defines a virtual double functor from $\mathbb{S}(\Sigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$ to $\mathbb{D}$. The actions on the objects, tight arrows, and loose arrows are straightforward using Definition 3.2.5. A cell of $\mathbb{S}(\Sigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$ of the form (3.5.1) is interpreted as the composite of the cartesian cell on the left and the cell $\llbracket\mu\rrbracket$ on the right, which is inductively defined in Definition 3.2.5.

$$\begin{tikzcd}
\llbracket\Gamma_0\rrbracket \arrow[r, "\llbracket\beta[S_0/\Delta_0 \fatsemi S_1/\Delta_1]\rrbracket", "|" marking] \arrow[d, "\llbracket S_0\rrbracket"'] \arrow[dr, phantom, "\mathtt{rest}"] & \llbracket\Gamma_1\rrbracket \arrow[d, "\llbracket S_1\rrbracket"] & , & \llbracket\Gamma_0\rrbracket \arrow[r, "\llbracket\alpha_1\rrbracket", "|" marking] \arrow[d, equal] & \cdots & \cdots \arrow[r, "\llbracket\alpha_n\rrbracket", "|" marking] & \llbracket\Gamma_n\rrbracket \arrow[d, equal] \\
\llbracket\Delta_0\rrbracket \arrow[r, "\llbracket\beta\rrbracket"', "|" marking] & \llbracket\Delta_1\rrbracket & & \llbracket\Gamma_0\rrbracket \arrow[rrr, "\llbracket\beta[S_0/\Delta_0 \fatsemi S_1/\Delta_1]\rrbracket"', "|" marking] \arrow[rrr, phantom, shift left=4ex, "\llbracket\mu\rrbracket"] & & & \llbracket\Gamma_n\rrbracket
\end{tikzcd}$$

These assignments are independent of the choice of terms and proterms since in $\mathbb{S}(\Sigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$, we take equivalence classes with respect to the equality judgments belonging to $\mathrm{E}_{\mathbb{D}}$. Proving that this defines a morphism in $\mathbf{FVDbl}^{\mathbf{split}}_{\mathbf{cart}}$ is a routine verification. For instance, it sends a chosen restriction $\alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]$ of α along S_0 and S_1 to $\llbracket\alpha[S_0/\Delta_0 \fatsemi S_1/\Delta_1]\rrbracket$, which is the same as $\llbracket\alpha\rrbracket[\llbracket S_0\rrbracket \fatsemi \llbracket S_1\rrbracket]$ by Lemma 3.2.7.

We show that $\varepsilon_{\mathbb{D}}$ is an equivalence as a virtual double functor. The surjectiveness part directly follows from the construction. The proofs of the fully-faithfulness on tight arrows and cells are parallel: if two terms or proterms in $\mathrm{Sp}(\mathbb{D})$ are interpreted as the same term or proterm in $\mathbb{D}$, then this equality is reflected in the equality judgments in $\mathrm{E}_{\mathbb{D}}$, and hence the terms or proterms are already derivably equal in $\mathrm{Sp}(\mathbb{D})$.

Now, we show that $\varepsilon_{\mathbb{D}}$ is a terminal object in the comma category $\mathbb{S} \downarrow \mathbb{D}$. Suppose we are given a morphism $F\colon \mathbb{S}(\Sigma, \mathrm{E}) \to \mathbb{D}$. If $\widehat{F}\colon (\Sigma, \mathrm{E}) \to \mathrm{Sp}(\mathbb{D})$ satisfies $\varepsilon_{\mathbb{D}} \circ \mathbb{S}(\widehat{F}) = F$, then it satisfies the following:

- $x : \widehat{F}(\sigma)$ is interpreted as $F(x : \sigma)$ in $\mathbb{D}$ for each category symbol σ,
- $(\widehat{F}(f))(x)$ is interpreted as $F(f(x))$ in $\mathbb{D}$ for each function symbol f,
- $(\widehat{F}(\rho))(x \fatsemi y)$ is interpreted as $F(\rho(x \fatsemi y))$ in $\mathbb{D}$ for each profunctor symbol ρ, and
- $(\widehat{F}(\kappa))(\overline{x_i})\{\overline{a_i}\}$ is interpreted as $F(\kappa(\overline{x_i})\{\overline{a_i}\})$ in $\mathbb{D}$ for each proterm symbol κ.

However, $\varepsilon_{\mathbb{D}}$ is injective on primitive contexts and procontexts, and also is injective on the terms and proterms by the fully-faithfulness. Hence, $\widehat{F}$ is uniquely determined for F by the above conditions:

$$\widehat{F}(\sigma) = \ulcorner F(x : \sigma)\urcorner, \quad \widehat{F}(f) = \ulcorner F(f(x))\urcorner, \quad \widehat{F}(\rho) = \ulcorner F(\rho(x, y))\urcorner, \quad \widehat{F}(\kappa) = \ulcorner F(\kappa(\overline{x_i})\{\overline{a_i}\})\urcorner.$$

Conversely, the assignment $\widehat{F}$ defined by the above gives a morphism $\widehat{F}\colon (\Sigma, \mathrm{E}) \to \mathrm{Sp}(\mathbb{D})$. The well-definedness of $\widehat{F}$ depends on the fact that a equality judgment in E induces an equality in $\mathbb{S}(\Sigma, \mathrm{E})$, which is sent to an equality in $\mathbb{D}$ by F. It also satisfies the equation $\varepsilon_{\mathbb{D}} \circ \mathbb{S}(\widehat{F}) = F$, which is confirmed by induction on the structure of the judgments in (Σ, E). Therefore, $\varepsilon_{\mathbb{D}}$ has the desired universal property. □

Remark 3.5.9. Owing to the splitness lemma Lemma 3.2.8, this adjunction achieves the desired syntax-semantics duality without loss of generality. It would be more precise to say that this 1-adjunction combines with the biequivalence between the 2-category of split CFVDCs and the 2-category of (cloven) CFVDCs to form a biadjunction. ⌟

3.5.3. Specifications with protype isomorphisms.

We can extend the biadjunction to the type theory with protype isomorphisms. First, we introduce a notion of specification with protype isomorphisms. We use the term "***multi-class***" to mean a class X with multiplicities $(\mathrm{M}_x)_{x \in \mathrm{X}}$, where M_x is a class. One can think of a multi-class as a (class-large) family of classes.

Definition 3.5.10. By a ***multi-class*** $(\mathrm{M})_x$, we mean a class X with multiplicities $(\mathrm{M}_x)_{x \in \mathrm{X}}$, where M_x is a class. A ***multi-class of isomorphism symbols*** for a signature Σ is a multi-class $\mathrm{PI}_{\rho,\omega}$

indexed by pairs of profunctor symbols (ρ, ω) of the same two-sided arity in Σ. We call the elements of $\mathrm{PI}_{\rho,\omega}$ ***isomorphism symbols***. ⌟

Definition 3.5.11. A ***specification with protype isomorphisms*** $(\Sigma, \mathrm{PI}, \mathrm{E})$ consists of

- a signature Σ,
- PI, a multi-class of isomorphism symbols for Σ, and
- a pair $(\mathrm{E}^{\mathsf{tm}}, \mathrm{E}^{\mathsf{ptm}})$ as in Definition 3.5.2, but the derivation of proterms can refer to the following rule.

$$\frac{m \in \mathrm{PI}_{\rho,\omega}}{x : \sigma \fatsemi y : \tau \vdash \Lambda_m : \rho(x \fatsemi y) \cong \omega(x \fatsemi y)}$$

A ***morphism of specifications with protype isomorphisms*** $\Phi\colon (\Sigma, \mathrm{PI}, \mathrm{E}) \to (\Sigma', \mathrm{PI}', \mathrm{E}')$ consists of a morphism of signatures $\Phi\colon \Sigma \to \Sigma'$ and a multi-class function $\breve{\Phi}\colon \mathrm{PI}_{\rho,\omega} \to \mathrm{PI}'_{\Phi(\rho),\Phi(\omega)}$ compatible with the index function of PI defined by Φ such that every judgment in E is translated to a judgment that is derivable from E' by $(\Phi, \breve{\Phi})$.

We write $\mathbf{Speci}^{\cong}$ for the 2-category of specifications with protype isomorphisms and morphisms between them. ⌟

We will construct a functor $\mathrm{Ufd}\colon \mathbf{Speci}^{\cong} \to \mathbf{Speci}$ which has a partial right adjoint. Since the right adjoint is defined on the image of Sp, we will obtain an adjunction between the category of specifications with protype isomorphisms and the category of split CFVDCs in the end.

Definition 3.5.12. We define a specification (without protype isomorphisms) $\mathrm{Ufd}(\Sigma, \mathrm{PI}, \mathrm{E})$ for a specification with protype isomorphisms $(\Sigma, \mathrm{PI}, \mathrm{E})$ as follows.

- the signature consists of data in Σ plus additional transformation symbols $\varphi_m\colon \rho \Rightarrow \omega$ and $\psi_m\colon \omega \Rightarrow \rho$ for each element $m \in \mathrm{PI}_{\rho,\omega}$,
- the equality judgments consist of the original equality judgments in E with all occurrences of protype isomorphisms inductively replaced by the corresponding proterms as shown in Figure 10, plus the following additional equality judgments:

$$x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \psi_m\{\varphi_m\{a\}\} \equiv a : \rho \quad \text{and} \quad x : \sigma \fatsemi y : \tau \mid b : \omega \vdash \varphi_m\{\psi_m\{b\}\} \equiv b : \omega \tag{3.5.2}$$

for each $m \in \mathrm{PI}_{\rho,\omega}$.

⌟

$$\begin{aligned} \mathsf{id}_\alpha\{a\} &\rightsquigarrow a & (\!|\mu, \nu|\!)\{a\} &\rightsquigarrow \mu\{a\} \\ \mathsf{id}_\alpha^{-1}\{a\} &\rightsquigarrow a & (\!|\mu, \nu|\!)^{-1}\{a\} &\rightsquigarrow \nu\{a\} \\ (\Omega \circ \Upsilon)\{a\} &\rightsquigarrow \Omega\{\Upsilon\{a\}\} & \Lambda_m\{a\} &\rightsquigarrow \varphi_m\{a\} \\ (\Omega \circ \Upsilon)^{-1}\{a\} &\rightsquigarrow \Upsilon^{-1}\{\Omega^{-1}\{a\}\} & \Lambda_m^{-1}\{a\} &\rightsquigarrow \psi_m\{a\} \end{aligned}$$

FIGURE 10. Translation of protype isomorphisms

Lemma 3.5.13. The assignment $(\Sigma, \mathrm{PI}, \mathrm{E}) \mapsto \mathrm{Ufd}(\Sigma, \mathrm{PI}, \mathrm{E})$ induces a functor $\mathrm{Ufd}\colon \mathbf{Speci}^{\cong} \to \mathbf{Speci}$. ⌟

PROOF SKETCH. For a morphism of specifications $\Phi\colon (\Sigma, \mathrm{E}) \to (\Sigma', \mathrm{E}')$, the assignment $\mathrm{Ufd}(\Phi)$ sends the transformation symbols φ_m and ψ_m to $\varphi_{\Phi(m)}$ and $\psi_{\Phi(m)}$. The equality judgments (3.5.2) are translated into the equality judgments of the same form and hence derivable from $\mathrm{Ufd}(\Sigma', \mathrm{E}')$. ☐

The functor does not have a right adjoint globally but a partial one.

Definition 3.5.14. A specification (Σ, E) is ***unary-cell-saturated*** if, for any proterm judgment $x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \vartheta : \omega$ derivable from E where $\sigma, \tau, \rho, \omega$ belongs to the signature Σ, there uniquely exists a transformation symbol $\kappa_\vartheta\colon \rho \Rightarrow \omega$ in Σ such that the equality judgment

$$x : \sigma \fatsemi y : \tau \mid a : \rho \vdash \kappa_\vartheta(x \fatsemi y)\{a\} \equiv \vartheta : \omega$$

is derivable from E. Let $\mathbf{Speci}_{\mathsf{sat}}$ be the full subcategory of $\mathbf{Speci}$ whose objects are unary-cell-saturated crude specifications. ⌟

It is easy to see that the associated specification $(\varSigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$ of a CFVDC $\mathbb{D}$ is unary-cell-saturated. A specification being saturated means that the symbols in the signature constitute a virtual double category that is equivalent to the syntactic VDC of the specification.

Proposition 3.5.15. The functor $\mathrm{Ufd}\colon \mathbf{Speci}^{\cong} \to \mathbf{Speci}$ has a relative right coadjoint Fd over the inclusion $J\colon \mathbf{Speci}_{\mathsf{sat}} \hookrightarrow \mathbf{Speci}$.

$$\begin{tikzcd}
\mathbf{Speci}^{\cong} \arrow[r, "\mathrm{Ufd}"] & \mathbf{Speci} \\
& \mathbf{Speci}_{\mathsf{sat}} \arrow[lu, "\mathrm{Fd}"] \arrow[u, hook, "J"'] \arrow[from=1-1, to=2-2, phantom, "\overset{\upsilon}{\Rightarrow}"]
\end{tikzcd} .$$

The components of the counit $\upsilon_{(P,\mathrm{D})}\colon \mathrm{Ufd}(\mathrm{Fd}(P,\mathrm{D})) \to (P,\mathrm{D})$ are sent to the equivalence by $\mathbb{S}$. ⌟

Here, the relative right coadjoint means that there is a natural isomorphism

$$\mathbf{Speci}(\mathrm{Ufd}(-), J(*)) \cong \mathbf{Speci}^{\cong}(-, \mathrm{Fd}(*))$$

induced by the υ.

Proof. For a unary-cell-saturated crude specification (P,D), a specification $\mathrm{Fd}(P,\mathrm{D})$ consists of the same signature P, the multi-class $\mathrm{D}^{\cong}$ defined from D by setting $\mathrm{D}^{\cong}_{\rho,\omega}$ to be the class of the pairs (ϑ, ς) of transformation symbols in D

$$\vartheta\colon \rho \Rightarrow \omega \quad \text{and} \quad \varsigma\colon \omega \Rightarrow \rho$$

for which D derives the equality judgments that express the two cells are inverses of each other, and the classes of term and proterm equality judgments in D plus the equality judgments

$$x : \sigma \fatsemi y : \tau \mid a : \rho(x \fatsemi y) \vdash \Lambda_{(\vartheta,\varsigma)}\{a\} \equiv \vartheta(x \fatsemi y)\{a\} : \omega(x \fatsemi y)$$
$$x : \sigma \fatsemi y : \tau \mid b : \omega(x \fatsemi y) \vdash \Lambda^{-1}_{(\vartheta,\varsigma)}\{b\} \equiv \varsigma(x \fatsemi y)\{b\} : \rho(x \fatsemi y)$$

for each isomorphism symbol (ϑ, ς) in $\mathrm{D}^{\cong}_{\rho,\omega}$. Then we will have a morphism of specifications $\upsilon_{(P,\mathrm{D})}$ that sends the new transformation symbols $\varphi_{(\vartheta,\varsigma)}$ and $\psi_{(\vartheta,\varsigma)}$ to the transformation symbols ϑ and ς. It follows that $\upsilon_{(P,\mathrm{D})}$ defines a morphism of specifications since the equality judgments in $\mathrm{Ufd}(\mathrm{Fd}(P,\mathrm{D}))$ are either in D or those of the form (3.5.2) for the pairs in $\mathrm{D}^{\cong}$, which are translated to equality judgments derivable from D.

We prove that this $\upsilon_{(P,\mathrm{D})}$ satisfies the universal property for the relative right coadjoint of Ufd. That is, for a morphism of specifications $\varPhi\colon \mathrm{Ufd}(\varSigma, \mathrm{PI}, \mathrm{E}) \to (P,\mathrm{D})$, there uniquely exists a morphism of specifications with protype isomorphisms $\widehat{\varPhi}\colon (\varSigma, \mathrm{PI}, \mathrm{E}) \to \mathrm{Fd}(P,\mathrm{D})$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Ufd}(\varSigma, \mathrm{PI}, \mathrm{E}) \arrow[d, "\mathrm{Ufd}(\widehat{\varPhi})"'] \arrow[dr, "\varPhi", "/\!/"'] & \\
\mathrm{Ufd}(\mathrm{Fd}(P,\mathrm{D})) \arrow[r, "\upsilon_{(P,\mathrm{D})}"'] & (P,\mathrm{D})
\end{tikzcd} \qquad \text{in } \mathbf{Speci}.$$

To make this diagram commute, the signature part of $\widehat{\varPhi}$ must be the same as $\varPhi$. Suppose we have a morphism $\widehat{\varPhi}$ and we determine how it should act on the isomorphism symbols in PI. Let (χ_m, λ_m) be the image of m under $\widehat{\varPhi}$. Then, the symbol χ_m equals to $\upsilon_{(P,\mathrm{D})}(\varphi_{(\chi_m,\lambda_m)}) = \upsilon_{(P,\mathrm{D})}(\varphi_{\widehat{\varPhi}(m)})$, which is the image of m under $\varPhi$. Similarly, we must have $\lambda_m = \varPhi(\psi_m)$. Therefore, the morphism $\widehat{\varPhi}$ must send m to the pair $(\varPhi(\varphi_m), \varPhi(\psi_m))$. This assignment $\widehat{\varPhi}$ is a morphism of specifications with protype isomorphisms since the equality judgments in E with the isomorphism symbols suitably replaced are translated by $\varPhi$ to the equality judgments provable from D. Note that the proterm $\Lambda_m\{a\}$ is sent to $\Lambda_{\widehat{\varPhi}(m)}\{a\}$, which behaves the same as $\varPhi(\varphi_m)(x \fatsemi y)\{a\}$ up to derivable equality in D.

To see that $\mathbb{S}(\upsilon_{(P,\mathrm{D})})$ is an equivalence, we confer Lemma 1.3.8. The equivalence on the tight part is apparent since $\upsilon_{(P,\mathrm{D})}$ does not change anything on types and terms. Next, for each loose arrow in $\mathbb{S}(\mathrm{Ufd}(\mathrm{Fd}(P,\mathrm{D})))$, we can find a corresponding loose arrow in $\mathbb{S}(\mathrm{Ufd}(\mathrm{Fd}(P,\mathrm{D})))$ by taking the protype with precisely the same presentation. Finally, when fixing a frame, the function on globular cells defined by $\upsilon_{(P,\mathrm{D})}$ sends proterm judgments with the additional transformation symbols $\varphi_{(\vartheta,\varsigma)}$ and $\psi_{(\vartheta,\varsigma)}$ to the proterm judgments without them by replacing those transformation symbols with ϑ and ς. The surjectiveness is checked similarly to the above argument. We can also see the injectiveness up to derivable equality by induction on the construction of the proterms. For instance, the equalities $\varphi_{(\vartheta,\varsigma)}(x \fatsemi y)\{a\} \equiv \vartheta(x \fatsemi y)\{a\}$ and $\psi_{(\vartheta,\varsigma)}(x \fatsemi y)\{a\} \equiv \varsigma(x \fatsemi y)\{a\}$ are already derivable from $\mathrm{Ufd}(\mathrm{Fd}(P, \mathrm{D}))$. ☐

Corollary 3.5.16. The composite $\$ \circ \mathrm{Ufd}\colon \mathbf{Speci}^{\cong} \to \mathbf{FVDbl}^{\mathbf{split}}_{\mathbf{cart}}$ has a right adjoint $\mathrm{Fd} \circ \mathrm{Sp}$:

$$\begin{tikzcd} \mathbf{Speci}^{\cong} \arrow[r, shift left, "\$\circ\mathrm{Ufd}"] \arrow[r, phantom, "\perp"] & \mathbf{FVDbl}^{\mathbf{split}}_{\mathbf{cart}} \arrow[l, shift left] \end{tikzcd}, \quad \text{given by} \quad \begin{tikzcd} \mathbf{Speci}^{\cong} \arrow[r, "\mathrm{Ufd}"] & \mathbf{Speci} \arrow[r, "\$"] \arrow[d, phantom, "\perp"] & \mathbf{FVDbl}^{\mathbf{split}}_{\mathbf{cart}} \arrow[dl, "\mathrm{Sp}"] \\ & \mathbf{Speci}_{\mathbf{sat}} \arrow[ul, "\mathrm{Fd}"] & \end{tikzcd}.$$

Moreover, the counit component of the adjunction is pointwise an equivalence as a virtual double functor. ⌟

Proof. Through Theorem 3.5.8 and Prop 3.5.15, the expected adjunction follows from the general theory of relative coadjunctions. Explicitly, for a specification S and a CFVDC $\mathbb{D}$,

$$\begin{aligned} \mathbf{FVDbl}^{\mathbf{split}}_{\mathbf{cart}}(\$(\mathrm{Ufd}(\mathrm{S})), \mathbb{D}) &\cong \mathbf{Speci}\,(\mathrm{Ufd}(\mathrm{S}), \mathrm{Sp}(\mathbb{D})) && \text{(by Theorem 3.5.8)} \\ &\cong \mathbf{Speci}^{\cong}\,(\mathrm{S}, \mathrm{Fd}(\mathrm{Sp}(\mathbb{D}))) && \text{(by Proposition 3.5.15)} \end{aligned}$$

The counit component of the adjunction is an equivalence by the construction of the adjunctions. ☐

Remark 3.5.17. The specification $\mathrm{Fd}(\mathrm{Sp}(\mathbb{D}))$ is not the same as the associated specification $(\Sigma_{\mathbb{D}}, \mathrm{E}_{\mathbb{D}})$ equipped with the isomorphism symbols, but the two give the equivalent virtual double categories. ⌟

Remark 3.5.18. For extensions of FVDblTT with additional constructors as in Section 3.3, we can obtain a similar biadjunction analogously once one determines the treatment of substitutions as explained Remark 3.3.3. The procedure goes as follows: (i) Prove the splitness lemma for CFVDCs with the additional structure of interest, where the splitness is defined in reflection of the treatment of substitutions, (ii) Construct the syntactic VDCs for the extended type theory and verify that they have the structures in question, and (iii) Prove the adjunction between the category of split CFVDCs with the additional structures and the category of specifications with the additional constructors in the same way as in Theorem 3.5.8. The biadjunction is again obtained by combining this adjunction with the biequivalence between the 2-categories of split and cloven CFVDCs with the structures. ⌟

3.6. Future Work

There are several directions for future work. First, we would like to extend the type theory FVDblTT to include more advanced structures that are studied in formal category theory using virtual double categories. In particular, we are interested in the extension of the type theory FVDblTT to *augmented* virtual double categories [**Kou20**, **Kou24**]. The latter paper conceptualizes the notion of a Kan extension and a Yoneda embedding inside this framework, and develops a formal category theory more flexibly than the original virtual double categories. Second, the dependent version of the type theory FVDblTT should be developed. There are several studies on directed type theory [**LH11**, **Nor19**, **ANv23**], and those are all based on dependent types. One of the primary objectives of those studies is to obtain a substantial type theory for higher categories as Martin-Löf type theory is for higher groupoids. The dependent version of the type theory FVDblTT might offer another candidate for this purpose using the unit protypes and the comprehension types. Finally, we are interested in the relationship between the type theory FVDblTT and other type theories or calculi for relations. In particular, we are interested in the connection to diagrammatic calculi for relations such as the one in [**BPS17**, **BDHS24**], or more directly, the string diagrams for double categories [**Mye18**]. They may be understood as a string diagrammatic presentation of the type theory FVDblTT. We hope to explore these connections in future work.

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