A Formal Theory of Anticolimits

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Introduction

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[rr, squiggly, "\exists\, ?", "\text{Taking colimit}"'] & & A \arrow[r] \arrow[d] & C \arrow[d] & & & C \arrow[d] \arrow[lll, squiggly, "\exists ?"', "\text{Taking anticolimit}"] \\
B & & & B \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] & & B \arrow[r] & D
\end{tikzcd}
$$

The central question is:

"How can we know if a cocone is a colimit of some diagram?"

(Tataru, Vicary. "The theory and applications of anticolimits" (2024))

So many examples are out there!

For categories, 2-categories, additive categories, ...

Goal : A conceptual understanding of anticolimits.

Structure

1. Elements of anticolimit.
2. Formal category theory via (virtual) double categories.
3. Formal theory of anticolimits : general theory
4. Formal theory of anticolimits : application

The slides for today.

Structure

1. Elements of anticolimit.
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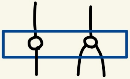
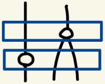
Original work

- Tataru and Vicary introduced anticolimits in the study of homotopy.io.

 is a "colimit" of 

- Their problem is like:

Given
$$\begin{tikzcd} A_0 \arrow[dr, bend right, "\varphi_0"'] & A_1 \arrow[d, "\varphi_1"] & \cdots & A_n \arrow[dll, "\varphi_n"] \\ & Z & & \end{tikzcd}$$
in a category $\mathcal{C}$,

find a diagram
$$\begin{tikzcd} & X_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & & \cdots & & X_n \arrow[dl, "l_n"'] \arrow[dr, "r_n"] & \\ A_0 & & A_1 & \cdots & & & A_n \end{tikzcd}$$
whose colimit is this.

- J : poset, $A : \max J \longrightarrow \mathcal{C}$, $c \in \mathcal{C}$, $\kappa : A \Rightarrow \Delta_c$

An anticolimit of κ is an extension $\tilde{A} : J \to \mathcal{C}$ of A such that κ induces a colimit cocone of $\tilde{A}$.

A recipe for anticolimits

J : poset, $X : \max \mathcal{J} \longrightarrow \mathcal{C}$, $c \in \mathcal{C}$, $\kappa : X \Rightarrow \Delta_c$

Def $\Pi_{\mathcal{J}}(\kappa) : \mathcal{J} \longrightarrow \mathcal{C}$ is defined (if possible) as follows :

- $j \in \mathcal{J}$ is mapped to a multiple pullback of $(X_i \xrightarrow{\kappa_i} c)_{i \geq j}$.
- $j \to j'$ in $\mathcal{J}$ is mapped to the canonical arrow in $\mathcal{C}$.

$$\mathcal{J} \quad \begin{tikzcd} & & \bullet \\ & \bullet \arrow[ur] \arrow[r] & \bullet \\ \bullet \arrow[ur] \arrow[rr, bend right] & & \bullet \end{tikzcd} \quad \longmapsto \quad \mathcal{C} \quad \begin{tikzcd} & & X_0 \arrow[dr, "\kappa_0"] & \\ & X_0 \times_c X_1 \arrow[ur] \arrow[r] & X_1 \arrow[r, "\kappa_1"'] & c \\ X_0 \times_c X_1 \times_c X_2 \arrow[ur] \arrow[rr] & & X_2 \arrow[ur, "\kappa_2"'] & \end{tikzcd}$$

Theorem [TV24] If $\Pi_{\mathcal{J}}(\kappa)$ exists and κ has an anticolimit, then $\Pi_{\mathcal{J}}(\kappa)$ is an anticolimit of κ.

When $\mathcal{C}$ has enough limits, whether κ has an anticolimit can be checked just by looking at $\Pi_{\mathcal{J}}(\kappa)$.

Further Examples

① Regular epi

Def $f : A \to B$ in a category $\mathcal{C}$ is a regular epimorphism if it is a coequalizer of some arrows.

$$X \underset{h}{\overset{k}{\rightrightarrows}} A \xrightarrow{f} B$$

Prop If $\mathcal{C}$ has pullbacks, f is a regular epimorphism iff it is a coequalizer of its kernel pair.

$$\begin{tikzcd} A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$\in$ an effective epimorphism

② Subcanonicity of sites

Def A site $(\mathcal{C}, T)$ is subcanonical if every representable presheaf is a sheaf w.r.t. T.

Prop TFAE when $\mathcal{C}$: complete.

(i) $(\mathcal{C}, T)$ is subcanonical.

(ii) For any T-covering $(A_i \xrightarrow{\kappa_i} C)_i$, C is a colimit of

$$\{\kappa_i\} \xrightarrow{\text{f.f.}} \mathcal{C}/C \xrightarrow{\text{dom}} \mathcal{C}$$

(iii) For any T-covering $(A_i \xrightarrow{\kappa_i} C)_i$, C is a colimit of

$$\left(\begin{tikzcd}[column sep=small] & A_i \times_C A_j \arrow[dl] \arrow[dr] & \\ A_i & & A_j \end{tikzcd}\right)_{i,j}.$$

Further Examples

③ Normal epi in Ab-cats

Def $f : A \to B$ in an Ab-cat. $\mathcal{C}$ is a normal epimorphism if it is a cokernel of some arrow.

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Prop In a finitely complete Ab-cat, an arrow is a normal epimorphism if it is a cokernel of its kernel.

④ Localization in 2-categories

Def An 1-cell is a localization if it is a coinverter of some 2-cell.

Prop A functor $F : \mathcal{C} \to \mathcal{D}$ is a localization iff

$$\begin{tikzcd} & \mathcal{C}[W^{-1}] \arrow[d, "\wr"] \\ \mathcal{C} \arrow[ur] \arrow[dr, "F"'] \arrow[r, phantom, "\circlearrowright"] & \mathcal{D} \end{tikzcd}$$

with $W := \{ f \mid Ff : \mathrm{iso} \}$.

⑤ Effective tabulator in double cats

(Strong)

Prop In a flat double cat $\mathbb{D}$ with tabulators,

Cells ≤1 for each frame

$A \overset{p}{\longrightarrow\!\!\!\!\!\!\mid\;\;} B$ is presented as

$$\begin{tikzcd} & \exists\, C \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\text{opc.}"] & \\ A \arrow[rr, "p"', "\shortmid" marking] & {} & B \end{tikzcd}$$

if it has an effective tabulator

$$\begin{tikzcd} & \{p\} \arrow[dl, "l_p"'] \arrow[dr, "r_p"] \arrow[d, phantom, "\text{opc.}"] & \\ A \arrow[rr, "p"', "\shortmid" marking] & {} & B \end{tikzcd}.$$

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Interlude: Formal Category Theory

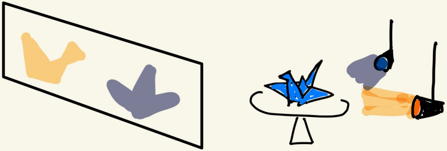
Formal Category Theory = Category Theory of Category Theories

Conceptual treatment
from an abstract viewpoint

- $\mathcal{V}$-enriched category theory
- $\mathcal{S}$-internal category theory
- $\mathcal{S}$-fibred category theory

It studies how one can develop category theory inside ~~2-categories~~ by imagining it as the ~~2-category~~ of categories. (Virtual) double cats

Mathematical Phenomena	Categorical Treatment of ⋯	Categories
Categorical Phenomena	Formal theory of ⋯	VDCs (or other structures)

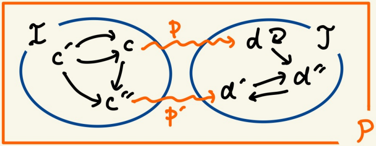


Goal : A formal theory of anticolimits.

Profunctors and Virtual equipments

Def A profunctor $P : \mathcal{I} \nrightarrow \mathcal{J}$ is a functor $P : \mathcal{I}^{op} \times \mathcal{J} \to \mathrm{Set}$.

⚠ Contravariant on its domain.



Prop The following correspond bijectively.

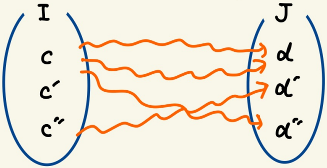
(i) Profunctos $P : \mathcal{I} \nrightarrow \mathcal{J}$

(ii) Pairs of embeddings $\mathcal{I} \xhookrightarrow{i} \mathcal{P} \xhookleftarrow{j} \mathcal{J}$

s.t. $\mathrm{ob}\,\mathcal{I} \sqcup \mathrm{ob}\,\mathcal{J} \xrightarrow[\cong]{\langle i,j \rangle} \mathrm{ob}\,\mathcal{P}$ and $\mathcal{E}(j(d), i(c)) = \emptyset \quad (\forall d, c)$

Ex • For a category $\mathcal{C}$, we have the hom-profunctor $\mathcal{C}(-,\bullet) : \mathcal{C} \nrightarrow \mathcal{C}$.

• For two sets I, J seen as discrete categories, a profunctor $I \nrightarrow J$ is a bipartite graph (or span).



Profunctors and Virtual equipments (continued)

A natural trans. $\begin{tikzcd} \mathcal{C} \arrow[d, bend right=50, "F"'] \arrow[d, bend left=50, "G"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ \mathcal{D} \end{tikzcd}$ is a natural family $(\alpha_c \in \mathcal{D}(Fc, Gc))_{c \in \mathcal{C}}$

$\Big\downarrow$ generalize

Naturality only involves the structure of the hom-profunctor $\mathcal{D}(-,\bullet)$.

A natural trans. $\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "\alpha"] & \\ \mathcal{D} \arrow[rr, "P"', "\shortmid"{marking}] & {} & \mathcal{D}' \end{tikzcd}$ is a natural family $(\alpha_c \in P(Fc, Gc))_{c \in \mathcal{C}}$

$(\alpha_{c,c'} : \mathcal{C}(c,c') \to P(Fc, Gc'))_{c,c'}$

$\leftrightsquigarrow$ $\begin{tikzcd} \mathcal{C} \arrow[r, "{\mathcal{C}(-,\bullet)}", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{D} \arrow[r, "P"', "\shortmid"{marking}] & \mathcal{D} \end{tikzcd}$

$\Big\downarrow$ generalize

A natural trans. $\begin{tikzcd} \mathcal{C}_0 \arrow[r, "Q_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "Q_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & \mathcal{C}_n \arrow[d, "G"] \\ \mathcal{D} \arrow[rrr, "P"', "\shortmid"{marking}] & & & \mathcal{D}' \end{tikzcd}$ is a natural family

$$\left(Q_0(c_0, c_1) \times \cdots \times Q_n(c_{n-1}, c_n) \xrightarrow{\alpha} P(Fc_0, Gc_n)\right)_{c_0, \ldots, c_n}$$

Profunctors and Virtual equipments (continued)

A natural trans.
$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "Q_1", "|" marking] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "|" marking] & \cdots \arrow[r, "Q_n", "|" marking] & \mathcal{C}_n \arrow[d, "G"] \\ \mathcal{D} \arrow[rrr, "P"', "|" marking] \arrow[urrr, phantom, "\alpha"] & & & \mathcal{D}' \end{tikzcd}$$
is a natural family

$$\left(Q_0(c_0, c_1) \times \cdots \times Q_n(c_{n-1}, c_n) \xrightarrow{\alpha} P(Fc_0, Gc_n)\right)_{c_0, \dots, c_n}$$

These natural transformations can be composed like
$$\begin{tikzcd} \bullet \arrow[r, "|" marking] \arrow[d] & \cdots \arrow[r, "|" marking] & \bullet \arrow[d] & \cdots & \bullet \arrow[r, "|" marking] \arrow[d] & \cdots \arrow[r, "|" marking] & \bullet \arrow[d] \\ \bullet \arrow[rr, "|" marking] \arrow[d] \arrow[urr, phantom, "\alpha_1"] & & \bullet & \cdots & \bullet \arrow[rr, "|" marking] \arrow[urr, phantom, "\alpha_n"] & & \bullet \arrow[d] \\ \bullet \arrow[rrrrrr, "|" marking] \arrow[urrrrrr, phantom, "\beta"] & & & & & & \bullet \end{tikzcd},$$
and constitute a virtual double category $\mathbb{PROF}$.

Def A virtual double category $\mathbb{X}$ consists of

- a category $\mathbb{X}_t$ of objects and tight arrows. $\begin{tikzcd} \bullet \arrow[d, "f"] \\ \bullet \end{tikzcd}$
- a family of classes of loose arrows $(\mathbb{X}(x, x'))_{x, x' \in \mathbb{X}}$ $\quad \bullet \xrightarrow{P} \bullet$ (loose)
- a family of classes of cells for each frame
$$\begin{tikzcd} \bullet \arrow[r, "P_1", "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "P_n", "|" marking] & \bullet \arrow[d, "g"] \\ \bullet \arrow[rr, "q"', "|" marking] \arrow[urr, phantom, "\alpha"] & & \bullet \end{tikzcd}.$$
- Data of composition and identities.

Profunctors and Virtual equipments (continued)

Def A restriction of

$$\begin{tikzcd} I \arrow[d, "f"'] & J \arrow[d, "g"] \\ I' \arrow[r, "p"', "|" marking] & J' \end{tikzcd}$$

is a cell

$$\begin{tikzcd} I \arrow[r, "{p[f;g]}", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{rest}"] & J \arrow[d, "g"] \\ I' \arrow[r, "p"', "|" marking] & J' \end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd} K \arrow[r, "|" marking] \arrow[d, "k"'] & \cdots \arrow[r, "|" marking] \arrow[d, phantom, "\forall"] & L \arrow[d, "h"] \\ I \arrow[d, "f"'] & \alpha & J \arrow[d, "g"] \\ I' \arrow[rr, "p"', "|" marking] & & J' \end{tikzcd} = \begin{tikzcd} K \arrow[r, "|" marking] \arrow[d, "k"'] & \cdots \arrow[r, "|" marking] \arrow[d, phantom, "\exists!\, \tilde{\alpha}"] & L \arrow[d, "h"] \\ I \arrow[rr, "{p[f;g]}"', "|" marking] \arrow[d, "f"'] & \mathrm{rest} & J \arrow[d, "g"] \\ I' \arrow[rr, "p"', "|" marking] & & J' \end{tikzcd}.$$

Def A (loose) unit on I is

a loose arrow $U_I : I \nrightarrow I$

together with a cell

$$\begin{tikzcd} & I \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_I"] & \\ I \arrow[rr, "|" marking] & {} & I \end{tikzcd}$$

with the following universal property:

$$\begin{tikzcd} K_n \arrow[r, "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "|" marking] & K_1 \arrow[r, "|" marking] & I \arrow[r, "|" marking] \arrow[d, phantom, "\forall\ \alpha"] & L_1 \arrow[r, "|" marking] & \cdots \arrow[r, "|" marking] & L_m \arrow[d, "g"] \\ N \arrow[rrrrrr, "p"', "|" marking] & & & & & & M \end{tikzcd}$$

$$\|$$

$$\begin{tikzcd} K_n \arrow[r, "|" marking] \arrow[d, equal, "\mathrm{id}"] & \cdots \arrow[r, "|" marking] & K_1 \arrow[r, "|" marking] \arrow[d, equal, "\mathrm{id}"] & I \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta_I"] \arrow[r, "|" marking] & L_1 \arrow[r, "|" marking] \arrow[d, equal, "\mathrm{id}"] & \cdots \arrow[r, "|" marking] & L_m \arrow[d, equal, "\mathrm{id}"] \\ K_n \arrow[r, "|" marking] \arrow[d, "f"'] & \cdots \arrow[r, "|" marking] & K_1 \arrow[r, "|" marking] & I \arrow[r, "|" marking] \arrow[d, phantom, "\exists!\, \tilde{\alpha}"] \arrow[r, "|" marking] & I \arrow[r, "|" marking] & L_1 \arrow[r, "|" marking] \cdots & L_m \arrow[d, "g"] \\ N \arrow[rrrrrr, "p"', "|" marking] & & & & & & M \end{tikzcd}$$

Def A virtual equipment is

a virtual double categories with restrictions and units on every object.

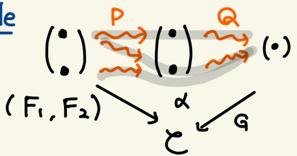
Colimits via profunctors

Natural transformations of the form

$$\begin{tikzcd} \mathcal{J} \arrow[r, "P", "|"{marking}] \arrow[dr, "F"'] & \mathcal{I} \arrow[r, "Q", "|"{marking}] \arrow[d, phantom, "\alpha"] & \mathcal{K} \arrow[dl, "G"] \\ & \mathcal{C} & \end{tikzcd} \qquad \left(= \begin{tikzcd} \mathcal{J} \arrow[r, "P", "|"{marking}] \arrow[d, "F"'] & \mathcal{I} \arrow[r, "Q", "|"{marking}] \arrow[d, phantom, "\alpha"] & \mathcal{K} \arrow[d, "G"] \\ \mathcal{C} \arrow[rr, "\mathcal{C}(-,\bullet)"', "|"{marking}] & & \mathcal{C} \end{tikzcd} \right)$$

will play a key role.

Example



The data above amounts to

$$\begin{tikzcd} F_1 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & G \\ F_2 \arrow[u, phantom] \arrow[ur, "h"'] & \end{tikzcd} \quad \text{in } \mathcal{C}$$

Colimits can be captured with triangle cells of this form:

Lem Let $F : \mathcal{I} \to \mathcal{C}$, $P : \underbrace{\mathcal{I} \nrightarrow 1}_{\mathcal{I}^{op} \to \mathrm{Set}}$

The weighted colimit $\mathrm{colim}^P F$ shows the following universal property

$$\begin{tikzcd} \mathcal{I} \arrow[r, "P", "|"{marking}] \arrow[dr, "F"'] & 1 \arrow[d, "\forall c"] \arrow[dl, phantom, "\forall\alpha"] \\ & \mathcal{C} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{I} \arrow[r, "P", "|"{marking}] \arrow[dr, "F"'] & 1 \arrow[d, "\mathrm{colim}^P F"'] \arrow[dl, phantom, "\lambda"] \arrow[d, bend left=60, "c"] \\ & \mathcal{C} \arrow[u, phantom, "\tilde{\alpha}\ \exists!"] \end{tikzcd}$$

$$\begin{tikzcd} F_i \arrow[dr, "\alpha_p"] \arrow[d, "Fu"'] & \\ F_{i'} \arrow[r, "\alpha_{p'}"'] & c \end{tikzcd} \quad = \quad \begin{tikzcd} F_i \arrow[dr, "\lambda_p"] \arrow[d, "Fu"'] & & \\ F_{i'} \arrow[r, "\lambda_{p'}"'] & \mathrm{colim}^P F \arrow[r, "\exists!\,\tilde{\alpha}"] & c \end{tikzcd}$$

⚠ This is not the "correct" notion of limit.

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Anticolimits via profunctors: balances

We take the example of regular epimorphisms as a model case,

and will solve the inverse problem of $X \overset{f}{\underset{g}{\rightrightarrows}} A \overset{\text{colimit}}{\longmapsto} A \xrightarrow{e} Z$

Fixed data

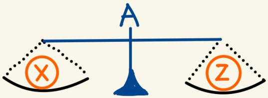
Def [Street. '80]

A *gamut* is a cell of the form

$$\begin{tikzcd} & \mathcal{I} \arrow[dr, "N", "|" marking] & \\ \mathcal{J} \arrow[ur, "M", "|" marking] \arrow[rr, bend right, "P"', "|" marking] \arrow[rr, phantom, "\mu"] & & \mathcal{K} \end{tikzcd}$$

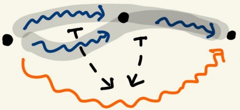
Def A *balance* on $\mathcal{C}$ consists of

a gamut μ and $\mathcal{I} \xrightarrow{A} \mathcal{C}$ (fulcrum)



Ex

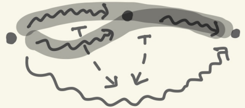
$$\mu_{coeq} := \begin{tikzcd} (\bullet) \arrow[r, "2", "|" marking] \arrow[rr, bend right, "1"', "|" marking] & (\bullet) \arrow[r, "1", "|" marking] \arrow[d, phantom, "\Downarrow"] & (\bullet) \\ & {} & \end{tikzcd}$$



Ex A fulcrum for μ_{coeq} is an object $A \in \mathcal{C}$.

Anticolimits via profunctors: diagrams

$$\begin{tikzcd} \mathcal{J} \arrow[r, "M", "|"{marking}] \arrow[rr, bend right, "P"', "|"{marking}] & \mathcal{I} \arrow[r, "N", "|"{marking}] \arrow[d, phantom, "\mu"] & \mathcal{K} \\ & {} & \end{tikzcd}$$



Def (μ, A) : balance

A left diagram is a pair (X, ξ).

$$\begin{tikzcd} \mathcal{J} \arrow[r, "M", "|"{marking}] \arrow[dr, bend right, "X"'] & \mathcal{I} \arrow[d, "A"] \arrow[dl, phantom, "\xi"] \\ & \mathcal{E} \end{tikzcd}$$

A right diagram is a pair (Z, ζ)

$$\begin{tikzcd} \mathcal{I} \arrow[r, "N", "|"{marking}] \arrow[d, "A"'] & \mathcal{K} \arrow[dl, bend left, "Z"] \arrow[dl, phantom, "\zeta"] \\ \mathcal{E} & \end{tikzcd}$$

Ex For (μ_{coeq}, A),

$$\left(\begin{array}{l}\text{Left}\\ \text{diagrams}\end{array}\right) = \left(\begin{array}{l}\text{Parallel arrows}\\ X \overset{f}{\underset{g}{\rightrightarrows}} A \quad \text{into } A\end{array}\right)$$

$$\left(\begin{array}{l}\text{Right}\\ \text{diagrams}\end{array}\right) = \left(\begin{array}{l}\text{Arrows}\\ A \xrightarrow{e} Z \quad \text{from } A\end{array}\right)$$

The left/right diagrams constitute categories $\mathbf{LD}(\mu, A)$ and $\mathbf{RD}(\mu, A)$.

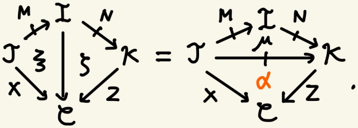
$(X, \xi) \xrightarrow{\sigma} (X', \xi')$ in $\mathbf{LD}(\mu, A)$ $\quad\Big\|\quad$ $\begin{tikzcd} & \mathcal{J} \arrow[d, bend right, "X"'] \arrow[d, bend left, "X'"] \arrow[d, phantom, "\sigma"] \\ & \mathcal{E} \end{tikzcd}$ s.t.

$$\begin{tikzcd} \mathcal{J} \arrow[r, "M", "|"{marking}] \arrow[dr, "X'"] \arrow[dr, bend right=60, "X"'] \arrow[dr, phantom, bend right=30, "\sigma"] & \mathcal{I} \arrow[d, "A"] \arrow[dl, phantom, "\xi'"] \\ & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{J} \arrow[r, "M", "|"{marking}] \arrow[dr, "X"'] & \mathcal{I} \arrow[d, "A"] \arrow[dl, phantom, "\xi"] \\ & \mathcal{E} \end{tikzcd}$$

Anticolimits via profunctors: bicones

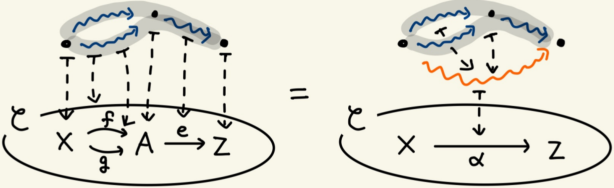
Def A <u>bicone</u> of (μ, A) is a triple of

- a left diagram $\xi = (X, \xi)$
- a right diagram $\zeta = (Z, \zeta)$, and
- a cell α such that



α is a bicone over ξ and ζ.

Ex For (μ_{coeq}, A),



α is unique if it exists.

Bicone-profunctor/General results on profunctors (1)

Prop Bicone : $LD(\mu, A)^{op} \times RD(\mu, A) \to Set$ is a functor,

$$(x, \xi) \quad , \quad (z, \zeta) \mapsto \left\{ \text{bicones over } \begin{matrix} (x,\xi) \\ (z,\zeta) \end{matrix} \right\}$$

which gives a profunctor Bicone : $LD(\mu, A) \nrightarrow RD(\mu, A)$.

Lem (Two-sided Grothendieck construction)

For a profunctor $T : A \nrightarrow B$, the category Υ defined by

- objects : $(a \in A, b \in B, t \in T(a,b))$
- arrows $\begin{matrix}(a,b,t) \\ \downarrow \\ (a',b',t')\end{matrix} : \left(\begin{matrix} a \\ f\downarrow \\ a' \end{matrix}, \begin{matrix} b \\ \downarrow g \\ b' \end{matrix} \right)$ s.t. $f \triangleleft t' = t \triangleright g$

induces the two-sided discrete fibration
$$\begin{tikzcd} & \Upsilon \arrow[dl, "L"'] \arrow[dr, "R"] & \\ A & & B \end{tikzcd} .$$

For BiCone, we write $\mathbf{Bico}(\mu, A)$ for this category Υ.

$$\begin{tikzcd} & \mathbf{Bico}(\mu,A) \arrow[dl] \arrow[dr] & \\ LD(\mu,A) & & RD(\mu,A) \end{tikzcd}$$

Notation
- ${}_a\Upsilon$: the fiber of $\Upsilon \xrightarrow{L} A$ at $a \in A$.
- Υ_b : the fiber of $\Upsilon \xrightarrow{R} B$ at $b \in B$.

Anticolimits via profunctors: limit/colimit bicone

Def A <u>colimit bicone</u> of $\xi \in LD(\mu, A)$ is an initial objects in ${}_{\xi}\mathbf{Bico}(\mu, A)$.

A <u>limit bicone</u> of $\zeta \in RD(\mu, A)$ is a terminal objects in $\mathbf{Bico}(\mu, A)_{\zeta}$

$\begin{pmatrix}\text{Left}\\ \text{diagrams}\end{pmatrix} = \begin{pmatrix}\text{Parallel arrows}\\ X \overset{f}{\underset{g}{\rightrightarrows}} A \quad \text{into } A\end{pmatrix}$

$\begin{pmatrix}\text{Right}\\ \text{diagrams}\end{pmatrix} = \begin{pmatrix}\text{Arrows}\\ A \xrightarrow{e} Z \quad \text{from } A\end{pmatrix}$

for (μ_{coeq}, A),

Ex The case of (μ_{coeq}, A).

• For $\xi := \left(X \overset{f}{\underset{g}{\rightrightarrows}} A\right)$, ${}_{\xi}\mathbf{Bico} \cong \left\{ \begin{matrix} A \\ \downarrow k \\ Y \end{matrix} \;\middle|\; \underbrace{X \overset{f}{\underset{g}{\rightrightarrows}} A \xrightarrow{k} Y}_{:\,\text{cofork}} \right\} \overset{\text{f.f.}}{\hookrightarrow} A/\mathcal{C}$.

A colimit bicone is a <u>coequalizer</u> diagram.

• For $\zeta := \left(A \xrightarrow{e} Z\right)$, $\mathbf{Bico}_{\zeta} \cong \left\{ A \xleftarrow{p} W \xrightarrow{q} A \;\middle|\; \underbrace{W \overset{p}{\underset{q}{\rightrightarrows}} A \xrightarrow{e} Z}_{:\,\text{cofork}} \right\} \overset{\text{f.f.}}{\hookrightarrow} {}_{A}\mathbf{Span}(\mathcal{C})_{A}$.

A limit bicone is a <u>kernel pair</u> diagram.

General results on profunctors (2)

Lem For a two-sided discrete fibration
$$\begin{tikzcd} & T \arrow[dl, "L"'] \arrow[dr, "R"] & \\ A & & B \end{tikzcd},$$

(i) L has a left adjoint iff ${}_aT$ has an initial for all $a \in A$.

(ii) R has a right adjoint iff T_b has a terminal for all $b \in B$.

The values of these adjoints are given by the initials/terminals.

The resulting adjoints are fully-faithful.

Assuming the existence of limit/colimit bicones, we obtain:

$$\boxed{\begin{tikzcd} LD(\mu, A) \arrow[r, shift left, "\mathrm{Colim}"] \arrow[r, phantom, "\perp"] & \mathrm{Bico}(\mu, A) \arrow[l, shift left, "L"] \arrow[r, shift left, "R"] \arrow[r, phantom, "\perp"] & RD(\mu, A) \arrow[l, shift left, "\mathrm{Lim}"] \end{tikzcd}.}$$

Ex For $(\mu_{\mathrm{coeq.}}, A)$, this adjunction is:

$$\begin{tikzcd} {}_A\mathrm{Span}(\mathcal{C})_A \arrow[r, shift left, "\mathrm{coeq.}"] \arrow[r, phantom, "\perp"] & A/\mathcal{C} \arrow[l, shift left, "\mathrm{kerpair.}"] \end{tikzcd}.$$

Remark Most of the results can be developed without assuming (co)limits; we could use relative adjoints on both sides.

General results on profunctors (3)

$$\boxed{\mathrm{LD}(\mu, A) \underset{L}{\overset{\mathrm{Colim}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Bico}(\mu, A) \underset{\mathrm{Lim}}{\overset{R}{\underset{\perp}{\rightleftarrows}}} \mathrm{RD}(\mu, A).}$$

Remark Idempotency of adjunctions are described in various ways.

$A \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} B$ is idempotent $\iff$ $F\eta$: iso $\iff$ $\mathrm{Fix}(FG) = \mathrm{Im}(F) \subseteq B$

$\iff \cdots$ $\qquad\qquad\qquad := \{ b \mid \varepsilon_b : \mathrm{iso} \}$

Prop If $T : A \nrightarrow B$ is a propositional profunctor and induces

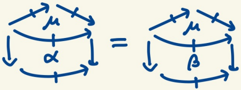
$:\iff \# T(a,b) \leqslant 1.$

the adjunction $A \underset{L}{\underset{\perp}{\rightleftarrows}} \Upsilon \overset{R}{\underset{\perp}{\rightleftarrows}} B$, then this is idempotent.

Ex If the gamut μ is epimorphic w.r.t. vertical composition,

Bicone is propositional.

μ_{coeq} is of this kind.



$\Rightarrow \alpha = \beta.$

Main result : anticolimit and effectiveness

$$\mathrm{LD}(\mu,A) \underset{L}{\overset{\mathrm{Colim}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Bico}(\mu,A) \underset{\mathrm{Lim}}{\overset{R}{\underset{\perp}{\rightleftarrows}}} \mathrm{RD}(\mu,A).$$

Thm Let (μ,A) be a balance.

If Bicone of this is propositional,

then the following are equivalent for $\zeta \in \mathrm{RD}(\mu,A)$:

(i) $\zeta \cong R\circ \mathrm{Colim}(\xi)$ for some $\xi \in \mathrm{LD}(\mu,A)$.

$:\Longleftrightarrow$ ζ has an anticolimit.

(ii) $\zeta \cong R\circ\mathrm{Colim}\circ L\circ \mathrm{Lim}(\zeta)$ (canonically)

ζ is "the colimit of the limit" of itself.

Ex For (μ_{coeq}, A),

(i) e : regular epi

(ii) e : coeq. of its kernel pair

Cor The adjunction reduces to an equivalence :

$$\left\{ \xi \in \mathrm{LD} \,\middle|\, \begin{array}{l}\xi \text{ has}\\ \text{an antilimit}\end{array} \right\} \simeq \left\{ \zeta \in \mathrm{RD} \,\middle|\, \begin{array}{l}\zeta \text{ has}\\ \text{an anticolimit}\end{array} \right\}.$$

Structure

1. Elements of anticolimit.
2. Formal category theory via (virtual) double categories.
3. Formal theory of anticolimits : general theory
4. Formal theory of anticolimits : application

Captured Examples (1)

① Set-enriched case

(i)
$$\mu_{pbpo} := \begin{tikzcd} & 2 \arrow[dr, "1"] \arrow[d, phantom, "\Downarrow !"] & \\ 1 \arrow[ur, "1"] \arrow[rr, "1"'] & {} & 1 \end{tikzcd}$$

$$\downarrow \mathrm{I}$$

$$\begin{tikzcd} & A \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & \\ X \arrow[ur] \arrow[dr] & {} & Z \\ & A' \arrow[ur] & \end{tikzcd} \quad \mathcal{C}$$

The resulting adj. is
$$\begin{tikzcd} {}_A\mathrm{Span}(\mathcal{C})_{A'} \arrow[d, bend left, "\text{p.o.}"] \\ {}_A\mathrm{Cospan}(\mathcal{C})_{A'} \arrow[u, bend left, "\text{p.b.}"] \arrow[u, phantom, "\vdash"] \end{tikzcd}$$

(ii) S : set

$$\mu_{mk,S} := \begin{tikzcd} & S \arrow[dr, "1"] \arrow[d, phantom, "\Downarrow !"] & \\ S \times S \arrow[ur, "M"] \arrow[rr, "1"'] & {} & 1 \end{tikzcd}$$

where M only has $(s,s') \overset{l_{s,s'}}{\rightsquigarrow} s$, $(s,s') \overset{r_{s,s'}}{\rightsquigarrow} s'$ $(\forall s,s')$

$(\zeta_s : A_s \to Z)_{s \in S}$ has an anticolimit

iff it is a colimit of
$$\left(\begin{tikzcd} & A_s \\ A_s \times_Z A_{s'} \arrow[ur] \arrow[dr] & \\ & A_{s'} \end{tikzcd} \right)_{s,s'}.$$

② Ab-enriched case

$$\mu_{ck} := \begin{tikzcd} & \Delta 1 \arrow[dr, "\mathbb{Z}"] \arrow[d, phantom, "\Downarrow !"] & \\ \Delta 1 \arrow[ur, "\mathbb{Z}"] \arrow[rr, "0"'] & {} & \Delta 1 \end{tikzcd}.$$

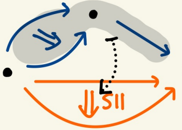
The resulting adjunction is
$$\begin{tikzcd} \mathcal{C}/A \arrow[r, shift left, "\mathrm{Coker}"] \arrow[r, phantom, "\perp"] & A/\mathcal{C} \arrow[l, shift left, "\mathrm{Ker}"] \end{tikzcd}.$$

Similar for $(\mathrm{Set}_*, \wedge)$-cats.

Captured Examples (2)

③ Cat-enriched case

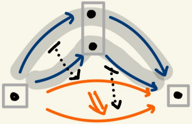
(i)



$$\left\{ \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & A \end{tikzcd} \right\} \begin{tikzcd} {} \arrow[r, shift left, "\text{“Coinvt”}"] \arrow[r, phantom, "\perp"] & {} \arrow[l, shift left] \end{tikzcd} A/\mathcal{K}_0$$

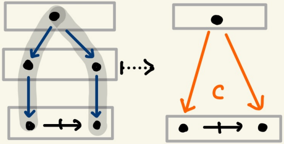
$$\begin{tikzcd} \mathrm{Lim}(\xi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z^{(\bullet \cong \bullet)} \arrow[d] & & A \arrow[d, "\xi"] \\ A^{(\bullet \to \bullet)} \arrow[r, "\xi^{\bullet\to\bullet}"'] & Z^{(\bullet \to \bullet)} & & Z \arrow[ll, mapsto, phantom] \end{tikzcd} \longleftarrow\!\!\shortmid$$

(ii) (⚠ Non-propositional)



$${}_A\mathrm{Span}(\mathcal{K})_B \begin{tikzcd} {} \arrow[r, shift left, "\text{Cocomma}"] \arrow[r, phantom, "\perp"] & {} \arrow[l, shift left, "\text{Comma}"] \end{tikzcd} {}_A\mathrm{Cospan}(\mathcal{K})_B$$

④ Double categories



$$\begin{tikzcd} {}_A\mathrm{Span}(\mathbb{D}_0)_B \arrow[r, shift left, "\text{Ext}"] \arrow[r, phantom, "\perp"] \arrow[dr, shift left] \arrow[dr, phantom, "\perp"] & \left\{ \begin{matrix} A & & B \\ \downarrow & & \downarrow \\ \bullet & \overset{\shortmid}{\longrightarrow} & \bullet \end{matrix} \text{ in } \mathbb{D} \right\} \arrow[l, shift left, "\text{Tab}"] \\ & \mathbb{D}(A,B) \arrow[ul, shift left] \arrow[u, hook] \end{tikzcd}$$

Future application

(i) Formal theory of homology?

For a gamut of the form

$$\begin{tikzcd} \mathcal{I} \arrow[r, "M"] \arrow[rr, bend right, "P"'] & \mathcal{I} \arrow[r, "M"] \arrow[d, phantom, "\mu"] & \mathcal{I} \\ & {} & \end{tikzcd}$$

(vertically epimorphic)

we can define chain complex as

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{I} \arrow[r, "M"] \arrow[drr, bend right, "c_2"'] & \mathcal{I} \arrow[r, "M"] \arrow[dr, "c_1"] & \mathcal{I} \arrow[r, "M"] \arrow[d, "c_0"] & \mathcal{I} \arrow[r] \arrow[dl, bend left, "c_{-1}"] & \cdots \\ & & \varphi_{21} \quad \varphi_{10} & \mathcal{C} \quad \varphi_{0,-1} & & \end{tikzcd}$$

s.t. the composites of two adjacent φ's factor through μ.

Ex

$$\begin{tikzcd} \Delta 1 \arrow[r, "\mathbb{Z}"] \arrow[rr, bend right, "0"'] & \Delta 1 \arrow[r, "\mathbb{Z}"] \arrow[d, phantom, "\Downarrow !"] & \Delta 1 \\ & {} & \end{tikzcd}$$

in **Ab**-$\mathbb{P}$rof

leads to a usual one.

(ii) Formal theory of regularity?

Bourke and Garner developed theory of 2-dimensional regularity based on kernel-quotient systems.

Def [BG14] A kernel-quotient system is an f.f. embedding $J : \underset{(0 \to 1)}{\overset{}{2}} \hookrightarrow F$

(with $2 = (0 \to 1)$)

Let $K := F \setminus \{J1\} \overset{\text{f.f.}}{\hookrightarrow} F$

With this, they construct an adjunction

$$\begin{tikzcd} {[K, \mathcal{C}]} \arrow[r, shift left, "\mathrm{quot}"] \arrow[r, phantom, "\perp"] & {[2, \mathcal{C}]} \arrow[l, shift left, "\mathrm{ker}"] \end{tikzcd}$$

We can recover this by taking μ as

$$\begin{tikzcd} F \setminus \mathrm{Im} J \arrow[r, "{F(-,J0)}"] \arrow[rr, bend right, "{F(-,J1)}"'] & 1 \arrow[r, "I"] \arrow[d, phantom, "\Downarrow"] & 1 \\ & {} & \end{tikzcd}$$

$(J0 \to J1)_*$

Future application

(i) Formal theory of homology?

For a gamut of the form

$$\begin{tikzcd} & \mathbb{I} \arrow[dr, "M"] & \\ \mathbb{I} \arrow[ur, "M"] \arrow[rr, "P"'] \arrow[rr, phantom, "\mu"] & & \mathbb{I} \end{tikzcd} \quad \left(\begin{array}{l}\text{vertically} \\ \text{epimorphic}\end{array}\right)$$

we can define chain complex as

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{I} \arrow[r, "M"] \arrow[drr, bend right, "C_2"'] & \mathbb{I} \arrow[r, "M"] \arrow[dr, "C_1"] & \mathbb{I} \arrow[r, "M"] \arrow[d, "C_0"] & \mathbb{I} \arrow[r] \arrow[dl, bend left, "C_{-1}"] & \cdots \\ & & & \mathcal{E} & & \end{tikzcd}$$

φ_{21}, φ_{10}, $\varphi_{0,-1}$

s.t. the composites of two adjacent φ's factor through μ.

Ex

$$\begin{tikzcd} & \Delta 1 \arrow[dr, "\mathbb{Z}"] & \\ \Delta 1 \arrow[ur, "\mathbb{Z}"] \arrow[rr, "0"'] \arrow[rr, phantom, "\Downarrow !"] & & \Delta 1 \end{tikzcd} \quad \text{in } \mathbf{Ab}\text{-}\mathbb{P}\mathrm{rof}$$

leads to a usual one.

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Let $\mathcal{K} := \mathcal{F} \setminus \{J1\} \overset{\text{f.f.}}{\hookrightarrow} \mathcal{F}$

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We can recover this by taking μ as

$$\begin{tikzcd} & 1 \arrow[dr, "\mathbb{I}"] & \\ \mathcal{F} \setminus \mathrm{Im} J \arrow[ur, "{\mathcal{F}(-, J0)}"] \arrow[rr, "{\mathcal{F}(-, J1)}"'] \arrow[rr, phantom, "\Downarrow"] & & 1 \end{tikzcd} \quad (J0 \to J1)_*$$

Final slide

Summary

- Anticolimit is the inverse problem of colimits.
  Solutions are occasionally given "effectively".
- We provided a general theory of anticolimits in virtual equipments.
- We constructed the (possibly relative) adjoint of limit and colimit from a data of diagram shape (= gamut).

$$\mathrm{LD}(\mu, A) \underset{L}{\overset{\mathrm{Colim}}{\rightleftarrows}} \mathrm{Bico}(\mu, A) \underset{\mathrm{Lim}}{\overset{R}{\rightleftarrows}} \mathrm{RD}(\mu, A).$$

What I couldn't include

- Pointwiseness of limits / colimits
- The cases of relative adjoints

What I want to look into

- Preservation / Reflection / Stability
- The two applications.

References

[TV24] Callin Tataru and Jamie Vicary. The theory and applications of anticolimits. 2024. preprint on arXiv.

[BG14] John Bourke and Richard Garner. Two-dimensional regularity and exactness. 2014. Journal of Pure and Applied Algebra.

[St80] Ross Street. Fibrations in bicategories. 1980, Cahier de Topologie et Géométrie Différentielle.

[nLab+] nLab page. Generalized kernels

Thank you!

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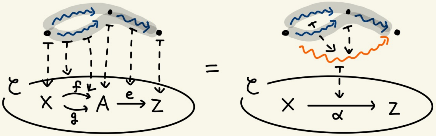
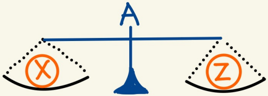
Please let me know if you hit upon any example of anticolimit!





$$\boxed{\mathrm{LD}(\mu, A) \underset{L}{\overset{\mathrm{Colim}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Bico}(\mu, A) \underset{\mathrm{Lim}}{\overset{R}{\underset{\perp}{\rightleftarrows}}} \mathrm{RD}(\mu, A).}$$