Numerical Methods Assignment:

Question 1:

We have:

$$f(x) = \cos x \tag{1}$$

Approximate $f'\left(\frac{\pi}{2}\right)$

a) Evaluate first order derivative, $f'\left(\frac{\pi}{4}\right)$, using:

i) First derivative formula:

First derivative is given by:

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \tag{2}$$

Let suppose $\epsilon = 10^{-6}$, the best value of h is $h = \sqrt{\epsilon} = \sqrt{10^{-6}} = 10^{-3}$.

We need to approximate $f'\left(\frac{\pi}{4}\right)$, which means $x = \frac{\pi}{4}$.

Putting values in Equation (1) in 2, we get:

$$f'\left(\frac{\pi}{4}\right) \approx \frac{\cos\left(\frac{\pi}{4} + 10^{-3}\right) - \cos\left(\frac{\pi}{4}\right)}{10^{-3}} \approx \frac{0.7063993 - 0.70710678}{10^{-3}} = -0.7074602$$

ii) Center-Difference Formula:

First-order derivative using center-difference is given by:

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h} \tag{3}$$

Putting values in Eq 3, we get:

$$f'\left(\frac{\pi}{4}\right) \approx \frac{cos\left(\frac{\pi}{4} + 10^{-3}\right) - cos\left(\frac{\pi}{4} - 10^{-3}\right)}{2 \times 10^{-3}}$$

$$\Rightarrow f'\left(\frac{\pi}{2}\right) = \frac{0.70639932 - 0.70781353}{2 \times 10^{-3}}$$

$$\Rightarrow f'\left(\frac{\pi}{2}\right) = -0.7071066$$

iii) Richardson Extrapolation Formula:

First of all we have to evaluate $\phi_0(h)$ and $\phi_0\left(\frac{h}{2}\right)$, given by:

$$\phi_0(h) = \frac{f(x+h) - f(x-h)}{h}$$

Author: Hayat Ullah
Email | LinkedIn | GitHub

$$\phi_0(h) = \frac{cos\left(\frac{\pi}{4} + 10^{-3}\right) - cos\left(\frac{\pi}{4} - 10^{-3}\right)}{2 \times 10^{-3}}$$

$$\Rightarrow \phi_0(h) = \frac{0.70639932 - 0.70781353}{2 \times 10^{-3}}$$

$$\Rightarrow \phi_0(h) = -0.7071066$$

Now evaluating $\phi_0\left(\frac{h}{2}\right)$:

$$\phi_0\left(\frac{h}{2}\right) = \frac{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}{h}$$

Putting values, we get:

$$\phi_0\left(\frac{h}{2}\right) = \frac{cos\left(\frac{\pi}{4} + 0.5 \times 10^{-3}\right) - cos\left(\frac{\pi}{4} - 0.5 \times 10^{-3}\right)}{1 \times 10^{-3}}$$

$$\Rightarrow \phi_0\left(\frac{h}{2}\right) = \frac{0.70675313 - 0.70746024}{1 \times 10^{-3}}$$

$$\Rightarrow \phi_0\left(\frac{h}{2}\right) = -0.70710675$$

Now evaluating $f'(x)$ by using:

$$f'(x) \approx \frac{4}{3}\phi_0\left(\frac{h}{2}\right) - \frac{1}{3}\phi_0(h)$$

$$= \frac{4}{3}(-0.7071066) - \frac{1}{3}(-0.70710675)$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = -0.70710655$$

b) Evaluating $f''\left(\frac{\pi}{4}\right)$ using Second Derivative formula:

Second derivative is approximated by:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Let h=0.001, and $x = \frac{\pi}{4}$, Putting values, we get:

$$f''\left(\frac{\pi}{4}\right) = \frac{cos\left(\frac{\pi}{4} + 0.001\right) - 2cos\left(\frac{\pi}{4}\right) + cos\left(\frac{\pi}{4} - 0.001\right)}{0.001^2}$$

$$f''\left(\frac{\pi}{4}\right) = \frac{0.70639932 - 2 \times 0.70710678 + 0.70781353}{0.000001}$$

$$f''\left(\frac{\pi}{4}\right) = -0.70710672228$$

c) MATLAB Code:

```
clc; clear; close all;
x = pi/4; h = 0.001;
```

Author: Hayat Ullah
Email | LinkedIn | GitHub

```
% Part a) i: Using First Derivative formula
df1 = (cos(x+h) - cos(x))/h;
er1 = abs((-sqrt(2)/2 - df1));
disp("a) i: Using First Derivative formula, f'(pi/4) = "+num2str(df1))
disp("Error = "+num2str(er1))
disp("_____________________________________________________")

% Part a) i: Using Center-Difference formula
df2 = (cos(x+h) - cos(x-h))/(2*h);
er2 = abs((-sqrt(2)/2 - df2));
disp("a) iii: Using Center-Difference formula, f'(pi/4) =
"+num2str(df2))
disp("Error = "+num2str(er2))
disp("_____________________________________________________")

% Part a) iii: Using Richardson Extrapolation formula
phi0h = (cos(x + h) - cos(x - h))/(2*h);
phi0hByTwo = (cos(x + h/2) - cos(x - h/2))/(h);
df3 = 4/3*phi0hByTwo - 1/3*phi0h;
er3 = abs((-sqrt(2)/2 - df3));
disp("a) iii: Using Richardson Extrapolation, f'(pi/4) =
"+num2str(df3))
disp("Error = "+num2str(er3))
disp("_____________________________________________________")

% Part b) Second Derivative formula:
ddf = (cos(x+h) - 2*cos(x) + cos(x-h))/(h^2);
err = abs((-sqrt(2)/2 - ddf));
disp("b) Using Second Derivative formula, f''(pi/4) = "+num2str(ddf))
disp("Error = "+num2str(err))
disp("_____________________________________________________")
```

Output:

```
a) i: Using First Derivative formula, f'(pi/4) = -0.70746
Error = 0.00035344
_____________________________________________________
a) iii: Using Center-Difference formula, f'(pi/4) = -0.70711
Error = 1.1785e-07
_____________________________________________________
a) iii: Using Richardson Extrapolation, f'(pi/4) = -0.70711
Error = 8.582e-14
_____________________________________________________
b) Using Second Derivative formula, f''(pi/4) = -0.70711
Error = 5.89e-08
_____________________________________________________
```

Richardson Extrapolation method gives the minimum error.

Author: Hayat Ullah
Email | LinkedIn | GitHub

Question 2:

Chebyshev polynomial is given by:

$$p(x) = \sum_{i=1}^{n} f(x_i) q_i(x)$$

$$q_i(x) = \prod_{k\neq i, k=1}^{n} \frac{x - x_k}{x_i - x_k}$$

Where $x_i = \pi \cos\left(\frac{k\pi}{n}\right)$ for $k = 1,2,3, \dots, n$.

Consider 10th degree polynomial, we have $n = 10 + 1 = 11$.

Evaluating x_i: consider this piece of MATLAB Code:

```
n = 11; k = 1:n; xi = pi*cos(k*pi/n)
```

The above code returns

$$x_i = [3.0143, \quad 2.6429, \quad 2.0573, 1.3051, \quad 0.4471, -0.4471, -1.3051, -2.0573, -2.6429, -3.0143, -3.1416]$$

Since $f(x) = cos\,(x)$, evaluating $f(x_i)$ using the following piece of matlab code:

```
fi = cos(xi)
```

$$f_i = [-0.9919, \quad -0.8782, \quad -0.4675, \quad 0.2626, \quad 0.9017, \quad 0.9017, \quad 0.2626, \quad -0.4675, \quad -0.8782, \quad -0.9919, \quad -1.0]$$

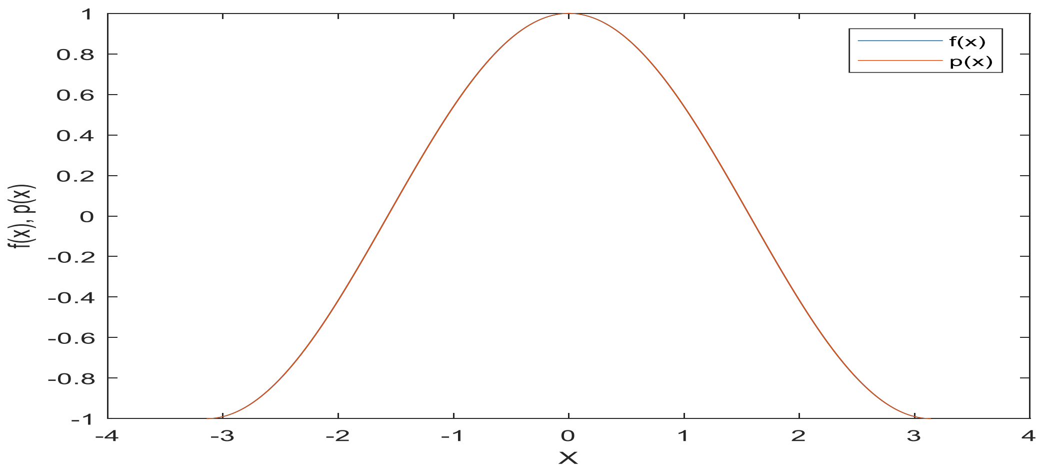
MATLAB Code for evaluating the Chebyshev Polynomial:

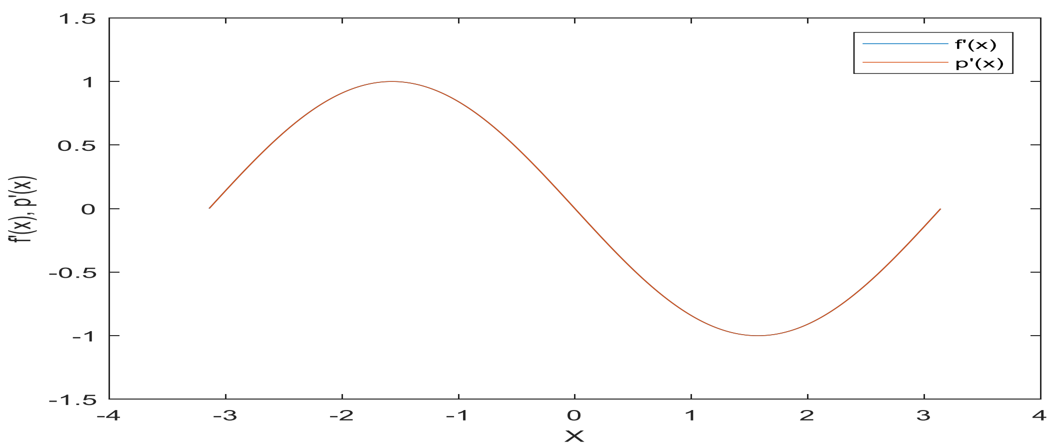
```
n = 11; k = 1:n; xi = pi*cos(k*pi/n); fi = cos(xi)
h = 0.001; xt = -pi:h:pi;
fv = lgrang(xi, fi, xt);

plot(xt, fv, xt, cos(xt))
xlabel('X'); ylabel('f(x)');
legend('f(x)', 'p(x)')
h = 0.001; xt = -pi:h:pi;
fv = lgrang(xi, fi, xt);
plot(xt, fv, xt, cos(xt))
xlabel('X'); ylabel('f(x), p(x)');
legend('f(x)', 'p(x)')

df = (cos(xt(2:end))-cos(xt(1:end-1)))/h;
dfv = (fv(2:end) - fv(1:end-1))/h;

% difference of derivatives:
```

Author: Hayat Ullah
Email | LinkedIn | GitHub

```
mdiffr = mean(abs(df(2:end) - dfv(2:end)));
disp("Mean difference is "+num2str(mdiffr));

figure;
plot(xt(1:end-1), df, xt(1:end-1), dfv)
xlabel("X"); ylabel("f'(x), p'(x)");
legend("f'(x)", "p'(x)")
```

From the above evaluation, we conclude that the average of absolute difference between $f'(x)$ and $p'(x)$ is 3.498×10^{-6} which is too small and can't be observed on the graph as the graphs are overlapping. Consider the figures below. First figure corresponds to $f(x), p(x)$ and the second one shows $f'(x), p(x)$






```
function p = lgrang(x, f, xt )
n = length(x); m = length(xt);
% Evaluates Lagrange interpolating poly
```

Author: Hayat Ullah
Email | LinkedIn | GitHub

```
% for the data vectors x and f,
% at the points xt(1), ..., xt(m),
% returning results in p(1), ..., p(m)
for i = 1 : n % determine weights
    w(i) = prod( x(i) - x([1:i-1 i+1:n]) );
end
for l = 1 : m % determine poly values at xt's
    wt = w.*( xt(l) - x );
    p(l) = sum(f./wt)/sum(1./wt);
end
```

Author: Hayat Ullah
Email | LinkedIn | GitHub