System of Ordinary Differential Equations (ODEs)

The given system of differential equation is given by:

$$\dot{x} = -z - y + 28$$
$$\dot{y} = x + \frac{y}{10}$$
$$\dot{z} = (x - 14)(z - 28) + \frac{1}{10}$$

Part 1:

For fixed points, $\dot{x} = \dot{y} = \dot{z} = 0$, the above system becomes:

$$0 = -z - y + 28 \quad \text{.........} \quad (1$$
$$0 = x + \frac{y}{10} \quad \text{.........} \quad (2$$
$$0 = (x - 14)(z - 28) + \frac{1}{10} \quad \text{.........} \quad (3$$

From first equation, $y + z = 28$, from second equation, $y = -10x \Rightarrow -10x + z = 28$,

Or $z = 10x + 28$, putting in third equation, we have:

$$0 = (x - 14)(10x + 28 - 28) + \frac{1}{10}$$
$$\Rightarrow 0 = 10(x - 14)(10x) + 1$$
$$\Rightarrow 0 = 100x^2 - 1400x + 1$$
$$\Rightarrow x = 13.9993 \text{ OR } x = 0.0007 \quad .$$

If x is $13.9993 \approx 14.0$, then $z = 10(13.9993) + 28 = 167.9930 \approx 168.0$ and $y = -10(13.9997) = -139.9970 \approx -140.0$.

If x is $0.0007 \approx 0.0$, then $z = 10(0.0007) + 28 = 28.0070 \approx 28.0$ and $y = -10(0.0007) = -0.0070 \approx 0.0$

So we have two fixed points i.e. $(14, -140, 168)$ & $(0, 0, 28)$.

Part 2:

MATLAB Code:

```
clc; clear; close all;

odefun = @(t, y) [-y(3) - y(2) + 28; ...
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              y(1) + 0.1*y(2); ...
              (y(1) - 14)*(y(3) - 28) + 0.1];

y0 = [10; 10; 10];
[t, y] = ode45(odefun, [0 1000], y0);

fixed_pts = [0,0,28; 14,-140,168];

plot3(y(:,1), y(:,2), y(:,3))
hold on
plot3(fixed_pts(:, 1), fixed_pts(:, 2), ...
    fixed_pts(:, 3), '*', 'MarkerSize', 8, 'LineWidth', 2)
xlabel('x')
ylabel('y')
zlabel('z')

plot3(y(:, 1), y(:, 2), -100*ones(size(y,1),1))
plot3(-50*ones(size(y,1),1), y(:, 2), y(:, 3))
plot3(y(:, 1), -140*ones(size(y,1),1), y(:, 3))

xlabel('x')
ylabel('y')
zlabel('z')

legend('Trajectory', 'xy Projection', ...
    'yz Projection', 'xz Projection');
grid on;

legend('Trajectory', 'Fixed Points', 'xy Projection', ...
    'yz Projection', 'xz Projection');
```
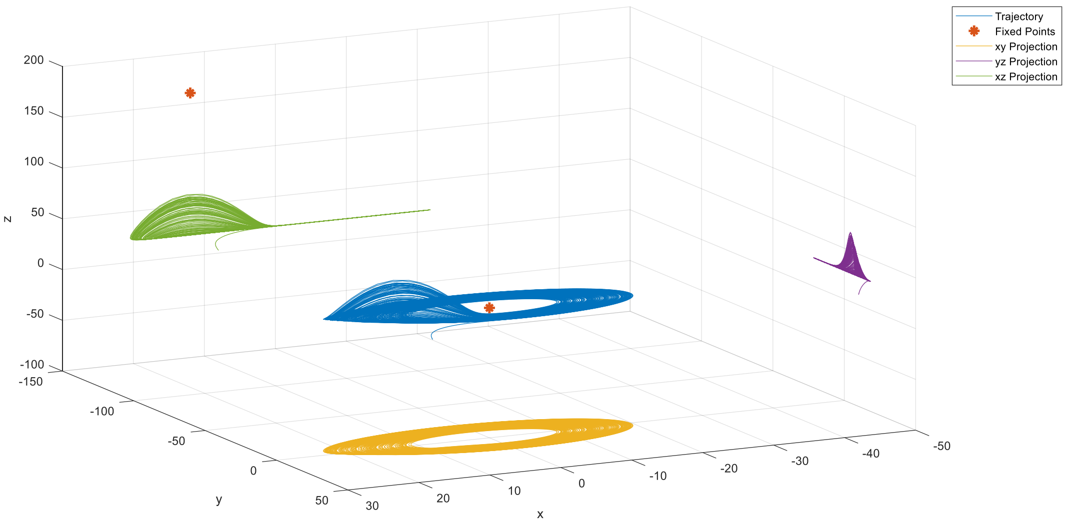



$$f_1(x,y,z) = -z - y + 28$$
$$f_2(x,y,z) = x + \frac{y}{10} = x + 0.1y$$
$$f_3(x,y,z) = (x-14)(z-28) + \frac{1}{10} = xz - 28x - 14z + 392.1$$

Part 3:

Stability of the system:

Find the Jacobian matrix, given by:

$$J(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.1 & 0 \\ z-28 & 0 & x-14 \end{bmatrix}$$

Evaluating Jacobian Matrix at the fixed points:

$$J(0,0,28) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.1 & 0 \\ 0 & 0 & -14 \end{bmatrix} = J_1$$
$$J\big((14,-140,168)\big) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.1 & 0 \\ 140 & 0 & 0 \end{bmatrix} = J_2$$

Evaluating eigenvalues of the Jacobian matrices:

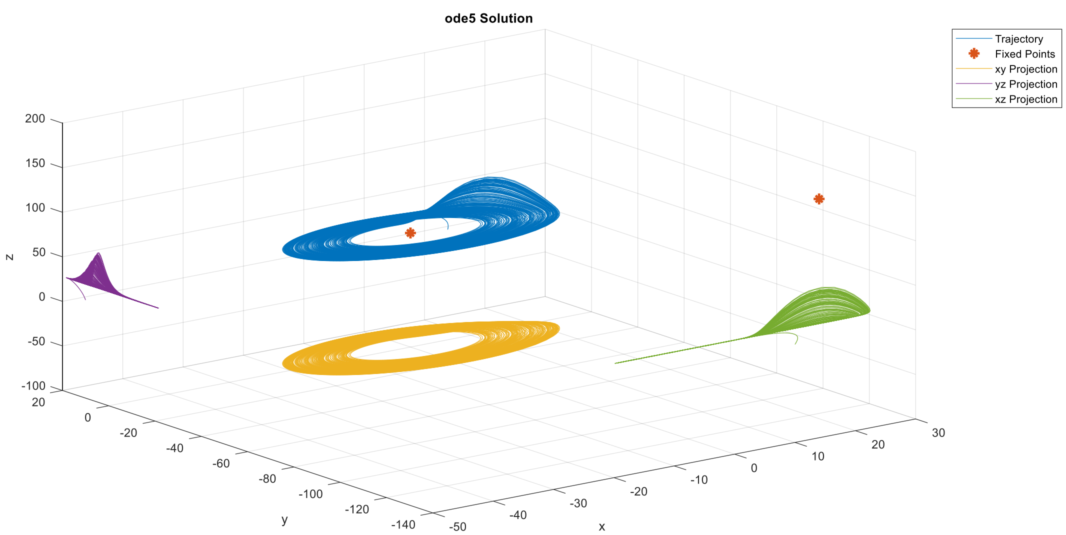
$$Eig(J_1) = \begin{bmatrix} 0.0500 \ + \ j0.9987 \\ 0.0500 \ + \ j0.9987 \\ -14 \end{bmatrix}$$

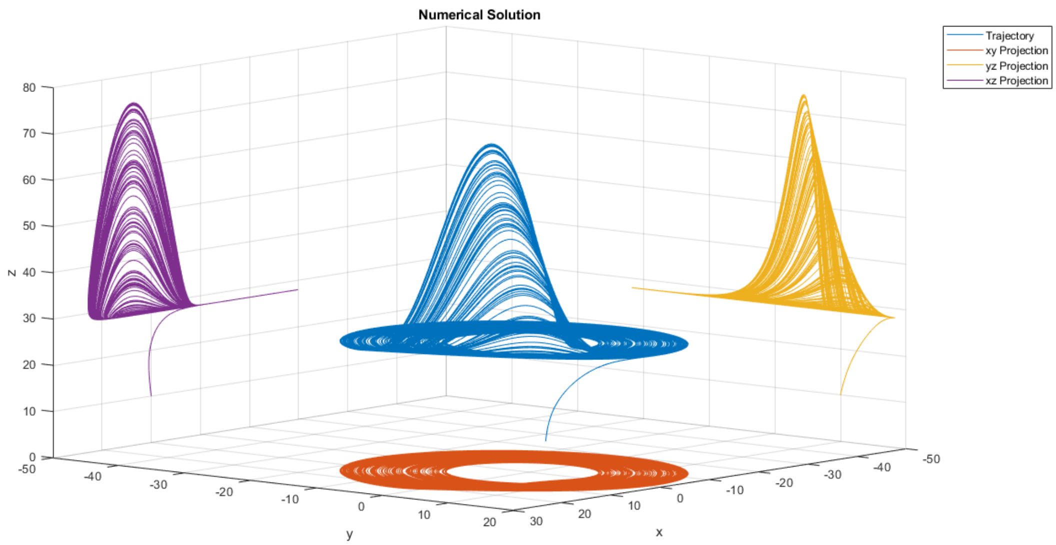
$$Eig(J_2) = \begin{bmatrix} 0.0004 \ + j11.8743 \\ 0.0004 \ + j11.8743 \\ 0.0993 \end{bmatrix}$$

Since both of the fixed points eigenvalues real component greater than 0, so the system is unstable at the fixed points.

Part 4:

Phase Diagrams:





Part 5:

Numerical Solution of the System:

Numerically derivative of a function, f(t), is given by:

$$\dot{f}(t) = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$$

The above system of ODEs becomes:

$$\begin{aligned} x(t+\delta t) &= (-z(t) - y(t) + 28)\delta t + x(t) \\ y(t+\delta t) &= \big(x(t) + 0.1y(t)\big)\delta t + y(t) \\ z(t+\delta t) &= (x(t) - 14)(z(t) - 28)\delta t + z(t) \end{aligned}$$

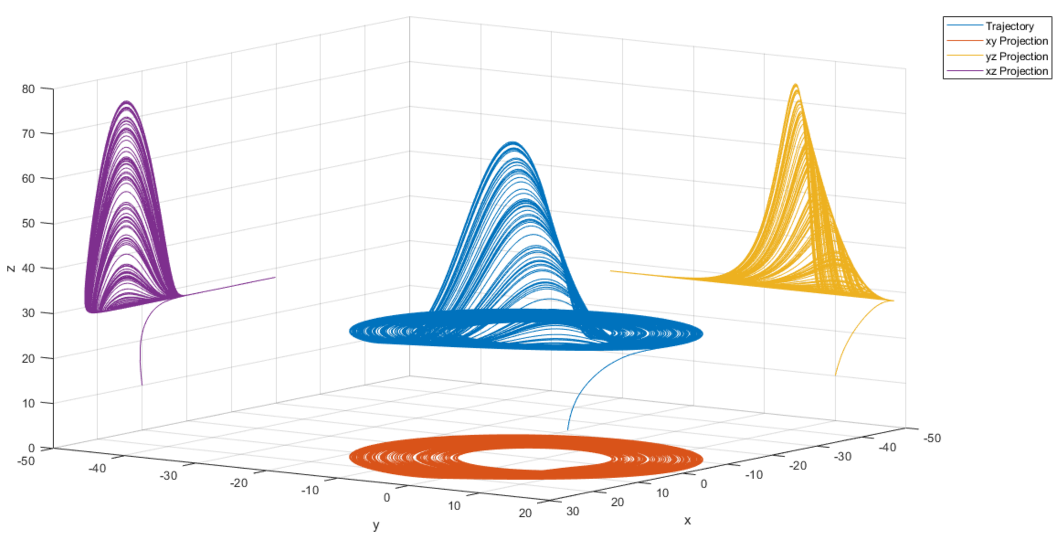
OR:

$$\begin{bmatrix} x(t+\delta t) \\ y(t+\delta t) \\ z(t+\delta t) \end{bmatrix} = \begin{bmatrix} -z(t) - y(t) + 28 \\ x + 0.1y(t) \\ (x(t) - 14)(z(t) - 28) + 0.1 \end{bmatrix} \delta t + \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

MATLAB Code:

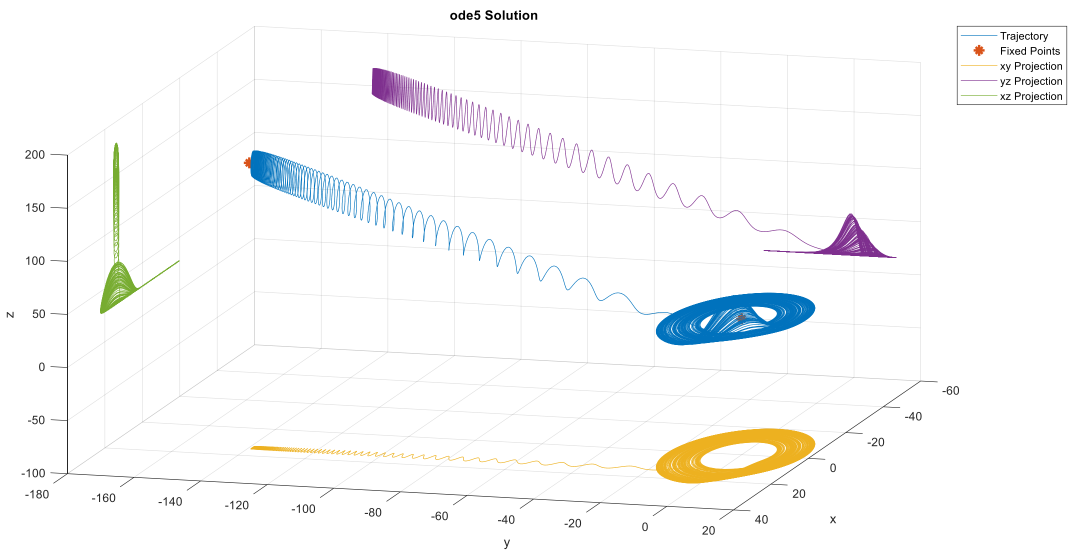
```
clc; close all; clear all;

func = @(Y) [-Y(3) - Y(2) + 28; ...
              Y(1) + 0.1*Y(2); ...
              (Y(1) - 14)*(Y(3) - 28) + 0.1]';
dt = 0.01;
endTime = 1000;
```

```
n = endTime/dt;
y = zeros(n, 3);

y(1, :) = [10 10 10]; % Initial Condition

for idx = 1:n-1
    y(idx + 1, :) = func(y(idx, :))*dt + y(idx, :);
end

plot3(y(:, 1), y(:, 2), y(:, 3))
hold on
plot3(y(:, 1), y(:, 2), zeros(size(y,1),1))
plot3(-50*ones(size(y,1),1), y(:, 2), y(:, 3))
plot3(y(:, 1), -50*ones(size(y,1),1), y(:, 3))

title('Numerical Solution')
xlabel('x')
ylabel('y')
zlabel('z')

legend('Trajectory', 'xy Projection', ...
    'yz Projection', 'xz Projection');
grid on;
```

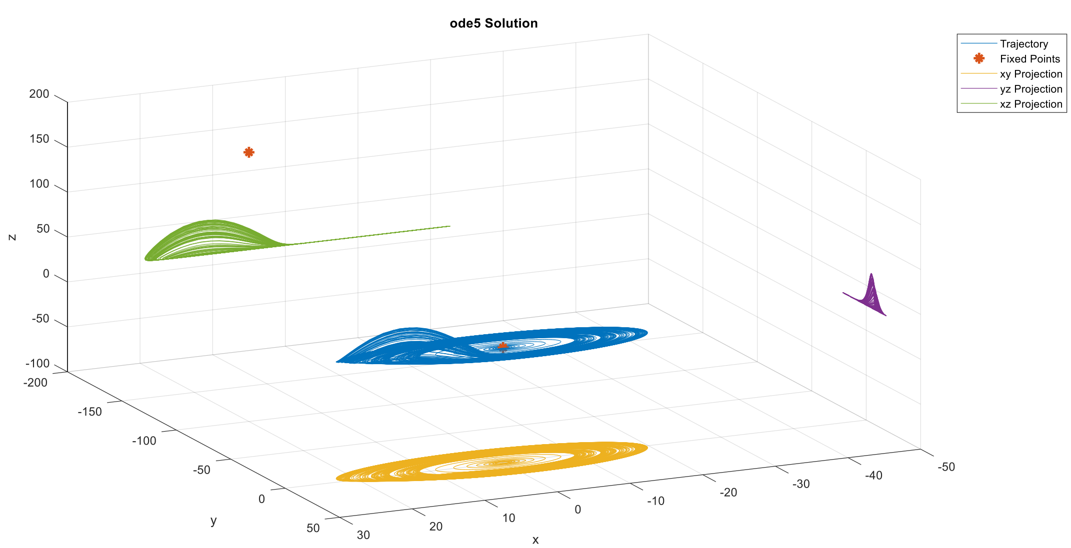


Since fixed points are: $(14, -140, 168)$ & $(0, 0, 28)$.

We use initial condition close to the first fixed point i.e. (13, -139, 169) instead of (10, 10, 10), and we get the following plot:



We use initial condition close to the second fixed point i.e. (0.5, 0.5, 27) instead of (10, 10, 10), and we get the following plot:



Part 6:

Rate of Change Code:

MATLAB Code:

```
clc; close all; clear all;

Part_5;

close all;
ydiff = diff(y)/dt;

plot3(ydiff(:, 1), ydiff(:, 2), ydiff(:, 3));
hold on
plot3(ydiff(:, 1), ydiff(:, 2), -250*ones(size(ydiff,1), 1));
plot3(ydiff(:, 1), -80*ones(size(ydiff,1), 1), ydiff(:, 3));
plot3(-80*ones(size(ydiff,1), 1), ydiff(:, 2), ydiff(:, 3));

xlabel('x'); ylabel('y'); zlabel('z');

legend('Rate of Change', 'xy Projection', ...
    'yz Projection', 'xz Projection');
grid on;
```