IV B-S on a tree

1. We now suppose that we have

   (i) Interest rate : r

   (ii) dividend rate : d

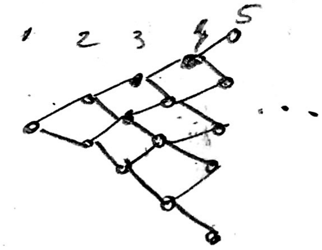
2. Then we have spot dynamics:

   (i) $dS = (r-d) S dt + \sigma S d W_t$ Brownian Motion

   (ii) Value of European option at Expiry T is:

$$e^{-rT} \mathbb{E}[C(S,T)],$$

where $C(S,T)$ is payoff at time T (before it was $f(S_T)$)

3. We want to price on a binomial Tree: 

...

We do:

(i) Divide time into steps and in each step we go a fix up or down

(ii) Then new dynamics are:

$$S_t = S_0 e^{(r-d-\frac{1}{2}\sigma^2)t + \sigma W_t}$$

4. We discretize W_t

(i) Take N steps from 0 to T.

$$0, \frac{T}{N}, 2\frac{T}{N}, \dots, (N-1)\frac{T}{N}, T$$

length $\frac{T}{N}$.

(ii) At step l we need to find.

1. $W_{(l+1)\frac{T}{N}} - W_{l\frac{T}{N}} = \sqrt{\frac{T}{N}}\, N(0,1)$

This is the amount we can go up or down.

2. We do this via a simple approx.:

i. We app. $\sqrt{\frac{T}{N}}\, N(0,1)$ with a r. var taking two values with mean 0 and Var $\frac{T}{N}$.

ii. Let X be a r. var taking ± 1 with Prob $\frac{1}{2}$. Then $\sqrt{\frac{T}{N}}\, X$ is the r. var we are looking for

(iii) We use ii to Approx W_l:

$$W_l = \sum_{j=1}^{l}\left(W_{j\frac{T}{N}} - W_{(j-1)\frac{T}{N}}\right) = \sum_{j=1}^{l}\sqrt{\frac{T}{N}}\, N(0,1) \simeq \underbrace{\sqrt{\frac{T}{N}}\sum_{j=1}^{l} X_j}$$

where X_j are i.i.d. dist. as X.

Y_l

iv) The Approx for $S_{\ell T/N}$ is:

(i) $$S_{\ell T/N} = S_0 e^{(r-d-\frac{1}{2}\sigma^2)\ell T/N + \sigma Y_\ell}$$

* Notice that the spot price do not depend on the path of W_t. It depends on t on the value of W_t at time t.

(i) We care about the sum Y_t, not about each individual X_i

(ii) $\Rightarrow$ recombining tree !!

v

5. Due to Martingale pricing we know that:

(i) Value at time k is equal to discounted val at time $k+1$.

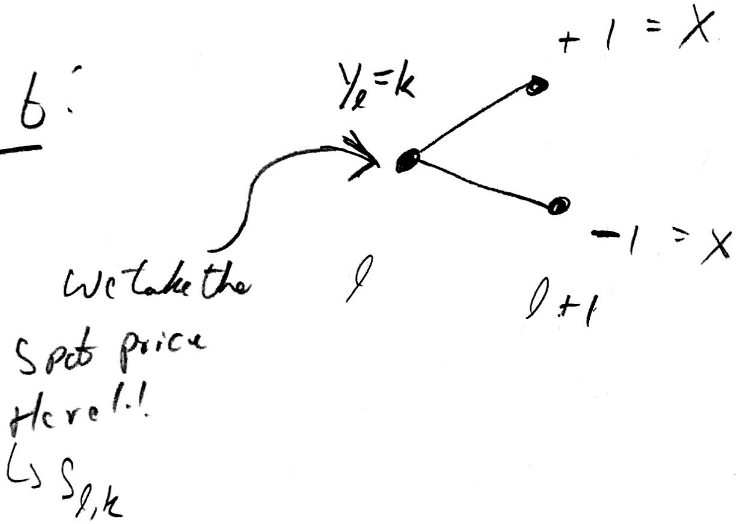
(ii) Let $S_{\ell,k}$ be te value of stock at time $\ell T/N$ if Y_ℓ is k Then:

iii) $C(S_{l,k}, lT/N) = e^{-rT/N} \mathbb{E}[S_{l+1}(Y_{l+1} \mid Y_l = k)]$

$$= \frac{1}{2} e^{-rT/N} \Big[C\big(S_{l,k} e^{(r-d-\frac{1}{2}\sigma^2)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}}, (l+1)\tfrac{T}{N}\big)$$

$$+ C\big(S_{l,k} e^{(r-d-\frac{1}{2}\sigma^2)\frac{T}{N} - \sigma\sqrt{\frac{T}{N}}}, (l+1)\tfrac{T}{N}\big)\Big]$$

Pic 6:



iv) This is how we can price European options:

(i) Plug the value at time T

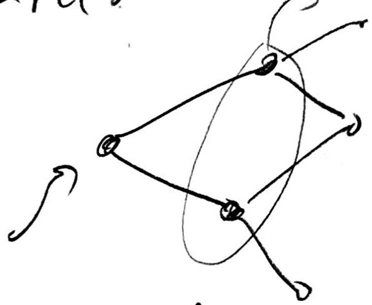
(ii) Use the formula to find value at time 0.

6. For American options and other exotics we do:

i. At any point in the tree we calculate as before the value going forward

ii. Then, the value at the point is the max of exercise value at that point and the expected value going forward.

iii:



do as before and find Expected value: $\tilde{C}(S_{9u}, 9T/N)$

assign value:

$\hookrightarrow \operatorname{Max}\{\max\{K-S,0\}, \tilde{C}(S_{9u}, 9\frac{T}{N})\}$.

* We do calculations in backwards approach starting at the end of the tree.

7. Algorithm:

1. Create final spot values of the form

$$S_0 e^{(r-d-\frac{1}{2}\sigma^2)T + \sigma\sqrt{\frac{T}{N}}\, j}, \quad j \in \{-N, \dots, N\}$$

2. For each spot value evaluate the payoff and store it

3. At previous time compute possible spot vals of form:

$$S_0 e^{(r-d-\frac{1}{2}\sigma^2)(N-1)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\, j}, \quad j \in \{-(N-1), \dots, N-1\}$$

4. For each of these spot, compute the payoff and take max with the discounted pay-off of the two possible values of next time.

5. repeat 3 and 4 until reaching 0

$\square$