

I. Black Scholes Model

(1) Basic assumptions:

(i) We have 1 Risky asset → stock

Riskless asset → money market

Cash

Bond

(ii) The log return of stock price is a random walk i.e. A Geometric Brownian motion s.t. $\begin{cases} \text{drift} \\ \text{volatility} \end{cases} \}$ constant

(iii) the rate of return of riskless asset is constant
so it is called risk-free interest rate

(iv) The stock pays no dividends

(2) Assumptions on the Markets:

(i) No arbitrage (so no riskless profit)

(ii) We can borrow and lend at will (of cash)

(iii) We can buy, sell, short any amount of stock

(iv) NO Transaction Cost

R.* Problem:

(i) There is a derivative security in the market

- 1. Payoff is known → depends on values of stock.
 - 2. Date of Payoff is fixed
- ↳ European call option

Conclusion of B-S Theorem:

(i) The derivative's Price is determined at current time (Even if we do not know the path the stock will take).

(ii) We do this by:

- (1) long position in the stock
- (2) short position in the option

(3) This portfolio value do not depend on the price of the stock.

II B-S: Equation

(i) Price of asset follows Brow. Motion:

so $\frac{dS}{S} = \mu dt + \sigma dW$, where W is a stochastic variable (Brow. Mot.)

(This is another way of writing:

$$S_{t+s} - S_t = \int_t^{t+s} \underbrace{\mu(S_u, u) du}_{\text{constant}} + \int_t^{t+s} \underbrace{\sigma(S_u, u) du}_{\text{constant}}$$

④ From $\frac{dS}{S} = \mu dt + \sigma dW$, we get

$$(1) E \rightarrow \mu dt$$

$$(2) \text{Var} \rightarrow \sigma^2 dt$$

(ii) Consider option $V(S, T)$: Payoff $\rightarrow ??$

We know by it's lemma that:

$$dV = \left(\frac{\partial V}{\partial S} \mu S dt + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW$$

(iii) Main part (Valuation by replication)

1. Consider a portfolio (called delta-hedge) consisting of :
 - (i) Short one option
 - (ii) Long $\frac{\partial V}{\partial S}$ shares at time t .

2) Value of holdings:

$$\Pi = -V + \frac{\partial V}{\partial S} S$$

short \nearrow long \nearrow

3. over Period $[t, t+\Delta t]$ Total profit loss is:

(assuming loss only from value of holdings)

$$\Delta \Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S$$

4. Discretize $\frac{dS}{S}$ and ∂V :

$$(i) \Delta S = \mu S \Delta t + \sigma S \Delta W$$

$$(ii) \Delta V = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \Delta t + \sigma S \frac{\partial V}{\partial S} \Delta W$$

$$\Delta \Pi = \left(-\frac{\partial U}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} \right) \Delta t$$

* So ΔU vanished !! \Rightarrow no dependence on
Brow. Motion !!
This is good !!!

So: (i) This is riskless portfolio \Rightarrow

1. Rate of return must be equal
to rate of return of any other
riskless instrument (If not we get)
arbitrage

6. Assuming risk-free rate of return is r we
must have over time period $[t, t+\Delta t]$:

$$r \Pi \Delta t = \Delta \Pi |^t_{t+\Delta t}$$

This is what we found
in 5.

○ This is riskless so holding it over Δt results in
this

7. Now Substituting formula for $\Delta \Pi$ and Π we get. 6

$$r\left(-V + S \frac{\partial V}{\partial S}\right)_{\Delta t} = \left(\frac{-\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)_{\Delta t}$$

\Rightarrow Simplifying:

B-S classic formula:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

* V needs to be twice diff. with respect to S and V is diff w.r.t. t .

* Changing $V \Rightarrow$ diff Price comes from diff Payoff function \underline{V}

B-S / Applied

0. Change notation to fit the book:

1. Model: $dS_t = \mu S_t dt + \sigma S_t dW_t$,

we have a riskless bond B .

2. Then B-S. the price of vanilla option with expiring T and payoff $f(S)$:

$$e^{-rT} \mathbb{E}(f(S_t)),$$

* Expectation is taken over risk-neutral process:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

2.1 use log and Ito's lemma:

$$d \log S_t = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t.$$

2.2. Solution:

$$\log S_t = \log S_0 + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t$$

3. Since W_t is Brown Motion \Rightarrow

W_T is $N(0, T)$ distributed

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3.2 Putting this into 2.2:

$$\log S_T = \log S_0 + (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} N(0, 1)$$

equivalently

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} N(0, 1)}$$

\Rightarrow 4. Price of vanilla option is.

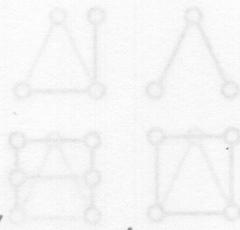
$$e^{-rT} \mathbb{E}[f(S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} N(0, 1)})]$$

II Monte Carlo Simulation:

(i) law of Large Numbers:

Y_1, Y_2, \dots, Y_n are i.i.d then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N Y_j \xrightarrow{\text{w.p.1}} \mathbb{E}[Y]$$



(2) Then it is clear what to do for pricing an option.

(i) draw Random Variable x from $N(0,1)$

(ii) Compute $f(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x})$

where $f(S) = (S - K)_+$ ↑ strike

(iii) Repeat many times and aggregate the average and mult. by e^{-rT}

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