I.1 Black Schoels Model

(1) Basic assumptions:

(i) We have 1. Risky asset → Stock
Riskless asset → money Market
Cash
Bond

(ii) The log return of stock price is a random walk i.e. A Geometric Brownion motion s.t. 1. drift, 2. Volatility } constant

(iii) the rate of return of riskless asset is <u>constant</u> so it is called risk-free interest rate

(iv) The stock pays <u>NO</u> dividends

(2) Assumptions on the Market:

(i) No arbitrage (so no riskless Profit)

(ii) We can borrow and lend at will (of Cash)

(iii) We can buy, sell, short any amount of <u>stock</u>

(iv) <u>NO</u> transaction cost

2.* Problem:

(i) There's a derivative security in the market

1. Payoff is known → depends on values of stock.
2. Date of payoff is fixed

→ European call option

Conclusion of B-S Theorem:

(i) The derivative's price is determined at current time (Even if we do not know the path the stock will take).

(ii) We do this by:

(1) long position in the stock
(2) Short position in the option

(3) This portfolio value do not depend on the price of the stock.

II B-S: Equation:

(i) Price of asset follows Brow. Motion:

so $\boxed{\frac{dS}{S} = \mu\, dt + \sigma\, dW}$, where W is a stochastic variable (Brow. Mot.)

(This is another way of writing:

$$S_{t+s} - S_t = \underbrace{\int_t^{t+s} \mu(S_u, u)\, du}_{\text{constant}} + \underbrace{\int_t^{t+s} \sigma(S_u, u)\, du}_{\text{constant}}$$

⊛ From $\frac{dS}{S} = \mu\, dt + \sigma\, dW$, we get

(1) $\mathbb{E} \to \mu\, dt$

(2) $Var \to \sigma^2 dt$

(ii) Consider option $V(S,T)$: Payoff $\to$??

We know by Itô's lemma that:

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{dV}{dt} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{dV}{dS} dW$$

(iii) Main Part (Valuation by replication)

1. Consider a portfolio (called delta-hedge) consisting of: (i) Short one option

(ii) long $\frac{\partial V}{\partial S}$ shares at time t.

2) Value of holdings:

$$\Pi = \underset{\text{short}}{-V} + \underset{\text{long}}{\frac{\partial V}{\partial S} S}$$

3. over period $[t, t+\Delta t]$ Total profit loss is:
(assuming loss only from value of holdings)

$$\Delta \Pi = -\Delta V + \frac{\partial V}{\partial S} \Delta S$$

4. Discretize $\frac{dS}{S}$ and ∂V.

(i) $\Delta S = \mu S \Delta t + \sigma S \Delta W$

(ii) $\Delta V = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \Delta t + \sigma S \frac{\partial V}{\partial S} \Delta W$

5. Substitute 1 into 3.

$$\Delta \Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \Delta t$$

* So ΔW Vanished !! $\Rightarrow$ no dependance on Brow. Motion !! This is good !!!!

So: (i) This is riskless portfolio $\Rightarrow$

1. rate of return must be equal to rate of return of <u>any</u> other riskless instrument (If not we get arbitrage)

6. Assuming risk-free rate of return is r we must have over time period $[t, t+\Delta t]$:

$$\boxed{r \Pi \Delta t = \Delta \Pi}$$

This is what we found in 5.

* Π is riskless so holding it over Δt results in this

7. Now Substituting formula for $\Delta\Pi$ and Π we get:

$$r\left(-V + S\frac{\partial V}{\partial S}\right)\Delta t = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S}\right)\Delta t$$

$\Rightarrow$ Simplifying:

B-S classic formula:

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0}$$

* V needs to be twice diff. with respect to S and V is diff with resp. to t.

* Changing $V \Rightarrow$ diff Price comes from diff payoff function $\underline{\underline{V}}$

III B-S Applied:

0. Change notation to fit the book:

1. Model: $dS_t = \mu S_t dt + \sigma S_t dW_t$,

We have a riskless bond B.

2. Then By B-S. the price of Vanilla option with expiry T and payoff f is:

$$e^{-rT}\mathbb{E}(f(S_t)),$$

* Expectation is taken over risk-neutral Process:

$$dS_t = rS_t d_t + \sigma S_t dW_t.$$

2.1 Use log and Ito's lemma:

$$d\log S_t = (r - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t.$$

2.2. Solution:

$$\log S_t = \log S_0 + (r - \tfrac{1}{2}\sigma^2)t + \sigma W_t$$

3. Since W_t is Brown Motion $\Longrightarrow$

W_T is $N(0,\overset{\text{variance}}{T})$ distributed

3.2 Putting this into 2.2:

$$\log S_T = \log S_0 + \left(r - \tfrac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\, N(0,1)$$

equivalently

$$S_T = S_0\, e^{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\,N(0,1)}$$

$\Longrightarrow$ 4. Price of vanilla option is:

$$e^{-rT}\,\mathbb{E}\left[f\left(S_0\, e^{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\,N(0,1)}\right)\right]$$

<u>II</u> We use Monte Carlo Simulation:

(i) law of Large Numbers:

$Y_1, Y_2, \ldots, Y_N$ are i.i.d then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} Y_j \xrightarrow{\text{w.p.1}} \mathbb{E}[Y_1]$$

(2) Then it is clear what to do for

Pricing an option:

(i) draw random variable x from $N(0,1)$

(ii) Compute $f\left(S_0\, e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\,x}\right)$

strike

where $f(S) = (S-K)_+$

(iii) repeat many times and aggragate the average and mult. by e^{-rT}

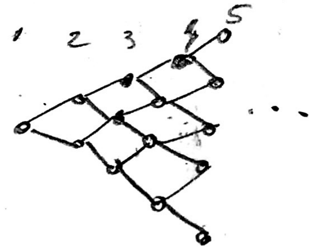
$\boxed{M}$

IV B-S on a tree

1. We now suppose that we have
   (i) Interest rate : r
   (ii) dividend rate : d

2. Then we have spot dynamics:
   (i) $dS = (r-d)\,S\,dt + \sigma S\, dW_t$ — Brownian Motion
   (ii) Value of European option at Expiry T is:
   $$e^{-rT}\,\mathbb{E}[C(S,T)],$$
   where $C(S,T)$ is payoff at time T (before it was $f(S_T)$)

3. We want to price on a binomial Tree:



We do:
   (i) Divide time into steps and in each step we go a fix up or down
   (ii) Then new dynamics are:
   $$S_t = S_0\, e^{(r-d-\frac{1}{2}\sigma^2)t + \sigma W_t}$$

4. We discretize W_t

(i) Take N steps from 0 to T.

$$0, \frac{T}{N}, 2\frac{T}{N}, \dots, (N-1)\frac{T}{N}, T$$

length $\frac{T}{N}$.

(ii) At step ℓ we need to find.

This is the amount we can go up or down.

1. $W_{(\ell+1)\frac{T}{N}} - W_{\ell\frac{T}{N}} = \sqrt{\frac{T}{N}}\, N(0,1)$

2. We do this via a simple approx.:

i We app. $\sqrt{\frac{T}{N}}\, N(0,1)$ with a r. var taking two values with mean 0 and Var $\frac{T}{N}$.

ii Let X be a r. var taking ± 1 with Prob $\frac{1}{2}$. Then $\sqrt{\frac{T}{N}}\, X$ is the r. var we are looking for

(iii) We use ii to Approx W_ℓ:

$$W_\ell = \sum_{j=1}^{\ell}\left(W_{j\frac{T}{N}} - W_{(j-1)\frac{T}{N}}\right) = \sum_{j=1}^{\ell}\sqrt{\frac{T}{N}}\, N(0,1) \simeq \underbrace{\sqrt{\frac{T}{N}}\sum_{j=1}^{\ell} X_j}_{Y_\ell}$$

where X_j ar i.i.d dist. as X.

iv) The Approx for $S_{\ell T/N}$ is:

(i) $$S_{\ell T/N} = S_0 e^{(r-d-\frac{1}{2}\sigma^2)\ell T/N + \sigma Y_\ell}$$

* Notice that the spot price do <u>not</u> depend on the path of W_t. It depends on t on the value of W_t at time t.

(i) We care about the sum Y_t, not about each individual X_i

(ii) $\Rightarrow$ <u>recombining tree !!</u>

v

5. Due to Martingale pricing we know that:

(i) Value at time k is equal to discounted val at time $k+1$.

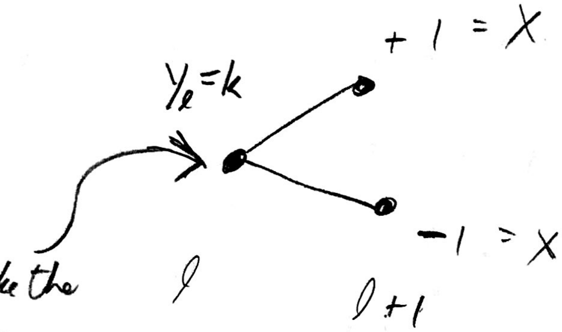
(ii) Let $S_{\ell,k}$ be the value of stock at time $\ell T/N$ if Y_ℓ is k Then:

iii) $c(S_{l,k}, lT/N) = e^{-rT/N}\mathbb{E}[S_{l+1}(Y_{l+1} \mid Y_l = k)]$

$$= \frac{1}{2} e^{-rT/N}\Big[c\big(S_{l,k}e^{(r-d-\frac{1}{2}\sigma^2)\frac{T}{N}+\sigma\sqrt{\frac{T}{N}}}, (l+1)\tfrac{T}{N}\big)$$

$$+ c\big(S_{l,k}e^{(r-d-\frac{1}{2}\sigma^2)\frac{T}{N}-\sigma\sqrt{\frac{T}{N}}}, (l+1)\tfrac{T}{N}\big)\Big]$$

Pic 6:



We take the spot price Here!!
↳ $S_{l,k}$

iv) This is how we can price European options:

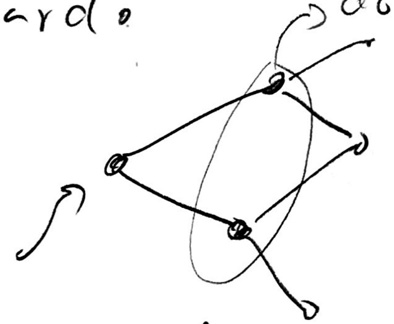
(i) Plug the value at time T

(ii) Use the formula to find value at time 0.

6. For American options and other exotics we do:

i. At any point in the tree we calculate as before the value going forward

ii. Then, the value at the point is the max of exercise value at that point and the expected value going forward.

iii. 

do as before and find Expected value: $\tilde{C}(S_{9u}, 9T/N)$

assign value:

$\hookrightarrow \max\left\{ \max\{K-S, 0\}, \tilde{C}\left(S_{9u}, 9\frac{T}{N}\right) \right\}$.

* We do calculations in backwards approach starting at the end of the tree.

7. Algorithm:

1. Create final spot values of the form

$$S_0 e^{(r-d-\frac{1}{2}\sigma^2)T + \sigma\sqrt{\frac{T}{N}}\, i}, \quad i \in \{-N, \dots, N\}$$

2. For each spot value evaluate the payoff and store it

3. At previous time compute possible spot vals of form:

$$S_0 e^{(r-d-\frac{1}{2}\sigma^2)(N-1)\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\, i}, \quad i \in \{-(N-1), \dots, N-1\}$$

4. For each of these spot, compute the payoff and take max with the discounted pay-off of the two possible values of next time.

5. repeat 3 and 4 until reaching 0

$\square$