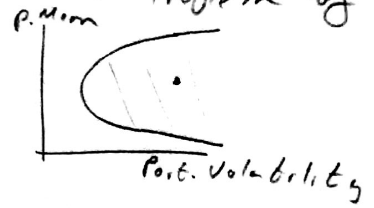
1/ We want to consider the Problem of Finding an "optimal" Portfolio: Pict.



Given Assets $S_1, \dots, S_n$
holding Period τ

* Classic Portfolio model: (i) $r_i = \dfrac{S_i(t+\tau) - S_i(t)}{S_i(t)}$, $i = 1, \dots, n$

(ii) Portfolio weights: $\{w_1, \dots, w_n\}$ s.t. $\sum_{i=1}^{n} w_i = 1 \to w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

(iii) return of T-day Portfolio: $r_p := \sum_{i=1}^{n} w_i r_i$

Matrix notation: $r_p = w^T r$ and $w^T \mathbb{1} = 1$

(iv) reward: $\mu_p = \mathbb{E}[r_p] = \sum_{i=1}^{n} w_i \mathbb{E}[r_i] = \sum_{i=1}^{n} w_i \mu_i$

Matrix notation: $\mu_p = w^T e$, where $e = \mathbb{E}[r] = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$

(v) Variance: Covariance Matrix:

$$V = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ & \vdots & & \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix} = \mathbb{E}\left[(r-e)(r-e)^T\right]$$

Variance: $\sigma_p^2 = w^T V w = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$

2. First task: Find Portfolio with fixed expected return that minimizes the variance: (2)

Problem: $\min \frac{1}{2} w^T V w$ ~ $\frac{1}{2}$Variance

s.t. $w^T e = \mu_p$ ~ expected return

$w^T \mathbb{1} = 1$ ~ feasible portfolio

* Let $P = \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n \\ 1 & 1 & \dots & 1 \end{pmatrix}$, $d = \begin{pmatrix} \mu \\ 1 \end{pmatrix}$

Then we get

$$\boxed{\begin{aligned} &\min \tfrac{1}{2} w^T V w \\ &\text{s.t. } Pw = d \end{aligned}}$$

3. Quadratic optimization

Problem: Minimize $\frac{1}{2} x^T A x + b^T x + c$ $(:= q(x))$

(1) s.t $Px = d$

where cols of P are independent [This happens iff $P^T \lambda = 0 \Leftrightarrow \lambda = 0$]

3.1 Solution

Define Lagrangian:

$$\mathcal{L}(x,\lambda) := \frac{1}{2}x^T A x + b^T x + c - \lambda^T(d - Px)$$

* $\mathcal{L}(x,\lambda) = q(x) \quad \forall x$ feasible for Problem (1)

(i) Find gradient of $\mathcal{L}$

$$\nabla_x \mathcal{L}(x,\lambda) = Ax + b + P^T\lambda$$

$$\nabla_\lambda \mathcal{L}(x,\lambda) = -(d - Px)$$

We care for: (x^*, λ^*) s.t. $\nabla_x \mathcal{L}(x^*,\lambda^*) = \nabla_\lambda (x^*,\lambda^*) = 0$.

This is given by:

$$\left.\begin{aligned} Ax^* + P^T\lambda^* &= -b \\ Px^* &= d \end{aligned}\right\} \overset{\text{matrix}}{\Longrightarrow} \begin{pmatrix} A & P^T \\ P & 0 \end{pmatrix}\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -b \\ d \end{pmatrix} \quad (2)$$

* We can show that if (2) has a unique solution then x^* is a max or min of (1)

3.2

<u>Th.</u> The matrix $\begin{pmatrix} A & P^T \\ P & 0 \end{pmatrix}$ is invertible if either A or $-A$ is positive definite on the $(n-m)$-dimensional subspace of $\mathbb{R}^n$ defined by $Z := \{x \in \mathbb{R}^n \mid Px = 0\}$

[* i.e. if $\forall\ x \in \mathbb{R}^n$ s.t. $Px = 0$, $x^T A x > 0$, then $(\cdot)$ is invert.]

<u>Proof</u>: Supp. A or $-A$ is positive definite on Z.

Then $(\cdot)$ is invt. $\Leftrightarrow \left[\begin{pmatrix} A & P^T \\ P & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right]$

(i) $\begin{pmatrix} A & P^T \\ P & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} Ax + P^T\lambda = 0 \\ Px = 0 \end{array}$

(ii) $Px = 0 \Rightarrow x \in Z$. So $0 = x^T A x + x^T P^T \lambda = x^T A x + (\lambda^T \underbrace{Px}_{0})^T = x^T A x$

and $A, -A$ are p.d. on Z, $x \in Z \Rightarrow x = 0$.

Then plug into first equation we get:

$$Ax + P^T\lambda = 0 \quad (\text{b.t } x = 0)$$

$$\Rightarrow \quad P^T\lambda = 0, \quad \text{but } P \text{ is invertible}$$

$\Rightarrow$ $\underline{\lambda = 0}$. So $x = 0$, $\lambda = 0$ ∎

3.3 So we can conclude:

~~If~~ conditions of Th. Hold: (i.e. $A \succ 0$ or $-A \succ 0$)

1. The system $\begin{pmatrix} A & P^T \\ P & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} -b \\ d \end{pmatrix}$ then it has a unique solution $(x^*, \lambda^*) \in \mathbb{R}^{n+m}$

2. ~~If~~ $A \succ 0$ then x^* is an opt. sol to:

$$\min \tfrac{1}{2} x^T A x + b^T x + c$$
$$\text{s.t. } Px = d$$

II Solving the Portfolio Problem (3)

1. Assumptions:

   0. Problem: $\min \tfrac{1}{2} w^T V w$ s.t. $Pw = d$ | where $P = \begin{pmatrix} \mu^T \\ \mathbb{1}^T \end{pmatrix}$, $d = \begin{pmatrix} \mu \\ 1 \end{pmatrix}$

   (i) Expected returns are different $\Rightarrow$ P is linearly ind.

   (ii) Covariance matrix V is positive definite

2. Solution: Apply quadratic th.

   (i). Consider the Lagrangian of (3). We seek a solution $(w_p, \lambda^*)^T \in \mathbb{R}^{n+2}$ to (looking at gradient=0)

$$\begin{pmatrix} V & P^T \\ P & 0 \end{pmatrix} \begin{pmatrix} w_p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} 0 \\ d \end{pmatrix}$$

$\boxed{* \; e = \mathbb{E}[r] = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}}$

i.e.: $Vw_p + P^T \lambda^* = 0 \Rightarrow Vw_p = -\lambda_1^* e - \lambda_2^* \mathbb{1}$ (4)

where $\lambda^* = (\lambda_1^*, \lambda_2^*)^T$

(ii) The 2nd inequality is $Pw_p = d \Rightarrow \begin{cases} e^T w_p = \mu_p \\ \mathbb{1}^T w_p = 1 \end{cases}$. (5)

Since V is positive definite, it is invertible so we can solve (4) to get:

$$w_p = -\lambda_1^* V^{-1} e - \lambda_2^* V^{-1} \mathbb{1}$$

Plug this into (5) to get:
$$\begin{cases} -(e^T V^{-1} e)\lambda_1^* - (e^T V^{-1} \mathbb{1})\lambda_2^* = \mu_p \\ -(\mathbb{1}^T V^{-1} e)\lambda_1^* - (\mathbb{1}^T V^{-1} \mathbb{1})\lambda_2^* = 1 \end{cases}$$

Set $\begin{pmatrix} B & A \\ A & C \end{pmatrix} = \begin{pmatrix} e^T V^{-1} e & e^T V^{-1} \mathbb{1} \\ \mathbb{1}^T V^{-1} e & \mathbb{1}^T V^{-1} \mathbb{1} \end{pmatrix}$, then this simplifies to:

$$-\begin{pmatrix} B & A \\ A & C \end{pmatrix} \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \end{pmatrix} = \begin{pmatrix} \mu_p \\ 1 \end{pmatrix} \quad (6)$$

(iii) Th. $D := \mathrm{Det}\begin{pmatrix} B & A \\ A & C \end{pmatrix} = BC - A^2$ is Positive

Proof: H.W. ∎

Then $\begin{pmatrix} B & A \\ A & C \end{pmatrix}$ is invertible and has a unique solution.

and: from (6) we get: $\begin{pmatrix} \lambda_1^* \\ \lambda_2^* \end{pmatrix} = \frac{-1}{D} \begin{pmatrix} C & -A \\ -A & B \end{pmatrix} \begin{pmatrix} \mu_p \\ 1 \end{pmatrix}$

This gives
$\Rightarrow$
$$\lambda_1^* = \frac{A - C\mu_p}{D} \quad \text{and} \quad \lambda_2^* = \frac{A\mu_p - B}{D}.$$

(iv) To get w_p from $\lambda^* = (\lambda_1^*, \lambda_2^*)$, we substitute λ^* into

$$w_p = -\lambda_1^* V^{-1} e - \lambda_2^* V^{-1} \mathbb{1}$$ and get:

$$w_p = \left(\frac{C\mu_p - A}{D}\right) V^{-1} e + \left(\frac{B - A\mu_p}{D}\right) V^{-1} \mathbb{1}$$

$$= \underbrace{\frac{1}{D}(B V^{-1}\mathbb{1} - A V^{-1} e)}_{g} + \underbrace{\frac{1}{D}(C V^{-1} e - A V^{-1}\mathbb{1})}_{h} \mu_p.$$

$$\Rightarrow w_p = g + h\mu_p$$

* Notice that g and h depend only on the covariance matrix V on an vector of expected returns e.

(v) How to get the "risk" value of the solution?

$$\sigma^2 = w^T V w = w^T V (g + h\mu_p) = w^T V \left[\frac{1}{D}(B V^{-1}\mathbb{1} - A V^{-1} e) + \mu (C V^{-1} e - A V^{-1}\mathbb{1})\right]$$

$$= \frac{1}{D} w^T \left([B\mathbb{1} - Ae] + \mu [Ce - A\mathbb{1}]\right) \quad (\text{Since } V^{-1}V = I_n)$$

$$= \frac{1}{D}(C\mu^2 - 2A\mu + B) \quad (\text{Since } w^T e = \mu \text{ and } w^T \mathbb{1} = 1$$

$$= \frac{C}{D}\left[\mu^2 - \frac{2A}{C}\mu + \frac{B}{C}\right] = \frac{C}{D}\left[\left(\mu - \frac{A}{C}\right)^2 + \frac{BC - A^2}{C^2}\right]$$

$$= \frac{C}{D}\left(\mu - \frac{A}{C}\right)^2 + \frac{1}{C} \quad (\text{Since } BC - A^2 = D)$$

So: If we want expected return μ_p, we solve for λ^* that gives w_p and we use this to get that σ^2

$$(7) \quad \sigma_p^2 = \frac{C}{D}\left(\mu_p - \frac{A}{C}\right)^2 + \frac{1}{C}$$ (is min possible variance with exp. μ_p)

(vi) This is called the Markowitz portfolio theory. We can derive the actual form of the frontier with it:

1. The risk-reward (σ-μ) relationship at the efficient frontier is given by points $(\sigma, \mu) \in \mathbb{R}^2$ s.t.

$$⑧ \quad \frac{\sigma^2}{1/C} - \frac{(\mu - A/C)^2}{D/C^2} = 1 \qquad (\text{from } ⑦)$$

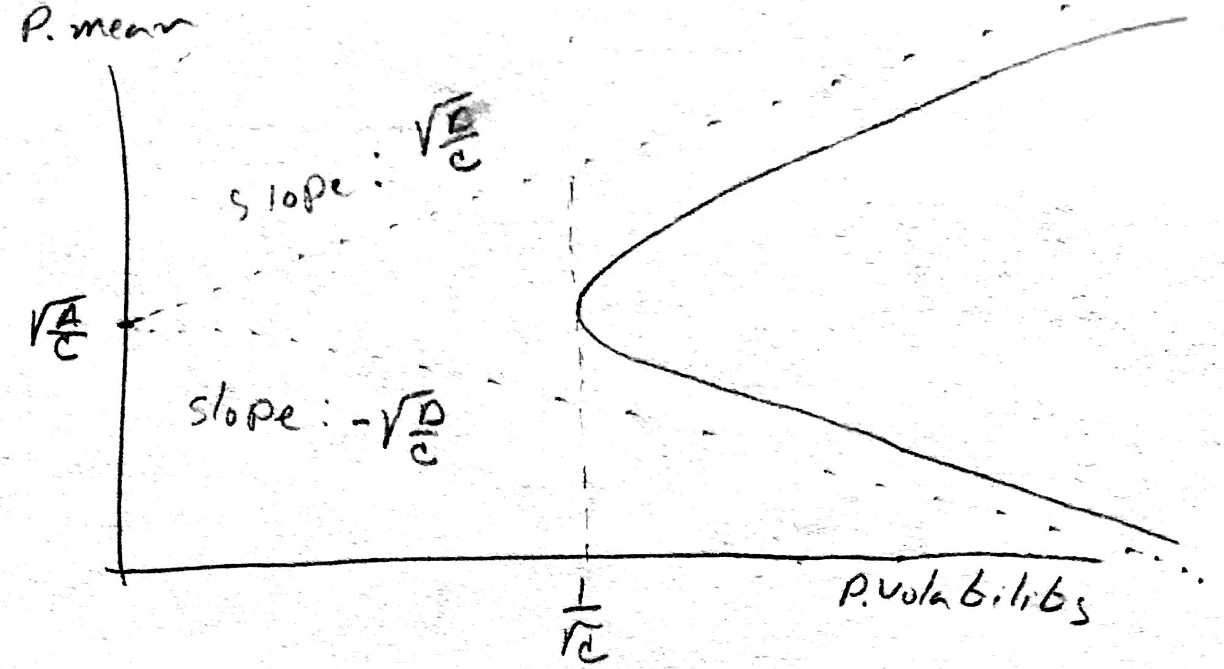
Note that C, A, D depend on e and V, so are set for all portfolios.

⑧ is a hyperbola with center $(0, A/C)$ and asymptotes:

$$\mu = \pm\sqrt{\frac{D}{C}}\,\sigma + \frac{A}{C} \quad \text{and vertex}$$

$$v = \left(\sqrt{\frac{1}{C}}, \frac{A}{C}\right)$$

Picture:



This allows for shortselling of Assets. If we do not want this then we need to solve.

$$\min \ \frac{1}{2} w^T V w$$

$$\text{s.t.} \quad Pw = d$$

$$w \geq 0$$

This problem can be solved via Lagrangian but does not lead to a close formula. We use general non-linear opt. algorithms for this ∎