Problem Set 5

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Abstract

This article contains my solutions to Problem Set 5 for the graduate Computational Physics course. Problem Set 3 investigates numerical integration and linear algebra.

1 Problem 1: The Gamma Function

Problem 1 investigates calculating the value of the gamma function by numerically evaluating the value of its integral. I first plotted the value of the integrand by creating a function that returned the integrand's value for a given value of a. As seen in Figure 1, the integrand starts at zero, rises to a maximum, then decays again for each curve.

The maximum of the integrand falls at $x = a - 1$ as shown in the following:

$$\Gamma(a) = \int_0^\infty f(x)\,dx, f(x) = x^{a-1}e^{-x}$$

Take $f'(x) = 0$ for the maximum:

$$f'(x) = -x^{a-1}e^{-x} + (a-1)x^{a-2}e^{-x} = 0$$

$$x^{a-1} = (a-1)x^{a-2}$$

$$x * x^{a-2} = (a-1)x^{a-2}$$

$$x = a - 1$$

I then needed to use the change of variables $z = \frac{x}{c+x}$ in order to evaluate the integral to infinity. To do so, I needed to find the value of x that gives $z = 1/2$ and the appropriate choice for the parameter c that puts the peak of the gamma function at $z = 1/2$:

$$x = \frac{zc}{1-z}$$

$$z = 1/2$$

$$x = \frac{c/2}{1/2} = c$$

So $x = c$ when $z = 1/2$. Further, the integrand's max occurs at $x = a - 1$:

$$x = a - 1 = c$$

and therefore the appropriate choice for the parameter c for the change of variables in the integral is $c = a - 1$.

Also, to avoid overflow/underflow errors, I rewrote the integrand as:

$$x^{a-1}e^{-x} = e^{(a-1)\ln x}e^{-x} = e^{(a-1)\ln x - x}$$

Now the computer performs arithmetic with values on the order of x and $\ln x$ rather than x^{a-1} and e^x. This arithmetic is with values that are less extremely different from one another, which will help to mitigate overflow and underflow errors.

Once the change of variables and overflow/underflow errors were taken care of, I performed the integral using Gaussian quadrature. The function worked properly, giving a value of 0.886 for $\Gamma(3/2)$, which is the correct value. The function also properly calculated $\Gamma(3), \Gamma(6), and \Gamma(10)$, returning values very very close to 2!, 5!, and 9!, respectively.

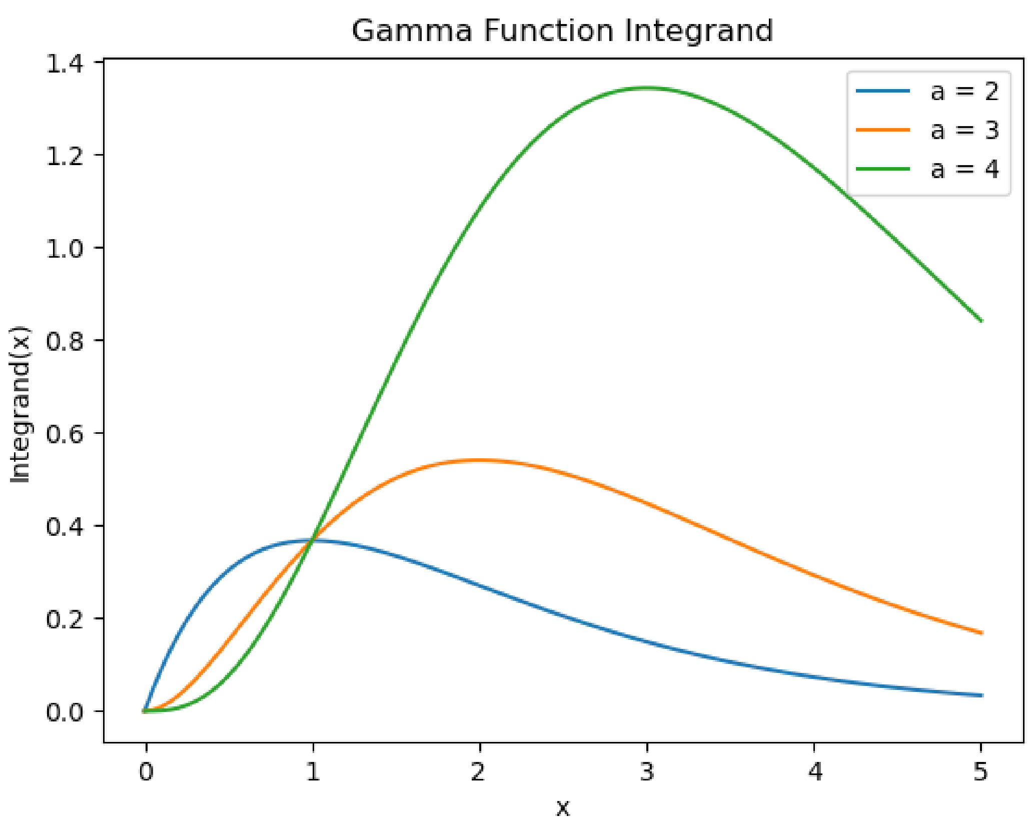

Figure 1: Plot of the Gamma function's integrand for different values of its parameter a.

2 Problem 2: Linear Algebra for Signal Analysis

In problem 2, I study the application of the SVD linear algebra technique to signal analysis. First, I read in the data from the .dat file, filter out unwanted elements of the file, and store the time and signal data in arrays. I then plot the data as seen in Figure 2.

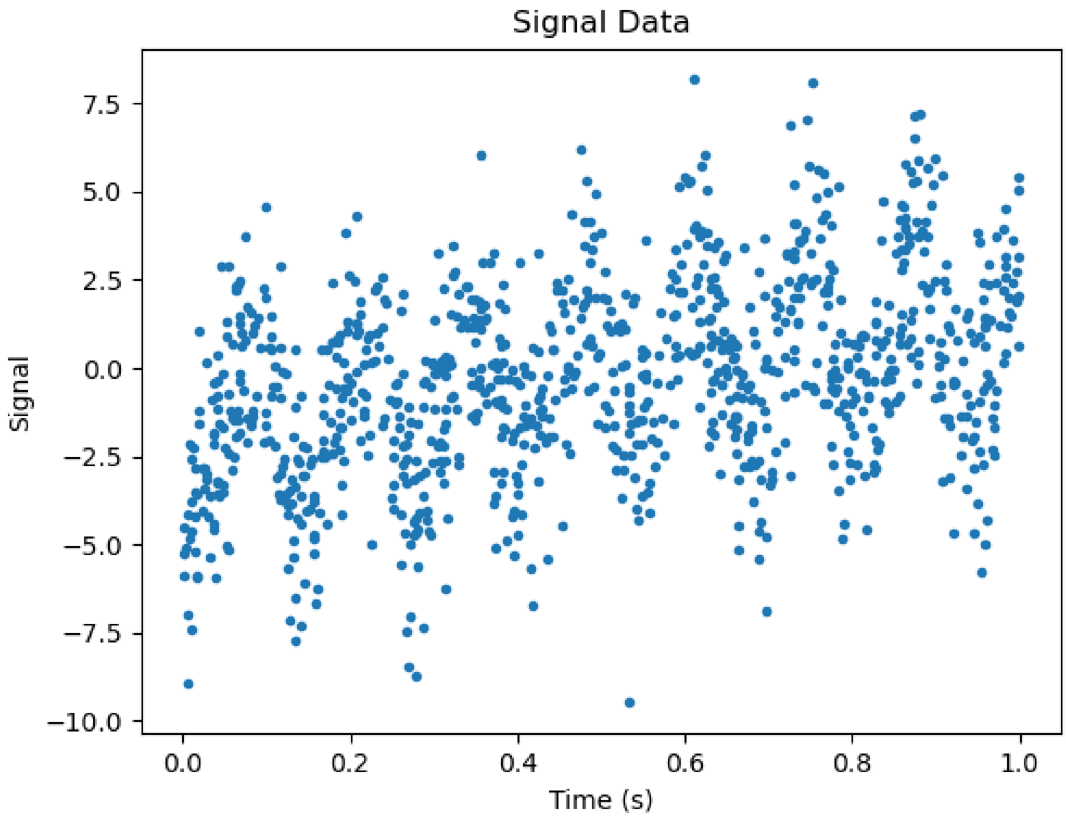

Figure 2: Plot of data read in from the signal.dat file.

I then used the SVD technique described in Mike's Jupyter notebook to fit the data to a third-order polynomial (Figure 3). Since the time was scaled to the order of 1E9, my higher order polynomials quickly blew up, so I rescaled the time by 1E-9. Next, I calculated the residuals of the data with respect to the model, which are shown in Figure 4. The plot of the residuals looks about the same as the original plot, so this clearly isn't a good explanation of the data.

I then tried to fit the data to polynomials of order 4 through 8 using the SVD technique with little success. There was no reasonable polynomial that fit the data well. As I increased the order of the fit, the condition number increased steadily. By the time I got to an eighth order polynomial, the greatest and smallest singular values were 4.15 and 6.1E-5, respectively, meaning that the condition number was on the order of 1E5. It only increased from there, so I stopped looking for higher order polynomials for fear of matrix A becoming singular. The fit to

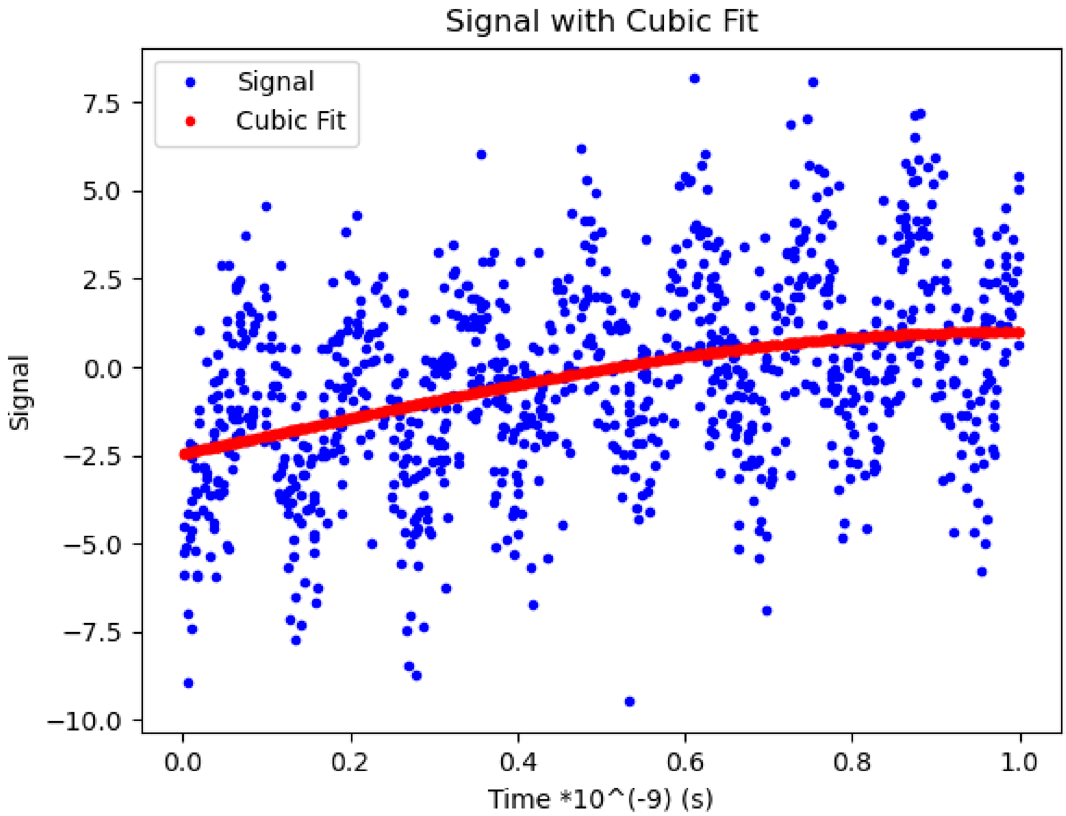

Figure 3: Plot of signal data with a third-order polynomial fit.

an eighth order polynomial can be seen in Figure 5.

Finally, I fit the data with a set of sine and cosine functions using a frequency from the period of half of the time span covered, $T = 0.5$ and $\omega = \frac{2\pi}{T}$, and its first 6 integer harmonics. I first subtracted out the linear component of the data, then performed the sine and cosine analysis on the "flat" signal. The fit is shown in Figure 6. An analysis of the residuals of this model shows that this model does a better job than the polynomial fit, but it is still not perfect. A periodicity with period of roughly 0.15 s is determined by observing the periodicity of this model.

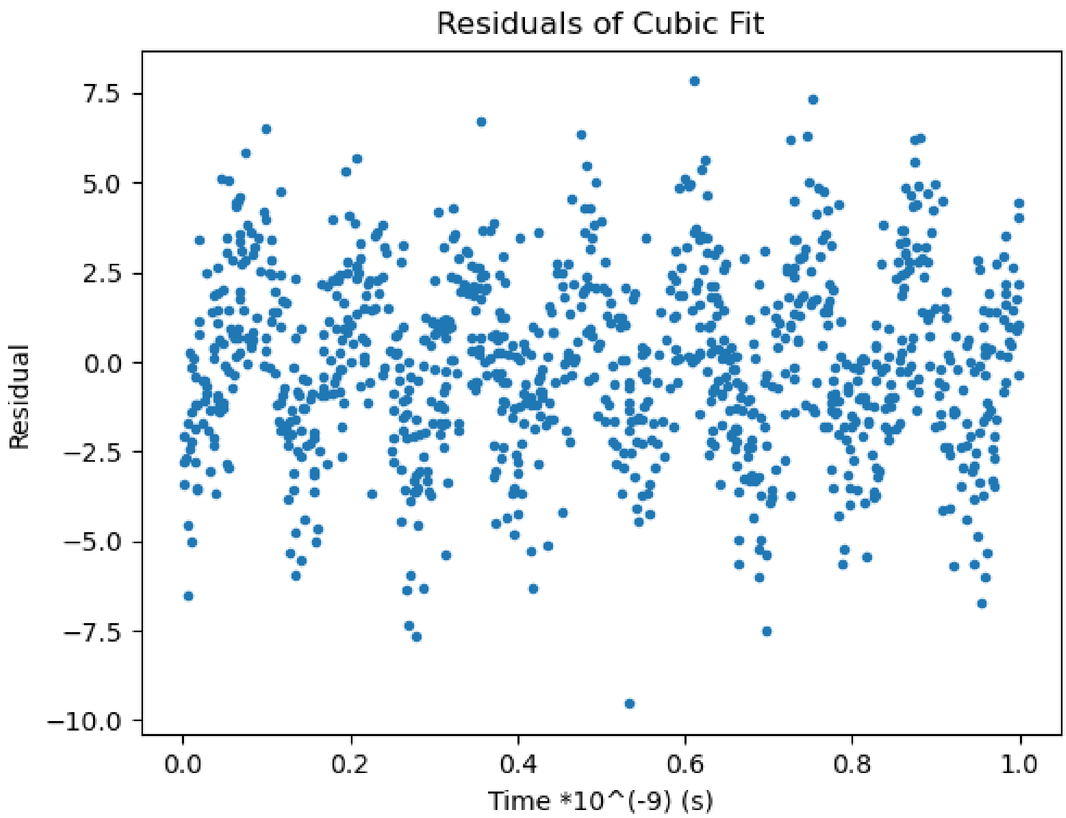

Figure 4: Plot of the residuals of the third-order polynomial fit. This looks about the same as the original data.

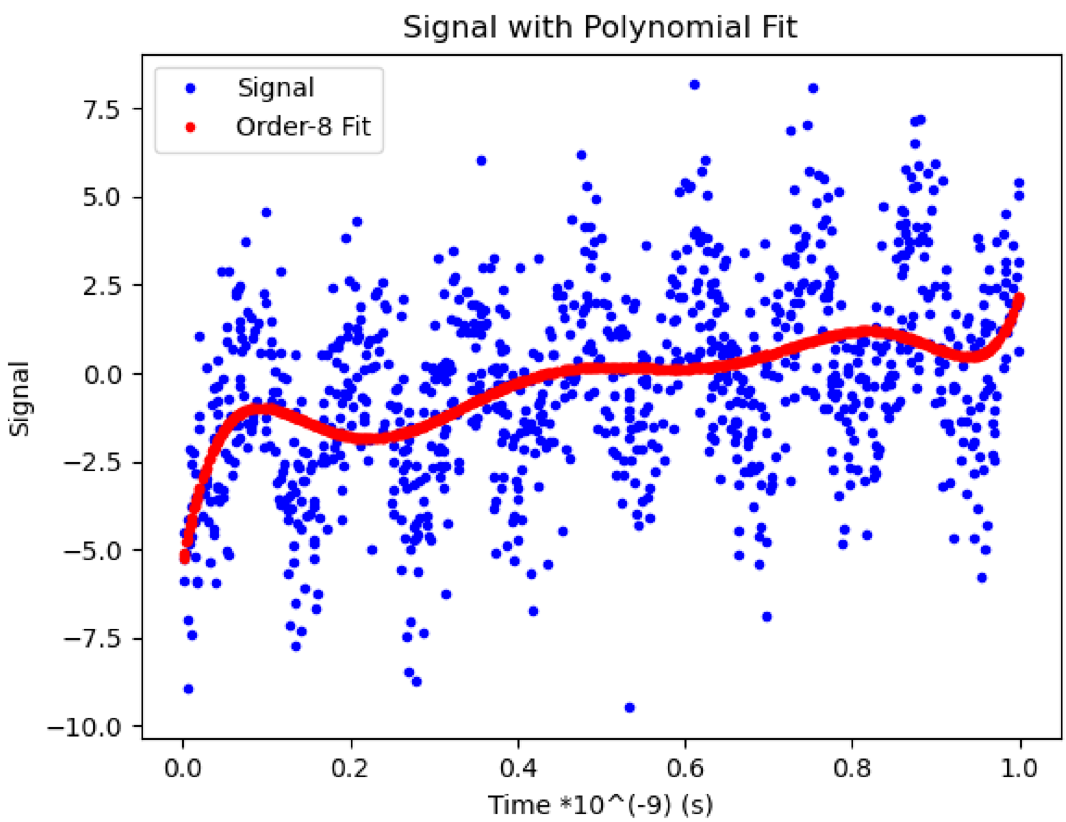

Figure 5: Plot of the signal with an eighth-order polynomial fit.

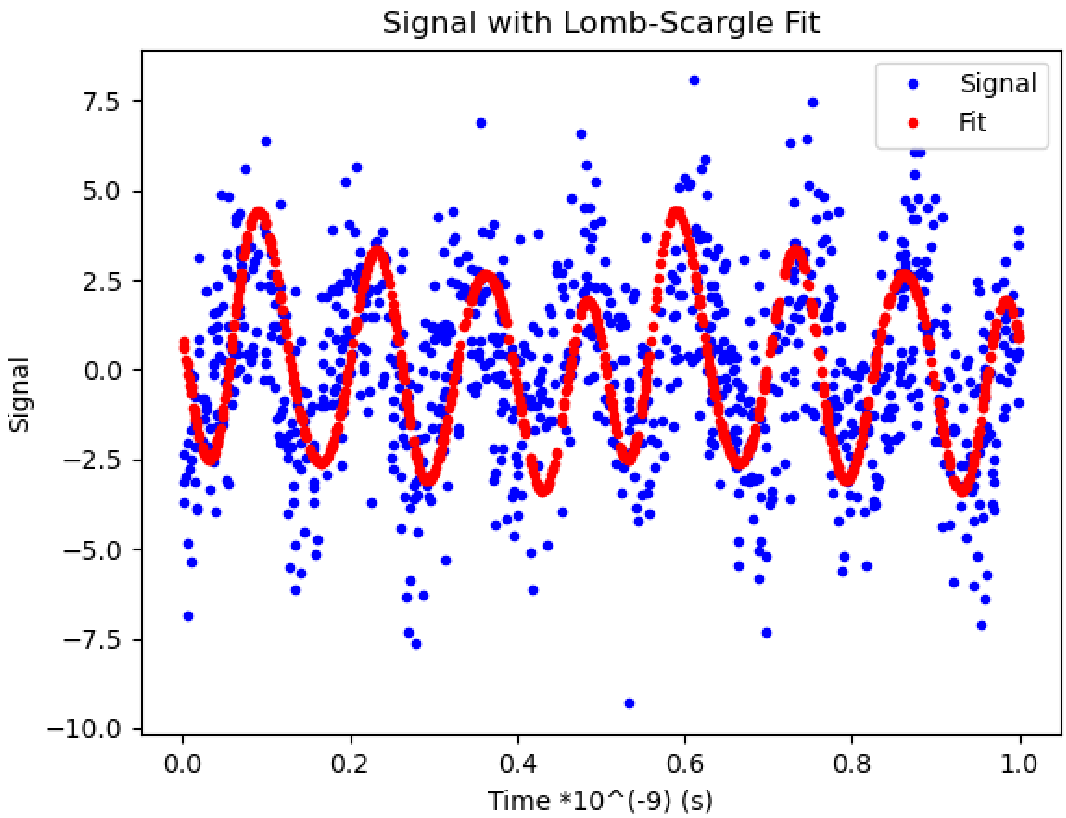

Figure 6: Plot of the signal with Sine and Cosine fit.