CMPSC 130B Winter 2021 Homework #1

Due Jan 20, 2021, 11:59 pm

1. (10 points) Suppose that available coins are denominations that are powers of c, i.e., the denominations are $c^0, c^1, c^2, .., c^k$ for some $c > 1$ and $k >= 1$. Does the greedy algorithm yield an optimal solution? If so, give a proof. If not, present a counterexample.

2. (10 points) You are given a set of n jobs, each of which takes one unit of time to complete. Each job i has an associated "latest completion time" t_i and finishing the job by then earns a reward of d_i dollars (nothing otherwise). Assume completion time $>= 0$ and that the rewards are distinct. It is possible for two jobs to have the same "latest completion time." Find an algorithm that runs in $O(n^2)$ time to maximize the reward.

   Example input:

Job	Latest completion time	Reward
1	5	40
2	5	50
3	4	60
4	4	30
5	3	15
6	3	5

3. (10 points) For the activity/interval scheduling problem, suppose we have k machines to schedule jobs. Derive an optimal algorithm that schedules the maximum number of jobs. Prove the correctness of your algorithm.

4. (10 points) Give an example of a graph with some negative edge weights where there are no negative-valued cycles but Dijkstra's algorithm fails to find the shortest path from a node s to another node t.

5. (10 points) Show that if k minimum cost edges in a graph span $k+1$ vertices then there exists a minimum spanning tree that includes all these k edges.

6. (20 points) You are asked to divide a given set of n 1D points into two clusters so that the inter-cluster distance (their separation) is maximized. Assume that the input points are unsorted. Your algorithm should run in $O(n)$ time.

7. (10 points) A spider needs to extend a web over n sequentially placed rafters that are not necessarily equally spaced. The spider can weave across a maximum distance of k each day, and it must end each day on a rafter so it can rest. The distance between consecutive rafters is at most k. Every night break costs the spider one spider-dollar (measuring risk from predators). Can you help the spider with an optimal algorithm that minimizes the number of spider-dollars? The distance between rafters is defined by array $D = [d_1, \ldots, d_{n-1}]$, which tells you that rafter $i+1$ is at

a distance of d_i from rafter i. The spider begins at rafter 1 and needs to travel to rafter n. Assume that every entry in array D is at most k. For full credit, your algorithm should run in time $O(n)$. Prove that your algorithm is correct, i.e. show that it always finds the lowest cost schedule. Justify the time complexity.

8. (10 points) Assume that a computer can hold 5 items in its cache. Suppose that the cache is empty initially and then the following items are requested in order:

   2, 3, 4, 1, 5, 6, 8, 2, 4, 5, 1, 6, 8, 2, 4, 1, 3, 6

   a. What is the minimum number of cache misses that any algorithm will have on the above schedule? Explain.
   b. What is the number of cache misses suffered by LRU (Least Recently Used)?

9. (10 points) Mister Genius thinks he has a solution to the prefix coding problem of k symbols into bits by examining the most frequent symbol first. Genius assigns 0 to the most frequent symbol, 10 to the next frequent symbol, .., $1^{k-1}0$ to the second least frequent symbol, and $1^{k-1}1$ to the least frequent symbol.

   a. Is this scheme a legal prefix coding?

   b. Is it optimal? If so, present a proof. If not, present a counterexample by showing that the expected cost of the encoding is higher than an optimal encoding.