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Neo smartpen

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1. Gauss Integral

Def 1 $s \in C^1(\mathbb{R})$ s.t. $|s(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$

$$Z(s) := \{s(x) = 0\}, \quad s'(x) \neq 0 \quad \forall x \in Z(s)$$

Gauss integral

$$Z = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2}s(x)^2}$$

Lemma $Z = \sum_{z \in Z(s)} \operatorname{sign} s'(z)$

Proof $\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2}s(x)^2} \quad \underline{y = \sqrt{t} s(x)}$

$$\begin{aligned} & \int_{\sqrt{t}s(-\infty)}^{\sqrt{t}s(+\infty)} \frac{dy}{\sqrt{2\pi t}} e^{-\frac{1}{2t}y^2} \\ &= \int_{\sqrt{t}s(-\infty)}^{\sqrt{t}s(+\infty)} \frac{dy}{\sqrt{2\pi t}} e^{-\frac{1}{2t}y^2} \end{aligned}$$

$$= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} s' e^{\frac{1}{2t}s(x)^2} = \int_{-\infty}^{+\infty} s'(x) \underbrace{\frac{e^{\frac{1}{2t}s(x)^2}}{\sqrt{2\pi t}}}_{\downarrow \delta} dx$$

$$= \sum_{z \in Z(s)} \frac{s'(z)}{|s'(z)|}$$

□

Def 3 Eucl. n -N general. for



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(2)

$$s \in C^1(\mathbb{R}^n)$$

$$Z = \int_{\mathbb{R}} \prod_{i=1}^n \frac{dx^i}{(2\pi)^{1/2}} \det \left(\frac{\partial s^j}{\partial x^k} \right) e^{-\frac{1}{2} s(x)^2}$$

$$= \sum_{z \in Z(s)} \text{sign} \det \left(\frac{\partial s^j}{\partial x^k} \right)$$

1. Z counts solution for $s^*(x) = 0$ with a product:

2. Z is an oriented intersection number

2. Superspace

- Superalgebra is a vect. space $A = A^0 \oplus A^1$ with a supercomm. multiplication, i.e.

$$\begin{matrix} f \in A^a \\ g \in A^b \end{matrix} \Rightarrow f \cdot g \in A^{a+b}, \quad fg = (-1)^{ab} gf$$

- Superspace $\mathbb{R}^{n|k}$ is defined by S.A. of coord functions

$$(C^\infty(\Omega)[\psi^1, \dots, \psi^k], +, \cdot) \text{ for open } \Omega \subset \mathbb{R}^n$$

$$\phi = \sum_{k=0}^n \sum_{i_1 < \dots < i_k} \phi_{i_1 \dots i_k} \psi^{i_1} \dots \psi^{i_k}$$

- For an n -dim mfd M the supermfd

$\hat{M}^{(k)}$ is given by the sheaf of SA $\mathcal{O}_{\hat{M}^{(k)}}$ on M . A2

For open $U \subset M$, $\Omega \subset \mathbb{R}^n$ with chart

$U \xrightarrow{\cong} \Omega$ we have $\mathcal{O}_{\hat{M}^{(k)}}(U) \cong C^\infty(\Omega)[\psi^1, \dots, \psi^k]$

Construction 5 For a rk k -VB $E \rightarrow M$ we have
in loc. trivialization $E|_U \cong U \times V$

$$C^\infty(U) \otimes \wedge^k V \cong \mathcal{O}_{\hat{M}^{(k)}}(U)$$

For $E = TM$ (T^*M) we ~~have~~ make an
identification $\psi^i \longleftrightarrow dx^i$ and

$$C^\infty(\hat{M}^{(k)}) \cong \Omega^k(M).$$

Def. 6 "Ghost number" of a superfield $\phi_\omega \in \hat{M}^{(k)}$

$$\deg \omega = gh \# \phi_\omega \in \mathbb{Z} \quad \Omega^k(M) \times (\Omega^k(M))^*$$

BRS-charge (coboundary op Q)

$$Q \phi_\omega \longleftrightarrow d\omega$$

in loc coord.

$$Q x^i = \psi^i, \quad Q \psi^i = 0.$$

Integration of superfields via $\text{ber}(x|\psi) \in \text{Ber}(\Omega^k)$

$$\int_M \omega = \int_{\hat{M}} \text{ber}(x|\psi) \phi_\omega \text{ with } \text{ber}(x|\psi) =$$

$$\text{ber}(x|\psi) = dx^1 \dots dx^n [d\psi^1 \dots d\psi^n]$$

Statement 7 We have $\int d\psi = 0$

and $\int [d\psi^1 \dots d\psi^n] \psi^1 \dots \psi^n = 1$.

3. SUSY Integral

Lemma 8 For $M \in \text{Mat}(n, \mathbb{R}) \exists$ a $\xi \in \{\pm 1\}$ s.t

$$\frac{\xi}{i^n} \int \left[\underbrace{d\psi^1}_{-1} \dots d\psi^n \underbrace{dx_1}_{1} \dots dx_n \right] e^{i \sum_{j,k} x_j M_{jk} \psi^k} = \det.$$

Proof

$$\frac{\xi}{i^n} \int [] \sum_{\ell=0}^{\infty} \left(\frac{i^\ell}{\ell!} \left(\sum_{j,k} x_j M_{jk} \psi^k \right)^\ell \right) =$$

$$= \frac{\xi}{i^n} \int [] \frac{i^n}{n!} \left(\sum_{j,k} x_j M_{jk} \psi^k \right)^n$$

$$= \xi \int [] \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n x_i \psi^{\sigma(i)} M_{i\sigma(i)}$$

$$= \pm \xi \int [] \frac{1}{n!} \sum_{\sigma \in S_n}$$

$$= \pm \xi \underbrace{\int [] x_n \dots x_1 \psi^1 \dots \psi^n}_1 \underbrace{\sum_{\sigma \in S_n} \text{sign } \sigma \prod_{i=1}^n M_{i\sigma(i)}}_{\det M}$$

Constr. 9

Recall $Z = \int_{\mathbb{R}^n} \frac{\prod dx^i}{(2\pi)^{n/2}} \det(\cdot) e^{-\dots}$

replace with the exp. in the Lemma

$$Z = \frac{1}{i} \int_{\hat{\mathbb{R}}^n} \int_{\prod_{a=1}^n (\mathbb{R}^n)^*} \frac{\text{ber}(x | \Psi, \gamma)}{(2\pi t)^{n/2}} e^{-\frac{1}{2t} S^j(x) S^j(x) + i \int_j \frac{\partial S^j}{\partial x^k}} dx$$

Want to make this Q-invariant.

$$gh \# \Psi = 1, \quad gh \# (\gamma_a) = -1$$

Need auxiliary field H_a with $gh \# (H_a) = e$

$$x_0 \in \mathbb{R}^n \quad \& (H_a) = H_a + i \frac{S^a(x_0)}{t} \quad \text{has one solution}$$

$$\int_{\mathbb{R}^n} \left(\frac{t}{2\pi}\right)^{n/2} \prod_{a=1}^n dH_a e^{-\frac{t}{2} \left(H_a + \frac{i S^a(x_0)}{t}\right)^2} = \pm 1$$

$$Z = \frac{1}{(2\pi i)^n} \int_{\hat{\mathbb{R}}^n \times (\hat{\mathbb{R}}^n)^*} \text{ber}(x, H | \Psi, \gamma) e^{-\frac{t}{2} H_j H_j - i H_j S^j + i \int_j \frac{\partial S^j}{\partial x^k}} dx$$

$$S = \frac{t}{2} H_j H_j + i H_j S^j - i \int_j \frac{\partial S^j}{\partial x^k} \Psi^k$$

(H_a, γ_a) antifield multiplet

Define $Q \gamma_a = H_a, \quad Q H_a = 0$

$$-S = Q(\Psi) \text{ for}$$

$$\Psi = -\frac{\hbar}{2} \gamma_a H_a - i \gamma_a S^a$$

Z is invariant under the change

$$S \rightarrow S - Q(\underbrace{\Delta\Psi}_{\text{variation of } \Psi}) \quad \left(\begin{array}{l} \text{provided the behaviour} \\ \text{at } \infty \text{ is not} \\ \text{changed} \end{array} \right)$$

" Z localizes to $Z(s)$ " $\Leftrightarrow \int$ is supported at $-S=0$

$$\Leftrightarrow \text{supp is at } Q(\Psi)=0 \Leftrightarrow$$

" Z localizes at Q -Fixedpoints".

4. SUSY correlation functions

Constr. 11 Define a superfield

$$\hat{Eul}_n : \hat{\mathbb{R}}^n \rightarrow \hat{\mathbb{R}}^n \quad (\text{rather } \hat{Eul}_n \in C^\infty(\hat{\mathbb{R}}^n)?)$$

$$\hat{Eul}_s := \int_{\hat{\mathbb{R}}^n} \frac{dH_1 \dots dH_n [d\gamma_1 \dots d\gamma_m]}{(2\pi i)^m} e^{Q(\Psi)}$$

with $S: \mathbb{R}^n \rightarrow V \cong \mathbb{R}^m, m \in \mathbb{N}$

$$\cdot gh \nmid \hat{Eul}_s = m$$

$$Q(\hat{Eul}_s) = 0 \Rightarrow \exists \hat{Eul}_s \in \mathcal{Q}^m(\mathbb{R}^n) \text{ s.t. } d(\hat{Eul}_s) = 0$$

Constr. 12

For $O_\omega: \widehat{\mathbb{R}^n} \rightarrow \widehat{\mathbb{R}^n} \times (\widehat{\mathbb{R}^n})^*$ with $QO_\omega = 0$

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \text{ber}(X|Y) O_\omega \widehat{\text{Eul}}_s$$

• If $Z \neq 0 \Leftarrow \text{gh}^\#(O_\omega) = n-m$

- $\langle O_\omega \rangle$ depends on the cohomology class only
- Localization identity

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \text{ber}(X|Y) O_\omega \widehat{\text{Eul}}_s = \int_{\mathbb{R}^n} \omega \wedge \text{Eul}_s$$

$$= \int_{Z(s)} i^* \omega$$

Talk 2 Evgenij Pascual

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \text{ber}(X, Y) O_\omega \widehat{\text{Eul}}_s$$

$$= \int_{\mathbb{R}^n} \omega \wedge \text{Eul}_s = \int_{Z(s)} i^* \omega$$

Thom Isomorphism

Poincaré lemma

$$H^*(M \times \mathbb{R}^n) = H^*(M)$$

$$H_c^*(M \times \mathbb{R}^n) = H_c^{*-n}(M)$$

For any VB $H_c^*(E) \cong H_c^{*-n}(M)$, $n = \dim E$

but in general ~~not~~ $H_c^*(E) \not\cong H_c^{*-n}(M)$

However, if M, E are orientable, then

$$H_c^*(E) \cong H_c^{*-n}(M)$$

Pf

$$H_c^*(E) \underset{\substack{\cong \\ \uparrow \\ \text{PD}}}{\cong} \left(H^{n+n-*}(E) \right)^* \cong \left(H^{n+n-*}(M) \right)^* \underset{\substack{\cong \\ \uparrow \\ \text{PD}}}{\cong} H_c^{*-n}(M). \quad \square$$

Compact support in vertical direction

$$\pi_*: \Omega_{cv}^*(E) \rightarrow \Omega^{*-n}(M) \quad \text{integration along fibers}$$

Projection formula

τ form on M , ω form on E , $\omega \in \Omega_{cv}(E)$

$$\text{Then } \pi_* (\pi^* \tau \wedge \omega) = \tau \wedge \pi_* \omega$$

Prop If E is oriented, then

$$\pi_{cv}^* : H_{cv}^*(E) \xrightarrow{\cong} H^{*-n}(M)$$

$\xleftarrow{\tau}$

Want to find π_*^{-1} (the Thom iso)

Consider $H^0(M) \ni 1$ has a well-defined image in $H_{cv}^n(E)$, which we call Φ , the Thom class of E .

$$\pi_* (\pi^* \omega \wedge \Phi) = \omega \wedge \pi_* \Phi = \omega.$$

$$\downarrow$$

$$\tau(\cdot) = \pi^*(\cdot) \wedge \Phi$$

Also important: Poincaré duality

Let S be a closed ^{oriented} submanifold of M , then its Poincaré dual γ_S is determined by

$$\int_S \omega = \int_M \omega \wedge \gamma_S \quad \forall \omega \in \Omega_c^*(M)$$

Prop γ_S is the same as Φ of the normal subbundle of S .

Euler class $E \xrightarrow{\pi} M$, $S_0 : M \rightarrow E$ zero section

$\Gamma(E) \ni S \wedge S_0 =: I$, Then

$$\dim I = \dim M - r, \quad r = \text{rk } E$$

and we set $e_s := PD(I)$

Now SUSY

For $\pi: E \rightarrow M$, E orientable, have the theorem

$$H^i(M) \cong H_{cv}^{i+m}(E)$$

Also, $s^* \Phi(E) = e_s$

$$\int_M \omega \wedge s^* \Phi(E) = \int_{Z(s)} L^* \omega$$

So need to replace $\frac{\partial s}{\partial x_j}$ by $\nabla_j s$

Let $\{e^a\}$ be an ON basis of E , then define ∇ by $\nabla e_a^i = dx^j \theta_j^{ab} e_b^i$

$$\text{Then } S = \frac{1}{2} H_j H_j + i H_j s^j - i \gamma_j \frac{\partial s^j}{\partial x_k} \psi^k$$

~~covariantize~~ $S = \frac{1}{2}$

$$S = -\frac{1}{2h} s^i s^j + i \gamma_j \frac{\partial s^j}{\partial x_k} \psi^k$$

$$\text{covariantize } S(x, \nabla) = -\frac{1}{2h} s^a s^a + i \gamma_a (\nabla_j s)^a \psi^j + \frac{1}{4} \gamma_a \gamma_b F_{ij}^{ab} \psi^i \psi^j$$

This form is obtained from the requirement

$S = Q(\Psi)$, where Ψ is covariantized

Define a superfield on \hat{M}

$$\hat{Eul}_s(E, \nabla) := \int \prod_{a=1}^n \frac{dx_a d\theta_a}{2\pi i} e^S$$

Have lin. op. $\nabla S : T_p M \rightarrow E_p$

Important that $Z(s)$ is a mfd.

This is the case if ∇ is ^{not} surjective.

How to get a localization formula for \hat{E} ?

Consider ex. seq. = Coker ∇S

$$0 \rightarrow \text{Im } \nabla S \rightarrow E \rightarrow \text{Cok } \nabla S \rightarrow 0$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad E / \text{Im } \nabla S$$

Then \cong

$$\int_{\hat{E}} \text{ber}(x, H | \psi, x) e^S 0 = \int_{\hat{M}} \text{ber}(x | \psi) \hat{Eul}_s(E, \nabla)$$

$$= \int_{Z(s)} i^* \omega_0 \wedge \text{Eul}(\text{Coker } \nabla S)$$

here we require that $\text{Coker}(\nabla S)$ is a bundle

Need to replace $H_{\text{ev}}^*(E) \rightarrow H_{\text{rd}}^*(E)$

rapidly decaying

Talk 3 Vector Pidsstrygach

Equivariant cohomology

Algebra A $\text{Der } A = \{ \delta: A \rightarrow A \text{ linear} \mid \delta(f \cdot g) = \delta(f) \cdot g + f \delta(g) \}$

For an A -module F

$\text{Der}(A, F) = \{ \delta: A \rightarrow F \mid \delta \text{ linear} \mid \delta(fg) = \delta(f) \cdot g + f \delta(g) \}$

Ex $A = C^\infty(\mathbb{R}^n)$ Exercise $[\delta_1, \delta_2] \in \text{Der}(A, F)$

$\text{Der}(A) = \{ \sum a_i \frac{\partial}{\partial x_i} \mid a_i \in C^\infty(\mathbb{R}^n) \}$

$\mathbb{R}^n \ni x \rightsquigarrow \mathfrak{m}_x = \{ f \in C^\infty(\mathbb{R}^n) \mid f(x) = 0 \}$

$C^\infty(\mathbb{R}^n) / \mathfrak{m}_x \cong \mathbb{R} \text{ field}$

$T_x^* \mathbb{R}^n = \mathfrak{m}_x / \mathfrak{m}_x^2, \quad T_x \mathbb{R}^n = (\mathfrak{m}_x / \mathfrak{m}_x^2)^\vee$
 $= \text{Der}(C^\infty(A), C^\infty(\mathbb{R}^n) / \mathfrak{m}_x)$

$C^\infty(\pi^* TM) = \Omega^*(M)$ graded algebra, superalgebra

$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$

d_{de} , X vector field

\mathcal{L}_X Lie derivative

i_X interior derivative

$$\deg d_{12} = 1$$

$$\deg \alpha_X = 0$$

$$\deg \iota_X = -1.$$

$$[\alpha_X, \alpha_Y] = \alpha_{[X, Y]} = \alpha_X \alpha_Y - \alpha_Y \alpha_X$$

$$[\alpha_X, \iota_Y] = \iota_{[X, Y]}$$

$$[\iota_X, \iota_Y] = \iota_X \iota_Y + \iota_Y \iota_X = 0$$

$$[d, \alpha_X] = 0$$

$$[d, d] = d \circ d + d \circ d = 0$$

$$\alpha_X = [d, \iota_X] = d \iota_X + \iota_X d$$

Lie gp G acts on M

$\mathfrak{g} = \text{Lie}(G) \ni \xi \longmapsto K^\xi$ fundamental vector field

$$K^\xi(p) = \left. \frac{d}{dt} \right|_{t=0} (e^{-t\xi} \cdot p)$$

$$\iota_{K^\xi}, \alpha_{K^\xi}, d$$

$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ supersymmetry of geometric symmetry

$$C(\text{spt}) := \mathbb{R}[\theta]/\theta^2 = \Lambda^1 \mathbb{R}^1$$

$$\Pi TM = \text{Map}(\text{spt}, M) = \text{Map}(C^\infty(M), \mathbb{R}[\theta]/\theta^2).$$

Ex $M = \mathfrak{g}$, G acts by the adj. action
 \Rightarrow fundamental vector fields

symmetries \mathcal{L}_{K^ξ} , \mathcal{I}_{K^ξ} , d_{dR} of $\Pi T\mathfrak{g}$

$$C^\infty(\Pi T^*\mathfrak{g}) = \Omega^*(\mathfrak{g}) \ni \left\{ \begin{array}{l} \text{diff. forms with} \\ \text{polynomial coeffic.} \end{array} \right\}$$

is generated by
 $l \otimes 1$, $1 \otimes l$

$\longleftarrow S^k \mathfrak{g}^\vee \otimes \wedge^l \mathfrak{g}^\vee$
 $l \in \mathfrak{g}^\vee$
 degree ≤ 1 diff. forms
 with constant coefficients

$$d_{dR}(l \otimes 1) = 1 \otimes l, \quad d_{dR}(1 \otimes l) = 0$$

$$K_\eta^\xi = [\xi, \eta] = \text{ad}_\xi(\eta)$$

$$\cancel{\mathcal{L}_{K^\xi}(1 \otimes l)(\eta) =}$$

$$\mathcal{L}_{K^\xi}(l \otimes 1) = l \circ \text{ad}_\xi \otimes 1 \neq$$

$$\mathcal{L}_{K^\xi}(1 \otimes l) = 1 \otimes l \circ \text{ad}_\xi$$

We also define another derivation d_K

$$d_K(1 \otimes l) = l \otimes 1, \quad d_K(l \otimes 1) = 0$$

$$[d_K, d_{dR}]|_{S^k \mathfrak{g}^\vee \otimes \wedge^l \mathfrak{g}^\vee} = (k+l) 1$$

$$k+l=0 \Leftrightarrow S^0 \mathfrak{g}^\vee \otimes \wedge^0 \mathfrak{g}^\vee = \mathbb{R}$$

$$H^i(\mathcal{K}(\mathfrak{g}), d_K) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i>0. \end{cases}$$

$$\begin{aligned} \hat{\tau}_{K^3}(1 \otimes 1) &= \hat{\tau}_{K^3} d_K(1 \otimes 1) = d_{K^3}(1 \otimes 1) - d_K \hat{\tau}_{K^3}(1 \otimes 1) \\ &= 1 \otimes 1 \circ d_{K^3} \quad (**) \end{aligned}$$

This defines a nonstandard \mathfrak{g} supersymmetry.

G acts freely on a manifold P s.t. $P \rightarrow P/G = M$ is a principal bundle

Connection 1-form $a \in \Omega^1(P; \mathfrak{g})^G \quad R_g^* a = \text{Ad}_{g^{-1}} a$

$$\Omega^*(P; V)_{\text{hor}} = \{ \alpha \in \Omega(P, V) \mid \iota_{K^\xi} \alpha = 0 \quad \forall \xi \in \mathfrak{g} \}$$

$$\Omega(P; V)^G = \{ \alpha \in \Omega(P, V) \mid R_g^* \alpha = \rho_g(\alpha) \}$$

Here V is any rep. of G

$$\Omega(P, V)_{\text{bas}} = \Omega(P, V)_{\text{hor}}^G \stackrel{\text{claim}}{=} \Omega^1(M, E)$$

$$E = P \times_G V$$

- Topological idea of equivar. cohomology,
1. find a contr. top. space EG with a free action of G
 2. G -action on $M \times EG$ is free, so compute
- $$H^*(M \times EG/G) = H_G^*(M)$$

~~Def~~ Let A be a superalgebra

$$\hat{y} := y_{-1} \oplus y_0 \oplus \langle d \rangle$$

$$\deg \quad \quad -1 \quad \quad 0 \quad \quad +1$$

Def Let A be a superalgebra with $\hat{y} \hookrightarrow \text{Der}(A)$

We say the pair has property C if

$$\exists \text{ elements } a_1, \dots, a_n \text{ s.t. } \hat{z}_{\hat{b}_i} a_j = \delta_{ij} \quad (*)$$

deg 0

Ex K_y - Koszul, $(**)$ implies the pair has property C

Def $H_G(A) = H(A \otimes W(y)_{\text{bas}}, \hat{d} + d_K)$

basic means $(\cap_{\deg} \text{Ker } z_{\frac{1}{2}}) \cap (\cap_{\deg} \text{Ker } d_{\frac{1}{2}})$

Weil algebra model

Claim $H(A \otimes W(y)_{\text{bas}}, \hat{d} + d_K) = H(A \otimes S^*(y^\vee), \hat{d} + z_y)$

Mathai-Quillen isomorphism

$A = \Omega^*(M)$ with a G -action and standard

$$L_K, \alpha_K, d_R$$

$$S^*(y^\vee) \otimes \Omega^*(M) = \text{Hom}(S_y, \Omega^*(M))$$

$$\downarrow \psi$$

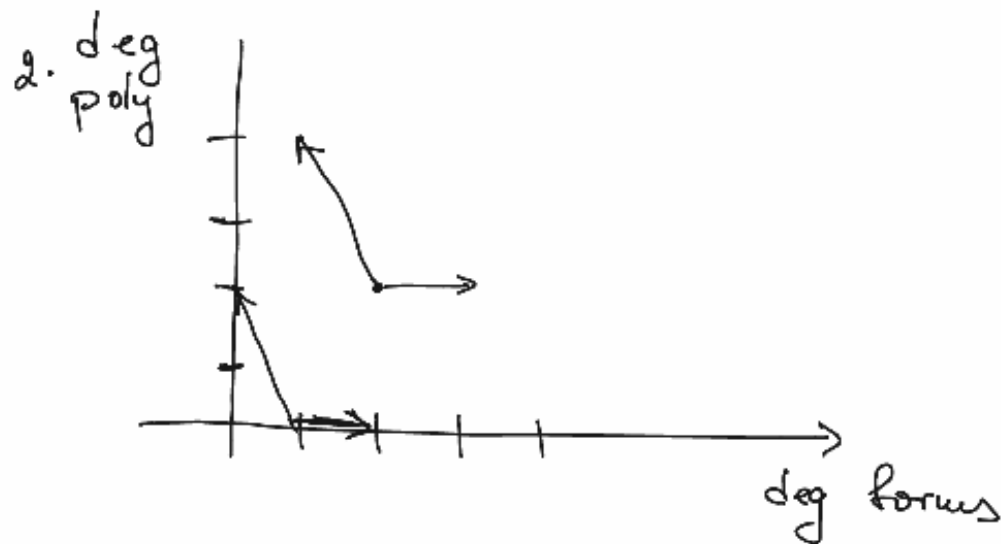
$$\alpha \in S^d(y^\vee) \otimes \Omega^k(M)$$

~~$(\hat{A})[\hat{s}] := d(\hat{\alpha}[\hat{s}])$~~

$$((d - 2\eta) \alpha) [\xi] = d(\alpha[\xi]) - 2\eta_{\xi}(\alpha[\xi])$$

$\underbrace{\quad}_{d\eta} \quad \underbrace{\quad}_d \quad \underbrace{\quad}_k$

form degree $d+1$ form degree $d-1$
 poly degree k poly degree $k+1$



$$\deg \alpha = \deg \text{form} + 2 \text{poly degree}$$

$$(d_{dR} - 2\eta_{\xi})^2 = \underbrace{d_{dR}^2}_0 - [d, 2\eta_{\xi}] + \underbrace{(2\eta_{\xi})^2}_0 = -\alpha_{\xi}$$

$$\alpha \text{ is invariant} \Rightarrow \alpha_{\xi} = 0 \Rightarrow (d_{dR} - 2\eta_{\xi})^2 = 0$$

Interpretation $\alpha \in S^*(\eta^{\vee}) \otimes \Omega(M)$

M closed $\int_M \alpha \in S^*(\eta^{\vee})$

$$\alpha = d_{\eta} \beta \Rightarrow \int_M \alpha = 0$$

$\alpha_{[d]}$ d -form component of α

$$d_{\eta} \alpha = 0 \Leftrightarrow \alpha_{[n]} = d \beta_{[n-1]} - i_{\xi}(\beta_{[n+1]})$$

$\underbrace{\quad}_0$

□ G acts on M , $d \in S^*(Y^v) \otimes \Omega^*(U)$

$$\sigma_y dy d = 0$$

$M_0 = \text{zeros of } K^\xi$.

Lemma $d(\xi)$ is exact on $M \setminus M_0$.

Proof Define 1-form $\Theta(X) = g(K^\xi, X)$

$$d_{K^\xi} \Theta = \underbrace{|K^\xi|^2}_{\deg 0} + \underbrace{d\Theta}_{\deg 2}$$

On $M \setminus M_0$ we have $d(\xi) = d_{K^\xi} \left(\frac{\Theta \wedge d(\xi)}{d_{K^\xi} \Theta(\xi)} \right)$

$$\text{since } (|K^\xi|^2 + d\Theta)^{-1} = \frac{1}{|K^\xi|^2} (1 - |K^\xi|^2 d\Theta + \dots)$$

$$\Rightarrow d(\xi)_{[n]} = d(\quad)_{[n-1]}$$

□

Then Assume K^ξ has isolated zeros

$Z(K^\xi) = \{p_0, \dots, p_n\}$. Then

$$\int_M d(\xi) = (-2\pi)^P \sum \frac{d(\xi)(p)}{\sqrt{\det(L_p)}}$$

where L_p - linearisation of K^ξ at p .

Pöschel's Equiv. Cohomology II 11.05.18 Göttingen

$$\underbrace{\wedge^* y^\vee \otimes S^*(y^\vee) \otimes y}_{W(y)} \supset y^\vee \otimes 1 \otimes y \ni \text{id} = a^{\text{st}} \text{ connected} \\
 \supset \wedge^2 y^\vee \otimes 1 \otimes y \ni [\cdot, \cdot] = F_a^{\text{st}} \text{ curved}$$

Mathai-Quillen iso:

$$\exp(\iota_{K^{\text{st}}}^M) : \Omega(M) \otimes W(y)_{\text{hor}} \rightarrow \Omega(M) \otimes W(y)_{\text{hor}} \\
 \text{and } d_{\text{dR}} + d^{\text{st}} \text{ to } d_{\text{dR}} - \iota_{K^{\text{st}}}^M + d_{K^{\text{st}}} \\
 \text{On } (\Omega(M) \otimes W(y))_{\text{hor}}^G \xrightarrow{\sim} (\Omega(M) \otimes W(y)_{\text{iso}})^G \\
 d_{\text{dR}} + d^{\text{st}} \rightsquigarrow d_{\text{dR}} - \iota_{K^{\text{st}}}^M := d_G$$

Want to have a formula

$$\int_Y \omega = \int_P \overset{=T^*\omega}{\tilde{\omega}} \cdot P(P \rightarrow Y)$$

$$\begin{array}{c} G \rightarrow P \\ \downarrow \\ Y \end{array}$$

$$\alpha \in (\Omega^*(M) \otimes S^*(y^\vee))^G$$

$$\int_P \alpha \in S^*(y^\vee)$$

not a number
so can't be
compared to
 $\int_Y \omega$ easily

$$\oint_{\varepsilon, P} \alpha := \frac{1}{\text{vol } G} \int_G \int_P \alpha$$

$$\int_G \int_P \alpha$$

Extend this integral to an integral over $P \times Y \times W(Y)^\vee$

$$P(P \rightarrow Y) = \int_{W(Y)^\vee} e^{Q(\Psi_{\text{proj}})} d\xi dy \quad (\text{Witten's answer})$$

$$\Psi_{\text{proj}} = i\langle \xi, \alpha \rangle + \langle \xi, [\xi, \eta] \rangle$$

$$W(Y) = \Lambda^1 Y^\vee \otimes S^1 Y^\vee \quad W(Y)^\vee = S^1 Y^\vee \otimes \Lambda^1 Y^\vee$$

-2 -1

$$Q(l \otimes 1) = 1 \otimes l \quad \equiv \quad Q\xi = \eta$$

$$Q(1 \otimes l) = \mathcal{L}^{\text{HK}}(l \otimes 1) \equiv Q\eta = -\mathcal{L}_{K\xi}^{\text{HK}} \xi$$

$$\int_Y \omega = \int_P \pi^* \omega \, P(P \rightarrow Y) = \int_{\substack{Y \times W(Y)^\vee \times P \\ \xi \in \xi, \eta}} \pi^* \omega \cdot e^{Q(\Psi_{\text{proj}}) - \frac{\varepsilon}{2} \langle \xi, \eta \rangle}$$

Emanuel Scheidegger Topological twist

4 - world, 6 - bundle w/conn

$$YM(A) = \int_M \text{tr}(F_A \wedge * F_A) \quad F = dA + [A, A]$$

In physics: YM theory

add SUSY \rightarrow Super-Yang-Mills theory

in various ways

$N=1 \rightarrow$ Kähler

$N=2 \rightarrow$ Donaldson invariants, SW inv

$N=4 \rightarrow$ Kapustin-Witten, geom. Langlands

To make mathematically well-defined:

top. twisted Super-Yang-Mills theory
is an example of $\begin{cases} \text{top. field theory} \\ \text{coh. field theory} \end{cases}$ of Witten's type (phys (math))

Basic Ingredients

- G cnp Lie gp, $\mathfrak{g} = \text{Lie}(G)$
gauge gp in physics
- (X, g) closed oriented 4-manifold
- $P \rightarrow X$ principal G -bundle

$$M \rightsquigarrow A := \text{Conn}(P)$$

$$E \rightarrow M \rightsquigarrow E := A \times \Omega^{2,+}(X, \text{ad } P)$$

$$\begin{array}{ccc} \uparrow & & \searrow \\ S & & S(A) := F_A^+ \end{array}$$

$$G \rightsquigarrow \mathcal{G} = \text{Aut}(P) \text{ (gauge gp in math)}$$

X

ψ

χ

H

ϕ

$\bar{\phi}$

γ

\rightsquigarrow

?

$N=2$ QFT in 4D

fields = (distributional)
representations of the $N=2$
Super-Poincaré alg.
(on vector bundles over X)

A non-ass super-aly (\mathbb{Z}_2 -graded)
over a comm. rly R is a
Lie superalgebra, if $\exists [\cdot, \cdot]: A \times A \rightarrow A$

$$\text{sd. } [X, Y] + (-1)^{|X||Y|} [Y, X] = 0$$

$$(-1)^{|Z||Y|} [Z, [X, Y]] + (-1)^{|X||Z|} [X, [Y, Z]] + (-1)^{|Y||X|} [Y, [Z, X]] = 0$$

$L = L_0 \oplus L_1 \Rightarrow L_0$ is a Lie alg.

If \exists at least one odd element, then

L_1 is an L_0 -module for $\text{ad } X \mapsto [X, X]$

If \exists at least two odd elements, then

$\exists L_0$ -equivariant ^{symm} map

$$\{ \cdot, \cdot \}: L_1 \otimes L_1 \rightarrow L_0 \text{ s.t.}$$

$$[\{X, Y\}, Z] + [\{Y, Z\}, X] + [\{Z, X\}, Y] = 0$$

$$\forall X, Y, Z \in L_1.$$

V real quadr. v. space (of dim 4)

$L'_0 = V \rtimes \underline{\text{so}}(V)$ Poincaré alg.

$L_0 := L'_0 \oplus \mathbb{R}$, \mathbb{R} any reductive Lie alg. (R-symm
(reductive: adj. repr. is completely reducible \Leftrightarrow
 $= s + a$, s semisimple & a abelian))

$\mathbb{R} = \underline{u}(1)$ $N=1$ extended super Poincaré algebra


$\mathbb{R} = \underline{su}(2) \oplus \underline{u}(1)$ $N=2$ — // — // — //

S real spinorial rep. of $\text{Spin}(V)$

\exists symm. nonzero map $\Gamma: S \times S \rightarrow V$, which is
equiv. wrt $\text{Spin}(V)$

S is an L_0 -module by requiring that

V acts as 0 on S , and is a rep. of \mathbb{R} .

$L_1 := \mathfrak{S}$ with $[S_1, S_2] := \Gamma(S_1, S_2)$  Neo smartpen

Special case $\dim V = 4$ (didn't use this above)

$$L_0 = \mathbb{R}^4 \rtimes (\mathfrak{su}(2)_- \oplus \mathfrak{su}(2)_+) \oplus \mathfrak{su}(2)_{\mathbb{R}} \oplus \mathfrak{u}(1)_{\mathbb{R}}$$

$$L_1 = (1, 2, 2)^1 \oplus (2, 1, 2)^{-1}$$

$$(1, 2, 2) = \underbrace{V_-}_{\substack{\nearrow \\ \mathfrak{su}(2)_- \\ \dim=1}} \otimes \underbrace{V_+}_{\substack{\uparrow \\ \mathfrak{su}(2)_+ \\ \dim=2}} \otimes \underbrace{V_{\mathbb{R}}}_{\nwarrow \mathfrak{su}(2)_{\mathbb{R}} \\ \dim=2}$$

$()^1$ weight of the $\mathfrak{u}(1)$ repres.

There is an L_0 equiv. map $\{ \cdot, \cdot \} : \text{Sym}^2(L_1) \rightarrow \mathbb{R}^4$

Physicists notations

Basis elements in $(1, 2, 2)^4; \ni \bar{Q}_{\dot{\alpha}}^A \quad \dot{\alpha} = 1, 2; A = 1, 2$

$(2, 1, 2)^4 \ni Q_{\alpha}^A \quad \alpha = 1, 2; A = 1, 2$

$$\{Q_{\alpha}^A, \bar{Q}_{\dot{\alpha}}^B\} = 2 \varepsilon_{\alpha\beta}^{\mu} \delta_{\dot{\alpha}\dot{\beta}}^{\nu} P_{\mu} \quad , \quad \mu = 1, 2, 3, 4$$

totally antisymm.

$$\varepsilon_{A_1 \dots A_n}^{\text{tensor}} = \begin{cases} +1, & \text{if } (A_1, \dots, A_n) = (1, \dots, N) \\ -1, & \text{if } (A_1, \dots, A_n) = (2, 1, \dots, 1) \\ 0, & \text{if } A_i = A_j \text{ for some } i \neq j. \end{cases}$$

$\sigma_{\alpha\beta}^{\mu}$ is the repr. matrix of the map Γ
= Pauli matrices.

Q's are called supercharges

P's are called momenta

Also, $\{Q_\alpha^A, Q_\beta^B\} = 0$.

$\Lambda = 2$ vector multiplet

Reps of Lie superalgebras are determined by
corr. reps of their even parts. Consider the following rep. of

Basis elements	$\underline{SU(2)}_- \oplus \underline{SU(2)}_+ \oplus \underline{SU(2)}_R$	$U(1)_R$	
A_μ	$(2, 2, 1)$	0	$\mu = 1, 2, 3, 4$
$\bar{\Psi}_\alpha^A$	$\oplus (1, 2, 2)$	-1	$A = 1, 2$
Ψ_α^A	$\oplus (2, 1, 2)$	+1	
Φ	$\oplus (1, 1, 1)$	2	
$\bar{\Phi}$	$\oplus (1, 1, 1)$	-2	
D^a	$\oplus (1, 1, 3)$	0	$a = 1, 2, 3$

D auxiliary field; appears only algebraically in the EL eqns; can be eliminated
ghost numbers (later)

Globalize $\mathbb{R}^4 \rightsquigarrow X^4$

vector spaces \rightsquigarrow vector bundles

We must introduce a new datum, the R-symmetry bundle $P_R \rightarrow X$ a principal $SU(2)_R$ -bundle.

$\Rightarrow A$ is a con-u.

$\phi, \bar{\phi}$ sections of $\text{ad } P \otimes \mathcal{O}$

$\mathcal{D} \quad \text{---} // \text{---} \quad \text{ad } P \otimes W \leftarrow \text{real } k=3 \text{ v.b. ass. to } P_R$

$$\psi, \bar{\psi} \in \Gamma \left(\underset{\substack{\uparrow \\ \text{spinor} \\ \text{bundles} \\ \text{of } X}}{S^{\pm}} \otimes \underset{\substack{\uparrow \\ \text{spinor} \\ \text{rep} \\ \text{of } \text{SU}(2)_R}}{S_R} \otimes \text{ad } P \right)$$

X does not need to be Spin!

In this case: $P_R \rightarrow X$ is an $\text{SO}(3)$ principal bundle s.t. $w_2(P_R) = w_2(X)$

Then $S^{\pm} \otimes S_R$ exist but S^{\pm} and S_R don't.

Moore: Choose $P_R \cong P_+ \xrightarrow{\text{s.t.}} P_+ \times_{\text{SO}(2)_+} \mathbb{R}^3 \cong \Lambda_+^2 T^*X$.

Action

Levi-Civita $\omega \in \Omega^1(P_{\text{SO}(4)}, \underset{\substack{\text{SO}(4) \\ \text{SO}(2)_+ \oplus \text{SU}(2)_-}}{\text{so}(4)})$

$$\omega = \omega_+ + \omega_-$$

Path Integral

$$Z(\omega_+, \omega_-, \phi_R) := \int [\mathcal{D}A \mathcal{D}\psi \dots \mathcal{D}\mathcal{D}] e^{-S_{\text{phys}}^{\text{YM}}} ()$$

$$S_{\text{phys}}^{\text{YM}} := - \int \frac{1}{g_0^2} \text{tr} \left(F \wedge * F \right) + \mathcal{D}\phi \wedge * \mathcal{D}\phi^* - \frac{1}{4} [\phi, \phi^*]^2 \omega$$

boronic part depends on g $\left(\begin{array}{l} + \frac{\theta_0}{8\pi^2} \int_X \text{tr} (F \wedge F) + \int_X \text{tr} (\bar{\psi} \not{D} \psi) + \bar{\psi} \psi \end{array} \right)$

topological term

$$+ \int_{\mathcal{M}} \frac{1}{2} \left(\bar{\psi} \not{D} \psi + \bar{\psi} \psi \bar{\psi} \psi + \bar{\phi} [\psi, \psi] + \right. \\ \left. + \phi [\psi, \bar{\psi}] \right)$$

The field \mathcal{D} does not appear here, since it is integrated out

$$\text{tr}(XY) = -\frac{1}{2h^\vee} \text{tr}_g(\text{ad}(X)\text{ad}(Y))$$