04.05.18

Andreas Ikkert

1. Gauss Integral

Def 1 $s \in C^1(\mathbb{R})$ s.t. $|s(x)| \to \infty$ as $|x| \to \infty$

$Z(s) := \{ s(x) = 0 \}$, $s'(x) \neq 0 \quad \forall x \in Z(s)$

Gauss integral

$$Z = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2} s(x)^2}$$

Lemma $$Z = \sum_{z \in Z(s)} \operatorname{sign} s'(z)$$

Proof
$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2} s(x)^2} \quad \underline{\underline{y = \sqrt{\hbar}\, s(x)}}$$

$$= \int_{\sqrt{\hbar} s(-\infty)}^{\sqrt{\hbar} s(+\infty)} \frac{dy}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar} y^2}$$

$$= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi\hbar}} s' e^{\frac{1}{2\hbar} s(x)^2} = \int_{-\infty}^{+\infty} s'(x) \underbrace{\frac{e^{\frac{1}{2\hbar} s(x)^2}}{\sqrt{2\pi\hbar}}}_{\downarrow\ \delta} dx$$

$$= \sum_{z \in Z(s)} \frac{s'(z)}{|s(z)|}$$

$\boxtimes$

Def 3 Eucl. n-N general. for

$s \in C^1(\mathbb{R}^n)$

$$Z = \int_{\mathbb{R}} \prod_{i=1}^{n} \frac{dx^i}{(2\pi)^{n/2}} \det\left(\frac{\partial s^j}{\partial x^k}\right) e^{-\frac{1}{2} s(x)^2}$$

$$= \sum_{z \in Z(s)} \operatorname{sign} \det\left(\frac{\partial s^j}{\partial x^k}\right)$$

1. Z Counts solution for $s^i(x) = 0$ with a prefact:
2. Z is an oriented intersection number

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2. Superspace

- Superalgebra is a vect. space $A = A^0 \oplus A^1$ with a supercomm. multiplication, i.e.

$$\begin{matrix} f \in A^a \\ g \in A^b \end{matrix} \Longrightarrow f \cdot g \in A^{a+b}, \quad fg = (-1)^{a+b} gf$$

- Superspace $\mathbb{R}^{n|k}$ is defined by S.A. of coord functions

$$(C^\infty(\Omega)[\psi^1, \dots, \psi^k], +, \cdot) \text{ for open } \Omega \subset \mathbb{R}^n$$

$$\Phi = \sum_{k=0}^{n} \sum_{i_1 < \dots < i_k} \phi_{i_1 \dots i_k} \psi^{i_1} \dots \psi^{i_k}$$

- For an n-dim mfld M the supermfld $\hat{M}^{(k)}$ is given by the sheaf of SA $\mathcal{O}_{\hat{M}^{(k)}}$ on M.

For open $U \subset M$, $\Omega \subset \mathbb{R}^n$ with chart

$U \xrightarrow{\cong} \Omega$ we have $\mathcal{O}_{\hat{M}(k)}(U) \simeq C^\infty(\Omega)[\psi^1, \dots, \psi^k]$

<u>Construction 5</u> For a rk k-VB $E \to M$ we have in loc. trivialization $E|_U \cong U \times V$

$C^\infty(U) \otimes \Lambda^* V \cong \mathcal{O}_{\hat{M}(k)}(U)$

For $E = TM$ (T^*M) we ~~have~~ make an identification $\psi^i \longleftrightarrow dx^i$ and

$$C^\infty(\hat{M}^{(k)}) \cong \Omega^*(M).$$

<u>Def. 6</u> "Ghost number" of a superfield $\mathcal{O}_\omega \in \hat{M}^{(k)}$

$$\deg \omega = gh\# \, \mathcal{O}_\omega \in \mathbb{Z} \qquad \Omega^*(M) \times (\Omega^*(M))^*$$

BRS-charge (coboundary op Q)

$$Q \mathcal{O}_\omega \longleftrightarrow d\omega$$

in loc coord.

$$Q x^i = \psi^i, \quad Q \psi^i = 0.$$

Integration of superfields via $\mathrm{ber}(x|\psi) \in \mathrm{Ber}(\Omega^*(M))$

$$\int_M \omega = \int_{\hat{M}} \mathrm{ber}(x|\psi)\, \mathcal{O}_\omega \quad \text{with} \quad \cancel{\mathrm{ber}(x|\psi)} =$$

$$\mathrm{ber}(x|\psi) = dx^1 \dots dx^n \, [d\psi^1 \dots d\psi^n]$$

Statement 7 We have $\int d\psi = 0$

and $\int [d\psi^1 \dots d\psi^n]\, \psi^1 \dots \psi^n = 1$.

3. SUSY Integral

Lemma 8 For $M \in Mat(n, \mathbb{R})$ $\exists$ a $\xi \in \{\pm 1\}$ s.t.

$$\frac{\xi}{i^n} \int [\overbrace{\underbrace{d\psi^1}_{-1}}^{1} \dots d\psi^n \overbrace{\underbrace{d\chi_1}_{1}}^{-1} \dots d\chi]\, e^{i \sum_{jk} \chi_j M_{jk} \psi^k} = \det.$$

Proof

$$\frac{\xi}{i^n} \int [\;] \sum_{\ell=0}^{\infty} \left(\frac{i^\ell}{\ell!} \left(\sum_{j,k}^{k} \chi_j M_{jk} \psi^k \right)^\ell \right) =$$

$$= \frac{\xi}{i^n} \int [\;] \frac{i^n}{n!} \left(\sum_{j,k}^{n} \chi_j M_{jk} \psi^k \right)^n$$

$$= \xi \int [\;] \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \chi_i \psi^{\sigma(i)} M_{i\sigma(i)}$$

$$\cancel{= \pm \xi \int [\;] \frac{1}{n!} \sum_{\sigma \in S_n}}$$

$$= \pm \xi \underbrace{\int [\;] \chi_n \cdots \chi_1 \psi^1 \dots \psi^n}_{1} \underbrace{\sum_{\sigma \in S_n} \operatorname{sign} \sigma \prod_{i=1}^{n} M_{i\sigma(i)}}_{\det M}$$

Constr. 9

Recall $\quad Z = \int_{\mathbb{R}^n} \frac{\prod dx^i}{(2\pi)^{n/2}} \underbrace{\det(\,)}_{} e^{\cdots}$

replace with the exp. in the Lemma

$$Z = \frac{\xi}{i} \int_{\hat{\mathbb{R}}^n} \int_{\cancel{(\hat{\mathbb{R}})^*}\ \Pi(\mathbb{R}^n)^*} \frac{\mathrm{ber}(x \mid \psi, \chi)}{(2\pi\hbar)^{n/2}} e^{-\frac{1}{2\hbar} S^j(x) S^j(x) + i \chi_j \frac{\partial S^j}{\partial x_i}}$$

Want to make this Q-invariant.

$gh\#\, \psi^i = 1 \quad , \quad gh\#(\chi_a) = -1$

Need auxiliary field H_a with $gh\#(H_a) = 0$

$x_0 \in \mathbb{R}^n \qquad f(H_a) = H_a + i \frac{S^a(x_0)}{\hbar}$ has one solution

$$\int_{\mathbb{R}^n} \left(\frac{\hbar}{2\pi}\right)^{n/2} \prod_{a=1}^{n} dH_a \, e^{-\frac{\hbar}{2}\left(H_a + \frac{i S^a(x_0)}{\hbar}\right)^2} = \pm 1$$

$$Z = \frac{\xi}{(2\pi i)^n} \int_{\hat{\mathbb{R}}^n \times (\hat{\mathbb{R}}^n)^*} \mathrm{ber}(x, H \mid \psi, \chi) e^{-\frac{\hbar}{2} H_j H_j - i H_j S^j + i \chi_j \frac{\partial S_j}{\partial x^k}}$$

$$S = \frac{\hbar}{2} H_j H_j + i H_j S^j - i \chi_j \frac{\partial S^j}{\partial x_k} \psi^k$$

(H_a, χ_a) antighost multiplet

Define $\; Q\chi_a = H_a \, , \quad Q H_a = 0$

$-S = Q(\Psi)$ for

$$\Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a$$

Z is invariant under the change

$$S \longrightarrow S - Q(\underbrace{\Delta\Psi}_{\text{variation of }\Psi})$$

(provided the behaviour at ∞ is not changed)

"Z localizes to $Z(s)$" $\iff$ ρ is supported at $-S=0$

$\iff$ supp is at $Q(\Psi)=0$ $\iff$

"Z localizes at Q-Fixedpoints".

4. SUSY correlation functions

Constr. 11 Define a superfield

$$\widehat{Eul}_n : \widehat{\mathbb{R}}^n \longrightarrow \widehat{\mathbb{R}}^n \quad (\text{rather } \widehat{Eul}_n \in C^\infty(\widehat{\mathbb{R}}^n)?)$$

$$\widehat{Eul}_s := \int_{\widehat{\mathbb{R}}^n} \frac{dH_1 \dots dH_n\,[d\chi_1 \dots d\chi_m]}{(2\pi i)^m}\, e^{Q(\Psi)}$$

with $s : \mathbb{R}^n \longrightarrow V \cong \mathbb{R}^m$, $m \in \mathbb{N}$

- $gh\# \ \widehat{Eul}_s = m$

$$Q(\widehat{Eul}_s) = 0 \implies \exists\ Eul_s \in \Omega^m(\mathbb{R}^n)$$

s.t. $d(Eul_s) = 0$

Constr. 12

For $\mathcal{O}_\omega : \widehat{\mathbb{R}^n} \longrightarrow \underset{\widehat{\mathbb{R}^n} \times \widehat{(\mathbb{R}^n)^*}}{\not{E}}$ with $Q\mathcal{O}_\omega = 0$

$$\langle \mathcal{O}_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(X|\Psi)\, \mathcal{O}_\omega\, \widehat{\mathrm{Eul}_s}$$

- If $Z \neq 0 \Leftarrow gh\#(\mathcal{O}_\omega) = n-m$
- $\langle \mathcal{O}_\omega \rangle$ depends on the cohomology class only
- Localization identity

$$\langle \mathcal{O}_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(X|\Psi)\, \mathcal{O}_\omega\, \widehat{\mathrm{Eul}_s} = \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s$$

$$= \int_{Z(s)} i^*\omega$$

Talk 2 Evgenij Pascual

$$\langle \mathcal{O}_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(X,\Psi)\, \mathcal{O}_\omega\, \widehat{\mathrm{Eul}_s}$$

$$= \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s = \int_{Z(s)} i^*\omega$$

Thom isomorphism

Poincaré lemma

$$H^*(M\times\mathbb{R}^n) = H^*(M)$$

$$H_c^*(M\times\mathbb{R}^n) = H_c^{*-n}(M)$$

For any VB $H^*(E) \cong H^{*-n}(M)$, $n = \operatorname{rk} E$

but in general ~~and~~ $H_c^*(E) \not\cong H_c^{*-n}(M)$

However, if M, E are orientable, then

$$H_c^*(E) \cong H_c^{*-n}(M)$$

Pf

$$H_c^*(E) \underset{\substack{\uparrow\\ PD}}{\cong} \left(H^{m+n-*}(E)\right)^* \cong \left(H^{m+n-*}(M)\right)^* \underset{\substack{\uparrow\\ PD}}{\cong}$$

$$\cong H_c^{*-n}(M).$$

Compact support in vertical direction

$\pi_*: \Omega_{cv}^*(E) \longrightarrow \Omega^{*-n}(M)$ integration along fibers

Projection formula

τ form on M, ω form on E, $\omega\in\Omega_{cv}(E)$

Then $\pi_*(\pi^*\tau\wedge\omega) = \tau\wedge\pi_*\omega$

Prop If E is oriented, then

$$\begin{tikzcd} \pi_{cv}^* : H^*_{cv}(E) \arrow[r, "\cong"] & H^{*-n}(M) \arrow[l, bend left, "\tau"] \end{tikzcd}$$

Want to find π_*^{-1} (the Thom iso)

Consider $H^0(M) \ni 1$ has a well-defined image in $H^n_{cv}(E)$, which we call Φ, the Thom class of E.

$$\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge \pi_* \Phi = \omega.$$

$$\Downarrow$$

$$\tau(\cdot) = \pi^*(\cdot) \wedge \Phi$$

<u>Also important: Poincaré duality</u>

Let S be a closed oriented submfld of M, then its Poincaré dual η_S is determined by

$$\int_S \omega = \int_M \omega \wedge \eta_S \quad \forall \omega \in \Omega_c^*(M)$$

Prop η_S is the same as Φ of the normal subbundle of S.

<u>Euler class</u> $E \xrightarrow{\pi} M$, $s_0 : M \to E$ zero section

$\Gamma(E) \ni s \pitchfork s_0 =: I$. Then

$$\dim I = \dim M - r, \quad r = \operatorname{rk} E$$

and we set $e_s := PD(I)$

Now SUSY

For $\pi : E \to M$, E orientable, have the Thom iso $H^i(M) \cong H^{i+m}_{cv}(E)$

Also, $s^* \Phi(E) = e_s$

$$\int_M \omega \wedge s^* \Phi(E) = \int_{Z(s)} \iota^* \omega$$

So need to replace $\frac{\partial s}{\partial x_j}$ by $\nabla_j s$

Let $\{e^a\}$ be an ON basis of E, then define ∇ by $\nabla e^a = dx^j \theta_j^{ab} e_b$

Then $S = \frac{\hbar}{2} H_j H_j + i H_j s^j - i \chi_j \frac{\partial s^j}{\partial x_k} \psi^k$

~~covariantize $S = \frac{\hbar}{2}$~~

$$S = -\frac{1}{2\hbar} s^j s^j + i \chi_j \frac{\partial s^j}{\partial x_k} \psi^k$$

covariantize $S(x, \nabla) = -\frac{1}{2\hbar} s^a s^a + i \chi_a (\nabla_j s)^a \psi^j +$

$$+ \frac{\hbar}{4} \chi_a \chi_b F_{ij}^{ab} \psi^i \psi^j$$

This form is obtained from the requirement $S = Q(\Psi)$, where Ψ is covariantized

Define a superfield on $\hat{M}$

$$\widehat{Eul}_s(E,\nabla) := \int \prod_{a=1}^{n} \frac{d\chi_a dH_a}{2\pi i} e^S$$

Have lin. op. $\nabla S : T_p M \to E_p$

Important that $Z(s)$ is a mfld.
This is the case if ∇ is not surjective.

How to get a localization formula for $\hat{E}$?

Consider ex. seq.

Coker ∇S

$$0 \to \mathrm{Im}\,\nabla S \to E \to \mathrm{Cok}\,\nabla S \to 0$$

$$\mathrm{Cok}\,\nabla S = E / \mathrm{Im}\,\nabla S$$

Then

$$\int_{\hat{E}} ber(X, H \mid \psi, \chi)\, e^S\, O = \int_{\hat{M}} ber(X \mid \psi)\, \widehat{Eul}_s(E,\nabla)$$

$$= \int_{Z(s)} i^* \omega_0 \wedge Eul(\mathrm{Coker}\,\nabla S)$$

Here we require that $\mathrm{Coker}(\nabla S)$ is a bundle

Need to replace $H^*_{cv}(E) \to H^*_{rd}(E)$
rapidly decaying

Talk 3 Victor Pidstrygach

Equivariant cohomology

Algebra A $\quad \mathrm{Der}\,A = \{ \delta : A \to A$ linear $|$
$\delta(f\cdot g) = \delta(f)\cdot g + f\,\delta(g) \}$

For an A-module F

$\mathrm{Der}(A,F) = \{ \delta : A \to F$ linear $| \, \delta(fg) = \delta(f)\cdot g + f\,\delta(g) \}$

Exercise $[\delta_1, \delta_2] \in \mathrm{Der}(A,F)$

Ex $\quad A = C^\infty(\mathbb{R}^n)$

$$\mathrm{Der}(A) = \left\{ \sum a_i \frac{\partial}{\partial x_i} \;\middle|\; a_i \in C^\infty(\mathbb{R}^n) \right\}$$

$$\mathbb{R}^n \ni x \rightsquigarrow m_x = \{ f \in C^\infty(\mathbb{R}^n) \mid f(x) = 0 \}$$

$C^\infty(\mathbb{R}^n)/m_x \cong \mathbb{R}$ field

$$T_x^\vee \mathbb{R}^n = m_x / m_x^2 \,, \quad T_x \mathbb{R}^n = \left(m_x / m_x^2 \right)^\vee$$

$$= \mathrm{Der}\left(C^\infty(A), C^\infty(\mathbb{R}^n)/m_x \right)$$

$C^\infty(\Pi T^* M) = \Omega^\bullet(M)$ graded algebra, superalgebra

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$$

d_{dR}, X vector field $\quad \mathcal{L}_X$ Lie derivative

ι_X interior derivative

$\deg d_{dR} = 1$ $\qquad \deg \iota_X = -1.$

$\deg \mathcal{L}_X = 0$

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$$

$$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$$

$$[\iota_X, \iota_Y] = \iota_X \iota_Y + \iota_Y \iota_X = 0$$

$$[d, \mathcal{L}_X] = 0$$

$$[d, d] = d\circ d + d\circ d = 0$$

$$\mathcal{L}_X = [d, \iota_X] = d\iota_X + \iota_X d$$

Lie gp G acts on M

$\mathfrak{g} = \mathrm{Lie}(G) \ni \xi \longmapsto K^{\xi}$ fundamental vector field

$$K^{\xi}(p) = \frac{d}{dt}\Big|_{t=0} \left(e^{-t\xi} \cdot p\right)$$

$$\iota_{K^{\xi}}, \quad \mathcal{L}_{K^{\xi}}, \quad d$$

$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ supersymmetry of geometric symmetry

$$C(spt) := \mathbb{R}[\theta]/\theta^2 = \Lambda^1 \mathbb{R}^1$$

$$\Pi TM = \mathrm{Map}(spt, M) = \mathrm{Map}\left(C^{\infty}(M), \mathbb{R}[\theta]/\theta^2\right).$$

Ex $M = \mathfrak{g}$, G acts by the adj. action

$\Rightarrow$ fundamental vector fields

symmetries ι_{K^ξ}, $\mathcal{L}_{K^\xi}$, d_{dR} of $\Pi T\mathfrak{g}$

$$C^\infty(\Pi T^*\mathfrak{g}) = \Omega^\bullet(\mathfrak{g}) \supset \left\{ \begin{matrix}\text{diff. forms with} \\ \text{polynomial coeffic.}\end{matrix} \right\}$$

$$\|$$

is generated by $\longleftarrow$ $S^\bullet\mathfrak{g}^\vee \otimes \Lambda^\bullet\mathfrak{g}^\vee$

$\ell \otimes 1$, $1 \otimes \ell$ $\quad$ $\ell \in \mathfrak{g}^\vee$

degree 1 diff. forms
with constant coefficients

$$d_{dR}(\ell \otimes 1) = 1 \otimes \ell \quad , \quad d_{dR}(1 \otimes \ell) = 0$$

$$K^\xi_\eta = [\xi, \eta] = ad_\xi(\eta)$$

~~$\iota_{K^\xi}(1 \otimes \ell)(\eta) =$~~

$$\iota_{K^\xi}(\ell \otimes 1) = \ell \circ ad_\xi \otimes 1 \quad *$$

$$\iota_{K^\xi}(1 \otimes \ell) = 1 \otimes \ell \circ ad_\xi$$

We also define another derivation d_K

$$d_K(1 \otimes \ell) = \ell \otimes 1 \qquad d_K(\ell \otimes 1) = 0$$

$$[d_K, d_{dR}]\Big|_{S^k\mathfrak{g}^\vee \otimes \Lambda^\ell \mathfrak{g}^\vee} = (k+\ell)\, 1$$

$$k+\ell = 0 \Leftrightarrow S^0\mathfrak{g}^\vee \otimes \Lambda^0\mathfrak{g}^\vee = \mathbb{R}$$

$$H^i(\mathcal{K}(\mathfrak{g}), d_K) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i>0. \end{cases}$$

$$\hat{\imath}_{K^\xi}(\ell\otimes 1) = \hat{\imath}_{K^\xi} d_K (1\otimes \ell) = L_{K^\xi}(1\otimes\ell) - d_K \hat{\imath}_{K^\xi}(1\otimes$$

$$= 1\otimes \ell\circ \mathrm{ad}_\xi \qquad (**)$$

This defines a nonstandard ~~sy~~ supersymmetry.

G acts freely on a mfld P s.t. $P \to P/G = M$ is a principal bundle

Connection 1-form $a \in \Omega^1(P;\mathfrak{g})^G$ $\quad R_g^* a = \mathrm{Ad}_{g^{-1}}($

$$a(K^\xi) = \xi$$

$$\Omega^\bullet(P;V)_{hor} = \{\alpha\in\Omega(P,V) \mid \imath_{K^\xi}\alpha = 0 \quad \forall \xi\in\mathfrak{g}\}$$

$$\Omega(P;V)^G = \{\alpha\in\Omega(P,V) \mid R_g^*\alpha = \rho_g(\alpha)\}$$

Here V is any rep. of G

$$\Omega(P,V)_{bas} = \Omega(P,V)^G_{hor} \overset{\text{Claim}}{=\!=} \Omega^1(M,E)$$

$$E = P\times_G V$$

Topological idea of equivar. cohomology,

1. find a contr. top. space EG with a free action of G
2. G-action on $N\times EG$ is free, so compute
$$H^\bullet(N\times EG/G) = H^\bullet_G(N)$$

~~Def Let A be a superalgebra~~

$\hat{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle$

deg $\quad -1 \quad 0 \quad +1$

Def Let A be a superalgebra with $\hat{\mathfrak{g}} \hookrightarrow \mathrm{Der}(A)$

We say the pair has property C if

$\exists$ elements $a_1, \dots, a_n$ s.t. $\hat{\iota}_{e_i} a_j = \delta_{ij}$ $\quad (*)$

deg 0

Ex $\mathcal{K}_{\mathfrak{g}}$ - Koszul, $(**)$ implies the pair has property C

Def $H_G(A) = H(A \otimes W(\mathfrak{g})_{bas}, \hat{d} + d_K)$

basic means $\left(\bigcap_{\xi \in \mathfrak{g}} \mathrm{Ker}\, \iota_\xi\right) \cap \left(\bigcap_{\xi \in \mathfrak{g}} \mathrm{Ker}\, \alpha_\xi\right)$

Weil algebra model

Claim $H(A \otimes W(\mathfrak{g})_{bas}, \hat{d} + d_K) = H(A \otimes S^\bullet(\mathfrak{g}^\vee)^G, \hat{d} + \iota_{\mathfrak{g}})$

Mathai - Quillen isomorphism

$A = \Omega^\bullet(M)$ with a G-action and standard

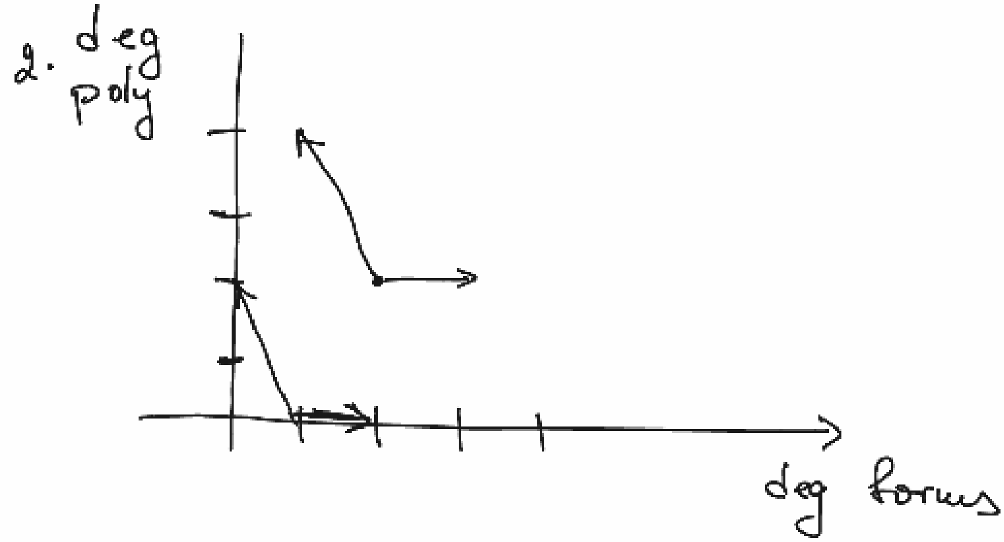
$\iota_{K^\xi}, \alpha_{K^\xi}, d_{dR}$

$S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M) = \mathrm{Hom}(S\mathfrak{g}, \Omega^\bullet(M))$

$\quad \alpha \in$ $\alpha \in S^d(\mathfrak{g}^\vee) \otimes \Omega^k(M)$

~~$(d\alpha)[\xi] := d(\alpha[\xi])$~~

$$\Big(\big(\underbrace{d - \iota_y}_{d_y}\big)\underset{d}{\alpha}\Big)[\underset{k}{\xi}] = \underset{\substack{\text{form degree } d+1 \\ \text{poly degree } k}}{d(\alpha[\xi])} - \underset{\substack{\text{form deg } d-1 \\ \text{poly deg } k+1}}{\iota_{K\xi}(\alpha[\xi])}$$



$$\deg \alpha = \text{deg form} + 2\,\text{poly degree}$$

$$(d_{dR} - \iota_{K\xi})^2 = \underset{\substack{\| \\ 0}}{d_{dR}^2} - [d, \iota_{K\xi}] + \underset{\substack{\| \\ 0}}{(\iota_{K\xi})^2} = -\alpha_{K\xi}$$

α is invariant $\Rightarrow$ $\alpha_{K\xi} = 0$ $\Rightarrow$ $(d_{dR} - \iota_{K\xi})^2 = 0$

<u>Interpretation</u> $\alpha \in S^\bullet(y^\vee) \otimes \Omega(M)$

M cmpt $\displaystyle\int_M \alpha \in S^\bullet(y^\vee)$

$\alpha = d_y \beta \Rightarrow \displaystyle\int_M \alpha = 0$

$\alpha_{[d]}$ d-form component of α

$d_y \alpha = 0 \Leftrightarrow \alpha_{[n]} = d\beta_{[n-1]} - i_{K\xi}\big(\underset{=0}{\cancel{\beta_{[n+1]}}}\big)$

G acts on M, $\alpha \in S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)$

$d_{\mathfrak{g}} \alpha = 0$

M_0 = zeros of K^ξ.

<u>Lemma</u> $\alpha(\xi)$ is exact on $M \setminus M_0$

<u>Pf</u> Define 1-form $\theta(X) = g(K^\xi, X)$

$$d_{K^\xi}\theta = \underbrace{|K^\xi|^2}_{\deg 0} + \underbrace{d\theta}_{\deg 2}$$

On $M \setminus M_0$ we have $\alpha(\xi) = d_{K^\xi}\left(\dfrac{\theta \wedge \alpha(\xi)}{d_{K^\xi}\theta(\xi)}\right)$

since $\left(|K^\xi|^2 + d\theta\right)^{-1} = \dfrac{1}{|K^\xi|^2}\left(1 - |K^\xi|^2 d\theta + \dots\right)$

$\Rightarrow \quad \alpha(\xi)_{[n]} = d(\quad)_{[n-1]}$ $\boxed{e}$

<u>Thm</u> Assume K^ξ has isolated zeros $Z(K^\xi) = \{p_0, \dots, p_n\}$. Then

$$\int_M \alpha(\xi) = (-2\pi)^p \sum \frac{\alpha(\xi)(p)}{\sqrt{\det(L_p)}}$$

where L_p - linearisation of K^ξ at p.