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1. Gauss Integral

Def 1 $s \in C^1(\mathbb{R})$ s.t. $|S(x)| \to \infty$ as $|x| \to \infty$

$Z(s) := \{ S(x) = 0 \}$, $S'(x) \neq 0 \quad \forall x \in Z(s)$

Gauss integral

$$Z = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} S'(x) e^{-\frac{1}{2} S(x)^2}$$

Lemma $Z = \sum_{z \in Z(s)} \operatorname{sign} s'(z)$

Proof

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} S'(x) e^{-\frac{1}{2} S(x)^2} \quad \underline{\underline{y = \sqrt{\hbar}\, S(x)}}$$

$$= \int_{\sqrt{\hbar} S(-\infty)}^{\sqrt{\hbar} S(+\infty)} \frac{dy}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar} y^2}$$

$$= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi\hbar}} s' e^{\frac{1}{2\hbar} S(x)^2} = \int_{-\infty}^{+\infty} s'(x) \underbrace{\frac{e^{\frac{1}{2\hbar} S(x)^2}}{\sqrt{2\pi\hbar}}}_{\downarrow \atop \delta} dx$$

$$= \sum_{z \in Z(s)} \frac{s'(z)}{|s(z)|}$$

▨

Def 3 Eucl. n-N general. for

$s \in C^1(\mathbb{R}^n)$

$$Z = \int_{R} \prod_{i=1}^{n} \frac{dx^i}{(2\pi)^{n/2}} \det\left(\frac{\partial s^j}{\partial x^k}\right) e^{-\frac{1}{2} s(x)^2}$$

$$= \sum_{z \in Z(s)} \operatorname{sign} \det\left(\frac{\partial s^j}{\partial x^k}\right)$$

1. Z Counts solution for $s^i(x) = 0$ with a prefact.
2. Z is an oriented intersection number

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2. Superspace

- Superalgebra is a vect. space $A = A^0 \oplus A^1$ with a supercomm. multiplication, i.e.

$$\begin{matrix} f \in A^a \\ g \in A^b \end{matrix} \implies f \cdot g \in A^{a+b}, \quad fg = (-1)^{a+b} gf$$

- Superspace $\mathbb{R}^{n|k}$ is defined by S.A. of coord functions

$$(C^\infty(\Omega)[\psi^1, \dots, \psi^k], +, \cdot) \text{ for open } \Omega \subset \mathbb{R}^n$$

$$\overset{\psi}{\phi} = \sum_{k=0}^{n} \sum_{i_1 < \dots < i_k} \phi_{i_1 \dots} \phi_{i_k} \psi^{i_1} \dots \psi^{i_k}$$

- For an n-dim mfld M the supermfld $\hat{M}^{(k)}$ is given by the sheaf of SA $\mathcal{O}_{\hat{M}^{(k)}}$ on M.

For open $U \subset M$, $\Omega \subset \mathbb{R}^n$ with chart
$U \xrightarrow{\cong} \Omega$ we have $\mathcal{O}_{\hat{M}^{(k)}}(U) \simeq C^\infty(\Omega)[\psi^1, .., \psi^k]$

<u>Construction 5</u> For a rk k-VB $E \to M$ we have in loc. trivialization $E|_U \cong U \times V$

$C^\infty(U) \otimes \Lambda^* V \cong \mathcal{O}_{\hat{M}^{(k)}}(U)$

For $E = TM$ (T^*M) we ~~have~~ make an identification $\psi^i \longleftrightarrow dx^i$ and

$$C^\infty(\hat{M}^{(k)}) \cong \Omega^*(M).$$

<u>Def. 6</u> "Ghost number" of a superfield $\mathcal{O}_\omega \in \hat{M}^{(k)}$

$$\deg \omega = gh \# \mathcal{O}_\omega \in \mathbb{Z} \qquad \Omega^*(M) \times (\Omega^*(M))^*$$

BRS-charge (coboundary op Q)

$$Q\,\mathcal{O}_\omega \longleftrightarrow d\omega$$

in loc coord.

$$Q x^i = \psi^i, \quad Q \psi^i = 0.$$

Integration of superfields via $\mathrm{ber}(x|\psi) \in \mathrm{Ber}(\Omega^*($

$$\int_M \omega = \int_{\hat{M}} \mathrm{ber}(x|\psi)\,\mathcal{O}_\omega \quad \text{with} \quad \cancel{\mathrm{ber}(x|\psi)} =$$

$$\mathrm{ber}(x|\psi) = dx^1 \dots dx^n \, [d\psi^1 .. d\psi^n]$$

Statement 7 We have $\int d\psi = 0$

and $\int [d\psi^1 \dots d\psi^n] \, \psi^1 \dots \psi^n = 1$.

3. SUSY Integral

Lemma 8 For $M \in \mathrm{Mat}(n, \mathbb{R})$ $\exists$ a $\xi \in \{\pm 1\}$ s.t

$$\frac{\xi}{i^n} \int [\underbrace{d\overbrace{\psi^1}^{1}}_{-1} \dots d\psi^n \underbrace{d\overbrace{\chi_1}^{-1}}_{1} \dots d\chi] \, e^{i \sum_{jk} \chi_j M_{jk} \psi^k} = \det$$

Proof

$$\frac{\xi}{i^n} \int [\;] \sum_{l=0}^{\infty} \left(\frac{i^l}{l!} \left(\sum_{j,k}^{k} \chi_j M_{jk} \psi^k \right)^l \right) =$$

$$= \frac{\xi}{i^n} \int [\;] \frac{i^n}{n!} \left(\sum_{j,k}^{n} \chi_j M_{jk} \psi^k \right)^n$$

$$= \xi \int [\;] \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \chi_i \psi^{\sigma(i)} M_{i\sigma(i)}$$

~~$= \pm \xi \int [\;] \frac{1}{n!} \sum_{\sigma \in S_n}$~~

$$= \pm \xi \underbrace{\int [\;] \chi_n \dots \chi_1 \, \psi^1 \dots \psi^n}_{1} \underbrace{\sum_{\sigma \in S_n} \operatorname{sign} \sigma \prod_{i=1}^{n} M_{i\sigma(i)}}_{\det M}$$

Constr. 9

Recall $$Z = \int_{\mathbb{R}^n} \frac{\prod dx^i}{(2\pi)^{n/2}} \underbrace{\det(\,)}_{} e^{\cdots}$$

replace with
the exp. in the Lemma

$$Z = \frac{\xi}{i} \int_{\widehat{\mathbb{R}}^n} \int_{\underset{\Pi(\mathbb{R}^n)^*}{\cancel{(\widehat{\mathbb{R}})^*}}} \frac{\operatorname{ber}(x \mid \psi, \chi)}{(2\pi\hbar)^{n/2}} e^{-\frac{1}{2\hbar} s^j(x) s^j(x) + i \chi_j \frac{\partial s^j}{\partial x_k}}$$

Want to make this Q-invariant.

$gh\#\, \psi^{\dot{a}} = 1$, $gh\#(\chi_a) = -1$

Need auxiliary field H_a with $gh\#(H_a) = 0$

$x_0 \in \mathbb{R}^n$ $\quad f(H_a) = H_a + i \frac{s^a(x_0)}{\hbar}$ has one solution

$$\int_{\mathbb{R}^n} \left(\frac{\hbar}{2\pi}\right)^{n/2} \prod_{a=1}^{n} dH_a\, e^{-\frac{\hbar}{2}\left(H_a + \frac{i s^a(x_0)}{\hbar}\right)^2} = \pm 1$$

$$Z = \frac{\xi}{(2\pi i)^n} \int_{\widehat{\mathbb{R}}^n \times (\widehat{\mathbb{R}}^n)^*} \operatorname{ber}(x, H \mid \psi, \chi)\, e^{-\frac{\hbar}{2} H_j H_j - i H_j s^j + i s_j \frac{\partial s_j}{\partial x^k}}$$

$$S = \frac{\hbar}{2} H_j H_j + i H_j s^j - i \chi_j \frac{\partial s^j}{\partial x_k} \psi^k$$

(H_a, χ_a) antighost multiplet

Define $Q \chi_a = H_a$, $Q H_a = 0$

$-S = Q(\Psi)$ for

$$\Psi = -\frac{\hbar}{2}\chi_a H_a - i\chi_a s^a$$

Z is invariant under the change

$$S \longrightarrow S - Q(\underbrace{\Delta\Psi}_{\text{variation of } \Psi})$$ (provided the behaviour at ∞ is not changed)

"Z localizes to Z(s)" $\Leftrightarrow$ $\int$ is supported at $-S=0$

$\Leftrightarrow$ supp is at $Q(\Psi)=0$ $\Leftrightarrow$

"Z localises at Q-Fixedpoints".

4. SUSY correlation functions

Constr. 11 Define a superfield

$$\widehat{Eul}_n : \widehat{\mathbb{R}}^n \longrightarrow \widehat{\mathbb{R}}^n \quad \left(\text{rather } \widehat{Eul}_n \in C^\infty(\widehat{\mathbb{R}}^n)?\right)$$

$$\widehat{Eul}_s := \int_{\widehat{\mathbb{R}}^n} \frac{dH_1 \dots dH_n [d\chi_1 \dots d\chi_m]}{(2\pi i)^m} e^{Q(\Psi)}$$

with $s : \mathbb{R}^n \longrightarrow V \cong \mathbb{R}^m$, $m \in \mathbb{N}$

- gh # $\widehat{Eul}_s = m$

$Q(\widehat{Eul}_s) = 0 \Longrightarrow \exists\ Eul_s \in \Omega^m(\mathbb{R}^n)$ s.t. $d(Eul_s) = 0$

Constr. 12

For $O_\omega : \widehat{\mathbb{R}^n} \longrightarrow \underset{\widehat{\mathbb{R}^n} \times \widehat{(\mathbb{R}^n)}^*}{\cancel{\hat{\mathbb{R}}}}$ with $Q O_\omega = 0$

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(x|\psi)\, O_\omega\, \widehat{\mathrm{Eul}_s}$$

- If $Z \neq 0 \Leftarrow gh\#(O_\omega) = n-m$
- $\langle O_\omega \rangle$ depends on the cohomology class only
- Localization identity

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(x|\psi)\, O_\omega\, \widehat{\mathrm{Eul}_s} = \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s$$

$$= \int_{Z(s)} i^* \omega$$

Talk 2 Evgenij Pascual

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \mathrm{ber}(x,\psi)\, O_\omega\, \widehat{\mathrm{Eul}_s}$$

$$= \int_{\mathbb{R}^n} \omega \wedge \mathrm{Eul}_s = \int_{Z(s)} i^* \omega$$

Thom isomorphism

Poincaré lemma

$$H^*(M\times\mathbb{R}^n) = H^*(M)$$

$$H^*_c(M\times\mathbb{R}^n) = H_c^{*-n}(M)$$

For any VB $\quad H^*(E) \cong H^{*-n}(M)$, $n = \mathrm{rk}\, E$

but in general $\quad H_c^*(E) \not\cong H_c^{*-n}(M)$

However, if M, E are orientable, then

$$H_c^*(E) \cong H_c^{*-n}(M)$$

Pf

$$H_c^*(E) \underset{\substack{\uparrow \\ PD}}{\cong} \left(H^{m+n-*}(E)\right)^* \cong \left(H^{m+n-*}(M)\right)^* \underset{\substack{\uparrow \\ PD}}{\cong}$$

$$\cong H_c^{*-n}(M).$$

▨

Compact support in vertical direction

$\pi_* : \Omega^*_{cv}(E) \longrightarrow \Omega^{*-n}(M)$ integration along fibers

Projection formula

τ form on M, ω form on E, $\omega \in \Omega_{cv}(E)$

Then $\pi_*(\pi^*\tau\wedge\omega) = \tau\wedge\pi_*\omega$

__Prop__ If E is oriented, then

$$\begin{tikzcd} \pi^*_{cv} : H^*_{cv}(E) \arrow[r, "\cong"] & H^{*-n}(M) \arrow[l, bend left, "\tau"] \end{tikzcd}$$

Want to find π_*^{-1} (the Thom iso)

Consider $H^0(M) \ni 1$ has a well-defined image in $H^n_{cv}(E)$, which we call Φ, the Thom class of E.

$$\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge \pi_*\Phi = \omega.$$

$$\Downarrow$$

$$\tau(\cdot) = \pi^*(\cdot) \wedge \Phi$$

__Also important: Poincaré duality__

Let S be a closed oriented submfld of M, then its Poincaré dual η_s is determined by

$$\int_S \omega = \int_M \omega \wedge \eta_s \quad \forall \omega \in \Omega^*_c(M)$$

__Prop__ η_s is the same as Φ of the normal subbundle of S.

__Euler class__ $E \xrightarrow{\pi} M$, $s_0 : M \to E$ zero section

$\Gamma(E) \ni s \pitchfork s_0 =: I$. Then

$\dim I = \dim M - r$, $r = \mathrm{rk}\, E$

and we set $e_s := PD(I)$

Now SUSY

For $\pi : E \to M$, E orientable, have the Thom iso $H^i(M) \cong H^{i+m}_{cv}(E)$

Also, $s^*\Phi(E) = e_s$

$$\int_M \omega \wedge s^*\Phi(E) = \int_{Z(s)} \iota^*\omega$$

So need to replace $\frac{\partial s}{\partial x_j}$ by $\nabla_j s$

Let $\{e^a\}$ be an ON basis of E, then define ∇ by $\nabla e^a = dx^j \theta_j^{ab} e_b$

Then $S = \frac{\hbar}{2} H_j H_j + i H_j s^j - i \chi_j \frac{\partial s^j}{\partial x_k} \psi^k$

~~covariantize $S = \frac{\hbar}{4}$~~

$$S = -\frac{1}{2\hbar} s^j s^j + i \chi_j \frac{\partial s^j}{\partial x_k} \psi^k$$

covariantize $S(x, \nabla) = -\frac{1}{2\hbar} s^a s^a + i \chi_a (\nabla_j s)^a \psi^j +$

$$+ \frac{\hbar}{4} \chi_a \chi_b F_{ij}^{ab} \psi^i \psi^j$$

This form is obtained from the requirement $S = Q(\Psi)$, where Ψ is covariantized

Define a superfield on $\hat{M}$

$$\widehat{Eul}_s(E,\nabla) := \int \prod_{a=1}^{n} \frac{d\chi_a \, dH_a}{2\pi i} \, e^{S}$$

Have lin. op. $\nabla S : T_p M \to E_p$

Important that $Z(s)$ is a mfld.

This is the case if ∇ is not surjective.

How to get a localization formula for $\hat{E}$?

Consider ex. seq.

$$0 \to \mathrm{Im}\,\nabla S \to E \to \overset{\text{Coker}\,\nabla S}{\underset{\underset{E/\mathrm{Im}\,\nabla S}{\parallel}}{\mathrm{Cok}\,\nabla S}} \to 0$$

Then

$$\int_{\hat{E}} ber(X, H \,|\, \Psi, X)\, e^{S} O = \int_{\hat{M}} ber(X|\Psi)\, \widehat{Eul}_s(E,\nabla)$$

$$= \int_{Z(s)} i^{*}\omega_0 \wedge Eul(\mathrm{Coker}\,\nabla S)$$

Here we require that $\mathrm{Coker}(\nabla S)$ is a bundle

Need to replace $H^{*}_{cv}(E) \longrightarrow \underset{\text{rapidly decaying}}{H^{*}_{rd}(E)}$

Talk 3 Victor Pidstrygach

Equivariant cohomology

Algebra A $\mathrm{Der}\, A = \{ \delta : A \to A$ linear $|$

$$\delta(f \cdot g) = \delta(f) \cdot g + f\,\delta(g)$$

For an A-module F

$\mathrm{Der}\,(A,F) = \{ \delta : A \to F \cancel{\}}$ linear $|\ \delta(fg) = \delta(f)\cdot g + \cancel{f}\,\delta(g)$

Exercise $[\delta_1, \delta_2] \in \mathrm{Der}\,(A,F)$

Ex $A = C^\infty(\mathbb{R}^n)$

$$\mathrm{Der}\,(A) = \{ \textstyle\sum a_i \frac{\partial}{\partial x_i} \mid a_i \in C^\infty(\mathbb{R}^n) \}$$

$$\mathbb{R}^n \ni x \rightsquigarrow m_x = \{ f \in C^\infty(\mathbb{R}^n) \mid f(x) = 0 \}$$

$C^\infty(\mathbb{R}^n)/m_x \cong \mathbb{R}$ field

$$T_x^\vee \mathbb{R}^n = m_x/m_x^2 \,, \quad T_x \mathbb{R}^n = (m_x/m_x^2)^\vee$$

$$= \mathrm{Der}\,(C^\infty(A), C^\infty(\mathbb{R}^n)/m_x)$$

$C^\infty(\Pi T^* M) = \Omega^\bullet(M)$ graded algebra, superalgebra

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$$

d_{dR}, X vector field $\mathcal{L}_X$ Lie derivative

ι_X interior derivative

$\deg d_{dR} = 1$ $\qquad \deg \iota_X = -1.$

$\deg \mathcal{L}_X = 0$

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$$

$$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$$

$$[\iota_X, \iota_Y] = \iota_X \iota_Y + \iota_Y \iota_X = 0$$

$$[d, \mathcal{L}_X] = 0$$

$$[d,d] = d \circ d + d \circ d = 0$$

$$\mathcal{L}_X = [d, \iota_X] = d\iota_X + \iota_X d$$

Lie gp G acts on M

$\mathfrak{g} = \mathrm{Lie}(G) \ni \xi \longmapsto K^{\xi}$ fundamental vector field

$$K^{\xi}(p) = \left.\frac{d}{dt}\right|_{t=0} \left(e^{-t\xi} \cdot p\right)$$

$$\iota_{K^{\xi}}, \quad \mathcal{L}_{K^{\xi}}, \quad d$$

$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ supersymmetry of geometric symmetry

$$C(\mathrm{spt}) := \mathbb{R}[\theta]/\theta^2 = \Lambda^1 \mathbb{R}^1$$

$$\Pi TM = \mathrm{Map}(\mathrm{spt}, M) = \mathrm{Map}\left(C^\infty(M), \mathbb{R}[\theta]/\theta^2\right).$$

Ex $M = \mathfrak{g}$, G acts by the adj. action

$\Rightarrow$ fundamental vector fields

symmetries α_{K^ξ}, ι_{K^ξ}, d_{dR} of $\Pi T\mathfrak{g}$

$$C^\infty(\Pi T^*\mathfrak{g}) = \Omega^\bullet(\mathfrak{g}) \supset \{\text{diff. forms with polynomial coeffic.}\}$$

$$\| $$

is generated by $\longleftarrow$ $S^\bullet\mathfrak{g}^\vee \otimes \Lambda^\bullet\mathfrak{g}^\vee$

$l\otimes 1$, $1\otimes l$ $\quad l \in \mathfrak{g}^\vee$

degree 1 diff. forms with constant coefficients

$$d_{dR}(l\otimes 1) = 1\otimes l, \quad d_{dR}(1\otimes l) = 0$$

$$K^\xi_\eta = [\xi,\eta] = ad_\xi(\eta)$$

~~$\alpha_{K^\xi_x}(1\otimes l)(\eta) =$~~

$$\alpha_{K^\xi}(l\otimes 1) = l\circ ad_\xi \otimes 1 \quad *$$

$$\alpha_{K^\xi}(1\otimes l) = 1\otimes l\circ ad_\xi$$

We also define another derivation d_K

$$d_K(1\otimes l) = l\otimes 1 \qquad d_K(l\otimes 1) = 0$$

$$[d_K, d_{dR}]\Big|^{*}_{S^k\mathfrak{g}^\vee\otimes\Lambda^l\mathfrak{g}^\vee} = (k+l)\,1$$

$$k+l = 0 \Leftrightarrow S^0\mathfrak{g}^\vee\otimes\Lambda^0\mathfrak{g}^\vee = \mathbb{R}$$

$$H^i(\mathcal{K}(\mathfrak{g}), d_K) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i>0. \end{cases}$$

$$\hat{\imath}_{K^\xi}(l\otimes 1) = \hat{\imath}_{K^\xi} d_K(1\otimes l) = \mathcal{L}_{K^\xi}(1\otimes l) - d_K \hat{\imath}_{K^\xi}(1\otimes$$

$$= 1\otimes l\circ \mathrm{ad}_\xi \qquad (**)$$

This defines a nonstandard supersymmetry.

G acts freely on a mfld P s.t. $P\to P/G = M$
is a principal bundle

Connection 1-form $a\in\Omega^1(P;\mathfrak{g})^G$ $\quad R_g^* a = \mathrm{Ad}_{g^{-1}}($

$$a(K^\xi) = \xi$$

$$\Omega^\bullet(P;V)_{hor} = \{\alpha\in\Omega(P,V) \mid \iota_{K^\xi}\alpha = 0 \ \forall \xi\in\mathfrak{g}\}$$

$$\Omega(P;V)^G = \{\alpha\in\Omega(P,V) \mid R_g^*\alpha = \rho_g(\alpha)\}$$

Here V is any rep. of G

$$\Omega(P,V)_{bas} = \Omega(P,V)^G_{hor} \overset{\text{Claim}}{=} \Omega^1(M,E)$$

$$E = P\times_G V$$

Topological idea of equivar. cohomology:

1. find a contr. top. space EG with a free action of G
2. G-action on $M\times EG$ is free, so compute
$$H^\bullet(N\times EG/G) = H^\bullet_G(N)$$

~~Def Let A be a superalgebra~~

$\hat{\mathfrak{g}} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle$

deg $\quad -1 \quad 0 \quad +1$

Def Let A be a superalgebra with $\hat{\mathfrak{g}} \hookrightarrow \mathrm{Der}(A)$

We say the pair has property C if

$\exists$ elements $a_1, \dots, a_n$ s.t. $\underbrace{\hat{\iota}_{\xi_i} a_j}_{\text{deg } 0} = \delta_{ij}$ $\quad (*)$

Ex $\mathcal{K}_{\mathfrak{g}}$ - Koszul, $(**)$ implies the pair has property C

Def $H_G(A) = H(A \otimes W(\mathfrak{g})_{bas}, \hat{d} + d_K)$

basic means $\left(\bigcap_{\xi \in \mathfrak{g}} \mathrm{Ker}\, \iota_\xi\right) \cap \left(\bigcap_{\xi \in \mathfrak{g}} \mathrm{Ker}\, \mathcal{L}_\xi\right)$

Weil algebra model

Claim $H(A \otimes W(\mathfrak{g})_{bas}, \hat{d} + d_K) = H(A \otimes S^\bullet(\mathfrak{g}^\vee)^G, \hat{d} + \iota_{\mathfrak{g}})$

Mathai-Quillen isomorphism

$A = \Omega^\bullet(M)$ with a G-action and standard

$\iota_{K^\xi}, \mathcal{L}_{K^\xi}, d_{dR}$

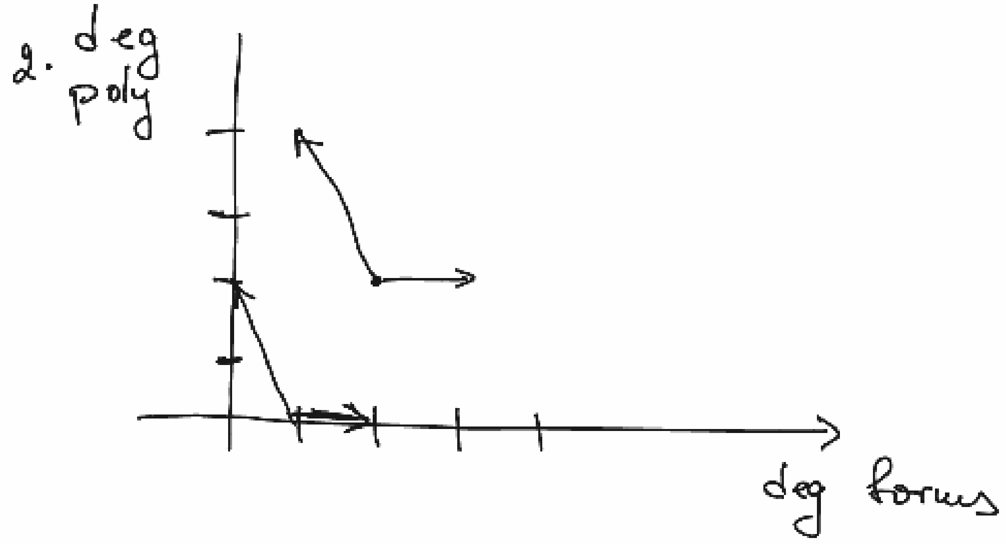
$\underset{\overset{\cup}{\alpha}}{S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)} = \mathrm{Hom}(S\mathfrak{g}, \Omega^\bullet(M))$

$\alpha \in S^d(\mathfrak{g}^\vee) \otimes \Omega^k(M)$

~~$(d\alpha)[\xi] := d(\alpha[\xi])$~~

$$\big(\underbrace{(d - \iota_y)}_{d_y}\underset{d}{\alpha}\big)\underset{k}{[\xi]} = d(\alpha[\xi]) - \iota_{K\xi}(\alpha[\xi])$$

form degree $d+1$ poly degree k — form deg $d-1$ poly deg $k+1$



deg α = deg form + 2 poly degree

$$(d_{dR} - \iota_{K\xi})^2 = \underset{\overset{\shortparallel}{0}}{d_{dR}^2} - [d, \iota_{K\xi}] + \underset{\overset{\shortparallel}{0}}{(\iota_{K\xi})^2} = -\alpha_{K\xi}$$

α is invariant $\Rightarrow$ $\alpha_{K\xi} = 0$ $\Rightarrow$ $(d_{dR} - \iota_{K\xi})^2 = 0$

Interpretation $\alpha \in S^\bullet(y^\vee) \otimes \Omega(M)$

M cmpt $\int_M \alpha \in S^\bullet(y^\vee)$

$\alpha = d_y \beta \Rightarrow \int_M \alpha = 0$

$\alpha_{[d]}$ d-form component of α

$$d_y \alpha = 0 \iff \alpha_{[n]} = d\beta_{[n-1]} - i_{K\xi}(\underset{\overset{\shortparallel}{0}}{\cancel{\beta_{[n+1]}}})$$

G acts on M, $\alpha \in S^\bullet(\mathfrak{g}^\vee) \otimes \Omega^\bullet(M)$

$d_{\mathfrak{g}}\alpha = 0$

M_0 = zeros of K^ξ.

Lemma $\alpha(\xi)$ is exact on $M \setminus M_0$

Pf Define 1-form $\theta(X) = g(K^\xi, X)$

$$d_{K^\xi}\theta = \underbrace{|K^\xi|^2}_{\deg 0} + \underbrace{d\theta}_{\deg 2}$$

On $M\setminus M_0$ we have $\alpha(\xi) = d_{K^\xi}\left(\frac{\theta \wedge \alpha(\xi)}{d_{K^\xi}\theta(\xi)}\right)$

since $\left(|K^\xi|^2 + d\theta\right)^{-1} = \frac{1}{|K^\xi|^2}\left(1 - |K^\xi|^2 d\theta + \dots\right.$

$$\Rightarrow \quad \alpha(\xi)_{[n]} = d\left(\quad\right)_{[n-1]} \qquad \square$$

Thm Assume K^ξ has isolated zeros $Z(K^\xi) = \{p_0, \dots, p_n\}$. Then

$$\int_M \alpha(\xi) = (-2\pi)^p \sum \frac{\alpha(\xi)(p)}{\sqrt{\det(L_p)}}$$

where L_p - linearisation of K^ξ at p.

Poddrygach Equiv. Cohomology II

11. 05. 18 Göttingen

$$\underbrace{\Lambda^\bullet \mathfrak{g}^\vee \otimes S^\bullet(\mathfrak{g}^\vee)}_{W^\bullet(\mathfrak{g})} \otimes \mathfrak{g} \supset \mathfrak{g}^\vee \otimes 1 \otimes \mathfrak{g} \ni id = a^* \text{ connection}$$

$$\supset \Lambda^2 \mathfrak{g}^\vee \otimes 1 \otimes \mathfrak{g} \ni [\cdot,\cdot] = F_{a^*} \text{ curvature}$$

Mathai - Quillen iso:

$$\exp(\iota^M_{K a^*}) : \Omega(M) \otimes W(\mathfrak{g})_{hor} \longrightarrow \Omega(M) \otimes W(\mathfrak{g})_{hor}$$

and $d_{dR} + d^*$ to $d_{dR} - \iota^M_{K F_{a^*}} + \alpha_{K a^*}$

On $\left(\Omega(M) \otimes W(\mathfrak{g})\right)^G_{hor}$ ~~to~~ $\longrightarrow \left(\Omega(M) \otimes W(\mathfrak{g})_{bas}\right)^G$

$$d_{dR} + d^* \rightsquigarrow d_{dR} - \iota^M_{K a^*} := d_G$$

Want to have a formula

$$\int_Y \omega = \int_P \overset{= \pi^*\omega}{\tilde{\omega}} \cdot \mathcal{P}(P \to Y)$$

$$\begin{tikzcd} G \arrow[r] & P \arrow[d] \\ & Y \end{tikzcd}$$

$\alpha \in \left(\Omega^\bullet(M) \otimes S^\bullet \mathfrak{g}^\vee\right)^G$ $\int_P \alpha \in S^\bullet(\mathfrak{g}^\vee)$

not a number so cant be compared to $\int_Y \omega$ easily

$$e^{-\frac{\varepsilon}{2}|\xi|^2}$$

$$\oint_{\varepsilon, P} \alpha := \frac{1}{\mathrm{vol}\, G} \int_{\mathfrak{g}} d\xi \int_P \alpha$$

~~$\alpha = d_G \beta \Rightarrow \int_P \alpha = 0$~~

~~Extend~~ this integral to an integral over ~~$P \times y \times W(y)^\vee$~~

$$P(P \to Y) = \int_{W(y)^\vee} e^{Q(\Psi_{proj})}\, d\bar\xi\, dy \quad \text{(Witten's answer)}$$

$$\Psi_{proj} = i\langle \xi, a\rangle + \langle \bar\xi, [\xi, \eta]\rangle$$

$$W(y) = \Lambda^\bullet y^\vee \otimes S^\bullet y^\vee \qquad W(y)^\vee = \underset{-2}{S^\bullet y^\vee} \otimes \underset{-1}{\Lambda^\bullet y^\vee}$$

$$Q(l \otimes 1) = 1 \otimes l \quad \equiv \quad Q\bar\xi = \eta$$

$$Q(1 \otimes l) = \mathcal{L}^{K}(l \otimes 1) \equiv Q\eta = -\mathcal{L}_{K^\xi}\bar\xi$$

$$\int_Y \omega = \int_P \pi^*\omega\, P(P \to Y) = \int_{\underset{\xi}{y} \times \underset{\bar\xi, \eta}{W(y)^\vee} \times P} \pi^*\omega \cdot e^{Q(\Psi_{proj}) - \frac{\varepsilon}{2}\langle \xi, d\bar\xi\rangle}$$

Emanuel Scheidegger Topological twist

4-mfld, G-bundle w/conn

$$YM(A) = \int_M tr(F_A \wedge * F_A) \qquad F = dA + [A, A]$$

In physics: YM theory

add SUSY ⟶ Super-Yang-Mills theory

in various ways

$N = 1 \rightsquigarrow$ Kähler

$N = 2 \rightsquigarrow$ Donaldson invariants, SW inv

$N = 4 \rightsquigarrow$ Kapustin-Witten, geom. Langlands

To make mathematically well-defined:

top. twisted Super-Yang-Mills thy is an example of $\begin{cases}\text{top. field theory of Witten's type (phy)} \\ \text{coh. field theory (math)}\end{cases}$

Basic ingredients

- G cmpt Lie gp, $\mathfrak{g} = \mathrm{Lie}(G)$
  gauge gp in physics
- (X, g) closed oriented 4-mfld
- $P \to X$ principal G-bundle

$$M \rightsquigarrow \mathcal{A} := \mathrm{Conn}(P)$$

$$\begin{tikzcd} E \arrow[r] & M \arrow[l, bend left, "s"] & \rightsquigarrow \ \mathcal{E} := \mathcal{A} \arrow[r, phantom, "\times"] \arrow[r, bend right, "s"'] & \Omega^{2,+}(X, \mathrm{ad} P) \end{tikzcd}$$

$$s(A) := F_A^+$$

$$G \rightsquigarrow \mathcal{G} = \mathrm{Aut}(P) \text{ (gauge gp in math)}$$

X

ψ

χ

$H \rightsquigarrow ?$

ϕ

$\bar{\phi}$

η

$N = 2$ QFT in 4D

fields = (distributional) representations of the $\boxed{N = 2 \text{ Super-Poincaré alg.}}$ (on vector bundles over X)

A non-ass super-alg ($\mathbb{Z}_2$-graded) over a comm. ring R is a Lie superalgebra, if $\exists\ [\cdot,\cdot] : A \times A \to$

s.t. $[X, Y] + (-1)^{|X||Y|}[Y, X] = 0$

$(-1)^{|Z||Y|}[Z, [X, Y]] + (-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|Y||X|}[Y, [Z, X]] =$

$L = L_0 \oplus L_1 \Rightarrow L_0$ is a Lie alg.

If $\exists$ at least one odd element, then

L_1 is an L_0-module for ad $X \longmapsto [Y, X]$

If $\exists$ at least two odd elements, then

$\exists$ L_0-equivariant $\overset{\text{symm}}{\vee}$ map

$\{\cdot,\cdot\} : L_1 \otimes L_1 \longrightarrow L_0$ s.t.

$$[\{X,Y\},Z] + [\{Y,Z\},X] + [\{Z,X\},Y] = 0$$
$$\forall X,Y,Z \in L_1.$$

V real quadr. v. space (of dim 4)

$L_0' = V \rtimes \underline{\mathfrak{so}}(V)$ Poincaré alg.

$L_0 := L_0' \oplus R$, R any reductive Lie alg. (R-symm)

(reductive: adj. repr. is completely reducible $\Leftrightarrow$
$= s + a$, s semisimple & a abelian)

$R = \underline{u}(1)$ $N=1$ extended super Poincaré-algebra

$R = \underline{su}(2) \oplus \underline{u}(1)$ $N=2$ ——//—— //——//——

S real spinorial rep. of $\mathrm{Spin}(V)$

$\exists$ symm. nonzero map $\Gamma : S \times S \to V$, which is equiv. wrt $\mathrm{Spin}(V)$

S is an L_0-module by requiring that V acts as 0 on S, and is a rep. of R.

$L_1 := S$ with $[s_1, s_2] := \Gamma(s_1, s_2)$

<u>Special case</u> $\dim V = 4$ (didn't use this above)

$$L_0 = \mathbb{R}^4 \rtimes (\mathfrak{su}(2)_- \oplus \mathfrak{su}(2)_+) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$$

$$L_1 = (1,2,2)^1 \oplus (2,1,2)^{-1}$$

$$(1,2,2) = \underset{\substack{\nearrow \\ \mathfrak{su}(2)_- \\ \dim = 1}}{V_-} \otimes \underset{\substack{\uparrow \\ \mathfrak{su}(2)_+ \\ \dim = 2}}{V_+} \otimes \underset{\substack{\nwarrow \\ \mathfrak{su}(2)_R \\ \dim = 2}}{V_R}$$

$(\ \ )^1$ weight of the $\mathfrak{u}(1)$ repres.

There is an L_0 equiv. map $\{\cdot,\cdot\} : \mathrm{Sym}^2(L_1) \to \mathbb{R}^4$

Physics notations

Basis elements in $(1,2,2)^1 \ni \bar{Q}^A_{\dot\alpha}$ $\quad \dot\alpha = 1,2$; $A = 1,2$

$(2,1,2)^{-1} \ni Q^A_\alpha$ $\quad \alpha = 1,2$; $A = 1,2$

$$\{Q^A_\alpha, \bar{Q}^B_\alpha\} = 2\underset{\substack{| \\ \text{totally antisymm.} \\ \text{tensor}}}{\varepsilon^{AB}} \sigma^\mu_{\alpha\beta} P_\mu \quad , \mu = 1,2,3,4$$

$$\varepsilon^{A_1 \dots A_n} = \begin{cases} +1, & \text{if } (A_1,\dots,A_n) = (1,\dots,N) \\ -1, & \text{if } (A_1,\dots,A_n) = (2,1,\dots, \\ 0, & \text{if } A_i = A_j \text{ for some } i \neq j. \end{cases}$$

$\sigma^\mu_{\alpha\beta}$ is the repr. matrix of the map Γ

= Pauli matrices.

Q's are called supercharges

P's are called momenta

Also, $\{Q_\alpha^A, Q_\beta^B\} = 0$.

$\mathcal{N}=2$ vector multiplet

Reps of Lie superalgebras are determined by corr. reps of their even parts. Consider the following rep. of l

Basis elements	$\underline{su}(2)_- \oplus \underline{su}(2)_+ \oplus \underline{su}(2)_R$	$u(1)_R$
A_μ	$(2,2,1)$	0
$\bar{\Psi}_{\dot{\alpha}}^A$	$\oplus$ $(1,2,2)$	-1
ψ_α^A	$\oplus$ $(2,1,2)$	$+1$
ϕ	$\oplus$ $(1,1,1)$	2
$\bar{\phi}$	$\oplus$ $(1,1,1)$	-2
D^a	$\oplus$ $(1,1,3)$	0

$\mu = 1,2,3,4$

$A = 1,2$

$a = 1,2,3$

$\uparrow$ ghost numbers (later)

D auxiliary field; appears only algebraically in the EL equs; can be eliminated

Globalize $\mathbb{R}^4 \rightsquigarrow X^4$

vector spaces $\rightsquigarrow$ vector bundles

We must introduce a new datum, the R-symmetry bundle $P_R \to X$ a principal $SU(2)_R$-bundle.

$\Rightarrow A$ is a con-n.

$\phi, \bar{\Phi}$ sections of $\mathrm{ad}P \otimes \mathbb{C}$

D —— // —— $\mathrm{ad}P \otimes W$ ← real rk = 3 v.b. ass. to P_R

$$\Psi, \bar{\Psi} \in \Gamma\big(\underbrace{S^{\pm}}_{\text{spinor bundles of } X} \otimes \underbrace{S_R}_{\text{spinor rep of } SU(2)_R} \otimes \mathrm{ad}P\big)$$

X does not need to be Spin!

In this case: $P_R \to X$ is an $SO(3)$ principal bundle s.t. $w_2(P_R) = w_2(X)$

Then $S^{\pm} \otimes S_R$ exist but $S^{\pm}$ and S_R don't.

Moore: Choose $P_R \cong P_+$ s.t. $P_+ \times_{SO(3)_+} \mathbb{R}^3 \cong \Lambda^2_+ T^* X$.

Action

Levi-Civita $\omega \in \Omega^1(P_{SO(4)}, \underset{\overset{\shortparallel}{SO(3)_+ \oplus su(2)_-}}{so(4)})$

$$\omega = \omega_+ + \omega_-$$

Path integral

$$Z(\omega_+, \omega_-, \cancel{\omega_R}) := \int [\mathcal{D}A\, \mathcal{D}\psi \ldots \mathcal{D}D]\, e^{-S^{YM}_{phys}} \quad (\quad)$$

$$S^{YM}_{phys} := \Big(- \int_X \frac{1}{g_0^2} \mathrm{tr}\Big(F \wedge *F + D\phi \wedge *D\phi^* - \frac{1}{4}[\phi, \phi^*]^2 \mathrm{vol}\Big)$$

$$\text{bosonic part, depends on } g \quad \Big(+ \underbrace{\frac{\theta_0}{8\pi^2} \int_X \mathrm{tr}(F \wedge F)}_{\text{topological term}} + \cancel{\int_X C \wedge (\bar{\Psi} D \Psi) + \bar{\Psi}\Psi\bar{\Psi}\Psi}$$

$$+ \int_X \frac{1}{g^2} \Big(\overline{\psi} \not{D} \psi + \overline{\psi}\psi\overline{\psi}\psi + \overline{\phi}[\psi,\psi] +$$

$$+ \phi[\psi,\overline{\psi}] \Big)$$

The field D does not appear here, since it is integrated out

$$\mathrm{tr}(XY) = -\frac{1}{2h^\vee} \mathrm{tr}_{\mathfrak{g}} \big(\mathrm{ad}(X)\, \mathrm{ad}(Y) \big)$$

$\mathcal{G} = \mathrm{Aut}(P)$ (symmetries)

$\mathcal{M} = \{ A \in \mathcal{A} \mid F_A^+ = 0 \} / \mathcal{G}$

$N=2$ vector multiplet

$SU(2)_{+} \oplus su(2)_{R} \oplus su(2)_{\phi}$
$\oplus\, u(1)_R$

A_μ, $(2,2,1)$ $\quad \phi$ $(1,1,1)_2$

$\overline{\psi}^A_\alpha$ $(1,2,2)_{-1}$ $\quad \overline{\phi}$ $(1,1,1)_{-2}$

ψ^A_α $(2,1,2)_{+1}$ $\quad D^a$ $(1,1,0)_0$

Path integral

$$Z[\omega_+, \omega_-, \omega_R] = \int [\mathcal{D}A \mathcal{D}\psi \ldots \mathcal{D}D]\, e^{-S^{SYM}_{phys}(A,..)}$$

$$S^{SYM}_{phys} := -\int_X \frac{1}{g_0^2} \mathrm{tr}\left(F_A \wedge * F_A\right) + D\phi \wedge * D\phi^*$$

$$- \frac{1}{4} [\phi, \phi^*]\, \mathrm{vol}_g + \frac{\theta_0}{8\pi^2} \int_X \mathrm{tr}(F \wedge F)$$

$$+ \frac{1}{g_0^2} \int_X \mathrm{tr}\Big(\overline{\psi} D \psi + \psi \overline{\psi} \psi \overline{\psi} + \overline{\phi}[\psi, \psi] +$$

$$+ \phi [\psi, \overline{\psi}] \Big).$$

$$\mathrm{tr}(XY) = -\frac{1}{2h^\vee} \mathrm{tr}_{\mathfrak{g}}\left(\mathrm{ad}(X) \circ \mathrm{ad}(Y)\right)$$

__Observation__: If $\omega^+ = \omega_R$ $(P^+ \cong P_R)$ then Z does not depend on ω_+, ω_- !

Witten's idea of the topological twist.

In fact: Choose a __new__ Lorentz subalgebra (instead of $\underline{su}(2)_- \oplus \underline{su}(2)_+$)

$$\underline{su}(2)_- \oplus \underline{\underline{su}}(2)_+ ,$$

where $\underline{\underline{su}}(2)_+ \underset{\text{diagonally}}{\subset} su(2)_+ \oplus su(2)_R$

A_μ $(2,2)_0$ $\quad \phi$ $(1,1)_2$

$\boxed{\overline{\psi}^A_{\dot\alpha} \ (1, 2\otimes 2)_{-1}}$ $\quad \overline{\phi}$ $(1,1)_{-2}$

ψ_μ — ψ^A_α $(2,2)_{+1}$ $\quad D^a$ $(1,3)_0$

Since $2 \otimes 2$ is __not__ irreducible, $\overline{\psi}^A_{\dot\alpha}$ decompose

$$(1, 2\otimes 2)_{-1} \begin{cases} (1,1)_{-1} & \eta \\ (1,3)_{-1} & \chi_{\mu\nu} = -\chi_{\cancel{\mu\nu}} \end{cases}$$

self dual 2-form

$\left(\begin{matrix}\text{here we use} \\ 3 \cong su(2)\otimes\mathbb{C} \\ \qquad \overset{\shortparallel}{\underline{so}(3)}\end{matrix}\right)$

$$Q^A_{\dot\alpha} \in (1,2,2)_{+1} \overset{\text{twist}}{\rightsquigarrow} (1, 2\otimes 2)_{+1} = (1,1)_{+1} \oplus (1,3)_1$$

$$Q^A_\alpha \in (2,1,1)_{-1} \overset{\text{twist}}{\rightsquigarrow} (2,1)_{-1} \qquad \uparrow \; Q := \delta^{\dot\alpha}_A \overline{Q}^A_{\dot\alpha}$$

Q is the projection of $Q^A_{\dot\alpha}$ to $\mathbf{1}$.

$\{\bar{Q}^A_\alpha , Q^B_\beta\} = 0 \Rightarrow Q^2 = 0.$

$\Rightarrow$ Q is a BRST operator.

$\Psi_\mu = QA_\mu \Rightarrow Q\Psi_\mu = 0.$

A general $\mathcal{N}=2$ action has the form

$$S = \int d^{2n}\theta \; \underbrace{K(\Phi,\bar{\Phi})}_{\text{Kähler potential}} + \int d^{n}\theta \; \underbrace{W(\Phi)}_{\substack{\uparrow \\ \text{superpotential,} \\ \text{holom. in } \Phi.}} + \int d^{n}\theta \, \bar{W}(\bar{\Phi}$$

Main consequences of the topological twist

1) $S^{SYM}_{top} = \{Q, \Psi\}$ action is Q-exact.

$$+ 2\pi i \tau_0 \int tr(F\wedge F) ,$$

where $\tau_0 := \frac{\theta_0}{2\pi} + \frac{4\pi i}{g_0^2}$

This is equivalent to

$$S^{SYM}_{top} = Q\Psi + 2\pi i \tau_0 \int tr(F\wedge F)$$

Useful identity

$$\int_X tr\, F^{\pm}\wedge F^{\pm} = 2\int_X tr(F\wedge * F) \pm tr(F\wedge F)$$

9) Energy-momentum tensor:

$$T_{\mu\nu} := \frac{1}{\sqrt{\det g}} \frac{\delta}{\delta g_{\mu\nu}} S^{SYM}_{top} = \{ Q , \Lambda_{\mu\nu} \}$$

for some $\Lambda_{\mu\nu}$

Hence, the path integral does not depend on the background metric.

Some comments on vdim Masd

Anomaly is proportional to $\text{Ind}(\not{D}_A) =$

= expression in Chern numbers of some spin bund

$= \underline{u}(1)_R$ charge

= ghost number

= degree of diff. forms on Masd.

Anomaly is a symmetry which does not survive any quantization procedure

V. Pidstrygach

Cohomological descent

$$W(\mathfrak{g}) = \Lambda^\bullet(\mathfrak{g}^\vee) \otimes S^\bullet(\mathfrak{g}^\vee)$$

Super-Lie-alg. $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle_1 = \tilde{\mathfrak{g}}$

$$\iota_\xi \qquad \mathcal{L}_\xi \qquad d \qquad\qquad \mathcal{L}_\xi = [d, \iota_\xi]$$

Ex 1 $G \circlearrowright M$ $\Omega^\bullet(M)$ is a representation of $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle_1$

Ex 2 Basis of $\mathfrak{g}^\vee$: Θ^i $\deg = 1$ $\|$ $\mathcal{L}_a = \mathcal{L}_K \Theta^a$

z^i $\deg = 2$ $\|$ $\iota_a = \iota_K \Theta^a$

$$d_K \Theta^i = z^i, \quad d_K z^i = 0$$

$$\mathcal{L}_a \Theta^b = -\sum_k c^b_{ak} \Theta^k \ ; \quad \mathcal{L}_a z^b = -\sum_k c^b_{ak} z^k$$

$$\iota_a \Theta^b = \delta_{ab} \qquad i_a z^b = \iota_a d \Theta^b = \mathcal{L}_a \Theta^b - d i_a \Theta^b =$$
$$= -\sum c^b_{ak} \Theta^k$$

A alg, $\tilde{\mathfrak{g}} \to \mathrm{Der}(A)$

$$A_{hor} = \{ \alpha \in A \mid i_a \alpha = 0 \ \forall a \}$$

$$A^G = \{ \alpha \in A \mid \mathcal{L}_a \alpha = 0 \ \forall a \}$$

G acts freely on P, then $\Omega^\bullet(P)^G_{hor} \cong \Omega^\bullet(P/G)$

$$\boxed{\mu^b = z^b + \frac{1}{2}\sum c^b_{jk}\theta^j\theta^k} \; (*) \quad \deg = 2 \text{ element}$$

Since $z^b = \mu^b - \frac{1}{2}\sum c^b_{jk}\theta^j\theta^k$,

$\{\mu^b, \theta^a\}$ can be taken as generators.

$$W(\mathfrak{g}) \cong \Lambda(\theta^1,..,\theta^n) \otimes S(\mu^1,..,\mu^n)$$

<u>Claim</u> μ^j are horizontal

$$i_a\mu^b = i_a z^b + \frac{1}{2}\sum_{jk} c^b_{jk}\left(i_a\theta^j\cdot\theta^k - \theta^j i_a\theta^k\right)$$

$$= -\sum_k c^b_{ak}\theta^k + \frac{1}{2}\sum_{jk}\left(\delta_{aj}\theta^k + \delta_{ak}\theta^j\right)$$

$$= -\sum_k c^b_{ak}\theta^k + \frac{1}{2}\sum c^b_{ak}\theta^k - \frac{1}{2}\sum \underset{\substack{\parallel \\ -c^b_{aj}}}{c^b_{ja}}\theta^j = 0.$$

$$d_K\theta^a = \mu^a - \frac{1}{2}\sum c^a_{jk}\theta^j\theta^k \quad (**)$$

$$\mathcal{L}_a\mu^l = \mathcal{L}_a z^l + \frac{1}{2}\mathcal{L}_a\sum_{jk} c^b_{jk}\theta^j\theta^k$$

$$\overset{\text{Jacobi}}{=} -\sum c^b_{ak}\mu^k$$

$$\boxed{d_K\mu^a = -\sum c^a_{jk}\theta^j\mu^b = \left(\sum \theta^b\mathcal{L}_b\right)\mu^a} \quad (***)$$

Bianchi identity

Since μ^a generate $\mathfrak{g}^\vee \subset S^\cdot(\mathfrak{g}^\vee) \Longrightarrow$

$\Rightarrow d_K \omega = \sum(\theta^b \alpha_b)\,\omega$

"Supersymmetric change of variables" (*)

Often, one takes (**) & (***) as definitions

If this is the approach, one has to prove

$d_K^2 = 0$ and acyclicity

$$H^i(W(\mathfrak{g}), d_K) = \begin{cases} \mathbb{R}, & i=0 \\ 0, & \text{otherwise.} \end{cases}$$

Mathai-Quillen iso Φ

$\mathfrak{g} \longrightarrow \operatorname{Der} A$

$\mathfrak{g} \longrightarrow \operatorname{Der} B$, A, B Algebras

Assume $\exists\, \hat{\theta}_b \in A$ s.t. $\iota_a \hat{\theta}^b = \delta_{ab}$

Derivation $\gamma = \sum \hat{\theta}^a \otimes \iota_a \in \operatorname{Der}(A\otimes B)$

If $n = \dim \mathfrak{g}$, then $\gamma^{n+1} = 0$

$\Phi = e^{\gamma}$ $\quad \operatorname{ad}_\gamma,\ \operatorname{conj}_\Phi \circlearrowleft \operatorname{Der}(A\otimes B)$

$\in$ $\operatorname{Aut}(A\otimes B)$ $\quad \operatorname{ad}_\gamma(\delta) = [\gamma, \delta]$, $\delta \in \operatorname{Der}(A\otimes B)$

$\operatorname{conj}_\Phi(\delta) = \Phi\,\delta\,\Phi^{-1}$

$\Phi(1\otimes \iota_a + \iota_a \otimes 1)\Phi^{-1} = \iota_a \otimes 1$

$$\Phi(d_A\otimes 1 + 1\otimes d_B)\Phi^{-1} = \underbrace{d - \sum \mu^k \otimes \iota_k + \sum \theta^k \otimes \alpha_k}_{\text{BRST}}$$

[Guillemin-Sternberg?]

$$\Phi (A \otimes B)_{hor} \Phi^{-1} = A_{hor} \otimes B$$

Apply this to $A = W(\mathfrak{g})$ $B = \Omega^\bullet(M)$, $G \circlearrowright M$

$$\left(W(\mathfrak{g}) \otimes \Omega^\bullet(M)\right)_{hor} = W(\mathfrak{g})_{hor} \otimes \Omega^\bullet(M)$$

$$= S^\bullet(\mathfrak{g}) \otimes \Omega^\bullet(M)$$

Furthermore

$$\left(W(\mathfrak{g}) \otimes \Omega^\bullet(M)\right)^G = \left(S^\bullet(\mathfrak{g}) \otimes \Omega^\bullet(M)\right)^G$$

$$d_K \otimes 1 + 1 \otimes d_{dR} - \sum \mu^k \otimes i_k + \sum \theta^k \otimes \mathcal{L}_k \Big|_{(\;)^G_{hor}} =$$

$$= d_{dR} - \sum \mu^k \otimes \iota_k$$

$$=: \left(d_{dR} - \iota_{\mathfrak{g}}\right)$$

This is a Cartan model.

$\mathcal{A}_P$ con-ns on $P \to X^4$

$\mathcal{G} = Aut(P)$

$C^\infty(\mathcal{A}_P) \ni A^a_\mu(x)$

$\Omega^1(\mathcal{A}_P ; \Omega^1(P; \mathfrak{g})^G_{hor}) \ni \Psi$ tautological 1-form

$\underbrace{d_C}_{\text{Cartan differential}} A = \Psi$, where $A \in \Omega^0(\mathcal{A}_P ; \Omega^1(P; \mathfrak{g})^G_{hor})$

$d_C \Psi = - D_A \phi$ $\qquad \phi \in C^\infty\big((\mathrm{Lie}(\mathcal{G}), \Omega^0(P,\mathfrak{g})^G\big)$

$d_c \phi = 0$ $\qquad \phi$ is the identity map.

$D_A \phi$ fundamental vector field of $\phi \in \mathrm{Lie}(\mathcal{G})$

Compare with the following:

$G \longrightarrow GL(\mathbb{R}^n)$

$\mathfrak{g} \longrightarrow \mathfrak{gl}_n(\mathbb{R})$

$$K^{\xi}_{X} = \overset{\xi_{ij}}{\xi \cdot X}$$

$$d_C^2(x^i) = d_C(dx^i) = (d - i_{K^\xi})dx^i$$
$$= - \sum \xi_{ij} x_j$$

BRST $\quad d_B$, B for BRST

$$d_B A = \Psi - D_A C$$

$$d_B \Psi = [\Psi, C] - D_A \phi$$

$$d_B C = \phi - \tfrac{1}{2}[C, C]$$

$$d_B \phi = -[C, \phi]$$

Pidstrygach 15.06 Göttingen

Cohomological descent II

$W^\bullet(\mathfrak{g}) \otimes \Omega^\bullet(M)$ G-action on M

$\exists$ automorphism of $W(\mathfrak{g}) \otimes \Omega^\bullet(M)$, which takes $(W(\mathfrak{g}) \otimes \Omega^\bullet(M))^G_{hor}$ to $(W(\mathfrak{g}) \otimes \Omega^\bullet(M))^G$

$$\| $$

$$\underbrace{(S^\bullet(\mathfrak{g}^\vee) \otimes \Omega(M))^G}_{\text{Cartan model}}$$

with the differential $d - i_{\mathfrak{g}}$

Free G-action on P (G princip. bundle)

$$\begin{tikzcd} (W(\mathfrak{g}) \otimes \Omega^\bullet(P))^G_{hor} \arrow[r, phantom, "\cong"] & (\underbrace{W(\mathfrak{g})}_{\text{contractible}} \otimes \Omega(P)_{hor})^G & \Omega(P)^G_{hor} \arrow[l] \end{tikzcd}$$

$\|$ $\qquad$ $\|$

$(\Lambda^\bullet \mathfrak{g}^\vee) \otimes S^\bullet(\mathfrak{g}^\vee)$ $\qquad$ induces an iso in cohomology $\qquad$ $\Omega^\bullet(P/G)$

$\theta^a \quad \mu^a$

Contraction! $\delta\mu^a = \theta^a$, $\delta\theta^a = 0$

Free action of the gauge gp $\mathcal{G}_P$ on $\mathcal{A}_P$

Cartan model

(A) $S^\bullet(\mathrm{Lie}(\mathcal{G})) \otimes \Omega^\bullet(\mathcal{A})$ $\quad d_c := d_{dR} - \iota_{\mathrm{Lie}(\mathcal{G})}$

$\sum p_i(\xi_r) \otimes \alpha_i$

$$\left(d - \iota_{\mathrm{Lie}(\mathcal{G})}\right) \sum_i p_i(\xi) \otimes \alpha_i = \sum p_i(\xi) \otimes d\alpha_i - \sum p_i(\xi) \otimes \iota_{K^\xi}$$

$\xi \in \mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathrm{ad}\, P)$

$P \to M$ $\qquad$ The fund. v.f. on $\mathcal{A}_P$: $K_a^\xi = D_a \xi$

Changing the notation $\xi \rightsquigarrow \phi$

$$(d - i_{Lie(\phi)})(\alpha) = \sum_i p_i(\phi) \otimes d\alpha_i - \sum_i p_i(\phi) \otimes i_{K\phi}(\alpha_i)$$

$A \in C^\infty(\mathcal{A}; \Omega^1(P, \mathfrak{g})^G_{hor})$

$\Psi \in \Omega^1(\mathcal{A}; \Omega^1(P, \mathfrak{g})^G_{hor})$

$d_C A = d_{dR} A = \Psi$

$d_C \Psi = -D_a \phi$

$d_C \phi = 0$

$\deg \phi =$

$\deg A =$

$\deg \psi =$

$\phi \in \mathrm{Hom}(Lie(\mathcal{G}), Lie(G))$ vector valued

identity map

G-invariant & closed elements of (A)

$\mathcal{O}_2^{(0)}(p) \quad \pi(p) \in M \quad (p \in P)$

$$\mathcal{O}_2^{(0)}(\pi(p)) := \frac{1}{8\pi^2} tr(\phi^2(p))$$

$$\mathcal{O}_2^{(0)}(x) \in S^2(Lie(G)^\vee)$$

$$\mathcal{O}_2^{(1)}(\gamma) = \frac{1}{4\pi^2} \int_\gamma tr(\phi\Psi) \qquad \mathcal{O}_2^{(1)} \in S^1(Lie(G)^\vee) \otimes \Omega^1(\mathcal{A}$$

$$\mathcal{O}_2^{(2)}(\Sigma) = \frac{1}{4\pi^2} \int_\Sigma tr(\phi F_a - \frac{1}{2}\psi \wedge \psi),$$

$$\mathcal{O}_2^{(2)} \in S^1(Lie(G)^\vee) \otimes \Omega^0(\mathcal{A}) \oplus \Omega^2(\mathcal{A})$$

All these elements are G-invariant

$$d_C\, tr(\phi^2(p)) = 2 tr(\phi(p)\, d_C \phi(p)) = 2 tr(\phi \underset{\overset{\shortparallel}{0}}{d_C \phi})(p) = 0$$

~~$2tr(\phi D_A \phi)(p) = d^X tr(\phi^2)(p)$~~

$$d_c \int_\gamma \mathrm{tr}(\phi\psi) = \int_\gamma \mathrm{tr}\left(d_c\phi \cdot \psi + \phi\, d_c\psi\right)$$

$$= \int_\gamma \mathrm{tr}\left(\phi\left(-D_a\phi\right)\right) = \int_\gamma d^M \mathrm{tr}(\phi^2) \overset{\text{Stokes}}{=\!=} 0$$

$$d_c \int_\Sigma \mathrm{tr}\left(\phi F_A - \tfrac{1}{2}\psi \wedge \psi\right) = \int_\Sigma \left(\phi\, d_c F_A - d_c\psi \wedge \psi\right)$$

$$= \int_\Sigma \mathrm{tr}\left(\phi \wedge D_A\psi + D_A\phi \wedge \psi\right) = -\int_\Sigma d^M \mathrm{tr}\left(\phi \wedge \psi\right).$$

Donaldson invariants

Motivation:

$$\mathcal{M}_k \subset \mathcal{B}_k^*$$

Can show: $[\mathcal{M}_k] \in H_d(\mathcal{B}_k^*)$ is well defined

$d = \mathrm{vdim}\, \mathcal{M}_k = 8k - 3(b_2^+ - b_1 + 1)$

$[\mathcal{M}_k]$ can be thought of as an invariant, but this is not conve

If $\omega \in H^d(\mathcal{B}_k^*)$, then $\langle \omega, [\mathcal{M}_k] \rangle \in \mathbb{Z}$

Q: $H^*(\mathcal{B}^*)$?

1. Slant product

Y, Z any top. spaces

$$/ : H^p(Z) \otimes H_q(Y \times Z) \to H_{q-p}(Y) \quad (*)$$

~~$\mathrm{Hom}(C_p(Z), \mathbb{Z}) \otimes C_q(Y \times Z)$~~ .

Fact ∃ natural chain map

$$\xi : C_q(Y \times Z) \longrightarrow \sum_{s+t=q} C_s(Y) \otimes C_t(Z)$$

which is a chain homotopy equivalence

In particular, ξ induces an iso

$$\xi : H_q(Y \times Z) \longrightarrow H_q(C_\bullet(Y) \otimes C_\bullet(Z))$$

$$\begin{tikzcd} & C_{q-p}Y \otimes C_p Z \arrow[d, phantom, "\cap"] & \\ \mathrm{Hom}(C_p \mathbb{Z}, \mathbb{Z}) \otimes C_q(Y \times Z) \arrow[r, "id\otimes\xi"] & \mathrm{Hom}(C_p Z, \mathbb{Z}) \otimes (C_\cdot Y \otimes C_\cdot Z)_q \arrow[r] & \mathbb{Z} \end{tikzcd}$$

$$\varphi \otimes (a \otimes b) \longmapsto \varphi(b)\, a$$

This is a chain map, hence we obtain $(\ast)$

Properties

1° Naturality

$f : Y \to Y'$ $\qquad u \in H^p(Z')$

$g : Z \to Z'$ $\qquad v \in H_q(Y \times Z)$

$$f_*\big(\underbrace{g^* u / v}_{H_{q-p}(Y)}\big) = u / (f \times g)_* v$$

2°. Cap product

$$\begin{tikzcd} & Y \times Z \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_Z"] & \\ Y & & Z \end{tikzcd}$$

$u \in H^p(Z)$

$v \in H_q(Y \times Z)$

$$u/v = (\pi_Y)_* (\pi_Z^* u \cap v)$$

Ex [illegible]

Z cmpt oriented mf'd of dim $= d$; Y c p' mfld of dim $= n$

$$H^d(Z) \otimes H_q(Y \times Z) \longrightarrow H_{q-d}(Y)$$

$$\begin{tikzcd} H^d(Z) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} \end{tikzcd} \qquad H_q(Y \times Z) \longrightarrow H_{q-d}(Y)$$

$$\begin{tikzcd}
H_q(Y\times Z) \arrow[r, dashed] \arrow[d, "PD"] & H_{q-d}(Y) \arrow[d, "PD"] \\
H^{n+d-q}(Y\times Z) \arrow[r, "\cong"] & H^{n-q}(Y)
\end{tikzcd}$$

integration along the fibers

de Rham - version :

$\omega \in \Omega^{k+d}(Y\times Z)$

$v_1,\dots, v_k \in \mathcal{X}(Y)$

$$\int_Z i_{v_1}\dots i_{v_k}\omega = \Omega(v_1,\dots,v_k)$$

$$\Omega^{k+d}(Y\times Z) \longrightarrow \Omega^k(Y) \qquad \text{induces} \quad \underbrace{H^{k+d}_{dR}(Y\times Z)\xrightarrow{\cong} H^k_{dR}(Y)}_{\text{integration along fibers.}}$$

$$\omega \longmapsto \Omega$$

2. The universal bundle

$P \to X \ni x_0$ basept

$\mathcal{G}_0 = \{ g\in Aut(P) \mid g_{x_0} = id\}$ based gauge gp

$\{1\} \to \mathcal{G}_0 \longrightarrow \mathcal{G} \xrightarrow{ev_{x_0}} Aut(P_{x_0}) = G \to 0$ exact

$\mathcal{G}_0$ acts freely on $\mathcal{A}$

$\tilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_0$ mfld

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$$G = \mathcal{G}/\mathcal{G}_0 \circlearrowright \tilde{\mathcal{B}} \longrightarrow \mathcal{B}$$

~~$\tilde{\mathcal{B}}/G$~~

$$\begin{tikzcd}
\mathcal{G} \circlearrowright \mathbb{P} := \pi_2^* P \arrow[r] \arrow[d] & P \arrow[d] \\
\mathcal{G} \circlearrowright \mathcal{A} \times X \arrow[r, "\pi_2"] & X
\end{tikzcd}$$

$$\tilde{\mathbb{P}} = \mathbb{P}/\mathcal{G}_0 \longrightarrow \mathcal{A} \times X/\mathcal{G}_0 =: \tilde{\mathcal{B}} \times X$$ universal bundle

$$/ : H^d(\underset{\substack{\rotatebox{90}{$\in$} \\ c_{d/2}(\tilde{\mathbb{P}})}}{\tilde{\mathcal{B}} \times X}) \times H_\ell(X) \longrightarrow H^{d-\ell}(\tilde{\mathcal{B}})$$

$$\tilde{\mu}_d : H_\ell(X) \longrightarrow H^{d-\ell}(\tilde{\mathcal{B}})$$

$$\alpha \longmapsto c_{d/2}(\tilde{\mathbb{P}})/\alpha$$

Prop

$P \to X$ SU(2) bundle

X 1-connected

$\Sigma_1, \dots, \Sigma_b$ basis of $H_2(X; \mathbb{Q})$

$\Rightarrow$ $H^*(\tilde{\mathcal{B}}; \mathbb{Q})$ is generated by $\tilde{\mu}_2(\Sigma_1), \dots, \tilde{\mu}_2(\Sigma_b)$ as a ring, i.e.

$H^*(\tilde{\mathcal{B}}; \mathbb{Q}) \cong S^*(H_2(X; \mathbb{Q}$

$H^{2k}(\tilde{\mathcal{B}}; \mathbb{Q}) \cong S^k(H_2(X;$

$$\tilde{\mu}_2(\Sigma_j) = c_2(\tilde{\mathbb{P}})/\Sigma_j \in H^2(\tilde{\mathcal{B}}; \mathbb{Q})$$

Qu : $H^*(\mathcal{B}^*; \mathbb{Q})$?

DIS

$\tilde{\mathcal{B}} \circlearrowleft \mathcal{G}$ the action of $\mathcal{G}/\{\pm 1\}$ is free only on $\hat{\mathcal{B}}^*$

$$\begin{tikzcd}
\mathbb{P}^{ad} \arrow[d, dotted, "i"] & \tilde{\mathbb{P}} \arrow[l, dashed] \arrow[d] \\
\mathcal{B}^* & \hat{\mathcal{B}}^* \arrow[l, dashed]
\end{tikzcd}$$

SO(3) bundle

~~[illegible]~~

Just like above.

$$\mu : H_2(X;\mathbb{Q}) \longrightarrow H^2(\mathcal{B}^*;\mathbb{Q})$$
$$\Sigma \longmapsto -\tfrac{1}{4} p_1(\mathbb{P}^{ad})/\Sigma$$

~~[illegible]~~

$$\begin{tikzcd}
SO(3) \arrow[r, hook] & \tilde{\mathcal{B}}^* \arrow[d] \\
& \mathcal{B}^*
\end{tikzcd}$$

$$\nu := -\tfrac{1}{4} p_1(\tilde{\mathcal{B}}^* \to \mathcal{B}^*) \in H^4(\mathcal{B}^*;\mathbb{Q})$$

Prop

$$H^*(\mathcal{B}^*;\mathbb{Q}) \cong \mathbb{Q}[\nu, \mu(\Sigma_1), \dots, \mu(\Sigma_b)] \text{ as rings.}$$

Rem. $$/ : \underset{-\frac{1}{4}p_1(\mathbb{P}^{ad})}{H^4(\mathcal{B}^* \times X)} \times H_0(X) \longrightarrow H^4(\mathcal{B}^*)$$

$$\mu_4 : H_0(X) \longrightarrow H^4(\mathcal{B}^*)$$
$$\alpha \longmapsto -\tfrac{1}{4} p_1(\mathbb{P}^{ad})/\alpha$$

$H_0(X) = \mathbb{Q}[x_0]$ & $\mu_4[x_0] = -\frac{1}{4} p_1(\mathbb{P}^{ad})/[x_0] = \nu$

3. The Donaldson invariant

Assume first $\mathrm{vdim}\, \mathcal{M}_k = 8k - 3(b^+ - b_1 + 1) = 0$

<u>Assume</u>

$\pi_1(X) = \{1\} \Rightarrow b_1 = 0$

$b_2^+ \geq 2$

Want:

(i) $\mathcal{M}_k$ contains no reducible solutions

(ii) $\mathcal{M}_k$ is a finite set of pts cut out transversally

(iii) Can attach signs ± 1 to each pt in $\mathcal{M}_k$

(i): holds for a generic metric on X, since $b_2^+ \geq 2$

(ii): $\bar{\mathcal{M}}_k = \mathcal{M}_k \cup (\mathcal{M}_{k-1} \cup X) \cup \dots$

$$\mathrm{vdim}\, \mathcal{M}_{k-1} = 8(k-1) - 3(b^+ + 1) = \underbrace{8k - 3(b^+ + 1)}_{\overset{\shortparallel}{0}} - 8 < 0$$

For <u>generic</u> g, $\mathcal{M}_{k-1} = \emptyset$

$\Rightarrow \bar{\mathcal{M}}_k = \mathcal{M}_k$ cmpt

g generic $\Rightarrow \mathcal{M}_k$ is cut out transversally

(iii) Roughly, $\varepsilon([A]) := \mathrm{sign}\, \underbrace{\det(d_A^+ + d_A^*)}_{\text{is } \underline{\text{not}} \text{ well-defined}}$

However, if A_0, A_1 are two solutions, we can define $\varepsilon(A_0, A_1) \in \{\pm 1\}$ s.t.

"$\mathrm{sign} \det(d_{A_1}^+ + d_{A_1}^*) = \varepsilon(A_0, A_1)\, \mathrm{sign} \det(d_{A_0}^+ + d_{A_0}^*)$"

so [illegible] A is [illegible] overall sign.

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$$\boxed{q - \sum_{[A]\in\mathcal{M}_k} -(A) = \mathbb{Z}}$$

The Donaldson invariant

More generally, ~~if vdim $\mathcal{M}_k > 0$, define~~ assume $2d = \text{vdim}\,\mathcal{M}_k > 0$, $d \in \mathbb{N}$

"Define"

$$q_k : S^k(H_2(X;\mathbb{Z})) \longrightarrow \mathbb{Z}$$

$$q_k(\Sigma_1, \ldots, \Sigma_d) = \langle \mu(\Sigma_1) \cup \ldots \cup \mu(\Sigma_d), [\mathcal{M}_k] \rangle$$

Main problem: $\mathcal{M}_k$ is noncompact.

Main property: Naturality

If $f : X \to Y$ orientation preserving diffeo,

$$q_k(f_*\Sigma_1, \ldots, f_*\Sigma_d) = q_k(\Sigma_1, \ldots, \Sigma_d)$$

Pidstrygach . 15. 06. Göttingen

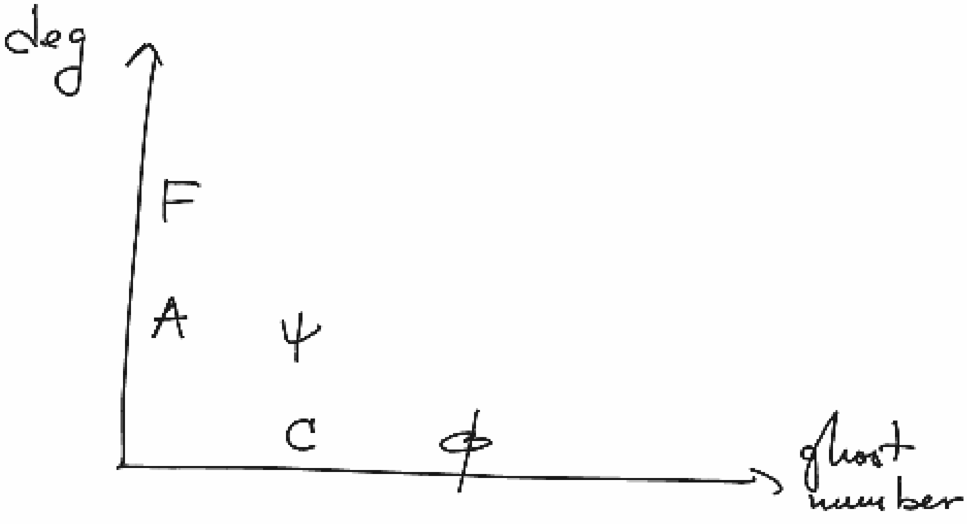
Donaldson polynomials & Observables

BRST A, ψ, ϕ as in the Cartan model

$c \in Lie(\mathcal{G}) = \Omega^0(P;\mathfrak{g})^G$, $\deg c = 1$

$$\left.\begin{aligned} d_B A &= \psi - D_A c \\ d_B \psi &= [\psi, c] - D_A \phi \\ d_B c &= \phi - \tfrac{1}{2}[c,c] \\ d_B \phi &= -[c,\phi] \end{aligned}\right\} (*)$$

$$\begin{tikzcd} \mathcal{E}^+ = \Omega^2_+(P;\mathfrak{g}) \arrow[d] \\ \mathcal{A}^* \arrow[u, dashed, bend left, "\mathcal{F}"] \arrow[d] \\ \mathcal{M} \subset \mathcal{A}^*/\mathcal{G} = \mathcal{B}^* \end{tikzcd}$$



Consider $\mathbb{A} = A + c$ as a new connection

$W(\mathfrak{g}^\vee) \longrightarrow \Omega^\bullet(P)$

$\theta^a \longmapsto$ comp. of a conn

$\mu^a \longmapsto$ comp. of curvature

$F_{\mathbb{A}} = F_A + \psi + \varphi$

$$\begin{cases} (d+d_B)\,\mathbb{A} = \mathbb{F} - \frac{1}{2}[\mathbb{A},\mathbb{A}] & \text{defn of curv} \\ (d+d_B)\,\mathbb{F} = [\mathbb{A},\mathbb{F}] & \text{Bianchi identity} \end{cases}$$

$\Longleftrightarrow (*)$

$$\begin{tikzcd} \mathbb{P} \arrow[d] \\ \mathcal{B} \times X \end{tikzcd}$$

$$c_2(\mathbb{P}) = \nu \otimes 1 + \sum_{\Sigma_i \in H_2(X;\mathbb{Z})} \mu_{\Sigma_i} \otimes [\Sigma_i]^{PD} + 1 \otimes c_2(\mathbb{P}_{Vel}$$

There is a conn $A + C$ on the principal bundle $\mathbb{P}$, which is $A + C \in \Omega^1(\mathcal{A} \times P, Lie(G) \oplus \mathfrak{g}$

The curvature is $\phi + \psi + F_A$

Andreas Ikkert The Donaldson-Witten partition function.

QFT-heuristics in $N=2$ SUSY-YM.

Def Corr. function

$$\langle \mathcal{F} \rangle = \langle \mathcal{F} \rangle_T = \int [dA \ldots] e^{S_T} \mathcal{F}$$

for

$$S_T = -\frac{1}{g_0^2} \int_X (-\mathrm{tr}\, F^+ \wedge F^+ + \ldots)$$

$$= -\frac{1}{g_0^2} \{Q, \Psi\}$$

Lemma a) $\langle \{Q, \mathcal{O}\} \rangle = 0$ for every field expressio

b) $\delta_{g_0^2} \langle \mathcal{O} \rangle = \langle \delta_{g_0^2} \mathcal{O} \rangle$

c) $\propto \{Q, \mathcal{O}\} \Rightarrow \delta_{g_{\mu\nu}}\langle \mathcal{O}\rangle = \langle \delta_{g_{\mu\nu}} \mathcal{O}\rangle$

When is $\langle \mathcal{O}\rangle$ invariant (under diffeomorphisms)?

$$\left.\begin{array}{ll}(i) & \{Q, \mathcal{O}\} = 0 \\ (ii) & \mathcal{O} \neq \mathcal{O}(g_{\mu\nu})\end{array}\right\} \Rightarrow \delta_{g_{\mu\nu}}\langle \mathcal{O}\rangle = 0.$$

(iii) If $\langle \mathcal{O}\rangle \neq 0 \Rightarrow \mathcal{O} \neq \{Q, C\}$ for some C

This is satisfied by $\mathcal{O}^{(0)}(p) = \mathrm{Tr}(\phi^2(p))$

Lemma $x_1 \longrightarrow x_2$ location change is Q-exact

$$\frac{\partial}{\partial x^\mu}\mathcal{O}^{(0)}(x) = 2i\{Q, \mathrm{Tr}\, \phi(x)\, \psi_\mu(x)\}$$

$$\mathcal{O}^{(0)}(p_1) - \mathcal{O}^{(0)}(p_2) = \int_\gamma \underbrace{\frac{\partial \mathcal{O}^{(0)}}{\partial x^\mu} dx^\mu}_{d\mathcal{O}}$$

$$= i\left\{Q, \underbrace{\int \mathrm{Tr}(\phi\psi_\mu)\, dx^\mu}_{\mathcal{O}^{(1)}}\right\}$$

Consider $d\mathcal{O}^{(i)} = i\{Q, \mathcal{O}^{(i+1)}\}$ $i = 0, \dots, 3$

$0 = i\{Q, \mathcal{O}^{(4)}\}$, $d\mathcal{O}^{(4)} = 0$

Explicitly,

$$\mathcal{O}^{(1)} = \mathrm{Tr}(\phi \wedge \psi)$$

$$\mathcal{O}^{(2)} = \mathrm{Tr}(\psi \wedge \psi + 2i\phi \wedge F)$$

$$\mathcal{O}^{(3)} = 2i\, \mathrm{Tr}(\psi \wedge F), \quad \mathcal{O}^{(4)} = -\mathrm{Tr}(F \wedge F)$$

For $\Sigma_i \in H_i(X)$ set $\mathcal{O}(\Sigma_i) := \int_{\Sigma_i} \mathcal{O}^{(i)}$

Fact a) $\mathcal{O}(\Sigma_i)$ are Q-closed

b) $gh \,\#\, \mathcal{O}^{(j)} = 4-j$

c) $\delta_{g_{\mu\nu}} \mathcal{O}(\Sigma_i) = 0$

Thm For $k_1, \dots, k_r \in \{0, \dots, 4\}$ and $\Sigma_{k_j} \in H_{k_j}(X)$

$$\delta_{g_{\mu\nu}} \langle \mathcal{O}(\Sigma_{k_1}) \cdots \mathcal{O}(\Sigma_{k_r}) \rangle = 0 .$$

In the weak coupling limit (g_0 small) the path integral is dominated by classical minima (tr $F^+ \wedge F^+$ pos. definite) $\Longleftrightarrow F^+ = 0$.

- path integral can be evaluated in $\mathcal{M}_{ASD}(P)$.

Lemma $n = \dim \mathcal{M}_{ASD}(P)$, in the wcl we have

a) $n = \#$ of 0 modes of ψ $- \underbrace{\# \text{ of } 0 \text{ modes } (\chi, \nu}_{=0, \; G = SU(2)}$

b) $\langle \mathcal{O} \rangle \neq 0 \Longleftrightarrow \langle \mathcal{O} \rangle = \langle \underbrace{\Theta_{i_1 \dots i_n}}_{\substack{n \text{ form} \\ \text{on the space} \\ \text{of connections}}} \; \underbrace{\psi^{i_1} \dots \psi^{i_n}}_{0 \text{ modes}} \rangle$

$$\Rightarrow [DA \dots] \to dA_1 \dots dA_n d\psi_1 \dots d\psi_n$$

Thm For $k_1, \dots, k_r \in \{0, \dots, 4\}$ and $\Sigma_{k_j} \in H_{k_j}(X)$ (so that $n = \sum (4 - k_j)$) there exists a k_j-form Θ_{Σ_j} on $\mathcal{M}$ s.t.

$$\langle \mathcal{O}(\Sigma_{k_1}) \dots \mathcal{O}(\Sigma_{k_n}) \rangle = \int_{\mathcal{M}} \Theta_{\Sigma_1} \wedge \dots \wedge \Theta_{\Sigma_k}$$

Proof Sketch

Problem: $\mathcal{O}^{(0)}$, $\mathcal{O}^{(1)}$, $\mathcal{O}^{(2)}$ contain ϕ! Eliminate ϕ by $\langle \phi \rangle$, which are expressable in terms of other fields

$$\Theta_{\Sigma_0} = \mathrm{Tr} \langle \phi(\Sigma_0) \rangle$$

$$\Theta_{\Sigma_1} = \alpha \int_{\Sigma} \mathrm{Tr} (\langle \phi \rangle \wedge \psi)$$

Witten shows

$$\langle \phi^a(x) \rangle = -i \int_X dy \underbrace{G^{ab}(x,y)}_{\text{Green's function}} [\psi_\alpha(y), \psi^\alpha(x)]$$

$$\Theta_{\Sigma_2} = \int_{\Sigma_2} \mathrm{Tr} (\psi \wedge \psi + 2i \langle \phi \rangle \wedge F)$$

$$\Theta_{\Sigma_3} = \alpha i \int_{\Sigma_3} \mathrm{Tr} (\psi \wedge F)$$

$$\Theta_{\Sigma_4} = -\zeta \int_X \mathrm{Tr} (F \wedge F) \quad , \zeta \in \mathbb{R}$$

---

2. Donaldson-Witten partition function

$$\mathcal{O} := \mathcal{O}(\Sigma_0) \, , \, \mathcal{O}(\Sigma) := \mathcal{O}(\Sigma_2)$$

Def For $p \in H_0(X)$ & $\Sigma \in H_2(X)$

$$Z_{DW}(p, \Sigma) := \overset{\text{instanton sector}}{\sum_m} \langle e^{p\mathcal{O} + \mathcal{O}(\Sigma)} \rangle = \sum_{l,z} \frac{p^l}{l!\, z!} \sum_m \langle \mathcal{O}^l \mathcal{O}(\Sigma)^z \rangle_m$$

Note For $\mathcal{O}^{l}\mathcal{O}(\Sigma)^{r}$ only the instanton sector with $\dim \mathcal{M}_m = 4l + 2r$ contributes **one** term

Wittens claim: $\mu_d(\underset{\Sigma_0}{\overset{\shortparallel}{\rho}}) \longleftrightarrow \mathcal{O}_\rho$ and

$\mu_d(\Sigma) \longleftrightarrow \mathcal{O}_\Sigma$ s.t.

$$Z_{DW}(p,s) := \sum_{l,r \geq 0} \frac{p^{\nu}}{l!\,r!} \underbrace{\langle \mathcal{O}^{l}\mathcal{O}(\Sigma)^{r} \rangle}_{P_W(\mathcal{O}^{l}\mathcal{O}^{r})}$$

$$= \frac{1}{2}\Lambda^{-3/4(\chi+\sigma)} \sum_{l,r\geq 0} \frac{p^{l}s^{r}}{l!\,r!}\Lambda^{2l+|r|} P_D(\rho^{l}\Sigma^{r})$$

$\rho \in H_0(X)$, $p \in H^0(X)$ dual element

$\mathcal{O}(\Sigma) = \sum_{r} s^{\mu}\mathcal{O}(\Sigma_r)$, $s^{\mu} \in H^2(X)$

$s^r = \prod_{\mu} (s^{\mu})^{r_\mu}$ with $r = \sum r_\mu$