Algebraic Topology

Lecture notes

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This is a draft. In particular, most of figures are missing. If you spot a mistake, please let me know.

TODO:
- Add an appendix on chain complexes.

Contents

Chapter 1

Introduction

The main purpose of this chapter is to explain informally the main ideas which will be developed in details later. In particular, the proofs are rather sketchy stressing main ideas only. More precise statements and proofs will be given in the subsequent chapters.

1.1 Differential forms, the theorems of Green and Stokes

Let $\omega = P(x,y)dx + Q(x,y)dy$ be a 1-form on an open subset $U \subset \mathbb{R}^2$. For example, if $f\colon U \to \mathbb{R}$ is a smooth map, then the differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is a 1-form.

Question 1.1. Under which circumstances does there exist some function f as above such that $\omega = df$?

Clearly, we have the following necessary condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1.2}$$

Proposition 1.3. *If U is convex, then (1.2) is also sufficient.*

Sketch of proof. Theorem of Green $\implies$ For any closed piecewise smooth curve $C \subset U$ without self-intersections we have

$$\int_C (P\,dx + Q\,dy) = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx\,dy = 0, \tag{1.4}$$

where D is the domain bounded by C. Notice that here we use the convexity of U, since otherwise C does not necessarily bound any domain.

Pick any $(x_0, y_0) \in U$. For any $(x,y) \in U$ choose a curve C' connecting (x_0, y_0) and (x,y). Define

$$f(x,y) := \int_{C'} P\,dx + Q\,dy.$$

Property (1.4) guaranties that f does not depend on the choice of C'. □

The following example shows that (1.2) is not sufficient for general U.

Example 1.5. Consider $U = \mathbb{R}^2 \setminus \{0\}$ and

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

If there were some f such that $\omega = df$, then we would have $\int_{S^1} \omega = 0$, where S^1 is the circle (for example, parametrized via $t \mapsto (\cos t, \sin t)$). This is a contradiction, since $\int_{S^1} \omega = 2\pi \neq 0$.

Notice that the proof of Proposition 1.2 does not work here, since the theorem of Green does not apply for (D, ω), where D is the unit disc.

Remark 1.6. One can show that for any closed piecewise smooth curve $C \subset \mathbb{R}^2 \setminus \{0\}$ we have

$$\frac{1}{2\pi} \int_C \Big(-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \Big)$$

is an integer.

Let U be an open subset of $\mathbb{R}^3$ and $\omega = P\,dx + Q\,dy + R\,dz$ be a 1-form. We can also ask whether $\omega = df$ for some $f \colon U \to \mathbb{R}$. Clearly, we have the following necessary condition:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \textit{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{1.7}$$

Proposition 1.8. *If U is convex, then* (1.7) *is also sufficient.*

The proof of this proposition is analogous to the proof of the previous one. Just instead of the theorem of Green we have to use the theorem of Stokes:

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_\Sigma \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\,dy\,dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\,dz\,dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy.$$

Proposition 1.9. *Condition* (1.7) *is also sufficient for* $\mathbb{R}^3 \setminus \{0\}$.

Sketch of proof. Let $C \subset \mathbb{R}^3$ be an arbitrary simple picewise smooth curve without self-intersections. Then there is a picewise smooth surface $\Sigma \subset \mathbb{R}^3$ such that $\partial\Sigma = C$. If $0 \in \Sigma$, a (small) perturbation yields a surface $\Sigma' \subset \mathbb{R}^3 \setminus \{0\}$ such that $\partial\Sigma' = C$. □

For a general U, Condition (1.7) is still insufficient, which is easily seen for the following example: $U = \mathbb{R}^3 \setminus \{z - \text{Axis}\}$ and

$$\omega = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy.$$

From this discussion we can make the following informal conclusion: Condition (1.7) is sufficient as long as U has no “holes” of codimension 2.

1.2 Ansatz of a construction.

Let $X \subset \mathbb{R}^n$ be an arbitrary subset, which is equipped with the induced topology. Define $Z_1(X)$ as a free Abelian group generated by (oriented) closed curves, i.e.,

$$C \in Z_1(X) \implies C = n_1 C_1 + \ldots n_k C_k, \tag{1.10}$$

where $n_j \in \mathbb{Z}$. Define

$$\int_C \omega := \sum n_k \int_{C_k} \omega.$$

Remark 1.11. If C_0 is a closed oriented curve, $2C_0$ can be understood as "running along C_0 twice in the same direction". Similarly, $-C_0$ can be understood as the curve C_0 with the opposite orientation. However, in most cases we treat (1.10) purely formally.

Assume temporarily that X is an *open* subset of $\mathbb{R}^2$. We would like to define an equivalence relation such that

$$C \sim C' \implies \int_C \omega = \int_{C'} \omega$$

holds for all $\omega = P\,dx + Q\,dy$ satisfying (1.2). The theorem of Green (or Stokes in the case $U \subset \mathbb{R}^3$) suggests the following:

$$C \sim C' \quad \Leftrightarrow \quad \exists \text{ a compact surface } \Sigma \text{ such that } \partial\Sigma = C \cup -C'. \tag{1.12}$$

Here C and C' are oriented curves and Σ is an oriented surface such that $\partial\Sigma = C \cup -C'$ as *oriented* curves. This definition also makes sense even in the case when X is not necessarily open.

More generally, a cycle $C = C_1 + \cdots + C_k$ is called *null homologous*, i.e., $C \sim 0$, if and only if

$$\exists \text{ a compact surface } \Sigma \text{ such that } \partial\Sigma = C_1 \cup \cdots \cup C_n.$$

Clearly, Condition (1.12) can be written as $C + (-C') \sim 0$.

Example 1.13. Null homologous cycles on the 2-sphere with 2 points removed (equivalently, $\mathbb{R}^2 \setminus \{0\}$).

Even more generally, each linear combination of null homologous cycles is also declared to be null homologous.

$Z_1(X) \supset B_1(X) = \{\text{null homologous cycles}\}$.

$H_1(X) := Z_1(X)/B_1(X)$ the first homology group of X.

Example 1.14. $H_1(S^2 \setminus \{p, q\}) \cong \mathbb{Z}$.

Problems: Curves C and surfaces Σ can have singularities and self-intersections.

More generally:

- $Z_n(X)$ freely generated by compact oriented n-dimensional "surfaces" without boundary.
- $Z_n(X) \supset B_n(X)$ the subgroup generated by the boundaries of compact oriented $(n+1)$-dimensional "surfaces".
- $H_n(X) := Z_n(X)/B_n(X)$ the nth homology group of X.

In general, we would like to associate to each topological space X a sequence of abelian groups $H_0(X), H_1(X), \dots, H_n(X), \dots$ such that the following holds:

(a) Each continuous map $f\colon X \to Y$ induces a sequence of homomorphisms $f_*\colon H_n(X) \to H_n(Y)$;

(b) $(f \circ g)_* = f_* \circ g_*, \quad id_* = id$.

(c) $H_0(\{pt\}) = \mathbb{Z}$ and $H_n(\{pt\}) = 0$ for all $n \geq 1$.

(d) $H_n(S^n) = \mathbb{Z}$ provided $n \geq 1$ and $H_k(S^n) = 0$ for all $k \geq n+1$ (More generally, for each compact connected oriented manifold M of dimension n the following holds: $H_n(M) = \mathbb{Z}$ and $H_k(M) = 0$ for all $k > n+1$).

(e) $f \simeq g \implies f_* = g_*$.

Here two continuous maps are said to be homotopic ($f \simeq g$), if there exists a continuous map $h\colon X \times [0,1]$, called homotopy, such that the following holds:

$$h|_{X\times 0} = f \qquad \text{and} \qquad h|_{X\times 1} = g.$$

Question 1.15. What does make Properties (a)-(e) interesting?

This question will be answered in the subsequent sections. We finish this section by the following facts, which will be useful below.

Proposition 1.16. *If f is a homeomorphism, then each $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. $id_{H_n} = id_* = (f \circ f^{-1})_* = f_* \circ (f^{-1})_* \implies f_*$ is an isomorphism and $(f_*)^{-1} = (f^{-1})_*$. □

1.3 The theorem of Brouwer

In this section we show that (a)-(e) imply the following famous result.

Theorem 1.17 (Brouwer)**.** *Any continuous map $f\colon B_n \to B_n$ has a fixed point.*

Proof. The proof consists of the following three steps.

Step 1. *For the ball $B_n := \{x \in \mathbb{R}^n \mid |x| \le 1\}$ we have $H_k(B_n) = 0$ for all $k \ge 1$.*

Let $c\colon B_n \to \{0\}$ be the constant map. The map $h(x,t) = tx,\ t \in [0,1]$ is a homotopy between id_B und $\imath \circ c$, where $\imath\colon \{0\} \to B_n$ is the inclusion. Thus, $id = \imath_* \circ c_* \implies H_k(B_n) = 0$ for all $k \ge 1$, since $\operatorname{Im} \imath_* = \{0\}$.

Step 2. *There is no continuous map $g\colon B_n \to \partial B_n = S^{n-1}$ such that $g(x) = x$ holds for all $x \in S^{n-1}$.*

Assume $n = 1$ first. In this case there is no continuous map $g\colon [-1,1] \to \{\pm 1\}$ as in the statement of this step, since the target space $\{\pm 1\}$ is disconnected, whereas the interval $[0,1]$ is connected.

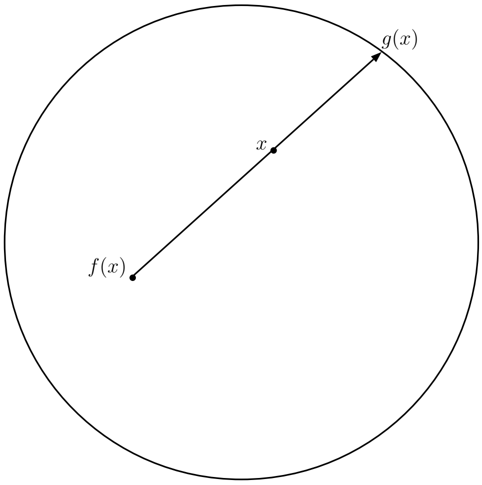
Let us consider now the case $n \ge 2$. Assume there is such $g\colon B_n \to S^{n-1}$. Then we have

$$\begin{aligned} id_{S^{n-1}} = g \circ \imath_{S^{n-1}} &\implies (id_{S^{n-1}})_* = g_* \circ (\imath_{S^{n-1}})_* = 0 \quad \text{on } H_{n-1}(S^{n-1}) \\ &\implies H_{n-1}(S^{n-1}) = 0. \end{aligned}$$

This contradiction proves Step 2.

Step 3. *We prove the theorem of Brower.*

Assume there exists a continuous map $f\colon B_n \to B_n$ without fixed points. Then there also exists a continuous map $g\colon B_n \to S^{n-1}$ such that $g|_{S^{n-1}} = id$:



This contradicts Step 2. ☐

1.4 The degree of a continuous map and the fundamental theorem of algebra

In this section we show that (a)-(e) imply that any non-constant polynomial with complex coefficients has at least one root. This statement is known as the fundamental theorem of algebra.

Thus, pick any $n \geq 1$ and choose a generator $\alpha \in H_n(S^n)$, i.e., an element α such that $H_n(S^n) = \mathbb{Z} \cdot \alpha$.

Definition 1.18. For any continuous map $f\colon S^n \to S^n$ define $\deg(f) \in \mathbb{Z}$ by

$$f_*\alpha = \deg(f)\alpha.$$

The degree of a map does not depend on the choice of a generator, since $f_*(-\alpha) = -f_*\alpha = -\deg(f)\alpha = \deg(f)(-\alpha)$.

Lemma 1.19. *The degree has the following properties:*

(i) $\deg(id) = 1$*;*

(ii) $\deg(f \circ g) = \deg f \cdot \deg g$*;*

(iii) $f \simeq g \implies \deg f = \deg g$*;*

(iv) $\deg(\textit{const. map}) = 0$*.*

☐

Lemma 1.20. *For* $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ *define* $f_n\colon S^1 \to S^1$ *by* $f_n(z) = z^n$*, where* $n \in \mathbb{Z}$*. Then we have*

$$\deg f_n = n.$$

Proof. The curve

$$\alpha\colon [0, 2\pi] \to S^1, \qquad \alpha(t) = \cos t + \sin t\, i = e^{ti},$$

generates $H_1(S^1)$. Since $f_n \circ \alpha(t) = e^{nti} = \cos(nt) + \sin(nt)i$, from the definition of the degree and Remark 1.11 we have $\deg f_n = n$. ☐

Theorem 1.21 (The fundamental theorem of Algebra)**.** *Each non-constant polynomial* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0,\ a_j \in \mathbb{C}$ *has at least one complex root.*

Proof. Identify S^1 with $S^1_r := \{z \in C \mid |z| = r\} \cong S^1$ with the help of the homeomorphism

$$S^1 \to S^1_r, \qquad z \mapsto rz.$$

The proof consists of the following three steps.

Step 1. *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *be a continuous map without zeros. Then for each* $r > 0$ *the map*

$$\frac{f}{|f|}\colon S^1_r \to S^1 \tag{1.22}$$

is homotopic to the constant map.

Indeed, a homotopy can be given explicitly by

$$F(z,t) = \frac{f(tz)}{|f(tz)|}, \qquad z \in S^1,\ t \in [0,r].$$

Step 2. *Let* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$ *be a polynomial without zeros. Then there exists some* $R > 0$ *such that the following holds:* $\forall r \geq R$ *the restriction of* $p/|p|$ *to* S^1_r *is homotopic to* f_n*.*

For all $z \in \mathbb{C}$ such that $|z| \geq 1$ we have

$$\begin{aligned} \left|a_{n-1}z^{n-1} + \dots a_1 z + a_0\right| &\leq |a_{n-1}||z|^{n-1} + \dots + |a_1||z| + |a_0| \\ &\leq n \max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}|z|^{n-1} \end{aligned}$$

Chose R so that $R > n \max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ and $R > 1$. For all $r \geq R$ and all $t \in [0,1]$ the polynomial

$$p_t(z) = z^n + t\big(a_{n-1}z^{n-1} + \dots + a_1 z + a_0\big)$$

has no zeros on S^1_r, since

$$\left|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\right| < Rr^{n-1} \leq r^n, \quad \text{provided} \quad |z| = r.$$

Then

$$P(z,t) = \frac{p_t(z)}{|p_t(z)|}\Big|_{S^1_r}$$

is a homotopy between $p/|p|$ and f_n viewed as a map on S^1_r.

Step 3. *We prove the fundamental theorem of algebra.*

Assume p is a non-constant polynomial without zeros. Denote

$$q_r(z) = \frac{p(z)}{|p(z)|}\Big|_{S^1_r},$$

where $r \geq R$. Step 2 $\implies$ $\deg q_r = n$. Step 1 $\implies$ $\deg q_r = 0$, i.e., $n = 0$. Thus, p is a constant polynomial, which is a contradiction. ∎