Global Analysis

Lecture notes

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This is a draft. If you spot a mistake, please let me know.

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Chapter 1

Introduction

1.1 The de Rham cohomology groups

Let M be a compact manifold of dimension n. Denote by $\Omega^k(M)$ the space of differential k-forms on M. Recall that there exist a unique $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) df is the differential of f if $f \in C^\infty(M) = \Omega^0(M)$;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^q \omega \wedge d\eta$ if $\eta \in \Omega^q(M)$;

(iii) $d^2 = 0$.

The last property simply means that $d(d\omega) = 0$ for each $\omega \in \Omega^k(M)$. This yields the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0 \tag{1.1}$$

Remark 1.2. The map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ depends on k, however this is suppressed in the notations.

Property (iii) means that (1.1) is a complex, that is the kernel of $d : \Omega^k \to \Omega^{k+1}$ contains the image of $d : \Omega^{k-1} \to \Omega^k$ and therefore we can define

$$H^k_{dR}(M) := \frac{\operatorname{Ker}\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\operatorname{Im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)} \tag{1.3}$$

This is called the k^{th} de Rham cohomology group, which is in fact a vector space. The number

$$b_k(M) := \dim H^k_{dR}(M)$$

is called the k^{th} Betti number of M and is an invariant of M. Notice that by the compactness of M we have $b_k(M) < \infty$.

Remark 1.4. While it is by no means obvious from the above description, Betti numbers are topological invariants of M, that is $b_k(M) = b_k(N)$ provided M and N are homeomorphic. In paticular, Betti numbers do not depend on the smooth structure.

Coming back to the de Rham cohomlogy groups, each element in $H^k_{dR}(M)$ is represented by the equivalence class

$$[\omega] = \left\{\omega + d\eta \mid \eta \in \Omega^{k-1}(M)\right\},$$

where $\omega \in \Omega^k(M)$ is closed: $d\omega = 0$. Hence, we may ask the following.

Question 1.5. What is the best representative in $[\omega]$?

Of course, at this point the above question is vague, since the notion of being "the best" is undefined. One possibility to convert this into a precise question is as follows. Just by its definition, the set $[\omega]$ is an affine subspace of $\Omega^k(M)$. We could call an element in $[\omega]$ "the best" if it is the closest one to the origin just as shown schematically on Figure 1.1.

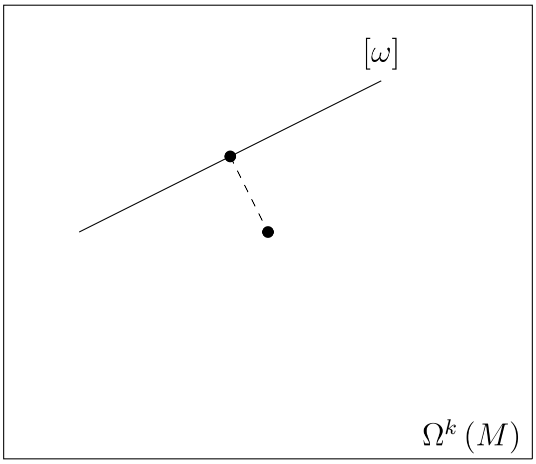

Figure 1.1: A choice of a representative in the de Rham cohomology class.

However, this raises our next question: How do we measure distance in $\Omega^k(M)$? A suitable answer to this question requires a detour, which we do next.

1.2 Some linear algebra

Let V be a vector space of dimension n over $\mathbb{R}$. Recall that for each basis $e = (e_1, \dots, e_n)$ of V there exist a unique basis $e^* = (e_1^*, \dots, e_n^*)$ of the dual vector space V^* such that

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

e^* is called the dual basis to e.

Assume $\langle \cdot, \cdot \rangle$ is a scalar product on V. If $e = (e_1, \dots, e_n)$ is an orthonormal basis of V, that is

$$\langle e_i, e_j \rangle = \delta_{ij},$$

then the dual basis e^* is given explicitly by

$$e_i^* := \langle e_i, \cdot \rangle \qquad \Longleftrightarrow \qquad e_i^*(v) = \langle e_i, v \rangle \quad \text{for } v \in V.$$

Then V^* has a unique scalar product such that $e^* = (e_1^*, \dots, e_n^*)$ is an orthonormal basis. Explicitly, for $\xi, \eta \in V^*$ define

$$\begin{matrix} \xi_i := \xi(e_i) \\ \eta_i := \eta(e_i) \end{matrix} \qquad \Longleftrightarrow \qquad \begin{matrix} \xi = \sum \xi_i \, e_i^* \\ \eta = \sum \eta_j \, e_j^* \end{matrix}$$

$$\Longrightarrow \langle \xi, \eta \rangle = \sum_{i=1}^{n} \xi_i \eta_i = \sum_{i=1}^{n} \xi(e_i)\, \eta(e_i).$$

To sum up, for any scalar product on V there exists a unique scalar product on V^* such that the dual basis of an orthonormal basis is itself orthonormal.

More generally, any basis e of V yields a basis of $\Lambda^k V^*$. Explicitly,

$$\Lambda^k e := \left\{ e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n \right\} \tag{1.6}$$

is a basis of $\Lambda^k V^*$ consisting of

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!k!}$$

elements. Just like in the case of $V^* = \Lambda^1 V^*$, we can define a scalar product on each $\Lambda^k V^*$ by declaring (1.6) to be an orthonormal basis.

Recall that any two bases e and f of V are related by a change-of-basis matrix A. This means

$$f = e \cdot A \qquad \Longleftrightarrow \qquad f_i = \sum_{j=1}^{n} a_{ij}\, e_j$$

Then e and f are said to be *cooriented*, if $\det A > 0$. It is easy to check that

$$e \sim f \qquad \equiv \qquad e \text{ and } f \text{ are cooriented}$$

yields an equivalence relation on the set of all bases of V. Moreover, there are exactly two equivalence classes represented by e and $\bar{e} = (-e_1, e_2, \ldots, e_n)$.

Definition 1.7. An orientation on V is a choice of an equivalence class of bases of V. Any basis in the chosen class is said to be positively oriented and any basis, which does not belong to the selected class is said to be negatively oriented.

Example 1.8. For $\mathbb{R}^n$ the class of the standard basis is called the standard orientation of $\mathbb{R}^n$.

Example 1.9. Any $\omega \in \Lambda^n V^*$, $\omega \neq 0$, determines an orientation of V by the rule: e is positively oriented if and only if

$$\omega(e_1, \ldots, e_n) > 0.$$

For example, if $e^* = (e_1^*, \ldots, e_n^*)$ is the dual basis to the standard one, then

$$\omega := e_1^* \wedge \ldots \wedge e_n^* \tag{1.10}$$

determines the standard orientation of $\mathbb{R}^n$.

Definition 1.11. Let V be an oriented Euclidean vector space of dimension n. An n-form ω is said to be the Euclidean volume form, if

$$\omega(e_1, \ldots, e_n) = 1 \tag{1.12}$$

holds for any positively oriented orthonormal basis e of V.

For example, in the case $V = \mathbb{R}^n$, which is equipped with the standard scalar product and orientation, (1.10) is the Euclidean volume form.

Example 1.13. Show that any oriented Euclidean vector space admits a unique Euclidean volume form. This is sometimes denoted by vol.

Proposition 1.14. *There is a unique linear map*

$$* : \Lambda^k V^* \to \Lambda^{n-k} V^* \qquad \textit{satisfying} \qquad \xi \wedge * \eta = \langle \xi, \eta \rangle \, vol \tag{1.15}$$

for all ξ, $\eta \in \Lambda^k V^*$.

Proof (Sketch). Let e be a positively oriented orthonormal basis of V. Set

$$\eta := e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} \in \Lambda^k e^*$$

Then assuming that $*$ exists, for

$$\xi = \sum \xi_{j_1 \ldots j_k} \, e^*_{j_1} \wedge \ldots \wedge e^*_{j_k}$$

we must have

$$\xi \wedge * \eta = \langle \xi, \eta \rangle \, vol = \xi_{j_1 \ldots j_k} \cdot e^*_1 \wedge \ldots \wedge e^*_n.$$

This yields

$$* \, \eta = * \left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} \right) = \varepsilon \cdot e^*_{l_1} \wedge \ldots \wedge e^*_{l_{n-k}}, \tag{1.16}$$

where $\varepsilon \in \{\pm 1\}$ and $1 \leq l_1 < \ldots < l_{n-k} \leq n$ consists of those integers in the interval $[1, n]$ which are complementary to $\{i_1, \ldots, i_k\}$.

For example, if $n = 6$ and $\eta = e_2 \wedge e_4$, then

$$* \, (e^*_2 \wedge e^*_4) = \varepsilon \, e^*_1 \wedge e^*_3 \wedge e^*_5 \wedge e^*_6 \, .$$

To determine ε, we compute

$$e^*_2 \wedge e^*_4 \wedge \varepsilon \, (e^*_1 \wedge e^*_3 \wedge e^*_5 \wedge e^*_6) = -\varepsilon \, vol \, ,$$
$$e^*_2 \wedge e^*_4 \wedge * \, (e^*_2 \wedge e^*_4) = \| \, e^*_2 \wedge e^*_4 \, \|^2 \cdot vol = vol,$$

which yields $\varepsilon = -1$ so that we finally obtain

$$* \, (e^*_2 \wedge e^*_4) = -e^*_1 \wedge e^*_3 \wedge e^*_5 \wedge e^*_6.$$

In general, ε is the signature of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Thus, (1.16) defines $*$ on the elements of the basis $\Lambda^k e$. This yields a unique linear map $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$, which satisfies (1.15). ☐

The map $*$ defined in the above proposition is called *the Hodge operator*.

Remark 1.17. It follows from the proof of the above proposition that the Hodge operator satisfies

$$* * \omega = (-1)^{k(n-k)} \, \omega \qquad \text{for all } \omega \in \Lambda^k V^*. \tag{1.18}$$

1.3 Riemannian metrics

Definition 1.19. A Riemannian metric g on M is a smooth section of $T^*M \otimes T^*M$ such that the following holds:

(i) g is symmetric, that is

$$g\,(v, w) = g\,(w, v) \qquad \text{for all } v, w \in T_m M \text{ and any } m \in M;$$

(ii) g is positive-definite, that is

$$g\left(v,v\right)>0 \qquad \text{for all } v\in T_mM, v\neq 0, \text{ and any } m\in M.$$

In other words, g is a family $\{g_m \mid m\in M\}$ of scalar products on each T_mM and g_m depends smoothly on m. In particular, each T_mM is an Euclidean vector space. Hence, each $\Lambda^k T_m^* M$ is also an Euclidean vector space.

An orientation of a manifold M is (informally speaking) a choice of coherent orientations of T_mM for each $m\in M$. More formally, we have the following.

Definition 1.20. A manifold M of dimension n is said to be orientable, if there exists $\omega\in\Omega^n\left(M\right)$ such that $\omega_m\neq 0$ for all $m\in M$.

By Example 1.9, for each $m\in M$ the n-form on T_mM determines a class of positively oriented bases of T_mM, that is an orientation. Notice that for any function f, which is positive everywhere, the forms ω and $f\cdot\omega$ determine the same orientation on each T_mM .

Albeit not all manifolds are orientable, orientability is a mild restriction. In particular, for any connected non-orientable manifold M there exists a unique double covering $M_2\to M$, which is orientable. The reader may find more information on this in [?].

Definition 1.21. An orientation of an n-manifold M is a class of n-forms $[\![\omega]\!]$, where

- ω is a nowhere vanishing n-form on M.
- $\omega_1\sim\omega_2$ if and only if there exists an everywhere positive function f such that $\omega_2=f\cdot\omega_1$.

Notice that $[\![\cdot]\!]$ above is *not* the de Rham cohomology class.

Just like in the preceding section, a differential n-form ω on M is said to be a Riemannian volume form, if

$$\omega_m\left(e_1,\dots,e_n\right)=1 \tag{1.22}$$

holds for any $m\in M$ and any oriented orthonormal basis $\left(e_1,\dots,e_n\right)$ of T_mM. Property 1.22 determines a Riemannian volume form uniquely. This volume form is denoted by vol.

Thus, by the preceding subsection, a Riemannian metric and orientation on M induce for each $k\leq n$ the Hodge operator $*\colon\Omega^k\left(M\right)\to\Omega^{n-k}\left(M\right)$ such that

$$\omega\wedge *\eta=\langle\omega,\eta\rangle\, vol \qquad \text{and} \qquad **\omega=\left(-1\right)^{k(n-k)}\omega$$

holds for all $\omega,\eta\in\Omega^k\left(M\right)$.

1.4 Harmonic forms

Let us come back to the original question about "the best" representatives of the de Rham cohomology classes. Thus, we define the L^2-scalar product on each $\Omega^k\left(M\right)$ by setting

$$\langle\omega,\eta\rangle_{L^2}:=\int_M\langle\omega_m,\eta_m\rangle\, vol_m=\int_M\omega\wedge *\eta.$$

With this at hand, we could call an element $\widehat{\omega}=\omega+d\eta\in[\omega]$ "the best", if $\widehat{\omega}$ minimizes the distance to the origin, that is if

$$\inf_{\eta\in\Omega^{k-1}(M)}\|\,\omega+d\eta\,\|_{L^2}^2=\|\,\widehat{\omega}\,\|_{L^2}. \tag{1.23}$$

Then, if (1.23) holds, for any $\eta \in \Omega^{k-1}(M)$ we must have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \| \,\widehat{\omega} + t\, d\eta\, \|_{L^2}^2 \\ &= \frac{d}{dt}\Big|_{t=0} \left(\| \,\widehat{\omega}\, \|_{L^2}^2 + 2t\langle \widehat{\omega}, d\eta\rangle_{L^2} + t^2 \| \,d\eta\, \|_{L^2}^2\right) \\ &= 2\langle \widehat{\omega}, d\eta\rangle_{L^2} \end{aligned} \tag{1.24}$$

Proposition 1.25. *Denote* $d^* \coloneqq (-1)^{nk+1} * d * \colon \Omega^{k+1}(M) \to \Omega^k(M)$.

(i) d^* *is the formal adjoint of* d*, that is*

$$\langle d\omega, \eta\rangle_{L^2} = \langle \omega, d^*\eta\rangle_{L^2} \qquad \textit{for all } \omega \in \Omega^k(M) \textit{ and } \eta \in \Omega^{k+1}(M). \tag{1.26}$$

(ii) (1.24) *is equivalent to*

$$d^*\widehat{\omega} = 0. \tag{1.27}$$

Proof. Notice first that (1.15) and (1.18) imply the equality

$$\omega \wedge \zeta = (-1)^{k(n-k)} \langle \omega, * \zeta\rangle vol \qquad \text{for all } \omega \in \Omega^k(M) \text{ and } \zeta \in \Omega^{n-k}(M).$$

Using this, we obtain

$$\begin{aligned} \langle \omega, d^*\eta\rangle_{L^2} &= (-1)^{kn+1} \int_M \langle \omega, * d * \eta\rangle\, vol \\ &= (-1)^{kn+1+k(n-k)} \int_M \omega \wedge d * \eta \\ &= (-1)^{1-k^2} \int_M (-1)^k \left(d(\omega \wedge * \eta) - d\omega \wedge * \eta\right) \end{aligned}$$

Here the last equality follows from the Leibnitz rule:

$$d(\omega \wedge \zeta) = d\omega \wedge \zeta + (-1)^k \omega \wedge d\zeta$$

provided $\omega \in \Omega^k(M)$ and $\zeta \in \Omega^l(M)$. Hence, by Stokes' theorem, we obtain

$$\begin{aligned} \langle \omega, d^*\eta\rangle_{L^2} &= (-1)^{k-k^2+1} \int_M d\omega \wedge * \eta \\ &= (-1)^{k-k^2} \langle d\omega, \eta\rangle_{L^2} \end{aligned}$$

By noticing that k^2 is even/odd if and only if k is even/odd, we arrive finally at (1.26).

To prove (ii), notice that

$$d^*\widehat{\omega} = 0 \qquad \Longrightarrow \qquad 0 = \langle d^*\widehat{\omega}, \eta\rangle_{L^2} = \langle \widehat{\omega}, d\eta\rangle_{L^2}.$$

Conversely, setting $\eta = d^*\widehat{\omega}$ in (1.24), we obtain

$$0 = \langle \widehat{\omega}, dd^*\widehat{\omega}\rangle_{L^2} = \langle d^*\widehat{\omega}, d^*\widehat{\omega}\rangle_{L^2} = \| \,d^*\omega\, \|_{L^2}^2 \qquad \Longrightarrow \qquad d^*\widehat{\omega} = 0.$$

□

Notice that (1.27) is nothing else but the Euler-Lagrange equation for the functional

$$f\colon \left\{\omega + d\eta \mid \eta \in \Omega^{k-1}(M)\right\} \longrightarrow \mathbb{R}, \qquad f(\omega + d\eta) = \|\, \omega + d\eta \,\|_{L^2}^2 \tag{1.28}$$

Definition 1.29. The map

$$\Delta = dd^* + d^*d\colon \Omega^k(M) \to \Omega^k(M)$$

is called *the Laplace operator* (or, simply, *the Laplacian*). A k-form ω such that $\Delta\omega = 0$ is called *harmonic*.

Proposition 1.30. *A k-form ω is harmonic if and only if*

$$d\omega = 0 \qquad \textit{and} \qquad d^*\omega = 0. \tag{1.31}$$

Proof. If (1.31) holds, then ω is clearly harmonic. To show the converse, consider

$$0 = \langle \Delta\omega, \omega\rangle_{L^2} = \langle dd^*\omega, \omega\rangle_{L^2} + \langle d^*d\omega, \omega\rangle = \|\, d^*\omega \,\|_{L^2}^2 + \|\, d\omega \,\|_{L^2}^2.$$

Since both summands are non-negative, we obtain (1.31). □

Theorem 1.32. *If a minimizer $\widehat{\omega}$ of (1.28) exists, then $\widehat{\omega}$ is harmonic. Moreover, if $\widehat{\omega}$ exists, then it is unique.*

Proof. Assume $\widehat{\omega}$ exists. Since $\widehat{\omega} = \omega + d\eta$ for some $\eta \in \Omega^{k-1}(M)$, we have

$$d\widehat{\omega} = d\omega + d(d\eta) = 0 + 0.$$

Combining this with (1.27), we obtain by Proposition 1.30, that $\widehat{\omega}$ is harmonic.

Furthermore, let $\widehat{\omega} = \omega + d\eta$ and $\widehat{\widehat{\omega}} = \omega + d\zeta$ be two harmonic representatives of $[\omega]$. Then

$$0 = d^*\widehat{\omega} - d^*\widehat{\widehat{\omega}} = d^*\left(\widehat{\omega} - \widehat{\widehat{\omega}}\right) = d^*d(\eta - \zeta).$$

Denoting temporarily $\xi \coloneqq \eta - \zeta$, we obtain

$$0 = \langle d^*d\xi, \xi\rangle_{L^2} = \langle d\xi, d\xi\rangle = \|\, d\xi \,\|_{L^2}^2 \quad \Longrightarrow \quad d\xi = 0 \quad \Longrightarrow \quad d\eta = d\zeta \quad \Longrightarrow \quad \widehat{\omega} = \widehat{\widehat{\omega}}.$$

This proves the uniquesness. □

Our aim is to prove the following.

Theorem 1.33. *(Hodge) Each de Rham cohomology class is represented by a unique harmonic form.*

Notice that since we have already proved the uniqueness, it is the existence, which remains to be proved. It turns out that this is somewhat harder and requires certain technology, which we will consider first.

Notice that for any oriented Riemannian manifold, the Laplacian on $\Omega^0(M) = C^\infty(M)$ is given by

$$\Delta f = d^*df = - * d * df.$$

Example 1.34. Consider the case $M = \mathbb{R}^3$ equipped with the standard Euclidean matric. If (x, y, z) are coordinates on $\mathbb{R}^3$, then

$$* df = * \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) = \frac{\partial f}{\partial x} dy \wedge dz - \frac{\partial f}{\partial y} dx \wedge dz + \frac{\partial f}{\partial z} dx \wedge dy.$$

Hence,

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz \qquad \Longrightarrow \qquad \Delta f = - \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right).$$

Sometimes this is called "the non-negative Laplacian", since its eigenvalues are non-negative.

More generally, if the Riemannian metric on $\mathbb{R}^n$ is given by a positive-definite matrix (g_{ij}), where $g_{ij} = g_{ij}(x)$, then the corresponding Laplacian on functions is given explicitly by

$$\Delta f = - \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|}\, g^{ij} \partial_j f \right), \tag{1.35}$$

where $|g| = |\det(g_{ij})|$ and $(g^{ij}) = (g_{ij})^{-1}$. This is sometimes called the Laplace-Beltrami operator.

Let M be a closed oriented Riemannian manifold.

Proposition 1.36 (Green's identity)**.** *For any $\omega, \eta \in \Omega^k(M)$ we have*

$$\langle \Delta\omega, \eta \rangle_{L^2} = \langle \omega, \Delta\eta \rangle_{L^2} = \langle d\omega, d\eta \rangle_{L^2} + \langle d^*\omega, d^*\eta \rangle.$$

Proof. By Proposition 1.25 *(i)*, we have

$$\langle dd^*\omega + d^*d\omega, \eta \rangle = \langle d^*\omega, d^*\eta \rangle + \langle d\omega, d\eta \rangle.$$

By the same token,

$$\langle \omega, dd^*\eta + d^*d\eta \rangle = \langle d^*\omega, d^*\eta \rangle + \langle d\omega, d\eta \rangle,$$

which yields the claim of this proposition. □

Corollary 1.37. *On a closed connected manifold any harmonic function is constant.*

Proof. If $f \in C^\infty(M)$ is harmonic, by Green's identity (with $\omega = \eta = f$) we obtain

$$0 = \langle \Delta f, f \rangle = \| df \|^2_{L^2} \qquad \Longrightarrow \qquad df = 0 \qquad \Longrightarrow \qquad f \text{ is constant.}$$

□

The Hodge theorem follows from the following more general result, which is also attributed to Hodge.

Theorem 1.38. *(Hodge) Let M be a closed oriented Riemannian manifold and $\eta \in \Omega^k(M)$. The equation $\Delta\omega = \eta$ has a solution if and only if*

$$\langle \eta, \omega_0 \rangle_{L^2} = 0 \tag{1.39}$$

for any harmonic k-form ω_0.

It is easy to see that (1.39) is a necessary condition. Indeed, if there is $\omega \in \Omega^k(M)$ such that $\Delta\omega = \eta$, then Green's identity yields

$$\langle \eta, \omega_0 \rangle_{L^2} = \langle \Delta\omega, \omega_0 \rangle = \langle \omega, \Delta\omega_0 \rangle = 0$$

for any harmonic k-form ω_0.

The proof that (1.39) is a sufficient condition will be given later. Taking this as granted for now, from Corollary 1.37 we obtain the following.

Corollary 1.40. *Let M be a closed oriented Riemannian manifold. The equation*

$$\Delta f = h, \qquad\qquad f, h \in C^\infty(M)$$

has a solution if and only if

$$\int_M h \cdot vol = 0.$$

Let me also show that Theorem 1.38 implies that any de Rham cohomolgy class is represented by a harmonic form. To this end, pick any closed k-form ω. Then by Proposition 1.30, $\omega + d\eta$ is harmonic if and only if

$$\begin{matrix} d(\omega + d\eta) = 0 \\ d^*(\omega + d\eta) = 0 \end{matrix} \qquad \Longleftrightarrow \qquad d^*(\omega + d\eta) = 0 \qquad \Longleftrightarrow \qquad d^*d\eta = -d^*\omega. \tag{1.41}$$

Furthermore, consider the equation

$$\Delta\eta = -d^*\omega. \tag{1.42}$$

If η_0 is any harmonic $(k-1)$-form, then

$$\langle d^*\omega, \eta_0 \rangle_{L^2} = \langle \omega, d\eta_0 \rangle = 0,$$

since η_0 is closed. Hence, Theorem 1.38 guarantees that (1.42) has a solution η. We have

$$\begin{aligned} \langle dd^*\eta, \Delta\eta \rangle_{L^2} &= \langle dd^*\eta, dd^*\eta \rangle_{L^2} + \langle dd^*\eta, d^*d\eta \rangle_{L^2} \\ &= \| dd^*\eta \|_{L^2}^2 + \langle d^2 d^*\eta, d\eta \rangle_{L^2} \\ &= \| dd^*\eta \|_{L^2}^2 . \end{aligned}$$

However, using (1.42) we obtain

$$\langle dd^*\eta, \Delta\eta \rangle_{L^2} = -\langle dd^*\eta, d^*\omega \rangle_{L^2} = -\langle d^2 d^*\eta, \omega \rangle_{L^2} = 0.$$

Hence, if η is a solution of (1.42), then in fact η solves (1.41) so that $\omega + d\eta$ is harmonic indeed.

With these preliminary considerations at hand we proceed by showing that Theorem 1.38 has useful applications, in particular in the theory of Riemann surfaces.

1.5 Riemann surfaces

Definition 1.43. A Riemann surface Σ is a complex manifold of dimension one. In particular, Σ admits an atlas $\{(\mathrm{U}_\alpha, \psi_\alpha) \mid \alpha \in A\}$, where $\psi_\alpha \colon \mathrm{U}_\alpha \longrightarrow \mathbb{C}$, such that each coordinate transformation map

$$\psi_\alpha \circ \psi_\beta^{-1} \colon \mathbb{C} \longrightarrow \mathbb{C}$$

is holomorphic on the domain of its definition.

Writing $z_\alpha = z = x + iy = \psi_\alpha$, we obtain a holomorphic coordinate on Σ (defined on U_α). Notice that

$$x = \operatorname{Re} z \qquad \text{and} \qquad y = \operatorname{Im} z$$

are real coordinates on U_α. In particular, for any $p \in \mathrm{U}_\alpha$ we have a basis

$$\partial^\alpha := \left(\partial_x, \partial_y\right)\Big|_p$$

of $T_p\Sigma$. Define a linear map

$$I_p \colon T_p\Sigma \longrightarrow T_p\Sigma \qquad \text{by} \qquad I_p\partial_x = \partial_y \quad \text{and} \quad I_p\partial_y = -\partial_x. \tag{1.44}$$

Using the fact that $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic, it is easy to check that I_p does not depend on the choice of a holomorphic coordinate. It follows also from (1.44) that $I_p^2 = -id$.

For any smooth (not necessarily holomorphic) function $f \colon \Sigma \longrightarrow \mathbb{C}$ define $\partial f, \bar\partial f \in \Omega^1(\Sigma; \mathbb{C})$ by

$$\begin{aligned} \partial f(v) &:= \tfrac{1}{2}\left(df(v) - i\, df(Iv)\right) \\ \bar\partial f(v) &:= \tfrac{1}{2}\left(df(v) + i\, df(Iv)\right) \end{aligned} \qquad \text{for } v \in T\Sigma.$$

In particular, we have $df = \partial f + \bar\partial f$ and f is holomorphic if and only if $\bar\partial f = 0$. If $f_\alpha := f \circ \psi_\alpha^{-1} \colon \mathbb{C} \longrightarrow \mathbb{C}$ is the local representation of f, then

$$\bar\partial f_\alpha = \frac{\partial f_\alpha}{\partial \bar z}\, d\bar z \qquad \text{and} \qquad \partial f_\alpha = \frac{\partial f_\alpha}{\partial z}\, dz$$

are local representations of $\bar\partial f$ and ∂f respectively, that is $\psi_\alpha^* \bar\partial f_\alpha = \bar\partial f$ and $\psi_\alpha^* \partial f_\alpha = \partial f$.

Furthermore, since

$$\begin{aligned} dz &= dx + i\, dy, \\ d\bar z &= dx - i\, dy, \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} dx &= \tfrac{1}{2}\left(dz + d\bar z\right), \\ dy &= \tfrac{1}{2i}\left(dz - d\bar z\right), \end{aligned}$$

any $\omega \in \Omega^1(\Sigma; \mathbb{C})$ can be written uniquely as $\omega = a\, dz + b\, d\bar z$ for some functions a and b. Denote by $\Omega^{1,0}(\Sigma)$ the space of all those complex valued 1-forms, whose local representation is $a\, dz$ for some $a \in C^\infty(\Sigma; \mathbb{C})$. More invariantly, we have

$$\Omega^{1,0}(\Sigma) = \left\{\omega \in \Omega^1(\Sigma; \mathbb{C}) \mid \omega(I\cdot) = i\,\omega(\cdot)\right\}.$$

Similarly, denote by $\Omega^{0,1}(\Sigma)$ the space of all those 1-forms, whose local representation is $b\, d\bar z$ for some complex valued function b, or equivalently,

$$\Omega^{0,1}(\Sigma) := \left\{\omega \in \Omega^1(\Sigma) \mid \omega(I\cdot) = -i\,\omega(\cdot)\right\}.$$

Thus, we obtain the decomposition

$$\Omega^1(\Sigma; \mathbb{C}) = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma).$$

Then ∂f and $\bar\partial f$ are nothing else but the components of df lying in $\Omega^{1,0}(\Sigma)$ and $\Omega^{0,1}(\Sigma)$ respectively.

Thus, in the case of a Riemann surface, the complexified de Rham complex has the following form:

$$\begin{tikzcd}[column sep=small] & & \Omega^{1,0}(\Sigma) \arrow[dr, "\bar\partial"] & & \\ 0 \arrow[r] & \Omega^0(\Sigma; \mathbb{C}) \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"'] & \oplus & \Omega^2(\Sigma, \mathbb{C}) \arrow[r] & 0 \\ & & \Omega^{0,1}(\Sigma) \arrow[ur, "\partial"'] & & \end{tikzcd} \tag{1.45}$$

Here $\bar{\partial}\colon \Omega^{1,0}(\Sigma) \to \Omega^2(\Sigma;\mathbb{C})$ is just the restriction of d to $\Omega^{1,0}(\Sigma)$ and $\partial\colon \Omega^{0,1}(\Sigma) \to \Omega^2(\Sigma,\mathbb{C})$ is the restriction of d to $\Omega^{0,1}(\Sigma)$. Locally, we have

$$\omega \in \Omega^{1,0}(\Sigma) \implies \omega = a\,dz \implies d\omega = da \wedge dz = \left(\frac{\partial a}{\partial z}dz + \frac{\partial a}{\partial \bar{z}}d\bar{z}\right) \wedge dz = -\frac{\partial a}{\partial \bar{z}}\,dz \wedge d\bar{z},$$
$$\omega \in \Omega^{0,1}(\Sigma) \implies \omega = b\,d\bar{z} \implies d\omega = db \wedge d\bar{z} = \left(\frac{\partial b}{\partial z}dz + \frac{\partial b}{\partial \bar{z}}d\bar{z}\right) \wedge d\bar{z} = \frac{\partial b}{\partial z}\,dz \wedge d\bar{z}.$$

Notice that we have

$$d^2 = 0 \qquad \iff \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

The splitting (1.45) of the de Rham complex yields the Dolbeault cohomology groups:

$$H^{1,0}(\Sigma) := \ker \bar{\partial},$$

$$H^0(\Sigma) := \ker \bar{\partial}, \qquad\qquad H^{1,1}(\Sigma) := \Omega^2 \big/ \operatorname{Im} \bar{\partial},$$

$$H^{0,1}(\Sigma) := \Omega^{0,1} \big/ \operatorname{Im} \bar{\partial}.$$

Clearly, $H^0(\Sigma)$ is the space of holomorphic functions on Σ. In particular, if Σ is compact and connected, then $H^0(\Sigma) \cong \mathbb{C}$.

$H^{1,0}(\Sigma)$ is the space of holomorphic differentials, that is 1-forms ω such that locally $\omega = a\,dz$ and a is a holomorphic function. The geometric meaning of $H^{0,1}(\Sigma)$ and $H^{1,1}(\Sigma)$ is somewhat less straightforward. An interested reader may wish to consult for example [??].

1.6 The Laplacian on Riemann surfaces

A peculiar feature of Riemann surfaces is that the Laplace-Beltrami operator can be defined without a reference to a Riemannian metric. Indeed, set

$$\Delta := 2i\,\bar{\partial}\partial = -2i\,\partial\bar{\partial}\colon \Omega^0(\Sigma;\mathbb{C}) \longrightarrow \Omega^2(\Sigma;\mathbb{C}).$$

If $z = x + iy$ is a local holomorphic coordinate as above, then we have

$$\begin{aligned}
\bar{\partial} f &= \frac{\partial f}{\partial \bar{z}}\,d\bar{z} = \frac{1}{2}\left(\partial_x f + \partial_y f\, i\right) d\bar{z} \qquad \Longrightarrow \\
\partial\bar{\partial} f &= \frac{1}{2}\frac{\partial}{\partial z}\left(\partial_x f + \partial_y f\, i\right) dz \wedge d\bar{z} \\
&= \frac{1}{4}\left(\partial_x - i\,\partial_y\right)\left(\partial_x f + \partial_y f\, i\right) dz \wedge d\bar{z} \\
&= \frac{1}{4}\left(\partial^2_{xx} f + \partial^2_{yy} f\right) dz \wedge d\bar{z}.
\end{aligned}$$

Furthermore, since

$$dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\,dx \wedge dy,$$

we obtain

$$\Delta f = -2i\,\partial\bar{\partial} f = -\left(\partial^2_{xx} f + \partial^2_{yy} f\right) dx \wedge dy.$$

Remark 1.46. To relate this operator to the Laplacian in the sense of Def. ??, let g be a Hermitian metric on Σ, that is $g(I\cdot, I\cdot) = g(\cdot,\cdot)$.

If $z = x + yi$ is a local holomorphic coordinate, then

$$(g_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad \Longrightarrow \qquad (g^{ij}) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some positive function $\lambda = \lambda(x, y)$.

Substituting this into (1.35), we obtain

$$\Delta f = \frac{1}{\lambda}\left(\partial_x\left(\lambda\cdot\lambda^{-1}\partial_x f\right) + \partial_y\left(\lambda\cdot\lambda^{-1}\partial_y f\right)\right) \;= \frac{1}{\lambda}\left(\partial^2_{xx} f + \partial^2_{yy} f\right).$$

Furthermore, the Riemannian volume form on Σ in terms of local coordinates (x, y) is

$$vol = \lambda\, dx \wedge dy.$$

Hence, $* \, vol = 1 \Longrightarrow *\,(dx \wedge dy) = \lambda^{-1}$. This yields

$$\Delta f = \frac{1}{\lambda}\left(\partial^2_{xx} f + \partial^2_{yy} f\right) = *\left(-2i\,\partial\bar\partial f\right).$$

Hence, up to the application of the isomorphism $*\colon \Omega^2(\Sigma) \to \Omega^0(\Sigma)$, the Laplacian coincides with $-2i\,\partial\bar\partial$ indeed.

In the current setting, Corollary 2.94 yields the following.

Theorem 1.47. *Let Σ be a compact connected Riemann surface. The equation $-2i\,\partial\bar\partial f = \eta$, where $\eta \in \Omega^2(\Sigma;\mathbb{C})$, has a solution if and only if $\int_\Sigma \eta = 0$.* □

1.7 Some consequences of Theorem 1.47

In this section we assume that Σ is a compact connected Riemann surface throughout.

We have a natural skew-symmetric pairing

$$\Omega^{1,0}(\Sigma) \times \Omega^{0,1}(\Sigma) \longrightarrow \mathbb{C}, \qquad (\omega, \eta) \longmapsto \int_\Sigma \omega \wedge \eta.$$

This yields a bilinear map

$$B\colon H^{1,0}(\Sigma) \times H^{0,1}(\Sigma) \longrightarrow \mathbb{C}, \qquad B(\omega, [\eta]) = \int_\Sigma \omega \wedge \eta. \tag{1.48}$$

Lemma 1.49. *B is well defined.*

Proof. Notice first that for any $\alpha, \beta \in \Omega^{1,0}(\Sigma)$ we have $\alpha \wedge \beta = 0$. Indeed, locally $\alpha = a \cdot dz$ and $\beta = b \cdot dz$ so that $\alpha \wedge \beta = ab\, dz \wedge dz = 0$. Using this observation, we obtain

$$\begin{aligned}
\int_\Sigma \omega \wedge \left(\eta + \bar\partial f\right) &= \int_\Sigma \omega \wedge \eta + \int_\Sigma \omega \wedge \left(\partial f + \bar\partial f\right) \\
&= \int_\Sigma \omega \wedge \eta + \int_\Sigma \omega \wedge df \\
&= \int_\Sigma \omega \wedge \eta - \int_\Sigma d(f \cdot \omega) + \int_\Sigma f d\omega \\
&= \int_\Sigma \omega \wedge \eta - 0 + 0.
\end{aligned}$$

In the last equality the second summand vanishes by the Stokes' thm; The last summand vanishes, since ω is closed: Locally $\omega = a(z)\, dz$ so that

$$d\omega = da \wedge dz = \left(\frac{\partial a}{\partial z} dz + \frac{\partial a}{\partial \bar{z}} d\bar{z}\right) \wedge dz = 0 + 0 = 0,$$

since a is a holomorphic. This finishes the proof of this lemma. $\square$

Notice that we have a natural map

$$i\colon H^{1,0}(\Sigma) \longrightarrow H^1(\Sigma; \mathbb{C}), \qquad \omega \longmapsto [\omega].$$

The class $[\omega]$ is well-defined, since ω is closed as it has been shown in the proof Lemma 1.49.

Furthermore, the conjugation $\Omega^1(\Sigma; \mathbb{C}) \longrightarrow \Omega^1(\Sigma, \mathbb{C}), \eta \longmapsto \bar{\eta}$, induces a map

$$\sigma\colon H^{1,0}(\Sigma) \longrightarrow H^{0,1}(\Sigma),$$

which is antilinear, that is $\sigma(i\eta) = -i\sigma(\eta)$.

Theorem 1.50.

(i) *σ is an isomorphism;*

(ii) *For a complex vector space V denote $V^* = \{\psi\colon V \to \mathbb{C} \mid \psi \text{ is antilinear}\}$. Then $B\colon H^{1,0}(\Sigma) \to H^{0,1}(\Sigma)^*$ is an isomorphism;*

(iii) *The map*

$$H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma) \longrightarrow H^1(\Sigma; \mathbb{C}), \qquad (\omega, [\eta]) \longmapsto i(\omega) + \overline{i(\sigma^{-1}([\eta]))} \tag{1.51}$$

is an isomorphism.

Proof. Notice that $\sigma\colon H^{1,0}(\Sigma) \longrightarrow H^{0,1}(\Sigma)$ is well-defined. We want to show that for each class $[\eta] \in H^{0,1}(\Sigma)$ there exists a unique $\omega \in H^{1,0}(\Sigma)$ such that $\sigma(\omega) = [\eta]$. Indeed, any representative of $[\eta] \in H^{0,1}(\Sigma)$ can be written as

$$\eta' = \eta + \bar{\partial} f$$

for some $f \in C^\infty(\Sigma; \mathbb{C})$. We want to find a representative η' such that

$$\partial\eta' = 0 \qquad \Longleftrightarrow \qquad \partial\bar{\partial} f = -\partial\eta. \tag{1.52}$$

The equation on the right hand side of (1.52) has a solution, since

$$\int_\Sigma \partial\eta = \int_\Sigma d\eta = 0.$$

In fact, f is defined uniquely by (1.52) up to the addition of a constant. Hence, for each class $[\eta] \in H^{0,1}(\Sigma)$ there exists a unique representative η' such that $\partial\eta' = 0$.

Define $\omega := \bar{\eta}'$. Then

$$\bar{\partial}\omega = \bar{\partial}\bar{\eta}' = \overline{\partial\eta'} = 0,$$

that is ω lies in $H^{1,0}(\Sigma)$ and is a unique preimage of $[\eta]$.

To prove *(ii)*, we only need to show that (1.48) is non-degenerate, that is

$$\forall\, \omega \in H^{1,0}(\Sigma) \qquad \exists\, [\eta] \in H^{0,1}(\Sigma) \qquad \text{such that } B(\omega, [\eta]) \neq 0.$$

Indeed, for $\omega \in H^{1,0}(\Sigma)$ set $\eta := \sigma(\omega) = [\bar\omega]$. Then

$$B(\omega, [\bar\omega]) = \int_\Sigma \omega \wedge \bar\omega \neq 0,$$

since $a\,dz \wedge \bar a\,d\bar z = |a|^2 dz \wedge d\bar z$.

Furthermore, the injectivity of (1.51) follows from *(ii)*. Indeed, notice first that $i(\omega)$ and $\overline{i(\sigma^{-1}([\eta]))}$ are always orthogonal with respect to B. Hence, $(\omega, [\eta])$ belongs to the kernel of (1.51) if and only if

$$i(\omega) = 0 \qquad \text{and} \qquad \overline{i(\sigma^{-1}([\eta]))} = 0. \tag{1.53}$$

However i is injective, since

$$B(i(\omega), \sigma(i(\omega))) = \int_\Sigma \omega \wedge \bar\omega \neq 0$$

provided $\omega \neq 0$. Hence, (1.53) yields $\omega = 0$ and $[\eta] = 0$, since all maps σ^{-1}, i, and the conjugation are injective.

To see that (1.51) is surjective, pick any $[\zeta] \in H^1_{dR}(\Sigma; \mathbb{C})$. I claim that there exists a representative $\zeta' = \zeta + df$ such that

$$\bar\partial \zeta' = 0 \qquad \Longleftrightarrow \qquad 0 = \bar\partial\zeta + \bar\partial df = \bar\partial\zeta - \partial\bar\partial f. \tag{1.54}$$

The existence of a function f satisfying (1.54) follows from Theorem 1.47 just like in the proof of *(i)*.

Furthermore, write

$$\zeta' = \omega + \eta, \qquad \text{where } \omega = \frac{1}{2}\left(\zeta' - i\zeta'(I\cdot)\right) \quad \text{and} \quad \eta = \frac{1}{2}\left(\zeta' + i\zeta'(I\cdot)\right).$$

Locally, we have

$$\zeta' = a\,dz + b\,d\bar z \qquad \Longrightarrow \qquad \omega = a\,dz \quad \text{and} \quad \eta = b\,d\bar z.$$

Since $\bar\partial\zeta' = 0$, we obtain $\frac{\partial a}{\partial \bar z} = 0$, that is ω is a holomorphic differential.

Furthermore,

$$\bar\partial\zeta' = 0 \quad \text{and} \quad d\zeta' = 0 \quad \Longrightarrow \quad \partial\zeta' = 0 \quad \Longrightarrow \quad \frac{\partial b}{\partial z} = 0 \quad \Longrightarrow \quad \partial\eta = 0$$
$$\Longrightarrow \quad \bar\partial\bar\eta = 0.$$

Hence, $\bar\eta$ is also a holomorphic differential. Combining this with *(i)*, we obtain *(iii)*. □

Corollary 1.55. *If $b_1(\Sigma) = 0$, then Σ admits a meromorphic function with a single simple pole.*

Proof. Pick any point $p \in \Sigma$ and a local holomorphic coordinate z centered at p. Let χ be a bump function at p, that is $\chi \equiv 1$ on a neighbourhood V of p and χ vanishes identically outside of a slightly larger neighbourhood $W \supset V$.

We wish to show that there exists a meromorphic function f on Σ with a unique simple pole at p. Assume for a moment, however, that such f does exist and consider

$$h := f - \chi\frac{a}{z},$$

where a is the residue of f at p. Then h is smooth everywhere, albeit may fail to be holomorphic on $W \backslash V$. In any case, we have

$$\bar{\partial} h = \bar{\partial} f - \frac{a}{z} \bar{\partial} \chi = -\frac{a}{z} \bar{\partial} \chi. \tag{1.56}$$

Notice that $\eta \coloneqq -\frac{a}{z} \bar{\partial} \chi$ is a smooth $(0,1)$-form supported in $W \backslash V$.

Conversely, if h is a smooth solution of (1.56), we can define f by

$$f \coloneqq h + \chi \frac{a}{z},$$

which is meromorphic and has a unique simple pole at p.

Thus, our task reduced to showing that (1.56) has a solution. By the definition of $H^{0,1}(\Sigma)$, this is the case if and only if the class of η in $H^{0,1}(\Sigma)$ vanishes. However, by (ii), we have

$$\dim_R H^{1,0}(\Sigma) + \dim_R H^{0,1}(\Sigma) = 2b_1(\Sigma) = 0 \qquad \Longrightarrow \qquad H^{0,1}(\Sigma) = \{0\}$$

so that $[\eta]$ vanishes trivially. This finishes the proof. $\square$

Theorem 1.57. *Let Σ be a closed Riemann surface. Then Σ is homeomorphic to $\mathbb{C}P^1 \cong S^2$ if and only if Σ is biholomorphic to $\mathbb{C}P^1$.*

Idea of the proof. By the classification theorem for closed surfaces, Σ is homeomorphic to S^2 if and only if $b_1(\Sigma) = 0$. By Corollary 1.55, there exists a meromorphic function f on Σ with a unique simple pole. We can view f as a holomorphic map $f \colon \Sigma \longrightarrow \mathbb{C}P^1$. A simple topological argument yields that f must be in fact bijective (this uses crucially that the pole of f is unique and simple). The reader can find the details of this topological argument in [Don, Sec 4.1]. $\square$

Another reformulation of the above theorem is that the sphere S^2 admits a unique structure of a complex manifold. This is in contrast with the torus $\mathbb{T}^2$ (or, in fact, any closed orientable surface with $b_1 \geq 2$), which admits continuous families of inequivalent complex structures.

Developing these ideas somewhat further one can obtain also a classification of all elliptic curves, that is Riemann surfaces homeomorphic to the torus. Or, one can show that any closed Riemann surface can be embedded into some projective space. However, this goes somewhat beyond the purposes of this course.

Chapter 2

Vector bundles, Sobolev spaces, and elliptic partial differential operators

2.1 Vector bundles

Basic definitions. Roughly speaking, a vector bundle is just a family of vector spaces parametrized by points of a manifold (or, more generally, of a topological space).

More formally, the notion of a vector bundle is defined as follows.

Definition 2.1. Choose a non-negative integer k. A real smooth vector bundle of rank k is a triple (π, E, M) such that the following holds:

(i) E and M are smooth manifolds, $\pi\colon E \to M$ is a smooth submersion (the differential is surjective at each point);

(ii) For each $m \in M$ *the fiber* $E_m := \pi^{-1}(m)$ has the structure of a vector space and $E_m \cong \mathbb{R}^k$;

(iii) For each $m \in M$ there is a neighborhood $U \ni m$ and a smooth map ψ_U such that the following diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes. Moreover, ψ_U is a fiberwise linear isomorphism.

The following terminology is commonly used: E is the total space, M is the base, π is the projection, and ψ_U is the local trivialization (over U).

It is worth pointing out that one can equally well talk about complex and quaternionic vector bundles. This requires only cosmetic changes, which are left to the reader. The preference for real vector bundles in this section is given for the sake of definitness mainly. I shall feel free to use complex vector bundles below without further explanations.

Example 2.2.

(a) The product bundle: $M \times \mathbb{R}^k$;

(b) The tangent bundle TM of any smooth manifold M.

Let E and F be two vector bundles over a common base M. A *homomorphism* between E and F is a smooth map $\varphi\colon E \to F$ such that the diagram

commutes and φ is a fiberwise linear map.

Two bundles E and F are said to be isomorphic, if there is a homomorphism φ, which is fiberwise an isomorphism.

A bundle E is said to be *trivial*, if E is isomorphic to the product bundle.

Operations on vector bundles. Let E and F be two vector bundles over a common base M. Then we can construct new bundles E^*, $\Lambda^p E$, $E \oplus F$, $E \otimes F$, and $\mathrm{Hom}(E,\, F)$ as follows:

($*$) $(E^*)_m = (E_m)^*$;

(Λ) $(\Lambda^p E)_m = \Lambda^p(E_m)$;

($\oplus$) $(E \oplus F)_m := E_m \oplus F_m$;

($\otimes$) $(E \otimes F)_m := E_m \otimes F_m$;

(Hom) $\mathrm{Hom}(E,\, F)_m := \mathrm{Hom}(E_m,\, F_m)$.

If $f\colon M' \to M$ is a smooth map, we can define the *pull-back* of $E \to M$ via

$$(f^*E)_{m'} := E_{f(m')}.$$

For example, if M' is an open subset of M and ι is the inclusion, then $E|_{M'} := \iota^* E$ is just the restriction of E to M'.

The reader should check that the families of vector spaces defined above satisfy the properties required by Definition 2.1.

Exercise 2.3. Prove that $E^* \otimes F$ is isomorphic to $\mathrm{Hom}(E, F)$.

Exercise 2.4. Prove that the tangent bundle of the 2-sphere is non-trivial. (Hint: Apply the hairy ball theorem).

Sections. Speaking informally, a section is an assignment of a vector in $s(m) \in E_m$ to each point $m \in M$ such that $s(m)$ depends smoothly on m. More formally, we have the following.

Definition 2.5. A smooth map $s\colon M \to E$ is called a section, if $\pi \circ s = \mathrm{id}_M$.

Sections of the tangent bundle TM are called vector fields. Sections of $\Lambda^p T^* M$ are called differential p-forms.

Exercise 2.6. Let $E \to M$ be a vector bundle of rank k and $U \subset M$ be an open subset. Prove that E is trivial over U if and only if there are k sections $e = (e_1, \dots, e_k)$, $e_j \in \Gamma(U;\, E)$, such that $e(m)$ is a basis of E_m for each $m \in U$. More precisely, given e show that ψ_U can be constructed according to the formula

$$\psi_U^{-1}\colon U \times \mathbb{R}^k \longrightarrow E|_U, \qquad (m, x) \mapsto e(m) \cdot x.$$

In fact this establishes a one-to-one correspondence between k-tuples of pointwise linearly independent sections and local trivializations of E.

We denote by $\Gamma(E) = \Gamma(M;\, E)$ the space of all smooth sections of E. Clearly, $\Gamma(E)$ is a vector space, where the addition and multiplication with a scalar are defined pointwise. In fact, $\Gamma(E)$ is a $C^\infty(M)$-module.

Given a local trivialization e over U (cf. Exercise 2.6) and a section s, we can write

$$s(m) = \sum_{j=1}^{k} \sigma_j(m) e_j(m)$$

for some functions $\sigma_j \colon U \to \mathbb{R}$. Thus, locally any section of a vector bundle can be thought of as a map $\sigma \colon U \to \mathbb{R}^k$.

It is important to notice that σ depends on the choice of a local trivialization. Indeed, if e' is another local trivialization of E over U', then there is a map

$$g \colon U \cap U' \longrightarrow \mathrm{GL}_n(\mathbb{R}) \quad \text{such that} \quad e = e' \cdot g. \tag{2.7}$$

If $\sigma' \colon U' \to \mathbb{R}^k$ is a local representation of s with respect to e', we have

$$s = e'\sigma' = eg^{-1}\sigma' = e\sigma \qquad \Longrightarrow \qquad \sigma' = g\sigma.$$

Covariant derivatives. The reader surely knows from the basic analysis course that the notion of the derivative is very useful. It is natural to ask whether there is a way to differentiate sections of bundles too.

To answer this question, recall the definition of the derivative of a function $f \colon M \to \mathbb{R}$. Namely, choose a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ and denote $m := \gamma(0),\ \mathrm{v} := \dot\gamma(0) \in T_m M$. Then

$$df(\mathrm{v}) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(m)}{t}. \tag{2.8}$$

Trying to replace f by a section s of a vector bundle, we immediately run into a problem, namely the difference $s(\gamma(t)) - s(m)$ is ill-defined in general since these two vectors may lie in different vector spaces.

Hence, instead of trying to mimic (2.8) we will define the derivatives of sections axiomatically, namely asking that the most basic property of the derivative—the Leibniz rule—holds.

Definition 2.9. Let $E \to M$ be a vector bundle. *A covariant derivative* is an $\mathbb{R}$-linear map $\nabla \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that

$$\nabla(fs) = df \otimes s + f\nabla s \tag{2.10}$$

holds for all $f \in C^\infty(M)$ and all $s \in \Gamma(E)$.

Example 2.11. Let $M \subset \mathbb{R}^N$ be an embedded submanifold. Then the tangent bundle TM is naturally a subbundle of the product bundle $\underline{\mathbb{R}}^N := M \times \mathbb{R}^N$. In particular, any section s of TM can be regarded as a map $M \to \mathbb{R}^N$. With this at hand we can define a connection on TM as follows

$$\nabla s := \mathrm{pr}(ds),$$

where pr is the orthogonal projection onto TM. A straightforward computation shows that this satisfies the Leibniz rule, i.e., ∇ is a connection indeed.

Theorem 2.12. *For any vector bundle $E \to M$ the space of all connections $\mathcal{A}(E)$ is an affine space modelled on $\Omega^1(\operatorname{End} E) = \Gamma\big(T^*M \otimes \operatorname{End}(E)\big)$.*

To be somewhat more concrete, the above theorem consists of the following statements:

(a) $\mathcal{A}(E)$ is non-empty.
(b) For any two connections ∇ and $\hat{\nabla}$ the difference $\nabla - \hat{\nabla}$ is a 1-form with values in $\mathrm{End}(E)$;
(c) For any $\nabla \in \mathcal{A}(E)$ and any $a \in \Omega^1(\mathrm{End}\, E)$ the following

$$(\nabla + a)s := \nabla s + as$$

is a connection.

For the proof of Theorem 2.12 we need the following elementary lemma, whose proof is left as an exercise.

Lemma 2.13. *Let $A\colon \Gamma(E) \to \Omega^p(F)$ be an $\mathbb{R}$-linear map, which is also $C^\infty(M)$-linear, i.e.,*

$$A(fs) = fA(s) \qquad \forall f \in C^\infty(M) \quad \textit{and} \quad \forall s \in \Gamma(E).$$

Then there exists $a \in \Omega^p\big(\mathrm{Hom}(E,F)\big)$ such that $A(s) = a \cdot s$. □

Proof of Theorem 2.12. Notice first that $\mathcal{A}(E)$ is convex, i.e., for any $\nabla, \hat{\nabla} \in \mathcal{A}(E)$ and any $t \in [0,1]$ the following $t\nabla + (1-t)\hat{\nabla}$ is also a connection.

If ψ_U is a local trivialization of E over U, then we can define a connection ∇_U on $E|_U$ by declaring

$$\nabla_U s := \psi_U^{-1}\, d(\psi_U(s)).$$

Using a partition of unity and the convexity property, a collection of these local covariant derivatives can be sewed into a global covariant derivative just like in the proof of the existence of Riemannian metrics on manifolds, cf. [BT03, Thm. 3.3.7]. This proves (a).

By (2.10), the difference $\nabla - \hat{\nabla}$ is $C^\infty(M)$–linear. Hence, (b) follows by Lemma 2.13.

The remaining step, namely (c), is straightforward. This finishes the proof of this theorem. □

While Theorem 2.12 answers the question of the existence of connections, the reader may wish to have a more direct way to put his hands on a connection. One way to do this is as follows.

Let e be a local trivialization. Since e is a pointwise base we can write

$$\nabla e = e \cdot A, \tag{2.14}$$

where $A = A(\nabla, e)$ is a $k \times k$-matrix, whose entries are 1-forms defined on U. A is called the connection matrix of ∇ with respect to e.

If σ is a local representation of a section s, then

$$\nabla s = \nabla(e\sigma) = \nabla(e)\sigma + e \otimes d\sigma = e\big(A\sigma + d\sigma\big).$$

Hence, it is common to say that locally

$$\nabla = d + A,$$

which means that $d\sigma + A\sigma$ is a local representation of ∇s. In particular, ∇ is uniquely determined by its connection matrix over U and any $A \in \Omega^1(U; \mathfrak{gl}_k(\mathbb{R}))$ appears as a connection matrix of some connection (cf. Theorem 2.12).

2.2 Sobolev spaces

2.2.1 Sobolev spaces on $\mathbb{R}^n$

Recall that the space of square-integrable functions is defined by

$$L^2(\mathbb{R}^n) := \Big\{ u \colon \mathbb{R}^n \to \mathbb{C} \mid u \text{ is measurable } \& \int_{\mathbb{R}^n} |u(x)|^2 dx < \infty \Big\}.$$

This is a (complex) Hilbert space with respect to the Hermitian scalar product

$$\langle u, v \rangle_{L^2} := \int_{\mathbb{R}^n} u(x)\, \bar{v}(x)\, dx.$$

Remark 2.15. Strictly speaking, we should also identify those functions, which differ only on a subset of measure zero so that $L^2(\mathbb{R}^n)$ consists of classes of functions. However, this will not be an issue for us and we shall treat square-integrable functions as honest functions.

The Fourier transform

For $f \colon \mathbb{R}^n \to \mathbb{C}$ decaying sufficiently fast at ∞ the Fourier transform $\widehat{f} \colon \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x)\, e^{-i\langle x, \xi\rangle} dx.$$

The Plancherel theorem states that

$$\|\widehat{f}\|_{L^2}^2 = \frac{1}{(2\pi)^n} \|f\|_{L^2}^2, \tag{2.16}$$

that is (an extension of) the map $f \mapsto \widehat{f}$ is essentially an isometry of $L^2(\mathbb{R}^n)$.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ be a multi-index. Denote

$$D^\alpha f := \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The basic property of the Fourier transform is

$$\widehat{D^\alpha f(\xi)} = \xi^\alpha \widehat{f}(\xi), \tag{2.17}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$.

Another basic property of the Fourier transform is the following.

Lemma 2.18 (Riemann-Lebesgue). *If $u \in L^1(\mathbb{R}^n)$, then $\widehat{u} \in C^0(\mathbb{R}^n)$ and $\widehat{u}$ decays at ∞.* □

Denote by $f^\vee(\xi) = (2\pi)^n \widehat{f}(-\xi) = \int f(x)\, e^{i\langle x, \xi\rangle} dx$.

Theorem 2.19 (The Fourier inversion theorem). *If $f \in L^1(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n)$, then f agrees almost everywhere with a continuous function f_0 and $\left(\widehat{f}\right)^\vee = f_0 = \left(\widehat{f^\vee}\right)$.* □

Sobolev spaces

Let $u \in L^2(\mathbb{R}^n)$. For $\alpha \in \mathbb{Z}^n_{\geq 0}$, we say that $v \in L^1_{loc}(\mathbb{R}^n)$ is the α^{th}-weak derivative if

$$\int_{\mathbb{R}^n} u \, \frac{\partial^{|\alpha|}\psi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v \cdot \psi$$

holds for all test functions ψ contained in

$$C_0^\infty(\mathbb{R}^n) := \{\psi \in C^\infty(\mathbb{R}^n) \mid \text{ supp } \psi \text{ is compact}\}.$$

The α^{th} weak derivative does not need to exist in general, however if it does exist we simply write

$$v = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

keeping in mind that the above equality holds in the weak sense.

With this understood, given any $k \geq 0$ we can define

$$W^{k,2}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \text{ exists and belongs to } L^2(\mathbb{R}^n) \ \forall \alpha \text{ s.t. } |\alpha| \leq k \right\}.$$

Somewhat less formally, this is expressed as follows:

$$u \in W^{k,2}(\mathbb{R}^n) \qquad \text{iff} \qquad \|u\|^2_{W^{k,2}} := \sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L^2} < \infty. \tag{2.20}$$

Properties (2.16) and (2.17) imply that $u \in W^{k,2}(\mathbb{R}^n)$ if and only if $(1+|\xi|^2)^{\frac{k}{2}}\, \widehat{u}(\xi) \in L^2(\mathbb{R}^n)$. Hence, we can equally well define $W^{k,p}(\mathbb{R}^n)$ by

$$W^{k,2}(\mathbb{R}^n) := \left\{ u \mid \|u\|^2_{W^{k,2}} := \int_{\mathbb{R}^n} (1+|\xi|^2)^k \, |\widehat{u}(\xi)|^2 d\xi < \infty \right\}.$$

Moreover, the norm appearing in the above definition is equivalent to (2.20). The advantage of the above definition is that this makes sense for any $k \in \mathbb{R}$, not just non-negative integers.

The following are basic results about Sobolev spaces.

Proposition 2.21. *Assume $u \in W^{k,2}(\mathbb{R}^n)$, where $\varkappa := k - \frac{n}{2} > l \in \mathbb{N}_0$. Then $u \in C^l(\mathbb{R}^n)$ after changing u on a subset of measure zero if necessary. Moreover, the natural inclusion*

$$W^{k,2}(\mathbb{R}^n) \longrightarrow C^l(\mathbb{R}^n)$$

is bounded, that is $\|u\|_{C^l} \leq C\|u\|_{W^{k,2}}$ for some positive constant C independent of u.

Proof. Using (2.17) we obtain

$$\begin{aligned}\int |\widehat{D^\alpha u}|(\xi)\, d\xi = \int |\xi^\alpha||\widehat{u}(\xi)|\, d\xi &\leq \int (1+|\xi|^2)^{\frac{|\alpha|}{2}} |\widehat{u}(\xi)|\, d\xi \\ &\leq \int (1+|\xi|^2)^{\frac{l}{2}} |\widehat{u}(\xi)|\, d\xi,\end{aligned}$$

provided $|\alpha| \leq l$.

Furthermore, by the Cauchy-Schwartz inequality we have

$$\int \left(1+|\xi|^2\right)^{\frac{l}{2}} |\widehat{u}\left(\xi\right)| d\xi = \int \left(1+|\xi|^2\right)^{\frac{k}{2}} |\widehat{u}\left(\xi\right)| \cdot \left(1+|\xi|^2\right)^{\frac{l-k}{2}} d\xi$$
$$\leq \left(\int \left(1+|\xi|^2\right)^k |\widehat{u}\left(\xi\right)|^2 d\xi\right)^{\frac{1}{2}} \left(\int \left(1+|\xi|^2\right)^{l-k} d\xi\right)^{\frac{1}{2}}$$

The last integral converges since $|\xi|^{2(l-k)} \cdot |\xi|^{n-1} = |\xi|^{\beta}$, where $\beta = 2\left(l-k\right)+n-1 < -n+n-1 = -1$. Hence, we obtain

$$\int |\widehat{D^\alpha u}|\left(\xi\right) d\xi \leq C\|u\|_{W^{k,2}}.$$

Hence, $D^\alpha u \in C^0$ by a combination of the Riemann-Lebesgue lemma and the Fourier inversion theorem.

Moreover,

$$\|D^\alpha u\|_{C^0} \leq \|\widehat{D^\alpha u}\|_{L^1} \leq C\|u\|_{W^{k,2}},$$

which finishes the proof. □

Proposition 2.22 (Rellich)**.** *Suppose $u_j \in W^{k,2}\left(\mathbb{R}^n\right)$ is a sequence such that there exists a compact subset $K \subset \mathbb{R}^n$ containing $\operatorname{supp} u_j$ for all j. If $\|u_j\|_{W^{k,2}}$ is bounded, then for any $s<k$ there is a subsequence u_{j_l}, which converges in $W^{s,2}\left(\mathbb{R}^n\right)$.* □

For any $p \in (1,+\infty)$ define

$$L^p\left(\mathbb{R}^n\right) := \left\{u \;|\; \|u\|_{L^p} := \left(\int_{\mathbb{R}^n} |u|^P\left(x\right) dx\right)^{\frac{1}{p}} < \infty\right\},$$

which is a Banach space. Using this, one can define the $W^{k,p}\left(\mathbb{R}^n\right)$-spaces in an obvious manner. For example, if k is an integer, we may set

$$W^{k,p}\left(\mathbb{R}^n\right) := \left\{u \;|\; \|u\|^p_{W^{k,p}} := \sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p} < \infty\right\}.$$

Theorem 2.23. *For any $k \geq 0$ the space $C_0^\infty\left(\mathbb{R}^n\right)$ is dense in $W^{k,p}\left(\mathbb{R}^n\right)$.* □

Hence, at least when k is a non-negative integer, we could define $W^{k,p}\left(\mathbb{R}^n\right)$ as the closure of $C_0^\infty\left(\mathbb{R}^n\right)$ with repsect to $\|\cdot\|_{W^{k,p}}$.

2.2.2 Sobolev spaces on manifolds

In the case when the base is a closed oriented Riemannian manifold M rather than $\mathbb{R}^n$, the definition of the L^p-spaces generalizes in a straightforward manner. Namely,

$$L^p\left(M\right) := \left\{u\colon M \longrightarrow \mathbb{R} \;|\; \|u\|_{L^p} := \left(\int_M |u|^p vol\right)^{\frac{1}{p}} < \infty\right\}.$$

Let E be an Euclidean vector bundle over a manifold M. That means that each fiber E_m is equipped with a scalar product $\langle\cdot,\cdot\rangle_m$ and this scalar product depends smoothly on m. More formally, just like in Definition 1.19, an Euclidean structure on E is a smooth section $\langle\cdot,\cdot\rangle$ of $E^*\otimes E^*$ such that

$$\langle v,w\rangle=\langle w,v\rangle \quad \text{and} \quad \langle v,v\rangle>0$$

holds for all $v,w\in E_m$ and all $m\in M$; In addition, in the last inequality we assume $v\neq 0$.

In any case, the definition of the L^p-spaces for sections of Euclidean bundles generalizes in a straightforward manner. Namely, if M is a compact oriented Riemannian manifold, then

$$L^p(E):=\left\{s \mid \ \|s\|_{L^p}:=\left(\int_M |s|^p\, vol\right)^{\frac{1}{p}}<\infty\right\}.$$

Furthermore, pick a connection $\nabla\in A(E)$. It is convinient to assume that ∇ is Euclidean, that is

$$d\langle s_1,s_2\rangle=\langle\nabla s_1,s_2\rangle+\langle s_1,\nabla s_2\rangle$$

holds for all smooth sections s_1 and s_2 of E.

Remark 2.24. Any Euclidean vector bundle admits an Euclidean connection. Moreover, the space of all Euclidean connections on E is an affine space modeled on

$$\Omega^1(M;\mathfrak{o}(E)):=\Gamma(T^*M\otimes\mathfrak{o}(E)), \quad \text{where} \quad \mathfrak{o}(E):=\{A\in End(E) \mid A^*=-A\}.$$

These facts can be established by a straightforward modification of the proof of Theorem 2.12.

We can define

$$W^{1,p}(E):=\left\{s \mid \ \|s\|_{W^{1,p}}:=(\|s\|^p_{L^p}+\|\nabla s\|^p_{L^p})^{\frac{1}{p}}<\infty\right\}.$$

Remark 2.25. If s is smooth, then ∇s is a 1-form with values in E. Then

$$|\nabla s|^2_m:=\sum_{i=1}^n|\nabla_{e_i}s|^2,$$

where $(e_1,\dots,e_n)$ is an orthonormal basis of T_mM. Thus, somewhat more precisely by $\|\nabla s\|_{L^p}$ we mean

$$\|\nabla s\|_{L^p}=\left(\int_m|\nabla s|^p_m\, vol_m\right)^{\frac{1}{p}}.$$

Just like in the case of $\mathbb{R}^n$, $W^{1,p}(E)$ can be understood in at least two following ways. First, we can define $W^{1,p}(E)$ as the completion of $\Gamma(E)$ with respect to the $\|\cdot\|_{W^{1,p}}$-norm. Secondly, for $s\in L^p(E)$ we can first define the weak covariant derivative ∇s as a functional acting on $\Omega^1(E)$ and ask ∇s to lie in $L^p(T^*M\otimes E)$.

To describe some details concerning the second approach, notice that akin to the de Rham complex for any connection ∇ we have the sequence

$$0\to\Omega^0(E)=\Gamma(E)\xrightarrow{\nabla=d_\nabla}\Omega^1(E)\xrightarrow{d_\nabla}\dots\xrightarrow{d_\nabla}\Omega^n(E)\to 0, \tag{2.26}$$

where for $\omega\otimes s\in\Omega^k(E)$ the map d_∇ is defined by

$$d_\nabla(\omega\otimes s)=(d\omega)\otimes s+(-1)^k\omega\wedge\nabla s.$$

Notice, however, that (2.26) is not a complex in general, that is $d_\nabla\circ d_\nabla$ does not necessarily vanish.

In any case, just like for the de Rham differential we can also define $d_\nabla^* \colon \Omega^{k+1}(E) \to \Omega^k(E)$ by $d_\nabla^* := (-1)^{nk+1} * d_\nabla *$, cf. Proposition 1.25. Here the Hodge operator acts as follows: $*(\omega \otimes s) = (*\omega) \otimes s$. Assuming ∇ is Euclidean, d_∇^* is the formal adjoint of d_∇, that is

$$\langle d_\nabla \alpha, \beta \rangle_{L^2} = \langle \alpha, d_\nabla^* \beta \rangle_{L^2}$$

holds for all $\alpha \in \Omega^k(E)$ and all $\beta \in \Omega^{k+1}(E)$.

With these preliminaries at hand, if $s \in L^p(E)$ the value of the weak derivative ∇s on $\psi \in \Omega^{k+1}(E)$ is declared to be

$$\langle \nabla s, \psi \rangle := \langle s, d_\nabla^* \psi \rangle_{L^2}.$$

Then we can define $W^{1,p}(E)$ as a subspace of $L^p(E)$ consisting of those s, whose weak derivative belongs to L^p, that is if there exists $w \in L^p(T^*M \otimes E)$ such that

$$\langle \omega, \psi \rangle_{L^2} = \langle s, d_\nabla^* \psi \rangle_{L^2}$$

holds for any $\psi \in \Gamma(E)$. Of course, in this case we must have $\omega = \nabla s$.

Our definition of $W^{1,p}(E)$ depends a priory on the choice of the connection ∇. It turns out that this dependence is not really essential as the following result shows.

Proposition 2.27. *Pick any two Euclidean connections ∇ and ∇' on E. Then, the corresponding norms $\|\cdot\|_{W^{1,p}}$ and $\|\cdot\|'_{W^{1,p}}$ are equivalent.*

Proof. By Theorem 2.12, we have $\nabla' = \nabla + a$, where $a \in \Omega^1(End(E))$ (in fact a takes values in $\mathfrak{o}(E)$, but this is immaterial here). Denote

$$A := \sup_{m \in M} |a_m| = \sup_{m \in M} \sup_{\substack{v \in T_m M \\ |v|=1}} |a_m(v)| \geq 0.$$

Then for any $v \in T_mM$ of unit norm and any smooth section s we have

$$\nabla'_v s = \nabla_v s + a(v) \cdot s \Longrightarrow |\nabla'_v s| \leq |\nabla_v s| + A|s|.$$

Hence, for any orthonormal basis $(e_1, \dots, e_n)$ of T_mM we obtain

$$\begin{aligned} |\nabla' s|_m^p = \sum_{i=1}^n |\nabla'_{e_i} s|^p \leq \sum_{i=1}^n (|\nabla_{e_i} s| + A|s|)^p \leq 2^{p-1} \left(\sum_{i=1}^n |\nabla_{e_i} s|^p + nA^p|s|^p \right) \\ = 2^{p-1}|\nabla s|_m^p + n2^{p-1}A^p|s|^p. \end{aligned} \tag{2.28}$$

Here to obtain the second inequality we used the following generalized mean inequality

$$\frac{a+b}{2} \leq \left(\frac{a^p + b^p}{2} \right)^{\frac{1}{p}},$$

where a and b are positive real numbers.

With the help of (2.28) we obtain

$$\|\nabla' s\|_{L^p} \leq C_1 \|\nabla s\|_{L^p} + C_2 \|s\|_{L^p}$$

for some positive constants C_1 and C_2 independent of s. This yields the claim of this proposition. $\square$

Our next aim is to define the Sobolev spaces $W^{k,p}(E)$ with $k \geq 1$ being an integer. To this end notice first the following.

Lemma 2.29.

(i) If ∇ is a connection on a vector bundle E, then there exists a unique connection ∇^* on E^* such that

$$d\langle \sigma, s\rangle = \langle \nabla^* \sigma, s\rangle + \langle \sigma, \nabla s\rangle \tag{2.30}$$

holds for all $\sigma \in \Gamma(E^*)$ and all $s \in \Gamma(E)$. Here $\langle \cdot, \cdot \rangle$ denotes the natural pointwise pairing $E^* \otimes E \to \mathbb{R}$.

(ii) If ∇^E and ∇^F are connections on vector bundles E and F respectively, then there exists a unique connection ∇ on $E \otimes F$ such that

$$\nabla (s \otimes t) = \nabla^E s \otimes t + s \otimes \nabla^F t \tag{2.31}$$

holds for all $s \in \Gamma(E)$ and all $t \in \Gamma(F)$.

The proof of this lemma can be obtained by a straightforward verification that ∇^* and ∇ defined by (2.30) and (2.31) respectively satisfies the Leibnitz rule. I leave this as an exercise to the reader.

With this understood, we can proceed as follows. Pick an Euclidean connection ∇^E on E and an Euclidean connection ∇^{TM} on TM. This yields a connection $\nabla = \nabla\left(\nabla^E, \nabla^{TM}\right)$ on $T^*M \otimes E$. Hence, for any smooth section s of E we can define the second derivative by

$$\nabla^2 s = \nabla\left(\nabla^E s\right) \in \Gamma\left(T^*M \otimes T^*M \otimes E\right).$$

In the less regular case, for example when $s \in W^{1,p}(E)$ we can still define the weak second derivative as a functional on $\Gamma(T^*M \otimes T^*M \otimes E)$ just like we defined the weak first derivative. Then we may set

$$\begin{aligned} W^{2,p}(E) &= \left\{ s \in W^{1,p}(E) \mid \nabla^2 s \in L^p(E) \right\}, \\ &= \left\{ s \mid \ \|s\|^p_{W^{2,p}} := \|s\|^p_{L^p} + \|\nabla s\|^p_{L^p} + \|\nabla^2 s\|^p_{L^p} < \infty \right\}, \end{aligned}$$

where the second equality should be treated with care just like in the case of functions.

Furthermore, we can define $W^{k,p}(E)$ for any integer $k \geq 2$ by induction. The details are left to the reader.

Alternatively, we can also define $W^{k,p}(E)$ as the closure of $\Gamma(E)$ with respect to $\|\cdot\|_{W^{k,p}}$, where

$$\|s\|^p_{W^{k,p}} := \sum_{j=0}^{k} \|\nabla^j s\|^p_{L^p}. \tag{2.32}$$

Yet another way to define Sobolev spaces, which works for any real k, is as follows. Since M is compact, we can pick a finite covering (U_α, ψ_α), where ψ_α is a trivialization of $E\big|_{U_\alpha}$ just like in Definition 2.1. Moreover, we can assume that each U_α is a coordinate chart. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, that is

- $\operatorname{supp} \rho_\alpha \subset U_\alpha$;
- $\rho_\alpha \geq 0$ everywhere;
- $\sum_\alpha \rho_\alpha(m) = 1$ for each $m \in M$.

If s is a section of E, over each U_α we can identify s with some $s_\alpha \colon U_\alpha \to \mathbb{R}^l$, where l is the rank of E, as follows:

$$\psi_\alpha \circ s = (id, s_\alpha)\,.$$

Hence, we can think of $\rho_\alpha \cdot s_\alpha$ as a map defined on $\mathbb{R}^n$ with compact support. Finally, we set

$$\|s\|^p_{W^{k,p}} := \sum_\alpha \|\rho_\alpha \cdot s_\alpha\|^p_{W^{k,p}}.$$

This norm depends on the choice of $\{(U_\alpha, \psi_\alpha, \rho_\alpha)\}$, however turns out to be equivalent to (2.32) if k is an integer. Hence, we can define $W^{k,p}(E)$ in the usual way, for example as the completion of $\Gamma(E)$ with respect to the above norm.

With this understood, for any $p > 1$ we have the sequence of inclusions

$$L^p(M;\, E) = W^{0,p}(M;\, E) \supset W^{1,p}(M;\, E) \supset W^{2,p}(M;\, E) \supset \dots$$

Relations between all these spaces is given by the following theorem, which is of fundamental importance in the theory of PDEs.

Theorem 2.33. *Let M be a compact manifold.*

(i) If $s \in W^{k,p}(M;\, E)$, then $s \in W^{l,q}(M;\, E)$ provided

$$k - \frac{n}{p} \geq l - \frac{n}{q} \quad \text{and} \quad k \geq l,$$

where $n = \dim M$, and there is a constant C independent of s such that $\|s\|_{W^{l,s}} \leq C\|s\|_{W^{k,p}}$. In other words, the natural embedding

$$j \colon W^{k,p}(M;\, E) \subset W^{l,q}(M;\, E)$$

is continuous.

(ii) j is a compact operator provided

$$k - \frac{n}{p} > l - \frac{n}{q} \quad \text{and} \quad k > l. \tag{2.34}$$

This means that any sequence bounded in $W^{k,p}$ has a subsequence, which converges in $W^{l,q}$ provided (2.34) holds.

(iii) We have a natural continuous embedding

$$W^{k,p}(M;\, E) \subset C^l(M;\, E)$$

provided $k - \frac{n}{p} > l$. In particular, if $s \in W^{k,p}(M;\, E)$ for some fixed p and for all $k \geq 0$, then $s \in C^\infty(M;\, E)$.

(iv) (a) In the case $kp > n$ the space $W^{k,p}(M;\, \mathbb{R})$ is an algebra.

(b) In the case $kp < n$, we have a bounded map

$$W^{k_1,p_1} \otimes W^{k_2,p_2} \to W^{k,p}, \qquad \text{provided} \quad k_1 - \frac{n}{p_1} + k_2 - \frac{n}{p_2} \geq k - \frac{n}{p}. \quad \square$$

2.3 Differential operators

Let E be a smooth complex vector bundle of rank a over M. This means that each fiber E_m has the structure of a complex vector space and its dimension equals a; Also, Property (iii) of Definition 2.1 should be read as follows: For each $m \in M$ there exists a neighbourhood $U \ni m$ and a smooth map ψ_U show that the following diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^a \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

commutes. Moreover, ψ_U is a fiberwise complex linear isomorphism.

Given a local trivialization ψ_U as above, any section $s \in \Gamma(E)$ can be identified with a smooth map $\sigma\colon U \to \mathbb{C}^a$ in the following sense:

$$\psi_U \circ s = (id_U, \sigma)\,.$$

In other words, we have a well-defined $C^\infty(M;\mathbb{C})$-linear isomorphism

$$\Psi_U\colon \Gamma\left(E|_U\right) \longrightarrow C^\infty(U;\mathbb{C}^a)\,.$$

Let F be another complex vector bundle over M of rank b. Then F admits a local trivialization over some neighbourhood U' of M. Replacing U and U' by $U \cap U'$ if necessary, we can assume $U' = U$. Notice also that by shrinking U further if necessary, we can assume that U is a chart. Denote by $(x_1, \dots x_n)$ local coordinates on U. Let ψ_U^F be the trivialization of F over U so that sections of F over U can be thought of as maps $U \longrightarrow \mathbb{C}^b$.

Definition 2.35. A $\mathbb{C}$-linear map $L : \Gamma(E) \to \Gamma(F)$ is called a differential operator of order at most l, if for each choice of local trivializations of E and F as above there is a differential operator $\tilde{L}\colon C^\infty(U;\mathbb{C}^a) \to C^\infty\left(U;\mathbb{C}^b\right)$ of order at most l such that the following diagram

$$\begin{tikzcd} \Gamma\left(E|_U\right) \arrow[r, "L"] \arrow[d, "\Psi_U"] & \Gamma\left(F|_U\right) \arrow[d, "\Psi_U^F"] \\ C^\infty(U;\mathbb{C}^a) \arrow[r, "\tilde{L}"] & C^\infty\left(U;\mathbb{C}^b\right) \end{tikzcd}$$

commutes. Here $\tilde{L}$ is said to be a differential operator of order at most l, if $\tilde{L}$ admits a representation

$$\tilde{L}\sigma = \sum_{|\alpha| \le l} a_\alpha(x) \frac{\partial^{|\alpha|}\sigma}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \tag{2.36}$$

where $a_\alpha \in C^\infty(U; M_{a\times b}(\mathbb{C}))$ and $M_{a\times b}(\mathbb{C})$ denotes the space of all $a \times b$-matrices with complex entries.

Remark 2.37. We can equally well define real differential operators acting on sections of real vector bundles. This requires cosmetic changes only. The choice to focus on complex differential operators will be somewhat clearer below. Roughly speaking, this is related to the fact, that the Fourier transform of a real valued function is a complex valued function.

Example 2.38. Let M be any manifold of dimension n. Recall that any chart $(U, x_1, \ldots x_n)$ yields a trivialization of T^*M. The corresponding map $\Psi_U : \Omega^1(U) \to C^\infty(U; \mathbb{R}^n)$ is given by

$$\omega = \sum_{i=1}^{n} \omega_i(x)\, dx_i \longmapsto \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}.$$

Consider the differential as the map $d\colon \Omega^0(M) = \Gamma(\underline{\mathbb{R}}) \to \Omega^1(M)$. Since $\underline{\mathbb{R}}$ is globally trivial, there is no need to choose an extra local trivialization of $\underline{\mathbb{R}}$. Then, relative to the above choice of the local trivialization of T^*M, the local representation of d is given by

$$f \longmapsto \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \frac{\partial f}{\partial x_1} + \ldots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \cdot \frac{\partial f}{\partial x_n}.$$

In particular, d is a first order (real) differential operator.

By the same token, $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a first order real differential operator for each k.

Let $\mathrm{Diff}_k(E, F)$ denote the vector space of all differential operators of order at most k.

Proposition 2.39. *Any $L \in \mathrm{Diff}_l(E, F)$ extends as a bounded map $L\colon W^{k,p}(E) \to W^{k-l,p}(F)$.*

The proof of this proposition follows immediately from the property $\widehat{D^\alpha f}(\xi) = \xi^\alpha \widehat{f}(\xi)$ and the definition of $W^{k,p}$-norm in terms of the Fourier transform.

2.3.1 Symbols of differential operators

Consider the following second order differential operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u, \tag{2.40}$$

where u is a function of n variables $(x_1, \ldots x_n)$. It is well-known from the theory of linear PDEs that in many cases the most essential properties of L depend on the highest order terms only, that is on

$$L^{(2)}u = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

It is convenient to represent this operator by the expression

$$\sigma(L) := -\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j,$$

where $\xi = (\xi_1, \ldots, \xi_n)$ can be understood as a formal variable and the negative sign is a convention. This leads to the notion of the symbol of a differential operator.

It is important to understand how the symbol changes if we change the variables, since on a manifold there are no preferred coordinates in general. Thus, let

$$\begin{aligned} y_1 &= y_1(x_1, \dots, x_n) \\ &\dots \\ y_n &= y_n(x_1, \dots, x_n) \end{aligned}$$

be new coordinates on $\mathbb{R}^n$. If $u(x) = v(y(x))$, then we have

$$\frac{\partial u}{\partial x_i} = \sum_{p=1}^n \frac{\partial v}{\partial y_p}\frac{\partial y_p}{\partial x_i} \qquad \Longrightarrow \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{p,q=1}^n \frac{\partial^2 v}{\partial y_p \partial y_q}\frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j} + \sum_{p=1}^n \frac{\partial v}{\partial y_p}\frac{\partial^2 y_p}{\partial x_i \partial x_j}.$$

Substituting this into (2.40) we obtain

$$Lu = \sum_{i,j,p,q=1}^n a_{ij}\frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j}\frac{\partial^2 v}{\partial y_p \partial y_q} + \dots =: \tilde{L}v$$

where "..." denotes the lower order terms. Therefore, the symbol of $\tilde{L}$ is given by

$$\sigma(\tilde{L})(\eta) = -\sum_{p,q=1}^n \left(\sum_{i,j=1}^n a_{ij}\frac{\partial y_p}{\partial x_i}\frac{\partial y_q}{\partial x_j} \right) \eta_p \eta_q.$$

Hence, if we set

$$\xi_i = \sum_{p=1}^n \frac{\partial y_p}{\partial x_i}\eta_p \qquad \Longrightarrow \qquad \xi_j = \sum_{q=1}^n \frac{\partial y_q}{\partial x_j}\eta_q,$$

then

$$\sigma(L)(\xi) = \sigma(\tilde{L})(J\xi)$$

holds identically for all $\xi \in \mathbb{R}^n$, where

$$J = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \dots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} = \left(\frac{\partial y}{\partial x} \right)^t.$$

This implies the following. Think of (x, ξ) as coordinates on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. If we change the coordinates x on $\mathbb{R}^n$ for y as above, we obtain new coordinates (y, η) on $T^*\mathbb{R}^n$. Moreover, these coordinates are related by

$$y = y(x) \qquad \text{and} \qquad \eta = \left(\frac{\partial y}{\partial x} \right)^t \cdot \xi.$$

Hence $\sigma(\tilde{L})(\eta) = \sigma(\tilde{L})(y, \eta)$ is just the expression for $\sigma(L)$ in these new coordinates. Put differently, $\sigma(L)(\xi) = \sigma(L)(x, \xi)$ is well-defined as a function on $T^*\mathbb{R}^n$.

Notice that $\sigma(L)(\xi)$ is homogeneous of degree 2 in ξ, that is

$$\sigma(L)(\lambda\xi) = \lambda^2 \sigma(L)(\xi)$$

for all $\lambda \in \mathbb{R}$.

In general, we can proceed as follows. For vector bundles E and F as above, consider $\pi^*E \longrightarrow T^*M$ and $\pi^*F \longrightarrow T^*M$, where $\pi\colon T^*M \longrightarrow M$ is the cotangent bundle. For any $l \in \mathbb{Z}$ set

$$\mathrm{Smb}_l(E;F) := \left\{\sigma \in \Gamma(\mathrm{Hom}(\pi^*E, \pi^*F)) \mid \sigma(m, \lambda\xi) = \lambda^l \sigma(m,\xi)\right\},$$

where $\lambda > 0$ and $(m,\xi) \in T^*M$.

For any linear differential operator $L\colon \Gamma(E) \longrightarrow \Gamma(F)$ of order at most l as above, its symbol $\sigma(L)$ is an element of $\mathrm{Smb}_l(E;F)$. This can be defined via the local representations of L just like we did above in the case of the second order operators acting on functions. To be more precise, if $\tilde{L}$ is a local representation of L just like in (2.36), define

$$\sigma\big(\tilde{L}\big)(x,\xi) := i^l \sum_{|\alpha|=l} a_\alpha(x)\,\xi^\alpha \in M_{a\times b}(\mathbb{C}).$$

This yields a map

$$\pi^{-1}(U) = U \times \mathbb{R}^n \longrightarrow M_{a\times b}(\mathbb{C}),$$

where $U \subset M$ is as in Theorem 2.35. A computation similar to the one we did above shows that $\sigma\big(\tilde{L}\big)$ is a local representation of a well-defined section $\sigma(L) \in \mathrm{Smb}_l(E;F)$.

Alternatively, somewhat more invariantly, one can also define the symbol as follows. Given $(m,\xi) \in T^*M$ and $e \in E_m$, pick $f \in C^\infty(M)$ and $s \in \Gamma(E)$ such that $f(m) = 0$, $df_m = \xi$, and $s(m) = e$. Define

$$\sigma(L)(m,\xi)\,e := L\left(\frac{i^l}{l!} f^l s\right)\bigg|_m \in F_m$$

It is then easy to check that this is equivalent to our definition of the symbol in terms of local representations of L.

Denote by $\mathrm{Diff}_l(E;F)$ the vector space of all linear differential operators $L\colon \Gamma(E) \to \Gamma(F)$ of order at most l. Thus, we obtain a linear map $\sigma\colon\ \mathrm{Diff}_l(E;F) \to \mathrm{Smb}_l(E;F)$, which has the following properties.

Proposition 2.41.

(i) For any $l \in \mathbb{Z}, l \geq 1$, the sequence

$$0 \longrightarrow \mathrm{Diff}_{l-1}(E,F) \longrightarrow \mathrm{Diff}_l(E,F) \longrightarrow \mathrm{Smb}_l(E;F)$$

is exact.

(ii) $\sigma(L_1 \circ L_2) = \sigma(L_1) \circ \sigma(L_2)$. □

Example 2.42. Recall that in local coordinates the Laplacian (acting on functions) is given by

$$\Delta f = -\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right).$$

Hence, for the symbol we have

$$\sigma(\Delta)(\xi) = +\frac{1}{\sqrt{|g|}}\xi_i\left(\sqrt{|g|}g^{ij}\xi_j\right) = g^{ij}\xi_i\xi_j = |\xi|^2.$$

Example 2.43. Consider $d\colon \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$, which is clearly the first order linear differential operator. Pick a point $m \in M$ and a function $f \in C^\infty(M)$ such that $f(m) = 0$. We have

$$d(f\omega)\big|_m = df \wedge \omega\big|_m + 0 \cdot d\omega\big|_m = df \wedge \omega\big|_m.$$

Hence, $\sigma(d)(m,\xi)(\alpha) = i\xi \wedge \alpha$.

2.3.2 The formal adjoint of a linear differential operator and its symbol

An important role in what follows is played by the adjoint of a differential operator. We first prove an auxiliary result, which will be useful below, define the (formal) adjoint operator, and prove its existence afterwards.

Let E be Hermitian vector bundle. The Hermitian structure is defined just like the Euclidean one, namely as a family of Hermitian scalar products $(\cdot,\cdot)_m$ on each fiber depending smoothly on m.

Lemma 2.44. *For any non-negative $k \in \mathbb{R}$, we have $\left(W^{k,2}(M;E)\right)^* \cong W^{-k,2}(M;E)$. In particular, the natural semilinear map*

$$s,t \longmapsto (s,t)_{L^2}\,, \quad s,t \in C^\infty(M;E)$$

extends to the semilinear map

$$W^{k,2}(M;E) \times W^{-k,2}(M;E) \longrightarrow \mathbb{C}$$

such that $|\,(s,t)\,| \le C\|s\|_{W^{k,2}}\|t\|_{W^{-k,2}}$.

Idea of the proof. The proof boils down to showing that for any $u,v \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ we have

$$\left|\int_{\mathbb{R}^n} u(x)\,\bar v(x)\,dx\right| \le C\|u\|_{W^{k,2}}\|v\|_{W^{-k,2}}.$$

To see that this inequality holds indeed, notice that the Plancherel theorem yields

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} u(x)\,\bar v(x)\,dx\right| &= (2\pi)^n\left|\int_{\mathbb{R}^n}\widehat u(\xi)\,\overline{\widehat v(\xi)}d\xi\right| \\
&\le (2\pi)^n\int_{\mathbb{R}^n}|\widehat u(\xi)|\big(1+|\xi|^2\big)^{\frac{k}{2}}\,|\widehat v(\xi)|\big(1+|\xi|^2\big)^{-\frac{k}{2}}d\xi \\
&\le (2\pi)^n\left(\int_{\mathbb{R}^n}|\widehat u(\xi)|^2\big(1+|\xi|^2\big)^{k}d\xi\right)^{\frac12}\left(\int_{\mathbb{R}^n}|\widehat v(\xi)|^2\big(1+|\xi|^2\big)^{-k}\right)^{\frac12} \\
&= (2\pi)^n\,\|u\|_{W^{k,2}}\cdot\|v\|_{W^{-k,2}},
\end{aligned}$$

where the last inequality follows from the Cauchy-Schwartz inequality. □

Let F be another Hermitian bundle. Slightly abusing notations, we denote the Hermitian structure of F still by $(\cdot,\cdot)$.

Definition 2.45. Let $L\colon \Gamma(E) \to \Gamma(F)$ be a $\mathbb{C}$-linear map. A $\mathbb{C}$-linear map $L^*\colon \Gamma(F) \to \Gamma(E)$ is called the formal adjoint of L, if

$$(Ls,t)_{L^2} = (s,L^*t)_{L^2} = \int_M (s,L^*t)_m\,\mathrm{vol}_m$$

holds for all $s \in \Gamma(E)$ and all $t \in \Gamma(F)$.

Proposition 2.46. *If $L \in \mathrm{Diff}_l(E;F)$, then L^* exists and is unique. Moreover, $L^* \in \mathrm{Diff}_l(F;E)$ and $\sigma(L^*) = \sigma(L)^*$, where $\sigma(L)^*$ denotes the pointwise Hermitian-dual map.*

Proof. The proof consists of the following steps.

Step 1. *The formal adjoint operator is unique.*

Indeed, if L_1^* and L_2^* are two formal adjoint operators, then for all s, t as in the definition we have

$$0 = \left(s, \left(L_1^* - L_2^*\right) t\right)_{L^2} \qquad \Longrightarrow \qquad L_1^* = L_2^*.$$

Step 2. *L^* exists.*

For each $t \in C^\infty(F)$ consider the functional

$$s \longmapsto \psi_t(s) = (Ls, t)_{L^2}.$$

Since

$$|\psi_t(s)| \leq \|Ls\|_{W^{-k,2}} \|t\|_{W^{k,2}} \leq C\|t\|_{W^{k,2}} \|s\|_{W^{l-k,2}}, \tag{2.47}$$

ψ_t can be viewed as a bounded functional on $W^{l-k,2}$. Hence, there exists a unique $u = u(t) \in W^{k-l,2}$ such that $\psi_t(s) = (s, u(t))$.

Since ψ_t depends semilinearly on t, the map $t \longmapsto u(t)$ is $\mathbb{C}$-linear. Moreover,

$$\|\psi_t\|_{W^{l-k,2}} = \sup_{s \neq 0} \frac{|\psi_t(s)|}{\|s\|_{W^{l-k,2}}} \leq C \|u(t)\|_{W^{k-l,2}}. \tag{2.48}$$

In fact, $\|\psi_t\| = C\|u(t)\|_{W^{k-l,2}}$, since[1], roughly speaking, (2.48) follows from the Cauchy-Schwartz inequality, which is optimal.

We set $L^*t := u(t)$. Then by (2.47) we obtain

$$\|L^*t\|_{W^{k-l,2}} = \|u(t)\|_{W^{k-l,2}} \leq C\|t\|_{W^{k,2}}$$

for any $k \in \mathbb{R}$. In particular, L^* yields a bounded map $W^{k,2} \to W^{k-l,2}$. If $t \in C^\infty$, then L^*t does not depend on k and belongs to $W^{k-l,2}$ for any $k \in \mathbb{R}$. Hence, $L^* \colon C^\infty(F) \to C^\infty(E)$.

Step 3. *Let $\tilde{L} \colon C^\infty(\mathbb{R}^n; \mathbb{C}^a) \to C^\infty\left(\mathbb{R}^n; \mathbb{C}^b\right)$ be a linear differential operator of order l. Define the weighted Hermitian L^2-scalar product on $C_0^\infty(\mathbb{R}^n; \mathbb{C}^a)$ by*

$$(s, \sigma)_{L^2, \rho} := \int_{\mathbb{R}^n} (s(x), \sigma(x))\, \rho(x)\, dx,$$

where ρ is a positive smooth function. Then $\tilde{L}^*$ defined by

$$\left(\tilde{L}s, t\right)_{L^2, \rho} = \left(s, \tilde{L}^*t\right)_{L^2, \rho} \qquad \forall s \in C_0^\infty(\mathbb{R}^n; \mathbb{C}^a) \text{ and } \forall t \in C_0^\infty\left(\mathbb{R}^n; \mathbb{C}^b\right)$$

is also a linear differential operator of order l. Moreover, $\sigma\left(\tilde{L}^*\right) = \sigma\left(\tilde{L}\right)^*$.

Write $\tilde{L} = \sum_{|\alpha| \leq l} a_\alpha(x) D^\alpha$ and consider the term corresponding to $\alpha = (1, 0, \ldots, 0)$. We have

$$\begin{aligned}
(a_\alpha D^\alpha s, t)_{L^2, \rho} &= \int_{\mathbb{R}^n} (a_\alpha D^\alpha s)^t\, \bar{t}\, \rho\, dx = i \int_{\mathbb{R}^n} (\partial_1 s)^{tr}\, a_\alpha^{tr} \bar{t}\, \rho\, dx \\
&= i \int_{\mathbb{R}^n} \left(\partial_1 \left(s^{tr} a_\alpha^{tr} \bar{t}\, \rho\right) - s^{tr} \partial_1 \left(a_\alpha^{tr} \bar{t}\, \rho\right)\right) dx \\
&= 0 - i \int_{\mathbb{R}^n} \left(s^{tr} a_\alpha^{tr} \left(\overline{\partial_1 t}\right) \rho\, dx - i \int_{\mathbb{R}^n} s^{tr} \partial_1 \left(a_\alpha^{tr} \rho\right) \bar{t}\, dx \right. \\
&= \int_{\mathbb{R}^n} \left(s^{tr} \left(\overline{a_\alpha^* D_\alpha t}\right) \rho\, dx - \int_{\mathbb{R}^n} s^{tr}\, \overline{b_\alpha t}\, \rho\, dx, \right.
\end{aligned}$$

[1]A proper justification of this equality requires an extra argument, which I omit here.

where $b_\alpha = -\frac{i}{\rho}\partial_1 \left(a_\alpha^* \rho\right)$ and $a_\alpha^* = \left(\bar{a}\right)^{tr}$ denotes the Hermitian dual matrix. In other words, for $\alpha = (1, 0, \ldots, 0)$ we have $\left(a_\alpha D^\alpha\right)^* = a_\alpha^* D_\alpha - b_\alpha$. Iterating this argument, we obtain $\left(a_\alpha D^\alpha\right)^* = a_\alpha^* D_\alpha + S_\alpha$, where $S_\alpha \in \mathrm{Diff}_{|\alpha|-1}$. This yields

$$\tilde{L}^* = \sum_{|\alpha|=l} a_\alpha^* \left(x\right) D_\alpha + S,$$

where $S \in \mathrm{Diff}_{l-1}$. In particular, we have $\sigma\left(\tilde{L}^*\right) = \sigma\left(\tilde{L}\right)^*$.

Step 4. *If $L \in \mathrm{Diff}_l\left(E; F\right)$, then $L^* \in \mathrm{Diff}_l\left(F; E\right)$ and $\sigma\left(L^*\right) = \sigma\left(L\right)^*$.*

Let ψ be a local trivialization of E over an open neighbourhood U. Recall that such a trivialization is given by a collection $\left(e_1, \ldots, e_a\right)$ of pointwise linearly independent sections over U. By applying the Gram-Schmidt orthogonalization process, we can assume that $\left(e_1\left(m\right), \ldots, e_a\left(m\right)\right)$ is an orthonormal basis of E_m for each m.

Assume U is a chart with local coordinates $\left(x_1, \ldots, x_n\right)$ and $\left(f_1, \ldots, f_b\right)$ is a trivialization of F such that $\left(f_1\left(m\right), \ldots, f_b\left(m\right)\right)$ is orthonormal at each $m \in U$. Let $\tilde{L} = \sum_{|\alpha|\leq l} a_\alpha\left(x\right) D^\alpha$ be the local representation of L over U. Write also $\mathrm{vol} = \rho\left(x\right) dx_1 \wedge \ldots \wedge dx_n = \rho\left(x\right) dx$, where ρ is a positive function on U. Then for any $s \in \Gamma\left(E\right)$ and $t \in \Gamma\left(F\right)$ such that $\operatorname{supp} s$ and $\operatorname{supp} t$ are contained in U, we have

$$\left(Ls, t\right)_{L^2} = \left(\tilde{L}\sigma, \tau\right)_{L^2, \rho} = \left(\sigma, \tilde{L}^*\tau\right)_{L^2, \rho},$$

where σ and τ are local representations of s and t respectively. Moreover, $\tilde{L}^* \in \mathrm{Diff}_l$. $\square$

2.4 Pseudodifferential operators

Before giving the formal definition of a pseudodifferential operator, let us consider the following model case. Assume we want to solve the Poisson equation

$$\Delta u\left(x\right) = f\left(x\right), \qquad x \in \mathbb{R}^n, \tag{2.49}$$

where f is a given function on $\mathbb{R}^n$. Assume $f \in L^2$ and we are looking for a solution $u \in W^{2,2}$ of (2.49). Applying the Fourier transform to both sides, we obtain

$$\begin{aligned} \left(\xi_1^2 + \ldots + \xi_n^2\right) \widehat{u}\left(\xi\right) = \widehat{f}\left(\xi\right) \qquad &\Longrightarrow \qquad \widehat{u}\left(\xi\right) = \frac{1}{|\xi|^2} \widehat{f}\left(\xi\right) \\ &\Longrightarrow \qquad u\left(x\right) = \int_{\mathbb{R}^n} \frac{e^i\langle x, \xi\rangle}{|\xi|^2} \widehat{f}\left(\xi\right) d\xi. \end{aligned}$$

In particular, the map, which assigns to $f \in L^2$ a solution $u \in W^{2,2}$ of (2.49) is not a differential operator. This leads to the concept of a pseudodifferential operator, which will be very useful below.

Let $U \subset \mathbb{R}^n$ be an open set and let l be an integer.

Definition 2.50. The class $\tilde{S}^l\left(U\right)$ consists of functions $p = p\left(x, \xi\right)$ on $U \times \mathbb{R}^n$ satisfying the following: For any compact $K \subset U$ we have

$$|D_x^\beta D_\xi^\alpha p\left(x, \xi\right)| \leq C_{\alpha,\beta,K}\left(1 + |\xi|\right)^{l-|\alpha|}, \qquad x \in K, \quad \xi \in \mathbb{R}^n, \tag{2.51}$$

where α and β are multiindices.

Definition 2.52. The class $S^l(U)$ consists of those $p \in \tilde{S}^l(U)$ satisfying the limit $\sigma_l(p)(x,\xi) = \lim_{\lambda\to\infty} \frac{p(x,\lambda\xi)}{\lambda^l}$ exists for $\xi \neq 0$ and

$$p(x,\xi) - \psi(\xi)\,\sigma_l(p)(\xi) \in \tilde{S}^{l-1}(U), \tag{2.53}$$

where ψ is a cut-off function such that $\psi \equiv 0$ near the origin and $\psi \equiv 1$ outside of the unit ball.

Example 2.54. Let $p(x,\xi) = \sum_{|\alpha|\le l} p_\alpha(x)\,\xi^\alpha$. If each p_α is bounded in any $C^r(U)$, then $p \in S^l(U)$.

Definition 2.55 (Local pseudodifferential operator)**.** For any $p \in \tilde{S}^l(U)$ and any $u \in C_0^\infty(U)$ set

$$L_p u(x) := \int_{\mathbb{R}^n} p(x,\xi)\,\widehat{u}(\xi)\,e^{i\langle x,\xi\rangle} d\xi.$$

Example 2.56. If $p(x,\xi) = |\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$, then $\xi_j^2\widehat{u}(\xi) = \widehat{D_j^2 u}$ so that $Lu(x) = -\sum \frac{\partial^2 u}{\partial x_j^2} = \Delta u$. More generally for any $p(x,\xi) = \sum_{|\alpha|\le l} p_\alpha(x)\,\xi^\alpha$ the map L_p is a linear differential operator of order l.

Example 2.57. Assume $K \in C^\infty(U\times U;\mathbb{C})$ and $\operatorname{supp} K(x,\cdot)$ is compact. Consider the map

$$Lu(x) = \int_U K(x,y)\,u(y)\,dy,$$

where $u \in C_0^\infty(U)$. We have

$$\begin{aligned}\int_U K(x,y)\,u(y)\,dy &= \int_U K(x,y)\left(\int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle}\widehat{u}(\xi)\,d\xi\right)dy\\ &= \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\left(\int_U e^{i\langle y-x,\xi\rangle} K(x,y)\,dy\right)\widehat{u}(\xi)\,d\xi.\end{aligned}$$

Setting $p(x,\xi) := \int_U e^{i\langle y-x,\xi\rangle} K(x,y)\,dy$, we see that $L = L_p^U$, that is integral operators are also pseudodifferential operators at least formally. In fact,

$$p(x,\xi) = e^{-i\langle x,\xi\rangle}\int_U e^{i\langle y,\xi\rangle} K(x,y)\,dy = e^{-i\langle x,\xi\rangle} K(x,\cdot)^\vee(\xi).$$

Hence $e^{i\langle x,\xi\rangle}p(x,\xi)$ is essentially the Fourier transform of a compactly supported function and therefore is rapidly decaying, that is $(1+|\xi|)^N\,|p(x,\xi)| \to 0$ as $\xi \to \infty$ for any $N > 0$.

Theorem 2.58. *Assume p belongs to the class*

$$\tilde{S}_0^l(U) := \left\{p \in \tilde{S}^l(U) \mid \exists \textit{ cmpt } K \subset U \textit{ with } \operatorname{supp} p(\cdot,\xi) \subset K\ \forall \xi \in \mathbb{R}^n\right\}.$$

Then L_p maps $C_0^\infty(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$ and its closure yields a bounded map $L_p\colon W^{k,2}(\mathbb{R}^n) \to W^{k-l,2}(\mathbb{R}^n)$.

Proof. The proof consists of the following steps.

Step 1. $L_p(C_0^\infty(\mathbb{R}^n)) \subset C^\infty(\mathbb{R}^n)$.

If $u \in C_0^\infty(\mathbb{R}^n)$, then

$$\xi^\alpha \widehat{u}(\xi) = \widehat{D^\alpha u}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} D^\alpha u(x) e^{-i\langle x,\xi\rangle} dx$$

yields that $|\xi^\alpha||\widehat{u}(\xi)| \le C_\alpha$, since $\operatorname{supp} u$ is compact. Hence, for any $N \in \mathbb{N}$ we have

$$|\widehat{u}(\xi)| \le C_N (1+|\xi|)^{-N}, \tag{2.59}$$

which implies in turn

$$|D_x^\beta p(x,\xi)\widehat{u}(\xi)| \le C_{N,\beta}(1+|\xi|)^l (1+|\xi|)^{-N}.$$

Hence, $\int p(x,\xi)\widehat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi$ is differentiable in x any number of times provided $N \gg 1$.

Step 2. *For any $\xi, \eta \in \mathbb{R}^n$ and any $k \ge 0$ we have the inequality*

$$\left(1+|\xi|^2\right)^{\frac{k}{2}} \le 2^{\frac{k}{2}} \left(1+|\xi-\eta|^2\right)^{\frac{k}{2}} \left(1+|\eta|^2\right)^{\frac{k}{2}}.$$

By the Cauchy-Schwartz inequality we have

$$1+|\zeta+\eta|^2 \;\le\; 1+(|\zeta|+|\eta|)^2 \;\le\; 1+2\left(|\zeta|^2+|\eta|^2\right) \;\le\; 2\left(1+|\zeta|^2\right)\left(1+|\eta|^2\right).$$

Substituting $\zeta = \xi - \eta$ we obtain $1+|\xi|^2 \le 2(1+|\xi-\eta|^2)(1+|\eta|^2)$, which implies the claim of this step.

Step 3. *We prove the inequality $\|L_p u\|_{W^{k,2}} \le C\|u\|_{W^{k+l,2}}$.*

First notice that we have

$$\begin{aligned}
\widehat{L_p u}(\xi) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \int_{\mathbb{R}^n} p(x,\eta)\widehat{u}(\eta)e^{i\langle x,\eta\rangle} d\eta dx \\
&= (2\pi)^{-n} \int \left(e^{-i\langle x,\xi-\eta\rangle} p(x,\eta)\,dx\right) \widehat{u}(\eta)\,d\eta \\
&= \int_{\mathbb{R}^n} \widehat{p}(\xi-\eta,\eta)\,\widehat{u}(\eta)\,d\eta.
\end{aligned}$$

Just like (2.59), we obtain

$$|\widehat{p}(\zeta,\eta)| \le C\left(1+|\zeta|^2\right)^{-N}\left(1+|\eta|^2\right)^{\frac{l}{2}}.$$

Furthermore, using Step 2 we have

$$\begin{aligned}
|\widehat{L_p u}(\xi)| &\le C\int \left(1+|\xi-\eta|^2\right)^{-N}\left(1+|\eta|^2\right)^{\frac{l}{2}} |\widehat{u}(\eta)|\,d\eta \\
&\le C\int \frac{\left(1+|\xi-\eta|^2\right)^{-N}}{\left(1+|\eta|^2\right)^{\frac{k}{2}}} \left(1+|\eta|^2\right)^{\frac{l+k}{2}} |\widehat{u}(\eta)|\,d\eta \\
&\le C\left(1+|\xi|^2\right)^{-\frac{k}{2}} \int \left(1+|\xi-\eta|^2\right)^{-N+\frac{k}{2}} \left(1+|\eta|^2\right)^{\frac{l+k}{2}} |\widehat{u}(\eta)|\,d\eta,
\end{aligned}$$

which yields in turn

$$|\widehat{L_p u}(\xi)|\left(1+|\xi|^2\right)^{\frac{k}{2}} \le C\int \left(1+|\xi-\eta|^2\right)^{-N+\frac{k}{2}} \left(1+|\eta|^2\right)^{\frac{l+k}{2}} |\widehat{u}(\eta)|\,d\eta.$$

Applying Young's inequality $\|f*g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p}$, for $N \gg 1$ we obtain

$$\|L_p u\|_{W^{k,2}} \le C\|\left(1+|\xi|^2\right)^{-N+\frac{k}{2}}\|_{L^1}\|u\|_{W^{k+l,2}}.$$

□

Let $L\colon C_0^\infty(M;\mathbb{C}) \to C^\infty(M;\mathbb{C})$ be a linear map.

Definition 2.60. L is said to be a pseudodifferential operator of order l if for any chart $U \subset M$ and any open subset $U' \subset U$ such that $\bar{U}' \subset U$ there exists some

$$p \in S_0^l(U) = \tilde{S}_0^l(U) \cap S^l(U)$$

with the following property: For any $u \in C_0^\infty(U')$ we have $Lu = L_p u$.

Remark 2.61. Pseudodifferential operators are non-local in general, that is $\operatorname{supp} Lu \not\subset \operatorname{supp} u$ in general. This lack of locality for pseudodifferential operators explains the appearence of U' in the above definition. In particular, the function p may depend on U'.

Just like in the case of the differential operators we can define the l-symbol of a pseudodifferential operator L by

$$\sigma_l(p)(x,\xi) = \lim_{\lambda\to\infty} \frac{p(x,\lambda\xi)}{\lambda^l}, \qquad \xi \neq 0$$

assuming that this limit exists.

Just like in the case of differential operators we have the following basic result.

Proposition 2.62. *Let U be an open bounded subset of $\mathbb{R}^n$ equipped with coordinates $(x_1,\dots,x_n)$ and let $p \in S_0^l(U)$. Let $y = f(x)$ be new coordinates on U. Let $\tilde{L}$ be the representation of $L = L_p$ in these new coordinates, i.e.,*

$$\tilde{L}v := L\left(v \circ f^{-1}\right) \circ f.$$

Then there is a function $q \in S_0^l(f(U))$ such that $\tilde{L} = L_q$. Moreover,

$$\sigma_l(q)\,\eta = \sigma_l(p)\left(\left(\frac{\partial y}{\partial x}\right)^t \eta\right).$$

□

Definition 2.63. L is said to be a pseudodifferential operator of order l, if locally $L = L_p$ and $p \in S^l$. The class of all pseudodifferential operators of order l is denoted by $\mathrm{PDiff}_l(M)$.

The above proposition yields that to any $L \in \mathrm{PDiff}_l(M)$ we can associate a well-defined (principal) symbol

$$\sigma_l(L) \in \mathrm{Smb}_l(\underline{\mathbb{C}};\underline{\mathbb{C}}) \subset C^\infty(T^*M;\mathbb{C}).$$

Notice, that if l is negative, the homogeneity property of the symbol implies that $\sigma_l(L)$ may be singular along the zero section. Thus, strictly speaking, σ_l is defined away from the zero section only.

A simple consequence of Theorem 2.58 is the following.

Proposition 2.64. *If M is compact, then the closure of any $L \in \mathrm{PDiff}_l(M)$ yields a linear bounded operator*

$$L\colon W^{k,2}(M) \longrightarrow W^{k-l,2}(M).$$

□

Let E and F be complex vector bundles over M.

Definition 2.65. A linear map $L\colon C_0^\infty(M;E)\to C^\infty(M;F)=\Gamma(F)$ is called a pseudodifferential operator if for any chart U such that E and F admit trivializations over U and any $U'\subset U$ such that $\bar{U}'\subset U$ there exists a matrix $p=(p_{ij})$ with $p_{ij}\in S_0^l(U)$ so that the diagram

$$\begin{tikzcd} C_0^\infty(U';E) \arrow[r, "L"] \arrow[d, "\Psi^E"] & C^\infty(U;F) \arrow[d, "\Psi^F"] \\ C_0^\infty(U';\mathbb{C}^a) \arrow[r, "L_p"] & C^\infty(U';\mathbb{C}^b) \end{tikzcd}$$

commutes. The matrix p is called the local symbol of L (this depends on the choices made). If $L\in \mathrm{PDiff}_l(E;F)$, then we have a well-defined principal symbol $\sigma_l(L)\in \mathrm{Smb}_l(E;F)$.

Theorem 2.66. *For any smooth manifold M the sequence*

$$0\longrightarrow K_{l-1}(E;F)\longrightarrow \mathrm{PDiff}_l(E;F)\xrightarrow{\sigma_l}\mathrm{Smb}_l(E;F)\longrightarrow 0$$

is exact. Here $K_{l-1}:=\operatorname{Ker}\sigma_l$. Morever, if M is compact, then

$$L\in K_{l-1}\qquad\Longrightarrow\qquad L\colon W^{k,2}(E)\longrightarrow W^{k-(l-1),2}(F)\ \text{ is bounded.}\tag{2.67}$$

Sketch of proof. We first prove (2.67). Thus, assume $L\in K_{l-1}$, that is $\sigma_l(L)=0$. Then locally we have $L=L_p$ and

$$\sigma_l(p)=0\qquad\Longrightarrow\qquad p\in\tilde{S}_0^{l-1}$$

by (2.53). Then (2.67) follows from Theorem 2.58.

We also need to show that $\sigma_l\colon \mathrm{PDiff}_l(E;F)\to\mathrm{Smb}_l(E;F)$ is surjective. Thus given $\sigma\in\mathrm{Smb}_l(E;F)$, pick a chart $(U,(x_1,\dots,x_n))$ such that both E and F admit trivializations over U. Hence, given these choices σ can be written as a matrix $p=(p_{ij}(x,\xi))$ so that the local pseudodifferential operator L_p can be defined. Then the global map $L\colon\Gamma(E)\to\Gamma(F)$ can be obtained with the help of a partition of unity. The reader may find the details in [Wel08, Thm. IV.3.16]. ☐

Just like in the case of linear differential operators we have the following result.

Theorem 2.68. *Let $E,F,$ and G be Hermitian vector bundles over a compact manifold M. Then the following holds:*

(a) If $L\in\mathrm{PDiff}_l(E;F)$ and $S\in\mathrm{PDiff}_s(F;G)$, then $S\circ L\in\mathrm{PDiff}_{l+s}(E;G)$ and $\sigma_{l+s}(S\circ L)=\sigma_s(S)\circ\sigma_l(L)$.

(b) If $L\in\mathrm{PDiff}(E;F)$, then the formal adjoint L^* exists and belongs to $\mathrm{PDiff}_l(F;E)$. Moreover, $\sigma_l(L^*)=\sigma_l(L)^*$.

The proof, which is omitted here, can be found in [Wel08, Thm. IV.3.17].

2.4.1 Elliptic operators and their parametrices

Definition 2.69. An operator $L\in\mathrm{PDiff}_l(E;F)$ is said to be elliptic, if for all $(m,\xi)\in T^*M$ such that $\xi\neq 0$ the symbol $\sigma(L)(m,\xi):E_m\to F_m$ is an isomorphism.

Example 2.70. Consider the Laplacian Δ acting on functions. By Example 2.42, $\sigma(\Delta)(m,\xi) = |\xi|^2$. Hence, Δ is elliptic.

More generally, consider the Laplacian $\Delta = dd^* + d^*d$ acting on p-forms. By Example 2.43, we know that $\sigma(d)(\xi)\alpha = i\,\xi\wedge\alpha$. For $\xi \in T^*M$ denote by $\xi^\# \in T_mM$ the metric dual of ξ, that is $\xi^\#$ is defined by requiring that the equality

$$g_m\left(\xi^\#, v\right) = \xi(v)$$

holds for any $v \in T_mM$. A computation yields that the adjoint of $\sigma(d)$ is given by

$$\sigma(d)^*(\xi)\beta = -i\left(\iota_{\xi^\#}\beta\right),$$

where $\imath_{\xi^\#}\colon \Lambda^{p+1}T^*_mM \to \Lambda^pT^*_mM$ is the contraction:

$$\imath_{\xi^\#}\omega = \omega\left(\xi^\#,\ \cdot\,,\dots,\ \cdot\,\right).$$

This yields in turn:

$$\begin{aligned}\sigma(\Delta)\omega &= \left(\sigma(d)\circ\sigma(d^*) + \sigma(d^*)\circ\sigma(d)\right)\omega = \xi\wedge\left(\iota_{\xi^\#}\omega\right) + \iota_{\xi^\#}\left(\xi\wedge\omega\right)\\ &= \xi\wedge\left(\iota_{\xi^\#}\omega\right) + |\xi|^2\,\omega - \xi\wedge i_{\xi^\#}\omega = |\xi|^2\,\omega.\end{aligned}$$

Thus, $\sigma(\Delta)$ is still given by the multiplication with $|\xi|^2$ and, hence, Δ is elliptic.

This example largely explains our interest to elliptic operators. It turns out that this class contains many other interesting operators and has particularly good properties, which we consider next.

Definition 2.71.

(i) If $L\colon C^\infty(E)\to C^\infty(F)$ is a linear map, then we say that $L\in \mathrm{OP}_l(E;F)$ if the closure of L yields a bounded map $W^{k,2}(E)\to W^{k-l,2}(F)$ for each $k\in\mathbb{R}$.

(ii) For $L\colon C^\infty(E)\to C^\infty(F)$, a linear map $S\colon C^\infty(F)\to C^\infty(E)$ is called a parametrix, if

$$L\circ S - id_F \in \mathrm{OP}_{-1}(F) \qquad\text{and}\qquad S\circ L - id_E\in \mathrm{OP}_{-1}(E).$$

For example, any pseudodifferential operator of order l on a compact manifold belongs to OP_l by Proposition 2.64. A parametrix should be understood as an "approximate inverse" in some sense and explicit examples are not easy to construct (this is not very surprising). One way to obtain a class of examples is as follows.

Suppose $\Omega\subset\mathbb{R}^n$ is a domain with smooth boundary $\partial\Omega$. Consider the classical Dirichlet boundary value problem

$$\begin{cases}\Delta u = f & \text{in }\Omega,\\ u\big|_{\partial\Omega} = 0,\end{cases}\tag{2.72}$$

where $\Delta = -\sum\frac{\partial^2}{\partial x_j^2}$ is the standard Laplacian acting on functions. I will use the well-known fact that the solution operator G, which assigns to f a unique solution $u = u_f$ of (2.72) belongs to OP_{-2} (we shall reprove this below). Taking this as granted, consider a slightly more general boundary value problem

$$\begin{cases}(\Delta + K)u = 0 & \text{in }\Omega,\\ u\big|_{\partial\Omega} = 0,\end{cases}$$

where $K\in\mathrm{Diff}_1$. Then for $L := \Delta + K$ we have $G\circ L - id = G\circ\Delta + G\circ K - id = G\circ K\in\mathrm{OP}_{-1}$ and also $L\circ G - id = K\circ G\in\mathrm{OP}_{-1}$. Thus, G is a parametrix for $\Delta + K$.

One of the main theorems about elliptic operators is the following.

Theorem 2.73. *For any elliptic $L \in \mathrm{PDiff}_l(E;F)$ there exists a parametrix $S \in \mathrm{PDiff}_{-l}(F;E)$.*

Proof. For any elliptic L the symbol $\sigma(L)$ is invertible at each (m,ξ), $\xi \neq 0$. Then by Theorem 2.66 there exists some $S \in \mathrm{PDiff}_{-l}$ such that $\sigma_{-l}(S) = \sigma_l(L)^{-1}$. We have

$$\sigma_0(L \circ S - id_F) = \sigma_l(L) \circ \sigma_{-l}(S) - \sigma_0(id_F) = 0,$$

so that $L \circ S - id_F \in \mathrm{OP}_{-1}$.

A similar argument yields that $S \circ L - id_E$ belongs to OP_{-1} too. $\square$

2.4.2 Elliptic estimate

Let B_1 and B_2 be two Banach spaces. Recall that an operator $K\colon B_1 \to B_2$ is said to be compact, if the image of the unit ball in B_1 is relatively compact in B_2. This means that for any sequence u_j in B_1 such that $\|u_j\|_{B_1} \leq 1$ there exists a subsequence u_{j_k} such that Ku_{j_k} converges in B_2.

Proposition 2.74. *Let M be a compact manifold, $L \in \mathrm{Diff}_l(E;F)$ an elliptic operator, and k a real number. Then the following holds:*

(i) $\mathcal{H}_L := \mathrm{Ker}\left(L\colon W^{k,2} \to W^{k-l,2}\right)$ is finite-dimensional and consists of smooth sections only; In particular, $\mathcal{H}_L$ does not depend on k.

(ii) Denote $V := \mathcal{H}_L^{\perp} \cap W^{k+l,2}(E)$, where $\perp$ means the orthogonal complement in $L^2(E)$. Then for all $k \geq 0$ there exists a constant $C_k > 0$ such that

$$\|Lv\|_{W^{k,2}} \geq C_k \|v\|_{W^{k+l,2}} \tag{2.75}$$

holds for all $v \in V$.

Proof. The proof consists of the following steps.

Step 1. *$K \in \mathrm{OP}_{-1}(E;F) \Longrightarrow K\colon W^{k,2}(E) \to W^{k,2}(F)$ is compact provided M is compact.*

Indeed, $K\colon W^{k,2}(E) \to W^{k+1,2}(F)$ is bounded and the embedding $j\colon W^{k+1,2}(F) \to W^{k,2}(F)$ is compact, since M is compact. Then $K\colon W^{k,2}(E) \to W^{k,2}(F)$ is compact as the composition of bounded and compact linear maps.

Step 2. *We prove (i).*

For the proof we need the following standard fact of functional analysis: The unit sphere in a Banach space B is compact if and only if B is finite-dimensional. With this understood, we obtain that for a compact operator $R\colon B \to B$, where B is a Banach space, any eigenspace $B_\lambda := \{u \mid Ru = \lambda u\}$ is finite dimensional provided $\lambda \neq 0$.

Assume $Lu = 0$. Applying a parametrix of L to both sides of this equation, we obtain $u + Ru = 0$, where $R = S \circ L - id\colon W^{k,2} \to W^{k,2}$ is compact. Hence, the space

$$\mathcal{H}_L^k := \left\{u \in W^{k,2} \mid Lu = 0\right\} = \left\{u \in W^{k,2} \mid Ru = -u\right\}$$

is finite-dimensional. Moreover, since $R \in \mathrm{OP}_{-1}$,

$$u \in W^{k,2} \quad \Longrightarrow \quad u \in W^{k+1,2} \quad \Longrightarrow \quad u \in W^{k+2,2} \quad \Longrightarrow \dots$$

so that $u \in \cap W^{s,2} = C^\infty$. In particular, each $u \in \mathcal{H}_L^k$ is smooth so that $\mathcal{H}_L^k$ in fact does not depend on k.

Step 3. *We prove* (2.75).

Assume (2.75) fails, that is there exists a sequence $v_j \in V$ such that

$$\|v_j\|_{W^{k+l,2}} = 1 \qquad \text{and} \qquad Lv_j \longrightarrow 0 \qquad \text{in } W^{k,2}.$$

Applying the parametrix S we obtain that $v_j + Sv_j \to 0$ in $W^{k+l,2}$. Since S is compact, v_j has a subsequence, which converges to some v in $W^{k+l,2}$. Then we must have

$$\|v\|_{W^{k+l,2}} = 1, \quad Lv = 0, \quad \text{and} \quad v \in (\operatorname{Ker} L)^{\perp_{L^2}}.$$

This contradiction finishes the proof. □

Corollary 2.76 (Elliptic estimate). *Let M be a compact manifold and $L \in \operatorname{Diff}_l(E;F)$ be an elliptic operator. Then for any $k \geq 0$ there exists a constant $C_k > 0$ such that the estimate*

$$\|u\|_{W^{k+l,2}} \leq C_k \left(\|Lu\|_{W^{k,2}} + \|u\|_{L^2} \right)$$

holds for any $u \in W^{k+l,2}$.

Proof. Since $\mathcal{H}_L \subset W^{k+l,2}$, for any $u \in W^{k+l,2}$ we can write $u = v + w$, where $v \in \mathcal{H}_L$ and $(v,w)_{L^2} = 0$. Notice that since v is smooth, $w = u - v \in W^{k+l,2}$, that is we have the decomposition

$$W^{k+l,2} = \mathcal{H}_L \oplus \mathcal{H}_L^{\perp_{L^2}} \cap W^{k+l,2}.$$

Since $\mathcal{H}_L$ is finite dimensional, the restriction of $\|\cdot\|_{W^{k+l,2}}$ to $\mathcal{H}_L$ is equivalent to the restriction of $\|\cdot\|_{L^2}$. Combining this with (2.75) we obtain

$$\begin{aligned} \|u\|_{W^{k+l,2}} &\leq \|v\|_{W^{k+l,2}} + \|w\|_{W^{k+l,2}} \leq C_k'\|v\|_{L^2} + C_k''\|Lw\|_{W^{k,2}} \\ &= C_k'\|v\|_{L^2} + C_k''\|Lu\|_{W^{k,2}} \leq C_k \left(\|Lu\|_{W^{k,2}} + \|u\|_{L^2} \right). \end{aligned}$$

This finishes the proof. □

Theorem 2.77. *Let M be a compact manifold and $L \in \operatorname{Diff}_l(E;F)$ be an elliptic operator. Then for any $k \in \mathbb{R}$ the operator $L \colon W^{k+l,p}(E) \to W^{k,p}(F)$ is Fredholm, i.e.,*

1. $\dim \operatorname{Ker} L < \infty$;
2. *The image of L is a closed subspace of $W^{k,p}(F)$;*
3. *The cokernel* $\operatorname{Coker} L := W^{k,p}(F) / \operatorname{Im} L$ *is finite dimensional.*

Proof. We shall give the proof only in the case $p = 2$ and $k \geq 0$ (mainly just because most of the results required were proved for these values).

We know already from Proposition 2.74, *(i)* that $\operatorname{Ker} L$ is finite dimensional. The rest of the proof consists of the following steps.

Step 1. $\operatorname{Im} L$ *is closed.*

Clearly $\operatorname{Im} L = L(V)$, where $V = \mathcal{H}_L^{\perp} \cap W^{k+l,2}(E)$. Assume w lies in the closure of $L(V)$, that is there exist a sequence $v_j \in V$ such that $Lv_j \to w$. Then by (2.75) we have

$$\|v_i - v_j\|_{W^{k+l,2}} \leq C\|Lv_i - Lv_j\|_{W^{k,2}} \leq C \left(\|Lv_i - w\|_{W^{k,2}} + \|Lv_j - w\|_{W^{k,2}} \right).$$

This yields that (v_j) is a Cauchy sequence, hence converges to some $v \in V$. Then we must have

$$Lv = \lim_{j \to \infty} Lv_j = w.$$

Step 2. *Let L^* be the formal adjoint of L. Then $w \in \operatorname{Im}(L)^{\perp_{L^2}} \cap W^{k,2}(F)$ if and only if $w \in \operatorname{Ker} L^*$.*

If $w \in \operatorname{Im}(L)^{\perp_{L^2}} \cap W^{k,2}(F)$, then for any $u \in W^{k+l,2}(E)$ we have

$$0 = (Lu, w)_{L^2} = (u, L^* w)_{L^2} \quad \Longrightarrow \quad L^* w = 0.$$

Conversely, if $L^* w = 0$, then $(Lu, w)_{L^2} = 0$ for any $u \in W^{k+l,2}(E)$.

Step 3. *We prove this theorem.*

It only remains to show that $\dim \operatorname{Coker} L < \infty$. To see this, identify $\operatorname{Coker} L$ with $(\operatorname{Im} L)^{\perp_{L^2}} \cap W^{k,2}(F)$ which equals $\operatorname{Ker} L^*$ by the preceding step. It remains to notice that L^* is elliptic too, so that $\dim \operatorname{Ker} L^* < \infty$ by Proposition 2.74, *(i)*. $\square$

The proof of the above theorem actually implies the following.

Corollary 2.78. *Let M be a compact manifold and $L \in \operatorname{Diff}_l(E; F)$ be an elliptic operator. Given $w \in W^{k,2}(F)$ the equation*

$$Lu = w \tag{2.79}$$

has a solution $u \in W^{k+l,2}(E)$ if and only if $w \perp \operatorname{Ker} L^*$. Moreover, if $w \in C^\infty(M; F)$, then any solution u of (2.79) is smooth.

Proof. We only need to show that $w \in C^\infty \Rightarrow u \in C^\infty$. However, this follows immediately from $\cap W^{k,2}(E) = C^\infty(E)$. $\square$

Corollary 2.80 (Fredholm's alternative)**.** *Let M be a compact manifold and $L \in \operatorname{Diff}_l(E; F)$ an elliptic operator. Then one and only one of the following statements holds:*

(a) The inhomogeneous equation $Lu = w$ has a solution for any $w \in \Gamma(F)$.

(b) The homogeneous equation $L^* v = 0$ has a non-trivial solution.

Corollary 2.78 implies in particular Theorem 1.38. To see this, it just suffices to recall that the Laplacian is formally self-adjoint, that is $\Delta^* = \Delta$, which is the content of Proposition 1.36.

In the case of the Laplacian, it is also possible to give a more detailed version of Corollary 2.78 as follows.

Theorem 2.81 (Hodge)**.** *Let M be a closed oriented Riemannian manifold. Denote $\mathcal{H}^k := \operatorname{Ker}\big(\Delta \colon \Omega^k(M) \to \Omega^k(M)\big)$. We have the following decomposition*

$$\Omega^k(M) = \mathcal{H}^k \oplus \operatorname{Im} d \oplus \operatorname{Im} d^*,$$

where all three spaces are orthogonal with respect to the L^2-scalar product.

Proof. First we show that $\mathcal{H}^k$ is orthogonal to $\operatorname{Im} d$ in the L^2-sense. Indeed, if $\omega \in \mathcal{H}^k$ and $\eta \in \Omega^{k-1}(M)$, then

$$\langle \omega, d\eta \rangle_{L^2} = \langle d^* \omega, \eta \rangle_{L^2} = 0,$$

since $d^* \omega = 0$ by Proposition 1.30.

The claims that $\mathcal{H}^k \perp \operatorname{Im} d^*$ and $\operatorname{Im} d \perp \operatorname{Im} d^*$ can be obtained by using similar arguments.

Furthermore, to prove the rest of the claim it suffices to show that $\operatorname{Im} \Delta = \operatorname{Im}(d) \oplus \operatorname{Im} d^*$. Clearly, we have $\operatorname{Im} \Delta \subset \operatorname{Im} d \oplus \operatorname{Im} d^*$ by the definition of Δ. The converse inclusion follows from Corollary 2.78 and the self-adjointness of Δ. $\square$

2.5 Elliptic complexes

The de Rham complex can be generalized as follows. Let $E_0, E_1, \dots, E_N$ be Hermitian vector bundles over a smooth manifold M. Assume we have the sequence

$$\Gamma(E_0) \xrightarrow{L_0} \Gamma(E_1) \xrightarrow{L_1} \Gamma(E_2) \longrightarrow \dots \xrightarrow{L_{N-1}} \Gamma(E_N), \tag{2.82}$$

where each $L_j \in \mathrm{Diff}_l(E_j; E_{j+1})$ and l does not depend on j. Assume also that (2.82) is a complex, i.e., $L_{j+1} \circ L_j = 0$.

Definition 2.83. (2.82) is said to be an elliptic complex, if the associated sequence of symbols

$$0 \longrightarrow \pi^* E_0 \xrightarrow{\sigma_l(L_0)} \pi^* E_1 \xrightarrow{\sigma_l(L_1)} \quad \dots \quad \xrightarrow{\sigma_l(L_{N-1})} \pi^* E_N \longrightarrow 0$$

is exact away from the zero section of T^*M.

Just like in the case of the de Rham complex, we can define the corresponding cohomology groups by

$$H^j(E) := \frac{\mathrm{Ker}\,(L_j \colon \Gamma(E_j) \longrightarrow \Gamma(E_{j+1}))}{\mathrm{Im}\,(L_{j-1} \colon \Gamma(E_{j-1}) \longrightarrow \Gamma(E_j))},$$

where $H^0(E) = \mathrm{Ker}\, L_0$ and $H^N(E) = \Gamma(E_N) \,/\, \mathrm{Im}\, L_{N-1}$.

Example 2.84. Any elliptic operator $L \colon \Gamma(E) \to \Gamma(F)$ is a (very short) elliptic complex.

Example 2.85. The de Rham complex of any smooth manifold.

New examples of elliptic complexes will be given below.
Associated to (2.82), is the (generalized) Laplacian

$$\Delta_j := L_j^* L_j + L_{j-1} L_{j-1}^* \colon \Gamma(E_j) \longrightarrow \Gamma(E_j),$$

which is clearly self-adjoint.

Exercise 2.86. Show that Δ_j is an elliptic operator provided (2.82) is an elliptic complex.

Any element of the space

$$\mathcal{H}(E_j) := \mathrm{Ker}\, \Delta_j \subset \Gamma(E_j)$$

is called harmonic.

Theorem 2.87. *If M is closed connected oriented Riemannian manifold, then the following holds:*

(i) $s \in \mathcal{H}(E_j) \iff L_j s = 0$ *and* $L_{j-1}^* s = 0$.

(ii) There is an orthogonal decomposition

$$\Gamma(E_j) = \mathcal{H}(E_j) \oplus \mathrm{Im}\, L_{j-1} \oplus \mathrm{Im}\, L_j^*.$$

(iii) $\dim \mathcal{H}(E_j) < \infty$ *and there is a canonical isomorphism*

$$\mathcal{H}(E_j) \longrightarrow H^j(E), \qquad s \longmapsto [s].$$

The proof of this theorem can be obtained by a straightforward modification of the proofs of the corressponding statements for the de Rham complex. I leave the details to the reader.

2.5.1 The Dolbeault complex

In this section M denotes a complex manifold of complex dimension n. This means that M is a real manifold (of dimension $2n$), each point admits a chart of the form $\psi\colon U \to \mathbb{C}^n$ and each change-of-coordinates map $\psi \circ \phi^{-1}\colon \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic.

Just like in the case of Riemann surfaces, writing $\psi = (z_1, \dots, z_n)$ we obtain real coordinates $(x_1, y_1, \dots, x_n, y_n)$, where

$$x_j = \operatorname{Re} z_j \qquad \text{and} \qquad y_j = \operatorname{Im} z_j.$$

Then we can define the complex structure $I\colon TM \longrightarrow TM$ by

$$I\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} \qquad \text{and} \qquad I\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}.$$

Using the fact that the change-of-coordinates maps are holomorphic, it can be shown that I is well-defined. Since $I^2 = -id$ by the very definition, we obtain that each tangent space T_mM is equipped with the structure of a complex vector space via

$$(a + bi) \cdot v = a\, v + b\, Iv, \qquad v \in T_mM.$$

Then the complexification of the cotangent space splits:

$$\begin{aligned}
T^*_mM \otimes \mathbb{C} &= (T^*_mM)^{1,0} \oplus (T^*_mM)^{0,1}\,,\\
(T^*_mM)^{1,0} &\coloneqq \{\psi\colon T_mM \longrightarrow \mathbb{C} \mid \psi\,(Iv) = i\psi\,(v)\}\,,\\
(T^*_mM)^{0,1} &\coloneqq \{\psi\colon T_mM \longrightarrow \mathbb{C} \mid \psi\,(Iv) = -i\psi\,(v)\}\,.
\end{aligned}$$

In other words, $(T^*_mM)^{1,0}$ consists of $\mathbb{C}$-linear functionals, whereas $(T^*_mM)^{0,1}$ consists of $\mathbb{C}$-antilinear ones.

Since $(T^*_mM)^{1,0}$ and $(T^*_mM)^{0,1}$ are isomorphic via $\psi \mapsto \bar{\psi}$, we obtain

$$\dim_{\mathbb{C}} (T^*_mM)^{1,0} = \dim_{\mathbb{C}} (T^*_mM)^{0,1} = \frac{1}{2}\dim_{\mathbb{C}} (T^*_mM \otimes \mathbb{C}) = n.$$

If $(z_1, \dots, z_n)$ are local holomorphic coordinates in a neighbourhood of m, denote

$$dz_j = dx_j + i\,dy_j \in (T^*_mM)^{1,0} \qquad \text{and} \qquad d\bar{z}_j = dx_j - i\,dy_j \in (T^*_mM)^{0,1}\,,$$

where $j = 1, \dots, n$. It is easy to see that $dz \coloneqq (dz_1, \dots dz_n)$ consists of linearly independent 1-forms. Hence, dz is a basis of $(T^*_mM)^{1,0}$ at each point m where the coordinates $(z_1, \dots, z_n)$ are defined. Likely, $d\bar{z} = (d\bar{z}_1, \dots, d\bar{z}_n)$ is a basis of $(T^*_mM)^{0,1}$.

Furthermore, using the isomorphism

$$\Lambda^k (U \oplus V) = \bigoplus_{\substack{p+q=k\\ p,q\geq 0}} \Lambda^p U \otimes \Lambda^q V$$

we obtain the decomposition

$$\Lambda^k (T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=k} \Lambda^p (T^*M)^{1,0} \otimes \Lambda^q (T^*M)^{0,1} =: \bigoplus_{p+q=k} \Lambda^{p,q}.$$

Denote

$$\Omega^{p,q}(M) \coloneqq \Gamma\left(\Lambda^{p,q}\right).$$

Thus, $\omega \in \Omega^{p,q}(M)$ if and only if ω is a smooth complex-valued differential form of degree $p+q$ and locally ω can be written in the following form

$$\omega = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_q}} \omega_{i_1\ldots i_p;j_1\ldots j_q}\, dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_q}.$$

For any $\omega \in \Omega^{0,0}(M) \cong C^\infty(M;\mathbb{C})$ we have $d\omega = \partial\omega + \bar\partial\omega$, where

$$\partial\omega = \sum_{j=1}^n \frac{\partial\omega}{\partial z_j} dz_j \in \Omega^{1,0}(M) \qquad \text{and} \qquad \bar\partial\omega = \sum_{j=1}^n \frac{\partial\omega}{\partial \bar z_j} d\bar z_j \in \Omega^{0,1}(M).$$

More generally, if $\omega \in \Omega^{p,q}(M)$, then we still have $d\omega = \partial\omega + \bar\partial\omega$, where

$$\partial\omega = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_q}} \partial\omega_{i_1\ldots i_p;j_1\ldots j_q}\wedge dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_q},$$

$$\bar\partial\omega = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_q}} \bar\partial\omega_{i_1\ldots i_p;j_1\ldots j_q}\wedge dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_q}.$$

In particular, $\partial\colon \Omega^{p,q}\to\Omega^{p+1,q}$ and $\bar\partial\colon \Omega^{p,q}\to\Omega^{p,q+1}$.

Notice that since $d = \partial + \bar\partial$, we have

$$d^2 = 0 \qquad \iff \qquad \partial^2 = 0, \quad \bar\partial^2 = 0, \quad \partial\bar\partial = -\bar\partial\partial.$$

Therefore, for any fixed p such that $0 \le p \le n$ we obtain the following complex

$$0 \longrightarrow \Omega^{p,0}(M) \xrightarrow{\bar\partial} \Omega^{p,1}(M) \xrightarrow{\bar\partial} \Omega^{p,2}(M) \xrightarrow{\bar\partial} \ldots \xrightarrow{\bar\partial} \Omega^{p,n}(M) \longrightarrow 0,$$

which is called the Dolbeault complex. The corresponding cohomology groups

$$H^{p,q}(M) := \frac{\operatorname{Ker}\left(\bar\partial\colon \Omega^{p,q}\longrightarrow\Omega^{p,q+1}\right)}{\operatorname{Im}\left(\bar\partial\colon \Omega^{p,q-1}\longrightarrow\Omega^{p,q}\right)}$$

are called the Dolbeault cohomolgy groups of M. Notice, however, that unlike the de Rham cohomology groups, the Dolbeault cohomology groups depend on the complex structure of M. In particular, they are *not* topological invariants of M.

Proposition 2.88. *For any complex manifold M the Dolbeault complex is elliptic.*

Proof. Just like in Example 2.43, for the symbol of the $\bar\partial$-operator we have

$$\sigma\left(\bar\partial\right)(\xi)\,\omega = i\,\xi^{0,1}\wedge\omega,$$

where $\xi^{0,1} = \frac{1}{2}\left(\xi + i\xi\left(I\cdot\right)\right)$ is the $(0,1)$-part of ξ and $\omega\in\Lambda^{p,q}$. We have to show the following

$$\xi^{0,1}\wedge\omega = 0 \qquad \Longleftrightarrow \qquad \omega = \xi^{0,1}\wedge\eta$$

for some $\eta\in\Lambda^{p,q-1}$, where $\xi\ne 0 \Leftrightarrow \xi^{0,1}\ne 0$.

Assume ξ is based at a point m. Since $\xi^{0,1}\ne 0$, we can find a complex basis $(\psi_1,\ldots,\psi_n)$ of $(T_m^*M)^{1,0}$ such that $\psi_1 = \overline{\xi^{0,1}}$. Then $\omega\in\Lambda^{p,q}_m$ can be uniquely written in the form

$$\omega = \sum_{\substack{i_1<\ldots<i_p \\ j_1<\ldots<j_q}} \omega_{i_1<\ldots<i_p;j_1<\ldots<j_q}\,\psi_{i_1}\wedge\ldots\wedge\psi_{i_p}\wedge\bar\psi_{j_1}\wedge\ldots\wedge\bar\psi_{j_q}.$$

Then $\xi^{0,1} \wedge \omega = 0 \;\Leftrightarrow\; \bar{\psi}_1 \wedge \omega = 0$ yields $\omega_{i_1,\ldots i_p;j_1,\ldots j_q} = 0$ provided $j_1 \neq 1$. Hence,

$$\begin{aligned}\omega &= \sum_{\substack{i_1<\ldots<i_p \\ j_1=1<j_2<\ldots<j_q}} \omega_{i_1\ldots i_p;1\,j_2\ldots j_q}\, \psi_{i_1} \wedge \ldots \wedge \psi_{i_p} \wedge \bar{\psi}_1 \wedge \bar{\psi}_{j_2} \wedge \ldots \wedge \bar{\psi}_{j_q} \\ &= (-1)^p\, \bar{\psi}_1 \wedge \sum_{\substack{i_1<\ldots<i_p \\ j_2<\ldots<j_q}} \omega_{i_1\ldots i_p;1\,j_2\ldots j_q}\, \psi_{i_1} \wedge \ldots \wedge \psi_{i_p} \wedge \bar{\psi}_{j_2} \wedge \ldots \wedge \bar{\psi}_{j_q} \\ &= \xi^{0,1} \wedge \eta.\end{aligned}$$

This finishes the proof of this proposition. □

Corollary 2.89. *If M is a compact complex manifold, then $h^{p,q}(M) := \dim_{\mathbb{C}} H^{p,q}(M) < \infty$.*

□

The integers $h^{p,q}(M)$ are called the Hodge numbers of M.

Let $\bar{\partial}^*$ be the formal adjoint of $\bar{\partial}$. The Laplacian corresponding to the Dolbeault complex is then

$$\square := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

which is called the Hodge-Laplacian. By Theorem 2.87 we have an isomorphism

$$H^{p,q}(M) \cong \mathrm{Ker}\left(\square \colon \Omega^{p,q}(M) \longrightarrow \Omega^{p,q}(M)\right).$$

Definition 2.90. Let g be a Riemannian metric on a complex manifold M such that $g(Iv, Iw) = g(v, w)$ for all v, w (such metrics always exist). Define a 2-form ω on M by

$$\omega(v, w) := g(Iv, w).$$

Then g is said to be Kähler, if ω is closed.

Kähler metrics should be understood as those metrics, which fit with the complex structure particularly well (this may not be obvious from the definition and I shall not try to justify this claim here).

One can show that the induced metric on an embedded complex submanifold of a Kähler manifold is itself Kähler. Furthermore, the induced metric on $\mathbb{C}P^n = S^{2n+1}/\mathrm{U}(1)$ is Kähler (this is known as the Fubini–Study metric). This yields plenty of examples of Kähler manifolds. Albeit these facts are rather elementary, I shall omit the proofs here.

Theorem 2.91. *If g is Kähler, then Δ maps $\Omega^{p,q}(M)$ into $\Omega^{p,q}(M)$ and $\Delta = 2\square$.*

In particular, if M is compact and Kähler, for any $k \geq 0$ we have

$$\mathcal{H}^k(M) \cong \bigoplus_{p+q=k} H^{p,q}(M) \qquad \text{and} \qquad H^{p,q}(M) \cong H^{q,p}(M) \tag{2.92}$$

for all p and q. □

Corollary 2.93. *If M is a compact Kähler manifold, then all odd Betti numbers of M are even.*

Proof. By (2.92) we have

$$\begin{aligned} b_{2k+1}(M) &= h^{0,2k+1} + h^{1,2k} + \ldots + h^{2k,1} + h^{2k+1,0} \\ &= 2\left(h^{0,2k+1} + h^{1,2k} + \ldots + h^{k,k+1}\right), \end{aligned}$$

where the last equality follows from $h^{p,q}(M) = h^{q,p}(M)$. □

Corollary 2.94. *There exist compact complex manifolds, which can not be embedded into any complex projective manifold.*

Sketch of the proof. Consider the following action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$:

$$a \cdot (z_1, z_2) = \left(e^{-a} z_1, e^{-a} z_2\right). \tag{2.95}$$

Think of $\mathbb{C}^2 \setminus \{0\}$ as $\mathbb{R} \times S^3$ via the diffeomorphism

$$\mathbb{R} \times S^3 \longrightarrow \mathbb{C}^2 \setminus \{0\}, \qquad (t, \sigma) \longmapsto e^t \cdot \sigma$$

Then the above action becomes $a \cdot (t, \sigma) = (t - a, \sigma)$. Hence, the quotient $M = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$ is diffeomorphic to $S^1 \times S^3$. In particular, M is compact and $b_1(M) = b_1(S^1 \times S^3) = 1$.

Furthermore, notice that M is a complex manifold, since action (2.92) preserves the standard complex structure of $\mathbb{C}^2 \setminus \{0\}$. However, M cannot be embedded as a complex submanifold into $\mathbb{C}P^N$, since otherwise M would be Kähler, which would contradict Corollary 2.93. □

Remark 2.96. It should be noticed, that any holomorphic function on a compact complex manifold is constant. This fact follows easily from the maximum modulus principle. Hence, a compact complex manifold can never be embedded (as a complex submanifold) into $\mathbb{C}^N$. Thus, $\mathbb{C}P^N$ can be seen as the "best" replacement for $\mathbb{C}^N$ for complex manifolds. Then Corollary 2.94 claims essentially that the Whitney embedding theorem is false in the context of complex manifolds.

Also, Corollary 2.94 should be contrasted with the fact that any compact Riemann surface can be embedded into some complex projective space.

Chapter 3

The Atiyah–Singer index theorem

3.1 Fredholm maps and their indices

Let H_1 and H_2 be separable Hilbert spaces. Recall that a linear bounded map $T\colon H_1 \to H_2$ is said to be Fredholm, if $\dim \operatorname{Ker} T < \infty, \operatorname{Im} T \subset H_2$ is a closed subspace, and $\dim \operatorname{Coker} T = \dim H_2 / \operatorname{Im} T < \infty$. Denote by $\mathcal{F}(H_1; H_2)$ the set of all Fredholm maps. Also, let $\mathcal{B}(H_1; H_2)$ denote the space of all linear bounded maps $H_1 \to H_2$ equipped with the norm

$$\|T\| := \sup_{h \in H_1 \setminus \{0\}} \frac{\|Th\|_{H_2}}{\|h\|_{H_1}}.$$

Theorem 3.1. *$\mathcal{F}(H_1; H_2)$ is an open subset of $\mathcal{B}(H_1; H_2)$. The map*

$$\operatorname{ind} : \mathcal{F}(H_1; H_2) \longrightarrow \mathbb{Z}, \qquad T \longmapsto \operatorname{ind} T := \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$$

is constant on each connected component of $\mathcal{F}(H_1; H_2)$.

Proof. The proof consists of the following steps.

Step 1. *$\mathcal{F}(H_1; H_2)$ is open in $\mathcal{B}(H_1; H_2)$.*

Pick $T_0 \in \mathcal{F}(H_1; H_2)$ and denote $V_0 := (\operatorname{Ker} T_0)^\perp \subset H_1$ and $W_0 = (\operatorname{Im} T_0)^\perp \cong \operatorname{Coker} T_0$. For any $T \in \mathcal{B}(H_1; H_2)$ consider the map

$$\tilde T\colon V \oplus W \longrightarrow H_2, \qquad \tilde T(v, w) = Tv + w.$$

Since $\tilde T_0$ is an isomorphism, $\tilde T$ is also an isomorphism provided T lies in a sufficiently small neighbourhood $U \subset \mathcal{B}(H_1, H_2)$ of T_0. In particular, for $T \in U$ we have

$$(\operatorname{Ker} T) \cap V = \{0\} \qquad \Longrightarrow \qquad \dim \operatorname{Ker} T < \infty.$$

Moreover, $T(V) = \tilde T(V) \subset H_2$ is a closed subspace of finite codimension. Hence, $\operatorname{Im} T$ is also closed and of finite codimension. Thus, T is Fredholm.

Step 2. ind *is constant on U, where U is as in the preceding step.*

Keeping the notations of the preceding step, we have the (non-orthogonal) decomposition

$$H_1 = \operatorname{Ker} T \oplus Z \oplus V,$$

where $Z := (\operatorname{Ker} T \oplus V)^{\perp}$. Then $T \colon Z \oplus V \to T(Z) \oplus T(V) = \operatorname{Im} T$ is an isomorphism. Moreover,

$$\operatorname{Ker} T_0 \cong H_1/V \cong \operatorname{Ker} T \oplus Z \qquad \text{and} \qquad \operatorname{Coker} T_0 \cong H_2/\operatorname{Im} T_0 = H_2/T_0(V). \tag{3.2}$$

Furthermore, notice that for any $T \in U$ we have

$$T(V) \cong \tilde{T}(V) \cong H_2/\tilde{T}(V) \cong W.$$

In particular, $T(V) \cong W \cong T_0(V)$ so that we obtain

$$\operatorname{Coker} T_0 \cong H_2/T(V) \cong (\operatorname{Im} T)^{\perp} \oplus T(Z) \cong \operatorname{Coker} T \oplus T(Z).$$

Combining this with (3.2), we obtain $\operatorname{ind} T = \operatorname{ind} T_0$. □

Remark 3.3. One can show that $\operatorname{ind} T_1$ equals $\operatorname{ind} T_2$ if and only if T_1 and T_2 lie in the same connected component of $\mathcal{F}(H_1; H_2)$.

Proposition 3.4. *If T is Fredholm and K is compact, then $T + K$ is also Fredholm and* $\operatorname{ind}(T + K) = \operatorname{ind}(T)$.

Proof. The proof consists of the following steps.

Step 1. *A bounded map $T \colon H_1 \to H_2$ is Fredholm if and only if there exist bounded maps $S_1, S_2 \colon H_1 \to H_1$ such that*

$$S_1 \circ T = \mathrm{id}_{H_1} + R_1 \qquad \textit{and} \qquad T \circ S_2 = \mathrm{id}_{H_2} + R_2,$$

where both R_1 and R_2 are compact.

The proof of this step is left as an exercise (see, however, the proof of Theorem 2.77).

Step 2. *We prove the statement of this proposition.*

Let T be a Fredholm map. Keeping the notations of the preceding step, we have

$$S_1(T + K) = S_1 T + S_1 K = \mathrm{id} + (R_1 + S_1 K),$$

where the map in parentheses is compact. Also, a similar argument yields that $(T + K) S_2 - \mathrm{id}$ is also compact. Hence, $T + K$ is Fredholm. Moreover, for each $\lambda \in [0, 1]$ the map $T + \lambda K$ is also Fredholm and therefore $\operatorname{ind}(T + \lambda K)$ is a locally constant function of λ. Hence, in fact $\operatorname{ind}(T + \lambda K)$ is constant in λ so that $\operatorname{ind}(T + K) = \operatorname{ind} T$. □

Corollary 3.5. *Let $L \in \operatorname{Diff}_l(E; F)$ be an elliptic differential operator. If M is closed, then*

$$\operatorname{ind}\big(L \colon W^{k+l,2}(E) \longrightarrow W^{k,2}(F)\big)$$

depends only on the principal symbol of L.

Proof. If $\sigma_l(L)$ is the principal symbol of L, pick any $L_0 \in \operatorname{Diff}_l(E; F)$ such that $\sigma_l(L_0) = \sigma_l(L)$. Then $K := L - L_0$ is of order at most $l - 1$. Hence, its closure (still denoted by the same letter)

$$K \colon W^{k+l,2}(E) \longrightarrow W^{k,2}(F)$$

is compact as a composition of a bounded map $W^{k+l,2}(E) \to W^{k+1,2}(F)$ and a compact one $W^{k+1,2}(F) \to W^{k,2}(F)$. Hence,

$$\operatorname{ind} L = \operatorname{ind}(L_0 + K) = \operatorname{ind} L_0.$$

□

Exercise 3.6. Find a mistake in the following "proof" of the claim that any two elliptic operators of the same order have equal indices. Thus, if L_0 and L_1 are two such operators, then $\operatorname{ind}((1-t)\,L_0 + tL_1)$ does not depend on t implying that $\operatorname{ind} L_0 = \operatorname{ind} L_1$.

Exercise 3.7. Show that the index of a formally self-adjoint differential operator vanishes.

The above corollary raises naturally the question how can one actually compute the index of a differential operator just in terms of its principal symbol. This task has been accomplished by Atiyah and Singer in 1960s. However, even to formulate the answer we shall need a detour.

3.2 Characteristic Classes

3.2.1 The curvature of a connection

Recall that given a vector bundle $E \to M$ equipped with a connection ∇ we have the sequence

$$\Omega^0(E) \xrightarrow{d_\nabla} \Omega^1(E) \xrightarrow{d_\nabla} \Omega^2(E) \xrightarrow{d_\nabla} \dots \xrightarrow{d_\nabla} \Omega^n(E) \longrightarrow 0,$$

see (2.26) for details. As it was already pointed out above, this sequence is not a complex in general, that is $d_\nabla^2 := d_\nabla \circ d_\nabla$ does not necessarily vanish. Nevertheless, d_∇^2 has the following interesting property.

Proposition 3.8. *$d_\nabla^2\colon \Omega^k(E) \longrightarrow \Omega^{k+2}(E)$ is $C^\infty(M)$-linear.*

Proof. Pick any $s \in \Gamma(E)$ and $f \in C^\infty(M)$. We have

$$\begin{aligned} d_\nabla d_\nabla(f\,s) &= d_\nabla(\nabla(f\,s)) = d_\nabla(df \otimes s + f\nabla s) \\ &= d(df) \otimes s - df \wedge \nabla s + df \wedge \nabla s + f d_\nabla(\nabla s) \\ &= f d_\nabla(\nabla s)\,. \end{aligned}$$

This proves this proposition for $k = 0$. The rest of the proof is left as an exercise to the reader. □

Remark 3.9. In our setting it is convenient to work over the field of complex numbers, although we could have considered $\mathbb{R}$ as a basic field equally well. Hence, we assume that ∇ is compatible with the complex structure of E, that is $\nabla(is) = i\nabla s$. In particular, in the above proposition $C^\infty(M)$ means $C^\infty(M;\mathbb{C})$. I shall not always be very picky concerning such details below.

Lemma 3.10. *Let $A\colon \Gamma(E) \to \Omega^k(M;F)$ be a $\mathbb{C}$-linear map, which is also $C^\infty(M;\mathbb{C})$-linear, that is*

$$A(f \cdot s) = f \cdot A(s) \qquad \forall f \in C^\infty(M;\mathbb{C}) \text{ and } s \in \Gamma(E)\,.$$

Then there exists $a \in \Omega^k(\operatorname{Hom}(E;F))$ such that $A(s) = a \cdot s$.

The proof of this proposition is also left as an exercise.

Corollary 3.11. *For any connection ∇ there exists a 2-form $F_\nabla \in \Omega^2(\operatorname{End}(E))$ such that for any $\omega \in \Omega^k(E)$ we have $d_\nabla(d_\nabla\omega) = F_\nabla \wedge \omega$.* □

Definition 3.12. The 2-form F_∇ as in the above corollary is called the curvature form of ∇.

To establish a geometric meaning of the curvature form, pick local coordinates $(x_1, \ldots x_n)$ on M. We could call

$$\nabla_j s = \imath_{\frac{\partial}{\partial x_j}} (\nabla s)$$

the j^{th} partial covariant derivative of $s \in \Gamma(E)$. In particular, $\nabla s = \sum_{j=1}^n dx_j \otimes \nabla_j s$. Hence,

$$\begin{aligned} d_\nabla (d_\nabla s) &= -\sum_{j=1}^n dx_j \wedge \nabla(\nabla_j s) = -\sum_{j=1}^n dx_j \wedge \left(\sum_{i=1}^n dx_i \otimes \nabla_i (\nabla_j s)\right) \\ &= \sum_{i<j} dx_i \wedge dx_j (\nabla_i(\nabla_j s) - \nabla_j(\nabla_i s)). \end{aligned}$$

In other words, by writing $F_\nabla = \sum_{i<j} F_{ij}\, dx_i \wedge dx_j$ where $F_{ij} \in \mathrm{End}(E)$, we have $F_{ij} s = \nabla_i(\nabla_j s) - \nabla_j(\nabla_i s)$. Thus, the curvature measures the extend to which partial covariant derivatives fail to commute.

3.2.2 Local representation of the curvature form

Recall that locally a connection can be identified with a 1-form. This means the following: Given an open subset U such that E admits a trivialization ψ over U, any section s over U can be identified with a map $\sigma\colon U \to \mathbb{C}^a$, where $a = \mathrm{rk}\, E$. Then

$$\nabla = d + A \qquad \Longleftrightarrow \qquad \nabla s \equiv d\sigma + A\sigma,$$

where $A \in \Omega^1(U; M_a(\mathbb{C}))$. Hence,

$$\begin{aligned} d_\nabla(\nabla s) &\equiv d(d\sigma + A\cdot\sigma) + A \wedge (d\sigma + A \cdot \sigma) \\ &= 0 + dA\cdot\sigma - A\wedge d\sigma + A \wedge d\sigma + A \wedge A \cdot \sigma \\ &= (dA + A\wedge A)\cdot \sigma. \end{aligned}$$

Hence, locally F_∇ can be identified with the 2-form $dA + A \wedge A$, which takes values in $M_a(\mathbb{C})$. Somewhat informally, one simply writes

$$F_\nabla = dA + A \wedge A \tag{3.13}$$

keeping in mind that the right hand side depends on the trivialization chosen.

Exercise 3.14. For any $A \in \Omega^1(M; M_a(\mathbb{C})) \cong M_a(\Omega^1(M))$ define $[A, A] \in \Omega^2(M; M_a(\mathbb{C}))$ by

$$[A, A](v, w) = A(v)\,A(w) - A(w)\,A(v) \in M_a(\mathbb{C}).$$

Show that $A \wedge A = \frac{1}{2}[A, A]$, where the factor $\frac{1}{2}$ stems from the following definition of the wedge-product for 1-forms:

$$\omega \wedge \eta = \frac{1}{2}(\omega \otimes \eta - \eta \otimes \omega) \qquad \Longleftrightarrow \qquad \omega \wedge \eta(v, w) = \frac{1}{2}(\omega(v)\,\eta(\omega) - \omega(w)\,\eta(v)).$$

In particular,

$$F_\nabla = dA + \frac{1}{2}[A, A].$$

Warning: Sometimes the wedge-product is defined without the factor $\frac{1}{2}$ in the literature. This may lead to confusions.

Exercise 3.15. Show that for anny $A, B \in \Omega^1(M; M_a(\mathbb{C}))$ we have

$$[A, B] = +[B, A]. \tag{3.16}$$

Exercise 3.17. If e' is another trivialization of E such that $e = e' \cdot g$ just like in Section 2.1, then

$$F'_\nabla = g^{-1} F_\nabla g, \tag{3.18}$$

where F'_∇ denotes the local representation of the curvature of ∇ with respect to e'.

Proposition 3.19 (Bianchi identity)**.** *If $A \in \Omega^1(U; M_a(\mathbb{C}))$ is a local representation of ∇ as above, then $dF_\nabla + [A, F_\nabla] = 0$.*

Sketch of proof. We have

$$\begin{aligned} d\left(dA + \frac{1}{2}[A, A]\right) + \left[A, dA + \frac{1}{2}[A, A]\right] &= \frac{1}{2}[dA, A] - \frac{1}{2}[A, dA] + [A, dA] + \frac{1}{2}[A, [A, A]] \\ &= -\frac{1}{2}[A, dA] - \frac{1}{2}[A, dA] + [A, dA] + \frac{1}{2}[A, [A, A]] \\ &= \frac{1}{2}[A, [A, A]], \end{aligned}$$

where the second equality follows from $[A, B] = -[B, A]$ if $A \in \Omega^1(U; M_a(\mathbb{C}))$ and $B \in \Omega^2(U; M_a(\mathbb{C}))$, cf. (3.16). The proof is finished by the following observation, whose proof is left to the reader: The Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

which is valid for any $X, Y, Z \in M_a(\mathbb{C})$, implies that $[A, [A, A]] = 0$. $\square$

Remark 3.20. More invariant interpretation of the Bianchi identity is as follows. By Lemma 2.29, ∇ induces a connection on E^*. Hence, using Lemma 2.29 again we obtain a connection, which is still denoted by the same letter, on $E^* \otimes E \cong \mathrm{End}(E)$. Then we also obtain the corresponding map $d_\nabla \colon \Omega^k(M; \mathrm{End}(E)) \longrightarrow \Omega^{k+1}(M; \mathrm{End}(E))$ and the above proposition may be restated simply as $d_\nabla F_\nabla = 0$.

Proposition 3.21. *If ∇ is Hermitian, then F_∇ takes values in skew-Hermitian endomorphisms. In other words, for any $v, w \in TM$ we have*

$$F_\nabla(v, w)^* = -F_\nabla(v, w).$$

Proof. Let $e = (e_1, \dots, e_a)$ be a local trivialization of E such that e is a pointwise orthonormal basis of the corresponding fiber. If A is the local representation of a Hermitian connection ∇ with respect to such a trivialization, then for any $\sigma_1, \sigma_2 \colon U \longrightarrow \mathbb{C}^a$ we have

$$d\langle \sigma_1, \sigma_2 \rangle = \langle (d + A)\sigma_1, \sigma_2 \rangle + \langle \sigma_1, (d + A)\sigma_2 \rangle,$$

since ∇ is Hermitian. Here $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian scalar product on $\mathbb{C}^a$. The right hand side of the above equality equals

$$d\langle \sigma_1, \sigma_2 \rangle + \langle A\sigma_1, \sigma_2 \rangle + \langle \sigma_1, A\sigma_2 \rangle,$$

so that we must have $\langle A\sigma_1, \sigma_2 \rangle = -\langle \sigma_1, A\sigma_2 \rangle$. Hence, $A \in \Omega^1(U; \mathfrak{u}(a))$ and therefore $F_\nabla = dA + \frac{1}{2}[A, A] \in \Omega^1(U; \mathfrak{u}(a))$, since the commutator of any pair of matrices from $\mathfrak{u}(a)$ belongs to $\mathfrak{u}(a)$. $\square$

3.2.3 Chern classes

Let $p\colon \mathfrak{u}(a) \to \mathbb{C}$ be a homogeneous polynomial of degree d. This means the following: Assume $\xi_1, \dots, \xi_n$ is a basis of $\mathfrak{u}(a)$ (in particular, $n = a^2$). If $\xi = \sum_{j=1}^n x_j \xi_j$, then $p(\xi) = p\left(\sum_{j=1}^n x_j\xi_j\right)$ is a polynomial of degree d in $(x_1, \dots, x_n)$. Assume also that p is invariant, that is

$$p\left(U\xi U^{-1}\right) = p(\xi) \qquad \text{for all } \xi \in \mathfrak{u}(a) \text{ and } U \in \mathrm{U}(a).$$

Example 3.22.

1) $p_d(\xi) := i \operatorname{tr} \xi^d$ is an invariant homogeneous polynomial of degree d.
2) $p(\xi) := \det \xi$ is an invariant homogeneous polynomial of degree a.
3) More generally, define polynomials $c_1, \dots, c_a$ on $\mathfrak{u}(a)$ by the equality

$$\det\left(\lambda \cdot \mathbb{1} + \frac{i}{2\pi}\xi\right) = \lambda^a + c_1(\xi)\lambda^{a-1} + \ldots + c_{a-1}(\xi)\lambda + c_a(\xi),$$

where $\lambda \in \mathbb{C}$ is a parameter and $\mathbb{1}$ is the identity matrix. For example, $c_1(\xi) = \frac{i}{2\pi}\operatorname{tr}\xi$ and $c_a(\xi) = \frac{i^a}{(2\pi)^a}\det\xi$. The reader should check that c_j is an invariant homogeneous polynomial of degree j.

Moreover, notice that the equality

$$\overline{\det\left(\lambda \cdot \mathbb{1} + \frac{i}{2\pi}\xi\right)} = \det\left(\bar\lambda \cdot \mathbb{1} + \frac{i}{2\pi}\xi\right)$$

implies that $\overline{c_j(\xi)} = c_j(\xi)$, that is each c_j takes values in $\mathbb{R}$ rather than $\mathbb{C}$.

Definition 3.23. Let E be a Hermitian vector bundle of rank a. Pick an invariant homogeneous polynomial p as above and a Hermitian connection ∇. In a local trivialization $e = (e_1, \dots, e_a)$ such that e is pointwise an orthonormal basis, think of F_∇ as a 2-form $F_\nabla^{\mathrm{loc}} = dA + \frac{1}{2}[A, A]$ with values in $\mathfrak{u}(a)$. Finally, set $p(F_\nabla) = p\left(F_\nabla^{\mathrm{loc}}\right) \in \Omega^{2d}(M;\mathbb{R})$.

Lemma 3.24. *The following holds:*

(i) *$p(F_\nabla)$ is well-defined.*
(ii) *$p(F_\nabla)$ is closed.*
(iii) *The de Rham cohomology class of $p(F_\nabla)$ depends neither on the choice of ∇ nor on the Hermitian scalar product on E.*

Proof. Claim *(i)* follows easily from (3.18) and the invariance of p. The rest of the proof consists of the following steps.

Step 1. *We prove (ii).*

Let $\xi, \xi_1, \dots, \xi_d$ be arbitrary elements of $\mathfrak{u}(a)$. Slightly abusing notations, denote by $p\colon \mathfrak{u}(a) \times \ldots \times \mathfrak{u}(a) \to \mathbb{C}$ the symmetrization of p, that is a multilinear symmetric map, whose restriction to the diagonal $\{(\xi, \dots, \xi)\}$ yields the original polynomial p.

Differentiating the equality

$$p\left(e^{t\xi}\xi_1 e^{-t\xi}, \dots, e^{t\xi}\xi_d e^{-t\xi}\right) = p(\xi_1, \dots, \xi_d)$$

with respect to t and setting $t = 0$, we obtain that the equality

$$p\left(\left[\xi,\xi_1\right],\xi_2,\dots,\xi_d\right) + p\left(\xi_1,\left[\xi,\xi_2\right],\dots,\xi_d\right) + \dots + p\left(\xi_1,\xi_2,\dots,\left[\xi,\xi_d\right]\right) = 0 \tag{3.25}$$

holds for any $\xi, \xi_1, \dots \xi_d \in \mathfrak{u}(a)$. This implies the following identity

$$p\left(\left[F_\nabla^{\mathrm{loc}}, A\right], A, \dots, A\right) + \dots + p\left(A, A, \dots, \left[F_\nabla^{\mathrm{loc}}, A\right]\right) = 0.$$

Hence,

$$\begin{aligned} dp\left(F_\nabla^{\mathrm{loc}}\right) &= p\left(dF_\nabla^{\mathrm{loc}}, F_\nabla^{\mathrm{loc}}, \dots, F_\nabla^{\mathrm{loc}}\right) + p\left(F_\nabla^{\mathrm{loc}}, dF_\nabla^{\mathrm{loc}}, \dots, F_\nabla^{\mathrm{loc}}\right) + \dots + p\left(F_\nabla^{\mathrm{loc}}, \dots, dF_\nabla^{\mathrm{loc}}\right) \\ &= p\left(\left[F_A^{\mathrm{loc}}, A\right], F_\nabla, \dots, F_\nabla\right) + p\left(F_\nabla^{\mathrm{loc}}, \left[F_\nabla^{\mathrm{loc}}, A\right], \dots, F_\nabla^{\mathrm{loc}}\right) + \dots \\ &\qquad + p\left(F_\nabla^{\mathrm{loc}}, F_\nabla^{\mathrm{loc}}, \dots, \left[F_\nabla^{\mathrm{loc}}, A\right]\right) \\ &= 0, \end{aligned}$$

where the first equality follows from the Bianchi identity and the second one from (3.25).

Step 2. *Let $I = [0,1]$ be the interval and $\imath_0, \imath_1 \colon M \to I$ be the natural inclusions corresponding to the endpoints of the interval. There exists a linear map $Q \colon \Omega^k(M \times I) \to \Omega^{k-1}(M)$ such that for any $\omega \in \Omega^k(M \times I)$ we have $\imath_1^*\omega - \imath_0^*\omega = dQ\omega - Qd\omega$.* ref

Step 3. *The cohomology class of $p(F_\nabla)$ does not depend on the choice of ∇.*

Pick any two Hermitian connections ∇_0 and ∇_1 and think of $\nabla_t := (1-t)\nabla_0 + t\nabla_1$ as a connection on $pr_1^*E \to M \times I$, where $pr_1 \colon M \times I \to M$ is the natural projection. Then we have

$$\begin{aligned} p\left(F_{\nabla_0}\right) - p\left(F_{\nabla_1}\right) &= \imath_1^* p\left(F_{\nabla_t}\right) - \imath_0^* p\left(F_{\nabla_t}\right) \\ &= dQp\left(F_{\nabla_t}\right) - Qdp\left(F_{\nabla_t}\right) \\ &= dQp\left(F_{\nabla_t}\right). \end{aligned}$$

Here the second equality follows from Step 2 and the third one from Step 1.

Step 4. *The cohomology class of $p(F_\nabla)$ does not depend on the Hermitian scalar product of E.*

The proof of this step is similar to the proof of the preceding one and follows in essense from the fact that the space of all Hermitian scalar products is convex. I leave the details to the reader. ☐

Let c_j be as in Example 3.22. Then by Lemma 3.24, $c_j(F_\nabla)$ is a closed real-valued form and the de Rham cohomology class of $c_j(F_\nabla)$ depends on E only.

Definition 3.26. The class $c_j(E) := [c_j(F_\nabla)] \in H^{2j}_{dR}(M;\mathbb{R})$ is called the j^{th} Chern class of E and

$$c(E) := 1 + c_1(E) + c_2(E) + \dots + c_a(E) \in H^\bullet(M;\mathbb{R})$$

is called the total Chern class of E.

Theorem 3.27. *The Chern classes satisfy the following properties:*

(i) $c_0(E) = 1$ *for any complex vector bundle* E*;*

(ii) The Chern classes depend on the isomorphism class of E *only;*

(iii) $c_j(f^*E) = f^*c_j(E)$ *for all complex vector bundles* $E \to M$ *and all (smooth) maps* $f \colon N \to M$*;*

(iv) $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$*;*

(v) *If* $E \cong E_1 \oplus \underline{\mathbb{C}^b}$*, then* $c_j(E) = 0$ *for* $j > \operatorname{rk} E - b$*. In particular, if* E *is trivial, then* $c_j(E) = 0$ *for all* $j > 0$.

Proof. Property *(i)* is just the definition of c_0. The rest of the proof consists of the following steps.

Step 1. *We prove (ii).*

Suppose $\psi\colon E_0 \to E_1$ is an isomorphism. Given a Hermitian scalar product $\langle\cdot,\cdot\rangle_1$ and a Hermitian connection ∇_1 on E_1 we can define a Hermitian scalar product $\langle\cdot,\cdot\rangle_0$ and a Hermitian connection ∇_0 on E_0 as follows:

$$\begin{aligned} \langle t_1, t_2\rangle_0 &= \langle \psi t_1,\ \psi t_2\rangle_1, && t_1, t_2 \in E_0.\\ \nabla_0 s_0 &:= \psi^{-1}\left(\nabla_1(\psi s_0)\right), && s_0 \in \Gamma(E_0). \end{aligned}$$

If $e^1 = (e^1_1, \ldots, e^1_a)$ is a local trivialization of E_1 such that e^1 is pointwise orthonormal, then $e^0 := (\psi^{-1}e^1_1, \ldots, \psi^{-1}e^1_a)$ is a local trivialization of E_0 with the same property. Hence, if $A \in \Omega^1(U;\mathfrak{u}(a))$ is a local representation of ∇_1, then the local representation of ∇_0 with respect to e^0 is also A, since by (2.14) we have

$$\nabla_1 e^1 = e^1 \cdot A \qquad \Longrightarrow \qquad \nabla_0 e^0 = \psi^{-1}\left(\nabla_1 e^1\right) = \psi^{-1}\left(e^1 \cdot A\right) = e^0 \cdot A.$$

This yields that the curvature forms of ∇_0 and ∇_1 with respect to the above trivializations coincide, so that we trivially have $c(E_0) = c(E_1)$.

Step 2. *We prove (iii).*

Given a Hermitian scalar product $\langle\cdot,\cdot\rangle$ on E and a local trivialization $e = (e_1, \ldots, e_a)$ of E over U such that e is pointwise an orthonormal basis, we can construct a Hermitian scalar product and a local trivialization of f^*E over $f^{-1}(U)$ by

$$\begin{aligned} \langle t_1, t_2\rangle_n &:= \langle t_1, t_2\rangle_{f(n)}, && t_1, t_2 \in E_n, n \in N\\ f^*e\big|_n &:= e\big|_{f(n)}. \end{aligned}$$

Furthermore, if a Hermitian connection ∇ on E is represented by some $A \in \Omega^1(U;\mathfrak{u}(a))$, then we can define a new connection $f^*\nabla$ on f^*E by declaring

$$(f^*\nabla)(f^*e) = (f^*e)\cdot f^*A,$$

that is f^*A is a local representation of $f^*\nabla$ with respect to f^*e. Then

$$F_{f^*\nabla} = d(f^*A) + \frac{1}{2}[f^*A, f^*A] = f^*\left(dA + \frac{1}{2}[A,A]\right) = f^*F_\nabla,$$

which implies *(iii)*.

Step 3. *We prove (iv).*

Let ∇_j be a connection on E_j, $j = 1, 2$. Define a connection ∇ on $E_1 \oplus E_2$ by

$$\nabla(t_1, t_2) = (\nabla_1 t_1, \nabla_2 t_2).$$

If e^j is a local trivialization of E_j, then $e = (e^1, e^2)$ is a local trivialization of $E_1 \oplus E_2$. With respect to these trivializations we have

$$F^{\mathrm{loc}}_\nabla = \left(\begin{array}{c|c} F^{\mathrm{loc}}_{\nabla_1} & 0 \\ \hline 0 & F^{\mathrm{loc}}_{\nabla_2} \end{array}\right),$$

where the right hand side is a block-matrix with non-trivial blocks in the upper left and lower right corners only.

Since for any block-matrix we have

$$\det \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = \det A \cdot \det B,$$

we obtain

$$c(F_\nabla) = \det \left(\begin{array}{c|c} \mathbb{1} + \frac{i}{2\pi} F_{\nabla_1} & 0 \\ \hline 0 & \mathbb{1} + \frac{i}{2\pi} F_{\nabla_2} \end{array}\right) = c(F_{\nabla_1}) \wedge c(F_{\nabla_2})$$

This implies *(iv)*.

Step 4. *We prove (v).*

If $E \cong \underline{\mathbb{C}^a}$, then we can use the trivial connection $\nabla = d$ to deduce that $c(E) = 1$. If $E \cong E_1 \oplus \underline{\mathbb{C}^b}$, by the preceding step we have

$$c(E) = c(E_1) \cup 1 = c(E_1),$$

which yields *(v)*. □

3.2.4 Other characteristic classes

Assume E can be written as the Whitney sum of line bundles: $E = L_1 \oplus \ldots \oplus L_a$. Then for the total Chern class of E we have

$$c(E) = c(L_1) \cdot \ldots \cdot c(L_a) = (1 + c_1(L)) \cdot \ldots \cdot (1 + c_1(L_a)),$$

where $\cdot$ denotes the product in $H^\bullet(M)$ i.e., the cup-product. Denoting $x_j \coloneqq c_1(L_j)$, we obtain

$$c(E) = \prod_{j=1}^{a} (1 + x_j) = 1 + \sigma_1 + \sigma_2 + \ldots + \sigma_a,$$

where $\sigma_k = \sigma_k(x_1, \ldots, x_a)$ is the k^{th} elementary symmetric fiunction of $x_1, \ldots, x_k$. In other words,

$$\begin{aligned} c_1(E) &= \sigma_1(x_1, \ldots, x_a) = x_1 + \ldots + x_a, \\ c_2(E) &= \sigma_2(x_1, \ldots, x_a) = x_1 x_2 + x_1 x_3 + \ldots + x_{a-1} x_a, \\ &\ldots \\ c_a(E) &= \sigma_a(x_1, \ldots, x_a) = x_1 x_2 \ldots x_a. \end{aligned}$$

The following algebraic fact will be useful below.

Fact: For any symmetric polynomial $p = p(x_1, \ldots, x_a)$ there exists a unique polynomial q in a variables such that

$$p(x_1, \ldots, x_a) = q(\sigma_1(x), \ldots, \sigma_a(x)).$$

This fact implies the following: Given any symmetric polynomial $p(x_1, \ldots, x_a)$ we can construct a characteristic class $p(E)$ by setting

$$p(E) \coloneqq q(c_1, c_2, \ldots, c_a) \in H^\bullet(M; \mathbb{R}).$$

For example, choose

$$
\begin{aligned}
\operatorname{ch}(x_1, \dots, x_a) &= \sum_{j=1}^{a} e^{x_j} \\
&= \left(1 + x_1 + \frac{1}{2}x_1^2 + \dots\right) + \dots + \left(1 + x_a + \frac{1}{2}x_a^2 + \dots\right) \\
&= a + (x_1 + \dots + x_a) + \frac{1}{2}\left(x_1^2 + \dots + x_a^2\right) + \dots \\
&= a + \sigma_1 + \frac{1}{2}\left(\sigma_1^2 - 2\sigma_2\right) + \dots
\end{aligned}
$$

Notice that here and below, we think of x_j as nilpotent elements of the ring $H^\bullet(M;\mathbb{R})$ so that all sums above are finite.

Thus we have

$$
\operatorname{ch}(E) = \operatorname{rk} E + c_1(E) + \frac{1}{2}\left(c_1^2(E) - 2c_2(E)\right) + \dots
$$

Remark 3.28. It is *not* true in general that any vector bundle can be written as the Whitney sum of line bundles. Nevertheless, this is true in some sense, which I shall not try to describe here. In particular, when manipulating with characteristic classes it is admissable to imagine that bundles decompose as Whitney sums of line bundles. This is called "the splitting principle", which is explained somewhat more concretely in the proof below.

Proposition 3.29. *The Chern character satisfies the following:*

(i) $\operatorname{ch}(E \oplus F) = \operatorname{ch}(E) \oplus \operatorname{ch}(F)$*;*

(ii) $\operatorname{ch}(E \otimes F) = \operatorname{ch}(E) \cdot \operatorname{ch}(F)$.

Proof. Assume that both E and F split as Whitney sums of line bundles:

$$
E = L_1 \oplus \dots \oplus L_a \qquad \text{and} \qquad F = \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_b.
$$

Denote $x_j := c_1(L_j)$ and $y_k := c_1(\mathcal{L}_k)$. The bundle $E \oplus F$ also splits as the Whitney sum of line bundles so that we have

$$
\operatorname{ch}(E \oplus F) = \sum_{j=1}^{a} e^{x_j} + \sum_{k=1}^{b} e^{y_j} = \operatorname{ch}(E) + \operatorname{ch}(F)
$$

for all bundles, which split as the sum of line bundles. This imples that we have the algebraic identity

$$
\operatorname{ch}(x_1, \dots x_a, y_1, \dots, y_b) = \operatorname{ch}(x_1, \dots, x_a) + \operatorname{ch}(y_1, \dots, y_b) \tag{3.30}
$$

where we think of ch simply as a polynomial in the corresponding number of variables. Notice that (3.30) can be easily established purely algebraically without any reference to vector bundles.

Furthermore, if $q_{a+b}(\sigma_1, \dots, \sigma_{a+b})$ is the expression of $\operatorname{ch}(x_1, \dots, x_a; y_1, \dots, y_b)$ in terms of elementary symmeric polynomials in $a + b$ variables, then we must have

$$
q_{a+b}(\sigma_1(x, y), \dots, \sigma_{a+b}(x, y)) = q_a(\sigma_1(x), \dots, \sigma_a(x)) + q_b(\sigma_1(y), \dots, \sigma_b(y)).
$$

This implies that *(i)* holds not only for those vector bundles, which split, but rather for all bundles.

The proof of *(ii)* goes along similar lines and the details are left to the reader. □

Definition 3.31. The Todd class $\operatorname{td}(E) \in H^\bullet(M;\mathbb{R})$ is the characteristic class corresponding to the polynomial

$$\operatorname{td}(x_1,\dots,x_a) := \prod_{j=1}^{a} \frac{x_j}{1-e^{-x_j}} = 1 + \frac{1}{2}\sigma_1 + \frac{1}{12}\left(\sigma_2 + \sigma_1^2\right) + \dots$$

3.3 A prototype of the index theorem

Denote

$$\Omega^{\mathrm{ev}}(M) := \Omega^0(M) \oplus \Omega^2(M) \oplus \dots \qquad \text{and} \qquad \Omega^{\mathrm{odd}}(M) := \Omega^1(M) \oplus \Omega^3(M) \oplus \dots$$

Exercise 3.32. Show that

$$D := d + d^* \colon \Omega^{\mathrm{ev}}(M) \longrightarrow \Omega^{\mathrm{odd}}(M) \tag{3.33}$$

is elliptic.

If M is compact, which is assumed throughout in this section, by Proposition 1.30 we have

$$\operatorname{Ker} D = \mathcal{H}^0(M) \oplus \mathcal{H}^2(M) \oplus \dots \cong H^{\mathrm{ev}}_{dR}(M).$$

Moreover, Theorem 2.81 implies

$$\operatorname{Coker} D = \mathcal{H}^1(M) \oplus \mathcal{H}^3(M) \oplus \dots \cong H^{\mathrm{odd}}_{dR}(M).$$

Therefore, for the index of D we obtain

$$\begin{aligned} \operatorname{ind} D &= \dim H^{\mathrm{ev}}_{dR}(M) - \dim H^{\mathrm{odd}}_{dR}(M) \\ &= b_0(M) + b_2(M) + \dots - b_1(M) - b_3(M) - \dots \\ &= \chi(M), \end{aligned}$$

where $\chi(M)$ is the Euler characteristic of M. Thus, we summarize.

Theorem 3.34. *The index of* (3.33) *equals* $\chi(M)$. □

This is an interesting theorem, since this is an example of a relation between solutions of PDEs and a purely topological quantity. For example, if $\chi(M) > 0$, we deduce that M must support at least one non-trivial harmonic form. Of course, the reader surely can give a more precise relation between topological invariants of M and the number of linearly independent harmonic forms on M, however notice the following:

- The Euler number is easier to compute than the individual Betti numbers.
- In more general situations the index is relatively easily computable, whereas more detailed information about solutions of PDEs is typically hard to obtain.

3.4 On the de Rham cohomolgy with compact supports

For non-compact manifolds the de Rham cohomology groups are typically poorly behaved. There are many ways to adapt the definition of the de Rham cohomology groups to this setting. One possibility is to replace each $\Omega^k(M;\mathbb{R})$ by

$$\Omega_0^k(M) := \left\{\omega \in \Omega^k(M) \,|\, \operatorname{supp}\omega \text{ is compact}\right\}.$$

That is instead of the de Rham complex (1.1) we consider

$$0 \to \Omega_0^0(M) \xrightarrow{d} \Omega_0^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega_0^n(M) \to 0.$$

Akin to (1.3), we define $H_0^k(M)$ to be the k^{th} cohomolgy group of the above complex.

Of course, if M is compact, we trivially have $H^k(M) = H_0^k(M)$, however for non-compact manifolds $H^k(M) \neq H_0^k(M)$ in general. For example, one can show that $H_0^n(\mathbb{R}^n) \cong \mathbb{R}$, see [BT82, Cor. 4.7.1].

The only property which will be of concern for us is the following. If M^n is oriented, the integration

$$\int : \Omega_0^n(M) \longrightarrow \mathbb{R}$$

is well-defined and yields an isomorphism $H_0^n(M) \cong \mathbb{R}$ [BT82, Cor. 5.8].

3.5 Basics of K-theory

Let N be a semiring, that is N is endowed with addition and multiplication just like a ring, however additive inverse of an element $n \in N$ does not need to exist.

Example 3.35. The set of natural numbers $\mathbb{N}$ is a semiring.

Any semiring N can be canonically 'enlarged' to a ring as follows. Denote

$$K(N) = \{n - m \,|\, n, m \in N\} / \sim,$$

where the difference $n - m$ is understood formally, and

$$n - m = n' - m' \qquad \Longleftrightarrow \qquad \exists\, k \in N \text{ such that } n + m' + k = n' + m + k.$$

Then $K(N)$ is a ring, since the additive inverse of $n - m$ exists and equals $m - n$.

Example 3.36. $K(\mathbb{N}) \cong \mathbb{Z}$.

Let M be a compact manifold. Then the set of isomorphism classes of (complex) vector bundles over M is a semiring with respect to $\oplus$ and $\otimes$ operations. Hence, we can construct a ring $K(M)$, which consists of (classes of) formal diferences $E - F$. By setting

$$\operatorname{ch}(E - F) := \operatorname{ch}(E) - \operatorname{ch}(F)$$

we obtain by Proposition 3.29 that $\operatorname{ch}\colon K(M) \longrightarrow H^\bullet(M)$ is a well-defined homomorphism of rings.

If M is non-compact, one has to modify the above construction slightly so that the resulting ring behaves nicely. Thus, for a non-compact M roughly speaking the ring $K(M)$ consists of classes of formal differences $E - F$ such that E and F are isomorphic on the complement of a compact subset $A \subset M$. By the proof of Theorem 3.27, there exist connections ∇^E and ∇^F on E and F respectively such that $F_{\nabla^E} = F_{\nabla^F}$ on $M \backslash A$. Hence, $c_j(E)$ and $c_j(F)$ are represented by $2j$-forms, which agree on $M \backslash A$ so that $\operatorname{ch}(E - F) \in H_0^\bullet(M)$.

The upshot of this section is that it is legitimate to consider formal differences of vector bundles and if E and F are isomorphic away from a compact subset of M, then $\operatorname{ch}(E \cdot F)$ takes values in the compactly supported cohomology ring of M.

Let me note in passing that the ring $K(M)$ is an important invariant of the underlying space M. This invariant has been extensively studied, however this goes beyond the scope of these lecture notes.

3.6 The symbol of an elliptic operator revisited

Let us consider first the following construction of vector bundles on the spheres. Denote

$$S^n_\pm := \{(x_0, \dots x_n) \in S^n \mid \pm x_n \geq 0\}$$

so that $S^n_+ \cap S^n_- = S^{n-1}$ is the equator. Given a (smooth) map $g\colon S^{n-1} \to \mathrm{GL}_a(\mathbb{C})$, we can construct a complex vector bundle E of rank a on S^n as follows. Declare E to be

$$\left(S^n_+ \times \mathbb{C}^a\right) \sqcup \left(S^n_- \times \mathbb{C}^a\right) \setminus \sim, \tag{3.37}$$

where $(x, v) \sim (x, g(x)v)$ if and only if $x \in S^{n-1}$. This is usually called the clutching construction, which yields all complex vector bundles on S^n up to an isomorphism.

Furthermore, pick $\sigma \in \mathrm{Smb}_l(E; F)$ and a point $m \in M$. We can do the following parameterized version of the clutching construction. Denote first by $T^*_m M^+$ the one-point compactification of $T^*_m M \cong \mathbb{R}^n$ so that $T^*_m M^+ \cong S^n$. We can think of $T^*_m M^+$ as the union of the "hemispheres"

$$S^-_m = \left\{\xi \in T^*_m M \mid |\xi| \leq 1\right\} \quad \text{and} \quad S^+_m = \left\{\xi \in T^*_m M \mid |\xi| \geq 1\right\} \cup \{\infty\}$$

so that $S^+_m \cap S^-_m = \{|\xi| = 1\} \subset T^*_m M$. Therefore, we can construct the bundle Σ_m by attaching $S^-_m \times E_m$ to $S^+_m \times F_m$ by means of σ. To be more precise,

$$\Sigma_m := \left(\left(S^-_m \times E_m\right) \sqcup \left(S^+_m \times F_m\right)\right) / \sim, \qquad \text{where} \qquad (\xi, v) \sim \left(\xi, \sigma(m, \xi)v\right)$$

provided $|\xi| = 1$. As m varies over M, we obtain a complex vector bundle Σ over $S^n M$, where $S^n M$ is a fibered space arising as pointwise one-point compactifications of the fibers of T^*M. Sometimes this vector bundle Σ is referred to as the symbol of $L \in \mathrm{Diff}_l(E; F)$ provided $\sigma = \sigma_l(L)$.

In any case, $\Sigma - \pi^* F$ can be viewed as an element of $K(S^n M)$. It will be more convenient for us to view $\Sigma - \pi^* F$ as an element of $K(T^*M)$. This makes sense, since Σ and $\pi^* F$ are isomorphic on the complement of the set $\{|\xi| \leq 1\}$, which is compact. Therefore, the class $\operatorname{ch}(\Sigma - F) \in H_0^\bullet(T^*M)$ is well defined.

With these preparations at hand, we can state the index theorem.

Theorem 3.38 (Atiyah-Singer). *Let M be a smooth closed oriented manifold of dimension n. Let $L \in \mathrm{Diff}_l(E; F)$ be an elliptic differential operator. Let Σ be a vector bundle associated with $\sigma_l(L)$ as above. Then*

$$\operatorname{ind} L = (-1)^n \int_{T^*M} \operatorname{ch}(\Sigma - \pi^* F) \cup \pi^* \operatorname{td}(TM \otimes \mathbb{C}). \tag{3.39}$$

Remark 3.40.

1) For any class $\omega = \sum w_j \in H_0^\bullet(M;\mathbb{R})$, where $w_j \in H_0^j(M;\mathbb{R})$, by definition we have $\int_M \omega = \int_M \omega_n$.
2) A priori it is by no means clear that the right hand side of (3.39) is an integer (though one can show that this is a rational number by purely topoligical means).
3) The right hand side of (3.39) can be expressed as an integral over M rather than T^*M. Most frequently one finds the Atiyah-Singer theorem in the literature in the form with the integration over M.

3.7 An application: The Riemann-Roch formula

Let $E \to \Sigma$ be a complex vector bundle over a Riemann surface Σ.

Definition 3.41. A holomorphic structure on E is an open covering $U = \{U_\alpha\}$ of Σ with the following properties:

- E admits a trivialization e^α over each U_α;
- On each $U_{\alpha\beta} := U_\alpha \cap U_\beta$ the corresponding transition maps $g_{\alpha\beta} \colon U_{\alpha\beta} \to \mathrm{GL}_a(\mathbb{C})$ defined by $e^\beta = e^\alpha \cdot g_{\alpha\beta}$ are holomorphic.

If E is a holomorphic vector bundle, that is a complex vector bundle equipped with a holomorphic structure, we can define the Dolbeault operator $\bar\partial \colon \Omega^0(E) \to \Omega^{0,1}(E)$ by the following rule: If $s = e^\alpha \cdot \sigma_\alpha$ over U_α, then $\bar\partial s := e^\alpha \cdot \bar\partial \sigma_\alpha$. In other words, we declare each e_j^α to be a local holomorphic section of E over U_α. The reader will have no difficulties to check that $\bar\partial$ is well-defined. Moreover, $\bar\partial$ is elliptic just like the Dolbeault operator considered in Section 2.5.1.

Clearly $H^0(E) := \operatorname{Ker} \bar\partial$ is the space of global holomorphic sections. Denote also $H^1(E) := \operatorname{Coker} \bar\partial$. If Σ is closed, the Atiyah-Singer index theorem yields an expression for

$$\operatorname{ind} \bar\partial = \dim H^0(E) - \dim H^1(E)$$

in terms of characteristic classes of E. Since Σ is one-dimensional, the only relevant class is $c_1(E)$. A computation shows that

$$\dim H^0(E) - \dim H^1(E) = a(1-g) + \int_\Sigma c_1(E),$$

where $a = \operatorname{rk} E$ and g is the genus of Σ. This is the celebrated Riemann-Roch formula. In particular, if

$$\int_\Sigma c_1(E) > (g-1)\,a,$$

then E admits a non-trivial holomorphic section.

For example, for $E = T^*\Sigma \cong (T^*\Sigma)^{1,0}$ we have

$$\int_\Sigma c_1(T^*\Sigma) = 2g - 2$$

so that the Riemann-Roch formula yields

$$\dim H^0(T^*\Sigma) - \dim H^1(T^*\Sigma) = 1 - g + 2g - 2 = g - 1.$$

Hence, a compact Riemann surface admits a non-trivial holomorphic $(1,0)-$ form provided $g(\Sigma) > 1$.

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