Algebraic Topology

Lecture notes

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This is a draft. In particular, most of figures are missing. If you spot a mistake, please let me know.

TODO:
• Add an appendix on chain complexes.

Contents

Chapter 1

Introduction

The main purpose of this chapter is to explain informally the main ideas which will be developed in details later. In particular, the proofs are rather sketchy stressing main ideas only. More precise statements and proofs will be given in the subsequent chapters.

1.1 Differential forms, the theorems of Green and Stokes

Let $\omega = P(x,y)dx + Q(x,y)dy$ be a 1-form on an open subset $U \subset \mathbb{R}^2$. For example, if $f\colon U \to \mathbb{R}$ is a smooth map, then the differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is a 1-form.

Question 1.1. Under which circumstances does there exist some function f as above such that $\omega = df$?

Clearly, we have the following necessary condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1.2}$$

Proposition 1.3. *If U is convex, then (1.2) is also sufficient.*

Sketch of proof. Theorem of Green $\implies$ For any closed piecewise smooth curve $C \subset U$ without self-intersections we have

$$\int_C (P\,dx + Q\,dy) = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx\,dy = 0, \tag{1.4}$$

where D is the domain bounded by C. Notice that here we use the convexity of U, since otherwise C does not necessarily bound any domain.

Pick any $(x_0, y_0) \in U$. For any $(x,y) \in U$ choose a curve C' connecting (x_0,y_0) and (x,y). Define

$$f(x,y) := \int_{C'} P\,dx + Q\,dy.$$

Property (1.4) guaranties that f does not depend on the choice of C'. □

The following example shows that (1.2) is not sufficient for general U.

Example 1.5. Consider $U = \mathbb{R}^2 \setminus \{0\}$ and

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

If there were some f such that $\omega = df$, then we would have $\int_{S^1} \omega = 0$, where S^1 is the circle (for example, parametrized via $t \mapsto (\cos t, \sin t)$). This is a contradiction, since $\int_{S^1} \omega = 2\pi \neq 0$.

Notice that the proof of Proposition 1.2 does not work here, since the theorem of Green does not apply for (D, ω), where D is the unit disc.

Remark 1.6. One can show that for any closed piecewise smooth curve $C \subset \mathbb{R}^2 \setminus \{0\}$ we have

$$\frac{1}{2\pi} \int_C \Big(-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \Big)$$

is an integer.

Let U be an open subset of $\mathbb{R}^3$ and $\omega = P\,dx + Q\,dy + R\,dz$ be a 1-form. We can also ask whether $\omega = df$ for some $f \colon U \to \mathbb{R}$. Clearly, we have the following necessary condition:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \textit{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{1.7}$$

Proposition 1.8. *If U is convex, then (1.7) is also sufficient.*

The proof of this proposition is analogous to the proof of the previous one. Just instead of the theorem of Green we have to use the theorem of Stokes:

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_\Sigma \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \Big)\, dy\,dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \Big)\, dz\,dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big)\, dx\,dy.$$

Proposition 1.9. *Condition (1.7) is also sufficient for $\mathbb{R}^3 \setminus \{0\}$.*

Sketch of proof. Let $C \subset \mathbb{R}^3$ be an arbitrary simple picewise smooth curve without self-intersections. Then there is a picewise smooth surface $\Sigma \subset \mathbb{R}^3$ such that $\partial \Sigma = C$. If $0 \in \Sigma$, a (small) perturbation yields a surface $\Sigma' \subset \mathbb{R}^3 \setminus \{0\}$ such that $\partial \Sigma' = C$. □

For a general U, Condition (1.7) is still insufficient, which is easily seen for the following example: $U = \mathbb{R}^3 \setminus \{z - \text{Axis}\}$ and

$$\omega = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy.$$

From this discussion we can make the following informal conclusion: Condition (1.7) is sufficient as long as U has no "holes" of codimension 2.

1.2 Ansatz of a construction.

Let $X \subset \mathbb{R}^n$ be an arbitrary subset, which is equipped with the induced topology. Define $Z_1(X)$ as a free Abelian group generated by (oriented) closed curves, i.e.,

$$C \in Z_1(X) \implies C = n_1 C_1 + \dots n_k C_k, \tag{1.10}$$

where $n_j \in \mathbb{Z}$. Define

$$\int_C \omega := \sum n_k \int_{C_k} \omega.$$

Remark 1.11. If C_0 is a closed oriented curve, $2C_0$ can be understood as "running along C_0 twice in the same direction". Similarly, $-C_0$ can be understood as the curve C_0 with the opposite orientation. However, in most cases we treat (1.10) purely formally.

Assume temporarily that X is an *open* subset of $\mathbb{R}^2$. We would like to define an equivalence relation such that

$$C \sim C' \implies \int_C \omega = \int_{C'} \omega$$

holds for all $\omega = P\,dx + Q\,dy$ satisfying (1.2). The theorem of Green (or Stokes in the case $U \subset \mathbb{R}^3$) suggests the following:

$$C \sim C' \quad \Leftrightarrow \quad \exists \text{ a compact oriented surface } \Sigma \text{ such that } \partial\Sigma = C \cup -C'. \tag{1.12}$$

Here C and C' are oriented curves and Σ is an oriented surface such that $\partial\Sigma = C \cup -C'$ as *oriented* curves. This definition also makes sense even in the case when X is not necessarily open.

More generally, a cycle $C = C_1 + \cdots + C_k$ is called *null homologous*, i.e., $C \sim 0$, if and only if

$$\exists \text{ a compact surface } \Sigma \text{ such that } \partial\Sigma = C_1 \cup \cdots \cup C_n.$$

Clearly, Condition (1.12) can be written as $C + (-C') \sim 0$.

Example 1.13. Null homologous cycles on the 2-sphere with 2 points removed (equivalently, $\mathbb{R}^2 \setminus \{0\}$).

Even more generally, each linear combination of null homologous cycles is also declared to be null homologous.
$Z_1(X) \supset B_1(X) = \{\text{null homologous cycles}\}$.
$H_1(X) := Z_1(X)/B_1(X)$ the first homology group of X.

Example 1.14. $H_1(S^2 \setminus \{p, q\}) \cong \mathbb{Z}$.

Problems: Curves C and surfaces Σ can have singularities and self-intersections.

More generally:

- $Z_n(X)$ freely generated by compact oriented n-dimensional "surfaces" without boundary.
- $Z_n(X) \supset B_n(X)$ the subgroup generated by the boundaries of compact oriented $(n+1)$-dimensional "surfaces".
- $H_n(X) := Z_n(X)/B_n(X)$ the nth homology group of X.

In general, we would like to associate to each topological space X a sequence of abelian groups $H_0(X), H_1(X), \dots, H_n(X), \dots$ such that the following holds:

(a) Each continuous map $f\colon X \to Y$ induces a sequence of homomorphisms $f_*\colon H_n(X) \to H_n(Y)$;

(b) $(f \circ g)_* = f_* \circ g_*, \quad id_* = id$.

(c) $H_0(\{pt\}) \cong \mathbb{Z}$ and $H_n(\{pt\}) = 0$ for all $n \geq 1$.

(d) $H_n(S^n) \cong \mathbb{Z}$ provided $n \geq 1$ and $H_k(S^n) = 0$ for all $k \geq n + 1$ (More generally, for each compact connected oriented manifold M of dimension n the following holds: $H_n(M) \cong \mathbb{Z}$ and $H_k(M) = 0$ for all $k > n + 1$).

(e) $f \simeq g \implies f_* = g_*$.

Here two continuous maps are said to be homotopic ($f \simeq g$), if there exists a continuous map $h\colon X \times [0,1]$, called homotopy, such that the following holds:

$$h|_{X\times 0} = f \qquad \text{and} \qquad h|_{X\times 1} = g.$$

Question 1.15. What does make Properties (a)-(e) interesting?

This question will be answered in the subsequent sections. We finish this section by the following facts, which will be useful below.

Proposition 1.16. *If f is a homeomorphism, then each $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. $id_{H_n} = id_* = (f \circ f^{-1})_* = f_* \circ (f^{-1})_* \implies f_*$ is an isomorphism and $(f_*)^{-1} = (f^{-1})_*$. □

1.3 The theorem of Brouwer

In this section we show that (a)-(e) imply the following famous result.

Theorem 1.17 (Brouwer)**.** *Any continuous map $f\colon B_n \to B_n$ has a fixed point.*

Proof. The proof consists of the following three steps.

Step 1. *For the ball $B_n := \{x \in \mathbb{R}^n \mid |x| \le 1\}$ we have $H_k(B_n) = 0$ for all $k \ge 1$.*

Let $c\colon B_n \to \{0\}$ be the constant map. The map $h(x,t) = tx,\ t \in [0,1]$ is a homotopy between id_B und $\imath \circ c$, where $\imath\colon \{0\} \to B_n$ is the inclusion. Thus, $id = \imath_* \circ c_* \implies H_k(B_n) = 0$ for all $k \ge 1$, since $\operatorname{Im} \imath_* = \{0\}$.

Step 2. *There is no continuous map $g\colon B_n \to \partial B_n = S^{n-1}$ such that $g(x) = x$ holds for all $x \in S^{n-1}$.*

Assume $n = 1$ first. In this case there is no continuous map $g\colon [-1,1] \to \{\pm 1\}$ as in the statement of this step, since the target space $\{\pm 1\}$ is disconnected, whereas the interval $[0,1]$ is connected.

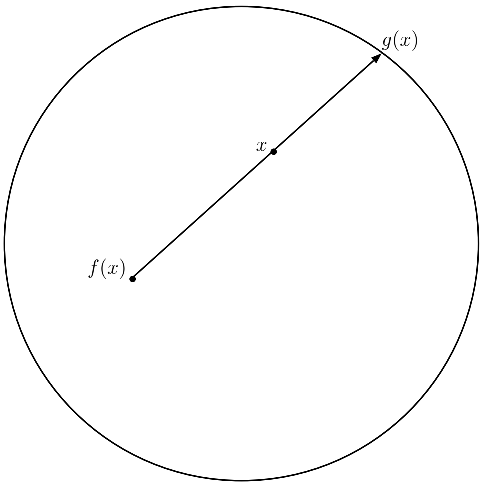
Let us consider now the case $n \ge 2$. Assume there is such $g\colon B_n \to S^{n-1}$. Then we have

$$\begin{aligned} id_{S^{n-1}} = g \circ \imath_{S^{n-1}} &\implies (id_{S^{n-1}})_* = g_* \circ (\imath_{S^{n-1}})_* = 0 \quad \text{on } H_{n-1}(S^{n-1}) \\ &\implies H_{n-1}(S^{n-1}) = 0. \end{aligned}$$

This contradiction proves Step 2.

Step 3. *We prove the theorem of Brower.*

Assume there exists a continuous map $f\colon B_n \to B_n$ without fixed points. Then there also exists a continuous map $g\colon B_n \to S^{n-1}$ such that $g|_{S^{n-1}} = id$:



This contradicts Step 2. □

1.4 The degree of a continuous map and the fundamental theorem of algebra

In this section we show that (a)-(e) imply that any non-constant polynomial with complex coefficients has at least one root. This statement is known as the fundamental theorem of algebra.

Thus, pick any $n \geq 1$ and choose a generator $\alpha \in H_n(S^n)$, i.e., an element α such that $H_n(S^n) = \mathbb{Z} \cdot \alpha$.

Definition 1.18. For any continuous map $f\colon S^n \to S^n$ define $\deg(f) \in \mathbb{Z}$ by

$$f_*\alpha = \deg(f)\alpha.$$

The degree of a map does not depend on the choice of a generator, since $f_*(-\alpha) = -f_*\alpha = -\deg(f)\alpha = \deg(f)(-\alpha)$.

Lemma 1.19. *The degree has the following properties:*

(i) $\deg(id) = 1$*;*

(ii) $\deg(f \circ g) = \deg f \cdot \deg g$*;*

(iii) $f \simeq g \implies \deg f = \deg g$*;*

(iv) $\deg(\textit{const. map}) = 0$*.*

□

Lemma 1.20. *For* $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ *define* $f_n\colon S^1 \to S^1$ *by* $f_n(z) = z^n$*, where* $n \in \mathbb{Z}$*. Then we have*

$$\deg f_n = n.$$

Idea of proof. The curve

$$\alpha\colon [0, 2\pi] \to S^1, \qquad \alpha(t) = \cos t + \sin t\, i = e^{ti},$$

generates $H_1(S^1)$. Since $f_n \circ \alpha(t) = e^{nti} = \cos(nt) + \sin(nt)i$, from the definition of the degree and Remark 1.11 we have $\deg f_n = n$. □

Theorem 1.21 (The fundamental theorem of Algebra)**.** *Each non-constant polynomial* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0,\ a_j \in \mathbb{C}$ *has at least one complex root.*

Proof. Identify S^1 with $S^1_r := \{z \in C \mid |z| = r\} \cong S^1$ with the help of the homeomorphism

$$S^1 \to S^1_r, \qquad z \mapsto rz.$$

The proof consists of the following three steps.

Step 1. *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *be a continuous map without zeros. Then for each* $r > 0$ *the map*

$$\frac{f}{|f|}\colon S^1_r \to S^1 \tag{1.22}$$

is homotopic to the constant map.

Indeed, a homotopy can be given explicitly by

$$F(z,t) = \frac{f(tz)}{|f(tz)|}, \qquad z \in S^1,\ t \in [0,r].$$

Step 2. *Let* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$ *be a polynomial without zeros. Then there exists some* $R > 0$ *such that the following holds:* $\forall r \geq R$ *the restriction of* $p/|p|$ *to* S^1_r *is homotopic to* f_n*.*

For all $z \in \mathbb{C}$ such that $|z| \geq 1$ we have

$$\begin{aligned}\bigl|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\bigr| &\leq |a_{n-1}||z|^{n-1} + \dots + |a_1||z| + |a_0| \\ &\leq n \max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}|z|^{n-1}\end{aligned}$$

Choose R so that $R > n\max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ and $R > 1$. For all $r \geq R$ and all $t \in [0,1]$ the polynomial

$$p_t(z) = z^n + t\bigl(a_{n-1}z^{n-1} + \dots + a_1 z + a_0\bigr)$$

has no zeros on S^1_r, since

$$\bigl|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\bigr| < Rr^{n-1} \leq r^n, \quad \text{provided} \quad |z| = r.$$

Then

$$P(z,t) = \frac{p_t(z)}{|p_t(z)|}\Big|_{S^1_r}$$

is a homotopy between $p/|p|$ and f_n viewed as a map on S^1_r.

Step 3. *We prove the fundamental theorem of algebra.*

Assume p is a non-constant polynomial without zeros. Denote

$$q_r(z) = \frac{p(z)}{|p(z)|}\Big|_{S^1_r},$$

where $r \geq R$. Step 2 $\implies$ $\deg q_r = n$. Step 1 $\implies$ $\deg q_r = 0$, i.e., $n = 0$. Thus, p is a constant polynomial, which is a contradiction. ☐

Chapter 2

Singular homology

2.1 Free abelian groups

An abelian group G is called free with a basis $A \subset G$, if $\forall g \in G$ there exists a unique representation $g = \sum_{a \in A} n_a a$, where $n_a \in \mathbb{Z}$ and $n_a \neq 0$ for finitely many $a \in A$ only.

Any set A generates an abelian group $F(A)$, which is free with a basis A. Indeed, define

$$F(A) := \big\{ f \colon A \to \mathbb{Z} \mid f(a) \neq 0 \text{ nur für endlich viele } a \in A \big\}.$$

Clearly, the functions

$$f_a(x) = \begin{cases} 1 & x = a, \\ 0 & \text{sonst}, \end{cases} \qquad a \in A$$

generate $F(A)$, that is $F(A)$ is free with a basis A.

Remark 2.1. For any $f \in F(A)$ we have

$$f = \sum_{a \in A} f(a) f_a.$$

In particular, $F(A)$ can be viewed as the group of all *finite* formal linear combinations $\sum_{a \in A} n_a a$, where $n_a \in \mathbb{Z}$.

2.2 Singular simplexes

Let $x_0, x_1, \ldots, x_k$ be arbitrary points in $\mathbb{R}^n$ such that $x_1 - x_0, \ldots, x_k - x_0$ are linearly independent.

Definition 2.2. The space

$$\Delta_k = \Delta(x_0, \ldots, x_k) = \Big\{ x = \sum_{i=0}^{k} t_i x_i \mid t_i \in [0,1], \quad \sum_{i=0}^{k} t_i = 1 \Big\}$$

is called *the (non-degenerate) k-simplex generated by* $x_0, \ldots, x_k$.

Example 2.3.
0) If $k = 0$, then $\Delta(x_0) = \{x_0\}$.
1) If $k = 1$, then $\Delta(x_0, x_1)$ is a segment $[x_0, x_1]$.
2) If $k = 2$, then $\Delta(x_0, x_1, x_3)$ is the triangle with the vertices x_0, x_1, x_2.
3) If $k = 3$, then $\Delta(x_0, x_1, x_3, x_4)$ is a tetrahedron with the vertices x_0, x_1, x_3, x_4.

Remark 2.4. The representation $x = \sum_{i=0}^{k} t_i x_i$ of a point in Δ_k is unique. Indeed, $\sum t_i x_i = \sum s_i x_i$, $\sum t_i = 1 = \sum s_i \implies$

$$0 = \sum (t_i - s_i) x_i = \sum (t_i - s_i) x_i - \sum (t_i - s_i) x_0 = \sum (t_i - s_i)(x_i - x_0) \implies t_i = s_i.$$

The coefficients $(t_0, t_1, \ldots, t_k) \in [0,1]^{k+1}$ are called *the barycentric coordinates* of the point $x \in \Delta_k$. In particular, each k-simplex is homeomorphic to the standard k-simplex

$$\Delta^k := \Delta(e_1, \ldots, e_k, e_{k+1}) \subset \mathbb{R}^{k+1},$$

where $e_1, \ldots, e_{k+1}$ is the standard basis of $\mathbb{R}^{k+1}$.

It is customary to drop the adjective "non-degenerate" when referring to simplexes. Sometimes degenerate simplexes (in the sense that $x_1 - x_0, \ldots, x_k - x_0$ may be linearly dependent) do appear below. Typically, this poses no problems, however the barycentric coordinates are ill defined in this case.

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From now on we pick one simplex in each dimension, for example the standard one.

Definition 2.5. Let X be a topological space. *A singular k-simplex* in X is a continuous map $f\colon \Delta^k \to X$.

In particular, a singular 0-simplex in X can be viewed as a point in X, a singular 1-simplex as a path in X etc.

Remark 2.6. The map f in the above definition does not need to be injective. In particular, the image of f may be (highly) singular.

For a singular k-simplex $f\colon \Delta^k \to X$ the $(k-1)$-simplex defined by

$$\partial^i f\colon \Delta^{k-1} \to X, \qquad \partial^i f(t_0, \ldots, t_{k-1}) = f(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1})$$

is called *the ith face* of f.

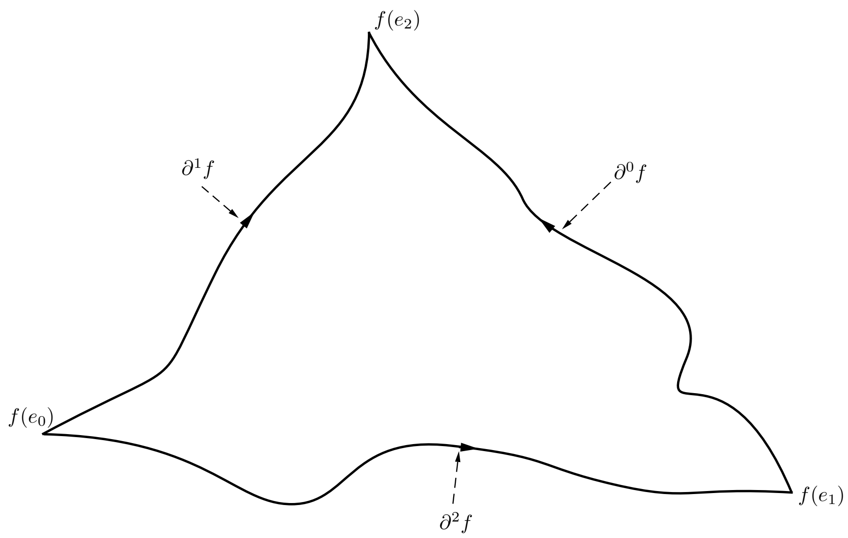

Figure 2.1: Faces of a singular simplex

Definition 2.7. Denote by $S_k(X)$ the free abelian group generated by all singular k-simplexes. Elements of $S_k(X)$ are formal linear combinations of the form

$$\sigma = \sum n_i f_i, \qquad n_i \in \mathbb{Z},$$

which are called *singular k-chains*. The $(k-1)$-chain

$$\begin{aligned} \partial f &= \partial^0 f - \partial^1 f + \partial^2 f - \cdots = \sum_{j=0}^{k} (-1)^j \partial^j f, \\ \partial \sigma &= \sum_i n_i \sum_j (-1)^j \partial^j f_i \end{aligned} \tag{2.8}$$

is called *the boundary* of f and σ respectively.

Proposition 2.9. *We have $\partial_{k-1} \circ \partial_k = 0$ (or, simply $\partial^2 = 0$) for all $k \geq 1$, i.e., the homomorphism*

$$S_k(X) \xrightarrow{\partial_k} S_{k-1}(X) \xrightarrow{\partial_{k-1}} S_{k-2}(X)$$

is trivial.

Proof. The proof consists of the following two steps.

Step 1. *Let f be a singular simplex. for each $j \geq i$ we have*

$$\partial^j \partial^i f = \partial^i \partial^{j+1} f.$$

Indeed,

$$\begin{aligned} \partial^j(\partial^i f)(t_0, \dots, t_{k-2}) &= \partial^i f(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}); \end{aligned}$$

$$\begin{aligned} \partial^i(\partial^{j+1} f)(t_0, \dots, t_{k-2}) &= \partial^{j+1} f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}). \end{aligned}$$

Step 2. *For each singular k-simplex we have $\partial(\partial f) = 0$.*

This follows from the following computation:

$$\begin{aligned} \partial(\partial f) &= \sum_{i=0}^{k} (-1)^i \partial^i (\partial f) = \sum_{i=0}^{k} \sum_{j=0}^{k} (-1)^{i+j} \partial^i \partial^j f = \sum_{j \geq i} + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{j \geq i} (-1)^{i+j} \partial^{j-1} \partial^i f + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{p+1 \geq q} (-1)^{p+q+1} \partial^p \partial^q f + \sum_{p>q} (-1)^{p+q} \partial^p \partial^q f \qquad p := j-1, \ q := i \\ &= 0. \end{aligned}$$

□

Corollary 2.10. $\operatorname{im} \partial_k \subset \ker \partial_{k-1}$.

The elements of $Z_{k-1}(X) := \ker \partial_{k-1}$ are called *cycles* and the elements of $B_{k-1}(X) := \operatorname{im} \partial_k$ are called *boundaries*.

Definition 2.11. The group

$$H_{k-1}(X) := \ker \partial_{k-1} / \operatorname{im} \partial_k = Z_{k-1}(X) / B_{k-1}(X)$$

is called the $(k-1)$ *th (singular) homology group* of X (with integer coefficients). In particular, $H_0(X) := S_0(X) / \operatorname{im} \partial_1$.

2.3 Some properties of the homology groups

Proposition 2.12.
$$X \textit{ path connected} \implies H_0(X) \cong \mathbb{Z}.$$

Proof. $S_0(X)$ is the free abelian group generated by the points of X. Let f be a singular 1-simplex, that is $f\colon [0,1] \to X$ is a path in X. By the definition of the boundary, $\partial f = x_1 - x_0$, where $x_1 = f(1)$ and $x_0 = f(0)$. By the hypothesis, we can connect any two points in X by a path, that is for any two points $x_0, x_1 \in X$ we have $[x_0] = [x_1] \in H_0(X)$.

Furthermore, define the homomorphism $\alpha\colon S_1(X) \to \mathbb{Z}$ by
$$\alpha\big(\sum n_i x_i\big) = \sum n_i.$$
Since $\alpha(\partial f) = 0$ for each singular 1-simplex, α yields a surjective homomorphism $H_0(X) \to \mathbb{Z}$, which is still denoted by α.

Suppose $\alpha([\sum n_i x_i]) = 0$. Then $[\sum n_i x_i] = \sum n_i [x_i] = \big(\sum n_i\big)[x_0] = 0$, that is α is injective. Thus, α is an isomorphism. ☐

Exercise 2.13. If X is not necessarily path connected, then the following holds: $H_0(X) \cong \mathbb{Z}^m$, where m is the number of path-components of X.

Proposition 2.14.
$$H_k(\{pt\}) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, \\ 0 & \textit{else.} \end{cases}$$

Proof. For $k = 0$ the statement of this proposition follows from the previous one. Hence, we may assume $k > 0$. For each such k there is exactly one k-simplex in $\{pt\}$, namely the constant map, which we denote by $c_k\colon \Delta^k \to \{pt\}$. For the boundary we have
$$\partial c_k = \sum_{i=0}^{k} (-1)^i \underbrace{c_k \circ d_i}_{c_{k-1}} = \begin{cases} 0, & \text{for } k \text{ odd}, \\ c_{k-1} & \text{for } k \text{ even}. \end{cases}$$
Hence,
$$Z_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even} \end{cases}$$
und
$$B_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even}. \end{cases}$$
Thus $H_k\big(\{pt\}\big) = Z_k\big(\{pt\}\big)/B_k\big(\{pt\}\big) = 0$. ☐

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Definition 2.15. A topological space X is said to be *contractible* if there is a point $x_0 \in X$ such that the identity map id_X is homotopic to the constant map c_{x_0}.

Proposition 2.16. *For a contractible space X we have*
$$H_k(X) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, \\ 0 & \textit{if } k \geq 1. \end{cases}$$

Proof. Since X is contractible, there exists a continuous map $h\colon X\times[0,1]\to X$ such that $h(x,0)=x$ and $h(x,1)=x_0$ hold for any $x\in X$. In particular, for a fixed $x\in X$ the path $t\mapsto h(t,x)$ connects x and x_0. This implies that X is path connected, hence $H_0(X)\cong\mathbb{Z}$ by Proposition 2.12.

Thus, we assume $k\ge 1$ in the sequel. Consider the quotient map

$$\begin{aligned}\pi\colon \Delta^{k-1}\times[0,1]\to\Delta^k\cong(\Delta^{k-1}\times[0,1])/(\Delta^{k-1}\times\{1\})\\ ((t_0,\dots,t_{k-1}),u)\mapsto(u,(1-u)t_0,\dots,(1-u)t_{k-1}).\end{aligned}$$

Let $h\colon X\times[0,1]\to X$ be a homotopy between id_X and c_{x_0}. Define $s\colon S_{k-1}(X)\to S_k(X)$ as follows: Since π is a quotient map and $h|_{X\times\{1\}}\equiv x_0$, for each singular $(k-1)$-simplex $\sigma\colon\Delta^{k-1}\to X$ there exists a unique map $s(\sigma)\colon\Delta^k\to X$ such that $h\circ(\sigma\times\mathrm{id})=s(\sigma)\circ\pi$. More explicitly,

$$s(\sigma)(t_0,t_1,\dots,t_k)=h\Big(\sigma\Big(\frac{t_1}{1-t_0},\dots,\frac{t_k}{1-t_0}\Big),\ t_0\Big)$$

whenever $t_0\ne 1$ and $s(\sigma)(t_1,\dots,t_k,1)=x_0$. Hence,

1. $\partial^0(s(\sigma))=\sigma$,
2. $\partial^i s(\sigma)=s(\partial^{i-1}\sigma)$ for $i>0$.

Therefore, for any $\sigma\in S_k(X)$ we have

$$\partial(s(\sigma))=\partial^0(s(\sigma))-\sum_{i=1}^{k}(-1)^{i-1}\partial^i(s(\sigma))=\sigma-\sum_{j=0}^{k-1}(-1)^j s(\partial^j\sigma)=\sigma-s(\partial\sigma). \tag{2.17}$$

This yields

$$\partial\circ s+s\circ\partial=\mathrm{id}.$$

Hence, if σ is a cycle, then $\sigma=\partial\big(s(\sigma)\big)+s\big(\partial\sigma\big)=\partial\big(s(\sigma)\big)$, i.e., any cycle is a boundary. In other words, $H_k(X)=0$ whenever $k\ge1$ as claimed. □

Theorem 2.18. *Let $f\colon X\to Y$ be a continuous map. Then for each $k\ge0$ the map f induces a group homomorphism*

$$f_*\colon H_k(X)\to H_k(Y)$$

and for any other continuous map $g\colon Y\to Z$ we have

$$(g\circ f)_*=g_*\circ f_*.$$

Finally, $(\mathrm{id}_X)_*=\mathrm{id}$.

Proof. Define first group homomorphisms $f_\#\colon S_k(X)\to S_k(Y)$, by declaring

$$\sigma\mapsto f\circ\sigma\qquad\text{for}\qquad\sigma\colon\Delta^k\to X.$$

Then for all singular k-simplexes $\sigma\colon\Delta^k\to X$ we have

$$\begin{aligned}(f_\#\partial^i(\sigma))(t_0,\dots,t_{k-1})&=f(\sigma(t_0,\dots,t_{i-1},0,t_i,\dots,t_{k-1}))\\&=(f_\#\sigma)(t_0,\dots,t_{i-1},0,t_i,\dots,t_{k-1})\\&=\partial^i(f_\#\sigma)(t_0,\dots,t_{k-1}),\end{aligned}$$

and therefore $f_\#\partial^i = \partial^i f_\#$, which yields in turn that $f_\#$ is a *chain map*, i.e.,

$$f_\#\partial = \partial f_\#.$$

This yields in particular that cycles are mapped to cycles and boundaries are mapped to boundaries:

$$f_\#(Z_k(X)) \subset Z_k(Y) \qquad \text{and} \qquad f_\#(B_k(X)) \subset B_k(Y).$$

Hence, we obtain a well defined group homomorphism:

$$f_*\colon H_k(X) = Z_k(X)/B_k(X) \to Z_k(Y)/B_k(Y) = H_k(Y)$$
$$f_*([\sigma]) := [f_\#(\sigma)].$$

Furthermore, for each singular k-simplex $\sigma\colon \Delta^k \to X$ we have

$$\begin{aligned}
&g_\# \circ f_\#(\sigma) = g_\#(f \circ \sigma) = g \circ f \circ \sigma = (g \circ f)_\#(\sigma),\\
&g_* \circ f_*([\sigma]) = g_*[f_\#(\sigma)] = [g_\# \circ f_\#(\sigma)] = [(g \circ f)_\#(\sigma)] = (g \circ f)_*([\sigma]),\\
&(\mathrm{id}_X)_\#(\sigma) = \sigma,\\
&(\mathrm{id}_X)_*([\sigma]) = [(\mathrm{id}_X)_\#(\sigma)] = [\sigma].
\end{aligned}$$

Therefore, $g_* \circ f_* = (g \circ f)_*$ and $(\mathrm{id}_X)_* = \mathrm{id}$. □

Corollary 2.19. *If $f\colon X \to Y$ is a homeomorphism, then $f_*\colon H_k(X) \to H_k(Y)$ is an isomorphism for each k.* □

2.4 Homotopies and homology groups

Satz 2.20. *If $f, g\colon X \to Y$ are homotopic maps, then the induced maps on the homology groups are equal:*

$$f \simeq g \qquad \Longrightarrow \qquad f_* = g_*.$$

Proof. The proof consists of the following three steps.

Step 1. *Define*

$$\eta_t\colon X \to X \times I, \qquad \eta_t(x) = (x, t).$$

For each continuous map $f\colon X \to Y$ we have $(f \times id)_\#\eta^X_{t\#} = \eta^Y_{t\#} \circ f_\#$.

This follows immediately from the observation that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\eta_t^X} & X \times I\\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times id}\\
Y & \xrightarrow{\eta_t^Y} & Y \times I
\end{array}$$

commutes.

Step 2. *There exists a sequence of homomorphisms $s_k^X\colon S_k(X) \to S_{k+1}(X \times I)$ satisfying*

$$\partial s_k^X + s_{k-1}^X\partial = \eta_{1\#} - \eta_{0\#}; \tag{2.21}$$
$$(f \times id_I)_\# \circ s_k^X = s_k^Y \circ f_\#. \tag{2.22}$$

Define $s_k = s_k^X$ recursively. For $k = 0$ and $x_0 \in X$, which we view as a 0-simplex, put

$$s_0\sigma\colon \Delta^1 \to X \times I, \qquad (t_0, t_1) \mapsto (x_0, t_1).$$

Then we have $\partial(s_0\sigma) = (x_0, 1) - (x_0, 0)$, i.e., (2.21) holds for $k = 0$. Equation (2.22) follows directly from the definition of s_0.

Suppose s_ℓ have been defined for all $\ell < k$. We define first s_k in a special case, namely for id_{Δ^k} viewed as an element $\imath_k \in S_k(\Delta^k)$. We have

$$\begin{aligned}\partial\big(\underbrace{\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k}_{\in S_k(\Delta^k \times I)}\big) &= \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \partial s_{k-1}\partial\imath_k \\ &\overset{(2.21)}{=} \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \big(\eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - s_{k-2}^{\Delta^k}\partial^2\imath_k\big) \\ &= 0.\end{aligned}$$

In this computation (2.21) is used for k replaced by $k-1$. Since $\Delta^k \times I$ is contractible, there exists some $a \in S_{k-1}(\Delta^k \times I)$ so that

$$\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k = \partial a.$$

Define $s_k(\imath_k) = a$. Then (2.21) holds for $\sigma = \imath_k$.

In general, define $s_k^X(\sigma) = (\sigma \times id)_\# a$. Then we have

$$\begin{aligned}\partial(s_k^X\sigma) &= \partial(\sigma \times id)_\# a = (\sigma \times id)_\#\partial a \\ &= (\sigma \times id)_\#\big(\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}^{\Delta^k}\partial\imath_k\big) \\ &= \eta_{1\#}\sigma_\#\imath_k - \eta_{0\#}\sigma_\#\imath_k - s_{k-1}^X\sigma_\#\partial\imath_k && (2.22) + \text{Step } 1 \\ &= \eta_{1\#}\sigma - \eta_{0\#}\sigma - s_{k-1}^X\partial\sigma.\end{aligned}$$

This proves (2.21).

We still have to show that (2.22) holds. Indeed,

$$(f \times id)_\# s_k\sigma = (f \times id)_\#(\sigma \times id)_\# a = \big((f \circ \sigma) \times id\big)_\# a = s_k(f\sigma) = s_k(f_\#\sigma).$$

Step 3. *We prove this theorem.*

Let h be a homotopy between f and g. From the following equalities

$$\partial(h_\# \circ s_k) + (h_\# \circ s_{k-1})\partial = h_\#\partial s_k + h_\#(s_{k-1}\partial) = h_\#(\eta_{1\#} - \eta_{0\#}) = f_\# - g_\#$$

we see that $f_\# - g_\# = \partial(h_\# \circ s_k)$ holds on $\ker\partial$. This shows that $f_* = g_*$. ☐

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Definition 2.23. A continuous map $f\colon X \to Y$ is called a homotopy equivalence, if there exists a continuous map $g\colon Y \to X$ such that the following holds:

$$g \circ f \simeq id_X \qquad \text{and} \qquad f \circ g \simeq id_Y.$$

In this case the spaces X and Y are called homotopy equivalent.

Example 2.24. (i) Any two homeomorphic spaces are homotopy equivalent.

(ii) $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$. More generally, any contractible space is homotopy equivalent to $\{pt\}$.

(iii) $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}.

To see (ii), let X be a contractible space and $\imath_{x_0}\colon \{x_0\} \to X$ be the embedding of the point x_0. Then $c_{x_0} \circ \imath_{x_0} = id_{x_0}$ and $\imath_{x_0} \circ c_{x_0} \simeq id_X$.

To see (iii), define $f\colon \mathbb{R}^n \setminus \{0\} \to S^n$ by $f(x) = x/|x|$. If $g\colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ denotes the inclusion, then $f \circ g = id_{S^{n-1}}$. Furthermore,

$$h(x,t) = \frac{1}{t + (1-t)|x|}x, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

is a homotopy between $g \circ f$ and $id_{\mathbb{R}^n \setminus \{0\}}$.

Corollary 2.25.

$$f \textit{ is a homotopy equivalence } \implies \forall k \quad f_*\colon H_k(X) \to H_k(Y) \textit{ is an isomorphism.}$$

Example 2.26.

$$H_k(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_k(\mathbb{R}^n \setminus \{pt\}) = H_k(S^{n-1}) = \begin{cases} \mathbb{Z} & k = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

2.5 Exact sequences and the Bockstein homomorphism

Definition 2.27. A sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A_{k+1} \xrightarrow{\alpha_{k+1}} A_k \xrightarrow{\alpha_k} A_{k-1} \longrightarrow \ldots \tag{2.28}$$

is called exact, if for all k the following holds: $\ker \alpha_k = \operatorname{im} \alpha_{k+1}$.

Some special cases:

(i) $0 \to A \xrightarrow{\alpha} B$ is exact $\quad\Leftrightarrow\quad$ α is injectiv;

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact $\quad\Leftrightarrow\quad$ α is surjectiv;

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\quad\Leftrightarrow\quad$ α is an isomorphism;

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact $\quad\Leftrightarrow\quad$ α is injectiv, β is surjectiv and $\ker \beta = \operatorname{im} \alpha$; In particular, β induces an isomorphism $C \cong B/A$.

The sequence (iv) is called a short exact sequence.

Example 2.29. $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a short exact sequence, where $\times n$ stands for the multiplication with a fixed $n \in \mathbb{Z}$.

Let A be a complex, that is A is a sequence

$$A: \qquad \cdots \longrightarrow A_{i+1} \xrightarrow{\partial} A_i \xrightarrow{\partial} A_{i-1} \longrightarrow \dots$$

such that $\partial^2 = 0$. Just like in the case of chain complexes, we define the kth homology group of A to be

$$H_k(A) := \frac{\ker\big(\partial\colon A_k \to A_{k-1}\big)}{\operatorname{im}\big(\partial\colon A_{k+1} \to A_k\big)}.$$

If A, B, and C are complexes, a sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of complexes is a commutative diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "\partial"] & A_{k+1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k+1}"'] & A_k \arrow[r, "\partial"] \arrow[d, "\alpha_k"'] & A_{k-1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k-1}"'] & \dots \\
\dots \arrow[r, "\partial"] & B_{k+1} \arrow[r, "\partial"] \arrow[d, "\beta_{k+1}"'] & B_k \arrow[r, "\partial"] \arrow[d, "\beta_k"'] & B_{k-1} \arrow[r, "\partial"] \arrow[d, "\beta_{k-1}"'] & \dots \\
\dots \arrow[r, "\partial"] & C_{k+1} \arrow[r, "\partial"] \arrow[d] & C_k \arrow[r, "\partial"] \arrow[d] & C_{k-1} \arrow[r, "\partial"] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.30}$$

Such a sequence is called *exact*, if each vertical sequence $0 \to A_k \to B_k \to C_k \to 0$ is exact.

Here of course we could equally well consider sequences of complexes consisting of more than 3 complexes.

Example 2.31. Let X, Y and Z be topological spaces and $f\colon X \to Y$, $g\colon Y \to Z$ continuous maps. Then one obtains a sequence of chain complexes

$$0 \to S_*(X) \xrightarrow{f_\#} S_*(Y) \xrightarrow{g_\#} S_*(Z) \to 0,$$

which is not necessarily exact. What conditions guarantee that the above sequence is exact will be considered below.

Proposition 2.32. *The maps α and β yield homomorphisms $\alpha\colon H_*(A) \to H_*(B)$ and $\beta\colon H_*(B) \to H_*(C)$ respectively.*

Proof. This follows immediately from the commutativity of (2.30). □

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Theorem 2.33. *A short exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ induces a (long) exact sequence of homology groups:*

$$\cdots \to H_k(A) \xrightarrow{\alpha} H_k(B) \xrightarrow{\beta} H_k(C) \xrightarrow{\delta} H_{k-1}(A) \xrightarrow{\alpha} H_{k-1}(B) \to \dots$$

Remark 2.34. The map δ is called the Bockstein homomorphism.

Proof. The proof consists of the following four steps.

Step 1. *We define* δ.

Pick $c \in C_k$, $\partial c = 0$. Since β_k is surjective, there exists some $b \in B_k$ such that $\beta(b) = c$. We have $\beta(\partial b) = \partial(\beta(b)) = \partial c = 0$. Since $\alpha\colon A_{k-1} \to \ker \beta_{k-1}$ is surjective, there is some $a \in A_{k-1}$ such that $\alpha(a) = \partial b$. Define

$$\delta[c] = [a].$$

We have to show that δ is well defined. Indeed, pick another representative $c' = c + \partial c''$ of the class $[c]$. For $c'' \in C_{k+1}$ there is some $b'' \in B_{k+1}$ such that $\beta(b'') = c'' \implies \beta(b + \partial b'') = c + \partial c''$. This yields $b' = b + \partial b'' + \alpha(a'')$, where $a'' \in A_k$. Furthermore, $\partial b' = \partial b + 0 + \alpha(\partial a')$. Since α is injective, we have $a' = a + \partial a''$, i.e., $[a] = [a']$.

Exercise 2.35. Check that δ is a group homomorphism.

Step 2. $\ker \alpha = \operatorname{im} \delta$.

Pick $a \in A_{k-1}$ such that $[a] \in \ker \alpha$, i.e., $\alpha(a) = \partial b$ for some $b \in B_k$. We have $\partial\beta(b) = \beta(\partial b) = \beta(\alpha(a)) = 0$. By the construction of δ, we obtain $\delta[\beta(b)] = [a]$. That is $\ker \alpha \subset \operatorname{im} \delta$.

If $a \in A_{k-1}$ is such that $[a] \in \operatorname{im} \delta$, then by the construction of δ, we have $\alpha(a) = \partial b \implies \alpha[a] = 0$.

Step 3. $\ker \delta = \operatorname{im} \beta$.

Pick some $[c] \in \ker \delta$. Using the notations of Step 1, we have $a = \partial a'$ for some $a' \in A_k$. The equations

$$\partial\big(b - \alpha(a')\big) = \partial b - \alpha(\partial a') = \partial b - \alpha(a) = 0;$$
$$\beta\big(b - \alpha(a')\big) = \beta(b) = c;$$

yield $\beta[b - \alpha(a')] = [c]$, i.e., $\ker \delta \subset \operatorname{im} \beta$.

The inclusion $\operatorname{im} \beta \subset \ker \delta$ follows immediately from the construction of δ.

Step 4. $\ker \beta = \operatorname{im} \alpha$.

Assume $b \in B_k$ satisfies $\beta[b] = 0$, that is $\partial b = 0$ and $\beta(b) = \partial c$ for some $c \in C_{k+1}$. Since β is surjective, there is some $\hat{b} \in B_{k+1}$ such that $\beta(\hat{b}) = c$. Furthermore,

$$\beta(b - \partial\hat{b}) = \beta(b) - \partial\beta(\hat{b}) = \beta(b) - \partial c = 0.$$

This yields that there exists some $a \in A_k$ such that $\alpha(a) = b - \partial\hat{b}$. Moreover,

$$\alpha(\partial a) = \partial\alpha(a) = \partial b - \partial^2\hat{b} = 0.$$

Since α is injective, we obtain $\partial a = 0$. This yields $\alpha[a] = [b - \partial\hat{b}] = [b]$, that is $\ker \beta \subset \operatorname{im} \alpha$.

The inclusion $\operatorname{im} \alpha \subset \ker \beta$ follows immediately from $\alpha \circ \beta = 0$. $\square$

2.6 Relative homology groups

For each subspace $A \subset X$ define

$$S_n(X, A) := S_n(X)/S_n(A).$$

The boundary map on $S_n(X)$ induces a boundary map on $S_n(X, A)$ and we obtain the following new chain complex:

$$\dots \to S_{n+1}(X, A) \xrightarrow{\partial} S_n(X, A) \xrightarrow{\partial} S_{n-1}(X, A) \to \dots$$

The homology groups of this complex are denoted by $H_*(X, A)$ and are called *the homology groups of X relative to A*, or, simply, *relative homology groups*. Let us provide some details of this definition:

- Elements of $H_n(X, A)$ are represented by *relative chains* $a \in S_n(X)$ such that $\partial a \in S_{n-1}(A)$;
- $[a] = 0 \in H_n(X, A) \quad \iff \quad a = \partial b + c, \quad b \in S_{n+1}(X),\ c \in S_n(A).$

By the very definition of $S_n(X, A)$, the sequence $0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$ is exact. Hence, Theorem 2.33 yields the following:

Theorem 2.36. *There is a long exact sequence of the homology groups*

$$\dots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \dots$$

Moreover, the following holds:

- *i_* is induced by the inclusion $i\colon A \subset X$;*
- *j_* is induced by the projection $S_n(X) \to S_n(X, A)$;*
- *$\delta[a] = [\partial a]$.*

□

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