Global Analysis

Lecture notes

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February 19, 2022

This is a draft. If you spot a mistake, please let me know.

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Chapter 1

Introduction

1.1 The de Rham cohomology groups

Let M be a compact manifold of dimension n. Denote by $\Omega^k(M)$ the space of differential k-forms on M. Recall that there exist a unique $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) df is the differential of f if $f \in C^\infty(M) = \Omega^0(M)$;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^q \omega \wedge d\eta$ if $\eta \in \Omega^q(M)$;

(iii) $d^2 = 0$.

The last property simply means that $d(d\omega) = 0$ for each $\omega \in \Omega^k(M)$. This yields the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0 \tag{1.1}$$

Remark 1.2. The map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ depends on k, however this is suppressed in the notations.

Property (iii) means that (1.1) is a complex, that is the kernel of $d : \Omega^k \to \Omega^{k+1}$ contains the image of $d : \Omega^{k-1} \to \Omega^k$ and therefore we can define

$$H^k_{dR}(M) := \frac{\operatorname{Ker}\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\operatorname{Im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)} \tag{1.3}$$

This is called the k^{th} de Rham cohomology group, which is in fact a vector space. The number

$$b_k(M) := \dim\ H^k_{dR}(M)$$

is called the k^{th} Betti number of M and is an invariant of M. Notice that by the compactness of M we have $b_k(M) < \infty$.

Remark 1.4. While it is by no means obvious from the above description, Betti numbers are topological invariants of M, that is $b_k(M) = b_k(N)$ provided M and N are homeomorphic. In paticular, Betti numbers do not depend on the smooth structure.

Coming back to the de Rham cohomlogy groups, each element in $H^k_{dR}(M)$ is represented by the equivalence class

$$[\omega] = \left\{\omega + d\eta \mid \eta \in \Omega^{k-1}(M)\right\},$$

where $\omega \in \Omega^k(M)$ is closed: $d\omega = 0$. Hence, we may ask the following.

Question 1.5. What is the best representative in $[\omega]$?

Of course, at this point the above question is vague, since the notion of being "the best" is undefined. One possibility to convert this into a precise question is as follows. Just by its definition, the set $[\omega]$ is an affine subspace of $\Omega^k(M)$. We could call an element in $[\omega]$ "the best" if it is the closest one to the origin just as shown schematically on Figure 1.1.

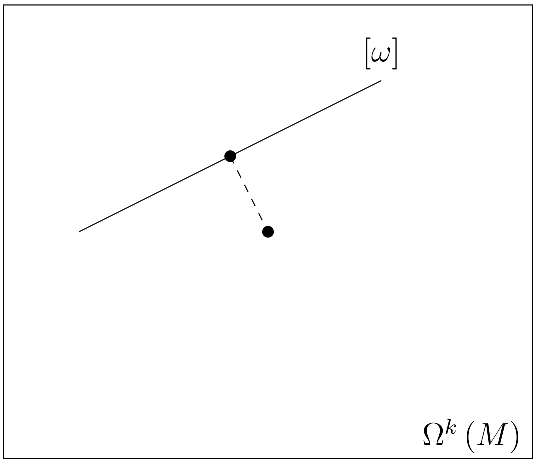

Figure 1.1: A choice of a representative in the de Rham cohomology class.

However, this raises our next question: How do we measure distance in $\Omega^k(M)$? A suitable answer to this question requires a detour, which we do next.

1.2 Some linear algebra

Let V be a vector space of dimension n over $\mathbb{R}$. Recall that for each basis $e = (e_1, \ldots, e_n)$ of V there exist a unique basis $e^* = (e_1^*, \ldots, e_n^*)$ of the dual vector space V^* such that

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

e^* is called the dual basis to e.

Assume $\langle \cdot, \cdot \rangle$ is a scalar product on V. If $e = (e_1, \ldots, e_n)$ is an orthonormal basis of V, that is

$$\langle e_i, e_j \rangle = \delta_{ij},$$

then the dual basis e^* is given explicitly by

$$e_i^* := \langle e_i, \cdot \rangle \qquad \Longleftrightarrow \qquad e_i^*(v) = \langle e_i, v \rangle \quad \text{for } v \in V.$$

Then V^* has a unique scalar product such that $e^* = (e_1^*, \ldots, e_n^*)$ is an orthonormal basis. Explicitly, for $\xi, \eta \in V^*$ define

$$\begin{matrix} \xi_i := \xi(e_i) \\ \eta_i := \eta(e_i) \end{matrix} \qquad \Longleftrightarrow \qquad \begin{matrix} \xi = \sum \xi_i \, e_i^* \\ \eta = \sum \eta_j \, e_j^* \end{matrix}$$

$$\implies \langle \xi, \eta \rangle = \sum_{i=1}^{n} \xi_i \eta_i = \sum_{i=1}^{n} \xi(e_i)\, \eta(e_i)\,.$$

To sum up, for any scalar product on V there exists a unique scalar product on V^* such that the dual basis of an orthonormal basis is itself orthonormal.

More generally, any basis e of V yields a basis of $\Lambda^k V^*$. Explicitly,

$$\Lambda^k e := \left\{ e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \mid 1 \le i_1 < i_2 < \ldots < i_k \le n \right\} \tag{1.6}$$

is a basis of $\Lambda^k V^*$ consisting of

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!k!}$$

elements. Just like in the case of $V^* = \Lambda^1 V^*$, we can define a scalar product on each $\Lambda^k V^*$ by declaring (1.6) to be an orthonormal basis.

Recall that any two bases e and f of V are related by a change-of-basis matrix A. This means

$$f = e \cdot A \qquad \iff \qquad f_i = \sum_{j=1}^{n} a_{ij}\, e_j$$

Then e and f are said to be *cooriented*, if $\det A > 0$. It is easy to check that

$$e \sim f \qquad \equiv \qquad e \text{ and } f \text{ are cooriented}$$

yields an equivalence relation on the set of all bases of V. Moreover, there are exactly two equivalence classes represented by e and $\bar{e} = (-e_1, e_2, \ldots, e_n)$.

Definition 1.7. An orientation on V is a choice of an equivalence class of bases of V. Any basis in the chosen class is said to be positively oriented and any basis, which does not belong to the selected class is said to be negatively oriented.

Example 1.8. For $\mathbb{R}^n$ the class of the standard basis is called the standard orientation of $\mathbb{R}^n$.

Example 1.9. Any $\omega \in \Lambda^n V^*$, $\omega \neq 0$, determines an orientation of V by the rule: e is positively oriented if and only if

$$\omega(e_1, \ldots, e_n) > 0.$$

For example, if $e^* = (e_1^*, \ldots, e_n^*)$ is the dual basis to the standard one, then

$$\omega := e_1^* \wedge \ldots \wedge e_n^* \tag{1.10}$$

determines the standard orientation of $\mathbb{R}^n$.

Definition 1.11. Let V be an oriented Euclidean vector space of dimension n. An n-form ω is said to be the Euclidean volume form, if

$$\omega(e_1, \ldots, e_n) = 1 \tag{1.12}$$

holds for any positively oriented orthonormal basis e of V.

For example, in the case $V = \mathbb{R}^n$, which is equipped with the standard scalar product and orientation, (1.10) is the Euclidean volume form.

Example 1.13. Show that any oriented Euclidean vector space admits a unique Euclidean volume form. This is sometimes denoted by vol.

Proposition 1.14. *There is a unique linear map*

$$* : \Lambda^k V^* \;\to\; \Lambda^{n-k} V^* \qquad \textit{satisfying} \qquad \xi \wedge * \, \eta = \langle \xi, \eta \rangle \, vol \tag{1.15}$$

for all ξ , $\eta \in \Lambda^k V^*$.

Proof (Sketch). Let e be a positively oriented orthonormal basis of V. Set

$$\eta := e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \in \Lambda^k e^*$$

Then assuming that $*$ exists, for

$$\xi = \sum \xi_{j_1 \ldots j_k} \, e_{j_1}^* \wedge \ldots \wedge e_{j_k}^*$$

we must have

$$\xi \wedge * \, \eta = \langle \xi, \eta \rangle \, vol = \xi_{j_1 \ldots j_k} \cdot e_1^* \wedge \ldots \wedge e_n^*.$$

This yields

$$* \;\; \eta = * \left(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \right) = \varepsilon \cdot e_{l_1}^* \wedge \ldots \wedge e_{l_{n-k}}^*, \tag{1.16}$$

where $\varepsilon \in \{\pm 1\}$ and $1 \leq l_1 < \ldots < l_{n-k} \leq n$ consists of those integers in the interval $[1, n]$ which are complementary to $\{i_1, \ldots, i_k\}$.

For example, if $n = 6$ and $\eta = e_2 \wedge e_4$, then

$$* \, (e_2^* \wedge e_4^*) = \varepsilon \; e_1^* \wedge e_3^* \wedge e_5^* \wedge e_6^* \, .$$

To determine ε, we compute

$$e_2^* \wedge e_4^* \wedge \varepsilon \, (e_1^* \wedge e_3^* \wedge e_5^* \wedge e_6^*) = -\varepsilon \; vol \, ,$$
$$e_2^* \wedge e_4^* \wedge * \, (e_2^* \wedge e_4^*) = \| \, e_2^* \wedge e_4^* \, \|^2 \cdot vol = vol,$$

which yields $\varepsilon = -1$ so that we finally obtain

$$* \, (e_2^* \wedge e_4^*) = -e_1^* \wedge e_3^* \wedge e_5^* \wedge e_6^*.$$

In general, ε is the signature of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Thus, (1.16) defines $*$ on the elements of the basis $\Lambda^k e$. This yields a unique linear map $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$, which satisfies (1.15). ☐

The map $*$ defined in the above proposition is called *the Hodge operator*.

Remark 1.17. It follows from the proof of the above proposition that the Hodge operator satisfies

$$* * \omega = (-1)^{k(n-k)} \, \omega \qquad \text{for all } \omega \in \Lambda^k V^*. \tag{1.18}$$

1.3 Riemannian metrics

Definition 1.19. A Riemannian metric g on M is a smooth section of $T^*M \otimes T^*M$ such that the following holds:

(i) g is symmetric, that is

$$g\,(v, w) = g\,(w, v) \qquad \text{for all } v, w \in T_m M \text{ and any } m \in M;$$

(ii) g is positive-definite, that is

$$g(v,v) > 0 \qquad \text{for all } v \in T_mM, v \neq 0, \text{ and any } m \in M.$$

In other words, g is a family $\{g_m \mid m \in M\}$ of scalar products on each T_mM and g_m depends smoothly on m. In particular, each T_mM is an Euclidean vector space. Hence, each $\Lambda^k T_m^* M$ is also an Euclidean vector space.

An orientation of a manifold M is (informally speaking) a choice of coherent orientations of T_mM for each $m \in M$. More formally, we have the following.

Definition 1.20. A manifold M of dimension n is said to be orientable, if there exists $\omega \in \Omega^n(M)$ such that $\omega_m \neq 0$ for all $m \in M$.

By Example 1.9, for each $m \in M$ the n-form on T_mM determines a class of positively oriented bases of T_mM, that is an orientation. Notice that for any function f, which is positive everywhere, the forms ω and $f \cdot \omega$ determine the same orientation on each T_mM .

Albeit not all manifolds are orientable, orientability is a mild restriction. In particular, for any connected non-orientable manifold M there exists a unique double covering $M_2 \to M$, which is orientable. The reader may find more information on this in [?].

Definition 1.21. An orientation of an n-manifold M is a class of n-forms $[\![\omega]\!]$, where

- ω is a nowhere vanishing n-form on M.
- $\omega_1 \sim \omega_2$ if and only if there exists an everywhere positive function f such that $\omega_2 = f \cdot \omega_1$.

Notice that $[\![\cdot]\!]$ above is *not* the de Rham cohomology class.

Just like in the preceding section, a differential n-form ω on M is said to be a Riemannian volume form, if

$$\omega_m(e_1, \dots, e_n) = 1 \tag{1.22}$$

holds for any $m \in M$ and any oriented orthonormal basis $(e_1, \dots, e_n)$ of T_mM. Property (1.22) determines a Riemannian volume form uniquely. This volume form is denoted by vol.

Thus, by the preceding subsection, a Riemannian metric and orientation on M induce for each $k \leq n$ the Hodge operator $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ such that

$$\omega \wedge * \eta = \langle \omega, \eta \rangle \; vol \qquad \text{and} \qquad * * \omega = (-1)^{k(n-k)} \omega$$

holds for all $\omega, \eta \in \Omega^k(M)$.

1.4 Harmonic forms

Let us come back to the original question about "the best" representatives of the de Rham cohomology classes. Thus, we define the L^2-scalar product on each $\Omega^k(M)$ by setting

$$\langle \omega, \eta \rangle_{L^2} := \int_M \langle \omega_m, \eta_m \rangle \; vol_m = \int_M \omega \wedge * \eta.$$

With this at hand, we could call an element $\widehat{\omega} = \omega + d\eta \in [\omega]$ "the best", if $\widehat{\omega}$ minimizes the distance to the origin, that is if

$$\inf_{\eta \in \Omega^{k-1}(M)} \| \, \omega + d\eta \, \|_{L^2}^2 = \| \, \widehat{\omega} \, \|_{L^2}. \tag{1.23}$$

Then, if (1.23) holds, for any $\eta \in \Omega^{k-1}(M)$ we must have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \| \,\widehat{\omega} + t\, d\eta\, \|_{L^2}^2 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\| \,\widehat{\omega}\, \|_{L^2}^2 + 2t\langle \widehat{\omega}, d\eta\rangle_{L^2} + t^2 \| \, d\eta\, \|_{L^2}^2\right) \\ &= 2\langle \widehat{\omega}, d\eta\rangle_{L^2} \end{aligned} \tag{1.24}$$

Proposition 1.25. *Denote* $d^* := (-1)^{nk+1} * d *\colon \Omega^{k+1}(M) \to \Omega^k(M)$.

(i) d^* *is the formal adjoint of* d*, that is*

$$\langle d\omega, \eta\rangle_{L^2} = \langle \omega, d^*\eta\rangle_{L^2} \qquad \textit{for all } \omega \in \Omega^k(M) \textit{ and } \eta \in \Omega^{k+1}(M). \tag{1.26}$$

(ii) (1.24) *is equivalent to*

$$d^*\widehat{\omega} = 0. \tag{1.27}$$

Proof. Notice first that (1.15) and (1.18) imply the equality

$$\omega \wedge \zeta = (-1)^{k(n-k)} \langle \omega, * \zeta\rangle vol \qquad \text{for all } \omega \in \Omega^k(M) \text{ and } \zeta \in \Omega^{n-k}(M).$$

Using this, we obtain

$$\begin{aligned} \langle \omega, d^*\eta\rangle_{L^2} &= (-1)^{kn+1} \int_M \langle \omega, * d * \eta\rangle \; vol \\ &= (-1)^{kn+1+k(n-k)} \int_M \omega \wedge d * \eta \\ &= (-1)^{1-k^2} \int_M (-1)^k \left(d(\omega \wedge * \eta) - d\omega \wedge * \eta\right) \end{aligned}$$

Here the last equality follows from the Leibnitz rule:

$$d(\omega \wedge \zeta) = d\omega \wedge \zeta + (-1)^k \omega \wedge d\zeta$$

provided $\omega \in \Omega^k(M)$ and $\zeta \in \Omega^l(M)$. Hence, by Stokes' theorem, we obtain

$$\begin{aligned} \langle \omega, d^*\eta\rangle_{L^2} &= (-1)^{k-k^2+1} \int_M d\omega \wedge * \eta \\ &= (-1)^{k-k^2} \langle d\omega, \eta\rangle_{L^2} \end{aligned}$$

By noticing that k^2 is even/odd if and only if k is even/odd, we arrive finally at (1.26).

To prove (ii), notice that

$$d^*\widehat{\omega} = 0 \qquad \Longrightarrow \qquad 0 = \langle d^*\widehat{\omega}, \eta\rangle_{L^2} = \langle \widehat{\omega}, d\eta\rangle_{L^2}.$$

Conversely, setting $\eta = d^*\widehat{\omega}$ in (1.24), we obtain

$$0 = \langle \widehat{\omega}, dd^*\widehat{\omega}\rangle_{L^2} = \langle d^*\widehat{\omega}, d^*\widehat{\omega}\rangle_{L^2} = \| \, d^*\omega \, \|_{L^2}^2 \qquad \Longrightarrow \qquad d^*\widehat{\omega} = 0.$$

□

Notice that (1.27) is nothing else but the Euler-Lagrange equation for the functional

$$f\colon \left\{\omega + d\eta \mid \eta \in \Omega^{k-1}(M)\right\} \to \mathbb{R}, \qquad f(\omega + d\eta) = \| \omega + d\eta \|_{L^2}^2 \tag{1.28}$$

Definition 1.29. The map

$$\Delta = dd^* + d^*d\colon \Omega^k(M) \to \Omega^k(M)$$

is called *the Laplace operator* (or, simply, *the Laplacian*). A k-form ω such that $\Delta\omega = 0$ is called *harmonic*.

Proposition 1.30. *A k-form ω is harmonic if and only if*

$$d\omega = 0 \qquad \textit{and} \qquad d^*\omega = 0. \tag{1.31}$$

Proof. If (1.31) holds, then ω is clearly harmonic. To show the converse, consider

$$0 = \langle \Delta\omega, \omega \rangle_{L^2} = \langle dd^*\omega, \omega \rangle_{L^2} + \langle d^*d\omega, \omega \rangle = \| d^*\omega \|_{L^2}^2 + \| d\omega \|_{L^2}^2.$$

Since both summands are non-negative, we obtain (1.31). □

Theorem 1.32. *If a minimizer $\widehat{\omega}$ of (1.28) exists, then $\widehat{\omega}$ is harmonic. Moreover, if $\widehat{\omega}$ exists, then it is unique.*

Proof. Assume $\widehat{\omega}$ exists. Since $\widehat{\omega} = \omega + d\eta$ for some $\eta \in \Omega^{k-1}(M)$, we have

$$d\widehat{\omega} = d\omega + d(d\eta) = 0 + 0.$$

Combining this with (1.27), we obtain by Proposition 1.30, that $\widehat{\omega}$ is harmonic.

Furthermore, let $\widehat{\omega} = \omega + d\eta$ and $\widehat{\widehat{\omega}} = \omega + d\zeta$ be two harmonic representatives of $[\omega]$. Then

$$0 = d^*\widehat{\omega} - d^*\widehat{\widehat{\omega}} = d^*\left(\widehat{\omega} - \widehat{\widehat{\omega}}\right) = d^*d(\eta - \zeta).$$

Denoting temporarily $\xi := \eta - \zeta$, we obtain

$$0 = \langle d^*d\xi, \xi \rangle_{L^2} = \langle d\xi, d\xi \rangle = \| d\xi \|_{L^2}^2 \Rightarrow d\xi = 0 \quad \Longrightarrow \quad d\eta = d\zeta \quad \Longrightarrow \quad \widehat{\omega} = \widehat{\widehat{\omega}}.$$

This proves the uniquesness. □

Our aim is to prove the following.

Theorem 1.33. *(Hodge) Each de Rham cohomology class is represented by a unique harmonic form.*

Notice that since we have already proved the uniqueness, it is the existence, which remains to be proved. It turns out that this is somewhat harder and requires certain technology, which we will consider first.

Notice that for any oriented Riemannian manifold, the Laplacian on $\Omega^0(M) = C^\infty(M)$ is given by

$$\Delta f = d^*df = - * d * df.$$

Example 1.34. Consider the case $M = \mathbb{R}^3$ equipped with the standard Euclidean matric. If (x, y, z) are coordinates on $\mathbb{R}^3$, then

$$* \, df = * \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) = \frac{\partial f}{\partial x} dy \wedge dz - \frac{\partial f}{\partial y} dx \wedge dz + \frac{\partial f}{\partial z} dx \wedge dy.$$

Hence,

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz \qquad \Longrightarrow \qquad \Delta f = - \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right).$$

Sometimes this is called "the non-negative Laplacian", since its eigenvalues are non-negative.

More generally, if the Riemannian metric on $\mathbb{R}^n$ is given by a positive-definite matrix (g_{ij}), where $g_{ij} = g_{ij}(x)$, then the corresponding Laplacian on functions is given explicitly by

$$\Delta f = - \frac{1}{\sqrt{|g|}} \, \partial_i \left(\sqrt{|g|} \, g^{ij} \partial_j f \right), \tag{1.35}$$

where $|g| = |\det (g_{ij})|$ and $(g^{ij}) = (g_{ij})^{-1}$. This is sometimes called the Laplace-Beltrami operator.

Let M be a closed oriented Riemannian manifold.

Proposition 1.36. *(Green's identity) For any $\omega, \eta \in \Omega^k(M)$ we have*

$$\langle \Delta \omega, \eta \rangle_{L^2} = \langle \omega, \Delta \eta \rangle_{L^2} = \langle d\omega, d\eta \rangle_{L^2} + \langle d^* \omega, d^* \eta \rangle.$$

Proof. By Proposition 1.25 *(i)*, we have

$$\langle dd^* \omega + d^* d\omega, \eta \rangle = \langle d^* \omega, d^* \eta \rangle + \langle d\omega, d\eta \rangle.$$

By the same token,

$$\langle \omega, dd^* \eta + d^* d\eta \rangle = \langle d^* \omega, d^* \eta \rangle + \langle d\omega, d\eta \rangle,$$

which yields the claim of this proposition. □

Corollary 1.37. *On a closed connected manifold any harmonic function is constant.*

Proof. If $f \in C^\infty(M)$ is harmonic, by Green's identity (with $\omega = \eta = f$) we obtain

$$0 = \langle \Delta f, f \rangle = \| \, df \, \|^2_{L^2} \qquad \Longrightarrow \qquad df = 0 \qquad \Longrightarrow \qquad f \text{ is constant.}$$

□

The Hodge theorem follows from the following more general result, which is also attributed to Hodge.

Theorem 1.38. *(Hodge) Let M be a closed oriented Riemannian manifold and $\eta \in \Omega^k(M)$. The equation $\Delta \omega = \eta$ has a solution if and only if*

$$\langle \eta, \omega_0 \rangle_{L^2} = 0 \tag{1.39}$$

for any harmonic k-form ω_0.