Global Analysis

Lecture notes

Andriy Haydys

March 30, 2022

This is a draft. If you spot a mistake, please let me know.

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Chapter 1

Introduction

1.1 The de Rham cohomology groups

Let M be a compact manifold of dimension n. Denote by $\Omega^k(M)$ the space of differential k-forms on M. Recall that there exist a unique $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) df is the differential of f if $f \in C^\infty(M) = \Omega^0(M)$;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^q\, \omega \wedge d\eta$ if $\eta \in \Omega^q(M)$;

(iii) $d^2 = 0$.

The last property simply means that $d(d\omega) = 0$ for each $\omega \in \Omega^k(M)$. This yields the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0 \tag{1.1}$$

Remark 1.2. The map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ depends on k, however this is suppressed in the notations.

Property (iii) means that (1.1) is a complex, that is the kernel of $d : \Omega^k \to \Omega^{k+1}$ contains the image of $d : \Omega^{k-1} \to \Omega^k$ and therefore we can define

$$H^k_{dR}(M) := \frac{\mathrm{Ker}\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\mathrm{Im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)} \tag{1.3}$$

This is called the k^{th} de Rham cohomology group, which is in fact a vector space. The number

$$b_k(M) := \dim\, H^k_{dR}(M)$$

is called the k^{th} Betti number of M and is an invariant of M. Notice that by the compactness of M we have $b_k(M) < \infty$.

Remark 1.4. While it is by no means obvious from the above description, Betti numbers are topological invariants of M, that is $b_k(M) = b_k(N)$ provided M and N are homeomorphic. In paticular, Betti numbers do not depend on the smooth structure.

Coming back to the de Rham cohomlogy groups, each element in $H^k_{dR}(M)$ is represented by the equivalence class

$$[\omega] = \left\{\omega + d\eta \mid \eta \in \Omega^{k-1}(M)\right\},$$

where $\omega \in \Omega^k(M)$ is closed: $d\omega = 0$. Hence, we may ask the following.

Question 1.5. What is the best representative in $[\omega]$?

Of course, at this point the above question is vague, since the notion of being "the best" is undefined. One possibility to convert this into a precise question is as follows. Just by its definition, the set $[\omega]$ is an affine subspace of $\Omega^k(M)$. We could call an element in $[\omega]$ "the best" if it is the closest one to the origin just as shown schematically on Figure 1.1.

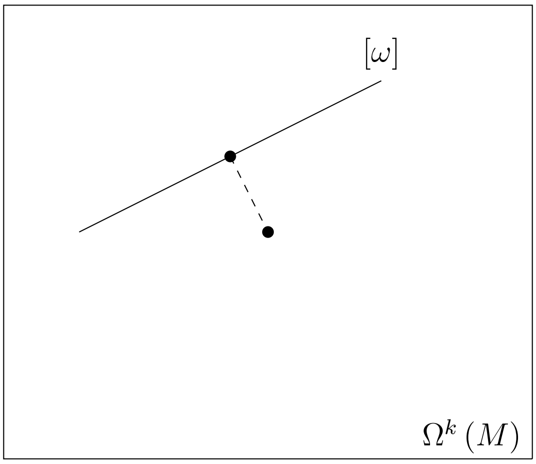

Figure 1.1: A choice of a representative in the de Rham cohomology class.

However, this raises our next question: How do we measure distance in $\Omega^k(M)$? A suitable answer to this question requires a detour, which we do next.

1.2 Some linear algebra

Let V be a vector space of dimension n over $\mathbb{R}$. Recall that for each basis $e = (e_1, \dots, e_n)$ of V there exist a unique basis $e^* = (e_1^*, \dots, e_n^*)$ of the dual vector space V^* such that

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

e^* is called the dual basis to e.

Assume $\langle \cdot, \cdot \rangle$ is a scalar product on V. If $e = (e_1, \dots, e_n)$ is an orthonormal basis of V, that is

$$\langle e_i, e_j \rangle = \delta_{ij},$$

then the dual basis e^* is given explicitly by

$$e_i^* := \langle e_i, \cdot \rangle \qquad \Longleftrightarrow \qquad e_i^*(v) = \langle e_i, v \rangle \quad \text{for } v \in V.$$

Then V^* has a unique scalar product such that $e^* = (e_1^*, \dots, e_n^*)$ is an orthonormal basis. Explicitly, for $\xi, \eta \in V^*$ define

$$\begin{matrix} \xi_i := \xi(e_i) \\ \eta_i := \eta(e_i) \end{matrix} \qquad \Longleftrightarrow \qquad \begin{matrix} \xi = \sum \xi_i \, e_i^* \\ \eta = \sum \eta_j \, e_j^* \end{matrix}$$

$$\implies \langle \xi, \eta \rangle = \sum_{i=1}^{n} \xi_i \eta_i = \sum_{i=1}^{n} \xi(e_i)\, \eta(e_i)\,.$$

To sum up, for any scalar product on V there exists a unique scalar product on V^* such that the dual basis of an orthonormal basis is itself orthonormal.

More generally, any basis e of V yields a basis of $\Lambda^k V^*$. Explicitly,

$$\Lambda^k e := \left\{ e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} \mid 1 \le i_1 < i_2 < \ldots < i_k \le n \right\} \tag{1.6}$$

is a basis of $\Lambda^k V^*$ consisting of

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!k!}$$

elements. Just like in the case of $V^* = \Lambda^1 V^*$, we can define a scalar product on each $\Lambda^k V^*$ by declaring (1.6) to be an orthonormal basis.

Recall that any two bases e and f of V are related by a change-of-basis matrix A. This means

$$f = e \cdot A \qquad \iff \qquad f_i = \sum_{j=1}^{n} a_{ij}\, e_j$$

Then e and f are said to be *cooriented*, if $\det A > 0$. It is easy to check that

$$e \sim f \qquad \equiv \qquad e \text{ and } f \text{ are cooriented}$$

yields an equivalence relation on the set of all bases of V. Moreover, there are exactly two equivalence classes represented by e and $\bar{e} = (-e_1, e_2, \ldots, e_n)$.

Definition 1.7. An orientation on V is a choice of an equivalence class of bases of V. Any basis in the chosen class is said to be positively oriented and any basis, which does not belong to the selected class is said to be negatively oriented.

Example 1.8. For $\mathbb{R}^n$ the class of the standard basis is called the standard orientation of $\mathbb{R}^n$.

Example 1.9. Any $\omega \in \Lambda^n V^*$, $\omega \neq 0$, determines an orientation of V by the rule: e is positively oriented if and only if

$$\omega(e_1, \ldots, e_n) > 0.$$

For example, if $e^* = (e^*_1, \ldots, e^*_n)$ is the dual basis to the standard one, then

$$\omega := e^*_1 \wedge \ldots \wedge e^*_n \tag{1.10}$$

determines the standard orientation of $\mathbb{R}^n$.

Definition 1.11. Let V be an oriented Euclidean vector space of dimension n. An n-form ω is said to be the Euclidean volume form, if

$$\omega(e_1, \ldots, e_n) = 1 \tag{1.12}$$

holds for any positively oriented orthonormal basis e of V.

For example, in the case $V = \mathbb{R}^n$, which is equipped with the standard scalar product and orientation, (1.10) is the Euclidean volume form.

Example 1.13. Show that any oriented Euclidean vector space admits a unique Euclidean volume form. This is sometimes denoted by vol.

Proposition 1.14. *There is a unique linear map*

$$* : \Lambda^k V^* \to \Lambda^{n-k} V^* \qquad \textit{satisfying} \qquad \xi \wedge * \eta = \langle \xi, \eta \rangle\, vol \tag{1.15}$$

for all ξ, $\eta \in \Lambda^k V^*$.

Proof (Sketch). Let e be a positively oriented orthonormal basis of V. Set

$$\eta := e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} \in \Lambda^k e^*$$

Then assuming that $*$ exists, for

$$\xi = \sum \xi_{j_1 \ldots j_k}\, e^*_{j_1} \wedge \ldots \wedge e^*_{j_k}$$

we must have

$$\xi \wedge * \eta = \langle \xi, \eta \rangle\, vol = \xi_{j_1 \ldots j_k} \cdot e^*_1 \wedge \ldots \wedge e^*_n.$$

This yields

$$* \, \eta = * \left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} \right) = \varepsilon \cdot e^*_{l_1} \wedge \ldots \wedge e^*_{l_{n-k}}, \tag{1.16}$$

where $\varepsilon \in \{\pm 1\}$ and $1 \le l_1 < \ldots < l_{n-k} \le n$ consists of those integers in the interval $[1, n]$ which are complementary to $\{i_1, \ldots, i_k\}$.

For example, if $n = 6$ and $\eta = e_2 \wedge e_4$, then

$$* \, (e^*_2 \wedge e^*_4) = \varepsilon\, e^*_1 \wedge e^*_3 \wedge e^*_5 \wedge e^*_6 \, .$$

To determine ε, we compute

$$e^*_2 \wedge e^*_4 \wedge \varepsilon\, (e^*_1 \wedge e^*_3 \wedge e^*_5 \wedge e^*_6) = -\varepsilon\, vol \, ,$$
$$e^*_2 \wedge e^*_4 \wedge * \, (e^*_2 \wedge e^*_4) = \| \, e^*_2 \wedge e^*_4 \, \|^2 \cdot vol = vol,$$

which yields $\varepsilon = -1$ so that we finally obtain

$$* \, (e^*_2 \wedge e^*_4) = -e^*_1 \wedge e^*_3 \wedge e^*_5 \wedge e^*_6.$$

In general, ε is the signature of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Thus, (1.16) defines $*$ on the elements of the basis $\Lambda^k e$. This yields a unique linear map $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$, which satisfies (1.15). ☐

The map $*$ defined in the above proposition is called *the Hodge operator*.

Remark 1.17. It follows from the proof of the above proposition that the Hodge operator satisfies

$$* * \omega = (-1)^{k(n-k)}\, \omega \qquad \text{for all } \omega \in \Lambda^k V^*. \tag{1.18}$$

1.3 Riemannian metrics

Definition 1.19. A Riemannian metric g on M is a smooth section of $T^*M \otimes T^*M$ such that the following holds:

(i) g is symmetric, that is

$$g\,(v, w) = g\,(w, v) \qquad \text{for all } v, w \in T_m M \text{ and any } m \in M;$$

(ii) g is positive-definite, that is

$$g(v,v) > 0 \qquad \text{for all } v \in T_mM, v \neq 0, \text{ and any } m \in M.$$

In other words, g is a family $\{g_m \mid m \in M\}$ of scalar products on each T_mM and g_m depends smoothly on m. In particular, each T_mM is an Euclidean vector space. Hence, each $\Lambda^k T_m^* M$ is also an Euclidean vector space.

An orientation of a manifold M is (informally speaking) a choice of coherent orientations of T_mM for each $m \in M$. More formally, we have the following.

Definition 1.20. A manifold M of dimension n is said to be orientable, if there exists $\omega \in \Omega^n(M)$ such that $\omega_m \neq 0$ for all $m \in M$.

By Example 1.9, for each $m \in M$ the n-form on T_mM determines a class of positively oriented bases of T_mM, that is an orientation. Notice that for any function f, which is positive everywhere, the forms ω and $f \cdot \omega$ determine the same orientation on each T_mM .

Albeit not all manifolds are orientable, orientability is a mild restriction. In particular, for any connected non-orientable manifold M there exists a unique double covering $M_2 \to M$, which is orientable. The reader may find more information on this in [?].

Definition 1.21. An orientation of an n-manifold M is a class of n-forms $[\![\omega]\!]$, where

- ω is a nowhere vanishing n-form on M.
- $\omega_1 \sim \omega_2$ if and only if there exists an everywhere positive function f such that $\omega_2 = f \cdot \omega_1$.

Notice that $[\![\cdot]\!]$ above is *not* the de Rham cohomology class.

Just like in the preceding section, a differential n-form ω on M is said to be a Riemannian volume form, if

$$\omega_m(e_1, \dots, e_n) = 1 \tag{1.22}$$

holds for any $m \in M$ and any oriented orthonormal basis $(e_1, \dots, e_n)$ of T_mM. Property 1.22 determines a Riemannian volume form uniquely. This volume form is denoted by vol.

Thus, by the preceding subsection, a Riemannian metric and orientation on M induce for each $k \leq n$ the Hodge operator $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ such that

$$\omega \wedge *\eta = \langle \omega, \eta \rangle \, vol \qquad \text{and} \qquad **\omega = (-1)^{k(n-k)}\omega$$

holds for all $\omega, \eta \in \Omega^k(M)$.

1.4 Harmonic forms

Let us come back to the original question about "the best" representatives of the de Rham cohomology classes. Thus, we define the L^2-scalar product on each $\Omega^k(M)$ by setting

$$\langle \omega, \eta \rangle_{L^2} := \int_M \langle \omega_m, \eta_m \rangle \, vol_m = \int_M \omega \wedge *\eta.$$

With this at hand, we could call an element $\widehat{\omega} = \omega + d\eta \in [\omega]$ "the best", if $\widehat{\omega}$ minimizes the distance to the origin, that is if

$$\inf_{\eta \in \Omega^{k-1}(M)} \| \, \omega + d\eta \, \|^2_{L^2} = \| \, \widehat{\omega} \, \|_{L^2}. \tag{1.23}$$

Then, if (1.23) holds, for any $\eta \in \Omega^{k-1}(M)$ we must have

$$0 = \frac{d}{dt}\bigg|_{t=0} \| \,\widehat{\omega} + t\, d\eta \,\|_{L^2}^2$$
$$= \frac{d}{dt}\bigg|_{t=0} \left(\| \,\widehat{\omega}\, \|_{L^2}^2 + 2t\langle \widehat{\omega}, d\eta\rangle_{L^2} + t^2 \|\, d\eta \,\|_{L^2}^2\right) \tag{1.24}$$
$$= 2\langle \widehat{\omega}, d\eta\rangle_{L^2}$$

Proposition 1.25. *Denote* $d^* := (-1)^{nk+1} * d\, *\colon \Omega^{k+1}(M) \to \Omega^k(M)$.

(i) d^* *is the formal adjoint of* d*, that is*

$$\langle d\omega, \eta\rangle_{L^2} = \langle \omega, d^*\eta\rangle_{L^2} \qquad \text{for all } \omega \in \Omega^k(M) \text{ and } \eta \in \Omega^{k+1}(M). \tag{1.26}$$

(ii) (1.24) *is equivalent to*

$$d^*\widehat{\omega} = 0. \tag{1.27}$$

Proof. Notice first that (1.15) and (1.18) imply the equality

$$\omega \wedge \zeta = (-1)^{k(n-k)} \langle \omega, *\zeta\rangle vol \qquad \text{for all } \omega \in \Omega^k(M) \text{ and } \zeta \in \Omega^{n-k}(M).$$

Using this, we obtain

$$\langle \omega, d^*\eta\rangle_{L^2} = (-1)^{kn+1} \int_M \langle \omega, * d * \eta\rangle \, vol$$
$$= (-1)^{kn+1+k(n-k)} \int_M \omega \wedge d * \eta$$
$$= (-1)^{1-k^2} \int_M (-1)^k \left(d\left(\omega \wedge *\eta\right) - d\omega \wedge *\eta\right)$$

Here the last equality follows from the Leibniz rule:

$$d(\omega \wedge \zeta) = d\omega \wedge \zeta + (-1)^k \omega \wedge d\zeta$$

provided $\omega \in \Omega^k(M)$ and $\zeta \in \Omega^l(M)$. Hence, by Stokes' theorem, we obtain

$$\langle \omega, d^*\eta\rangle_{L^2} = (-1)^{k-k^2+1} \int_M d\omega \wedge *\eta$$
$$= (-1)^{k-k^2} \langle d\omega, \eta\rangle_{L^2}$$

By noticing that k^2 is even/odd if and only if k is even/odd, we arrive finally at (1.26).

To prove (ii), notice that

$$d^*\widehat{\omega} = 0 \qquad \Longrightarrow \qquad 0 = \langle d^*\widehat{\omega}, \eta\rangle_{L^2} = \langle \widehat{\omega}, d\eta\rangle_{L^2}.$$

Conversely, setting $\eta = d^*\widehat{\omega}$ in (1.24), we obtain

$$0 = \langle \widehat{\omega}, dd^*\widehat{\omega}\rangle_{L^2} = \langle d^*\widehat{\omega}, d^*\widehat{\omega}\rangle_{L^2} = \|\, d^*\omega \,\|_{L^2}^2 \qquad \Longrightarrow \qquad d^*\widehat{\omega} = 0.$$

□

Notice that (1.27) is nothing else but the Euler-Lagrange equation for the functional

$$f\colon \left\{\omega + d\eta \mid \eta \in \Omega^{k-1}(M)\right\} \longrightarrow \mathbb{R}, \qquad f(\omega + d\eta) = \| \omega + d\eta \|_{L^2}^2 \tag{1.28}$$

Definition 1.29. The map

$$\Delta = dd^* + d^*d\colon \Omega^k(M) \to \Omega^k(M)$$

is called *the Laplace operator* (or, simply, *the Laplacian*). A k-form ω such that $\Delta\omega = 0$ is called *harmonic*.

Proposition 1.30. *A k-form ω is harmonic if and only if*

$$d\omega = 0 \qquad \textit{and} \qquad d^*\omega = 0. \tag{1.31}$$

Proof. If (1.31) holds, then ω is clearly harmonic. To show the converse, consider

$$0 = \langle \Delta\omega, \omega \rangle_{L^2} = \langle dd^*\omega, \omega \rangle_{L^2} + \langle d^*d\omega, \omega \rangle = \| d^*\omega \|_{L^2}^2 + \| d\omega \|_{L^2}^2.$$

Since both summands are non-negative, we obtain (1.31). □

Theorem 1.32. *If a minimizer $\widehat{\omega}$ of (1.28) exists, then $\widehat{\omega}$ is harmonic. Moreover, if $\widehat{\omega}$ exists, then it is unique.*

Proof. Assume $\widehat{\omega}$ exists. Since $\widehat{\omega} = \omega + d\eta$ for some $\eta \in \Omega^{k-1}(M)$, we have

$$d\widehat{\omega} = d\omega + d(d\eta) = 0 + 0.$$

Combining this with (1.27), we obtain by Proposition 1.30, that $\widehat{\omega}$ is harmonic.

Furthermore, let $\widehat{\omega} = \omega + d\eta$ and $\widehat{\widehat{\omega}} = \omega + d\zeta$ be two harmonic representatives of $[\omega]$. Then

$$0 = d^*\widehat{\omega} - d^*\widehat{\widehat{\omega}} = d^*\left(\widehat{\omega} - \widehat{\widehat{\omega}}\right) = d^*d(\eta - \zeta).$$

Denoting temporarily $\xi := \eta - \zeta$, we obtain

$$0 = \langle d^*d\xi, \xi \rangle_{L^2} = \langle d\xi, d\xi \rangle = \| d\xi \|_{L^2}^2 \quad \Longrightarrow \quad d\xi = 0 \quad \Longrightarrow \quad d\eta = d\zeta \quad \Longrightarrow \quad \widehat{\omega} = \widehat{\widehat{\omega}}.$$

This proves the uniquesness. □

Our aim is to prove the following.

Theorem 1.33. *(Hodge) Each de Rham cohomology class is represented by a unique harmonic form.*

Notice that since we have already proved the uniqueness, it is the existence, which remains to be proved. It turns out that this is somewhat harder and requires certain technology, which we will consider first.

Notice that for any oriented Riemannian manifold, the Laplacian on $\Omega^0(M) = C^\infty(M)$ is given by

$$\Delta f = d^*df = - * d * df.$$

Example 1.34. Consider the case $M = \mathbb{R}^3$ equipped with the standard Euclidean matric. If (x, y, z) are coordinates on $\mathbb{R}^3$, then

$$* df = * \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) = \frac{\partial f}{\partial x} dy \wedge dz - \frac{\partial f}{\partial y} dx \wedge dz + \frac{\partial f}{\partial z} dx \wedge dy.$$

Hence,

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz \qquad \Longrightarrow \qquad \Delta f = - \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right).$$

Sometimes this is called "the non-negative Laplacian", since its eigenvalues are non-negative.

More generally, if the Riemannian metric on $\mathbb{R}^n$ is given by a positive-definite matrix (g_{ij}), where $g_{ij} = g_{ij}(x)$, then the corresponding Laplacian on functions is given explicitly by

$$\Delta f = - \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|}\, g^{ij} \partial_j f \right), \tag{1.35}$$

where $|g| = |\det(g_{ij})|$ and $(g^{ij}) = (g_{ij})^{-1}$. This is sometimes called the Laplace-Beltrami operator.

Let M be a closed oriented Riemannian manifold.

Proposition 1.36. *(Green's identity) For any $\omega, \eta \in \Omega^k(M)$ we have*

$$\langle \Delta\omega, \eta \rangle_{L^2} = \langle \omega, \Delta\eta \rangle_{L^2} = \langle d\omega, d\eta \rangle_{L^2} + \langle d^*\omega, d^*\eta \rangle.$$

Proof. By Proposition 1.25 *(i)*, we have

$$\langle dd^*\omega + d^*d\omega, \eta \rangle = \langle d^*\omega, d^*\eta \rangle + \langle d\omega, d\eta \rangle.$$

By the same token,

$$\langle \omega, dd^*\eta + d^*d\eta \rangle = \langle d^*\omega, d^*\eta \rangle + \langle d\omega, d\eta \rangle,$$

which yields the claim of this proposition. □

Corollary 1.37. *On a closed connected manifold any harmonic function is constant.*

Proof. If $f \in C^\infty(M)$ is harmonic, by Green's identity (with $\omega = \eta = f$) we obtain

$$0 = \langle \Delta f, f \rangle = \| df \|^2_{L^2} \qquad \Longrightarrow \qquad df = 0 \qquad \Longrightarrow \qquad f \text{ is constant.}$$

□

The Hodge theorem follows from the following more general result, which is also attributed to Hodge.

Theorem 1.38. *(Hodge) Let M be a closed oriented Riemannian manifold and $\eta \in \Omega^k(M)$. The equation $\Delta\omega = \eta$ has a solution if and only if*

$$\langle \eta, \omega_0 \rangle_{L^2} = 0 \tag{1.39}$$

for any harmonic k-form ω_0.

It is easy to see that (1.39) is a necessary condition. Indeed, if there is $\omega \in \Omega^k(M)$ such that $\Delta\omega = \eta$, then Green's identity yields

$$\langle \eta, \omega_0 \rangle_{L^2} = \langle \Delta\omega, \omega_0 \rangle = \langle \omega, \Delta\omega_0 \rangle = 0$$

for any harmonic k-form ω_0.

The proof that (1.39) is a sufficient condition will be given later. Taking this as granted for now, from Corollary 1.37 we obtain the following.

Corollary 1.40. *Let M be a closed oriented Riemannian manifold. The equation*

$$\Delta f = h, \qquad\qquad f, h \in C^\infty(M)$$

has a solution if and only if

$$\int_M h \cdot vol = 0.$$

Let me also show that Theorem 1.38 implies that any de Rham cohomolgy class is represented by a harmonic form. To this end, pick any closed k-form ω. Then by Proposition ??, $\omega + d\eta$ is harmonic if and only if

$$\begin{matrix} d(\omega + d\eta) = 0 \\ d^*(\omega + d\eta) = 0 \end{matrix} \quad \Longleftrightarrow \quad d^*(\omega + d\eta) = 0 \quad \Longleftrightarrow \quad d^* d\eta = -d^*\omega. \tag{1.41}$$

Furthermore, consider the equation

$$\Delta\eta = -d^*\omega. \tag{1.42}$$

If η_0 is any harmonic $(k-1)$-form, then

$$\langle d^*\omega, \eta_0 \rangle_{L^2} = \langle \omega, d\eta_0 \rangle = 0,$$

Since η_0 is closed. Hence, Theorem 1.38 guarantees that (1.42) has a solution η. We have

$$\begin{aligned} \langle dd^*\eta, \Delta\eta \rangle_{L^2} &= \langle dd^*\eta, dd^*\eta \rangle_{L^2} + \langle dd^*\eta, dd^*\eta \rangle_{L^2} \\ &= \| dd^*\eta \|_{L^2}^2 + \langle d^2 d^*\eta, d\eta \rangle_{L^2} \\ &= \| dd^*\eta \|_{L^2}^2 . \end{aligned}$$

However, using (1.42) we obtain

$$\langle dd^*\eta, \Delta\eta \rangle_{L^2} = -\langle dd^*\eta, d^*\omega \rangle_{L^2} = -\langle d^2 d^*\eta, \omega \rangle_{L^2} = 0.$$

Hence, if η is a solution of (1.42), then in fact η solves (1.41) so that $\omega + d\eta$ is harmonic indeed.

With these preliminary considerations at hand we proceed by showing that Theorem 1.38 has useful applications, in particular in the theory of Riemann surfaces.

1.5 Riemann surfaces

Definition 1.43. A Riemann surface Σ is a complex manifold of dimension one. In particular, Σ admits an atlas $\{(\mathrm{U}_\alpha, \psi_\alpha) \mid \alpha \in A\}$, where $\psi_\alpha \colon \mathrm{U}_\alpha \longrightarrow \mathbb{C}$, such that each coordinate transformation map

$$\psi_\alpha \circ \psi_\beta^{-1} \colon \mathbb{C} \longrightarrow \mathbb{C}$$

is holomorphic on the domain of its definition.

Writing $z_\alpha = z = x + iy = \psi_\alpha$, we obtain a holomorphic coordinate on Σ (defined on U_α). Notice that

$$x = \operatorname{Re} z \qquad \text{and} \qquad y = \operatorname{Im} z$$

are real coordinates on U_α. In particular, for any $p \in \mathrm{U}_\alpha$ we have a basis

$$\partial^\alpha := (\partial_x, \partial_y)\big|_p$$

of $T_p\Sigma$. Define a linear map

$$I_p \colon T_p\Sigma \longrightarrow T_p\Sigma \qquad \text{by} \qquad I_p\partial_x = \partial_y \quad \text{and} \quad I_p\partial_y = -\partial_x. \tag{1.44}$$

Using the fact that $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic, it is easy to check that I_p does not depend on the choice of a holomorphic coordinate. It follows also from (1.44) that $I_p^2 = -id$.

For any smooth (not necessarily holomorphic) function $f \colon \Sigma \longrightarrow \mathbb{C}$ define $\partial f, \bar\partial f \in \Omega^1(\Sigma; \mathbb{C})$ by

$$\begin{aligned} \partial f(v) &:= \tfrac{1}{2}\left(df(v) - i\, df(Iv)\right) \\ \bar\partial f(v) &:= \tfrac{1}{2}\left(df(v) + i\, df(Iv)\right) \end{aligned} \qquad \text{for } v \in T\Sigma.$$

In particular, we have $df = \partial f + \bar\partial f$ and f is holomorphic if and only if $\bar\partial f = 0$. If $f_\alpha := f \circ \psi_\alpha^{-1} \colon \mathbb{C} \longrightarrow \mathbb{C}$ is the local representation of f, then

$$\bar\partial f_\alpha = \frac{\partial f_\alpha}{\partial \bar z}\, d\bar z \qquad \text{and} \qquad \partial f_\alpha = \frac{\partial f_\alpha}{\partial z}\, dz$$

are local representations of $\bar\partial f$ and ∂f respectively, that is $\psi_\alpha^* \bar\partial f_\alpha = \bar\partial f$ and $\psi_\alpha^* \partial f_\alpha = \partial f$.

Furthermore, since

$$\begin{aligned} dz &= dx + i\, dy, \\ d\bar z &= dx - i\, dy, \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} dx &= \tfrac{1}{2}(dz + d\bar z), \\ dy &= \tfrac{1}{2i}(dz - d\bar z), \end{aligned}$$

any $\omega \in \Omega^1(\Sigma; \mathbb{C})$ can be written uniquely as $\omega = a\, dz + b\, d\bar z$ for some functions a and b. Denote by $\Omega^{1,0}(\Sigma)$ the space of all those complex valued 1-forms, whose local representation is $a\, dz$ for some $a \in C^\infty(\Sigma; \mathbb{C})$. More invariantly, we have

$$\Omega^{1,0}(\Sigma) = \left\{\omega \in \Omega^1(\Sigma; \mathbb{C}) \;\middle|\; \omega(I\cdot) = i\,\omega(\cdot)\right\}.$$

Similarly, denote by $\Omega^{0,1}(\Sigma)$ the space of all those 1-forms, whose local representation is $b\, d\bar z$ for some complex valued function b, or equivalently,

$$\Omega^{0,1}(\Sigma) := \left\{\omega \in \Omega^1(\Sigma) \;\middle|\; \omega(I\cdot) = -i\,\omega(\cdot)\right\}.$$

Thus, we obtain the decomposition

$$\Omega^1(\Sigma; \mathbb{C}) = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma).$$

Then ∂f and $\bar\partial f$ are nothing else but the components of df lying in $\Omega^{1,0}(\Sigma)$ and $\Omega^{0,1}(\Sigma)$ respectively.

Thus, in the case of a Riemann surface, the complexified de Rham complex has the following form:

$$\begin{tikzcd}[column sep=small] & & \Omega^{1,0}(\Sigma) \arrow[dr, "\bar\partial"] & & \\ 0 \arrow[r] & \Omega^0(\Sigma; \mathbb{C}) \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"'] & \oplus & \Omega^2(\Sigma, \mathbb{C}) \arrow[r] & 0 \\ & & \Omega^{0,1}(\Sigma) \arrow[ur, "\partial"'] & & \end{tikzcd} \tag{1.45}$$

Here $\bar{\partial}\colon \Omega^{1,0}(\Sigma) \to \Omega^2(\Sigma;\mathbb{C})$ is just the restriction of d to $\Omega^{1,0}(\Sigma)$ and $\partial\colon \Omega^{0,1}(\Sigma) \to \Omega^2(\Sigma,\mathbb{C})$ is the restriction of d to $\Omega^{0,1}(\Sigma)$. Locally, we have

$$\omega \in \Omega^{1,0}(\Sigma) \implies \omega = a\,dz \implies d\omega = da \wedge dz = \left(\frac{\partial a}{\partial z}dz + \frac{\partial a}{\partial \bar z}d\bar z\right) \wedge dz = -\frac{\partial a}{\partial \bar z}\,dz \wedge d\bar z,$$
$$\omega \in \Omega^{0,1}(\Sigma) \implies \omega = b\,d\bar z \implies d\omega = db \wedge d\bar z = \left(\frac{\partial b}{\partial z}dz + \frac{\partial b}{\partial \bar z}d\bar z\right) \wedge d\bar z = \frac{\partial b}{\partial z}\,dz \wedge d\bar z.$$

Notice that we have
$$d^2 = 0 \qquad \iff \qquad \partial\bar\partial + \bar\partial\partial = 0.$$

The splitting (1.45) of the de Rham complex yields the Dolbeault cohomology groups:

$$H^{1,0}(\Sigma) := \ker\bar\partial,$$

$$H^0(\Sigma) := \ker\bar\partial, \qquad\qquad H^{1,1}(\Sigma) := \Omega^2\big/\operatorname{Im}\bar\partial,$$

$$H^{0,1}(\Sigma) := \Omega^{0,1}\big/\operatorname{Im}\bar\partial.$$

Clearly, $H^0(\Sigma)$ is the space of holomorphic functions on Σ. In particular, if Σ is compact and connected, then $H^0(\Sigma) \cong \mathbb{C}$.

$H^{1,0}(\Sigma)$ is the space of holomorphic differentials, that is 1-forms ω such that locally $\omega = a\,dz$ and a is a holomorphic function. The geometric meaning of $H^{0,1}(\Sigma)$ and $H^{1,1}(\Sigma)$ is somewhat less straightforward. An interested reader may wish to consult for example [??].

1.6 The Laplacian on Riemann surfaces

A peculiar feature of Riemann surfaces is that the Laplace-Beltrami operator can be defined without a reference to a Riemannian metric. Indeed, set

$$\Delta := 2i\,\bar\partial\partial = -2i\,\partial\bar\partial\colon \Omega^0(\Sigma;\mathbb{C}) \longrightarrow \Omega^2(\Sigma;\mathbb{C}).$$

If $z = x + iy$ is a local holomorphic coordinate as above, then we have

$$\begin{aligned} \bar\partial f &= \frac{\partial f}{\partial \bar z}\,d\bar z = \frac{1}{2}(\partial_x f + \partial_y f\,i)\,d\bar z \qquad \implies \\ \partial\bar\partial f &= \frac{1}{2}\frac{\partial}{\partial z}(\partial_x f + \partial_y f\,i)\,dz \wedge d\bar z \\ &= \frac{1}{4}(\partial_x - i\,\partial_y)(\partial_x f + \partial_y f\,i)\,dz \wedge d\bar z \\ &= \frac{1}{4}\left(\partial^2_{xx} f + \partial^2_{yy} f\right) dz \wedge d\bar z. \end{aligned}$$

Furthermore, since

$$dz \wedge d\bar z = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\,dx \wedge dy,$$

we obtain
$$\Delta f = -2i\,\partial\bar\partial f = -\left(\partial^2_{xx} f + \partial^2_{yy} f\right) dx \wedge dy.$$

Remark 1.46. To relate this operator to the Laplacian in the sense of Def. ??, let g be a Hermitian metric on Σ, that is $g(I\cdot, I\cdot) = g(\cdot,\cdot)$.

If $z = x + yi$ is a local holomorphic coordinate, then

$$(g_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad \Longrightarrow \qquad (g^{ij}) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some positive function $\lambda = \lambda(x, y)$.

Substituting this into (1.35), we obtain

$$\Delta f = \frac{1}{\lambda}\left(\partial_x \left(\lambda \cdot \lambda^{-1} \partial_x f\right) + \partial_y \left(\lambda \cdot \lambda^{-1} \partial_y f\right)\right) \quad = \frac{1}{\lambda}\left(\partial^2_{xx} f + \partial^2_{yy} f\right).$$

Furthermore, the Riemannian volume form on Σ in terms of local coordinates (x, y) is

$$vol = \lambda\, dx \wedge dy.$$

Hence, $* \, vol = 1 \Longrightarrow *\,(dx \wedge dy) = \lambda^{-1}$. This yields

$$\Delta f = \frac{1}{\lambda}\left(\partial^2_{xx} f + \partial^2_{yy} f\right) = *\left(-2i\, \partial\bar\partial f\right).$$

Hence, up to the application of the isomorphism $*\colon \Omega^2(\Sigma) \to \Omega^0(\Sigma)$, the Laplacian coincides with $-2i\,\partial\bar\partial$ indeed.

In the current setting, Corollary 1.40 yields the following.

Theorem 1.47. *Let Σ be a compact connected Riemann surface. The equation $-2i\,\partial\bar\partial f = \eta$, where $\eta \in \Omega^2(\Sigma; \mathbb{C})$, has a solution if and only if $\int_\Sigma \eta = 0$.* □

1.7 Some consequences of Theorem 1.47

In this section we assume that Σ is a compact connected Riemann surface throughout.

We have a natural skew-symmetric pairing

$$\Omega^{1,0}(\Sigma) \times \Omega^{0,1}(\Sigma) \longrightarrow \mathbb{C}, \qquad (\omega, \eta) \longmapsto \int_\Sigma \omega \wedge \eta.$$

This yields a bilinear map

$$B\colon H^{1,0}(\Sigma) \times H^{0,1}(\Sigma) \longrightarrow \mathbb{C}, \qquad B(\omega, [\eta]) = \int_\Sigma \omega \wedge \eta. \tag{1.48}$$

Lemma 1.49. *B is well defined.*

Proof. Notice first that for any $\alpha, \beta \in \Omega^{1,0}(\Sigma)$ we have $\alpha \wedge \beta = 0$. Indeed, locally $\alpha = a \cdot dz$ and $\beta = b \cdot dz$ so that $\alpha \wedge \beta = ab\, dz \wedge dz = 0$. Using this observation, we obtain

$$\begin{aligned} \int_\Sigma \omega \wedge \left(\eta + \bar\partial f\right) &= \int_\Sigma \omega \wedge \eta + \int_\Sigma \omega \wedge \left(\partial f + \bar\partial f\right) \\ &= \int_\Sigma \omega \wedge \eta + \int_\Sigma \omega \wedge df \\ &= \int_\Sigma \omega \wedge \eta - \int_\Sigma d(f \cdot \omega) + \int_\Sigma f d\omega \\ &= \int_\Sigma \omega \wedge \eta - 0 + 0. \end{aligned}$$

In the last equality the second summand vanishes by the Stokes' thm; The last summand vanishes, since ω is closed: Locally $\omega = a(z)\, dz$ so that

$$d\omega = da \wedge dz = \left(\frac{\partial a}{\partial z} dz + \frac{\partial a}{\partial \bar{z}} d\bar{z}\right) \wedge dz = 0 + 0 = 0,$$

since a is a holomorphic. This finishes the proof of this lemma. □

Notice that we have a natural map

$$i\colon H^{1,0}(\Sigma) \longrightarrow H^1(\Sigma; \mathbb{C}), \qquad \omega \longmapsto [\omega].$$

The class $[\omega]$ is well-defined, since ω is closed as it has been shown in the proof Lemma 1.49.

Furthermore, the conjugation $\Omega^1(\Sigma; \mathbb{C}) \longrightarrow \Omega^1(\Sigma, \mathbb{C}), \eta \longmapsto \bar{\eta}$, induces a map

$$\sigma\colon H^{1,0}(\Sigma) \longrightarrow H^{0,1}(\Sigma),$$

which is antilinear, that is $\sigma(i\eta) = -i\sigma(\eta)$.

Theorem 1.50.

(i) *σ is an isomorphism;*

(ii) *For a complex vector space V denote $V^* = \{\psi\colon V \to \mathbb{C} \mid \psi \text{ is antilinear}\}$. Then $B\colon H^{1,0}(\Sigma) \to H^{0,1}(\Sigma)^*$ is an isomorphism;*

(iii) *The map*

$$H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma) \longrightarrow H^1(\Sigma; \mathbb{C}), \qquad (\omega, [\eta]) \longmapsto i(\omega) + \overline{i(\sigma^{-1}([\eta]))} \tag{1.51}$$

is an isomorphism.

Proof. Notice that $\sigma\colon H^{1,0}(\Sigma) \longrightarrow H^{0,1}(\Sigma)$ is well-defined. We want to show that for each class $[\eta] \in H^{0,1}(\Sigma)$ there exists a unique $\omega \in H^{1,0}(\Sigma)$ such that $\sigma(\omega) = [\eta]$. Indeed, any representative of $[\eta] \in H^{0,1}(\Sigma)$ can be written as

$$\eta' = \eta + \bar{\partial} f$$

for some $f \in C^\infty(\Sigma; \mathbb{C})$. We want to find a representative η' such that

$$\partial \eta' = 0 \qquad \Longleftrightarrow \qquad \partial\bar{\partial} f = -\partial\eta. \tag{1.52}$$

The equation on the right hand side of (1.52) has a solution, since

$$\int_\Sigma \partial\eta = \int_\Sigma d\eta = 0.$$

In fact, f is defined uniquely by (1.52) up to the addition of a constant. Hence, for each class $[\eta] \in H^{0,1}(\Sigma)$ there exists a unique representative η' such that $\partial\eta' = 0$.

Define $\omega := \bar{\eta}'$. Then

$$\bar{\partial}\omega = \bar{\partial}\bar{\eta}' = \overline{\partial\eta'} = 0,$$

that is ω lies in $H^{1,0}(\Sigma)$ and is a unique preimage of $[\eta]$.

To prove *(ii)*, we only need to show that (1.48) is non-degenerate, that is

$$\forall\, \omega \in H^{1,0}(\Sigma) \qquad \exists\, [\eta] \in H^{0,1}(\Sigma) \qquad \text{such that } B(\omega, [\eta]) \neq 0.$$

Indeed, for $\omega \in H^{1,0}(\Sigma)$ set $\eta := \sigma(\omega) = [\bar{\omega}]$. Then

$$B(\omega, [\bar{\omega}]) = \int_{\Sigma} \omega \wedge \bar{\omega} \neq 0,$$

since $a\,dz \wedge \bar{a}\,d\bar{z} = |a|^2 dz \wedge d\bar{z}$.

Furthermore, the injectivity of (1.51) follows from *(ii)*. Indeed, notice first that $i(\omega)$ and $\overline{i(\sigma^{-1}([\eta]))}$ are always orthogonal with respect to B. Hence, $(\omega, [\eta])$ belongs to the kernel of (1.51) if and only if

$$i(\omega) = 0 \qquad \text{and} \qquad \overline{i(\sigma^{-1}([\eta]))} = 0. \tag{1.53}$$

However i is injective, since

$$B(i(\omega), \sigma(i(\omega))) = \int_{\Sigma} \omega \wedge \bar{\omega} \neq 0$$

provided $\omega \neq 0$. Hence, (1.53) yields $\omega = 0$ and $[\eta] = 0$, since all maps σ^{-1}, i, and the conjugation are injective.

To see that (1.51) is surjective, pick any $[\zeta] \in H^1_{dR}(\Sigma; \mathbb{C})$. I claim that there exists a representative $\zeta' = \zeta + df$ such that

$$\bar{\partial}\zeta' = 0 \qquad \Longleftrightarrow \qquad 0 = \bar{\partial}\zeta + \bar{\partial}df = \bar{\partial}\zeta - \partial\bar{\partial}f. \tag{1.54}$$

The existence of a function f satisfying (1.54) follows from Theorem 1.47 just like in the proof of *(i)*.

Furthermore, write

$$\zeta' = \omega + \eta, \qquad \text{where } \omega = \frac{1}{2}(\zeta' - i\zeta'(I\cdot)) \qquad \text{and} \qquad \eta = \frac{1}{2}(\zeta' + i\zeta'(I\cdot)).$$

Locally, we have

$$\zeta' = a\,dz + b\,d\bar{z} \qquad \Longrightarrow \qquad \omega = a\,dz \quad \text{and} \quad \eta = b\,d\bar{z}.$$

Since $\bar{\partial}\zeta' = 0$, we obtain $\frac{\partial a}{\partial \bar{z}} = 0$, that is ω is a holomorphic differential.

Furthermore,

$$\bar{\partial}\zeta' = 0 \quad \text{and} \quad d\zeta' = 0 \quad \Longrightarrow \quad \partial\zeta' = 0 \quad \Longrightarrow \quad \frac{\partial b}{\partial z} = 0 \quad \Longrightarrow \quad \partial\eta = 0$$
$$\Longrightarrow \quad \bar{\partial}\bar{\eta} = 0.$$

Hence, $\bar{\eta}$ is also a holomorphic differential. Combining this with *(i)*, we obtain *(iii)*. $\square$

Corollary 1.55. *If $b_1(\Sigma) = 0$, then Σ admits a meromorphic function with a single simple pole.*

Proof. Pick any point $p \in \Sigma$ and a local holomorphic coordinate z centered at p. Let χ be a bump function at p, that is $\chi \equiv 1$ on a neighbourhood V of p and χ vanishes identically outside of a slightly larger neighbourhood $W \supset V$.

We wish to show that there exists a meromorphic function f on Σ with a unique simple pole at p. Assume for a moment, however, that such f does exist and consider

$$h := f - \chi\frac{a}{z},$$

where a is the residue of f at p. Then h is smooth everywhere, albeit may fail to be holomorphic on $W\backslash V$. In any case, we have

$$\bar{\partial} h = \bar{\partial} f - \frac{a}{z}\bar{\partial}\chi = -\frac{a}{z}\bar{\partial}\chi. \tag{1.56}$$

Notice that $\eta \coloneqq -\frac{a}{z}\bar{\partial}\chi$ is a smooth $(0,1)$-form supported in $W\backslash V$.

Conversely, if h is a smooth solution of (1.56), we can define f by

$$f \coloneqq h + \chi\frac{a}{z},$$

which is meromorphic and has a unique simple pole at p.

Thus, our task reduced to showing that (1.56) has a solution. By the definition of $H^{0,1}(\Sigma)$, this is the case if and only if the class of η in $H^{0,1}(\Sigma)$ vanishes. However, by Theorem 1.50 *(ii)*, we have

$$\dim_R H^{1,0}(\Sigma) + \dim_R H^{0,1}(\Sigma) = 2b_1(\Sigma) = 0 \qquad \Longrightarrow \qquad H^{0,1}(\Sigma) = \{0\}$$

so that $[\eta]$ vanishes trivially. This finishes the proof. □

Theorem 1.57. *Let Σ be a closed Riemann surface. Then Σ is homeomorphic to $\mathbb{C}P^1 \cong S^2$ if and only if Σ is biholomorphic to $\mathbb{C}P^1$.*

Idea of the proof. By the classification theorem for closed surfaces, Σ is homeomorphic to S^2 if and only if $b_1(\Sigma) = 0$. By Corollary 1.55, there exists a meromorphic function f on Σ with a unique simple pole. We can view f as a holomorphic map $f\colon \Sigma \longrightarrow \mathbb{C}P^1$. A simple topological argument yields that f must be in fact bijective (this uses crucially that the pole of f is unique and simple). The reader can find the details of this topological argument in [Don, Sec 4.1]. □

Another reformulation of the above theorem is that the sphere S^2 admits a unique structure of a complex manifold. This is in contrast with the torus $\mathbb{T}^2$ (or, in fact, any closed orientable surface with $b_1 \geq 2$), which admits continuous families of inequivalent complex structures.

Developing these ideas somewhat further one can obtain also a classification of all elliptic curves, that is Riemann surfaces homeomorphic to the torus. Or, one can show that any closed Riemann surface can be embedded into some projective space. However, this goes somewhat beyond the purposes of this course.

Chapter 2

Vector bundles, Sobolev spaces, and elliptic partial differential operators

2.1 Vector bundles

Basic definitions. Roughly speaking, a vector bundle is just a family of vector spaces parametrized by points of a manifold (or, more generally, of a topological space).

More formally, the notion of a vector bundle is defined as follows.

Definition 2.1. Choose a non-negative integer k. A real smooth vector bundle of rank k is a triple (π, E, M) such that the following holds:

(i) E and M are smooth manifolds, $\pi\colon E \to M$ is a smooth submersion (the differential is surjective at each point);

(ii) For each $m \in M$ *the fiber* $E_m := \pi^{-1}(m)$ has the structure of a vector space and $E_m \cong \mathbb{R}^k$;

(iii) For each $m \in M$ there is a neighborhood $U \ni m$ and a smooth map ψ_U such that the following diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes. Moreover, ψ_U is a fiberwise linear isomorphism.

The following terminology is commonly used: E is the total space, M is the base, π is the projection, and ψ_U is the local trivialization (over U).

It is worth pointing out that one can equally well talk about complex and quaternionic vector bundles. This requires only cosmetic changes, which are left to the reader. The preference for real vector bundles in this section is given for the sake of definitness mainly. I shall feel free to use complex vector bundles below without further explanations.

Example 2.2.

(a) The product bundle: $M \times \mathbb{R}^k$;

(b) The tangent bundle TM of any smooth manifold M.

Let E and F be two vector bundles over a common base M. A *homomorphism* between E and F is a smooth map $\varphi\colon E \to F$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

commutes and φ is a fiberwise linear map.

Two bundles E and F are said to be isomorphic, if there is a homomorphism φ, which is fiberwise an isomorphism.

A bundle E is said to be *trivial*, if E is isomorphic to the product bundle.

Operations on vector bundles. Let E and F be two vector bundles over a common base M. Then we can construct new bundles E^*, $\Lambda^p E$, $E \oplus F$, $E \otimes F$, and $\mathrm{Hom}(E, F)$ as follows:

($*$) $(E^*)_m = (E_m)^*$;

(Λ) $(\Lambda^p E)_m = \Lambda^p(E_m)$;

($\oplus$) $(E \oplus F)_m := E_m \oplus F_m$;

($\otimes$) $(E \otimes F)_m := E_m \otimes F_m$;

(Hom) $\mathrm{Hom}(E, F)_m := \mathrm{Hom}(E_m, F_m)$.

If $f\colon M' \to M$ is a smooth map, we can define the *pull-back* of $E \to M$ via

$$(f^*E)_{m'} := E_{f(m')}.$$

For example, if M' is an open subset of M and ι is the inclusion, then $E|_{M'} := \iota^* E$ is just the restriction of E to M'.

The reader should check that the families of vector spaces defined above satisfy the properties required by Definition 2.1.

Exercise 2.3. Prove that $E^* \otimes F$ is isomorphic to $\mathrm{Hom}(E, F)$.

Exercise 2.4. Prove that the tangent bundle of the 2-sphere is non-trivial. (Hint: Apply the hairy ball theorem).

Sections. Speaking informally, a section is an assignment of a vector in $s(m) \in E_m$ to each point $m \in M$ such that $s(m)$ depends smoothly on m. More formally, we have the following.

Definition 2.5. A smooth map $s\colon M \to E$ is called a section, if $\pi \circ s = \mathrm{id}_M$.

Sections of the tangent bundle TM are called vector fields. Sections of $\Lambda^p T^*M$ are called differential p-forms.

Exercise 2.6. Let $E \to M$ be a vector bundle of rank k and $U \subset M$ be an open subset. Prove that E is trivial over U if and only if there are k sections $e = (e_1, \dots, e_k)$, $e_j \in \Gamma(U; E)$, such that $e(m)$ is a basis of E_m for each $m \in U$. More precisely, given e show that ψ_U can be constructed according to the formula

$$\psi_U^{-1}\colon U \times \mathbb{R}^k \longrightarrow E|_U, \qquad (m, x) \mapsto e(m) \cdot x.$$

In fact this establishes a one-to-one correspondence between k-tuples of pointwise linearly independent sections and local trivializations of E.

We denote by $\Gamma(E) = \Gamma(M;\, E)$ the space of all smooth sections of E. Clearly, $\Gamma(E)$ is a vector space, where the addition and multiplication with a scalar are defined pointwise. In fact, $\Gamma(E)$ is a $C^\infty(M)$-module.

Given a local trivialization e over U (cf. Exercise 2.6) and a section s, we can write

$$s(m) = \sum_{j=1}^{k} \sigma_j(m) e_j(m)$$

for some functions $\sigma_j \colon U \to \mathbb{R}$. Thus, locally any section of a vector bundle can be thought of as a map $\sigma \colon U \to \mathbb{R}^k$.

It is important to notice that σ depends on the choice of a local trivialization. Indeed, if e' is another local trivialization of E over U', then there is a map

$$g \colon U \cap U' \longrightarrow \mathrm{GL}_n(\mathbb{R}) \quad \text{such that} \quad e = e' \cdot g. \tag{2.7}$$

If $\sigma' \colon U' \to \mathbb{R}^k$ is a local representation of s with respect to e', we have

$$s = e'\sigma' = eg^{-1}\sigma' = e\sigma \qquad \Longrightarrow \qquad \sigma' = g\sigma.$$

Covariant derivatives. The reader surely knows from the basic analysis course that the notion of the derivative is very useful. It is natural to ask whether there is a way to differentiate sections of bundles too.

To answer this question, recall the definition of the derivative of a function $f \colon M \to \mathbb{R}$. Namely, choose a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ and denote $m := \gamma(0),\ \mathrm{v} := \dot\gamma(0) \in T_mM$. Then

$$df(\mathrm{v}) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(m)}{t}. \tag{2.8}$$

Trying to replace f by a section s of a vector bundle, we immediately run into a problem, namely the difference $s(\gamma(t)) - s(m)$ is ill-defined in general since these two vectors may lie in different vector spaces.

Hence, instead of trying to mimic (2.8) we will define the derivatives of sections axiomatically, namely asking that the most basic property of the derivative—the Leibnitz rule—holds.

Definition 2.9. Let $E \to M$ be a vector bundle. *A covariant derivative* is an $\mathbb{R}$-linear map $\nabla \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that

$$\nabla(fs) = df \otimes s + f\nabla s \tag{2.10}$$

holds for all $f \in C^\infty(M)$ and all $s \in \Gamma(E)$.

Example 2.11. Let $M \subset \mathbb{R}^N$ be an embedded submanifold. Then the tangent bundle TM is naturally a subbundle of the product bundle $\underline{\mathbb{R}}^N := M \times \mathbb{R}^N$. In particular, any section s of TM can be regarded as a map $M \to \mathbb{R}^N$. With this at hand we can define a connection on TM as follows

$$\nabla s := \mathrm{pr}(ds),$$

where pr is the orthogonal projection onto TM. A straightforward computation shows that this satisfies the Leibniz rule, i.e., ∇ is a connection indeed.

Theorem 2.12. *For any vector bundle $E \to M$ the space of all connections $\mathcal{A}(E)$ is an affine space modelled on* $\Omega^1(\mathrm{End}\, E) = \Gamma\big(T^*M \otimes \mathrm{End}(E)\big)$.

To be somewhat more concrete, the above theorem consists of the following statements:

(a) $\mathcal{A}(E)$ is non-empty.
(b) For any two connections ∇ and $\hat{\nabla}$ the difference $\nabla - \hat{\nabla}$ is a 1-form with values in $\mathrm{End}(E)$;
(c) For any $\nabla \in \mathcal{A}(E)$ and any $a \in \Omega^1(\mathrm{End}\, E)$ the following
$$(\nabla + a)s := \nabla s + as$$
is a connection.

For the proof of Theorem 2.12 we need the following elementary lemma, whose proof is left as an exercise.

Lemma 2.13. *Let $A\colon \Gamma(E) \to \Omega^p(F)$ be an $\mathbb{R}$-linear map, which is also $C^\infty(M)$-linear, i.e.,*
$$A(fs) = fA(s) \qquad \forall f \in C^\infty(M) \quad \text{and} \quad \forall s \in \Gamma(E).$$
Then there exists $a \in \Omega^p\big(\mathrm{Hom}(E, F)\big)$ such that $A(s) = a \cdot s$. □

Proof of Theorem 2.12. Notice first that $\mathcal{A}(E)$ is convex, i.e., for any $\nabla, \hat{\nabla} \in \mathcal{A}(E)$ and any $t \in [0, 1]$ the following $t\nabla + (1 - t)\hat{\nabla}$ is also a connection.

If ψ_U is a local trivialization of E over U, then we can define a connection ∇_U on $E|_U$ by declaring
$$\nabla_U s := \psi_U^{-1}\, d(\psi_U(s)).$$
Using a partition of unity and the convexity property, a collection of these local covariant derivatives can be sewed into a global covariant derivative just like in the proof of the existence of Riemannian metrics on manifolds, cf. [?BardenThomas_IntroDiffMflds, Thm. 3.3.7]. This proves (a).

By (2.10), the difference $\nabla - \hat{\nabla}$ is $C^\infty(M)$–linear. Hence, (b) follows by Lemma 2.13.

The remaining step, namely (c), is straightforward. This finishes the proof of this theorem. □

While Theorem 2.12 answers the question of the existence of connections, the reader may wish to have a more direct way to put his hands on a connection. One way to do this is as follows.

Let e be a local trivialization. Since e is a pointwise base we can write
$$\nabla e = e \cdot A, \tag{2.14}$$
where $A = A(\nabla, e)$ is a $k \times k$-matrix, whose entries are 1-forms defined on U. A is called the connection matrix of ∇ with respect to e.

If σ is a local representation of a section s, then
$$\nabla s = \nabla(e\sigma) = \nabla(e)\sigma + e \otimes d\sigma = e\big(A\sigma + d\sigma\big).$$
Hence, it is common to say that locally
$$\nabla = d + A,$$
which means that $d\sigma + A\sigma$ is a local representation of ∇s. In particular, ∇ is uniquely determined by its connection matrix over U and any $A \in \Omega^1(U; \mathfrak{gl}_k(\mathbb{R}))$ appears as a connection matrix of some connection (cf. Theorem 2.12).

2.2 Sobolev spaces

2.2.1 Sobolev spaces on $\mathbb{R}^n$

Recall that the space of square-integrable functions is defined by

$$L^2(\mathbb{R}^n) := \Big\{ u \colon \mathbb{R}^n \to \mathbb{C} \mid u \text{ is measurable } \& \int_{\mathbb{R}^n} |u(x)|^2 dx < \infty \Big\}.$$

This is a (complex) Hilbert space with respect to the Hermitian scalar product

$$\langle u, v \rangle_{L^2} := \int_{\mathbb{R}^n} u(x)\, \bar{v}(x)\, dx.$$

Remark 2.15. Strictly speaking, we should also identify those functions, which differ only on a subset of measure zero so that $L^2(\mathbb{R}^n)$ consists of classes of functions. However, this will not be an issue for us and we shall treat square-integrable functions as honest functions.

The Fourier transform

For $f \colon \mathbb{R}^n \to \mathbb{C}$ decaying sufficiently fast at ∞ the Fourier transform $\widehat{f} \colon \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x)\, e^{-i\langle x, \xi \rangle} dx.$$

The Plancherel theorem states that

$$\|\widehat{f}\|_{L^2}^2 = \frac{1}{(2\pi)^n} \|f\|_{L^2}^2, \tag{2.16}$$

that is (an extension of) the map $f \mapsto \widehat{f}$ is essentially an isometry of $L^2(\mathbb{R}^n)$.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ be a multi-index. Denote

$$D^\alpha f := \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The basic property of the Fourier transform is

$$\widehat{D^\alpha f(\xi)} = \xi^\alpha \widehat{f}(\xi), \tag{2.17}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

Another basic property of the Fourier transform is the following.

Lemma 2.18 (Riemann-Lebesgue)**.** *If $u \in L^1(\mathbb{R})$, then $\widehat{u} \in C^0(\mathbb{R}^n)$ and $\widehat{u}$ decays at ∞.* □

Denote by $f^\vee(\xi) = (2\pi)^n \widehat{f}(-\xi) = \int f(x)\, e^{i\langle x, \xi \rangle} dx$.

Theorem 2.19 (The Fourier inversion theorem)**.** *If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R}^n)$, then f agrees almost everywhere with a continuous function f_0 and $\left(\widehat{f}\right)^\vee = f_0 = \left(\widehat{f^\vee}\right)$.* □

Sobolev spaces

Let $u \in L^2(\mathbb{R}^n)$. For $\alpha \in \mathbb{Z}^n_{\geq 0}$, we say that $v \in L^1_{loc}(\mathbb{R}^n)$ is the α^{th}-weak derivative if

$$\int_{\mathbb{R}^n} u \, \frac{\partial^{|\alpha|}\psi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = (-1)^{|\alpha|} \int_{\mathbb{R}^n} v \cdot \psi$$

holds for all test functions ψ contained in

$$C_0^\infty(\mathbb{R}^n) := \{\psi \in C^\infty(\mathbb{R}^n) \mid \text{ supp } \psi \text{ is compact}\}.$$

The α^{th} weak derivative does not need to exist in general, however if it does exist we simply write

$$v = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

keeping in mind that the above equality holds in the weak sense.

With this understood, given any $k \geq 0$ we can define

$$W^{k,2}(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \text{ exists and belongs to } L^2(\mathbb{R}^n) \ \forall \alpha \text{ s.t. } |\alpha| \leq k \right\}.$$

Somewhat less formally, this is expressed as follows:

$$u \in W^{k,2}(\mathbb{R}^n) \qquad \text{iff} \qquad \|u\|^2_{W^{k,2}} := \sum_{|\alpha| \leq k} \|D^\alpha u\|^2_{L^2} < \infty. \tag{2.20}$$

Properties (2.16) and (2.17) imply that $u \in W^{k,2}(\mathbb{R}^n)$ if and only if $(1 + |\xi|^2)^{\frac{k}{2}} \, \widehat{u}(\xi) \in L^2(\mathbb{R}^n)$. Hence, we can equally well define $W^{k,p}(\mathbb{R}^n)$ by

$$W^{k,2}(\mathbb{R}^n) := \left\{ u \mid \|u\|^2_{W^{k,2}} := \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^k |\widehat{u}(\xi)|^2 d\xi < \infty \right\}.$$

Moreover, the norm appearing in the above definition is equivalent to (2.20). The advantage of the above definition is that this makes sense for any $k \in \mathbb{R}$, not just non-negative integers.

The following are basic results about Sobolev spaces.

Proposition 2.21. *Assume $u \in W^{k,2}(\mathbb{R}^n)$, where $\varkappa := k - \frac{n}{2} > l \in \mathbb{N}_0$. Then $u \in C^l(\mathbb{R}^n)$ after changing u on a subset of measure zero if necessary. Moreover, the natural inclusion*

$$W^{k,2}(\mathbb{R}^n) \longrightarrow C^l(\mathbb{R}^n)$$

is bounded, that is $\|u\|_{C^l} \leq C\|u\|_{W^{k,2}}$ for some positive constant C independent of u.

Proof. Using (2.17) we obtain

$$\begin{aligned} \int |\widehat{D^\alpha u}|(\xi) \, d\xi = \int |\xi^\alpha| |\widehat{u}(\xi)| d\xi &\leq \int \left(1 + |\xi|^2\right)^{\frac{|\alpha|}{2}} |\widehat{u}(\xi)| d\xi \\ &\leq \int \left(1 + |\xi|^2\right)^{\frac{l}{2}} |\widehat{u}(\xi)| d\xi, \end{aligned}$$

provided $|\alpha| \leq l$.

Furthermore, by the Cauchy-Schwartz inequality we have

$$\begin{aligned}\int \left(1+|\xi|^2\right)^{\frac{l}{2}} |\widehat{u}\left(\xi\right)| d\xi &= \int \left(1+|\xi|^2\right)^{\frac{k}{2}} |\widehat{u}\left(\xi\right)| \cdot \left(1+|\xi|^2\right)^{\frac{l-k}{2}} d\xi \\ &\leq \left(\int \left(1+|\xi|^2\right)^k |\widehat{u}\left(\xi\right)|^2 d\xi\right)^{\frac{1}{2}} \left(\int \left(1+|\xi|^2\right)^{l-k} d\xi\right)^{\frac{1}{2}}\end{aligned}$$

The last integral converges since $|\xi|^{2(l-k)} \cdot |\xi|^{n-1} = |\xi|^{\beta}$, where $\beta = 2\left(l-k\right)+n-1 < -n+n-1 = -1$. Hence, we obtain

$$\int |\widehat{D^\alpha u}|\left(\xi\right) d\xi \leq C\|u\|_{W^{k,2}}.$$

Hence, $D^\alpha u \in C^0$ by a combination of the Riemann-Lebesgue lemma and the Fourier inversion theorem.

Moreover,

$$\|D^\alpha u\|_{C^0} \leq \|\widehat{D^\alpha u}\|_{L^1} \leq C\|u\|_{W^{k,2}},$$

which finishes the proof. □

Proposition 2.22 (Rellich)**.** *Suppose $u_j \in W^{k,2}\left(\mathbb{R}^n\right)$ is a sequence such that there exists a compact subset $K \subset \mathbb{R}^n$ containing $\operatorname{supp} u_j$ for all j. If $\|u_j\|_{W^{k,2}}$ is bounded, then for any $s < k$ there is a subsequence u_{j_l}, which converges in $W^{s,2}\left(\mathbb{R}^n\right)$.* □

For any $p \in (1, +\infty)$ define

$$L^p\left(\mathbb{R}^n\right) := \left\{u \mid \ \|u\|_{L^p} := \left(\int_{\mathbb{R}^n} |u|^P\left(x\right) dx\right)^{\frac{1}{p}} < \infty\right\},$$

which is a Banach space. Using this, one can define the $W^{k,p}\left(\mathbb{R}^n\right)$-spaces in an obvious manner. For example, if k is an integer, we may set

$$W^{k,p}\left(\mathbb{R}^n\right) := \left\{u \mid \ \|u\|^p_{W^{k,p}} := \sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p} < \infty\right\}.$$

Theorem 2.23. *For any $k \geq 0$ the space $C^\infty_0\left(\mathbb{R}^n\right)$ is dense in $W^{k,p}\left(\mathbb{R}^n\right)$.* □

Hence, at least when k is a non-negative integer, we could define $W^{k,p}\left(\mathbb{R}^n\right)$ as the closure of $C^\infty_0\left(\mathbb{R}^n\right)$ with repsect to $\|\cdot\|_{W^{k,p}}$.

2.2.2 Sobolev spaces on manifolds

In the case when the base is a closed oriented Riemannian manifold M rather than $\mathbb{R}^n$, the definition of the L^p-spaces generalizes in a straightforward manner. Namely,

$$L^p\left(M\right) := \left\{u \colon M \longrightarrow \mathbb{R} \mid \ \|u\|_{L^p} := \left(\int_M |u|^p vol\right)^{\frac{1}{p}} < \infty\right\}.$$

Let E be an Euclidean vector bundle over a manifold M. That means that each fiber E_m is equipped with a scalar product $\langle\cdot,\cdot\rangle_m$ and this scalar product depends smoothly on m. More formally, just like in Definition 1.19, an Euclidean structure on E is a smooth section $\langle\cdot,\cdot\rangle$ of $E^*\otimes E^*$ such that

$$\langle v,w\rangle=\langle w,v\rangle \quad \text{and} \quad \langle v,v\rangle>0$$

holds for all $v,w\in E_m$ and all $m\in M$; In addition, in the last inequality we assume $v\neq 0$.

In any case, the definition of the L^p-spaces for sections of Euclidean bundles generalizes in a straightforward manner. Namely, if M is a compact oriented Riemannian manifold, then

$$L^p(E):=\left\{s \mid \ \|s\|_{L^p}:=\left(\int_M |s|^p\, vol\right)^{\frac{1}{p}}<\infty\right\}.$$

Furthermore, pick a connection $\nabla\in A(E)$. It is convinient to assume that ∇ is Euclidean, that is

$$d\langle s_1,s_2\rangle=\langle\nabla s_1,s_2\rangle+\langle s_1,\nabla s_2\rangle$$

holds for all smooth sections s_1 and s_2 of E.

Remark 2.24. Any Euclidean vector bundle admits an Euclidean connection. Moreover, the space of all Euclidean connections on E is an affine space modeled on

$$\Omega^1(M;\mathfrak{o}(E)):=\Gamma(T^*M\otimes\mathfrak{o}(E)), \quad \text{where} \quad \mathfrak{o}(E):=\{A\in End(E) \mid A^*=-A\}.$$

These facts can be established by a straightforward modification of the proof of Theorem 2.12.

We can define

$$W^{1,p}(E):=\left\{s \mid \ \|s\|_{W^{k,p}}:=(\|s\|^p_{L^p}+\|\nabla s\|^p_{L^p})^{\frac{1}{p}}<\infty\right\}.$$

Remark 2.25. If s is smooth, then ∇s is a 1-form with values in E. Then

$$|\nabla s|^2_m:=\sum_{i=1}^n|\nabla_{e_i}s|^2,$$

where $(e_1,\dots,e_n)$ is an orthonormal basis of T_mM. Thus, somewhat more precisely by $\|\nabla s\|_{L^p}$ we mean

$$\|\nabla s\|_{L^p}=\left(\int_m|\nabla s|^p_m\, vol_m\right)^{\frac{1}{p}}.$$

Just like in the case of $\mathbb{R}^n$, $W^{1,p}(E)$ can be understood in at least two following ways. First, we can define $W^{1,p}(E)$ as the completion of $\Gamma(E)$ with respect to the $\|\cdot\|_{W^{1,p}}$-norm. Secondly, for $s\in L^p(E)$ we can first define the weak covariant derivative ∇s as a functional acting on $\Omega^1(E)$ and ask ∇s to lie in $L^p(T^*M\otimes E)$.

To describe some details concerning the second approach, notice that akin to the de Rham complex for any connection ∇ we have the sequence

$$0\to\Omega^0(E)=\Gamma(E)\xrightarrow{\nabla=d_\nabla}\Omega^1(E)\xrightarrow{d_\nabla}\dots\xrightarrow{d_\nabla}\Omega^n(E)\to 0, \tag{2.26}$$

where for $\omega\otimes s\in\Omega^k(E)$ the map d_∇ is defined by

$$d_\nabla(\omega\otimes s)=(d\omega)\otimes s+(-1)^k\omega\wedge\nabla s.$$

Notice, however, that (2.26) is not a complex in general, that is $d_\nabla\circ d_\nabla$ does not necessarily vanish.

In any case, just like for the de Rham differential we can also define $d^*_\nabla\colon \Omega^{k+1}(E)\to\Omega^k(E)$ by $d^*_\nabla := (-1)^{nk+1} * d_\nabla *$, cf. Proposition 1.25. Here the Hodge operator acts as follows: $*(\omega\otimes s) = (*\omega)\otimes s$. Assuming ∇ is Euclidean, d^*_∇ is the formal adjoint of d_∇, that is

$$\langle d_\nabla\alpha,\beta\rangle_{L^2} = \langle\alpha, d^*_\nabla\beta\rangle_{L^2}$$

holds for all $\alpha\in\Omega^k(E)$ and all $\beta\in\Omega^{k+1}(E)$.

With these preliminaries at hand, if $s\in L^p(E)$ the value of the weak derivative ∇s on $\psi\in\Omega^{k+1}(E)$ is declared to be

$$\langle\nabla s,\psi\rangle := \langle s, d^*_\nabla\psi\rangle_{L^2}.$$

Then we can define $W^{1,p}(E)$ as a subspace of $L^p(E)$ consisting of those s, whose weak derivative belongs to L^p, that is if there exists $w\in L^p(T^*M\otimes E)$ such that

$$\langle\omega,\psi\rangle_{L^2} = \langle s, d^*_\nabla\psi\rangle_{L^2}$$

holds for any $\psi\in\Gamma(E)$. Of course, in this case we must have $\omega=\nabla s$.

Our definition of $W^{1,p}(E)$ depends a priory on the choice of the connection ∇. It turns out that this dependence is not really essential as the following result shows.

Proposition 2.27. *Pick any two Euclidean connections ∇ and ∇' on E. Then, the corresponding norms $\|\cdot\|_{W^{1,p}}$ and $\|\cdot\|'_{W^{1,p}}$ are equivalent.*

Proof. By Theorem 2.12, we have $\nabla' = \nabla + a$, where $a\in\Omega^1(End(E))$ (in fact a takes values in $\mathfrak{o}(E)$, but this is immaterial here). Denote

$$A := \sup_{m\in M}|a_m| = \sup_{m\in M}\sup_{\substack{v\in T_mM\\|v|=1}}|a_m(v)|\ge 0.$$

Then for any $v\in T_mM$ of unit norm and any smooth section s we have

$$\nabla'_v s = \nabla_v s + a(v)\cdot s \Longrightarrow |\nabla'_v s|\le|\nabla_v s| + A|s|.$$

Hence, for any orthonormal basis $(e_1,\dots,e_n)$ of T_mM we obtain

$$\begin{aligned}|\nabla' s|^p_m = \sum_{i=1}^n|\nabla'_{e_i}s|^p \le \sum_{i=1}^n(|\nabla_{e_i}s| + A|s|)^p &\le 2^{p-1}\left(\sum_{i=1}^n|\nabla_{e_i}s|^p + nA^p|s|^p\right)\\ &= 2^{p-1}|\nabla s|^p_m + n2^{p-1}A^p|s|^p.\end{aligned}\tag{2.28}$$

Here to obtain the second inequality we used the following generalized mean inequality

$$\frac{a+b}{2}\le\left(\frac{a^p+b^p}{2}\right)^{\frac1p},$$

where a and b are positive real numbers.

With the help of (2.28) we obtain

$$\|\nabla' s\|_{L^p}\le C_1\|\nabla s\|_{L^p} + C_2\|s\|_{L^p}$$

for some positive constants C_1 and C_2 independent of s. This yields the claim of this proposition. □

Our next aim is to define the Sobolev spaces $W^{k,p}(E)$ with $k \geq 1$ being an integer. To this end notice first the following.

Lemma 2.29.

(i) If ∇ is a connection on a vector bundle E, then there exists a unique connection ∇^* on E^* such that

$$d\langle \sigma, s\rangle = \langle \nabla^* \sigma, s\rangle + \langle \sigma, \nabla s\rangle \tag{2.30}$$

holds for all $\sigma \in \Gamma(E^*)$ and all $s \in \Gamma(E)$. Here $\langle \cdot, \cdot \rangle$ denotes the natural pointwise pairing $E^* \otimes E \to \mathbb{R}$.

(ii) If ∇^E and ∇^F are connections on vector bundles E and F respectively, then there exists a unique connection ∇ on $E \otimes F$ such that

$$\nabla(s \otimes t) = \nabla^E s \otimes t + s \otimes \nabla^F t \tag{2.31}$$

holds for all $s \in \Gamma(E)$ and all $t \in \Gamma(F)$.

The proof of this lemma can be obtained by a straightforward verification that ∇^* and ∇ defined by (2.30) and (2.31) respectively satisfies the Leibnitz rule. I leave this as an exercise to the reader.

With this understood, we can proceed as follows. Pick an Euclidean connection ∇^E on E and an Euclidean connection ∇^{TM} on TM. This yields a connection $\nabla = \nabla\left(\nabla^E, \nabla^{TM}\right)$ on $T^*M \otimes E$. Hence, for any smooth section s of E we can define the second derivative by

$$\nabla^2 s = \nabla\left(\nabla^E s\right) \in \Gamma(T^*M \otimes T^*M \otimes E).$$

In the less regular case, for example when $s \in W^{1,p}(E)$ we can still define the weak second derivative as a functional on $\Gamma(T^*M \otimes T^*M \otimes E)$ just like we defined the weak first derivative. Then we may set

$$\begin{aligned} W^{2,p}(E) &= \left\{ s \in W^{1,p}(E) \mid \nabla^2 s \in L^p(E) \right\}, \\ &= \left\{ s \mid \; \|s\|^p_{W^{2,p}} := \|s\|^p_{L^p} + \|\nabla s\|^p_{L^p} + \|\nabla^2 s\|^p_{L^p} < \infty \right\}, \end{aligned}$$

where the second equality should be treated with care just like in the case of functions.

Furthermore, we can define $W^{k,p}(E)$ for any integer $k \geq 2$ by induction. The details are left to the reader.

Alternatively, we can also define $W^{k,p}(E)$ as the closure of $\Gamma(E)$ with respect to $\|\cdot\|_{W^{k,p}}$, where

$$\|s\|^p_{W^{k,p}} := \sum_{j=0}^{k} \|\nabla^j s\|^p_{L^p}. \tag{2.32}$$

Yet another way to define Sobolev spaces, which works for any real k, is as follows. Since M is compact, we can pick a finite covering (U_α, ψ_α), where ψ_α is a trivialization of $E\big|_{U_\alpha}$ just like in Definition 2.1. Moreover, we can assume that each U_α is a coordinate chart. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, that is

- $\operatorname{supp} \rho_\alpha \subset U_\alpha$;
- $\rho_\alpha \geq 0$ everywhere;
- $\sum_\alpha \rho_\alpha(m) = 1$ for each $m \in M$.

If s is a section of E, over each U_α we can identify s with some $s_\alpha\colon U_\alpha \to \mathbb{R}^l$, where l is the rank of E, as follows:

$$\psi_\alpha \circ s = (id, s_\alpha)\,.$$

Hence, we can think of $\rho_\alpha \cdot s_\alpha$ as a map defined on $\mathbb{R}^n$ with compact support. Finally, we set

$$\|s\|^p_{W^{k,p}} := \sum_\alpha \|\rho_\alpha \cdot s_\alpha\|^p_{W^{k,p}}.$$

This norm depends on the choice of $\{(U_\alpha, \psi_\alpha, \rho_\alpha)\}$, however turns out to be equivalent to (2.32) if k is an integer. Hence, we can define $W^{k,p}(E)$ in the usual way, for example as the completion of $\Gamma(E)$ with respect to the above norm.

With this understood, for any $p > 1$ we have the sequence of inclusions

$$L^p(M;\, E) = W^{0,p}(M;\, E) \supset W^{1,p}(M;\, E) \supset W^{2,p}(M;\, E) \supset \dots$$

Relations between all these spaces is given by the following theorem, which is of fundamental importance in the theory of PDEs.

Theorem 2.33. *Let M be a compact manifold.*

(i) If $s \in W^{k,p}(M;\, E)$, then $s \in W^{l,q}(M;\, E)$ provided

$$k - \frac{n}{p} \geq l - \frac{n}{q} \quad \textit{and} \quad k \geq l,$$

where $n = \dim M$, and there is a constant C independent of s such that $\|s\|_{W^{l,s}} \leq C\|s\|_{W^{k,p}}$. In other words, the natural embedding

$$j\colon W^{k,p}(M;\, E) \subset W^{l,q}(M;\, E)$$

is continuous.

(ii) j is a compact operator provided

$$k - \frac{n}{p} > l - \frac{n}{q} \quad \textit{and} \quad k > l. \tag{2.34}$$

This means that any sequence bounded in $W^{k,p}$ has a subsequence, which converges in $W^{l,q}$ provided (2.34) holds.

(iii) We have a natural continuous embedding

$$W^{k,p}(M;\, E) \subset C^l(M;\, E)$$

provided $k - \frac{n}{p} > l$. In particular, if $s \in W^{k,p}(M;\, E)$ for some fixed p and for all $k \geq 0$, then $s \in C^\infty(M;\, E)$.

(iv) (a) In the case $kp > n$ the space $W^{k,p}(M;\, \mathbb{R})$ is an algebra.

(b) In the case $kp < n$, we have a bounded map

$$W^{k_1,p_1} \otimes W^{k_2,p_2} \to W^{k,p}, \qquad \textit{provided} \quad k_1 - \frac{n}{p_1} + k_2 - \frac{n}{p_2} \geq k - \frac{n}{p}. \quad \square$$

2.3 Differential operators

Let E be a smooth complex vector bundle of rank a over M. This means that each fiber E_m has the structure of a complex vector space and its dimension equals a; Also, Property (iii) of Definition 2.1 should be read as follows: For each $m \in M$ there exists a neighbourhood $U \ni m$ and a smooth map ψ_U show that the following diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^a \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

commutes. Moreover, ψ_U is a fiberwise complex linear isomorphism.

Given a local trivialization ψ_U as above, any section $s \in \Gamma(E)$ can be identified with a smooth map $\sigma \colon U \to \mathbb{C}^a$ in the following sense:

$$\psi_U \circ s = (id_U, \sigma).$$

In other words, we have a well-defined $C^\infty(M; \mathbb{C})$-linear isomorphism

$$\Psi_U \colon \Gamma\left(E|_U\right) \longrightarrow C^\infty(U; \mathbb{C}^a).$$

Let F be another complex vector bundle over M of rank b. Then F admits a local trivialization over some neighbourhood U' of M. Replacing U and U' by $U \cap U'$ if necessary, we can assume $U' = U$. Notice also that by shrinking U further if necessary, we can assume that U is a chart. Denote by $(x_1, \ldots x_n)$ local coordinates on U. Let ψ_U^F be the trivialization of F over U so that sections of F over U can be thought of as maps $U \longrightarrow \mathbb{C}^b$.

Definition 2.35. A $\mathbb{C}$-linear map $L : \Gamma(E) \to \Gamma(F)$ is called a differential operator of order at most l, if for each choice of local trivializations of E and F as above there is a differential operator $\widetilde{L} \colon C^\infty(U; \mathbb{C}^a) \to C^\infty\left(U; \mathbb{C}^b\right)$ of order at most l such that the following diagram

$$\begin{tikzcd} \Gamma\left(E|_U\right) \arrow[r, "L"] \arrow[d, "\Psi_U"] & \Gamma\left(F|_U\right) \arrow[d, "\Psi_U^F"] \\ C^\infty(U; \mathbb{C}^a) \arrow[r, "\tilde{L}"] & C^\infty\left(U; \mathbb{C}^b\right) \end{tikzcd}$$

commutes. Here $\widetilde{L}$ is said to be a differential operator of order at most l, if $\widetilde{L}$ admits a representation

$$\widetilde{L}\sigma = \sum_{|\alpha| \leq l} a_\alpha(x) \frac{\partial^{|\alpha|}\sigma}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \tag{2.36}$$

where $a_\alpha \in C^\infty(U; M_{a \times b}(\mathbb{C}))$ and $M_{a \times b}(\mathbb{C})$ denotes the space of all $a \times b$-matrices with complex entries.

Remark 2.37. We can equally well define real differential operators acting on sections of real vector bundles. This requires cosmetic changes only. The choice to focus on complex differential operators will be somewhat clearer below. Roughly speaking, this is related to the fact, that the Fourier transform of a real valued function is a complex valued function.

Example 2.38. Let M be any manifold of dimension n. Recall that any chart $(U, x_1, \dots x_n)$ yields a trivialization of T^*M. The corresponding map $\Psi_U \colon \Omega^1(U) \to C^\infty(U; \mathbb{R}^n)$ is given by

$$\omega = \sum_{i=1}^n \omega_i(x)\, dx_i \longmapsto \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}.$$

Consider the differential as the map $d \colon \Omega^0(M) = \Gamma(\underline{\mathbb{R}}) \to \Omega^1(M)$. Since $\underline{\mathbb{R}}$ is globally trivial, there is no need to choose an extra local trivialization of $\underline{\mathbb{R}}$. Then, relative to the above choice of the local trivialization of T^*M, the local representation of d is given by

$$f \longmapsto \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \frac{\partial f}{\partial x_1} + \dots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \cdot \frac{\partial f}{\partial x_n}.$$

In particular, d is a first order (real) differential operator.

By the same token, $d \colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a first order real differential operator for each k.

Let $\operatorname{Diff}_k(E, F)$ denote the vector space of all differential operators of order at most k.

Proposition 2.39. *Any $L \in \operatorname{Diff}_l(E, F)$ extends as a bounded map $L \colon W^{k,p}(E) \to W^{k-l,p}(F)$.*

The proof of this proposition follows immediately from the property $\widehat{D^\alpha f}(\xi) = \xi^\alpha \widehat{f}(\xi)$ and the definition of $W^{k,p}$-norm in terms of the Fourier transform.

2.3.1 Symbols of differential operators

Consider the following second order differential operator

$$Lu = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)\, u, \tag{2.40}$$

where u is a function of n variables $(x_1, \dots x_n)$. It is well-known from the theory of linear PDEs that in many cases the most essential properties of L depend on the highest order terms only, that is on

$$L^{(2)}u = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

It is convenient to represent this operator by the expression

$$\sigma(L) \coloneqq \sum_{i,j=1}^n a_{ij}(x)\, \xi_i \xi_j,$$

where $\xi = (\xi_1, \dots, \xi_n)$ can be understood as a formal variable. This leads to the notion of the symbol of a differential operator.

It is important to understand how the symbol changes if we change the variables, since on a manifold there are no preferred coordinates in general. Thus, let

$$
\begin{gathered}
y_1 = y_1(x_1, \dots, x_n) \\
\dots \\
y_n = y_n(x_1, \dots, x_n)
\end{gathered}
$$

be new coordinates on $\mathbb{R}^n$. If $u(x) = v(y(x))$, then we have

$$
\frac{\partial u}{\partial x_i} = \sum_{p=1}^{n} \frac{\partial v}{\partial y_p} \frac{\partial y_p}{\partial x_i} \qquad \Longrightarrow \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{p,q=1}^{n} \frac{\partial^2 v}{\partial y_p \partial y_q} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} + \sum_{p=1}^{n} \frac{\partial v}{\partial y_p} \frac{\partial^2 y_p}{\partial x_i \partial x_j}.
$$

Substituting this into (2.40) we obtain

$$
Lu = \sum_{i,j,p,q=1}^{n} a_{ij} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} \frac{\partial^2 v}{\partial y_p \partial y_q} + \dots =: \tilde{L} v
$$

where "..." denotes the lower order terms. Therefore, the symbol of $\tilde{L}$ is given by

$$
\sigma(\tilde{L})(\eta) = \sum_{p,q=1}^{n} \left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} \right) \eta_p \eta_q .
$$

Hence, if we set

$$
\xi_i = \sum_{p=1}^{n} \frac{\partial y_p}{\partial x_i} \eta_p \qquad \Longrightarrow \qquad \xi_j = \sum_{q=1}^{n} \frac{\partial y_q}{\partial x_j} \eta_q ,
$$

then

$$
\sigma(L)(\xi) = \sigma(\tilde{L})(J\xi)
$$

holds identically for all $\xi \in \mathbb{R}^n$, where

$$
J = \begin{pmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_1} \\
\frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_2} \\
\dots & \dots & \dots & \dots \\
\frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \dots & \frac{\partial y_n}{\partial x_n}
\end{pmatrix} = \left(\frac{\partial y}{\partial x} \right)^t .
$$

This implies the following. Think of (x, ξ) as coordinates on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. If we change the coordinates x on $\mathbb{R}^n$ for y as above, we obtain new coordinates (y, η) on $T^*\mathbb{R}^n$. Moreover, these coordinates are related by

$$
y = y(x) \qquad \text{and} \qquad \eta = \left(\frac{\partial y}{\partial x} \right)^t \cdot \xi .
$$

Hence $\sigma(\tilde{L})(\eta) = \sigma(\tilde{L})(y, \eta)$ is just the expression for $\sigma(L)$ in these new coordinates. Put differently, $\sigma(L)(\xi) = \sigma(L)(x, \xi)$ is well-defined as a function on $T^*\mathbb{R}^n$.

Notice that $\sigma(L)(\xi)$ is homogeneous of degree 2 in ξ, that is

$$
\sigma(L)(\lambda \xi) = \lambda^2 \sigma(L)(\xi)
$$

for all $\lambda \in \mathbb{R}$.