Differential Geometry I

Lecture notes

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This is a draft. If you spot a mistake, please let me know.

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Chapter 1

Smooth surfaces

1.1 The notion of a smooth surface

Let $\mathrm{U} \subset \mathbb{R}^n$ be an open subset and $f \in C^1(\mathrm{U})$. It is known from analysis that $x_0 \in \mathrm{U}$ is a point of extremum for f if

$$\frac{\partial f}{\partial x_i}(x_0) = 0$$

holds for all $i = 1, \ldots, n$. Notice that this is a necessary condition, which is not sufficient in general.

A more general type of problems does not fit into this scheme. For example, consider the following.

Problem. Among all rectangular parallelepipeds, whose diagonal has a fixed length, say 1, find the one with maximal volume.

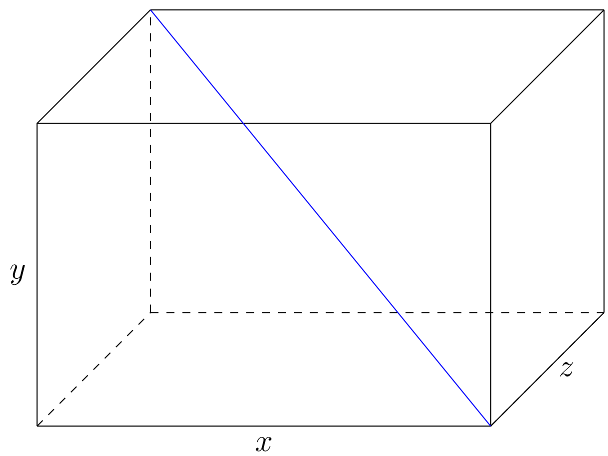


Figure 1.1: A parallelepiped

Thus, we want to find a point of maximum of the function $f(x, y, z) = xyz$ on the set

$$\mathrm{V} = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0 \quad \text{and} \quad x^2 + y^2 + z^2 = 1\} \subset S^2. \tag{1.1}$$

However, V is *not* an open subset of $\mathbb{R}^3$ so that the receipy known from the analysis course is not readily applicable.

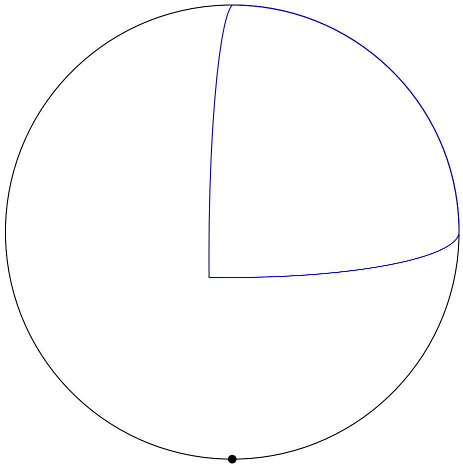


Figure 1.2: The spherical triangle $x, y, z > 0$

This problem is relatively easy to solve, however. Indeed, since $z > 0$, we obtain $z = \sqrt{1 - x^2 - y^2}$ so that we are essentially interested in the function

$$F(x, y) := f\big(x, y, \sqrt{1 - x^2 - y^2}\big) = xy\sqrt{1 - x^2 - y^2}.$$

More precisely, we want to find points of maximum of F on the set $\{(x, y) \mid x^2 + y^2 < 1, x > 0, y > 0\}$, which is an open subset of $\mathbb{R}^2$.

We compute

$$\begin{aligned} \frac{\partial F}{\partial x} &= y\sqrt{1 - x^2 - y^2} - xy\frac{x}{\sqrt{1 - x^2 - y^2}} = 0, \\ \frac{\partial F}{\partial y} &= x\sqrt{1 - x^2 - y^2} - xy\frac{y}{\sqrt{1 - x^2 - y^2}} = 0. \end{aligned} \tag{1.2}$$

Since $x \neq 0$ and $y \neq 0$, we have

$$\begin{aligned} (1.2) \quad &\iff \quad \begin{matrix} 1 - x^2 - y^2 = x^2 \\ 1 - x^2 - y^2 = y^2 \end{matrix} \quad \implies \quad x^2 = y^2 \quad \implies \quad x = y \\ &\implies \quad 3x^2 = 1 \quad \implies \quad x = y = \frac{1}{\sqrt{3}} \\ &\implies \quad z = \frac{1}{\sqrt{3}}. \end{aligned}$$

Hence, if there is a parallelepiped maximizing the volume among all rectangular parallelepipeds with the given length of the diagonal, this must be the cube.

Exercise 1.3. Show that $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a point of maximum indeed.

Consider a more general problem of constrained maximum/minimum. Given $f, \varphi \in C^\infty(\mathbb{R}^n)$ find a point of maximum/minimum of f on the set

$$S := \{x \in \mathbb{R}^n \quad | \quad \varphi(x) = 0\}.$$

Proposition 1.4. *Assume that for $p \in S$ we have*

$$\frac{\partial \varphi}{\partial x_n}(p) \neq 0. \tag{1.5}$$

Then there is a neighbourhood W of p in $\mathbb{R}^n$, an open subset $\mathrm{V} \subset \mathbb{R}^{n-1}$, and a smooth function $\psi\colon \mathrm{V} \to \mathbb{R}$ such that for $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we have

$$x \in S \cap \mathrm{W} \qquad \iff \qquad y \in \mathrm{V} \quad \textit{and} \quad z = \psi(y).$$

This is a celebrated implicit function theorem, whose proof was given in the analysis course.

Theorem 1.6. *Let $p \in S$ be a point of (local) maximum of f on S. If (1.5) holds, then there exists some $\lambda \in \mathbb{R}$ such that*

$$\frac{\partial f}{\partial x_j}(p) = \lambda \frac{\partial \varphi}{\partial x_j}(p) \qquad \iff \qquad \nabla f(p) = \lambda \nabla \varphi(p) \tag{1.7}$$

holds for each $j = 1, \ldots, n$.

Proof. Let $p = (y_0, z_0)$ be a local maximum for f on S. Hence, y_0 is a local maximum for the function

$$F\colon \mathrm{V} \to \mathbb{R}, \qquad F(y) := f(y, \psi(y))$$

This yields

$$\frac{\partial F}{\partial y_j}(y_0) = \frac{\partial f}{\partial y_j}(p) + \frac{\partial f}{\partial x_n}(p)\frac{\partial \psi}{\partial y_j}(y_0) = 0$$

for all $j \leq n - 1$.

Furthermore, since $\varphi(y, \psi(y)) \equiv 0$, we have

$$\frac{\partial \varphi}{\partial y_j} + \frac{\partial \varphi}{\partial x_n}\frac{\partial \psi}{\partial y_j} \equiv 0.$$

This yields in turn

$$\frac{\partial \psi}{\partial y_j}(y_0) = -\frac{\partial \varphi}{\partial y_j}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p) \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y_j}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)\right) \cdot \frac{\partial \varphi}{\partial y_j}(p).$$

Thus, (1.7) holds for all $j \leq n - 1$ with $\lambda := \frac{\partial f}{\partial x_n}(p) / \frac{\partial \varphi}{\partial x_n}(p)$ independent of j.

For $j = n$ we have

$$\frac{\partial f}{\partial x_n}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)\right) \cdot \frac{\partial \varphi}{\partial x_n}(p) = \lambda \frac{\partial \varphi}{\partial x_n}(p).$$

Thus, (1.7) holds also for $j = n$ with the same λ. $\square$

Let us come back to the example about maximal value of parallelepipeds with a fixed length of the diagonal. Thus, if (x, y, z) is a point of maximum of f on (1.1), then there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z \end{aligned} \qquad \Longrightarrow \qquad (xyz)^2 = 8\lambda^3 xyz \qquad \Longrightarrow \qquad xyz = 8\lambda^3.$$

This yields in turn

$$8\lambda^3 = xyz = x(yz) = 2\lambda x^2.$$

Notice that $\lambda \neq 0$, since otherwise $x = 0$ or $y = 0$ or $z = 0$. Hence, we obtain $x = 2\lambda$.

A similar argument yields also $y = 2\lambda$ and $z = 2\lambda$. Therefore we obtain

$$4\lambda^2 + 4\lambda^2 + 4\lambda^2 = 1 \qquad \Longrightarrow \qquad \lambda = \frac{1}{2\sqrt{3}} \qquad \Longrightarrow \qquad x = y = z = \frac{1}{\sqrt{3}},$$

which is in agreement with our previous computation.

Coming back to Proposition 1.4, it is clear that it is only important that one of the partial derivatives of φ does not vanish. This leads to the following definition.

Definition 1.8 (Surface)**.** A non-empty set $S \subset \mathbb{R}^3$ is called a (smooth) *surface*, if for any $p \in S$ there exists an open set $\mathrm{V} \subset \mathbb{R}^2$ and a smooth map $\psi : \mathrm{V} \to \mathbb{R}^3$ such that the following holds:

(i) $\psi(\mathrm{V}) =: \mathrm{U}$ is a neighbourhood of p in S; in particular, $\psi(\mathrm{V}) \subset S$.

(ii) $\psi \colon \mathrm{V} \to \mathrm{U}$ is a homeomorphism.

(iii) $D_q\psi \colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective $\forall q \in \mathrm{V}$.

Example 1.9. Assume $\varphi \in C^\infty(\mathbb{R}^3)$ satisfies

$$\frac{\partial \varphi}{\partial z}(p) \neq 0 \qquad \text{for all} \quad p \in S := \varphi^{-1}(0).$$

Let ψ be as in Proposition 1.28. Define $\Psi(x,y) := (x, y, \psi(x,y))$. If U and V are also as in Proposition 1.28, then $\Psi \colon \mathrm{V} \to S \cap \mathrm{U}$ is a homeomorphism, since $\pi \colon S \cap \mathrm{U} \to \mathrm{V}, \pi(x,y,z) = (x,y)$ is a continuous inverse. Furthermore,

$$D\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_x\psi & \partial_y\psi \end{pmatrix}$$

is clearly injective at all points. Hence, S is a surface.

Again, the same conclusion holds if we assume only that $\nabla\varphi(p) \neq 0$ for all $p \in \varphi^{-1}(0)$. In particular,

- the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$
- the cylinder $C = \{(x,y,z) \mid x^2 + y^2 = 1\}$
- the hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$

are surfaces

Example 1.10 (Torus)**.** Let C be the circle of radius r in the yz-plane centered at the point $(0, a, 0)$ as shown on Fig. 1.4, where $a > r$.

More formally,

$$T := \left\{ \left(\sqrt{x^2+y^2} - a\right)^2 + z^2 = r^2 \right\}.$$

Exercise 1.11. Check that T is a surface indeed.

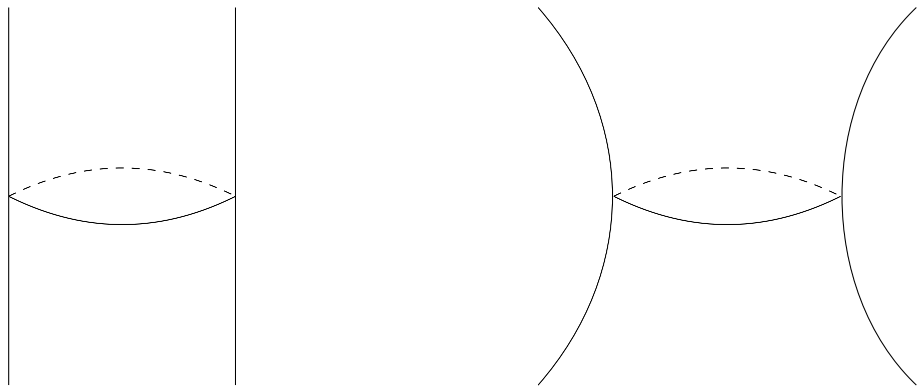


Figure 1.3: The cylinder and hyperboloid

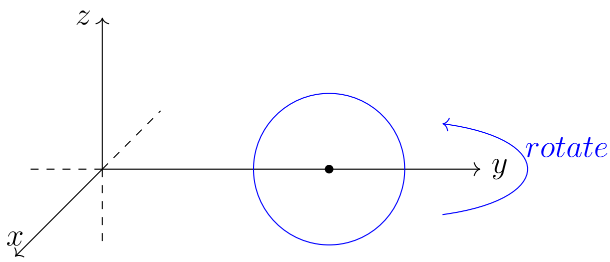


Figure 1.4: The torus as a circle rotated with respect to an axis

Example 1.12 (A non-example)**.** The double cone $C_0 := \{x^2 + y^2 - z^2 = 0\}$ is not a surface. Indeed, assume C_0 is a surface. Then the tip of the cone p must have a neighbourhood U homeomorphic to an open disc in $\mathbb{R}^2$.

Let $f \colon \mathrm{U} \to D$ be a homeomorphism. Then $f \colon \mathrm{U}\backslash\{p\} \to D\backslash\{f(p)\}$ is also a homeomorphism. However, this is impossible, since the punctured disc is connected but $\mathrm{U}\backslash\{p\}$ is disconnected. Hence, p does not have a neighbourhood homeomorphic to a disc (or any open subset of $\mathbb{R}^2$).

Exercise 1.13. Show that a straight line is not a surface.

Remark 1.14.

1) The map ψ in the definition of the surface is called a *parametrization*.
2) Condition (iii) is equivalent to the following:

$$\partial_u \psi \quad \text{and} \quad \partial_v \psi \quad \text{are linearly independent}$$

at each point $(u, v) \in \mathrm{V}$.

Proposition 1.15. *Let S be a surface. For any $p \in S$ there exists a neighbourhood $W \subset \mathbb{R}^3$ and $\varphi \in C^\infty(W)$ such that*

$$S \cap W = \{x \in W \mid \varphi(x) = 0\} \quad \textit{and} \quad \nabla\varphi(x) \neq 0$$

for any $x \in S \cap W$.

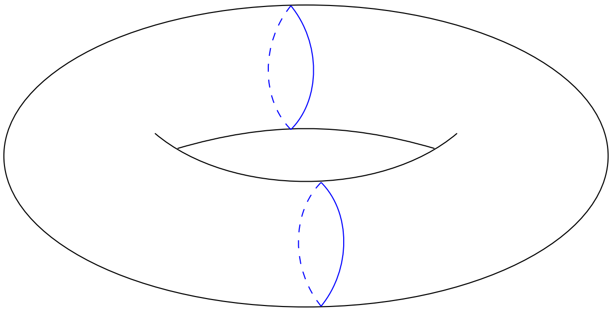


Figure 1.5: The torus

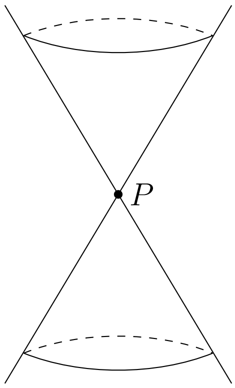


Figure 1.6: The double cone

Proof. Choose a parametrization $\psi\colon \mathrm{V} \to \mathrm{U} \subset S$. Let $(u_0, v_0) \in \mathrm{V}$ be a unique point such that $\psi(u_0, v_0) = p$. Choose a vector $n \in \mathbb{R}^3$ such that

$$\partial_u \psi(u_0, v_0), \quad \partial_v \psi(u_0, v_0), \quad n \tag{1.16}$$

are linearly independent. Consider the map

$$\Psi\colon \mathrm{V} \times \mathbb{R} \to \mathbb{R}^3, \qquad \Psi(u, v, w) = \psi(u, v) + w \cdot n$$

The linear independence of (1.16) yields $\det D\Psi(u_0, v_0, 0) \neq 0$. By the inverse map theorem, there exists an open neighbourhood $W \subset \mathbb{R}^3$ of p and a smooth map $\Phi\colon W \to V \times \mathbb{R} \subset \mathbb{R}^3$ such that

$$\Psi \circ \Phi(x) = x \qquad \forall\, x \in W.$$

If $\Phi = (\varphi_1, \varphi_2, \varphi_3)$, then

$$\Psi \circ \Phi(x) = \psi(\varphi_1(x), \varphi_2(x)) + \varphi_3(x) \cdot n = x.$$

Observe that

$$x \in S \cap W \qquad \Longleftrightarrow \qquad \exists\, (u, v) \in \mathrm{V} \textit{ such that } \psi(u, v) = x$$

and consequently

$$\Psi(u, v, 0) = \psi(u, v) = x = \Psi(\varphi_1(x), \varphi_2(x), \varphi_3(x)).$$

Since Ψ is injective (on an open neighbourhood of $(u_0, v_0, 0)$), we have

$$x \in S \cap W \qquad \Longleftrightarrow \qquad \varphi_3\left(x\right) = 0.$$

Furthermore, since $\det D\Phi\left(x\right) \neq 0$ for all $x \in W$, the vectors $\nabla\varphi_1\left(x\right), \nabla\varphi_2\left(x\right), \nabla\varphi_3\left(x\right)$ are linearly independent at each $x \in W$. In particular, $\nabla\varphi_3\left(x\right) \neq 0$ for all $x \in W$. □

The following corollary follows immediately from Proposition 1.15.

Corollary 1.17. *Any surface is locally the graph of a smooth function.* □

Example 1.18 (A non-example)**.** The union of two intersecting planes in $\mathbb{R}^3$ is *not* a surface. Indeed, assume that

$$S \coloneqq \{z = 0\} \cup \{x = 0\}$$

is a surface. Then there exists a smooth function φ defined in a neighbourhood W of the origin such that φ vanishes on S and $\nabla\varphi(0) \neq 0$ by Proposition 1.15. Notice that φ vanishes identically along S, hence φ vanishes identically along all three coordinate axes (at least in a neighbourhood of the origin). This yields in turn $\nabla\varphi\left(0\right) = 0$, which is a contradiction.

Exercise 1.19. Show that the cone $C \coloneqq \{x^2 + y^2 - z^2 = 0,\ z \geq 0\}$ is not a smooth surface, cf. Example 1.12 above.

1.2 The change of coordinates maps

Neither parametrizations, nor local functions as in the Proposition 1.15 are unique. Our next goal is to understand a relation between different parametrizations.

Thus, let

$$\psi_1 \colon \mathrm{V}_1 \longrightarrow \mathrm{U}_1 \subset S \qquad \text{and} \qquad \psi_2 \quad\colon \mathrm{V}_2 \longrightarrow \mathrm{U}_2 \subset S$$

be two parametrizations such that $\mathrm{U}_1 \cap \mathrm{U}_2 \neq 0$. Since both ψ_1 and ψ_2 are homeomorphisms, we have a well-defined continuous map

$$\psi_{21} \coloneqq \psi_2^{-1} \circ \psi_1 \colon \mathrm{V}_{12} \longrightarrow \mathrm{V}_{21}$$

which is called "a transition map" or "a change of coordinates map".

Notice that ψ_{21} is a map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined on an open subset. Therefore, transition maps can be studied by the tools familiar from the analysis course.

Example 1.20. Consider the sphere S^2, which can be covered by the images of two parametrizations as follows. The inverse of the steregraphic projection from the north pole N is given by

$$(u, v) \longmapsto \psi_N\left(u, v\right) = \frac{1}{1 + u^2 + v^2}\left(2u, 2v, -1 + u^2 + v^2\right)$$

This is a homeomorphism viewed as a map $\mathbb{R}^2 \longrightarrow S^2 \backslash \{N\}$ and is clearly smooth.

Exercise 1.21. Show that $D\psi_N$ is injective at each point.

Thus, ψ_N is a parametrization (at each point $p \in S^2 \backslash \{N\}$). Of course, we have also the inverse ψ_S of the stereographic projection from the south pole S. The images of these two parametrizations cover together the whole sphere S^2. A straightforward computation shows that the change of coordinates map $\psi_{SN} \coloneqq \psi_S^{-1} \circ \psi_N \colon \mathbb{R}^2 \backslash \{0\} \longrightarrow \mathbb{R}^2 \backslash \{0\}$ is given by

$$\psi_{SN}\left(u, v\right) = \frac{1}{u^2 + v^2}\left(u, v\right)$$

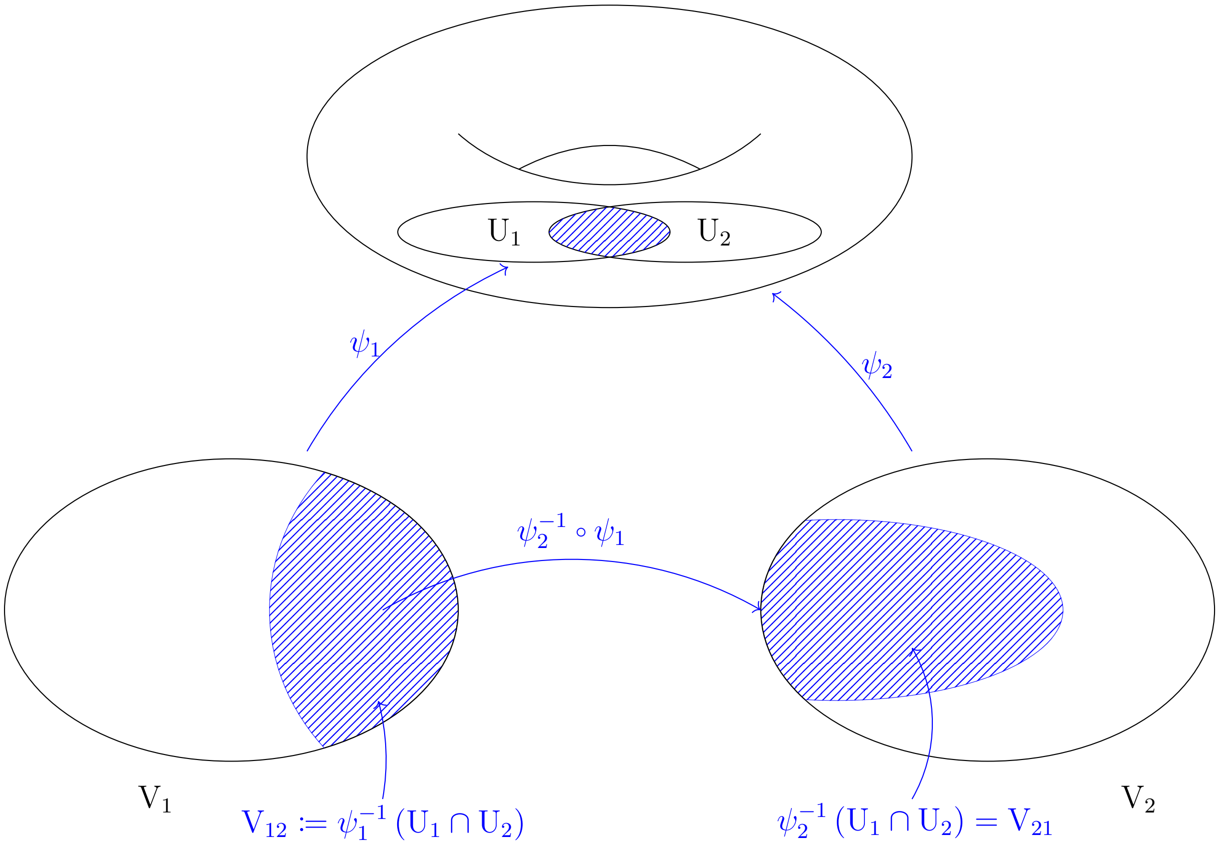


Figure 1.7: The transition map

Exercise 1.22. Show that the sphere can not be covered by the image of a single parametrization.

Theorem 1.23. *Let S be a surface. For any two parametrizations ψ_1 and ψ_2 as above, the change of coordinates map ψ_{12} is smooth.*

Proof. Since smoothness is a local property, it suffices to show that for all $(u_0, v_0) \in \mathrm{V}_{12}$ there exists a neighbourhood $\mathrm{V}_0 \subset \mathrm{V}_{12}$ such that $\psi_{21}|_{\mathrm{V}_0}$ is smooth.

Thus, set $p_0 \coloneqq \psi_1(u_0, v_0)$. For this p_0 and ψ_2 construct a smooth map $\Phi_2 \colon W \longrightarrow \mathrm{V}_2 \times \mathbb{R}$ as in the proof of the Proposition 1.15. Recall that

$$\Phi_2|_{S\cap W} \colon S \cap W \longrightarrow \mathrm{V}_2 \times \{0\} = \mathrm{V}_2$$

equals ψ_2^{-1}.

The map $\Phi_2 \circ \psi_1 \colon \psi_1^{-1}(S \cap W) \to \mathrm{V}_2$ is clearly smooth as a composition of smooth maps. Set $\mathrm{V}_0 \coloneqq \mathrm{V}_{12} \cap \psi_1^{-1}(S \cap W)$. Since the image of ψ_1 lies in S, we obtain that

$$\Phi_2 \circ \psi_1|_{\mathrm{V}_0} = \psi_2^{-1} \circ \psi_1|_{\mathrm{V}_0} = \psi_{21}|_{\mathrm{V}_0}$$

is smooth. $\square$

1.3 Smooth functions on surfaces

Definition 1.24. Let S be a surface. A function $f \colon S \to \mathbb{R}$ is said to be smooth, if for any parametrization $\psi \colon \mathrm{V} \to \mathrm{U}$ the composition

$$F \coloneqq f \circ \psi \colon \mathrm{V} \longrightarrow \mathbb{R}$$

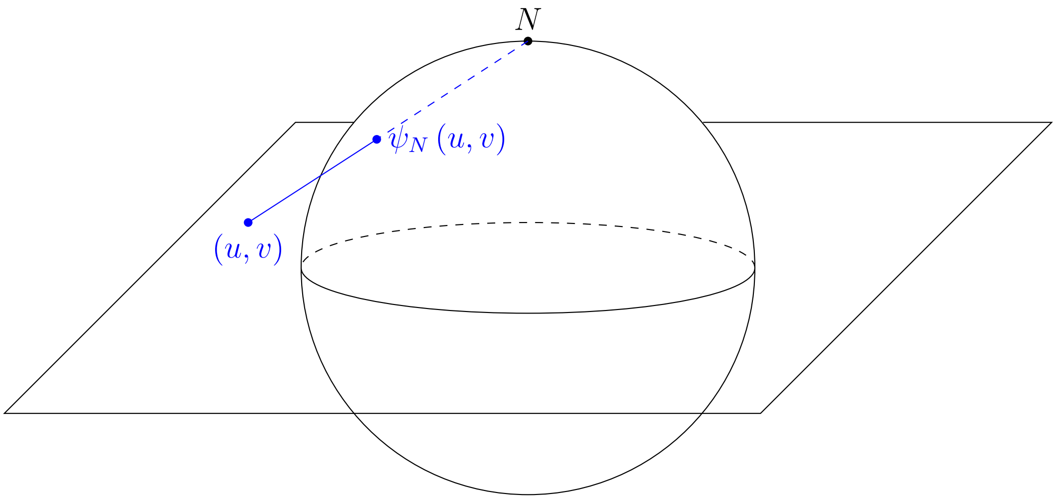


Figure 1.8: The inverse of the stereographic projection

is smooth. The function $F := f \circ \psi$ is called a local (coordinate) representation of f.

Remark 1.25. Theorem 1.23 imples that if $f \circ \psi_1$ is smooth, then $f \circ \psi_2$ is also smooth on $\mathrm{V}_{21} = \psi_2^{-1}(\mathrm{U}_1 \cap \mathrm{U}_2)$. Indeed,

$$f \circ \psi_2 = f \circ \psi_1 \circ \left(\psi_1^{-1} \circ \psi_2\right) = (f \circ \psi_1) \circ \psi_{12}$$

$f \circ \psi_1$ and ψ_{12} are smooth. Hence, if (V_i, ψ_i) is a collection of parametrizations such that $\psi_i(\mathrm{V}_i)$ covers all of S, it suffices to check that $f \circ \psi_i$ is smooth for all i.

Example 1.26. Let $h\colon \mathbb{R}^3 \to \mathbb{R}$ be an arbitrary smooth function. Define $f\colon S \to \mathbb{R}$ as the restriction of h. Then f is smooth, since for any parametrization ψ we have $f \circ \psi = h \circ \psi$ and the right hand side is clearly smooth.

For example, for any fixed $a \in \mathbb{R}^3$ the height function

$$f_a(x) = \langle a, x\rangle \qquad x \in S$$

is a smooth function on S. In particular, set $S = S^2$ and $h(x,y,z) = z$. Then the coordinate representation of $f = h\big|_{S^2}$ with respect to ψ_N is

$$F(u,v) = f \circ \psi_N(u,v) = \frac{-1 + u^2 + v^2}{1 + u^2 + v^2}.$$

This can be seen as a sanity check: This function is smooth indeed.

Example 1.27. Let $\psi\colon \mathrm{V} \to \mathrm{U}$ be a parametrization of a surface S. Since ψ is a homeomorphism, we have the inverse map

$$\varphi := \psi^{-1}\colon \mathrm{U} \longrightarrow \mathrm{V}.$$

Since U itself is a surface (with a single parametrization ψ), it makes sense to ask if φ viewed as a map $\mathrm{U} \to \mathbb{R}^2$ is smooth, which means by definition that both components of φ are smooth functions. This is the case indeed, since the local representation of φ is nothing else but $\varphi \circ \psi = \mathrm{id}$, which is surely smooth. Any such pair (U, φ) is called *a chart* on S.

Proposition 1.28. *Let S be a surface. Then the set $C^\infty(S)$ of all smooth functions on S is a vector space, that is*

$$\begin{matrix} f, g \in C^\infty(S) \\ \lambda, \mu \in \mathbb{R} \end{matrix} \qquad \Longrightarrow \qquad \lambda f + \mu g \in C^\infty(S).$$

In fact, we also have

$$f, g \in C^\infty(S) \qquad \Longrightarrow \qquad f \cdot g \in C^\infty(S),$$

where $f \cdot g$ is the product-function $p \mapsto f(p) \cdot g(p)$.

Proof. We prove the last statement only, while the first one is left as an exercise to the reader. If $\psi \colon \mathrm{U} \to \mathrm{V}$ is a parametrization, then $(f \cdot g) \circ \psi = (f \circ \psi) \cdot (g \circ \psi)$. Since $(f \circ \psi) \in C^\infty(\mathrm{V})$ and $(g \circ \psi) \in C^\infty(\mathrm{V})$, the function $(f \cdot g) \circ \psi$ is smooth as the product of smooth functions of two variables. $\square$

Let $W \subset \mathbb{R}^n$ be an open set.

Definition 1.29. A continuous map $f \colon W \longrightarrow S$, where S is a surface, is called *smooth*, if for any parametrization $\psi \colon \mathrm{V} \to \mathrm{U} \subset S$ the map

$$\varphi \circ f = \psi^{-1} \circ f \colon f^{-1}(\mathrm{U}) \longrightarrow \mathrm{V} \subset \mathbb{R}^2$$

is smooth.

In the above definition we require that f is continuous to ensure that $f^{-1}(\mathrm{U})$ is an open subset so that it makes sense to talk about smoothness of the coordinate representation $\varphi \circ f$.

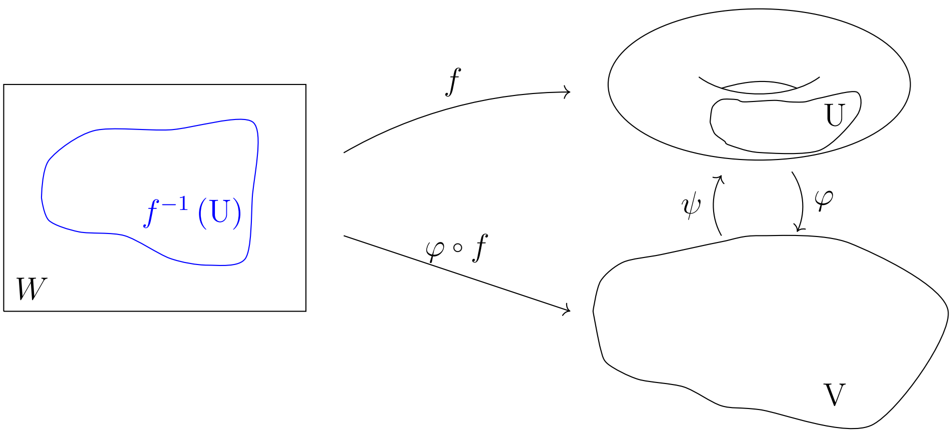


Figure 1.9: A map into a surface and its coordinate representation

Proposition 1.30. *$f \colon W \to S$ is smooth if and only if f is smooth as a map $W \to \mathbb{R}^3$. More formally, this means the following: If $\iota \colon S \to \mathbb{R}^3$ denotes the natural inclusion map, then*

$$f \in C^\infty(W; S) \qquad \Longleftrightarrow \qquad \iota \circ f \in C^\infty\left(W; \mathbb{R}^3\right)$$

Proof. Pick a parametrization ψ of S and construct a smooth map $\Phi\colon X \to \mathbb{R}^3$ just as in the proof of Proposition 1.15, where $X \subset \mathbb{R}^3$ is an open set. Assume $f\colon W \to \mathbb{R}^3$ is smooth. Then $\Phi \circ f$ is also smooth as the composition of smooth maps. However, since f takes values in S and $\Phi|_S = \varphi = \psi^{-1}$, we obtain that $\varphi \circ f = \Phi \circ f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is smooth.

Conversely, assume that $f\colon W \to S$ is smooth. Then

$$f\big|_{f^{-1}(\mathrm{U})} = (\psi \circ \varphi) \circ f\big|_{f^{-1}(\mathrm{U})} = \psi \circ (\varphi \circ f)\big|_{f^{-1}(\mathrm{U})}$$

is again smooth as the composition of smooth maps. □

The following class of maps will be particularly important in the sequel.

Definition 1.31. Let $I \subset \mathbb{R}$ be an (open) interval. A smooth map $\gamma\colon I \to S$ is called a smooth curve on S.

If $0 \in I$, we say that γ is a smooth curve through $p := \gamma(0) \in S$.

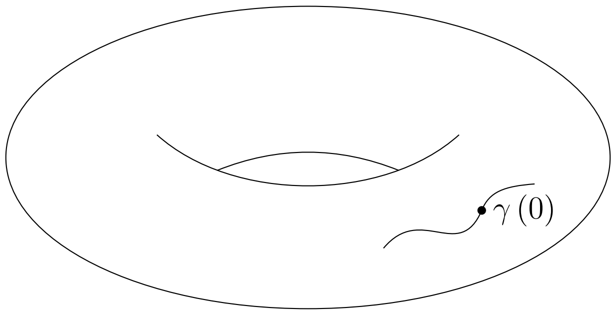


Figure 1.10: A smooth curve on a surface

Example 1.32. Let $p \in S^2$ and $v \in \mathbb{R}^3$ such that $\langle p, v\rangle = 0$ and $\|v\| = 1$. Define $\gamma_v\colon \mathbb{R} \to \mathbb{R}^3$ by $\gamma_v(t) = (\cos t) \cdot p + (\sin t) \cdot v$. Since

$$\begin{aligned} \|\gamma_v(t)\|^2 &= \langle \cos t \cdot p + \sin t\, v, \cos t\, p + \sin t \cdot v\rangle \\ &= \cos^2 t \cdot \|p\|^2 + 0 + \sin^2 t \cdot \|v\|^2 \\ &= \cos^2 t + \sin^2 t = 1, \end{aligned}$$

we obtain that $\gamma_v\colon \mathbb{R} \to S^2$ is a smooth curve through p. Of course, the image of γ_v is a great circle on S^2.

Even more generally, we can define smooth maps between surfaces as follows.

Definition 1.33. Let S_1 and S_2 be two surfaces. A continuous map $f\colon S_1 \to S_2$ is said to be smooth, if for any parametrizations $\psi\colon \mathrm{V} \to \mathrm{U} \subset S_1$ and $\chi\colon W \to X \subset S_2$ the map

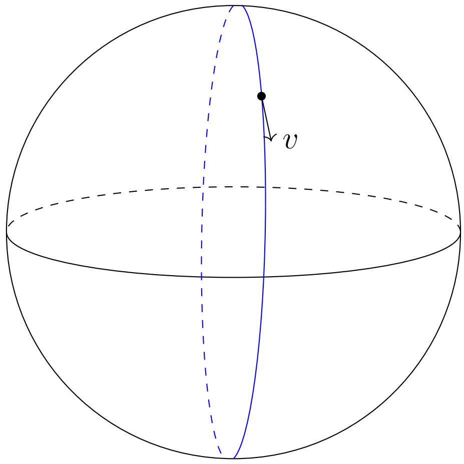
$$\chi^{-1} \circ f \circ \psi\colon \psi^{-1}\left(f^{-1}(X)\right) \longrightarrow W \tag{1.34}$$

is smooth. Just like in the case of functions, (1.34) is called the coordinate (or local) representation of f.

Remark 1.35. Since parametrizations and charts contain the same amount of information, we can also define smoothness of a map $f\colon S_1 \to S_2$ in terms of charts as follows: f is smooth if and only if for any chart (U, φ) on S_1 and any chart (X, ξ) on S_2 the map

$$\xi \circ f \circ \varphi^{-1}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is smooth (on an open subset where defined). The map $\xi \circ f \circ \varphi^{-1}$ is also called a coordinate representation of f (with respect to charts (U, φ) and (X, ξ)).



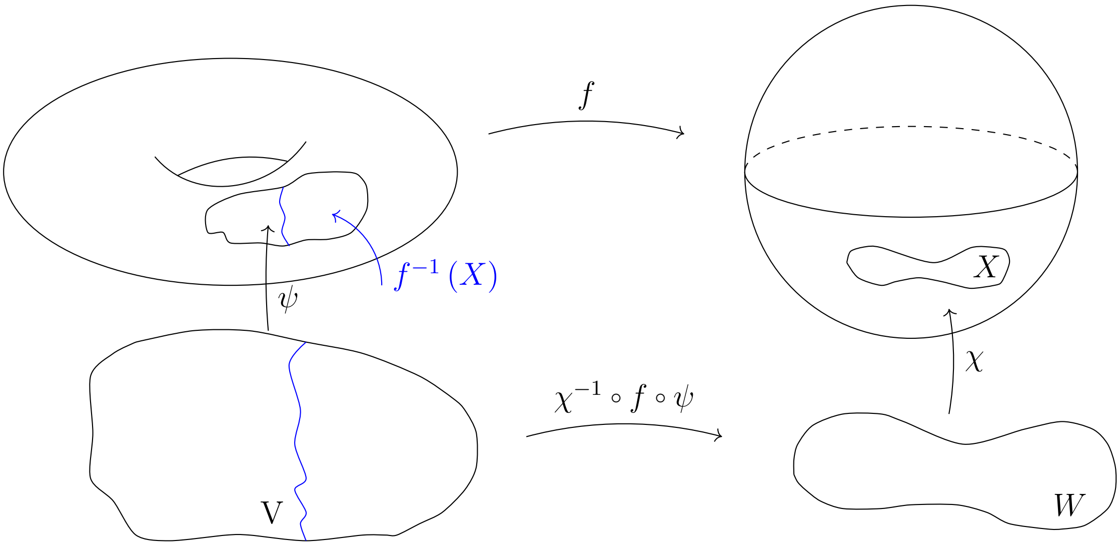


Figure 1.11: A smooth map between surfaces and its coordinate representation

Remark 1.36. Just like in the case of functions, it suffices to find two collections $\{\psi_i \colon \mathrm{V}_i \to \mathrm{U}_i\}$ and $\{\chi_j \colon W_j \to X_j\}$ of parametrizations such that

$$\bigcup_i \mathrm{U}_i = S_1 \qquad \textit{and} \qquad \bigcup_j X_j = S_2$$

and check that all coordinate representations $\chi_j^{-1} \circ f \circ \psi_i$ are smooth.

Consider the antipodal map

$$a \colon S^2 \to S^2, \quad a(x) = -x.$$

For any $(u, v) \in \mathbb{R}^2$ we have

$$a \circ \psi_N (u, v) = -\frac{1}{1 + u^2 + v^2} \left(2u, 2v, -1 + u^2 + v^2\right)$$

Since $\psi_S^{-1} \colon S^2 \backslash \{S\} \to \mathbb{R}^2$ is given by

$$(x, y, z) \longmapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right),$$

we obtain

$$\begin{aligned}\psi_S^{-1} \circ a \circ \psi_N (u, v) &= \frac{1}{1 + \frac{1-u^2-v^2}{1+u^2+v^2}} \left(-\frac{2u}{1+u^2+v^2}, -\frac{2v}{1+u^2+v^2} \right) \\ &= -\frac{1+u^2+v^2}{2} \left(\frac{2u}{1+u^2+v^2}, \frac{-2v}{1+u^2+v^2} \right) \\ &= -(u, v)\end{aligned}$$

It follows in a similar manner, that $\psi_S^{-1} \circ a \circ \psi_S$, $\psi_N^{-1} \circ a \circ \psi_N$, and $\psi_N^{-1} \circ a \circ \psi_S$ are also smooth. Hence, a is smooth.

Proposition 1.37. *Let $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth map such that $h(S_1) \subset S_2$, where S_1 and S_2 are surfaces. Then $h\big|_{S_1} : S_1 \to S_2$ is also smooth.*

The proof of this proposition is similar to the proof of Proposition 1.30 and is left as an exercise to the reader.

To construct a more interesting example, pick a polynomial

$$p(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

with complex coefficients. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can view p as a smooth map $\mathbb{R}^2 \to \mathbb{R}^2$. Define $f\colon S^2 \to S^2$ by

$$f(p) = \begin{cases} \psi_N \circ p \circ \psi_N^{-1}(p) & \text{if } p \neq N, \\ N & \text{if } p = N. \end{cases} \tag{1.38}$$

I claim that f is smooth. Indeed, since by the construction of f, the coordinate representation of f with respect to the pair $(\mathbb{R}^2, \psi_N)$ and $(\mathbb{R}^2, \psi_N)$ of parametrizations (the first one on the source of f, the second one on the target), is

$$\psi_N^{-1} \circ f \circ \psi_N = \underbrace{\psi_N^{-1} \circ \psi_N}_{\text{id}} \circ p \circ \underbrace{\psi_N^{-1} \circ \psi_N}_{\text{id}} = p.$$

Hence f is smooth at each point $p \in S^2 \backslash \{N\}$. To check that f is also smooth at N too, consider

$$\psi_S \circ f \circ \psi_S^{-1}(z) = \begin{cases} \psi_S \circ \psi_N^{-1} \circ p \circ \psi_N \circ \psi_S^{-1} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

We know that

$$\begin{aligned}\psi_{SN}(z) = \psi_S \circ \psi_N^{-1}(z) = \frac{1}{|z|^2} z = \frac{1}{z \cdot \bar{z}} \cdot z = \frac{1}{\bar{z}} \\ \implies \quad \psi_{NS}(z) = \psi_{SN}^{-1}(z) = \frac{1}{\bar{z}}\,.\end{aligned}$$

Hence, we compute

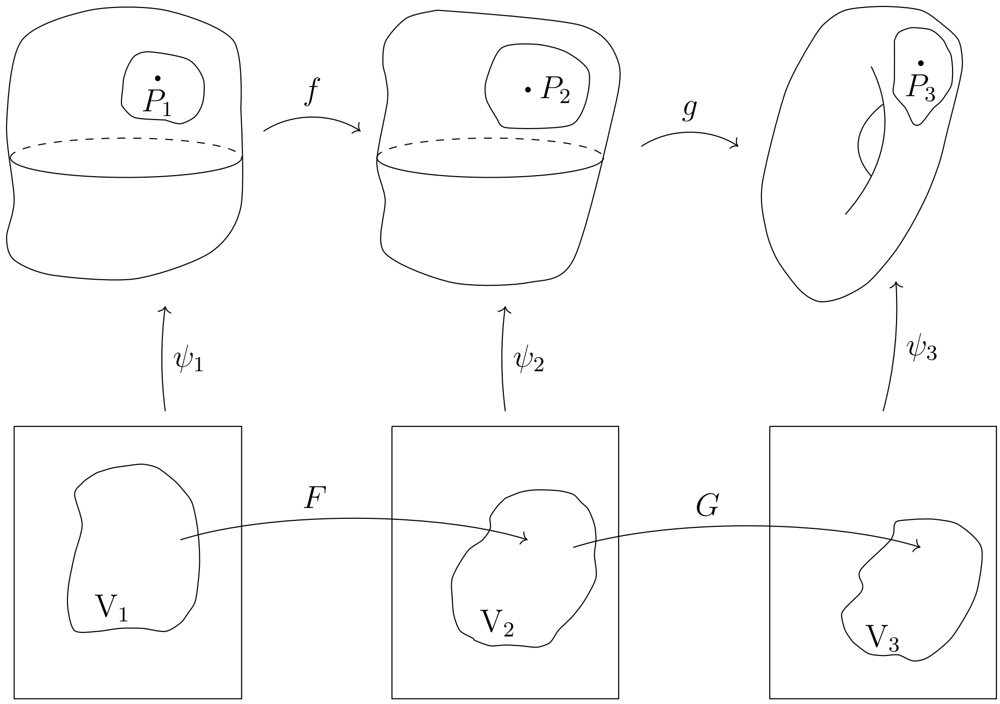
$$\begin{aligned}\psi_{SN} \circ p \circ \psi_{NS}(z) &= \psi_{SN} \left(\frac{1}{\bar{z}^n} + \frac{a_{n-1}}{\bar{z}^{n-1}} + \ldots + a_0 \right) \\ &= \psi_{SN} \left(\frac{1 + a_{n-1}\bar{z} + \ldots + a_0 \bar{z}^n}{\bar{z}^n} \right) \\ &= \frac{z^n}{1 + \bar{a}_{n-1} z + \ldots + \bar{a}_0 z^n}\,, \qquad \text{if } z \neq 0.\end{aligned}$$

This yields that $\psi_S \circ f \circ \psi_S^{-1}$ is smooth even at $z = 0$, that is f is smooth everywhere on S (or, simply, f is smooth).

Theorem 1.39. *Suppose $f\colon S_1 \to S_2$ and $g\colon S_2 \to S_3$ are smooth maps between surfaces. Then $g \circ f\colon S_1 \to S_3$ is also smooth.*

Proof. Pick a point $p_1 \in S_1$ and denote $p_2 := f(p_1) \in S_2$, $p_3 := g(p_2) = g(f(p_1)) \in S_3$. Pick parametrizations

$$\psi_j\colon \mathrm{V}_j \longrightarrow \mathrm{U}_j \subset S_j.$$



In a sufficiently small neighbourhood of p_1 we have

$$\psi_3^{-1} \circ (g \circ f) \circ \psi_1 = \underbrace{\psi_3^{-1} \circ g \circ \psi_2}_{G \in C^\infty} \circ \underbrace{\psi_2^{-1} \circ f \circ \psi_1}_{F \in C^\infty}.$$

Hence, $g \circ f$ is smooth in a neighbourhood of p_1. Since p_1 was arbitrary, $g \circ f$ is smooth everywhere. ☐

Remark 1.40. The proof shows that the coordinate representation of the composition is the composition of coordinate representations.

Notice that Theorem 1.39 yields in particular the following: If $\gamma\colon I \to S_1$ is a smooth curve and $f\colon S_1 \to S_2$ is a smooth map, then $f \circ \gamma\colon I \to S_2$ is also a smooth curve.

Definition 1.41. A smooth map $f\colon S_1 \to S_2$ is called a diffeomorphism, if there exists a smooth map $g\colon S_2 \to S_1$ such that

$$g \circ f = \mathrm{id}_{S_1} \qquad \textit{and} \qquad f \circ g = \mathrm{id}_{S_2}$$

Example 1.42. The antipodal map $a\colon S^2 \to S^2$ is a diffeomorphism.

Example 1.43. The hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$ and cylinder $C = \{x^2 + y^2 = 1\}$ are diffeomorphic, that is there exists a diffeomorphism $f\colon H \to C$. Explicitly, define

$$h\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \textit{by} \quad h(x, y, z) = \left(\frac{x}{\sqrt{1+z^2}}, \frac{y}{\sqrt{1+z^2}}, z\right)$$

Clearly, $h \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$. If $(x, y, z) \in H$, then $\left(\frac{x}{\sqrt{1+z^2}}\right)^2 + \left(\frac{y}{\sqrt{1+z^2}}\right)^2 = \frac{x^2+y^2}{1+z^2} = 1$, that is $f := h\big|_H : H \to C$ is smooth.

Exercise 1.44. Show that the restriction of $h^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ given explicitly by

$$h^{-1}(u, v, w) = \left(\sqrt{1+w^2}\, u, \sqrt{1+w^2}\, v, w\right)$$

yields a smooth inverse of f.

Remark 1.45. A map $f : S_1 \to S_2$ may fail to be a diffeomorphism in the following two ways: either f^{-1} does not exist or f^{-1} exists but is not smooth.

Example 1.46 (A non-example)**.** Consider a map

$$f : C \longrightarrow C, \quad f(x, y, z) = \left(x, y, z^3\right),$$

which is smooth. The inverse $f^{-1} : C \to C$ exists:

$$f^{-1}(x, y, z) = \left(x, y, \sqrt[3]{z}\right).$$

It is continuous, but fails to be smooth.

Exercise 1.47. Compute a coordinate representation of f^{-1} and check that this fails to be smooth indeed.

Example 1.48. Let S be a smooth surface and let $\psi : \mathrm{V} \to \mathrm{U}$ be any parametrization. Consider U as a surface covered by the image of a single parametrization ψ. Then $\varphi = \psi^{-1}$ exists and is smooth as we have seen in Example 1.27. That is U is diffeomorphic to V, which is an open subset of $\mathbb{R}^2$. Summing up, we see that any surface is locally diffeomorphic to an open subset of $\mathbb{R}^2$.

Exercise 1.49.

(i) Show that the disc $D := \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}$ is diffeomorphic to $\mathbb{R}^2$, that is there exists a smooth bijective map $f : D \to \mathbb{R}^2$ such that $f^{-1} : \mathbb{R}^2 \to D$ is also smooth.

(ii) Show that any smooth surface is locally diffeomorphic to $\mathbb{R}^2$, that is any point $p \in S$ has a neighbourhood U diffeomorphic to $\mathbb{R}^2$.

1.4 The tangent plane

Let S be a surface.

Definition 1.50. A vector $v \in \mathbb{R}^3$ is said to be tangent to S at p, if there exists a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to S$ such that

$$\gamma(0) = p \quad \textit{and} \quad \dot{\gamma}(0) = v.$$

Notice that when computing the tangent vector of γ we think of γ as a curve in $\mathbb{R}^3$.

The set T_pS of all vectors tangent to S at the point p is called *the tangent space* of S at p.

Example 1.51. For $S = S^2$ and an arbitrary point p we have the curve

$$\gamma\colon \mathbb{R} \to S^2, \qquad \gamma_v(t) = \cos t \cdot p + \sin t \cdot v,$$

where $\|v\| = 1$ and $v \perp p$ just as in in Example 1.32. Then $\dot\gamma_v(0) = v$. Hence, v is tangent to S^2 at p.

In fact, any vector v which is orthogonal to p is tangent to S^2 at p. Indeed, set $\lambda := \|v\|$ and $v_1 := \lambda^{-1} v$, and

$$\gamma\colon \mathbb{R} \to S^2, \qquad \gamma(t) = \gamma_{v_1}(\lambda t).$$

Then $\gamma(0) = p$ and $\dot\gamma(0) = \lambda\dot\gamma_{v_1}(0) = v$.

Proposition 1.52. *Let $\psi\colon \mathrm{V} \to \mathrm{U}$ be a parametrization such that $\psi(u_0, v_0) = p$. Then*

$$T_pS = \operatorname{Im} D_{(u_0,v_0)}\psi.$$

In particular, T_pS is a vector space of dimension 2.

Proof. The proof consists of the following steps.

Step 1. *We have* $\operatorname{Im} D_{(u_0,v_0)}\psi \subset T_pS$.

Assume $v \in \operatorname{Im} D_{(u_0,v_0)}\psi$. Then there exists a vector $w \in \mathbb{R}^2$ such that $D_{(u_0,v_0)}\psi(w) = v$. Consider the smooth curve $\beta\colon (-\varepsilon, \varepsilon) \to \mathrm{V}$

$$\beta(t) = (u_0, v_0) + t \cdot w.$$

Then $\gamma(t) := \psi \circ \beta(t)$ is a smooth curve in S such that

$$\gamma(0) = \psi(\beta(0)) = \psi(u_0, v_0) = p \qquad \text{and} \qquad \dot\gamma(0) = D_{(u_0,v_0)}\psi(w) = v.$$

Hence, $v \in T_pS$.

Step 2. $T_pS \subset \operatorname{Im} D(u_0, v_0)\psi$

If $v \in T_pS$, then there exists $\gamma\colon (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Can assume $\operatorname{Im}\gamma \subset \mathrm{U}$ by choosing ε smaller if necessary. If $\varphi = \psi^{-1}$, then $\beta(t) := \varphi \circ \gamma(t)$ is a smooth curve in $\mathrm{V} \subset \mathbb{R}^2$ such that $\beta(0) = (u_0, v_0)$. Denote $w := \dot\beta(0) \in \mathbb{R}^2$. Then we have

$$\begin{aligned} v = \dot\gamma(0) &= \frac{d}{dt}\Big|_{t=0} \big(\psi_0 \circ \beta\big)(t) = \big(D_{(u_0,v_0)}\psi\big)\,\big(\dot\beta(0)\big) \\ &= D_{(u_0,v_0)}\psi(w) \in \operatorname{Im} D_{(u_0,v_0)}\psi. \end{aligned}$$

Step 3. $\dim T_pS = 2$.

This follows immediately from the injectivity of $D_{(u_0,v_0)}\psi$. ∎

Proposition 1.53. *Pick $p \in S$ and recall that there exists a neighbourhood $W \subset \mathbb{R}^3$ of p and a smooth function $\varphi\colon W \to \mathbb{R}$ such that*

$$S \cap W = \{q \in W \mid \varphi(q) = 0\} \quad \text{and} \quad \nabla\varphi(q) \neq 0 \quad \forall\, q \in W.$$

Then $T_pS = \nabla\varphi(p)^{\perp}$.

Proof. If γ is any curve in S through p, then

$$\varphi \circ \gamma(t) = 0 \quad \forall t \qquad \Longrightarrow \qquad \left.\frac{d}{dt}\right|_{t=0} \varphi(\gamma(t)) = 0.$$

Therefore, we obtain

$$0 = \left.\frac{d}{dt}\right|_{t=0} \varphi(\gamma(t)) = \langle \nabla\varphi(p), \dot\gamma(0)\rangle \qquad \Longrightarrow \qquad T_pS \subset \nabla\varphi(p)^\perp.$$

Since both T_pS and $\nabla\varphi(p)^\perp$ are two-dimensional, these spaces must be equal in fact. $\square$

Example 1.54. Set $\varphi(x,y,z) = \big(x^2+y^2+z^2-1\big)/2$. Then $\varphi^{-1}(0) = S^2$ and

$$\nabla\varphi(p) = p \neq 0 \ \text{ if } \ p \in S^2 \qquad \Longrightarrow \qquad T_pS^2 = p^\perp.$$

This is consistent with Example 1.51.

Example 1.55. Set $\varphi(x,y,z) = \big(x^2+y^2-z^2-1\big)/2$. If $p = (x,y,z) \in H =: \varphi^{-1}(0)$, then $\nabla\varphi(p) = (x,y,-z) \neq 0$ and therefore

$$T_pH = (x,y,-z)^\perp = \big\{v = (v_1,v_2,v_3) \in \mathbb{R}^3 \mid xv_1 + yv_2 - zv_3 = 0\big\}.$$

Example 1.56. Set $\varphi(x,y,z) := \big(x^2+y^2-1\big)/2, \quad C = \varphi^{-1}(0) \ni p = (x,y,z)$. Then

$$T_pC = \big\{v = (v_1,v_2,v_3) \mid xv_1 + yv_2 = 0, v_3 \text{ is arbitrary }\big\}.$$

1.5 The differential of a smooth map

Just as in calculus of several variables, we wish to study smooth functions, or, more generally, smooth maps, by approximating those by linear ones. This leads to the concept of the differential, which we define first for the case of functions. The more general case of smooth maps is considered below.

Definition 1.57 (Differential of a smooth function)**.** Let S be a surface and $f \in C^\infty(S)$. Define a map $d_pf\colon T_pS \to \mathbb{R}$ as follows: for $v \in T_pS$ choose a smooth curve γ throught p with $\dot\gamma(0) = v$ and set

$$d_pf(v) = \left.\frac{d}{dt}\right|_{t=0} f\circ\gamma(t). \tag{1.58}$$

Proposition 1.59. *d_pf is a well-defined linear map.*

Proof. Pick a parametrization $\psi\colon \mathrm{V} \to \mathrm{U} \ni p$. Without loss of generality we can assume that $\psi^{-1}(p) = 0 \in \mathrm{V}$.

If γ_1 and γ_2 are two curves through p such that $\dot\gamma_1(0) = v = \dot\gamma_2(0)$, then for $\beta_j := \psi^{-1}\circ\gamma_j$ we have

$$\gamma_j(t) = \psi\circ\beta_j(t) \qquad \Longrightarrow \qquad v = D_0\psi\big(\dot\beta_1(0)\big) = D_0\psi\big(\dot\beta_2(0)\big).$$

Since $D_0\psi$ is injective, we obtain $\dot\beta_1(0) = \dot\beta_2(0) =: w$. Furthermore,

$$\left.\frac{d}{dt}\right|_{t=0} f\circ\gamma_1(t) = \left.\frac{d}{dt}\right|_{t=0} \big(f\circ\psi\circ\psi^{-1}\circ\gamma_1(t)\big) = \left.\frac{d}{dt}\right|_{t=0} \big(F\circ\beta_1(t)\big) = D_0F(w).$$

Likewise, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} f\circ\gamma_2(t) = D_0F(w) \qquad\Longrightarrow\qquad \left.\frac{d}{dt}\right|_{t=0}\left(f\circ\gamma_1(t)\right) = \left.\frac{d}{dt}\right|_{t=0}\left(f\circ\gamma_2(t)\right).$$

Hence, d_pf is well-defined and, moreover, we have the equality

$$d_pf\circ D_0\psi = D_0F,$$

where $F := f\circ\psi$ is the coordinate representation of f. Since both $D_0\psi$ and D_0F are linear, so is d_pf. $\square$

Exercise 1.60. Think of $\mathbb{R}^2$ as a surface in $\mathbb{R}^3$ (for example, as $\mathbb{R}^2\times\{0\}\subset\mathbb{R}^3$). Let $f\colon \mathbb{R}^2\to\mathbb{R}^2$ be any smooth map. Show that the differential of f in the sense of Definition 1.57 coincides with the one known from the analysis course.

Exercise 1.61. If $h\in C^\infty(\mathbb{R}^3)$ and $f = h\big|_S$, then for all $p\in S$ we have

$$d_pf = D_ph\big|_{T_pS}.$$

Definition 1.62. A point $p\in S$ is called critical for $f\in C^\infty(S)$, if $d_pf = 0$, that is $d_pf(v) = 0$ for all $v\in T_pS$.

Proposition 1.63. *If p is a point of local maximum (minimum) for f, then p is critical for f.*

Proof. If p is a point of local maximum for f, then for any curve γ through p, 0 is a point of local maximum for $f\circ\gamma$. Hence, $\frac{d}{dt}\big|_{t=0} f\circ\gamma(t) = 0$. $\square$

Proposition 1.64. *Let $h,\varphi\in C^\infty(\mathbb{R}^3)$. Assume $\nabla\varphi(p)\neq 0$ for any $p\in S = \varphi^{-1}(0)$. If $p\in S$ is a point of local maximum for $f = h\big|_S$, then*

$$\nabla h(p) = \lambda\nabla\varphi(p) \tag{1.65}$$

for some $\lambda\in\mathbb{R}$.

Proof. Our hypothesis implies that S is a surface and $T_pS = \big(\nabla\varphi(p)\big)^\perp$, see Example 1.9 and Proposition 1.53. Hence,

$$d_pf = 0 \qquad\Longleftrightarrow\qquad D_ph\big|_{T_pS} = 0 \qquad\Longleftrightarrow\qquad \langle v, \nabla h(p)\rangle = 0 \quad \forall\, v\in T_pS.$$

In other words, $\nabla h(p)$ is orthogonal to T_pS. However, $T_pS^\perp$ is one-dimensional and contains $\nabla\varphi(p)\neq 0$. This implies (1.65). $\square$

Remark 1.66. This proof is in a sense more conceptual than the proof of Theorem 1.6.

More generally, for any $f\in C^\infty(S;\mathbb{R}^n)$ the differential $d_pf\colon T_pS\to\mathbb{R}^n$ is defined by (1.58) too. This yields immediately the following: If f is written in components as $f = (f_1,\dots,f_n)$, then d_pf can be written in components as

$$d_pf = \big(d_pf_1,\dots,d_pf_n\big).$$

Also, the differential is well-defined for maps $f\colon\mathbb{R}^n\to S$ and is a linear map of the form $d_pf\colon\mathbb{R}^n\to T_{f(p)}S$. For maps $f\colon S_1\longrightarrow S_2$ between surfaces we define

$$d_pf\colon T_pS_1\longrightarrow T_{f(p)}S_2$$

essentially by the same rule: If $\dot\gamma(0) = v\in T_pS_1$, then $d_pf(v) := \frac{d}{dt}\big|_{t=0}\left(f\circ\gamma(t)\right)$. This yields again a well-defined linear map as the reader can easily check.

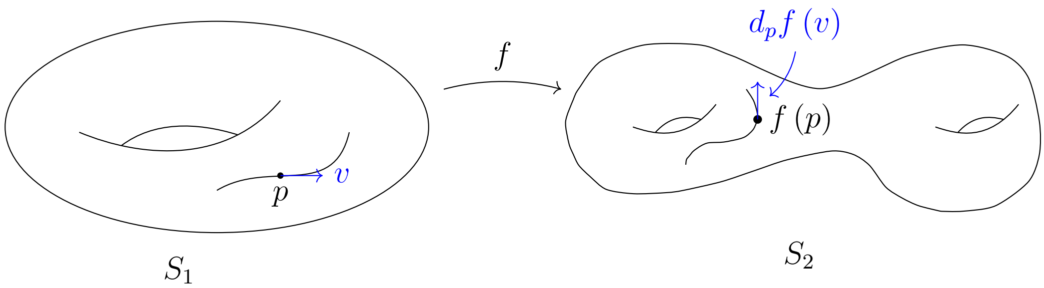


Figure 1.12: The differential of a smooth map

Proposition 1.67. *Let S_1, S_2, S_3 be smooth surfaces. For any smooth maps $f\colon S_1 \to S_2$ and $g\colon S_2 \to S_3$ and any point $p \in S_1$ we have*

$$D_p(g \circ f) = D_{f(p)}g \circ D_pf.$$

This also holds if any of S_i is replaces by an open subset of $\mathbb{R}^n$.

Proof. Let γ_1 be any smooth curve in S_1 through p. Denote $\gamma_2 = f \circ \gamma$, which is a smooth curve in S_2 through $f(p)$. If $\dot{\gamma_1}(0) = v_1$, then $v_2 := \dot{\gamma_2}(0) = D_pf(v_1)$ by the definition of D_pf. Hence,

$$\begin{aligned} D_p(g \circ f)(v_1) &= \frac{d}{dt}\Big|_{t=0}\Big(g \circ \underbrace{f \circ \gamma_1}_{\gamma_2}(t)\Big) = \frac{d}{dt}\Big|_{t=0}(g \circ \gamma_2(t)) = D_{f(p)}g(v_2) \\ &= D_{f(p)}g\big(D_pf(v_1)\big). \end{aligned}$$

□

Corollary 1.68. *If $f\colon S_1 \to S_2$ is a diffeomorphism, then $d_pf\colon T_pS_1 \to T_{f(p)}S_2$ is an isomorphism for any $p \in S_1$.* □

Definition 1.69. A map $f\colon S_1 \to S_2$ is called *a local diffeomorphism* if for any $p \in S_1$ there exists a neighbourhood $\mathrm{U}_1 \subset S_1$ and a neighbourhood $\mathrm{U}_2 \subset S_2$ of $f(p)$ such that $f\colon \mathrm{U}_1 \to \mathrm{U}_2$ is a diffeomorphism.

Theorem 1.70. *Let $f\colon S_1 \to S_2$ be a smooth map such that $d_pf\colon T_pS_1 \to T_{f(p)}S_2$ is an isomorphism for all $p \in S_1$. Then f is a local diffeomorphism.*

Proof. Pick any $p \in S_1$ and parametrizations $\psi_1\colon \mathrm{V}_1 \to W_1 \subset S_1$ and $\psi_2\colon \mathrm{V}_2 \to W_2 \subset S_2$. Without loss of generality we can assume that $\psi_1(0) = p$ and $\psi_2(0) = f(p)$.

Recall that the coordinate representation of f is $F = \psi_2^{-1} \circ f \circ \psi_1$, see Fig. 1.13. Hence, by Proposition 1.67 we obtain $d_0F = d_{f(p)}\psi_2^{-1} \circ d_pf \circ d_0\psi_1$. Furthermore, since all of the following linear maps

$$d_0\psi_1\colon \mathbb{R}^2 \longrightarrow T_pS_1, \qquad d_{f(p)}\psi_2\colon T_{f(p)}S_2 \longrightarrow \mathbb{R}^2, \qquad \textit{and} \qquad d_pf\colon T_pS_1 \to T_pS_2$$

are isomorphisms, we conclude that d_0F is an isomorphism too.

From the analysis course it is known that there exists a neighbourhood $\widetilde{\mathrm{V}}_1 \subset \mathrm{V}_1$ of the origin and a neighbourhood $\widetilde{\mathrm{V}}_2 \subset \mathrm{V}_2$ of the origin such that $F\colon \widetilde{\mathrm{V}}_1 \to \widetilde{\mathrm{V}}_2$ is a diffeomorphism. Denoting $\mathrm{U}_1 = \psi_1(\widetilde{\mathrm{V}}_1)$ and $\mathrm{U}_2 = \psi_2(\widetilde{\mathrm{V}}_2)$, we have

$$f\big|_{\mathrm{U}_1} = \psi_2 \circ F \circ \psi_1^{-1}\big|_{\mathrm{U}_1}\colon \mathrm{U}_1 \to \mathrm{U}_2$$

is a diffeomorphism, since it is a composition of diffeomorphisms. □

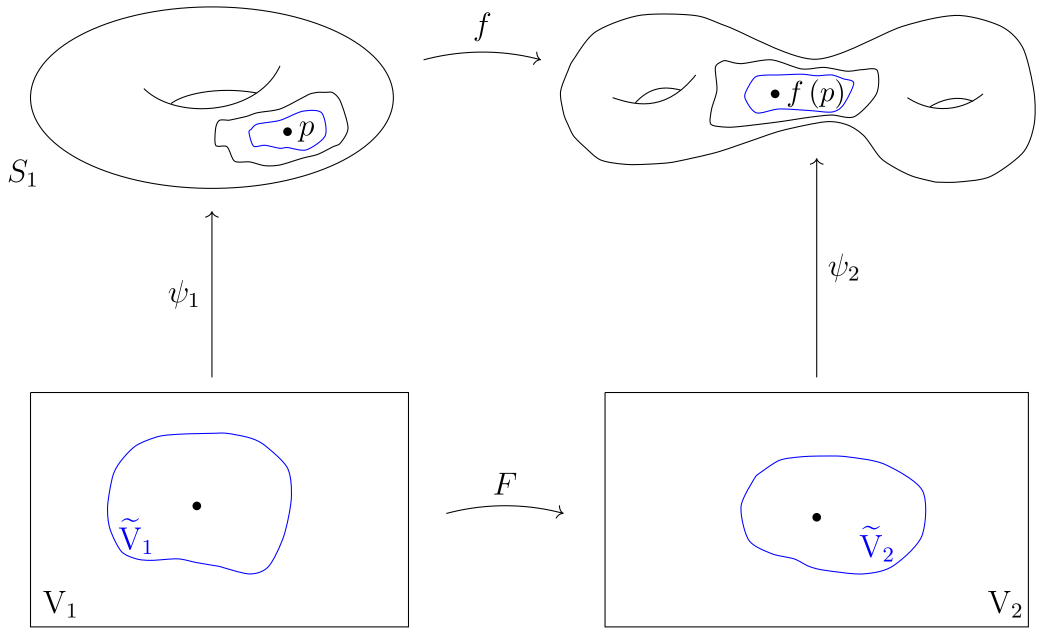


Figure 1.13: Illustration for the proof of Theorem 1.70

Remark 1.71. It follows from the proof of Theorem 1.70, that

$$d_p f = d_0\psi_2 \circ d_0 F \circ d_p\psi_1^{-1},$$

where both $d_0\psi_2$ and $d_p\psi_1^{-1}$ are linear isomorphisms.

In particular, this implies that the following holds:

- $d_p f$ is injective $\iff$ $D_{\psi_1(p)}F$ is injective;
- $d_p f$ is surjective $\iff$ $D_{\psi_1(p)}F$ is surjective;
- $d_p f$ is an isomorphism $\iff$ $D_{\psi_1(p)}F$ is an isomorphism.

Definition 1.72. For $f \in C^\infty(S_1; S_2)$ a point $p \in S_1$ is called *a critical point* of f if $d_p f$ is not surjective.

Since $\dim T_pS_1 = \dim T_{f(p)}S_2$, a simple argument from linear algebra yields:

$$d_p f \text{ is non-surjective} \iff d_p f \text{ is non-injective} \iff d_p f \text{ is not an isomorphism.} \qquad (1.73)$$

Notice, however, that Definition 1.72 makes sense in more general situations where, for example, the target S_2 (and/or the source S_1) is replaced by $\mathbb{R}^n$. However, (1.73) is false in general for those more general cases.

To see that Definition 1.72 coincides with the previous one in the case of function, suppose p is a critical point of a smooth function $f\colon S_1 \to \mathbb{R}$ in the sense of Definition 1.72. If there exists $v \in T_pS_1$ such that $d_pf(v) \neq 0$, then the linearity of d_pf yields immediately that d_pf is surjective. Hence, d_pf is non-surjective if and only if it vanishes, cf. Definition 1.62.

Definition 1.74. A point $q \in S_2$ is called a *regular value* of f, if any $p \in f^{-1}(q)$ is a regular (that is non-critical) point of f, i.e., if for all $p \in f^{-1}(q)$ the differential d_pf is surjective.

The argument demonstrating (1.73) yields also the following:

$$d_p f \text{ is surjective} \iff d_p f \text{ is injective} \iff d_p f \text{ is an isomorphism.}$$

Example 1.75. Identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider the map $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^n$, where $n \in \mathbb{Z}, n \geq 2$. It is known from analysis that $d_z f\colon \mathbb{C} \to \mathbb{C}$ can be identified with the map $h \mapsto f'(z) \cdot h$. Hence, z is critical if and only if $f'(z) = 0 \Leftrightarrow nz^{n-1} = 0 \Leftrightarrow z = 0$. Hence, f has a single critical point $z = 0$ and a single critical value, the zero. All other points are regular and any non-zero value is also regular.

Viewing f as a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, we obtain an example of a local diffeomorphism, which is not a diffeomorphism (assuming $n \geq 2$).

Theorem 1.76 (The fundamental theorem of algebra)**.** *Let $q(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with complex coefficients. Then p has at least one complex root.*

Proof. First recall that the map $f\colon S^2 \to S^2$,

$$f(p) = \begin{cases} N & p = N, \\ \psi_N \circ q \circ \psi_N^{-1}, & p \neq N, \end{cases}$$

is smooth. Indeed, the details of this claim are spelled on Page 14. The rest of the proof consists of the following steps.

Step 1. *f has at most n critical points (values).*

Indeed, a point $p \in S^2 \setminus \{N\}$ is critical for f if and only if $z := \psi_n(p)$ is critical for q. Hence, in this case $q'(z) = 0$, that is z is a root of the polynomial $nz^{n-1} + (n-1)\, a_{n-1} z^{n-2} + \ldots + a_1$, which can have at most $(n-1)$ roots.

Step 2. *Denote by $R(f)$ the set of regular values of f. Then for any $r \in R(f)$ the set $f^{-1}(r)$ is finite and the map $R(f) \to \mathbb{Z}_{\geq 0}, r \mapsto \# f^{-1}(r)$ is constant.*

Pick any $r \in R(f)$ and any $p \in f^{-1}(r)$. Then $f(p) = r$ and $d_p f$ is an isomorphism. Hence, by Theorem 1.70 there exists a neighbourhood U_p of p and a neighbourhood W_r such that $f\colon \mathrm{U}_p \to W_r$ is a diffeomorphism. In particular, $f^{-1}(r) \cap \mathrm{U}_p = \{p\}$, that is $f^{-1}(r)$ is discrete. Since $f^{-1}(r)$ is a closed subset of S^2, $f^{-1}(r)$ is compact. But a compact discrete set must be finite.

Denote $f^{-1}(r) = \{p_1, \ldots p_m\}$ and the corresponding neighbourhoods $\mathrm{U}_1, \ldots \mathrm{U}_m$ and $\mathrm{W}_1, \ldots \mathrm{W}_m$. Set $\mathrm{W} := \mathrm{W}_1 \cap \ldots \cap \mathrm{W}_m$ and $\widetilde{\mathrm{U}}_j := f^{-1}(\mathrm{W}) \cap \mathrm{U}_j$. Then for each $j \leq m$ the map $f\colon \widetilde{U}_j \to W$ is a diffeomorphism. In particular, for all $r' \in W$ there exists a unique $p_j' \in \widetilde{\mathrm{U}}_j$ such that $f(p_j') = r'$. Hence, $\# f^{-1}(r') = \# f^{-1}(r)$ for all $r' \in W$, so that the function

$$R(f) \longrightarrow \mathbb{Z}, \quad r \longmapsto \# f^{-1}(r) \tag{1.77}$$

is locally constant.

However $R(f)$ is the complement of a finite number of points in S^2, hence connected. Therefore (1.77) is (globally) constant.

Step 3. *We prove this theorem.*

Pick any pairwise distinct points $p_1, \ldots, p_{n+1} \in S^2 \setminus \{N\}$ such that $f(p_1), \ldots, f(p_{n+1})$ are also pairwise distinct. Since f has at most n critical values, at least one of those points is a regular value of f and (1.77) does not vanish at this point. Hence, (1.77) vanishes nowhere on $R(f)$.

If the south pole S is a critical value of f, then $f^{-1}(S) \neq \varnothing$, since $f^{-1}(S)$ contains a critical point. However,

$$f^{-1}(S) \neq \varnothing \quad \Longleftrightarrow \quad q^{-1}(0) \neq \varnothing.$$

If S is a regular value, then Step 2 yields $\# f^{-1}(S) \geq 1$. This yields in turn $q^{-1}(0) \neq \varnothing$, which finishes this proof. $\square$