Algebraic Topology

Lecture notes

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This is a draft. In particular, most of figures are missing. If you spot a mistake, please let me know.

TODO:
- Add an appendix on chain complexes.

Contents

Chapter 1

Introduction

The main purpose of this chapter is to explain informally the main ideas which will be developed in details later. In particular, the proofs are rather sketchy stressing main ideas only. More precise statements and proofs will be given in the subsequent chapters.

1.1 Differential forms, the theorems of Green and Stokes

Let $\omega = P(x,y)dx + Q(x,y)dy$ be a 1-form on an open subset $U \subset \mathbb{R}^2$. For example, if $f\colon U \to \mathbb{R}$ is a smooth map, then the differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is a 1-form.

Question 1.1. Under which circumstances does there exist some function f as above such that $\omega = df$?

Clearly, we have the following necessary condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1.2}$$

Proposition 1.3. *If U is convex, then (1.2) is also sufficient.*

Sketch of proof. Theorem of Green $\implies$ For any closed piecewise smooth curve $C \subset U$ without self-intersections we have

$$\int_C (P\,dx + Q\,dy) = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx\,dy = 0, \tag{1.4}$$

where D is the domain bounded by C. Notice that here we use the convexity of U, since otherwise C does not necessarily bound any domain.

Pick any $(x_0, y_0) \in U$. For any $(x,y) \in U$ choose a curve C' connecting (x_0,y_0) and (x,y). Define

$$f(x,y) := \int_{C'} P\,dx + Q\,dy.$$

Property (1.4) guaranties that f does not depend on the choice of C'. □

The following example shows that (1.2) is not sufficient for general U.

Example 1.5. Consider $U = \mathbb{R}^2 \setminus \{0\}$ and

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

If there were some f such that $\omega = df$, then we would have $\int_{S^1} \omega = 0$, where S^1 is the circle (for example, parametrized via $t \mapsto (\cos t, \sin t)$). This is a contradiction, since $\int_{S^1} \omega = 2\pi \neq 0$.

Notice that the proof of Proposition 1.2 does not work here, since the theorem of Green does not apply for (D, ω), where D is the unit disc.

Remark 1.6. One can show that for any closed piecewise smooth curve $C \subset \mathbb{R}^2 \setminus \{0\}$ we have

$$\frac{1}{2\pi} \int_C \Big(-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \Big)$$

is an integer.

Let U be an open subset of $\mathbb{R}^3$ and $\omega = P\,dx + Q\,dy + R\,dz$ be a 1-form. We can also ask whether $\omega = df$ for some $f \colon U \to \mathbb{R}$. Clearly, we have the following necessary condition:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \textit{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{1.7}$$

Proposition 1.8. *If U is convex, then (1.7) is also sufficient.*

The proof of this proposition is analogous to the proof of the previous one. Just instead of the theorem of Green we have to use the theorem of Stokes:

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_\Sigma \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\, dy\,dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\, dz\,dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx\,dy.$$

Proposition 1.9. *Condition (1.7) is also sufficient for $\mathbb{R}^3 \setminus \{0\}$.*

Sketch of proof. Let $C \subset \mathbb{R}^3$ be an arbitrary simple picewise smooth curve without self-intersections. Then there is a picewise smooth surface $\Sigma \subset \mathbb{R}^3$ such that $\partial\Sigma = C$. If $0 \in \Sigma$, a (small) perturbation yields a surface $\Sigma' \subset \mathbb{R}^3 \setminus \{0\}$ such that $\partial\Sigma' = C$. $\square$

For a general U, Condition (1.7) is still insufficient, which is easily seen for the following example: $U = \mathbb{R}^3 \setminus \{z - \text{Axis}\}$ and

$$\omega = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy.$$

From this discussion we can make the following informal conclusion: Condition (1.7) is sufficient as long as U has no "holes" of codimension 2.

1.2 Ansatz of a construction.

Let $X \subset \mathbb{R}^n$ be an arbitrary subset, which is equipped with the induced topology. Define $Z_1(X)$ as a free Abelian group generated by (oriented) closed curves, i.e.,

$$C \in Z_1(X) \implies C = n_1 C_1 + \ldots n_k C_k, \tag{1.10}$$

where $n_j \in \mathbb{Z}$. Define

$$\int_C \omega := \sum n_k \int_{C_k} \omega.$$

Remark 1.11. If C_0 is a closed oriented curve, $2C_0$ can be understood as "running along C_0 twice in the same direction". Similarly, $-C_0$ can be understood as the curve C_0 with the opposite orientation. However, in most cases we treat (1.10) purely formally.

Assume temporarily that X is an *open* subset of $\mathbb{R}^2$. We would like to define an equivalence relation such that

$$C \sim C' \implies \int_C \omega = \int_{C'} \omega$$

holds for all $\omega = P\,dx + Q\,dy$ satisfying (1.2). The theorem of Green (or Stokes in the case $U \subset \mathbb{R}^3$) suggests the following:

$$C \sim C' \quad \Leftrightarrow \quad \exists \text{ a compact oriented surface } \Sigma \text{ such that } \partial\Sigma = C \cup -C'. \tag{1.12}$$

Here C and C' are oriented curves and Σ is an oriented surface such that $\partial\Sigma = C \cup -C'$ as *oriented* curves. This definition also makes sense even in the case when X is not necessarily open.

More generally, a cycle $C = C_1 + \cdots + C_k$ is called *null homologous*, i.e., $C \sim 0$, if and only if

$$\exists \text{ a compact surface } \Sigma \text{ such that } \partial\Sigma = C_1 \cup \cdots \cup C_n.$$

Clearly, Condition (1.12) can be written as $C + (-C') \sim 0$.

Example 1.13. Null homologous cycles on the 2-sphere with 2 points removed (equivalently, $\mathbb{R}^2 \setminus \{0\}$).

Even more generally, each linear combination of null homologous cycles is also declared to be null homologous.

$Z_1(X) \supset B_1(X) = \{\text{null homologous cycles}\}$.

$H_1(X) := Z_1(X)/B_1(X)$ the first homology group of X.

Example 1.14. $H_1(S^2 \setminus \{p, q\}) \cong \mathbb{Z}$.

Problems: Curves C and surfaces Σ can have singularities and self-intersections.

More generally:

- $Z_n(X)$ freely generated by compact oriented n-dimensional "surfaces" without boundary.
- $Z_n(X) \supset B_n(X)$ the subgroup generated by the boundaries of compact oriented $(n+1)$-dimensional "surfaces".
- $H_n(X) := Z_n(X)/B_n(X)$ the nth homology group of X.

In general, we would like to associate to each topological space X a sequence of abelian groups $H_0(X), H_1(X), \ldots, H_n(X), \ldots$ such that the following holds:

(a) Each continuous map $f\colon X \to Y$ induces a sequence of homomorphisms $f_*\colon H_n(X) \to H_n(Y)$;

(b) $(f \circ g)_* = f_* \circ g_*, \quad id_* = id.$

(c) $H_0(\{pt\}) \cong \mathbb{Z}$ and $H_n(\{pt\}) = 0$ for all $n \geq 1$.

(d) $H_n(S^n) \cong \mathbb{Z}$ provided $n \geq 1$ and $H_k(S^n) = 0$ for all $k \geq n + 1$ (More generally, for each compact connected oriented manifold M of dimension n the following holds: $H_n(M) \cong \mathbb{Z}$ and $H_k(M) = 0$ for all $k > n + 1$).

(e) $f \simeq g \implies f_* = g_*$.

Here two continuous maps are said to be homotopic ($f \simeq g$), if there exists a continuous map $h\colon X \times [0,1]$, called homotopy, such that the following holds:

$$h|_{X\times 0} = f \qquad \text{and} \qquad h|_{X\times 1} = g.$$

Question 1.15. What does make Properties (a)-(e) interesting?

This question will be answered in the subsequent sections. We finish this section by the following facts, which will be useful below.

Proposition 1.16. *If f is a homeomorphism, then each $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. $id_{H_n} = id_* = (f \circ f^{-1})_* = f_* \circ (f^{-1})_* \implies f_*$ is an isomorphism and $(f_*)^{-1} = (f^{-1})_*$. □

1.3 The theorem of Brouwer

In this section we show that (a)-(e) imply the following famous result.

Theorem 1.17 (Brouwer)**.** *Any continuous map $f\colon B_n \to B_n$ has a fixed point.*

Proof. The proof consists of the following three steps.

Step 1. *For the ball $B_n := \{x \in \mathbb{R}^n \mid |x| \le 1\}$ we have $H_k(B_n) = 0$ for all $k \ge 1$.*

Let $c\colon B_n \to \{0\}$ be the constant map. The map $h(x,t) = tx,\ t \in [0,1]$ is a homotopy between id_B und $\imath \circ c$, where $\imath\colon \{0\} \to B_n$ is the inclusion. Thus, $id = \imath_* \circ c_* \implies H_k(B_n) = 0$ for all $k \ge 1$, since $\operatorname{Im} \imath_* = \{0\}$.

Step 2. *There is no continuous map $g\colon B_n \to \partial B_n = S^{n-1}$ such that $g(x) = x$ holds for all $x \in S^{n-1}$.*

Assume $n = 1$ first. In this case there is no continuous map $g\colon [-1,1] \to \{\pm 1\}$ as in the statement of this step, since the target space $\{\pm 1\}$ is disconnected, whereas the interval $[0,1]$ is connected.

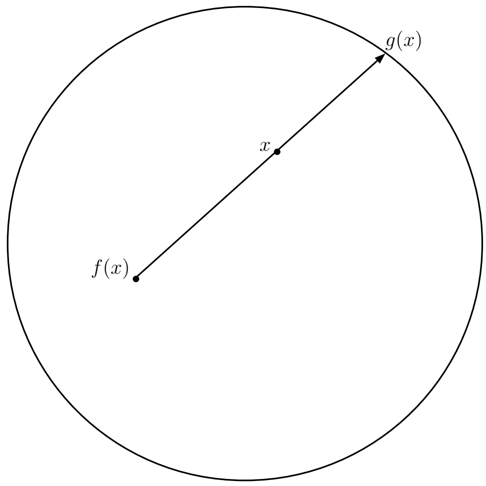
Let us consider now the case $n \ge 2$. Assume there is such $g\colon B_n \to S^{n-1}$. Then we have

$$\begin{aligned} id_{S^{n-1}} = g \circ \imath_{S^{n-1}} &\implies (id_{S^{n-1}})_* = g_* \circ (\imath_{S^{n-1}})_* = 0 \quad \text{on } H_{n-1}(S^{n-1}) \\ &\implies H_{n-1}(S^{n-1}) = 0. \end{aligned}$$

This contradiction proves Step 2.

Step 3. *We prove the theorem of Brower.*

Assume there exists a continuous map $f\colon B_n \to B_n$ without fixed points. Then there also exists a continuous map $g\colon B_n \to S^{n-1}$ such that $g|_{S^{n-1}} = id$:



This contradicts Step 2. □

1.4 The degree of a continuous map and the fundamental theorem of algebra

In this section we show that (a)-(e) imply that any non-constant polynomial with complex coefficients has at least one root. This statement is known as the fundamental theorem of algebra.

Thus, pick any $n \geq 1$ and choose a generator $\alpha \in H_n(S^n)$, i.e., an element α such that $H_n(S^n) = \mathbb{Z} \cdot \alpha$.

Definition 1.18. For any continuous map $f\colon S^n \to S^n$ define $\deg(f) \in \mathbb{Z}$ by

$$f_*\alpha = \deg(f)\alpha.$$

The degree of a map does not depend on the choice of a generator, since $f_*(-\alpha) = -f_*\alpha = -\deg(f)\alpha = \deg(f)(-\alpha)$.

Lemma 1.19. *The degree has the following properties:*

(i) $\deg(id) = 1$*;*

(ii) $\deg(f \circ g) = \deg f \cdot \deg g$*;*

(iii) $f \simeq g \implies \deg f = \deg g$*;*

(iv) $\deg(\textit{const. map}) = 0$*.*

□

Lemma 1.20. *For* $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ *define* $f_n\colon S^1 \to S^1$ *by* $f_n(z) = z^n$*, where* $n \in \mathbb{Z}$*. Then we have*

$$\deg f_n = n.$$

Idea of proof. The curve

$$\alpha\colon [0, 2\pi] \to S^1, \qquad \alpha(t) = \cos t + \sin t\, i = e^{ti},$$

generates $H_1(S^1)$. Since $f_n \circ \alpha(t) = e^{nti} = \cos(nt) + \sin(nt)i$, from the definition of the degree and Remark 1.11 we have $\deg f_n = n$. □

Theorem 1.21 (The fundamental theorem of Algebra)**.** *Each non-constant polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0,\ a_j \in \mathbb{C}$ has at least one complex root.*

Proof. Identify S^1 with $S^1_r := \{z \in C \mid |z| = r\} \cong S^1$ with the help of the homeomorphism

$$S^1 \to S^1_r, \qquad z \mapsto rz.$$

The proof consists of the following three steps.

Step 1. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a continuous map without zeros. Then for each $r > 0$ the map*

$$\frac{f}{|f|}\colon S^1_r \to S^1 \tag{1.22}$$

is homotopic to the constant map.

Indeed, a homotopy can be given explicitly by

$$F(z,t) = \frac{f(tz)}{|f(tz)|}, \qquad z \in S^1,\ t \in [0,r].$$

Step 2. *Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$ be a polynomial without zeros. Then there exists some $R > 0$ such that the following holds: $\forall r \geq R$ the restriction of $p/|p|$ to S^1_r is homotopic to f_n.*

For all $z \in \mathbb{C}$ such that $|z| \geq 1$ we have

$$\begin{aligned} \left|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\right| &\leq |a_{n-1}||z|^{n-1} + \dots + |a_1||z| + |a_0| \\ &\leq n \max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}|z|^{n-1} \end{aligned}$$

Choose R so that $R > n\max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ and $R > 1$. For all $r \geq R$ and all $t \in [0,1]$ the polynomial

$$p_t(z) = z^n + t\big(a_{n-1}z^{n-1} + \dots + a_1 z + a_0\big)$$

has no zeros on S^1_r, since

$$\left|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\right| < Rr^{n-1} \leq r^n, \quad \text{provided} \quad |z| = r.$$

Then

$$P(z,t) = \frac{p_t(z)}{|p_t(z)|}\Big|_{S^1_r}$$

is a homotopy between $p/|p|$ and f_n viewed as a map on S^1_r.

Step 3. *We prove the fundamental theorem of algebra.*

Assume p is a non-constant polynomial without zeros. Denote

$$q_r(z) = \frac{p(z)}{|p(z)|}\Big|_{S^1_r},$$

where $r \geq R$. Step 2 $\Longrightarrow$ $\deg q_r = n$. Step 1 $\Longrightarrow$ $\deg q_r = 0$, i.e., $n = 0$. Thus, p is a constant polynomial, which is a contradiction. $\square$

Chapter 2

Singular homology

2.1 Free abelian groups

An abelian group G is called free with a basis $A \subset G$, if $\forall g \in G$ there exists a unique representation $g = \sum_{a\in A} n_a a$, where $n_a \in \mathbb{Z}$ and $n_a \neq 0$ for finitely many $a \in A$ only.

Any set A generates an abelian group $F(A)$, which is free with a basis A. Indeed, define

$$F(A) := \big\{ f\colon A \to \mathbb{Z} \mid f(a) \neq 0 \text{ nur für endlich viele } a \in A \big\}.$$

Clearly, the functions

$$f_a(x) = \begin{cases} 1 & x = a, \\ 0 & \text{sonst}, \end{cases} \qquad a \in A$$

generate $F(A)$, that is $F(A)$ is free with a basis A.

Remark 2.1. For any $f \in F(A)$ we have

$$f = \sum_{a\in A} f(a) f_a.$$

In particular, $F(A)$ can be viewed as the group of all *finite* formal linear combinations $\sum_{a\in A} n_a a$, where $n_a \in \mathbb{Z}$.

2.2 Singular simplexes

Let $x_0, x_1, \dots, x_k$ be arbitrary points in $\mathbb{R}^n$ such that $x_1 - x_0, \dots, x_k - x_0$ are linearly independent.

Definition 2.2. The space

$$\Delta_k = \Delta(x_0, \dots, x_k) = \Big\{ x = \sum_{i=0}^{k} t_i x_i \mid t_i \in [0,1], \quad \sum_{i=0}^{k} t_i = 1 \Big\}$$

is called *the (non-degenerate) k-simplex generated by* $x_0, \dots, x_k$.

Example 2.3.
0) If $k = 0$, then $\Delta(x_0) = \{x_0\}$.
1) If $k = 1$, then $\Delta(x_0, x_1)$ is a segment $[x_0, x_1]$.
2) If $k = 2$, then $\Delta(x_0, x_1, x_3)$ is the triangle with the vertices x_0, x_1, x_2.
3) If $k = 3$, then $\Delta(x_0, x_1, x_3, x_4)$ is a tetrahedron with the vertices x_0, x_1, x_3, x_4.

Remark 2.4. The representation $x = \sum_{i=0}^{k} t_i x_i$ of a point in Δ_k is unique. Indeed, $\sum t_i x_i = \sum s_i x_i$, $\sum t_i = 1 = \sum s_i \implies$

$$0 = \sum (t_i - s_i)x_i = \sum (t_i - s_i)x_i - \sum (t_i - s_i)x_0 = \sum (t_i - s_i)(x_i - x_0) \implies t_i = s_i.$$

The coefficients $(t_0, t_1, \dots, t_k) \in [0,1]^{k+1}$ are called *the barycentric coordinates* of the point $x \in \Delta_k$. In particular, each k-simplex is homeomorphic to the standard k-simplex

$$\Delta^k := \Delta(e_1, \dots, e_k, e_{k+1}) \subset \mathbb{R}^{k+1},$$

where $e_1, \dots, e_{k+1}$ is the standard basis of $\mathbb{R}^{k+1}$.

It is customary to drop the adjective "non-degenerate" when referring to simplexes. Sometimes degenerate simplexes (in the sense that $x_1 - x_0, \dots, x_k - x_0$ may be linearly dependent) do appear below. Typically, this poses no problems, however the barycentric coordinates are ill defined in this case.

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From now on we pick one simplex in each dimension, for example the standard one.

Definition 2.5. Let X be a topological space. *A singular k-simplex* in X is a continuous map $f\colon \Delta^k \to X$.

In particular, a singular 0-simplex in X can be viewed as a point in X, a singular 1-simplex as a path in X etc.

Remark 2.6. The map f in the above definition does not need to be injective. In particular, the image of f may be (highly) singular.

For a singular k-simplex $f\colon \Delta^k \to X$ the $(k-1)$-simplex defined by

$$\partial^i f\colon \Delta^{k-1} \to X, \qquad \partial^i f(t_0, \dots, t_{k-1}) = f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1})$$

is called *the ith face* of f.

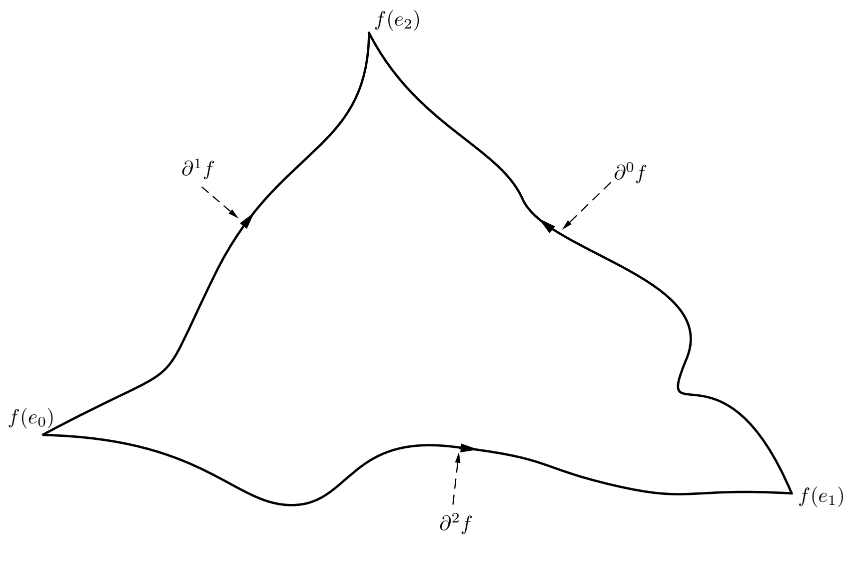

Figure 2.1: Faces of a singular simplex

Definition 2.7. Denote by $S_k(X)$ the free abelian group generated by all singular k-simplexes. Elements of $S_k(X)$ are formal linear combinations of the form

$$\sigma = \sum n_i f_i, \qquad n_i \in \mathbb{Z},$$

which are called *singular k-chains*. The $(k-1)$-chain

$$\begin{aligned} \partial f &= \partial^0 f - \partial^1 f + \partial^2 f - \cdots = \sum_{j=0}^{k} (-1)^j \partial^j f, \\ \partial \sigma &= \sum_i n_i \sum_j (-1)^j \partial^j f_i \end{aligned} \tag{2.8}$$

is called *the boundary* of f and σ respectively.

Proposition 2.9. *We have $\partial_{k-1} \circ \partial_k = 0$ (or, simply $\partial^2 = 0$) for all $k \geq 1$, i.e., the homomorphism*

$$S_k(X) \xrightarrow{\partial_k} S_{k-1}(X) \xrightarrow{\partial_{k-1}} S_{k-2}(X)$$

is trivial.

Proof. The proof consists of the following two steps.

Step 1. *Let f be a singular simplex. for each $j \geq i$ we have*

$$\partial^j \partial^i f = \partial^i \partial^{j+1} f.$$

Indeed,

$$\begin{aligned} \partial^j (\partial^i f)(t_0, \dots, t_{k-2}) &= \partial^i f(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}); \end{aligned}$$

$$\begin{aligned} \partial^i (\partial^{j+1} f)(t_0, \dots, t_{k-2}) &= \partial^{j+1} f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}). \end{aligned}$$

Step 2. *For each singular k-simplex we have $\partial(\partial f) = 0$.*

This follows from the following computation:

$$\begin{aligned} \partial(\partial f) &= \sum_{i=0}^{k} (-1)^i \partial^i (\partial f) = \sum_{i=0}^{k} \sum_{j=0}^{k} (-1)^{i+j} \partial^i \partial^j f = \sum_{j \geq i} + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{j \geq i} (-1)^{i+j} \partial^{j-1} \partial^i f + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{p+1 \geq q} (-1)^{p+q+1} \partial^p \partial^q f + \sum_{p>q} (-1)^{p+q} \partial^p \partial^q f \qquad p := j-1,\ q := i \\ &= 0. \end{aligned}$$

□

Corollary 2.10. $\operatorname{im} \partial_k \subset \ker \partial_{k-1}$.

The elements of $Z_{k-1}(X) := \ker \partial_{k-1}$ are called *cycles* and the elements of $B_{k-1}(X) := \operatorname{im} \partial_k$ are called *boundaries*.

Definition 2.11. The group

$$H_{k-1}(X) := \ker \partial_{k-1} / \operatorname{im} \partial_k = Z_{k-1}(X)/B_{k-1}(X)$$

is called the $(k-1)$ *th (singular) homology group* of X (with integer coefficients). In particular, $H_0(X) := S_0(X)/\operatorname{im} \partial_1$.

2.3 Some properties of the homology groups

Proposition 2.12.

$$X \text{ path connected} \implies H_0(X) \cong \mathbb{Z}.$$

Proof. $S_0(X)$ is the free abelian group generated by the points of X. Let f be a singular 1-simplex, that is $f\colon [0,1] \to X$ is a path in X. By the definition of the boundary, $\partial f = x_1 - x_0$, where $x_1 = f(1)$ and $x_0 = f(0)$. By the hypothesis, we can connect any two points in X by a path, that is for any two points $x_0, x_1 \in X$ we have $[x_0] = [x_1] \in H_0(X)$.

Furthermore, define the homomorphism $\alpha\colon S_1(X) \to \mathbb{Z}$ by

$$\alpha\Big(\sum n_i x_i\Big) = \sum n_i.$$

Since $\alpha(\partial f) = 0$ for each singular 1-simplex, α yields a surjective homomorphism $H_0(X) \to \mathbb{Z}$, which is still denoted by α.

Suppose $\alpha([\sum n_i x_i]) = 0$. Then $[\sum n_i x_i] = \sum n_i [x_i] = \big(\sum n_i\big)[x_0] = 0$, that is α is injective. Thus, α is an isomorphism. □

Exercise 2.13. If X is not necessarily path connected, then the following holds: $H_0(X) \cong \mathbb{Z}^m$, where m is the number of path-components of X.

Proposition 2.14.

$$H_k(\{pt\}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{else.} \end{cases}$$

Proof. For $k = 0$ the statement of this proposition follows from the previous one. Hence, we may assume $k > 0$. For each such k there is exactly one k-simplex in $\{pt\}$, namely the constant map, which we denote by $c_k\colon \Delta^k \to \{pt\}$. For the boundary we have

$$\partial c_k = \sum_{i=0}^{k} (-1)^i \underbrace{c_k \circ d_i}_{c_{k-1}} = \begin{cases} 0, & \text{for } k \text{ odd}, \\ c_{k-1} & \text{for } k \text{ even}. \end{cases}$$

Hence,

$$Z_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even} \end{cases}$$

und

$$B_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even}. \end{cases}$$

Thus $H_k\big(\{pt\}\big) = Z_k\big(\{pt\}\big)/B_k\big(\{pt\}\big) = 0$. □

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Definition 2.15. A topological space X is said to be *contractible* if there is a point $x_0 \in X$ such that the identity map id_X is homotopic to the constant map c_{x_0}.

Proposition 2.16. *For a contractible space X we have*

$$H_k(X) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

Proof. Since X is contractible, there exists a continuous map $h\colon X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ hold for any $x \in X$. In particular, for a fixed $x \in X$ the path $t \mapsto h(t,x)$ connects x and x_0. This implies that X is path connected, hence $H_0(X) \cong \mathbb{Z}$ by Proposition 2.12.

Thus, we assume $k \geq 1$ in the sequel. Consider the quotient map

$$\begin{aligned}\pi\colon \Delta^{k-1} \times [0,1] &\to \Delta^k \cong (\Delta^{k-1} \times [0,1])/(\Delta^{k-1} \times \{1\})\\ ((t_0,\dots,t_{k-1}),u) &\mapsto (u,(1-u)t_0,\dots,(1-u)t_{k-1}).\end{aligned}$$

Let $h\colon X \times [0,1] \to X$ be a homotopy between id_X and c_{x_0}. Define $s\colon S_{k-1}(X) \to S_k(X)$ as follows: Since π is a quotient map and $h|_{X\times\{1\}} \equiv x_0$, for each singular $(k-1)$-simplex $\sigma\colon \Delta^{k-1} \to X$ there exists a unique map $s(\sigma)\colon \Delta^k \to X$ such that $h \circ (\sigma \times \mathrm{id}) = s(\sigma) \circ \pi$. More explicitly,

$$s(\sigma)(t_0,t_1,\dots,t_k) = h\Big(\sigma\Big(\frac{t_1}{1-t_0},\dots,\frac{t_k}{1-t_0}\Big),\ t_0\Big)$$

whenever $t_0 \neq 1$ and $s(\sigma)(t_1,\dots,t_k,1) = x_0$. Hence,

1. $\partial^0(s(\sigma)) = \sigma$,
2. $\partial^i s(\sigma) = s(\partial^{i-1}\sigma)$ for $i > 0$.

Therefore, for any $\sigma \in S_k(X)$ we have

$$\partial(s(\sigma)) = \partial^0(s(\sigma)) - \sum_{i=1}^{k}(-1)^{i-1}\partial^i(s(\sigma)) = \sigma - \sum_{j=0}^{k-1}(-1)^j s(\partial^j\sigma) = \sigma - s(\partial\sigma). \tag{2.17}$$

This yields

$$\partial \circ s + s \circ \partial = \mathrm{id}.$$

Hence, if σ is a cycle, then $\sigma = \partial\big(s(\sigma)\big) + s\big(\partial\sigma\big) = \partial\big(s(\sigma)\big)$, i.e., any cycle is a boundary. In other words, $H_k(X) = 0$ whenever $k \geq 1$ as claimed. □

Theorem 2.18. *Let $f\colon X \to Y$ be a continuous map. Then for each $k \geq 0$ the map f induces a group homomorphism*

$$f_*\colon H_k(X) \to H_k(Y)$$

and for any other continuous map $g\colon Y \to Z$ we have

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, $(\mathrm{id}_X)_* = \mathrm{id}$.

Proof. Define first group homomorphisms $f_\#\colon S_k(X) \to S_k(Y)$, by declaring

$$\sigma \mapsto f \circ \sigma \quad \text{for} \quad \sigma\colon \Delta^k \to X.$$

Then for all singular k-simplexes $\sigma\colon \Delta^k \to X$ we have

$$\begin{aligned}(f_\#\partial^i(\sigma))(t_0,\dots,t_{k-1}) &= f(\sigma(t_0,\dots,t_{i-1},0,t_i,\dots,t_{k-1}))\\ &= (f_\#\sigma)(t_0,\dots,t_{i-1},0,t_i,\dots,t_{k-1})\\ &= \partial^i(f_\#\sigma)(t_0,\dots,t_{k-1}),\end{aligned}$$

and therefore $f_\# \partial^i = \partial^i f_\#$, which yields in turn that $f_\#$ is a *chain map*, i.e.,

$$f_\# \partial = \partial f_\#.$$

This yields in particular that cycles are mapped to cycles and boundaries are mapped to boundaries:

$$f_\#(Z_k(X)) \subset Z_k(Y) \qquad \text{and} \qquad f_\#(B_k(X)) \subset B_k(Y).$$

Hence, we obtain a well defined group homomorphism:

$$\begin{gathered} f_*\colon H_k(X) = Z_k(X)/B_k(X) \to Z_k(Y)/B_k(Y) = H_k(Y) \\ f_*([\sigma]) := [f_\#(\sigma)]. \end{gathered}$$

Furthermore, for each singular k-simplex $\sigma\colon \Delta^k \to X$ we have

$$\begin{aligned} & g_\# \circ f_\#(\sigma) = g_\#(f \circ \sigma) = g \circ f \circ \sigma = (g \circ f)_\#(\sigma), \\ & g_* \circ f_*([\sigma]) = g_*[f_\#(\sigma)] = [g_\# \circ f_\#(\sigma)] = [(g \circ f)_\#(\sigma)] = (g \circ f)_*([\sigma]), \\ & (\mathrm{id}_X)_\#(\sigma) = \sigma, \\ & (\mathrm{id}_X)_*([\sigma]) = [(\mathrm{id}_X)_\#(\sigma)] = [\sigma]. \end{aligned}$$

Therefore, $g_* \circ f_* = (g \circ f)_*$ and $(\mathrm{id}_X)_* = \mathrm{id}$. □

Corollary 2.19. *If $f\colon X \to Y$ is a homeomorphism, then $f_*\colon H_k(X) \to H_k(Y)$ is an isomorphism for each k.* □

2.4 Homotopies and homology groups

Satz 2.20. *If $f, g\colon X \to Y$ are homotopic maps, then the induced maps on the homology groups are equal:*

$$f \simeq g \qquad \Longrightarrow \qquad f_* = g_*.$$

Proof. The proof consists of the following three steps.

Step 1. *Define*

$$\eta_t\colon X \to X \times I, \qquad \eta_t(x) = (x, t).$$

For each continuous map $f\colon X \to Y$ we have $(f \times id)_\# \eta^X_{t\#} = \eta^Y_{t\#} \circ f_\#$.

This follows immediately from the observation that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_t^X} & X \times I \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f\times id} \\ Y & \xrightarrow{\eta_t^Y} & Y \times I \end{array}$$

commutes.

Step 2. *There exists a sequence of homomorphisms $s^X_k\colon S_k(X) \to S_{k+1}(X \times I)$ satisfying*

$$\partial s^X_k + s^X_{k-1}\partial = \eta_{1\#} - \eta_{0\#}; \tag{2.21}$$

$$(f \times id_I)_\# \circ s^X_k = s^Y_k \circ f_\#. \tag{2.22}$$

Define $s_k = s_k^X$ recursively. For $k = 0$ and $x_0 \in X$, which we view as a 0-simplex, put

$$s_0\sigma \colon \Delta^1 \to X \times I, \qquad (t_0, t_1) \mapsto (x_0, t_1).$$

Then we have $\partial(s_0\sigma) = (x_0, 1) - (x_0, 0)$, i.e., (2.21) holds for $k = 0$. Equation (2.22) follows directly from the definition of s_0.

Suppose s_ℓ have been defined for all $\ell < k$. We define first s_k in a special case, namely for id_{Δ^k} viewed as an element $\imath_k \in S_k(\Delta^k)$. We have

$$\begin{aligned}
\partial\big(\underbrace{\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k}_{\in S_k(\Delta^k \times I)}\big) &= \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \partial s_{k-1}\partial\imath_k \\
&\overset{(2.21)}{=} \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \big(\eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - s_{k-2}^{\Delta^k}\partial^2\imath_k\big) \\
&= 0.
\end{aligned}$$

In this computation (2.21) is used for k replaced by $k-1$. Since $\Delta^k \times I$ is contractible, there exists some $a \in S_{k-1}(\Delta^k \times I)$ so that

$$\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k = \partial a.$$

Define $s_k(\imath_k) = a$. Then (2.21) holds for $\sigma = \imath_k$.

In general, define $s_k^X(\sigma) = (\sigma \times id)_\# a$. Then we have

$$\begin{aligned}
\partial(s_k^X\sigma) &= \partial(\sigma \times id)_\# a = (\sigma \times id)_\#\partial a \\
&= (\sigma \times id)_\#\big(\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}^{\Delta^k}\partial\imath_k\big) \\
&= \eta_{1\#}\sigma_\#\imath_k - \eta_{0\#}\sigma_\#\imath_k - s_{k-1}^X\sigma_\#\partial\imath_k && (2.22) + \text{Step } 1 \\
&= \eta_{1\#}\sigma - \eta_{0\#}\sigma - s_{k-1}^X\partial\sigma.
\end{aligned}$$

This proves (2.21).

We still have to show that (2.22) holds. Indeed,

$$(f \times id)_\# s_k\sigma = (f \times id)_\#(\sigma \times id)_\# a = \big((f \circ \sigma) \times id\big)_\# a = s_k(f\sigma) = s_k(f_\#\sigma).$$

Step 3. *We prove this theorem.*

Let h be a homotopy between f and g. From the following equalities

$$\partial(h_\# \circ s_k) + (h_\# \circ s_{k-1})\partial = h_\#\partial s_k + h_\#(s_{k-1}\partial) = h_\#(\eta_{1\#} - \eta_{0\#}) = f_\# - g_\#$$

we see that $f_\# - g_\# = \partial(h_\# \circ s_k)$ holds on $\ker\partial$. This shows that $f_* = g_*$. $\square$

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Definition 2.23. A continuous map $f \colon X \to Y$ is called a homotopy equivalence, if there exists a continuous map $g \colon Y \to X$ such that the following holds:

$$g \circ f \simeq id_X \qquad \text{and} \qquad f \circ g \simeq id_Y.$$

In this case the spaces X and Y are called homotopy equivalent.

Example 2.24. (i) Any two homeomorphic spaces are homotopy equivalent.

(ii) $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$. More generally, any contractible space is homotopy equivalent to $\{pt\}$.

(iii) $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}.

To see (ii), let X be a contractible space and $\imath_{x_0} \colon \{x_0\} \to X$ be the embedding of the point x_0. Then $c_{x_0} \circ \imath_{x_0} = id_{x_0}$ and $\imath_{x_0} \circ c_{x_0} \simeq id_X$.

To see (iii), define $f \colon \mathbb{R}^n \setminus \{0\} \to S^n$ by $f(x) = x/|x|$. If $g \colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ denotes the inclusion, then $f \circ g = id_{S^{n-1}}$. Furthermore,

$$h(x,t) = \frac{1}{t + (1-t)|x|} x, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

is a homotopy between $g \circ f$ and $id_{\mathbb{R}^n \setminus \{0\}}$.

Corollary 2.25.

$$f \text{ is a homotopy equivalence} \quad \Longrightarrow \quad \forall k \quad f_* \colon H_k(X) \to H_k(Y) \text{ is an isomorphism.}$$

Example 2.26.

$$H_k(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_k(\mathbb{R}^n \setminus \{pt\}) = H_k(S^{n-1}) = \begin{cases} \mathbb{Z} & k = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

2.5 Exact sequences and the Bockstein homomorphism

Definition 2.27. A sequence of homomorphisms of abelian groups

$$\dots \longrightarrow A_{k+1} \xrightarrow{\alpha_{k+1}} A_k \xrightarrow{\alpha_k} A_{k-1} \longrightarrow \dots \tag{2.28}$$

is called exact, if for all k the following holds: $\ker \alpha_k = \operatorname{im} \alpha_{k+1}$.

Some special cases:

(i) $0 \to A \xrightarrow{\alpha} B$ is exact $\quad \Leftrightarrow \quad$ α is injectiv;

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact $\quad \Leftrightarrow \quad$ α is surjectiv;

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\quad \Leftrightarrow \quad$ α is an isomorphism;

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact $\quad \Leftrightarrow \quad$ α is injectiv, β is surjectiv and $\ker \beta = \operatorname{im} \alpha$; In particular, β induces an isomorphism $C \cong B/A$.

The sequence (iv) is called a short exact sequence.

Example 2.29. $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a short exact sequence, where $\times n$ stands for the multiplication with a fixed $n \in \mathbb{Z}$.

Let A be a complex, that is A is a sequence

$$A: \qquad \cdots \longrightarrow A_{i+1} \xrightarrow{\partial} A_i \xrightarrow{\partial} A_{i-1} \longrightarrow \ldots$$

such that $\partial^2 = 0$. Just like in the case of chain complexes, we define the kth homology group of A to be

$$H_k(A) := \frac{\ker\left(\partial\colon A_k \to A_{k-1}\right)}{\operatorname{im}\left(\partial\colon A_{k+1} \to A_k\right)}.$$

If A, B, and C are complexes, a sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of complexes is a commutative diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r, "\partial"] & A_{k+1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k+1}"'] & A_k \arrow[r, "\partial"] \arrow[d, "\alpha_k"'] & A_{k-1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k-1}"'] & \ldots \\
\ldots \arrow[r, "\partial"] & B_{k+1} \arrow[r, "\partial"] \arrow[d, "\beta_{k+1}"'] & B_k \arrow[r, "\partial"] \arrow[d, "\beta_k"'] & B_{k-1} \arrow[r, "\partial"] \arrow[d, "\beta_{k-1}"'] & \ldots \\
\ldots \arrow[r, "\partial"] & C_{k+1} \arrow[r, "\partial"] \arrow[d] & C_k \arrow[r, "\partial"] \arrow[d] & C_{k-1} \arrow[r, "\partial"] \arrow[d] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.30}$$

Such a sequence is called *exact*, if each vertical sequence $0 \to A_k \to B_k \to C_k \to 0$ is exact.

Here of course we could equally well consider sequences of complexes consisting of more than 3 complexes.

Example 2.31. Let X, Y and Z be topological spaces and $f\colon X \to Y$, $g\colon Y \to Z$ continuous maps. Then one obtains a sequence of chain complexes

$$0 \to S_*(X) \xrightarrow{f_\#} S_*(Y) \xrightarrow{g_\#} S_*(Z) \to 0,$$

which is not necessarily exact. What conditions guarantee that the above sequence is exact will be considered below.

Proposition 2.32. *The maps α and β yield homomorphisms $\alpha\colon H_*(A) \to H_*(B)$ and $\beta\colon H_*(B) \to H_*(C)$ respectively.*

Proof. This follows immediately from the commutativity of (2.30). □

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Theorem 2.33. *A short exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ induces a (long) exact sequence of homology groups:*

$$\cdots \to H_k(A) \xrightarrow{\alpha} H_k(B) \xrightarrow{\beta} H_k(C) \xrightarrow{\delta} H_{k-1}(A) \xrightarrow{\alpha} H_{k-1}(B) \to \ldots$$

Remark 2.34. The map δ is called the Bockstein homomorphism.

Proof. The proof consists of the following four steps.

Step 1. *We define* δ.

Pick $c \in C_k$, $\partial c = 0$. Since β_k is surjective, there exists some $b \in B_k$ such that $\beta(b) = c$. We have $\beta(\partial b) = \partial(\beta(b)) = \partial c = 0$. Since $\alpha\colon A_{k-1} \to \ker \beta_{k-1}$ is surjective, there is some $a \in A_{k-1}$ such that $\alpha(a) = \partial b$. Define

$$\delta[c] = [a].$$

We have to show that δ is well defined. Indeed, pick another representative $c' = c + \partial c''$ of the class $[c]$. For $c'' \in C_{k+1}$ there is some $b'' \in B_{k+1}$ such that $\beta(b'') = c'' \implies \beta(b + \partial b'') = c + \partial c''$. This yields $b' = b + \partial b'' + \alpha(a'')$, where $a'' \in A_k$. Furthermore, $\partial b' = \partial b + 0 + \alpha(\partial a')$. Since α is injective, we have $a' = a + \partial a''$, i.e., $[a] = [a']$.

Exercise 2.35. Check that δ is a group homomorphism.

Step 2. $\ker \alpha = \operatorname{im} \delta$.

Pick $a \in A_{k-1}$ such that $[a] \in \ker \alpha$, i.e., $\alpha(a) = \partial b$ for some $b \in B_k$. We have $\partial \beta(b) = \beta(\partial b) = \beta(\alpha(a)) = 0$. By the construction of δ, we obtain $\delta[\beta(b)] = [a]$. That is $\ker \alpha \subset \operatorname{im} \delta$.

If $a \in A_{k-1}$ is such that $[a] \in \operatorname{im} \delta$, then by the construction of δ, we have $\alpha(a) = \partial b \implies \alpha[a] = 0$.

Step 3. $\ker \delta = \operatorname{im} \beta$.

Pick some $[c] \in \ker \delta$. Using the notations of Step 1, we have $a = \partial a'$ for some $a' \in A_k$. The equations

$$\begin{aligned} \partial\big(b - \alpha(a')\big) &= \partial b - \alpha(\partial a') = \partial b - \alpha(a) = 0; \\ \beta\big(b - \alpha(a')\big) &= \beta(b) = c; \end{aligned}$$

yield $\beta[b - \alpha(a')] = [c]$, i.e., $\ker \delta \subset \operatorname{im} \beta$.

The inclusion $\operatorname{im} \beta \subset \ker \delta$ follows immediately from the construction of δ.

Step 4. $\ker \beta = \operatorname{im} \alpha$.

Assume $b \in B_k$ satisfies $\beta[b] = 0$, that is $\partial b = 0$ and $\beta(b) = \partial c$ for some $c \in C_{k+1}$. Since β is surjective, there is some $\hat{b} \in B_{k+1}$ such that $\beta(\hat{b}) = c$. Furthermore,

$$\beta(b - \partial \hat{b}) = \beta(b) - \partial \beta(\hat{b}) = \beta(b) - \partial c = 0.$$

This yields that there exists some $a \in A_k$ such that $\alpha(a) = b - \partial \hat{b}$. Moreover,

$$\alpha(\partial a) = \partial \alpha(a) = \partial b - \partial^2 \hat{b} = 0.$$

Since α is injective, we obtain $\partial a = 0$. This yields $\alpha[a] = [b - \partial \hat{b}] = [b]$, that is $\ker \beta \subset \operatorname{im} \alpha$.

The inclusion $\operatorname{im} \alpha \subset \ker \beta$ follows immediately from $\alpha \circ \beta = 0$. □

2.6 Relative homology groups

For each subspace $A \subset X$ define

$$S_n(X, A) := S_n(X)/S_n(A).$$

The boundary map on $S_n(X)$ induces a boundary map on $S_n(X, A)$ and we obtain the following new chain complex:

$$\cdots \to S_{n+1}(X, A) \xrightarrow{\partial} S_n(X, A) \xrightarrow{\partial} S_{n-1}(X, A) \to \dots$$

The homology groups of this complex are denoted by $H_*(X, A)$ and are called *the homology groups of X relative to A*, or, simply, *relative homology groups*. Let us provide some details of this definition:

- Elements of $H_n(X, A)$ are represented by *relative chains* $a \in S_n(X)$ such that $\partial a \in S_{n-1}(A)$;
- $[a] = 0 \in H_n(X, A) \quad \iff \quad a = \partial b + c, \quad b \in S_{n+1}(X),\ c \in S_n(A)$.

By the very definition of $S_n(X, A)$, the sequence $0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$ is exact. Hence, Theorem 2.33 yields the following:

Theorem 2.36. *There is a long exact sequence of the homology groups*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \dots$$

Moreover, the following holds:

- *i_* is induced by the inclusion $i\colon A \subset X$;*
- *j_* is induced by the projection $S_n(X) \to S_n(X, A)$;*
- $\delta[a] = [\partial a]$.

□ L 6

Suppose $A \subset X$ and $B \subset Y$. A map between pairs of spaces (X, A) and (Y, B) is a map $f\colon X \to Y$ such that $f(A) \subset B$.

Proposition 2.37. *Each map $f\colon (X, A) \to (Y, B)$ induces a homomorphism of relative homology groups $H_*(X, A) \to H_*(Y, B)$.* □

Two continuous maps $f, g\colon (X, A) \to (X, B)$ are called homotopic (as maps between pairs of spaces), if there exists a continuous map $h\colon (X \times I, A \times I) \to (Y, B)$, such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. Notice that the homotopy h in this definition satisfies $h(A \times I) \subset B$.

Two pairs (X, A) and (Y, B) are said to be homotopy equivalent, if there exist $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, A)$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$, where id_X is viewed as a map of pairs $(X, A) \to (X, A)$ (and similarly for id_Y). Just like in the situation of Corollary 2.25, we have the following result.

Proposition 2.38. *If (X, A) and (Y, B) are homotopy equivalent, then $H_k(X, A)$ and $H_k(Y, B)$ are isomorphic for all k.* □

The following theorem, whose proof will be given in Section **??** below, turns out to be a useful tool for the computations of relative homology groups. For the time being, we take Theorem 2.39 as granted. Ref

Theorem 2.39 (Excision)**.** *Assume the subspaces $Z \subset A \subset X$ satisfy $\bar{Z} \subset \operatorname{Int} A$. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism of relative homology groups:*

$$H_*(X \setminus Z, A \setminus Z) \cong H_*(X, A).$$

2.7 The homology groups of the spheres

Theorem 2.40. *The following holds:*

$$H_k(S^0) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } k=0;\\ 0 & \text{else;}\end{cases} \qquad \text{and for } n\geq 1 \qquad H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k=0,n;\\ 0 & \text{else.}\end{cases}$$

Proof. Denote

$$S^n = \{x=(x_0,\dots,x_{n+1})\in S^{n+1} \mid x_{n+1}=0\},$$
$$S^{n+1}_+ := \{x\in S^{n+1} \mid x_{n+1}\geq 0\}, \qquad S^{n+1}_- := \{x\in S^{n+1} \mid x_{n+1}\leq 0\}.$$

Notice that $S^{n+1}_\pm$ is homeomorphic to $B_{n+1}=\{x\in\mathbb{R}^{n+2} \mid |x|\leq 1, x_{n+1}=0\}$. In particular, $S^{n+1}_\pm$ is contractible.

Step 1. *The map* $\delta\colon H_{k+1}(S^{n+1}_-, S^n)\to H_k(S^n)$ *is an isomorphism provided* $k\geq 1$.

By the long exact sequence of the pair (S^{n+1}_-, S^n) we have

$$0 = H_{k+1}(S^{n+1}_-) \to H_{k+1}(S^{n+1}_-, S^n) \xrightarrow{\delta} H_k(S^n) \to H_k(S^{n+1}_-) = 0. \tag{2.41}$$

Hence, δ is an isomorphism.

Step 2. *Define*

$$\tilde H_0(S^n) := \ker\big(H_0(S^n)\to H_0(S^{n+1}_-)\big) \cong \begin{cases}\mathbb{Z} & \text{if } n=0,\\ 0 & \text{else.}\end{cases}$$

Then $\delta\colon H_1(S^{n+1}_-, S^n)\to \tilde H_0(S^n)$ *is an isomorphism.*

Recall that for a connected space X a generator of $H_0(X)$ is the class of any point. Hence, if $n>0$, then the homomorphism $H_0(S^n)\to H_0(S^{n+1}_-)$ induced by the inclusion is in fact an isomorphism. In particular, $\tilde H_0(S^n)=0$ in this case. However, if $n=0$, S^0 consists of two points (in particular, has two connected components), whereas S^1_- is connected. Hence, the inclusion $\{-1\}\to S^1_-$ induces an isomorphism on H_0, however, the generator corresponding to the point $+1\in S^0$ is in the kernel of the homomorphism $H_0(S^n)\to H_0(S^{n+1}_-)$. In particular, $\tilde H_0(S^0)\cong\mathbb{Z}$.

Furthermore, just like in the previous step, the long exact sequence of the pair (S^{n+1}_-, S^n) yields

$$0 = H_1(S^{n+1}_-) \to H_1(S^{n+1}_-, S^n) \xrightarrow{\delta} H_0(S^n) \to H_0(S^{n+1}_-).$$

In particular, δ is injective and, hence, an isomorphism onto its image in $H_0(S^n)$, which is the kernel of $H_0(S^n)\to H_0(S^{n+1}_-)$, that is $\tilde H_0(S^0)$.

Step 3. *For all* $k\geq 0$ *and* $n\geq 0$ *the map*

$$j_*\colon H_{k+1}(S^{n+1}) \to H_{k+1}(S^{n+1}, S^{n+1}_+) \tag{2.42}$$

is an isomorphism.

For $k > 0$, this follows from the long exact sequence of the pair (S^{n+1}, S_+^{n+1}):

$$0 = H_{k+1}(S_+^{n+1}) \to H_{k+1}(S^{n+1}) \xrightarrow{j_*} H_{k+1}(S^{n+1}, S_+^{n+1}) \to H_k(S_+^{n+1}) = 0$$

For $k = 0$, we have

$$0 = H_1(S_+^{n+1}) \to H_1(S^{n+1}) \xrightarrow{j_*} H_1(S^{n+1}, S_+^{n+1}) \to \underbrace{H_0(S_+^{n+1}) \to H_0(S^{n+1})}_{\text{isomorphism}} = \mathbb{Z}.$$

Hence, the third arrow represents the zero homomorphism and, therefore, j_* is surjective. Since j_* is injective, this is an isomorphism.

Step 4. *For all $k \geq 0$ the inclusion $p\colon (S_-^{n+1}, S^n) \cong (S^{n+1}, S_+^{n+1})$ induces the isomorphism*

$$p_*\colon H_{k+1}(S_-^{n+1}, S^n) \to H_{k+1}(S^{n+1}, S_+^{n+1}). \tag{2.43}$$

Indeed, denote

$$Z := \left\{x \in S^{n+1} \mid x_{n+2} \geq \tfrac{1}{2}\right\}.$$

Then the homomorphism $H_{k+1}(S_-^{n+1}, S^n) \to H_{k+1}(S^{n+1} \setminus Z,\ S_+^{n+1} \setminus Z)$ induced by the inclusion $(S_-^{n+1}, S^n) \to (S^{n+1} \setminus Z,\ S_+^{n+1} \setminus Z)$ is an isomorphism, since the pairs (S_-^{n+1}, S^n) and $(S^{n+1} \backslash Z,\ S_+^{n+1} \backslash Z)$ are homotopy equivalent. Theorem 2.39 yields that the homomorphism $H_{k+1}(S^{n+1}, S_+^{n+1}) \to H_{k+1}(S^{n+1} \backslash Z,\ S_+^{n+1} \backslash Z)$ induced by the inclusion is also an isomorphism. This proves (2.43).

Step 5. *We prove this theorem*

A combination of the previous steps yields the sequence of isomorphisms

$$H_{k+1}(S^{n+1}) \xrightarrow{j_*} H_{k+1}(S^{n+1}, S_+^{n+1}) \xrightarrow{p_*^{-1}} H_{k+1}(S_-^{n+1}, S^n) \xrightarrow{\delta} \tilde{H}_k(S^n),$$

where

$$\tilde{H}_k(S^n) = \begin{cases} \tilde{H}_0(S^n), & \text{if } k = 0, \\ H_k(S^n), & \text{if } k > 0. \end{cases}$$

This implies the statement of this theorem. □

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Corollary 2.44. *The n-sphere S^n is not contractible for all $n \geq 0$.* □

For a general topological space X define also

$$\tilde{H}_0(X) := \ker \varepsilon, \qquad \text{wobei} \quad \varepsilon\colon H_0(X) \to \mathbb{Z}, \quad \varepsilon\left[\sum n_i x_i\right] := \sum n_i,$$

and $\tilde{H}_k(X) = H_k(X)$ für $k \geq 1$. Using these notations we have

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n; \\ 0 & \text{else,} \end{cases}$$

for all n.

2.8 The hairy ball theorem

Recall (cf. Definition 1.18) that the degree $\deg f$ of a continuous map $f\colon S^n \to S^n$ is an integer, which is determined by the property

$$f_* a = (\deg f)\cdot a \qquad \text{for all } a \in H_n(S^n).$$

Define *the suspension* $\Sigma f\colon S^{n+1} \to S^{n+1}$ of f via

$$\Sigma f(x_0,\dots,x_{n+1}) = \begin{cases} (0,\dots,0,x_{n+1}) & \text{if } |x_{n+1}| = 1,\\ \big(tf(\frac{x_0}{t},\dots,\frac{x_n}{t}), x_{n+1}\big) & \text{if } |x_{n+1}| < 1, \end{cases}$$

where $t = \sqrt{1 - x_{n+1}^2}$.

Proposition 2.45. $\deg \Sigma f = \deg f$.

Proof. By the proof of Theorem 2.40 we have the following commutative diagram

$$\begin{tikzcd} H_{n+1}(S^{n+1}) \arrow[r, "j_*"] \arrow[d, "\Sigma f_*"'] & H_{n+1}(S^{n+1}, S^{n+1}_+) \arrow[r, "p_*^{-1}"] \arrow[d, "\Sigma f_*"'] & H_{n+1}(S^{n+1}_-, S^n) \arrow[r, "\delta"] \arrow[d, "\Sigma f_*"'] & H_n(S_n) \arrow[d, "f_*"'] \\ H_{n+1}(S^{n+1}) \arrow[r, "j_*"] & H_{n+1}(S^{n+1}, S^{n+1}_+) \arrow[r, "p_*^{-1}"] & H_{n+1}(S^{n+1}_-, S^n) \arrow[r, "\delta"] & H_n(S_n). \end{tikzcd}$$

Denoting $\alpha := \delta \circ p_*^{-1} \circ j_*$, we obtain

$$\Sigma f_*(a) = \alpha^{-1}\circ f_* \circ \alpha(x) = \alpha^{-1}\big((\deg f)\cdot \alpha(a)\big) = (\deg f)\cdot a \implies \deg \Sigma f = \deg f.$$

$\square$

Theorem 2.46. *There is no continuous map $f\colon S^{2n} \to \mathbb{R}^{2n+1}\setminus\{0\}$ such that $f(x) \perp x$ holds for all $x \in S^{2n}$.*

Proof. The proof consists of the following steps.

Step 1. *Let*

$$s_0\colon S^n \to S^n, \qquad (x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n),$$

be the restriction of the reflection in the hyperplane $\{x_0 = 0\}$. Then $\deg s_0 = -1$.

The sequence of isomorphisms

$$H_1(S^1) \xrightarrow{j_*} H_1(S^1, S^1_+) \xrightarrow{p_*^{-1}} H_1(S^1_-, S^0) \xrightarrow{\delta} \tilde H_0(S_0)$$

shows that

$$\sigma(t) = (\sin 2\pi t, \cos 2\pi t)$$

is a generator of $H_1(S^1)$. Since $s\circ\sigma(t) = \sigma(-t)$, we have $s_*[\sigma] = -[\sigma]$ and therefore the claim of this step holds for $n = 1$.

If s_0 is the reflection on S^n, then Σs_0 is the reflection on S^{n+1}. The induction with respect to n yields the proof for all $n > 1$.

Step 2. *For the antipodal map $A\colon S^n \to S^n,\ A(x) = -x$ we have* $\deg A = (-1)^{n+1}$.

The antipodal map on S^n is the composition of $n+1$ reflections.

Step 3. *If $f\colon S^n \to S^n$ is a continuous map without fixed points, then $f \simeq A$.*

The map
$$F(x,t) := \frac{tf(x) + (t-1)x}{|tf(x) + (t-1)x|}$$
is a well-defined homotopy between f and A.

Step 4. *If $f\colon S^n \to S^n$ is a continuous map such that $f(x) \neq -x$ for all $x \in S^n$, then f is homotopic to the identity map.*

$$\begin{aligned} f(x) \neq -x \implies & A \circ f \text{ has no fixed points } \implies A \circ f \simeq A \implies A \circ A \circ f \simeq A \circ A \\ \implies & \quad f \simeq id. \end{aligned}$$

Step 5. *We prove the hairy ball theorem.*

Assume there exists a continuous map $f\colon S^{2n} \to \mathbb{R}^{2n+1} \setminus \{0\}$ such that $f(x) \perp x$. By renormalizing we can assume without loss of generality that $f\colon S^{2n} \to S^{2n}$. The assumption $f(x) \perp x$ yields in particular that f has no fixed points. By Step 3, f is homotopic to A.
On the other hand, f is homotopic to id by Step 4. This yields a contradiction since
$$A \simeq f \simeq id \implies 1 = \deg id = \deg A = (-1)^{2n+1} = -1.$$
□

This theorem is often informally formulated as follows.

Corollary 2.47. *One can not comb a hairy ball flat without creating a cowlick.* □

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Remark 2.48. Each sphere of odd dimension $2n-1 \geq 1$ admits a continuous map $f\colon S^{2n-1} \to \mathbb{R}^{2n} \setminus \{0\}$ such that $f(x) \perp x$ holds for all $x \in S^{2n-1}$. Indeed,
$$\begin{aligned} S^{2n-1} &= \left\{x = (x_0, x_1, x_2, x_3, \ldots, x_{2n-2}, x_{2n-1}) \mid \sum x_i^2 = 1\right\} \\ f(x) &= (x_1, -x_0, x_3, -x_2, \ldots, x_{2n-1}, -x_{2n-2}). \end{aligned}$$

Proposition 2.49. *Let $[S^n, S^n]$ be the set of all homotopy classes of continuous maps $S^n \to S^n$, where $n \geq 1$. The map*
$$[S^n, S^n] \to \mathbb{Z}, \qquad [f] \mapsto \deg f \tag{2.50}$$
is surjective.

Proof. If $n = 1$, for each $k \in \mathbb{Z}$ we have an explicit continuous map $f_k\colon S^1 \to S^1$ of degree k, namely $f_k(z) := z^k$. If $n = 2$, we have $\deg \Sigma f_k = \deg f_k = k$. The induction with respect to n finishes the proof. □

Remark 2.51. It can be shown that (2.50) is even bijective (Theorem of Hopf). Also, $[S^n, S^n]$ is a group and (2.50) is an isomorphism of groups.

2.9 Group actions on the spheres

Let G be a group. We say that G acts on a set X if a homomorphism $\rho\colon G \to \operatorname{Aut}(X)$ is given, where $\operatorname{Aut}(X)$ is the group of all bijective maps $X \to X$. An action is called *free* whenever the following holds:

$$\forall x \in X \quad \operatorname{Stab}_x := \{g \in G \mid \rho(g)(x) = x\} = \{e\}.$$

If X is in addition a topological space, then we require also that for each $g \in G$ the map $\rho(g)$ is a homeomorphism.

Theorem 2.52. *$\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that acts freely on S^{2n}.*

Proof. Assume that $G \neq \{e\}$ acts on S^{2n} freely. Consider the map

$$d\colon G \to \{\pm 1\}, \qquad d(g) = \deg(\rho(g)).$$

Here d takes values in $\{\pm 1\}$, since each $\rho(g)$ is a homeomorphism. Furthermore, $d(gh) = \deg(\rho(g)\rho(h)) = d(g)d(h)$, that is d is a group homomorphism.

If $g \neq e$, then $\rho(g)$ has no fixed points. By Step 4 in the proof of Theorem 2.46, the following holds: $\deg \rho(g) = \deg A = -1$, i.e., d has a trivial kernel and is surjective.

Clealy $\mathbb{Z}/2\mathbb{Z}$ acts freely on S^{2n}:

$$\rho(e) = \mathrm{id}, \qquad \rho(1) := A,$$

where A is the antipodal map. $\square$

Remark 2.53. On the odd-dimensional spheres other non-trivial groups may act freely. For example, $\mathrm{U}(1) := \{z \in \mathbb{C} \mid |z| = 1\} \cong S^1$ acts on

$$S^{2n-1} = \{(z_0, \dots, z_n) \in \mathbb{C}^n \mid \sum |z_j|^2 = 1\}$$

via the homomorphism

$$w \mapsto f_w, \qquad f_w(z) = (wz_0, \dots, wz_n).$$

2.10 Homology groups of graphs

Definition 2.54. A (finite topological) graph is a pair (G, V), where G is a Hausdorff space and $G \supset V$ is a finite subset. The elements of V are called vertices of G. Besides, we require that the following holds:

- $G \setminus V$ consists of finitely many path components $\mathring{e}_1, \dots, \mathring{e}_J$. The closure e_j of each component $\mathring{e}_j$ is homeomorphic to the interval $[0, 1]$ and is called an edge of G;
- $e_j \setminus \mathring{e}_j$ consists of two different vertices.

The aim of this section is to prove the following result.

Theorem 2.55. *The group $H_1(G)$ is free and finitely generated. Moreover, the following holds:*

$$\operatorname{rk} H_0(G) - \operatorname{rk} H_1(G) = \#\,\textit{vertices} - \#\,\textit{edges} =: \chi(G).$$

The number $\chi(G)$ is called the Euler characteristic of G.

The proof requires some notions and auxiliary claims that we consider first. The proof of Theorem 2.55 can be found at the end of this section.

Definition 2.56. A subset $A \subset B$ is called a deformation retract of B, if the following holds: There exists a continuous map $r\colon B \to A$, which is called *a retraction*, such that the following holds:

$$r \circ \imath = \mathrm{id}_A \quad \text{and} \quad \imath \circ r \simeq id_B,$$

where $\imath\colon A \subset B$ is the inclusion.

It follows immediately from the above definition that the induced maps

$$\imath_*\colon H_*(A) \to H_*(B) \quad \text{and} \quad r_*\colon H_*(B) \to H_*(A)$$

are mutually inverse. In particular, both maps are isomorphisms.

Lemma 2.57. *Let A be a deformation retract of B, where $A \subset B \subset X$. Then the inclusion $\imath\colon (X, A) \to (X, B)$ induces an isomorphism*

$$\imath_*\colon H_*(X, A) \to H_*(X, B).$$

Proof. The proof of this lemma hinges on the following algebraic fact.

Lemma 2.58 ("Five lemma")**.** *Assume the horizontal sequences in the commutative diagram of abelian groups*

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

are exact. Furthermore, assume that f_2 and f_4 are isomorphisms, f_1 is an epimorphism, and f_5 is a monomorphism. Then f_3 is an isomorphism. □

Consider the commutative diagram

$$\begin{array}{ccccccccc}
H_k(A) & \longrightarrow & H_k(X) & \longrightarrow & H_k(X, A) & \longrightarrow & H_{k-1}(A) & \longrightarrow & H_{k-1}(X) \\
{\scriptstyle \imath_*}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle \imath_*}\downarrow & & {\scriptstyle \imath_*}\downarrow & & {\scriptstyle id}\downarrow \\
H_k(B) & \longrightarrow & H_k(X) & \longrightarrow & H_k(X, B) & \longrightarrow & H_{k-1}(B) & \longrightarrow & H_{k-1}(X).
\end{array}$$

Here the horizontal sequences are long exact sequences of the pairs (X, A) and (X, B). Furthermore, the first two vertical arrows and the last two ones represent isomorphisms. The proof now follows from the five lemma. □

From the long exact sequence of the pair $\big([0, 1], \{0, 1\}\big)$ we obtain the following result.

Lemma 2.59. *The following holds:*

$$H_k\big([0, 1], \{0, 1\}\big) \cong \begin{cases} \mathbb{Z} & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

□

Proposition 2.60. *The inclusion $\imath_j \colon (e_j, \partial e_j) \to (G, V)$ induces a monomorphism*

$$\imath_{j*} \colon H_k(e_j, \partial e_j) \to H_k(G, V).$$

Moreover, the following holds:

$$H_k(G, V) = \bigoplus_j \operatorname{im} \imath_{j*} \cong \begin{cases} \mathbb{Z}^J & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

Proof. Let $f_j \colon [0,1] \to e_j$ be a homeomorphism, $a_j := f(\frac{1}{2})$, and $d_j := f([\frac{1}{4}, \frac{3}{4}])$. Denote also $A = \{a_1, \dots, a_J\}$ and $D = d_1 \sqcup \dots \sqcup d_J$. Consider the commutative diagram

$$\begin{array}{ccccc} H_k\big(d_j, d_j \setminus \{a_j\}\big) & \xrightarrow{\alpha_1} & H_k(e_j, e_j \setminus \{a_j\}) & \xleftarrow{\beta_1} & H_k(e_j, \partial e_j) \\ \downarrow & & \downarrow & & \downarrow \\ H_k(D, D \setminus A) & \xrightarrow{\alpha_2} & H_k(G, G \setminus A) & \xleftarrow{\beta_2} & H_k(G, V). \end{array}$$

All four horizontal homomorphisms are in fact isomorphisms. Indeed, α_1 and α_2 are isomorphisms by excision, β_1 and β_2 by Lemma 2.57.

Since

$$H_k(D, D \setminus A) = \bigoplus_{j=1}^{J} H_k(d_j, d_j \setminus \{a_j\}) \cong \bigoplus_{j=1}^{J} H_k(e_j, \partial e_j),$$

we obtain the claim of this proposition. □

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Proof of Theorem 2.55. For the proof we need the following algebraic fact.

Lemma 2.61. *Any subgroup of a free abelian group is also free.* □

The remaining part of the proof consists of the following three steps.

Step 1. *$H_1(G)$ is free.*

The long exact sequence of the pair (G, V) yields:

$$0 \to H_1(G) \to H_1(G, V) \to H_0(V) \to H_0(G) \to 0. \tag{2.62}$$

$H_1(G, V)$ is free $\implies$ $H_1(G)$ is free.

Step 2. *Let $f \colon A \to F$ be an epimorphism between two finitely generated free abelian groups. Then*

$$A = \ker f \oplus A_0,$$

where $f \colon A_0 \to F$ is an isomorphism and $\ker f$ is free.

Let $f_1, \dots, f_n$ be generators of F. Choose $b_1, \dots, b_n \in A$ such that $f(b_j) = f_j$. Since $\ker f \subset A$ and A is free, $\ker A$ is also free. Pick generators $a_1, \dots, a_k$ of $\ker f$. Then we have $A = \mathbb{Z}[a_1, \dots, a_k, b_1, \dots b_n]$. Indeed, for an arbitrary element $a \in A$ we have

$$f(a) \in F \implies f(a) = \sum m_j f_j \implies a - \sum m_j b_j \in \ker f \implies a - \sum m_j b_j = \sum p_i a_i.$$

Moreover, the representation $a = \sum m_j b_j + \sum p_i a_i$ is unique.

Step 3. *We prove this theorem.*

Without loss of generality we can assume that G is path connected. Then (2.62) yields

$$0 \to H_1(G) \to H_1(G,V) \to \tilde{H}_0(V) \to 0,$$

i.e., $H_1(G,V) \cong H_1(G) \oplus \tilde{H}_0(V)$. This yields in turn

$$\#\,\text{edges} = \operatorname{rk} H_1(G,V) = \operatorname{rk} H_1(G) + \operatorname{rk} \tilde{H}_0(V) = \operatorname{rk} H_1(G) + \#\,\text{vertices} - 1.$$

□

Example 2.63. The circle $G = e_0 \cup e_1$, $V = \{v_1, v_2\}$. We have $\chi(G) = 0 \implies rkH_1(G) = \operatorname{rk} H_0(G) = 1$.

Example 2.64. The wedge product of two circles. $G = e_0 \cup \cdots \cup e_4$, $V = \{v_1, v_2, v_3\}$.

Picture

$\chi(G) = -1 \implies \operatorname{rk} H_1(G) = 2$.

Definition 2.65. A graph (G,V) is called *planar*, if there is an embedding of G into $\mathbb{R}^2$, that is if G can be drawn on the plane such that edges are represented by simple continuous curves that intersect only at the vertices.

Each connected planar graph decomposes $\mathbb{R}^2$ into a finite number of bounded domains, which are called *faces*, and an unbounded domain, which is also called a face. Moreover, each bounded domain is homeomorphic to a disc (a theorem of Schoenflies).

Theorem 2.66 (Euler)**.** *For any planar connected graph G we have*

$$\#\,\textit{vertices} - \#\,\textit{edges} + \#\,\textit{faces} = 2. \tag{2.67}$$

Notice that the unbounded face also counts in (2.67).

Proof. By means of the stereographic projection we can view G as a subspace of S^2. Notice that the unbounded face together with the point at infinity is mapped to a face on S^2.

Just like in the proof of Proposition 2.60 we obtain

$$H_2(S^2,G) \cong \mathbb{Z}^F \qquad \text{and} \qquad H_k(S^2,G) = 0 \quad \text{for all } k \notin \{0,\,2\},$$

where F is the number of faces. From the long exact sequence of the pair (S^2,G) we have

$$0 \to H_2(S^2) \to H_2(S^2,G) \to H_1(G) \to H_1(S^2) = 0,$$

which yields

$$\mathbb{Z}^F \cong \mathbb{Z} \oplus H_1(G) \implies F = 1 + \operatorname{rk} H_0(G) - \#\,\text{vertices} + \#\,\text{edges}$$

by Theorem 2.55. Since G is connected by the hypothesis, we have $\operatorname{rk} H_0(G) = 1$ and therefore (2.67) holds. □

Exercise 2.68. Solve the "Three utilities problem": Suppose there are three cottages on a plane and each needs to be connected to the water, gas, and electricity companies. Without using a third dimension or sending any of the connections through another company or cottage, is there a way to make all nine connections without any of the lines crossing each other?

Hint: to obtain a solution consider the graph $K_{3,3}$:

Image of $K_{3,3}$

Assuming $K_{3,3}$ is planar, show that the following holds:

(i) $\#$ faces $\leq \frac{1}{2}\#$ edges;

(ii) $\#$ edges $\leq 2\#$vertices $- 4$.

Deduce from the last property that $K_{3,3}$ is non-planar.

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2.11 Homology groups of surfaces

2.11.1 The torus

The torus $\mathbb{T}^2$ can be understood as a square with opposite sides being glued.

Figure

Let $f\colon R \to \mathbb{T}^2$ be the quotient map. Then $f(\partial R)$ consists of two circles A and B intersecting at a point.

Theorem 2.69.

$$H_k(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 2; \\ \mathbb{Z}^2 & \text{for } k = 1; \\ 0 & \text{else}. \end{cases}$$

Proof. The proof consists of the following three steps.

Step 1. *The map $f\colon (R, \partial R) \to (\mathbb{T}^2, A \cup B)$ induces an isomorphism*

$$f_*\colon H_*(R, \partial R) \to H_*(\mathbb{T}^2, A \cup B).$$

Let m be the center of the square R and D a disc centered at m contained in the interior of R. Just like in the proof of Proposition 2.60 one obtains that all horizontal arrows of the commutative diagram

$$\begin{tikzcd} H_k\big(R, \partial R\big) \arrow[r] \arrow[d, "f_*"'] & H_k(R, R \setminus \{m\}) \arrow[d] & H_k(D, D \setminus \{m\}) \arrow[l] \arrow[d, "f_*"] \\ H_k(\mathbb{T}^2, A \cup B) \arrow[r] & H_k\big(\mathbb{T}^2, \mathbb{T}^2 \setminus \{f(m)\}\big) & H_k\big(f(D), f(D) \setminus \{f(m)\}\big) \arrow[l] \end{tikzcd}$$

represent isomorphisms (to prove this one needs in particular that $A \cup B$ is a deformation retract of $\mathbb{T}^2 \setminus \{m\}$). Since the right vertical arrow represents an isomorphism, we obtain that the leftmost vertical arrow represents an isomorphism too.

Step 2. *If $k \geq 1$, then*

$$H_k(\mathbb{T}^2, A \cup B) \cong \begin{cases} \mathbb{Z} & \text{for } k = 2, \\ 0 & \text{else}. \end{cases}$$

The statement of this step follows from the long exact sequence of the pair $(R, \partial R)$.

Step 3. *We prove this theorem.*

The non-trivial part of the long exact sequence of the pair $(\mathbb{T}^2, A \cup B)$ has the following form

$$0 \to H_2(\mathbb{T}^2) \to H_2(\mathbb{T}^2, A \cup B) \xrightarrow{\delta} H_1(A \cup B) \to H_1(\mathbb{T}^2) \to 0,$$

where $H_2(\mathbb{T}^2, A \cup B) \cong \mathbb{Z}$ and $H_1(A \cup B) \cong \mathbb{Z}^2$ by Example 2.64.

To determine δ, consider the commutative diagram

$$\begin{tikzcd} H_2\big(R, \partial R\big) \arrow[r, "\delta'"] \arrow[d, "f_*"'] & H_1(\partial R) \arrow[d, "f'_*"] \\ H_2(\mathbb{T}^2, A \cup B) \arrow[r, "\delta"] & H_1\big(A \cup B\big), \end{tikzcd}$$

where $f' \colon \partial R \to A \cup B$ is the restriction of f. The induced map f'_* is trivial (*Why?*). Since f_* and δ' are isomorphisms, δ must be also trivial. This yields

$$H_2(\mathbb{T}^2) \cong \ker \delta = H_2(\mathbb{T}^2, A \cup B) \cong \mathbb{Z} \qquad \text{and} \qquad H_1(\mathbb{T}^2) \cong H_1(A \cup B) \cong \mathbb{Z}^2.$$

This finishes the proof. □

In fact, tracing through the above proof we can work out the generators of $H_1(\mathbb{T}^2)$. Indeed, it was shown that the inclusion $A \cup B \subset \mathbb{T}^2$ induces an isomorphism $H_1(A \cup B) \to H_1(\mathbb{T}^2)$. Hence, the circles A and B generate $H_1(\mathbb{T}^2)$. L 11