List of Problems in Global Analysis

1. Let M be a closed oriented Riemannian manifold.

   (a) Show that any solution $\omega \in \Omega^k(M)$ of the equation $\Delta\omega = d\eta$, where $\eta \in \Omega^{k-1}(M)$, is closed.

   (b) Show that any solution $\omega \in \Omega^k(M)$ of the equation $\Delta\omega = d^*\eta$, where $\eta \in \Omega^{k+1}(M)$, is co-closed.

2. Prove that any cohomology class in $H^1_{dR}\big(\mathbb{R}^2 \setminus \{0\}\big)$ is represented by a harmonic 1-form.

3. Prove that on any closed connected oriented Riemannian n-manifold, any harmonic n-form is proportional to the volume form.

4. Let D be the disc in $\mathbb{R}^2$ of unit radius centered at the origin. Find the dimension of the space
$$\big\{\omega \in \Omega^1(D) \mid d\omega = 0 = d^*\omega, \ \omega(\partial_n) = 0\big\},$$
   where ∂_n is the unit normal field along ∂D. Also, show that the space
$$\big\{\omega \in \Omega^1(D) \mid \Delta\omega = 0, \ \omega(\partial_n) = 0\big\},$$
   is infinite dimensional.

5. Let M^2 be an oriented surface equipped with a Riemannian metric g. For an everywhere positive function u, denote $g_u := u^2 g$. Show that if $\omega \in \Omega^k(M)$, $k = 0, 1, 2$, we have
$$\Delta_g\omega = 0 \qquad \Longleftrightarrow \qquad \Delta_{g_u}\omega = 0.$$

6. Show that a k-form ω is harmonic if and only if $*\omega$ is harmonic.

7. Let M be a closed oriented Riemannian four-manifold. Since for the $*$-operator acting on $\Lambda^2 T^*M$ we have $*^2 = id$, the bundle of 2-forms splits into the ± 1-eigenspaces: $\Lambda^2 T^*M = \Lambda^2_+ T^*M \oplus \Lambda^2_- T^*M$.

   (a) Show that $\dim \Lambda^2_\pm T^*_m M = 3$ for each $m \in M$;

   (b) Show that $H^2_{dR}(M)$ splits as $H^2_+(M) \oplus H^2_-(M)$, where $H^2_\pm(M)$ is the maximal positive/negative subspace of the symmetric bilinear form
$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta;$$

   (c) Show that the sequence
$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^\pm} \Omega^2_\pm(M) \to 0 \tag{1}$$
   is a complex, whose homology groups are isomorphic to
$$H^0(M), \qquad H^1(M), \qquad \text{and} \qquad H^2_\pm(M).$$

8. Denote by $\mathcal{H}^k(M)$ the space of all harmonic k-forms on a closed oriented Riemannian manifold M. Assuming that $\mathcal{H}^k(M)$ is finite-dimensional, show that we have the decomposition

$$\Omega^k(M) = \operatorname{Im} d \oplus \mathcal{H}^k \oplus \operatorname{Im} d^*,$$

which is in fact L^2-orthogonal.

9. Let Σ be a Riemann surface.

   (i) Show that for any holomorphic $(1,0)$ form ζ, the real 1-forms $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ are harmonic.

   (ii) Show that for any real harmonic 1-form ω there exists a holomorphic $(1,0)$ form ζ such that $\operatorname{Re}\zeta = \omega$.

10. Let M and N be two closed orientable manifolds of the same dimension n. Assume, moreover, that N is connected. Pick $\omega \in \Omega^n(N)$ such that $\int_N \omega = 1$ and define $\deg f = \int_M f^*\omega$.

   (i) Show that $\deg f$ is well-defined;

   (ii) Show that $\deg f$ is in fact an integer;

   (iii) Show that f is surjective whenever $\deg f \neq 0$;

   (iv) Let $f\colon \Sigma_1 \to \Sigma_2$ be a holomorphic map between compact Riemann surfaces and let $F(z)$ be a coordinate representation of f with respect to a local holomorphic coordinate z centered at some $p \in \Sigma_1$ (and a local holomorphic coordinate on Σ_2). A non-negative integer $m = m(p)$ is said to be a multiplicity of f at p if F can be represented in the form $F(z) = F(0) + z^m F_1(z)$, where $F_1(0) \neq 0$. Show that for *any* $q \in \Sigma_2$ we have
   $$\deg f = \sum_{p \in f^{-1}(q)} m(q).$$

   (v) Let $f\colon \Sigma_1 \to \Sigma_2$ be a holomorphic map between compact Riemann surfaces. Show that $\deg f \geq 0$. Moreover, $\deg f = 0$ if and only if f is constant and $\deg f = 1$ if and only if f is a biholomorphism.

   (vi) Prove the following: If a compact Riemann surface Σ admits a meromorphic function with a unique simple pole, then Σ is biholomorphic to $\mathbb{CP}^1$.

11. Let Σ be a Riemann surface diffeomorphic (homeomorphic) to the torus. Prove that for any two distinct points $p_1, p_2 \in \Sigma$ there exists a meromorphic function f on Σ such that both p_1 and p_2 are simple poles of f and f is holomorphic on $\Sigma \setminus \{p_1, p_2\}$. Also, show that the following limiting case holds: for any $p \in \Sigma$ there exists a meromorphic function g with a unique pole at p of order 2.

12. Prove that the wedge-product of harmonic forms does not need to be harmonic (*Hint:* Take a compact Riemann surface Σ of genus ≥ 2. Pick a non-trivial holomorphic $(1,0)$ form ζ. Show that $\operatorname{Re}\zeta \wedge \operatorname{Im}\zeta \neq 0$ must vanish somewhere and therefore cannot be harmonic.)

13. Prove that the tangent bundle of the 2-sphere is non-trivial.

14. Show that there is a non-trivial bundle $E \to M$ such that $E \oplus \mathbb{R}^k$ is trivial, where $\mathbb{R}^k$ denotes the trivial vector bundle of rank k.

15. Let $E \to I = [0, 1]$ be a vector bundle.

    (a) Pick a connection ∇ on E. Show that for any $v \in E_0$ there exists a unique section s_v such that $\nabla s_v = 0$ and $s_v(0) = v$.

    (b) Show that any bundle $E \to I$ is trivial.

    (c) Show that for any vector bundle $E \to M \times I$ we have $E \cong \pi_1^* E|_{M \times \{0\}} = E|_{M \times \{0\}} \times I$, where $\pi_1 \colon M \times I \to M$ is the natural projection.

    (d) Let $f_0, f_1 \colon M \to N$ be two smoothly homotopic maps. Show that $f_0^* E \cong f_1^* E$ for any vector bundle $E \to N$.

    (e) Show that any vector bundle over a contractible base is trivial.

16. Denote
$$L = \big\{ ([z], w) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid w = 0 \text{ or } [w] = [z] \big\}.$$
Define the projection map $\pi \colon L \to \mathbb{CP}^1$ by $\big([z], w \big) \mapsto [z]$. Show that L is a complex vector bundle of rank 1 over $\mathbb{CP}^1 \cong S^2$. This is called the tautological line bundle of $\mathbb{CP}^1$.

17. Let L be a complex line bundle bundle, that is a complex vector bundle of rank 1, over S^2 such that L admits a trivialization σ_N over $S^2 \setminus \{N\}$ and a trivialization σ_S over $S^2 \setminus \{S\}$, where $N = -S$ is the northern pole[1]. This yields a map $g \colon S^2 \setminus \{S, N\} \to \mathbb{C}^*$ defined by
$$\sigma_S(m) = g(m) \sigma_N(m).$$
The degree of the map $g/|g| \colon S^1 \to S^1$, where the source $S^1 \subset S^2 \setminus \{S, N\}$ is thought of as the equator, is called the degree of L. Show that the following holds:

    (i) The degree of a complex line bundle is well-defined and depends on the isomorphism class of L only.

    (ii) The degree of the tautological bundle equals -1.

    (iii) The degree of $T^* S^2$ equals 2. Here $T^* S^2$ is viewed as a complex line bundle as follows: The Hodge operator on $T^* S^2$ satisfies $*^2 = -id$. Hence, elements of $T^* S^2$ can be multiplied by complex numbers: $(a + bi) \cdot \omega := a\omega + b * \omega$.

    (iv) $\deg(L_1 \otimes L_2) = \deg L_1 + \deg L_2$.

    (v) $\deg L^* = -\deg L$, where $L^* = \mathrm{Hom}(L, \mathbb{C})$ is the dual line bundle.

    (vi) For any integer n there exists a complex line bundle L_n such that $\deg L_n = n$.

    (vii) Two line bundles are isomorphic if and only if their degrees are equal.

    (viii) Prove that the tangent bundle of S^2 is non-trivial.

18. (a) Let $E \to M$ be a vector bundle and $F \to M$ be a subbundle of E. Show that there is a subbundle $G \subset E$ such that $E = F \oplus G$. In other words, any short exact sequence of vector bundles $0 \to F \to E \to G \to 0$ splits.

    (b) Prove that for any vector bundle E over a compact[2] base M there exists a vector bundle F such that $E \oplus F$ is trivial.

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[1] In fact any vector bundle has this property by Problem 15 (e).

[2] This is true for any manifold, however, a solution is slightly simpler if one assumes the base to be compact.

(c) Denote by $\mathrm{Gr}_k(\mathbb{R}^n)$ the Grassmannian of all k-dimensional subspaces in $\mathbb{R}^n$. Let

$$E_{k,n} := \left\{ (x, V) \in \mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n) \mid x \in V \right\}$$

be the tautological bundle over $\mathrm{Gr}_k(\mathbb{R}^n)$, cf. Problem 16. Show that for any rank k vector bundle E over a compact manifold M there exists a (smooth) map $f \colon M \to \mathrm{Gr}_k(\mathbb{R}^n)$ such that $E \cong f^* E_{k,n}$.

19. Show that any function $f \in H^1(0,1)$ is continuous without using the Sobolev embedding theorem.

20. Show that the function

   (i) $f(x) = |x|$ belongs to $H^1(-1,1)$;

   (ii) $f(x) = |x|^{1/2}$ does not belong to $H^1(-1,1)$.

21. For which values of $a \in \mathbb{R}$ does the function $f(x) = |x|^a$ belong to $H^k(\mathbb{R}^n)$?

22. Show that there exists a function $f \in H^1(\mathbb{R}^2)$, which is not continuous.

23. (a) Prove the following simple version of the Sobolev embedding theorem: Show that $H^1(0,1)$ embeds into the Hölder space $C^{0,1/2}(0,1)$.

   (b) Prove that the embedding $H^1(0,1) \to C^0(0,1)$ is compact.

24. Let $q \colon H \times H \to \mathbb{R}$ be a symmetric bilinear form on a Hilbert space H with the following property: for any $u \in H$ there is some constant $C(u) > 0$ such that $|q(u,v)| \leq C(u)\|v\|$ for all $v \in H$. Show that there exists a symmetric bounded linear operator $Q \colon H \to H$ such that $\langle Qu, v \rangle = q(u,v)$.

25. (a) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (with a smooth boundary). Prove that the Poincaré inequality

   $$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

   holds for any $u \in C_0^1(\Omega) := \{u \in C^1(\mathbb{R}^n) \mid \mathrm{supp}\, u \subset \Omega\}$. Here the constant C may depend[3] on Ω, but not on u.
   Remark: One can show that for any $u \in H^1(\Omega)$ the trace $u|_{\partial\Omega}$ is well defined as an L^2-function, for example. With this at hand, one can define $H_0^1(\Omega)$ as a subspace of H^1-functions with vanishing trace on the boundary.

   (b) Define $H_0^1(\Omega)$ as a completion of $C_0^1(\Omega)$ with respect to the H^1-norm: $H_0^1(\Omega) := \overline{C_0^1(\Omega)}_{\|\cdot\|_{H^1}}$. Show that

   $$a(u,v) := \int_\Omega \langle \nabla u, \nabla v \rangle$$

   is a scalar product on $H_0^1(\Omega)$ equivalent to the standard one, i.e., there exist positive constants c and C such that

   $$c\|u\|_{H^1} \leq a(u,u)^{1/2} \leq C\|u\|_{H^1}$$

   holds for all $u \in H_0^1(\Omega)$.

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[3]One can show that C may be chosen to be independent of Ω, but the proof of this is somewhat more elaborate.

(c) A function $u \in H_0^1(\Omega)$ is called *a weak solution* of the Poisson equation

$$\Delta u = f, \qquad u|_{\partial\Omega} = 0$$

if $a(u, \varphi) = \langle f, \varphi \rangle_{L^2}$ holds for any $\varphi \in C_0^\infty(\Omega)$, where $f \in C^0(\bar{\Omega})$. Show that any strong solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of the Poisson equation is its weak solution.

(d) Show that the Poisson equation has a weak solution $u \in H_0^1(\Omega)$.

(e) Show that the energy functional $E_f \colon H_0^1(\Omega) \to \mathbb{R}$, $E_f(u) = \|\nabla u\|_{L^2}^2 - \langle f, u \rangle_{L^2}$ is bounded from below and E_f attains its infimum. Moreover, if u is a point of minimum of E_f and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then u is a strong solution of the Poisson equation.

26. Let $E \to M$ be a vector bundle. Compute the symbol of an arbitrary connection ∇ on E.

27. Compute the symbol of $d^* \colon \Omega^{k+1}(M) \to \Omega^k(M)$.

28. For a complex manifold M, compute the symbol of the Dolbeault operator $\bar{\partial} \colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$.

29. Show that the operator

$$L \colon C^\infty(\mathbb{R}^3; \mathbb{H}) \to C^\infty(\mathbb{R}^3; \mathbb{H}), \qquad Lu = i\, \partial_x u + j\, \partial_y u + k\, \partial_z u$$

is elliptic, where $\mathbb{H}$ denotes the algebra of quaternions.

30. Is the bi-Laplacian $u \mapsto \Delta(\Delta u)$, $u \in C^\infty(\mathbb{R}^n)$, an elliptic operator? Is $d + d^* \colon \Omega^k(M) \to \Omega^{k+1}(M) \oplus \Omega^{k-1}(M)$ elliptic? Is $d + d^* \colon \Omega^{\mathrm{even}}(M) \to \Omega^{\mathrm{odd}}(M)$ elliptic, where $\Omega^{\mathrm{even}}(M) := \Omega^0 \oplus \Omega^2 \oplus \ldots$?

31. Show that any pseudo-differential operator acting on $C_0^\infty(\mathbb{R}^n)$, say, is an integral operator, that is of the form

$$u \mapsto \int_{\mathbb{R}^n} K(x, y) u(y)\, dy.$$

Compute K for the inverse of the standard Laplacian on $\mathbb{R}^n$.

32. Let

$$\Gamma(E_0) \xrightarrow{L_0} \Gamma(E_1) \xrightarrow{L_1} \Gamma(E_2) \tag{2}$$

be a complex, where both L_0 and L_1 are differential operators. Show that (2) is an elliptic complex if and only if the operator $L_1 + L_0^* \colon \Gamma(E_1) \to \Gamma(E_2) \oplus \Gamma(E_0)$ is elliptic.

33. Show that Atiyah's complex (1) is elliptic.

34. Prove that a bounded linear operator $T \colon H_1 \to H_2$, where H_1 and H_2 are Hilbert spaces, is Fredholm if and only if there exist bounded linear maps $S_1, S_2 \colon H_2 \to H_1$ such that

$$S_1 \circ T = \mathrm{id}_{H_1} + R_1 \qquad \text{and} \qquad T \circ S_2 = \mathrm{id}_{H_2} + R_2,$$

where both R_1 and R_2 are compact.