Algebraic Topology

Lecture notes

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This is a draft. If you spot a mistake, please let me know.

TODO:
- Add an appendix on chain complexes.

Contents

Chapter 1

Introduction

The main purpose of this chapter is to explain informally the main ideas which will be developed in details later. In particular, the proofs are rather sketchy stressing main ideas only. More precise statements and proofs will be given in the subsequent chapters.

1.1 Differential forms, the theorems of Green and Stokes

Let $\omega = P(x,y)dx + Q(x,y)dy$ be a 1-form on an open subset $U \subset \mathbb{R}^2$. For example, if $f\colon U \to \mathbb{R}$ is a smooth map, then the differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is a 1-form.

Question 1.1. Under which circumstances does there exist some function f as above such that $\omega = df$?

Clearly, we have the following necessary condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1.2}$$

Proposition 1.3. *If U is convex, then (1.2) is also sufficient.*

Sketch of proof. Theorem of Green $\implies$ For any closed piecewise smooth curve $C \subset U$ without self-intersections we have

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0, \tag{1.4}$$

where D is the domain bounded by C. Notice that here we use the convexity of U, since otherwise C does not necessarily bound any domain.

Pick any $(x_0, y_0) \in U$. For any $(x,y) \in U$ choose a curve C' connecting (x_0, y_0) and (x,y). Define

$$f(x,y) := \int_{C'} P\,dx + Q\,dy.$$

Property (1.4) guaranties that f does not depend on the choice of C'. $\square$

The following example shows that (1.2) is not sufficient for general U.

Example 1.5. Consider $U = \mathbb{R}^2 \setminus \{0\}$ and

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

If there were some f such that $\omega = df$, then we would have $\int_{S^1} \omega = 0$, where S^1 is the circle (for example, parametrized via $t \mapsto (\cos t, \sin t)$). This is a contradiction, since $\int_{S^1} \omega = 2\pi \neq 0$.

Notice that the proof of Proposition 1.2 does not work here, since the theorem of Green does not apply for (D, ω), where D is the unit disc.

Remark 1.6. One can show that for any closed piecewise smooth curve $C \subset \mathbb{R}^2 \setminus \{0\}$ we have

$$\frac{1}{2\pi} \int_C \Big(-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \Big)$$

is an integer.

Let U be an open subset of $\mathbb{R}^3$ and $\omega = P\,dx + Q\,dy + R\,dz$ be a 1-form. We can also ask whether $\omega = df$ for some $f\colon U \to \mathbb{R}$. Clearly, we have the following necessary condition:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \textit{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{1.7}$$

Proposition 1.8. *If U is convex, then (1.7) is also sufficient.*

The proof of this proposition is analogous to the proof of the previous one. Just instead of the theorem of Green we have to use the theorem of Stokes:

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_\Sigma \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\,dy\,dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\,dz\,dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy.$$

Proposition 1.9. *Condition (1.7) is also sufficient for $\mathbb{R}^3 \setminus \{0\}$.*

Sketch of proof. Let $C \subset \mathbb{R}^3$ be an arbitrary simple picewise smooth curve without self-intersections. Then there is a picewise smooth surface $\Sigma \subset \mathbb{R}^3$ such that $\partial \Sigma = C$. If $0 \in \Sigma$, a (small) perturbation yields a surface $\Sigma' \subset \mathbb{R}^3 \setminus \{0\}$ such that $\partial \Sigma' = C$. $\square$

For a general U, Condition (1.7) is still insufficient, which is easily seen for the following example: $U = \mathbb{R}^3 \setminus \{z - \text{Axis}\}$ and

$$\omega = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy.$$

From this discussion we can make the following informal conclusion: Condition (1.7) is sufficient as long as U has no "holes" of codimension 2.

1.2 Ansatz of a construction.

Let $X \subset \mathbb{R}^n$ be an arbitrary subset, which is equipped with the induced topology. Define $Z_1(X)$ as a free Abelian group generated by (oriented) closed curves, i.e.,

$$C \in Z_1(X) \implies C = n_1 C_1 + \dots n_k C_k, \tag{1.10}$$

where $n_j \in \mathbb{Z}$. Define

$$\int_C \omega := \sum n_k \int_{C_k} \omega.$$

Remark 1.11. If C_0 is a closed oriented curve, $2C_0$ can be understood as "running along C_0 twice in the same direction". Similarly, $-C_0$ can be understood as the curve C_0 with the opposite orientation. However, in most cases we treat (1.10) purely formally.

Assume temporarily that X is an *open* subset of $\mathbb{R}^2$. We would like to define an equivalence relation such that

$$C \sim C' \implies \int_C \omega = \int_{C'} \omega$$

holds for all $\omega = P\,dx + Q\,dy$ satisfying (1.2). The theorem of Green (or Stokes in the case $U \subset \mathbb{R}^3$) suggests the following:

$$C \sim C' \quad \Leftrightarrow \quad \exists \text{ a compact oriented surface } \Sigma \text{ such that } \partial\Sigma = C \cup -C'. \tag{1.12}$$

Here C and C' are oriented curves and Σ is an oriented surface such that $\partial\Sigma = C \cup -C'$ as *oriented* curves. This definition also makes sense even in the case when X is not necessarily open.

More generally, a cycle $C = C_1 + \cdots + C_k$ is called *null homologous*, i.e., $C \sim 0$, if and only if

$$\exists \text{ a compact surface } \Sigma \text{ such that } \partial\Sigma = C_1 \cup \cdots \cup C_n.$$

Clearly, Condition (1.12) can be written as $C + (-C') \sim 0$.

Example 1.13. Null homologous cycles on the 2-sphere with 2 points removed (equivalently, $\mathbb{R}^2 \setminus \{0\}$).

Even more generally, each linear combination of null homologous cycles is also declared to be null homologous.

$Z_1(X) \supset B_1(X) = \{\text{null homologous cycles}\}$.

$H_1(X) := Z_1(X)/B_1(X)$ the first homology group of X.

Example 1.14. $H_1(S^2 \setminus \{p, q\}) \cong \mathbb{Z}$.

Problems: Curves C and surfaces Σ can have singularities and self-intersections.

More generally:

- $Z_n(X)$ freely generated by compact oriented n-dimensional "surfaces" without boundary.
- $Z_n(X) \supset B_n(X)$ the subgroup generated by the boundaries of compact oriented $(n+1)$-dimensional "surfaces".
- $H_n(X) := Z_n(X)/B_n(X)$ the nth homology group of X.

In general, we would like to associate to each topological space X a sequence of abelian groups $H_0(X), H_1(X), \ldots, H_n(X), \ldots$ such that the following holds:

(a) Each continuous map $f\colon X \to Y$ induces a sequence of homomorphisms $f_*\colon H_n(X) \to H_n(Y)$;

(b) $(f \circ g)_* = f_* \circ g_*, \quad id_* = id$.

(c) $H_0(\{pt\}) \cong \mathbb{Z}$ and $H_n(\{pt\}) = 0$ for all $n \geq 1$.

(d) $H_n(S^n) \cong \mathbb{Z}$ provided $n \geq 1$ and $H_k(S^n) = 0$ for all $k \geq n+1$ (More generally, for each compact connected oriented manifold M of dimension n the following holds: $H_n(M) \cong \mathbb{Z}$ and $H_k(M) = 0$ for all $k > n+1$).

(e) $f \simeq g \implies f_* = g_*$.

Here two continuous maps are said to be homotopic ($f \simeq g$), if there exists a continuous map $h\colon X \times [0,1] \to Y$, called homotopy, such that the following holds:

$$h|_{X\times 0} = f \qquad \text{and} \qquad h|_{X\times 1} = g.$$

Question 1.15. What does make Properties (a)-(e) interesting?

This question will be answered in the subsequent sections. We finish this section by the following fact, which will be useful below.

Proposition 1.16. *If f is a homeomorphism, then each $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. $id_{H_n} = id_* = (f \circ f^{-1})_* = f_* \circ (f^{-1})_* \implies f_*$ is an isomorphism and $(f_*)^{-1} = (f^{-1})_*$. □

1.3 The theorem of Brouwer

In this section we show that (a)-(e) imply the following famous result.

Theorem 1.17 (Brouwer)**.** *Any continuous map $f\colon B_n \to B_n$ has a fixed point.*

Proof. The proof consists of the following three steps.

Step 1. *For the ball $B_n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ we have $H_k(B_n) = 0$ for all $k \geq 1$.*

Let $c\colon B_n \to \{0\}$ be the constant map. The map $h(x,t) = tx,\ t \in [0,1]$ is a homotopy between id_B und $\imath \circ c$, where $\imath\colon \{0\} \to B_n$ is the inclusion. Thus, $id = \imath_* \circ c_* \implies H_k(B_n) = 0$ for all $k \geq 1$, since $\operatorname{Im} \imath_* = \{0\}$.

Step 2. *There is no continuous map $g\colon B_n \to \partial B_n = S^{n-1}$ such that $g(x) = x$ holds for all $x \in S^{n-1}$.*

Assume $n = 1$ first. In this case there is no continuous map $g\colon [-1,1] \to \{\pm 1\}$ as in the statement of this step, since the target space $\{\pm 1\}$ is disconnected, whereas the interval $[0,1]$ is connected.

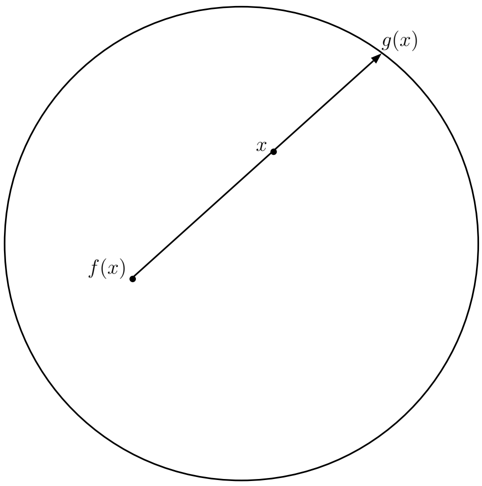
Let us consider now the case $n \geq 2$. Assume there is such $g\colon B_n \to S^{n-1}$. Then we have

$$\begin{aligned} id_{S^{n-1}} = g \circ \imath_{S^{n-1}} &\implies (id_{S^{n-1}})_* = g_* \circ (\imath_{S^{n-1}})_* = 0 \quad \text{on } H_{n-1}(S^{n-1}) \\ &\implies H_{n-1}(S^{n-1}) = 0. \end{aligned}$$

This contradiction proves Step 2.

Step 3. *We prove the theorem of Brower.*

Assume there exists a continuous map $f\colon B_n \to B_n$ without fixed points. Then there also exists a continuous map $g\colon B_n \to S^{n-1}$ such that $g|_{S^{n-1}} = id$:



This contradicts Step 2. $\square$

1.4 The degree of a continuous map and the fundamental theorem of algebra

In this section we show that (a)-(e) imply that any non-constant polynomial with complex coefficients has at least one root. This statement is known as the fundamental theorem of algebra.

Thus, pick any $n \geq 1$ and choose a generator $\alpha \in H_n(S^n)$, i.e., an element α such that $H_n(S^n) = \mathbb{Z} \cdot \alpha$.

Definition 1.18. For any continuous map $f\colon S^n \to S^n$ define $\deg(f) \in \mathbb{Z}$ by

$$f_*\alpha = \deg(f)\alpha.$$

The degree of a map does not depend on the choice of a generator, since $f_*(-\alpha) = -f_*\alpha = -\deg(f)\alpha = \deg(f)(-\alpha)$.

Lemma 1.19. *The degree has the following properties:*

(i) $\deg(id) = 1$*;*

(ii) $\deg(f \circ g) = \deg f \cdot \deg g$*;*

(iii) $f \simeq g \implies \deg f = \deg g$*;*

(iv) $\deg(\textit{const. map}) = 0$*.*

$\square$

Lemma 1.20. *For* $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ *define* $f_n\colon S^1 \to S^1$ *by* $f_n(z) = z^n$*, where* $n \in \mathbb{Z}$*. Then we have*

$$\deg f_n = n.$$

Idea of proof. The curve

$$\alpha\colon [0, 2\pi] \to S^1, \qquad \alpha(t) = \cos t + \sin t\, i = e^{ti},$$

generates $H_1(S^1)$. Since $f_n \circ \alpha(t) = e^{nti} = \cos(nt) + \sin(nt)i$, from the definition of the degree and Remark 1.11 we have $\deg f_n = n$. $\square$

Theorem 1.21 (The fundamental theorem of Algebra)**.** *Each non-constant polynomial* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0,\ a_j \in \mathbb{C}$ *has at least one complex root.*

Proof. Identify S^1 with $S^1_r := \{z \in C \mid |z| = r\} \cong S^1$ with the help of the homeomorphism

$$S^1 \to S^1_r, \qquad z \mapsto rz.$$

The proof consists of the following three steps.

Step 1. *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *be a continuous map without zeros. Then for each* $r > 0$ *the map*

$$\frac{f}{|f|}\colon S^1_r \to S^1 \tag{1.22}$$

is homotopic to the constant map.

Indeed, a homotopy can be given explicitly by

$$F(z,t) = \frac{f(tz)}{|f(tz)|}, \qquad z \in S^1,\ t \in [0,r].$$

Step 2. *Let* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$ *be a polynomial without zeros. Then there exists some* $R > 0$ *such that the following holds:* $\forall r \geq R$ *the restriction of* $p/|p|$ *to* S^1_r *is homotopic to* f_n*.*

For all $z \in \mathbb{C}$ such that $|z| \geq 1$ we have

$$\begin{aligned} \left|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\right| &\leq |a_{n-1}||z|^{n-1} + \dots + |a_1||z| + |a_0| \\ &\leq n \max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}|z|^{n-1} \end{aligned}$$

Choose R so that $R > n\max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ and $R > 1$. For all $r \geq R$ and all $t \in [0,1]$ the polynomial

$$p_t(z) = z^n + t\big(a_{n-1}z^{n-1} + \dots + a_1 z + a_0\big)$$

has no zeros on S^1_r, since

$$\left|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\right| < Rr^{n-1} \leq r^n, \quad \text{provided} \quad |z| = r.$$

Then

$$P(z,t) = \frac{p_t(z)}{|p_t(z)|}\Big|_{S^1_r}$$

is a homotopy between $p/|p|$ and f_n viewed as a map on S^1_r.

Step 3. *We prove the fundamental theorem of algebra.*

Assume p is a non-constant polynomial without zeros. Denote

$$q_r(z) = \frac{p(z)}{|p(z)|}\Big|_{S^1_r},$$

where $r \geq R$. Step 2 $\Longrightarrow$ $\deg q_r = n$. Step 1 $\Longrightarrow$ $\deg q_r = 0$, i.e., $n = 0$. Thus, p is a constant polynomial, which is a contradiction. ∎

Chapter 2

Singular homology

2.1 Free abelian groups

An abelian group G is called free with a basis $A \subset G$, if $\forall g \in G$ there exists a unique representation $g = \sum_{a \in A} n_a a$, where $n_a \in \mathbb{Z}$ and $n_a \neq 0$ for finitely many $a \in A$ only.

Any set A generates an abelian group $F(A)$, which is free with a basis A. Indeed, define

$$F(A) := \big\{ f \colon A \to \mathbb{Z} \mid f(a) \neq 0 \text{ for finitely many } a \in A \text{ only} \big\}.$$

Clearly, the functions

$$f_a(x) = \begin{cases} 1 & x = a, \\ 0 & \text{otherwise}, \end{cases} \qquad a \in A$$

generate $F(A)$, that is $F(A)$ is free with a basis A.

Remark 2.1. For any $f \in F(A)$ we have

$$f = \sum_{a \in A} f(a) f_a.$$

In particular, $F(A)$ can be viewed as the group of all *finite* formal linear combinations $\sum_{a \in A} n_a a$, where $n_a \in \mathbb{Z}$.

2.2 Singular simplexes

Let $x_0, x_1, \dots, x_k$ be arbitrary points in $\mathbb{R}^n$ such that $x_1 - x_0, \dots, x_k - x_0$ are linearly independent.

Definition 2.2. The space

$$\Delta_k = \Delta(x_0, \dots, x_k) = \Big\{ x = \sum_{i=0}^k t_i x_i \mid t_i \in [0,1], \quad \sum_{i=0}^k t_i = 1 \Big\}$$

is called *the (non-degenerate) k-simplex generated by* $x_0, \dots, x_k$.

Example 2.3.
0) If $k = 0$, then $\Delta(x_0) = \{x_0\}$.
1) If $k = 1$, then $\Delta(x_0, x_1)$ is a segment $[x_0, x_1]$.
2) If $k = 2$, then $\Delta(x_0, x_1, x_3)$ is the triangle with the vertices x_0, x_1, x_2.
3) If $k = 3$, then $\Delta(x_0, x_1, x_3, x_4)$ is a tetrahedron with the vertices x_0, x_1, x_3, x_4.

Remark 2.4. The representation $x = \sum_{i=0}^k t_i x_i$ of a point in Δ_k is unique. Indeed, $\sum t_i x_i = \sum s_i x_i$, $\sum t_i = 1 = \sum s_i \implies$

$$0 = \sum (t_i - s_i) x_i = \sum (t_i - s_i) x_i - \sum (t_i - s_i) x_0 = \sum (t_i - s_i)(x_i - x_0) \implies t_i = s_i.$$

The coefficients $(t_0, t_1, \dots, t_k) \in [0,1]^{k+1}$ are called *the barycentric coordinates* of the point $x \in \Delta_k$. In particular, each k-simplex is homeomorphic to the standard k-simplex

$$\Delta^k := \Delta(e_1, \dots, e_k, e_{k+1}) \subset \mathbb{R}^{k+1},$$

where $e_1, \dots, e_{k+1}$ is the standard basis of $\mathbb{R}^{k+1}$.

It is customary to drop the adjective "non-degenerate" when referring to simplexes. Sometimes degenerate simplexes (in the sense that $x_1 - x_0, \dots, x_k - x_0$ may be linearly dependent) do appear below. Typically, this poses no problems, however the barycentric coordinates are ill defined in this case. L 2

From now on we pick one simplex in each dimension, for example the standard one.

Definition 2.5. Let X be a topological space. *A singular k-simplex* in X is a continuous map $f \colon \Delta^k \to X$.

In particular, a singular 0-simplex in X can be viewed as a point in X, a singular 1-simplex as a path in X etc.

Remark 2.6. The map f in the above definition does not need to be injective. In particular, the image of f may be (highly) singular.

For a singular k-simplex $f \colon \Delta^k \to X$ the $(k-1)$-simplex defined by

$$\partial^i f \colon \Delta^{k-1} \to X, \qquad \partial^i f(t_0, \dots, t_{k-1}) = f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1})$$

is called *the ith face* of f.

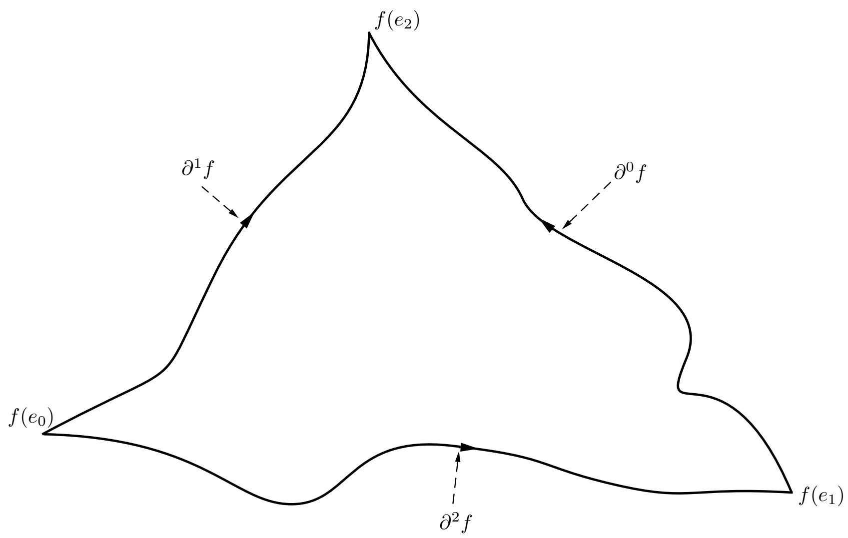


Figure 2.1: Faces of a singular simplex

Definition 2.7. Denote by $S_k(X)$ the free abelian group generated by all singular k-simplexes. Elements of $S_k(X)$ are formal linear combinations of the form

$$\sigma = \sum n_i f_i, \qquad n_i \in \mathbb{Z},$$

which are called *singular k-chains*. The $(k-1)$-chain

$$\begin{aligned} \partial f &= \partial^0 f - \partial^1 f + \partial^2 f - \dots = \sum_{j=0}^{k} (-1)^j \partial^j f, \\ \partial \sigma &= \sum_i n_i \sum_j (-1)^j \partial^j f_i \end{aligned} \tag{2.8}$$

is called *the boundary* of f and σ respectively.

Proposition 2.9. *The homomorphism*

$$S_k(X) \xrightarrow{\partial_k} S_{k-1}(X) \xrightarrow{\partial_{k-1}} S_{k-2}(X)$$

is trivial, that is $\partial_{k-1} \circ \partial_k = 0$ (or, simply $\partial^2 = 0$) for all $k \geq 1$.

Proof. The proof consists of the following two steps.

Step 1. *Let f be a singular simplex. for each $j \geq i$ we have*

$$\partial^j \partial^i f = \partial^i \partial^{j+1} f.$$

Indeed,

$$\begin{aligned} \partial^j (\partial^i f)(t_0, \dots, t_{k-2}) &= \partial^i f(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}); \end{aligned}$$

$$\begin{aligned} \partial^i (\partial^{j+1} f)(t_0, \dots, t_{k-2}) &= \partial^{j+1} f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}). \end{aligned}$$

Step 2. *For each singular k-simplex we have $\partial(\partial f) = 0$.*

This follows from the following computation:

$$\begin{aligned} \partial(\partial f) &= \sum_{i=0}^{k-1} (-1)^i \partial^i (\partial f) = \sum_{i=0}^{k-1} \sum_{j=0}^{k} (-1)^{i+j} \partial^i \partial^j f = \sum_{j \geq i} + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{j \geq i} (-1)^{i+j} \partial^{j-1} \partial^i f + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{p+1 \geq q} (-1)^{p+q+1} \partial^p \partial^q f + \sum_{p>q} (-1)^{p+q} \partial^p \partial^q f \qquad p := j-1,\ q := i \\ &= 0. \end{aligned}$$

□

Corollary 2.10. $\operatorname{im} \partial_k \subset \ker \partial_{k-1}$.

The elements of $Z_{k-1}(X) := \ker \partial_{k-1}$ are called *cycles* and the elements of $B_{k-1}(X) := \operatorname{im} \partial_k$ are called *boundaries*.

Definition 2.11. The group

$$H_{k-1}(X) := \ker \partial_{k-1} / \operatorname{im} \partial_k = Z_{k-1}(X)/B_{k-1}(X)$$

is called the $(k-1)$ *th (singular) homology group* of X (with integer coefficients). In particular, $H_0(X) := S_0(X)/\operatorname{im} \partial_1$.

2.3 Some properties of the homology groups

Proposition 2.12.

$$X \textit{ path connected} \implies H_0(X) \cong \mathbb{Z}.$$

Proof. $S_0(X)$ is the free abelian group generated by the points of X. Let f be a singular 1-simplex, that is $f\colon [0,1] \to X$ is a path in X. By the definition of the boundary, $\partial f = x_1 - x_0$, where $x_1 = f(1)$ and $x_0 = f(0)$. By the hypothesis, we can connect any two points in X by a path, that is for any two points $x_0, x_1 \in X$ we have $[x_0] = [x_1] \in H_0(X)$.

Furthermore, define the homomorphism $\alpha\colon S_0(X) \to \mathbb{Z}$ by

$$\alpha\Big(\sum n_i x_i\Big) = \sum n_i.$$

Since $\alpha(\partial f) = 0$ for each singular 1-simplex (hence, for each singular 1-chain), α yields a surjective homomorphism $H_0(X) \to \mathbb{Z}$, which is still denoted by α.

Suppose $\alpha([\sum n_i x_i]) = 0$. Then $[\sum n_i x_i] = \sum n_i [x_i] = \big(\sum n_i\big)[x_0] = 0$, that is α is injective. Thus, α is an isomorphism. □

Exercise 2.13. If X is not necessarily path connected, then the following holds: $H_0(X) \cong \mathbb{Z}^m$, where m is the number of path-components of X.

Proposition 2.14.

$$H_k(\{pt\}) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, \\ 0 & \textit{else.} \end{cases}$$

Proof. For $k = 0$ the statement of this proposition follows from the previous one. Hence, we may assume $k > 0$. For each such k there is exactly one k-simplex in $\{pt\}$, namely the constant map, which we denote by $c_k\colon \Delta^k \to \{pt\}$. For the boundary we have

$$\partial c_k = \sum_{i=0}^{k} (-1)^i \underbrace{\partial^i c_k}_{c_{k-1}} = \begin{cases} 0, & \text{for } k \text{ odd}, \\ c_{k-1} & \text{for } k \text{ even.} \end{cases}$$

Hence,

$$Z_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even} \end{cases}$$

and

$$B_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd}, \\ 0 & \text{for } k \text{ even.} \end{cases}$$

Thus $H_k\big(\{pt\}\big) = Z_k\big(\{pt\}\big)/B_k\big(\{pt\}\big) = 0$. □

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Definition 2.15. A topological space X is said to be *contractible* if there is a point $x_0 \in X$ such that the identity map id_X is homotopic to the constant map c_{x_0}.

Proposition 2.16. *For a contractible space X we have*

$$H_k(X) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, \\ 0 & \textit{if } k \geq 1. \end{cases}$$

Proof. Since X is contractible, there exists a continuous map $h\colon X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ hold for any $x \in X$. In particular, for a fixed $x \in X$ the path $t \mapsto h(t,x)$ connects x and x_0. This implies that X is path connected, hence $H_0(X) \cong \mathbb{Z}$ by Proposition 2.12.

Thus, we assume $k \geq 1$ in the sequel. Consider the quotient map

$$\pi\colon \Delta^{k-1} \times [0,1] \to \Delta^k \cong (\Delta^{k-1} \times [0,1])/(\Delta^{k-1} \times \{1\})$$
$$((t_0,\ldots,t_{k-1}),u) \mapsto (u,(1-u)t_0,\ldots,(1-u)t_{k-1}).$$

Define $s\colon S_{k-1}(X) \to S_k(X)$ as follows: Since π is a quotient map and $h|_{X\times\{1\}} \equiv x_0$, by the universal property of the quotient map for each singular $(k-1)$-simplex $\sigma\colon \Delta^{k-1} \to X$ there exists a unique map $s(\sigma)\colon \Delta^k \to X$ such that $h \circ (\sigma \times \mathrm{id}) = s(\sigma) \circ \pi$. More explicitly,

$$s(\sigma)(t_0,t_1,\ldots,t_k) = h\Big(\sigma\Big(\frac{t_1}{1-t_0},\ldots,\frac{t_k}{1-t_0}\Big),\ t_0\Big)$$

whenever $t_0 \neq 1$ and $s(\sigma)(t_1,\ldots,t_k,1) = x_0$. Hence,

1. $\partial^0(s(\sigma)) = \sigma$,
2. $\partial^i s(\sigma) = s(\partial^{i-1}\sigma)$ for $i > 0$.

Extending s by linearity to all of $S_{k-1}(X)$, for any $\sigma \in S_k(X)$ we have

$$\partial(s(\sigma)) = \partial^0(s(\sigma)) - \sum_{i=1}^{k}(-1)^{i-1}\partial^i(s(\sigma)) = \sigma - \sum_{j=0}^{k-1}(-1)^j s(\partial^j\sigma) = \sigma - s(\partial\sigma). \tag{2.17}$$

This yields

$$\partial \circ s + s \circ \partial = \mathrm{id}.$$

Hence, if σ is a cycle, then $\sigma = \partial\big(s(\sigma)\big) + s\big(\partial\sigma\big) = \partial\big(s(\sigma)\big)$, i.e., any cycle is a boundary. In other words, $H_k(X) = 0$ whenever $k \geq 1$ as claimed. $\square$

Theorem 2.18. *Let $f\colon X \to Y$ be a continuous map. Then for each $k \geq 0$ the map f induces a group homomorphism*

$$f_*\colon H_k(X) \to H_k(Y)$$

and for any other continuous map $g\colon Y \to Z$ we have

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, $(\mathrm{id}_X)_* = \mathrm{id}$.

Proof. Define first group homomorphisms $f_\#\colon S_k(X) \to S_k(Y)$, by declaring

$$\sigma \mapsto f \circ \sigma \quad \text{for} \quad \sigma\colon \Delta^k \to X.$$

Then for all singular k-simplexes $\sigma\colon \Delta^k \to X$ we have

$$\begin{aligned}(f_\#\partial^i(\sigma))(t_0,\ldots,t_{k-1}) &= f(\sigma(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{k-1})) \\ &= (f_\#\sigma)(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{k-1}) \\ &= \partial^i(f_\#\sigma)(t_0,\ldots,t_{k-1}),\end{aligned}$$

and therefore $f_\#\partial^i = \partial^i f_\#$, which yields in turn that $f_\#$ is a *chain map*, i.e.,

$$f_\#\partial = \partial f_\#.$$

This yields in particular that cycles are mapped to cycles and boundaries are mapped to boundaries:

$$f_\#(Z_k(X)) \subset Z_k(Y) \qquad \text{and} \qquad f_\#(B_k(X)) \subset B_k(Y).$$

Hence, we obtain a well defined group homomorphism:

$$\begin{gathered} f_*\colon H_k(X) = Z_k(X)/B_k(X) \to Z_k(Y)/B_k(Y) = H_k(Y) \\ f_*([\sigma]) := [f_\#(\sigma)]. \end{gathered}$$

Furthermore, for each singular k-simplex $\sigma\colon \Delta^k \to X$ we have

$$\begin{aligned} &g_\# \circ f_\#(\sigma) = g_\#(f \circ \sigma) = g \circ f \circ \sigma = (g \circ f)_\#(\sigma), \\ &g_* \circ f_*([\sigma]) = g_*[f_\#(\sigma)] = [g_\# \circ f_\#(\sigma)] = [(g \circ f)_\#(\sigma)] = (g \circ f)_*([\sigma]), \\ &(\mathrm{id}_X)_\#(\sigma) = \sigma, \qquad (\mathrm{id}_X)_*([\sigma]) = [(\mathrm{id}_X)_\#(\sigma)] = [\sigma]. \end{aligned}$$

Therefore, $g_* \circ f_* = (g \circ f)_*$ and $(\mathrm{id}_X)_* = \mathrm{id}$. □

Corollary 2.19. *If $f\colon X \to Y$ is a homeomorphism, then $f_*\colon H_k(X) \to H_k(Y)$ is an isomorphism for each k.* □

2.4 Homotopies and homology groups

Theorem 2.20. *If $f, g\colon X \to Y$ are homotopic maps, then the induced maps on the homology groups are equal:*

$$f \simeq g \qquad \Longrightarrow \qquad f_* = g_*.$$

Proof. The proof consists of the following three steps.

Step 1. *Define*

$$\eta_t\colon X \to X \times I, \qquad \eta_t(x) = (x, t).$$

For each continuous map $f\colon X \to Y$ we have $(f \times id)_\# \eta^X_{t\#} = \eta^Y_{t\#} \circ f_\#$.

This follows immediately from the observation that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_t^X} & X \times I \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f \times id} \\ Y & \xrightarrow{\eta_t^Y} & Y \times I \end{array}$$

commutes.

Step 2. *There exists a sequence of homomorphisms $s^X_k\colon S_k(X) \to S_{k+1}(X \times I)$ satisfying*

$$\partial s^X_k + s^X_{k-1}\partial = \eta_{1\#} - \eta_{0\#}; \tag{2.21}$$

$$(f \times id_I)_\# \circ s^X_k = s^Y_k \circ f_\#. \tag{2.22}$$

Define $s_k = s_k^X$ recursively. For $k = 0$ and $x_0 \in X$, which we view as a 0-simplex, put

$$s_0\sigma\colon \Delta^1 \to X \times I, \qquad (t_0, t_1) \mapsto (x_0, t_1).$$

Then we have $\partial(s_0\sigma) = (x_0, 1) - (x_0, 0)$, i.e., (2.21) holds for $k = 0$. Equation (2.22) follows directly from the definition of s_0.

Suppose s_ℓ have been defined for all $\ell < k$. We define first s_k in a special case, namely for id_{Δ^k} viewed as an element $\imath_k \in S_k(\Delta^k)$. We have

$$\begin{aligned}
\partial\big(\underbrace{\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k}_{\in S_k(\Delta^k\times I)}\big) &= \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \partial s_{k-1}\partial\imath_k \\
&\overset{(2.21)}{=} \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \big(\eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - s_{k-2}^{\Delta^k}\partial^2\imath_k\big) \\
&= 0.
\end{aligned}$$

In this computation (2.21) is used with k replaced by $k - 1$. Since $\Delta^k \times I$ is contractible, there exists some $a \in S_{k-1}(\Delta^k \times I)$ so that

$$\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k = \partial a.$$

Define $s_k(\imath_k) = a$. Then (2.21) holds for $\sigma = \imath_k$.

In general, define $s_k^X(\sigma) = (\sigma \times id)_\# a$. Then we have

$$\begin{aligned}
\partial(s_k^X\sigma) &= \partial(\sigma \times id)_\# a = (\sigma \times id)_\#\partial a \\
&= (\sigma \times id)_\#\big(\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}^{\Delta^k}\partial\imath_k\big) \\
&= \eta_{1\#}\sigma_\#\imath_k - \eta_{0\#}\sigma_\#\imath_k - s_{k-1}^X\sigma_\#\partial\imath_k && \text{(2.22) + Step 1} \\
&= \eta_{1\#}\sigma - \eta_{0\#}\sigma - s_{k-1}^X\partial\sigma.
\end{aligned}$$

This proves (2.21).

We still have to show that (2.22) holds. Indeed,

$$(f \times id)_\# s_k\sigma = (f \times id)_\#(\sigma \times id)_\# a = \big((f\circ\sigma) \times id\big)_\# a = s_k(f\circ\sigma) = s_k(f_\#\sigma).$$

Step 3. *We prove this theorem.*

Let h be a homotopy between f and g. From the following equalities

$$\partial(h_\# \circ s_k) + (h_\# \circ s_{k-1})\partial = h_\#\partial s_k + h_\#(s_{k-1}\partial) = h_\#(\eta_{1\#} - \eta_{0\#}) = f_\# - g_\#$$

we see that $f_\# - g_\# = \partial(h_\# \circ s_k)$ holds on $\ker\partial$. This shows that $f_* = g_*$. $\square$

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Definition 2.23. A continuous map $f\colon X \to Y$ is called a homotopy equivalence, if there exists a continuous map $g\colon Y \to X$ such that the following holds:

$$g \circ f \simeq id_X \qquad \text{and} \qquad f \circ g \simeq id_Y.$$

In this case the spaces X and Y are called homotopy equivalent.

Example 2.24. (i) Any two homeomorphic spaces are homotopy equivalent.

(ii) $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$. More generally, any contractible space is homotopy equivalent to $\{pt\}$.

(iii) $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}.

To see (ii), let X be a contractible space and $\imath_{x_0}\colon \{x_0\} \to X$ be the embedding of the point x_0. Then $c_{x_0} \circ \imath_{x_0} = id_{x_0}$ and $\imath_{x_0} \circ c_{x_0} \simeq id_X$.

To see (iii), define $f\colon \mathbb{R}^n \setminus \{0\} \to S^n$ by $f(x) = x/|x|$. If $g\colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ denotes the inclusion, then $f \circ g = id_{S^{n-1}}$. Furthermore,

$$h(x,t) = \frac{1}{t + (1-t)|x|}x, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

is a homotopy between $g \circ f$ and $id_{\mathbb{R}^n \setminus \{0\}}$.

Corollary 2.25.

$$f \text{ is a homotopy equivalence} \implies \forall k \quad f_*\colon H_k(X) \to H_k(Y) \text{ is an isomorphism.}$$

Example 2.26. Since $\mathbb{R}^n$ is homotopy equivalent to a point, we have

$$H_k(\mathbb{R}^n) \cong H_k\big(\{pt\}\big) \cong \begin{cases} \mathbb{Z} & k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Assuming the homology groups of the n-sphere are known, we have

$$H_k(\mathbb{R}^n \setminus \{pt\}) \cong H_k(S^{n-1}) \cong \begin{cases} \mathbb{Z} & k = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the latter isomorphism is established in Theorem 2.41 below.

2.5 Exact sequences and the Bockstein homomorphism

Definition 2.27. A sequence of homomorphisms of abelian groups

$$\dots \longrightarrow A_{k+1} \xrightarrow{\alpha_{k+1}} A_k \xrightarrow{\alpha_k} A_{k-1} \longrightarrow \dots \tag{2.28}$$

is called exact, if for all k the following holds: $\ker \alpha_k = \operatorname{im} \alpha_{k+1}$.

Some special cases:

(i) $0 \to A \xrightarrow{\alpha} B$ is exact $\quad\Leftrightarrow\quad$ α is injectiv;

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact $\quad\Leftrightarrow\quad$ α is surjectiv;

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\quad\Leftrightarrow\quad$ α is an isomorphism;

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact $\quad\Leftrightarrow\quad$ α is injectiv, β is surjectiv and $\ker \beta = \operatorname{im} \alpha$; In particular, β induces an isomorphism $C \cong B/A$.

The sequence (iv) is called a short exact sequence.

Example 2.29. $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a short exact sequence, where $\times n$ stands for the multiplication with a fixed $n \in \mathbb{Z}$.

Let A be a complex, that is A is a sequence

$$A: \qquad \cdots \longrightarrow A_{k+1} \xrightarrow{\partial} A_k \xrightarrow{\partial} A_{k-1} \longrightarrow \ldots$$

such that $\partial^2 = 0$. Just like in the case of chain complexes, we define the kth homology group of A to be

$$H_k(A) := \frac{\ker\big(\partial\colon A_k \to A_{k-1}\big)}{\operatorname{im}\big(\partial\colon A_{k+1} \to A_k\big)}.$$

Notice the following: Assuming (2.28) is a complex, i.e., $\alpha_{k+1} \circ \alpha_k = 0$ holds for all k, we obtain that (2.28) is exact if and only if $H_k(A) = \{0\}$ for all k.

If A, B, and C are complexes, a sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of complexes is a commutative diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r, "\partial"] & A_{k+1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k+1}"'] & A_k \arrow[r, "\partial"] \arrow[d, "\alpha_k"'] & A_{k-1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k-1}"'] & \ldots \\
\ldots \arrow[r, "\partial"] & B_{k+1} \arrow[r, "\partial"] \arrow[d, "\beta_{k+1}"'] & B_k \arrow[r, "\partial"] \arrow[d, "\beta_k"'] & B_{k-1} \arrow[r, "\partial"] \arrow[d, "\beta_{k-1}"'] & \ldots \\
\ldots \arrow[r, "\partial"] & C_{k+1} \arrow[r, "\partial"] \arrow[d] & C_k \arrow[r, "\partial"] \arrow[d] & C_{k-1} \arrow[r, "\partial"] \arrow[d] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.30}$$

Such a sequence is called *exact*, if each vertical sequence $0 \to A_k \to B_k \to C_k \to 0$ is exact.

Here of course we could equally well consider sequences of complexes consisting of more than 3 complexes.

Example 2.31. Let X, Y and Z be topological spaces and $f\colon X \to Y$, $g\colon Y \to Z$ continuous maps. Then one obtains a sequence of chain complexes

$$0 \to S_*(X) \xrightarrow{f_\#} S_*(Y) \xrightarrow{g_\#} S_*(Z) \to 0,$$

which is not necessarily exact. What conditions guarantee that the above sequence is exact will be considered below.

Proposition 2.32. *For any homomorphism of complexes $\alpha\colon A \to B$ we have a homomorphism $\alpha\colon H_*(A) \to H_*(B)$ of homology groups, which is still denoted by the same letter.*

Proof. This follows immediately from the commutativity of (the upper part of) (2.30). ☐

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Theorem 2.33. *A short exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ induces a (long) exact sequence of homology groups:*

$$\cdots \to H_k(A) \xrightarrow{\alpha} H_k(B) \xrightarrow{\beta} H_k(C) \xrightarrow{\delta} H_{k-1}(A) \xrightarrow{\alpha} H_{k-1}(B) \to \ldots$$

Remark 2.34. The map δ is called the Bockstein homomorphism.

Proof. The proof consists of the following four steps.

Step 1. *We define* δ.

Pick $c \in C_k$, $\partial c = 0$. Since β_k is surjective, there exists some $b \in B_k$ such that $\beta(b) = c$. We have $\beta(\partial b) = \partial(\beta(b)) = \partial c = 0$. Since $\alpha\colon A_{k-1} \to \ker \beta_{k-1}$ is surjective, there is some $a \in A_{k-1}$ such that $\alpha(a) = \partial b$. Define

$$\delta[c] = [a].$$

We have to show that δ is well defined. Indeed, pick another representative $c' = c + \partial c''$ of the class $[c]$. For $c'' \in C_{k+1}$ there is some $b'' \in B_{k+1}$ such that $\beta(b'') = c'' \implies \beta(b + \partial b'') = c + \partial c''$. This yields $b' = b + \partial b'' + \alpha(a'')$, where $a'' \in A_k$. Furthermore, $\partial b' = \partial b + 0 + \alpha(\partial a')$. Since α is injective, we have $a' = a + \partial a''$, i.e., $[a] = [a']$.

Exercise 2.35. Check that δ is a group homomorphism.

Step 2. $\ker \alpha = \operatorname{im} \delta$.

Pick $a \in A_{k-1}$ such that $[a] \in \ker \alpha$, i.e., $\alpha(a) = \partial b$ for some $b \in B_k$. We have $\partial\beta(b) = \beta(\partial b) = \beta(\alpha(a)) = 0$. By the construction of δ, we obtain $\delta[\beta(b)] = [a]$. That is $\ker \alpha \subset \operatorname{im} \delta$.

If $a \in A_{k-1}$ is such that $[a] \in \operatorname{im} \delta$, then by the construction of δ, we have $\alpha(a) = \partial b \implies \alpha[a] = 0$.

Step 3. $\ker \delta = \operatorname{im} \beta$.

Pick some $[c] \in \ker \delta$. Using the notations of Step 1, we have $a = \partial a'$ for some $a' \in A_k$. The equations

$$\begin{aligned} \partial\big(b - \alpha(a')\big) &= \partial b - \alpha(\partial a') = \partial b - \alpha(a) = 0; \\ \beta\big(b - \alpha(a')\big) &= \beta(b) = c; \end{aligned}$$

yield $\beta[b - \alpha(a')] = [c]$, i.e., $\ker \delta \subset \operatorname{im} \beta$.

The inclusion $\operatorname{im} \beta \subset \ker \delta$ follows immediately from the construction of δ.

Step 4. $\ker \beta = \operatorname{im} \alpha$.

Assume $b \in B_k$ satisfies $\beta[b] = 0$, that is $\partial b = 0$ and $\beta(b) = \partial c$ for some $c \in C_{k+1}$. Since β is surjective, there is some $\hat{b} \in B_{k+1}$ such that $\beta(\hat{b}) = c$. Furthermore,

$$\beta(b - \partial\hat{b}) = \beta(b) - \partial\beta(\hat{b}) = \beta(b) - \partial c = 0.$$

This yields that there exists some $a \in A_k$ such that $\alpha(a) = b - \partial\hat{b}$. Moreover,

$$\alpha(\partial a) = \partial\alpha(a) = \partial b - \partial^2\hat{b} = 0.$$

Since α is injective, we obtain $\partial a = 0$. This yields $\alpha[a] = [b - \partial\hat{b}] = [b]$, that is $\ker \beta \subset \operatorname{im} \alpha$.

The inclusion $\operatorname{im} \alpha \subset \ker \beta$ follows immediately from $\alpha \circ \beta = 0$. $\square$

2.6 Relative homology groups

For each subspace $A \subset X$ define

$$S_n(X, A) := S_n(X)/S_n(A).$$

The boundary map on $S_n(X)$ induces a boundary map on $S_n(X, A)$ and we obtain the following new chain complex:

$$\dots \to S_{n+1}(X, A) \xrightarrow{\partial} S_n(X, A) \xrightarrow{\partial} S_{n-1}(X, A) \to \dots$$

The homology groups of this complex are denoted by $H_*(X, A)$ and are called *the homology groups of X relative to A*, or, simply, *relative homology groups*. Let us provide some details of this definition:

- Elements of $H_n(X, A)$ are represented by *relative chains* $a \in S_n(X)$ such that $\partial a \in S_{n-1}(A)$;
- $[a] = 0 \in H_n(X, A) \quad \iff \quad a = \partial b + c, \quad b \in S_{n+1}(X),\ c \in S_n(A).$

By the very definition of $S_n(X, A)$, the sequence $0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$ is exact. Hence, Theorem 2.33 yields the following:

Theorem 2.36. *There is a long exact sequence of the homology groups*

$$\dots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \dots$$

Moreover, the following holds:

- *i_* is induced by the inclusion $i\colon A \subset X$;*
- *j_* is induced by the projection $S_n(X) \to S_n(X, A)$;*
- $\delta[a] = [\partial a]$.

□

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Suppose $A \subset X$ and $B \subset Y$. A map between pairs of spaces (X, A) and (Y, B) is a map $f\colon X \to Y$ such that $f(A) \subset B$.

Proposition 2.37. *Each map $f\colon (X, A) \to (Y, B)$ induces a homomorphism of relative homology groups $H_*(X, A) \to H_*(Y, B)$.* □

Exercise 2.38. Show that the Bockstein homomorphism is natural in the following sense. Let f be as in Proposition 2.37. Denote by $\hat{f}\colon A \to B$ the restriction of f to A. Then the diagram

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\delta"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "\hat{f}_*"] \\
H_n(Y, B) \arrow[r, "\delta"] & H_{n-1}(B)
\end{tikzcd}$$

commutes.

Two continuous maps $f, g\colon (X, A) \to (X, B)$ are called homotopic (as maps between pairs of spaces), if there exists a continuous map $h\colon (X \times I, A \times I) \to (Y, B)$, such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. Notice that the homotopy h in this definition satisfies $h(A \times I) \subset B$.

Two pairs (X, A) and (Y, B) are said to be homotopy equivalent, if there exist $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, A)$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$, where id_X is viewed as a map of pairs $(X, A) \to (X, A)$ (and similarly for id_Y). Just like in the situation of Corollary 2.25, we have the following result.

Proposition 2.39. *If (X, A) and (Y, B) are homotopy equivalent, then $H_k(X, A)$ and $H_k(Y, B)$ are isomorphic for all k.* □

The following theorem, whose proof will be given in Section 2.14 below, turns out to be a useful tool for the computations of relative homology groups. For the time being, we take Theorem 2.40 as granted.

Theorem 2.40 (Excision)**.** *Assume the subspaces $Z \subset A \subset X$ satisfy $\bar{Z} \subset \operatorname{Int} A$. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism of relative homology groups:*

$$H_*(X \setminus Z, A \setminus Z) \cong H_*(X, A).$$

2.7 The homology groups of the spheres

Theorem 2.41. *The following holds:*

$$H_k(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \textit{if } k = 0; \\ 0 & \textit{else;} \end{cases} \qquad \textit{and for } n \geq 1 \qquad H_k(S^n) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, n; \\ 0 & \textit{else.} \end{cases}$$

Proof. Denote

$$S^n = \{x = (x_0, \dots, x_{n+1}) \in S^{n+1} \mid x_{n+1} = 0\},$$
$$S^{n+1}_+ := \{x \in S^{n+1} \mid x_{n+1} \geq 0\}, \qquad S^{n+1}_- := \{x \in S^{n+1} \mid x_{n+1} \leq 0\}.$$

Notice that $S^{n+1}_\pm$ is homeomorphic to $B_{n+1} = \{x \in \mathbb{R}^{n+2} \mid |x| \leq 1, x_{n+1} = 0\}$. In particular, $S^{n+1}_\pm$ is contractible.

Step 1. *The map $\delta\colon H_{k+1}(S^{n+1}_-, S^n) \to H_k(S^n)$ is an isomorphism provided $k \geq 1$.*

By the long exact sequence of the pair (S^{n+1}_-, S^n) we have

$$0 = H_{k+1}(S^{n+1}_-) \to H_{k+1}(S^{n+1}_-, S^n) \xrightarrow{\delta} H_k(S^n) \to H_k(S^{n+1}_-) = 0. \tag{2.42}$$

Hence, δ is an isomorphism.

Step 2. *Define*

$$\tilde{H}_0(S^n) := \ker\bigl(H_0(S^n) \to H_0(S^{n+1}_-)\bigr) \cong \begin{cases} \mathbb{Z} & \textit{if } n = 0, \\ 0 & \textit{else.} \end{cases}$$

Then $\delta\colon H_1(S^{n+1}_-, S^n) \to \tilde{H}_0(S^n)$ is an isomorphism.

Recall that for a connected space X a generator of $H_0(X)$ is the class of any point. Hence, if $n > 0$, then the homomorphism $H_0(S^n) \to H_0(S^{n+1}_-)$ induced by the inclusion is in fact an isomorphism. In particular, $\tilde{H}_0(S^n) = 0$ in this case. However, if $n = 0$, S^0 consists of two points (in particular, has two connected components), whereas S^1_- is connected. Hence, the homomorphism $H_0(S^0) \to H_0(S^1_-)$ is of the form

$$\mathbb{Z}^2 \to \mathbb{Z}, \qquad (a, b) \mapsto a + b$$

and its kernel is $\tilde{H}_0(S^0) = \{(a, -a) \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$.

Furthermore, just like in the previous step, the long exact sequence of the pair (S^{n+1}_-, S^n) yields

$$0 = H_1(S^{n+1}_-) \to H_1(S^{n+1}_-, S^n) \xrightarrow{\delta} H_0(S^n) \to H_0(S^{n+1}_-).$$

In particular, δ is injective and, hence, an isomorphism onto its image in $H_0(S^n)$, which is the kernel of $H_0(S^n) \to H_0(S^{n+1}_-)$, that is $\tilde{H}_0(S^0)$.

Step 3. *For all $k \geq 0$ and $n \geq 0$ the map*

$$j_*\colon H_{k+1}(S^{n+1}) \to H_{k+1}(S^{n+1}, S^{n+1}_+) \tag{2.43}$$

is an isomorphism.

For $k > 0$, this follows from the long exact sequence of the pair (S^{n+1}, S^{n+1}_+):

$$0 = H_{k+1}(S^{n+1}_+) \to H_{k+1}(S^{n+1}) \xrightarrow{j_*} H_{k+1}(S^{n+1}, S^{n+1}_+) \to H_k(S^{n+1}_+) = 0$$

For $k = 0$, we have

$$0 = H_1(S^{n+1}_+) \to H_1(S^{n+1}) \xrightarrow{j_*} H_1(S^{n+1}, S^{n+1}_+) \to \underbrace{H_0(S^{n+1}_+) \to H_0(S^{n+1})}_{\text{isomorphism}} = \mathbb{Z}.$$

Hence, the third arrow represents the zero homomorphism and, therefore, j_* is surjective. Since j_* is injective, this is an isomorphism.

Step 4. *For all $k \geq 0$ the inclusion $p\colon (S^{n+1}_-, S^n) \to (S^{n+1}, S^{n+1}_+)$ induces the isomorphism*

$$p_*\colon H_{k+1}(S^{n+1}_-, S^n) \to H_{k+1}(S^{n+1}, S^{n+1}_+). \tag{2.44}$$

Indeed, denote

$$Z := \left\{x \in S^{n+1} \mid x_{n+1} \geq \tfrac{1}{2}\right\}.$$

Then the homomorphism $H_{k+1}(S^{n+1}_-, S^n) \to H_{k+1}(S^{n+1} \setminus Z,\ S^{n+1}_+ \setminus Z)$ induced by the inclusion $(S^{n+1}_-, S^n) \to (S^{n+1} \setminus Z,\ S^{n+1}_+ \setminus Z)$ is an isomorphism, since the pairs (S^{n+1}_-, S^n) and $(S^{n+1} \setminus Z,\ S^{n+1}_+ \setminus Z)$ are homotopy equivalent. Theorem 2.40 yields that the homomorphism $H_{k+1}(S^{n+1}, S^{n+1}_+) \to H_{k+1}(S^{n+1} \setminus Z,\ S^{n+1}_+ \setminus Z)$ induced by the inclusion is also an isomorphism. This proves (2.44).

Step 5. *We prove this theorem*

A combination of the previous steps yields the sequence of isomorphisms

$$H_{k+1}(S^{n+1}) \xrightarrow{j_*} H_{k+1}(S^{n+1}, S^{n+1}_+) \xrightarrow{p_*^{-1}} H_{k+1}(S^{n+1}_-, S^n) \xrightarrow{\delta} \tilde{H}_k(S^n),$$

where

$$\tilde{H}_k(S^n) = \begin{cases} \tilde{H}_0(S^n), & \text{if } k = 0, \\ H_k(S^n), & \text{if } k > 0. \end{cases}$$

This implies the statement of this theorem. $\square$

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Corollary 2.45. *The n-sphere S^n is not contractible for all $n \geq 0$.* $\square$

For a general topological space X define also

$$\tilde{H}_0(X) := \ker \varepsilon, \qquad \text{where} \quad \varepsilon\colon H_0(X) \to \mathbb{Z}, \quad \varepsilon\Big[\sum n_i x_i\Big] := \sum n_i,$$

and $\tilde{H}_k(X) = H_k(X)$ for $k \geq 1$. Using these notations we have

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n; \\ 0 & \text{else,} \end{cases}$$

for all n.

2.8 The hairy ball theorem

Recall (cf. Definition 1.18) that the degree $\deg f$ of a continuous map $f\colon S^n \to S^n$ is an integer, which is determined by the property

$$f_* a = (\deg f) \cdot a \qquad \text{for all } a \in H_n(S^n).$$

Define *the suspension* $\Sigma f\colon S^{n+1} \to S^{n+1}$ of f via

$$\Sigma f(x_0, \dots, x_{n+1}) = \begin{cases} (0, \dots, 0, x_{n+1}) & \text{if } |x_{n+1}| = 1, \\ \big(tf(\frac{x_0}{t}, \dots, \frac{x_n}{t}), x_{n+1}\big) & \text{if } |x_{n+1}| < 1, \end{cases}$$

where $t = \sqrt{1 - x_{n+1}^2}$.

Proposition 2.46. $\deg \Sigma f = \deg f$.

Proof. By the proof of Theorem 2.41 we have the following commutative diagram

$$\begin{CD} H_{n+1}(S^{n+1}) @>{j_*}>> H_{n+1}(S^{n+1}, S^{n+1}_+) @>{p_*^{-1}}>> H_{n+1}(S^{n+1}_-, S^n) @>{\delta}>> H_n(S_n) \\ @V{\Sigma f_*}VV @V{\Sigma f_*}VV @V{\Sigma f_*}VV @V{f_*}VV \\ H_{n+1}(S^{n+1}) @>{j_*}>> H_{n+1}(S^{n+1}, S^{n+1}_+) @>{p_*^{-1}}>> H_{n+1}(S^{n+1}_-, S^n) @>{\delta}>> H_n(S_n). \end{CD}$$

Denoting $\alpha := \delta \circ p_*^{-1} \circ j_*$, we obtain

$$\Sigma f_*(a) = \alpha^{-1} \circ f_* \circ \alpha(x) = \alpha^{-1}\big((\deg f) \cdot \alpha(a)\big) = (\deg f) \cdot a \implies \deg \Sigma f = \deg f.$$

$\square$

Theorem 2.47. *There is no continuous map $f\colon S^{2n} \to \mathbb{R}^{2n+1} \setminus \{0\}$ such that $f(x) \perp x$ holds for all $x \in S^{2n}$.*

Proof. The proof consists of the following steps.

Step 1. *Let*

$$s_0\colon S^n \to S^n, \qquad (x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n),$$

be the restriction of the reflection in the hyperplane $\{x_0 = 0\}$. Then $\deg s_0 = -1$.

The sequence of isomorphisms

$$H_1(S^1) \xrightarrow{j_*} H_1(S^1, S^1_+) \xrightarrow{p_*^{-1}} H_1(S^1_-, S^0) \xrightarrow{\delta} \tilde H_0(S_0)$$

shows that

$$\sigma(t) = (\sin 2\pi t, \cos 2\pi t)$$

is a generator of $H_1(S^1)$. Since $s \circ \sigma(t) = \sigma(-t)$, we have $s_*[\sigma] = -[\sigma]$ and therefore the claim of this step holds for $n = 1$.

If s_0 is the reflection on S^n, then Σs_0 is the reflection on S^{n+1}. The induction with respect to n yields the proof for all $n > 1$.

Step 2. *For the antipodal map $A\colon S^n \to S^n,\ A(x) = -x$ we have $\deg A = (-1)^{n+1}$.*

The antipodal map on S^n is the composition of $n + 1$ reflections.

Step 3. *If $f\colon S^n \to S^n$ is a continuous map without fixed points, then $f \simeq A$.*

The map

$$F(x,t) := \frac{tf(x) + (t-1)x}{|tf(x) + (t-1)x|}$$

is a well-defined homotopy between f and A.

Step 4. *If $f\colon S^n \to S^n$ is a continuous map such that $f(x) \neq -x$ for all $x \in S^n$, then f is homotopic to the identity map.*

$$\begin{aligned} f(x) \neq -x \implies & A \circ f \text{ has no fixed points } \implies A \circ f \simeq A \implies A \circ A \circ f \simeq A \circ A \\ \implies & \quad f \simeq id. \end{aligned}$$

Step 5. *We prove the hairy ball theorem.*

Assume there exists a continuous map $f\colon S^{2n} \to \mathbb{R}^{2n+1} \setminus \{0\}$ such that $f(x) \perp x$. By renormalizing we can assume without loss of generality that $f\colon S^{2n} \to S^{2n}$. The assumption $f(x) \perp x$ yields in particular that f has no fixed points. By Step 3, f is homotopic to A.
On the other hand, f is homotopic to id by Step 4. This yields a contradiction since

$$A \simeq f \simeq id \implies 1 = \deg id = \deg A = (-1)^{2n+1} = -1.$$

□

This theorem is often informally formulated as follows.

Corollary 2.48. *One can not comb a hairy ball flat without creating a cowlick.* □

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Remark 2.49. Each sphere of odd dimension $2n-1 \geq 1$ admits a continuous map $f\colon S^{2n-1} \to \mathbb{R}^{2n} \setminus \{0\}$ such that $f(x) \perp x$ holds for all $x \in S^{2n-1}$. Indeed,

$$\begin{aligned} S^{2n-1} &= \left\{x = (x_0, x_1, x_2, x_3, \dots, x_{2n-2}, x_{2n-1}) \mid \sum x_i^2 = 1\right\} \\ f(x) &= (x_1, -x_0, x_3, -x_2, \dots, x_{2n-1}, -x_{2n-2}). \end{aligned}$$

Proposition 2.50. *Let $[S^n, S^n]$ be the set of all homotopy classes of continuous maps $S^n \to S^n$, where $n \geq 1$. The map*

$$[S^n, S^n] \to \mathbb{Z}, \qquad [f] \mapsto \deg f \tag{2.51}$$

is surjective.

Proof. If $n = 1$, for each $k \in \mathbb{Z}$ we have an explicit continuous map $f_k\colon S^1 \to S^1$ of degree k, namely $f_k(z) := z^k$. If $n = 2$, we have $\deg \Sigma f_k = \deg f_k = k$. The induction with respect to n finishes the proof. □

Remark 2.52. It can be shown that (2.51) is even bijective (Theorem of Hopf). Also, $[S^n, S^n]$ is a group and (2.51) is an isomorphism of groups.

2.9 Group actions on the spheres

Let G be a group. We say that G acts on a set X if a homomorphism $\rho\colon G \to \mathrm{Aut}(X)$ is given, where $\mathrm{Aut}(X)$ is the group of all bijective maps $X \to X$. An action is called *free* whenever the following holds:

$$\forall x \in X \quad \mathrm{Stab}_x := \{g \in G \mid \rho(g)(x) = x\} = \{e\}.$$

If X is in addition a topological space, then we require also that for each $g \in G$ the map $\rho(g)$ is a homeomorphism.

Theorem 2.53. *$\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that acts freely on S^{2n}.*

Proof. Assume that $G \neq \{e\}$ acts on S^{2n} freely. Consider the map

$$d\colon G \to \{\pm 1\}, \qquad d(g) = \deg(\rho(g)).$$

Here d takes values in $\{\pm 1\}$, since each $\rho(g)$ is a homeomorphism. Furthermore, $d(gh) = \deg(\rho(g)\rho(h)) = d(g)d(h)$, that is d is a group homomorphism.

If $g \neq e$, then $\rho(g)$ has no fixed points. By Steps 2 and 3 in the proof of Theorem 2.47, the following holds: $\deg \rho(g) = \deg A = -1$, i.e., d has a trivial kernel and is surjective.

Clealy $\mathbb{Z}/2\mathbb{Z}$ acts freely on S^{2n}:

$$\rho(e) = \mathrm{id}, \qquad \rho(1) := A,$$

where A is the antipodal map. □

Remark 2.54. On the odd-dimensional spheres other non-trivial groups may act freely. For example, $\mathrm{U}(1) := \{z \in \mathbb{C} \mid |z| = 1\} \cong S^1$ acts on

$$S^{2n-1} = \{(z_0, \dots, z_n) \in \mathbb{C}^n \mid \sum |z_j|^2 = 1\}$$

via the homomorphism

$$w \mapsto f_w, \qquad f_w(z) = (wz_0, \dots, wz_n).$$

2.10 Homology groups of graphs

Definition 2.55. A (finite topological) graph is a pair (G, V), where G is a Hausdorff space and $G \supset V$ is a finite subset. The elements of V are called vertices of G. Besides, we require that the following holds:

- $G \setminus V$ consists of finitely many path components $\mathring{e}_1, \dots, \mathring{e}_J$. The closure e_j of each component $\mathring{e}_j$ is homeomorphic to the interval $[0, 1]$ and is called an edge of G;
- $e_j \setminus \mathring{e}_j$ consists of two different vertices.

The aim of this section is to prove the following result.

Theorem 2.56. *The group $H_1(G)$ is free and finitely generated. Moreover, the following holds:*

$$\mathrm{rk}\, H_0(G) - \mathrm{rk}\, H_1(G) = \#\,\textit{vertices} - \#\,\textit{edges} =: \chi(G).$$

The number $\chi(G)$ is called the Euler characteristic of G.

The proof requires some notions and auxiliary claims that we consider first. The proof of Theorem 2.56 can be found at the end of this section.

Definition 2.57. A subset $A \subset B$ is called a deformation retract of B, if the following holds: There exists a continuous map $r\colon B \to A$, which is called *a retraction*, such that the following holds:

$$r \circ \imath = \mathrm{id}_A \quad \text{and} \quad \imath \circ r \simeq id_B,$$

where $\imath\colon A \subset B$ is the inclusion.

It follows immediately from the above definition that the induced maps

$$\imath_*\colon H_*(A) \to H_*(B) \quad \text{and} \quad r_*\colon H_*(B) \to H_*(A)$$

are mutually inverse. In particular, both maps are isomorphisms.

Lemma 2.58. *Let A be a deformation retract of B, where $A \subset B \subset X$. Then the inclusion $\imath\colon (X, A) \to (X, B)$ induces an isomorphism*

$$\imath_*\colon H_*(X, A) \to H_*(X, B).$$

Proof. The proof of this lemma hinges on the following algebraic fact.

Lemma 2.59 ("Five lemma")**.** *Assume the horizontal sequences in the commutative diagram of abelian groups*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

are exact. Furthermore, assume that f_2 and f_4 are isomorphisms, f_1 is an epimorphism, and f_5 is a monomorphism. Then f_3 is an isomorphism. □

Consider the commutative diagram

$$\begin{tikzcd}
H_k(A) \arrow[r] \arrow[d, "\imath_*"'] & H_k(X) \arrow[r] \arrow[d, "id"'] & H_k(X, A) \arrow[r] \arrow[d, "\imath_*"'] & H_{k-1}(A) \arrow[r] \arrow[d, "\imath_*"'] & H_{k-1}(X) \arrow[d, "id"'] \\
H_k(B) \arrow[r] & H_k(X) \arrow[r] & H_k(X, B) \arrow[r] & H_{k-1}(B) \arrow[r] & H_{k-1}(X).
\end{tikzcd}$$

Here the horizontal sequences are long exact sequences of the pairs (X, A) and (X, B). Furthermore, the first two vertical arrows and the last two ones represent isomorphisms. The proof now follows from the five lemma. □

From the long exact sequence of the pair $\big([0, 1], \{0, 1\}\big)$ we obtain the following result.

Lemma 2.60. *The following holds:*

$$H_k\big([0, 1], \{0, 1\}\big) \cong \begin{cases} \mathbb{Z} & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

□

Proposition 2.61. *The inclusion* $\imath_j\colon (e_j, \partial e_j) \to (G, V)$ *induces a monomorphism*

$$\imath_{j*}\colon H_k(e_j, \partial e_j) \to H_k(G, V).$$

Moreover, the following holds:

$$H_k(G,V) = \bigoplus_j \operatorname{im} \imath_{j*} \cong \begin{cases} \mathbb{Z}^J & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

Proof. Let $f_j\colon [0,1] \to e_j$ be a homeomorphism, $a_j := f(\frac{1}{2})$, and $d_j := f([\frac{1}{4}, \frac{3}{4}])$. Denote also $A = \{a_1, \dots, a_J\}$ and $D = d_1 \sqcup \dots \sqcup d_J$. Consider the commutative diagram

$$\begin{tikzcd}
H_k\big(d_j, d_j \setminus \{a_j\}\big) \arrow[r, "\alpha_1"] \arrow[d] & H_k(e_j, e_j \setminus \{a_j\}) \arrow[d] & H_k(e_j, \partial e_j) \arrow[l, "\beta_1"'] \arrow[d] \\
H_k(D, D \setminus A) \arrow[r, "\alpha_2"] & H_k(G, G \setminus A) & H_k(G, V). \arrow[l, "\beta_2"']
\end{tikzcd}$$

All four horizontal homomorphisms are in fact isomorphisms. Indeed, α_1 and α_2 are isomorphisms by excision, β_1 and β_2 by Lemma 2.58.

Since

$$H_k(D, D \setminus A) = \bigoplus_{j=1}^J H_k(d_j, d_j \setminus \{a_j\}) \cong \bigoplus_{j=1}^J H_k(e_j, \partial e_j),$$

we obtain the claim of this proposition. $\square$

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Proof of Theorem 2.56. For the proof we need the following algebraic fact.

Lemma 2.62. *Any subgroup of a free abelian group is also free.* $\square$

The remaining part of the proof consists of the following three steps.

Step 1. $H_1(G)$ *is free.*

The long exact sequence of the pair (G, V) yields:

$$0 \to H_1(G) \to H_1(G,V) \to H_0(V) \to H_0(G) \to 0. \tag{2.63}$$

$H_1(G,V)$ is free $\implies$ $H_1(G)$ is free.

Step 2. *Let* $f\colon A \to F$ *be an epimorphism between two finitely generated free abelian groups. Then*

$$A = \ker f \oplus A_0,$$

where $f\colon A_0 \to F$ *is an isomorphism and* $\ker f$ *is free.*

Let $f_1, \dots, f_n$ be generators of F. Choose $b_1, \dots, b_n \in A$ such that $f(b_j) = f_j$. Since $\ker f \subset A$ and A is free, $\ker A$ is also free. Pick generators $a_1, \dots, a_k$ of $\ker f$. Then we have $A = \mathbb{Z}[a_1, \dots, a_k, b_1, \dots b_n]$. Indeed, for an arbitrary element $a \in A$ we have

$$f(a) \in F \implies f(a) = \sum m_j f_j \implies a - \sum m_j b_j \in \ker f \implies a - \sum m_j b_j = \sum p_i a_i.$$

Moreover, the representation $a = \sum m_j b_j + \sum p_i a_i$ is unique.

Step 3. *We prove this theorem.*

Without loss of generality we can assume that G is path connected. Then (2.63) yields

$$0 \to H_1(G) \to H_1(G,V) \to \tilde{H}_0(V) \to 0,$$

i.e., $H_1(G,V) \cong H_1(G) \oplus \tilde{H}_0(V)$. This yields in turn

$$\#\,\text{edges} = \operatorname{rk} H_1(G,V) = \operatorname{rk} H_1(G) + \operatorname{rk} \tilde{H}_0(V) = \operatorname{rk} H_1(G) + \#\,\text{vertices} - 1.$$

□

Example 2.64. The circle $G = e_0 \cup e_1$, $V = \{v_1, v_2\}$. We have $\chi(G) = 0 \implies rkH_1(G) = \operatorname{rk} H_0(G) = 1$.

Example 2.65. The wedge product of two circles is a graph shown on Fig. 2.2. Since $\chi(G) = -1$, we have $\operatorname{rk} H_1(G) = 2$.

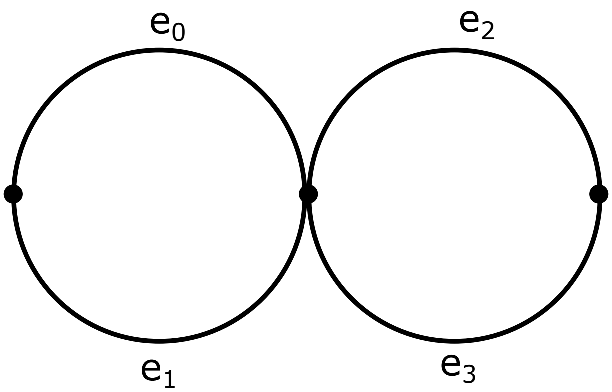

Figure 2.2: The wedge product of two circles.

Definition 2.66. A graph (G,V) is called *planar*, if there is an embedding of G into $\mathbb{R}^2$, that is if G can be drawn on the plane such that edges are represented by simple continuous curves that intersect only at the vertices.

Each connected planar graph decomposes $\mathbb{R}^2$ into a finite number of bounded domains, which are called *faces*, and an unbounded domain, which is also called a face. Moreover, each bounded domain is homeomorphic to a disc (a theorem of Schoenflies).

Theorem 2.67 (Euler)**.** *For any planar connected graph G we have*

$$\#\,\textit{vertices} - \#\,\textit{edges} + \#\,\textit{faces} = 2. \tag{2.68}$$

Notice that the unbounded face also counts in (2.68).

Proof. By means of the stereographic projection we can view G as a subspace of S^2. Notice that the unbounded face together with the point at infinity is mapped to a face on S^2.

Just like in the proof of Proposition 2.61 we obtain

$$H_2(S^2, G) \cong \mathbb{Z}^F \qquad \text{and} \qquad H_k(S^2, G) = 0 \quad \text{for all } k \notin \{0,\, 2\},$$

where F is the number of faces. From the long exact sequence of the pair (S^2, G) we have

$$0 \to H_2(S^2) \to H_2(S^2, G) \to H_1(G) \to H_1(S^2) = 0,$$

which yields

$$\mathbb{Z}^F \cong \mathbb{Z} \oplus H_1(G) \implies F = 1 + \operatorname{rk} H_0(G) - \#\,\text{vertices} + \#\,\text{edges}$$

by Theorem 2.56. Since G is connected by the hypothesis, we have $\operatorname{rk} H_0(G) = 1$ and therefore (2.68) holds. $\square$

Exercise 2.69. Solve the "Three utilities problem": Suppose there are three cottages on a plane and each needs to be connected to the water, gas, and electricity companies. Without using a third dimension or sending any of the connections through another company or cottage, is there a way to make all nine connections without any of the lines crossing each other?

Hint: to obtain a solution consider the graph $K_{3,3}$:

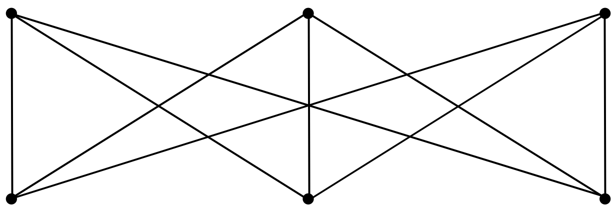

Figure 2.3: Graph $K_{3,3}$.

Assuming $K_{3,3}$ is planar, show that the following holds:

(i) $\#\,\text{faces} \leq \frac{1}{2}\#\,\text{edges}$;

(ii) $\#\,\text{edges} \leq 2\#\text{vertices} - 4$.

Deduce from the last property that $K_{3,3}$ is non-planar.

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2.11 Homology groups of surfaces

2.11.1 The torus

The torus $\mathbb{T}^2$ can be understood as a square R with opposite sides being glued as shown on Fig 2.4.

Let $f\colon R \to \mathbb{T}^2$ be the quotient map. Then $f(\partial R)$ consists of two circles A and B intersecting at a point.

Theorem 2.70.

$$H_k(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & \textit{for } k = 0, 2; \\ \mathbb{Z}^2 & \textit{for } k = 1; \\ 0 & \textit{else}. \end{cases}$$

Proof. The proof consists of the following three steps.

Step 1. *The map* $f\colon (R,\partial R)\to(\mathbb{T}^2, A\cup B)$ *induces an isomorphism*

$$f_*\colon H_*(R,\partial R)\to H_*(\mathbb{T}^2, A\cup B).$$

Let m be the center of the square R and D a disc centered at m contained in the interior of R. Just like in the proof of Proposition 2.61 one obtains that all horizontal arrows of the commutative diagram

$$\begin{tikzcd} H_k\big(R,\partial R\big) \arrow[r] \arrow[d, "f_*"'] & H_k(R, R\setminus\{m\}) \arrow[d] & H_k(D, D\setminus\{m\}) \arrow[l] \arrow[d, "f_*"] \\ H_k(\mathbb{T}^2, A\cup B) \arrow[r] & H_k\big(\mathbb{T}^2,\mathbb{T}^2\setminus\{f(m)\}\big) & H_k\big(f(D), f(D)\setminus\{f(m)\}\big) \arrow[l] \end{tikzcd}$$

represent isomorphisms (to prove this one needs in particular that $A\cup B$ is a deformation retract of $\mathbb{T}^2\setminus\{m\}$). Since the right vertical arrow represents an isomorphism, we obtain that the leftmost vertical arrow represents an isomorphism too.

Step 2. *If* $k\geq 1$, *then*

$$H_k(\mathbb{T}^2, A\cup B)\cong\begin{cases}\mathbb{Z} & \textit{for } k=2,\\ 0 & \textit{else.}\end{cases}$$

The statement of this step follows from the long exact sequence of the pair $(R,\partial R)$.

Step 3. *We prove this theorem.*

The non-trivial part of the long exact sequence of the pair $(\mathbb{T}^2, A\cup B)$ has the following form

$$0\to H_2(\mathbb{T}^2)\to H_2(\mathbb{T}^2, A\cup B)\xrightarrow{\delta} H_1(A\cup B)\to H_1(\mathbb{T}^2)\to 0,$$

where $H_2(\mathbb{T}^2, A\cup B)\cong\mathbb{Z}$ and $H_1(A\cup B)\cong\mathbb{Z}^2$ by Example 2.65.

To determine δ, consider the commutative diagram

$$\begin{tikzcd} H_2\big(R,\partial R\big) \arrow[r, "\delta'"] \arrow[d, "f_*"'] & H_1(\partial R) \arrow[d, "f'_*"] \\ H_2(\mathbb{T}^2, A\cup B) \arrow[r, "\delta"] & H_1\big(A\cup B\big), \end{tikzcd}$$

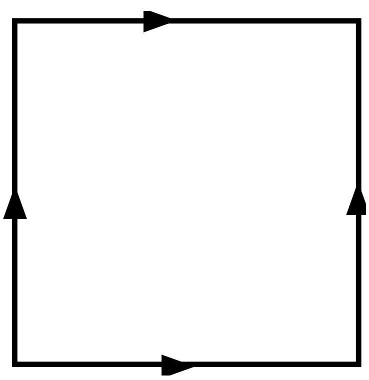

Figure 2.4: The torus as a square with opposite sides being glued.

where $f' \colon \partial R \to A \cup B$ is the restriction of f. The induced map f'_* is trivial (*Why?*). Since f_* and δ' are isomorphisms, δ must be also trivial. This yields

$$H_2(\mathbb{T}^2) \cong \ker \delta = H_2(\mathbb{T}^2, A \cup B) \cong \mathbb{Z} \qquad \text{and} \qquad H_1(\mathbb{T}^2) \cong H_1(A \cup B) \cong \mathbb{Z}^2.$$

This finishes the proof. □

In fact, tracing through the above proof we can work out the generators of $H_1(\mathbb{T}^2)$. Indeed, it was shown that the inclusion $A \cup B \subset \mathbb{T}^2$ induces an isomorphism $H_1(A \cup B) \to H_1(\mathbb{T}^2)$. Hence, the circles A and B generate $H_1(\mathbb{T}^2)$. L 11

2.11.2 The projective plane

The projective plane $\mathbb{RP}^2$ can be defined as a square R with the opposite sides being glued as shown on Figure 2.5.

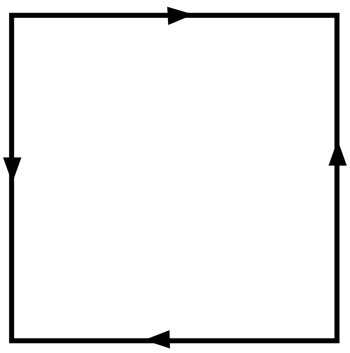

Figure 2.5: The real projective plane as a square with opposite sides being glued.

Let $f \colon R \to \mathbb{RP}^2$ be the quotient map. Then, unlike in the case of the torus, $A := f(\partial R)$ is a circle in $\mathbb{RP}^2$.

Theorem 2.71.

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & \textit{for } k = 0; \\ \mathbb{Z}/2\mathbb{Z} & \textit{for } k = 1; \\ 0 & \textit{else}. \end{cases}$$

Proof. Just like in the proof of Theorem 2.70 we obtain that

$$f_* \colon H_*(R, \partial R) \to H_*(\mathbb{RP}^2, A)$$

is an isomorphism. The non-trivial part of the long exact sequence of the pair $(\mathbb{RP}^2, A)$ is of the following form:

$$0 \to H_2(\mathbb{RP}^2) \to H_2(\mathbb{RP}^2, A) \xrightarrow{\delta} H_1(A) \xrightarrow{i_*} H_1(\mathbb{RP}^2) \to 0.$$

To determine the Bockstein homomorphism δ, consider the commutative diagram

$$\begin{tikzcd} H_2(R, \partial R) \arrow[r, "\delta'"] \arrow[d, "f_*"'] & H_1(\partial R) \arrow[d, "f'_*"] \\ H_2(\mathbb{RP}^2, A) \arrow[r, "\delta"] & H_1(A). \end{tikzcd}$$

A short thought yields that f'_* is a multiplikation with ± 2 (*Why?*), i.e., δ is injective and $H_1(A)/\operatorname{im}\delta \cong \mathbb{Z}/2\mathbb{Z}$. In particular, $H_2(\mathbb{RP}^2) \cong \ker\delta = \{0\}$ and $i_* \colon H_1(A)/\operatorname{im}\delta \to H_1(\mathbb{RP}^2)$ is an isomorphism $\square$

2.11.3 The Klein bottle

Just like torus and projective plane, the Klein bottle K can be also defined as a square R with glued opposite sides as shown on Figure 2.6.

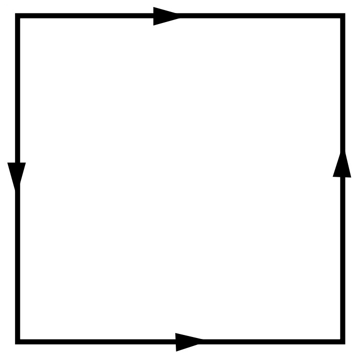

Figure 2.6: The Klein bottle as a square with opposite sides being glued.

Theorem 2.72.

$$H_k(K) = \begin{cases} \mathbb{Z} & \textit{for } k = 0; \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \textit{for } k = 1; \\ 0 & \textit{else}. \end{cases}$$

The proof of this theorem is left as an exercise. L 12

2.11.4 Connected sum of manifolds

Let me recall the definition of a manifold.

Definition 2.73. A (topological) manifold of dimension n is a Hausdorff space[1] M such that for each point $m \in M$ there exists a neighborhood, which is homeomorphic to an open subset in $\mathbb{R}^n$.

Manifolds of dimension 1 are usually called *curves* and manifolds of dimension two *surfaces*.

Exercise 2.74. Show that for each $x_0 \in \mathbb{R}^n$ and $r > 0$ the open ball $\mathring{B}_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$ is homeomorphic to $\mathbb{R}^n$. Furthermore, using this show that each point of a manifold has a neighborhood homeomorphic to $\mathbb{R}^n$.

Example 2.75.

[1] In addition, it is required that M satisfies the second countability axiom, i.e., M has at most countable basis of its topology. This is not crucial for the arguments used below, hence I do not mention this explicitly in the definition.

- $\mathbb{R}^n$ is an n-manifold; More generally, any open subset of $\mathbb{R}^n$ is an n-manifold;
- S^n is an n-manifold;
- The torus, projective plane, and Klein bottle are surfaces;

Let M_1 and M_2 be two connected manifolds of dimension n. Choose $m_j \in M_j$ and homeomorphisms $\varphi_j\colon B_1(0) \to U_j \subset M_j$ such that $\varphi_j(0) = m_j$. With the help of the identification $B_1(0) \setminus \{0\} \cong S^{n-1} \times (0,1)$, φ_j induces a homeomorphism $S^{n-1} \times (0,1) \to U_j \setminus \{m_j\}$.

Definition 2.76. The space

$$\begin{aligned} &M_1 \# M_2 := \big(M_1 \setminus \{m_1\} \sqcup M_2 \setminus \{m_2\}\big)/\sim, \quad \text{where} \\ &\varphi_1(x,r) \sim \varphi_2(x, 1-r), \qquad x \in S^{n-1} \text{ and } r \in (0,1), \end{aligned}$$

is called *the connected sum* of M_1 and M_2.

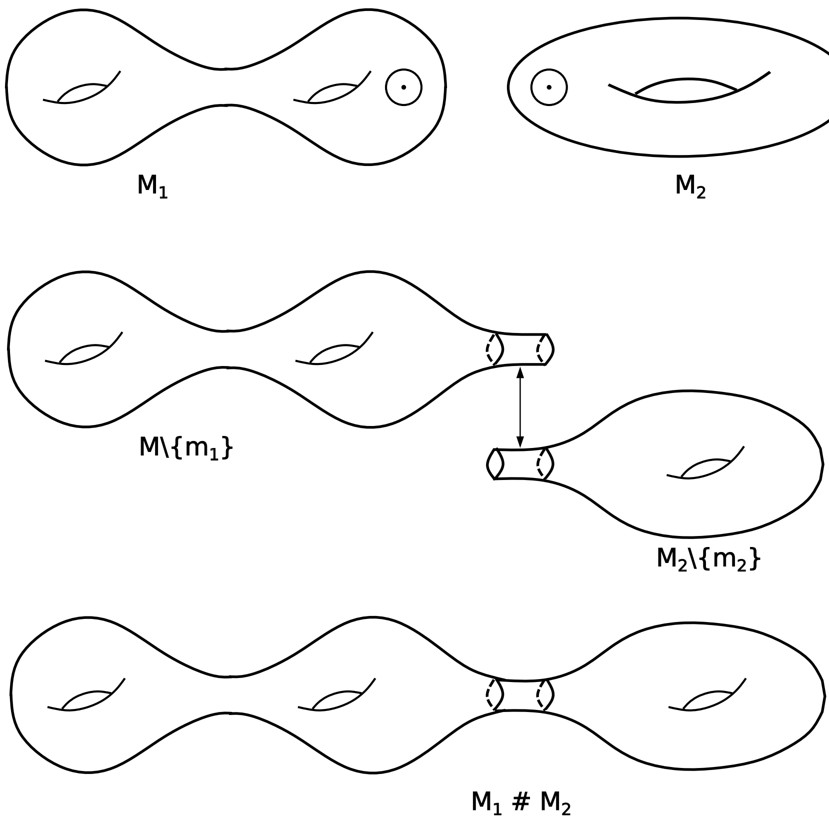

Figure 2.7: Connected sum of two surfaces.

Exercise 2.77. Show that $M_1 \# M_2$ is a manifold of dimension n and does not depend on the choices involved in the construction (meaning the following: For any other choice of points m_j and homeomorphisms φ_j the results of the above construction are homeomorphic).

2.11.5 Compact surfaces

Denote

$$\Sigma_0 = S^2, \quad \Sigma_1 = \mathbb{T}^2, \quad \Sigma_2 = \mathbb{T}^2 \# \mathbb{T}^2, \ \ldots, \quad \Sigma_g = \#_g \mathbb{T}^2.$$

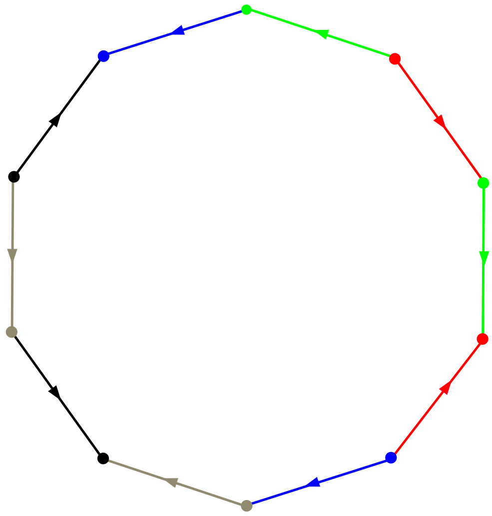

Figure 2.8: Σ_2 from a decagon.

Proposition 2.78. *The surface Σ_2 can be constructed from the Decagon via gluing of sides as indicated on Fig. 2.8.*

Proof. First construct the "connected sum of squares" as shown on Figure 2.9. To obtain Σ_2 from this we still need to glue the opposite sides of the two "squares" as indicated on the picture.

Pick a segment connecting two vertices of the squares as shown on the Figure 2.9 (the colored segment) and cut the "connected sum" along this segment. The result of this is a decagon. This means that we can obtain Σ_2 after gluing appropriate sides of this decagon. $\square$

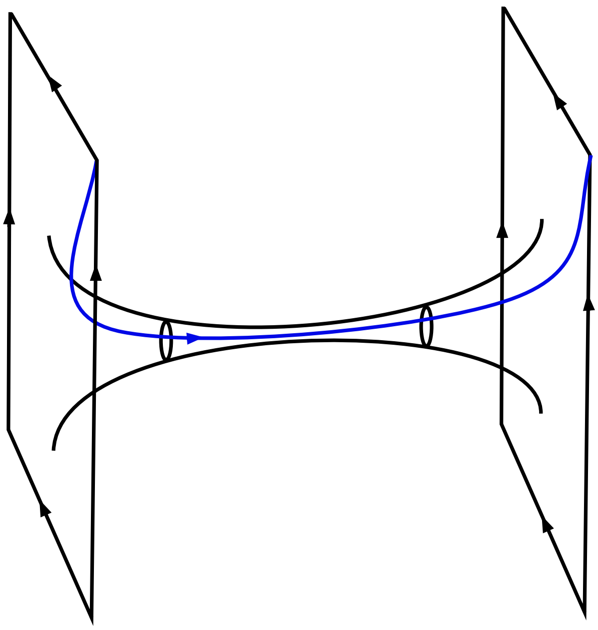

Figure 2.9: The connected sum of two tori represented by squares.

Induction with respect to g yields the following.

Corollary 2.79. *For each $g \geq 1$ the surface Σ_g can be constructed from $(6g-2)$-gon R_{6g-2} via gluing of sides.* □

Remark 2.80. The representation of Σ_g in the above corollary is not optimal in the following sense: Σ_g can be obtained from a $(2g+2)$-gon via gluing of sides. For our purposes the existence of some representation will suffice.

By the inspection of the construction of Σ_g from R_{6g-2} just like in the proof of Step 3 of Theorem 2.70, we obtain the following.

Proposition 2.81. *If $f\colon R_{6g-2} \to \Sigma_g$ denotes the quotient map, then the induced homomorphism $H_1(\partial R_{6g-2}) \to H_1\big(f(\partial R_{6g-2})\big)$ is trivial.* □

Theorem 2.82. *We have*

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, 2; \\ \mathbb{Z}^{2g} & \textit{if } k = 1; \\ 0 & \textit{else.} \end{cases} \tag{2.83}$$

□

The proof of this theorem uses Proposition 2.81 and the argument is parallel to the one used in the proof of Theorem 2.70. The details are left to the reader.

Denote also

$$S_1 := \mathbb{RP}^2, \quad S_2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \quad \text{und} \quad S_g = S_{g-1} \# \mathbb{RP}^2.$$

Just like in Theorem 2.82 one can show, that the homology groups of S_g are given by

$$H_k(S_g) = \begin{cases} \mathbb{Z} & \text{if } k = 0; \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k = 1; \\ 0 & \text{else.} \end{cases}$$

In particular, the computations above yield the following.

Proposition 2.84. *The surfaces*

$$\Sigma_0,\ \Sigma_1, \dots, \Sigma_g, \dots, \quad S_1,\ S_2, \dots, S_g, \dots \tag{2.85}$$

are pairwise non-homeomorphic. □

Theorem 2.86 (Classification of curves)**.** *Each connected curve (i.e., 1-manifold) is homeomorphic either to the interval $(0, 1)$ or to the circle S^1.*

Proof. See [Mil65] or [GP74]. □

Theorem 2.87 (Classification of compact surfaces)**.** *Each compact connected surface is homeomorphic to Σ_g or S_g for some $g \geq 0$, that is (2.85) is a complete list of all compact surfaces up to homeomorphisms.* □

2.12 The Meyer–Vietoris sequence

Let $A, B \subset X$ be two subsets. Consider the homomorphisms

$$i_*\colon H_*(A\cap B) \to H_*(A), \quad j_*\colon H_*(A\cap B)\to H_*(B),$$
$$k_*\colon H_*(A)\to H_*(X) \quad \text{and} \quad l_*\colon H_*(B)\to H_*(X).$$

Furthermore, define

$$\begin{aligned} &\varphi\colon H_*(A\cap B)\to H_*(A)\oplus H_*(B), && \varphi(x) = (i_*(x), j_*(x)) \quad \text{and} \\ &\psi\colon H_*(A)\oplus H_*(B)\to H_*(X), && \psi(u,v) = k_*(u) - l_*(v). \end{aligned} \tag{2.88}$$

Theorem 2.89. *If $X = \operatorname{Int}(A)\cup\operatorname{Int}(B)$, then for all $k\in\mathbb{N}$ there is a natural homomorphism*

$$\Delta\colon H_k(X)\to H_{k-1}(A\cap B)$$

such that the sequence

$$\dots \to H_k(A\cap B) \xrightarrow{\varphi} H_k(A)\oplus H_k(B) \xrightarrow{\psi} H_k(X) \xrightarrow{\Delta} H_{k-1}(A\cap B) \to \dots \tag{2.90}$$

is exact. This sequence is also exact for $\tilde{H}_*$ whenever $A\cap B\neq\emptyset$.

We postpone the proof of this theorem till Section 2.14 below and take this result as granted for the time being.

Example 2.91 (The spheres). Define

$$S^n = \bigl\{(x_0,\dots,x_n) \mid \sum x_i^2 = 1\bigr\},$$
$$A := S^n\setminus\{(0,\dots,0,1)\}\cong\mathbb{R}^n, \quad B := S^n\setminus\{(0,\dots,0,-1)\}\cong\mathbb{R}^n.$$

Since $A\cap B\cong\mathbb{R}^n\setminus\{0\}$ and S^{n-1} is a deformation retract of $\mathbb{R}^n\setminus\{0\}$, we have the following exact sequence:

$$0\to\tilde{H}_k(S^n)\to\tilde{H}_{k-1}(S^{n-1})\to 0.$$

This yields immediately that the homology groups of the spheres are as described in Theorem 2.41. L 13

Example 2.92 (The torus). Let $D_1\subset D_2\subset\operatorname{Int}(R)$ be two discs with the same center. Setting $A := \mathbb{T}^2\setminus D_1$ and $B := D_2$, the following holds:

- The wedge product of two circles ($A\cup B$ in the notation of Subsection 2.11.1) is a deformation retract of $\mathbb{T}^2\setminus D_1$;
- S^1 is the deformation retract of $A\cap B$.

Using these properties and the Mayer–Vietoris sequence, we have:

$$0\to H_2(\mathbb{T}^2)\to H_1(S^1)\xrightarrow{\varphi} H_1(\mathbb{T}^2\setminus D_1)\oplus 0\to H_1(\mathbb{T}^2)\to\tilde{H}_0(S^1) = 0.$$

Since φ is the zero homomorphism (*why?*), we obtain:

$$H_2(\mathbb{T}^2)\cong H_1(S^1)\cong\mathbb{Z} \quad\text{and}\quad H_1(\mathbb{T}^2)\cong H_1(S^1\vee S^1)\cong\mathbb{Z}^2.$$

Exercise 2.93. Compute the homology groups of the projective plane and the Klein bottle using the Meyer–Vietoris sequence.

Definition 2.94. Let X and Y be two topological spaces with chosen points $x_0 \in X$ and $y_0 \in Y$. The space

$$X \vee Y = \big(X \sqcup Y\big)/\{x_0, y_0\}$$

is called *the wedge product* of (X, x_0) and (Y, y_0).

Proposition 2.95. *If x_0 is a deformation retract of a neighborhood $U \subset X$ and y_0 is a deformation retract of a neighborhood $V \subset Y$, then*

$$\tilde{H}_*\big(X \vee Y\big) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y).$$

Proof. Set $A = X \cup V$ and $B = Y \cup U$. Then $U \cup V$ retracts onto the point $[x_0] = [y_0]$ in $X \vee Y$. One obtains the claim of this proposition immediately from the Meyer–Vietoris sequence. □

Corollary 2.96. *For all $n \geq 1$ we have*

$$\tilde{H}_k\Big(\bigvee_{j=1}^{N} S^n\Big) \cong \begin{cases} \mathbb{Z}^N & \textit{if } k = n, \\ 0 & \textit{else.} \end{cases}$$

□

2.13 Homology groups of a pair and a quotient

Let G be an abelian group and $K \subset H \subset G$ subgroups. Recall that this yields the following exact sequence:

$$0 \to H/K \to G/K \to G/H \to 0$$

For $B \subset A \subset X$, this yields the following exact sequence

$$0 \to S_*(A, B) \to S_*(X, B) \to S_*(X, A) \to 0.$$

By Theorem 2.33 we obtain *the long exact sequence of the triple* (X, A, B):

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \dots$$

Theorem 2.97. *Let $A \subset X$ be a closed subset such that A is a deformation retract of a neighborhood $U \supset A$. Then the quotient map $q\colon (X, A) \to (X/A, A/A)$ induces an isomorphism*

$$q_*\colon H_*(X, A) \to H_*(X/A, A/A) \cong \tilde{H}_*(X/A).$$

Proof. The proof consists of the following two steps.

Step 1. *$\imath_*\colon H_*(X, A) \to H_*(X, U)$ is an isomorphism.*

Since A is a deformation retract of U, we have that the map $H_*(A) \to H_*(U)$ induced by the inclusion is an isomorphism. From the long exact sequence of the pair (U, A) we obtain that $H_*(U, A)$ is trivial. An application of the long exact sequence of the triple (X, U, A)

$$0 = H_n(U, A) \to H_n(X, A) \to H_n(X, U) \to H_{n-1}(U, A) = 0$$

finishes the proof of this step.

Step 2. *We prove this theorem.*

Consider the commutative diagram

$$\begin{tikzcd}
H_k(X,A) \arrow[r] \arrow[d, "q_*"'] & H_k(X,U) \arrow[d] & H_k(X\setminus A,\ U\setminus A) \arrow[l] \arrow[d, "q_*"] \\
H_k(X/A, A/A) \arrow[r] & H_k(X/A, U/A) & H_k(X/A\setminus A/A,\ U/A\setminus A/A). \arrow[l]
\end{tikzcd}$$

By Step 1, the two left horizontal arrows represent isomorphisms. The right horizontal arrows also represent isomorphisms by excision. The right vertical arrow also represents an isomorphism, since the restriction of q to the complement of A is a homeomorphism. Hence, q_* on the left is also an isomorphism.

Finally, the long exact sequence of the pair (X, x_0), where $x_0 \in X$, shows that $\tilde{H}_*(X)$ and $H_*(X/A, A/A)$ are isomorphic. □

2.14 Proof of the exactness of the Mayer–Vietoris sequence and excision

Let $\mathcal{U} = \{U_j\}$ be a family of subsets of X such that $\{\mathrm{Int}(U_j)\}$ is a covering of X. Denote

$$S^{\mathcal{U}}_*(X) := \Big\{\sum_i n_i\sigma_i \mid \forall i \quad \exists j \quad \text{such that } \operatorname{im}\sigma_i \subset U_j\Big\}.$$

Clearly, $S^{\mathcal{U}}_*(X)$ is a subcomplex of $S_*(X)$. Denote by $H^{\mathcal{U}}_*(X)$ the homology groups of this complex. The main step in the proof of the excision theorem is the following.

Proposition 2.98. *The inclusion $\imath\colon S^{\mathcal{U}}_*(X) \to S_*(X)$ is a chain homotopy equivalence. In particular, $H^{\mathcal{U}}_*(X) \cong H_*(X)$.*

Chain homotopy equivalence is not yet defined.

For the proof of this proposition we need some auxiliary claim and constructions. The proof itself can be found on Page 39 below.

Let $\Delta = \Delta(x_0,\dots,x_k)$ be a simplex in an Euclidean space V. For an arbitrary $b \in V$ define the cone of Δ by the formula

$$C_b(\Delta) = \Delta(b, x_0, \dots, x_k). \tag{2.99}$$

Geometrically $C_b(\Delta)$ is the cone of Δ (at least in the case when b is not contained in the affine subspace generated by $x_0,\dots,x_k$).

The point

$$b = b(\Delta) := \frac{1}{k+1}\sum x_j$$

is called *the barycenter* of Δ. *The barycentric subdivision* $\mathrm{Sd}(\Delta)$ is a chain in V, which is defined recursively in k, namely:

$$\begin{aligned} \mathrm{Sd}\big(\Delta(x_0)\big) &= \Delta(x_0) && \text{if } k = 0,\\ \mathrm{Sd}(\Delta) &= C_{b(\Delta)}\big(\mathrm{Sd}(\partial\Delta)\big) && \text{if } k > 0. \end{aligned} \tag{2.100}$$

For example, the barycentric subdivision of the standard 2-simplex is shown on Fig. 2.10.

For an arbitrary subset $A \subset \mathbb{R}^n$ *the diameter* of A is defined by

$$\operatorname{diam} A := \sup_{x,y\in A} |x-y|.$$

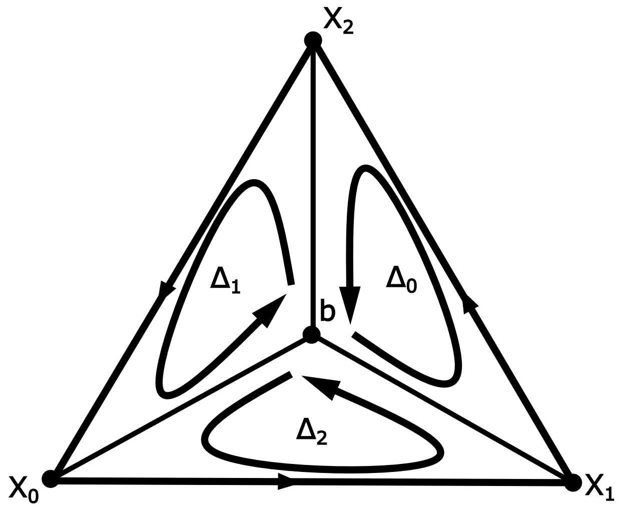

Figure 2.10: The barycentric subdivision of the standard 2-simplex.

Lemma 2.101. *For each simplex Δ', which appears in the representation of* $\mathrm{Sd}(\Delta)$ *as a chain, we have*

$$\operatorname{diam}\Delta' \le \frac{k}{k+1}\operatorname{diam}\Delta. \tag{2.102}$$

Proof. The proof consists of the following two steps.

Step 1. *For $\Delta = \Delta(x_0, \ldots, x_k)$ we have*

$$\operatorname{diam}\Delta = \max_{i,j}|x_i - x_j|$$

Pick $x \in \Delta$ and set $y = \sum t_j x_j \in \Delta$, where $\sum t_j = 1,\ t_j \in [0,1]$. We have

$$\begin{aligned} |x-y| &= \left|x - \sum t_j x_j\right| = \left|\sum t_j(x-x_j)\right| \le \sum t_j|x-x_j| \\ &\le \max_j |x-x_j|. \end{aligned} \tag{2.103}$$

This yields

$$|x-y| \le \max_j |x-x_j| \le \max_{i,j}|x_i - x_j|.$$

Step 2. *We prove this lemma.*

We apply induction with respect to k. For $k = 0$ Inequality (2.102) clearly holds. Furthermore, we assume that this inequality also holds for all $(k-1)$-simplexes in V. Let Δ' be a simplex, which appears in the representation of $\mathrm{Sd}(\Delta)$, that is $\Delta' = \big(b(\Delta), y_0, \ldots, y_{k-1}\big)$, where all y_j are contained in some face $\partial_j\Delta$ of Δ. By Step 1, we obtain

$$\operatorname{diam}\Delta' \le \max\big\{|y_i - y_j|, |b - y_i|\big\}.$$

Furthermore, we have

$$\begin{aligned} |y_i - y_j| &\le \operatorname{diam}\Delta(y_0, \ldots, y_{k-1}) \\ &\le \frac{k-1}{k}\operatorname{diam}\partial_j\Delta && \text{by the induction hypothesis} \\ &\le \frac{k-1}{k}\operatorname{diam}\Delta && \partial_j\Delta \subset \Delta \\ &\le \frac{k}{k+1}\operatorname{diam}\Delta && \text{since } x \mapsto x/(x+1) \text{ is increasing.} \end{aligned}$$

It remains to show that the inequality

$$|b - y_i| \le \frac{k}{k+1} \operatorname{diam} \Delta$$

also holds. Indeed,

$$\begin{aligned}
|b - y_i| &\le |b - x_j| && \text{for some } j \text{ by (2.103)} \\
&= \Big| \frac{1}{k+1} \sum_i x_i - x_j \Big| = \Big| \frac{1}{k+1} \sum_i (x_i - x_j) \Big| \\
&\le \frac{k}{k+1} \max_i |x_i - x_j| \\
&\le \frac{k}{k+1} \operatorname{diam} \Delta.
\end{aligned}$$

Here we have also used the fact that the second sum in the second line has at most k non-trivial summands. □

Let X be a convex subset of an Euclidean space and $\Delta_k \subset \mathbb{R}^{k+1}$ be the standard k-simplex. A map $f\colon \Delta_k \to X$ such that

$$f\Big(\sum t_i y_i\Big) = \sum t_i f(y_i) \qquad \text{for all } y_i \in \Delta_k \ \text{ and all } t_i \ge 0,\ \sum t_i = 1$$

is called *an affine simplex* in X. Clearly, any affine simplex $\Delta_k \to X$ in X is uniquely determined by the images of the vertices. In particular, each affine simplex can be identified with $\Delta(x_0, \dots, x_k)$, where $x_i = f(e_i) \in X$.

Denote by $AS_k(X)$ the free abelian group, which is generated by all affine k-simplexes. Formula (2.8) defines the boundary map on AS_*, that is (AS_*, ∂) is a chain map. Besides, define $AS_{-1}(X) := \mathbb{Z}[\varnothing]$ and $\partial\Delta(x_0) = [\varnothing]$ for all 0-simplexes $\Delta(x_0)$.

Proposition 2.104. *Map* (2.100) *together with* $\operatorname{Sd}(\varnothing) := \varnothing$ *determines a chain map* $\operatorname{Sd}\colon AS_* \to AS_*$ *with the following properties:*

(i) Sd *is chain homotopic to the identity homomorphism;*

(ii) For each simplex Δ'*, which appears in* $\operatorname{Sd}(\Delta)$*, we have* $\operatorname{diam}\Delta' \le \frac{k}{k+1} \operatorname{diam}\Delta$.

Proof. The proof consists of the following three steps.

Step 1. *For each* $b \in X$ *the homomorphism*

$$C_b\colon AS_k(X) \to AS_{k+1}(X),$$

which is determined by (2.99) *and* $C_b(\varnothing) = \{b\}$*, is a chain homotopy between* id *and the trivial homomorphism, that is*

$$\partial C_b + C_b \partial = id. \tag{2.105}$$

The claim of this step follows from the following simple observation:

$$\partial C_b\big(\Delta(x_0, \dots, x_k)\big) = \Delta(x_0, \dots, x_k) - \partial C_b\big(\partial\Delta(x_0, \dots, x_k)\big).$$

Step 2. Sd *is a chain homomorphism.*

Define additionally $\mathrm{Sd}(\varnothing) = \varnothing$. To show that Sd is a chain homomorphism, observe first that $\mathrm{Sd} = id$ on AS_{-1} and AS_0 and therefore we have

$$\partial \circ \mathrm{Sd} = \mathrm{Sd} \circ \partial \tag{2.106}$$

on AS_{-1}. For $k \geq 0$ the proof of (2.106) is obtained by induction:

$$\begin{aligned}
\partial\,\mathrm{Sd}\,\Delta &= \partial\, C_b\, \mathrm{Sd}\,\partial\Delta \\
&= \mathrm{Sd}\,\partial\Delta - C_b(\partial\,\mathrm{Sd}\,\partial\Delta) && (2.105) \\
&= \mathrm{Sd}\,\partial\Delta - C_b(\mathrm{Sd}\,\partial\,\partial\Delta) && \text{by the induction hypothesis} \\
&= \mathrm{Sd}\,\partial\Delta && \partial^2 = 0.
\end{aligned}$$

Step 3. Sd *is chain homotopic to the identity homomorphism.*

Define $T\colon AS_k \to AS_{k+1}$ recursively in k, namely

$$T(\varnothing) = 0 \qquad \text{and} \qquad T\Delta = C_{b(\Delta)}\big(\Delta - T\,\partial\Delta\big).$$

The property

$$T\,\partial + \partial\, T = id - \mathrm{Sd}$$

holds clearly on AS_{-1}. For $k \geq 0$ the proof goes just like above by the induction:

$$\begin{aligned}
\partial\, T\Delta &= \partial\, C_b\big(\Delta - T\,\partial\,\Delta\big) \\
&= \Delta - T\,\partial\,\Delta - C_b\big(\partial\Delta - \partial\, T\,\partial\,\Delta\big) && (2.105) \\
&= \Delta - T\,\partial\,\Delta - C_b\big(\partial\Delta - \partial\Delta + \mathrm{Sd}\,\partial\Delta - T\,\partial\,\partial\,\Delta\big) && \text{by the induction hypothesis} \\
&= \Delta - T\,\partial\,\Delta - \mathrm{Sd}\,\Delta && (2.100).
\end{aligned}$$

To finish the proof of this proposition, it remains only to notice that *(ii)* follows immediately from (2.100) and Lemma 2.101. □

Proof of Proposition 2.98. The proof consists of the following four steps.

Step 1. *Define*

$$\mathrm{Sd}\colon S_*(X) \to S_*(X) \qquad \textit{by} \qquad \mathrm{Sd}(\sigma) = \sigma_\#\big(\mathrm{Sd}(\Delta_k)\big)$$

and similarly also T. Then we have

$$\mathrm{Sd} \circ \partial = \partial \circ \mathrm{Sd} \qquad \textit{and} \qquad T\,\partial + \partial\, T = id - \mathrm{Sd}.$$

The proof is a simple exercise.

Step 2. *(Lebegue's lemma) Let $\mathcal{V}$ be an arbitrary open covering of a compact metric space Y. There is a number $\varepsilon = \varepsilon(\mathcal{V})$ with the following property: Each subset $Z \subset Y$ such that* $\operatorname{diam} Z \leq \varepsilon$ *is contained in some $V_i \in \mathcal{V}$.*

Indeed, by the compactness of Y we obtain that there is an open finite covering of Y by balls $B_{r_i}(y_i)$ such that each ball $B_{2r_i}(y_i)$ is contained in some $V_j \in \mathcal{V}$. Let ε be smaller than the minimum of all r_i.

Furthermore, for any two points $z_1, z_2 \in Y$ such that $d_Y(z_1, z_2) \leq \varepsilon$ we have

$$\exists B_{r_i}(y_i) \ni z_1 \implies d_Y(z_2, y_i) \leq d_Y(z_2, z_1) + d_Y(z_1, y_i) \leq \varepsilon + r_i \leq 2r_i.$$

This shows that $z_2 \in B_{2r_i}(y_i) \subset V_j$.

Step 3. *The following holds:*

(i) Sd^m *is chain homotopic to the identity homomorphism for all* $m \in \mathbb{N}$*;*

(ii) For all $\sigma\colon \Delta_k \to X$ *there exists some* $m \in \mathbb{N}$ *such that* $\mathrm{Sd}^m(\sigma) \in C_k^{\mathcal{U}}(X)$*.*

Define

$$D_m := \sum_{i=0}^{m-1} T \circ \mathrm{Sd}^i.$$

The first claim follows from the following computation:

$$\begin{aligned} \partial\, D_m + D_m\, \partial &= \sum_{i=0}^{m-1} (\partial\, T\, \mathrm{Sd}^i + T\, \mathrm{Sd}^i \partial) = \sum_{i=0}^{m-1} (\partial\, T\, \mathrm{Sd}^i + T\, \partial\, \mathrm{Sd}^i) \\ &= \sum_{i=0}^{m-1} (id - \mathrm{Sd})\mathrm{Sd}^i = id - \mathrm{Sd}^m. \end{aligned}$$

The second claim follows from a combination of Step 2 and Proposition 2.104.

Step 4. *For each* $\sigma\colon \Delta_k \to X$ *let* $m = m(\sigma) \in \mathbb{N}$ *be the minimal integer such that (ii) from Step 3 above holds. Define*

$$D\colon S_k(X) \to S_{k+1}(X), \qquad D\sigma = D_{m(\sigma)}\sigma.$$

Then there exists a chain homomorphism $\rho\colon S_*(X) \to S_*^{\mathcal{U}}(X)$ *such that*

$$D\partial + \partial D = id - \imath\, \rho \qquad \textit{and} \qquad \rho\, \imath = id, \tag{2.107}$$

where $\imath\colon S_*^{\mathcal{U}}(X) \to S_*(X)$ *is the inclusion.*

Define ρ by the equality

$$\partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma) \qquad \Longleftrightarrow \qquad \rho(\sigma) = \sigma - \partial D\sigma - D\partial\sigma.$$

Using the equality $\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) = \sigma - \mathrm{Sd}^{m(\sigma)}\sigma$, we obtain

$$\rho(\sigma) = \mathrm{Sd}^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma).$$

From the inequality $m(\sigma) \geq m(\partial_j\sigma)$, which is valid for all $j \in \{0, \ldots, k\}$, we obtain

$$\begin{aligned} D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) &= \sum_{j=0}^{k} (-1)^j \Big(D_{m(\sigma)}(\partial_j\sigma) - D(\partial_j\sigma) \Big) \\ &= \sum_{j=0}^{k} (-1)^j \sum_{i \geq m(\partial_j\sigma)} T\, \mathrm{Sd}^i(\partial_j\sigma) \quad \in C_k^{\mathcal{U}}(X). \end{aligned}$$

This yields that $\rho(\sigma)$ lies in $C_k^{\mathcal{U}}(X)$ too, since $\mathrm{Sd}^{m(\sigma)}\sigma \in C_k^{\mathcal{U}}(X)$.

Besides, ρ is a chain homomorphism:

$$\partial\, \rho\, \sigma = \partial\sigma - \partial\, \partial\, D\, \sigma - \partial\, D\, \partial\, \sigma = \rho(\partial\sigma).$$

The fact that ρ takes values in $C_*^{\mathcal{U}}(X)$, yields that the first equation of (2.107) holds. One obtains the second equation by observing that for all $\sigma \in C_*^{\mathcal{U}}(X)$ we have $m(\sigma) = 0 \implies D\sigma = 0 \implies \rho(\sigma) = \sigma$. This finishes the proof of Step 4 and simultaneously also the proof of this proposition, since (2.107) implies that $\imath_*\colon H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism. $\square$

With this understood, we can give the proof of the excision theorem.

Proof of Theorem 2.40. The proof consists of the following two steps.

Step 1. *For any subsets $A, B \subset X$ such that $X = \mathrm{Int} A \cup \mathrm{Int} B$ the inclusion $(B, A \cap B) \to (X, A)$ induces an isomorphism*

$$H_*(B, A \cap B) \to H_*(X, A).$$

Set $\mathcal{U} = \{A, B\}$. All maps, which appear in (2.107), preserve $S_*(A)$. This yields that the inclusion

$$\imath\colon S_*^{\mathcal{U}}(X)/S_*(A) \to S_*(X)/S_*(A)$$

induces an isomorphism on the homology groups, since for the induced maps D and ρ Relations (2.107) are also satisfied.

Furthermore, we have

$$S_*^{\mathcal{U}}(X)/S_*(A) = \big(S_*(A) + S_*(B)\big)/S_*(A) \cong S_*(B)/S_*(A \cap B).$$

Moreover, this isomorphism is induced by the inclusion $S_*(B)/S_*(A \cap B) \to S_*^{\mathcal{U}}(X)/S_*(A)$.

Step 2. *The claim of Step 1 is equivalent to the claim of the excision theorem.*

Setting

$$B := X \setminus Z \quad \text{and} \quad Z := X \setminus B,$$

we have $A \cap B = A \setminus Z$. Moreover, the condition $\bar{Z} \subset \mathrm{Int}(A)$ is equivalent to $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. $\square$

Proposition 2.98 also allows us to prove the exactness of the Mayer–Vietoris sequence as follows.

Proof of Theorem 2.89. Set $\mathcal{U} = \{A, B\}$. It is easy to check that the sequence of chain complexes

$$0 \to S_*(A \cap B) \xrightarrow{\varphi} S_*(A) \oplus S_*(B) \xrightarrow{\psi} S_*^{\mathcal{U}}(X) = S_*(A) + S_*(B) \to 0$$

is exact, where[2] $\varphi(x) = (x, x)$ and $\psi(u, v) = u - v$, cf. (2.88). The long exact sequence of the homology groups combined with Proposition 2.98 yield Mayer–Vietoris sequence (2.90). $\square$

The homomorphism $\Delta\colon H_k(X) \to H_{k-1}(A \cap B)$, which appears in the Mayer–Vietoris sequence, can be given explicitly. Namely, let $z \in S_k(X)$ be an arbitrary chain. It follows from the proof that there is a decomposition $z = x + y$, where $x \in S_k(A)$ and $y \in S_k(B)$. Besides, $\partial x + \partial y = \partial z = 0$. Notice however, that neither x nor y must be a chain. Then we have $\Delta([z]) = [\partial x] = -[\partial y]$. Details are left to the reader.

The above implies in particular that Δ is natural in the following sense. Let X, A, B and X', A', B' be as in Theorem 2.89. Furthermore, let $f\colon X \to X'$ be a continuous map such that $f(A) \subset A'$ and $f(B) \subset B'$. Then the diagram

$$\begin{tikzcd}
H_k(A \cap B) \arrow[r] \arrow[d, "f_*"'] & H_k(A) \oplus H_k(B) \arrow[r] \arrow[d, "f_*"'] & H_k(X) \arrow[r, "\Delta"] \arrow[d, "f_*"'] & H_{k-1}(A \cap B) \arrow[d, "f_*"'] \\
H_k(A' \cap B') \arrow[r] & H_k(A') \oplus H_k(B') \arrow[r] & H_k(X') \arrow[r, "\Delta"] & H_{k-1}(A' \cap B')
\end{tikzcd}$$

is commutative.

Sometimes the following relative version of the Mayer–Vietoris sequence is also useful.

[2]Here we omitted the natural inclusions in the notations.

Proposition 2.108. *Assume the following holds: $X = \mathrm{Int}A \cup \mathrm{Int}B$, $X \supset Y = \mathrm{Int}C \cup \mathrm{Int}D$, $C \subset A$, and $D \subset B$. Then the sequence*

$$\dots \to H_k(A \cap B, C \cap D) \xrightarrow{\Phi} H_k(A, C) \oplus H_k(B, D) \xrightarrow{\Psi} H_k(X, Y) \xrightarrow{\Delta} H_{k-1}(A \cap B, C \cap D) \to \dots$$

is exact.

Proof. Let $\mathcal{U} = \{A, B\}$ and $\mathcal{V} = \{C, D\}$ be coverings of X and Y respectively. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_k(C \cap D) \arrow[r, "\varphi"] \arrow[d] & S_k(C) \oplus S_k(D) \arrow[r, "\psi"] \arrow[d] & S_k^{\mathcal{V}}(Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_k(A \cap B) \arrow[r, "\varphi"] \arrow[d] & S_k(A) \oplus S_k(B) \arrow[r, "\psi"] \arrow[d] & S_k^{\mathcal{U}}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_k\big(A \cap B,\ C \cap D\big) \arrow[r, "\varphi"] \arrow[d] & S_k(A, C) \oplus S_k(B, D) \arrow[r, "\psi"] \arrow[d] & S_k^{\mathcal{U},\mathcal{V}}(X, Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here $S_k^{\mathcal{U},\mathcal{V}}(X, Y) = S_k^{\mathcal{U}}(X)/S_k^{\mathcal{V}}(Y)$ by definition and the homomorphisms φ and ψ in the last row are induced by φ and ψ in the middle raw.

Furthermore, the first two raws are exact. In particular, we have $\psi \circ \varphi = 0$ in the middle raw. This equality must still hold in the third raw, that is the third raw is a chain complex. The corresponding long exact sequence is of the following form

$$\dots \longrightarrow H_k(Z_1) \longrightarrow H_k(Z_2) \longrightarrow H_k(Z_3) \longrightarrow H_{k-1}(Z_1) \longrightarrow \dots,$$

where Z_j stands for the complex of the jth raw. This yields

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_k(Z_3) \longrightarrow 0 \longrightarrow \dots$$

That is the homology groups of Z_3 are trivial, so that the third raw is also exact. □

2.A Poincaré conjectures

Conjecture 2.109 (Poincaré)**.** *A compact n-manifold that is homotopy equivalent to the n-sphere is homeomorphic to the n-sphere.*

For $n = 1$ and $n = 2$ this conjecture follows from the classification theorems of Section 2.11.5. Stephen Smale proved this conjecture for $n \geq 5$ in 1960. Later in 1982 Michael Freedman proved also the conjecture in the case $n = 4$. Only in 2002 the case $n = 3$ was published by Grigori Perelman.

Let M be a manifold of dimension n. An open subset $U \subset M$ together with a homeomorphism φ between U and an open subset of $\mathbb{R}^n$ is called *a chart*. A set

$$\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$$

consisting of charts, which cover all of M, is called *an atlas*.

Example 2.110. The sphere S^n has an atlas consisting of two charts. This was given in Example 2.91.

An atlas is called *smooth*, if each *coordinates change map*

$$\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is smooth. The coordinates change maps are maps between open subsets of $\mathbb{R}^n$ and smoothness means that each component is differentiable to any order. *A smooth manifold* is a topological manifold[3] together with a smooth atlas.

Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be two smooth manifolds. A map $f \colon M \to N$ is said to be smooth, if all coordinate representations of f, that is the maps

$$\psi_j \circ f \circ \varphi_i^{-1} \colon \mathbb{R}^n \to \mathbb{R}^m,$$

are smooth (these maps are possibly defined on open subsets of $\mathbb{R}^n$ only). Here (V_j, ψ_j) is a chart on N.

Exercise 2.111.

- Show that S^n has no atlas consisting of a single chart;
- Construct a smooth atlas on $\mathbb{T}^2$ and $\mathbb{RP}^2$.

Two manifolds M and N are called *diffeomorphic*, if there exists a bijection $f \colon M \to N$, so that both f and f^{-1} are smooth. In this case f is called a diffeomorphism.

Theorem 2.112 (Milnor)**.** *There exist* 7*-manifolds, which are homeomorphic but not diffeomorphic to the* 7*-sphere.*

It was shown later that there are exactly 28 smooth manifolds (up to a diffeomorphism), which are homeomorphic to the 7-sphere.

Equivalently, one can reformulate the above theorem somewhat more intrinsically using the notion of a smooth structure. Namely, two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are called equivalent, if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. A maximal atlas on M is called *a smooth structure*. In other words, a smooth structure is an equivalence class of smooth atlases.

Proposition 2.113. *Let* M *be a topological manifold.* M *admits at least two inequivalent smooth structures if and only if there exists a smooth manifold* N*, which is homeomorphic but not diffeomorphic to* M*.*

Proof. Let $\mathcal{A}$ be a smooth atlas on M. Assume there exist a smooth manifold $(N, \mathcal{B})$ and a homeomorphism $f \colon M \to N$, which is not a diffeomorphism. Define a new atlas $\mathcal{B}'$ on M by

$$\mathcal{B}' := \big\{ (f^{-1}(V_j), \psi_j \circ f) \mid (V_j, \psi_j) \in \mathcal{B} \big\}.$$

The atlases $\mathcal{A}$ and $\mathcal{B}'$ are *not* equivalent, since otherwise f would be a diffeomorphism.

If M admits two inequivalent smooth atlases $\mathcal{A}$ and $\mathcal{A}'$, then $id_M \colon (M, \mathcal{A}) \to (M, \mathcal{A}')$ is a homeomorphism, which is not a diffeomorphism. □

Remark 2.114. There are examples of (compact) topological manifolds, which do not admit any smooth structure.

Conjecture 2.115 ("Smooth Poincaré conjecture")**.** *The natural smooth structure on the* 4*-sphere is unique.*

It is not known up to now whether this conjecture is true or false. At the same time, it is known that $\mathbb{R}^4$ admits infinitely many (even uncountably many) smooth structures. Examples of smooth 4-manifolds admitting several smooth structures are also known.

[3]Technically, certain axioms are also required to hold, but this will not be a concern for us.

Chapter 3

CW complexes and cellular homology

3.1 Attaching topological spaces

Let X be a topological space. *The cone* of X is the space

$$CX := X \times [0,1]/\sim, \qquad (x_1, 0) \sim (x_2, 0) \quad \forall x_1, x_2 \in X.$$

Exercise 3.1. Show that the tip of the cone $\{p\} := [X \times \{0\}]$ is a deformation retract of the cone. In particular, cones are contractible.

Let X, Y be topological spaces such that $X \cap Y = \varnothing$, $A \subset X$ and $f\colon A \to Y$ a continuous map. We say that the space

$$X \cup_f Y = \big(X \sqcup Y\big)/\sim, \qquad \text{where} \quad a \sim f(a) \quad \forall a \in A$$

is obtained by attaching X to Y via f.

Some properties considered in the previous chapter can be elegantly expressed in terms of the above attaching construction. For example, consider the space $X \cup CA$, where the attaching map is the inclusion $a \mapsto (a, 1)$. We have

$$\begin{aligned}
\tilde H_*(X \cup CA) &\cong H_*(X \cup CA, CA) && \text{by the LES of the pair } (X \cup CA, CA)\\
&\cong H_*\big(X \cup CA \setminus \{p\}, CA \setminus \{p\}\big) && \text{by excision}\\
&\cong H_*(X, A) && A \subset CA \setminus \{p\} \quad \text{is a deform. retract.}
\end{aligned}$$

This means that the relative homology groups can be represented as the absolute homology groups of the space $X \cup CA$. Here one does not need to impose any assumptions on A, cf. Theorem 2.97. L 14

Let $\varphi_\gamma\colon S^{n-1} \to X,\ \gamma \in \Gamma$, be a family of continuous maps. We say that the space

$$\Big(X \bigsqcup_{\gamma \in \Gamma} B_{n,\gamma}\Big)/\sim, \qquad \text{where} \quad y \sim \varphi_\gamma(y) \quad \forall y \in \partial B_{n,\gamma}$$

is obtained from X by attaching of n-cells and $\Phi_\gamma\colon B_{n,\gamma} \to X \bigsqcup B_{n,\gamma}/\sim$ is called *the characteristic map*. The restriction of Φ_γ to the interior $\mathring{B}_{n,\gamma}$ of the ball is a homeomorphism onto its image e^n_γ, which is referred to as an n-cell.

Definition 3.2. A structure of a CW complex on a Hausdorff space X is a sequence of closed subspaces

$$X^0 \subset X^1 \subset \dots \subset X^n \subset \dots$$

such that the following holds:

(i) $X = \cup_n X^n$;

(ii) X^0 is a discrete space;

(iii) X^n is obtained from X^{n-1} by attaching of n-cells;

(iv) A subset $A \subset X$ is closed (open) in X if and only if $A \cap X^n$ is closed (open) in X^n.

The subspace X^n is called the n-skeleton of X.

A CW structure is called finite if it consists of finitely many cells.

Proposition 3.3. *Let X be a topological space equipped with a CW structure. The following holds:*

- *$X \supset A$ is closed (open) $\Longleftrightarrow \Phi_\gamma^{-1}(A) \subset B_n$ is closed (open);*
- *For finite CW structures (iv) of the definition above holds automatically.*

Proof. The continuity of Φ_γ yields immediately the proof of the first statement in one direction. To show the other direction, assume that $A \cap X^{n-1}$ is closed. Then $A \cap X^n$ is closed in X^n by the definition of the quotient topology.

Assume $A \subset X$ is closed. Since each X^n is closed, the set $X^n \cap A$ is also closed for any CW complex. If the CW structure is finite, then $A = \cup(A \cap \bar{e}^n_\gamma)$ is compact as a finite union of compact subsets. Since X is a Hausdorff space, A is closed. $\square$

Example 3.4. A finite topological graph is a CW complex.

Example 3.5. Each compact surface admits a CW structure. This follows for example from Corollary 2.79.

Example 3.6. The sphere $S^n = B_n/\partial B_n$ has a CW structure, which consists of one 0-cell and one n-cell:

$$X^0 = \cdots = X^{n-1} = \{pt\}, \quad X^n = S^n = \{pt\} \cup B_n,$$

where $\varphi\colon \partial B_n \to \{pt\}$ is necessarily the constant map.

Example 3.7. (Non-Example) Consider the space

$$X := \bigcup_{n \in \mathbb{N}} X_n$$

where X_n is the circle in $\mathbb{R}^2$ of radius $1/n$ centered at $(0, 1/n)$. We define the topology on X as the one inherited from $\mathbb{R}^2$. Then $X \setminus \{0\}$ consists of infinitely many intervals, however this is not a CW structure (*Why?*).

Example 3.8 (Real projective space)**.**

$$\begin{aligned}
\mathbb{RP}^n &= \text{ the space of all lines in } \mathbb{R}^{n+1} \text{ through the origin} \\
&= S^n/\sim, && \text{where } x \sim -x \quad \forall x \in S^n, \\
&= S^n_-/\sim, && \text{where } x \sim -x \quad \forall x \in \partial S^n_-, \\
&= \mathbb{RP}^{n-1} \cup e^n.
\end{aligned}$$

The attaching map $\varphi\colon S^{n-1} \to \mathbb{RP}^{n-1}$ is the quotient map (in particular, this is a 2-to-1 map). This yields a finite CW structure on $\mathbb{RP}^n$:

$$X^n = \mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n.$$

Example 3.9 (Complex projective space)**.**

$$\begin{aligned}\mathbb{CP}^n &= \{\mathbb{C}\text{-lines} \subset \mathbb{C}^{n+1} \text{ through } 0\} \\ &= \big(\mathbb{C}^{n+1} \setminus 0\,\big)/\sim & (z_0,\dots,z_n) &\sim (\lambda z_0,\dots,\lambda z_n), \quad \lambda \in \mathbb{C}\setminus 0, \\ &= S^{2n+1}/\sim & (z_0,\dots,z_n) &\sim (\lambda z_0,\dots,\lambda z_n), \quad |z|=1,\ |\lambda|=1, \\ &= B_{2n}/\sim & z' &\sim \lambda z' \quad \forall z' \in \partial B_{2n},\ |\lambda|=1.\end{aligned}$$

To see the last equality, notice first that for any non-zero $z_0 \in \mathbb{C}$ there exists a unique $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\lambda z_0 \in \mathbb{R}_{>0}$. Hence, for any $(z_0, z_1, \dots, z_n) \in S^{2n+1}$ with $z_0 \neq 0$ there exists a unique $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $r := \lambda z_0 \in \mathbb{R}_{>0}$. Hence,

$$\begin{aligned}\big\{(z_0,z_1,\dots,z_n) \in S^{2n+1} \mid z_0 \neq 0\big\}/\sim \quad &\cong \big\{(r,z_1,\dots,z_n) \mid |z|^2 = 1-r^2,\ r \in (0,1]\,\big\} \\ &\cong B^{2n} \setminus \partial B^{2n}.\end{aligned}$$

This yields in turn $\mathbb{CP}^n = e^{2n} \cup \big(\partial B^{2n}/\sim\,\big) = e^{2n} \cup \mathbb{CP}^{n-1}$. Moreover, the attaching map is the projection $S^{2n-1} \to \mathbb{CP}^{n-1}$ (the Hopf map). This yields a CW structure on $\mathbb{CP}^n$:

$$\mathbb{CP}^n = e^0 \cup e^2 \cup \dots \cup e^{2n}.$$

Example 3.10 (Quaternion-projective space)**.** Replacing $\mathbb{R}$ or $\mathbb{C}$ by quaterions in the constructions above, we obtain the quaternion-projective space:

$$\mathbb{HP}^n = \big(\mathbb{H}^{n+1} \setminus 0\big)/(\mathbb{H} \setminus 0) = e^0 \cup e^4 \cup \dots \cup e^{4n}.$$

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Proposition 3.11. *We have*

$$H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & k = 0,2,\dots,2n, \\ 0 & \textit{else} \end{cases} \quad \textit{and} \quad H_k(\mathbb{HP}^n) \cong \begin{cases} \mathbb{Z} & k = 0,4,\dots,4n, \\ 0 & \textit{else}. \end{cases} \tag{3.12}$$

Proof. By the induction on n we show that $H_k(\mathbb{CP}^n)$ are indeed given by (3.12). The proof for $\mathbb{HP}^n$ can be obtained along similar lines.

For $n = 0$ we have $\mathbb{CP}^0 = \{pt\}$ and therefore (3.12) holds in this case.

The long exact sequence of the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ yields

$$\dots \to H_{k+1}(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to H_k(\mathbb{CP}^{n-1}) \to H_k(\mathbb{CP}^n) \to H_k(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to \dots \tag{3.13}$$

We also have $\mathbb{CP}^n/\mathbb{CP}^{n-1} = e^{2n}/\partial e^{2n} = S^{2n}$.

Exercise 3.14. Show that $\mathbb{CP}^{n-1}$ is a deformation retract of a neighborhood in $\mathbb{CP}^n$.

For $k < 2n$ (3.13) yields $H_k(\mathbb{CP}^n) \cong H_k(\mathbb{CP}^{n-1})$. For $k = 2n$ we obtain

$$0 = H_{2n}(\mathbb{CP}^{n-1}) \to H_{2n}(\mathbb{CP}^n) \to H_{2n}(S^{2n}) \to H_{2n-1}(\mathbb{CP}^{n-1}) = 0,$$

that is $H_{2n}(\mathbb{CP}^n) \cong \mathbb{Z}$. □

3.2 Operations on CW complexes

Product. If $X = \cup e^n_\gamma$ and $Y = \cup e^m_\beta$ are CW complexes, then

$$X \times Y = \bigcup_{k=m+n} \bigcup_{\gamma,\beta} e^n_\gamma \times e^m_\beta.$$

This yields a CW structure on $X \times Y$, since $B_n \times B_m$ is homeomorphic to B_{n+m} (*Why?*).

Example 3.15. $S^1 = e^0 \cup e^1 \implies \mathbb{T}^2 = S^1 \times S^1 = e^0 \cup (e^1_1 \cup e^1_2) \cup e^2 = \{pt\} \cup (A \cup B) \cup \text{disc}$, cf. Section 2.11.1.

Quotient. A subcomplex A of a CW complex X is a closed subset, which is a union of cells in X. Under these circumstances (X, A) is called a CW pair.

The CW complex X/A consists of cells of $X \setminus A$ and an additional 0-cell $[A]$. For an n cell with an attaching map $\varphi_\gamma : S^{n-1} \to X^{n-1}$ the corresponding attaching map is given by the composition $S^{n-1} \to X^{n-1} \to X^{n-1}/(X^{n-1} \cap A)$.

Example 3.16. Consider the torus $\mathbb{T}^2 = e^0 \cup (e_1^1 \cup e_2^1) \cup e^2$ and set $A = e^0 \cup (e_1^1 \cup e_2^1) = S^1 \vee S^1$. Then we have $\mathbb{T}^2/A = e^0 \cup e^2 = S^2$.

Suspension. The space

$$SX := \big(X \times I / X \times \{0\}\big)/X \times \{1\} = C_1 X \cup_X C_2 X$$

is called the suspension of X. In particular, when X is a CW complex the suspension SX is also a CW complex.

For example, we have $S(S^n) \cong S^{n+1}$.

Smash product. Let (X, x_0) and (Y, y_0) be pointed topological spaces. The wedge product $X \vee Y$ kan be identified with the subspace

$$X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y.$$

The space

$$X \wedge Y := X \times Y / X \vee Y = X \times Y / \big(X \times \{y_0\} \cup \{x_0\} \times Y\big)$$

is called the smash product of (X, x_0) and (Y, y_0). If X and Y are CW complexes such that x_0 and y_0 are 0 cells of X and Y respectively, then $X \wedge Y$ is a (pointed) CW complex.

Example 3.17. Consider the spheres as CW complexes as follows: $S^n = e^0 \cup e^n$ and $S^m = e^0 \cup e^m$. Then

$$S^m \times S^n = e^0 \cup e^m \cup e^n \cup e^{m+n} \supset e^0 \cup e^m \cup e^n = S^m \vee S^n.$$

This yields $S^m \wedge S^n = e^0 \cup e^{m+n} = S^{m+n}$.

Reduced suspension. Let (X, x_0) be a pointed topological space. The space

$$\Sigma X = X \times I / \big(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I\big) = X \wedge S^1$$

is called the reduced suspension of X. For example, $\Sigma S^n \cong S^{n+1}$.

The following observation will be useful in the sequel. First notice that the (non-reduced) suspension of the n-ball is clearly homeomorphic to the $(n+1)$-ball. By collapsing an interval on the boundary, we obtain a topological space, which is still homeomorphic to the $(n+1)$-ball, that is $\Sigma B_n \cong B_{n+1}$. This yields in turn the following: If X is a CW complex, then ΣX is a CW complex too and each n-cell in X corresponds to an $(n+1)$-cell in ΣX:

$$X = \bigcup_{n \geq 0} \bigcup_{\gamma} e_\gamma^n \qquad \Longrightarrow \qquad \Sigma X = \bigcup_{n \geq 0} \bigcup_{\gamma} e_\gamma^{n+1}. \tag{3.18}$$

3.3 Homotopy extension property

Let X be a topological space and $A \subset X$. Recall that a continuous map $r\colon X \to A$ is called a retraction if $r|_A = r \circ \imath_A = id_A$. Also, A is called the deformation retract of X if id_X is homotopic to a retraction $r\colon X \to A$, cf. Definition 2.57.

Definition 3.19. We say that the pair (X, A) has the homotopy extension property (HEP for short), if the following holds: If a continuous map $f\colon X \to Y$ and a homotopy $h\colon A \times I \to Y$ of $f|_A = f \circ \imath_A$ are given, then there is a homotopy $H\colon X \times I \to Y$ such that $H \circ (\imath_A \times id) = h$.

Lemma 3.20. *A pair (X, A) has the HEP if and only if $X \times \{0\} \cup A \times I \subset X \times I$ admits a retraction.*

Proof. The following observation is useful for the proof: The data consisting of a continuous map $f\colon X \to Y$ together with a homotopy of $f \circ \imath_A$ is equivalent to a continuous map $X \times \{0\} \cup A \times I \to Y$.

If there exists a retraction $r\colon X \times I \to X \times \{0\} \cup A \times I$, then $H := h \circ (r \times id)$ is an extension of h.

If (X, A) has the HEP, then for $id\colon X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ there exists an extension $r\colon X \times I \to X \times \{0\} \cup A \times I$, which is the required extension. □

Let X be a CW complex and Y a topological space. For the proof of the next proposition we need the following observation: A continuous map $f\colon X \to Y$ is the same as the sequence $f_n\colon X^n \to Y$ of continuous maps such that $f_n|_{X^k} = f_k$ provided $k \le n$. Indeed, given a continuous map $f\colon X \to Y$, the corresponding sequence is constructed simply by setting $f_n = f|_{X^n}$. If a sequence f_n is given, we can define a map $f\colon X \to Y$ by

$$f(x) = f_n(x) \qquad \text{provided} \quad x \in X^n.$$

This map is continuous, since for each open subset $U \subset Y$ the subset $f^{-1}(U) \cap X^n = f_n^{-1}(U)$ is open and therefore also the subset $f^{-1}(U)$ is open in X.

Proposition 3.21. *If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I \subset X \times I$ is a deformation retract. In particular, each CW pair has the HEP.*

The proof of this proposition is given after the proof of Lemma 3.22.

Consider $[0, \infty) = \cup_{i \in \mathbb{N}} [i-1, i]$ as a CW complex. The CW subcomplex

$$T = \bigcup_i X^i \times [i, \infty) \subset X \times [0, \infty)$$

is called *the telescope* of X.

Lemma 3.22. *T is homotopy equivalent to X.*

Proof. Since X is a deformation retract of $X \times [0, \infty)$, it is enough to show that T is also a deformation retract of $X \times [0, \infty)$.

Set $Y_i := T \cup \big(X \times [i, \infty)\big)$. By Proposition 3.21, $X^i \times [i, i+1] \cup X \times \{i+1\}$ is a deformation retract of $X \times [i, i+1]$. This yields that Y_{i+1} is a deformation retract of Y_i. Denote by $h_{i,t}$ a homotopy between id and the retraction $Y_i \to Y_{i+1}$.

Define $f_t\colon X \times [0, \infty) \to T$ by

$$f_t(x,\tau) = \begin{cases} h_{0,2t}(x,\tau) & t \in [0, \frac{1}{2}], \\ h_{1,4t-2} \circ r_0(x,\tau) & t \in [\frac{1}{2}, \frac{3}{4}], \\ \quad \dots & \\ h_{i,\rho_i(t)} \circ r_{i-1} \circ \dots \circ r_0(x,\tau) & t \in [1-2^{-i}, 1-2^{-i-1}], \\ \quad \dots & \dots \end{cases}$$

where $\rho_i\colon [1-\frac{1}{2^i}, 1-\frac{1}{2^{i+1}}] \to [0,1]$ is a homeomorphism, for example $\rho_i(t) = 2^{i+1}t - 2^{i+1} - 2$. Then f_t is a map $X \times [0,\infty) \to T$ such that $f_t|_{X^i \times [0,\infty)} = id$ for $t \geq 1 - \frac{1}{2^{i+1}}$. Moreover, f_t is continuous, since f_t is continuous on each $X^i \times [i, i+1]$. This yields the claim. □

Proof of Proposition 3.21. Notice that there exists a retraction $r\colon B_n \times I \to B_n \times \{0\} \cup \partial B_n \times I$. This can be obtained for example as the projection from the point $(0,2) \in B_n \times \mathbb{R}$.

This yields a retraction $r_n\colon X^n \times I \to X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$, where $A^n := X^n \cap A$. Indeed, $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching of $B_n \times I$ along $B_n \times \{0\} \cup \partial B_n \times I$.

Let $h_{n,t}$ be a homotopy between r_n and $id_{X^n \times I}$. Just like in the proof of Lemma 3.22, the composition of $\{h_{n,t}\}$ yields the required retraction. □ L 16

3.4 Cellular homology

Consider the sequence

$$\dots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \dots, \qquad (3.23)$$

where the homomorphisms d_{n+1} are defined as the composition

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}} H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1})$$

(these maps are part of the long exact sequence of the pair (X^{n+1}, X^n) and (X^n, X^{n-1})). This yields

$$d_n \circ d_{n+1} = j_{n-1} \circ (\delta_n \circ j_n) \circ \delta_{n+1} = 0,$$

since $\delta_n \circ j_n = 0$ as the composition of two homomorphisms in the long exact sequence of the pair (X^n, X^{n-1}). Hence, (3.23) is a chain complex. The homology groups of (3.23) are called *the cellular homology groups of* X.

Theorem 3.24. *The cellular homology groups are isomorphic to the singular homology groups.*

The proof of the above theorem requires certain auxiliary statements, which are proved first.

Definition 3.25. If $X = X^n$ for some n, then X is called *finite dimensional.* A minimal n such that $X = X^n$ is called *the dimension* of X.

Lemma 3.26. *Let* (X_α, x_α) *be a family of pointed spaces such that each* x_α *is a deformation retract of a neighborhood in* X_α*. Then the following holds*

$$\tilde{H}_*\Big(\bigvee_\alpha X_\alpha\Big) \cong \bigoplus_\alpha \tilde{H}_*(X_\alpha),$$

where the isomorphism is induced by the inclusions $\imath_\alpha\colon X_\alpha \to \bigvee X_a$*.*

Proof. This follows from Theorem 2.97:

$$\bigoplus_\alpha \tilde{H}_*(X_\alpha) \cong \bigoplus_\alpha H_*(X_\alpha, \{x_\alpha\}) \cong H_*\big(\sqcup X_\alpha, \sqcup\{x_\alpha\}\big)$$
$$\cong \tilde{H}_*\big(\sqcup X_\alpha / \sqcup \{x_\alpha\}\big) = \tilde{H}_*\Big(\bigvee_\alpha X_\alpha\Big).$$

□

Lemma 3.27. *For any CW complex X the following holds:*

(a) $H_k(X^n, X^{n-1})$ *is a free abelian group generated by the n cells of X for $k = n$ and trivial for $k > 0,\ k \neq n$;*

(b) $H_k(X^n) = 0$ *for $k > n$.*

Proof. Claim (a) follows from the following observations: $X^{n-1} \subset X^n$ is a deformation retract of a neighborhood and X^n/X^{n-1} is the wedge product of n-spheres.

Claim (b) is left as an exercise. □

Proof of Theorem 3.24. The proof consists of four steps.

Step 1. *For any finite dimensional CW complex X such that $X^n = \{pt\}$ for some $n \in \mathbb{N}$ we have $\tilde{H}_k(X) = 0$ for all $k \leq n$.*

Consider the sequence of homomorphisms

$$H_k(X^k) \to H_k(X^{k+1}) \to H_k(X^{k+2}) \to \dots,$$

which are induced by the inclusions. The long exact sequence of the pair (X^{k+m+1}, X^{k+m}) yields that any homomorphism appearing in this sequence is surjective. This implies the claim of this step.

Step 2. *For any CW complex X such that $X^n = \{pt\}$ for some $n \in \mathbb{N}$ we have $\tilde{H}_k(X) = 0$ for all $k \leq n$.*

Set $R := X^0 \times [0, \infty) \subset T$, where T is the telescope of X. Denote also $Z := R \cup_i X^i \times \{i\}$. Then Z/R is homeomorphic to $\bigvee_i X^i$. Using the previous step, we obtain $\tilde{H}_k(Z/R) = 0$ for all $k \leq n$. The long exact sequence of the pair (Z, R) yields $\tilde{H}_k(Z) = 0$ for all $k \leq n$.

Furthermore, we have

$$T/Z = \big(T/\sqcup X^i \times \{i\}\big)/R = \big(\cup_i SX^i\big)/R = \bigvee_i \Sigma X^i.$$

Moreover, the $(n+1)$ skeleton of ΣX^i is a point, cf. (3.18). This yields $\tilde{H}_k(T/Z) = 0$ for $k \leq n+1$. From the long exact sequence of the pair (T, Z) we obtain $\tilde{H}_k(T) = 0$ for $k \leq n$. The claim of this step now follows from Lemma 3.22.

Step 3. *The map $H_k(X^n) \to H_k(X)$ induced by the inclusion is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

This follows immediately from Step 2 by using the long exact sequence of the pair (X, X^n).

Step 4. *We prove this theorem.*

By the long exact sequence of the pair (X^{n-1}, X^{n-2}) we have

$$0 = H_{n-1}(X^{n-2}) \to H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}, X^{n-2}).$$

Since j_{n-1} is injective, we obtain $\ker d_n = \ker(j_{n-1} \circ \delta_n) = \ker \delta_n = \operatorname{im} j_n \cong H_n(X^n)$.

Since j_n is injective, we have $j_n(\operatorname{im} \delta_{n+1}) = \operatorname{im}(j_n \circ \delta_{n+1}) = \operatorname{im} d_{n+1}$. This yields that j_n induces an isomorphism $H_n(X^n)/\operatorname{im} \delta_{n+1} \cong \ker d_n / \operatorname{im} d_{n+1}$.

Furthermore, by the long exact sequence of the pair (X^{n+1}, X^n) we obtain

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}} H_n(X^n) \longrightarrow H_n(X^{n+1}) \to 0.$$

In particular, we have $H_n(X^n)/\operatorname{im} \delta_{n+1} \cong H_n(X^{n+1})$. The claim of this theorem follows now from the observation that $H_n(X^{n+1}) \cong H_n(X)$ by Step 3. □

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Corollary 3.28. *Let k be the number of the n cells of some CW structure of X. Then $H_n(X)$ has at most k generators. In particular, if there are no n cells, then $H_n(X) = 0$.* □

Theorem 3.29. *Consider e^n_γ es a generator of $H_n(X^n, X^{n-1})$. The homomorphism d_n in (3.23) is given by*

$$d_n(e^n_\gamma) = \sum_\mu d_{\gamma\mu} e^{n-1}_\mu, \tag{3.30}$$

$d_{\gamma\mu}$ is the degree of the map

$$S^{n-1} = \partial e^n_\gamma \to X^{n-1} \to X^{n-1}/\big(X^{n-1} \setminus e^{n-1}_\mu\big) = S^{n-1}.$$

Moreover, the sum in (3.30) is finite.

Proof.

$$\begin{tikzcd}
H_n(B_{n,\gamma}, \partial B_{n,\gamma}) \arrow[r, "\delta", "\cong"'] \arrow[d, "\Phi_{\gamma *}"] & \tilde H_{n-1}(\partial B_{n,\gamma}) \arrow[rr, "\Delta_*"] \arrow[d, "\varphi_{\gamma *}"] & & \tilde H_{n-1}(S^{n-1}_\mu) \\
H_n(X^n, X^{n-1}) \arrow[r, "\delta"] \arrow[dr, "d_n"] & \tilde H_{n-1}(X^{n-1}) \arrow[rr, "q_*"] \arrow[d, "j_{n-1}"] & & \tilde H_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\mu *}"'] \arrow[d, "\cong"] \\
& H_{n-1}(X^{n-1}, X^{n-2}) \arrow[rr, "\cong"] & & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}),
\end{tikzcd}$$

where the following notations are used:

- Φ_γ is the characteristic map of e_γ;
- $\varphi_\gamma \colon \partial B_{n,\gamma} \to X^{n-1}$ is the attaching map of e_γ;
- $q \colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the projection;
- $q_\mu \colon X^{n-1}/X^{n-2} \to X^{n-1}/\big(X^{n-1} \setminus e^{n-1}_\mu\big) \cong S^{n-1}$ is the projection;
- $\Delta := q_\mu \circ q \circ \varphi_\gamma$.

The generator $e_\gamma \in H_n(X^n, X^{n-1})$ is $\Phi_{\gamma *} \circ \delta^{-1}(a)$, where a is the generator of $\tilde H_{n-1}(\partial B_{n,\gamma})$. The commutativity of the diagram yields the equality

$$d_{\mu\gamma} a = q_{\mu *}(d_n(e_\gamma)) = \Delta_* a = (\deg \Delta) a.$$

Here the first equality follows from the following observation: $q_{\mu *}$ maps e^{n-1}_μ to a and vanishes on all other generators. This yields (3.30). □

Example 3.31. (Homology groups of real projective spaces) We begin with some observations. A map $f\colon S^n \vee S^n \to S^n$ can be understood as a pair (f_1, f_2) of maps $S^n \to S^n$. Then for the induced map we have $f_*(x, y) = f_{1*}x + f_{2*}y$ (this follows from the fact that the projection $S^n \sqcup S^n \to S^n \vee S^n$ induces an isomorphism on $\tilde{H}_*$).

Another observation is as follows. Let $F\colon S^n \to S^n \vee S^n$ be a map with the property: F maps S^n_+ on one copy of S^n and S^n_- on the other one (the image of the equator must be the point in $S^n \vee S^n$). Then we have $F_*a = (f_{+*}a, f_{-*}a)$, where $f_\pm\colon S^n_\pm/\partial S^n_\pm \to S^n$ is defined as the restriction of F.

Furthermore, let us proceed to the computation of the homology groups of $\mathbb{RP}^n$. We know from Example 3.8 that

$$\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n,$$

where the attaching map φ is the projection $S^{n-1} \to \mathbb{RP}^{n-1}$. Consider the commutative diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\varphi"] \arrow[d] & \mathbb{RP}^{n-1} \arrow[d] \\
S^{n-1}/S^{n-2} \arrow[r] \arrow[d, "\cong"'] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \arrow[d, "\cong"] \\
S^{n-1} \vee S^{n-1} \arrow[r, "\psi"] & S^{n-1}.
\end{tikzcd}$$

Here the components (ψ_1, ψ_2) of ψ satisfy the relation $\psi_2 = \psi_1 \circ A$, where A is the antipodal map. By the construction of cells in $\mathbb{RP}^n$, we can assume that ψ_1 is the identity map.

The map $\Delta\colon S^{n-1} \to S^{n-1}$ (the diagonal in the above diagram) induces Δ_*. We have

$$\Delta_* a = a + A_* a = \big(1 + (-1)^n\big)a, \qquad a \in H_{n-1}(S^{n-1})$$

that is $\deg \Delta = 1 + (-1)^n$.

This yields that (3.23) for $\mathbb{RP}^n$ has the following form:

$$\begin{aligned}
&0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2\times} \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 &&\text{if } n \text{ is even},\\
&0 \to \mathbb{Z} \xrightarrow{0\times} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0\times} \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 &&\text{if } n \text{ is odd}.
\end{aligned}$$

This implies in turn that the homology groups of $\mathbb{RP}^n$ are given by

$$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0 \quad \text{and } k = n \text{ provided } n \text{ is odd;} \\ \mathbb{Z}/2\mathbb{Z} & \text{for } k \text{ odd, } k < n; \\ 0 & \text{else.} \end{cases}$$

3.5 The Euler characteristics

For any topological space X we set

$$b_k(X) := \operatorname{rk} H_k(X) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

This is called the *kth* Betti number of X.

Assume that all Betti numbers of X are finite and only finitely many are non-zero. Under these circumstances the integer

$$\chi(X) := \sum_k (-1)^k b_k(X)$$

is called the Euler characteristic of X. For example, by Corollary 3.28 the Euler characteristic of a finite CW complex is well defined.

Theorem 3.32. *For a finite CW complex X we have*

$$\chi(X) = \sum_n (-1)^n c_n,$$

where c_n is the number of n-cells of X.

Proof. First notice that by Step 2 in the proof of Theorem 2.56 we obtain the following fact: If $0 \to A \to B \to C \to 0$ is an exact sequence of finitely generated abelian groups, then $\operatorname{rk} B = \operatorname{rk} A + \operatorname{rk} C$.

Let

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

be Complex (3.23) for X. Denote

$$Z_k := \ker d_k, \quad B_k := \operatorname{im} d_{k+1}, \quad \text{and} \quad H_k := Z_k / B_k.$$

We have

$$\begin{aligned} &0 \to B_k \to Z_k \to H_k \to 0 && \text{is exact} && \Longrightarrow && \operatorname{rk} Z_k = \operatorname{rk} B_k + \operatorname{rk} H_k; \\ &0 \to Z_k \to C_k \to B_{k-1} \to 0 && \text{is exact} && \Longrightarrow && \operatorname{rk} C_k = \operatorname{rk} Z_k + \operatorname{rk} B_{k-1}. \end{aligned}$$

Hence, we obtain: $\operatorname{rk} C_k = \operatorname{rk} B_k + \operatorname{rk} B_{k-1} + \operatorname{rk} H_k \implies \sum (-1)^k \operatorname{rk} C_k = \sum (-1)^k \operatorname{rk} H_k$. $\square$

This theorem generalizes Theorem 2.56 for arbitrary dimensions.

Remark 3.33 (Another proof of Theorem 2.67). A planar graph yields a CW structure on $S^2 = \mathbb{R}^2 \cup \{\infty\}$. By Theorem 3.32 we have

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = \chi(S^2) = 2.$$

Chapter 4

The fundamental group

4.1 Basic constructions

The following terminology will be useful in the sequel.

Definition 4.1. For $A \subset X$ we say that two continuous maps of pairs $f_0, f_1\colon (X,A) \to (Y,B)$ are homotopic *relative to* A, if there exists a continuous map of pairs $h\colon (X \times I, A \times I) \to (Y,B)$ such that

$$h\big|_{X\times\{0\}} = f_0 \qquad \text{and} \qquad h\big|_{X\times\{1\}} = f_1.$$

To elaborate, the above definition means that h is a homotopy between f_0 and f_1 such that

$$h(a,t) \in B \qquad \text{for all } a \in A \text{ and all } t \in I.$$

In this case we write

$$f_0 \simeq f_1 \text{ rel } A.$$

In the particular case $A = \{x_0\}$, $B = \{y_0\}$ we write simply $f_0 \simeq f_1 \operatorname{rel} x_0$. This means, that there is a homotopy h between f_0 and f_1, such that $h(x_0,t) = y_0$ for all $t \in I$.

Let X be a topological space. For two continuous paths $u, v\colon I \to X$ such that $u(1) = v(0)$ define the concatenation (product) by the formula

$$u * v\,(t) := \begin{cases} u(2t) & \text{for } t \in [0, \frac{1}{2}],\\ v(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Pick any basepoint x_0 and denote

$$\Omega(X,x_0) := \big\{u\colon I \to X \text{ is continuous } \mid u(0) = x_0 = u(1)\big\}.$$

Elements of $\Omega(X,x_0)$ are called loops in X based at x_0. Two loops u_0 and u_1 are said to be equivalent ($u_0 \sim u_1$), if u_0 and u_1 are homotopic relative to the basepoint. Define

$$\pi_1(X,x_0) := \Omega(X,x_0)/\sim.$$

The above concatenation operation yields a well-defined map $*\colon \Omega(X,x_0) \times \Omega(X,x_0) \to \Omega(X,x_0)$. Since

$$u_0 \sim u_1 \text{ and } v_0 \sim v_1 \qquad \Longrightarrow \qquad u_0 * v_0 \sim u_1 * v_1,$$

we obtain a well-defined map

$$\pi_1(X,x_0) \times \pi_1(X,x_0) \to \pi_1(X,x_0), \qquad [u]\cdot[v] = [u*v]. \tag{4.2}$$

Theorem 4.3. *$\pi_1(X, x_0)$ is a group with respect to the product given by (4.2).*

Proof. The proof consists of the following steps.

Step 1. *The constant loop $c(t) = x_0$ is the identity element in $\pi_1(X, x_0)$, that is for any $u \in \Omega(X, x_0)$ we have $u * c \sim u$ and $c * u \sim u$.*

A homotopy between $u * c$ and u can be constructed explicitly, namely

$$h(t,s) = \begin{cases} u\big(2t/(1+s)\big) & \text{if } t \in \big[0,\, \frac{1+s}{2}\big], \\ x_0 & \text{if } t \in \big[\frac{1+s}{2},\, 1\big]. \end{cases}$$

A homotopy between $c * u$ and u can be given by a similar formula.

Step 2. *For $u \in \Omega(X, x_0)$ define $\bar{u} \in \Omega(X, x_0)$ by $\bar{u}(t) = u(1-t)$. The map $u \mapsto \bar{u}$ yields a well-defined map $\pi_1(X, x_0) \to \pi_1(X, x_0)$ such that $[u] \cdot [\bar{u}] = [c] = [\bar{u}] \cdot [u]$.*

We have to show that $u * \bar{u}$ is homotopic to c. The required homotopy is given again by the following explicit formula:

$$h(t,s) = \begin{cases} u\big(2t(1-s)\big) & \text{if } t \in \big[0,\, \frac{1}{2}\big], \\ u\big((2-2t)(1-s)\big) & \text{if } t \in \big[\frac{1}{2},\, 1\big]. \end{cases}$$

Step 3. *For any $u, v, w \in \Omega(X, x_0)$ we have $(u * v) * w \sim u * (v * w)$. In particular, the product on $\pi_1(X, x_0)$ is associative.*

Again, one can construct the explicit homotopy as follows:

$$h(t,s) := \begin{cases} u\big(\frac{4t}{1+s}\big) & \text{if } t \in \big[0,\, \frac{1+s}{4}\big], \\ v(4t - 1 - s) & \text{if } t \in \big[\frac{1+s}{4},\, \frac{2+s}{4}\big], \\ w\big(1 - \frac{4-4t}{2-s}\big) & \text{if } t \in \big[\frac{2+s}{4},\, 1\big]. \end{cases}$$

Finally, a combination of Steps 1–3 yields that $\pi_1(X, x_0)$ is a group. Indeed, the last step yields associativity, the first one existence of the identity element, and the second one the existence of the inverse. $\square$

It is worthwhile to note, that the proof of the above theorem yields an explicit expression for the inverse element of $[u] \in \pi_1(X, x_0)$, namely

$$[u]^{-1} = [\bar{u}].$$

Definition 4.4. The group $\pi_1(X, x_0)$ is called the fundamental group of X (relative to the basepoint x_0).

Example 4.5. If X is contractible, then any loop is homotopic to the constant one. In other words, $\pi_1(X, x_0) = \{1\}$ for any basepoint x_0. For example, $\pi_1(\mathbb{R}^n, x_0) = \{1\}$.

It is natural to ask whether the fundamental group depends on the basepoint. An answer to this question is given by the following result.

Proposition 4.6. *If X is path connected, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic for any $x_0, x_1 \in X$.*

Proof. Pick a curve w connecting x_0 and x_1. Define the map

$$P_w\colon \pi_1(X,x_0)\to\pi_1(X,x_1) \qquad P_w([u]) = [\bar w * u * w].$$

By the proof of Theorem 4.3 we have

$$\begin{aligned} P_w\big([u][v]\big) &= [\bar w * u * v * w] = [\bar w * u * (w * \bar w) * v * w] = \big[(\bar w * u * w) * (\bar w * v * w)\big] \\ &= P_w([u])\cdot P_w([v]). \end{aligned}$$

Hence, P_w is a group homomorphism.

Denoting by $P_{\bar w}$ the corresponding homomorphism $\pi_1(X,x_1)\to\pi_1(X,x_0)$, we obtain

$$P_{\bar w}\circ P_w([u]) = \big[w * (\bar w * u * w) * \bar w\big] = \big[(w * \bar w) * u * (w * \bar w)\big] = [u].$$

Hence, $P_{\bar w}\circ P_w = id$. A similar argument shows that $P_w\circ P_{\bar w} = id$. In other words, P_w is an isomorphism whose inverse is $P_{\bar w}$. □

Thus, if X is path connected, the isomorphism class of the fundamental group is independent of the basepoint. Somewhat loosely speaking, in this case one usually drops the basepoint from the notation of the fundamental group and calls this "the fundamental group of X".

Proposition 4.7. *Any continuous map $f\colon (X,x_0)\to(Y,y_0)$ induces the group homomorphism*

$$f_*\colon \pi_1(X,x_0)\to\pi_1(Y,y_0), \qquad f_*[u] = [f\circ u]$$

with the following properties:

(i) $id_* = id$;

(ii) $(g\circ f)_* = g_*\circ f_*$;

(iii) $f\simeq g \operatorname{rel} x_0 \quad\Longrightarrow\quad f_* = g_*$;

(iv) (X,x_0) *and* (Y,y_0) *are homotopy equivalent* $\quad\Longrightarrow\quad \pi_1(X,x_0)\cong\pi_1(Y,y_0)$.

□

Just like in the case of homology groups, Properties (i) and (ii) mean that the fundamental group is functorial. In (iv) X and Y are meant to be homotopy equivalent as *pointed* topological spaces. The proof is left as an exercise to the reader.

Notice also that the first two properties of the above theorem imply that f_* is an isomorphism provided f is a homeomorphism. In other words, the fundamental group is an invariant of (pointed) topological spaces (more precisely, the isomorphism class of the fundamental group is an invariant). Notice also, that nevertheless, it may happen that f is injective and f_* is not. Likewise, f may be surjective and f_* may fail to be surjective.

We finish this section by the following elementary fact.

Theorem 4.8. *For any two pointed topological spaces (X,x_0) and (Y,y_0) we have a natural isomorphism*

$$\pi_1\big(X\times Y,\,(x_0,y_0)\big)\cong\pi_1(X,x_0)\times\pi_1(Y,y_0).$$

Proof. The proof follows immediately from the following elementary observations:

- $\Omega\big(X\times Y,\,(x_0,y_0)\big) = \Omega(X,x_0)\times\Omega(Y,y_0)$;

- $\Omega\big(X \times Y, (x_0, y_0)\big) \ni (u, v) \simeq (u_1, v_1) \text{ rel } (x_0, y_0) \iff u \simeq u_1 \text{ rel } x_0$ and $v \simeq v_1 \text{ rel } x_1$.

These two observations imply that the natural map $[(u, v)] \mapsto \big([u], [v]\big)$ is an isomorphism. □

The reader may suspect that the fundamental group is intimately related to the first homology group, since, after all, both are constructed starting from continuous maps $I \to X$. This is true indeed and the precise relation is known as the Hurewicz homomorphism, which is described below.

Theorem 4.9 (Hurewicz homomorphism)**.** *Let X be a path-connected topological space. The map*

$$\Omega(X, x_0) \to S_1(X), \qquad u \mapsto u$$

induces a well-defined homomorphism (the Hurewicz homomorphism)

$$h\colon \pi_1(X, x_0) \to H_1(X)$$

with the following properties: h is surjective and $\ker h = [\pi_1, \pi_1]$, where we abbreviated $\pi_1 := \pi_1(X, x_0)$ for brevity. In particular, $H_1(X)$ is the abelianization of π_1, that is $\pi_1/[\pi_1, \pi_1] \cong H_1(X)$. □

The proof of this theorem can be found for example in [Hat02, Thm. 2A.1]. L 18

4.2 Coverings

It is not easy to compute the fundamental group of a topological space just from the definition. For example, even for the very simple topological space S^1 it is not so clear what is its fundamental group. However, a loop in X can (and should) be viewed as a continuous map $S^1 \to X$. Since we are working in the category of pointed spaces, we require also $u(1) = x_0$, where S^1 is thought of as the set of complex numbers of absolute value 1. In any case, if $X = S^1$, we have a well-defined map

$$\deg\colon \pi_1(S^1) \to \mathbb{Z} \qquad \deg[u] = \deg u,$$

where $\deg u$ is the degree of u in the sense of Definition 1.18. We already know that this map is surjective and we shall show below that this is in fact an isomorphism. However, the proof of this fact requires the notion of a covering, which we consider next.

Definition 4.10. *A covering* of a topological space X consists of a topological space Y and a map $p\colon Y \to X$ with the following property: For any $x \in X$ there exists a neigbourhood $U \ni x$ such that

$$p^{-1}(U) = \bigsqcup_{y \in p^{-1}(x)} V_y \qquad \text{and} \qquad p\big|_{V_y}\colon V_y \to U \text{ is a homeomorphism} \tag{4.11}$$

for each $y \in p^{-1}(x)$.

We always assume that X and Y are (path)-connected. Otherwise we can consider coverings of connected components individually.

Notice that the definition yields that each fiber $p^{-1}(x)$ is a discrete set, since each V_y contains a unique point from $p^{-1}(x)$, namely y. If this set is finite for any $x \in X$, then $\#p^{-1}(x)$ is constant over U. Hence, $x \mapsto \#p^{-1}(x)$ is a locally constant function and, therefore, is constant. Denoting this common value by n, we say that Y is an n-sheeted covering of X.

Example 4.12.

(i) The map $\exp\colon \mathbb{R} \to S^1$, $\exp(x) = e^{2\pi i x}$ satisfies (4.11) demonstrating that $\mathbb{R}$ is a covering of S^1. Furthermore, $p^{-1}(1) = \mathbb{Z}$ and for any $U \subsetneq S^1$ we have

$$\exp^{-1}(U) = \bigsqcup_{i\in\mathbb{Z}} V_i \qquad \text{such that} \qquad \exp\colon V_i \to U \text{ is a homeomorphism.}$$

(ii) Consider the map $p_2\colon S^1 \to S^1$, $p_2(z) := z^2$. The preimage of each point consists of exactly two points, which differ by the sign. Furthermore, for $U = \{z \in S^1 \mid -\frac{\pi}{2} < \arg z < \frac{\pi}{2}\}$ we have

$$p^{-1}(U) = V_+ \cup V_-, \qquad \text{where } V_+ := \Big\{ -\frac{\pi}{4} < \arg z < \frac{\pi}{4} \Big\} \text{ and } V_- := -V_+.$$

This demonstrates that any point in U has a neighbourhood such that (4.11) holds. It is then easy to see that in fact any point in S^1 has this property so that S^1 is a 2–sheeted (=double) covering of itself.

Moreover, it is clear that for any $n \in \mathbb{N}$ the map $p_n\colon S^1 \to S^1$, $p_n(z) = z^n$ satisfies (4.11). Thus, S^1 is also an n–sheeted covering of itself.

(iii) Consider the natural projection $\pi\colon S^n \to \mathbb{RP}^n$, $x \mapsto \pi(x) = \mathbb{R}\cdot x$. For any $V \subset S^n$ we have $\pi^{-1}\big(\pi(V)\big) = V \cup \big(-V\big)$. Hence, for any $p \in \mathbb{RP}^n$ we can pick a point $p_+ \in \pi^{-1}(p)$ and a small neighbourhood V_+ of p_+ such that

$$\pi^{-1}(U) = V_+ \sqcup V_-, \qquad \text{where } U := \pi(V_+) \text{ and } V_- := -V_+.$$

Moreover, in this case $\pi\colon V_\pm \to U$ is a homeomorphism so that S^n is a double covering of $\mathbb{RP}^n$.

Notice that the natural projection $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is *not* a covering, since, for example, the fibers of this map are not discrete. Nor is the map $p_n\colon \mathbb{C} \to \mathbb{C}$ a covering for $n \neq 1$, since $\#p_n^{-1}(1) = n$ and $\#p_n^{-1}(0) = 1$.

The following terminology will be useful in the sequel.

Definition 4.13. A map $\tilde f\colon Z \to Y$ is said to be a lift of $f\colon Z \to X$ if $p \circ \tilde f = f$.

Theorem 4.14. *Let $p\colon Y \to X$ be a covering.*

(i) For any path $u\colon I \to X$ starting at some $x_0 \in X$ and any $y_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde u\colon I \to Y$ starting at y_0.

(ii) For each homotopy $h\colon I \times I \to X$ such that $h(0,s) = x_0$ for all $s \in I$ there is a unique lift $\tilde h\colon I \times I \to Y$ such that $\tilde h(0,s) = y_0$ for all $s \in I$.

Proof. Since I is compact, there exists a partition $t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = 1$ with the following property: For each $k \in \mathbb{N}_0$, $k \le n-1$ there exists $U_k \subset X$ such that

(a) $u\big([t_k, t_{k+1}]\big) \subset U_k$;

(b) $p^{-1}(U_k) = \sqcup_j V_{kj}$ and $p\colon V_{kj} \to U_k$ is a homeomorphism.

We construct a lift of u by the induction on k. The initial step proceeds as follows. Since $x_0 = u(0) \in U_0$, there exists some V_{0j} such that $y_0 \in V_{0j}$. Hence, using the fact that $p\colon V_{0j} \to U_0$ is a homeomorphism, for $t \in [0, t_1]$ we define

$$\tilde{u}(t) = p|_{V_{0j}}^{-1} \circ u\,(t).$$

Furthermore, suppose that $\tilde{u}\colon [0, t_k] \to Y$ has been constructed. Since $u(t_k) \in U_k$, there exists some $j = j(k)$ such that $\tilde{u}(t_k) \in V_{kj}$. By (a) combined with the fact that $p\colon V_{kj} \to U_k$ is a homeomorphism, we obtain an extension of $\tilde{u}$ to $[t_k, t_{k+1}]$ by setting

$$\tilde{u}(t) = p|_{V_{kj}}^{-1} \circ u\,(t), \qquad \text{for } t \in [t_k, t_{k+1}].$$

This finishes the proof of the existence of $\tilde{u}$.

To prove the uniqueness, assume that $\tilde{u}$ and $\hat{u}$ are two lifts of u. Denote

$$\bar{\tau} = \sup\big\{\tau \in I \;\mid\; \tilde{u} = \hat{u} \text{ on } [0, \tau\,]\big\}.$$

If $\hat{\tau} = 1$ we are done, otherwise there exists a unique $k \le n-1$ such that $\bar{\tau} \in [t_k, t_{k+1})$. Since $\tilde{u}(t_k) = \hat{u}(t_k)$, we must have

$$\tilde{u}(t) = p|_{V_{kj}}^{-1} \circ u\,(t) = \hat{u}(t) \qquad \text{for all } t \in [t_k, t_{k+1}]$$

contradicting the definition of $\bar{\tau}$. This contradiction finishes the proof of *(i)*. The proof of *(ii)* is similar and is left to the reader. □

Corollary 4.15. *If $p\colon Y \to X$ is a covering and $p(y_0) = x_0$, then $p_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective. Moreover,*

$$\operatorname{Im} p_* = \big\{u \in \Omega(X, x_0) \mid \tilde{u} \in \Omega(Y, y_0)\big\}.$$

Proof. Assume $v \in \Omega(Y, y_0)$ represents an element in $\ker p_*$. This means that $u = p \circ v$ is homotopic to the constant path x_0. If h is a homotopy between u and x_0, let $\tilde{h}$ be the lift provided by Theorem 4.14, *(ii)*. Since $h(1, s) = x_0$, we have $\tilde{h}(1, s) \in p^{-1}(x_0)$. By recalling that $p^{-1}(x_0)$ is discrete, we obtain that the map $s \mapsto \tilde{h}(1, s)$ is constant, since this is a continuous map. Furthermore, by the uniqueness of the lift we have

$$\tilde{u} = v = \tilde{h}(\cdot, 0) \qquad \Longrightarrow \qquad \tilde{h}(1, 0) = v(1) = y_0 \qquad \Longrightarrow \qquad \tilde{h}(1, s) = y_0 \quad \forall s \in I.$$

Hence, $\tilde{h}$ is a homotopy between v and the constant loop so that $\ker p_*$ is trivial indeed.

Furthermore, if $[u] \in \operatorname{Im} p_*$, then there exists some $v \in \Omega(Y, y_0)$ such that $p \circ v$ is homotopic to u. Arguing just like above, we obtain a homotopy $\tilde{h}$ between the lift $\tilde{u}$ starting at y_0 in Y and the lift of $p \circ v$, that is v. Moreover, $\tilde{h}(1, s)$ is constant, hence $\tilde{h}(1, s) = \tilde{h}(1, 1) = v(1) = y_0$. Hence, $\tilde{u}(1) = \tilde{h}(1, 0) = y_0$, that is $\tilde{u}$ is a loop based at y_0. □

Corollary 4.16. *The fundamental group of the circle is infinite cyclic, that is $\pi_1(S^1, 1) \cong \mathbb{Z}$.*

Proof. Recall that the circle S^1 is covered by $\mathbb{R}$, see Example 4.12, (i). Hence, any loop u in S^1 based at 1 admits a lift $\tilde{u}\colon I \to \mathbb{R}$ starting at the origin. Consider the map

$$e\colon \Omega(S^1, 1) \to p^{-1}(1) = \mathbb{Z}, \qquad u \mapsto \tilde{u}(1).$$

Since $\mathbb{R}$ is path-connected, this map is surjective.

By the proof of Corollary 4.15, we obtain that if u is homotopic to v, then $\tilde{u}(1) = \tilde{v}(1)$. Hence, the map e yields a well-defined surjective map (still denoted by the same letter)

$$e\colon \pi_1(S^1, 1) \to \mathbb{Z}, \qquad u \mapsto \tilde{u}(1).$$

In fact, e is a group homomorphism. To see this, notice that if $u, v \in \Omega(S^1, 1)$, then the lift of $u * v$ starting at the origin is the curve

$$t \mapsto \begin{cases} \tilde{u}(2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \tilde{v}(2t-1) + \tilde{u}(1) & \text{if } t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

so that $e(u * v) = e(u) + e(v)$.

Furthermore, if $[u] \in \ker e$, then the lift $\tilde{u}$ is a loop in $\mathbb{R}$. Since $\mathbb{R}$ is contractible, we have $[\tilde{u}] = 0 \implies [u] = \exp_*[\tilde{u}] = 0$. Hence, e is injective. However we have seen above that e is also surjective. Thus, e is an isomorphism. □

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4.3 Uniqueness of coverings

In this section I show that a covering $p\colon Y \to X$ is uniquely determined (in a suitable sense) by the image of the fundamental group of Y in the fundamental group of X. To this end, the following will be useful.

Lemma 4.17. *Let $p\colon (Y, y_0) \to (X, x_0)$ be a covering, where both X and Y are connected. For any continuous map $f\colon (Z, z_0) \to (X, x_0)$, where Z is path-connected and locally path-connected the following holds:*

$$\exists! \text{ lift } \tilde{f}\colon (Z, z_0) \to (Y, y_0) \qquad \Longleftrightarrow \qquad f_*\big(\pi_1(Z, z_0)\big) \subset p_*\big(\pi_1(Y, y_0)\big). \tag{4.18}$$

Sketch of proof. If there exists a lift, then $p \circ \tilde{f} = f \implies \operatorname{Im} f_* \subset \operatorname{im} p_* \subset \pi_1(X, x_0)$.

Furthermore, we need to show that the lift does exist and is unique. To this end, assume first that $\tilde{f}$ exists. Since Z is path-connected, for any $z \in Z$ we can find a path u connecting z_0 with z. Then $\tilde{f} \circ u$ is a path in Y projecting to $f \circ u$. In other words, $\tilde{f} \circ u$ is the unique lift of $f \circ u$ beginning at y_0. In particular, at the terminal point we must have

$$\tilde{f}(z) = \widetilde{f \circ u}\,(1). \tag{4.19}$$

In particular, if $\tilde{f}$ exists, it is unique.

The idea is to utilize (4.19) *to define* $\tilde{f}$. To explain, let u be a path in Z connecting z_0 with z as above. Define $\tilde{f}$ by (4.19). To show that this is well defined, let v be any other path connecting z_0 with z. Then $u * \bar{v}$ is a loop based at z_0 and therefore by (4.18) and Corollary 4.15, the lift of $f \circ (u * \bar{v})$ is a loop in Y based at y_0. This implies

$$\widetilde{f \circ u}(1) = \big(\widetilde{f \circ v}\big)(1),$$

thus proving that $\tilde{f}$ is well defined.

It is also pretty clear that $\tilde{f}$ is continuous, since essentially $\tilde{f}$ is obtained as a composition of f and p^{-1} restricted to a sufficiantly small open subset. The details can be found for example in [Mas91, P. 129] (this uses the local path-connectedness of Z) □

Let $p_1: (Y_1, y_{01}) \to (X, x_0)$ and $p_2: (Y_2, y_{02}) \to (X, x_0)$ be two covering spaces.

Definition 4.20. *A homomorphism* of Y_1 into Y_2 is a continuous map $\varphi: Y_1 \to Y_2$ such that the diagram

$$\begin{tikzcd} Y_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes. A homomorphism φ is called *an isomorphism* is there exists a homomorphism $\psi: Y_2 \to Y_1$ such that $\psi \circ \varphi = id_{Y_1}$ and $\varphi \circ \psi = id_{Y_2}$. In the case $Y_1 = Y_2 = Y$ and $p_1 = p_2$, an isomorphism $\varphi: Y \to Y$ is called *a deck transformation*.

Corollary 4.21. *Let $p_1: (Y_1, y_{01}) \to (X, x_0)$ and $p_2: (Y_2, y_{02}) \to (X, x_0)$ be two path-connected and locally path-connected covering spaces. Then Y_1 and Y_2 are isomorphic if and only if $p_{1*}\big(\pi_1(Y_1, y_{01})\big)$ and $p_{2*}\big(\pi_1(Y_2, y_{02})\big)$ are conjugate in $\pi_1(X, x_0)$.*

Proof. Let u be a loop in X based at x_0 such that

$$p_{2*}\big(\pi_1(Y_2, y_{02})\big) = [\bar u] \cdot p_{1*}\big(\pi_1(Y_1, y_{01})\big) \cdot [u]. \tag{4.22}$$

If $\tilde u$ is the lift of u starting at y_{01}, denote by y_{01}' the terminal point of $\tilde u$. By the proof of Proposition 4.6, the map

$$P_{\tilde u}: \Omega(Y_1, y_{01}') \to \Omega(Y_1, y_{01}), \qquad v \mapsto \tilde u * v * \bar{\tilde u}$$

induces an isomorphism $\pi_1(Y_1, y_{01}') \to \pi_1(Y_1, y_{01})$. Combining this with (4.22) we obtain

$$p_{1*}\big(\pi_1(Y_1, y_{01})\big) = [u] \cdot p_{1*}\big(\pi_1(Y_1, y_{01}')\big) \cdot [\bar u] \quad \Longrightarrow \quad p_{1*}\big(\pi_1(Y_1, y_{01}')\big) = p_{2*}\big(\pi_1(Y_2, y_{02})\big).$$

The statement of this corollary follows from Lemma 4.17. □

Corollary 4.23. *Let $p: (Y, y_0) \to (X, x_0)$ be a path-connected and locally path-connected covering. Then for any $y_1, y_2 \in p^{-1}(x_0)$ there exists a unique deck transformation φ such that $\varphi(y_1) = y_2$.* □

4.4 The universal covering space and the classification of the covering spaces

Definition 4.24. A path-connected topological space Y is said to be *simply connected*, if $\pi_1(Y, y_0)$ is trivial for some ($\Rightarrow$ any) basepoint y_0.

A simply-connected covering space of X is called *the universal covering of X* and is typically denoted by $\tilde X$. It follows from Corollary 4.21 that for a path-connected and locally path-connected space X if the universal covering exists, it is unique up to an isomorphism.

It turns out that the universal covering plays a very particular rôle. Our aim in this section is to show that simply connected coverings exist. This in turn will allow us to strengthen Corollary 4.21 substantially.

Lemma 4.25 (A necessary condition for the existence of the universal covering)**.** *Assume a path-connected and locally path-connected space X admits a simply connected covering $p\colon \tilde X \to X$. Then for any $x \in X$ there exists a neighbourhood U of x such that*

$$\imath_*\colon \pi_1(U, x) \to \pi_1(X, x)$$

is the trivial homomorphism. This means that any loop in U based at x can be homotoped in X to the constant loop.

Proof. Let U be a neighbourhood of x as in the definition of the covering. Pick any $\tilde x \in p^{-1}(x)$ and denote by V the component of $p^{-1}(U)$ containing $\tilde x$. Consider the commutative diagram

$$\begin{tikzcd} \pi_1(V, \tilde x) \arrow[r] \arrow[d, "(p|_V)_*"'] & \pi_1(\tilde X, \tilde x) \arrow[d] \\ \pi_1(U, x) \arrow[r, "\imath_*"] & \pi_1(X, x). \end{tikzcd}$$

Notice that the homomorphism represented by the left vertical arrow is in fact an isomorphism. Since $\pi_1(\tilde X, \tilde x)$ is trivial, the image of $\imath_*$ must be trivial, that is $\imath_*$ is the trivial homomorphism. □

Definition 4.26. A space X such that for any $x \in X$ there exists a neighbourhood U of x such that $\imath_*\colon \pi_1(U, x) \to \pi_1(X, x)$ is the trivial homomorphism is called *semilocally simply connected*.

The infinite union of shrinking circles as in Example 3.7 yields an example of a space, which is path-connected, locally path-connected, but not semilocally simply connected.

Theorem 4.27. *Any path-connected, locally path-connected, and semilocally simply connected space X admits a universal covering space $\tilde X$.*

Sketch of proof. Assume first that X admits a universal covering $\tilde X$. Denote by $p\colon \tilde X \to X$ the projection and pick points $x_0 \in X$ and $\tilde x_0 \in p^{-1}(x_0) \subset \tilde X$. Since $\tilde X$ is path-connected, for any $\tilde x \in \tilde X$ there is a path $\tilde u$ connecting $\tilde x_0$ with $\tilde x$.

If $\tilde v$ is any other path with the starting point $\tilde x_0$ and the terminal point $\tilde x$, then we have

$$\tilde v \;\simeq\; \tilde v * (\bar{\tilde u} * \tilde u) \;\simeq\; (\tilde v * \bar{\tilde u}) * \tilde u \;\simeq\; \tilde u. \tag{4.28}$$

Here the first and the second relations follow by the proof of Theorem 4.3, whereas the last one follows from simply connectedness of $\tilde X$. Notice that all homotopies in (4.28) preserve the ends of the corresponding paths, that is $\tilde u$ and $\tilde v$ are homotopic relative to the endpoints.

Thus, for any $\tilde x \in \tilde X$ there is a unique equivalence class of paths connecting $\tilde x_0$ with $\tilde x$. However, each $\tilde u$ as above is the unique lift of $u := p \circ \tilde u$ starting at $\tilde x_0$. Therefore, we have a natural bijective map

$$\tilde X_0 := \big\{ [u] \mid u \text{ is a path in } X \text{ starting at } x_0 \big\} \to \tilde X, \qquad [u] \mapsto \tilde u(1),$$

where the equivalence relation for $\tilde X_0$ is given by the existence of homotopies relative to the endpoints.

The idea is now to *define* the universal covering as $\tilde X_0$. Notice that we have a natural map

$$p\colon \tilde X_0 \to X, \qquad [u] \mapsto u(1).$$

It can be shown that $\tilde{X}_0$ admits a unique topology such that the above map is a covering [Mas91, P. 143–144].

To show that $\tilde{X}_0$ is path-connected, for any $[u] \in \tilde{X}_0$ consider the map

$$s \mapsto u_s(t) := \begin{cases} u(t) & \text{if } t \leq s \\ u(s) & \text{if } t \geq s. \end{cases}$$

This yields a path in X_0 between the constant path x_0 and $[u]$.

It remains to show that $\pi_1(\tilde{X}_0, x_0)$ is trivial. Since p_* is injective, it suffices to show that the image of $\pi_1(\tilde{X}, x_0)$ in $\pi_1(X, x_0)$ is trivial. Any element in $\operatorname{Im} p_*$ is represented by $[u] \in \pi_1(X, x_0)$ such that u lifts to a loop in $\tilde{X}_0$. By the uniqueness of the lift, the curve

$$s \mapsto [u_s] \tag{4.29}$$

is the lift of u starting at the constant loop x_0. This curve is a loop in $\tilde{X}_0$ if $[u_1] = x_0$, that is $[u] = x_0$. This finishes the proof. □

Theorem 4.30. *Let (X, x_0) be a path-connected, locally path-connected, and semilocally path-connected space. There is a natural bijective correspondence between the set of all path-connected coverings of X up to isomorphisms and the conjugacy classes of subgroups in $\pi_1(X, x_0)$.*

Proof. Given a path-connected covering $p\colon Y \to X$, pick any $y_0 \in p^{-1}(x_0)$ and associate the conjugacy class $p_*(\pi_1(Y, y_0))$ to Y. This is well defined and injective by Corollary 4.21.

Thus, we need to show that for any subgroup H in $\pi_1(X, x_0)$ there exists a covering (Y, y_0) such that $p_*\big(\pi_1(Y, y_0)\big)$ is conjugate to H. Let $\tilde{X}_0$ be defined as in the proof of Theorem 4.27. Define an equivalence relation on $\tilde{X}_0$ by

$$[u] \sim [v] \qquad \Longleftrightarrow \qquad u(1) = v(1) \quad \text{and} \quad [u * \bar{v}] \in H.$$

The fact that H is a group implies that $\sim$ is an equivalence relation.

Denote $Y := \tilde{X}/\sim$. We still have a natural map

$$q\colon Y \to X, \qquad q\big([u]\big) = u(1),$$

which can be shown to be a covering.

Just like in the proof of Theorem 4.27, one can show that for any loop u in X based at x_0 the lift $\tilde{u}$ to Y is given by (4.29). This is a loop in Y if and only if

$$[u] = [u_1] \sim x_0,$$

where x_0 denotes the class of the constant loop. This is clearly equivalent to saying that $[u] \in H$. In other words, by Corollary 4.15 we have

$$[u] \in q_*\big(\pi_1(Y, x_0)\big) \qquad \Longleftrightarrow \qquad [u] \in H,$$

which proves the existence part. □

Let me note in passing that the hypotheses of Theorem 4.30 are not very restrictive. In practice, one is usually interested in covering spaces of reasonably nice spaces, for example manifolds. In this category, the hypotheses of being locally path-connected and semilocally simply connected are satisfied automatically. Thus, for any path-connected ($\Leftrightarrow$ connected) manifold M there is a bijective correspondence between conjugacy classes of subgroups of $\pi_1(M)$ and its coverings. L 20

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