MATH-F310: Differential Geometry I
- Assignment 2 -

Charts and parametrisations

1. $\diamondsuit$[1] *Stereographic projection:*

   (a) Let S^2 be the standard 2- dimensional sphere of radius 1 centered at the origin in $\mathbb{R}^3$. Describe the two stereographic projections from the north and south poles (call them N and S) respectively.

   (b) Let $\varphi_N : S^2 \backslash \{N\} \to \mathbb{R}^2$ be the stereographic projection from N. We define $\psi_N := \varphi_N^{-1} : \mathbb{R}^2 \to S^2 \backslash \{N\} \subset \mathbb{R}^3$. Show that $D\psi_N$ is injective at each point. Thus ψ_N is a parametrization at each point of $S^2 \backslash \{N\}$.

   (c) Denote by $\psi_S \colon \mathbb{R}^2 \to S^2 \backslash \{S\}$ the inverse of the stereographic projection from the south pole. Prove that:

   $$\psi_{SN} := \psi_S^{-1} \circ \psi_N : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2 \backslash \{0\} \text{ is given by}$$
   $$\psi_{SN}(u, v) = \frac{1}{u^2 + v^2}(u, v)$$

   (d) Show that any circle that does not pass through the north pole is mapped to a circle under φ_N.

   (e) † Stereographic projections preserves angles, in the sense that if two curves intersect at an angle θ on the sphere, so do their images curves on the plane $z = 0$. (To be done after some more classes maybe)

2. $\diamondsuit$ Show that the cylinder in $\mathbb{R}^3$: $\{(x, y, z) : x^2 + y^2 = 1\}$ can be described locally as a graph of a function from $\mathbb{R}^2 \to \mathbb{R}$. Hence it's a surface.

3. $\diamondsuit$ Consider the ellipsoid $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$. Write down local charts for this

[1] Exercises marked by a $\diamondsuit$ will be done in class (if time permits).
Exercises marked by a $\spadesuit$ are to prepare at home for the first test (8th of October).
Exercises marked by a † are extra exercises.

surface in $\mathbb{R}^3$. Now consider the function $(x, y, z) \mapsto x$ on this surface. Write down the coordinate representation of this function with respect to the charts.

4. ♠ Consider the hyperboloid $x^2 + \frac{y^2}{4} - \frac{z^2}{4} = 1$. For any point on the hyperboloid, find a local chart and compute the transition map between two non-equal overlapping charts. Now consider the function $(x, y, z) \mapsto y^2$ on this surface. Write down the coordinate representation of this function with respect to two different charts of your choice.

5. ♠ Let $S \subset \mathbb{R}^3$ be a surface. Show that for any smooth function $f : S \to \mathbb{R}$ and any $p \in S, \exists$ an open neighbourhood $W \subset \mathbb{R}^3$ around p and a smooth function $h : W \to \mathbb{R}$ such that $h|_S = f|_W$.

6. † Consider the firgure ∞ as the image of the following curve

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, t \mapsto \left(\cos(t - \frac{\pi}{2}), \sin(2t)\right).$$

Show that $\psi : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3, (t, u) \mapsto (\gamma(t), u)$ is an immersion (i.e. that $D\psi$ is injective everywhere). Show however that the image of ψ is not a surface of $\mathbb{R}^3$. What goes wrong ?