Differential Geometry I

Lecture notes

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Chapter 1

Introduction

A substantial part of mathematics is related to solving equations of various types. Given any equation, we may try to analyze this by studying the following sequence of questions:

1. Does there exist a solution (a root)?
2. If the answer to the previous question is affirmative, how many solutions does the equation have?
3. If there are finitely many solutions, can we find all of them?

For example, the reader learned at school the properties of the quadratic equation $ax^2+bx+c=0$. In this case the above questions are easy to settle and the answers are well known to the reader.

Sometimes an equation may have an infinite number of solutions. If there are only countably many roots, the last question from the list above still makes sense. For example, all solutions of the equation $\sin x = 0$ are given by a simple formula: $x_n = \pi n,\ n \in \mathbb{Z}$.

In many cases, however, equations have uncountably many solutions so that asking to find all solutions is not really meaningful. Instead, it turns out to be more interesting to replace Question 3 by the following one:

3′. What are the properties of the set of *all solutions*?

Which particular properties we are interested in may depend on the context. The property most relevant to the content of this course is concerned with the local structure of the set of all solutions.

Let us consider an example. The equation

$$x_1^2 + x_2^2 + x_3^2 = 1, \tag{1.1}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, clearly has uncountably many solutions.

Denote $S^2 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, that is S^2 is the set of all solutions of (1.1). Of course, S^2 is the sphere of radius 1, however let us pretend for a moment that we do not know this. As a subset of $\mathbb{R}^3$, S^2 is a topological space. It turns out that this topological space has a very particular property, which we consider in some detail next.

The familiar stereographic projection from the north pole $N := (0,0,1)$ is given by

$$\varphi_N \colon S^2 \setminus \{N\} \to \mathbb{R}^2, \qquad \varphi_N(x) = \Big(\frac{x_1}{1-x_3},\ \frac{x_2}{1-x_3}\Big).$$

This is in fact a homeomorphism with the inverse

$$\varphi_N^{-1}(y) = \frac{1}{1+y_1^2+y_2^2}\big(2y_1,\ 2y_2,\ -1+y_1^2+y_2^2\big), \qquad y = (y_1, y_2) \in \mathbb{R}^2.$$

We can also define a stereographic projection from the south pole $S := (0, 0, -1)$ by

$$\varphi_S \colon S^2 \setminus \{S\} \to \mathbb{R}^2, \qquad \varphi_S(x) = \Big(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\Big),$$

which is also a homeomorphism.

Since any point on the sphere lies either in $S^2 \setminus \{N\}$ or $S^2 \setminus \{S\}$ (or both), any point on the sphere has a neighbourhood, which is homeomorphic to an open subset of $\mathbb{R}^n$ (of course, $n = 2$ in our particular example and the open subset is $\mathbb{R}^2$ itself). This property leads to the notion of *a manifold*, which will play a cenral rôle in the course. We will see below, that this property is *not* specific to Equation (1.1). On the contrary, for any smooth map $F \colon \mathbb{R}^k \to \mathbb{R}^\ell$ and almost any $c \in \mathbb{R}^\ell$ the set of all solutions to the equation $F(x) = c$ is a manifold. That is, there is a huge pull of examples of manifolds and many objects of particular interest in mathematics turn out to be manifolds.

Coming back to our example, we compute:

$$\varphi_S \circ \varphi_N^{-1}(y) = \Big(\frac{y_1}{|y|^2}, \frac{y_2}{|y|^2} \Big). \tag{1.2}$$

Hence, $\varphi_S \circ \varphi_N^{-1}$ is *smooth* on an open subset $\mathbb{R}^2 \setminus \{0\}$ and a similar computation yields that this is also true for $\varphi_N \circ \varphi_S^{-1}$. This property can be used to study smooth functions on the sphere directly without reference to the ambient space. More importantly, in more general situations where the ambient Euclidean space may be simply absent, an analogue of this property allows one to apply familiar tools of analysis to functions defined on more sophisticated objects than just subsets of an Euclidean space. In some sense, this constitutes the core of differential geometry.

Summing up, the aim of these notes is to transfer familiar tools of mathematical analysis to a more geometric setting where the underlying domain of a function (map) is not just an open subset of $\mathbb{R}^n$, but rather a manifold. The benefits of doing so are ubiquitous, but explaining this in some detail requires a bit of work. It is my hope to convey that the notion of a manifold is useful and well worth studying further.

Chapter 2

Smooth manifolds

2.1 Basic definitions and examples

Recall that a topological space M is called Hausdorff, if for any two distinct points $m_1, m_2 \in M$ there are neighbourhoods $U_1 \ni m_1$ and $U_2 \ni m_2$ such that $U_1 \cap U_2 = \varnothing$. If the topology of M admits a countable base, then M is said to be second countable. For example, $\mathbb{R}^k$ is both Hausdorff and second countable.

Definition 2.1. A Hausdorff second countable topological space M is called *a topological manifold* of dimension k, if M is locally homeomorphic to $\mathbb{R}^k$.

To explain, this means that any point $m \in M$ admits a neighbourhood U and a homeomorphism $\varphi\colon U \to V$, where V is an open subset of $\mathbb{R}^k$. The pair (U, φ) (or, sometimes just U) is called a chart on M near m.

Notice that the requirements that a manifold is Hausdorff and second countable are to a great extent of technical nature, whereas being locally homeomorphic to $\mathbb{R}^k$ is a crucial property of manifolds.

Clearly, $\mathbb{R}^k$ and in fact any open subset of $\mathbb{R}^k$ are examples of topological manifolds of dimension k. As we have established in the introduction, 2-spheres are manifolds of dimension two. Similar arguments yield in fact that the k-sphere

$$S^k := \Big\{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid \sum_{j=1}^{k+1} x_j^2 = 1\Big\}$$

is a k-manifold.

Somewhat special is the case of dimension zero. Since $\mathbb{R}^0$ is by definition a single point, the above definition requires that each point of M has a neighborhood consisting only of this point. In other words, M is a countable discrete space.

Definition 2.2. A collection of charts $\mathcal{U} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is called a C^0-*atlas*, if $\bigcup_{\alpha\in A} U_\alpha = M$, that is if any point of M is contained in some chart. Here A is an arbitrary index set.

For example, $\mathbb{R}^k$ admits a C^0-atlas consisting of a single chart $(\mathbb{R}^k, id)$. In the introduction we have constructed a C^0-atlas on the 2-sphere consisting of two charts. However, there is no C^0-atlas on S^2 consisting of a single chart, since S^2 is not homeomorphic to an open subset of $\mathbb{R}^2$ (why?).

Given a C^0-atlas $\mathcal{U}$, pick any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) such that $U_\alpha \cap U_\beta \neq \varnothing$. The map

$$\theta_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta), \tag{2.3}$$

which a homeomorphism between two open subsets of $\mathbb{R}^k$, is called *a coordinate transformation*[1]. Notice that $\theta_{\beta\alpha}$ is the inverse map to $\theta_{\alpha\beta}$. In particular, $\theta_{\alpha\beta}$ is a homeomorphism between $\varphi_\beta(U_\alpha \cap U_\beta)$ and $\varphi_\alpha(U_\alpha \cap U_\beta)$.

It is a common practice to suppress the domain and the target of $\theta_{\alpha\beta}$ writing simply $\theta_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. While this may be confusing at first, the advantage is that this allows us to suppress less important details so that the most essential features are clearer. If in doubt, the reader should write the domain and target explicitly.

Definition 2.4. A C^0-atlas $\mathcal{U}$ is called *smooth*, if all coordinate transformation maps $\theta_{\alpha\beta}$, $\alpha, \beta \in A$, are smooth.

Remark 2.5. Equally well, we can say that $\mathcal{U}$ is a C^ℓ-atlas, if all coordinate transformation maps belong to $C^\ell(\mathbb{R}^n; \mathbb{R}^n)$ (keep in mind that these are defined on open subsets of $\mathbb{R}^n$ only) for some fixed natural number ℓ. The theory does not depend much on the choice of ℓ as long as ℓ is not too small. In practice $\ell \geq 3$ would suffice in most of the cases, however to avoid non-essential details it is convenient to put $\ell = \infty$ from the very beginning.

Two charts (U, φ) and any (V, ψ) not necessarily from the same atlas are said to be *smoothly compatible* if the maps

$$\varphi \circ \psi^{-1} \qquad \text{and} \qquad \psi \circ \varphi^{-1} \tag{2.6}$$

are smooth, compare with (2.3). We consider two atlases $\mathcal{U}$ and $\mathcal{V}$ as "essentially equal", if all charts from $\mathcal{U}$ are smoothly compatible with all charts in $\mathcal{V}$. More formally, we have the following definition.

Definition 2.7. Two atlases $\mathcal{U}$ and $\mathcal{V}$ on the same underlying topological space M are called equivalent, if $\mathcal{U} \cup \mathcal{V}$ is a smooth atlas on M, that is if all charts from $\mathcal{U}$ are smoothly compatible with all charts in $\mathcal{V}$. An equivalence class of atlases is called *a smooth structure* on M. *A smooth manifold* consists of a Hasdorff second countable topological space and a smooth structure.

To explain the point of the above definition, consider the 2-sphere. In the introduction we constructed a smooth atlas on S^2, namely $\mathcal{U} := \big\{ (S^2 \setminus \{N\},\ \varphi_N),\ (S^2 \setminus \{S\},\ \varphi_S) \big\}$. However, there are many ways to construct another smooth atlas, for example as follows:

$$\mathcal{U}' := \big\{ S^2 \setminus \{N\}, \varphi_N \big\} \cup \big\{ (S^2_+, \varphi_+) \big\}.$$

Here $S^2_+ := \big\{ x \in S^2 \mid x_3 > 0 \big\}$ and $\varphi_+(x) = (x_1, x_2)$.

Exercise 2.8. Check that $\mathcal{U}'$ is a smooth atlas equivalent to $\mathcal{U}$.

It should be intuitively clear, that the description of S^2 via smooth atlases $\mathcal{U}$ and $\mathcal{U}'$ are 'essentially equal'. Hence, it is natural to identify $(S^2, \mathcal{U})$ and $(S^2, \mathcal{U}')$.

An atlas $\mathcal{U}$ is called *maximal*, if for any chart (V, ψ) smoothly compatible with all charts in $\mathcal{U}$ is already contained in $\mathcal{U}$.

The importance of maximal atlases stems from the following result.

Lemma 2.9. *Each equivalence class of smooth atlases is represented by a unique maximal atlas.*

[1]The origin of this terminology will be clear below.

Proof. For a smooth atlas $\mathcal{U}$ on M define

$$\mathcal{U}_{max} := \big\{(V,\psi) \text{ is a chart on } M \text{ s.t. (2.6) are both smooth for all } (U,\varphi)\in\mathcal{U}\big\}.$$

Exercise 2.10. Check that $\mathcal{U}_{max}$ is a smooth atlas on M.

By the construction of $\mathcal{U}_{max}$, we have $\mathcal{U}\subset\mathcal{U}_{max}$. Hence, any chart smoothly compatible with any chart in $\mathcal{U}_{max}$ is also smoothly compatible with any chart in $\mathcal{U}$ and therefore is contained in $\mathcal{U}_{max}$. Hence, $\mathcal{U}_{max}$ is maximal. Clearly, $\mathcal{U}$ and $\mathcal{U}_{max}$ represent the same smooth structure. □

By the above lemma, a smooth manifold may be considered as being equipped with a maximal atlas. In particular, if $\mathcal{U}$ is any smooth atlas on M, we may freely add any chart smoothly compatible with all charts in $\mathcal{U}$ without changing the smooth structure. For example, if (U,φ) is a chart near m_0, then $(U,\hat\varphi)$ with

$$\hat\varphi(m)=\varphi(m)-\varphi(m_0)$$

is also a chart near $m_0\in M$ smoothly compatible with all charts in $\mathcal{U}$. The chart $(U,\hat\varphi)$ satisfies

$$\hat\varphi(m_0)=0,$$

which is commonly expressed by saying that $(U,\hat\varphi)$ is *centered at* m_0.

Remark 2.11. In what follows only smooth manifolds will be considered. Therefore, by saying that M is a manifold, we always mean a *smooth* manifold, unless explicitly stated otherwise.

Let us finish this section with some further examples of manifolds.

Example 2.12 (Products)**.** Let M and N be smooth manifolds of dimensions k and ℓ respectively. Let $\mathcal{U}=\big\{(U_\alpha,\varphi_\alpha)\mid\alpha\in A\big\}$ and $\mathcal{V}=\big\{(V_\lambda,\psi_\lambda)\mid\lambda\in\Lambda\big\}$ be smooth atlases on M and N respectively. Then the product $M\times N$ is a Hausdorff second countable topological space. We define a C^0-atlas on $M\times N$ by setting

$$\mathcal{W}:=\Big\{\big(U_\alpha\times V_\lambda,\ \varphi_\alpha\times\psi_\lambda\big)\mid\alpha\in A,\ \lambda\in\Lambda\Big\}.$$

Given any two charts $\big(U_\alpha\times V_\lambda,\ \varphi_\alpha\times\psi_\lambda\big)$ and $\big(U_\beta\times V_\mu,\ \varphi_\beta\times\psi_\mu\big)$ the corresponding coordinate transformation is given by $\theta_{\alpha\beta}\times\eta_{\lambda\mu}$, where $\theta_{\alpha\beta}=\varphi_\alpha\circ\varphi_\beta^{-1}$ and $\eta_{\lambda\mu}=\psi_\lambda\circ\psi_\mu^{-1}$ are smooth maps. More precisely, this means the following:

$$\begin{aligned}\theta_{\alpha\beta}\times\eta_{\lambda\mu}\colon\mathbb{R}^k\times\mathbb{R}^\ell&\to\mathbb{R}^k\times\mathbb{R}^\ell,\\ \theta_{\alpha\beta}\times\eta_{\lambda\mu}(x,y)=\big(\theta_{\alpha\beta}(x),\ \eta_{\lambda\mu}(y)\big),&\quad x\in\mathbb{R}^k,\ y\in\mathbb{R}^\ell.\end{aligned}$$

In particular, $\theta_{\alpha\beta}\times\eta_{\lambda\mu}$ is a smooth map, which means that the atlas constructed above is smooth. Hence, $M\times N$ is a smooth manifold of dimension $k+\ell$. This yields in particular that the following

(i) the k-dimensional torus $\mathbb{T}^k:=S^1\times\cdots\times S^1$ and

(ii) the cylinder $\mathbb{R}\times S^1$

are smooth manifolds. In the latter case, the dimension of $\mathbb{R}\times S^1$ equals 2.

Example 2.13 (Real projective spaces)**.** The real projective space $\mathbb{RP}^k$ of dimension k is defined to be the set of all lines in $\mathbb{R}^{k+1}$ through the origin. Since each line through the origin is uniquely determined by a point on this line distinct from the origin, we have

$$\mathbb{RP}^k = \big(\mathbb{R}^{k+1} \setminus \{0\}\big)/\sim,$$

where $x, y \in \mathbb{R}^{k+1} \setminus \{0\}$ are defined to be equivalent if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $y = \lambda x$. In particular, we have the canonical surjective quotient map

$$\pi \colon \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{RP}^k, \qquad \pi(x) = [x].$$

If $x = (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\}$, it is customary to write $[x_0 : x_1 : \dots : x_k]$ for $[x]$.

We endow $\mathbb{RP}^k$ with the quotient topology, that is $U \subset \mathbb{RP}^k$ is open if and only $\pi^{-1}(U)$ is open in $\mathbb{R}^{k+1} \setminus \{0\}$. It is straighforward to check that this yields a Hausdorff second countable topological space.

To construct a C^0-atlas on $\mathbb{RP}^k$, observe that each

$$U_j := \big\{[x_0 : x_1 : \dots : x_k] \in \mathbb{RP}^k \mid x_j \neq 0\big\}, \qquad j = 0, 1, \dots, k,$$

is an open subset of $\mathbb{RP}^k$. Indeed, this follows from the fact that

$$\pi^{-1}(U_j) = \big\{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\} \mid x_j \neq 0\big\}$$

is an open subset of $\mathbb{R}^{k+1} \setminus \{0\}$.

The map

$$\begin{aligned} &\varphi_j \colon U_j \to \mathbb{R}^k, \\ &\varphi_j\big[x_0 : x_1 : \dots : x_{j-1} : x_j : x_{j+1} : \dots : x_k\big] = \Big(\frac{x_0}{x_j}, \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_k}{x_j}\Big) \end{aligned}$$

is well-defined and continuous. Moreover, the map

$$\psi_j \colon \mathbb{R}^k \to U_j, \qquad \psi_j\big(y_0, y_1, \dots, y_{k-1}\big) = \big[y_0 \colon y_1 \colon \dots \colon y_{j-1} \colon 1 \colon y_j \colon \dots \colon y_{k-1}\big]$$

is a continuous inverse of φ_j, that is φ_j is a homeomorphism. Since the collection $U_0, \dots, U_k$ clearly covers all of $\mathbb{RP}^k$, $\mathcal{U} := \big\{(U_j, \varphi_j) \mid j = 0, 1, \dots, k\big\}$ is a C^0-atlas on $\mathbb{RP}^k$.

Next, let us consider the coordinate transformations. To simplify the notations we consider only the map $\theta_{01} = \varphi_0 \circ \varphi_1^{-1} = \varphi_0 \circ \psi_1$. We have

$$\theta_{01}(y_0, \dots, y_{k-1}) = \varphi_0\big([y_0 : 1 : y_1, \dots, y_{k-1}]\big) = \Big(\frac{1}{y_0}, \frac{y_1}{y_0}, \dots, \frac{y_{k-1}}{y_0}\Big),$$

which is smooth on

$$\varphi_1\big(U_0 \cap U_1\big) = \big\{y \in \mathbb{R}^k \mid y_0 \neq 0\big\}.$$

A similar argument yields that all coordinate transformations $\theta_{ij} = \varphi_i \circ \psi_j$ are smooth on their domains of definition. Thus, $\mathcal{U}$ is a smooth atlas and $\mathbb{RP}^k$ is a smooth manifold of dimension k.

It may be useful to keep some non-examples of manifolds in mind.

(a) The set $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$ consisting of two straight lines $y = \pm x$ intersecting at the origin, is not a manifold. Indeed, if M were a manifold, its dimension must be one. However, the origin does not have a neighbourhood in M homeomorphic to an open subset of $\mathbb{R}^1$ (Why?).

(b) A disjoint union of manifolds is a manifold. However, a disjoint uncountable union of non-empty manifolds is *not* a manifold, since the second countability axiom is violated. For example,

$$N := \bigsqcup_{\alpha \in (0,1)} \mathbb{R}_\alpha$$

is not a manifold. Notice that the above example is *not* homeomorphic to $(0,1) \times \mathbb{R}$, which is a manifold indeed, since, for example, each line $\mathbb{R}_\alpha \subset N$ is an open subset.

(c) Consider the following line "with a double point":

$$L := (-\infty, 0) \cup \{a, b\} \cup (0, +\infty).$$

Here $\{a, b\}$ is understood as a set consisting of two distinct elements. The following two subsets

$$U_a := (-\infty, 0) \cup \{a\} \cup (0, +\infty) \qquad \text{and} \qquad U_b := (-\infty, 0) \cup \{b\} \cup (0, +\infty)$$

cover all of L. Define $\varphi_a \colon U_a \to \mathbb{R}$ by $\varphi_a(x) = x$ if $x \neq a$ and $\varphi_a(a) = 0$. By the same token we can define $\varphi_b \colon U_b \to \mathbb{R}$.

A topology on L is defined simply by saying that V is open if and only if $\varphi_a(V \cap U_a)$ and $\varphi_b(V \cap U_b)$ are open in $\mathbb{R}$.

This yields a second countable topological space with a smooth atlas. However, L is non-Hausdorff.