

Manifolds

There are a number of ways we could generalize our discussion of surfaces.

1. Hypersurfaces These are subsets $S \subset \mathbb{R}^{k+1}$ admitting (smooth) parametrizations $\psi: V \rightarrow V \subset S$, $V \subset \mathbb{R}^k$ in a nbhd of each pt just as in the definition of the surface. Virtually all notions and theorems about surfaces generalize immediately to this case (together with proofs), since the condition $k=2$ was never used in an essential way.

2. Embedded submanifolds. Roughly speaking these are "k-dimensional surfaces in \mathbb{R}^{k+l} ". More formally, we could call $S \subset \mathbb{R}^{k+l}$ a k-dimensional submanifold, if S admits parametrizations $\psi: V \rightarrow V \subset S$, where $V \subset \mathbb{R}^k$ is open and $D\psi$ is injective at each pt. Most of the statements about surfaces we have seen generalize to this case too (and rather trivially) except the very last section involving the Gauss map. It generalizes too, however, this requires some extra work

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and, more importantly, not all statements made for surfaces hold true in this case.

Abstract manifolds

Nonetheless, even higher degree of abstraction is required for applications. Hence, we adapt the following approach.

Def A Hausdorff topological space M is said to be a topological manifold of dimension k , $k \in \mathbb{N}$, if M is locally homeomorphic to \mathbb{R}^k .

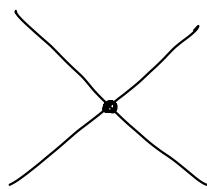
To explain : $\forall m \in M \quad \exists$ a nbhd $U \subset M$ and a homeomorphism $\varphi : U \rightarrow V \subset \mathbb{R}^k$, where V is open. A pair (U, φ) is called a chart on M .

Ex 1) Any surface is a topological mfld of dimension $k = 2$.

2) S^k is a topological mfld of dimension n . This can be seen by covering S^k by two charts just in the case $k = 2$.

A non-example : $\{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$

is not a topological mfld.



Def. A collection $\mathcal{U} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ (3)
of charts on M is called a C° -atlas, if

$$\bigcup_{\alpha \in A} U_\alpha = M.$$

- A C° -atlas is called smooth, if each coordinate transformation

$$\Theta_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \overset{\cap}{\underset{\mathbb{R}^k}{\varphi_\beta(U_\alpha \cap U_\beta)}} \rightarrow \overset{\cap}{\underset{\mathbb{R}^k}{\varphi_\alpha(U_\alpha \cap U_\beta)}}$$

is smooth.

- A smooth manifold is a topological mfld together with a smooth atlas.

Examples

- \mathbb{R}^k with a single chart $(\mathbb{R}^k, \text{id})$ is a smooth mfld.
More generally, any open subset of \mathbb{R}^k is a smooth mfld.
- Any surface is a smooth mfld of dim. = 2.
- S^k is a smooth mfld of dim. k .
- $\mathbb{R}\mathbb{P}^k =$ the set of all lines in \mathbb{R}^{k+1} through the origin
 $= \mathbb{R}^{k+1} \setminus \{0\} / \sim$ $(x_0, \dots, x_k) \sim (\lambda x_0, \dots, \lambda x_k)$
 $\lambda \in \mathbb{R} \setminus \{0\}$.
- $= S^k / \sim$ $x \sim -x$

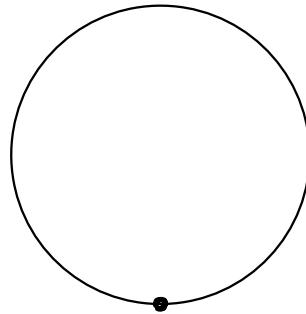
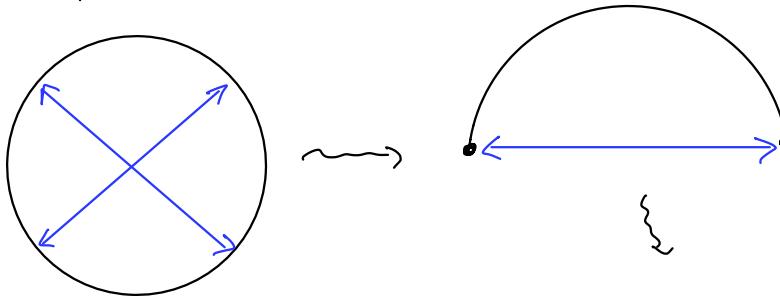
Define the topology on $\mathbb{R}\mathbb{P}^k$ as the quotient

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topology of $\mathbb{R}^{k+1} \setminus \{0\}$, that is

$U \subset \mathbb{RP}^k$ is open $\iff \pi^{-1}(U) \subset \mathbb{R}^{k+1} \setminus \{0\}$ is open
 where $\pi: \mathbb{R}^{k+1} \setminus \{0\} \rightarrow \mathbb{R}^{k+1} \setminus \{0\}/\sim = \mathbb{RP}^k$ is the quotient map.

For example, if $k=1$ \mathbb{RP}^1 is homeomorphic to S^1 :



Define

$$U_j := \{ [x_0 : \dots : x_n] \in \mathbb{RP}^k \mid x_j \neq 0 \} \quad j=0, \dots, k$$

U_j is open, since

$$\pi^{-1}(U_j) = \{ (x_0, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\} \mid x_j \neq 0 \} \text{ is open.}$$

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Consider the map $\varphi_j : U_j \rightarrow \mathbb{R}^k$

$$\varphi_j([x]) = \left(\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_k}{x_j} \right)$$

Then the map $\psi_j : \mathbb{R}^k \rightarrow U_j$

$$\psi_j(y_0, \dots, y_{k-1}) = [y_0 : \dots : y_{j-1} : 1 : y_j : \dots : y_{k-1}]$$

is the inverse of φ_j . In particular, ψ_j is a homeomorphism. Thus,

$$\mathcal{U} = \{ (U_j, \psi_j) \mid j=0, \dots, k \}$$

is a C^∞ -atlas on \mathbb{RP}^k .

Consider the coordinate transformation

$$\theta_{01} = \varphi_0 \circ \varphi_1^{-1} = \varphi_0 \circ \psi_1$$

$$\theta_{01}(y_0, \dots, y_{k-1}) = \varphi_0([y_0 : 1 : y_1 : \dots : y_{k-1}])$$

$$= \left(\frac{1}{y_0}, \frac{y_1}{y_0}, \frac{y_2}{y_0}, \dots, \frac{y_{k-1}}{y_0} \right)$$

which is smooth on

$$\{ y \in \mathbb{R}^k \mid y_0 \neq 0 \} = \varphi_0(U_0 \cap U_1)$$

The reader should check that in fact each $\theta_{ij} = \varphi_i \circ \varphi_j^{-1}$ is smooth. Thus, \mathcal{U} is in fact a smooth atlas.

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Rem For $k = 2$ we obtain a smooth mfld of dimension 2, however it turns out that \mathbb{RP}^2 cannot be represented as a surface in \mathbb{R}^3 . We would have discovered this mfld if we would consider non-orientable surfaces more carefully. Indeed, the Gauss map of a non-orientable surface $S \subset \mathbb{R}^3$ is naturally defined as a map

$$S \ni p \mapsto (T_p S)^\perp \in \mathbb{RP}^2.$$

4) Products If M and N are smooth manifolds of dimensions k and l respectively, then $M \times N$ is a smooth mfld of dimension $k+l$. Indeed, if $\mathcal{U} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ is a smooth atlas on M and $\mathcal{V} = \{(V_\beta, \xi_\beta) \mid \beta \in B\}$ is a smooth atlas on N , then

$$\mathcal{W} := \{(U_\alpha \times V_\beta, \varphi_\alpha \times \xi_\beta) \mid \alpha \in A, \beta \in B\}$$

is a smooth atlas on $M \times N$.

Exercise Find the coordinate transformations of the atlas \mathcal{W} and show that these are smooth indeed.

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In particular,

(i) $T^k = S^1 \times \dots \times S^1$ is a smooth manifold of dimension k

(ii) the cylinder $\mathbb{R} \times S^1$ is a smooth manifold of dimension 2.

A smooth atlas does not need to be unique. For example, on S^2 we can choose

$$\mathcal{U} = \{(S^2 \setminus \{N\}, \varphi_N), (S^2 \setminus \{S\}, \varphi_S)\}$$

or the atlas consisting of 6 hemispheres. This leads to the following definition.

Def Two atlases $\mathcal{U} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ and $\mathcal{V} = \{(V_\beta, \xi_\beta) \mid \beta \in B\}$ on the same top. space M are said to be equivalent if $\mathcal{U} \cup \mathcal{V}$ is also a smooth atlas, that is if

$$\xi_\beta \circ \varphi_\alpha^{-1} \quad \text{and} \quad \varphi_\alpha \circ \xi_\beta^{-1}$$

are smooth $\forall \alpha \in A$ and $\forall \beta \in B$.

Def A smooth structure is an equivalence class of atlases.

Rem In what follows we shall feel free to replace an atlas by an equivalent one.

Smooth maps

Let (M, u) be a smooth manifold.

Def A function $f: M \rightarrow \mathbb{R}$ is called smooth, if $\forall \alpha \in A$

$$F_\alpha := f \circ \varphi_\alpha^{-1}: \mathbb{R}^k \longrightarrow \mathbb{R}$$

is smooth.

Just as in the case of surfaces, F_α is called the coordinate representation of f with respect to the chart $(U_\alpha, \varphi_\alpha)$.

Also, just as in the case of surfaces each F_α is defined on an open subset of \mathbb{R}^k , namely $\varphi_\alpha(U_\alpha)$. This should be clear by now and will not be mentioned explicitly below unless really necessary.

Exercise Let $f: M \rightarrow \mathbb{R}$ be a function.

If $U \sim V$, show that

f is smooth wrt $U \iff f$ is smooth wrt V .

Prop The set $C^\infty(M)$ of all smooth functions on a smooth manifold is an algebra, that is

- $f, g \in C^\infty(M) \quad \left\{ \begin{array}{l} \\ \lambda, \mu \in \mathbb{R} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda f + \mu g \in C^\infty(M) \\ f \cdot g \in C^\infty(M) \end{array} \right.$ □

The proof of this proposition is similar to the proof of the analogous proposition for the surfaces.

More generally, let (M, u) and (N, v) be two smooth manifolds of dimensions k and l respectively.

Def A map $f: M \rightarrow N$ is said to be smooth, if each coordinate representation

$$\tilde{\varphi}_p \circ f \circ \varphi_\alpha^{-1}: \mathbb{R}^k \longrightarrow \mathbb{R}^l$$

is smooth.

Prop

$$\left. \begin{array}{l} f \in C^\infty(M; N) \\ g \in C^\infty(N; L) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} g \circ f \in C^\infty(M; L) \end{array} \right.$$
 □

Again, the proof of this proposition (10) is a verbatim repetition of the corresp. proof for the surfaces.

Also, just in the case of surfaces, we have the notions of a diffeomorphism and a local diffeomorphism.

The tangent space

If M is an abstract manifold and $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ is a smooth curve, then $\dot{\gamma}(0)$ does not make sense in any obvious way. Hence, our definition of the tangent space does not immediately generalize to the present setting.

To come up with a suitable generalization, observe the following: $v \in \mathbb{R}^k$ is the tangent vector of a curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^k$, $\gamma(0)=p$, if and only if

$$\gamma(t) = p + v \cdot t + o(t) \quad \text{as } t \rightarrow 0.$$

Hence we may consider the following equivalence relation: two smooth curves $\gamma_1, \gamma_2: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^k$, $\gamma_1(0)=p=\gamma_2(0)$,

are said to be equivalent if

$$\gamma_1(t) - \gamma_2(t) = o(t).$$

Our observation above yields immediately.

Prop

$$\gamma_1 \sim \gamma_2 \iff \dot{\gamma}_1(0) = \dot{\gamma}_2(0).$$
\(\blacksquare\)

This equivalence relation makes sense on manifolds as follows.

Def Let M be a smooth manifold of dimension k . Pick a pt $m \in M$. Two smooth curves $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \rightarrow M$ s.t. $\gamma_1(0) = m = \gamma_2(0)$ are said to be equivalent, if for any chart (U, φ) s.t. $m \in U$ we have

$$\varphi \circ \gamma_1 \sim_{\mathbb{R}^k} \varphi \circ \gamma_2 \iff \varphi \circ \gamma_1(t) - \varphi \circ \gamma_2(t) = o(t) \quad (*)$$

An equivalence class of curves is called a tangent vector at the pt m .

Lemma If $(*)$ holds for some chart (U, φ) s.t. $m \in U$, then $(*)$ holds for any chart containing m .

Proof Let $(\hat{U}, \hat{\varphi})$ be any other chart s.t. $m \in \hat{U}$. Denote $p := \varphi(m)$. Then

$$\begin{aligned}\hat{\varphi} \circ \gamma_1(t) - \hat{\varphi} \circ \gamma_2(t) &= \underbrace{\hat{\varphi} \circ \varphi^{-1}}_{\Theta} \circ \underbrace{\varphi \circ \gamma_1(t)}_{\beta_1} - \underbrace{\hat{\varphi} \circ \varphi^{-1}}_{\Theta} \circ \underbrace{\varphi \circ \gamma_2(t)}_{\beta_2} \\ &= \Theta \circ \beta_1(t) - \Theta \circ \beta_2(t)\end{aligned}$$

Since Θ is smooth, Θ is Lipschitz, that is $\exists L > 0$ s.t.

$$|\Theta(x) - \Theta(y)| \leq L|x-y| \quad \forall x, y \in B_\delta(p)$$

where $\delta > 0$.

Hence,

$$|\Theta \circ \beta_1(t) - \Theta \circ \beta_2(t)| \leq L |\beta_1(t) - \beta_2(t)| = o(t).$$

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Def $T_m M := \{[\gamma] \mid \gamma \text{ is a smooth curve through } m\}$

is called the tangent space of M at m .

Prop / Defn $T_m M$ is a vector space of dim k .

Proof Pick a chart (U, φ) containing m and denote $\varphi(m) = p \in \mathbb{R}^k$. Consider the map

$$\{ \gamma \mid \gamma(0) = m \} \longrightarrow \mathbb{R}^k$$

(*)

smooth
 curve
 in M . $\gamma \longmapsto \left. \frac{d}{dt} \right|_{t=0} \underbrace{\varphi \circ \gamma(t)}_{\text{curve in } \mathbb{R}^k}.$

Exercise Show that this map is surjective.

If $\gamma_1 \sim \gamma_2$, then $\beta_1 := \varphi \circ \gamma_1 \sim \beta_2 := \varphi \circ \gamma_2$

by the proof of the lemma on P. 11. Hence,

$\dot{\beta}_1(0) = \dot{\beta}_2(0)$ so that we have a
well-defined surjective map

$$\varphi_*: T_m M \longrightarrow \mathbb{R}^k.$$

In fact, φ_* is bijective:

$$\varphi_*[\gamma_1] = \varphi_*[\gamma_2] \iff \beta_1 \sim \beta_2 \iff \dot{\beta}_1(0) = \dot{\beta}_2(0)$$

Thus, we define the structure of a vector space on $T_m M$ so that φ_* is a linear isomorphism.

Exercise Show that the following holds:

(i) If $\gamma \in T_m M$ and $\lambda \in \mathbb{R}$, then

$$\lambda[\gamma] = [\gamma(\lambda \cdot)]$$

(ii) For two curves γ_1, γ_2 through m define

$$\gamma(t) = \varphi^{-1}(\beta_1(t) + \beta_2(t)),$$

where $\beta_j := \varphi \circ \gamma_j$. Show that γ is a smooth curve through m and

$$[\gamma_1] + [\gamma_2] = [\gamma].$$

We still need to show that the structure of the vector space on $T_m M$ does not depend on the choice of the chart (U, φ) . To this end, let $(\hat{U}, \hat{\varphi})$ be another chart s.t. $m \in \hat{U}$. Let

$$\hat{\varphi}_*: T_m M \rightarrow \mathbb{R}^k, [\gamma] \mapsto \left. \frac{d}{dt} \right|_{t=0} \hat{\varphi} \circ \gamma(t)$$

be the corresponding map. Denoting temporarily by $+_\varphi$ the addition obtained via φ_* , we obtain

$$[\gamma] = [\gamma_1] +_\varphi [\gamma_2] \Leftrightarrow \dot{\beta}(0) = \dot{\beta}_1(0) + \dot{\beta}_2(0)$$

where $\beta(t) = \varphi \circ \gamma(t)$ and $\beta_j(t) = \varphi \circ \gamma_j(t)$.

Denote $\hat{\beta}(t) = \hat{\varphi} \circ \gamma(t)$ and $\hat{\beta}_j(t) = \hat{\varphi} \circ \gamma_j(t)$.

Then

$$\hat{\beta} = \hat{\varphi} \cdot \gamma = \hat{\varphi} \circ \varphi^{-1} \circ \varphi \circ \gamma = \Theta \circ \beta \Rightarrow$$

$$\dot{\hat{\beta}}(0) = D_p \Theta(\dot{\beta}(0)).$$

Similarly, we have $\dot{\hat{\beta}}_j(0) = D_p \Theta(\dot{\beta}_j(0))$.

Since $D_p \Theta$ is a linear map, we have

$$\begin{aligned}\dot{\hat{\beta}}(0) &= D_p \Theta(\dot{\beta}_1(0) + \dot{\beta}_2(0)) \\ &= D_p \Theta(\dot{\beta}_1(0)) + D_p \Theta(\dot{\beta}_2(0)) \\ &= \dot{\hat{\beta}}_1(0) + \dot{\hat{\beta}}_2(0)\end{aligned}$$

$$\Leftrightarrow [\gamma] = [\gamma_1] +_{\hat{\varphi}} [\gamma_2]$$

$$[\gamma_1] +_{\varphi} [\gamma_2].$$

The fact that the multiplication with scalars is independent of the choice of a chart follows immediately from Part (i) of the exercise on P. 13.

Prop If $S \subset \mathbb{R}^3$ is a smooth surface, then $T_p S$ in the sense of the definition on p. 12 is naturally isomorphic to the tangent plane of S .

Proof Denote temporarily the tangent plane of S at p by E_p . Consider the map

$$\begin{aligned} T_p S &\longrightarrow E_p \\ [\gamma] &\longmapsto \dot{\gamma}(0) \end{aligned} \tag{*}$$

Exercise Check that this map is well-defined, that is independent of the choice of the representative.

This map is linear. Indeed, if ψ is a parametrization at p such that $\psi(0)=p$ and

$$\begin{aligned} \gamma_1 &:= \psi \circ \beta_1, \quad \gamma_2 := \psi \circ \beta_2 \\ \text{then } [\gamma_1] + [\gamma_2] &\text{ is represented by} \\ \text{the curve } t &\mapsto \psi(\beta_1(t) + \beta_2(t)) = \gamma(t) \end{aligned}$$

Hence

$$\begin{aligned} \dot{\gamma}(0) &= D_0 \psi(\dot{\beta}_1(t) + \dot{\beta}_2(t)) \\ &= D_0 \psi(\dot{\beta}_1(t)) + D_0 \psi(\dot{\beta}_2(t)) \\ &= \dot{\gamma}_1(0) + \dot{\gamma}_2(0). \end{aligned}$$

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That is $[\gamma_1] + [\gamma_2] \in T_p S$ is mapped onto $\dot{\gamma}_1(0) + \dot{\gamma}_2(0)$.

Since $\lambda[\gamma] = [\gamma(\lambda \cdot)] \longleftrightarrow \frac{d}{dt} \Big|_{t=0} \gamma(\lambda t) = \lambda \dot{\gamma}(0)$, we see that (14.*) is linear.

Since this map is clearly surjective and $T_p S$ and F_p have equal dimension, (14.*) is an isomorphism. \square

Rem The proof of the prop/defn on P. 12 implies the following. Let (U, φ) be a chart on M and $m \in U$. Denote $\varphi(m) = p \in \mathbb{R}^k$ and define $\gamma_j : (-\varepsilon, \varepsilon) \rightarrow U$, $\gamma_j(0) = m$ by

$$\varphi \circ \gamma_j(t) = p + (0, \dots, 0, t, 0, \dots, 0).$$

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Then the classes $[\gamma_1], \dots, [\gamma_k]$ constitute a basis of $T_m M$.

The differential of a smooth map.

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