Quadratic forms on surfaces

Let S be a surface.

Def A Riemannian metric on S is a family of scalar products $\langle \cdot, \cdot \rangle_p$ on each tangent space T_pS, $p \in S$, such that $\langle \cdot, \cdot \rangle_p$ depends smoothly on p.

To explain, let $\varphi : V \to U$ be a parametrization. If $q \in V$ and $p = \varphi(q)$, then T_pS has a basis $(\partial_u \varphi, \partial_v \varphi)$. Hence, the scalar product $\langle \cdot, \cdot \rangle_p$ is represented by its Gram matrix

$$M = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \begin{aligned} E &= \langle \partial_u \varphi, \partial_u \varphi \rangle_p \\ F &= \langle \partial_u \varphi, \partial_v \varphi \rangle_p \\ G &= \langle \partial_v \varphi, \partial_v \varphi \rangle_p \end{aligned}$$

We say, that $\langle \cdot, \cdot \rangle_p$ depends smoothly on p, if all 3 functions E, F, G are smooth (on V, where they are defined).

Ex For any $p \in S$ we have $T_pS \subset \mathbb{R}^3$. Since $\mathbb{R}^3$ is equipped with the standard scalar product

$$\langle x, y \rangle_{st} := x_1 y_1 + x_2 y_2 + x_3 y_3$$

we can restrict $\langle\cdot,\cdot\rangle_{st}$ to T_pS to obtain a scalar product on T_pS. This is a Riemannian metric on S, since

$$E(u,v) = \langle \partial_u \psi, \partial_u \psi \rangle_S = \langle \partial_u \psi, \partial_u \psi \rangle_{st}$$

is a smooth function of (u,v) (and similarly for F and G).

This particular Riemannian metric on S is called the first fundamental form of S in the classical theory of surfaces.

Exercise Let $\langle\cdot,\cdot\rangle$ be the first fundamental form of S and $f : S \to S$ be a diffeomorphism. For $v, w \in T_pS$ define a new scalar product

$$\langle v, w \rangle_f := \langle \underset{\substack{\cap \\ T_{f(p)}S}}{d_pf(v)}, \underset{\substack{\cap \\ T_{f(p)}S}}{d_pf(w)} \rangle_{f(p)}$$

Show that $\langle\cdot,\cdot\rangle_f$ is a Riemannian metric on S.

For the sake of simplicity of exposition, assume S is oriented and let n be the unit normal field. We can view n as a smooth map

$$n : S \longrightarrow S^2,$$

which is called the Gauss map. Then $\forall p \in S$ we have

$$d_p n : T_p S \longrightarrow T_{n(p)} S^2 = n(p)^{\perp} = T_p S.$$

This is called the shape operator.

As a linear map in a 2-dimensional vector space, the shape operator has two invariants:

$$\boxed{K(p) := \det(d_p n)} \quad \text{and} \quad H(p) := -\frac{1}{2} \operatorname{tr}(d_p n)$$

Def $K(p)$ is called the Gauss curvature and $H(p)$ is called the mean curvature of S at p.

K, H are smooth functions on S.

Ex 1 $S = \mathbb{R}^2 \equiv \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Gauss map $n(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ constant

Shape operator $d_p n \equiv 0$

$\Rightarrow \quad K \equiv 0.$

Ex 2 $\quad S^2_r := \{ x \in \mathbb{R}^3 \mid |x|^2 = r^2 \}$

Gauss map $\quad n(p) = \frac{1}{r} p$

The shape operator: $\quad d_p n(v) = \frac{1}{r} v \Rightarrow d_p n = \frac{1}{r}\, id$

$\Rightarrow \quad K(p) = \frac{1}{r^2}$ is constant on S^2

If $r \to \infty$, $K(p) \to 0$ and the sphere looks more and more flat in a nbhd of each point (that is why our Earth is "flat").

Thus, we can view the Gauss curvature as a measure of flatness of S.

Lemma The shape operator is symmetric, that is

$$\langle d_p n(v), w \rangle = \langle v, d_p n(w) \rangle$$

$\forall p \in S$ and $\forall v, w \in T_p S$.

Proof Let $\varphi : V \to S$ be a parametrization s.t. $\varphi(0) = p$. Then $(\partial_u \varphi, \partial_v \varphi)|_{(u,v)=0}$ is a basis of $T_p S$. Hence, it suffices to show the equaility

$$\langle d_p n(\partial_u \varphi), \partial_v \varphi \rangle = \langle \partial_u \varphi, d_p n(\partial_v \varphi) \rangle, \quad (*)$$

where the derivatives are evaluated at the origin.

To this end, notice that by the definition of n we have

$$\langle n(\varphi(u,v)), \partial_u \varphi(u,v) \rangle = 0 \qquad \forall (u,v) \in V$$

Differentiating this equality with respect to v and setting $(u,v)=0$, we obtain

$$\langle d_p n(\partial_u \varphi), \partial_v \varphi \rangle + \langle n(p), \partial_{uv} \varphi \rangle = 0$$

Similarly, we obtain

$$\langle \partial_u \varphi, d_p n(\partial_v \varphi) \rangle + \langle \partial_{uv} \varphi, n(p) \rangle = 0.$$

Subtracting these two equalities, we arrive at (4.*). ▨

<u>Def</u> The bilinear symmetric map

$$\begin{aligned} \mathrm{II} : T_pS \times T_pS &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto \langle v, d_p n(w) \rangle_p \end{aligned}$$

is called <u>the second fundamental form</u> of S (at the point p).

Notice that II is smooth, that is for any parametrization φ

$\mathrm{II}(\partial_u \varphi(u,v), \partial_u \varphi(u,v))$, $\mathrm{II}(\partial_u \varphi, \partial_v \varphi)$,

$\mathrm{II}(\partial_v \varphi, \partial_v \varphi)$

are smooth functions of (u,v).

<u>Rem</u> One can recover the shape operator from the second fundamental form, that is these two objects contain the same ammount of information.

The geometric meaning of the Gauss curvature.

Let $p \in S$ be a critical pt of $f \in C^\infty(S)$. Given $v \in T_pS$, pick $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t. $\gamma(0) = p$ and $\dot\gamma(p) = v$.

<u>Def</u> The map

$$\mathrm{Hess}_p f : T_pS \to \mathbb{R}, \quad \mathrm{Hess}_p f\,(v) = \frac{d}{dt}\Big|_{t=0} \big(f \circ \gamma(t)\big)$$

is called the Hessian of f at p.

<u>Prop</u>

(i) $\mathrm{Hess}_p f$ is a well-defined quadratic map;

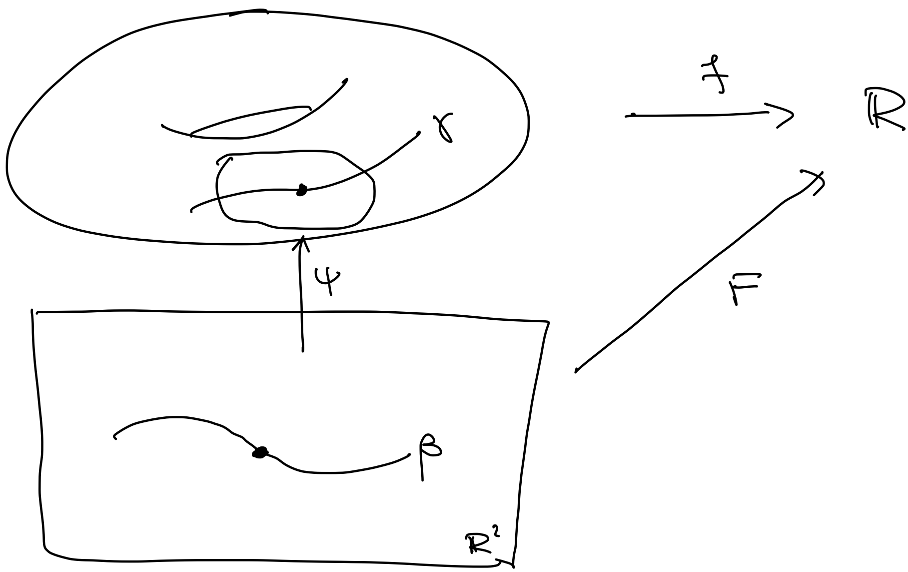
(ii) If p is a pt of loc. minimum, then $\mathrm{Hess}_p(f)\,(v) \geq 0 \quad \forall v \in T_pS$. If p is a pt of loc. maximum, then $\mathrm{Hess}_p f\,(v) \leq 0$.

(iii) If $\mathrm{Hess}_p f\,(v) > 0 \quad \forall v \neq 0$, then p is a pt of loc minimum. If $\mathrm{Hess}_p f\,(v) < 0 \quad \forall v \neq 0$, then p is a pt of loc. maximum.

Proof

Choose a parametrization ψ s.t. $\psi(0)=p$ and denote

$$F := f \circ \psi \qquad \beta := \varphi \circ \gamma = \psi^{-1} \circ \gamma.$$



Then if $\beta(t) = (\beta_1(t), \beta_2(t))$, we have

$$f \circ \gamma(t) = F \circ \beta(t) = F(\beta_1(t), \beta_2(t))$$

$$\Rightarrow \frac{d}{dt} f \circ \gamma(t) = \partial_u F(\beta(t))\, \beta_1'(t) + \partial_v F(\beta(t))\, \beta_2'(t)$$

Notice that $\beta(0) = 0$ and $\partial_u F(0) = 0 = \partial_v F(0)$.
Furthermore we have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} f \circ \gamma(t) = \partial^2_{uu} F(0)\, \beta_1'(0)^2 + 2\partial^2_{uv} F(0)\, \beta_1'(0)\beta_2'(0) + \partial^2_{vv} F(0)\, \beta_2'(0)^2. \qquad (*)$$

Recalling that $\beta'(0) = d_p\varphi(v)$, we see that the right-hand-side of (7.*) depends only on $\beta'(0)$ and not on the choice of γ.

Moreover, (7.*) also shows that $\mathrm{Hess}_p f\,(v)$ is a quadratic form of v.

In fact we have shown that $\mathrm{Hess}_p f$ corresponds to the Hessian of the loc. representation F of f in the following sense: The diagram

$$\begin{tikzcd}
T_pS \arrow[rr, "\mathrm{Hess}_p f"] \arrow[d, "d_p\varphi"'] & & \mathbb{R} \\
\mathbb{R}^2 \arrow[urr, "\mathrm{Hess}_{\varphi(p)} F"'] & &
\end{tikzcd}$$

commutes. That is we can identify $\mathrm{Hess}_p f$ with $\mathrm{Hess}_{\varphi(p)} F$ by means of the isomorphism $d_p\varphi : T_pS \longrightarrow \mathbb{R}^2$. This immediately implies (ii) and (iii). $\square$

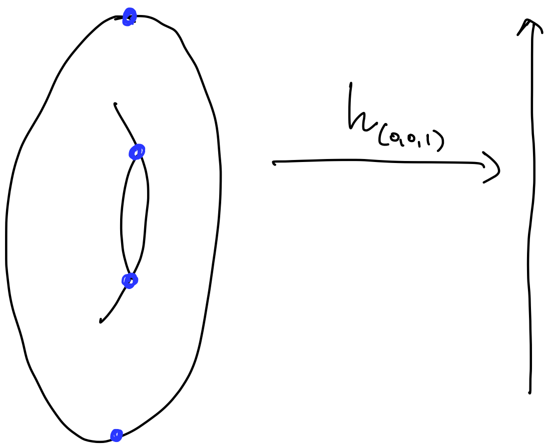
Let $a \in \mathbb{R}^3$ be any fixed vector, $a \neq 0$.
Let $h_a : S \to \mathbb{R}$ be the restriction of

$$\mathbb{R}^3 \to \mathbb{R} \quad , \quad x \longmapsto \langle x, a \rangle .$$

Then h_a is called the height function on S in the direction of a.

Notice that p is a critical pt of h_a if and only if $T_pS \perp a$.

<u>Ex</u> For $a = (0,0,1)$ we have the standard height function



<u>Prop</u> Let n be an orientation of S. Then for any $p \in S$ we have

$$\mathrm{II}_p = - \mathrm{Hess}_p (h_{n(p)})$$

<u>Proof</u> Observe first that

$T_pS \perp n(p)$ that is p is a critical pt of $h_{n(p)}$.

Given $v \in T_pS$ choose a curve $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t. $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then

$$\mathrm{Hess}_p(h_{n(p)}) = \frac{d^2}{dt^2}\Big|_{t=0} \langle \gamma(t), n(p) \rangle$$

$$= \langle \ddot\gamma(0), n(p) \rangle$$

However, $\gamma(t) \in S \Rightarrow \dot\gamma(t) \in T_{\gamma(t)}S \quad \forall t$

$$\Rightarrow \langle \dot\gamma(t), n(\gamma(t)) \rangle = 0 \quad \forall t$$

$$\frac{d}{dt}\Big|_{t=0} \qquad \langle \ddot\gamma(0), n(p) \rangle + \underbrace{\langle \dot\gamma(0), d_p n(\dot\gamma(0)) \rangle}_{\mathrm{II}_p(v)} = 0$$

This yields $\mathrm{II}_p(v) = -\langle \ddot\gamma(0), n(p) \rangle$

$$= \mathrm{Hess}_p(h_{n(p)}) \qquad \boxtimes$$

Fix $p \in S$. Without loss of generality assume that

$$p = 0 \in \mathbb{R}^3 \quad \text{and} \quad n(0) = (0,0,1).$$

This can be always achieved by applying

a translation and a rotation in $\mathbb{R}^3$.

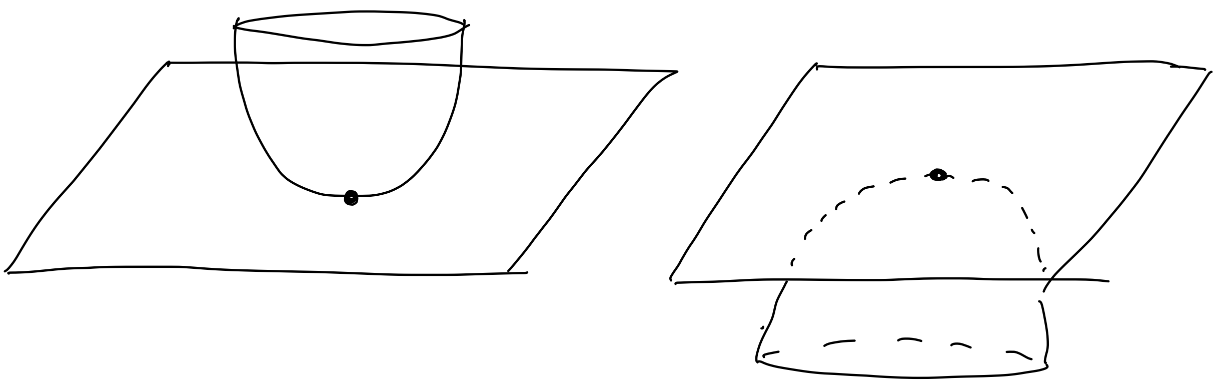
Since the shape operator $d_o n : \underset{\mathbb{R}^2}{\underset{\shortparallel}{T_o S}} \to \underset{\mathbb{R}^2}{\underset{\shortparallel}{T_o S}}$

is symmetric, $d_o n$ has two real eigenvalues, say k_1 and k_2.

Consider the following cases:

A) $K(p) > 0 \implies k_1 \cdot k_2 > 0 \implies$

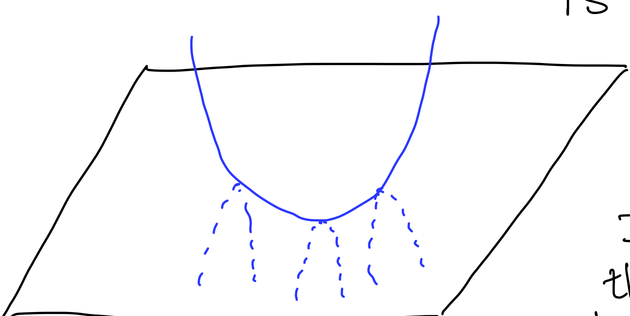
$\mathrm{Hess}_o(h_{n(o)})$ is either positive-definite or negative definite



B) $K(p) < 0 \implies z\big|_S$ attains both positive and negative values

$\Downarrow$

In any nbhd of p there are pts in S above and below $T_p S$.



<u>Rem</u> If $K(p) = 0$, in general one cannot say anything about the position of S relative to T_pS.