Differential Geometry I

Lecture notes

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This is a draft. If you spot a mistake, please let me know.

Contents

Chapter 1

Smooth surfaces

1.1 The notion of a smooth surface

Let $\mathrm{U} \subset \mathbb{R}^n$ be an open subset and $f \in C^1(\mathrm{U})$. It is known from analysis that $x_0 \in \mathrm{U}$ is a point of extremum for f if

$$\frac{\partial f}{\partial x_i}(x_0) = 0$$

holds for all $i = 1, \ldots, n$. Notice that this is a necessary condition, which is not sufficient in general.

A more general type of problems does not fit into this scheme. For example, consider the following.

Problem. Among all rectangular parallelepipeds, whose diagonal has a fixed length, say 1, find the one with maximal volume.

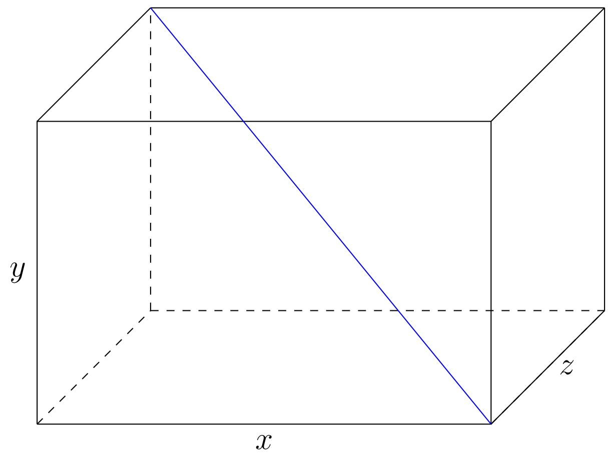

Figure 1.1: A parallelepiped

Thus, we want to find a point of maximum of the function $f(x, y, z) = xyz$ on the set

$$\mathrm{V} = \{(x, y, z) \in \mathbb{R}^3 \,|\, x > 0, y > 0, z > 0 \quad \text{and} \quad x^2 + y^2 + z^2 = 1\} \subset S^2. \qquad (1.1)$$

However, V is *not* an open subset of $\mathbb{R}^3$ so that the receipy known from the analysis course is not readily applicable.

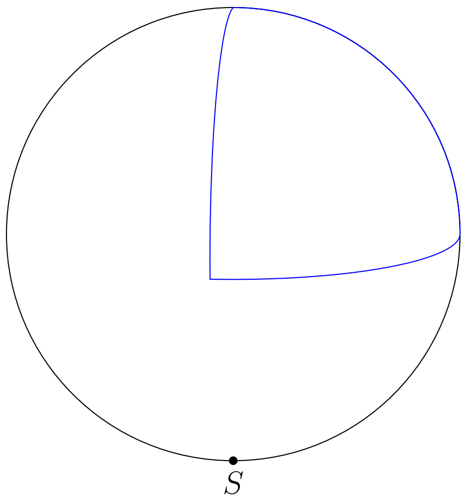

Figure 1.2: The spherical triangle $x, y, z > 0$

This problem is relatively easy to solve, however. Indeed, since $z > 0$, we obtain $z = \sqrt{1 - x^2 - y^2}$ so that we are essentially interested in the function

$$F(x, y) := f\left(x, y, \sqrt{1 - x^2 - y^2}\right) = xy\sqrt{1 - x^2 - y^2}.$$

More precisely, we want to find points of maximum of F on the set $\{(x, y) \mid x^2 + y^2 < 1, x > 0, y > 0\}$, which is an open subset of $\mathbb{R}^2$.

We compute

$$\begin{aligned} \frac{\partial F}{\partial x} &= y\sqrt{1 - x^2 - y^2} - xy\frac{x}{\sqrt{1 - x^2 - y^2}} = 0, \\ \frac{\partial F}{\partial y} &= x\sqrt{1 - x^2 - y^2} - xy\frac{y}{\sqrt{1 - x^2 - y^2}} = 0. \end{aligned} \tag{1.2}$$

Since $x \neq 0$ and $y \neq 0$, we have

$$\begin{aligned} (1.2) \quad &\Longleftrightarrow \quad \begin{array}{l} 1 - x^2 - y^2 = x^2 \\ 1 - x^2 - y^2 = y^2 \end{array} \quad \Longrightarrow \quad x^2 = y^2 \quad \Longrightarrow \quad x = y \\ &\Longrightarrow \quad 3x^2 = 1 \quad \Longrightarrow \quad x = y = \frac{1}{\sqrt{3}} \\ &\Longrightarrow \quad z = \frac{1}{\sqrt{3}}. \end{aligned}$$

Hence, if there is a parallelepiped maximizing the volume among all rectangular parallelepipeds with the given length of the diagonal, this must be the cube.

Exercise 1.3. Show that $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a point of maximum indeed.

Consider a more general problem of constrained maximum/minimum. Given $f, \varphi \in C^\infty(\mathbb{R}^n)$ find a point of maximum/minimum of f on the set

$$S := \{x \in \mathbb{R}^n \quad | \quad \varphi(x) = 0\}.$$

Proposition 1.4. *Assume that for $p \in S$ we have*

$$\frac{\partial \varphi}{\partial x_n}(p) \neq 0. \tag{1.5}$$

Then there is a neighbourhood W of p in $\mathbb{R}^3$, an open subset $\mathrm{V} \subset \mathbb{R}^{n-1}$, and a smooth function $\psi\colon \mathrm{V} \to \mathbb{R}$ such that for $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we have

$$x \in S \cap \mathrm{W} \qquad \Longleftrightarrow \qquad y \in \mathrm{V} \quad \textit{and} \quad z = \psi(y).$$

This is a celebrated implicit function theorem, whose proof was given in the analysis course.

Theorem 1.6. *Let $p \in S$ be a point of (local) maximum of f on S. If (1.5) holds, then there exists some $\lambda \in \mathbb{R}$ such that*

$$\frac{\partial f}{\partial x_j}(p) = \lambda \frac{\partial \varphi}{\partial x_j}(p) \qquad \Longleftrightarrow \qquad \nabla f(p) = \lambda \nabla \varphi(p) \tag{1.7}$$

holds for each $j = 1, \dots, n$.

Proof. Let $p = (y_0, z_0)$ be a local maximum for f on S. Hence, y_0 is a local maximum for the function

$$F\colon \mathrm{V} \to \mathbb{R}, \qquad F(y) := f(y, \psi(y))$$

This yields

$$\frac{\partial F}{\partial y_j}(y_0) = \frac{\partial f}{\partial y_j}(p) + \frac{\partial f}{\partial x_n}(p)\frac{\partial \psi}{\partial y_j}(y_0) = 0$$

for all $j \leq n-1$.

Furthermore, since $\varphi(y, \psi(y)) \equiv 0$, we have

$$\frac{\partial \varphi}{\partial y_j} + \frac{\partial \varphi}{\partial x_n}\frac{\partial \psi}{\partial y_j} \equiv 0.$$

This yields in turn

$$\frac{\partial \psi}{\partial y_j}(y_0) = -\frac{\partial \varphi}{\partial y_j}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p) \qquad \Longrightarrow \qquad \frac{\partial f}{\partial y_j}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)\right) \cdot \frac{\partial \varphi}{\partial y_j}(p).$$

Thus, (1.7) holds for all $j \leq n-1$ with $\lambda := \frac{\partial f}{\partial x_n}(p) / \frac{\partial \varphi}{\partial x_n}(p)$ independent of j.

For $j = n$ we have

$$\frac{\partial f}{\partial x_n}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)\right) \cdot \frac{\partial \varphi}{\partial x_n}(p) = \lambda \frac{\partial \varphi}{\partial x_n}(p).$$

Thus, (1.7) holds also for $j = n$ with the same λ. $\square$

Let us come back to the example about maximal value of parallelepipeds with a fixed length of the diagonal. Thus, if (x, y, z) is a point of maximum of f on (1.1), then there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z \end{aligned} \qquad \Longrightarrow \qquad (xyz)^2 = 8\lambda^3 xyz \qquad \Longrightarrow \qquad xyz = 8\lambda^3.$$

This yields in turn
$$8\lambda^3 = xyz = x(yz) = 2\lambda x^2.$$
Notice that $\lambda \neq 0$, since otherwise $x = 0$ or $y = 0$ or $z = 0$. Hence, we obtain $x = 2\lambda$.

A similar argument yields also $y = 2\lambda$ and $z = 2\lambda$. Therefore we obtain
$$4\lambda^2 + 4\lambda^2 + 4\lambda^2 = 1 \qquad \Longrightarrow \qquad \lambda = \frac{1}{2\sqrt{3}} \qquad \Longrightarrow \qquad x = y = z = \frac{1}{\sqrt{3}},$$
which is in agreement with our previous computation.

Coming back to Proposition 1.4, it is clear that it is only important that one of the partial derivatives of φ does not vanish. This leads to the following definition.

Definition 1.8 (Surface)**.** A non-empty set $S \subset \mathbb{R}^3$ is called a (smooth) *surface*, if for any $p \in S$ there exists an open set $\mathrm{V} \subset \mathbb{R}^2$ and a smooth map $\psi : \mathrm{V} \to S$ such that the following holds:

(i) $\psi(\mathrm{V}) =: \mathrm{U}$ is a neighbourhood of p in S.

(ii) $\psi\colon \mathrm{V} \to \mathrm{U}$ is a homeomorphism.

(iii) $D_q\psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective $\forall q \in \mathrm{V}$.

Example 1.9. Assume $\varphi \in C^\infty(\mathbb{R}^3)$ satisfies
$$\frac{\partial \varphi}{\partial z}(p) \neq 0 \qquad \text{for all} \quad p \in S := \varphi^{-1}(0).$$
Let ψ be as in Proposition 1.4. Define $\Psi(x,y) := (x, y, \psi(x,y))$. If U and V are also as in Proposition 1.4, then $\Psi\colon \mathrm{V} \to S \cap \mathrm{U}$ is a homeomorphism, since $\pi\colon S \cap \mathrm{U} \to \mathrm{V}, \pi(x,y,z) = (x,y)$ is a continuous inverse. Furthermore,
$$D\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_x \psi & \partial_y \psi \end{pmatrix}$$
is clearly injective at all points. Hence, S is a surface.

Again, the same conclusion holds if we assume only that $\nabla\varphi(p) \neq 0$ for all $p \in \varphi^{-1}(0)$. In particular,

- the sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$
- the cylinder $C = \{(x,y,z) \mid x^2 + y^2 = 1\}$
- the hyperboloid $H = \{x^2 + y^2 - z^2 = 1\}$

are surfaces

Example 1.10 (Torus)**.** Let C be the circle of radius r in the yz-plane centered at the point $(0, a, 0)$ as shown on Fig. 1.4, where $a > r$.

More formally,
$$T := \left\{ \left(\sqrt{x^2 + y^2} - a \right)^2 + z^2 = r^2 \right\}.$$
Exercise 1.11. Check that T is a surface indeed.

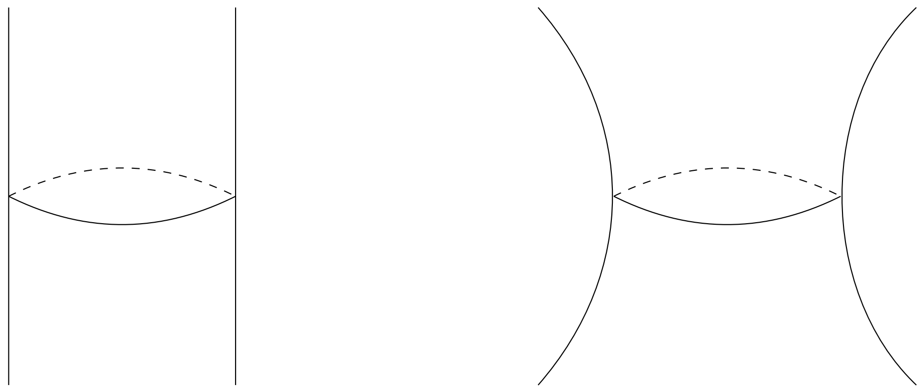

Figure 1.3: The cylinder and hyperboloid

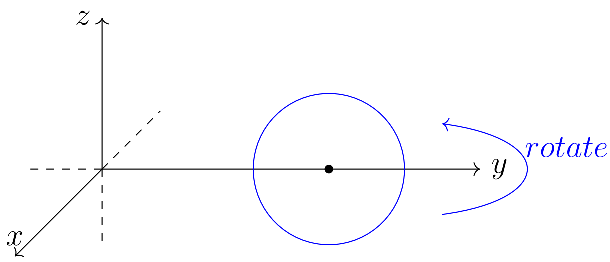

Figure 1.4: The torus as a circle rotated with respect to an axis

Example 1.12 (A non-example)**.** A double cone $C_0 := \{x^2 + y^2 - z^2 = 0\}$ is not a surface. Indeed, assume C_0 is a surface. Then the tip of the cone p must have a neighbourhood U homeomorphic to an open disc in $\mathbb{R}^2$.

Let $f\colon \mathrm{U} \to D$ be a homeomorphism. Then $f\colon \mathrm{U}\backslash\{p\} \to D\backslash\{f(p)\}$ is also a homeomorphism. However, this is impossible, since the punctured disc is connected but $\mathrm{U}\backslash\{p\}$ is disconnected. Hence, p does not have a neighbourhood homeomorphic to a disc (or any open subset of $\mathbb{R}^2$).

Exercise 1.13. Show that a straight line is not a surface.

Remark 1.14.

1) The map ψ in the definition of the surface is called a *parametrization*.
2) Condition (iii) is equivalent to the following:

$$\partial_u \psi \quad \text{and} \quad \partial_v \psi \quad \text{are linearly independent}$$

at each point $(u, v) \in \mathrm{V}$.

Proposition 1.15. *Let S be a surface. For any $p \in S$ there exists a neighbourhood $W \subset \mathbb{R}^3$ and $\varphi \in C^\infty(W)$ such that*

$$S \cap W = \{x \in W \mid \varphi(x) = 0\} \quad \textit{and} \quad \nabla\varphi(x) \neq 0$$

for any $x \in S \cap W$.

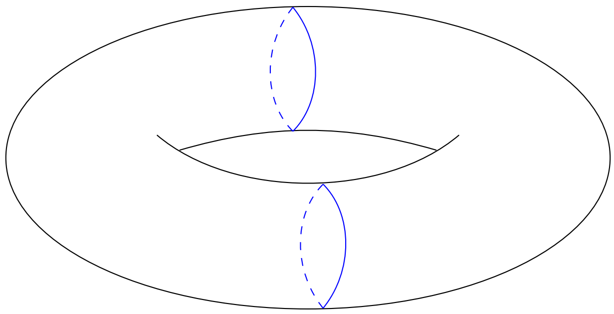

Figure 1.5: The torus

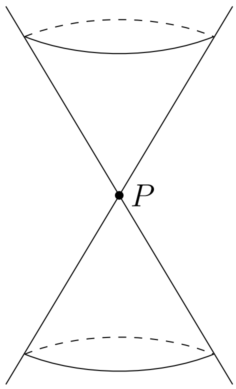

Figure 1.6: The double cone

Proof. Choose a parametrization $\psi\colon \mathrm{V} \to \mathrm{U} \subset S$. Let $(u_0, v_0) \in \mathrm{V}$ be a unique point such that $\psi(u_0, v_0) = p$. Choose a vector $n \in \mathbb{R}^3$ such that

$$\partial_u \psi(u_0, v_0), \quad \partial_v \psi(u_0, v_0), \quad n \tag{1.16}$$

are linearly independent. Consider the map

$$\Psi\colon \mathrm{V} \times \mathbb{R} \to \mathbb{R}^3, \qquad \Psi(u, v, w) = \psi(u, v) + w \cdot n$$

The linear independence of (1.16) yields $\det D\Psi(u_0, v_0, 0) \neq 0$. By the inverse map theorem, there exists an open neighbourhood $W \subset \mathbb{R}^3$ of p and a smooth map $\Phi\colon W \to V \times \mathbb{R} \subset \mathbb{R}^3$ such that

$$\Psi \circ \Phi(x) = x \qquad \forall x \in W.$$

If $\Phi = (\varphi_1, \varphi_2, \varphi_3)$, then

$$\Psi \circ \Phi(x) = \psi(\varphi_1(x), \varphi_2(x)) + \varphi_3(x) \cdot n = x.$$

Observe that

$$x \in S \cap W \qquad \Longleftrightarrow \qquad \exists (u, v) \in \mathrm{V} \textit{ such that } \psi(u, v) = x$$

and consequently

$$\Psi(u, v, 0) = \psi(u, v) = x = \Psi(\varphi_1(x), \varphi_2(x), \varphi_3(x)).$$

Since Ψ is injective (on an open neighbourhood of $(u_0, v_0, 0)$), we have

$$x \in S \cap W \qquad \Longleftrightarrow \qquad \varphi_3(x) = 0.$$

Furthermore, since $\det D\Phi(x) \neq 0$ for all $x \in W$, the vectors $\nabla\varphi_1(x), \nabla\varphi_2(x), \nabla\varphi_3(x)$ are linearly independent at each $x \in W$. In particular, $\nabla\varphi_3(x) \neq 0$ for all $x \in W$. □

The following corollary follows immediately from Proposition 1.15.

Corollary 1.17. *Any surface is locally the graph of a smooth function.* □

Example 1.18 (A non-example)**.** The union of two intersecting planes in $\mathbb{R}^3$ is *not* a surface. Indeed, assume that

$$S := \{z = 0\} \cup \{x = 0\}$$

is a surface. Then there exists a smooth function φ defined in a neighbourhood W of the origin such that φ vanishes on S and $\nabla\varphi(0) \neq 0$ by Proposition 1.15. Notice that φ vanishes identically along S, hence φ vanishes identically along all three coordinate axes (at least in a neighbourhood of the origin). This yields in turn $\nabla\varphi(0) = 0$, which is a contradiction.