

## 1. Gauss Integral

Def 1  $s \in C^1(\mathbb{R})$  s.t.  $|s(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$

$$Z(s) := \{s(x) = 0\}, \quad s'(x) \neq 0 \quad \forall x \in Z(s)$$

Gauss integral

$$Z = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2} s(x)^2}$$

Lemma  $Z = \sum_{z \in Z(s)} \operatorname{sign} s'(z)$

Proof

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} s'(x) e^{-\frac{1}{2} s(x)^2} \quad \underline{y = \sqrt{t} s(x)}$$

$$= \int \frac{dy}{\sqrt{2\pi t}} e^{-\frac{1}{2t} y^2}$$

$$\sqrt{t} s(+\infty)$$

$$= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\pi t}} s' e^{\frac{1}{4t} s(x)^2}$$

$$= \int_{-\infty}^{+\infty} s'(x) \underbrace{\frac{e^{\frac{1}{4t} s(x)^2}}{\sqrt{\pi t}} dx}_{\downarrow S}$$

$$= \sum_{z \in Z(s)} \frac{s'(z)}{|s(z)|}$$

□

Def 3 Eucl.  $n$ -N general. for

$s \in C^1(\mathbb{R}^n)$

$$Z = \int_{\mathbb{R}^n} \prod_{i=1}^n \frac{dx^i}{(2\pi)^{n/2}} \det \left( \frac{\partial s^j}{\partial x^k} \right) e^{-\frac{1}{2} s(x)^2}$$

$$= \sum_{z \in Z(s)} \text{sign} \det \left( \frac{\partial s^j}{\partial x^k} \right)$$

1.  $Z$  counts solution for  $s'(x) = 0$  with a pre-factor.
2.  $Z$  is an oriented intersection number

## 2. Superspace

- Superalgebra is a vect. space  $A = A^0 \oplus A^1$  with a supercomm. multiplication, i.e.

$$\begin{matrix} f \in A^a \\ g \in A^b \end{matrix} \Rightarrow f \cdot g \in A^{a+b}, \quad fg = (-1)^{a+b} gf$$

- Superspace  $\mathbb{R}^{n|k}$  is defined by S.A. of word functions

$(C^\infty(\Omega)[\psi^1, \dots, \psi^k], +, \cdot)$  for open  $\Omega \subset \mathbb{R}^n$

$$\psi = \sum_{k=0}^n \sum_{i_1 < \dots < i_k} \phi_{i_1} \dots \phi_{i_k} \psi^{i_1} \dots \psi^{i_k}$$

- For an  $n$ -dim manifold  $M$  the supervielbein  $\hat{M}^{(k)}$  is given by the sheaf of SA  $\mathcal{O}_{X^{(k)}}$  on  $M$ .

For open  $U \subset M$ ,  $\Omega \subset \mathbb{R}^n$  with chart  Neo smartpen  
 $U \xrightarrow{\cong} \Omega$  we have  $\mathcal{O}_{\hat{M}^{(k)}}(U) \simeq C^\infty(\Omega)[\psi^1, \dots, \psi^k]$

Construction 5 For a  $\leq k$  k-VB  $E \rightarrow M$  we have  
 in loc. trivialization  $E|_U \cong U \times V$

$$C^\infty(U) \otimes \Lambda^* V \cong \mathcal{O}_{\hat{M}^{(k)}}(U)$$

For  $E = TM$  ( $T^*M$ ) we have make an  
 identification  $\psi^i \leftrightarrow dx^i$  and

$$C^\infty(\hat{M}^{(k)}) \cong \Omega^*(M).$$

Def. 6 "Ghost number" of a superfield  $\phi_\omega \in \hat{M}^{(k)}$   
 $\deg \omega = gh \# \phi_\omega \in \mathbb{Z}$   $\Omega^*(M) \times (\Omega^*(M))^*$

BRS-charge (coboundary op  $Q$ )

$$Q \phi_\omega \leftrightarrow d\omega$$

in loc. coord.

$$Q x^i = \psi^i, Q \psi^i = 0.$$

Integration of superfields via  $\text{ber}(x|\psi) \in \text{Ber}(\Omega^*)$

$$\int_M \omega = \int_{\hat{M}} \text{ber}(x|\psi) \phi_\omega \text{ with } \cancel{\text{ber}(x|\psi)} =$$

$$\text{ber}(x|\psi) = dx^1 \dots dx^n [d\psi^1 \dots d\psi^n]$$

Statement 7 We have  $\int d\psi = 0$

and  $\int [d\psi' \dots d\psi^n] \psi' \dots \psi^n = 1$

### 3. SUSY Integral

Lemma 8 For  $M \in \text{Mat}(n, \mathbb{R})$   $\exists \alpha \xi \in \mathbb{H}^{\pm 1}$  s.t

$$\frac{\xi}{i^n} \int_{-1}^1 [d\psi^1 \dots d\psi^n] \left[ \begin{array}{c} \int_{-1}^1 \dots \int_{-1}^1 \\ \vdots \\ \int_1^{-1} \dots \int_1^{-1} \end{array} \right] e^{i \sum_{j,k} y_j M_{jk} \psi^k} = \det M$$

Proof

$$\frac{\xi}{i^n} \int [ ] \sum_{\ell=0}^{\infty} \left( \frac{i^\ell}{\ell!} \left( \sum_{j,k} y_j M_{jk} \psi^k \right)^\ell \right) =$$

$$= \frac{\xi}{i^n} \int [ ] \frac{i^n}{n!} \left( \sum_{j,k} y_j M_{jk} \psi^k \right)^n$$

$$= \xi \int [ ] \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n y_i \psi^{\sigma(i)} M_{i\sigma(i)}$$

$$= \cancel{\pm \xi} \int [ ] \frac{1}{n!} \sum_{\sigma \in S_n}$$

$$= \pm \xi \underbrace{\int [ ] y_n \dots y_1 \psi^1 \dots \psi^n}_{2} \underbrace{\sum_{\sigma \in S_n} \text{sign } \sigma \prod_{i=1}^n M_{i\sigma(i)}}_{\det M}$$

## Constr. 9

Recall  $Z = \int_{\mathbb{R}^n} \frac{\prod dx^i}{(2\pi)^{n/2}} \det(\ ) e^{-\dots}$

replace with  
the exp. in the Lemma

$$Z = \frac{1}{i} \int_{\mathbb{R}^n} \int_{\overset{\wedge}{\mathbb{R}^n} \times (\overset{\wedge}{\mathbb{R}^n})^*} \frac{\text{ber}(x | \psi, \chi)}{(2\pi i)^{n/2}} e^{-\frac{1}{2\pi} S^i(x) S^i(x) + i \int_j \frac{\partial S^j}{\partial x^i}}$$

Want to make this Q-invariant.

$$gh\# \psi^k = 1, \quad gh\#(\varphi_a) = -1$$

Need auxiliary field  $H_a$  with  $gh\#(H_a) = 0$

$$x_0 \in \mathbb{R}^n \quad f(H_a) = H_a + i \frac{S^a(x_0)}{t} \quad \text{has one solution}$$

$$\int_{\mathbb{R}^n} \left( \frac{t}{2\pi} \right)^{n/2} \prod_{a=1}^n dH_a e^{-\frac{t}{2} (H_a + \frac{i S^a(x_0)}{t})^2} = \pm 1$$

$$Z = \frac{1}{(2\pi i)^n} \int_{\overset{\wedge}{\mathbb{R}^n} \times (\overset{\wedge}{\mathbb{R}^n})^*} \text{ber}(x, H | \psi, \chi) e^{-\frac{t}{2} H_j H_j - i H_j S^j + i S^j \frac{\partial S^j}{\partial x^k} \psi^k}$$

$(H_a, \varphi_a)$  antighost multiplet

$$\text{Define } Q \varphi_a = H_a, \quad Q H_a = 0$$

$-S = Q(\Psi)$  for

$$\cdot \Psi = -\frac{t}{2} \bar{x}_a H_a - i \bar{x}_a S^a$$

$Z$  is invariant under the change

$$S \rightarrow S - Q(\underbrace{\Delta \Psi}_{\text{variation of } \Psi})$$

(provided the behaviour at  $\infty$  is not changed)

" $Z$  localizes to  $Z(s)$ "  $\Leftrightarrow$   $\int$  is supported at  $-S = 0$

$\Leftrightarrow$  supp is at  $Q(\Psi) = 0 \Leftrightarrow$

" $Z$  localizes at  $Q$ -Fixedpoints".

#### 4. SUSY correlation functions

Constr. 11 Define a superfield

$$\widehat{\text{Eul}}_n : \widehat{\mathbb{R}^n} \rightarrow \widehat{\mathbb{R}^n} \quad (\text{rather } \widehat{\text{Eul}}_n \in C^\infty(\widehat{\mathbb{R}^n}))?$$

$$\widehat{\text{Eul}}_s := \int_{\widehat{\mathbb{R}^n}} \frac{dH_1 \dots dH_n [dx_1 \dots dx_m]}{(2\pi i)^m} e^{Q(\Psi)}$$

with  $S : \mathbb{R}^n \rightarrow \mathbb{V} \cong \mathbb{R}^m$ ,  $m \in \mathbb{N}$

$$\cdot gh \# \widehat{\text{Eul}}_s = m$$

$$Q(\widehat{\text{Eul}}_s) = 0 \Rightarrow \exists \text{ s.t. } \widehat{\text{Eul}}_s \in \mathcal{Q}^m(\mathbb{R}^n)$$

s.t.  $d(\widehat{\text{Eul}}_s) = 0$

## Constr. 12

For  $O_\omega : \widehat{\mathbb{R}^n} \rightarrow \widehat{\mathbb{E}}$  with  $Q O_\omega = 0$   
 $\widehat{\mathbb{R}^n} \times \widehat{(\mathbb{R}^n)^*}$

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \text{ber}(x, \gamma) O_\omega \widehat{\text{Eul}_s}$$

• If  $z \neq 0 \Leftrightarrow gh \#(O_\omega) = n - m$

- $\langle O_\omega \rangle$  depends on the cohomology class only
- Localization identity

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \text{ber}(x, \gamma) O_\omega \widehat{\text{Eul}_s} = \int_{\mathbb{R}^n} \omega \wedge \widehat{\text{Eul}_s}$$

$$= \int_{Z(s)} i^* \omega$$

## Talk 2 Evgenij Pascaal

$$\langle O_\omega \rangle = \int_{\widehat{\mathbb{R}^n}} \text{ber}(x, \gamma) O_\omega \widehat{\text{Eul}_s}$$

$$= \int_{\mathbb{R}^n} \omega \wedge \widehat{\text{Eul}_s} = \int_{Z(s)} z^* \omega$$

## Thom Isomorphism

### Poincaré' lemma

$$H^*(M \times \mathbb{R}^n) = H^*(M)$$

$$H_c^*(M \times \mathbb{R}^n) = H_c^{*-n}(M)$$

For any VB  $H_c^*(E) \cong H^{*-n}(M)$ ,  $n = \text{rk } E$

but in general  ~~$H_c^*(E) \not\cong H_c^{*-n}(M)$~~

However, if  $M, E$  are orientable, then

$$H_c^*(E) \cong H_c^{*-n}(M)$$

Pf

$$\begin{aligned} H_c^*(E) &\stackrel{\text{PD}}{\cong} \left( H^{n+n-*}(E) \right)^* \cong \left( H^{n+n-*}(M) \right)^* \stackrel{\text{PD}}{\cong} \\ &\cong H_c^{*-n}(M). \end{aligned}$$

□

### Compact support in vertical direction

$$\pi_*: \Omega_{cv}^*(E) \longrightarrow \Omega^{*-n}(M) \quad \begin{matrix} \text{integration} \\ \text{along fibers} \end{matrix}$$

### Projection formula

$\tau$  form on  $M$ ,  $\omega$  form on  $E$ ,  $\omega \in \Omega_{cv}(E)$

$$\text{Then } \pi_*(\pi^*\tau \wedge \omega) = \tau \wedge \pi_*\omega$$

Prop If  $E$  is oriented, then

$$\pi_{\text{cur}}^*: H_{\text{cur}}^*(E) \xrightarrow{\cong} H^{*-n}(M)$$

Want to find  $\pi_*^{-1}$  (the Thom iso)

Consider  $H^0(M) \ni 1$  has a well-defined image in  $H_{\text{cur}}^*(E)$ , which we call  $\Phi$ , the Thom class of  ~~$\# E$~~ .

$$\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge \pi_* \Phi = \omega.$$

↓

$$\tau(\cdot) = \pi^*(\cdot) \wedge \Phi$$

Also important: Poincaré duality

Let  $S$  be a closed <sup>oriented</sup> subbundle of  $M$ , then its Poincaré dual  $y_S$  is determined by

$$\int_S \omega = \int_M \omega \wedge y_S \quad \forall \omega \in \Omega_c^*(M)$$

Prop  $y_S$  is the same as  $\Phi$  of the normal subbundle of  $S$ .

Euler class  $E \xrightarrow{\pi} M$ ,  $s_0: M \rightarrow E$  zero section

$$\Gamma(E) \ni s \wedge s_0 =: I. \text{ Then}$$

$$\dim I = \dim M - r, \quad r = rk E$$

and we set  $e_s := \text{PD}(I)$

### Now SUSY

For  $\pi: E \rightarrow M$ ,  $E$  orientable, have the

Then is  $H^i(M) \cong H_{\text{cov}}^{i+m}(E)$

Also,  $s^* \bar{\Phi}(E) = e_s$

$$\int_M \omega \wedge s^* \bar{\Phi}(E) = \int_{Z(s)} \iota^* \omega$$

So need to replace  $\frac{\partial s}{\partial x_j}$  by  $\nabla_j s$

Let  $\{e^\alpha\}$  be an ON basis of  $E$ , then

define  $\nabla$  by  $\nabla e_\alpha^\alpha = dx^j \theta_{\alpha}^{\alpha b} e_b$

then  $S = \frac{\hbar}{2} H_j H_j + i H_j s^j - i \gamma_j \frac{\partial s^j}{\partial x_k} \psi^k$

covariantize  $S = \frac{\hbar}{2} \nabla^j \nabla_j$

$$S = -\frac{1}{2\hbar} s^j s^j + i \gamma_j \frac{\partial s^j}{\partial x_k} \psi^k$$

covariantize  $S(x, \nabla) = -\frac{1}{2\hbar} s^\alpha s^\alpha + i \gamma_\alpha (\nabla_j s)^\alpha \psi^j +$   
 $+ \frac{\hbar}{4} \gamma_a \gamma_b F_{ij}^{ab} \psi^i \psi^j$

This form is obtained from the requirement

$$S = Q(\Psi), \text{ where } \Psi \text{ is covariantized}$$

Define a superfield on  $\hat{M}$

$$\widehat{\text{Eul}}_s(E, \nabla) := \int \prod_{a=1}^n \frac{dx_a d\theta_a}{2\pi i} e^S$$

Have lin. op.  $\nabla S : T_p M \rightarrow E_p$

Important that  $Z(S)$  is a field.

This is the case if  $\nabla$  is <sup>not</sup> surjective.

How to get a localization formula for  $\widehat{E}$ ?

Consider ex. seq.

$\xrightarrow{\text{Coker } \nabla S}$

$$0 \rightarrow \text{Im } \nabla S \rightarrow E \rightarrow \text{Coker } \nabla S \rightarrow 0$$

" "

$$E / \text{Im } \nabla S$$

Then  $\widehat{E}$

$$\begin{aligned} \int_{\widehat{E}} \text{ber}(x, H | \psi, x) e^S &= \int_{\widehat{M}} \text{ber}(x | \psi) \widehat{\text{Eul}}_s(E, \nabla) \\ &= \int_{Z(S)} i^* \omega_0 \wedge \text{Eul}(\text{Coker } \nabla S) \end{aligned}$$

here we require that  $\text{Coker } (\nabla S)$  is a bundle  
Need to replace  $H_{\text{cr}}^*(E) \rightarrow H_{\text{rd}}^*(E)$   
rapidly decaying

## Equivariant cohomology

Algebra A  $\text{Der } A = \{ \delta: A \rightarrow A \text{ linear} \mid \delta(f \cdot g) = \delta(f) \cdot g + f \delta(g) \}$

For an A-module F

$\text{Der}(A, F) = \{ \delta: A \rightarrow F \text{ linear} \mid \delta(fg) = \delta(f) \cdot g + f \delta(g) \}$

Ex  $A = C^\infty(\mathbb{R}^n)$  Exercise  $[\delta_1, \delta_2] \in \text{Der}(A, F)$

$\text{Der}(A) = \left\{ \sum a_i \frac{\partial}{\partial x_i} \mid a_i \in C^\infty(\mathbb{R}^n) \right\}$

$\mathbb{R}^n \ni x \rightsquigarrow m_x = \{ f \in C^\infty(\mathbb{R}^n) \mid f(x) = 0 \}$

$C^\infty(\mathbb{R}^n)/m_x \cong \mathbb{R}$  field

$T_x^* \mathbb{R}^n = m_x/m_x^2, T_x \mathbb{R}^n = (m_x/m_x^2)^\sim$   
 $= \text{Der}(C^\infty(A), C^\infty(\mathbb{R}^n)/m_x)$

$C^\infty(\Pi T^* M) = \Omega^*(M)$  graded algebra, superalgebra

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$$

$d_{de} \rightarrow X$  vector field  $L_X$  Lie derivative  
 $i_X$  interior derivative

$$\begin{array}{l} \deg d_{\alpha} = 1 \\ \deg \alpha_x = 0 \end{array} \quad \begin{array}{l} \deg \alpha_x = -1 \end{array}$$

$$[\alpha_x, \alpha_y] = \alpha_{[x,y]} = \alpha_x \alpha_y - \alpha_y \alpha_x$$

$$[\alpha_x, \alpha_y] = \alpha_{[x,y]}$$

$$[\alpha_x, \alpha_y] = \alpha_x \alpha_y + \alpha_y \alpha_x = 0$$

$$[d, \alpha_x] = 0$$

$$[d, d] = d \circ d + d \circ d = 0$$

$$\alpha_x = [d, \alpha_x] = d \alpha_x + \alpha_x d$$

Lie gp  $G$  acts on  $M$

$$\text{if } \mathfrak{g} = \text{Lie}(G) \Rightarrow \xi \mapsto K^{\xi} \text{ fundamental vector field}$$

$$K^{\xi}(p) = \left. \frac{d}{dt} \right|_{t=0} (e^{-t\xi} \cdot p)$$

$$\alpha_{K^{\xi}}, \quad \alpha_{K^{\xi}}, \quad d$$

$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  supersymmetry or  
geometric symmetry

$$C(\text{spt}) := \mathbb{R}[\theta]/\theta^2 = \wedge^1 \mathbb{R}^1$$

$$\pi^* TM = \text{Map}(\text{spt}, M) = \text{Map}(C^\infty(M), \mathbb{R}[\theta]/\theta^2).$$

Ex  $M = \mathbb{M}$ ,  $G$  acts by the adj. action  
 $\Rightarrow$  fundamental vector fields

symmetries  $\mathcal{L}_{K^3}$ ,  $\mathcal{L}_{K^3}$ ,  $d_{dR}$  of  $\Pi T^*M$

$C^\infty(\Pi T^*M) = \Omega^*(\mathbb{M}) \ni \{\text{diff. forms with polynomial coeffic.}\}$

is generated by

$l \otimes 1$ ,  $1 \otimes l$

$l \in \mathbb{M}^*$

degree 1 diff. forms  
with constant coefficients

$S^*y^* \otimes A^*y^*$

$$d_{dR}(l \otimes 1) = 1 \otimes l \quad , \quad d_{dR}(1 \otimes l) = 0$$

$$K_y^3 = [\xi, y] = \text{ad}_\xi(y)$$

~~$\mathcal{L}_{K_x^3}(1 \otimes l)(y) =$~~

$$\mathcal{L}_{K_x^3}(l \otimes 1) = l \circ \text{ad}_\xi \otimes 1 \neq$$

$$\mathcal{L}_{K_x^3}(1 \otimes l) = 1 \otimes l \otimes \text{ad}_\xi$$

We also define another derivation  $d_K$

$$d_K(1 \otimes l) = l \otimes 1 \quad d_K(l \otimes 1) = 0$$

$$[d_K, d_{dR}] \Big|_{S^*y^* \otimes A^*y^*} = (k+l) 1$$

$$k+l=0 \Leftrightarrow S^*y^* \otimes A^*y^* = \mathbb{R}$$

$$H^i(K(y), d_K) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i>0. \end{cases}$$

$$\hat{\iota}_{K^3}(l \otimes 1) = \hat{\iota}_{K^3} d_K(1 \otimes l) = d_{K^3}(1 \otimes l) - d_K \hat{\iota}_{K^3}(1 \otimes l)$$

$$= 1 \otimes l \circ ad_{\hat{\iota}_{K^3}} \quad (**)$$

This defines a nonstandard supersymmetry.

$G$  acts freely on a wld  $P$  s.t.  $P \rightarrow P/G = M$  is a principal bundle

Connection 1-form  $\alpha \in \Omega^1(P; \mathfrak{g})^G$   $R_g^* \alpha = Ad_{g^{-1}}(\alpha)$

$$\alpha(K^\xi) = \xi$$

$$\Omega^*(P; V)_{\text{hor}} = \{ \alpha \in \Omega(P, V) \mid \iota_{K^3} \alpha = 0 \quad \forall \xi \in \mathfrak{g} \}$$

$$\Omega(P; V)^G = \{ \alpha \in \Omega(P, V) \mid R_g^* \alpha = p_g(\alpha) \}$$

Here  $V$  is any rep. of  $G$

$$\Omega(P, V)_{\text{bas}} = \Omega(P, V)_{\text{hor}}^G \xrightarrow{\text{claim}} \Omega^1(M, E)$$

$$E = P \times_G V$$

Topological idea of equivariant cohomology:

1. find a contr. top. space  $EG$  with a free action of  $G$

2.  $G$ -action on  $M \times EG$  is free, to compute

$$H^*(M \times EG/G) = H_G^*(M)$$

Def Let  $A$  be a superalgebra

$$\hat{y} = \sum_{\deg} \begin{matrix} y_{-1} & \oplus & y_0 & \oplus & y_1 \end{matrix}$$

Def Let  $A$  be a superalgebra with  $\hat{y} \in \text{Der}(A)$

We say the pair has property C if

$$\exists \text{ elements } a_1, \dots, a_n \text{ s.t. } \hat{y}_{[i]} a_j = \delta_{ij} \quad (*)$$

$\deg 0$

Ex  $K_y$  - Koszul,  $(*)$  implies the pair  
has property C

$$\text{Def } H_G(A) = H(A \otimes W(y)_{\text{bas}}, \overset{\uparrow}{d + d_K})$$

basic means  $(\bigcap_{\substack{S \in \mathcal{Y} \\ S \neq y}} \ker z_S) \cap (\bigcap_{\substack{S \in \mathcal{Y} \\ S \neq y}} \ker d_S)$

Weil algebra model

$$\text{Claim } H(A \otimes W(y)_{\text{bas}}, \overset{\uparrow}{d + d_K}) = H(A \otimes S^*(y^*), \overset{\uparrow}{d + z_y})$$

Mathai-Quillen isomorphism

$A = \Omega^\bullet(M)$  with a  $G$ -action and standard  
 $z_K$ ,  $d_K$ ,  $d_R$

$$S^*(y^*) \otimes \Omega^\bullet(M) = \text{Hom}(S^y, \Omega^\bullet(M))$$

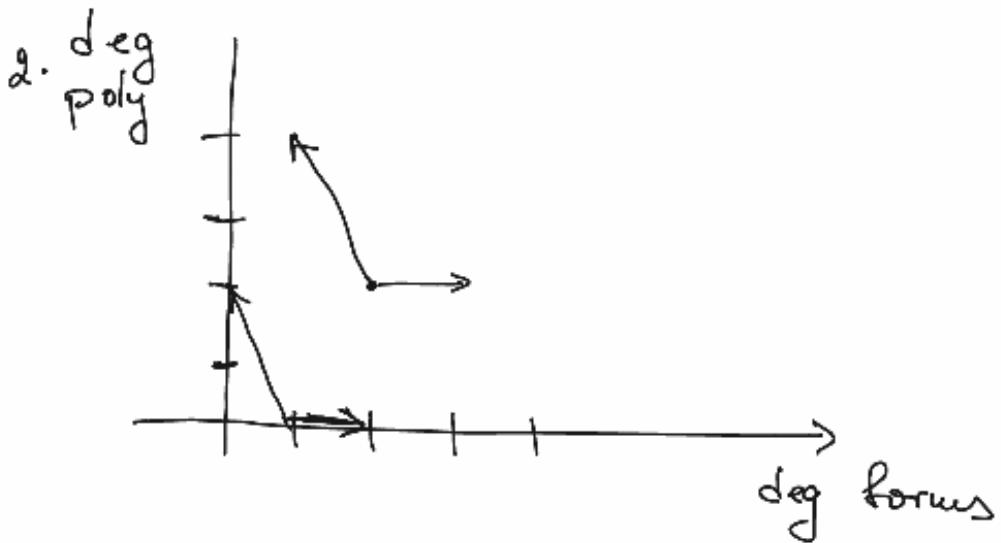
$$d \in S^d(y^*) \otimes \Omega^k(M)$$

$$(d)(z) := d(z)$$

$$((d - i_{k^3})\alpha)[\xi] = d(\alpha[\xi]) - i_{k^3}(\alpha[\xi])$$

Neo smartpen

form degree  $d+1$   
 poly degree  $k$ 
form deg  $d$ ,  
poly deg  $k+1$



$$\deg \alpha = \deg \text{form} + 2 \text{poly degree}$$

$$(d - i_{k^3})^2 = d^2 - [d, i_{k^3}] + (i_{k^3})^2 = - i_{k^3}$$

"                         "  
 0                         0

$$\alpha \text{ is invariant} \Rightarrow i_{k^3} = 0 \Rightarrow (d - i_{k^3})^2 = 0$$

Interpretation  $\alpha \in S^*(M) \otimes \Omega(M)$

$$M \text{ cpt} \quad \int_M \alpha \in S^*(M)$$

$$\alpha = d_y \beta \Rightarrow \int_M \alpha = 0$$

$\alpha_{[d]}$   $d$ -form component of  $\alpha$

$$d_y \alpha = 0 \Leftrightarrow \alpha_{[n]} = d \beta_{[n-1]} - i_{k^3}(\beta_{[n+1]})$$

" 0 "

$\mathcal{G}$  acts on  $M$ ,  $\alpha \in S^*(\mathbb{R}^n) \otimes \mathcal{Q}^*(\mu)$

$$\int_M d\alpha = 0$$

$M_0 = \text{zeros of } K^\xi$ .

Lemma  $\alpha(\xi)$  is exact on  $M \setminus M_0$ .

Def Define 1-form  $\Theta(X) = g(K^\xi, X)$

$$d_{K^\xi} \Theta = \underbrace{|K^\xi|^2}_{\deg 0} + \underbrace{d\Theta}_{\deg 2}$$

On  $M \setminus M_0$  we have  $\alpha(\xi) = d_{K^\xi} \left( \frac{\Theta \wedge \alpha(\xi)}{d_{K^\xi} \Theta(\xi)} \right)$

since  $(|K^\xi|^2 + d\Theta)^{-1} = \frac{1}{|K^\xi|^2} (1 - |K^\xi|^2) d\Theta + \dots$

$$\Rightarrow \alpha(\xi)_{[u]} = d(\ )_{[u-1]}$$



Then Assume  $K^\xi$  has isolated zeros

$Z(K^\xi) = \{p_0, \dots, p_n\}$ . Then

$$\int_M \alpha(\xi) = (2\pi)^n \sum \frac{\alpha(\xi)(p)}{\sqrt{\det(L_p)}}$$

where  $L_p$ -linearisation of  $K^\xi$  at  $p$ .

# Paddigfach Equiv. Cohomology II

11.05.18 Göttingen

 Neo smartpen

$$\underbrace{K^y \otimes S^*(y^\vee)}_{\kappa(y)} \otimes y \supset y^\vee \otimes 1 \otimes y \ni \text{id} = a^{**} \text{ connected} \\ \supset 1^2 y^\vee \otimes 1 \otimes y \ni [\cdot, \cdot] = F_a^{**} \text{ curved}$$

Mashai-Quillen iso:

$$\exp(\mathcal{L}_{K^{**}}^M) : (\Omega(M) \otimes W(y))_{hor} \rightarrow (\Omega(M) \otimes W(y))_{hor}$$

$$\text{and } d_{dR} + d^{**} \text{ to } d_{dR} - \mathcal{L}_{K^{**}}^M + \mathcal{L}_{K^{**}}^M$$

$$\text{on } (\Omega(M) \otimes W(y))_{hor}^G \xrightarrow{\cong} (\Omega(M) \otimes W(y))_{hor}^G$$

$$d_{dR} + d^{**} \rightsquigarrow d_{dR} - \mathcal{L}_{K^{**}}^M := d_G$$

Want to have a formula

$$\int_Y \omega = \int_P \tilde{\omega} \circ P(P \rightarrow Y) \quad G \rightarrow P \downarrow Y$$

$$\alpha \in (\Omega^*(M) \otimes S^*(y^\vee))^G \quad \int_P \alpha \in S^*(y^\vee)$$

e<sup>-\frac{\alpha}{2} |\beta|^2</sup>

$$\oint_{\varepsilon, P} \alpha := \frac{1}{\text{vol } G} \int_M d\beta \int_P \alpha$$

not a number  
so can't be  
compared to  
 $\int_Y \omega$  easily

~~so it's not also~~

Extend this integral to an integral

over  $P \times \bar{y} \times W(\bar{y})^\vee$

$$P(P \rightarrow Y) = \int_{W(\bar{y})^\vee} e^{Q(\Psi_{\text{proj}})} d\bar{y} dy \quad (\text{Witten's answer})$$

$$\Psi_{\text{proj}} = i\langle \xi, \alpha \rangle + \langle \bar{\xi}, [\xi, \eta] \rangle$$

$$W(\bar{y}) = \Lambda^1 \bar{y}^\vee \otimes S^1 \bar{y}^\vee \quad W(\bar{y})^\vee = S^1 \bar{y}^\vee \otimes \Lambda^1 \bar{y}^\vee$$

$$Q(\ell \otimes 1) = 1 \otimes \ell = Q\bar{\xi} = \gamma$$

$$Q(1 \otimes \ell) = \mathcal{L}_{K^*}^*(\ell \otimes 1) = Q\gamma = -\mathcal{L}_{K^*\bar{\xi}}^*\bar{\xi}$$

$$\int_Y \omega = \int_P \pi^* \omega P(P \rightarrow Y) = \int_{\substack{y \in \bar{y} \times W(\bar{y})^\vee \times P \\ \xi, \eta}} \pi^* \omega \cdot e^{Q(\Psi_{\text{proj}}) - \frac{i}{2}\langle \xi, \bar{\xi} \rangle}$$

## Emanuel Scheidegger: Topological twist

4-world, 6-bundle w/ conn

$$YM(A) = \int_M \text{tr}(F_A \wedge * F_A) \quad F = dA + [A, A]$$

In physics: YM theory

add SUSY  $\rightarrow$  Super-Yang-Mills theory

in various ways  $N=1 \rightarrow$  Kähler

$N=2 \rightarrow$  Donaldson invariants, SW in

$N=4 \rightarrow$  Kapustin-Witten, geom. Langlands

▷ make mathematically well-defined:

Neo smartpen

top. twisted Super-Yang-Mills theory  
is an example of { top. field theory  
coh. field theory }  
of Witten's type (phy  
(math))

### Basic ingredients

- $G$  cpt Lie gp,  $\mathfrak{g} = \text{Lie}(G)$   
gauge gp in physics
- $(X, g)$  closed oriented 4-mfd
- $P \rightarrow X$  principal  $G$ -bundle

$$M \rightsquigarrow A := \text{Conn}(P)$$

$$\begin{array}{c} E \rightarrow M \rightsquigarrow \mathcal{E} := A \times \Omega^{2,+}(X, \text{ad } P) \\ \downarrow s \\ s(A) := F_A^+ \end{array}$$

$$G \rightsquigarrow G = \text{Aut}(P) \text{ (gauge gp in math)}$$

X		<u><math>N=2</math> QFT in 4D</u>
$\psi$		Fields = (distributonal) representations of the $N=2$
$X$		Super-Poincaré alg. (on vector bundles over $X$ )
H		
$\phi$		
$\bar{\phi}$		
$\gamma$		
		A non-ass super-alg ( $\mathbb{Z}_2$ -graded) over a comm. ring $R$ is a Lie superalgebra, if $\exists [ \cdot, \cdot ] : A \times A \rightarrow A$
sat.	$[X, Y] + (-1)^{ X  Y } [Y, X] = 0$	
	$(-1)^{ Z  Y } [Z, [X, Y]] + (-1)^{ X  Z } [X, [Y, Z]] + (-1)^{ Y  X } [Y, [Z, X]] = 0$	A21.

$L = L_0 \oplus L_1 \Rightarrow L_0$  is a Lie alg.

If  $\exists$  at least one odd element, then

$L_1$  is an  $L_0$ -module for ad  $X \mapsto [X, X]$

If  $\exists$  at least two odd elements, then

$\exists$   $L_0$ -equivariant <sup>symm</sup> map

$$\{ \cdot, \cdot \}: L_1 \otimes L_1 \rightarrow L_0 \text{ s.t.}$$

$$[\{x, y\}, z] + [\{y, z\}, x] + [\{z, x\}, y] = 0 \\ \forall x, y, z \in L_1.$$

$V$  real quadr. v. space (of dim 4)

$L'_0 = V \times \underline{\mathfrak{so}(V)}$  Poincaré alg.

$L_0 := L'_0 \oplus \mathbb{R}$ ,  $\mathbb{R}$  any reductive Lie alg. ( $R$ -hyperfine  
(reductive: adj. repr. is completely reducible  $\Leftrightarrow$ )  
 $= s+a$ ,  $s$  semisimple &  $a$  abelian)

$\mathbb{R} = \underline{\mathfrak{u}(1)}$   $N=1$  extended super Poincaré algebra

$\mathbb{R} = \underline{\mathfrak{su}(2)} \oplus \underline{\mathfrak{u}(1)}$   $N=2$  — // — / — //

$S$  real spinorial rep. of  $\mathrm{Spin}(V)$

$\exists$  symm. nonzero map  $\Gamma: S \times S \rightarrow V$ , which is equiv. wrt  $\mathrm{Spin}(V)$

$S$  is an  $L_0$ -module by requiring that  $V$  acts as 0 on  $S$ , and is a rep. of  $R$ .

$$L_1 := S \text{ with } [S_1, S_2] := \Gamma(S_1, S_2) \quad \text{Neo smartpen}$$

Special case  $\dim V = 4$  (didn't use this above)

$$L_0 = \mathbb{R}^4 \times (\mathfrak{so}(2)_- \oplus \mathfrak{su}(2)_+) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$$

$$L_1 = (1, 2, 2)^1 \oplus (2, 1, 2)^{-1}$$

$$(1, 2, 2) = V_- \otimes V_+ \otimes V_R$$

$$\begin{matrix} & \nearrow \\ \mathfrak{su}(2)_- & \uparrow \\ \dim = 1 & \dim = 2 \end{matrix} \quad \begin{matrix} & \nwarrow \\ \mathfrak{su}(2)_R & \\ \dim = 2 & \end{matrix}$$

$(\ )^1$  weight of the  $u(1)$  repres.

There is an  $L_0$  equiv. map  $\{ \cdot, \cdot \} : \text{Sym}^2(L_1) \rightarrow \mathbb{R}^4$   
Physics notations

Basic elements in  $(1, 2, 2)^1 \Rightarrow \bar{Q}_\alpha^A \quad \alpha = 1, 2; A = 1, 2$

$(2, 1, 2)^1 \Rightarrow Q_\alpha^A \quad \alpha = 1, 2; A = 1, 2$

$$\{ Q_\alpha^A, \bar{Q}_\beta^B \} = 2 \varepsilon^{AB} \delta_{\alpha\beta}^\mu P_\mu, \quad \mu = 1, 2, 3, 4$$

totally antisym.

$$\varepsilon^{A_1 \dots A_n} \stackrel{\text{tensor}}{=} \begin{cases} +1, & \text{if } (A_1, \dots, A_n) = (1, \dots, N) \\ -1, & \text{if } (A_1, \dots, A_n) = (2, 1, \dots, 1) \\ 0, & \text{if } A_i = A_j \text{ for some } i \neq j. \end{cases}$$

$\delta_{ij}^\mu$  is the repr. matrix of the map  $\Gamma$   
= Pauli matrices.

$Q$ 's are called supercharges

$P$ 's are called momenta

Also,  $\{Q_\alpha^A, Q_\beta^B\} = 0$ .

$N=2$  vector multiplet

Reps of Lie superalgebras are determined by corr. reps of their even parts. Consider the following rep. of?

Barts elements	$\underline{\mathfrak{su}(2)}_- \oplus \underline{\mathfrak{su}(2)}_+ \oplus \underline{\mathfrak{u}(2)}_R$	$U(1)_R$	
$A_\mu$	(2, 2, 1)	0	$\mu = 1, 2, 3, 4$
$\Psi_A^\pm$	(1, 2, 2)	-1	$A = 1, 2$
$\psi_d^A$	(2, 1, 2)	+1	
$\phi$	(1, 1, 1)	2	
$\bar{\phi}$	(1, 1, 1)	-2	
$D^a$	(1, 1, 3)	0	$a = 1, 2, 3$

↑  
ghost numbers (later)

⇒ auxiliary field; appears only algebraically in the EL eqns; can be eliminated

Globalize  $\mathbb{R}^4 \rightsquigarrow X^4$

vector spaces  $\rightsquigarrow$  vector bundles

We must introduce a new datum, the  $R$ -symmetry bundle  $P_R \rightarrow X$  a principal  $\mathfrak{su}(2)_R$ -bundle.

$\Rightarrow A$  is a con-u.



Neo smartpen

$\phi, \bar{\phi}$  sections of  $\text{ad } P \otimes \mathbb{O}$

$$\mathcal{D} \longrightarrow // \longrightarrow \text{ad } P \otimes W$$

$\psi, \bar{\psi} \in \Gamma(S^{\pm} \otimes S_R \otimes \text{ad } P)$

spinor  
bundles  
of  $X$

spinor  
rep  
of  $SU(2)_R$

real  $k=3$  v.b.  
ass. to  $P_R$

$X$  does not need to be Spin!

In this case:  $P_R \rightarrow X$  is an  $SO(3)$  principal bundle s.t.  $w_2(P_R) = w_2(X)$

Then  $S^{\pm} \otimes S_R$  exist but  $S^{\pm}$  and  $S_R$  don't.

Moore: Choose  $P_R \cong P_+$  s.t.  $P_+ \times_{SO(2)_+} \mathbb{R}^3 \cong \Lambda^2 T^* X$ .

### Action

Levi-Civita  $\omega \in \Omega^1(P_{SO(4)}, \overset{SO(4)}{\mathfrak{so}(4)})$   
 $\mathfrak{so}(3)_+ \oplus \mathfrak{su}(2)_-$

$$\omega = \omega_+ + \omega_-$$

### Path integral

$$Z(\omega_+, \omega_-, \mathbf{x}) := \int [dA d\bar{A} \dots dD] e^{-S_{\text{phys}}^{\text{YM}}} \quad ( )$$

$$S_{\text{phys}}^{\text{YM}} := - \int \frac{1}{g_0^2} \text{tr} (F_A * F_B) + D\phi A * D\phi^* - \frac{1}{4} [\phi, \phi^*]_{V_0}$$

bononic part, depending on  $\mathbf{x}$

$$+ \frac{\theta_2}{8\pi^2} \int_X \text{tr} (F_A F_B) + \frac{8\pi C_F}{\theta_2} (\bar{\psi} \not{\partial} \psi + \bar{\psi} \not{\partial} \psi)$$

$\underbrace{\quad}_{\text{topological term}}$

$$+ \int_{\Sigma} \left( \bar{\psi} D \psi + \bar{\psi} \psi \bar{\psi} \psi + \bar{\phi} [\psi, \psi] + \right. \\ \left. + \phi [\psi, \bar{\psi}] \right)$$

The field  $D$  does not appear here, since it is integrated out

$$\text{tr}(XY) = -\frac{1}{2h^v} \text{tr}_g (\text{ad}(X)\text{ad}(Y))$$

$G = \text{Aut}(P)$  (symmetries)

$$\mathcal{M} = \{A \in A \mid F_A^+ = 0\}/G$$

$N=2$  vector multiplet

$$A_\mu, (2, 2, 1) \quad \not\propto \quad (1, 1, 1)_2$$

$$\bar{\Psi}_\alpha^A \quad (1, 2, 2)_{-1} \quad \not\propto \quad (1, 1, 1)_{-2}$$

$$\Psi_\alpha^A \quad (2, 1, 2)_{+1} \quad \mathcal{D}^\alpha \quad (1, 1, 0)_0$$

$$su(2) \oplus su(2) \oplus su(2)$$

$$\oplus u(1)_R$$

Path integral

$$\mathcal{Z}[\omega_+, \omega_-, \omega_R] = \int [dA d\bar{\psi} \psi_- d\phi^* \bar{\phi}] e^{-S_{\text{phys}}^{\text{SYM}}(A, \phi)}$$

$$\begin{aligned} S_{\text{phys}}^{\text{SYM}} : &= - \int_X \frac{1}{g^2} \text{tr} (F_A \wedge *F_A) + D\phi^* \wedge D\phi^* \\ &\quad - \frac{1}{4} [\phi, \phi^*] \text{vol}_g + \frac{\Theta_0}{8\pi^2} \int_X \text{tr}(F_A F) \\ &\quad + \frac{1}{g^2} \int_X \text{tr} (\bar{\psi} D\psi + \psi \bar{D}\psi + \bar{\psi} \psi + \bar{\phi} [\psi, \bar{\psi}]_+ \\ &\quad + \phi [\psi, \bar{\psi}]). \end{aligned}$$

$$\text{tr}(XY) = - \frac{1}{2h^v} \text{tr}_g (\text{ad}(X) \circ \text{ad}(Y))$$

Observation: If  $\omega^+ = \omega_R$  ( $P^+ \cong P_R$ )  
 then  $Z$  does not depend on  $\omega_+, \omega_-$ !

Witten's idea of the topological twist.

In fact: Choose a new Lorentz subalgebra  
 (instead of  $\underline{\text{SU}}(2)_- \oplus \underline{\text{SU}}(2)_+$ )

$$\underline{\text{SU}}(2)_- \oplus \underline{\text{SU}}(2)_+,$$

where  $\underline{\text{SU}}(2)_+ \subset_{\text{diagonally}} \underline{\text{SU}}(2)_L \oplus \underline{\text{SU}}(2)_R$

$$A_\mu (z, z)_0 \neq (1, 1)_z$$

$$\boxed{\bar{\Psi}_\alpha^A (1, 2 \otimes 2)_{-1} \neq (1, 1)_{-z}}$$

$$\psi_\mu^A (2, 2)_{+1} \stackrel{D^a}{\rightarrow} (1, 3)_0$$

Since  $2 \otimes 2$  is not irreducible.,  $\bar{\Psi}_\alpha^A$  decompose

$$(1, 2 \otimes 2)_{-1} \begin{cases} (1, 1)_{-1}, & \gamma \\ (1, 3)_{-1}, & \text{self dual } z\text{-form} \end{cases}$$

$$Q_\alpha^A \in (1, 2, 2)_{+1} \xrightarrow{\text{twist}} (1, 2 \otimes 2)_{+1} = (1, 1)_{+1} \oplus (1, 3)$$

$$Q_\alpha^A \in (2, 1, 1)_{-1} \xrightarrow{\text{twist}} (2, 1)_{-1}, \quad Q := S_A^\alpha \bar{Q}_\alpha^A$$

$Q$  is the projection of  $Q_\alpha^A$  to 1.

$$\{ \bar{Q}_\alpha^A, Q_\beta^B \} = 0 \Rightarrow Q^2 = 0.$$

$\Rightarrow Q$  is a BRST operator.

$$\psi_\mu = QA_\mu \Rightarrow Q\psi_\mu = 0.$$

A general  $N=2$  action has the form

$$S = \int d^2\theta K(\Phi, \bar{\Phi}) + \int d^2\theta W(\Phi) + \int d^2\theta \bar{W}(\bar{\Phi})$$

Kähler potential ↑  
superpotential,  
indom. in  $\bar{\Phi}$ .

Main consequences of the topological twist

$$1) S_{\text{top}}^{\text{SYM}} = \{ Q, \Psi \} \quad \text{action is } Q\text{-exact.}$$

$$+ 2\pi i \tau_0 \int \text{tr}(F \wedge F),$$

$$\text{where } \tau_0 := \frac{\Theta_0}{2\pi} + \frac{4\pi i}{g^2}$$

This is equivalent to

$$S_{\text{top}}^{\text{SYM}} = Q\Psi + 2\pi i \tau_0 \int \text{tr}(F \wedge F)$$

Useful identity

$$\int_X \text{tr} F^\pm \wedge F^\pm = 2 \int_X \text{tr}(F \wedge \star F) \pm \text{tr}(F \wedge F)$$

g) Energy-momentum tensor:

$$T_{\mu\nu} := \frac{1}{\sqrt{\det g}} \frac{\delta}{\delta g^{\mu\nu}} S_{\text{top}}^{\text{SYM}} = \{ Q, -A_{\mu\nu} \}$$

for some  $A_{\mu\nu}$

Hence, the path integral does not depend on the background metric.

Some comment on vdim Mass

Anomaly is proportional to  $\text{ind}(\not D_A) =$   
= expression in Chern numbers of some spin  
=  $U(1)_R$  charge  
= ghost number  
= degree of diff. forms on Mass.

Anomaly is a symmetry which does not survive any quantization procedure

## Cohomological descent

$$W(y) = \wedge^{\bullet}(y^\vee) \otimes S^{\bullet}(y^\vee)$$

Super-Lie-alg.  $\mathfrak{g} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle_1 = \tilde{\mathfrak{g}}$

$$\mathfrak{g}_{-1} \quad \mathfrak{g}_0 \quad d \quad \mathcal{L}_{\mathfrak{g}} = [d, \mathfrak{g}_0]$$

Ex 1 GGM  $\Omega^\bullet(M)$  is a representation  
of  $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \langle d \rangle_1$

Ex 2 Basis of  $y^\vee$ :  $\theta^i \quad \deg = 1 \quad \mathcal{L}_a = \mathcal{L}_k \theta^a$   
 ~~$z^i \quad \deg = 2 \quad \iota_a = \iota_k z^k$~~

$$d_{\mathfrak{g}_K} \theta^i = z^i, \quad d_K z^i = 0$$

$$\mathcal{L}_a \theta^b = - \sum_k c_{ak}^b \theta^k; \quad \mathcal{L}_a z^b = - \sum_k c_{ak}^b z^k$$

$$\iota_a \theta^b = \delta_{ab} \quad \iota_a z^b = \iota_a d \theta^b = \mathcal{L}_a \theta^b - d_a^b \theta^b = - \sum_k c_{ak}^b \theta^k$$

A alg,  $\tilde{\mathfrak{g}} \rightarrow \text{Der}(A)$

$$A_{\text{hor}} = \{ \alpha \in A \mid i_a \alpha = 0 \ \forall a \}$$

$$A_{\text{vert}}^G = \{ \alpha \in A \mid \mathcal{L}_a \alpha = 0 \ \forall a \}$$

$G$  acts freely on  $\Phi$ , then  $\Omega^\bullet(P)^G_{\text{hor}} \cong \Omega^\bullet(P/G)$

$$\boxed{\mu^b = z^b + \frac{1}{2} \sum c_{jk}^b \theta^j \theta^k}$$

deg = 2  
element

Since  $z^b = \mu^b - \frac{1}{2} \sum c_{jk}^b \theta^j \theta^k$ ,

$\{\mu^b, \theta^a\}$  can be taken as generators.

$$W(y) \approx \Lambda(\theta', \dots, \theta^n) \otimes S(\mu^1, \dots, \mu^n)$$

Claim  $\mu^i$  are horizontal

$$i_a \mu^b = i_a z^b + \frac{1}{2} \sum_{jk} c_{jk}^b (i_a \theta^j \cdot \theta^k - \theta^j i_a \theta^k)$$

$$= - \sum_k c_{ak}^b \theta^k + \frac{1}{2} \sum_{j,k} (\delta_{aj} \theta^k + \delta_{ak} \theta^j)$$

$$= - \sum_k c_{ak}^b \theta^k + \frac{1}{2} \sum c_{ak}^b \theta^k - \frac{1}{2} \sum c_{ja}^b \theta^j = 0.$$

$- c_{aj}^b$

$$d_k \theta^a = \mu^a - \frac{1}{2} \sum c_{jk}^a \theta^j \theta^k \quad (**)$$

~~defn~~

$$\begin{aligned} \mathcal{L}_a \mu^b &= \mathcal{L}_a z^b + \frac{1}{2} \mathcal{L}_a \sum_{jk} c_{jk}^b \theta^j \theta^k \\ &\stackrel{\text{Jacobi}}{=} - \sum c_{ak}^b \mu^k \end{aligned}$$

$$d_k \mu^a = - \sum c_{jk}^a \theta^j \mu^b = \left( \sum \theta^b \mathcal{L}_b \right) \mu^a \quad (***)$$

Since  $\mu^a$  generate  $y^v \in S(y^v) \Rightarrow$  Bianchi identity

$$\Rightarrow d_K \omega = \sum (\theta^k \omega_k) \omega$$

"Supersymmetric change of variables" (\*)

Often, one takes (\*) & (\*\*\*\*) as definitions

If this is the approach, one has to prove

$d_K^2 = 0$  and acyclicity

$$H^i(W(y), d_K) = \begin{cases} R, & i=0 \\ 0, & \text{otherwise.} \end{cases}$$

Mathai-Quillen iso  $\oplus$

$\tilde{y} \rightarrow \text{Der } A$ ,  $A, B$  Algebras  
 $\tilde{y} \rightarrow \text{Der } B$

Assume  $\exists \hat{\Theta}_b \in A$  s.t.  $\iota_a \hat{\Theta}^b = \delta_{ab}$

Derivation  $\gamma = \sum \hat{\Theta}^a \otimes \iota_a \in \text{Der}(A \otimes B)$

If  $n = \dim y$ , then  $\gamma^{n+1} = 0$

$\underline{\Phi} = e^\gamma \quad \text{adj}_\gamma, \text{conj}_{\underline{\Phi}} \in \text{Der}(A \otimes B) \ni$

$\text{Aut}(A \otimes B) \quad \text{adj}_\gamma(\delta) = [\gamma, \delta], \quad \delta \in \text{Der}(A \otimes B)$

$\text{conj}_{\underline{\Phi}}(\delta) = \underline{\Phi} \delta \underline{\Phi}^{-1}$

$$\underline{\Phi} (1 \otimes i_a + i_a \otimes 1) \underline{\Phi}^{-1} = i_a \otimes 1$$

$$\underline{\Phi} (d_A \otimes 1 + 1 \otimes d_B) \underline{\Phi}^{-1} = \underbrace{d - \sum \mu^k \otimes i_k + \sum \theta^k \otimes \iota_k}_{\text{BRST}}$$

[Guillemin-Sternberg?]

$$\Phi(A \otimes B)_{hor} \Phi^{-1} = A_{hor} \otimes B$$

Apply this to  $A = W(y)$ ,  $B = \Omega^*(M)$ ,  $C \in M$

$$(W(y) \otimes \Omega^*(M))_{hor} = W(y)_{hor} \otimes \Omega^*(M)$$

$$= S^*(y) \otimes \Omega^*(M)$$

~~Moreover~~ Furthermore

$$(W(y) \otimes \Omega^*(M))^G = (S^*(y) \otimes \Omega^*(M))^G$$

$$d_x \otimes 1 + 1 \otimes d_{dR} - \sum \mu^k \otimes i_k + \sum \theta^k \otimes \omega_k \Big|_{( )^G_{hor}} =$$

$$= d_{dR} - \sum \mu^k \otimes i_k$$

$$= : (d_{dR} - \iota_y)$$

This is a Cartan model.

$A_p$  comes on  $P \rightarrow X^4$

$$G = \text{Aut}(P)$$

$$C^\infty(A_p) \ni A_\mu^a(x)$$

$\Omega^1(A_p; \Omega^1(P; y)_{hor}^G) \ni \psi$  tautological 1-form

$d_C A = \psi$ , where  $A \in \Omega^1(A_p; \Omega^1(P; y)_{hor}^G)$   
Cartan differential

$$d_C \phi = -D_A \phi \quad \phi \in C^\infty(\text{Lie}(G), \Omega^0(P, \mathfrak{g})^*)$$

$$d_C \phi = 0 \quad \phi \text{ is the identity map.}$$

$D_A \phi$  fundamental vector field of  $\phi \in \text{Lie}(G)$

Compare with the following:

$$G \rightarrow GL(\mathbb{R}^n)$$

$$\mathfrak{g} \rightarrow \mathfrak{gl}_n(\mathbb{R})$$

$$K_x^\xi = \sum_{ij} \xi_{ij} x_j$$

$$\begin{aligned} d_C^2(x^i) &= d_C(dx^i) = (d - i_{K^\xi}) dx^i \\ &= - \sum \xi_{ij} x_j \end{aligned}$$

BRST  $d_B$ ,  $B$  for BRST

$$d_B A = \psi - D_A C$$

$$d_B \psi = [\psi, c] - D_A \phi$$

$$d_B^2 c = \phi - \frac{1}{2} [c, c]$$

$$d_B \phi = -[c, \phi]$$

## Cohomological descent II

$W(y) \otimes \Omega^*(M)$  G-action on M

$\exists$  automorphism of  $W(y) \otimes \Omega^*(M)$ , which takes  $(W(y) \otimes \Omega^*(M))^G_{\text{hor}}$  to  $(W(y) \otimes \Omega^*(M))^G$

with the differential  $d - iy$

$(S^*(y^*) \otimes \Omega(M))^G$

Cartan model

Free G-action on P (6 princip. bundle)

$$(W(y) \otimes \Omega^*(P))_{\text{hor}}^G \simeq \left( \underbrace{W(y) \otimes \Omega(P)}_{\text{contractible}} \right)^G \xleftarrow{\quad \text{induces} \quad} \Omega(P)^G_{\text{hor}}$$

$\Downarrow$

$(1^*y^*) \otimes S^*(y^*)$

Contraction:  $\delta \mu^\alpha = \theta^\alpha, \delta \theta^\alpha = 0$

$\Omega^*(P/G)$

Free action of the gauge gp  $G_P$  on  $A_p$

Cartan model

$$(A) S^*(\text{Lie}(G)) \otimes \Omega^*(A) \quad d := d_{dR} - \iota_{\text{Lie}(G)}$$

$$\sum p_i(\xi) \otimes d_i$$

$$(d - \iota_{\text{Lie}(G)}) \sum_i p_i(\xi) \otimes d_i = \sum p_i(\xi) \otimes dd_i - \sum p_i(\xi) \otimes i_{K\xi}$$

$$\xi \in \text{Lie}(G) = \Omega^*(\text{ad } P)$$

$$P \rightarrow M \quad \text{The fund. v.f. on } A_p : K_a^\xi = D_a \xi$$

Changing the notation is as follows

$$(d - i_{\pi(\phi)})(\alpha) = \sum_i p_i(\phi) \otimes d\alpha_i - \sum_i p_i(\phi) \otimes i_{\pi(\phi)}(\alpha_i)$$

$A \in C^*(A; \Omega^1(P, g)^G_{hor})$	$d_c A = d_{de} A = \psi$	$\deg \phi = *$
$\psi \in \Omega^1(A; \Omega^1(P, g)^G_{hor})$	$d_c \psi = - D_A \phi$	$\deg A = *$
	$d_c \phi = 0$	$\deg \psi = *$

$\phi \in \text{Hom}(\text{Lie}(G), \text{Lie}(S))$  vector valued

'identity map'

$G$ -invariant & closed elements of  $(A)$

~~$\Omega^{(0)}_x(\phi)$~~   $\Omega^{(0)}_x(\phi) \quad \pi(p) \in M \quad (p \in P \text{ } \exists)$

$$\Omega^{(0)}_x(\phi) := \frac{1}{8\pi^2} \text{tr}(\phi^2(p))$$

$$\Omega^{(0)}_x(x) \in S^2(\text{Lie}(S)^\vee)$$

$$\Omega^{(1)}_x(\gamma) = \frac{1}{4\pi^2} \int_\gamma \text{tr}(\phi \gamma) \quad \Omega^{(1)}_x \in S^2(\text{Lie}(S)^\vee) \otimes \Omega^1(A)$$

$$\Omega^{(2)}_x(\Sigma) = \frac{1}{\pi^2} \int_\Sigma \text{tr}(\phi F_a - \frac{1}{2} \psi_a +), \quad \Omega^{(2)}_x \in S^2(\text{Lie}(S)^\vee) \otimes \Omega^2(A)$$

$$\Omega^{(k)}_x \in S^k(\text{Lie}(S)^\vee) \otimes \Omega^k(A) \oplus \Omega^2(A)$$

All these elements are  $G$ -invariant

$$d_c \text{tr}(\phi^2(p)) = 2 \text{tr}(\phi(p) d_c \phi(p)) = 2 \text{tr}(\phi d_c \phi)(p) = 0$$

$$\text{tr}(\phi D_A \phi)(p) = \frac{1}{2} \text{tr}(\phi^2)(p)$$

$$d_c \int_Y \text{tr}(\phi \psi) = \int_Y \text{tr}(d_c \phi \cdot \psi + \phi d_c \psi) \quad \begin{matrix} \text{Neo smartpen} \\ \text{Logo} \end{matrix}$$

$$= \int_Y \text{tr}(\phi (-D_a \phi)) = \int_Y d^M \text{tr}(\phi^c) \stackrel{\text{Stokes}}{=} 0$$

$$d_c \int_{\Sigma} \text{tr}(\phi F_A - \frac{1}{2} \psi \wedge \psi) = \int_{\Sigma} (\phi d_c F_A - d_c \psi \wedge \psi)$$

$$= \int_{\Sigma} \text{tr}(\phi \wedge D_A \psi + D_A \phi \wedge \psi) = - \int_{\Sigma} d^M \text{tr}(\phi \wedge \psi)$$



## Donaldson invariant

Motivation:

$$M_k \subset \mathcal{B}_k^*$$

Can show:  $[M_k] \in H_d(\mathcal{B}_k^*)$  is well defined

$$d = \text{vdim } M_k = 8k - 3(b_2^+ - b_1 + 1)$$

$[M_k]$  can be thought of as an invariant, but this is not convex

If  $\omega \in H^d(\mathcal{B}_k^*)$ , then  $\langle \omega, [M_k] \rangle \in \mathbb{Z}$

Q:  $H^*(\mathcal{B}^*)$ ?

1. Slant product

$X, Z$  any top. spaces

$$\wedge : H^p(Z) \otimes H_q(Y \times Z) \rightarrow H_{q-p}(Y) \quad (*)$$

~~Hom(C\_p(Z), Z)  $\otimes$  C\_q(Y  $\times$  Z)~~

Fact ~~as~~  $\exists$  natural chain map

$$\xi : C_q(Y \times Z) \rightarrow \sum_{s+t=q} C_s(Y) \otimes C_t(Z)$$

which is a chain homotopy equivalence

In particular,  $\xi$  induces an iso

$$\xi : H_q(Y \times Z) \rightarrow H_q(C_*(Y) \otimes C_*(Z))$$

$$\text{Hom}(C_p\mathbb{Z}, \mathbb{Z}) \otimes C_q(Y \times Z) \quad \xrightarrow{\quad \partial \quad} \quad C_{q-p}\mathbb{Y} \otimes C_p\mathbb{Z}$$

$$\xrightarrow{\text{def}} \text{Hom}(C_p Z, Z) \otimes (C_p Y \otimes C_p Z) \longrightarrow \mathbb{Z}$$

$$\varphi \otimes (a \otimes b) \longmapsto \varphi(b)a$$

This is a chain map, hence we obtain (2)

## Properties

## 1° Naturality

$$f: Y \rightarrow Y' \qquad u \in H^*(Z')$$

$$g: Z \rightarrow Z' \quad w \in H_g(Y \times Z)$$

$$f_* \left( \frac{g^* u}{v} \right) = u / (f \times g)_* v$$

$$H_{g-p}(Y)$$

## 2°. Cap product

$$Y \times Z \quad u \in H^p(Z)$$

$$v \in H_p(Y \times Z)$$

$$w/v = (\pi_Y)_*(\pi_z^* w \cap v)$$

$$\frac{1}{\parallel \Sigma} \left( \bar{z} \right) \otimes H_p(Y \times \bar{z}) \rightarrow H_{p-d}(Y)$$

$$Z \quad H_q(Y \times Z) \rightarrow H_{q-d}(Y)$$

$$H_*(Y \times Z) \longrightarrow H_{*-1}(Y)$$

$$\begin{array}{ccc} & \downarrow \text{PD} & \\ H^{n+d-q}(Y \times Z) & \xrightarrow{\cong} & H^{n-q}(Y) \\ \text{integration along the fibers} \end{array}$$

de Rham - version :

$$\omega \in \Omega^{k+d}(Y \times Z)$$

$$v_1, \dots, v_k \in \mathcal{X}(Y)$$

$$\int_{v_1} \dots i_{v_k} \omega = \Omega(v_1, \dots, v_k)$$

$$\begin{array}{ccc} \Omega^{k+d}(Y \times Z) & \longrightarrow & \Omega^k(Y) \\ \omega & \longmapsto & \Omega \end{array} \quad \text{induces} \quad \underbrace{H_{dR}^{k+d}(Y \times Z) \xrightarrow{\cong} H_{dR}^k(Y)}_{\text{integration along fibers.}}$$

## 2. The universal bundle

$P \rightarrow X \ni x_0$  base pt

$G_0 = \{g \in \text{Aut}(P) \mid g|_{x_0} = \text{id}\}$  based gauge gp

$$\{ \xrightarrow{\pi} G_0 \longrightarrow G \xrightarrow{\text{ev}_{x_0}} \text{Aut}(P_{x_0}) = G \rightarrow 0 \quad \text{exact}$$

i.e. acts freely on  $A$

$$\tilde{B} = A/G_0 \text{ mod}$$

$$G = G/G_0 \quad G \xrightarrow{\tilde{\beta}} \mathcal{B}$$

~~Ex/Ex~~

$$\begin{array}{ccc} G/G_0 & \xrightarrow{\quad \quad} & \mathcal{B} \\ \downarrow & & \downarrow \\ G/G_0 \times X & \xrightarrow{\pi_2} & X \end{array}$$

$$\tilde{\mathbb{P}} = \mathbb{P}/G_0 \rightarrow A \times X/G_0 = \tilde{\mathcal{B}} \times X \quad \text{universal bundle}$$

$$\iota : H^d(\tilde{\mathcal{B}} \times X) \times H_\ell(X) \rightarrow H_{\ell-d}(\tilde{\mathcal{B}})$$

$\Downarrow$

$$c_{d\ell}(\tilde{\mathbb{P}})$$

$$\tilde{M}_\ell : H_\ell(X) \rightarrow H^{d-\ell}(\tilde{\mathcal{B}})$$

$$\alpha \mapsto c_{d\ell}(\tilde{\mathbb{P}})/\alpha$$

Prop

$$P \rightarrow X \quad \text{SU}(2) \text{ bundle}$$

$X$  1-connected

$\Sigma_1, \dots, \Sigma_6$  basis of  $H_1(X; \mathbb{Q})$

$\left. \begin{array}{l} P \rightarrow X \quad \text{SU}(2) \text{ bundle} \\ X \quad 1\text{-connected} \\ \Sigma_1, \dots, \Sigma_6 \text{ basis of } H_1(X; \mathbb{Q}) \end{array} \right\} \Rightarrow \left. \begin{array}{l} H^*(\tilde{\mathcal{B}}; \mathbb{Q}) \text{ is generated} \\ \text{by } \tilde{\mu}_2(\Sigma_1), \dots, \tilde{\mu}_2(\Sigma_6) \\ \text{as a ring, i.e.} \\ H^*(\tilde{\mathcal{B}}; \mathbb{Q}) \cong S^*(H_2(X; \mathbb{Q})) \\ H^{2k}(\tilde{\mathcal{B}}; \mathbb{Q}) \cong S^k(H_2(X; \mathbb{Q})) \end{array} \right\}$

$$\tilde{\mu}_2(\Sigma_j) = c_2(\tilde{\mathbb{P}})/\Sigma_j \in H^2(\tilde{\mathcal{B}}; \mathbb{Q})$$

$$\therefore \text{rel}: H^*(\mathcal{B}^*; \mathbb{Q}) ?$$

$\tilde{B} \supseteq \gamma$  the action of  $G/\{ \pm 1 \}$   
 $\int P^{\text{ad}} \subset \tilde{P}$  is free only on  $\tilde{B}^*$

so  $i$ :  $B^* \leftarrow -\tilde{B}^*$

in the above.

$$\mu: H_*(X; \mathbb{Q}) \rightarrow H^*(B^*; \mathbb{Q})$$

$$\Sigma \mapsto -\frac{1}{4} p_1(P^{\text{ad}}) / \Sigma$$

~~exact sequence~~

$$SO(B) \hookrightarrow \tilde{B}^*$$

$$\downarrow$$

$$\nu = -\frac{1}{4} p_1(\tilde{B}^* \rightarrow B^*) \in H^4(B^*; \mathbb{Q})$$

$$B^*$$

Prop

$$H^*(B^*; \mathbb{Q}) \cong \mathbb{Q}[\nu, \mu(\Sigma_1), \dots, \mu(\Sigma_e)] \text{ as rings.}$$

$$\text{Rev: } \mu: H^*(B^* \times V) \times H_*(X) \xrightarrow{\quad} H^*(B^* \times -\frac{1}{4} p_1(P^{\text{ad}}))$$

$$\mu_4: H_*(X) \rightarrow H^*(B^*)$$

$$\alpha \mapsto -\frac{1}{4} p_1(P^{\text{ad}}) / \alpha$$

$$H_*(X) = \mathbb{Q}[x_0] \quad \& \quad \mu_4[x_0] = -\frac{1}{4} p_1(P^{\text{ad}}) / [x_0] = \nu$$

### 3. The Donaldson invariant

SDG 6

$$\text{Assume first value } M_k = 8k - 3(b^+ - b^- + 1) = 0$$



Neo smartpen

Assume

$$\pi_1(X) = \{1\} \Rightarrow b_1 = 0$$

$$b_2^+ \geq 2$$

Want:

- (i)  $M_k$  contains no reducible solutions
- (ii)  $M_k$  is a finite set of pts cut out transversally
- (iii) Can attach signs  $\pm$  to each pt in  $M_k$

(i): holds for a generic metric on  $X$ , since  $b_2^+ \geq 2$

$$(ii): \bar{M}_k = M_k \cup (M_{k-1} \cup X) \cup \dots$$

$$\text{vdim } M_{k-1} = 8(k-1) - 3(b^+ + 1) = \underbrace{8k - 3(b^+ + 1)}_{> 0} - 8 < 0$$

For generic  $g$ ,  $M_{k-1} = \emptyset$

$$\Rightarrow \bar{M}_k = M_k \text{ except}$$

$g$  generic  $\Rightarrow M_k$  is cut out transversally

(iii) Roughly,  $\varepsilon([A]) := \text{sign}_{\mathbb{Z}} \det \left( c_A^{i-} + d_A^{j+} \right)$   
 is well-defined

however, if  $A_0, A_1$  are two solutions, we can  
 define  $\varepsilon(A_0, A_1) \in \{\pm 1\}$  s.t.

$$\text{sign} \det \left( d_{A_1}^{i+} + d_{A_2}^{j+} \right) = \varepsilon(A_0, A_1) \text{ sign} \det \left( d_{A_0}^{i+} + d_{A_1}^{j+} \right)$$

$\Rightarrow -A$  is infra-red  $\Rightarrow$  an overall sign.

$$q - \sum_{[A] \in M_k} -(A) = \mathbb{Z}$$



The Donaldson invariant

More generally, assume  $\dim M_k > 0$ ,  $d \in \mathbb{N}$

"Define"

$$q_k: S^k(H_2(X; \mathbb{Z})) \rightarrow \bigoplus \mathbb{Z}$$

$$q_k(\Sigma_1, \dots, \Sigma_d) = \langle \mu(\Sigma_1) \cup \dots \cup \mu(\Sigma_d), [M_k] \rangle$$

Main problem:  $M_k$  is noncompact.

Main property: Naturality

If  $f: X \rightarrow Y$  orientation preserving diff.,

$$q_k(f_* \Sigma_1, \dots, f_* \Sigma_d) = q_k(\Sigma_1, \dots, \Sigma_d)$$

## Donaldson polynomials & Observables

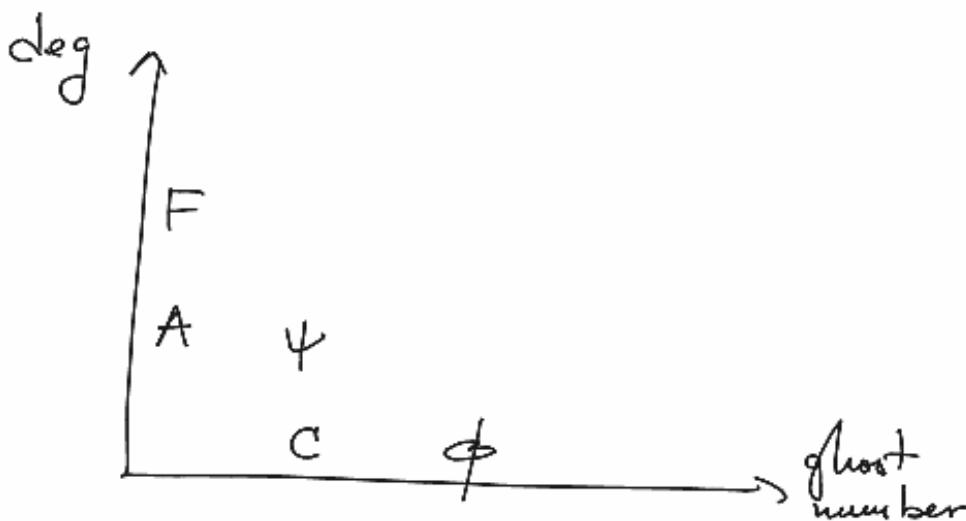
BRST  $A, \psi, \phi$  as in the Cartan model

$$\left. \begin{array}{l} c \in \text{Lie}(G) = \Omega^0(P; \mathfrak{g})^F, \deg c = 1 \\ d_B A = \psi - D_A c \\ d_B \psi = [\psi, c] - D_A \phi \\ d_B c = \phi - \frac{1}{2}[c, c] \\ d_B \phi = -[c, \phi] \end{array} \right\} (*)$$

$\mathcal{E}^+ = \Omega^2(P; \mathfrak{g})$

$\mathcal{F} \downarrow \quad \downarrow A^*$

$M \subset A^*/G = \mathcal{B}^*$



Consider  $A = A + c$  as a new connection

$$W(y^\nu) \rightarrow \Omega^0(P)$$

$\theta^\alpha \longleftrightarrow$  comp. of a conn.

$\mu^\alpha \longleftrightarrow$  comp. of curvature

$$F_A = F_A + \psi + \phi$$

$$\left\{ \begin{array}{l} (d + d_B) A = F - \frac{1}{2} [A, A] \quad \text{defn of curv} \\ (d + d_B) F = [A, F] \quad \text{Bianchi identity} \end{array} \right. \Leftrightarrow (*)$$



Neo smartpen

$$\begin{matrix} \mathbb{P} \\ \downarrow \\ \mathcal{B} \times X \end{matrix} \quad C_2(\mathbb{P}) = \gamma \otimes 1 + \sum_{\Sigma_i \in H_2(X; \mathbb{Z})} \mu_{\Sigma_i} \otimes [\Sigma_i]^{\text{PD}} + 1 \otimes c_2(\mathbb{P})$$

There is a conn \$A+C\$ on the principal bundle \$\mathbb{P}\$, which is \$A+C \in \Omega^1(\mathcal{B} \times \mathbb{P}, \text{Lie}(G)) \otimes \mathbb{C}\_y\$

The curvature is \$\phi + \psi + F\_A\$

Andreas Ikkert The Donaldson-Witten partition function.

QFT-heuristics in \$N=2\$ SUSY-YM.

Def Corr. function

$$\langle F \rangle = \langle F \rangle_T = \int [dA \dots] e^{S_T - F}$$

$$\text{for } S_T = -\frac{1}{g_0^2} \int_X (-\text{tr } F^+ \wedge F^+ + \dots)$$

$$= -\frac{1}{g_0^2} \{ Q, \Psi \}$$

Lemma a) \$\langle \{Q, \bullet \circ \mathcal{O}\} \rangle = 0\$ for every field expression  
 b) \$\delta\_{g\_0} \langle \mathcal{O} \rangle = \langle \delta\_{g\_0} \mathcal{O} \rangle\$

$$c) \{Q, \vartheta\} \Rightarrow \delta_{g_{\mu\nu}} \langle \vartheta \rangle = \langle \delta_{g_{\mu\nu}} \vartheta \rangle$$

When is  $\langle \vartheta \rangle$  invariant (under diffeomorphisms)?

$$\left. \begin{array}{l} (i) \{Q, \vartheta\} = 0 \\ (ii) \vartheta \neq \vartheta(g_{\mu\nu}) \end{array} \right\} \Rightarrow \delta_{g_{\mu\nu}} \langle \vartheta \rangle = 0.$$

(iii) If  $\langle \vartheta \rangle \neq 0 \Rightarrow \vartheta \neq \{Q, e\}$  for some  $e$

This is satisfied by  $\vartheta^{(0)}(p) = \text{Tr}(\phi^z(p))$

Lemma  $x_1 \rightarrow x_2$  location change  $\rightarrow Q \rightarrow \infty$

$$\frac{\partial}{\partial x^m} \vartheta^{(0)}(x) = i \{Q, \text{Tr} \phi(x) \psi_m(x)\}$$

$$\vartheta^{(0)}(p_1) - \vartheta^{(0)}(p_2) = \int \underbrace{\frac{\partial Q^{(0)}}{\partial x^m} dx^m}_{dQ} \vartheta$$

$$= i \left\{ Q, \int \underbrace{\text{Tr}(\phi \psi_m) dx^m}_{Q \in \Omega} \right\}$$

Constr  $d\vartheta^{(i)} = i \{Q, \vartheta^{(i+1)}\} \quad i=0, \dots, 3$

$$0 = i \{Q, \vartheta^{(4)}\}, \quad dQ^{(4)} = 0$$

Explicitly,

$$\vartheta^{(0)} = \text{Tr}(\phi_1 \psi)$$

$$\vartheta^{(1)} = \text{Tr}(\psi_1 \psi + z_i \phi_1 F)$$

$$\vartheta^{(2)} = z_i \text{Tr}(\psi_1 F), \quad \vartheta^{(3)} = -\text{Tr}(F_1 F)$$

For  $\Sigma_i \in H_i(X)$  set  $\mathcal{O}(\Sigma_i) := \sum_i \mathcal{O}^{(i)}$



Neo smartpen

Fact a)  $\mathcal{O}(\Sigma_i)$  are Q-closed

$$b) gh \# \mathcal{O}^{(j)} = 4-j$$

$$c) \delta_{g_{\mu\nu}} \mathcal{O}(\Sigma_i) = 0$$

Then For  $k_1, \dots, k_n \in \{0, \dots, 4\}$  and  $\Sigma_{k_j} \in H_{k_j}(X)$

$$\delta_{g_{\mu\nu}} \langle \mathcal{O}(\Sigma_{k_1}) \dots \mathcal{O}(\Sigma_{k_n}) \rangle = 0.$$

In the weak coupling limit ( $g_0$  small) the path integral is dominated by classical minima ( $\text{tr } F^+ \wedge F^+$  pos. definite)  $\Leftrightarrow F^+ = 0$ .

- path integral can be evaluated in  $M_{ASD}(\mathbb{P})$ .

Lemma  $n = \dim M_{ASD}(\mathbb{P})$ , in the wcl we have

$$a) n = \# \text{ of } \mathbb{O} \text{ modes of } \Psi - \underbrace{\# \text{ of } 0 \text{ modes } (\gamma, v)}_{= 0, \quad G = SU(2)}$$

$$b) \langle \mathcal{O} \rangle \neq 0 \Leftrightarrow \langle \mathcal{O} \rangle = \left\langle \underbrace{\Theta_{i_1 \dots i_n}}_{n \text{ form on the space of connections}} \underbrace{\psi^{i_1} \dots \psi^{i_n}}_{0 \text{ modes}} \right\rangle$$

$$\Rightarrow [dA \dots] \rightarrow dA_1 \dots dA_n d\psi_1 \dots d\psi_n$$

Th For  $k_1, \dots, k_n \in \{0, \dots, 4\}$  and  $\Sigma_{k_j} \in H_{k_j}(X)$  ( $\Rightarrow$  that  $n = \sum (4 - k_j)$ ) there exists a  $k_j$ -form  $\Theta_{\Sigma_j}$  on  $M$  s.t.

$$\langle \mathcal{O}(\Sigma_1) \dots \mathcal{O}(\Sigma_n) \rangle = \sum_M \Theta_{\Sigma_1} \dots \Theta_{\Sigma_n}$$



Neo smartpen

### Proof sketch

Problem:  $\mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}$  contain  $\phi$ !  
~~so we can't do it~~ Eliminate  $\phi$  by  $\langle \phi \rangle$ ,

$$\Theta_{\Sigma_0} = \text{Tr} \langle \phi(\Sigma_0) \rangle$$

which are expressible  
in terms of other  
fields

$$\Theta_{\Sigma_1} = \text{Tr} \int_{\Sigma} \langle \phi \rangle \wedge F$$

Witten shows

$$\langle \phi^a(x) \rangle = -i \int_X dy G^{ab}(x, y) [\psi_a(y), \psi^b(x)]$$

Green's  
function

$$\Theta_{\Sigma_2} = \int_{\Sigma_2} \text{Tr} (\psi_1 \psi + 2i \langle \phi \rangle \wedge F)$$

$$\Theta_{\Sigma_3} = i \int_{\Sigma_3} \text{Tr} (\psi \wedge F)$$

$$\Theta_{\Sigma_4} = - \int_X \text{Tr} (F \wedge F), \quad \zeta \in \mathbb{R}$$

2. Donaldson-Witten partition function

$$\mathcal{O} := \mathcal{O}(\Sigma_0), \quad \mathcal{O}(\Sigma) := \mathcal{O}(\Sigma_2)$$

Def For  $P \in H_0(X)$  &  $\Sigma \in H_2(X)$

$$Z_{DW}(P, \Sigma) := \sum_m \langle e^{P \mathcal{O} + \mathcal{O}(\Sigma)} \rangle = \sum_{l, \ell} \frac{P^\ell}{\ell! 2^l} \sum_m \langle \mathcal{O}^\ell \mathcal{O}(\Sigma) \rangle$$

Note For  $\mathcal{O}^l \mathcal{O}(\Sigma)^r$  only the instanton sector with  $\dim M_m = 4l + 2r$  contributes one term

 Neo smartpen

Witten's claim:  $M_d(p) \leftrightarrow \Theta_p$  and

$$M_d(\Sigma) \leftrightarrow \Theta_\Sigma \text{ sat.}$$

$$\begin{aligned} Z_{DW}(p, s) &:= \sum_{l, r \geq 0} \frac{\vec{P}}{l! r!} \underbrace{\langle \mathcal{O}^l \mathcal{O}(\Sigma)^r \rangle}_{P_W(\mathcal{O}^l \mathcal{O}^r)} \\ &= \frac{1}{2} \Delta^{-\frac{3}{4}(g+5)} \sum_{l, r \geq 0} \frac{p^l s^r}{l! r!} \Delta^{2l+|e|} P_D(p^l \Sigma^r) \end{aligned}$$

$p \in H_0(X)$ ,  $p \in H^0(X)$  dual element

$$\mathcal{O}(\Sigma) = \sum_r \mathcal{O}(\Sigma_r), \quad \mathcal{O}^r \in H^2(X)$$

$$\Sigma^r = \prod_M (\zeta^\mu)^{r_\mu} \quad \text{with} \quad r = \sum r_\mu$$

## Monopole eqns

### Gauge theory

$G$  Lie gp (structure  $\mathfrak{g}$ ) ,  $\mathfrak{g}_y = \text{Lie}(G)$

$P \rightarrow M$  principal  $G$ -bundle

$S = \text{Aut}(P)$  gauge gp

$A = \text{Conn}(P)$

Cartan model  $A \in C^\infty(A, \Omega^1(P; \mathfrak{g})_{\text{hor}}^G)$

$\psi \in \Omega^1(A, \Omega^1(P; \mathfrak{g})_{\text{hor}}^G)$

$A$  gauge field,  $\psi$  superpartner (fermion field)

$\phi \in \text{Hom}(\text{Lie}(S), \text{Lie}(G))$  scalar field

$D$  auxiliary field.

Today  $D \sim F^+ = \mu(M) ?$

Math: consider another rep. of  $G$

Phys: Add another content

We can extend the  $N=1$  theory to  $N=2$

Consider repr. of  $SU(2)_- \boxtimes SU(2)_+ \boxtimes SU(2) \boxtimes U(1)$ ,

Pphys	Math	
Vect. $A_\mu$	$(2, 2, 1)^0$	
Spinor $(\bar{\Psi}_\alpha^A, \Psi_\alpha^A)$	$(1, 2, 2)^{-1} \oplus (2, 1, 2)^1$	
scalar $(\phi, \bar{\phi})$	$(1, 1, 1)^2 \oplus (1, 1, 1)^{-2}$	
$D$		

Vector multiplet  
(VM)

Partition f-n  $Z = \int [d\psi..] e^S$

Want to compute after the topological twist:

~~$SU(2)_+ \times SU(2)_- \times U(1)_R$~~



$$SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1)_R$$

$$(A, B, z) \mapsto (A, B, A, z)$$

After the top. twist spinors become vectors  
(some) scalars spinors

### Hypermultiplets

Take a ex rep. R of G

$$R^\vee = R^* = \text{Hom}(R, \mathbb{C})$$

Define  $W = R \oplus R^\vee$ . On this we define a  
ex. str.  $J$  by the following rule

$$\varphi \in R^\vee, \varphi = \langle \cdot, r_\varphi \rangle \quad J(r, \varphi) = (r_\varphi, -\langle \cdot, r \rangle)$$

With  $I = i$  from  $R$  and  $K = IJ$   
 we gain a quaternionic structure on  $W$ .  Neo smartpen

Instead of  $R \oplus R^\vee$  take  $(r, r_4) = :M \in R \oplus \bar{R}.$

$M$  is transforming in  $R$  (scalar)

$P \xrightarrow{\sim} G$  Define  $\mathcal{R} := P \times_G R$   
 $\downarrow$   
 $X$

Now call  $S_\pm$  the  $SU(2)_\pm$  spin bundles

Together and after the twist we get  $\hat{M} = (M_+, M_-)$

$$(M_+, M_-) \in \Gamma(S^+ \otimes \mathcal{R}) \oplus \Gamma(S^- \otimes \mathcal{R})$$

- If  $X$  is not spin, we need  $w_2(\mathcal{R}) = w_2(X)$
- Even if there is no  $\mathcal{R}$  as bundle, we can construct  $S^+ \otimes \mathcal{R}$ .

Moment map (after twist also auxiliary field)

Recall:  $\forall \xi \in \mathfrak{g} \quad K_\xi \in \mathfrak{X}(X) \quad , \quad K_\xi \stackrel{d}{=} \frac{d}{dt}|_{t=0} (\exp(t\xi))^{-1}$

for  $I, J, K$  we define 2-forms

$$\omega_I(\cdot, \cdot) = \langle I \cdot, \cdot \rangle$$

$$\omega_J(\cdot, \cdot) = \langle J \cdot, \cdot \rangle$$

$$\omega_K(\cdot, \cdot) = \langle K \cdot, \cdot \rangle$$

Since  $i_{K_\xi} \omega_{I, J, K}$  is exact, we have  $\mu_\xi^i$  s.t.  $d\mu_\xi^i = i_{K_\xi} \omega$

Together these define a moment map  $\mu: W \rightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$

Now we can write down the monopole equations (Q fixed pts)

$$\begin{cases} F^+ = \mu(M) \\ \nabla M = 0 \end{cases} \quad \nabla : \Gamma(S^* \otimes R) \rightarrow \Gamma(S \otimes R)$$

Example  $G = U(1)$ , then  $R = L$  line bundle and the monopole eqns are classical Seiberg-Witten eqns.

$$\begin{cases} F_{\alpha\beta}^+ = \bar{M}_{(\alpha} M_{\beta)} \\ \nabla_{\alpha\beta} M^{\beta} = 0 \end{cases}$$

$$\begin{aligned} \varepsilon_{\alpha\beta} M^{\beta} &= M_2 \\ \begin{pmatrix} \text{diag} \\ \text{diag} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} &= \begin{pmatrix} -\mu \\ \mu \end{pmatrix} \end{aligned}$$

Recall: For  $S \sim \frac{1}{e^2} \int$  coupling constant = action

$e$  is dependent on additional data  $\Lambda$ , "energy scale". The evolution of  $e$  is given by  $\beta$ -function:  $\Lambda \frac{d}{d\Lambda} e = \beta$

From physics we know  $\beta$  is non-positive (optically free theory)

In our case

$$\beta \sim -2h^v + c_2(R)$$

Fix generators  $X_i$  of  $\mathfrak{g}$   $c_2(R) = \sum_i x_i x_i$   
So we should have  $2h^v < c_2(R)$

Outlook We can compute Donaldson's invariants with the new theory.



For this we need to map these UV Ideas to IR (low  $\Lambda$  or worms at the fields)

infrared (IR) limit  $\rightarrow$  weak or integrable systems

~~scribble~~

Elliptic curves

▷ all pure or  $G$  is Riemann surface  $= \mathbb{C}/\Lambda$ ,

$\Lambda$  is a lattice  $\Lambda = \mathbb{Z}(a, b)$ ,  $a, b$  rdg. of  $\mathbb{Q}\mathbb{R}$

$$s.t. Ra + Rb = C$$

$$\bullet P_{(2)} = \frac{1}{z^2} + \frac{1}{w^2} - \frac{1}{(z-w)^2} - \frac{1}{w^2}$$

property: meromorphic on  $\mathbb{C}$ , poles at  $w \in \Lambda$ , order 2  
(with mult 0)

$$\bullet P_{(2)}(w) = \delta_{(2)} \quad \forall w \in \Lambda$$

[const.]

$$\bullet 2g_2 = g_2(\Lambda), g_3 = g_3(\Lambda) \text{ s.t.}$$

$$P_{(2)}^4 = 4(P_{(2)}^3 - g_2 P_{(2)} - g_3)$$

$$E = \{(x, y; 1) \in \mathbb{C}\mathbb{P}^2 \mid y^2 = 4x^3 - g_2 x - g_3\}$$

Then The map  $\mathbb{C} \rightarrow E$

$$z \mapsto [P_{(2)}; P_{(2)}'; 1]$$

is bijective and induces  $P_{(2)} \cong E$

(concrectly,  $\forall g_2, g_3 \in \mathbb{C} : A = -16(-4g_2^3 + 27g_3^2) \neq 0$ )

(i.e.  $E$  is smooth)  $\exists \Lambda \subset \mathbb{C} \text{ s.t. } E = \mathbb{C}/\Lambda$

genus  $E = 1 \Rightarrow \dim H^0(E; \mathbb{Q}) \geq 1$   
 $\omega \in \omega \neq 0$  closed ball

line up  $\eta: H_1(E; \mathbb{Z}) \rightarrow \mathbb{C}$

$$x \mapsto e^{2\pi i x}$$

One can show  $H_1(E; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ ,

then  $\eta \circ \eta = \lambda$  is a lattice in  $\mathbb{C}$

$$\eta: E \rightarrow \mathbb{C}, p \mapsto \int_p^{\infty}$$

Reur. 1)  $\omega$  can be given explicitly:  $\omega = \frac{dx}{y}$

$$\Rightarrow \pi_1(\mathbb{C}) = \mathbb{Z} \Rightarrow \pi_1(E) = \lambda = H_1(E; \mathbb{Z})$$

$$H_1(E; \mathbb{Z}) \times H_1(E; \mathbb{Z}) \rightarrow \mathbb{Z}$$

$(a, b) \mapsto$  sum of signs of oriented  
intersection points

$\Rightarrow \lambda \& \lambda'$  are equivalent  $\Leftrightarrow \exists b \in \mathbb{C}^*: \lambda = b \cdot \lambda'$

$$\lambda = \langle a, b \rangle = a^\top \langle \alpha, 1, \beta \rangle = \langle \lambda, \gamma \rangle$$

it is oriented  $\Leftrightarrow b \in \mathbb{H} = \{b \in \mathbb{C}: \operatorname{Im} b > 0\}$

$$\lambda_2 \in \lambda_1 \Leftrightarrow \exists A \in \operatorname{SL}_2 \mathbb{Z}, \tau = Az = \frac{az + b}{cz + d}$$

Prop

$\operatorname{SL}_2 \mathbb{Z}$  classifies elliptic curves over  $\mathbb{C}$

$\mathbb{C}$

Rem We could define all. curves by

$$y^2 = P_3(x), \deg P_3 = 3$$

We could do other normal forms for  $P_3$ , e.g.

$$\text{Bis } P_3(x) = x(x-1)(x-w)$$

Special Kähler str.

( $\mathbb{H}, \mathbb{I}$ ) or inf

► Sp. Kähler str. is special ( $M, g_M, \mathbb{I}, \mathbb{D}$ ) etc.

1)  $(g_M, \mathbb{I})$  Kähler str.

2)  $\nabla$  conn (not necessarily  $\nabla^{\perp}$ )  $\nabla_{\partial} = 0$ , flat  
beside free

$$3) (\nabla_{\mathbb{I}})^{\perp} X = (\nabla_X \mathbb{I}) X$$

►  $(x_1, \dots, x_n, y_1, \dots, y_m)$  flat Darboux sys.

1)  $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$

$$2) \nabla dx_j = \nabla dy_j = 0$$

Prop  $z_j$  fibration flat  $z_j$  onto  $\mathbb{R}^m$  &  $w_j$  on  $\mathbb{H}$

$$\text{Re } z_j = x_j$$

$$\text{Re } w_j = y_j$$

$(z_1, \dots, z_n)$  loc. coord.  $\mathbb{R}^n$   $(w_1, \dots, w_m)$  sp.-loc. coord.  
sp. loc. coord.

(2), (6) unique sp. loc. coord.,  $z_{ij} = \frac{\partial z_i}{\partial z_j}$

$z_{ij} = \bar{z}_{ji} \Rightarrow \exists f(z_1, \dots, z_n) \text{ s.t. } y_j = \frac{\partial f}{\partial z_j}$   
† loc. prepotentiel

$\omega = \frac{i}{2} \sum_{i,j} \text{Im } z_{ij} dz_i d\bar{z}_j,$   
(Im  $z_{ij}$ ) is pos. def.

$\mathbb{P}$  alg. integrable sys:

$\pi: \Sigma^n \rightarrow M$  bolo,  $\mathbb{R}M$  canfd

a)  $\Sigma$  is a conf,  $\eta \in \Omega^{2,0}(\Sigma)$  ex sympl.

$$(\Rightarrow \eta^M \neq 0)$$

b)  $Z_m = \pi^*(m)$  - ex Lagrangian submf  
cpt ex boms

c)  $\exists$  a family  $\{\rho_i\}_{i=1}^n$   $\subset H^1(Z_m) \cap H^2(Z_m; \mathbb{R})$   
positive polarization of  $Z_m$

Slogan: "alg. intg. systems" = "sp. holo sh. on  $\Sigma$ "

Then the base of any alg. intg. system carries  
a sp. holo sh.

$n=1$  (is enough for us)  $\Sigma$  ex surface

$M$  Riemann surface

$$p \in \Sigma, V_p = T_p \Sigma \cong_{\text{holo}} \text{vertical span}$$

$$\cong \ker D_p \pi$$

say sympl. form  $\eta$  does

$$V_p \xrightarrow{\cong} T_p M, \omega: T_p M \rightarrow V_p$$

$$V_p \xrightarrow{\cong} \eta(V_p)$$

$$f: M \rightarrow \mathbb{C}, \eta((\text{conf}), \Sigma) = \pi^* df(\Sigma)$$

$$\exp: T_p M \rightarrow \Sigma$$

$$V_p$$

$N_m = \text{Ker-Slab } p, p \in \text{"canfd"}$

Fact:  $T_m M, N_m$  is orthogonal to  $T_p M$ .

$$\lambda \subset T_p M, \text{st } N_m = \lambda \cap T_p M$$

$\lambda$  is a Lagrangian submf wth the canonical sympl str.

There  $M$  close  $U_m$  holds  $n=1$

$(x_1, \dots, x_m)$  s.t.  $x_i(x) \in \Lambda^m$

1 form local triv of  $T^*M$  on  $U_m$

Defin  $\nabla$  by  $\nabla x_j = 0$  - flat on  $T^*M$

as  $\Lambda^k T^*M$  is  $\text{Span} \cong \nabla$  basis free

$\text{Hol}_\nabla \subset \text{Spf } \Lambda^k \Lambda_m (\cong \text{SL}_2 \mathbb{Z} \text{ if } n=1)$

(\*) gives [good] s.t.  $[p_m]$  is gen by  $H^2(\mathbb{P}^m \mathbb{Z})$

Think of  $p$  as a 2-form on  $\mathbb{Z}$  s.t.

$p|_W$  is an invrnt volume form

$T^*M = V$  yields a sym. str on  $T_m M$

$T_m^*M$

Skew-sym. integer-valued bilin map  $\Lambda^m \times \Lambda_m \rightarrow \mathbb{Z}$

$\langle \cdot, \cdot \rangle$

Prop 3 Darboux coord. on  $\Lambda_m$ , i.e.

$x_1, \dots, x_n, \delta_1, \dots, \delta_n$  s.t.  $\langle x_i, x_j \rangle = \langle \delta_i, \delta_j \rangle = \delta_{ij} 0$   
 $\langle \delta_i + \delta_j \rangle = \overline{\delta_j}$

$\Lambda^k T^*M$  basis  $= x_j = dx_j, \delta_j = dy_j$

Define  $w = dx_1 dy_1 + \dots + dx_n dy_n$

Thm 1 ( $g = w(\cdot, \cdot), I, \omega, \nabla$ ) is special kahler

Thm 2 For any sp.kahler on  $M$   $\exists$  ~~an~~ dg. integr. pt.

which bears the sp.kahler from Thm 1

provided that  $V \subset \text{SL}_2 \mathbb{Z}$

How do we get sp. loc. coord?

$$\text{Put: } dz := \int_{\gamma} \eta \quad d\omega := - \int_{\gamma} \eta$$

The pentagon example:  $M = \mathbb{C} \setminus \{\pm 2\lambda^3 y, \lambda \in \mathbb{C}\}$   
may change  $\lambda = 1$

$$Z_w = \{y^2 = z^3 - 3\lambda^2 z + w\} \text{ smooth on } M$$

$$\lambda = ydz, z = \{y, z, w\} \mid \dots y \in \mathbb{C}^{2 \times 2}$$

$$\lambda_w = \lambda_w(\lambda_w; \mathbf{z}) \text{ lattice in } \mathbb{C}$$

Choose  $x_1, x_2$ ,  $\langle x_i, x_j \rangle = \pm 1$

Nearest  $z_c = \sqrt[3]{\lambda}$  is sp. loc. coord.

## ~~ghosts~~ Ghosts

Cartan model of equiv. cohomology  $\mathbb{C}^{\times}V \rtimes G$

$(S(y^*) \otimes \Omega^*(V))^G$  cohomology of this  
are equiv. cohomol. of  $V$ .

$$(d - i_y) \sum p_i(\xi) \otimes d_i = \sum (p_i(\xi) \otimes dd - p_i(\xi) \otimes i_y)$$

Consider the case when  $V$  is a vector space  
equipped with a  $G$ -invariant metric

$$\rho: G \rightarrow \text{Aut}(V)$$

$$\hat{\rho}: \mathfrak{g}^* \rightarrow \text{End}(V)$$

$$dim V = m$$

$$dim G = d$$

$G$  compact Lie gp.

$$U := \frac{1}{\pi^m} e^{-\|x\|^2} \int (e^{x^t \hat{\rho}(\xi) y} + i dx^t)^{-1} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \\ y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$U := \pi^{-m} e^{-\|x\|^2} \int \exp\left(\frac{1}{4} x^t \hat{\rho}(\xi) y + i dx^t y\right) \mathcal{D}x$$

$U$  is the Mathai-Quillen element

$P$  a conn

$$\downarrow \quad a \in \Omega^1(P; \mathfrak{g})^G, \quad a(k^\xi) = \xi$$

$$F_a = da + \frac{1}{2} [a, a] \in \Omega^2(P; \mathfrak{g})^G$$

If we have a metric on  $P$ , we can define  
a by the orthogonal complement construction.

Define  $\nu \in \Omega^1(P; \mathfrak{g})^*$  via

$$\langle \nu(v), \xi \rangle := g(K^\xi, v)$$

Then  $\langle \nu(K^\eta), \xi \rangle = g(K^\eta, K^\xi)$   
"  $\langle S\gamma, \xi \rangle$  for some  $S = S_p$

$S^{-1}v_\tau = a_p$  is a connection

$$d_a = d(S^{-1}\nu) = dS^{-1} \cdot \nu + S^{-1}d\nu$$

Need to know  $d_a|_{sp} = f_a|_{sp}$

$$\cancel{dS^{-1}\nu|_{sp}} + S^{-1}(d\nu|_{sp})$$

Fourier-transform

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int f(x) e^{ix\xi} dx$$

$$f(x) = \frac{1}{\sqrt{\pi}} \int e^{-i \frac{x}{\lambda} \xi} e^{ix\xi} \hat{f}(\xi) d\xi$$

$b \approx b/\lambda$   
 $\xi \approx \xi/\lambda$

$$x \in \mathbb{R}^n, \xi \in \mathbb{R}^n$$

$$\langle \nu S^{-1}b \rangle = (\det S)^{-1} \int e^{-i \langle b, \xi \rangle} e^{i \langle x, S\xi \rangle} \hat{f}(\xi) d\xi$$