Differential Geometry I

Lecture notes

Andriy Haydys

October 11, 2021

This is a draft. If you spot a mistake, please e-mail me: andriy(DOT)haydys@ulb.be.

Contents

Chapter 1

Introduction

A substantial part of mathematics is related to solving equations of various types. Given any equation, we may try to analyze this by studying the following sequence of questions:

1. Does there exist a solution (a root)?
2. If the answer to the previous question is affirmative, how many solutions does the equation have?
3. If there are finitely many solutions, can we find all of them?

For example, the reader learned at school the properties of the quadratic equation $ax^2+bx+c=0$. In this case the above questions are easy to settle and the answers are well known to the reader.

Sometimes an equation may have an infinite number of solutions. If there are only countably many roots, the last question from the list above still makes sense. For example, all solutions of the equation $\sin x = 0$ are given by a simple formula: $x_n = \pi n,\ n \in \mathbb{Z}$.

In many cases, however, equations have uncountably many solutions so that asking to find all solutions is not really meaningful. Instead, it turns out to be more interesting to replace Question 3 by the following one:

3′. What are the properties of the set of *all solutions*?

Which particular properties we are interested in may depend on the context. The property most relevant to the content of this course is concerned with the local structure of the set of all solutions.

Let us consider an example. The equation

$$x_1^2 + x_2^2 + x_3^2 = 1, \tag{1.1}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, clearly has uncountably many solutions.

Denote $S^2 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, that is S^2 is the set of all solutions of (1.1). Of course, S^2 is the sphere of radius 1, however let us pretend for a moment that we do not know this. As a subset of $\mathbb{R}^3$, S^2 is a topological space. It turns out that this topological space has a very particular property, which we consider in some detail next.

The familiar stereographic projection from the north pole $N := (0,0,1)$ is given by

$$\varphi_N \colon S^2 \setminus \{N\} \to \mathbb{R}^2, \qquad \varphi_N(x) = \Big(\frac{x_1}{1-x_3},\ \frac{x_2}{1-x_3}\Big).$$

This is in fact a homeomorphism with the inverse

$$\varphi_N^{-1}(y) = \frac{1}{1+y_1^2+y_2^2}\big(2y_1,\ 2y_2,\ -1+y_1^2+y_2^2\big), \qquad y = (y_1, y_2) \in \mathbb{R}^2.$$

We can also define a stereographic projection from the south pole $S := (0, 0, -1)$ by

$$\varphi_S\colon S^2 \setminus \{S\} \to \mathbb{R}^2, \qquad \varphi_S(x) = \Big(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}\Big),$$

which is also a homeomorphism.

Since any point on the sphere lies either in $S^2 \setminus \{N\}$ or $S^2 \setminus \{S\}$ (or both), any point on the sphere has a neighbourhood, which is homeomorphic to an open subset of $\mathbb{R}^n$ (of course, $n = 2$ in our particular example and the open subset is $\mathbb{R}^2$ itself). This property leads to the notion of *a manifold*, which will play a cenral rôle in the course. We will see below, that this property is *not* specific to Equation (1.1). On the contrary, for any smooth map $F\colon \mathbb{R}^k \to \mathbb{R}^\ell$ and almost any $c \in \mathbb{R}^\ell$ the set of all solutions to the equation $F(x) = c$ is a manifold. That is, there is a huge pull of examples of manifolds and many objects of particular interest in mathematics turn out to be manifolds.

Coming back to our example, we compute:

$$\varphi_S \circ \varphi_N^{-1}(y) = \Big(\frac{y_1}{|y|^2}, \frac{y_2}{|y|^2}\Big). \tag{1.2}$$

Hence, $\varphi_S \circ \varphi_N^{-1}$ is *smooth* on an open subset $\mathbb{R}^2 \setminus \{0\}$ and a similar computation yields that this is also true for $\varphi_N \circ \varphi_S^{-1}$. This property can be used to study smooth functions on the sphere directly without reference to the ambient space. More importantly, in more general situations where the ambient Euclidean space may be simply absent, an analogue of this property allows one to apply familiar tools of analysis to functions defined on more sophisticated objects than just subsets of an Euclidean space. In some sense, this constitutes the core of differential geometry.

Summing up, the aim of these notes is to transfer familiar tools of mathematical analysis to a more geometric setting where the underlying domain of a function (map) is not just an open subset of $\mathbb{R}^n$, but rather a manifold. The benefits of doing so are ubiquitous, but explaining this in some detail requires a bit of work. It is my hope to convey that the notion of a manifold is useful and well worth studying further.

Chapter 2

Smooth manifolds

2.1 Basic definitions and examples

Recall that a topological space M is called Hausdorff, if for any two distinct points $m_1, m_2 \in M$ there are neighbourhoods $U_1 \ni m_1$ and $U_2 \ni m_2$ such that $U_1 \cap U_2 = \varnothing$. If the topology of M admits a countable base, then M is said to be second countable. For example, $\mathbb{R}^k$ is both Hausdorff and second countable.

Definition 2.1. A Hausdorff second countable topological space M is called *a topological manifold* of dimension k, if M is locally homeomorphic to $\mathbb{R}^k$.

To explain, this means that any point $m \in M$ admits a neighbourhood U and a homeomorphism $\varphi\colon U \to V$, where V is an open subset of $\mathbb{R}^k$. The pair (U, φ) (or, sometimes just U) is called a chart on M near m.

Notice that the requirements that a manifold is Hausdorff and second countable are to a great extent of technical nature, whereas being locally homeomorphic to $\mathbb{R}^k$ is a crucial property of manifolds.

Clearly, $\mathbb{R}^k$ and in fact any open subset of $\mathbb{R}^k$ are examples of topological manifolds of dimension k. As we have established in the introduction, 2-spheres are manifolds of dimension two. Similar arguments yield in fact that the k-sphere

$$S^k := \Big\{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid \sum_{j=1}^{k+1} x_j^2 = 1\Big\}$$

is a k-manifold.

Somewhat special is the case of dimension zero. Since $\mathbb{R}^0$ is by definition a single point, the above definition requires that each point of M has a neighborhood consisting only of this point. In other words, M is a countable discrete space.

Definition 2.2. A collection of charts $\mathcal{U} = \{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is called a C^0-*atlas*, if $\bigcup_{\alpha \in A} U_\alpha = M$, that is if any point of M is contained in some chart. Here A is an arbitrary index set.

For example, $\mathbb{R}^k$ admits a C^0-atlas consisting of a single chart $(\mathbb{R}^k, id)$. In the introduction we have constructed a C^0-atlas on the 2-sphere consisting of two charts. However, there is no C^0-atlas on S^2 consisting of a single chart, since S^2 is not homeomorphic to an open subset of $\mathbb{R}^2$ (why?).

Given a C^0-atlas $\mathcal{U}$, pick any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) such that $U_\alpha \cap U_\beta \neq \varnothing$. The map

$$\theta_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta), \tag{2.3}$$

which a homeomorphism between two open subsets of $\mathbb{R}^k$, is called *a coordinate transformation*[1]. Notice that $\theta_{\beta\alpha}$ is the inverse map to $\theta_{\alpha\beta}$. In particular, $\theta_{\alpha\beta}$ is a homeomorphism between $\varphi_\beta(U_\alpha \cap U_\beta)$ and $\varphi_\alpha(U_\alpha \cap U_\beta)$.

It is a common practice to suppress the domain and the target of $\theta_{\alpha\beta}$ writing simply $\theta_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. While this may be confusing at first, the advantage is that this allows us to suppress less important details so that the most essential features are clearer. If in doubt, the reader should write the domain and target explicitly.

Definition 2.4. A C^0-atlas $\mathcal{U}$ is called *smooth*, if all coordinate transformation maps $\theta_{\alpha\beta}$, $\alpha, \beta \in A$, are smooth.

Remark 2.5. Equally well, we can say that $\mathcal{U}$ is a C^ℓ-atlas, if all coordinate transformation maps belong to $C^\ell(\mathbb{R}^n; \mathbb{R}^n)$ (keep in mind that these are defined on open subsets of $\mathbb{R}^n$ only) for some fixed natural number ℓ. The theory does not depend much on the choice of ℓ as long as ℓ is not too small. In practice $\ell \geq 3$ would suffice in most of the cases, however to avoid non-essential details it is convenient to put $\ell = \infty$ from the very beginning.

Two charts (U, φ) and any (V, ψ) not necessarily from the same atlas are said to be *smoothly compatible* if the maps

$$\varphi \circ \psi^{-1} \qquad \text{and} \qquad \psi \circ \varphi^{-1} \tag{2.6}$$

are smooth, compare with (2.3). We consider two atlases $\mathcal{U}$ and $\mathcal{V}$ as "essentially equal", if all charts from $\mathcal{U}$ are smoothly compatible with all charts in $\mathcal{V}$. More formally, we have the following definition.

Definition 2.7. Two atlases $\mathcal{U}$ and $\mathcal{V}$ on the same underlying topological space M are called equivalent, if $\mathcal{U} \cup \mathcal{V}$ is a smooth atlas on M, that is if all charts from $\mathcal{U}$ are smoothly compatible with all charts in $\mathcal{V}$. An equivalence class of atlases is called *a smooth structure* on M. *A smooth manifold* consists of a Hausdorff second countable topological space and a smooth structure.

To explain the point of the above definition, consider the 2-sphere. In the introduction we constructed a smooth atlas on S^2, namely $\mathcal{U} := \big\{(S^2 \setminus \{N\},\ \varphi_N),\ (S^2 \setminus \{S\},\ \varphi_S)\big\}$. However, there are many ways to construct another smooth atlas, for example as follows:

$$\mathcal{U}' := \big\{S^2 \setminus \{N\}, \varphi_N\big\} \cup \big\{(S^2_+, \varphi_+)\big\}.$$

Here $S^2_+ := \big\{x \in S^2 \mid x_3 > 0\big\}$ and $\varphi_+(x) = (x_1, x_2)$.

Exercise 2.8. Check that $\mathcal{U}'$ is a smooth atlas equivalent to $\mathcal{U}$.

It should be intuitively clear, that the description of S^2 via smooth atlases $\mathcal{U}$ and $\mathcal{U}'$ are 'essentially equal'. Hence, it is natural to identify $(S^2, \mathcal{U})$ and $(S^2, \mathcal{U}')$.

An atlas $\mathcal{U}$ is called *maximal*, if for any chart (V, ψ) smoothly compatible with all charts in $\mathcal{U}$ is already contained in $\mathcal{U}$.

The importance of maximal atlases stems from the following result.

Lemma 2.9. *Each equivalence class of smooth atlases is represented by a unique maximal atlas.*

[1]The origin of this terminology will be clear below.

Proof. For a smooth atlas $\mathcal{U}$ on M define

$$\mathcal{U}_{max} := \big\{(V,\psi) \text{ is a chart on } M \text{ s.t. (2.6) are both smooth for all } (U,\varphi) \in \mathcal{U}\big\}.$$

Exercise 2.10. Check that $\mathcal{U}_{max}$ is a smooth atlas on M.

By the construction of $\mathcal{U}_{max}$, we have $\mathcal{U} \subset \mathcal{U}_{max}$. Hence, any chart smoothly compatible with any chart in $\mathcal{U}_{max}$ is also smoothly compatible with any chart in $\mathcal{U}$ and therefore is contained in $\mathcal{U}_{max}$. Hence, $\mathcal{U}_{max}$ is maximal. Clearly, $\mathcal{U}$ and $\mathcal{U}_{max}$ represent the same smooth structure. □

By the above lemma, a smooth manifold may be considered as being equipped with a maximal atlas. In particular, if $\mathcal{U}$ is any smooth atlas on M, we may freely add any chart smoothly compatible with all charts in $\mathcal{U}$ without changing the smooth structure. For example, if (U,φ) is a chart near m_0, then $(U,\hat\varphi)$ with

$$\hat\varphi(m) = \varphi(m) - \varphi(m_0)$$

is also a chart near $m_0 \in M$ smoothly compatible with all charts in $\mathcal{U}$. The chart $(U,\hat\varphi)$ satisfies

$$\hat\varphi(m_0) = 0,$$

which is commonly expressed by saying that $(U,\hat\varphi)$ is *centered at* m_0.

Remark 2.11. In what follows only smooth manifolds will be considered. Therefore, by saying that M is a manifold, we always mean a *smooth* manifold, unless explicitly stated otherwise.

Let us finish this section with some further examples of manifolds.

Example 2.12 (Products)**.** Let M and N be smooth manifolds of dimensions k and ℓ respectively. Let $\mathcal{U} = \big\{(U_\alpha,\varphi_\alpha) \mid \alpha \in A\big\}$ and $\mathcal{V} = \big\{(V_\lambda,\psi_\lambda) \mid \lambda \in \Lambda\big\}$ be smooth atlases on M and N respectively. Then the product $M \times N$ is a Hausdorff second countable topological space. We define a C^0-atlas on $M \times N$ by setting

$$\mathcal{W} := \Big\{\big(U_\alpha \times V_\lambda,\ \varphi_\alpha \times \psi_\lambda\big) \mid \alpha \in A,\ \lambda \in \Lambda\Big\}.$$

Given any two charts $\big(U_\alpha \times V_\lambda,\ \varphi_\alpha \times \psi_\lambda\big)$ and $\big(U_\beta \times V_\mu,\ \varphi_\beta \times \psi_\mu\big)$ the corresponding coordinate transformation is given by $\theta_{\alpha\beta} \times \eta_{\lambda\mu}$, where $\theta_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ and $\eta_{\lambda\mu} = \psi_\lambda \circ \psi_\mu^{-1}$ are smooth maps. More precisely, this means the following:

$$\begin{aligned} \theta_{\alpha\beta} \times \eta_{\lambda\mu} \colon \mathbb{R}^k \times \mathbb{R}^\ell &\to \mathbb{R}^k \times \mathbb{R}^\ell, \\ \theta_{\alpha\beta} \times \eta_{\lambda\mu}(x,y) = \big(\theta_{\alpha\beta}(x),\ \eta_{\lambda\mu}(y)\big), &\quad x \in \mathbb{R}^k,\ y \in \mathbb{R}^\ell. \end{aligned}$$

In particular, $\theta_{\alpha\beta} \times \eta_{\lambda\mu}$ is a smooth map, which means that the atlas constructed above is smooth. Hence, $M \times N$ is a smooth manifold of dimension $k+\ell$. This yields in particular that the following

(i) the k-dimensional torus $\mathbb{T}^k := S^1 \times \cdots \times S^1$ and

(ii) the cylinder $\mathbb{R} \times S^1$

are smooth manifolds. In the latter case, the dimension of $\mathbb{R} \times S^1$ equals 2.

Example 2.13 (Real projective spaces)**.** The real projective space $\mathbb{RP}^k$ of dimension k is defined to be the set of all lines in $\mathbb{R}^{k+1}$ through the origin. Since each line through the origin is uniquely determined by a point on this line distinct from the origin, we have

$$\mathbb{RP}^k = \big(\mathbb{R}^{k+1} \setminus \{0\}\big)/\sim,$$

where $x, y \in \mathbb{R}^{k+1} \setminus \{0\}$ are defined to be equivalent if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $y = \lambda x$. In particular, we have the canonical surjective quotient map

$$\pi\colon \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{RP}^k, \qquad \pi(x) = [x].$$

If $x = (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\}$, it is customary to write $[x_0 : x_1 : \dots : x_k]$ for $[x]$.

We endow $\mathbb{RP}^k$ with the quotient topology, that is $U \subset \mathbb{RP}^k$ is open if and only $\pi^{-1}(U)$ is open in $\mathbb{R}^{k+1} \setminus \{0\}$. It is straighforward to check that this yields a Hausdorff second countable topological space.

To construct a C^0-atlas on $\mathbb{RP}^k$, observe that each

$$U_j := \big\{[x_0 : x_1 : \dots : x_k] \in \mathbb{RP}^k \mid x_j \neq 0\big\}, \qquad j = 0, 1, \dots, k,$$

is an open subset of $\mathbb{RP}^k$. Indeed, this follows from the fact that

$$\pi^{-1}(U_j) = \big\{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} \setminus \{0\} \mid x_j \neq 0\big\}$$

is an open subset of $\mathbb{R}^{k+1} \setminus \{0\}$.

The map

$$\begin{aligned}
&\varphi_j\colon U_j \to \mathbb{R}^k,\\
&\varphi_j\big[x_0 : x_1 : \dots : x_{j-1} : x_j : x_{j+1} : \dots : x_k\big] = \Big(\frac{x_0}{x_j}, \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_k}{x_j}\Big)
\end{aligned}$$

is well-defined and continuous. Moreover, the map

$$\psi_j\colon \mathbb{R}^k \to U_j, \qquad \psi_j\big(y_0, y_1, \dots, y_{k-1}\big) = \big[y_0\colon y_1\colon \dots : y_{j-1}\colon 1\colon y_j\colon \dots : y_{k-1}\big] \tag{2.14}$$

is a continuous inverse of φ_j, that is φ_j is a homeomorphism. Since the collection $U_0, \dots, U_k$ clearly covers all of $\mathbb{RP}^k$, $\mathcal{U} := \big\{(U_j, \varphi_j) \mid j = 0, 1, \dots, k\big\}$ is a C^0-atlas on $\mathbb{RP}^k$.

Next, let us consider the coordinate transformations. To simplify the notations we consider only the map $\theta_{01} = \varphi_0 \circ \varphi_1^{-1} = \varphi_0 \circ \psi_1$. We have

$$\theta_{01}(y_0, \dots, y_{k-1}) = \varphi_0\big([y_0 : 1 : y_1, \dots, y_{k-1}]\big) = \Big(\frac{1}{y_0}, \frac{y_1}{y_0}, \dots, \frac{y_{k-1}}{y_0}\Big),$$

which is smooth on

$$\varphi_1\big(U_0 \cap U_1\big) = \big\{y \in \mathbb{R}^k \mid y_0 \neq 0\big\}.$$

A similar argument yields that all coordinate transformations $\theta_{ij} = \varphi_i \circ \psi_j$ are smooth on their domains of definition. Thus, $\mathcal{U}$ is a smooth atlas and $\mathbb{RP}^k$ is a smooth manifold of dimension k.

It may be useful to keep some non-examples of manifolds in mind.

(a) The set $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$ consisting of two straight lines $y = \pm x$ intersecting at the origin, is not a manifold. Indeed, if M were a manifold, its dimension must be one. However, the origin does not have a neighbourhood in M homeomorphic to an open subset of $\mathbb{R}^1$ (Why?).

(b) A disjoint union of manifolds is a manifold. However, a disjoint uncountable union of non-empty manifolds is *not* a manifold, since the second countability axiom is violated. For example,

$$N := \bigsqcup_{\alpha \in (0,1)} \mathbb{R}_\alpha$$

is not a manifold. Notice that the above example is *not* homeomorphic to $(0,1) \times \mathbb{R}$, which is a manifold indeed, since, for example, each line $\mathbb{R}_\alpha \subset N$ is an open subset.

(c) Consider the following line "with a double point":

$$L := (-\infty, 0) \cup \{a, b\} \cup (0, +\infty).$$

Here $\{a, b\}$ is understood as a set consisting of two distinct elements. The following two subsets

$$U_a := (-\infty, 0) \cup \{a\} \cup (0, +\infty) \qquad \text{and} \qquad U_b := (-\infty, 0) \cup \{b\} \cup (0, +\infty)$$

cover all of L. Define $\varphi_a \colon U_a \to \mathbb{R}$ by $\varphi_a(x) = x$ if $x \neq a$ and $\varphi_a(a) = 0$. By the same token we can define $\varphi_b \colon U_b \to \mathbb{R}$.

A topology on L is defined simply by saying that V is open if and only if $\varphi_a(V \cap U_a)$ and $\varphi_b(V \cap U_b)$ are open in $\mathbb{R}$.

This yields a second countable topological space with a smooth atlas. However, L is non-Hausdorff.

L 1

2.2 Smooth maps

Given a smooth structure on M, we can make sense of smoothness of functions defined on M as follows.

Definition 2.15. Let M be a manifold with a smooth structure represented by a smooth atlas $\mathcal{U} = \{(U_\alpha, \varphi_a)\}$. A function $f \colon M \to \mathbb{R}$ is said to be smooth, if for any chart (U_α, φ_a) the function $f \circ \varphi_\alpha^{-1} \colon \mathbb{R}^k \to \mathbb{R}$ is smooth.

Notice that since an open subset V of M is again a smooth manifold, it makes sense to say that a function is smooth on V. The smoothness of functions is then a local property in the following sense: f is smooth if and only if the restriction of f to any open subset of M is smooth. In particular, if $\{V_\alpha \mid \alpha \in A\}$ is an open covering of M and f is smooth on each V_α, then f is smooth on M.

Strictly speaking, we still have to show that the notion of smoothness in Definition 2.15 is independent of the choice of an atlas. Indeed, assume that f is smooth with respect to $\mathcal{U}$ and pick an atlas $\mathcal{V} = \{(V_\mu, \psi_\mu)\}$ equivalent to $\mathcal{U}$. Then on $\psi_\mu(U_\alpha \cap V_\mu) \subset \mathbb{R}^n$ we have

$$f \circ \psi_\mu^{-1}\big|_{\psi_\mu(U_\alpha \cap V_\mu)} = f \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \psi_\mu^{-1}\big|_{\psi_\mu(U_\alpha \cap V_\mu)} = f \circ \varphi_\alpha^{-1} \circ \theta_{\alpha\mu}\big|_{\psi_\mu(U_\alpha \cap V_\mu)},$$

where $\theta_{\alpha\mu} = \varphi_\alpha \circ \psi_\mu^{-1}$ is a smooth map. Hence, f is smooth with respect to $\mathcal{V}$ on any subset $U_\alpha \cap V_\mu$. Since these subsets cover all of M, f is smooth on M with respect to $\mathcal{V}$.

Example 2.16. Let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be any smooth function. Define $f\colon S^2 \to \mathbb{R}$ as the restriction of F to S^2. I claim that f is a smooth function on S^2. Indeed, let (U, φ) be any chart on S^2 constructed in the introduction. For concreteness, let us pick the chart $\big(S^2 \setminus \{N\}, \varphi_N\big)$. Then

$$f \circ \varphi_N^{-1}(y_1, y_2) = F\Big(\frac{2y_1}{1+y_1^2+y_2^2}, \frac{2y_2}{1+y_1^2+y_2^2}, \frac{-1+y_1^2+y_2^2}{1+y_1^2+y_2^2}\Big), \qquad (y_1, y_2) \in \mathbb{R}^2.$$

Hence, $f \circ \varphi^{-1}$ is smooth and it is clear that this is also the case for $\big(S^2 \setminus \{S\}, \varphi_S\big)$. Thus, f is a smooth function on S^2.

Example 2.17. Let $F\colon \mathbb{R}^{k+1} \setminus \{0\} \to \mathbb{R}$ be a smooth homogeneous function of degree 0, that is $F(\lambda x) = F(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{k+1} \setminus \{0\}$. Define $f\colon \mathbb{RP}^k \to \mathbb{R}$ by setting $f\big([x]\big) = F(x)$. This yields a well-defined function, which I claim is smooth. Indeed, pick any chart (U_j, φ_j) constructed in Example 2.13. Using (2.14), we obtain

$$f \circ \varphi_j^{-1}(y_0, \dots, y_{k-1}) = F(y_0, \dots, y_{j-1}, 1, y_j, \dots, y_{k-1}),$$

which is smooth everywhere on $\mathbb{R}^k$. Hence, f is smooth.

Proposition 2.18. *The set $C^\infty(M)$ of all smooth functions on a manifold M is an algebra, that is*

- $f, g \in C^\infty(M)$, $\lambda, \mu \in \mathbb{R} \implies \lambda f + \mu g \in C^\infty(M)$;
- $f, g \in C^\infty(M) \implies f \cdot g \in C^\infty(M)$.

Proof. Let f, g be any two smooth functions and λ, μ two real numbers. For any chart (U, φ) the functions

$$(\lambda f + \mu g) \circ \varphi^{-1} = \lambda\big(f \circ \varphi^{-1}\big) + \mu\big(g \circ \varphi^{-1}\big),$$
$$(f \cdot g) \circ \varphi^{-1} = f \circ \varphi^{-1} \cdot g \circ \varphi^{-1}$$

are clearly smooth, hence $\lambda f + \mu g$ and $f \cdot g$ are smooth functions on M. □

Let $f\colon M \to \mathbb{R}^\ell$ be a map, which can be written as an ℓ-tuple of functions: $f = (f_1, \dots, f_\ell)$. We say that f is smooth, if each component f_j is a smooth function on M.

It is also possible to define the notion of smoothness for maps between manifolds. To this end, let M and N be two manifolds of dimensions k and ℓ respectively. Pick an atlas $\mathcal{U}$ on M and an atlas $\mathcal{V}$ on N.

Definition 2.19. A continuous map $f\colon M \to N$ is said to be smooth, if for any $(U, \varphi) \in \mathcal{U}$ and any $(V, \psi) \in \mathcal{V}$ the map

$$\psi \circ f \circ \varphi^{-1}\colon \mathbb{R}^k \to \mathbb{R}^\ell$$

is smooth.

Remark 2.20. The requirement that f is continuous in the above definition is only needed to ensure that $\psi \circ f \circ \varphi^{-1}$ is defined on an open subset of $\mathbb{R}^k$. The map $\psi \circ f \circ \varphi^{-1}$ is called *the coordinate presentation of f* (with respect to charts (U, φ) and (V, ψ)).

The argument used to verify that the notion of smoothness of a function is well-defined is very common in the theory of manifolds and will be typically omitted below. However, the reader may wish to prove the following proposition as an exercise.

Proposition 2.21. *If $f\colon M \to N$ and $g\colon N \to L$ are smooth maps between smooth manifolds, then $g \circ f$ is also smooth.* □

Definition 2.22. A smooth map $f\colon M\to N$ such that $f^{-1}\colon N\to M$ exists and is also smooth is called *a diffeomorphism.*

Observe that if (U,φ) is a chart, then $\varphi\colon U\to\varphi(U)\subset\mathbb{R}^k$ is a diffeomorphism.

To obtain a somewhat non-trivial example of a diffeomorphism, consider the tangent function:

$$\tan\colon\big(-\pi/2,\ \pi/2\,\big)\to\mathbb{R},\qquad \tan(x)=\frac{\sin x}{\cos x}.$$

This is smooth, bijective, the inverse function $\arctan$ exists and is smooth. Hence the interval $(-\pi/2,\ \pi/2)$ is diffeomorphic to $\mathbb{R}$. In fact any open interval is diffeomorphic to $\mathbb{R}$ (Why?).

A standard non-example is given by the map

$$f\colon\mathbb{R}\to\mathbb{R},\qquad f(x)=x^3,$$

which is clearly smooth and bijective. The inverse map however fails to be smooth at the origin so that f is *not* a diffeomorphism.

If there exists a diffeomorphism between M and N, we say that M and N are diffeomorphic. Notice that in this case we must have $k=\dim M=\dim N=\ell$. Indeed, if f is a diffeomorphism between M and N, then $F:=\psi\circ f\circ\varphi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^k$ and $\mathbb{R}^\ell$. Let us denote $G:=F^{-1}=\varphi\circ f^{-1}\circ\psi^{-1}$, which is smooth by the definition of smoothness for f^{-1}. Since $G\circ F=\mathrm{id}_{\mathbb{R}^k}$, for the differentials DG and DF we have

$$D_xG\cdot D_{G(x)}F=D_x(\,id_{\mathbb{R}^k})=id_{\mathbb{R}^k},$$

where $id_{\mathbb{R}^k}$ is the identity map. In particular, DG is surjective at each point. Furthermore, by a similar argument applied to the identity $F\circ G=\mathrm{id}_{\mathbb{R}^\ell}$, we obtain that DG is injective at each point. In other words, $D_yG\colon\mathbb{R}^\ell\to\mathbb{R}^k$ is a linear isomorphism, which is only possible if $k=\ell$.

Definition 2.23. A map $f\colon M\to N$ is called *a local diffeomorphism*, if for any point $m\in M$ there exists an open neighbourhood $U\ni m$ in M and an open neighbourhood $V\ni f(m)$ in N such that

$$f\big|_U\colon U\to V$$

is a diffeomorphism.

A non-trivial example of a local diffeomorphism can be obtained as follows. The map

$$f\colon\mathbb{R}\to S^1,\qquad f(x)=\big(\sin x,\ \cos x\big)$$

is a local diffeomorphism (why?), which is not a diffeomorphism, since $f(0)=f(\pm 2\pi)=f(\pm 4\pi)=\dots$

A non-trivial result from the course of analysis we need here is the following.

Theorem 2.24. *Let U be open in $\mathbb{R}^n$ and $f\colon U\to\mathbb{R}^k$ be smooth. Assume that at some $x\in U$ the differential D_xf of f is invertible. Then $n=k$ and f is a local diffeomorphism at x, that is there exist open subsets $U'\ni x$ and $V\ni f(x)$ such that*

$$f\big|_{U'}\colon U'\to V$$

is a diffeomorphism.

A proof of this theorem can be found for example in [?BardenThomas_IntroDiffMflds, Thm 9.4.1].

2.3 The fundamental theorem of algebra

As an application of the notions introduced in the preceding sections, we prove the fundamental theorem of algebra in this section. This requires some additional notions and constructions, which are of independent interest.

Let M and N be two manifolds of dimensions k and ℓ respectively. Pick a smooth map $f\colon M \to N$, a point $m \in M$ and charts (U_0, φ_0) and (V_0, ψ_0) such that $m \in U$ and $f(m) \in V$.

Definition 2.25. We say that m is *a critical point of* f if the differential of the coordinate representation

$$D_{\varphi_0(m)}\big(\psi_0 \circ f \circ \varphi_0^{-1}\big) \tag{2.26}$$

is non-surjective at $\varphi_0(m)$.

Lemma 2.27. *The notion of a critical point is well-defined, i.e., this is independent of the choice of charts.*

Proof. Pick any charts (U_1, φ_1) and (V_1, ψ_1) such that $m \in U_1$ and $f(m) \in V_1$. We have

$$\psi_0 \circ f \circ \varphi_0^{-1} = \psi_0 \circ \psi_1^{-1} \circ \psi_1 \circ f \circ \varphi_1^{-1} \circ \varphi_1 \circ \varphi_0^{-1} = \theta_{01}^{\psi} \circ \big(\psi_1 \circ f \circ \varphi_1^{-1}\big) \circ \theta_{10}^{\varphi},$$

which is valid on an open subset containing m. Hence, by the chain rule, we obtain for the differentials

$$D\big(\psi_0 \circ f \circ \varphi_0^{-1}\big) = D\theta_{01}^{\psi} \circ D\big(\psi_1 \circ f \circ \varphi_1^{-1}\big) \circ D\theta_{10}^{\varphi}.$$

Since $D\theta_{01}^{\psi}$ and $D\theta_{10}^{\varphi}$ are invertible everywhere on the domain of their definition, we obtain that $D\big(\psi_0 \circ f \circ \varphi_0^{-1}\big)$ is non-surjective at $\varphi_0(m)$ if and only if $D\big(\psi_1 \circ f \circ \varphi_1^{-1}\big)$ is non-surjective at $\varphi_1(m)$. This finishes the proof of this lemma. $\square$

Definition 2.28. Any non-critical point is called *regular*. We say that $n \in N$ is *a regular value* of f, if $f^{-1}(n)$ consists of regular points only. If $f^{-1}(n)$ contains at least one singular point, then n is called *a singular value* of f.

Let me stress that any point, which does not lie in the image of f, is a regular value of f (this fact of course follows from the definition but may be easily missed at first).

Notice that in the particular case $k = \ell$, (2.26) is a linear map $\mathbb{R}^k \to \mathbb{R}^k$. Hence, (2.26) is non-surjective if and only if it has a non-trivial kernel, or, still if and only if $\det D_{\varphi_0(m)}\big(\psi_0 \circ f \circ \varphi_0^{-1}\big) = 0$.

With these preliminaries at hand, we can prove the following.

Theorem 2.29 (The fundamental theorem of algebra)**.** *Let* $p(z) := a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$ *be a polynomial with complex coefficients of degree* $k \geq 1$. *Then* p *has at least one (complex) root.*

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$ by writing

$$y = (y_1, y_2) \equiv y_1 + y_2 i = z.$$

For a fixed polynomial $p = a_k z^k + \cdots + a_1 z + a_0$ such that $a_k \neq 0$, where $k \geq 1$, define a map $f\colon S^2 \to S^2$ by the rule

$$f(x) = \begin{cases} N, & \text{if } x = N, \\ \varphi_N^{-1} \circ p \circ \varphi_N(x), & \text{if } x \neq N. \end{cases}$$

The proof proceeds in a number of steps.

Step 1. *f is smooth.*

It is enough to check that f is smooth near N, i.e., that $\varphi_S \circ f \circ \varphi_S^{-1}$ is smooth. To see this, consider

$$\varphi_S \circ f \circ \varphi_S^{-1} = \varphi_S \circ \varphi_N^{-1} \circ \varphi_N \circ f \circ \varphi_N^{-1} \circ \varphi_N \circ \varphi_S^{-1} = \theta_{SN} \circ p \circ \theta_{SN}^{-1},$$

which is valid on $\mathbb{C} \setminus \{0\}$. By (1.2),

$$\theta_{SN}(z) = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}\,.$$

Consequently, for $z \neq 0$ we have

$$\varphi_S \circ f \circ \varphi_S^{-1}(z) = \frac{1}{\overline{p(1/\bar{z})}} = \frac{1}{\frac{\bar{a}_k}{z^k} + \cdots + \frac{\bar{a}_1}{z} + \bar{a}_0} = \frac{z^k}{\bar{a}_k + \cdots + \bar{a}_1 z^{k-1} + \bar{a}_0 z^k}.$$

Since $\varphi_S \circ f \circ \varphi_S^{-1}(0) = 0$, $\varphi_S \circ f \circ \varphi_S^{-1}$ is clearly smooth on $\mathbb{C}$. Hence, f is smooth as claimed.

Step 2. *The differential of the map $z \mapsto p(z)$ at the point z can be identified with $h \mapsto p'(z)h$, where*

$$p'(z) = ka_k z^{k-1} + \ldots 2a_2 z + a_1.$$

Denote $p(z) = u(y_1, y_2) + v(y_1, y_2)i$. Since p is a holomorphic function of z, by the Cauchy-Riemann equations we have

$$\begin{aligned}
D_z p(h) &= \begin{pmatrix} \dfrac{\partial u}{\partial y_1} & \dfrac{\partial u}{\partial y_2} \\ \dfrac{\partial v}{\partial y_1} & \dfrac{\partial v}{\partial y_2} \end{pmatrix} \begin{pmatrix} h_1 \\ \\ h_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial y_1} h_1 - \dfrac{\partial v}{\partial y_1} h_2 \\ \dfrac{\partial v}{\partial y_1} h_1 + \dfrac{\partial u}{\partial y_1} h_2 \end{pmatrix} \\
&\equiv \Big(\frac{\partial u}{\partial y_1} h_1 - \frac{\partial v}{\partial y_1} h_2 \Big) + \Big(\frac{\partial v}{\partial y_1} h_1 + \frac{\partial u}{\partial y_1} h_2 \Big) i \\
&= \frac{\partial p}{\partial y_1}\, h = p'(z)\, h.
\end{aligned}$$

Step 3. *The set of critical values of f is finite.*

Indeed, any critical point of f is either N or of the form $\varphi_N^{-1}(z)$, where z is a zero of the polynomial

$$p'(z) = ka_k z^{k-1} + \ldots 2a_2 z + a_1.$$

Hence, the number of critical points of f is finite and therefore the number of critical values is also finite.

Remark 2.30. Notice that N is a critical point of f as long as $k \geq 2$, which we can assume without loss of generality, since any polynomial of degree 1 obviously has a root.

Step 4. *For any regular value $x \in S^2$, the number of points in $f^{-1}(x)$ is finite and independent of x.*

First notice that the set $\mathcal{R}(f)$ of regular values is open and connected as a complement of a finite number of points in S^2.

Furthermore, for any $n \in \mathcal{R}(f)$ and any $m \in f^{-1}(n)$, the map $\varphi_N \circ f \circ \varphi_N^{-1} = p$ is a local diffeomorphism at $\varphi(m)$ by Theorem 2.24. Since φ_N is a homeomorphism, f is a local homeomorphism at m. In particular, there is a neighbourhood $U' \ni m$ such that $U' \cap f^{-1}(n) =$

$\{m\}$. Thus, $f^{-1}(n)$ is discreet. Since S^2 is compact, $f^{-1}(n)$ is also compact as a closed subset of a compact space. Hence, $f^{-1}(n)$ must in fact be a finite set.

The above argument actually shows that the map

$$\mathcal{R}(f) \ni x \mapsto \# f^{-1}(x) \in \mathbb{N} \tag{2.31}$$

is locally constant. Since $\mathcal{R}(f)$ is connected, this function must be constant.

Step 5. *We prove this theorem.*

Observe first that (2.31) cannot vanish everywhere on $\mathcal{R}(f)$. Indeed, the image of f is obviously infinite, whereas the set of critical values is finite by Step 3. Hence, there are regular values, which are in the image of f.

In fact, (2.31) vanishes nowhere as a locally constant function on an open connected space. Hence, $f^{-1}(S) \neq \varnothing$ as long as $S \in \mathcal{R}(f)$. Also, if S is a critical value, then $f^{-1}(S)$ contains at least one critical point. In either case, $f^{-1}(S)$ is non-empty, which means that p has at least one root. $\square$

The above proof turns out to contain a few ideas which can be used in other circumstances too. However, this requires some technical results, which are proved first. L 2

2.4 Tangent spaces

We begin with the following consideration. Let γ be a smooth curve in $\mathbb{R}^k$ through some $p \in \mathbb{R}^k$, that is a smooth map $\gamma\colon (a,b) \to \mathbb{R}^k$, such that $\gamma(t_0) = p$ for some $t_0 \in (a,b)$. Recall that the tangent vector of γ at p is

$$\dot\gamma(t_0) := \frac{d}{dt}\Big|_{t=t_0} \gamma(t) \in \mathbb{R}^k.$$

Let now γ be a smooth curve on the 2-sphere through some $p \in S^2$. Since S^2 is a subset of $\mathbb{R}^3$, we may think of γ as a curve in $\mathbb{R}^3$ satisfying

$$\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t) = 1 \qquad \forall t \in (a,b). \tag{2.32}$$

It is reasonable to call the set

$$T_pS^2 := \big\{ \mathrm{v} \in \mathbb{R}^3 \mid \mathrm{v} \text{ is the tangent vector of some smooth curve on } S^2 \text{ through } p \big\}$$

the tangent space of S^2 at the point p.

To determine T_pS^2 more explicitly, differentiate (2.32) with respect to t and set $t = t_0$:

$$\gamma_1(t_0)\dot\gamma_1(t_0) + \gamma_2(t_0)\dot\gamma_2(t_0) + \gamma_3(t_0)\dot\gamma_3(t_0) = 0 \qquad \Longleftrightarrow \qquad \langle p, \dot\gamma(t_0)\rangle = 0.$$

In other words, the tangent vector of any smooth curve on S^2 through p is necessarily orthogonal to p. Moreover, it is clear that any vector orthogonal to p arises in this way (consider all great circles through p). Hence,

$$T_pS^2 = p^\perp.$$

However, for an abstract manifold M a smooth curve γ on M does not lie in an Euclidean space in any obvious way so that the above definition of T_pS^2 does not immediately generalize. A nice workaround goes as follows.

Let γ be a smooth curve through $m \in M$. We can assume that γ is defined on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and $\gamma(0) = m$.

Definition 2.33. Two smooth curves γ_1 and γ_2 through m as above are said to be equivalent, if for some chart (U, φ) such that $m \in U$ we have

$$\frac{d}{dt}\Big|_{t=0} \big(\varphi \circ \gamma_1(t)\big) = \frac{d}{dt}\Big|_{t=0} \big(\varphi \circ \gamma_2(t)\big). \tag{2.34}$$

In other words, $\gamma_1 \sim \gamma_2$ if and only if the smooth curves $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_1$ in $\mathbb{R}^k$ (!) have the same tangent vector.

Furthermore, an equivalence class of smooth curves is called *a tangent vector* at m. The set $T_m M$ of all tangent vectors at m is called *the tangent space* at m.

Exercise 2.35. Show that the above equivalence relation is independent of the choice of a chart.

Exercise 2.36. Show that the tangent space to S^2 at some $p \in S^2$ in the sense of Definition 2.33 is $p^\perp$.

While Definition 2.33 is has a clear geometric meaning, the algebraic structure of $T_m M$ is opaque in this approach. For this reason we adopt an alternative definition, which is more algebraic.

First notice that given any tangent vector $[\gamma]$ through m we can define the map $\partial_{[\gamma]} \colon C^\infty(M) \to \mathbb{R}$ by setting

$$\partial_{[\gamma]}(f) = \frac{d}{dt}\Big|_{t=0} f\big(\gamma(t)\big) = \lim_{t\to 0} \frac{f\big(\gamma(t)\big) - f(m)}{t}.$$

Proposition 2.37. *The map $\partial_{[\gamma]}$ is well defined and has the following properties:*

(i) $\partial_{[\gamma]}$ *is $\mathbb{R}$-linear, that is*

$$\partial_{[\gamma]}\big(\lambda f + \mu g\big) = \lambda\, \partial_{[\gamma]}(f) + \mu\, \partial_{[\gamma]}(g)$$

holds for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in C^\infty(M)$.

(ii) $\partial_{[\gamma]}$ *satisfies*

$$\partial_{[\gamma]}(fg) = \partial_{[\gamma]}(f)g(m) + f(m)\partial_{[\gamma]}(g)$$

for all $f, g \in C^\infty(M)$.

Proof. We only need to prove that $\partial_{[\gamma]}$ is well-defined, since *(i)* and *(ii)* are clear from the definition.

Thus, pick two smooth equivalent curves γ_1 and γ_2 through m. Pick also a chart (U, φ) near m and denote $F := f \circ \varphi^{-1} \colon \mathbb{R}^k \to \mathbb{R}$ and $\beta_j := \varphi \circ \gamma_j \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k$. Notice, that

$$\dot\beta_1(0) = \dot\beta_2(0) =: \mathrm{v},$$

since γ_1 and γ_2 are equivalent.

We have

$$\frac{d}{dt}\Big|_{t=0} \Big(f \circ \gamma_1(t)\Big) = \frac{d}{dt}\Big|_{t=0} \Big(f \circ \varphi^{-1} \circ \varphi \circ \gamma_1(t)\Big) = \frac{d}{dt}\Big|_{t=0} \Big(F \circ \beta_1(t)\Big) = D_{\mathrm{v}} F\big(\varphi(m)\big),$$

where $D_{\mathrm{v}} F$ is the derivative of F in the direction of v, that is

$$D_{\mathrm{v}} F = \sum_{j=1}^{k} \frac{\partial F}{\partial x_j} \mathrm{v}_j = \langle \nabla F, \mathrm{v} \rangle. \tag{2.38}$$

A similar computation yields also

$$\frac{d}{dt}\Big|_{t=0}\Big(f\circ\gamma_2(t)\Big) = D_{\mathrm{v}}F\big(\varphi(m)\big),$$

thus demonstrating that $\partial_{[\gamma]}$ depends on the equivalence class of γ only as the notation suggests. $\square$

Motivated by the above proposition we give the following

Definition 2.39. An $\mathbb{R}$-linear map $\partial\colon C^\infty(M)\to\mathbb{R}$ satisfying the Leibnitz rule

$$\partial(fg) = \partial(f)\,g(m) + f(m)\,\partial(g) \qquad \forall f,g\in C^\infty(M)$$

is called *a derivation at m*.

Notice that a constant function, which takes value 1 everywhere on M, is annihilated by any derivation. Indeed, this follows from the following computation:

$$\partial(1) = \partial(1^2) = \partial(1)\cdot 1 + 1\cdot\partial(1) = 2\,\partial(1).$$

By the linearity of derivations, any constant function is annihilated by each derivation.

For the proof of Proposition 2.41 below, we need the following technical result, whose proof is deferred till the next section.

Proposition 2.40. *For any manifold M and any point $m_0\in M$ the following holds.*

(i) Suppose f is a smooth function defined on a neighborhood U of m_0. Then there is a smooth function $\hat f$ defined everywhere on M and a neighborhood $\hat U\subset U$ of m_0 such that f and $\hat f$ coincide everywhere on $\hat U$.

(ii) Let ∂ be a derivation at m_0. If the restrictions of f and $\hat f$ to some neighborhood $\hat U$ of m_0 are equal, then $\partial(f)=\partial(\hat f)$.

Thus, for any tangent vector at m we constructed an explicit derivation at m. It turns out that this map is a bijection as we show next.

Denote temporarily by Der_mM the set of all derivations at m.

Proposition 2.41. *The map*

$$T_mM\to Der_mM,\qquad [\gamma]\mapsto\partial_{[\gamma]} \tag{2.42}$$

is a bijection.

Proof. We continue to use notations of the proof of Proposition 2.37. In addition, the chart (U,φ) is assumed to be centered at m_0.

Step 1. (2.42) *is injective.*

Assume $\partial_{[\gamma_1]}=\partial_{[\gamma_2]}$, that is $\partial_{[\gamma_1]}(f)=\partial_{[\gamma_2]}(f)$ holds for any $f\in C^\infty(M)$. This implies in turn that

$$D_{\mathrm{v}_1}F\big(\varphi(m)\big) = D_{\mathrm{v}_2}F\big(\varphi(m)\big)$$

holds for any $F\in C^\infty(\mathbb{R}^k)$, where $\mathrm{v}_j=\dot\beta_j(0)$. Substituting $F=x_j$ in the above equality, we obtain that the jth components of v_1 and v_2 are equal for any j, i.e., $\mathrm{v}_1=\mathrm{v}_2$, which yields in turn that γ_1 and γ_2 are equivalent. Hence, the injectivity of (2.42) follows.

Step 2. *For any $F \in C^\infty(\mathbb{R}^k)$ there exist smooth functions $G_1, \dots, G_k$ such that*

$$F(x) = F(0) + \sum_{j=1}^{k} x_j G_j(x). \tag{2.43}$$

This follows by the following computation:

$$F(x) - F(0) = \int_0^1 \frac{d}{dt} F(tx)\, dt = \int_0^1 \langle \nabla F(tx), x \rangle\, dt,$$

which yields (2.43) with $G_j(x) = \int_0^1 \frac{\partial F}{\partial x_j}(tx)\, dt$.

Step 3. (2.42) *is surjective.*

Denote the jth component of φ by x_j so that $\varphi = (x_1, \dots, x_k)$. Notice that each x_j is a smooth function defined on U.

By Proposition 2.40, we can find $\hat{U} \subset U$ and a smooth function $\hat{x}_j$ defined everywhere on U such that x_j and $\hat{x}_j$ coincide on $\hat{U}$.

Pick any $\partial \in Der_m M$ and define

$$\mathrm{v}_j := \partial(\hat{x}_j) \in \mathbb{R}, \qquad \mathrm{v} = (\mathrm{v}_1, \dots, \mathrm{v}_k),$$

Notice that by Proposition 2.40, *(ii)*, v_j does not depend on the choice of $\hat{x}_j$.

Furthermore, define

$$\begin{aligned} &\beta \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^k, \qquad && \beta(t) = \big(\mathrm{v}_1 t, \dots, \mathrm{v}_k t\big), \\ &\gamma \colon (-\varepsilon, \varepsilon) \to M, && \gamma := \varphi^{-1} \circ \beta. \end{aligned}$$

By the previous step, there exist some functions $G_j \colon \mathbb{R}^k \to \mathbb{R}$ such that $f \circ \varphi^{-1}(x) = \sum x_j G_j(x)$. Hence,

$$f = f(m) + \sum_{j=1}^{k} x_j g_j, \tag{2.44}$$

where we think of x_j as a function on U and $g_j = G_j \circ \varphi$. In particular, $x_j(m) = 0$ for all $j = 1, \dots, k$.

Applying Proposition 2.40, *(i)* again, we can find some $\hat{g}_j$ defined globally on M and a neighbourhood[2] $\hat{U} \subset U$ such that

$$f\big|_{\hat{U}} = f(m) + \sum_{j=1}^{k} \hat{x}_j \hat{g}_j\big|_{\hat{U}} = f(m) + \sum_{j=1}^{k} x_j g_j\big|_{\hat{U}}.$$

By Proposition 2.40, *(ii)* we obtain

$$\begin{aligned} \partial(f) &= \partial\big(f(m)\big) + \sum_{j=1}^{k} \partial(x_j) g_j(m) + x_j(m) \partial(g_j) \\ &= 0 + \sum_{j=1}^{k} \mathrm{v}_j g_j(m) + 0 = \sum_{j=1}^{k} \mathrm{v}_j g_j(m). \end{aligned} \tag{2.45}$$

[2]Shrinking the neighbourhoods if necessary, without loss of generality we can assume that $\hat{U}$ is the same neighbourhood for all $\hat{x}_j$ and g_j.

Furthermore, recalling (2.38), we obtain

$$\partial_{[\gamma]}(f) = \sum_{j=1}^{k} \mathrm{v}_j \frac{\partial F}{\partial x_j}(0) = \sum_{j=1}^{k} \mathrm{v}_j \frac{\partial}{\partial x_j}\Big|_{x=0} \Big(f(m) + \sum_{i=1}^{k} x_i G_i(x) \Big)$$
$$= \sum_{j=1}^{k} \mathrm{v}_j \Big(0 + \sum_{i=1}^{k} \delta_{ij} G_i(0) \Big) = \sum_{j=1}^{k} \mathrm{v}_j \, g_j(m).$$

Comparing this with (2.45), we conclude that $\partial(f) = \partial_{[\gamma]}(f)$ holds for any $f \in C^\infty(M)$. Hence, $\partial = \partial_{[\gamma]}$, which finishes the proof of this step and the proof of this proposition too. □

We use the bijective map of Proposition 2.41 to identify T_mM with Der_mM. Since Der_mM is clearly a vector space, we obtain the structure of a vector space on T_mM in this way. Also, in view of this identification, we drop the notation Der_mM in favour of T_mM and we will switch freely between the two interpretations of tangent vectors as classes of curves and derivations.

Proposition 2.46. *For any $m \in M$ the tangent space T_mM is a vector space of dimension $k = \dim M$.*

Proof. Pick a chart (U, φ), $\varphi = (x_1, \dots, x_k)$ centered at m as in the proof of Proposition 2.37. For each $j = 1, \dots, k$ define a curve γ_j by

$$\varphi \circ \gamma_j(t) = (0, \dots, 0, t, 0, \dots, 0),$$

where the only non-trivial component is on the jth place. Correspondingly, we have k derivations:

$$\partial_j := \partial_{[\gamma_j]}.$$

Notice that if F is the coordinate representation of f, we have

$$\partial_j(f) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma_j\,(t) = \frac{d}{dt}\Big|_{t=0} F(0, \dots, 0, t, 0, \dots, 0) = \frac{\partial F}{\partial x_j}(0).$$

We want to show that $\partial_1, \dots, \partial_k$ is a basis of T_mM.

To show that $\partial_1, \dots, \partial_k$ are linearly independent, assume there are some real numbers $\lambda_1, \dots, \lambda_k$ such that $\lambda_1\,\partial_1 + \dots + \lambda_k\,\partial_k = 0$, that is

$$\lambda_1\,\partial_1(f) + \dots + \lambda_k\,\partial_k(f) = 0$$

holds for any $f \in C^\infty(M)$. Substituting $f = x_j$ in the above equality[3], we obtain $\lambda_j = 0$. Hence, $\partial_1, \dots, \partial_k$ are linearly independent indeed.

Let us show that any derivation ∂ at m can be represented as a linear combination of $\partial_1, \dots, \partial_k$. By Proposition 2.41, there exists a curve γ through m such that $\partial = \partial_{[\gamma]}$. If $\beta = \varphi \circ \gamma$ is a coordinate representation of γ and $\mathrm{v} = \dot\beta(0)$, then

$$\partial(f) = D_{\mathrm{v}}F = \sum \mathrm{v}_j \frac{\partial F}{\partial x_j}(0) = \sum \mathrm{v}_j\,\partial_j(f).$$

Hence, $\partial = \mathrm{v}_1\,\partial_1 + \dots + \mathrm{v}_k\,\partial_k$. □

Notice that the proof of the above proposition yields in fact a basis of T_mM for any choice of a chart (U, φ) such that $m \in U$. In fact, we have shown that $\partial_1, \dots, \partial_k$ is a basis of T_mM, where

$$\partial_j f = \frac{\partial}{\partial x_j}\Big|_{x=\varphi(m)} \Big(f \circ \varphi^{-1}(x) \Big).$$

L 3

[3] Technically, we should first fix an "extension" $\hat x_j$ of x_j as a in the proof of Proposition 2.41 above. However, this should be clear by now and we will omit this sort of argument below.