Algebraic Topology

Lecture notes

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This is a draft. In particular, most of figures are missing. If you spot a mistake, please let me know.

TODO:
- Add an appendix on chain complexes.

Contents

Chapter 1

Introduction

The main purpose of this chapter is to explain informally the main ideas which will be developed in details later. In particular, the proofs are rather sketchy stressing main ideas only. More precise statements and proofs will be given in the subsequent chapters.

1.1 Differential forms, the theorems of Green and Stokes

Let $\omega = P(x,y)dx + Q(x,y)dy$ be a 1-form on an open subset $U \subset \mathbb{R}^2$. For example, if $f \colon U \to \mathbb{R}$ is a smooth map, then the differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is a 1-form.

Question 1.1. Under which circumstances does there exist some function f as above such that $\omega = df$?

Clearly, we have the following necessary condition:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1.2}$$

Proposition 1.3. *If U is convex, then (1.2) is also sufficient.*

Sketch of proof. Theorem of Green $\implies$ For any closed piecewise smooth curve $C \subset U$ without self-intersections we have

$$\int_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0, \tag{1.4}$$

where D is the domain bounded by C. Notice that here we use the convexity of U, since otherwise C does not necessarily bound any domain.

Pick any $(x_0, y_0) \in U$. For any $(x,y) \in U$ choose a curve C' connecting (x_0, y_0) and (x,y). Define

$$f(x,y) := \int_{C'} P\,dx + Q\,dy.$$

Property (1.4) guaranties that f does not depend on the choice of C'. $\square$

The following example shows that (1.2) is not sufficient for general U.

Example 1.5. Consider $U = \mathbb{R}^2 \setminus \{0\}$ and

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

If there were some f such that $\omega = df$, then we would have $\int_{S^1} \omega = 0$, where S^1 is the circle (for example, parametrized via $t \mapsto (\cos t, \sin t)$). This is a contradiction, since $\int_{S^1} \omega = 2\pi \neq 0$.

Notice that the proof of Proposition 1.2 does not work here, since the theorem of Green does not apply for (D, ω), where D is the unit disc.

Remark 1.6. One can show that for any closed piecewise smooth curve $C \subset \mathbb{R}^2 \setminus \{0\}$ we have

$$\frac{1}{2\pi} \int_C \Big(-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \Big)$$

is an integer.

Let U be an open subset of $\mathbb{R}^3$ and $\omega = P\,dx + Q\,dy + R\,dz$ be a 1-form. We can also ask whether $\omega = df$ for some $f \colon U \to \mathbb{R}$. Clearly, we have the following necessary condition:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{1.7}$$

Proposition 1.8. *If U is convex, then (1.7) is also sufficient.*

The proof of this proposition is analogous to the proof of the previous one. Just instead of the theorem of Green we have to use the theorem of Stokes:

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_\Sigma \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \Big)\, dy\,dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \Big)\, dz\,dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big)\, dx\,dy.$$

Proposition 1.9. *Condition (1.7) is also sufficient for $\mathbb{R}^3 \setminus \{0\}$.*

Sketch of proof. Let $C \subset \mathbb{R}^3$ be an arbitrary simple picewise smooth curve without self-intersections. Then there is a picewise smooth surface $\Sigma \subset \mathbb{R}^3$ such that $\partial\Sigma = C$. If $0 \in \Sigma$, a (small) perturbation yields a surface $\Sigma' \subset \mathbb{R}^3 \setminus \{0\}$ such that $\partial\Sigma' = C$. $\square$

For a general U, Condition (1.7) is still insufficient, which is easily seen for the following example: $U = \mathbb{R}^3 \setminus \{z - \text{Axis}\}$ and

$$\omega = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy.$$

From this discussion we can make the following informal conclusion: Condition (1.7) is sufficient as long as U has no "holes" of codimension 2.

1.2 Ansatz of a construction.

Let $X \subset \mathbb{R}^n$ be an arbitrary subset, which is equipped with the induced topology. Define $Z_1(X)$ as a free Abelian group generated by (oriented) closed curves, i.e.,

$$C \in Z_1(X) \implies C = n_1 C_1 + \dots n_k C_k, \tag{1.10}$$

where $n_j \in \mathbb{Z}$. Define

$$\int_C \omega := \sum n_k \int_{C_k} \omega.$$

Remark 1.11. If C_0 is a closed oriented curve, $2C_0$ can be understood as "running along C_0 twice in the same direction". Similarly, $-C_0$ can be understood as the curve C_0 with the opposite orientation. However, in most cases we treat (1.10) purely formally.

Assume temporarily that X is an *open* subset of $\mathbb{R}^2$. We would like to define an equivalence relation such that

$$C \sim C' \implies \int_C \omega = \int_{C'} \omega$$

holds for all $\omega = P\,dx + Q\,dy$ satisfying (1.2). The theorem of Green (or Stokes in the case $U \subset \mathbb{R}^3$) suggests the following:

$$C \sim C' \quad \Leftrightarrow \quad \exists \text{ a compact oriented surface } \Sigma \text{ such that } \partial\Sigma = C \cup -C'. \tag{1.12}$$

Here C and C' are oriented curves and Σ is an oriented surface such that $\partial\Sigma = C \cup -C'$ as *oriented* curves. This definition also makes sense even in the case when X is not necessarily open.

More generally, a cycle $C = C_1 + \cdots + C_k$ is called *null homologous*, i.e., $C \sim 0$, if and only if

$$\exists \text{ a compact surface } \Sigma \text{ such that } \partial\Sigma = C_1 \cup \cdots \cup C_n.$$

Clearly, Condition (1.12) can be written as $C + (-C') \sim 0$.

Example 1.13. Null homologous cycles on the 2-sphere with 2 points removed (equivalently, $\mathbb{R}^2 \setminus \{0\}$).

Even more generally, each linear combination of null homologous cycles is also declared to be null homologous.

$Z_1(X) \supset B_1(X) = \{\text{null homologous cycles}\}$.

$H_1(X) := Z_1(X)/B_1(X)$ the first homology group of X.

Example 1.14. $H_1(S^2 \setminus \{p, q\}) \cong \mathbb{Z}$.

Problems: Curves C and surfaces Σ can have singularities and self-intersections.

More generally:

- $Z_n(X)$ freely generated by compact oriented n-dimensional "surfaces" without boundary.
- $Z_n(X) \supset B_n(X)$ the subgroup generated by the boundaries of compact oriented $(n+1)$-dimensional "surfaces".
- $H_n(X) := Z_n(X)/B_n(X)$ the nth homology group of X.

In general, we would like to associate to each topological space X a sequence of abelian groups $H_0(X), H_1(X), \dots, H_n(X), \dots$ such that the following holds:

(a) Each continuous map $f\colon X \to Y$ induces a sequence of homomorphisms $f_*\colon H_n(X) \to H_n(Y)$;

(b) $(f \circ g)_* = f_* \circ g_*, \quad id_* = id$.

(c) $H_0(\{pt\}) \cong \mathbb{Z}$ and $H_n(\{pt\}) = 0$ for all $n \geq 1$.

(d) $H_n(S^n) \cong \mathbb{Z}$ provided $n \geq 1$ and $H_k(S^n) = 0$ for all $k \geq n+1$ (More generally, for each compact connected oriented manifold M of dimension n the following holds: $H_n(M) \cong \mathbb{Z}$ and $H_k(M) = 0$ for all $k > n+1$).

(e) $f \simeq g \implies f_* = g_*$.

Here two continuous maps are said to be homotopic ($f \simeq g$), if there exists a continuous map $h\colon X \times [0,1]$, called homotopy, such that the following holds:

$$h|_{X\times 0} = f \qquad \text{and} \qquad h|_{X\times 1} = g.$$

Question 1.15. What does make Properties (a)-(e) interesting?

This question will be answered in the subsequent sections. We finish this section by the following facts, which will be useful below.

Proposition 1.16. *If f is a homeomorphism, then each $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. $id_{H_n} = id_* = (f \circ f^{-1})_* = f_* \circ (f^{-1})_* \implies f_*$ is an isomorphism and $(f_*)^{-1} = (f^{-1})_*$. ∎

1.3 The theorem of Brouwer

In this section we show that (a)-(e) imply the following famous result.

Theorem 1.17 (Brouwer)**.** *Any continuous map $f\colon B_n \to B_n$ has a fixed point.*

Proof. The proof consists of the following three steps.

Step 1. *For the ball $B_n := \{x \in \mathbb{R}^n \mid |x| \le 1\}$ we have $H_k(B_n) = 0$ for all $k \ge 1$.*

Let $c\colon B_n \to \{0\}$ be the constant map. The map $h(x,t) = tx,\ t \in [0,1]$ is a homotopy between id_B und $\imath \circ c$, where $\imath\colon \{0\} \to B_n$ is the inclusion. Thus, $id = \imath_* \circ c_* \implies H_k(B_n) = 0$ for all $k \ge 1$, since $\operatorname{Im} \imath_* = \{0\}$.

Step 2. *There is no continuous map $g\colon B_n \to \partial B_n = S^{n-1}$ such that $g(x) = x$ holds for all $x \in S^{n-1}$.*

Assume $n = 1$ first. In this case there is no continuous map $g\colon [-1,1] \to \{\pm 1\}$ as in the statement of this step, since the target space $\{\pm 1\}$ is disconnected, whereas the interval $[0,1]$ is connected.

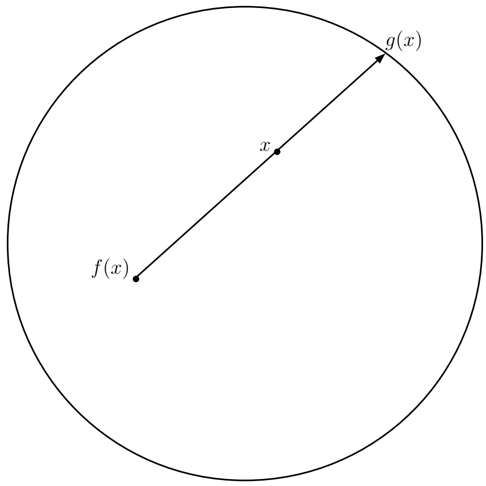
Let us consider now the case $n \ge 2$. Assume there is such $g\colon B_n \to S^{n-1}$. Then we have

$$\begin{aligned} id_{S^{n-1}} = g \circ \imath_{S^{n-1}} &\implies (id_{S^{n-1}})_* = g_* \circ (\imath_{S^{n-1}})_* = 0 \quad \text{on } H_{n-1}(S^{n-1}) \\ &\implies H_{n-1}(S^{n-1}) = 0. \end{aligned}$$

This contradiction proves Step 2.

Step 3. *We prove the theorem of Brower.*

Assume there exists a continuous map $f\colon B_n \to B_n$ without fixed points. Then there also exists a continuous map $g\colon B_n \to S^{n-1}$ such that $g|_{S^{n-1}} = id$:



This contradicts Step 2. $\square$

1.4 The degree of a continuous map and the fundamental theorem of algebra

In this section we show that (a)-(e) imply that any non-constant polynomial with complex coefficients has at least one root. This statement is known as the fundamental theorem of algebra.

Thus, pick any $n \geq 1$ and choose a generator $\alpha \in H_n(S^n)$, i.e., an element α such that $H_n(S^n) = \mathbb{Z} \cdot \alpha$.

Definition 1.18. For any continuous map $f\colon S^n \to S^n$ define $\deg(f) \in \mathbb{Z}$ by

$$f_*\alpha = \deg(f)\alpha.$$

The degree of a map does not depend on the choice of a generator, since $f_*(-\alpha) = -f_*\alpha = -\deg(f)\alpha = \deg(f)(-\alpha)$.

Lemma 1.19. *The degree has the following properties:*

(i) $\deg(id) = 1$*;*

(ii) $\deg(f \circ g) = \deg f \cdot \deg g$*;*

(iii) $f \simeq g \implies \deg f = \deg g$*;*

(iv) $\deg(\textit{const. map}) = 0$*.*

$\square$

Lemma 1.20. *For* $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ *define* $f_n\colon S^1 \to S^1$ *by* $f_n(z) = z^n$*, where* $n \in \mathbb{Z}$*. Then we have*

$$\deg f_n = n.$$

Idea of proof. The curve

$$\alpha\colon [0, 2\pi] \to S^1, \qquad \alpha(t) = \cos t + \sin t\, i = e^{ti},$$

generates $H_1(S^1)$. Since $f_n \circ \alpha(t) = e^{nti} = \cos(nt) + \sin(nt)i$, from the definition of the degree and Remark 1.11 we have $\deg f_n = n$. $\square$

Theorem 1.21 (The fundamental theorem of Algebra)**.** *Each non-constant polynomial* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0,\ a_j \in \mathbb{C}$ *has at least one complex root.*

Proof. Identify S^1 with $S^1_r := \{z \in C \mid |z| = r\} \cong S^1$ with the help of the homeomorphism

$$S^1 \to S^1_r, \qquad z \mapsto rz.$$

The proof consists of the following three steps.

Step 1. *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *be a continuous map without zeros. Then for each* $r > 0$ *the map*

$$\frac{f}{|f|}\colon S^1_r \to S^1 \tag{1.22}$$

is homotopic to the constant map.

Indeed, a homotopy can be given explicitly by

$$F(z,t) = \frac{f(tz)}{|f(tz)|}, \qquad z \in S^1,\ t \in [0,r].$$

Step 2. *Let* $p(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$ *be a polynomial without zeros. Then there exists some* $R > 0$ *such that the following holds:* $\forall r \geq R$ *the restriction of* $p/|p|$ *to* S^1_r *is homotopic to* f_n*.*

For all $z \in \mathbb{C}$ such that $|z| \geq 1$ we have

$$\begin{aligned}\left|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\right| &\leq |a_{n-1}||z|^{n-1} + \dots + |a_1||z| + |a_0| \\ &\leq n \max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}|z|^{n-1}\end{aligned}$$

Choose R so that $R > n\max\{|a_{n-1}|, \dots, |a_1|, |a_0|\}$ and $R > 1$. For all $r \geq R$ and all $t \in [0,1]$ the polynomial

$$p_t(z) = z^n + t\big(a_{n-1}z^{n-1} + \dots + a_1 z + a_0\big)$$

has no zeros on S^1_r, since

$$\left|a_{n-1}z^{n-1} + \dots + a_1 z + a_0\right| < Rr^{n-1} \leq r^n, \quad \text{provided} \quad |z| = r.$$

Then

$$P(z,t) = \frac{p_t(z)}{|p_t(z)|}\Big|_{S^1_r}$$

is a homotopy between $p/|p|$ and f_n viewed as a map on S^1_r.

Step 3. *We prove the fundamental theorem of algebra.*

Assume p is a non-constant polynomial without zeros. Denote

$$q_r(z) = \frac{p(z)}{|p(z)|}\Big|_{S^1_r},$$

where $r \geq R$. Step 2 $\implies \deg q_r = n$. Step 1 $\implies \deg q_r = 0$, i.e., $n = 0$. Thus, p is a constant polynomial, which is a contradiction. $\square$

Chapter 2

Singular homology

2.1 Free abelian groups

An abelian group G is called free with a basis $A \subset G$, if $\forall g \in G$ there exists a unique representation $g = \sum_{a\in A} n_a a$, where $n_a \in \mathbb{Z}$ and $n_a \neq 0$ for finitely many $a \in A$ only.

Any set A generates an abelian group $F(A)$, which is free with a basis A. Indeed, define

$$F(A) := \{f\colon A \to \mathbb{Z} \mid f(a) \neq 0 \text{ nur für endlich viele } a \in A\}.$$

Clearly, the functions

$$f_a(x) = \begin{cases} 1 & x = a, \\ 0 & \text{sonst}, \end{cases} \qquad a \in A$$

generate $F(A)$, that is $F(A)$ is free with a basis A.

Remark 2.1. For any $f \in F(A)$ we have

$$f = \sum_{a\in A} f(a) f_a.$$

In particular, $F(A)$ can be viewed as the group of all *finite* formal linear combinations $\sum_{a\in A} n_a a$, where $n_a \in \mathbb{Z}$.

2.2 Singular simplexes

Let $x_0, x_1, \ldots, x_k$ be arbitrary points in $\mathbb{R}^n$ such that $x_1 - x_0, \ldots, x_k - x_0$ are linearly independent.

Definition 2.2. The space

$$\Delta_k = \Delta(x_0, \ldots, x_k) = \Big\{x = \sum_{i=0}^k t_i x_i \mid t_i \in [0,1], \quad \sum_{i=0}^k t_i = 1\Big\}$$

is called *the (non-degenerate) k-simplex generated by* $x_0, \ldots, x_k$.

Example 2.3.
0) If $k = 0$, then $\Delta(x_0) = \{x_0\}$.
1) If $k = 1$, then $\Delta(x_0, x_1)$ is a segment $[x_0, x_1]$.
2) If $k = 2$, then $\Delta(x_0, x_1, x_3)$ is the triangle with the vertices x_0, x_1, x_2.
3) If $k = 3$, then $\Delta(x_0, x_1, x_3, x_4)$ is a tetrahedron with the vertices x_0, x_1, x_3, x_4.

Remark 2.4. The representation $x = \sum_{i=0}^k t_i x_i$ of a point in Δ_k is unique. Indeed, $\sum t_i x_i = \sum s_i x_i$, $\sum t_i = 1 = \sum s_i \implies$

$$0 = \sum (t_i - s_i)x_i = \sum (t_i - s_i)x_i - \sum (t_i - s_i)x_0 = \sum (t_i - s_i)(x_i - x_0) \implies t_i = s_i.$$

The coefficients $(t_0, t_1, \ldots, t_k) \in [0,1]^{k+1}$ are called *the barycentric coordinates* of the point $x \in \Delta_k$. In particular, each k-simplex is homeomorphic to the standard k-simplex

$$\Delta^k := \Delta(e_1, \ldots, e_k, e_{k+1}) \subset \mathbb{R}^{k+1},$$

where $e_1, \ldots, e_{k+1}$ is the standard basis of $\mathbb{R}^{k+1}$.

It is customary to drop the adjective "non-degenerate" when referring to simplexes. Sometimes degenerate simplexes (in the sense that $x_1 - x_0, \ldots, x_k - x_0$ may be linearly dependent) do appear below. Typically, this poses no problems, however the barycentric coordinates are ill defined in this case.

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From now on we pick one simplex in each dimension, for example the standard one.

Definition 2.5. Let X be a topological space. *A singular k-simplex* in X is a continuous map $f\colon \Delta^k \to X$.

In particular, a singular 0-simplex in X can be viewed as a point in X, a singular 1-simplex as a path in X etc.

Remark 2.6. The map f in the above definition does not need to be injective. In particular, the image of f may be (highly) singular.

For a singular k-simplex $f\colon \Delta^k \to X$ the $(k-1)$-simplex defined by

$$\partial^i f\colon \Delta^{k-1} \to X, \qquad \partial^i f(t_0, \ldots, t_{k-1}) = f(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1})$$

is called *the ith face* of f.

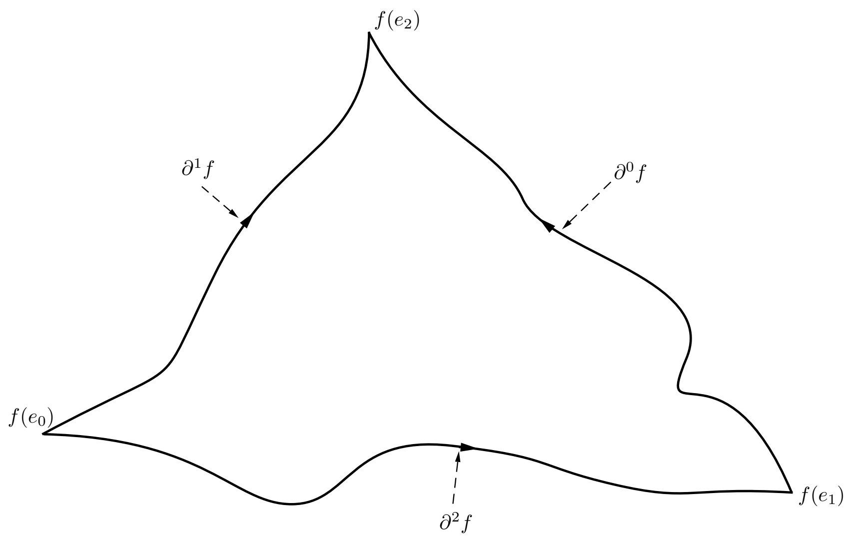

Figure 2.1: Faces of a singular simplex

Definition 2.7. Denote by $S_k(X)$ the free abelian group generated by all singular k-simplexes. Elements of $S_k(X)$ are formal linear combinations of the form

$$\sigma = \sum n_i f_i, \qquad n_i \in \mathbb{Z},$$

which are called *singular k-chains*. The $(k-1)$-chain

$$\begin{aligned} \partial f &= \partial^0 f - \partial^1 f + \partial^2 f - \cdots = \sum_{j=0}^{k} (-1)^j \partial^j f, \\ \partial \sigma &= \sum_i n_i \sum_j (-1)^j \partial^j f_i \end{aligned} \tag{2.8}$$

is called *the boundary* of f and σ respectively.

Proposition 2.9. *We have $\partial_{k-1} \circ \partial_k = 0$ (or, simply $\partial^2 = 0$) for all $k \geq 1$, i.e., the homomorphism*

$$S_k(X) \xrightarrow{\partial_k} S_{k-1}(X) \xrightarrow{\partial_{k-1}} S_{k-2}(X)$$

is trivial.

Proof. The proof consists of the following two steps.

Step 1. *Let f be a singular simplex. for each $j \geq i$ we have*

$$\partial^j \partial^i f = \partial^i \partial^{j+1} f.$$

Indeed,

$$\begin{aligned} \partial^j (\partial^i f)(t_0, \dots, t_{k-2}) &= \partial^i f(t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}); \end{aligned}$$

$$\begin{aligned} \partial^i (\partial^{j+1} f)(t_0, \dots, t_{k-2}) &= \partial^{j+1} f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-2}) \\ &= f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{k-2}). \end{aligned}$$

Step 2. *For each singular k-simplex we have $\partial(\partial f) = 0$.*

This follows from the following computation:

$$\begin{aligned} \partial(\partial f) &= \sum_{i=0}^{k} (-1)^i \partial^i (\partial f) = \sum_{i=0}^{k} \sum_{j=0}^{k} (-1)^{i+j} \partial^i \partial^j f = \sum_{j \geq i} + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{j \geq i} (-1)^{i+j} \partial^{j-1} \partial^i f + \sum_{j<i} (-1)^{i+j} \partial^i \partial^j f \\ &= \sum_{p+1 \geq q} (-1)^{p+q+1} \partial^p \partial^q f + \sum_{p>q} (-1)^{p+q} \partial^p \partial^q f \qquad p := j-1, \ q := i \\ &= 0. \end{aligned}$$

□

Corollary 2.10. $\operatorname{im} \partial_k \subset \ker \partial_{k-1}$.

The elements of $Z_{k-1}(X) := \ker \partial_{k-1}$ are called *cycles* and the elements of $B_{k-1}(X) := \operatorname{im} \partial_k$ are called *boundaries*.

Definition 2.11. The group

$$H_{k-1}(X) := \ker \partial_{k-1} / \operatorname{im} \partial_k = Z_{k-1}(X)/B_{k-1}(X)$$

is called the $(k-1)$ *th (singular) homology group* of X (with integer coefficients). In particular, $H_0(X) := S_0(X)/\operatorname{im} \partial_1$.

2.3 Some properties of the homology groups

Proposition 2.12.
$$X \textit{ path connected } \implies H_0(X) \cong \mathbb{Z}.$$

Proof. $S_0(X)$ is the free abelian group generated by the points of X. Let f be a singular 1-simplex, that is $f\colon [0,1] \to X$ is a path in X. By the definition of the boundary, $\partial f = x_1 - x_0$, where $x_1 = f(1)$ and $x_0 = f(0)$. By the hypothesis, we can connect any two points in X by a path, that is for any two points $x_0, x_1 \in X$ we have $[x_0] = [x_1] \in H_0(X)$.

Furthermore, define the homomorphism $\alpha\colon S_1(X) \to \mathbb{Z}$ by

$$\alpha\big(\sum n_i x_i\big) = \sum n_i.$$

Since $\alpha(\partial f) = 0$ for each singular 1-simplex, α yields a surjective homomorphism $H_0(X) \to \mathbb{Z}$, which is still denoted by α.

Suppose $\alpha([\sum n_i x_i]) = 0$. Then $[\sum n_i x_i] = \sum n_i[x_i] = \big(\sum n_i\big)[x_0] = 0$, that is α is injective. Thus, α is an isomorphism. □

Exercise 2.13. If X is not necessarily path connected, then the following holds: $H_0(X) \cong \mathbb{Z}^m$, where m is the number of path-components of X.

Proposition 2.14.
$$H_k(\{pt\}) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, \\ 0 & \textit{else.} \end{cases}$$

Proof. For $k = 0$ the statement of this proposition follows from the previous one. Hence, we may assume $k > 0$. For each such k there is exactly one k-simplex in $\{pt\}$, namely the constant map, which we denote by $c_k\colon \Delta^k \to \{pt\}$. For the boundary we have

$$\partial c_k = \sum_{i=0}^{k} (-1)^i \underbrace{c_k \circ d_i}_{c_{k-1}} = \begin{cases} 0, & \text{for } k \text{ odd,} \\ c_{k-1} & \text{for } k \text{ even.} \end{cases}$$

Hence,

$$Z_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd,} \\ 0 & \text{for } k \text{ even} \end{cases}$$

und

$$B_k\big(\{pt\}\big) = \begin{cases} S_k\big(\{pt\}\big) & \text{for } k \text{ odd,} \\ 0 & \text{for } k \text{ even.} \end{cases}$$

Thus $H_k\big(\{pt\}\big) = Z_k\big(\{pt\}\big)/B_k\big(\{pt\}\big) = 0$. □

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Definition 2.15. A topological space X is said to be *contractible* if there is a point $x_0 \in X$ such that the identity map id_X is homotopic to the constant map c_{x_0}.

Proposition 2.16. *For a contractible space X we have*

$$H_k(X) = \begin{cases} \mathbb{Z} & \textit{if } k = 0, \\ 0 & \textit{if } k \geq 1. \end{cases}$$

Proof. Since X is contractible, there exists a continuous map $h\colon X\times[0,1]\to X$ such that $h(x,0)=x$ and $h(x,1)=x_0$ hold for any $x\in X$. In particular, for a fixed $x\in X$ the path $t\mapsto h(t,x)$ connects x and x_0. This implies that X is path connected, hence $H_0(X)\cong\mathbb{Z}$ by Proposition 2.12.

Thus, we assume $k\geq 1$ in the sequel. Consider the quotient map

$$\begin{aligned}\pi\colon \Delta^{k-1}\times[0,1]&\to\Delta^k\cong(\Delta^{k-1}\times[0,1])/(\Delta^{k-1}\times\{1\})\\ ((t_0,\dots,t_{k-1}),u)&\mapsto(u,(1-u)t_0,\dots,(1-u)t_{k-1}).\end{aligned}$$

Let $h\colon X\times[0,1]\to X$ be a homotopy between id_X and c_{x_0}. Define $s\colon S_{k-1}(X)\to S_k(X)$ as follows: Since π is a quotient map and $h|_{X\times\{1\}}\equiv x_0$, for each singular $(k-1)$-simplex $\sigma\colon\Delta^{k-1}\to X$ there exists a unique map $s(\sigma)\colon\Delta^k\to X$ such that $h\circ(\sigma\times\mathrm{id})=s(\sigma)\circ\pi$. More explicitly,

$$s(\sigma)(t_0,t_1,\dots,t_k)=h\Big(\sigma\Big(\frac{t_1}{1-t_0},\dots,\frac{t_k}{1-t_0}\Big),\ t_0\Big)$$

whenever $t_0\neq 1$ and $s(\sigma)(t_1,\dots,t_k,1)=x_0$. Hence,

1. $\partial^0(s(\sigma))=\sigma$,
2. $\partial^i s(\sigma)=s(\partial^{i-1}\sigma)$ for $i>0$.

Therefore, for any $\sigma\in S_k(X)$ we have

$$\partial(s(\sigma))=\partial^0(s(\sigma))-\sum_{i=1}^{k}(-1)^{i-1}\partial^i(s(\sigma))=\sigma-\sum_{j=0}^{k-1}(-1)^j s(\partial^j\sigma)=\sigma-s(\partial\sigma).\qquad(2.17)$$

This yields

$$\partial\circ s+s\circ\partial=\mathrm{id}.$$

Hence, if σ is a cycle, then $\sigma=\partial\big(s(\sigma)\big)+s\big(\partial\sigma\big)=\partial\big(s(\sigma)\big)$, i.e., any cycle is a boundary. In other words, $H_k(X)=0$ whenever $k\geq 1$ as claimed. □

Theorem 2.18. *Let $f\colon X\to Y$ be a continuous map. Then for each $k\geq 0$ the map f induces a group homomorphism*

$$f_*\colon H_k(X)\to H_k(Y)$$

and for any other continuous map $g\colon Y\to Z$ we have

$$(g\circ f)_*=g_*\circ f_*.$$

Finally, $(\mathrm{id}_X)_*=\mathrm{id}$.

Proof. Define first group homomorphisms $f_\#\colon S_k(X)\to S_k(Y)$, by declaring

$$\sigma\mapsto f\circ\sigma\quad\text{ for }\quad\sigma\colon\Delta^k\to X.$$

Then for all singular k-simplexes $\sigma\colon\Delta^k\to X$ we have

$$\begin{aligned}(f_\#\partial^i(\sigma))(t_0,\dots,t_{k-1})&=f(\sigma(t_0,\dots,t_{i-1},0,t_i,\dots,t_{k-1}))\\ &=(f_\#\sigma)(t_0,\dots,t_{i-1},0,t_i,\dots,t_{k-1})\\ &=\partial^i(f_\#\sigma)(t_0,\dots,t_{k-1}),\end{aligned}$$

and therefore $f_\# \partial^i = \partial^i f_\#$, which yields in turn that $f_\#$ is a *chain map*, i.e.,

$$f_\# \partial = \partial f_\#.$$

This yields in particular that cycles are mapped to cycles and boundaries are mapped to boundaries:

$$f_\#(Z_k(X)) \subset Z_k(Y) \qquad \text{and} \qquad f_\#(B_k(X)) \subset B_k(Y).$$

Hence, we obtain a well defined group homomorphism:

$$\begin{gathered} f_*\colon H_k(X) = Z_k(X)/B_k(X) \to Z_k(Y)/B_k(Y) = H_k(Y) \\ f_*([\sigma]) := [f_\#(\sigma)]. \end{gathered}$$

Furthermore, for each singular k-simplex $\sigma\colon \Delta^k \to X$ we have

$$\begin{aligned} & g_\# \circ f_\#(\sigma) = g_\#(f \circ \sigma) = g \circ f \circ \sigma = (g \circ f)_\#(\sigma), \\ & g_* \circ f_*([\sigma]) = g_*[f_\#(\sigma)] = [g_\# \circ f_\#(\sigma)] = [(g \circ f)_\#(\sigma)] = (g \circ f)_*([\sigma]), \\ & (\mathrm{id}_X)_\#(\sigma) = \sigma, \\ & (\mathrm{id}_X)_*([\sigma]) = [(\mathrm{id}_X)_\#(\sigma)] = [\sigma]. \end{aligned}$$

Therefore, $g_* \circ f_* = (g \circ f)_*$ and $(\mathrm{id}_X)_* = \mathrm{id}$. □

Corollary 2.19. *If $f\colon X \to Y$ is a homeomorphism, then $f_*\colon H_k(X) \to H_k(Y)$ is an isomorphism for each k.* □

2.4 Homotopies and homology groups

Satz 2.20. *If $f, g\colon X \to Y$ are homotopic maps, then the induced maps on the homology groups are equal:*

$$f \simeq g \qquad \Longrightarrow \qquad f_* = g_*.$$

Proof. The proof consists of the following three steps.

Step 1. *Define*

$$\eta_t\colon X \to X \times I, \qquad \eta_t(x) = (x, t).$$

For each continuous map $f\colon X \to Y$ we have $(f \times id)_\# \eta^X_{t\#} = \eta^Y_{t\#} \circ f_\#$.

This follows immediately from the observation that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_t^X} & X \times I \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f \times id} \\ Y & \xrightarrow{\eta_t^Y} & Y \times I \end{array}$$

commutes.

Step 2. *There exists a sequence of homomorphisms $s_k^X\colon S_k(X) \to S_{k+1}(X \times I)$ satisfying*

$$\partial s_k^X + s_{k-1}^X \partial = \eta_{1\#} - \eta_{0\#}; \tag{2.21}$$

$$(f \times id_I)_\# \circ s_k^X = s_k^Y \circ f_\#. \tag{2.22}$$

Define $s_k = s_k^X$ recursively. For $k = 0$ and $x_0 \in X$, which we view as a 0-simplex, put

$$s_0\sigma\colon \Delta^1 \to X \times I, \qquad (t_0, t_1) \mapsto (x_0, t_1).$$

Then we have $\partial(s_0\sigma) = (x_0, 1) - (x_0, 0)$, i.e., (2.21) holds for $k = 0$. Equation (2.22) follows directly from the definition of s_0.

Suppose s_ℓ have been defined for all $\ell < k$. We define first s_k in a special case, namely for id_{Δ^k} viewed as an element $\imath_k \in S_k(\Delta^k)$. We have

$$\begin{aligned}\partial\big(\underbrace{\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k}_{\in S_k(\Delta^k\times I)}\big) &= \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \partial s_{k-1}\partial\imath_k \\ &\overset{(2.21)}{=} \eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - \big(\eta_{1\#}\partial\imath_k - \eta_{0\#}\partial\imath_k - s_{k-2}^{\Delta^k}\partial^2\imath_k\big) \\ &= 0.\end{aligned}$$

In this computation (2.21) is used for k replaced by $k-1$. Since $\Delta^k \times I$ is contractible, there exists some $a \in S_{k-1}(\Delta^k \times I)$ so that

$$\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}\partial\imath_k = \partial a.$$

Define $s_k(\imath_k) = a$. Then (2.21) holds for $\sigma = \imath_k$.

In general, define $s_k^X(\sigma) = (\sigma \times id)_\# a$. Then we have

$$\begin{aligned}\partial(s_k^X\sigma) &= \partial(\sigma\times id)_\# a = (\sigma\times id)_\#\partial a \\ &= (\sigma\times id)_\#\big(\eta_{1\#}\imath_k - \eta_{0\#}\imath_k - s_{k-1}^{\Delta^k}\partial\imath_k\big) \\ &= \eta_{1\#}\sigma_\#\imath_k - \eta_{0\#}\sigma_\#\imath_k - s_{k-1}^X\sigma_\#\partial\imath_k && (2.22) + \text{Step } 1 \\ &= \eta_{1\#}\sigma - \eta_{0\#}\sigma - s_{k-1}^X\partial\sigma.\end{aligned}$$

This proves (2.21).

We still have to show that (2.22) holds. Indeed,

$$(f\times id)_\# s_k\sigma = (f\times id)_\#(\sigma\times id)_\# a = \big((f\circ\sigma)\times id\big)_\# a = s_k(f\sigma) = s_k(f_\#\sigma).$$

Step 3. *We prove this theorem.*

Let h be a homotopy between f and g. From the following equalities

$$\partial(h_\#\circ s_k) + (h_\#\circ s_{k-1})\partial = h_\#\partial s_k + h_\#(s_{k-1}\partial) = h_\#(\eta_{1\#} - \eta_{0\#}) = f_\# - g_\#$$

we see that $f_\# - g_\# = \partial(h_\#\circ s_k)$ holds on $\ker\partial$. This shows that $f_* = g_*$. $\square$

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Definition 2.23. A continuous map $f\colon X \to Y$ is called a homotopy equivalence, if there exists a continuous map $g\colon Y \to X$ such that the following holds:

$$g\circ f \simeq id_X \qquad \text{and} \qquad f\circ g \simeq id_Y.$$

In this case the spaces X and Y are called homotopy equivalent.

Example 2.24. (i) Any two homeomorphic spaces are homotopy equivalent.

(ii) $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$. More generally, any contractible space is homotopy equivalent to $\{pt\}$.

(iii) $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}.

To see (ii), let X be a contractible space and $\imath_{x_0} \colon \{x_0\} \to X$ be the embedding of the point x_0. Then $c_{x_0} \circ \imath_{x_0} = id_{x_0}$ and $\imath_{x_0} \circ c_{x_0} \simeq id_X$.

To see (iii), define $f \colon \mathbb{R}^n \setminus \{0\} \to S^n$ by $f(x) = x/|x|$. If $g \colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ denotes the inclusion, then $f \circ g = id_{S^{n-1}}$. Furthermore,

$$h(x,t) = \frac{1}{t + (1-t)|x|} x, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

is a homotopy between $g \circ f$ and $id_{\mathbb{R}^n \setminus \{0\}}$.

Corollary 2.25.

$$f \textit{ is a homotopy equivalence} \implies \forall k \quad f_* \colon H_k(X) \to H_k(Y) \textit{ is an isomorphism.}$$

Example 2.26.

$$H_k(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_k(\mathbb{R}^n \setminus \{pt\}) = H_k(S^{n-1}) = \begin{cases} \mathbb{Z} & k = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

2.5 Exact sequences and the Bockstein homomorphism

Definition 2.27. A sequence of homomorphisms of abelian groups

$$\dots \longrightarrow A_{k+1} \xrightarrow{\alpha_{k+1}} A_k \xrightarrow{\alpha_k} A_{k-1} \longrightarrow \dots \tag{2.28}$$

is called exact, if for all k the following holds: $\ker \alpha_k = \operatorname{im} \alpha_{k+1}$.

Some special cases:

(i) $0 \to A \xrightarrow{\alpha} B$ is exact $\quad\Leftrightarrow\quad$ α is injectiv;

(ii) $A \xrightarrow{\alpha} B \to 0$ is exact $\quad\Leftrightarrow\quad$ α is surjectiv;

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\quad\Leftrightarrow\quad$ α is an isomorphism;

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact $\quad\Leftrightarrow\quad$ α is injectiv, β is surjectiv and $\ker \beta = \operatorname{im} \alpha$; In particular, β induces an isomorphism $C \cong B/A$.

The sequence (iv) is called a short exact sequence.

Example 2.29. $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a short exact sequence, where $\times n$ stands for the multiplication with a fixed $n \in \mathbb{Z}$.

Let A be a complex, that is A is a sequence

$$A: \qquad \cdots \longrightarrow A_{i+1} \xrightarrow{\partial} A_i \xrightarrow{\partial} A_{i-1} \longrightarrow \ldots$$

such that $\partial^2 = 0$. Just like in the case of chain complexes, we define the kth homology group of A to be

$$H_k(A) := \frac{\ker\left(\partial\colon A_k \to A_{k-1}\right)}{\operatorname{im}\left(\partial\colon A_{k+1} \to A_k\right)}.$$

If A, B, and C are complexes, a sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of complexes is a commutative diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ldots \arrow[r, "\partial"] & A_{k+1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k+1}"'] & A_k \arrow[r, "\partial"] \arrow[d, "\alpha_k"'] & A_{k-1} \arrow[r, "\partial"] \arrow[d, "\alpha_{k-1}"'] & \ldots \\
\ldots \arrow[r, "\partial"] & B_{k+1} \arrow[r, "\partial"] \arrow[d, "\beta_{k+1}"'] & B_k \arrow[r, "\partial"] \arrow[d, "\beta_k"'] & B_{k-1} \arrow[r, "\partial"] \arrow[d, "\beta_{k-1}"'] & \ldots \\
\ldots \arrow[r, "\partial"] & C_{k+1} \arrow[r, "\partial"] \arrow[d] & C_k \arrow[r, "\partial"] \arrow[d] & C_{k-1} \arrow[r, "\partial"] \arrow[d] & \ldots \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.30}$$

Such a sequence is called *exact*, if each vertical sequence $0 \to A_k \to B_k \to C_k \to 0$ is exact.

Here of course we could equally well consider sequences of complexes consisting of more than 3 complexes.

Example 2.31. Let X, Y and Z be topological spaces and $f\colon X \to Y$, $g\colon Y \to Z$ continuous maps. Then one obtains a sequence of chain complexes

$$0 \to S_*(X) \xrightarrow{f_\#} S_*(Y) \xrightarrow{g_\#} S_*(Z) \to 0,$$

which is not necessarily exact. What conditions guarantee that the above sequence is exact will be considered below.

Proposition 2.32. *The maps α and β yield homomorphisms $\alpha\colon H_*(A) \to H_*(B)$ and $\beta\colon H_*(B) \to H_*(C)$ respectively.*

Proof. This follows immediately from the commutativity of (2.30). $\square$

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Theorem 2.33. *A short exact sequence of complexes $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ induces a (long) exact sequence of homology groups:*

$$\cdots \to H_k(A) \xrightarrow{\alpha} H_k(B) \xrightarrow{\beta} H_k(C) \xrightarrow{\delta} H_{k-1}(A) \xrightarrow{\alpha} H_{k-1}(B) \to \ldots$$

Remark 2.34. The map δ is called the Bockstein homomorphism.

Proof. The proof consists of the following four steps.

Step 1. *We define* δ.

Pick $c \in C_k$, $\partial c = 0$. Since β_k is surjective, there exists some $b \in B_k$ such that $\beta(b) = c$. We have $\beta(\partial b) = \partial(\beta(b)) = \partial c = 0$. Since $\alpha\colon A_{k-1} \to \ker \beta_{k-1}$ is surjective, there is some $a \in A_{k-1}$ such that $\alpha(a) = \partial b$. Define

$$\delta[c] = [a].$$

We have to show that δ is well defined. Indeed, pick another representative $c' = c + \partial c''$ of the class $[c]$. For $c'' \in C_{k+1}$ there is some $b'' \in B_{k+1}$ such that $\beta(b'') = c'' \implies \beta(b + \partial b'') = c + \partial c''$. This yields $b' = b + \partial b'' + \alpha(a'')$, where $a'' \in A_k$. Furthermore, $\partial b' = \partial b + 0 + \alpha(\partial a')$. Since α is injective, we have $a' = a + \partial a''$, i.e., $[a] = [a']$.

Exercise 2.35. Check that δ is a group homomorphism.

Step 2. $\ker \alpha = \operatorname{im} \delta$.

Pick $a \in A_{k-1}$ such that $[a] \in \ker \alpha$, i.e., $\alpha(a) = \partial b$ for some $b \in B_k$. We have $\partial\beta(b) = \beta(\partial b) = \beta(\alpha(a)) = 0$. By the construction of δ, we obtain $\delta[\beta(b)] = [a]$. That is $\ker \alpha \subset \operatorname{im} \delta$.

If $a \in A_{k-1}$ is such that $[a] \in \operatorname{im} \delta$, then by the construction of δ, we have $\alpha(a) = \partial b \implies \alpha[a] = 0$.

Step 3. $\ker \delta = \operatorname{im} \beta$.

Pick some $[c] \in \ker \delta$. Using the notations of Step 1, we have $a = \partial a'$ for some $a' \in A_k$. The equations

$$\partial\big(b - \alpha(a')\big) = \partial b - \alpha(\partial a') = \partial b - \alpha(a) = 0;$$
$$\beta\big(b - \alpha(a')\big) = \beta(b) = c;$$

yield $\beta[b - \alpha(a')] = [c]$, i.e., $\ker \delta \subset \operatorname{im} \beta$.

The inclusion $\operatorname{im} \beta \subset \ker \delta$ follows immediately from the construction of δ.

Step 4. $\ker \beta = \operatorname{im} \alpha$.

Assume $b \in B_k$ satisfies $\beta[b] = 0$, that is $\partial b = 0$ and $\beta(b) = \partial c$ for some $c \in C_{k+1}$. Since β is surjective, there is some $\hat{b} \in B_{k+1}$ such that $\beta(\hat{b}) = c$. Furthermore,

$$\beta(b - \partial\hat{b}) = \beta(b) - \partial\beta(\hat{b}) = \beta(b) - \partial c = 0.$$

This yields that there exists some $a \in A_k$ such that $\alpha(a) = b - \partial\hat{b}$. Moreover,

$$\alpha(\partial a) = \partial\alpha(a) = \partial b - \partial^2\hat{b} = 0.$$

Since α is injective, we obtain $\partial a = 0$. This yields $\alpha[a] = [b - \partial\hat{b}] = [b]$, that is $\ker \beta \subset \operatorname{im} \alpha$.

The inclusion $\operatorname{im} \alpha \subset \ker \beta$ follows immediately from $\alpha \circ \beta = 0$. $\square$

2.6 Relative homology groups

For each subspace $A \subset X$ define

$$S_n(X, A) := S_n(X)/S_n(A).$$

The boundary map on $S_n(X)$ induces a boundary map on $S_n(X, A)$ and we obtain the following new chain complex:

$$\cdots \to S_{n+1}(X, A) \xrightarrow{\partial} S_n(X, A) \xrightarrow{\partial} S_{n-1}(X, A) \to \dots$$

The homology groups of this complex are denoted by $H_*(X, A)$ and are called *the homology groups of X relative to A*, or, simply, *relative homology groups*. Let us provide some details of this definition:

- Elements of $H_n(X, A)$ are represented by *relative chains* $a \in S_n(X)$ such that $\partial a \in S_{n-1}(A)$;
- $[a] = 0 \in H_n(X, A) \quad \iff \quad a = \partial b + c, \quad b \in S_{n+1}(X),\ c \in S_n(A)$.

By the very definition of $S_n(X, A)$, the sequence $0 \to S_*(A) \to S_*(X) \to S_*(X, A) \to 0$ is exact. Hence, Theorem 2.33 yields the following:

Theorem 2.36. *There is a long exact sequence of the homology groups*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \to \dots$$

Moreover, the following holds:

- *i_* is induced by the inclusion $i\colon A \subset X$;*
- *j_* is induced by the projection $S_n(X) \to S_n(X, A)$;*
- *$\delta[a] = [\partial a]$.*

□ L 6

Suppose $A \subset X$ and $B \subset Y$. A map between pairs of spaces (X, A) and (Y, B) is a map $f\colon X \to Y$ such that $f(A) \subset B$.

Proposition 2.37. *Each map $f\colon (X, A) \to (Y, B)$ induces a homomorphism of relative homology groups $H_*(X, A) \to H_*(Y, B)$.* □

Two continuous maps $f, g\colon (X, A) \to (X, B)$ are called homotopic (as maps between pairs of spaces), if there exists a continuous map $h\colon (X \times I, A \times I) \to (Y, B)$, such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. Notice that the homotopy h in this definition satisfies $h(A \times I) \subset B$.

Two pairs (X, A) and (Y, B) are said to be homotopy equivalent, if there exist $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (X, A)$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$, where id_X is viewed as a map of pairs $(X, A) \to (X, A)$ (and similarly for id_Y). Just like in the situation of Corollary 2.25, we have the following result.

Proposition 2.38. *If (X, A) and (Y, B) are homotopy equivalent, then $H_k(X, A)$ and $H_k(Y, B)$ are isomorphic for all k.* □

The following theorem, whose proof will be given in Section **??** below, turns out to be a useful tool for the computations of relative homology groups. For the time being, we take Theorem 2.39 as granted. Ref

Theorem 2.39 (Excision)**.** *Assume the subspaces $Z \subset A \subset X$ satisfy $\bar{Z} \subset \operatorname{Int} A$. Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism of relative homology groups:*

$$H_*(X \setminus Z, A \setminus Z) \cong H_*(X, A).$$

2.7 The homology groups of the spheres

Theorem 2.40. *The following holds:*

$$H_k(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 0; \\ 0 & \text{else;} \end{cases} \qquad \text{and for } n \geq 1 \qquad H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n; \\ 0 & \text{else.} \end{cases}$$

Proof. Denote

$$S^n = \{x = (x_0, \dots, x_{n+1}) \in S^{n+1} \mid x_{n+1} = 0\},$$
$$S^{n+1}_+ := \{x \in S^{n+1} \mid x_{n+1} \geq 0\}, \qquad S^{n+1}_- := \{x \in S^{n+1} \mid x_{n+1} \leq 0\}.$$

Notice that $S^{n+1}_\pm$ is homeomorphic to $B_{n+1} = \{x \in \mathbb{R}^{n+2} \mid |x| \leq 1, x_{n+1} = 0\}$. In particular, $S^{n+1}_\pm$ is contractible.

Step 1. *The map $\delta\colon H_{k+1}(S^{n+1}_-, S^n) \to H_k(S^n)$ is an isomorphism provided $k \geq 1$.*

By the long exact sequence of the pair (S^{n+1}_-, S^n) we have

$$0 = H_{k+1}(S^{n+1}_-) \to H_{k+1}(S^{n+1}_-, S^n) \xrightarrow{\delta} H_k(S^n) \to H_k(S^{n+1}_-) = 0. \tag{2.41}$$

Hence, δ is an isomorphism.

Step 2. *Define*

$$\tilde{H}_0(S^n) := \ker\big(H_0(S^n) \to H_0(S^{n+1}_-)\big) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{else.} \end{cases}$$

Then $\delta\colon H_1(S^{n+1}_-, S^n) \to \tilde{H}_0(S^n)$ is an isomorphism.

Recall that for a connected space X a generator of $H_0(X)$ is the class of any point. Hence, if $n > 0$, then the homomorphism $H_0(S^n) \to H_0(S^{n+1}_-)$ induced by the inclusion is in fact an isomorphism. In particular, $\tilde{H}_0(S^n) = 0$ in this case. However, if $n = 0$, S^0 consists of two points (in particular, has two connected components), whereas S^1_- is connected. Hence, the inclusion $\{-1\} \to S^1_-$ induces an isomorphism on H_0, however, the generator corresponding to the point $+1 \in S^0$ is in the kernel of the homomorphism $H_0(S^n) \to H_0(S^{n+1}_-)$. In particular, $\tilde{H}_0(S^0) \cong \mathbb{Z}$.

Furthermore, just like in the previous step, the long exact sequence of the pair (S^{n+1}_-, S^n) yields

$$0 = H_1(S^{n+1}_-) \to H_1(S^{n+1}_-, S^n) \xrightarrow{\delta} H_0(S^n) \to H_0(S^{n+1}_-).$$

In particular, δ is injective and, hence, an isomorphism onto its image in $H_0(S^n)$, which is the kernel of $H_0(S^n) \to H_0(S^{n+1}_-)$, that is $\tilde{H}_0(S^0)$.

Step 3. *For all $k \geq 0$ and $n \geq 0$ the map*

$$j_*\colon H_{k+1}(S^{n+1}) \to H_{k+1}(S^{n+1}, S^{n+1}_+) \tag{2.42}$$

is an isomorphism.

For $k > 0$, this follows from the long exact sequence of the pair (S^{n+1}, S^{n+1}_+):

$$0 = H_{k+1}(S^{n+1}_+) \to H_{k+1}(S^{n+1}) \xrightarrow{j_*} H_{k+1}(S^{n+1}, S^{n+1}_+) \to H_k(S^{n+1}_+) = 0$$

For $k = 0$, we have

$$0 = H_1(S^{n+1}_+) \to H_1(S^{n+1}) \xrightarrow{j_*} H_1(S^{n+1}, S^{n+1}_+) \to \underbrace{H_0(S^{n+1}_+) \to H_0(S^{n+1})}_{\text{isomorphism}} = \mathbb{Z}.$$

Hence, the third arrow represents the zero homomorphism and, therefore, j_* is surjective. Since j_* is injective, this is an isomorphism.

Step 4. *For all $k \geq 0$ the inclusion $p\colon (S^{n+1}_-, S^n) \cong (S^{n+1}, S^{n+1}_+)$ induces the isomorphism*

$$p_*\colon H_{k+1}(S^{n+1}_-, S^n) \to H_{k+1}(S^{n+1}, S^{n+1}_+). \tag{2.43}$$

Indeed, denote

$$Z := \left\{x \in S^{n+1} \mid x_{n+2} \geq \tfrac{1}{2}\right\}.$$

Then the homomorphism $H_{k+1}(S^{n+1}_-, S^n) \to H_{k+1}(S^{n+1} \setminus Z,\ S^{n+1}_+ \setminus Z)$ induced by the inclusion $(S^{n+1}_-, S^n) \to (S^{n+1} \setminus Z,\ S^{n+1}_+ \setminus Z)$ is an isomorphism, since the pairs (S^{n+1}_-, S^n) and $(S^{n+1}\setminus Z,\ S^{n+1}_+\setminus Z)$ are homotopy equivalent. Theorem 2.39 yields that the homomorphism $H_{k+1}(S^{n+1}, S^{n+1}_+) \to H_{k+1}(S^{n+1}\setminus Z,\ S^{n+1}_+\setminus Z)$ induced by the inclusion is also an isomorphism. This proves (2.43).

Step 5. *We prove this theorem*

A combination of the previous steps yields the sequence of isomorphisms

$$H_{k+1}(S^{n+1}) \xrightarrow{j_*} H_{k+1}(S^{n+1}, S^{n+1}_+) \xrightarrow{p_*^{-1}} H_{k+1}(S^{n+1}_-, S^n) \xrightarrow{\delta} \tilde{H}_k(S^n),$$

where

$$\tilde{H}_k(S^n) = \begin{cases} \tilde{H}_0(S^n), & \text{if } k = 0, \\ H_k(S^n), & \text{if } k > 0. \end{cases}$$

This implies the statement of this theorem. □

L 7

Corollary 2.44. *The n-sphere S^n is not contractible for all $n \geq 0$.* □

For a general topological space X define also

$$\tilde{H}_0(X) := \ker \varepsilon, \qquad \text{wobei} \quad \varepsilon\colon H_0(X) \to \mathbb{Z}, \quad \varepsilon\left[\sum n_i x_i\right] := \sum n_i,$$

and $\tilde{H}_k(X) = H_k(X)$ für $k \geq 1$. Using these notations we have

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n; \\ 0 & \text{else}, \end{cases}$$

for all n.

2.8 The hairy ball theorem

Recall (cf. Definition 1.18) that the degree $\deg f$ of a continuous map $f\colon S^n \to S^n$ is an integer, which is determined by the property

$$f_* a = (\deg f)\cdot a \qquad \text{for all } a \in H_n(S^n).$$

Define *the suspension* $\Sigma f\colon S^{n+1} \to S^{n+1}$ of f via

$$\Sigma f(x_0,\dots,x_{n+1}) = \begin{cases} (0,\dots,0,x_{n+1}) & \text{if } |x_{n+1}| = 1,\\ \big(tf(\frac{x_0}{t},\dots,\frac{x_n}{t}), x_{n+1}\big) & \text{if } |x_{n+1}| < 1,\end{cases}$$

where $t = \sqrt{1 - x_{n+1}^2}$.

Proposition 2.45. $\deg \Sigma f = \deg f$.

Proof. By the proof of Theorem 2.40 we have the following commutative diagram

$$\begin{tikzcd} H_{n+1}(S^{n+1}) \arrow[r, "j_*"] \arrow[d, "\Sigma f_*"'] & H_{n+1}(S^{n+1}, S^{n+1}_+) \arrow[r, "p_*^{-1}"] \arrow[d, "\Sigma f_*"'] & H_{n+1}(S^{n+1}_-, S^n) \arrow[r, "\delta"] \arrow[d, "\Sigma f_*"'] & H_n(S_n) \arrow[d, "f_*"'] \\ H_{n+1}(S^{n+1}) \arrow[r, "j_*"] & H_{n+1}(S^{n+1}, S^{n+1}_+) \arrow[r, "p_*^{-1}"] & H_{n+1}(S^{n+1}_-, S^n) \arrow[r, "\delta"] & H_n(S_n). \end{tikzcd}$$

Denoting $\alpha := \delta\circ p_*^{-1}\circ j_*$, we obtain

$$\Sigma f_*(a) = \alpha^{-1}\circ f_*\circ\alpha(x) = \alpha^{-1}\big((\deg f)\cdot\alpha(a)\big) = (\deg f)\cdot a \implies \deg\Sigma f = \deg f.$$

□

Theorem 2.46. *There is no continuous map $f\colon S^{2n} \to \mathbb{R}^{2n+1}\setminus\{0\}$ such that $f(x)\perp x$ holds for all $x\in S^{2n}$.*

Proof. The proof consists of the following steps.

Step 1. *Let*

$$s_0\colon S^n\to S^n, \qquad (x_0,x_1,\dots,x_n)\mapsto(-x_0,x_1,\dots,x_n),$$

be the restriction of the reflection in the hyperplane $\{x_0 = 0\}$. Then $\deg s_0 = -1$.

The sequence of isomorphisms

$$H_1(S^1) \xrightarrow{j_*} H_1(S^1, S^1_+) \xrightarrow{p_*^{-1}} H_1(S^1_-, S^0) \xrightarrow{\delta} \tilde H_0(S_0)$$

shows that

$$\sigma(t) = (\sin 2\pi t, \cos 2\pi t)$$

is a generator of $H_1(S^1)$. Since $s\circ\sigma(t) = \sigma(-t)$, we have $s_*[\sigma] = -[\sigma]$ and therefore the claim of this step holds for $n=1$.

If s_0 is the reflection on S^n, then Σs_0 is the reflection on S^{n+1}. The induction with respect to n yields the proof for all $n>1$.

Step 2. *For the antipodal map $A\colon S^n\to S^n,\ A(x) = -x$ we have $\deg A = (-1)^{n+1}$.*

The antipodal map on S^n is the composition of $n+1$ reflections.

Step 3. *If $f\colon S^n \to S^n$ is a continuous map without fixed points, then $f \simeq A$.*

The map
$$F(x,t) := \frac{tf(x)+(t-1)x}{|tf(x)+(t-1)x|}$$
is a well-defined homotopy between f and A.

Step 4. *If $f\colon S^n \to S^n$ is a continuous map such that $f(x) \neq -x$ for all $x \in S^n$, then f is homotopic to the identity map.*

$$\begin{aligned} f(x) \neq -x &\implies A \circ f \text{ has no fixed points } \implies A\circ f \simeq A \implies A\circ A\circ f \simeq A \circ A \\ &\implies \quad f \simeq id. \end{aligned}$$

Step 5. *We prove the hairy ball theorem.*

Assume there exists a continuous map $f\colon S^{2n} \to \mathbb{R}^{2n+1} \setminus \{0\}$ such that $f(x) \perp x$. By renormalizing we can assume without loss of generality that $f\colon S^{2n} \to S^{2n}$. The assumption $f(x) \perp x$ yields in particular that f has no fixed points. By Step 3, f is homotopic to A.
On the other hand, f is homotopic to id by Step 4. This yields a contradiction since
$$A \simeq f \simeq id \implies 1 = \deg id = \deg A = (-1)^{2n+1} = -1.$$
□

This theorem is often informally formulated as follows.

Corollary 2.47. *One can not comb a hairy ball flat without creating a cowlick.* □

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Remark 2.48. Each sphere of odd dimension $2n-1 \geq 1$ admits a continuous map $f\colon S^{2n-1} \to \mathbb{R}^{2n} \setminus \{0\}$ such that $f(x) \perp x$ holds for all $x \in S^{2n-1}$. Indeed,
$$\begin{aligned} S^{2n-1} &= \Big\{x = (x_0, x_1, x_2, x_3, \dots, x_{2n-2}, x_{2n-1}) \mid \sum x_i^2 = 1\Big\} \\ f(x) &= (x_1, -x_0, x_3, -x_2, \dots, x_{2n-1}, -x_{2n-2}). \end{aligned}$$

Proposition 2.49. *Let $[S^n, S^n]$ be the set of all homotopy classes of continuous maps $S^n \to S^n$, where $n \geq 1$. The map*
$$[S^n, S^n] \to \mathbb{Z}, \qquad [f] \mapsto \deg f \tag{2.50}$$
is surjective.

Proof. If $n = 1$, for each $k \in \mathbb{Z}$ we have an explicit continuous map $f_k\colon S^1 \to S^1$ of degree k, namely $f_k(z) := z^k$. If $n = 2$, we have $\deg \Sigma f_k = \deg f_k = k$. The induction with respect to n finishes the proof. □

Remark 2.51. It can be shown that (2.50) is even bijective (Theorem of Hopf). Also, $[S^n, S^n]$ is a group and (2.50) is an isomorphism of groups.

2.9 Group actions on the spheres

Let G be a group. We say that G acts on a set X if a homomorphism $\rho\colon G \to \mathrm{Aut}(X)$ is given, where $\mathrm{Aut}(X)$ is the group of all bijective maps $X \to X$. An action is called *free* whenever the following holds:

$$\forall x \in X \quad \mathrm{Stab}_x := \{g \in G \mid \rho(g)(x) = x\} = \{e\}.$$

If X is in addition a topological space, then we require also that for each $g \in G$ the map $\rho(g)$ is a homeomorphism.

Theorem 2.52. *$\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that acts freely on S^{2n}.*

Proof. Assume that $G \neq \{e\}$ acts on S^{2n} freely. Consider the map

$$d\colon G \to \{\pm 1\}, \qquad d(g) = \deg(\rho(g)).$$

Here d takes values in $\{\pm 1\}$, since each $\rho(g)$ is a homeomorphism. Furthermore, $d(gh) = \deg(\rho(g)\rho(h)) = d(g)d(h)$, that is d is a group homomorphism.

If $g \neq e$, then $\rho(g)$ has no fixed points. By Step 4 in the proof of Theorem 2.46, the following holds: $\deg \rho(g) = \deg A = -1$, i.e., d has a trivial kernel and is surjective.

Clealy $\mathbb{Z}/2\mathbb{Z}$ acts freely on S^{2n}:

$$\rho(e) = \mathrm{id}, \qquad \rho(1) := A,$$

where A is the antipodal map. □

Remark 2.53. On the odd-dimensional spheres other non-trivial groups may act freely. For example, $\mathrm{U}(1) := \{z \in \mathbb{C} \mid |z| = 1\} \cong S^1$ acts on

$$S^{2n-1} = \{(z_0, \dots, z_n) \in \mathbb{C}^n \mid \sum |z_j|^2 = 1\}$$

via the homomorphism

$$w \mapsto f_w, \qquad f_w(z) = (wz_0, \dots, wz_n).$$

2.10 Homology groups of graphs

Definition 2.54. A (finite topological) graph is a pair (G, V), where G is a Hausdorff space and $G \supset V$ is a finite subset. The elements of V are called vertices of G. Besides, we require that the following holds:

- $G \setminus V$ consists of finitely many path components $\mathring{e}_1, \dots, \mathring{e}_J$. The closure e_j of each component $\mathring{e}_j$ is homeomorphic to the interval $[0,1]$ and is called an edge of G;
- $e_j \setminus \mathring{e}_j$ consists of two different vertices.

The aim of this section is to prove the following result.

Theorem 2.55. *The group $H_1(G)$ is free and finitely generated. Moreover, the following holds:*

$$\mathrm{rk}\, H_0(G) - \mathrm{rk}\, H_1(G) = \#\, \textit{vertices} - \#\, \textit{edges} =: \chi(G).$$

The number $\chi(G)$ is called the Euler characteristic of G.

The proof requires some notions and auxiliary claims that we consider first. The proof of Theorem 2.55 can be found at the end of this section.

Definition 2.56. A subset $A \subset B$ is called a deformation retract of B, if the following holds: There exists a continuous map $r\colon B \to A$, which is called *a retraction*, such that the following holds:

$$r \circ \imath = \mathrm{id}_A \quad \text{and} \quad \imath \circ r \simeq id_B,$$

where $\imath\colon A \subset B$ is the inclusion.

It follows immediately from the above definition that the induced maps

$$\imath_*\colon H_*(A) \to H_*(B) \quad \text{and} \quad r_*\colon H_*(B) \to H_*(A)$$

are mutually inverse. In particular, both maps are isomorphisms.

Lemma 2.57. *Let A be a deformation retract of B, where $A \subset B \subset X$. Then the inclusion $\imath\colon (X, A) \to (X, B)$ induces an isomorphism*

$$\imath_*\colon H_*(X, A) \to H_*(X, B).$$

Proof. The proof of this lemma hinges on the following algebraic fact.

Lemma 2.58 ("Five lemma"). *Assume the horizontal sequences in the commutative diagram of abelian groups*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

are exact. Furthermore, assume that f_2 and f_4 are isomorphisms, f_1 is an epimorphism, and f_5 is a monomorphism. Then f_3 is an isomorphism. □

Consider the commutative diagram

$$\begin{tikzcd}
H_k(A) \arrow[r] \arrow[d, "\imath_*"'] & H_k(X) \arrow[r] \arrow[d, "id"'] & H_k(X, A) \arrow[r] \arrow[d, "\imath_*"'] & H_{k-1}(A) \arrow[r] \arrow[d, "\imath_*"'] & H_{k-1}(X) \arrow[d, "id"'] \\
H_k(B) \arrow[r] & H_k(X) \arrow[r] & H_k(X, B) \arrow[r] & H_{k-1}(B) \arrow[r] & H_{k-1}(X).
\end{tikzcd}$$

Here the horizontal sequences are long exact sequences of the pairs (X, A) and (X, B). Furthermore, the first two vertical arrows and the last two ones represent isomorphisms. The proof now follows from the five lemma. □

From the long exact sequence of the pair $\big([0, 1], \{0, 1\}\big)$ we obtain the following result.

Lemma 2.59. *The following holds:*

$$H_k\big([0, 1], \{0, 1\}\big) \cong \begin{cases} \mathbb{Z} & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

□

Proposition 2.60. *The inclusion* $\imath_j\colon (e_j, \partial e_j) \to (G, V)$ *induces a monomorphism*

$$\imath_{j*}\colon H_k(e_j, \partial e_j) \to H_k(G, V).$$

Moreover, the following holds:

$$H_k(G, V) = \bigoplus_j \operatorname{im} \imath_{j*} \cong \begin{cases} \mathbb{Z}^J & \textit{if } k = 1, \\ 0 & \textit{if } k > 1. \end{cases}$$

Proof. Let $f_j\colon [0,1] \to e_j$ be a homeomorphism, $a_j := f(\frac{1}{2})$, and $d_j := f([\frac{1}{4}, \frac{3}{4}])$. Denote also $A = \{a_1, \dots, a_J\}$ and $D = d_1 \sqcup \dots \sqcup d_J$. Consider the commutative diagram

$$\begin{tikzcd}
H_k\big(d_j, d_j \setminus \{a_j\}\big) \arrow[r, "\alpha_1"] \arrow[d] & H_k(e_j, e_j \setminus \{a_j\}) \arrow[d] & H_k(e_j, \partial e_j) \arrow[l, "\beta_1"'] \arrow[d] \\
H_k(D, D \setminus A) \arrow[r, "\alpha_2"] & H_k(G, G \setminus A) & H_k(G, V). \arrow[l, "\beta_2"']
\end{tikzcd}$$

All four horizontal homomorphisms are in fact isomorphisms. Indeed, α_1 and α_2 are isomorphisms by excision, β_1 and β_2 by Lemma 2.57.

Since

$$H_k(D, D \setminus A) = \bigoplus_{j=1}^J H_k(d_j, d_j \setminus \{a_j\}) \cong \bigoplus_{j=1}^J H_k(e_j, \partial e_j),$$

we obtain the claim of this proposition. $\square$ L 9

Proof of Theorem 2.55. For the proof we need the following algebraic fact.

Lemma 2.61. *Any subgroup of a free abelian group is also free.* $\square$

The remaining part of the proof consists of the following three steps.

Step 1. $H_1(G)$ *is free.*

The long exact sequence of the pair (G, V) yields:

$$0 \to H_1(G) \to H_1(G, V) \to H_0(V) \to H_0(G) \to 0. \tag{2.62}$$

$H_1(G, V)$ is free $\implies$ $H_1(G)$ is free.

Step 2. *Let* $f\colon A \to F$ *be an epimorphism between two finitely generated free abelian groups. Then*

$$A = \ker f \oplus A_0,$$

where $f\colon A_0 \to F$ *is an isomorphism and* $\ker f$ *is free.*

Let $f_1, \dots, f_n$ be generators of F. Choose $b_1, \dots, b_n \in A$ such that $f(b_j) = f_j$. Since $\ker f \subset A$ and A is free, $\ker A$ is also free. Pick generators $a_1, \dots, a_k$ of $\ker f$. Then we have $A = \mathbb{Z}[a_1, \dots, a_k, b_1, \dots b_n]$. Indeed, for an arbitrary element $a \in A$ we have

$$f(a) \in F \implies f(a) = \sum m_j f_j \implies a - \sum m_j b_j \in \ker f \implies a - \sum m_j b_j = \sum p_i a_i.$$

Moreover, the representation $a = \sum m_j b_j + \sum p_i a_i$ is unique.

Step 3. *We prove this theorem.*

Without loss of generality we can assume that G is path connected. Then (2.62) yields

$$0 \to H_1(G) \to H_1(G, V) \to \tilde{H}_0(V) \to 0,$$

i.e., $H_1(G, V) \cong H_1(G) \oplus \tilde{H}_0(V)$. This yields in turn

$$\#\,\text{edges} = \operatorname{rk} H_1(G, V) = \operatorname{rk} H_1(G) + \operatorname{rk} \tilde{H}_0(V) = \operatorname{rk} H_1(G) + \#\,\text{vertices} - 1.$$

□

Example 2.63. The circle $G = e_0 \cup e_1$, $V = \{v_1, v_2\}$. We have $\chi(G) = 0 \implies rkH_1(G) = \operatorname{rk} H_0(G) = 1$.

Example 2.64. The wedge product of two circles. $G = e_0 \cup \cdots \cup e_4$, $V = \{v_1, v_2, v_3\}$.

Picture

$\chi(G) = -1 \implies \operatorname{rk} H_1(G) = 2$.

Definition 2.65. A graph (G, V) is called *planar*, if there is an embedding of G into $\mathbb{R}^2$, that is if G can be drawn on the plane such that edges are represented by simple continuous curves that intersect only at the vertices.

Each connected planar graph decomposes $\mathbb{R}^2$ into a finite number of bounded domains, which are called *faces*, and an unbounded domain, which is also called a face. Moreover, each bounded domain is homeomorphic to a disc (a theorem of Schoenflies).

Theorem 2.66 (Euler)**.** *For any planar connected graph G we have*

$$\#\,\textit{vertices} - \#\,\textit{edges} + \#\,\textit{faces} = 2. \tag{2.67}$$

Notice that the unbounded face also counts in (2.67).

Proof. By means of the stereographic projection we can view G as a subspace of S^2. Notice that the unbounded face together with the point at infinity is mapped to a face on S^2.

Just like in the proof of Proposition 2.60 we obtain

$$H_2(S^2, G) \cong \mathbb{Z}^F \qquad \text{and} \qquad H_k(S^2, G) = 0 \quad \text{for all } k \notin \{0,\, 2\,\},$$

where F is the number of faces. From the long exact sequence of the pair (S^2, G) we have

$$0 \to H_2(S^2) \to H_2(S^2, G) \to H_1(G) \to H_1(S^2) = 0,$$

which yields

$$\mathbb{Z}^F \cong \mathbb{Z} \oplus H_1(G) \implies F = 1 + \operatorname{rk} H_0(G) - \#\,\text{vertices} + \#\,\text{edges}$$

by Theorem 2.55. Since G is connected by the hypothesis, we have $\operatorname{rk} H_0(G) = 1$ and therefore (2.67) holds. □

Exercise 2.68. Solve the “Three utilities problem”: Suppose there are three cottages on a plane and each needs to be connected to the water, gas, and electricity companies. Without using a third dimension or sending any of the connections through another company or cottage, is there a way to make all nine connections without any of the lines crossing each other?

Hint: to obtain a solution consider the graph $K_{3,3}$:

Image of $K_{3,3}$

Assuming $K_{3,3}$ is planar, show that the following holds:

(i) $\#\,\text{faces} \leq \frac{1}{2}\#\,\text{edges}$;

(ii) $\#\,\text{edges} \leq 2\#\text{vertices} - 4$.

Deduce from the last property that $K_{3,3}$ is non-planar.

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2.11 Homology groups of surfaces

2.11.1 The torus

The torus $\mathbb{T}^2$ can be understood as a square R with opposite sides being glued as shown on Fig 2.2.

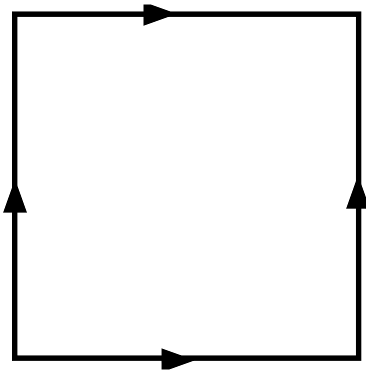

Figure 2.2: The torus as a square with opposite sides being glued.

Let $f\colon R \to \mathbb{T}^2$ be the quotient map. Then $f(\partial R)$ consists of two circles A and B intersecting at a point.

Theorem 2.69.

$$H_k(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & \textit{for } k = 0, 2; \\ \mathbb{Z}^2 & \textit{for } k = 1; \\ 0 & \textit{else}. \end{cases}$$

Proof. The proof consists of the following three steps.

Step 1. *The map $f\colon (R, \partial R) \to (\mathbb{T}^2, A \cup B)$ induces an isomorphism*

$$f_*\colon H_*(R, \partial R) \to H_*(\mathbb{T}^2, A \cup B).$$

Let m be the center of the square R and D a disc centered at m contained in the interior of R. Just like in the proof of Proposition 2.60 one obtains that all horizontal arrows of the commutative diagram

$$\begin{tikzcd}
H_k\big(R, \partial R\big) \arrow[r] \arrow[d, "f_*"'] & H_k\big(R, R \setminus \{m\}\big) \arrow[d] & H_k\big(D, D \setminus \{m\}\big) \arrow[l] \arrow[d, "f_*"] \\
H_k(\mathbb{T}^2, A \cup B) \arrow[r] & H_k\big(\mathbb{T}^2, \mathbb{T}^2 \setminus \{f(m)\}\big) & H_k\big(f(D), f(D) \setminus \{f(m)\}\big) \arrow[l]
\end{tikzcd}$$

represent isomorphisms (to prove this one needs in particular that $A \cup B$ is a deformation retract of $\mathbb{T}^2 \setminus \{m\}$). Since the right vertical arrow represents an isomorphism, we obtain that the leftmost vertical arrow represents an isomorphism too.

Step 2. *If $k \geq 1$, then*

$$H_k(\mathbb{T}^2, A \cup B) \cong \begin{cases} \mathbb{Z} & \text{for } k = 2, \\ 0 & \text{else.} \end{cases}$$

The statement of this step follows from the long exact sequence of the pair $(R, \partial R)$.

Step 3. *We prove this theorem.*

The non-trivial part of the long exact sequence of the pair $(\mathbb{T}^2, A \cup B)$ has the following form

$$0 \to H_2(\mathbb{T}^2) \to H_2(\mathbb{T}^2, A \cup B) \xrightarrow{\delta} H_1(A \cup B) \to H_1(\mathbb{T}^2) \to 0,$$

where $H_2(\mathbb{T}^2, A \cup B) \cong \mathbb{Z}$ and $H_1(A \cup B) \cong \mathbb{Z}^2$ by Example 2.64.

To determine δ, consider the commutative diagram

$$\begin{tikzcd} H_2\big(R, \partial R\big) \arrow[r, "\delta'"] \arrow[d, "f_*"'] & H_1(\partial R) \arrow[d, "f'_*"] \\ H_2(\mathbb{T}^2, A \cup B) \arrow[r, "\delta"] & H_1\big(A \cup B\big), \end{tikzcd}$$

where $f' \colon \partial R \to A \cup B$ is the restriction of f. The induced map f'_* is trivial (*Why?*). Since f_* and δ' are isomorphisms, δ must be also trivial. This yields

$$H_2(\mathbb{T}^2) \cong \ker \delta = H_2(\mathbb{T}^2, A \cup B) \cong \mathbb{Z} \qquad \text{and} \qquad H_1(\mathbb{T}^2) \cong H_1(A \cup B) \cong \mathbb{Z}^2.$$

This finishes the proof. □

In fact, tracing through the above proof we can work out the generators of $H_1(\mathbb{T}^2)$. Indeed, it was shown that the inclusion $A \cup B \subset \mathbb{T}^2$ induces an isomorphism $H_1(A \cup B) \to H_1(\mathbb{T}^2)$. Hence, the circles A and B generate $H_1(\mathbb{T}^2)$. L 11

2.11.2 The projective plane

The projective plane $\mathbb{RP}^2$ can be defined as a square R with the opposite sides being glued as shown on Figure 2.3.

Let $f \colon R \to \mathbb{RP}^2$ be the quotient map. Then, unlike in the case of the torus, $A := f(\partial R)$ is a circle in $\mathbb{RP}^2$.

Theorem 2.70.

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & \text{for } k = 0; \\ \mathbb{Z}/2\mathbb{Z} & \text{for } k = 1; \\ 0 & \text{else.} \end{cases}$$

Proof. Just like in the proof of Theorem 2.69 we obtain that

$$f_* \colon H_*(R, \partial R) \to H_*(\mathbb{RP}^2, A)$$

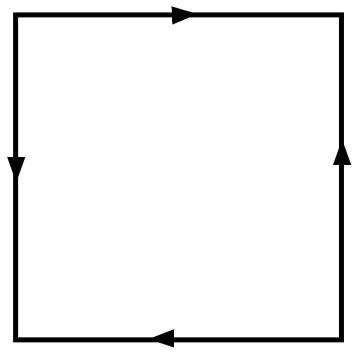

Figure 2.3: The real projective plane as a square with opposite sides being glued.

is an isomorphism. The non-trivial part of the long exact sequence of the pair $(\mathbb{RP}^2, A)$ is of the following form:

$$0 \to H_2(\mathbb{RP}^2) \to H_2(\mathbb{RP}^2, A) \xrightarrow{\delta} H_1(A) \xrightarrow{i_*} H_1(\mathbb{RP}^2) \to 0.$$

To determine the Bockstein homomorphism δ, consider the commutative diagram

$$\begin{tikzcd} H_2\big(R, \partial R\big) \arrow[r, "\delta'"] \arrow[d, "f_*"'] & H_1(\partial R) \arrow[d, "f'_*"] \\ H_2(\mathbb{RP}^2, A) \arrow[r, "\delta"] & H_1\big(A\big). \end{tikzcd}$$

A short thought yields that f'_* is a multiplikation with ± 2 (*Why?*), i.e., δ is injective and $H_1(A)/\operatorname{im}\delta \cong \mathbb{Z}/2\mathbb{Z}$. In particular, $H_2(\mathbb{RP}^2) \cong \ker\delta = \{0\}$ and $i_*\colon H_1(A)/\operatorname{im}\delta \to H_1(\mathbb{RP}^2)$ is an isomorphism $\square$

2.11.3 The Klein bottle

Just like torus and projective plane, the Klein bottle K can be also defined as a square R with glued opposite sides as shown on Figure 2.4.

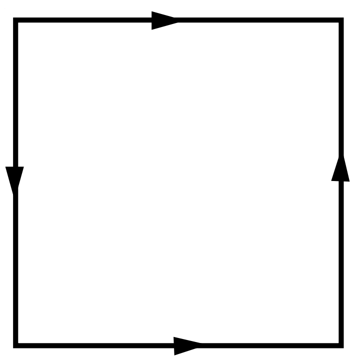

Figure 2.4: The Klein bottle as a square with opposite sides being glued.

Theorem 2.71.

$$H_k(K) = \begin{cases} \mathbb{Z} & \text{for } k = 0; \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{for } k = 1; \\ 0 & \text{else}. \end{cases}$$

The proof of this theorem is left as an exercise. L 12

2.11.4 Connected sum of manifolds

Let me recall the definition of a manifold.

Definition 2.72. A (topological) manifold of dimension n is a Hausdorff space[1] M such that for each point $m \in M$ there exists a neighborhood, which is homeomorphic to an open subset in $\mathbb{R}^n$.

Manifolds of dimension 1 are usually called *curves* and manifolds of dimension two *surfaces*.

Exercise 2.73. Show that for each $x_0 \in \mathbb{R}^n$ and $r > 0$ the open ball $\mathring{B}_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$ is homeomorphic to $\mathbb{R}^n$. Furthermore, using this show that each point of a manifold has a neighborhood homeomorphic to $\mathbb{R}^n$.

Example 2.74.

- $\mathbb{R}^n$ is an n-manifold; More generally, any open subset of $\mathbb{R}^n$ is an n-manifold;
- S^n is an n-manifold;
- The torus, projective plane, and Klein bottle are surfaces;

Let M_1 and M_2 be two connected manifolds of dimension n. Choose $m_j \in M_j$ and homeomorphisms $\varphi_j \colon B_1(0) \to U_j \subset M_j$ such that $\varphi_j(0) = m_j$. With the help of the identification $B_1(0) \setminus \{0\} \cong S^{n-1} \times (0,1)$, φ_j induces a homeomorphism $S^{n-1} \times (0,1) \to U_j \setminus \{m_j\}$.

Definition 2.75. The space

$$\begin{aligned} &M_1 \# M_2 := \big(M_1 \setminus \{m_1\} \sqcup M_2 \setminus \{m_2\}\big)/\sim, \quad \text{where} \\ &\varphi_1(x,r) \sim \varphi_2(x, 1-r), \qquad x \in S^{n-1} \text{ and } r \in (0,1), \end{aligned}$$

is called *the connected sum* of M_1 and M_2.

Figure.

Exercise 2.76. Show that $M_1 \# M_2$ is a manifold of dimension n and does not depend on the choices involved in the construction (meaning the following: For any other choice of points m_j and homeomorphisms φ_j the results of the above construction are homeomorphic).

[1] In addition, it is required that M satisfies the second countability axiom, i.e., M has at most countable basis of its topology. This is not crucial for the arguments used below, hence I do not mention this explicitly in the definition.

2.11.5 Compact surfaces

Denote

$$\Sigma_0 = S^2, \quad \Sigma_1 = \mathbb{T}^2, \quad \Sigma_2 = \mathbb{T}^2 \# \mathbb{T}^2, \ \dots, \quad \Sigma_g = \#_g \mathbb{T}^2.$$

Proposition 2.77. *The surface Σ_2 can be constructed from the Decagon*

Figure

via gluing of sides. □

Proof. First construct the "connected sum of squares" as shown on Figure 2.5. To obtain Σ_2

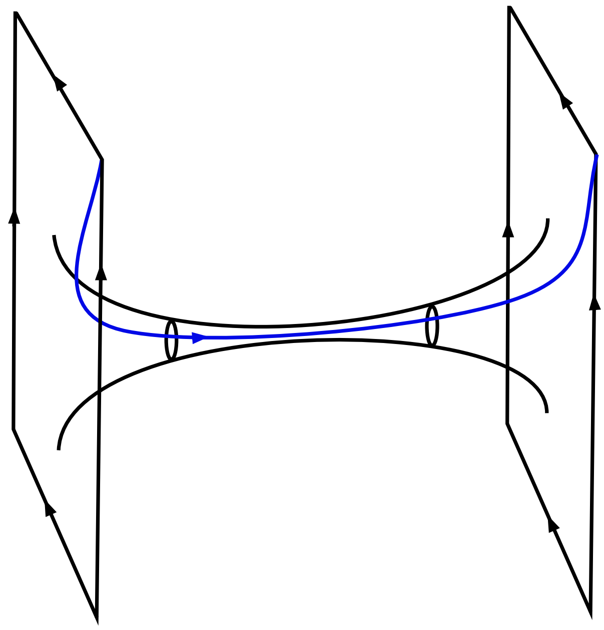

Figure 2.5: The connected sum of two tori represented by squares.

from this we still need to glue the opposite sides of the two "squares" as indicated on the picture.

Pick a segment connecting two vertices of the squares as shown on the Figure 2.5 (the colored segment) and cut the "connected sum" along this segment. The result of this is a decagon. This means that we can obtain Σ_2 after gluing appropriate sides of this decagon. □

Induction with respect to g yields the following.

Corollary 2.78. *For each $g \geq 1$ the surface Σ_g can be constructed from $(6g-2)$-gon R_{6g-2} via gluing of sides.* □

Remark 2.79. The representation of Σ_g in the above corollary is not optimal in the following sense: Σ_g can be obtained from a $(2g+2)$-gon via gluing of sides. For our purposes the existence of some representation will suffice.

By the inspection of the construction of Σ_g from R_{6g-2} just like in the proof of Step 3 of Theorem 2.69, we obtain the following.

Proposition 2.80. *If $f\colon R_{6g-2} \to \Sigma_g$ denotes the quotient map, then the induced homomorphism $H_1(\partial R_{6g-2}) \to H_1\big(f(\partial R_{6g-2})\big)$ is trivial.* □

Theorem 2.81. *We have*

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2; \\ \mathbb{Z}^{2g} & \text{if } k = 1; \\ 0 & \text{else.} \end{cases} \tag{2.82}$$

□

The proof of this theorem uses Proposition 2.80 and the argument is parallel to the one used in the proof of Theorem 2.69. The details are left to the reader.

Denote also

$$S_1 := \mathbb{RP}^2, \quad S_2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \quad \text{und} \quad S_g = S_{g-1} \# \mathbb{RP}^2.$$

Just like in Theorem 2.81 one can show, that the homology groups of S_g are given by

$$H_k(S_g) = \begin{cases} \mathbb{Z} & \text{if } k = 0; \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k = 1; \\ 0 & \text{else.} \end{cases}$$

In particular, the computations above yield the following.

Proposition 2.83. *The surfaces*

$$\Sigma_0,\ \Sigma_1, \dots, \Sigma_g, \dots, \quad S_1,\ S_2, \dots, S_g, \dots \tag{2.84}$$

are pairwise non-homeomorphic. □

Theorem 2.85 (Classification of curves)**.** *Each connected curve (i.e., 1-manifold) is homeomorphic either to the interval $(0, 1)$ or to the circle S^1.*

Proof. See [Mil65] or [GP74]. □

Theorem 2.86 (Classification of compact surfaces)**.** *Each compact connected surface is homeomorphic to Σ_g or S_g for some $g \geq 0$, that is (2.84) is a complete list of all compact surfaces up to homeomorphisms.* □

2.12 The Meyer–Vietoris sequence

Let $A, B \subset X$ be two subsets. Consider the homomorphisms

$$i_*\colon H_*(A \cap B) \to H_*(A), \quad j_*\colon H_*(A \cap B) \to H_*(B),$$
$$k_*\colon H_*(A) \to H_*(X) \quad \text{and} \quad l_*\colon H_*(B) \to H_*(X).$$

Furthermore, define

$$\begin{aligned} \varphi&\colon H_*(A \cap B) \to H_*(A) \oplus H_*(B), & \varphi(x) &= (i_*(x), j_*(x)) \quad \text{and} \\ \psi&\colon H_*(A) \oplus H_*(B) \to H_*(X), & \psi(u, v) &= k_*(u) - l_*(v). \end{aligned} \tag{2.87}$$

Theorem 2.88. *If $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$, then for all $k \in \mathbb{N}$ there is a natural homomorphism*

$$\Delta \colon H_k(X) \to H_{k-1}(A \cap B)$$

such that the sequence

$$\cdots \to H_k(A \cap B) \xrightarrow{\varphi} H_k(A) \oplus H_k(B) \xrightarrow{\psi} H_k(X) \xrightarrow{\Delta} H_{k-1}(A \cap B) \to \dots \tag{2.89}$$

is exact. This sequence is also exact for $\tilde{H}_*$ whenever $A \cap B \neq \emptyset$.

We postpone the proof of this theorem till Section 2.14 below and take this result as granted for the time being.

Example 2.90 (The spheres). Define

$$S^n = \{(x_0, \dots, x_n) \mid \sum x_i^2 = 1\},$$
$$A := S^n \setminus \{(0, \dots, 0, 1)\} \cong \mathbb{R}^n, \quad B := S^n \setminus \{(0, \dots, 0, -1)\} \cong \mathbb{R}^n.$$

Since $A \cap B \cong \mathbb{R}^n \setminus \{0\}$ and S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$, we have the following exact sequence:

$$0 \to \tilde{H}_k(S^n) \to \tilde{H}_{k-1}(S^{n-1}) \to 0.$$

This yields immediately that the homology groups of the spheres are as described in Theorem 2.40. L 13

Example 2.91 (The torus). Let $D_1 \subset D_2 \subset \mathrm{Int}(R)$ be two discs with the same center. Setting $A := \mathbb{T}^2 \setminus D_1$ and $B := D_2$, the following holds:

- The wedge product of two circles ($A \cup B$ in the notation of Subsection 2.11.1) is a deformation retract of $\mathbb{T}^2 \setminus D_1$;
- S^1 is the deformation retract of $A \cap B$.

Using these properties and the Mayer–Vietoris sequence, we have:

$$0 \to H_2(\mathbb{T}^2) \to H_1(S^1) \xrightarrow{\varphi} H_1(\mathbb{T}^2 \setminus D_1) \oplus 0 \to H_1(\mathbb{T}^2) \to \tilde{H}_0(S^1) = 0.$$

Since φ is the zero homomorphism (*why?*), we obtain:

$$H_2(\mathbb{T}^2) \cong H_1(S^1) \cong \mathbb{Z} \quad \text{and} \quad H_1(\mathbb{T}^2) \cong H_1(S^1 \vee S^1) \cong \mathbb{Z}^2.$$

Exercise 2.92. Compute the homology groups of the projective plane and the Klein bottle using the Meyer–Vietoris sequence.

Definition 2.93. Let X and Y be two topological spaces with chosen points $x_0 \in X$ and $y_0 \in Y$. The space

$$X \vee Y = \big(X \sqcup Y\big)/\{x_0, y_0\}$$

is called *the wedge product* of (X, x_0) and (Y, y_0).

Proposition 2.94. *If x_0 is a deformation retract of a neighborhood $U \subset X$ and y_0 is a deformation retract of a neighborhood $V \subset Y$, then*

$$\tilde{H}_*\big(X \vee Y\big) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y).$$

Proof. Set $A = X \cup V$ and $B = Y \cup U$. Then $U \cup V$ retracts onto the point $[x_0] = [y_0]$ in $X \vee Y$. One obtains the claim of this proposition immediately from the Meyer–Vietoris sequence. □

Corollary 2.95. *For all $n \geq 1$ we have*

$$\tilde{H}_k\Big(\bigvee_{j=1}^{N} S^n\Big) \cong \begin{cases} \mathbb{Z}^N & \text{if } k = n, \\ 0 & \text{else.} \end{cases}$$

□

2.13 Homology groups of a pair and a quotient

Let G be an abelian group and $K \subset H \subset G$ subgroups. Recall that this yields the following exact sequence:

$$0 \to H/K \to G/K \to G/H \to 0$$

For $B \subset A \subset X$, this yields the following exact sequence

$$0 \to S_*(A,B) \to S_*(X,B) \to S_*(X,A) \to 0.$$

By Theorem 2.33 we obtain *the long exact sequence of the triple* (X,A,B):

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \dots$$

Theorem 2.96. *Let $A \subset X$ be a closed subset such that A is a deformation retract of a neighborhood $U \supset A$. Then the quotient map $q\colon (X,A) \to (X/A, A/A)$ induces an isomorphism*

$$q_*\colon H_*(X,A) \to H_*(X/A, A/A) \cong \tilde{H}_*(X).$$

Proof. The proof consists of the following two steps.

Step 1. *$\imath_*\colon H_*(X,A) \to H_*(X,U)$ is an isomorphism.*

Since A is a deformation retract of U, we have that the map $H_*(A) \to H_*(U)$ induced by the inclusion is an isomorphism. From the long exact sequence of the pair (U,A) we obtain that $H_*(U,A)$ is trivial. An application of the long exact sequence of the triple (X,U,A)

$$0 = H_n(U,A) \to H_n(X,A) \to H_n(X,U) \to H_{n-1}(U,A) = 0$$

finishes the proof of this step.

Step 2. *We prove this theorem.*

Consider the commutative diagram

$$\begin{tikzcd}
H_k(X,A) \arrow[r] \arrow[d, "q_*"'] & H_k(X,U) \arrow[d] & H_k(X \setminus A,\, U \setminus A) \arrow[l] \arrow[d, "q_*"] \\
H_k(X/A, A/A) \arrow[r] & H_k(X/A, U/A) & H_k(X/A \setminus A/A,\, U/A \setminus A/A). \arrow[l]
\end{tikzcd}$$

By Step 1, the two left horizontal arrows represent isomorphisms. The right horizontal arrows also represent isomorphisms by excision. The right vertical arrow also represents an isomorphism, since the restriction of q to the complement of A is a homeomorphism. Hence, q_* on the left is also an isomorphism.

Finally, the long exact sequence of the pair (X,x_0), where $x_0 \in X$, shows that $\tilde{H}_*(X)$ and $H_*(X/A, A/A)$ are isomorphic. □

2.14 Proof of the exactness of the Mayer–Vietoris sequence and excision

Let $\mathcal{U} = \{U_j\}$ be a family of subsets of X such that $\{\mathrm{Int}(U_j)\}$ is a covering of X. Denote

$$S_*^{\mathcal{U}}(X) := \Big\{ \sum_i n_i \sigma_i \mid \forall i \quad \exists j \quad \text{mit der Eigenschaft: } \operatorname{im} \sigma_i \subset U_j \Big\}.$$

Clearly, $S_*^{\mathcal{U}}(X)$ is a subcomplex of $S_*(X)$. Denote by $H_*^{\mathcal{U}}(X)$ the homology groups of this complex. The main step in the proof of the excision theorem is the following.

Proposition 2.97. *The inclusion $\imath\colon S^{\mathcal{U}}_*(X) \to S_*(X)$ is a chain homotopy equivalence. In particular, $H^{\mathcal{U}}_*(X) \cong H_*(X)$.*

Chain homotopy equivalence is not yet defined.

For the proof of this proposition we need some auxiliary claim and constructions. The proof itself can be found on Page 37 below.

Let $\Delta = \Delta(x_0, \dots, x_k)$ be a simplex in an Euclidean space V. For an arbitrary $b \in V$ define the cone of Δ by the formula

$$C_b(\Delta) = \Delta(b, x_0, \dots, x_k). \tag{2.98}$$

Geometrically $C_b(\Delta)$ is the cone of Δ (at least in the case when b is not contained in the affine subspace generated by $x_0, \dots, x_k$).

The point

$$b = b(\Delta) := \frac{1}{k+1} \sum x_j$$

is called *the barycenter* of Δ. *The barycentric subdivision* $\mathrm{Sd}(\Delta)$ is a chain in V, which is defined recursively in k, namely:

$$\begin{aligned} &\mathrm{Sd}\big(\Delta(x_0)\big) = \Delta(x_0) && \text{if } k = 0,\\ &\mathrm{Sd}(\Delta) = C_{b(\Delta)}\big(\mathrm{Sd}(\partial\Delta)\big) && \text{if } k > 0. \end{aligned} \tag{2.99}$$

For an arbitrary subset $A \subset \mathbb{R}^n$ *the diameter* of A is defined by

$$\operatorname{diam} A := \sup_{x,y\in A} |x - y|.$$

Lemma 2.100. *For each simplex Δ', which appears in the representation of $\mathrm{Sd}(\Delta)$ as a chain, we have*

$$\operatorname{diam} \Delta' \le \frac{k}{k+1} \operatorname{diam} \Delta. \tag{2.101}$$

Proof. The proof consists of the following two steps.

Step 1. *For $\Delta = \Delta(x_0, \dots, x_k)$ we have*

$$\operatorname{diam} \Delta = \max_{i,j} |x_i - x_j|$$

Pick $x \in \Delta$ and set $y = \sum t_j x_j \in \Delta$, where $\sum t_j = 1,\ t_j \in [0, 1]$. We have

$$\begin{aligned} |x - y| &= \big|x - \sum t_j x_j\big| = \big|\sum t_j (x - x_j)\big| \le \sum t_j |x - x_j| \\ &\le \max_j |x - x_j|. \end{aligned} \tag{2.102}$$

This yields

$$|x - y| \le \max_j |x - x_j| \le \max_{i,j} |x_i - x_j|.$$

Step 2. *We prove this lemma.*

We apply induction with respect to k. For $k = 0$ Inequality (2.101) clearly holds. Furthermore, we assume that this inequality also holds for all $(k-1)$-simplexes in V. Let Δ' be a simplex, which appears in the representation of $\mathrm{Sd}(\Delta)$, that is $\Delta' = \big(b(\Delta), y_0, \ldots, y_{k-1}\big)$, where all y_j are contained in some face $\partial_j\Delta$ of Δ. By Step 1, we obtain

$$\operatorname{diam}\Delta' \leq \max\big\{|y_i - y_j|, |b - y_i|\big\}.$$

Furthermore, we have

$$\begin{aligned}
|y_i - y_j| &\leq \operatorname{diam}\Delta(y_0, \ldots, y_{k-1}) \\
&\leq \frac{k-1}{k}\operatorname{diam}\partial_j\Delta && \text{by the induction hypothesis} \\
&\leq \frac{k-1}{k}\operatorname{diam}\Delta && \partial_j\Delta \subset \Delta \\
&\leq \frac{k}{k+1}\operatorname{diam}\Delta && \text{since } x \mapsto x/(x+1) \text{ is increasing.}
\end{aligned}$$

It remains to show that the inequality

$$|b - y_i| \leq \frac{k}{k+1}\operatorname{diam}\Delta$$

also holds. Indeed,

$$\begin{aligned}
|b - y_i| &\leq |b - x_j| && \text{for some } j \text{ by (2.102)} \\
&= \Big|\frac{1}{k+1}\sum_i x_i - x_j\Big| = \Big|\frac{1}{k+1}\sum_i (x_i - x_j)\Big| \\
&\leq \frac{k}{k+1}\max_i |x_i - x_j| \\
&\leq \frac{k}{k+1}\operatorname{diam}\Delta.
\end{aligned}$$

Here we have also used the fact that the second sum in the second line has at most k non-trivial summands. $\square$

Let X be a convex subset of an Euclidean space and $\Delta_k \subset \mathbb{R}^{k+1}$ be the standard k-simplex. A map $f\colon \Delta_k \to X$ such that

$$f\Big(\sum t_i y_i\Big) = \sum t_i f(y_i) \qquad \text{for all } y_i \in \Delta_k \text{ and all } t_i \geq 0,\ \sum t_i = 1$$

is called *an affine simplex* in X. Clearly, any affine simplex $\Delta_k \to X$ in X is uniquely determined by the images of the vertices. In particular, each affine simplex can be identified with $\Delta(x_0, \ldots, x_k)$, where $x_i = f(e_i) \in X$.

Denote by $AS_k(X)$ the free abelian group, which is generated by all affine k-simplexes. Formula (2.8) defines the boundary map on AS_*, that is (AS_*, ∂) is a chain map. Besides, define $AS_{-1}(X) := \mathbb{Z}[\varnothing]$ and $\partial\Delta(x_0) = [\varnothing]$ for all 0-simplexes $\Delta(x_0)$.

Proposition 2.103. *Map* (2.99) *together with* $\mathrm{Sd}(\varnothing) := \varnothing$ *determines a chain map* $\mathrm{Sd}\colon AS_* \to AS_*$ *with the following properties:*

(i) Sd *is chain homotopic to the identity homomorphism;*

(ii) For each simplex Δ'*, which appears in* $\mathrm{Sd}(\Delta)$*, we have* $\operatorname{diam}\Delta' \leq \frac{k}{k+1}\operatorname{diam}\Delta$.

Proof. The proof consists of the following three steps.

Step 1. *For each $b \in X$ the homomorphism*

$$C_b \colon AS_k(X) \to AS_{k+1}(X),$$

which is determined by (2.98) and $C_b(\varnothing) = \{b\}$, is a chain homotopy between id and the trivial homomorphism, that is

$$\partial C_b + C_b \partial = id. \tag{2.104}$$

The claim of this step follows from the following simple observation:

$$\partial C_b\big(\Delta(x_0, \dots, x_k)\big) = \Delta(x_0, \dots, x_k) - \partial C_b\big(\partial \Delta(x_0, \dots, x_k)\big).$$

Step 2. Sd *is a chain homomorphism.*

Define additionally $\mathrm{Sd}(\varnothing) = \varnothing$. To show that Sd is a chain homomorphism, observe first that $\mathrm{Sd} = id$ on AS_{-1} and AS_0 and therefore we have

$$\partial \circ \mathrm{Sd} = \mathrm{Sd} \circ \partial \tag{2.105}$$

on AS_{-1}. For $k \geq 0$ the proof of (2.105) is obtained by induction:

$$\begin{aligned}
\partial\, \mathrm{Sd}\, \Delta &= \partial\, C_b\, \mathrm{Sd}\, \partial \Delta \\
&= \mathrm{Sd}\, \partial \Delta - C_b(\partial\, \mathrm{Sd}\, \partial \Delta) && \text{(2.104)} \\
&= \mathrm{Sd}\, \partial \Delta - C_b(\mathrm{Sd}\, \partial\, \partial \Delta) && \text{by the induction hypothesis} \\
&= \mathrm{Sd}\, \partial \Delta && \partial^2 = 0.
\end{aligned}$$

Step 3. Sd *is chain homotopic to the identity homomorphism.*

Define $T \colon AS_k \to AS_{k+1}$ recursively in k, namely

$$T(\varnothing) = 0 \qquad \text{and} \qquad T\Delta = C_{b(\Delta)}\big(\Delta - T\, \partial \Delta\big).$$

The property

$$T\, \partial + \partial\, T = id - \mathrm{Sd}$$

holds clearly on AS_{-1}. For $k \geq 0$ the proof goes just like above by the induction:

$$\begin{aligned}
\partial\, T \Delta &= \partial\, C_b\big(\Delta - T\, \partial\, \Delta\big) \\
&= \Delta - T\, \partial\, \Delta - C_b\big(\partial \Delta - \partial\, T\, \partial\, \Delta\big) && \text{(2.104)} \\
&= \Delta - T\, \partial\, \Delta - C_b\big(\partial \Delta - \partial \Delta + \mathrm{Sd}\, \partial \Delta - T\, \partial\, \partial\, \Delta\big) && \text{by the induction hypothesis} \\
&= \Delta - T\, \partial\, \Delta - \mathrm{Sd}\, \Delta && \text{(2.99)}.
\end{aligned}$$

To finish the proof of this proposition, it remains only to notice that *(ii)* follows immediately from (2.99) and Lemma 2.100. □

Proof of Proposition 2.97. The proof consists of the following four steps.

Step 1. *Define*

$$\mathrm{Sd} \colon S_*(X) \to S_*(X) \qquad \text{by} \qquad \mathrm{Sd}(\sigma) = \sigma_\#\big(\mathrm{Sd}(\Delta_k)\big)$$

and similarly also T. Then we have

$$\mathrm{Sd} \circ \partial = \partial \circ \mathrm{Sd} \qquad \text{and} \qquad T\, \partial + \partial\, T = id - \mathrm{Sd}.$$

The proof is a simple exercise.

Step 2. *(Lebegue's lemma) Let $\mathcal{V}$ be an arbitrary open covering of a compact metric space Y. There is a number $\varepsilon = \varepsilon(\mathcal{V})$ with the following property: Each subset $Z \subset Y$ such that $\operatorname{diam} Z \leq \varepsilon$ is contained in some $V_i \in \mathcal{V}$.*

Indeed, by the compactness of Y we obtain that there is an open finite covering of Y by balls $B_{r_i}(y_i)$ such that each ball $B_{2r_i}(y_i)$ is contained in some $V_j \in \mathcal{V}$. Let ε be smaller than the minimum of all r_i.

Furthermore, for any two points $z_1, z_2 \in Y$ such that $d_Y(z_1, z_2) \leq \varepsilon$ we have

$$\exists B_{r_i}(y_i) \ni z_1 \implies d_Y(z_2, y_i) \leq d_Y(z_2, z_1) + d_Y(z_1, y_i) \leq \varepsilon + r_i \leq 2r_i.$$

This shows that $z_2 \in B_{2r_i}(y_i) \subset V_j$.

Step 3. *The following holds:*

(i) Sd^m *is chain homotopic to the identity homomorphism for all $m \in \mathbb{N}$;*

(ii) For all $\sigma\colon \Delta_k \to X$ there exists some $m \in \mathbb{N}$ such that $\operatorname{Sd}^m(\sigma) \in C_k^{\mathcal{U}}(X)$.

Define

$$D_m := \sum_{i=0}^{m-1} T \circ \operatorname{Sd}^i.$$

The first claim follows from the following computation:

$$\begin{aligned}\partial D_m + D_m \partial &= \sum_{i=0}^{m-1} (\partial T \operatorname{Sd}^i + T \operatorname{Sd}^i \partial) = \sum_{i=0}^{m-1} (\partial T \operatorname{Sd}^i + T \partial \operatorname{Sd}^i) \\ &= \sum_{i=0}^{m-1} (id - \operatorname{Sd}) \operatorname{Sd}^i = id - \operatorname{Sd}^m.\end{aligned}$$

The second claim follows from a combination of Step 2 and Proposition 2.103.

Step 4. *For each $\sigma\colon \Delta_k \to X$ let $m = m(\sigma) \in \mathbb{N}$ be the minimal integer such that (ii) from Step 3 above holds. Define*

$$D\colon S_k(X) \to S_{k+1}(X), \qquad D\sigma = D_{m(\sigma)}\sigma.$$

Then there exists a chain homomorphism $\rho\colon S_*(X) \to S_*^{\mathcal{U}}(X)$ such that

$$D\partial + \partial D = id - \imath\,\rho \qquad \text{and} \qquad \rho\,\imath = id, \tag{2.106}$$

where $\imath\colon S_*^{\mathcal{U}}(X) \to S_*(X)$ is the inclusion.

Define ρ by the equality

$$\partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma) \qquad \iff \qquad \rho(\sigma) = \sigma - \partial D\sigma - D\partial\sigma.$$

Using the equality $\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) = \sigma - \operatorname{Sd}^{m(\sigma)}\sigma$, we obtain

$$\rho(\sigma) = \operatorname{Sd}^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma).$$

From the inequality $m(\sigma) \ge m(\partial_j\sigma)$, which is valid for all $j \in \{0, \dots, k\}$, we obtain

$$\begin{aligned} D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) &= \sum_{j=0}^{k} (-1)^j \Big(D_{m(\sigma)}(\partial_j\sigma) - D(\partial_j\sigma) \Big) \\ &= \sum_{j=0}^{k} (-1)^j \sum_{i \ge m(\partial_j\sigma)} T \,\mathrm{Sd}^i(\partial_j\sigma) \quad \in \; C_k^{\mathcal{U}}(X). \end{aligned}$$

This yields that $\rho(\sigma)$ lies in $C_k^{\mathcal{U}}(X)$ too, since $\mathrm{Sd}^{m(\sigma)}\sigma \in C_k^{\mathcal{U}}(X)$.

Besides, ρ is a chain homomorphism:

$$\partial\,\rho\,\sigma = \partial\sigma - \partial\,\partial\,D\,\sigma - \partial\,D\,\partial\,\sigma = \rho(\partial\sigma).$$

The fact that ρ takes values in $C_*^{\mathcal{U}}(X)$, yields that the first equation of (2.106) holds. One obtains the second equation by observing that for all $\sigma \in C_*^{\mathcal{U}}(X)$ we have $m(\sigma) = 0 \implies D\sigma = 0 \implies \rho(\sigma) = \sigma$. This finishes the proof of Step 4 and simultaneously also the proof of this proposition, since (2.106) implies that $\imath_* \colon H_*^{\mathcal{U}}(X) \to H_*(X)$ is an isomorphism. $\square$

With this understood, we can give the proof of the excision theorem.

Proof of Theorem 2.39. The proof consists of the following two steps.

Step 1. *For any subsets $A, B \subset X$ such that $X = \mathrm{Int}A \cup \mathrm{Int}B$ the inclusion $(B, A\cap B) \to (X, A)$ induces an isomorphism*

$$H_*(B, A\cap B) \to H_*(X, A).$$

Set $\mathcal{U} = \{A, B\}$. All maps, which appear in (2.106), preserve $S_*(A)$. This yields that the inclusion

$$\imath\colon\, S_*^{\mathcal{U}}(X)/S_*(A) \to S_*(X)/S_*(A)$$

induces an isomorphism on the homology groups, since for the induced maps D and ρ Relations (2.106) are also satisfied.

Furthermore, we have

$$S_*^{\mathcal{U}}(X)/S_*(A) = \big(S_*(A) + S_*(B)\big)/S_*(A) \cong S_*(B)/S_*(A\cap B).$$

Moreover, this isomorphism is induced by the inclusion $S_*(B)/S_*(A\cap B) \to S_*^{\mathcal{U}}(X)/S_*(A)$.

Step 2. *The claim of Step 1 is equivalent to the claim of the excision theorem.*

Setting

$$B := X \setminus Z \quad \text{and} \quad Z := X \setminus B,$$

we have $A \cap B = A \setminus Z$. Moreover, the condition $\bar{Z} \subset \mathrm{Int}(A)$ is equivalent to $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. $\square$

Proposition 2.97 also allows us to prove the exactness of the Mayer–Vietoris sequence as follows.

Proof of Theorem 2.88. Set $\mathcal{U} = \{A, B\}$. It is easy to check that the sequence of chain complexes

$$0 \to S_*(A\cap B) \xrightarrow{\varphi} S_*(A)\oplus S_*(B) \xrightarrow{\psi} S_*^{\mathcal{U}}(X) = S_*(A)+S_*(B) \to 0$$

is exact, where[2] $\varphi(x) = (x, x)$ and $\psi(u, v) = u - v$, cf. (2.87). The long exact sequence of the homology groups combined with Proposition 2.97 yield Mayer–Vietoris sequence (2.89). $\square$

The homomorphism $\Delta\colon H_k(X) \to H_{k-1}(A\cap B)$, which appears in the Mayer–Vietoris sequence, can be given explicitly. Namely, let $z \in S_k(X)$ be an arbitrary chain. It follows from the proof that there is a decomposition $z = x + y$, where $x \in S_k(A)$ and $y \in S_k(B)$. Besides, $\partial x + \partial y = \partial z = 0$. Notice however, that neither x nor y must be a chain. Then we have $\Delta([z]) = [\partial x] = -[\partial y]$. Details are left to the reader.

The above implies in particular that Δ is natural in the following sense. Let X, A, B and X', A', B' be as in Theorem 2.88. Furthermore, let $f\colon X \to X'$ be a continuous map such that $f(A) \subset A'$ and $f(B) \subset B'$. Then the diagram

$$\begin{tikzcd}
H_k(A\cap B) \arrow[r] \arrow[d, "f_*"'] & H_k(A)\oplus H_k(B) \arrow[r] \arrow[d, "f_*"'] & H_k(X) \arrow[r, "\Delta"] \arrow[d, "f_*"'] & H_{k-1}(A\cap B) \arrow[d, "f_*"'] \\
H_k(A'\cap B') \arrow[r] & H_k(A')\oplus H_k(B') \arrow[r] & H_k(X') \arrow[r, "\Delta"] & H_{k-1}(A'\cap B')
\end{tikzcd}$$

is commutative.

Sometimes the following relative version of the Mayer–Vietoris sequence is also useful.

Proposition 2.107. *Assume the following holds: $X = \mathrm{Int}A \cup \mathrm{Int}B$, $X \supset Y = \mathrm{Int}C \cup \mathrm{Int}D$, $C \subset A$, and $D \subset B$. Then the sequence*

$$\dots \to H_k(A\cap B, C\cap D) \xrightarrow{\Phi} H_k(A, C)\oplus H_k(B, D) \xrightarrow{\Psi} H_k(X, Y) \xrightarrow{\Delta} H_{k-1}(A\cap B, C\cap D) \to \dots$$

is exact.

Proof. Let $\mathcal{U} = \{A, B\}$ and $\mathcal{V} = \{C, D\}$ be coverings of X and Y respectively. Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_k(C\cap D) \arrow[r, "\varphi"] \arrow[d] & S_k(C)\oplus S_k(D) \arrow[r, "\psi"] \arrow[d] & S_k^{\mathcal{V}}(Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_k(A\cap B) \arrow[r, "\varphi"] \arrow[d] & S_k(A)\oplus S_k(B) \arrow[r, "\psi"] \arrow[d] & S_k^{\mathcal{U}}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_k\big(A\cap B,\ C\cap D\big) \arrow[r, "\varphi"] \arrow[d] & S_k(A, C)\oplus S_k(B, D) \arrow[r, "\psi"] \arrow[d] & S_k^{\mathcal{U},\mathcal{V}}(X, Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

[2] Here we omitted the natural inclusions in the notations.

Here $S_k^{\mathcal{U},\mathcal{V}}(X,Y) = S_k^{\mathcal{U}}(X)/S_k^{\mathcal{V}}(Y)$ by definition and the homomorphisms φ and ψ in the last row are induced by φ and ψ in the middle raw.

Furthermore, the first two raws are exact. In particular, we have $\psi \circ \varphi = 0$ in the middle raw. This equality must still hold in the third raw, that is the third raw is a chain complex. The corresponding long exact sequence is of the following form

$$\dots \longrightarrow H_k(Z_1) \longrightarrow H_k(Z_2) \longrightarrow H_k(Z_3) \longrightarrow H_{k-1}(Z_1) \longrightarrow \dots,$$

where Z_j stands for the complex of the jth raw. This yields

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_k(Z_3) \longrightarrow 0 \longrightarrow \dots$$

That is the homology groups of Z_3 are trivial, so that the third raw is also exact. □

2.A Poincaré conjectures

Conjecture 2.108 (Poincaré)**.** *A compact n-manifold that is homotopy equivalent to the n-sphere is homeomorphic to the n-sphere.*

For $n = 1$ and $n = 2$ this conjecture follows from the classification theorems of Section 2.11.5. Stephen Smale proved this conjecture for $n \geq 5$ in 1960. Later in 1982 Michael Freedman proved also the conjecture in the case $n = 4$. Only in 2002 the case $n = 3$ was published by Grigori Perelman.

Let M be a manifold of dimension n. An open subset $U \subset M$ together with a homeomorphism φ between U and an open subset of $\mathbb{R}^n$ is called *a chart*. A set

$$\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$$

consisting of charts, which cover all of M, is called *an atlas*.

Example 2.109. The sphere S^n has an atlas consisting of two charts. This was given in Example 2.90.

An atlas is called *smooth*, if each *coordinates change map*

$$\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is smooth. The coordinates change maps are maps between open subsets of $\mathbb{R}^n$ and smoothness means that each component is differentiable to any order. *A smooth manifold* is a topological manifold[3] together with a smooth atlas.

Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be two smooth manifolds. A map $f \colon M \to N$ is said to be smooth, if all coordinate representations of f, that is the maps

$$\psi_j \circ f \circ \varphi_i^{-1} \colon \mathbb{R}^n \to \mathbb{R}^m,$$

are smooth (these maps are possibly defined on open subsets of $\mathbb{R}^n$ only). Here (V_j, ψ_j) is a chart on N.

Exercise 2.110.

- Show that S^n has no atlas consisting of a single chart;

[3]Technically, certain axioms are also required to hold, but this will not be a concern for us.

- Construct a smooth atlas on $\mathbb{T}^2$ and $\mathbb{RP}^2$.

Two manifolds M and N are called *diffeomorphic*, if there exists a bijection $f\colon M \to N$, so that both f and f^{-1} are smooth. In this case f is called a diffeomorphism.

Theorem 2.111 (Milnor)**.** *There exist 7-manifolds, which are homeomorphic but not diffeomorphic to the 7-sphere.*

It was shown later that there are exactly 28 smooth manifolds (up to a diffeomorphism), which are homeomorphic to the 7-sphere.

Equivalently, one can reformulate the above theorem somewhat more intrinsically using the notion of a smooth structure. Namely, two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are called equivalent, if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. A maximal atlas on M is called *a smooth structure*. In other words, a smooth structure is an equivalence class of smooth atlases.

Proposition 2.112. *Let M be a topological manifold. M admits at least two inequivalent smooth structures if and only if there exists a smooth manifold N, which is homeomorphic but not diffeomorphic to M.*

Proof. Let $\mathcal{A}$ be a smooth atlas on M. Assume there exist a smooth manifold $(N, \mathcal{B})$ and a homeomorphism $f\colon M \to N$, which is not a diffeomorphism. Define a new atlas $\mathcal{B}'$ on M by

$$\mathcal{B}' := \big\{ (f^{-1}(V_j), \psi_j \circ f) \mid (V_j, \psi_j) \in \mathcal{B} \big\}.$$

The atlases $\mathcal{A}$ and $\mathcal{B}'$ are *not* equivalent, since otherwise f would be a diffeomorphism.

If M admits two inequivalent smooth atlases $\mathcal{A}$ and $\mathcal{A}'$, then $id_M\colon (M, \mathcal{A}) \to (M, \mathcal{A}')$ is a homeomorphism, which is not a diffeomorphism. $\square$

Remark 2.113. There are examples of (compact) topological manifolds, which do not admit any smooth structure.

Conjecture 2.114 ("Smooth Poincaré conjecture")**.** *The natural smooth structure on the 4-sphere is unique.*

It is not known up to now whether this conjecture is true or false. At the same time, it is known that $\mathbb{R}^4$ admits infinitely many (even uncountably many) smooth structures. Examples of smooth 4-manifolds admitting several smooth structures are also known.

Chapter 3

CW complexes and cellular homology

3.1 Attaching topological spaces

Let X be a topological space. *The cone* of X is the space

$$CX := X \times [0,1]/\sim, \qquad (x_1,0) \sim (x_2,0) \quad \forall x_1, x_2 \in X.$$

Exercise 3.1. Show that the tip of the cone $\{p\} := [X \times \{0\}]$ is a deformation retract of the cone. In particular, cones are contractible.

Let X, Y be topological spaces such that $X \cap Y = \varnothing$, $A \subset X$ and $f\colon A \to Y$ a continuous map. We say that the space

$$X \cup_f Y = \big(X \sqcup Y\big)/\sim, \qquad \text{where} \quad a \sim f(a) \quad \forall a \in A$$

is obtained by attaching X to Y via f.

Some properties considered in the previous chapter can be elegantly expressed in terms of the above attaching construction. For example, consider the space $X \cup CA$, where the attaching map is the inclusion $a \mapsto (a, 1)$. Es gilt:

$$\begin{aligned} \tilde H_*(X \cup CA) &\cong H_*(X \cup CA, CA) && \text{by the LES of the pair } (X \cup CA, CA) \\ &\cong H_*\big(X \cup CA \setminus \{p\}, CA \setminus \{p\}\big) && \text{by excision} \\ &\cong H_*(X, A) && A \subset CA \setminus \{p\} \quad \text{is a deform. retract.} \end{aligned}$$

This means that the relative homology groups can be represented as the absolute homology groups of the space $X \cup CA$. Here one does not need to impose any assumptions on A, cf. Theorem 2.96. L 14

Let $\varphi_\gamma\colon S^{n-1} \to X,\ \gamma \in \Gamma$, be a family of continuous maps. We say that the space

$$\Big(X \bigsqcup_{\gamma \in \Gamma} B_{n,\gamma}\Big)/\sim, \qquad \text{where} \quad y \sim \varphi_\gamma(y) \quad \forall y \in \partial B_{n,\gamma}$$

is obtained from X by attaching of n-cells and $\Phi_\gamma\colon B_{n,\gamma} \to X \bigsqcup B_{n,\gamma}/\sim$ is called *the characteristic map*. The restriction of Φ_γ to the interior $\mathring{B}_{n,\gamma}$ of the ball is a homeomorphism onto its image e^n_γ, which is referred to as an n-cell.

Definition 3.2. A structure of a CW complex on a Hausdorff space X is a sequence of closed subspaces

$$X^0 \subset X^1 \subset \dots \subset X^n \subset \dots$$

such that the following holds:

(i) $X = \cup_n X^n$;

(ii) X^0 is a discrete space;

(iii) X^n is obtained from X^{n-1} by attaching of n-cells;

(iv) A subset $A \subset X$ is closed (open) in X if and only if $A \cap X^n$ is closed (open) in X^n.

The subspace X^n is called the n-skeleton of X.

A CW structure is called finite if it consists of finitely many cells.

Proposition 3.3. *Let X be a topological space equipped with a CW structure. The following holds:*

- *$X \supset A$ is closed (open) $\iff \Phi_\gamma^{-1}(A) \subset B_n$ is closed (open);*
- *For finite CW structures (iv) of the definition above holds automatically.*

Proof. The continuity of Φ_γ yields immediately the proof of the first statement in one direction. To show the other direction, assume that $A \cap X^{n-1}$ is closed. Then $A \cap X^n$ is closed in X^n by the definition of the quotient topology.

Assume $A \subset X$ is closed. Since each X^n is closed, the set $X^n \cap A$ is also closed for any CW complex. If the CW structure is finite, then $A = \cup(A \cap \bar{e}^n_\gamma)$ is compact as a finite union of compact subsets. Since X is a Hausdorff space, A is closed. □

Example 3.4. A finite topological graph is a CW complex.

Example 3.5. Each compact surface admits a CW structure. This follows for example from Corollary 2.78.

Example 3.6. The sphere $S^n = B_n/\partial B_n$ has a CW structure, which consists of one 0-cell and one n-cell:

$$X^0 = \cdots = X^{n-1} = \{pt\}, \quad X^n = S^n = \{pt\} \cup B_n,$$

where $\varphi \colon \partial B_n \to \{pt\}$ is necessarily the constant map.

Example 3.7. (Non-Example) Consider the space

$$X := \bigcup_{n\in\mathbb{N}} X_n$$

where X_n is the circle in $\mathbb{R}^2$ of radius $1/n$ centered at $(0, 1/n)$. Then $X \setminus 0$ consists of infinitely many intervals, however this is not a CW structure (*Why?*).

Example 3.8 (Real projective space)**.**

$$\begin{aligned}
\mathbb{RP}^n &= \text{ the space of all lines in } \mathbb{R}^{n+1} \text{ through the origin} \\
&= S^n/\sim, && \text{where } x \sim -x \quad \forall x \in S^n, \\
&= S^n_-/\sim, && \text{where } x \sim -x \quad \forall x \in \partial S^n_-, \\
&= \mathbb{RP}^{n-1} \cup e^n.
\end{aligned}$$

The attaching map $\varphi \colon S^{n-1} \to \mathbb{RP}^{n-1}$ is the quotient map (in particular, this is a 2-to-1 map). This yields a finite CW structure on $\mathbb{RP}^n$:

$$X^n = \mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n.$$

Example 3.9 (Complex projective space)**.**

$$
\begin{aligned}
\mathbb{CP}^n &= \big\{\mathbb{C}\text{-lines} \subset \mathbb{C}^{n+1} \text{ through } 0\big\} \\
&= \big(\mathbb{C}^{n+1} \setminus 0\,\big)/\sim && (z_0,\dots,z_n) \sim (\lambda z_0,\dots,\lambda z_n), \quad \lambda \in \mathbb{C}\setminus 0, \\
&= S^{2n+1}/\sim && (z_0,\dots,z_n) \sim (\lambda z_0,\dots,\lambda z_n), \quad |z|=1,\ |\lambda|=1, \\
&= B_{2n}/\sim && z' \sim \lambda z' \quad \forall z' \in \partial B_{2n},\ |\lambda| = 1.
\end{aligned}
$$

To see the last equality, notice first that for any non-zero $z_0 \in \mathbb{C}$ there exists a unique $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\lambda z_0 \in \mathbb{R}_{>0}$. Hence, for any $(z_0, z_1, \dots, z_n) \in S^{2n+1}$ with $z_0 \neq 0$ there exists a unique $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $r := \lambda z_0 \in \mathbb{R}_{>0}$. Hence,

$$
\begin{aligned}
\big\{(z_0,z_1,\dots,z_n) \in S^{2n+1} \mid z_0 \neq 0\big\}/\sim \quad &\cong \big\{(r, z_1,\dots,z_n) \mid |z|^2 = 1 - r^2,\ r \in (0,1]\,\big\} \\
&\cong B^{2n} \setminus \partial B^{2n}.
\end{aligned}
$$

This yields in turn $\mathbb{CP}^n = e^{2n} \cup \big(\partial B^{2n}/\sim\,\big) = e^{2n} \cup \mathbb{CP}^{n-1}$. Moreover, the attaching map is the projection $S^{2n-1} \to \mathbb{CP}^{n-1}$ (the Hopf map). This yields a CW structure on $\mathbb{CP}^n$:

$$
\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}.
$$

Example 3.10 (Quaternion-projective space)**.** Replacing $\mathbb{R}$ or $\mathbb{C}$ by quaterions in the constructions above, we obtain the quaternion-projective space:

$$
\mathbb{HP}^n = \big(\mathbb{H}^{n+1} \setminus 0\big)/(\mathbb{H} \setminus 0) = e^0 \cup e^4 \cup \cdots \cup e^{4n}.
$$

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