The tangent bundle

Some elements of linear algebra

Let V be a vector space, $\dim V =: k$.
Any basis $v = (v_1, \dots, v_k)$ of V yields an iso

$$\mathbb{R}^k \longrightarrow V, \quad y \longmapsto \sum_{j=1}^{k} y_j v_j = v \cdot y$$

Conversely, if $\varphi : \mathbb{R}^k \longrightarrow V$ is a linear isomorphism, then the image of the standard basis of $\mathbb{R}^k$ is a basis of V. This yields a bijective correspondence between the set of all bases of V and the set of all isomorphisms $\mathbb{R}^k \to V$.

If $w = (w_1, \dots, w_k)$ is another basis of V, we obtain the change-of-basis matrix B as follows. Writing

$$w_j = \sum_{i=1}^{k} b_{ij} v_i \qquad (*)$$

we set $B = (b_{ij})$. Then $(*)$ is equivalent to

$$w = \underbrace{v \cdot B}_{\text{matrix multiplication}}$$

Let M be a manifold of dimension k.
Pick a pt $m \in M$ and a chart (U, φ) s.t. $m \in U$.
Denote $p := \varphi(m) \in \mathbb{R}^k$. We obtain a basis of $T_m M$ as follows:

$$v_\varphi = v := ([\gamma_1], \dots, [\gamma_k]), \text{ where } \gamma_j(t) = \varphi^{-1}(p + te_j)$$

and $e = (e_1, \dots, e_k)$ is the standard basis of $\mathbb{R}^k$, see the remark on P. 15 of Part 4.
If $(\hat{U}, \hat{\varphi})$ is another chart s.t. $m \in \hat{U}$, we obtain another basis

$$v_{\hat{\varphi}} = \hat{v} := ([\hat{\gamma}_1], \dots, [\hat{\gamma}_k]), \text{ where } \hat{\gamma}_j(t) = \hat{\varphi}^{-1}(\hat{p} + te_j)$$

and $\hat{p} = \hat{\varphi}(m)$.

<u>Prop</u> Let $\Theta := \hat{\varphi} \circ \varphi^{-1} : \mathbb{R}^k \to \mathbb{R}^k$ be the coordinate transformation map. Then the change-of-basis matrix between v and $\hat{v}$ is $D_p \Theta$:

$$v = \hat{v} \cdot D_p \Theta.$$

<u>Proof</u> Without loss of generality we can assume $p = 0 = \hat{p}$.

We have

$$\hat{\varphi} \circ \gamma_j(t) = \underbrace{\hat{\varphi} \circ \varphi^{-1}}_{\Theta} \circ \underbrace{\varphi \circ \gamma_j(t)}_{te_j} = \Theta(te_j).$$

Hence,

$$d_m \hat{\varphi}\,[\gamma_j] = \frac{d}{dt}\Big|_{t=0} \theta(te_j) = \sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j} e_i , \quad (*)$$

where the partial derivatives are evaluated at the origin (suppressed in the notations).

Notice, however,

$$\hat{\varphi} \circ \hat{\gamma}_i(t) = te_i \implies d_m \hat{\varphi}\,[\hat{\gamma}_i] = e_i .$$

Hence, by (*) we obtain

$$d_m \hat{\varphi}\,[\gamma_j] = \sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j} d_m \hat{\varphi}\,([\hat{\gamma}_i])$$

$d_m\hat{\varphi}$ is linear

$$= d_m \hat{\varphi} \left(\sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j} [\hat{\gamma}_i] \right).$$

Since $\hat{\varphi} : \hat{U} \to \hat{\varphi}(\hat{U}) \subset \mathbb{R}^k$ is a diffeomorphism, $d_m\hat{\varphi}$ is an isomorphism. Hence,

$$[\gamma_j] = \sum_{i=1}^{k} \frac{\partial \theta_i}{\partial x_j} [\hat{\gamma}_i] . \qquad \blacksquare$$

Consider the set

$$TM = \bigsqcup_{m \in M} T_m M ,$$

where the symbol $\bigsqcup$ denotes the disjoint union.

This comes equipped with the map

$$\pi : TM \to M \quad , \quad \pi(v) = m \iff v \in T_m M.$$

<u>Example</u> If V is a vector space, we have a canonical identification $T_m V \cong V$ for each $m \in V$, see Assignment 8, Problem 1. Hence,

$$TV = \bigsqcup_{m \in V} \{m\} \times V = V \times V$$

and $\pi(m, v) = m$ is the projection onto the first component.

Furthermore, for any chart (U, φ) on M we have a basis $V_\varphi(m)$ of $T_m M$ for each $m \in U$. Therefore, we obtain the bijection

$$U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U) = \bigsqcup_{m \in U} T_m M$$

$$(m, y) \longmapsto V_\varphi(m) \cdot y = \sum_{j=1}^{k} y_j \, [\gamma_j^m],$$

where $\gamma_j^m(t) = \varphi^{-1}(\varphi(m) + t e_j)$. Combining this with $\varphi : U \to \varphi(U)$, which is also a bijection, we obtain a bijective map

$$\tau = \tau_\varphi : \varphi(U) \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$$

$$(x, y) \longmapsto V_\varphi(\varphi^{-1}(x)) \cdot y = \sum y_j \, [\gamma_j^m]$$

$$m = \varphi^{-1}(x).$$

<u>Thm</u> Let $\mathcal{U} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ be a smooth atlas on M. There exists a unique Hausdorff topology on TM such that

$$\mathcal{V} = \{(\pi^{-1}(U_\alpha), \tau_\alpha^{-1}) \mid \alpha \in A\}$$

is a C^0-atlas on TM, where $\tau_\alpha = \tau_{\varphi_\alpha}$. In fact, $\mathcal{V}$ is a smooth atlas so that TM is a smooth mfld of dimension $2k$. Moreover, π is a smooth map with surjective differential at each point.

<u>Proof</u> The proof consists of the following steps.

<u>Step 1</u> For the coordinate transformation $\Theta_{\alpha\beta} = \tau_\alpha^{-1} \circ \tau_\beta$ on TM we have

$$\Theta_{\alpha\beta}(x, y) = (\theta_{\alpha\beta}(x), D_x\theta_{\alpha\beta} \cdot y).$$

In particular, $\Theta_{\alpha\beta}$ is smooth.

Denote $\tau_\beta(x, y) = v \implies$

$$\varphi_\beta(\pi(v)) = x \quad \text{and} \quad v = V_\beta(\underbrace{\varphi_\beta^{-1}(x)}_{=:m}) \cdot y.$$

By the proposition on P. 2 we have

$$V_\beta(m) = V_\alpha(m)\, D_x\theta_{\alpha\beta}$$

Denote $\tau_\alpha^{-1}(v) = (s, t) \in \mathbb{R}^k \times \mathbb{R}^k.$

Then

$$s = \varphi_\alpha^{-1}(\pi(v)) = \varphi_\alpha(\varphi_\beta^{-1}(x)) = \theta_{\alpha\beta}(x)$$

$$t = D\theta_{\alpha\beta} \cdot y \qquad \text{since} \qquad v = v_\beta(m) \cdot y = v_\alpha \cdot D\theta_{\alpha\beta} \cdot y$$

<u>Step 2</u> We construct the topology on TM.
Declare a set $V \subset TM$ to be open if and only if $\tau_\alpha^{-1}(V)$ is open in $\mathbb{R}^{2k}$ for any $\alpha \in A$.
We have

(i) $\emptyset$ is open and $\tau_\alpha^{-1}(TM) = \varphi_\alpha(U_\alpha) \times \mathbb{R}^k$ is open.

(ii) V_1, V_2 are open $\Rightarrow$ $\tau_\alpha^{-1}(V_1 \cap V_2) = \tau_\alpha^{-1}(V_1) \cap \tau_\alpha^{-1}(V_2)$ is open $\Rightarrow$ $V_1 \cap V_2$ is open

(iii) Each V_β, $\beta \in B$, is open $\Rightarrow$

$$\Rightarrow \tau_\alpha^{-1}\Big(\bigcup_{\beta\in B} V_\beta\Big) = \bigcup_{\beta \in B} \tau_\alpha^{-1}(V_\beta) \quad \text{is open}$$

$$\Rightarrow \bigcup_{\beta \in B} V_\beta \quad \text{is open.}$$

Hence, we obtain a topology on TM s.t. π is continuous. Moreover, each $(\pi^{-1}(U_\alpha), \tau_\alpha^{-1})$ is a chart on TM.

This topology is Hausdorff. Indeed, pick $v_1, v_2 \in TM$, $v_1 \neq v_2$.

(a) If $\pi(v_1) \neq \pi(v_2)$, choose open subs. $U_1, U_2 \subset M$ s.t. U_1 and U_2 separate $\pi(v_1)$ and $\pi(v_2)$. Then $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ separate $\pi(v_1)$ and $\pi(v_2)$.

(b) If $\pi(v_1) = \pi(v_2) =: m$. Pick any chart (U, φ) s.t. $m \in U$. Then $\tau(U \times V_1)$ and $\tau(U \times V_2)$ separate v_1 and v_2 if $V_1, V_2 \subset \mathbb{R}^k$ separate $\pi_2(\tau^{-1}(v_1))$ and $\pi_2(\tau^{-1}(v_2))$.

Step 3 We finish the proof of this theorem.

Pick a chart $(U_\alpha, \varphi_\alpha)$ on M, hence also a chart $(\pi^{-1}(U_\alpha), \tau_\alpha^{-1})$ on TM. The coordinate representation of π with respect to these charts is

$$\varphi_\alpha \circ \pi \circ \tau_\alpha \; (x, y) = \varphi_\alpha(\varphi_\alpha^{-1}(x)) = x$$

$$\Rightarrow \quad \varphi_\alpha \circ \pi \circ \tau_\alpha = \pi_1$$

$\Rightarrow$ π is smooth and $d_v \pi$ is surjective. $\blacksquare$

Example 1) For $M = S^k$ the tangent bundle can be identified with

$$\{ (x, y) \in S^k \times \mathbb{R}^{k+1} \mid \langle x, y \rangle = 0 \} \subset \mathbb{R}^{2k+2}.$$

If $k = 1$, this yields TS^1 as a submanifold in $\mathbb{R}^4$. In fact, we can realize TS^1 as a surface in $\mathbb{R}^3$ as follows. Consider the map

$$f : S^1 \times \mathbb{R} \longrightarrow \mathbb{R}^4, \quad f(x_0, x_1; t) = (x_0, x_1; t x_1, -t x_0)$$

The reader should check that f is a diffeomorphism between the cylinder $S^1 \times \mathbb{R}$ and $TS^1 \subset \mathbb{R}^4$.

2) If $S \subset \mathbb{R}^3$ is a surface, then the tangent bundle of S can be identified with

$$\{(x,y) \in S \times \mathbb{R}^3 \mid y \in T_x S\} \subset \mathbb{R}^6.$$

Vector fields and their integral curves

<u>Defn</u> A smooth map $v: M \to TM$ such that

$$\pi \circ v = id_M \iff v(m) \in T_m M \quad \forall m \in M$$

is called a (smooth) vector field on M.

<u>Example</u> 1) Let $S \subset \mathbb{R}^3$ be a surface and v be a smooth vector field on S. This means that $v(p) \in T_p S$, i.e., we can view v as a map $S \to \mathbb{R}^3$ s.t. $v(p) \in T_p S$ $\forall p \in S$. It is an exercise to check that v is smooth if and only if v is smooth as a map $S \to \mathbb{R}^3$.

2) Just like in the case of surfaces, we can view a vector field on S^k as a map $v: S^k \to \mathbb{R}^{k+1}$ s.t. $v(x) \in T_x S^k \iff \langle v(x), x \rangle = 0$. For example, if $k=1$

$$v(x) = (-x_1, x_0)$$

is a vector field on S^1.

Denote $\mathfrak{X}(M) := \{ v \mid v \text{ is a vector field on } M \}$.
Define the structure of a vector space on $\mathfrak{X}(M)$ by setting

- $(v_1 + v_2)(m) = v_1(m) + v_2(m)$;
- $(\lambda v)(m) = \lambda v(m)$.

Here, $v, v_1, v_2 \in \mathfrak{X}(M)$, $\lambda \in \mathbb{R}$, $m \in M$.
In fact, any vector field can be multiplied by a smooth function:

$$(f \cdot v)(m) = f(m) \cdot v(m)$$

Thus, $\mathfrak{X}(M)$ is a $C^\infty(M)$-module.

<u>Example</u> Let V be a real vector space of dimension k. By the example on p. 4 we have a diffeomorphism

$$TV \cong V \times V$$
$$[\gamma] \longmapsto (\gamma(0), \dot{\gamma}(0))$$

Hence, $v \in \mathfrak{X}(V)$ can be thought of as a map

$$v : V \longrightarrow V \times V \qquad \text{s.t.} \quad \pi_1 \circ v = \mathrm{id}_V$$

$$\Longleftrightarrow \quad v(m) = (m, V(m)),$$

where $V : V \to V$ is a smooth map. Thus, for a vector space V we have a natural isomorphism of $C^\infty(V)$-modules:

$$\mathfrak{X}(V) \cong C^\infty(V; V).$$

Generalizing this example slightly, pick a chart (U, φ) on a manifold M. Recall that this yields a basis $V_\varphi(m)$ of $T_m M$ at each $m \in U$. Hence, $\forall m \in U$ $\exists\, y(m) \in \mathbb{R}^k$ s.t.

$$v(m) = V_\varphi(m) \cdot y(m) = \sum_{j=1}^{k} y_j(m)\, [\gamma_j^m]$$

The map $y : U \to \mathbb{R}^k$ is called the coordinate representation of v with respect to (U, φ). Notice that the coord. repr. is well-defined even if v is not necessarily smooth.

<u>Prop</u> A map $v : M \to TM$ s.t. $\pi \circ v = id_M$ is a smooth vector field if and only if each coordinate repr. of v is smooth.

<u>Proof</u> Pick a chart (U, φ) on M. This yields a chart $(\pi^{-1}(U), \tau_\varphi^{-1})$ on TM. With respect to these charts, the coord. representation of v is

$$\tau_\varphi^{-1} \circ v \circ \varphi^{-1}(x) = (x, y \circ \varphi^{-1}(x)).$$

Hence, v is smooth $\iff$ y is smooth. $\boxed{}$

Thus, locally over each chart vector fields can be identified with smooth vector-valued maps.

WARNING: This identification is false globally in general, i.e., it is not true that any vector field on a k-mfld M can be thought of as a map $M \to \mathbb{R}^k$.

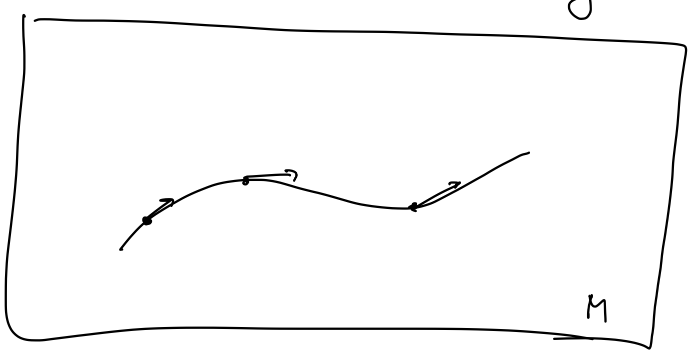
If $\gamma : (a, b) \to M$ is a smooth curve, then the tangent vector $\dot{\gamma}(t) \in T_{\gamma(t)} M$ is defined by

$$\dot{\gamma}(t) := [s \mapsto \gamma_t(s)]_{s=0}, \text{ where } \gamma_t(s) = \gamma(t+s).$$

Example Let $S \subset \mathbb{R}^3$ be a surface. Then for a smooth curve γ on S we have

$$\dot{\gamma}(t) = [\gamma_t(s)]_{s=0} \equiv \frac{d}{ds}\Big|_{s=0} \gamma_t(s)$$
$$= \frac{d}{ds}\Big|_{s=0} \gamma(t+s)$$

is the tangent vector as used in the theory of surfaces. This also holds for any submanifold $M \subset \mathbb{R}^n$.



Def A (smooth) curve $\gamma : (a, b) \to M$ is called an integral curve of a vector field v, if

$$\dot{\gamma}(t) = v(\gamma(t)) \qquad \forall t \in (a, b).$$

Example $M = S^1$, $v(x_0, x_1) = (-x_1, x_0)$, where $v : S^1 \to \mathbb{R}^2$ is thought of as a vector field on S^1 just as in the example on P. 8.
For $\gamma(t) := (\cos t, \sin t)$ we have

$$\dot{\gamma}(t) = (-\sin t, \cos t),$$

$$v(\gamma(t)) = (-\sin t, \cos t).$$

Hence, $\gamma : \mathbb{R} \to S^1$ is an integral curve of v.

Example $M = \mathbb{R}^k$, $v \equiv y : \mathbb{R}^k \to \mathbb{R}^k$
A curve $\gamma : (a, b) \to \mathbb{R}^k$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_k(t))$ is an integral curve of v iff

$$\begin{cases} \dot{\gamma}_1(t) = y_1(\gamma_1(t), \dots, \gamma_k(t)) \\ \vdots \quad - \; - \; - \; - \; - \; - \; - \; - \\ \dot{\gamma}_k(t) = y_k(\gamma_1(t), \dots, \gamma_k(t)) \end{cases} \qquad (*)$$

Notice that $(*)$ is an autonomous system of ODEs. Conversely, any solution of an autonomous system of ODEs corresponds to an integral curve of a vector field on $\mathbb{R}^k$.

<u>Exercise</u> Assume $y \in C^\infty(\mathbb{R}^k; \mathbb{R}^k)$. If $\gamma \in C^1(\mathbb{R}; \mathbb{R}^k)$ satisfies $(*)$, then γ is smooth