MATH-F310: Differential Geometry I
- Assignment 1 -

Smooth surfaces and Lagrange's multipliers theorem

1. $\diamondsuit$[1] Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ where λ_1 and λ_2 are eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.
2. $\diamondsuit$ For the following functions $f : \mathbb{R}^3 \to \mathbb{R}$ determine all values $c \in \mathbb{R}$ for which $f^{-1}(c)$ is a surface in $\mathbb{R}^3$.
   (a) $f(x, y, z) = x^2 + y^2 + z^2$.
   (b) $f(x, y, z) = x^2 + y^2 - z^2$.
   (c) $f(x, y, z) = xyz$.
3. $\diamondsuit$ Let C be the circle of radius r in the yz-plane centered at the point $(0, R, 0)$, where $R > r$. And let T be the torus obtained by rotating C around the x-axis. More formally
$$T := \left\{ \left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2 \right\}$$
   Show that T is indeed a surface.
4. $\spadesuit$ Show that $\{(x, y, z) \in \mathbb{R}^3 \mid y^2 = x^3\}$ is not a smooth surface in $\mathbb{R}^3$.
5. $\spadesuit$ Show that the maximum and the minimum values of the function $g(x_1, \ldots, x_{n+1}) = \sum_{i,j=1}^{n+1} a_{ij} x_i x_j$ on the unit n-sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$, where (a_{ij}) is a symmetric $n \times n$ matrix of real numbers, are eigenvalues of the matrix (a_{ij}).

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[1] Exercises marked by a $\diamondsuit$ will be done in class (if time permits).
Exercises marked by a $\spadesuit$ are to prepare at home for the first test (8th of October).
Exercises marked by a † are extra exercises.

6. † *The Klein bottle:* For $(u,v) \in \mathbb{R}^2$ and $R > r$, consider the following functions

$$\begin{aligned} x(u,v) &= (R + r\cos u)\cos v \\ y(u,v) &= (R + r\cos u)\sin v \\ \zeta(u,v) &= r\sin(u)e^{iv/2} \in \mathbb{C}. \end{aligned}$$

Show that the image of the map $f : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{C}, (u,v) \mapsto (x,y,\zeta)(u,v)$ is a surface of $\mathbb{R}^2 \times \mathbb{C} \cong \mathbb{R}^4$. This surface is called the Klein bottle. Note that

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + |\zeta|^2 = r^2,$$

hence you can think of it as a torus for which the last coordinate does a half twist when v goes from 0 to 2π.