The tangent plane

Let S be a surface and $p \in S$.

Def A vector $v \in \mathbb{R}^3$ is said to be tangent to S at p, if $\exists$ a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to S$ s.t.

$$\gamma(0) = p \quad \text{and} \quad \dot{\gamma}(0) = v.$$

When computing the tangent vector of γ we think of γ as a curve in $\mathbb{R}^3$.

Ex $S = S^2 \ni p$ arbitrary. Recall the curve

$$\gamma_v(t) = \cos t \cdot p + \sin t \cdot v,$$

where $\|v\| = 1$ and $v \perp p$. Then

$\dot{\gamma}_v(0) = v$. Hence, v is tangent to S^2 at p.

T_pS = the set of all tangent vectors to S at p.

Prop Let $\varphi : V \to U$ be a parametrization such that $\varphi(u_0, v_0) = p$. Then

$$T_pS = \operatorname{Im} D_{(u_0,v_0)}\varphi.$$

In particular, T_pS is a vector space of dim. 2.

Proof

Step 1 $\quad \operatorname{Im} D_{(u_0,v_0)}\psi \subset T_pS$

Assume $v \in \operatorname{Im} D_{(u_0,v_0)}\psi \implies$

$\exists\, w \in \mathbb{R}^2$ s.t. $D_{(u_0,v_0)}\psi(w) = v$

Consider the $\underbrace{\text{curve}}_{}$ smooth $\beta : (-\varepsilon, \varepsilon) \to V$

$\beta(t) = (u_0, v_0) + t \cdot w.$

Then $\gamma(t) := \psi \circ \beta(t)$ is a smooth curve in S s.t.

$\gamma(0) = \psi(\beta(0)) = \psi(u_0, v_0) = p,$

$\dot{\gamma}(0) = D_{(u_0,v_0)}\psi(w) = v.$

$\implies v \in T_pS.$

Step 2 $\quad T_pS \subset \operatorname{Im} D_{(u_0,v_0)}\psi$

If $v \in T_pS$, then $\exists\, \gamma : (-\varepsilon, \varepsilon) \to S$ s.t.

$\gamma(0) = p$ & $\dot{\gamma}(0) = v$. Can assume

$\operatorname{Im}\gamma \subset U$ by choosing ε smaller if necessary.

If $\varphi = \psi^{-1}$, then

$\beta(t) := \varphi \circ \gamma(t)$ is a smooth curve in $V \subset \mathbb{R}^2$ s.t. $\beta(0) = (u_0, v_0)$.

Denote $w := \dot\beta(0) \in \mathbb{R}^2$. Then we have

$$v = \dot\gamma(0) = (\underbrace{\Psi \circ \beta}_{\Psi \circ \Psi^{-1} \circ \gamma}\dot{)}(0) = (D_{(u_0,v_0)} \Psi)(\dot\beta(0))$$

$$= D_{(u_0,v_0)} \Psi(w) \in \operatorname{Im} D_{(u_0,v_0)} \Psi.$$

<u>Step 3</u> $\dim T_p S = 2$.

This follows immediately from the injectivity of $D_{(u_0,v_0)} \Psi$.

$\square$

<u>Exercise</u> Assume $v \perp p \in S^2$ and $\|v\| = 2$. Find a curve in S^2 through p with the tangent vector v.

<u>Prop</u> Pick $p \in S$ and recall that there exists a nbhd $W \subset \mathbb{R}^3$ of p and a smooth function $\varphi : W \to \mathbb{R}$ s.t.

$$S \cap W = \{ q \in W \mid \varphi(q) = 0 \} \text{ and } \nabla\varphi(q) \neq 0 \quad \forall q \in W.$$

Then

$$T_p S = \nabla\varphi(p)^{\perp}.$$

Proof If γ is any curve in S through p, then

$$\varphi\circ\gamma(t)=0 \quad \forall t \Rightarrow \underbrace{\frac{d}{dt}\Big|_{t=0}\varphi(\gamma(t))}_{\langle\nabla\varphi(p),\,\dot\gamma(0)\rangle}=0$$

$$\Rightarrow T_pS\subset\nabla\varphi(p)^\perp$$

both have dimension 2 $\Rightarrow T_pS=\nabla\varphi(p)^\perp$

$\blacksquare$

Ex 1) $\forall p\in S^2 \quad T_pS^2=p^\perp$

Indeed, for $\varphi(x,y,z)=x^2+y^2+z^2-1$ we have $\nabla\varphi(x,y,z)=2(x,y,z)$.

2) $p=(x,y,z)\in H=\{\underbrace{x^2+y^2-z^2-1}_{\varphi(x,y,z)}=0\}$

$$\nabla\varphi(p)=2(x,y,-z)\neq0$$

$$\Rightarrow T_pH=(x,y,-z)^\perp$$

$$=\{V=(v_1,v_2,v_3)\in\mathbb{R}^3\mid xv_1+yv_2-zv_3=0\}.$$

3) $p=(x,y,z)\in C=\{x^2+y^2=1\}$

$$T_pC=\{V=(v_1,v_2,v_3)\mid xv_1+yv_2=0,\ v_3 \text{ arbitrary}\}.$$

Differential of a smooth map

Let S be a surface and $f \in C^\infty(S)$.
Define a map $d_pf : T_pS \longrightarrow \mathbb{R}$ as follows:

for $v \in T_pS$ choose a smooth curve γ throught p with $\dot\gamma(0) = v$ and set

$$d_pf(v) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma(t).$$

<u>Prop</u> d_pf is a well-defined linear map.

<u>Proof</u>

<u>Step 1</u> d_pf is well-defined.

If γ_1 and γ_2 are two curves through p s.t. $\dot\gamma_1(0) = v = \dot\gamma_2(0)$, then for $\beta_j := \varphi^{-1} \circ \gamma_j$ we have

$$\gamma_j(t) = \varphi \circ \beta_j(t) \Rightarrow v = D_0\varphi(\dot\beta_1(0)) = D_0\varphi(\dot\beta_2(0))$$

$$\underbrace{}_{\text{injective}}$$

$$\Rightarrow \dot\beta_1(0) = \dot\beta_2(0) =: w$$

$$\frac{d}{dt}\Big|_{t=0} f \circ \gamma_1(t) = \frac{d}{dt}\Big|_{t=0} \left(f \circ \varphi \circ \varphi^{-1} \circ \gamma_1(t) \right)$$

$$= \frac{d}{dt} \left(F \circ \beta_1(t) \right)$$

$$= D_0F(w)$$

Similarly, $\frac{d}{dt}\Big|_{t=0} f\circ\gamma_2(t) = D_0F(w)$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0}(f\circ\gamma_1(t)) = \frac{d}{dt}\Big|_{t=0}(f\circ\gamma_2(t)).$$

<u>Step 2</u> $d_pf\circ D_0\Psi = D_0F$, where $F := f\circ\Psi$.

This follows from the pf of step 1.

<u>Step 3</u> d_pf is linear

$$\underbrace{d_pf\circ D_0\Psi = D_0F}_{\text{linear}} \implies d_pf \text{ is linear}$$

$\boxed{\checkmark}$

<u>Exercise</u> If $h\in C^\infty(\mathbb{R}^3)$ and $f = h|_S$, then $\forall p\in S$ we have

$$d_pf = D_ph|_{T_pS}.$$

<u>Def</u> A point $p\in S$ is called critical for $f\in C^\infty(S)$, if $d_pf = 0$, that is $d_pf(v)=0$ $\forall v\in T_pS$.

<u>Prop</u> If p is a pt of loc. max. (min) for f, then p is critical for f.

<u>Proof</u> p is a pt of loc. max for f $\implies$

$\forall$ curve γ through p, 0 is a pt of loc. max for $f\circ\gamma$

$\implies \frac{d}{dt}\Big|_{t=0} f\circ\gamma = 0$ $\boxed{\checkmark}$

<u>Prop</u> Let $h, \varphi \in C^\infty(\mathbb{R}^3)$. Assume $\nabla\varphi(p) \neq 0$ for any $p \in S = \varphi^{-1}(0)$. If $p \in S$ is a pt of loc. max. for $f = h|_S$, then

$$\nabla h(p) = \lambda \nabla\varphi(p)$$

for some $\lambda \in \mathbb{R}$.

<u>Proof</u> Notice: S is a surface and

$$T_pS = \{ v \in \mathbb{R}^3 \mid \langle v, \nabla\varphi(p)\rangle = 0 \}$$

$$d_pf = 0 \iff D_ph\big|_{T_pS} = 0$$

$$\iff \langle v, \nabla h(p)\rangle = 0 \quad \forall v \in T_pS$$

$$\iff \nabla h(p) = \lambda \nabla\varphi(p) \text{ for some } \lambda \in \mathbb{R} \quad \blacksquare$$

<u>Rem</u> This proof is in a sense more conceptual then the pf of Thm 1 on P. 4 of Part 1.

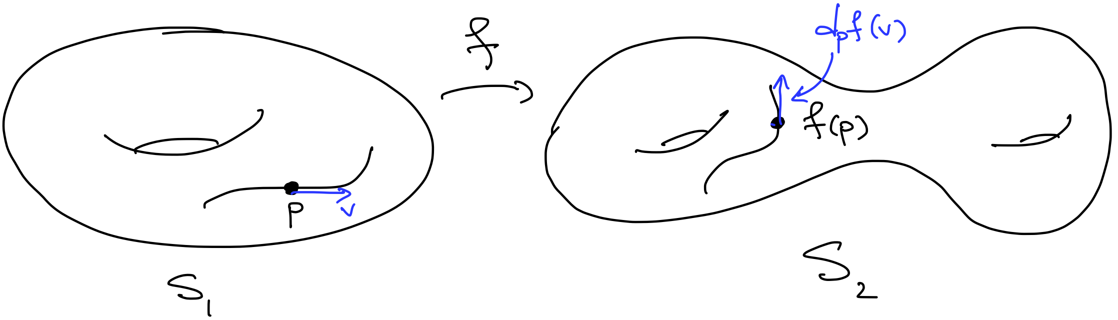
More generally, for any $f \in C^\infty(S; \mathbb{R}^n)$ the differential $d_pf : T_pS \to \mathbb{R}^n$ is defined by the same formula.

Also, the differential is well-defined for maps $f : \mathbb{R}^n \to S$, $d_pf : \mathbb{R}^n \to T_{f(p)}S$

$$f : S_1 \to S_2, \quad d_pf : T_pS_1 \to T_{f(p)}S_2$$

In the latter case, if $\dot\gamma(0) = v \in T_pS_1$, then

$$d_p f(v) = \frac{d}{dt}\Big|_{t=0} \left(f \circ \gamma(t) \right)$$



<u>Prop</u> Let S_1, S_2, S_3 be smooth surfaces. For any smooth maps $f: S_1 \to S_2$ and $g: S_2 \to S_3$ and any pt $p \in S_1$ we have

$$\boxed{D_p(g \circ f) = D_{f(p)} g \circ D_p f.}$$

This also holds if any of S_i is replaces by an open subset of $\mathbb{R}^n$.

<u>Proof</u> Let γ_1 be any smooth curve in S_1 through p. Denote $\gamma_2 = f \circ \gamma$, which is a smooth curve in S_2 through $f(p)$. If $\dot\gamma_1(0) = v_1$, then $v_2 := \dot\gamma_2(0) = D_p f(v_1)$ by the definition of $D_p f$. Hence,

$$D_p(g \circ f)(v_1) = \frac{d}{dt}\Big|_{t=0} \big(g \circ \underbrace{f \circ \gamma_1}_{\gamma_2}(t) \big)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(g \circ \gamma_2(t) \right) = D_{f(p)} g(v_2)$$

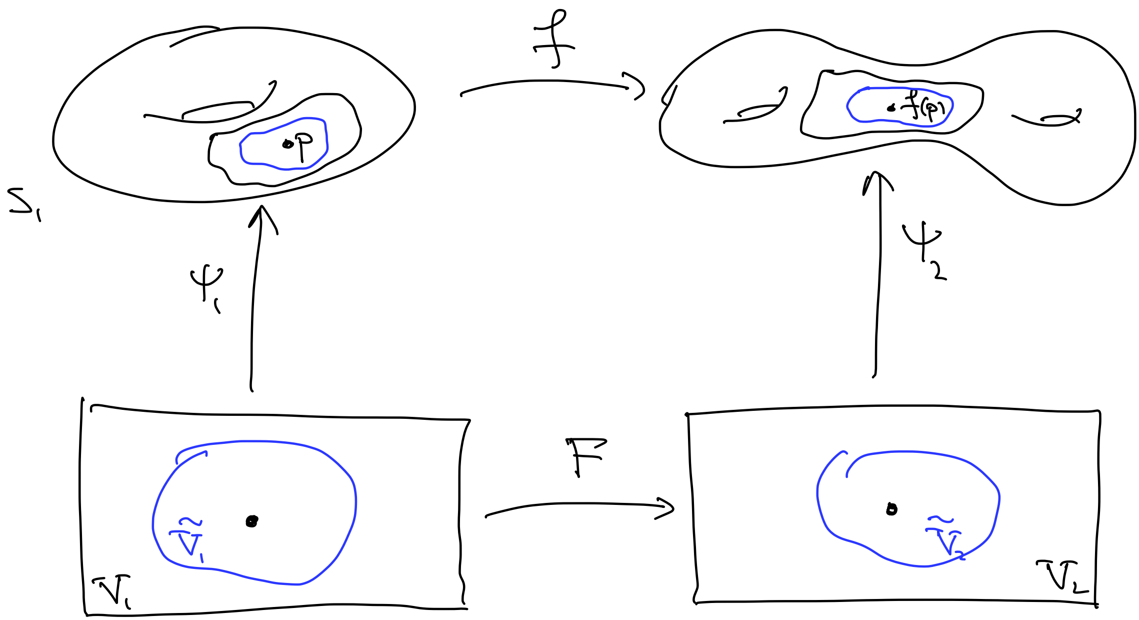
$$= D_{f(p)} g\left(D_p f(v_1) \right)$$

Cor If $f: S_1 \to S_2$ is a diffeomorphism, then $\forall p \in S_1$ $d_p f: T_p S_1 \longrightarrow T_{f(p)} S_2$ is an isomorphism. ▨

Def A map $f: S_1 \to S_2$ is called a <u>local diffeomorphism</u> if $\forall p \in S_1$ $\exists$ a nbhd $U_1 \subset S_1$ and a nbhd $U_2 \subset S_2$ of $f(p)$ s.t. $f: U_1 \longrightarrow U_2$ is a diffeomorphism.

Thm Let $f: S_1 \to S_2$ be a smooth map s.t. $\forall p \in S_1$ $d_p f: T_p S_1 \to T_{f(p)} S_2$ is an isomorphism. Then f is a local diffeomorphism.

Proof Pick any $p \in S_1$ and parametrizations $\varphi_1: V_1 \longrightarrow W_1 \subset S_1$ and $\varphi_2: V_2 \subset W_2 \subset S_2$



Without loss of generality $\varphi_1(0)=p$ and $\varphi_2(0)=f(p)$.

$$F=\varphi_2^{-1}\circ f\circ\varphi_1 \implies d_0F = d_{f(p)}\varphi_2^{-1}\circ d_pf\circ d_0\varphi_1$$

$d_0\varphi_1 : \mathbb{R}^2 \to T_pS_1$ is an iso.

$d_{f(p)}\varphi_2 : T_{f(p)}S_2 \to \mathbb{R}^2$ is an iso.

d_pf is an iso $\implies d_0F$ is an iso.

From analysis, it is known that $\exists$ a nbhd $0\in\tilde{V}_1\subset V_1$ and a nbhd $\tilde{V}_2\subset V_2$ of 0 such that $F:\tilde{V}_1\to\tilde{V}_2$ is a diffeomorphism.

Denote $U_1=\varphi_1(\tilde{V}_1)$, $U_2=\varphi_2(\tilde{V}_2)$.

Then

$$f\big|_{U_1} = \varphi_2\circ F\circ\varphi_1^{-1}\big|_{U_1} : U_1\to U_2$$

is a diffeomorphism, since it is a composition of diffeomorphisms. $\blacksquare$

<u>Rem</u> It follows from the proof, that

$$d_pf = \underbrace{d_0\varphi_2\circ d_0F\circ d_p\varphi^{-1}}_{\text{iso}}.$$

In particular,

$d_p f$ is injective $\Longleftrightarrow$ $D_{\varphi_1(p)}$ is injective

surj. $\Longleftrightarrow$ surj.

iso $\Longleftrightarrow$ iso

Let $f \in C^\infty(S_1; S_2)$.

<u>Def</u> 1) A pt $p \in S_1$ is called a critical pt of f if $d_p f$ is not surjective $\Longleftrightarrow$ $d_p f$ is not injective $\Longleftrightarrow$ $d_p f$ is not an iso.

2) $q \in S_2$ is called a <u>regular value</u> of f, if $\forall p \in f^{-1}(q)$ is regular (non-critical), i.e., if $\forall p \in f^{-1}(q)$ $d_p f$ is surjective $\Longleftrightarrow$ $d_p f$ is injective $\Longleftrightarrow$ $d_p f$ is an iso.

<u>Ex</u> $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$, $f(z) = z^n$, $n \in \mathbb{Z}$, $n \geq 2$. It is known from analysis that $D_z f : \mathbb{C} \to \mathbb{C}$ can be identified with the map $h \longmapsto f'(z) \cdot h$. Hence, z is critical iff $f'(z) = 0 \Longleftrightarrow n z^{n-1} = 0$

$\Longleftrightarrow z = 0$. Hence, f has a single critical pt $z = 0$ and a single critical value, the zero. All other pts are regular and any non-zero value is also regular.

<u>Thm</u> (The fundamental theorem of algebra)
Let $q(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with cx. coefficients. Then p has at least one cx. root.

<u>Proof</u> Recall that the map $f : S^2 \to S^2$,

$$f(p) = \begin{cases} N & p = N, \\ \varphi_N \circ q \circ \varphi_N^{-1}, & p \neq N, \end{cases}$$

is smooth.

<u>Step 1</u> f has has at most n-1 crit pts (values).

$p \in S^2 \setminus \{N\}$ is critical $\iff$ $z := \varphi_N(p)$ is critical for q $\iff$ $q'(z) = 0$, that is

$$n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \dots + a_1 = 0$$

This can have at most $(n-1)$ roots.

<u>Step 2</u> Denote by $\mathcal{R}(f)$ the set of regular values of f. Then for any $r \in \mathcal{R}(f)$ the set $f^{-1}(r)$ is finite and the map

$$r \longmapsto \# f^{-1}(r)$$

is constant.

Let $p \in f^{-1}(r)$, $r \in \mathcal{R}(f) \Longrightarrow$

$f(p) = r$ & $d_p f$ is an iso

$\Rightarrow \exists$ a nbhd U_p of p and a nbhd W_r s.t.

$f: U_p \longrightarrow W_r$ is a diffeo. In particular, $f^{-1}(r) \cap U_p = \{p\} \Longrightarrow$

$f^{-1}(r)$ is discrete.

However, $f^{-1}(r)$ is a closed subset of S^2, hence compact. But a compact discrete set must be finite.

Denote $f^{-1}(r) = \{p_1, \dots, p_m\}$ and the corresponding nbhds $U_1, \dots, U_m$, $W_1, \dots, W_m$.

Set $W := W_1 \cap \dots \cap W_m$ and $\tilde{U}_j := f^{-1}(W) \cap U_j$. Then for each $j \leq m$ the map $f: \tilde{U}_j \longrightarrow W$ is a diffeomorphism.

In particular, $\forall\, r' \in W$ $\exists !\, p_j' \in \tilde{U}_j$ s.t. $f(p_j') = r'$. Hence, $\# f^{-1}(r') = \# f^{-1}(r)$ $\forall\, r' \in W$, so that the function

$$R(f) \longrightarrow \mathbb{Z}, \quad r \longmapsto \# f^{-1}(r) \qquad (*)$$

is locally constant.

However, $R(f)$ is the complement of a finite number of pts in S^2, hence connected. Therefore, $(*)$ is (globally) constant.

Step 3 We prove this thm.

Pick any pairwise distinct pts $p_1, \dots, p_n \in S^2 \setminus \{N\}$ s.t. $f(p_1), \dots, f(p_n)$ are also pairwise distinct. Since f has at most $n-1$ critical values, at leas one pt from $\{f(p_1), \dots, f(p_n)\}$ is a regular value and $(*)$ does not vanish at this pt. Hence, $(*)$ vanishes nowhere on $\mathcal{R}(f)$.

If S is a critical value of S, then $f^{-1}(S) \neq \emptyset$, since $f^{-1}(S)$ contains a critical pt. However,

$$f^{-1}(S) \neq \emptyset \iff g^{-1}(0) \neq 0.$$

If S is a regular value, then by Step 2 $\# f^{-1}(S) \geq 1 \implies f^{-1}(S) \neq \emptyset$.

This finishes the proof. $\blacksquare$