Differential Geometry I

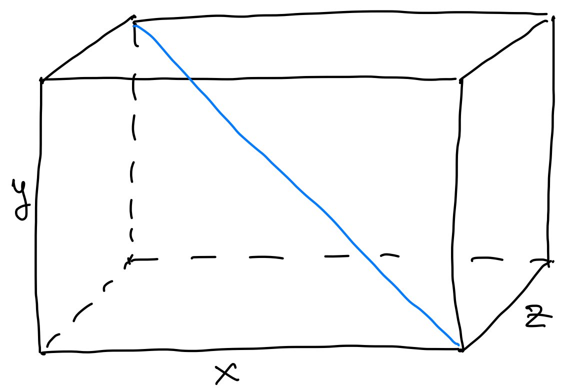
Let $U \subset \mathbb{R}^n$ be an open subset and $f \in C^1(U)$. It is known from analysis that $x_0 \in U$ is a point of extremum for f if

$$\frac{\partial f}{\partial x_i}(x_0) = 0 \qquad \forall i = 1, \dots, n.$$

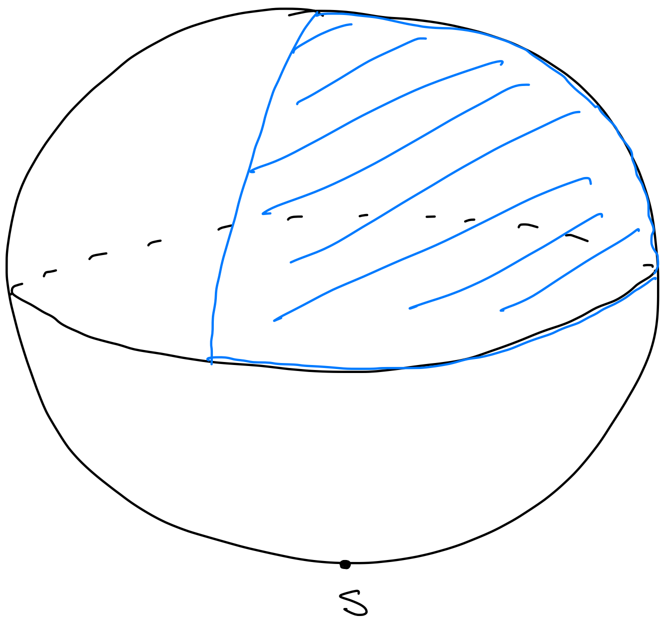
Notice that this is a necessary condition, which is not sufficient in general.

A more general type of problems does not fit into this scheme. For example, consider the following.

<u>Problem</u> Among all rectangular parallelepipeds, whose diagonal has a fixed length 1, find the one with maximal volume.



Thus, we want to find a point of maximum 2
of the function $f(x,y,z) = xyz$ on the set
$V := \{(x,y,z) \in \mathbb{R}^3 \mid x>0, y>0, z>0 \text{ and } x^2+y^2+z^2=1\} \subset S^2$



However, V is <u>not</u> an open subset of $\mathbb{R}^3$ so that the receipy known from the analysis course is not applicable.

This problem is relatively easy to solve, however. Indeed, since $z>0 \Longrightarrow z = \sqrt{1-x^2-y^2}$ so that

$$f(x,y,\sqrt{1-x^2-y^2}) = \underbrace{xy\sqrt{1-x^2-y^2}}_{F(x,y)}, \qquad x^2+y^2<1$$

Hence, we want to find points of maximum of the function F on the set $\{(x,y) \mid x^2+y^2<1,\ x>0,\ y>0\}$, which is an open subset of $\mathbb{R}$.

We compute

$$\frac{\partial F}{\partial x} = y\sqrt{1-x^2-y^2} - xy\frac{x}{\sqrt{1-x^2-y^2}} = 0$$

$$(*)$$

$$\frac{\partial F}{\partial y} = x\sqrt{1-x^2-y^2} - xy\frac{y}{\sqrt{1-x^2-y^2}} = 0$$

Since $x \neq 0$ and $y \neq 0$, we have

$$(*) \iff \begin{matrix} 1-x^2-y^2 = x^2 \\ 1-x^2-y^2 = y^2 \end{matrix} \implies x^2 = y^2 \implies x = y$$

$$\implies 3x^2 = 1 \implies x = y = \frac{1}{\sqrt{3}}$$

$$\implies z = \frac{1}{\sqrt{3}}$$

Hence, among all rectangular parallelepipeds with the given length of the diagonal the cube maximazes the volume.

<u>Exercise</u> Show that $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a point of maximum indeed.

Consider a more general problem of constrained maximum / minimum. Given $f, \varphi \in C^\infty(\mathbb{R}^n)$ find a point of maximum / minimum of f on the set

$$S := \{ x \in \mathbb{R}^n \mid \varphi(x) = 0 \}.$$

<u>Prop</u> 1 Assume that for $p \in S$ we have

$$\frac{\partial \varphi}{\partial x_n}(p) \neq 0. \qquad (*)$$

Then $\exists$ a neighbourhood U of p in S, an open subset $V \subset \mathbb{R}^{n-1}$, and a smooth function $\psi : V \to \mathbb{R}$ such that

$$\underset{\mathbb{R}^{n-1}\ \ \ \mathbb{R}}{x = (y, z)} \in S \cap U \iff z = \psi(y), \ y \in V.$$

This is a celebrated implicit function theorem, whose proof was given in the analysis course.

<u>Thm</u> 1 Let $p \in S$ be a point of (local) maximum of f on S. If $(*)$ holds, then $\exists \lambda \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x_j}(p) = \lambda \frac{\partial \varphi}{\partial x_j}(p) \iff \nabla f(p) = \lambda \nabla \varphi(p)$$

holds for each $j = 1, \dots, n$.

<u>Proof</u> Let $p = (y_0, z_0)$.

p is a (loc.) maximum for $f|_S \iff$

y_0 is a loc. maximum for a function

$F : V \to \mathbb{R}$, $F(y) := f(y, \varphi(y))$

$$\Rightarrow \frac{\partial F}{\partial y_j}(y_0) = \frac{\partial f}{\partial y_j}(p) + \frac{\partial f}{\partial x_n}(p) \cdot \frac{\partial \varphi}{\partial y_j}(y_0) = 0$$

$\forall j \le n-1$

$$\varphi(y, \varphi(y)) = 0 \Rightarrow \frac{\partial \varphi}{\partial y_j} + \frac{\partial \varphi}{\partial x_n} \frac{\partial \varphi}{\partial y_j} \equiv 0$$

$$\Rightarrow \frac{\partial \varphi}{\partial y_j}(y_0) = -\frac{\partial \varphi}{\partial y_j}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p)$$

substitute

$$\Rightarrow \frac{\partial f}{\partial y_j}(p) = \underbrace{\left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p) \right)}_{=\lambda \text{ does not depend on } j} \cdot \frac{\partial \varphi}{\partial y_j}(p)$$

For $j = n$ we have

$$\frac{\partial f}{\partial x_n}(p) = \left(\frac{\partial f}{\partial x_n}(p) \Big/ \frac{\partial \varphi}{\partial x_n}(p) \right) \cdot \frac{\partial \varphi}{\partial x_n}(p) \quad \checkmark$$

$\blacksquare$

Let us come back to the example about maximal volume of parallelepipeds with a fixed length of the diagonal. Thus, if (x,y,z) is a point of maximum of f on

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ x,y,z>0\},$$

then there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z \end{aligned} \quad \Rightarrow \quad (xyz)^2 = 8\lambda^3 xyz$$

$\lambda \neq 0$, since otherwise $x=0$ or $y=0$ or $z=0$.

$$\Rightarrow \quad \underset{\displaystyle 2\lambda x^2}{\underset{\|}{xyz}} = 8\lambda^3 \quad \Rightarrow \quad x = 2\lambda$$

using the first eqn

A similar argument yields also $y = 2\lambda$ and $z = 2\lambda$

$$\Rightarrow \quad 4\lambda^2 + 4\lambda^2 + 4\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \quad x = y = z = \frac{1}{\sqrt{3}}$$

Coming back to Prop. 1 on P. 4, it is clear that it is only important that one of the partial derivatives of φ does not vanish. This leads to the following definition.

<u>Def</u> (Surface) A non-empty set $S \subset \mathbb{R}^3$ is called a (smooth) surface, if for any $p \in S$ $\exists$ an open set $V \subset \mathbb{R}^2$ and a smooth map $\psi : V \to S$ such that the following holds:

(i) $\psi(V) =: U$ is a neighbourhood of p in S.

(ii) $\psi : V \to U$ is a homeomorphism.

(iii) $D_q \psi : \mathbb{R}^2 \to \mathbb{R}^3$ is injective $\forall q \in V$.

<u>Ex</u> Assume $\varphi \in C^\infty(\mathbb{R}^3)$ satisfies

$$\frac{\partial \varphi}{\partial z}(p) \neq 0 \qquad \forall p \in S = \{ \varphi(x,y,z) = 0 \}.$$

Let ψ be as in Prop 1 on P. 4. Define $\Psi(x,y) := (x, y, \psi(x,y))$. If U and V are also as in Prop. 1 on P4, then $\Psi : V \to S \cap U$ is a homeomorphism, since $\pi : S \cap U \to V$, $\pi(x,y,z) = (x,y)$

is a continuous inverse. Furthermore,

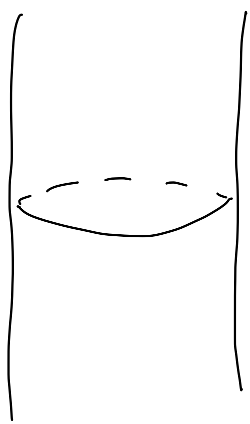
$$D\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{pmatrix}$$ is clearly injective at all points.

Hence, S is a surface.

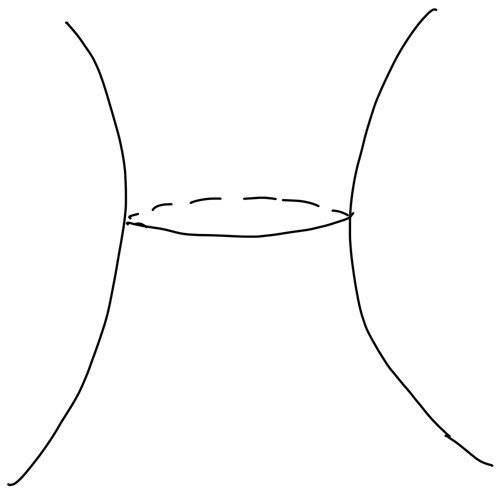
In particular,

- the sphere $S^2 = \{ x^2+y^2+z^2=1 \}$
- the cylinder $C = \{ (x,y,z) \mid x^2+y^2=1 \}$
- the hyperboloid $H = \{ x^2+y^2-z^2=1 \}$

are surfaces

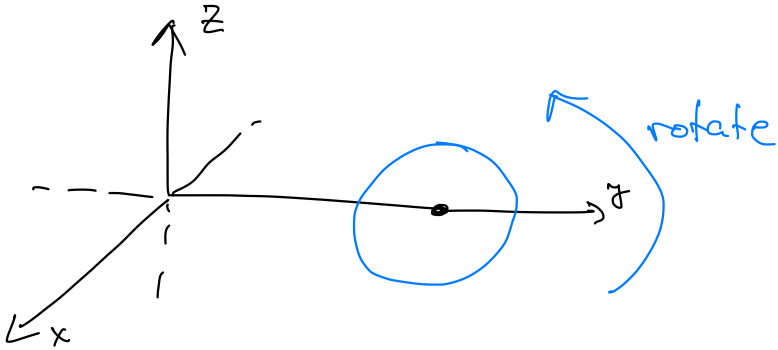


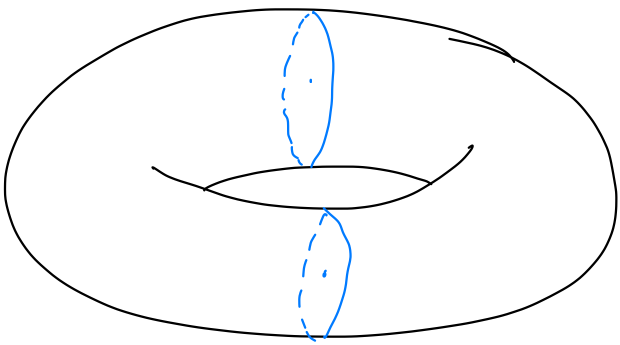
C



H

Ex (Torus) Let C be the circle of radius r in the yz-plane centered at the point $(0, a, 0)$, where $a > r$





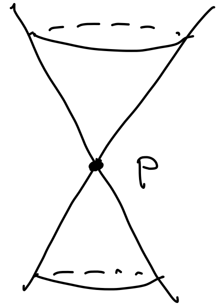
Torus

More formally,

$$T := \{ (\sqrt{x^2+y^2} - a)^2 + z^2 = r^2 \}$$

Exercise Check that T is a surface indeed.

<u>A non-example</u> A double cone



$C_0 := \{x^2 + y^2 - z^2 = 0\}$

is not a surface.

Indeed, assume C_0 is a surface. Then the tip of the cone p must have a neighbourhood U homeomorphic to an open disc in $\mathbb{R}^2$.

Let $f : U \to D$ be a homeomorphism.

Then $f : U \setminus \{p\} \to D \setminus f(p)$ is also a homeo.

$\quad\quad\quad\quad$ disconnected $\quad\quad\quad\quad$ connected

Hence, p does not have a nbhd homeomorphic to a disc (or any open subset of $\mathbb{R}^2$).

<u>Exercise</u> Show that a straight line is not a surface.

<u>Rem</u> 1) The map φ in the definition of the surface is called a <u>parametrization</u>.

2) Condition (iii) is equivalent to

$$\frac{\partial \varphi}{\partial u} \quad \& \quad \frac{\partial \varphi}{\partial v}$$

are linearly independent at each pt $(u,v) \in V$.

Prop Let S be a surface. For any $p \in S$ $\exists$ a nbhd $W \subset \mathbb{R}^3$ and $\varphi \in C^\infty(W)$ such that

$$S \cap W = \{ x \in W \mid \varphi(x) = 0 \}$$

and $\nabla \varphi(x) \neq 0 \quad \forall x \in S \cap W$.

Proof Choose a parametrization $\psi : \underset{\mathbb{R}^2}{\overset{\cap}{V}} \to \underset{S}{\overset{\cap}{U}}$.

Let $\psi(u_0, v_0) = p$ and choose a vector $n \in \mathbb{R}^3$ such that

$$\frac{\partial \psi}{\partial u}(u_0, v_0) \quad , \quad \frac{\partial \psi}{\partial v}(u_0, v_0) \quad , \quad n \qquad (*)$$

are linearly independent. Consider the map

$$\Psi : V \times \mathbb{R} \to \mathbb{R}^3, \quad \Psi(u, v, w) = \psi(u, v) + w \cdot n$$

By $(*)$, $\det D\Psi(u_0, v_0, 0) \neq 0$. By the inverse map theorem, $\exists$ open neighbourhood $W \subset \mathbb{R}^3$ of p and a smooth map $\Phi : W \to V \times \mathbb{R} \subset \mathbb{R}^3$ such that

$$\Psi \circ \Phi(x) = x \qquad \forall x \in W$$

If $\Phi = (\varphi_1, \varphi_2, \varphi_3)$, then

$$\Psi \circ \Phi(x) = \psi(\varphi_1(x), \varphi_2(x)) + \varphi_3(x) \cdot n = x$$

Observe that

$$x \in S \cap W \iff \exists (u,v) \in V \text{ s.t. } \varphi(u,v) = x$$

$$x = \varphi(u,v) = \Psi(u,v,0)$$
$$= \Psi(\varphi_1(x), \varphi_2(x), \varphi_3(x))$$

Since Ψ is injective (on an open nbhd of $(u_0, v_0, 0)$), we have

$$x \in S \cap W \iff \varphi_3(x) = 0.$$

Furthermore, $\det D\Phi(x) \neq 0 \quad \forall x \in W$

$\iff \nabla\varphi_1(x), \nabla\varphi_2(x), \nabla\varphi_3(x)$ are linearly independent $\forall x \in W$

$\implies \nabla\varphi_3(x) \neq 0 \quad \forall x \in W.$ ▯

<u>Corollary</u> Any surface is locally the graph of a smooth function.

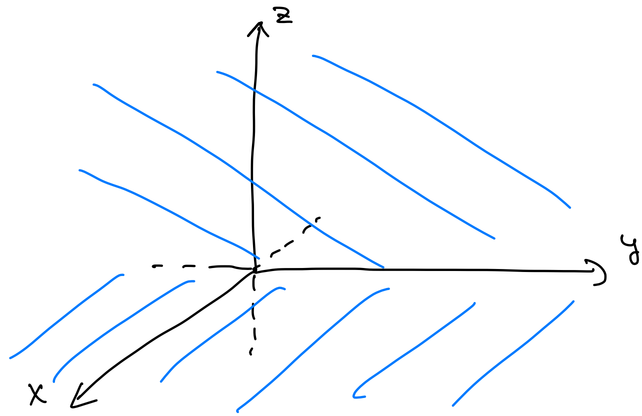
The proof follows from Prop 1 on P. 4.

<u>A non-example</u> A union of two intersecting planes is <u>not</u> a surface

Indeed, assume that

$S := \{z=0\} \cup \{x=0\}$

is a surface.



Then $\exists$ a smooth function φ defined in a nbhd W of the origin such that

$$\varphi(x,y,z) = 0 \quad \text{on} \quad S$$

$$\Rightarrow \quad \nabla\varphi(0) = 0.$$

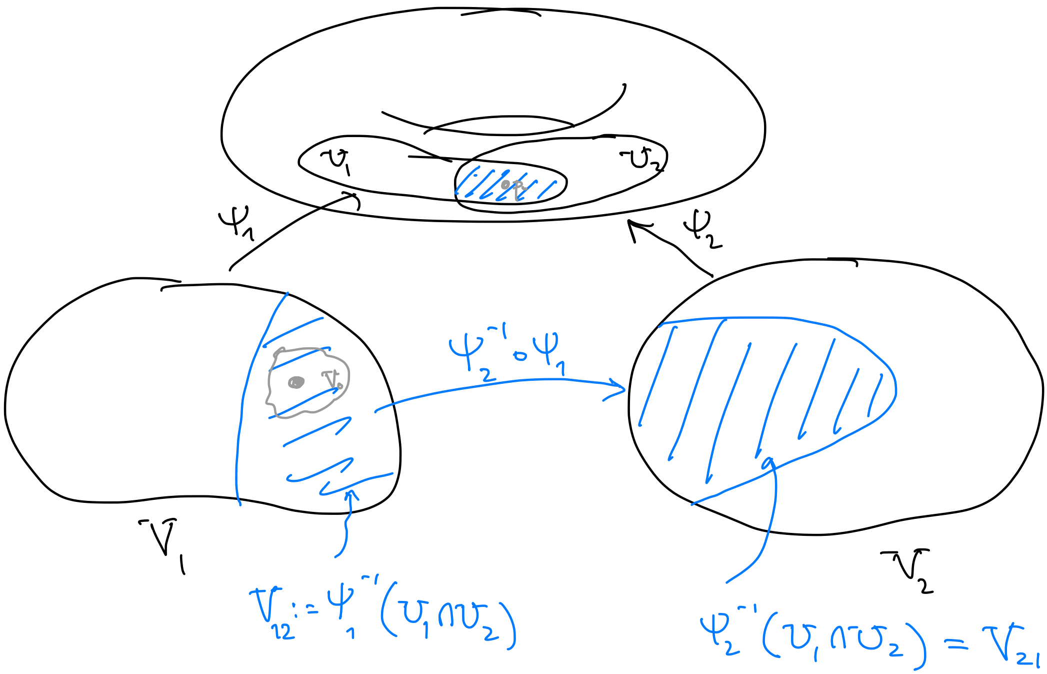
Thus, S is <u>not</u> a surface.

<u>Remark</u> Neither parametrizations, nor local functions as in the Proposition on P. 11 are unique. Our goal is to understand a relation between different parametrizations.

Thus, let $\varphi_1 : V_1 \longrightarrow U_1 \subset S$

$\varphi_2 : V_2 \longrightarrow U_2 \subset S$

be two parametrizations s.t. $U_1 \cap U_2 \neq 0$



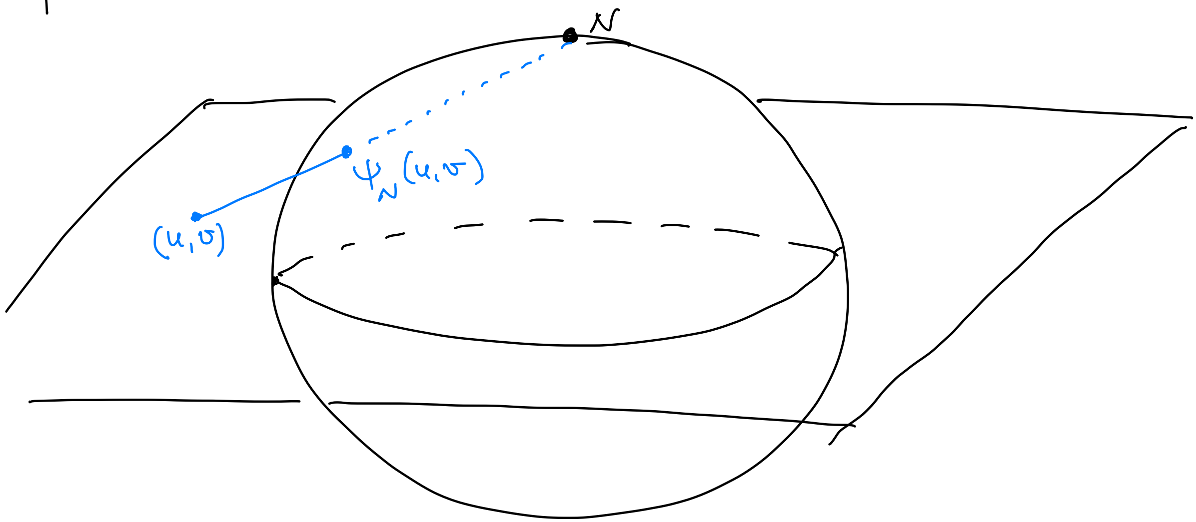
Since φ_1 & φ_2 are homeomorphisms, we have a well-defined continuous map

$$\varphi_{21} := \varphi_2^{-1} \circ \varphi_1 : V_{12} \longrightarrow V_{21}$$

which is called "transition map" or "change of coordinates map".

Notice that φ_{21} is a map $\mathbb{R}^2 \to \mathbb{R}^2$ defined on an open subset.

Ex Consider the sphere S^2, which can be covered by the images of two parametrizations as follows.



The inverse of the steregraphic projection from the north pole N is given by

$$(u,v) \longmapsto \varphi_N(u,v) = \frac{1}{1+u^2+v^2}\left(2u,\ 2v,\ -1+u^2+v^2\right)$$

This is a homeomorphism viewed as a map $\mathbb{R}^2 \to S^2 \setminus \{N\}$ and is clearly smooth.

Exercise Show that $D\varphi_N$ is injective at each point.

Thus φ_N is a parametrization (at each point $p \in S^2 \setminus \{N\}$).

Of course, we have also the inverse ψ_S of the stereographic projection from the south pole S. The images of these two parametrizations cover together the whole sphere S^2.

<u>Rem</u> A computation shows that the change of coordinates map $\psi_{SN} := \psi_S^{-1} \circ \psi_N : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}$ is given by

$$\psi_{SN}(u,v) = \frac{1}{u^2+v^2}(u,v)$$

<u>Exercise</u> Show that the sphere can not be covered by the image of a single parametr.

<u>Thm</u> The change of coordinates map is smooth.

<u>Proof</u> Since smoothness is a local property, it suffices to show that $\forall\,(u_0,v_0) \in V_{12}$ $\exists$ a nbhd $V_0 \subset V_{12}$ such that $\psi_{21}|_{V_0}$ is smooth.

Thus, set $p_0 := \psi_1(u_0,v_0)$. For this p_0 and ψ_2 construct a smooth map $\Phi_2 : W \longrightarrow V_2 \times \mathbb{R}$ as in the proof of the Proposition on P. 11.

Recall that

$$\Phi_2\big|_{S\cap W} : S\cap W \longrightarrow V_2 \times \{0\} = V_2$$

equals φ_2^{-1}.

The map $\Phi_2 \circ \varphi_1 : \varphi_1^{-1}(S\cap W) \to V_2$ is clearly smooth as a composition of smooth maps. Set $V_0 := V_{12} \cap \varphi_1^{-1}(S\cap W)$.

Since the image of φ_1 lies in S, we have

$$\Phi_2 \circ \varphi_1 \big|_{V_0} = \varphi_2^{-1} \circ \varphi_1 \big|_{V_0} = \varphi_{21}\big|_{V_0}$$

is smooth. ∎

<u>Def</u> Let S be a surface.
A function $f : S \to \mathbb{R}$ is said to be smooth, if for any parametrization $\varphi : V \to U$ the composition

$$F := f \circ \varphi : V \to \mathbb{R}$$

is smooth. The function $F := f\circ\varphi$ is called a local (coordinate) representation of f.

<u>Rem</u> The theorem on P. 15 implies that if $f \circ \varphi_1$ is smooth, then $f \circ \varphi_2$ is also smooth on $V_{21} = \varphi_2^{-1}(U_1 \cap U_2)$.
Indeed,

$$f \circ \varphi_2 = f \circ \varphi_1 \circ (\varphi_1^{-1} \circ \varphi_2) = \underbrace{(f \circ \varphi_1)}_{\text{smooth}} \circ \underbrace{\varphi_{12}}_{\text{smooth}}$$

Hence, if (V_i, φ_i) is a collection of parametrizations such that $\varphi_i(V_i)$ covers all of S, it suffices to check that $f \circ \varphi_i$ is smooth $\forall i$.

<u>Ex</u> Let $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be an arbitrary smooth function. Define $f : S \to \mathbb{R}$ as the restriction of h. Then f is smooth, since for any parametrization φ we have

$$f \circ \varphi = \underbrace{h \circ \varphi}_{\text{smooth}} .$$

For example, for any fixed $a \in \mathbb{R}^3$ the height function

$$f_a(x) = \langle a, x \rangle \qquad x \in S$$

is a smooth function on S.

In particular, set $S = S^2$ and $h(x,y,z) = z$. Then the coordinate representation of $f = h|_{S^2}$ with respect to ψ_N is

$$F(u,v) = f \circ \psi_N(u,v) = \frac{-1+u^2+v^2}{1+u^2+v^2}.$$

<u>Ex</u> Let $\psi : V \to U$ be a parametrization of a surface S. Since ψ is a homeomorphism, we have the inverse map

$$\varphi := \psi^{-1} : U \to V.$$

Since U itself is a surface (with a single parametrization ψ), it makes sense to ask if φ viewed as a map $U \to \mathbb{R}^2$ is smooth, which means by definition that both components of φ are smooth functions.

This is the case indeed, since the local representation of φ is nothing else but

$$\varphi \circ \psi = id,$$

which is certainly smooth.

Any pair (U, φ) is called <u>a chart</u> on S.

<u>Prop 1</u> Let S be a surface. Then the set $C^\infty(S)$ of all smooth functions on S is a vector space, that is

$$\left.\begin{array}{l} f, g \in C^\infty(S) \\ \lambda, \mu \in \mathbb{R} \end{array}\right\} \Longrightarrow \lambda f + \mu g \in C^\infty(S).$$

In fact, we also have

$$f, g \in C^\infty(S) \Longrightarrow f \cdot g \in C^\infty(S)$$

<u>Proof</u> We prove the last statement only.

Let $\psi : U \to V$ be a parametrization.

Then
$$(f \cdot g) \circ \psi = \underbrace{\underbrace{(f \circ \psi)}_{\in C^\infty(V)} \cdot \underbrace{(g \circ \psi)}_{\in C^\infty(V)}}_{\in C^\infty(V).}$$

$\square$

Let $W \subset \mathbb{R}^n$ be an open set.

<u>Def</u> A cont. map $f : W \longrightarrow S$, where S is a surface, is called <u>smooth</u>, if for any parametrization $\psi : V \to U \subset S$ the map

$$\varphi \circ f = \psi^{-1} \circ f : \quad f^{-1}(U) \longrightarrow \underset{\mathbb{R}^2}{V \cap}$$

is smooth.