Global Analysis

Lecture notes

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This is a draft. If you spot a mistake, please let me know.

Contents

Chapter 1

Introduction

1.1 The de Rham cohomology groups

Let M be a compact manifold of dimension n. Denote by $\Omega^k(M)$ the space of differential k-forms on M. Recall that there exist a unique $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ with the following properties:

(i) df is the differential of f if $f \in C^\infty(M) = \Omega^0(M)$;

(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^q \omega \wedge d\eta$ if $\eta \in \Omega^q(M)$;

(iii) $d^2 = 0$.

The last property simply means that $d(d\omega) = 0$ for each $\omega \in \Omega^k(M)$. This yields the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(M) \to 0 \tag{1.1}$$

Remark 1.2. The map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ depends on k, however this is suppressed in the notations.

Property (iii) means that (1.1) is a complex, that is the kernel of $d : \Omega^k \to \Omega^{k+1}$ contains the image of $d : \Omega^{k-1} \to \Omega^k$ and therefore we can define

$$H_{dR}^k(M) := \frac{\text{Ker}\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\text{Im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)} \tag{1.3}$$

This is called the kth de Rham cohomology group, which is in fact a vector space. The number

$$b_k(M) := \dim H_{dR}^k(M)$$

is called the kth Betti number of M and is an invariant of M. Notice that by the compactness of M we have $b_k(M) < \infty$.

Remark 1.4. While it is by no means obvious from the above description, Betti numbers are topological invariants of M, that is $b_k(M) = b_k(N)$ provided M and N are homeomorphic. In paticular, Betti numbers do not depend on the smooth structure.

Coming back to the de Rham cohomlogy groups, each element in $H_{dR}^k(M)$ is represented by the equivalence class

$$[\omega] = \left\{\omega + d\eta \mid \eta \in \Omega^{k-1}(M)\right\},$$

where $\omega \in \Omega^k(M)$ is closed: $d\omega = 0$. Hence, we may ask the following.

Question 1.5. What is the best representative in $[\omega]$?

Of course, at this point the above question is vague, since the notion of being "the best" is undefined. One possibility to convert this into a precise question is as follows. Just by its definition, the set $[\omega]$ is an affine subspace of $\Omega^k(M)$. We could call an element in $[\omega]$ "the best" if it is the closest one to the origin just as shown schematically on Figure 1.1.
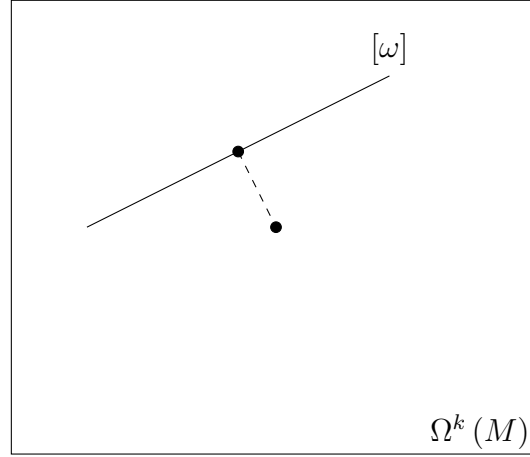

Figure 1.1: A choice of a representative in the de Rham cohomology class.

However, this raises our next question: How do we measure distance in $\Omega^k(M)$? A suitable answer to this question requires a detour, which we do next.

1.2   Some linear algebra

Let V be a vector space of dimension n over $\mathbb{R}$. Recall that for each basis $e = (e_1, \ldots, e_n)$ of V there exist a unique basis $e^* = (e_1^*, \ldots, e_n^*)$ of the dual vector space V^* such that

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

e^* is called the dual basis to e.

Assume $\langle \cdot, \cdot \rangle$ is a scalar product on V. If $e = (e_1, \ldots, e_n)$ is an orthonormal basis of V, that is

$$\langle e_i, e_j \rangle = \delta_{ij},$$

then the dual basis e^* is given explicitly by

$$e_i^* := \langle e_i, \cdot \rangle \qquad \Longleftrightarrow \qquad e_i^*(v) = \langle e_i, v \rangle \quad \text{for } v \in V.$$

Then V^* has a unique scalar product such that $e^* = (e_1^*, \ldots, e_n^*)$ is an orthonormal basis. Explicitly, for $\xi, \eta \in V^*$ define

$$\begin{aligned} \xi_i &:= \xi(e_i) \\ \eta_i &:= \eta(e_i) \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} \xi &= \sum \xi_i\, e_i^* \\ \eta &= \sum \eta_j\, e_j^* \end{aligned}$$

$$\Longrightarrow \langle \xi, \eta \rangle = \sum_{i=1}^{n} \xi_i \eta_i = \sum_{i=1}^{n} \xi(e_i)\, \eta(e_i).$$

To sum up, for any scalar product on V there exists a unique scalar product on V^* such that the dual basis of an orthonormal basis is itself orthonormal.

More generally, any basis e of V yields a basis of $\Lambda^k V^*$. Explicitly,

$$\Lambda^k e := \left\{ e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n \right\} \tag{1.6}$$

is a basis of $\Lambda^k V^*$ consisting of

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

elements. Just like in the case of $V^* = \Lambda^1 V^*$, we can define a scalar product on each $\Lambda^k V^*$ by declaring (1.6) to be an orthonormal basis.

Recall that any two bases e and f of V are related by a change-of-basis matrix A. This means

$$f = e \cdot A \qquad \Longleftrightarrow \qquad f_i = \sum_{j=1}^{n} a_{ij}\, e_j$$

Then e and f are said to be *cooriented*, if $\det A > 0$. It is easy to check that

$$e \sim f \qquad \equiv \qquad e \text{ and } f \text{ are cooriented}$$

yields an equivalence relation on the set of all bases of V. Moreover, there are exactly two equivalence classes represented by e and $\bar{e} = (-e_1, e_2, \ldots, e_n)$.

Definition 1.7. An orientation on V is a choice of an equivalence class of bases of V. Any basis in the chosen class is said to be positively oriented and any basis, which does not belong to the selected class is said to be negatively oriented.

Example 1.8. For $\mathbb{R}^n$ the class of the standard basis is called the standard orientation of $\mathbb{R}^n$.

Example 1.9. Any $\omega \in \Lambda^n V^*$, $\omega \neq 0$, determines an orientation of V by the rule: e is positively oriented if and only if

$$\omega(e_1, \ldots, e_n) > 0.$$

For example, if $e^* = (e_1^*, \ldots, e_n^*)$ is the dual basis to the standard one, then

$$\omega := e_1^* \wedge \ldots \wedge e_n^* \tag{1.10}$$

determines the standard orientation of $\mathbb{R}^n$.

Definition 1.11. Let V be an oriented Euclidean vector space of dimension n. An n-form ω is said to be the Euclidean volume form, if

$$\omega(e_1, \ldots, e_n) = 1 \tag{1.12}$$

holds for any positively oriented orthonormal basis e of V.

For example, in the case $V = \mathbb{R}^n$, which is equipped with the standard scalar product and orientation, (1.10) is the Euclidean volume form.

Example 1.13. Show that any oriented Euclidean vector space admits a unique Euclidean volume form. This is sometimes denoted by *vol*.

Proposition 1.14. *There is a unique linear map*

$$* : \Lambda^k V^* \to \Lambda^{n-k} V^* \qquad satisfying \qquad \xi \wedge * \eta = \langle \xi, \eta \rangle \, vol \qquad (1.15)$$

for all $\xi, \eta \in \Lambda^k V^*$.

Proof (Sketch). Let e be a positively oriented orthonormal basis of V. Set

$$\eta := e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \in \Lambda^k e^*$$

Then assuming that $*$ exists, for

$$\xi = \sum \xi_{j_1 \ldots j_k} \, e_{j_1}^* \wedge \ldots \wedge e_{j_k}^*$$

we must have

$$\xi \wedge * \eta = \langle \xi, \eta \rangle \, vol = \xi_{j_1 \ldots j_k} \cdot e_1^* \wedge \ldots \wedge e_n^*.$$

This yields

$$* \eta = * \left(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \right) = \varepsilon \cdot e_{l_1}^* \wedge \ldots \wedge e_{l_{n-k}}^*, \qquad (1.16)$$

where $\varepsilon \in \{\pm 1\}$ and $1 \le l_1 < \ldots < l_{n-k} \le n$ consists of those integers in the interval $[1, n]$ which are complementary to $\{i_1, \ldots, i_k\}$.

For example, if $n = 6$ and $\eta = e_2 \wedge e_4$, then

$$* \left(e_2^* \wedge e_4^* \right) = \varepsilon \, e_1^* \wedge e_3^* \wedge e_5^* \wedge e_6^* \, .$$

To determine ε, we compute

$$e_2^* \wedge e_4^* \wedge \varepsilon \left(e_1^* \wedge e_3^* \wedge e_5^* \wedge e_6^* \right) = -\varepsilon \, vol \, ,$$
$$e_2^* \wedge e_4^* \wedge * \left(e_2^* \wedge e_4^* \right) = \| \, e_2^* \wedge e_4^* \, \|^2 \cdot vol = vol,$$

which yields $\varepsilon = -1$ so that we finally obtain

$$* \left(e_2^* \wedge e_4^* \right) = -e_1^* \wedge e_3^* \wedge e_5^* \wedge e_6^*.$$

In general, ε is the signature of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

Thus, (1.16) defines $*$ on the elements of the basis $\Lambda^k e$. This yields a unique linear map $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$, which satisfies (1.15). $\qquad \square$

The map $*$ defined in the above proposition is called *the Hodge operator*.

Remark 1.17. It follows from the proof of the above proposition that the Hodge operator satisfies

$$* * \omega = (-1)^{k(n-k)} \, \omega \qquad \text{for all } \omega \in \Lambda^k V^*. \qquad (1.18)$$

1.3 Riemannian metrics

Definition 1.19. A Riemannian metric g on M is a smooth section of $T^* M \otimes T^* M$ such that the following holds:

(i) g is symmetric, that is

$$g(v, w) = g(w, v) \qquad \text{for all } v, w \in T_m M \text{ and any } m \in M;$$

(ii) g is positive-definite, that is

$$g(v,v) > 0 \qquad \text{for all } v \in T_m M, v \neq 0, \text{ and any } m \in M.$$

In other words, g is a family $\{g_m \mid m \in M\}$ of scalar products on each $T_m M$ and g_m depends smoothly on m. In particular, each $T_m M$ is an Euclidean vector space. Hence, each $\Lambda^k T_m^* M$ is also an Euclidean vector space.

An orientation of a manifold M is (informally speaking) a choice of coherent orientations of $T_m M$ for each $m \in M$. More formally, we have the following.

Definition 1.20. A manifold M of dimension n is said to be orientable, if there exists $\omega \in \Omega^n(M)$ such that $\omega_m \neq 0$ for all $m \in M$.

By Example 1.9, for each $m \in M$ the n-form on $T_m M$ determines a class of positively oriented bases of $T_m M$, that is an orientation. Notice that for any function f, which is positive everywhere, the forms ω and $f \cdot \omega$ determine the same orientation on each $T_m M$.

Albeit not all manifolds are orientable, orientability is a mild restriction. In particular, for any connected non-orientable manifold M there exists a unique double covering $M_2 \to M$, which is orientable. The reader may find more information on this in [?].

Definition 1.21. An orientation of an n-manifold M is a class of n-forms $[\![\omega]\!]$, where

- ω is a nowhere vanishing n-form on M.
- $\omega_1 \sim \omega_2$ if and only if there exists an everywhere positive function f such that $\omega_2 = f \cdot \omega_1$.

Notice that $[\![\cdot]\!]$ above is *not* the de Rham cohomology class.

Just like in the preceding section, a differential n-form ω on M is said to be a Riemannian volume form, if

$$\omega_m(e_1, \ldots, e_n) = 1 \tag{1.22}$$

holds for any $m \in M$ and any oriented orthonormal basis $(e_1, \ldots, e_n)$ of $T_m M$. Property 1.22 determines a Riemannian volume form uniquely. This volume form is denoted by vol.

Thus, by the preceding subsection, a Riemannian metric and orientation on M induce for each $k \leq n$ the Hodge operator $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ such that

$$\omega \wedge *\eta = \langle \omega, \eta \rangle \, vol \qquad \text{and} \qquad **\omega = (-1)^{k(n-k)} \omega$$

holds for all $\omega, \eta \in \Omega^k(M)$.

1.4 Harmonic forms

Let us come back to the original question about "the best" representatives of the de Rham cohomology classes. Thus, we define the L^2-scalar product on each $\Omega^k(M)$ by setting

$$\langle \omega, \eta \rangle_{L^2} := \int_M \langle \omega_m, \eta_m \rangle \, vol_m = \int_M \omega \wedge *\eta.$$

With this at hand, we could call an element $\widehat{\omega} = \omega + d\eta \in [\omega]$ "the best", if $\widehat{\omega}$ minimizes the distance to the origin, that is if

$$\inf_{\eta \in \Omega^{k-1}(M)} \| \omega + d\eta \|_{L^2}^2 = \| \widehat{\omega} \|_{L^2}. \tag{1.23}$$

Then, if (1.23) holds, for any $\eta \in \Omega^{k-1}(M)$ we must have

$$
\begin{aligned}
0 &= \frac{d}{dt}\bigg|_{t=0} \| \widehat{\omega} + t\, d\eta \|_{L^2}^2 \\
&= \frac{d}{dt}\bigg|_{t=0} \left(\| \widehat{\omega} \|_{L^2}^2 + 2t\langle \widehat{\omega}, d\eta \rangle_{L^2} + t^2 \| d\eta \|_{L^2}^2 \right) \\
&= 2\langle \widehat{\omega}, d\eta \rangle_{L^2}
\end{aligned}
\tag{1.24}
$$

Proposition 1.25. *Denote* $d^* := (-1)^{nk+1} * d * \colon \Omega^{k+1}(M) \to \Omega^k(M)$.

(i) d^* *is the formal adjoint of d, that is*

$$
\langle d\omega, \eta \rangle_{L^2} = \langle \omega, d^*\eta \rangle_{L^2} \qquad \text{for all } \omega \in \Omega^k(M) \text{ and } \eta \in \Omega^{k+1}(M).
\tag{1.26}
$$

(ii) (1.24) *is equivalent to*

$$
d^*\widehat{\omega} = 0.
\tag{1.27}
$$

Proof. Notice first that (1.15) and (1.18) imply the equality

$$
\omega \wedge \zeta = (-1)^{k(n-k)} \langle \omega, *\zeta \rangle \, vol \qquad \text{for all } \omega \in \Omega^k(M) \text{ and } \zeta \in \Omega^{n-k}(M).
$$

Using this, we obtain

$$
\begin{aligned}
\langle \omega, d^*\eta \rangle_{L^2} &= (-1)^{kn+1} \int_M \langle \omega, *d*\eta \rangle \, vol \\
&= (-1)^{kn+1+k(n-k)} \int_M \omega \wedge d*\eta \\
&= (-1)^{1-k^2} \int_M (-1)^k \left(d(\omega \wedge *\eta) - d\omega \wedge *\eta \right)
\end{aligned}
$$

Here the last equality follows from the Leibnitz rule:

$$
d(\omega \wedge \zeta) = d\omega \wedge \zeta + (-1)^k \omega \wedge d\zeta
$$

provided $\omega \in \Omega^k(M)$ and $\zeta \in \Omega^l(M)$. Hence, by Stokes' theorem, we obtain

$$
\begin{aligned}
\langle \omega, d^*\eta \rangle_{L^2} &= (-1)^{k-k^2+1} \int_M d\omega \wedge *\eta \\
&= (-1)^{k-k^2} \langle d\omega, \eta \rangle_{L^2}
\end{aligned}
$$

By noticing that k^2 is even/odd if and only if k is even/odd, we arrive finally at (1.26).

To prove (ii), notice that

$$
d^*\widehat{\omega} = 0 \qquad \Longrightarrow \qquad 0 = \langle d^*\widehat{\omega}, \eta \rangle_{L^2} = \langle \widehat{\omega}, d\eta \rangle_{L^2}.
$$

Conversely, setting $\eta = d^*\widehat{\omega}$ in (1.24), we obtain

$$
0 = \langle \widehat{\omega}, dd^*\widehat{\omega} \rangle_{L^2} = \langle d^*\widehat{\omega}, d^*\widehat{\omega} \rangle_{L^2} = \| d^*\omega \|_{L^2}^2 \qquad \Longrightarrow \qquad d^*\widehat{\omega} = 0.
$$

$\square$

Notice that (1.27) is nothing else but the Euler-Lagrange equation for the functional

$$f \colon \left\{ \omega + d\eta \mid \eta \in \Omega^{k-1}\left(M\right) \right\} \longrightarrow \mathbb{R}, \qquad f\left(\omega + d\eta\right) = \parallel \omega + d\eta \parallel_{L^2}^2 \qquad (1.28)$$

Definition 1.29. The map

$$\Delta = dd^* + d^*d \colon \Omega^k\left(M\right) \to \Omega^k\left(M\right)$$

is called *the Laplace operator* (or, simply, *the Laplacian*). A k-form ω such that $\Delta\omega = 0$ is called *harmonic*.

Proposition 1.30. *A k-form ω is harmonic if and only if*

$$d\omega = 0 \qquad and \qquad d^*\omega = 0. \qquad (1.31)$$

Proof. If (1.31) holds, then ω is clearly harmonic. To show the converse, consider

$$0 = \langle \Delta\omega, \omega \rangle_{L^2} = \langle dd^*\omega, \omega \rangle_{L^2} + \langle d^*d\omega, \omega \rangle = \parallel d^*\omega \parallel_{L^2}^2 + \parallel d\omega \parallel_{L^2}^2.$$

Since both summands are non-negative, we obtain (1.31). $\qquad\square$

Theorem 1.32. *If a minimizer $\widehat{\omega}$ of (1.28) exists, then $\widehat{\omega}$ is harmonic. Moreover, if $\widehat{\omega}$ exists, then it is unique.*

Proof. Assume $\widehat{\omega}$ exists. Since $\widehat{\omega} = \omega + d\eta$ for some $\eta \in \Omega^{k-1}\left(M\right)$, we have

$$d\widehat{\omega} = d\omega + d\left(d\eta\right) = 0 + 0.$$

Combining this with (1.27), we obtain by Proposition 1.30, that $\widehat{\omega}$ is harmonic.

Furthermore, let $\widehat{\omega} = \omega + d\eta$ and $\widehat{\widehat{\omega}} = \omega + d\zeta$ be two harmonic representatives of $[\omega]$. Then

$$0 = d^*\widehat{\omega} - d^*\widehat{\widehat{\omega}} = d^*\left(\widehat{\omega} - \widehat{\widehat{\omega}}\right) = d^*d\left(\eta - \zeta\right).$$

Denoting temporarily $\xi := \eta - \zeta$, we obtain

$$0 = \langle d^*d\xi, \xi \rangle_{L^2} = \langle d\xi, d\xi \rangle = \parallel d\xi \parallel_{L^2}^2 \quad \Longrightarrow \quad d\xi = 0 \quad \Longrightarrow \quad d\eta = d\zeta \quad \Longrightarrow \quad \widehat{\omega} = \widehat{\widehat{\omega}}.$$

This proves the uniquesness. $\qquad\square$

Our aim is to prove the following.

Theorem 1.33. *(Hodge) Each de Rham cohomology class is represented by a unique harmonic form.*

Notice that since we have already proved the uniqueness, it is the existence, which remains to be proved. It turns out that this is somewhat harder and requires certain technology, which we will consider first.

Notice that for any oriented Riemannian manifold, the Laplacian on $\Omega^0\left(M\right) = C^\infty\left(M\right)$ is given by

$$\Delta f = d^*df = -*d*df.$$

Example 1.34. Consider the case $M = \mathbb{R}^3$ equipped with the standard Euclidean matric. If (x, y, z) are coordinates on $\mathbb{R}^3$, then

$$* df = * \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) = \frac{\partial f}{\partial x} dy \wedge dz - \frac{\partial f}{\partial y} dx \wedge dz + \frac{\partial f}{\partial z} dx \wedge dy.$$

Hence,

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz \implies \Delta f = - \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right).$$

Sometimes this is called "the non-negative Laplacian", since its eigenvalues are non-negative.

More generally, if the Riemannian metric on $\mathbb{R}^n$ is given by a positive-definite matrix (g_{ij}), where $g_{ij} = g_{ij}(x)$, then the corresponding Laplacian on functions is given explicitly by

$$\Delta f = - \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} \, g^{ij} \partial_j f \right), \tag{1.35}$$

where $|g| = |\det (g_{ij})|$ and $(g^{ij}) = (g_{ij})^{-1}$. This is sometimes called the Laplace-Beltrami operator.

Let M be a closed oriented Riemannian manifold.

Proposition 1.36. *(Green's identity) For any $\omega, \eta \in \Omega^k(M)$ we have*

$$\langle \Delta \omega, \eta \rangle_{L^2} = \langle \omega, \Delta \eta \rangle_{L^2} = \langle d\omega, d\eta \rangle_{L^2} + \langle d^* \omega, d^* \eta \rangle.$$

Proof. By Proposition 1.25 *(i)*, we have

$$\langle dd^* \omega + d^* d\omega, \eta \rangle = \langle d^* \omega, d^* \eta \rangle + \langle d\omega, d\eta \rangle.$$

By the same token,

$$\langle \omega, dd^* \eta + d^* d\eta \rangle = \langle d^* \omega, d^* \eta \rangle + \langle d\omega, d\eta \rangle,$$

which yields the claim of this proposition. $\qquad \square$

Corollary 1.37. *On a closed connected manifold any harmonic function is constant.*

Proof. If $f \in C^\infty(M)$ is harmonic, by Green's identity (with $\omega = \eta = f$) we obtain

$$0 = \langle \Delta f, f \rangle = \| \, df \, \|_{L^2}^2 \implies df = 0 \implies f \text{ is constant.}$$

$\qquad \square$

The Hodge theorem follows from the following more general result, which is also attributed to Hodge.

Theorem 1.38. *(Hodge) Let M be a closed oriented Riemannian manifold and $\eta \in \Omega^k(M)$. The equation $\Delta \omega = \eta$ has a solution if and only if*

$$\langle \eta, \omega_0 \rangle_{L^2} = 0 \tag{1.39}$$

for any harmonic k-form ω_0.

It is easy to see that (1.39) is a necessary condition. Indeed, if there is $\omega \in \Omega^k(M)$ such that $\Delta \omega = \eta$, then Green's identity yields

$$\langle \eta, \omega_0 \rangle_{L^2} = \langle \Delta \omega, \omega_0 \rangle = \langle \omega, \Delta \omega_0 \rangle = 0$$

for any harmonic k-form ω_0.

The proof that (1.39) is a sufficient condition will be given later. Taking this as granted for now, from Corollary 1.37 we obtain the following.

Corollary 1.40. *Let M be a closed oriented Riemannian manifold. The equation*

$$\Delta f = h, \qquad\qquad f, h \in C^\infty(M)$$

has a solution if and only if

$$\int_M h \cdot vol = 0.$$

Let me also show that Theorem 1.38 implies that any de Rham cohomolgy class is represented by a harmonic form. To this end, pick any closed k-form ω. Then by Proposition ??, $\omega + d\eta$ is harmonic if and only if

$$\begin{aligned} d(\omega + d\eta) &= 0 \\ d^*(\omega + d\eta) &= 0 \end{aligned} \qquad \Longleftrightarrow \qquad d^*(\omega + d\eta) = 0 \qquad \Longleftrightarrow \qquad d^*d\eta = -d^*\omega. \quad (1.41)$$

Furthermore, consider the equation

$$\Delta \eta = -d^*\omega. \tag{1.42}$$

If η_0 is any harmonic $(k-1)$-form, then

$$\langle d^*\omega, \eta_0 \rangle_{L^2} = \langle \omega, d\eta_0 \rangle = 0,$$

Since η_0 is closed. Hence, Theorem 1.38 guarantees that (1.42) has a solution η. We have

$$\begin{aligned} \langle dd^*\eta, \Delta\eta \rangle_{L^2} &= \langle dd^*\eta, dd^*\eta \rangle_{L^2} + \langle dd^*\eta, dd^*\eta \rangle_{L^2} \\ &= \| dd^*\eta \|_{L^2}^2 + \langle d^2 d^*\eta, d\eta \rangle_{L^2} \\ &= \| dd^*\eta \|_{L^2}^2 . \end{aligned}$$

However, using (1.42) we obtain

$$\langle dd^*\eta, \Delta\eta \rangle_{L^2} = -\langle dd^*\eta, d^*\omega \rangle_{L^2} = -\langle d^2 d^*\eta, \omega \rangle_{L^2} = 0.$$

Hence, if η is a solution of (1.42), then in fact η solves (1.41) so that $\omega + d\eta$ is harmonic indeed.

With these preliminary considerations at hand we proceed by showing that Theorem 1.38 has useful applications, in particular in the theory of Riemann surfaces.

1.5   Riemann surfaces

Definition 1.43. A Riemann surface Σ is a complex manifold of dimension one. In particular, Σ admits an atlas $\{(U_\alpha, \psi_\alpha) \mid \alpha \in A\}$, where $\psi_\alpha \colon U_\alpha \longrightarrow \mathbb{C}$, such that each coordinate transformation map

$$\psi_\alpha \circ \psi_\beta^{-1} \colon \mathbb{C} \longrightarrow \mathbb{C}$$

is holomorphic on the domain of its definition.

Writing $z_\alpha = z = x + iy = \psi_\alpha$, we obtain a holomorphic coordinate on Σ (defined on U_α). Notice that

$$x = \operatorname{Re} z \qquad \text{and} \qquad y = \operatorname{Im} z$$

are real coordinates on U_α. In particular, for any $p \in U_\alpha$ we have a basis

$$\partial^\alpha := (\partial_x, \partial_y) \big|_p$$

of $T_p\Sigma$. Define a linear map

$$I_p \colon T_p\Sigma \longrightarrow T_p\Sigma \qquad \text{by} \qquad I_p\partial_x = \partial_y \quad \text{and} \quad I_p\partial_y = -\partial_x. \qquad (1.44)$$

Using the fact that $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic, it is easy to check that I_p does not depend on the choice of a holomorphic coordinate. It follows also from (1.44) that $I_p^2 = -id$.

For any smooth (not necessarily holomorphic) function $f \colon \Sigma \longrightarrow \mathbb{C}$ define $\partial f, \bar\partial f \in \Omega^1(\Sigma; \mathbb{C})$ by

$$\begin{aligned} \partial f(v) &:= \tfrac{1}{2}(df(v) - i\, df(Iv)) \\ \bar\partial f(v) &:= \tfrac{1}{2}(df(v) + i\, df(Iv)) \end{aligned} \qquad \text{for } v \in T\Sigma.$$

In particular, we have $df = \partial f + \bar\partial f$ and f is holomorphic if and only if $\bar\partial f = 0$. If $f_\alpha := f \circ \psi_\alpha^{-1} \colon \mathbb{C} \longrightarrow \mathbb{C}$ is the local representation of f, then

$$\bar\partial f_\alpha = \frac{\partial f_\alpha}{\partial \bar z}\, d\bar z \qquad \text{and} \qquad \partial f_\alpha = \frac{\partial f_\alpha}{\partial z}\, dz$$

are local representations of $\bar\partial f$ and ∂f respectively, that is $\psi_\alpha^* \bar\partial f_\alpha = \bar\partial f$ and $\psi_\alpha^* \partial f_\alpha = \partial f$.

Furthermore, since

$$\begin{aligned} dz &= dx + i\, dy, \\ d\bar z &= dx - i\, dy, \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} dx &= \tfrac{1}{2}(dz + d\bar z), \\ dy &= \tfrac{1}{2i}(dz - d\bar z), \end{aligned}$$

any $\omega \in \Omega^1(\Sigma; \mathbb{C})$ can be written uniquely as $\omega = a\, dz + b\, d\bar z$ for some functions a and b. Denote by $\Omega^{1,0}(\Sigma)$ the space of all those complex valued 1-forms, whose local representation is $a\, dz$ for some $a \in C^\infty(\Sigma; \mathbb{C})$. More invariantly, we have

$$\Omega^{1,0}(\Sigma) = \left\{ \omega \in \Omega^1(\Sigma; \mathbb{C}) \mid \omega(I\cdot) = i\, \omega(\cdot) \right\}.$$

Similarly, denote by $\Omega^{0,1}(\Sigma)$ the space of all those 1-forms, whose local representation is $b\, d\bar z$ for some complex valued function b, or equivalently,

$$\Omega^{0,1}(\Sigma) := \left\{ \omega \in \Omega^1(\Sigma) \mid \omega(I\cdot) = -i\, \omega(\cdot) \right\}.$$

Thus, we obtain the decomposition

$$\Omega^1(\Sigma; \mathbb{C}) = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma).$$

Then ∂f and $\bar\partial f$ are nothing else but the components of df lying in $\Omega^{1,0}(\Sigma)$ and $\Omega^{0,1}(\Sigma)$ respectively.

Thus, in the case of a Riemann surface, the complexified de Rham complex has the following form:

$$
\begin{array}{ccccc}
 & & \Omega^{1,0}(\Sigma) & & \\
 & \overset{\partial}{\nearrow} & & \overset{\bar\partial}{\searrow} & \\
0 \longrightarrow \Omega^0(\Sigma; \mathbb{C}) & & \oplus & & \Omega^2(\Sigma, \mathbb{C}) \longrightarrow 0 \\
 & \underset{\bar\partial}{\searrow} & & \underset{\partial}{\nearrow} & \\
 & & \Omega^{0,1}(\Sigma) & &
\end{array}
\qquad (1.45)
$$

Here $\bar{\partial} \colon \Omega^{1,0}(\Sigma) \to \Omega^2(\Sigma; \mathbb{C})$ is just the restriction of d to $\Omega^{1,0}(\Sigma)$ and $\partial \colon \Omega^{0,1}(\Sigma) \to \Omega^2(\Sigma, \mathbb{C})$ is the restriction of d to $\Omega^{0,1}(\Sigma)$. Locally, we have

$$\omega \in \Omega^{1,0}(\Sigma) \implies \omega = a\,dz \implies d\omega = da \wedge dz = \left(\frac{\partial a}{\partial z} dz + \frac{\partial a}{\partial \bar{z}} d\bar{z} \right) \wedge dz = -\frac{\partial a}{\partial \bar{z}}\, dz \wedge d\bar{z},$$

$$\omega \in \Omega^{0,1}(\Sigma) \implies \omega = b\,d\bar{z} \implies d\omega = db \wedge d\bar{z} = \left(\frac{\partial b}{\partial z} dz + \frac{\partial b}{\partial \bar{z}} d\bar{z} \right) \wedge d\bar{z} = \frac{\partial b}{\partial z}\, dz \wedge d\bar{z}.$$

Notice that we have

$$d^2 = 0 \qquad \Longleftrightarrow \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

The splitting (1.45) of the de Rham complex yields the Dolbeault cohomology groups:

$$H^{1,0}(\Sigma) := \ker \bar{\partial},$$

$$H^0(\Sigma) := \ker \bar{\partial}, \qquad\qquad\qquad\qquad H^{1,1}(\Sigma) := \Omega^2 / \operatorname{Im} \bar{\partial},$$

$$H^{0,1}(\Sigma) := \Omega^{0,1} / \operatorname{Im} \bar{\partial}.$$

Clearly, $H^0(\Sigma)$ is the space of holomorphic functions on Σ. In particular, if Σ is compact and connected, then $H^0(\Sigma) \cong \mathbb{C}$.

$H^{1,0}(\Sigma)$ is the space of holomorphic differentials, that is 1-forms ω such that locally $\omega = a\,dz$ and a is a holomorphic function. The geometric meaning of $H^{0,1}(\Sigma)$ and $H^{1,1}(\Sigma)$ is somewhat less straightforward. An interested reader may wish to consult for example [??].

1.6 The Laplacian on Riemann surfaces

A peculiar feature of Riemann surfaces is that the Laplace-Beltrami operator can be defined without a reference to a Riemannian metric. Indeed, set

$$\Delta := 2i\,\bar{\partial}\partial = -2i\,\partial\bar{\partial} \colon \Omega^0(\Sigma; \mathbb{C}) \longrightarrow \Omega^2(\Sigma; \mathbb{C}).$$

If $z = x + iy$ is a local holomorphic coordinate as above, then we have

$$\bar{\partial}f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z} = \frac{1}{2}(\partial_x f + \partial_y f\, i)\, d\bar{z} \qquad \Longrightarrow$$

$$\partial\bar{\partial}f = \frac{1}{2}\frac{\partial}{\partial z}(\partial_x f + \partial_y f\, i)\, dz \wedge d\bar{z}$$

$$= \frac{1}{4}(\partial_x - i\,\partial_y)(\partial_x f + \partial_y f\, i)\, dz \wedge d\bar{z}$$

$$= \frac{1}{4}\left(\partial^2_{xx} f + \partial^2_{yy} f \right) dz \wedge d\bar{z}.$$

Furthermore, since

$$dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\,dx \wedge dy,$$

we obtain

$$\Delta f = -2i\,\partial\bar{\partial}f = -\left(\partial^2_{xx} f + \partial^2_{yy} f \right) dx \wedge dy.$$

Remark 1.46. To relate this operator to the Laplacian in the sense of Def. ??, let g be a Hermitian metric on Σ, that is $g\left(I\cdot, I\cdot\right) = g\left(\cdot, \cdot\right)$.

If $z = x + yi$ is a local holomorphic coordinate, then

$$(g_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad \Longrightarrow \qquad (g^{ij}) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

for some positive function $\lambda = \lambda\left(x, y\right)$.

Substituting this into (1.35), we obtain

$$\Delta f = \frac{1}{\lambda}\left(\partial_x\left(\lambda \cdot \lambda^{-1}\partial_x f\right) + \partial_y\left(\lambda \cdot \lambda^{-1}\partial_y f\right)\right) \quad = \frac{1}{\lambda}\left(\partial_{xx}^2 f + \partial_{yy}^2 f\right).$$

Furthermore, the Riemannian volume form on Σ in terms of local coordinates (x, y) is

$$vol = \lambda\, dx \wedge dy.$$

Hence, $* vol = 1 \implies *\left(dx \wedge dy\right) = \lambda^{-1}$. This yields

$$\Delta f = \frac{1}{\lambda}\left(\partial_{xx}^2 f + \partial_{yy}^2 f\right) = *\left(-2i\, \partial\bar{\partial} f\right).$$

Hence, up to the application of the isomorphism $*\colon \Omega^2\left(\Sigma\right) \to \Omega^0\left(\Sigma\right)$, the Laplacian coincides with $-2i\, \partial\bar{\partial}$ indeed.

In the current setting, Corollary 1.40 yields the following.

Theorem 1.47. *Let Σ be a compact connected Riemann surface. The equation $-2i\, \partial\bar{\partial} f = \eta$, where $\eta \in \Omega^2\left(\Sigma; \mathbb{C}\right)$, has a solution if and only if $\int_\Sigma \eta = 0$.* $\square$

1.7   Some consequences of Theorem 1.47

In this section we assume that Σ is a compact connected Riemann surface throughout.

We have a natural skew-symmetric pairing

$$\Omega^{1,0}\left(\Sigma\right) \times \Omega^{0,1}\left(\Sigma\right) \longrightarrow \mathbb{C}, \qquad (\omega, \eta) \longmapsto \int_\Sigma \omega \wedge \eta.$$

This yields a bilinear map

$$B\colon H^{1,0}\left(\Sigma\right) \times H^{0,1}\left(\Sigma\right) \longrightarrow \mathbb{C}, \qquad B\left(\omega, [\eta]\right) = \int_\Sigma \omega \wedge \eta. \tag{1.48}$$

Lemma 1.49. *B is well defined.*

Proof. Notice first that for any $\alpha, \beta \in \Omega^{1,0}\left(\Sigma\right)$ we have $\alpha \wedge \beta = 0$. Indeed, locally $\alpha = a \cdot dz$ and $\beta = b \cdot dz$ so that $\alpha \wedge \beta = ab\, dz \wedge dz = 0$. Using this observation, we obtain

$$\int_\Sigma \omega \wedge \left(\eta + \bar{\partial} f\right) = \int_\Sigma \omega \wedge \eta + \int_\Sigma \omega \wedge \left(\partial f + \bar{\partial} f\right)$$

$$= \int_\Sigma \omega \wedge \eta + \int_\Sigma \omega \wedge df$$

$$= \int_\Sigma \omega \wedge \eta - \int_\Sigma d\left(f \cdot \omega\right) + \int_\Sigma f d\omega$$

$$= \int_\Sigma \omega \wedge \eta - 0 + 0.$$

In the last equality the second summand vanishes by the Stokes' thm; The last summand vanishes, since ω is closed: Locally $\omega = a(z)\,dz$ so that

$$d\omega = da \wedge dz = \left(\frac{\partial a}{\partial z} dz + \frac{\partial a}{\partial \bar{z}} d\bar{z} \right) \wedge dz = 0 + 0 = 0,$$

since a is a holomorphic. This finishes the proof of this lemma. $\qquad\square$

Notice that we have a natural map

$$i\colon H^{1,0}(\Sigma) \longrightarrow H^1(\Sigma;\mathbb{C}), \qquad \omega \longmapsto [\omega].$$

The class $[\omega]$ is well-defined, since ω is closed as it has been shown in the proof Lemma 1.49.

Furthermore, the conjugation $\Omega^1(\Sigma;\mathbb{C}) \longrightarrow \Omega^1(\Sigma,\mathbb{C}), \eta \longmapsto \bar{\eta}$, induces a map

$$\sigma\colon H^{1,0}(\Sigma) \longrightarrow H^{0,1}(\Sigma),$$

which is antilinear, that is $\sigma(i\eta) = -i\sigma(\eta)$.

Theorem 1.50.

(i) σ is an isomorphism;

(ii) For a complex vector space V denote $V^* = \{\psi\colon V \to \mathbb{C} \mid \psi \text{ is antilinear}\}$. Then $B\colon H^{1,0}(\Sigma) \to H^{0,1}(\Sigma)^*$ is an isomorphism;

(iii) The map

$$H^{1,0}(\Sigma) \oplus H^{0,1}(\Sigma) \longrightarrow H^1(\Sigma;\mathbb{C}), \qquad (\omega, [\eta]) \longmapsto i(\omega) + \overline{i(\sigma^{-1}([\eta]))} \qquad (1.51)$$

is an isomorphism.

Proof. Notice that $\sigma\colon H^{1,0}(\Sigma) \longrightarrow H^{0,1}(\Sigma)$ is well-defined. We want to show that for each class $[\eta] \in H^{0,1}(\Sigma)$ there exists a unique $\omega \in H^{1,0}(\Sigma)$ such that $\sigma(\omega) = [\eta]$. Indeed, any representative of $[\eta] \in H^{0,1}(\Sigma)$ can be written as

$$\eta' = \eta + \bar{\partial}f$$

for some $f \in C^\infty(\Sigma;\mathbb{C})$. We want to find a representative η' such that

$$\partial\eta' = 0 \qquad \Longleftrightarrow \qquad \partial\bar{\partial}f = -\partial\eta. \qquad (1.52)$$

The equation on the right hand side of (1.52) has a solution, since

$$\int_\Sigma \partial\eta = \int_\Sigma d\eta = 0.$$

In fact, f is defined uniquely by (1.52) up to the addition of a constant. Hence, for each class $[\eta] \in H^{0,1}(\Sigma)$ there exists a unique representative η' such that $\partial\eta' = 0$.

Define $\omega := \bar{\eta}'$. Then

$$\bar{\partial}\omega = \bar{\partial}\bar{\eta}' = \overline{\partial\eta'} = 0,$$

that is ω lies in $H^{1,0}(\Sigma)$ and is a unique preimage of $[\eta]$.

To prove *(ii)*, we only need to show that (1.48) is non-degenerate, that is

$$\forall\, \omega \in H^{1,0}(\Sigma) \qquad \exists\, [\eta] \in H^{0,1}(\Sigma) \qquad \text{such that } B(\omega, [\eta]) \neq 0.$$

Indeed, for $\omega \in H^{1,0}(\Sigma)$ set $\eta := \sigma(\omega) = [\bar{\omega}]$. Then

$$B(\omega, [\bar{\omega}]) = \int_\Sigma \omega \wedge \bar{\omega} \neq 0,$$

since $a\, dz \wedge \bar{a}\, d\bar{z} = |a|^2 dz \wedge d\bar{z}$.

Furthermore, the injectivity of (1.51) follows from *(ii)*. Indeed, notice first that $i(\omega)$ and $\overline{i(\sigma^{-1}([\eta]))}$ are always orthogonal with respect to B. Hence, $(\omega, [\eta])$ belongs to the kernel of (1.51) if and only if

$$i(\omega) = 0 \qquad \text{and} \qquad \overline{i(\sigma^{-1}([\eta]))} = 0. \qquad (1.53)$$

However i is injective, since

$$B(i(\omega), \sigma(i(\omega))) = \int_\Sigma \omega \wedge \bar{\omega} \neq 0$$

provided $\omega \neq 0$. Hence, (1.53) yields $\omega = 0$ and $[\eta] = 0$, since all maps σ^{-1}, i, and the conjugation are injective.

To see that (1.51) is surjective, pick any $[\zeta] \in H^1_{dR}(\Sigma; \mathbb{C})$. I claim that there exists a representative $\zeta' = \zeta + df$ such that

$$\bar{\partial}\zeta' = 0 \qquad \Longleftrightarrow \qquad 0 = \bar{\partial}\zeta + \bar{\partial}df = \bar{\partial}\zeta - \partial\bar{\partial}f. \qquad (1.54)$$

The existence of a function f satisfying (1.54) follows from Theorem 1.47 just like in the proof of *(i)*.

Furthermore, write

$$\zeta' = \omega + \eta, \qquad \text{where } \omega = \frac{1}{2}(\zeta' - i\zeta'(I\cdot)) \quad \text{and} \quad \eta = \frac{1}{2}(\zeta' + i\zeta'(I\cdot)).$$

Locally, we have

$$\zeta' = a\, dz + b\, d\bar{z} \qquad \Longrightarrow \qquad \omega = a\, dz \quad \text{and} \quad \eta = b\, d\bar{z}.$$

Since $\bar{\partial}\zeta' = 0$, we obtain $\frac{\partial a}{\partial \bar{z}} = 0$, that is ω is a holomorphic differential.

Furthermore,

$$\bar{\partial}\zeta' = 0 \quad \text{and} \quad d\zeta' = 0 \implies \partial\zeta' = 0 \implies \frac{\partial b}{\partial z} = 0 \implies \partial\eta = 0$$
$$\implies \bar{\partial}\bar{\eta} = 0.$$

Hence, $\bar{\eta}$ is also a holomorphic differential. Combining this with *(i)*, we obtain *(iii)*. $\qquad \square$

Corollary 1.55. *If $b_1(\Sigma) = 0$, then Σ admits a meromorphic function with a single simple pole.*

Proof. Pick any point $p \in \Sigma$ and a local holomorphic coordinate z centered at p. Let χ be a bump function at p, that is $\chi \equiv 1$ on a neighbourhood V of p and χ vanishes identically outside of a slightly larger neighbourhood $W \supset V$.

We wish to show that there exists a meromorphic function f on Σ with a unique simple pole at p. Assume for a moment, however, that such f does exist and consider

$$h := f - \chi\frac{a}{z},$$

where a is the residue of f at p. Then h is smooth everywhere, albeit may fail to be holomorphic on $W \backslash V$. In any case, we have

$$\bar{\partial} h = \bar{\partial} f - \frac{a}{z} \bar{\partial} \chi = -\frac{a}{z} \bar{\partial} \chi. \tag{1.56}$$

Notice that $\eta := -\frac{a}{z} \bar{\partial} \chi$ is a smooth $(0,1)$-form supported in $W \backslash V$.

Conversely, if h is a smooth solution of (1.56), we can define f by

$$f := h + \chi \frac{a}{z},$$

which is meromorphic and has a unique simple pole at p.

Thus, our task reduced to showing that (1.56) has a solution. By the definition of $H^{0,1}(\Sigma)$, this is the case if and only if the class of η in $H^{0,1}(\Sigma)$ vanishes. However, by Theorem 1.50 *(ii)*, we have

$$\dim_R H^{1,0}(\Sigma) + \dim_R H^{0,1}(\Sigma) = 2b_1(\Sigma) = 0 \qquad \implies \qquad H^{0,1}(\Sigma) = \{0\}$$

so that $[\eta]$ vanishes trivially. This finishes the proof. $\qquad\square$

Theorem 1.57. *Let Σ be a closed Riemann surface. Then Σ is homeomorphic to $\mathbb{C}P^1 \cong S^2$ if and only if Σ is biholomorphic to $\mathbb{C}P^1$.*

Idea of the proof. By the classification theorem for closed surfaces, Σ is homeomorphic to S^2 if and only if $b_1(\Sigma) = 0$. By Corollary 1.55, there exists a meromorphic function f on Σ with a unique simple pole. We can view f as a holomorphic map $f : \Sigma \longrightarrow \mathbb{C}P^1$. A simple topological argument yields that f must be in fact bijective (this uses crucially that the pole of f is unique and simple). The reader can find the details of this topological argument in [Don, Sec 4.1]. $\quad\square$

Another reformulation of the above theorem is that the sphere S^2 admits a unique structure of a complex manifold. This is in contrast with the torus $\mathbb{T}^2$ (or, in fact, any closed orientable surface with $b_1 \geq 2$), which admits continuous families of inequivalent complex structures.

Developing these ideas somewhat further one can obtain also a classification of all elliptic curves, that is Riemann surfaces homeomorphic to the torus. Or, one can show that any closed Riemann surface can be embedded into some projective space. However, this goes somewhat beyond the purposes of this course.

Chapter 2

Vector bundles, Sobolev spaces, and elliptic partial differential operators

2.1 Vector bundles

Basic definitions. Roughly speaking, a vector bundle is just a family of vector spaces parametrized by points of a manifold (or, more generally, of a topological space).

More formally, the notion of a vector bundle is defined as follows.

Definition 2.1. Choose a non-negative integer k. A real smooth vector bundle of rank k is a triple (π, E, M) such that the following holds:

(i) E and M are smooth manifolds, $\pi \colon E \to M$ is a smooth submersion (the differential is surjective at each point);

(ii) For each $m \in M$ *the fiber* $E_m := \pi^{-1}(m)$ has the structure of a vector space and $E_m \cong \mathbb{R}^k$;

(iii) For each $m \in M$ there is a neighborhood $U \ni m$ and a smooth map ψ_U such that the following diagram

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\ \psi_U\ } & U \times \mathbb{R}^k \\
& {\scriptstyle \pi}\searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\
& U &
\end{array}
$$

commutes. Moreover, ψ_U is a fiberwise linear isomorphism.

The following terminology is commonly used: E is the total space, M is the base, π is the projection, and ψ_U is the local trivialization (over U).

It is worth pointing out that one can equally well talk about complex and quaternionic vector bundles. This requires only cosmetic changes, which are left to the reader. The preference for real vector bundles in this section is given for the sake of definitness mainly. I shall feel free to use complex vector bundles below without further explanations.

Example 2.2.

(a) The product bundle: $M \times \mathbb{R}^k$;

(b) The tangent bundle TM of any smooth manifold M.

Let E and F be two vector bundles over a common base M. A *homomorphism* between E and F is a smooth map $\varphi \colon E \to F$ such that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\;\;\varphi\;\;} & F \\
& \searrow{\scriptstyle \pi_E} \quad \swarrow{\scriptstyle \pi_F} & \\
& M &
\end{array}
$$

commutes and φ is a fiberwise linear map.

Two bundles E and F are said to be isomorphic, if there is a homomorphism φ, which is fiberwise an isomorphism.

A bundle E is said to be *trivial*, if E is isomorphic to the product bundle.

Operations on vector bundles. Let E and F be two vector bundles over a common base M. Then we can construct new bundles E^*, $\Lambda^p E$, $E \oplus F$, $E \otimes F$, and $\mathrm{Hom}(E, F)$ as follows:

(∗) $(E^*)_m = (E_m)^*$;

(Λ) $(\Lambda^p E)_m = \Lambda^p(E_m)$;

(⊕) $(E \oplus F)_m := E_m \oplus F_m$;

(⊗) $(E \otimes F)_m := E_m \otimes F_m$;

(Hom) $\mathrm{Hom}(E, F)_m := \mathrm{Hom}(E_m, F_m)$.

If $f \colon M' \to M$ is a smooth map, we can define the *pull-back* of $E \to M$ via

$$
(f^* E)_{m'} := E_{f(m')}.
$$

For example, if M' is an open subset of M and ι is the inclusion, then $E|_{M'} := \iota^* E$ is just the restriction of E to M'.

The reader should check that the families of vector spaces defined above satisfy the properties required by Definition 2.1.

Exercise 2.3. Prove that $E^* \otimes F$ is isomorphic to $\mathrm{Hom}(E, F)$.

Exercise 2.4. Prove that the tangent bundle of the 2-sphere is non-trivial. (Hint: Apply the hairy ball theorem).

Sections. Speaking informally, a section is an assignment of a vector in $s(m) \in E_m$ to each point $m \in M$ such that $s(m)$ depends smoothly on m. More formally, we have the following.

Definition 2.5. A smooth map $s \colon M \to E$ is called a section, if $\pi \circ s = \mathrm{id}_M$.

Sections of the tangent bundle TM are called vector fields. Sections of $\Lambda^p T^* M$ are called differential p-forms.

Exercise 2.6. Let $E \to M$ be a vector bundle of rank k and $U \subset M$ be an open subset. Prove that E is trivial over U if and only if there are k sections $e = (e_1, \ldots, e_k)$, $e_j \in \Gamma(U; E)$, such that $e(m)$ is a basis of E_m for each $m \in U$. More precisely, given e show that ψ_U can be constructed according to the formula

$$
\psi_U^{-1} \colon U \times \mathbb{R}^k \longrightarrow E|_U, \qquad (m, x) \mapsto e(m) \cdot x.
$$

In fact this establishes a one-to-one correspondence between k-tuples of pointwise linearly independent sections and local trivializations of E.

We denote by $\Gamma(E) = \Gamma(M;\, E)$ the space of all smooth sections of E. Clearly, $\Gamma(E)$ is a vector space, where the addition and multiplication with a scalar are defined pointwise. In fact, $\Gamma(E)$ is a $C^\infty(M)$-module.

Given a local trivialization e over U (cf. Exercise 2.6) and a section s, we can write

$$s(m) = \sum_{j=1}^{k} \sigma_j(m) e_j(m)$$

for some functions $\sigma_j \colon U \to \mathbb{R}$. Thus, locally any section of a vector bundle can be thought of as a map $\sigma \colon U \to \mathbb{R}^k$.

It is important to notice that σ depends on the choice of a local trivialization. Indeed, if e' is another local trivialization of E over U', then there is a map

$$g \colon U \cap U' \longrightarrow \mathrm{GL}_n(\mathbb{R}) \quad \text{such that} \quad e = e' \cdot g. \tag{2.7}$$

If $\sigma' \colon U' \to \mathbb{R}^k$ is a local representation of s with respect to e', we have

$$s = e'\sigma' = eg^{-1}\sigma' = e\sigma \qquad \Longrightarrow \qquad \sigma' = g\sigma.$$

Covariant derivatives. The reader surely knows from the basic analysis course that the notion of the derivative is very useful. It is natural to ask whether there is a way to differentiate sections of bundles too.

To answer this question, recall the definition of the derivative of a function $f \colon M \to \mathbb{R}$. Namely, choose a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ and denote $m := \gamma(0)$, $\mathrm{v} := \dot\gamma(0) \in T_m M$. Then

$$df(\mathrm{v}) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(m)}{t}. \tag{2.8}$$

Trying to replace f by a section s of a vector bundle, we immediately run into a problem, namely the difference $s(\gamma(t)) - s(m)$ is ill-defined in general since these two vectors may lie in different vector spaces.

Hence, instead of trying to mimic (2.8) we will define the derivatives of sections axiomatically, namely asking that the most basic property of the derivative—the Leibnitz rule—holds.

Definition 2.9. Let $E \to M$ be a vector bundle. *A covariant derivative is an $\mathbb{R}$-linear map $\nabla \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that*

$$\nabla(fs) = df \otimes s + f\nabla s \tag{2.10}$$

holds for all $f \in C^\infty(M)$ and all $s \in \Gamma(E)$.

Example 2.11. Let $M \subset \mathbb{R}^N$ be an embedded submanifold. Then the tangent bundle TM is naturally a subbundle of the product bundle $\underline{\mathbb{R}}^N := M \times \mathbb{R}^N$. In particular, any section s of TM can be regarded as a map $M \to \mathbb{R}^N$. With this at hand we can define a connection on TM as follows

$$\nabla s := \mathrm{pr}(ds),$$

where pr is the orthogonal projection onto TM. A straightforward computation shows that this satisfies the Leibniz rule, i.e., ∇ is a connection indeed.

Theorem 2.12. *For any vector bundle $E \to M$ the space of all connections $\mathcal{A}(E)$ is an affine space modelled on $\Omega^1(\mathrm{End}\, E) = \Gamma\big(T^*M \otimes \mathrm{End}(E)\big)$.*

To be somewhat more concrete, the above theorem consists of the following statements:

(a) $\mathcal{A}(E)$ is non-empty.
(b) For any two connections ∇ and $\hat{\nabla}$ the difference $\nabla - \hat{\nabla}$ is a 1-form with values in $\mathrm{End}(E)$;
(c) For any $\nabla \in \mathcal{A}(E)$ and any $a \in \Omega^1(\mathrm{End}\, E)$ the following

$$(\nabla + a)s := \nabla s + as$$

is a connection.

For the proof of Theorem 2.12 we need the following elementary lemma, whose proof is left as an exercise.

Lemma 2.13. *Let* $A \colon \Gamma(E) \to \Omega^p(F)$ *be an* $\mathbb{R}$*-linear map, which is also* $C^\infty(M)$*-linear, i.e.,*

$$A(fs) = f A(s) \qquad \forall f \in C^\infty(M) \quad and \quad \forall s \in \Gamma(E).$$

Then there exists $a \in \Omega^p\big(\mathrm{Hom}(E, F)\big)$ *such that* $A(s) = a \cdot s$. $\qquad\square$

Proof of Theorem 2.12. Notice first that $\mathcal{A}(E)$ is convex, i.e., for any $\nabla, \hat{\nabla} \in \mathcal{A}(E)$ and any $t \in [0, 1]$ the following $t\nabla + (1 - t)\hat{\nabla}$ is also a connection.

If ψ_U is a local trivialization of E over U, then we can define a connection ∇_U on $E|_U$ by declaring

$$\nabla_U s := \psi_U^{-1}\, d(\psi_U(s)).$$

Using a partition of unity and the convexity property, a collection of these local covariant derivatives can be sewed into a global covariant derivative just like in the proof of the existence of Riemannian metrics on manifolds, cf. [?BardenThomas_IntroDiffMflds, Thm. 3.3.7]. This proves (a).

By (2.10), the difference $\nabla - \hat{\nabla}$ is $C^\infty(M)$–linear. Hence, (b) follows by Lemma 2.13.

The remaining step, namely (c), is straightforward. This finishes the proof of this theorem. $\qquad\square$

While Theorem 2.12 answers the question of the existence of connections, the reader may wish to have a more direct way to put his hands on a connection. One way to do this is as follows.

Let e be a local trivialization. Since e is a pointwise base we can write

$$\nabla e = e \cdot A, \tag{2.14}$$

where $A = A(\nabla, e)$ is a $k \times k$-matrix, whose entries are 1-forms defined on U. A is called the connection matrix of ∇ with respect to e.

If σ is a local representation of a section s, then

$$\nabla s = \nabla(e\sigma) = \nabla(e)\sigma + e \otimes d\sigma = e\big(A\sigma + d\sigma\big).$$

Hence, it is common to say that locally

$$\nabla = d + A,$$

which means that $d\sigma + A\sigma$ is a local representation of ∇s. In particular, ∇ is uniquely determined by its connection matrix over U and any $A \in \Omega^1(U; \mathfrak{gl}_k(\mathbb{R}))$ appears as a connection matrix of some connection (cf. Theorem 2.12).

2.2 Sobolev spaces

2.2.1 Sobolev spaces on $\mathbb{R}^n$

Recall that the space of square-integrable functions is defined by

$$L^2\left(\mathbb{R}^n\right) := \left\{ u \colon \mathbb{R}^n \to \mathbb{C} \mid u \text{ is measurable } \& \int_{\mathbb{R}^n} |u\left(x\right)|^2 dx < \infty \right\}.$$

This is a (complex) Hilbert space with respect to the Hermitian scalar product

$$\langle u, v \rangle_{L^2} := \int_{\mathbb{R}^n} u\left(x\right) \bar{v}\left(x\right) dx.$$

Remark 2.15. Strictly speaking, we should also identify those functions, which differ only on a subset of measure zero so that $L^2\left(\mathbb{R}^n\right)$ consists of classes of functions. However, this will not be an issue for us and we shall treat square-integrable functions as honest functions.

The Fourier transform

For $f \colon \mathbb{R}^n \to \mathbb{C}$ decaying sufficiently fast at ∞ the Fourier transform $\widehat{f} \colon \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\widehat{f}\left(\xi\right) := \frac{1}{\left(2\pi\right)^n} \int_{\mathbb{R}^n} f\left(x\right) e^{-i\langle x, \xi \rangle} dx.$$

The Plancherel theorem states that

$$\| \widehat{f} \|_{L^2}^2 = \frac{1}{\left(2\pi\right)^n} \| f \|_{L^2}^2, \tag{2.16}$$

that is (an extension of) the map $f \mapsto \widehat{f}$ is essentially an isometry of $L^2\left(\mathbb{R}^n\right)$.

Let $\alpha = \left(\alpha_1, \ldots, \alpha_n\right) \in \mathbb{Z}_{\geq 0}^n$ be a multi-index. Denote

$$D^\alpha f := \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The basic property of the Fourier transform is

$$\widehat{D^\alpha f}\left(\xi\right) = \xi^\alpha \widehat{f}\left(\xi\right), \tag{2.17}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Another basic property of the Fourier transform is the following.

Lemma 2.18 (Riemann-Lebesgue). *If $u \in L^1(\mathbb{R})$, then $\widehat{u} \in C^0(\mathbb{R}^n)$ and $\widehat{u}$ decays at ∞.* $\quad\square$

Denote by $f^\vee\left(\xi\right) = \left(2\pi\right)^n \widehat{f}\left(-\xi\right) = \int f\left(x\right) e^{i\langle x, \xi \rangle} dx$.

Theorem 2.19 (The Fourier inversion theorem). *If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R}^n)$, then f agrees almost everywhere with a continuous function f_0 and $\left(\widehat{f}\right)^\vee = f_0 = \left(\widehat{f^\vee}\right)$.* $\quad\square$