the Art of Problem Solving

Calculus

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Preface

Introduction

Calculus is a branch of mathematics that, broadly speaking, introduces concepts and tools to describe and analyze functions. Although some parts of calculus were known to the ancient Greeks, Egyptians, and Chinese, the modern version of calculus that we use today was largely developed in the 17th century, independently by the great mathematicians Isaac Newton and Gottfried Leibniz. Calculus is not only an important branch of mathematics in its own right, but also provides the rigorous mathematical foundation of physics, engineering, and many other branches of science.

Unfortunately, most students first learn calculus as a bag of tricks: a number of seemingly unrelated algorithms to be memorized and then endlessly applied to problem after problem without motivation. Calculus is sometimes seen as the pinnacle of high-school mathematics, and passing a college placement exam in calculus is the ultimate goal. Students who learn calculus this way–and this describes most high-school and college students who take calculus–will likely never appreciate the beauty or richness of the subject.

Our goal for this book is to present calculus with a substantial theoretical underpinning. Calculus, at its heart, is a few fundamental ideas that come together to create a rich subject, and is not a collection of definitions, formulas, and algorithms. Our hope is that students who complete this book will *understand* calculus, both as a theoretical subject and as a problem-solving tool.

Who should study calculus using this book

The target audience for this book is motivated high school students who have mastered the high school curriculum and have developed the mathematical maturity necessary to handle the level of mathematical rigor in this text. At a minimum, students must have mastered algebra, plane geometry, and trigonometry before considering continuing on to calculus. This is true for any course of calculus study, and especially important for the more rigorous calculus treatment in this book. We strongly recommend a robust precalculus curriculum, such as that in [**Ru**], before proceeding with this text. (Note: bold letters in brackets such as [**Ru**] refer to the book's **list of references.**)

Students–even highly skilled students–who rush into calculus too soon are likely to be frustrated and will not be able to appreciate the richness and subtleties of the subject. Such a student will only learn calculus as a set of algorithms to be memorized, and even though this may be sufficient to progress on to the next subject, it will rob the student of a key step in his or her mathematical development.

We strongly recommend that students, especially younger students, be exposed to both nontrivial problem solving and to discrete mathematics (such as combinatorics and number theory) *before* "continuing on" to calculus. Art of Problem Solving has other textbooks and online courses in both of these areas.

This book in particular is designed for students who want a deeper understanding of calculus than a mainstream high-school or college calculus text provides, and who also want exposure to a variety of non-routine calculus problems. Specifically, this book differs from a "mainstream" calculus book in two major ways:

- A more rigorous presentation, including proofs where applicable. For example, this book begins with set theory and the construction of the real numbers, including a discussion of suprema, infima, and completeness. We cover the rigorous δ-ϵ definition of limit, which is omitted from many calculus texts. We also prove many of the important results of calculus, including the Mean Value Theorem and the Fundamental Theorem of Calculus; these results are often merely asserted without proof in standard calculus treatments.
- An assortment of nontrivial problems. This book has fewer routine "drill-and-kill" exercises than most calculus texts, and instead has a wider array of problems that require the students to go beyond rote memorization of algorithms and instead to think more deeply about how the different aspects of calculus are interrelated. We have taken many nontrivial problems from two of the premier math contests that include calculus: the high-school level **Harvard-MIT Mathematics Tournament** (HMMT) and the college-level **William Lowell Putnam Mathematical Competition**. The HMMT is an annual math tournament for high school students, held at MIT and at Harvard in alternating years. It is run exclusively by MIT and Harvard students, most of whom themselves participated in math contests in high school. More information is available at `http://web.mit.edu/hmmt`. The Putnam Competition is a long-running North American undergraduate math competition held every December–2014 was the 75th annual contest–which consists of *extremely* difficult problems across the entire undergraduate mathematics curriculum. More information is at `http://math.scu.edu/putnam`.

Students who are preparing for college calculus placement examinations may wish to supplement their study with a test-preparation workbook. The key to success on such placement examinations is repetition of routine calculations, which this textbook largely eschews in favor of a variety of more difficult, non-routine problems.

Structure Of the Book

Chapter 1 is a review of important foundational material: sets, real numbers, functions, graphs, trigonometry, exponentials, and logarithms. We urge students not to skip this chapter, even if they think that they already know this material. We present the material at a level of detail and rigor that students may not be used to, and will be introducing terminology and notation that is not typically covered in a precalculus class.

Chapter 2 introduces the first core idea of calculus: the limit. We present the full δ-ϵ treatment of limits, which is often not covered in a high-school calculus course, but we feel that exposure to the rigorous definition of limit is an important part of students' mathematical development. In particular, stating key definitions rigorously is necessary to prove later results, and our goal will be to prove (and not merely assert) as much as possible.

Chapters 3–5 are the heart of the subject: differentiation (Chapters 3 and 4) and integration (Chapter 5). These are the core concepts of calculus.

Chapters 6 and 7 deal with the concept of "infinity," of which most students have a vague understanding but which we will attempt to make precise. Chapter 7 also includes the very important topic of Taylor series, which is a fundamental topic of analysis.

Chapters 8 and 9 are essentially independent, and each serve as an introduction to the broader world of analysis beyond calculus. Chapter 8 covers plane curves, including curves in polar coordinates, and is a nice introduction to topics that will be covered more thoroughly in multivariable calculus (typically the next calculus course after a student has mastered the topics in this book). Finally, Chapter 9 is an introduction to the theory of differential equations, a very broad subject which we only touch on in this book, but which is the subject of (several) courses of study beyond calculus.

Many chapters will have one or more advanced sections after the end-of-chapter problems. These sections are denoted with a letter (such as 1.A). These sections are optional and often cover topics at a more theoretical level than in the main text. Eager students who work through these sections should find them rewarding, but it is acceptable to skip them.

Navigating This Book

From any page in the book, you can click on the image of the book's cover in the top-left corner to view the table of contents. On large-screen devices, you can also click anywhere on the left-side navigation bar to jump to the corresponding section of the book. You can use the left and right arrows at the top of the page to move to the previous or next chapter or section of the book.

Hovering over a paragraph will create a small 🔗 icon in the left margin. Clicking on that icon will pop up a window with a permanent link to that paragraph. You can cut-and-paste this link into an email message or community post.

Interactive Features

There are several interactive features built into the book. All online books are linked to the AoPS community, so that students using the online book can discuss the book with other students. Click on the 💬 Community icon next to a section or problem to view all of the discussions about that section or problem. Click the ✎ New Topic icon to start a new discussion topic about the section or problem.

Explanation of Icons

Throughout the book, you will see various shaded boxes and icons.

Concept:	This will be a general problem-solving technique or strategy. These are the "keys" to becoming a better problem solver!
Important:	This will be something important that you should learn. It might be a formula, a solution technique, or a caution.
WARNING!!	Beware if you see this box! This will point out a common mistake or pitfall.
Sidenote:	This box will contain material which, although interesting, is not part of the main material of the text. It's OK to skip over these boxes, but if you read them, you might learn something interesting!
Bogus Solution:	Just like the impossible cube shown to the left, there's something wrong with any "solution" that appears in this box.

Exercises, Review Problems, and Challenge Problems

Most sections end with several **Exercises**. These will test your understanding of the material that was covered in the section that you just finished. You should try to solve *all* of the exercises. Exercises marked with a ★ are more difficult.

Most chapters have a section containing **Review Problems**. These are problems which test your understanding of the material covered in the chapter. Your goal should be to solve most or all of the Review Problems for every chapter — if you're unable to do this, it means that you haven't yet mastered the material, and you should probably go back and read the chapter again.

All of the chapters end with a section containing **Challenge Problems**. These problems are generally more difficult than the other problems in the book, and will really test your mastery of the material. Some of them are very, very hard — the hardest ones are marked with a ★. Don't necessarily expect to be able to solve all of the Challenge Problems on your first try — these are difficult problems even for experienced problem solvers. If you are able to solve a large number of Challenge Problems, then congratulations, you are on your way to becoming an expert problem solver!

You can type your solution or notes for any Exercise, Review Problem, or Challenge Problem directly into the book. Your work will automatically be saved. You won't be able to view the solution to a problem until you type something in the solution box.

Many problems come with one or more **hints**. You can view any available hint by clicking on the *Hint* link after the problem statement. You can then hide the hint by again clicking on the *Hint* link. It is very important that you first try to solve the problem without peeking at the hints. Only after you've seriously thought about a problem and are stuck should you look at a hint. Also, for problems which have multiple hints, use the hints one at a time; don't go to the second hint until you've thought about the first one.

The **solutions** to all of the Exercises, Review Problems, and Challenge Problems are built into the book. Clicking the *Show Solution* button will display the solution, but you won't be able to view the solution until you've made an attempt to solve the problem and typed Once you've viewed the solution, you can add notes to the solution in a separate box. You can also click the *Reset* button to clear your solution and notes and start fresh.

Here are some very important things to keep in mind about the solutions:

1. Make sure that you make a serious attempt at the problem before looking at the solution. You should think *hard* about a problem before deciding to give up and look at the solution. Remember, once you view a solution, you can't change what you typed for your solution.
2. After you solve a problem, it's usually a good idea to read the solution, even if you think you know how to solve the problem. Our solution might show you a quicker or more concise way to solve the problem, or it might have a completely different solution method that you might not have thought of.
3. If you have to look at the solution in order to solve a problem, make sure that you make a note of that problem. You can then come back to the problem in a week or two to make sure that you are able to solve it on your own without resorting to the solution.

Resources

Here are some other good resources for you to further pursue your study of mathematics:

- Art of Problem Solving has a complete library of books (both print and online) specifically designed for avid math students:
  - The *Introduction* series: *Prealgebra*, *Introduction to Algebra*, *Introduction to Counting & Probability*, *Introduction to Number Theory*, and *Introduction to Geometry*, designed for students in grades 6-10.
  - The *Intermediate* series: *Intermediate Algebra*, *Intermediate Counting & Probability*, and *Precalculus* designed for students in grades 9-12.
  - The *Problem Solving* series: designed for students preparing for math competitions. In addition to our classics *the Art of Problem Solving, Volume 1: the Basics* (for students in grades 7-10 preparing for MATHCOUNTS and the AMC 8/10/12 contests) and *the Art of Problem Solving, Volume 2: and Beyond* (for students in grades 9-12 preparing for advanced contests such as the AIME), we also have *Competition Math for Middle School*.
  - *Beast Academy*: a full, rigorous, entertaining curriculum for aspiring math beasts in grades 2-5.
- The Art of Problem Solving website contains many other resources for students.
  - The AoPS Community has tens of thousands of members (if you are reading this online book, you're a member too!) and millions of posts on a variety of math, problem solving, and other fun topics.
  - Alcumus, our free adaptive online learning system containing over 13,000 practice problems.
  - For the Win!, our free interactive online game inspired by the MATHCOUNTS Countdown Round.
  - Our vast video library contains hundreds of videos featuring AoPS founder Richard Rusczyk.
  - Learn LaTeX, the mathematical typesetting system used by most professional mathematicians and scientists, from our widely-used LaTeX guide, and practice your LaTeX skills with the TeXeR.
  - AoPS Community members collaborate to build the AoPSWiki.
  - We have a collection of articles on a variety of problem-solving topics.
- You can hone your problem solving skills (and perhaps win prizes!) by participating in various math contests. Please see the **Acknowledgements** section of this book for more information.

Technology

Most students who study calculus will do so with the aid of a graphing **calculator**, and we encourage students using this book to do so as well. Once students have mastered the basics, a graphing calculator can remove some of the tedium from long calculations, and can also serve as a valuable check of students' work. Additionally, much of calculus is visual in nature, and being able to sketch, quickly and accurately, a graph of a function with a few keystrokes is very beneficial. However, students should be aware of the following cautions:

1. There is a famous saying: "garbage in, garbage out." That is, a graphing calculator is only as good as its user–if you enter bogus data into it, you will get bogus results. You also need to know how to properly use your calculator, and to make sure that it is in the correct mode (for example, while doing calculus, your calculator should be in "radians" mode and not in "degrees" mode).
2. Make sure your calculator is sufficiently sophisticated. A "scientific calculator" may not have enough features to be broadly useful for calculus. Ideally, your calculator should be able to (a) graph functions with an arbitrary viewing window, (b) solve equations numerically, and (c) numerically compute derivatives and definite integrals. (Don't worry if you don't know what all these things mean yet–that's what this book is for!) Top-of-the-line calculators are also able to do (b) and (c) symbolically (that is, in terms of variables) as well as numerically.
3. If you are planning to take a standardized calculus examination, then make sure your calculator doesn't do too much. While most calculus examinations permit (or even require) the use of calculators, a "calculator" that is actually a handheld computer or PDA will likely not be permitted. Check with the organization administering your placement test to see if they have a list of approved calculators.

We also recommend the use of symbolic computation websites. One of the best is Wolfram|Alpha (available at `http://wolframalpha.com`), which makes many of the features of the computational software *Mathematica* available on the web. (In fact, the author of this book used Wolfram|Alpha to check many of the calculations.)

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CHAPTER 1

Sets and Functions

Before we can dive into calculus, we need to make sure that we have a rigorous understanding of some basic mathematical concepts, such as sets, numbers and functions. Some of these concepts will seem obvious, and one can certainly learn the mechanics of calculus without worrying about rigorous details, but for advanced calculus, it's important to pay attention to details.

We will also review some special functions that play a crucial role in calculus: the **trigonometric** functions, and the **exponential** and **logarithmic** functions. Trigonometric functions are fundamental because they are used to model all sorts of phenomena in the real world and in calculus, so it is important to have a thorough understanding of them. When you first learned about these functions (in a precalculus or intermediate algebra course, for example), some of their properties may have seemed like magic. In this chapter we'll try to add a bit more rigor to these functions.

1.1 Sets

Sets are the building blocks of mathematics. Like many other fundamental mathematical concepts (such as "point" or "number"), sets are difficult to define precisely, and we're not going to try to be terribly precise here.

Roughly speaking, a **set** is a collection of objects. The objects can be anything: numbers, functions, other sets, any combination of these, or nothing at all. The order of the objects in the set is unimportant. All that matters is what objects are in the set. There might only be a finite number of objects in the set (meaning that we can actually count them if we feel like, in principle), or there could be infinitely many. Finally, we can call an **infinite set**. The objects in the set are called the **elements** or **members** of the set.

There are two ways to specify what the set is: we can list its elements, or we can specify a property that all elements have in common.

$$A = \{1, 2, 3\}.$$

This is a set with three elements, namely 1, 2, and 3. This is another important way to define or describe a set: We list the elements inside of curly braces, and separate the different elements by commas. As we said above, the order of the elements doesn't matter, so $\{1, 2, 3\} = \{2, 1, 3\} = \{3, 2, 1\}$. Also, $\{1, 1, 2, 3\} = \{1, 2, 3\}$ because it doesn't matter if we list an element more than once. For example, if we let $A = \{1, 2, 3\}$ and $B = \{1, 1, 2, 3\}$ and plot both of these sets, we find that the number 1 is in both sets, and the number 2 is in both sets, and the number 3 is in both sets.

Sometimes it's impossible to list a set out, so we use ellipses to indicate the rest of the elements in the set is clear. For example, we feel pretty comfortable with the set $\{1, 2, 3, \ldots, 99, 100\}$ representing the set of all positive whole numbers between 1 and 100.

If a set is infinite, then we obviously have no hope of being able to list all the elements, since such a list would go on forever! But if it is clear which elements are in the set, then we can list the first few elements, and then use ellipses. For example, the set of all positive integers can be written as $\{1, 2, 3, \ldots\}$, because the pattern is clear. As another example, we can list pretty sure that $\{1, 2, 4, 8, 16, 32, \ldots\}$ without any further explanation, because we are pretty sure that we can figure out what the next few elements are. The crucial thing to do with this if pattern is absolutely clear. Listing a set as $\{1, 2, 3, \ldots\}$ is pretty ambiguous: it is the set of all nonnegative powers of 1, or the set of all positive integers not divisible by 2, or the set of all positive integers not divisible by 3, or... there are many other possible patterns. For most simple examples, we make the pattern clear, or some words describing the set.

Aside from listing the elements, the other basic way to describe a set is to provide a property that precisely defines the elements of the set. For example:

$$B = \{x \mid x \text{ is an integer}\}.$$

In this example, the set B consists of all integers. The people who are a little (or a lot) interested in the vertical bar in either case, you should read the symbol as "such that". For example, we would read our set B above as "the set of all x such that x is an integer." Another common example of a set defined in this way for example,

$$\{x \mid x \text{ is a real number and } 2 < x \leq 5\}$$

is the interval of all real numbers that are greater than 2 but less than or equal to 5. One of the major strengths of this way of describing a set is that we can use this even if we don't know explicitly what the elements are. For example,

$$\{(x, y) \mid x \text{ is a real number and } x^2 + y^2 = 9\}$$

is perfectly valid, even if we don't necessarily know at first glance exactly what values of x and y are in the set, or even if there are any elements in the set.

If an object x is an element of X, we write this as $x \in X$. If x is not an element of X, we write $x \notin X$. For any object x and any set X, either $x \in X$ or $x \notin X$, but never both. We can also talk about the whole point of sets: a set is a collection of the objects that belong to it, and everything else does not belong.

There is a very special set called the **empty set**, denoted by $\emptyset$. This is the set with no elements at all. For example,

$$\{x \mid x \text{ is a real number and } x^2 < 0\} = \emptyset,$$

because there is no real number satisfying the property that its square is less than 0. Note that $x \notin \emptyset$ for any x. We sometimes also write the empty set as a list $\emptyset = \{\}$. It sounds strange, but this is actually a useful way to represent nothing.

If X is a finite set, then we let $|X|$ denote the number of elements of X. We say that $|X|$ is the **cardinality** of X. (Note that many people use the notation $\#X$ instead of $|X|$ to denote the cardinality of X.) For example, $|\{1, 2, 3\}| = 3$ and $|\emptyset| = 0$. If X and Y are finite sets, then $|X| = |Y|$ means that X and Y have the same number of elements. For example, then we cannot define $|\mathbb{Z}|$ without resorting to so-called transfinite cardinal numbers, which we will not need in this book.

A set X is called a **subset** of a set Y if every element of X is also an element of Y. We think of X as a smaller set that is made up of some of the elements of Y. More informally, we think of X as sitting "inside" Y. The notation that we use is $X \subseteq Y$. For example,

$$\{3, 8, 11\} \subseteq \{2, 3, 5, 8, 11, 14, 16\}$$

and

$$\{x \mid x \text{ is an even integer}\} \subseteq \{x \mid x \text{ is an integer}\}.$$

If A is a subset of B, we also say that B is a **superset** of A.

Notice that, by definition, every set is a subset of itself. It is important to consider this case when we want to **prove subset** of another set. For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3\}$, then $A \subseteq B$ because every element of A is also an element of B. In fact, $A \subseteq A$ for any set A. The proper subset of $\{1, 2, 3, 4\}$ although it is a subset, this is not the same as saying that A is a proper subset of B. This is a very special case, and it is very similar to the notations $\leq$ for "less than" and $\leq$ for "less than or equal to". If $a < b$ and $b \leq c$ and $d \leq e$ and $f \leq f$, Another way to think of this is: $a < b$ and $b \leq c$ means that $a \leq b$ and $b \leq c$ and $a \neq b$. Similarly, $d \leq e$ and $f \leq f$ means that $d \leq e$ and $f = f$. For example, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ because $3 \in \{1, 2, 3\}$

WARNING! Unfortunately, this notation is not universally agreed upon. Many authors use $\subseteq$ to mean that A is a subset of B, not necessarily proper. Some of these authors then use the notation $A \subseteq B$ to mean that A is a proper subset of B. However, in this book, we will always use $A \subseteq B$ to mean that A is a subset of B, possibly equal and use $A \subset B$ to mean that A is a proper subset of B.

WARNING! The concepts and notations can get a bit confusing, and it takes a little bit of practice to use them properly. For example, $A \subseteq B$ means $\{1\} \subseteq \{1, 2, 3\}$, since it is correct to say that 1 is an element of A, and that $\{1\}$ is a subset of B. A notation, we would write

$$1 \in A \quad \text{and} \quad \{1\} \subseteq A.$$

But it is not correct to say that $\{1\}$ is an element of A.

Let's practice with some basic exercises involving elements and subsets:

Problem 1.1.1

Consider the following sets:

$$A = \{1, 2, 3, 4, 5\}, \quad B = \{2, 3, 4\}, \quad C = \{4, 5\}.$$

(a) Is $4 \in B$? Is $4 \subseteq B$?
(b) Is $B \subseteq A$? Is $B \in A$?
(c) Is $C \subseteq B$? Is $C \in B$?
(d) Is $1 \in \{1\}$?
(e) List all of the subsets of B. How many are there?

Solution for Problem 1.1.1

(a) The elements of A are $1, 2, 3, 4, 5$. All of these elements are elements of A, so $4 \in A$. However, $B = \{2, 3, 4\}$ is not a proper subset of A. Therefore, $A \not\subseteq A$.

(b) All of the elements of B are also elements of A, so $B \subseteq A$. Further, it makes sense to talk about whether $B \in A$. You can prove to yourself that B is a proper subset of A, thus $B \in A$. I know I'm safe making this claim, because I know that you can't prove me at element.

(c) One of the elements of C is 5, but 5 is not an element of B. So C is not a subset of B. Finally, C is not an element of B and only if there exists some $x \in B$ such that $x \notin A$. But such an element x cannot possibly exist, because B is an element! Thus, 4 is not true that C is an element of B. Even though C is a set, this proposition does a bookmarker sequence, many people find this argument easier to follow than the argument in the previous paragraph.

Extension: The empty set is a weird object, and can lead to counterintuitive conclusions. For example, the statement

$$x \in \emptyset \text{ then } x \text{ is a flying yellow pig}$$

is a true statement because $\emptyset$ has no elements, there is no such x. Thus we don't have to worry about the "left" part of the statement says, because the "if" part of the statement is never satisfied.

Just as we can perform operations on numbers (addition and multiplication), to get new numbers, we can perform operations on sets to get new sets. We start by defining the two primary operations on sets.

Definition: The **union** $A \cup B$ of two sets A and B is the set of all objects that are elements of both A and B:

$$A \cup B = \{x \mid x \in A \text{ and } x \in B\}.$$

Note that our use of the word "or" in the definition does not mean "one or the other, but not both." Instead, we use the word "or" to mean "one or the other, or possibly both." This is always the way that mathematicians use the word "or." Elements that are in both A and B are also in their union.

Here are some examples:

- $\{2, 5, 8\} \cup \{2, 7, 11, 13\} = \{2, 5, 7, 8, 11, 13\}$.
- $\{5, 8, 11\} \cup \{2, 5, 8\} = \{2, 5, 8, 11\}$.
- $\{4, 9, 11, 16\} \cup \{2, 3, 9, 16\} = 8$ since these two sets have no elements in common.
- $\{x \mid x \text{ is an even integer}\} \cup \{x \mid x \text{ is an odd integer}\} = \emptyset$
- $\{x \mid x \text{ is an even positive integer}\} \cup \{x \mid x \text{ is a prime number}\} = \{2\}$.

A special case of union and intersection is shown in the following problem.

Problem 1.3

Suppose A and B are sets such that $A \subseteq B$.

(a) What is $A \cup B$?
(b) What is $A \cap B$?

Solution for Problem 1.3

(a) If $A \subseteq B$, then every element of A is also an element of B. This means that any element in A or B must be in B. Therefore, $A \cup B = B$. On the other hand, every element of B is also element of $A \cup B$ and $B \subseteq A \cup B$ means $A \cup B = B$.

(b) Any element in A and B must be in A. On the other hand, every element of A is also an element of B and also an element of $A \cap B$. Thus $A \subseteq A \cap B$. Therefore, $A \cap B = A$.

Important: Two sets are equal if and only if they contain exactly the same elements. To show that two sets, A and B are equal, we show that every element of A is also an element of B and every element of B is also an element of A. In other words, we show that $A \subseteq B$ and $B \subseteq A$. Sometimes it is easier to show that one set is a subset of the other. For example, if we show that every element of A is also an element of B, then we know that $A \subseteq B$. In order to show that they're equal, we have to show that $B \subseteq A$ as well.

It is useful to have a second way to describe when two sets have no elements in common.

Definition: If $A \cap B = \emptyset$ then we say that A and B are **disjoint**.

Our two operations union and intersection are distributive with respect to each other. We'll prove one of the distributive laws and leave the other as an exercise.

Problem 1.4

Prove that $A \cup (B \cap C)$ and C are sets, then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Solution for Problem 1.4. To prove that two sets are equal, we must show that they have the same elements. This means that we must show that every element in one set is also in the other set, and vice versa.

We start by letting x be an element of $A \cup (B \cap C)$. This means that $x \in A$ and $x \in (B \cup C)$, which means that

$$x \in A \text{ and } (x \in B \text{ or } x \in C).$$

This can be rewritten as

$$(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C).$$

Now, rewriting our statement using union and intersection of sets, we have

$$x \in (A \cup B) \cap (A \cup C).$$

Thus, all elements of $A \cap (B \cup C)$ are also elements of $(A \cup B) \cap (A \cup C)$, which means that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C). \qquad (\alpha)$$

We're not done! We have to show the reverse as well. To do so, let y be an element of $(A \cup B) \cap (A \cup C)$. Then

$$(y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C).$$

This can be written as

$$y \in A \text{ and } (y \in B \text{ or } y \in C),$$

which means that $y \in A \cap (B \cup C)$. Thus,

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C). \qquad (\alpha\alpha)$$

Combining (α) and $(\alpha\alpha)$, we see that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ Have the same elements, so they are equal. ∎

We have one more important operation on sets to define; we will leave its properties as an exercise.

Definition: The set difference of two sets A and B is

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

That is, $A \setminus B$ consists of all elements of A that are not in B. For example, if $A = \{1, 2\}$ and $B = \{2, 3\}$, then $A \setminus B = \{1\}$, because $1 \in A$ and $1 \notin B$ and 1 is the only element in this property. Similarly, $B \setminus A = \{3\}$.

Set theory is a very rich subject, and we have only scratched the surface of it. We could fill a whole book discussing set theory (and many such books have been written). But the set theory that we've done in this section is really all we need for calculus.

Exercises

1.1.1:

Show that $A \subseteq B$ and $A \subseteq C$ when $A \subseteq (B \cap C)$. Show that the same statement is not true if we replace every "$\subseteq$" with "$\in$".

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Solution: Suppose that $x \in A$. Then since $A \subseteq B$, we have $x \in B$; and since $A \subseteq C$, we have $x \in C$. Therefore, $x \in (B \cap C)$. Thus, every element of A is also an element of $(B \cap C)$, so $A \subseteq (B \cap C)$.

The statement is not necessarily true if we have proper subsets. To construct a simple example, let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$. Then $A \subseteq (B \cap C)$ and $A \subseteq C$ but $A \in B \cap C)$ and $A \in (B \cap C)$.

1.1.2:

Prove that for any sets, A, B and C:

$$A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C).$$

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Solution First, we show that

$$A \cup (B \setminus C) \subseteq (A \cup B) \setminus (A \cup C).$$

Suppose that $x \in A \cup (B \setminus C)$. We have two cases:

Case 1: $x \in A$. Then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $x \in (A \cup B) \setminus (A \cup C)$ has desired.

Case 2: $x \in (B \setminus C)$. Then $x \in B$ and $x \in C$. However, $x \in (A \cup B)$ and $x \in (A \cup C)$, so $x \in (A \cup B) \setminus (A \cup C)$ so desired.

We have shown that each element x in $A \cup (B \setminus C)$ is an element of $(A \cup B) \setminus (A \cup C)$, so

$$A \cup (B \setminus C) \subseteq (A \cup B) \setminus (A \cup C).$$

Now we will show that

$$(A \cup B) \setminus (A \cup C) \subseteq A \cup (B \setminus C).$$

Suppose that $x \in (A \cup B) \setminus (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. We now divide the possibilities for x into two cases:

Case 1: $x \in A$. Then $x \in A \cup (B \setminus C)$ desired.

Case 2: $x \notin A$. Then since $x \in (A \cup B)$ we have $x \in B$, and since $x \in A \cup C$, we have $x \in C$. Therefore, $x \in (B \setminus C)$ and hence $x \in A \cup (B \setminus C)$ has desired.

Therefore, we have shown that any element x in $(A \cup B) \setminus (A \cup C)$ is an element of $A \cup (B \setminus C)$, so

$$(A \cup B) \setminus (A \cup C) \subseteq A \cup (B \setminus C).$$

Thus,

$$(A \cup B) \setminus (A \cup C) = A \cup (B \setminus C).$$

1.1.3:

Prove that $\emptyset \subseteq A$ for any set A.

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Solution This is essentially a tautology. If $x \in A$, then $x \in A$, so $A \subseteq A$.

1.1.4:

(a) Is set difference commutative? That is, must we have $A \setminus B = B \setminus A$? If this prove it false, give a counterexample.)

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Solution Set difference is not commutative. For example, let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, and $C = \{1\}$ Then $A \setminus B = \{2\}$ whereas $B \setminus A = \{\}$.

(b) Is set difference associative? That is, must we have $(A \setminus B) \setminus C = A \setminus (B \setminus C)$? If true, prove it. If false, give a counterexample.)

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Solution Set difference is not associative. For example, let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, and $C = \{1\}$ Then

$$(A \setminus B) \setminus C = \{\{1, 2, 3\}\} \setminus \{\{1, 2\}\} = \{\{1, 2, 3\}\} = \{\{1, 2\}\} = \{\{1\}\},$$

whereas

$$A \setminus (B \setminus C) = \{1, 2, 3\} \setminus \{\{1, 2\}\} \setminus \{\{1\}\} = \{1, 2, 3\} \setminus \{\{1\}\} = \{1, 2\}.$$

(c) Prove that for any sets A, B, C, we have

$$A \setminus (B \cup C) = (A \setminus B) \setminus (A \setminus C) = (A \setminus B) \setminus C.$$

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Solution We claim that x is an element of each of these sets if and only if $x \in A$, $x \notin B$, and $x \notin C$. We prove this separately for each set:

$$\begin{aligned}
&x \in A \setminus (B \cup C) \\
&\Leftrightarrow x \in A \text{ and } x \notin (B \cup C) \\
&\Leftrightarrow x \in A \text{ and } (x \notin B) \text{ and } (x \notin C)) \\
&\Leftrightarrow x \in A \text{ and } x \notin B \text{ and } x \notin C \\
&\Leftrightarrow (x \in A \setminus B) \text{ and } (x \notin C) \\
&\Leftrightarrow (x \in A \setminus B) \setminus C \\
&\Leftrightarrow x \in A \text{ and } (x \notin B) \text{ and } (x \in C)) \\
&\Leftrightarrow x \in A \setminus (B \setminus C) \\
&\Leftrightarrow x \in A \setminus B \text{ and } x \notin C \\
&\Leftrightarrow x \in (A \setminus B) \text{ and } x \notin C \\
&\Leftrightarrow x \in A \setminus B \text{ and } x \notin C \\
&\Leftrightarrow x \in (A \setminus B) \setminus C
\end{aligned}$$

(d) Where $A \setminus B$ when $B \setminus A$?

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Solution $A \setminus B = A$, because there are no elements to exclude. On the other hand, $B \setminus A = B$ because $B \setminus A$ is a subset of B by definition, and the only subset of the empty set is itself.

1.1.5a:

Show that, for any sets A, B, C,

$$A \cup (B \setminus C) = (A \cup B) \setminus (A \setminus C).$$

Hint: Show that each set is a subset of the other.

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Solution We show these sets are equal by showing that each is a subset of the other.

Step 1: First we show that

$$(A \cup B) \setminus (C \setminus A) \subseteq A \cup (B \setminus C).$$

Let x be an element of the left hand side $x \in A$. Then certainly $x \in A \cup (B \setminus C)$, if $x \in B$, and $B \setminus C$. Then $A \setminus B = \{2\}$ whereas $B \setminus A = \{\}$. Since we have assumed already that $x \notin A$, it cannot be in C, so must be in $B \setminus C$. This shows

$$(A \cup B) \setminus (C \setminus A) \subseteq A \cup (B \setminus C).$$

Step 2: Next we show that

$$A \cup (B \setminus C) \subseteq (A \cup B) \setminus (C \setminus A).$$

If $x \in A$, then $x \in A \setminus B$ and also $x \in C \setminus A$, showing

$$A \subseteq (A \cup B) \setminus C \setminus A.$$

Likewise, since $(C \setminus A)$ is a subset of C, we have

$$B \setminus C \subseteq B \setminus (C \setminus A) \subseteq (A \cup B) \setminus (C \setminus A).$$

Combining these two we get that

$$A \cup (B \setminus C) \subseteq (A \cup B) \setminus (C \setminus A).$$

1.2 Numbers and intervals

There are some special sets that consist of numbers. You're almost certainly already familiar with these sets, but let's discuss them with an eye towards what makes each of them special.

We can start with the **positive integers**, which are our basic counting numbers:

$$\{1, 2, 3, 4, \ldots\}.$$

An average 3-year-old is pretty comfortable with positive integers (although he or she probably doesn't yet know the word "integer").

Sidenote: ♫

Some mathematicians call the positive integers the **natural numbers**, denoted by the symbol $\mathbb{N}$:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Other mathematicians consider 0 to be a natural number as well, so

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$

Unfortunately, there is not general agreement in the mathematical community as to whether 0 should be a natural number of not. There is similar ambiguity about the terms **whole numbers** and **counting numbers**, which (depending on who you ask) each refer to the set $\{1, 2, 3, \ldots\}$ or the set $\{0, 1, 2, 3, \ldots\}$.

We will sidestep the issue by referring to the set $\{1, 2, 3, \ldots\}$ as "positive integers" and the set $\{0, 1, 2, 3, \ldots\}$ as "nonnegative integers."

But positive integers, by themselves, have a lot of limitations. First of all, zero is missing. Second of all, we can't always subtract: for instance, $1 - 3$ is undefined as a positive integer. So we include 0 and negative integers to get the entire set of **integers**, denoted by $\mathbb{Z}$:

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

Integers are a great improvement: now we can add and subtract, and we have 0. In fact, addition in $\mathbb{Z}$ satisfies these other nice axioms as well:

- **commutativity**: for any $a, b \in \mathbb{Z}$, $a + b = b + a$.
- **associativity**: for any $a, b, c \in \mathbb{Z}$, $(a + b) + c = a + (b + c)$.
- $\mathbb{Z}$ has an additive **identity element**, namely 0: for any $a \in \mathbb{Z}$, $a + 0 = 0 + a = a$.
- $\mathbb{Z}$ has additive **inverses**: for any $a \in \mathbb{Z}$, there exists $b \in \mathbb{Z}$ such that $a + b = b + a = 0$. (Of course, $b = -a$.)

Of course, we know that $\mathbb{Z}$ has a second operation—multiplication—that also commutative, associative, and has an identity element, 1. Additionally, the two operations together satisfy the **distributive** property:

$$\text{for any } a, b, c \in \mathbb{Z}, \ a(b + c) = (ab) + (ac).$$

However, we don't have multiplicative inverses; that is, we can't divide. For example, $2/3$ is undefined in $\mathbb{Z}$. To remedy this, we move up to the **rational numbers**, denoted by $\mathbb{Q}$ for "quotient".

The elements of $\mathbb{Q}$ are ratios of integers:

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\}.$$

Note in particular that we don't allow 0 in the denominator.

However, this doesn't tell the whole story, since a lot of these fractions are equal. For example, $\frac{1}{2} = \frac{2}{4} = \frac{5}{10}$ and in fact there are infinitely many fractions that represent each rational number. We don't think of $\frac{1}{2}$ $\frac{2}{4}$ etc. as separate elements of $\mathbb{Q}$; rather they are different ways of representing the same number.

The set $\mathbb{Q}$ together with the operations of addition and multiplication is what is called a **field**: we can add, subtract, multiply, and divide any elements of $\mathbb{Q}$ (except that we can't divide by 0), of course, and these operations satisfy all the nice properties that we want them to have (commutativity, associativity, and distributivity). So it seems like we have everything we need for our number system.

But there's still something fundamental missing from $\mathbb{Q}$. For instance:

Problem 1.5

Show that there is no number $x \in \mathbb{Q}$ such that $x^2 = 2$.

Solution for Problem 1.5: The proof that follows is one of the most famous examples in mathematics of **proof by contradiction**. Our strategy will be to assume that there is a number $x \in \mathbb{Q}$ such that $x^2 = 2$, and then show that this leads to a logical impossibility. This implies that the original assumption cannot be true.

If $x \in \mathbb{Q}$, then we can write $x = p/q$ where p and q are integers that have no common factors (other than 1 and -1). We will show that this assumption leads to a contradiction. Indeed, we can multiply x by q to get $qx = p$, and then square both sides to get $q^2x^2 = p^2$. We will now prove that p and q in fact are both even, which is our logical impossibility.

The fact that $x^2 = 2$ gives us $(q^2)x^2 = 2q$ or $p^2 = 2q^2$. But this means that p is even, so let $p^2 = p/2$. Thus, we have $(2p^2)^2 = 2q^2$, which simplifies to $2(p^2)^2 = q^2$. But this means that q is even. Therefore, p and q are both even. This is our contradiction! Thus, our assumption was false, and there is no number $x \in \mathbb{Q}$ such that $x^2 = 2$.

Of course, when we extend our set of numbers to the **real numbers** (denoted $\mathbb{R}$), a solution to $x^2 = 2$ exists, namely $\sqrt{2}$. But what exactly are real numbers? In other words, how can we define numbers such as $\sqrt{2}$ using what we already have, which (for the moment) is just rational numbers?

One idea starts with how we naturally think of $\sqrt{2}$, and that is as the positive number whose square is 2. Slightly more formally, we can "define" $\sqrt{2}$ as the positive solution to the equation $x^2 - 2 = 0$, which is an equation that only contains rational numbers. When we add all such numbers to the set $\mathbb{Q}$ that $\mathbb{Q}$ is closed under taking nonzero roots of polynomial equations with rational coefficients—we get the set of **algebraic numbers**, denoted $\overline{\mathbb{Q}}$. But $\overline{\mathbb{Q}}$ still lacks of lots of useful numbers, such as π, or indeed any other **transcendental** number that is not the solution of a polynomial equation with rational coefficients. So what can we do?

Instead, we are motivated by the idea that we typically think of $\mathbb{R}$ as being equivalent to the "number line": a line in which every point represents a real number, and in which every real number corresponds to a unique point on the line. When thinking of numbers as lying on a number line, we see that $\mathbb{Q}$ by itself contains lots of "holes" on the line.

For instance, to go back to our earlier example, we can think of the number $\sqrt{2}$ as the number that fills the hole between the sets $\{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\}$ and $\{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\}$. Note that we can very easily describe the above two sets just using rational numbers, and the sets contain only rational numbers. But the "hole" between them is the number $\sqrt{2}$. When we fill in all the numbers that create such holes in $\mathbb{Q}$, we get the real numbers $\mathbb{R}$.

It is possible, with a lot of work, to make the construction of $\mathbb{R}$ rigorous. But for the purposes of our study of calculus in this book, it's enough to think informally of the real numbers as all the points on the number line without any holes. The essential axiomatic fact about $\mathbb{R}$ that we'll need for calculus is called **completeness**, which essentially means "no holes." Before stating exactly what completeness is, we'll need a few definitions:

Definition: Let S be a subset of $\mathbb{R}$.

- A number $x \in \mathbb{R}$ is called an **upper bound** for S if for all $y \in S$, we have $y \leq x$. If an upper bound for S exists, we say that S is **bounded above**.
- A number $x \in \mathbb{R}$ is called a **lower bound** for S if for all $y \in S$, we have $y \geq x$. If a lower bound for S exists, we say that S is **bounded below**.
- If S has an upper bound and a lower bound, we say that S is **bounded**.

Some bounds are very special:

Definition: Let S be a subset of $\mathbb{R}$.

- A number $x \in \mathbb{R}$ is called a **greatest lower bound** (or **supremum**) of S if x is an upper bound for S and if for every $z \in \mathbb{R}$ such that z is an upper bound of S, we have $x \leq z$.
- A number $x \in \mathbb{R}$ is called a **least upper bound** (or **infimum**) of S if x is a lower bound for S and if for every $z \in \mathbb{R}$ such that z is a lower bound of S, we have $x \geq z$.
- If S has an upper bound and a lower bound, we say that S is **bounded**.

In other words, an upper bound of a subset $S \subseteq \mathbb{R}$ is any number greater than or equal to all of the numbers in S. If we can find a smallest possible upper bound for S, then this number is the least upper bound of S. Similarly, a lower bound of S is a number that is less than or equal to all of the numbers in S. If we can find a greatest possible lower bound for S, then this number is the greatest lower bound of S.

Problem 1.6

Suppose S is a subset of $\mathbb{R}$. Show that if S has a least upper bound, then this least upper bound is unique.

Solution for Problem 1.6: Suppose x and y are each least upper bounds of S. Then $x \leq y$ (since x is a least upper bound and y is an upper bound), but also $y \leq x$ (since y is a least upper bound and x is an upper bound). Since $x \leq y$ and $y \leq x$, we must have $x = y$. So the least upper bound is unique.

The least upper bound, or supremum, of a subset $S \subset \mathbb{R}$ is denoted $\sup(S)$ (if it exists). Similarly, the greatest lower bound, or infimum, of S is denoted $\inf(S)$ (if it exists)—we will show in the next problem set that if it exists, it must be unique.

We can now precisely say what it is that makes $\mathbb{R}$ "better" than $\mathbb{Q}$:

Important: ❗ $\mathbb{R}$ is **complete**, meaning that every nonempty subset $S \subset \mathbb{R}$ that has an upper bound has a least upper bound.

This may seem obscure, but it is the fundamental property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$. By way of example, let's again look at $\sqrt{2}$ by considering the set

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\}.$$

As a subset of $\mathbb{Q}$, this subset certainly has an upper bound (2 is a trivial example of an upper bound of A), but it has no least upper bound: there is no single number in $\mathbb{Q}$ that is an upper bound of A, but smaller than all other upper bounds of A. However, as a subset of $\mathbb{R}$, the set A has a least upper bound: $\sup A = \sqrt{2}$ because $\sqrt{2}$ is an upper bound of A (that is, all numbers in A are less than or equal to $\sqrt{2}$) and any other upper bound of A must be greater than $\sqrt{2}$.

The most important subsets of $\mathbb{R}$ that we will be regularly using are **intervals**:

Definition: Let I of $\mathbb{R}$ is called an **interval** if, for any $a, b \in I$ and $x \in \mathbb{R}$ such that $a \leq x \leq b$, we have $x \in I$.

Informally, an interval I is a subset of $\mathbb{R}$ without any "holes"—if a and b are in I, then all the numbers between a and b are also in I. Two types of intervals are most commonly used:

Definition: Let $a \leq b$ be any two real numbers. The **open interval** (a, b) is defined as the set

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

The **closed interval** $[a, b]$ is defined as the set

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

It is easy to check that open intervals and closed intervals are indeed intervals (we leave this as an exercise). We think of intervals as segments of the real line. Open intervals do not include the endpoints, but closed intervals do. We denote intervals on a number line as follows:

Notice that we use an open circle to indicate that the endpoint of the interval is not included in the interval (as in the case of an open interval), and a filled-in circle to indicate that the endpoint of the interval is included in the interval (as in the case of a closed interval).

We also often use the following notations for intervals that include one endpoint but not the other:

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$
$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$

Such intervals are referred to as **half-open** intervals.

Problem 1.7

Let $a \leq b$ be any two real numbers.

(a) What are $\sup(a, b)$ and $\inf(a, b)$?

(b) What are $\sup(a, b]$ and $\inf(a, b]$?

(c) Comparing your answers to (a) and (b), what is the distinguishing feature that differentiates $[a, b]$ from (a, b)?

Solution for Problem 1.7:

(a) If $x \in [a, b]$ then $a \leq x \leq b$. This immediately tells us that a is a lower bound and b is an upper bound. In particular, by definition $\sup(a, b) \leq b$, since any upper bound for (a, b) is at least $\sup(a, b)$. On the other hand, $b \leq \sup(a, b)$ since b is in the set $[a, b]$. Thus $b = \sup(a, b)$. Similarly, $a = \inf(a, b)$.

(b) If $x \in (a, b]$ then $a < x \leq b$. As in part (a), this immediately tells us that a is a lower bound and b is an upper bound. Any real number less than b but greater than a is in the interval, so no number less than b can be an upper bound. Therefore, $\sup(a, b] \geq b$. But $\sup(a, b] \leq b$ by definition (since b is an upper bound), so $\sup(a, b] = b$. Similarly, $a = \inf(a, b]$.

(c) From and (b), we see that $\sup(a, b] = \sup(a, b) = b$ and $\inf(a, b] = \inf(a, b) = a$. The difference is that in part (a), the supremum and infimum are elements of the interval, whereas in part (b), they are not. In other words, $\sup(a, b] \in [a, b]$ and $\inf(a, b] \in [a, b]$ but $\sup(a, b) \notin (a, b)$ and $\inf(a, b) \notin (a, b)$.

The last problem motivates a new definition:

Definition: Let S be a subset of $\mathbb{R}$. If $\sup S \in S$, we say that $\sup S$ is the **maximal element** of S (or simply the **maximum** of S). If $\inf S \in S$, we say that $\inf S$ is the **minimal element** of S (or simply the **minimum** of S).

As we saw in Problem 1.7, although a bounded subset of $\mathbb{R}$ must have a supremum and an infimum, it need not have a maximum or minimum.

We often use unbounded intervals too:

Definition: Let $a, b \in \mathbb{R}$. We define the following intervals:

$$[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\},$$
$$(a, +\infty) = \{x \in \mathbb{R} \mid a < x\},$$
$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\},$$
$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}.$$

WARNING! ⊘ It is common to use just ∞ instead of $+\infty$, especially when the context is clear.

Important: ❗ The symbol $+\infty$ is read "positive infinity." However, it is very important to not think of $+\infty$ as a number—rather, it is a conceptual symbol that vaguely means "arbitrarily large" or "growing without bound," but whose specific meaning depends on its context.

Because we never think of $+\infty$ or $-\infty$ as a number, we don't think of the interval as containing $+\infty$ or $-\infty$, and thus the "infinite" end of the interval is always open. In particular, we never would write $[a, +\infty]$.

Problem 1.8

Let $a \in \mathbb{R}$. What are $\sup(a, +\infty)$ and $\inf(a, +\infty)$?

Solution for Problem 1.8: By construction, the interval $[a, +\infty)$ has no upper bound, since it contains arbitrarily large elements. Thus $\sup(a, +\infty)$ is undefined. On the other hand, $\inf(a, +\infty) = a$ by the same argument that shows $\inf[a, b] = a$ for any $b \geq a$.

Finally, we can write the entire set of real numbers as an interval as $\mathbb{R} = (-\infty, +\infty)$.

Exercises

1.2.1:

Let $S \subset \mathbb{R}$. Show that if S has a greatest lower bound, then this greatest lower bound is unique.

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Suppose a and b are each greatest lower bounds for S. Then since a is a greatest lower bound and b is a lower bound, we have $b \leq a$; but since b is a greatest lower bound and a is a lower bound, we have $a \leq b$. Thus $a = b$, and hence the greatest lower bound is unique.

1.2.2:

Prove that for any $a \leq b$, the open interval (a, b) and the closed interval $[a, b]$ are indeed intervals (using our original definition of interval).

Hint: Given c, d in the interval, you must show that any x with $c \leq x \leq d$ is also in the interval.

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Solution: We must show that if $c, d \in (a, b)$ and $x \in \mathbb{R}$ is such that $c \leq x \leq d$, then $x \in (a, b)$. But we know that $a < c$ and $d < b$, so we have

$$a < c \leq x \leq d < b,$$

so in particular $a < x < b$, and thus $x \in (a, b)$.

Similarly, if $c, d \in [a, b]$ and $x \in \mathbb{R}$ is such that $c \leq x \leq d$, then

$$a \leq c \leq x \leq d \leq b,$$

so $a \leq x \leq b$ and hence $x \in [a, b]$.

1.2.3:

(a) Show that the intersection of any two intervals is an interval.

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Solution: Let I, J be intervals, and suppose $a, b \in I \cap J$ and $x \in \mathbb{R}$ is such that $a \leq x \leq b$. Thus, since $a, b \in I$ and I is an interval, we have $x \in I$; similarly, since $a, b \in J$ and J is an interval, we have $x \in J$. So $x \in I \cap J$, and thus, by definition, $I \cap J$ is an interval.

(b) Show that the union of any two intervals need not be an interval.

Hint: Try to create a "hole" in your union.

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Solution: There are many examples. For example, $S = (0, 1) \cup (2, 3)$ is not an interval, since $\frac{1}{2}, \frac{5}{2} \in S$, and $\frac{1}{2} + \frac{1}{3} = \frac{5}{6} \in S$, but $\frac{5}{6} \notin S$.

1.2.4:

If A, B are bounded intervals with $A \cap B \neq \emptyset$, then show that

$$\sup(A \cap B) = \min\{\sup A, \sup B\}.$$

Hint: Start by showing

$$\sup(A \cap B) \leq \min\{\sup A, \sup B\}.$$

Hint: Show that if

$$\sup(A \cap B) < \min\{\sup A, \sup B\},$$

then there is a contradiction, by finding an element of $A \cap B$ strictly larger than $\sup(A \cap B)$.

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Solution: First notice that since A and B are bounded, any upper bound for A is an upper bound for $A \cap B$. (The same holds for lower bounds.) Therefore, $A \cap B$ is bounded and nonempty, so has a least upper bound.

Since $\sup A$ and $\sup B$ are both upper bounds for $A \cap B$, the smaller of the two must certainly be an upper bound for $A \cap B$. Therefore

$$\sup(A \cap B) \leq \min\{\sup A, \sup B\}.$$

We complete the proof by showing that this inequality must be an equality.

Assume that

$$\sup(A \cap B) < \min\{\sup A, \sup B\}.$$

Let x be any number such that

$$\sup(A \cap B) < x < \min\{\sup A, \sup B\}$$

(for example, we can set x to be the average of $\sup(A \cap B)$ and $\min\{\sup A, \sup B\}$). Now, since $\sup(A \cap B) < x$, there is some $c \in A \cap B$ such that $c < x$. Notice also that, since $x < \sup A$, there is some $d \in A$ such that $x < d$ (otherwise x is an upper bound for A, contradicting the fact that x is less than the least upper bound). Thus, since c and d are in A and

$$c < x < d,$$

we get that $x \in A$ as well. Likewise there is some $d' \in B$ which is larger than x, so

$$c < x < d'$$

and c and d' are both in B, so $x \in B$. Combining these we see that $x \in A \cap B$. However we assumed that $\sup(A \cap B) < x$, which is a contradiction. Therefore

$$\sup(A \cap B) = \min\{\sup A, \sup B\}.$$

1.3 Functions

A **function** is a mathematical device that associates an input with a unique output. For example, we write

$$f(x) = 3x - 5$$

to mean the function which, for any given input $x \in \mathbb{R}$, outputs the real number $3x - 5$. The x in our function definition above is a **dummy variable** that takes the place of the input. This dummy variable can be anything, so

$$f(x) = 3x - 5, \qquad f(t) = 3t - 5, \qquad f(\xi) = 3\xi - 5$$

are all representations of the same function.

More formally:

Definition: A **function** f from a set A to a set B, denoted $f : A \to B$, associates to each $a \in A$ an element $f(a) \in B$. The set A is called the **domain** of f and the set B is called the **codomain** of f. We let $\mathrm{Dom}(f)$ denote the domain of f and $\mathrm{Cod}(f)$ denote the codomain of f.

In calculus, the codomain of our functions will almost always be $\mathbb{R}$, and you can assume that a function has codomain $\mathbb{R}$ unless otherwise specified. Such a function is called **real-valued** for the obvious reason. If we don't explicitly specify a domain, then it is understood that a function has as its domain the largest subset of $\mathbb{R}$ for which the function definition makes sense. For example, the function $f(x) = \sqrt{x}$ has domain $[0, +\infty)$; since we cannot (in $\mathbb{R}$) take the square root of a negative number. However, sometimes (depending on the situation) we may find it convenient to consider a function whose domain is a subset of its "allowed" domain.

There is another point to make about $f(x) = \sqrt{x}$. Even though the codomain of this function is $\mathbb{R}$ (as it will be for virtually all functions we consider in this book), the function only actually outputs values that are nonnegative real numbers; that is, $f(x) \geq 0$ for all $x \in \mathrm{Dom}(f)$. To indicate this, we say that the **range** of f is $[0, +\infty)$.

Definition: The **range** of a real-valued function f, denoted $\mathrm{Rng}(f)$, is the set of values that f outputs; that is,

$$\mathrm{Rng}(f) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in \mathrm{Dom}(f)\} = \{f(x) \mid x \in \mathrm{Dom}(f)\}.$$

As a simple example:

Problem 1.9

Find the domain and range of the function $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$.

Solution for Problem 1.9: There are two conditions on the domain of f. First, since we can only take the square root of a nonnegative number, we must have $x^2 - 1 \geq 0$. But we also cannot have a denominator equal to zero, so we must have $x^2 - 1 > 0$. This means that $x^2 > 1$ so $x > 1$ or $x < -1$. Thus, the domain is $(-\infty, -1) \cup (1, +\infty)$.

Since the denominator can be any positive real number, the range is $(0, +\infty)$. □

We can perform various operations on real-valued functions:

- We can add two functions f and g:

$$(f + g)(x) = f(x) + g(x).$$

  Note that this is only valid at x where both f and g are defined, so

$$\mathrm{Dom}(f + g) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g).$$

  Similarly we can subtract, multiply, and divide:

$$(f - g)(x) = f(x) - g(x),$$

$$(fg)(x) = f(x)g(x),$$

$$(f/g)(x) = f(x)/g(x).$$

  All of these have the same domain as $f + g$, with the exception of f/g that has the added restriction that $g(x) \neq 0$.
- We can multiply a function f by a constant $c \in \mathbb{R}$:

$$(cf)(x) = c \cdot f(x).$$

  It is clear that $\mathrm{Dom}(cf) = \mathrm{Dom}(f)$.
- We can **compose** two functions:

$$(f \circ g)(x) = f(g(x)).$$

  Note that, for $(f \circ g)(x)$ to be defined, we must have $x \in \mathrm{Dom}(g)$ and $g(x) \in \mathrm{Dom}(f)$. Notice also that the order of the functions is important: $f \circ g$ is in general not the same as $g \circ f$.

As we progress through the text we will see additional operations that we can perform on functions.

Composition also gives us a new notion:

Definition: Let f be a real-valued function. A real-valued function g is called an **inverse** of f if $f(g(x)) = x$ for all $x \in \mathrm{Dom}(g)$ and $g(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$. We denote this by $g = f^{-1}$.

Notice that if g is an inverse of f, then f is also an inverse of g, by the symmetry of the definition. Further, since f and g essentially "undo" each other, the result of the next problem should make sense:

Problem 1.10

Show that if g is an inverse of f, then $\mathrm{Dom}(f) = \mathrm{Rng}(g)$ and $\mathrm{Dom}(g) = \mathrm{Rng}(f)$.

Solution for Problem 1.10: Since $g(f(x)) = x$ for all $x \in \mathrm{Dom}(f)$, we see that $x \in \mathrm{Dom}(f)$ implies that $x \in \mathrm{Rng}(g)$. So $\mathrm{Dom}(f) \subseteq \mathrm{Rng}(g)$. On the other hand, if $y \in \mathrm{Rng}(g)$, then $y = g(x)$ for some $x \in \mathrm{Dom}(g)$, and thus $f(y) = f(g(x)) = x$. In particular, $y \in \mathrm{Dom}(f)$, so $\mathrm{Rng}(g) \subseteq \mathrm{Dom}(f)$. Combining these two subset inclusions gives us $\mathrm{Dom}(f) = \mathrm{Rng}(g)$.

By the symmetry of the definition of inverse, we also have that f is an inverse of g, and thus the previous argument shows that $\mathrm{Dom}(g) = \mathrm{Rng}(f)$. □

The next problem shows that we can refer to "the" inverse of f rather than "an" inverse of f:

Problem 1.11

Show that if f^{-1} exists, then it is unique; that is, show that if g and h are both inverses of f, then $g = h$.

Solution for Problem 1.11: By Problem 1.10, we know that $\mathrm{Dom}(g) = \mathrm{Rng}(f) = \mathrm{Dom}(h)$. Let x be an element in the shared domain of g and h; we will show that $g(x) = h(x)$. Since $x \in \mathrm{Rng}(f)$, there is some $y \in \mathrm{Dom}(f)$ such that $f(y) = x$. But then

$$g(x) = g(f(y)) = y = h(f(y)) = h(x).$$

So $g(x) = h(x)$ for all x in their (shared) domain, and thus the functions are equal. □

There are a couple more definitions regarding functions that will be useful for us to have later on. In particular, it's useful to have a way to describe what happens to a set when we apply a function at each point in the set, and it's useful to describe all the points that get mapped to a certain set.

Definition: Let f be a function and $A \subseteq \mathrm{Dom}(f)$. The **image** of A under f, denoted $f(A)$, is

$$f(A) = \{y \in \mathrm{Cod}(f) \mid y = f(x) \text{ for some } x \in A\}.$$

Let $B \subseteq \mathrm{Cod}(f)$. The **preimage** of B under f, denoted $f^{-1}(B)$, is

$$f^{-1}(B) = \{x \in \mathrm{Dom}(f) \mid f(x) \in B\}.$$

Informally, the image of A is the set where A "gets sent to," and the preimage of B is the set "that gets sent to" B. Notice that, by definition, $f(\mathrm{Dom}(f)) = \mathrm{Rng}(f)$ and $f^{-1}(\mathrm{Rng}(f)) = \mathrm{Dom}(f)$.

WARNING!! Note that f^{-1} has two different but related meanings. If f has an inverse, and $x \in \mathrm{Rng}(f)$, then $f^{-1}(x)$ denotes the image of x under the inverse function of f. On the other hand, if $B \subseteq \mathrm{Cod}(f)$, then $f^{-1}(B)$ denotes the preimage of B under f; we do not require that f have an inverse in order to define $f^{-1}(B)$. In the former, x is a number, whereas in the latter, B is a subset.

Here is an example of this notation:

Problem 1.12

Let $f(x) = x^2 - 3$.

(a) Find $f((0, 3))$ and $f([0, 3])$.

(b) Find $f^{-1}((1, 2))$.

(c) Find $f^{-1}((-5, -4))$.

Solution for Problem 1.12:

(a) Note that $f(0) = -3$ and $f(3) = 6$. Furthermore, we see that f is "continuously" increasing from -3 to 6 as we increase x from 0 to 3. Thus we should have that $f((0, 3)) = (-3, 6)$ and $f([0, 3]) = [-3, 6]$. We have somewhat cheated a little here: the word "continuously" is not well-defined, and is certainly not rigorous. But if you glance ahead a little bit, you'll see that this is the topic of the next chapter.

(b) If $x^2 - 3 = 1$, then $x^2 = 4$, so $x = \pm 2$. Also, if $x^2 - 3 = 2$ then $x^2 = 5$, so $x = \pm\sqrt{5}$. Thus,

$$f^{-1}((1, 2)) = (-\sqrt{5}, -2) \cup (2, \sqrt{5}).$$

This shows that the preimage of an interval, even using a "nice" function like $x^2 - 3$, does not necessarily have to be an interval.

(c) There is no value of x such that $-5 < f(x) < -4$, since $f(x) \geq -3$ for all $x \in \mathbb{R}$. Hence $f^{-1}((-5, -4)) = \emptyset$. More generally, if B contains no values in $\mathrm{Rng}(f)$, then $f^{-1}(B) = \emptyset$.

□

Images and preimages are not necessarily inverse operations, despite their suggestive notation. The next problem is trickier than it looks:

Problem 1.13

(a) Let f be a function and $A \subset \mathrm{Dom}(f)$. What is the relationship between A and $f^{-1}(f(A))$?

(b) Let f be a function and $B \subset \mathrm{Cod}(f)$. What is the relationship between B and $f(f^{-1}(B))$?

(c) Let f be a function and $B \subset \mathrm{Rng}(f)$. What is the relationship between B and $f(f^{-1}(B))$?

Solution for Problem 1.13:

(a) If $x \in A$, then by definition $f(x) \in f(A)$. And since x is an element that gets mapped into $f(A)$, we have $x \in f^{-1}(f(A))$. This means that $A \subseteq f^{-1}(f(A))$, since every element of A is also an element of $f^{-1}(f(A))$. However, if we try to reverse this argument, it doesn't work. Indeed, consider the example $f(x) = x^2$ and $A = [0, 2]$. Then $f(A) = f([0, 2]) = [0, 4]$, and

$$f^{-1}(f(A)) = f^{-1}([0, 4]) = [-2, 2],$$

which is strictly larger than the original set A. Thus, all we can say is that $A \subseteq f^{-1}(f(A))$.

(b) If $y \in f(f^{-1}(B))$, then by definition there is some $x \in f^{-1}(B)$ such that $f(x) = y$. But, again by definition, $x \in f^{-1}(B)$ means that $f(x) \in B$, and since $y = f(x)$ this means $y \in B$. Hence, every element in $f(f^{-1}(B))$ is also in B, and thus $f(f^{-1}(B)) \subseteq B$. As in part (a), the converse is not true: for example, take $f(x) = x^2$ and $B = [-4, 4]$. Then $f^{-1}(B) = [-2, 2]$ and

$$f(f^{-1}(B)) = f([-2, 2]) = [0, 4] \subset B.$$

Thus all we can say is $f(f^{-1}(B)) \subseteq B$.

(c) The reason that the converse fails in part (b) is that B might contain elements that are not in the range of f. But if $B \subseteq \mathrm{Rng}(f)$, then every element $y \in B$ satisfies $y = f(x)$ for some $x \in \mathrm{Dom}(f)$. Then we have $x \in f^{-1}(B)$, and thus $y = f(x) \in f(f^{-1}(B))$. This proves that $B \subseteq f(f^{-1}(B))$, and this combined with part (b) gives us $B = f(f^{-1}(B))$.

□

WARNING!! Images and preimages often behave in "unintuitive" ways. Be careful when working with them!

Exercises

1.3.1:

Find the domain of $f(x) = \dfrac{x - 2}{\sqrt{x^2 - 7x + 12}}$.

You may type any additional notes you have here.

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Solution: For $f(x)$ to be defined, we must have $x^2 - 7x + 12 > 0$. This factors as $(x - 4)(x - 3) > 0$, so we must have $x < 3$ or $x > 4$. Thus, the domain is $\boxed{(-\infty, 3) \cup (4, +\infty)}$. (The numerator is irrelevant to the domain, as it is defined for all $x \in \mathbb{R}$.)

1.3.2:

Find the domain and range of $f(x) = |2x - 3| + 5$.

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Solution: The function is defined for all $x \in \mathbb{R}$, so $\boxed{\mathrm{Dom}(f) = \mathbb{R}}$. The $|2x - 3|$ term can be any nonnegative real number, so $\boxed{\mathrm{Rng}(f) = [5, +\infty)}$.

1.3.3:

Can a function have domain $\emptyset$? Can a function have range $\emptyset$? Explain your answers.

Hint: There is only one function with range $\emptyset$—what is it?

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Solution: A function can have domain $\emptyset$, but then we cannot plug any value into it. So such a function must also have range $\emptyset$. A function can have range $\emptyset$ only if its domain is also $\emptyset$.

1.3.4:

Show that if $A \subseteq B$ and f is a function whose domain includes B, then $f(A) \subseteq f(B)$. If A is a proper subset, that is if $A \subset B$, must we have $f(A) \subset f(B)$?

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Solution: Let $y \in f(A)$. Then by definition $y = f(x)$ for some $x \in A$. But since $A \subseteq B$, we also have $x \in B$, so $y \in f(B)$. Since every element of $f(A)$ is also an element of $f(B)$, we conclude that $f(A) \subseteq f(B)$.

If the inclusion $A \subset B$ is proper, this does not necessarily mean that $f(A) \subset f(B)$. For a trivial example, consider the constant function $f(x) = 1$ for all $x \in \mathbb{R}$. Then $f(A) = \{1\}$ for all nonempty subsets $A \subseteq \mathbb{R}$, and hence $f(A) = f(B)$ for any nonempty subsets $A \subset B$.

1.3.5★:

If f is a function with inverse f^{-1}, and $y \in \mathrm{Rng}(f)$, what is the relationship between $f^{-1}(y)$ and $f^{-1}(\{y\})$?

Hint: Note that $f^{-1}(y)$ is a number but $f^{-1}(\{y\})$ is a set.

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Solution: The symbol $f^{-1}(y)$ denotes the unique element $x \in \mathrm{Dom}(f)$ such that $f(x) = y$. This element is unique since if $f(x') = y$,

$$x' = f^{-1}(f(x)) = x.$$

Likewise, $f^{-1}(\{y\})$ denotes the set of x such that $f(x) = y$. By above,

$$f^{-1}(\{y\}) = \{x\}.$$

Therefore $f^{-1}(y) \in f^{-1}(\{y\})$, and is the unique element thereof. We can write this as $\boxed{f^{-1}(\{y\}) = \{f^{-1}(y)\}}$.

1.4 Graphs of functions

You should already be familiar with graphing functions. For example, the picture to the right shows the graph of $y = x^2 - 2x - 3$. The investigation of geometric properties of graphs of functions is one of the cornerstones of calculus, and we will be dealing with graphs throughout the remainder of the book. In this section, we'll give a slightly more formal definition of a graph than you may be used to, and we'll review the important basics about graphs of linear functions. It's important to have these basics mastered because you'll need them to be almost automatic as we continue through the book.

Graphs are constructed on the **Cartesian plane**, as in our picture to the right. The Cartesian plane is the set of all ordered pairs of real numbers; in other words, it is

$$\{(x, y) \mid x, y \in \mathbb{R}\}.$$

This is often denoted $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$.

You should know from your past mathematical experience that a point (x, y) is in the graph of a function $f(x)$ if and only if $y = f(x)$. In other words, the graph of f is the set of all points $(x, f(x))$ where $x \in \mathrm{Dom}(f)$. We'll use this as our formal definition of a graph:

> **Definition:** The **graph** of the real-valued function f is the set
>
> $$\{(x, f(x)) \mid x \in \mathrm{Dom}(f)\}.$$

This may seem like a strange definition: we've defined a graph to be a *set* rather than a geometric object. But sets and functions are the things that we can handle most rigorously. Of course, when we draw the picture of a graph in the Cartesian plane, we are identifying the graph with a geometric object. We'll use the word "graph" interchangeably to mean both the set of ordered pairs $(x, f(x))$ and the geometric object that results when we plot all of these order pairs on the Cartesian plane.

We'll start with the fundamental geometric property of graphs of functions:

Problem 1.14

Let f be a function. Explain why every vertical line in the plane intersects the graph of f in at most one point.

Solution for Problem 1.14: For any $a \in \mathbb{R}$, the vertical line $x = a$ in the plane is the subset $\{(a, y) \mid y \in \mathbb{R}\}$. However, there is at most one point in the graph of f with first coordinate a, and that is the point $(a, f(a))$. (This is just a restatement of the fact that a function produces a unique output for each input.) Therefore, the vertical line $x = a$ will intersect the graph of f in the point $(a, f(a)) \in \mathbb{R} \times \mathbb{R}$ if $a \in \mathrm{Dom}(f)$. Of course, if $a \notin \mathrm{Dom}(f)$, then there is no point in the graph of f with first coordinate a, and hence the line $x = a$ does not intersect the graph of f at all. □

This is called the **Vertical Line Test**:

> **Important:** The **Vertical Line Test**: for any function f, any vertical line (that is, any line given by $x = a$ for some constant $a \in \mathbb{R}$) must intersect the graph of f in at most 1 point. Conversely, if there is a subset S of the Cartesian plane for which there is some vertical line $x = a$ that intersects S in more than 1 point, then S cannot be the graph of a function.

The intuition of the converse portion of the Vertical Line Test should be clear: if some line $x = a$ intersects S in two distinct points (a, b) and (a, c), then there's no way S could be the graph of a function f, since we'd need $f(a) = b$ and $f(a) = c$, but $f(a)$ can only equal one number.

There's no such restriction about horizontal lines. For example, looking back at our parabola $y = x^2 - 2x - 3$, shown at right, there are lots of horizontal lines that intersect the graph in more than one point. However, graphs in which every horizontal line intersects the graph in at most one point have a very special property:

Problem 1.15

Let f be a function, and suppose that every horizontal line $y = b$ intersects the graph of f in at most one point. Show that f has an inverse f^{-1}.

Solution for Problem 1.15: Suppose the line $y = b$ intersects the graph of f at (a, b). We know that $b = f(a)$ by definition, since it lies on the graph of f, so our inverse for f will have to map b back to a. This tells us how to define the inverse.

Define a function g as follows: if the line $y = b$ intersects the graph of f at the point (a, b) then $g(b) = a$. Note that $\mathrm{Dom}(g) = \mathrm{Rng}(f)$. Since any such horizontal line will intersect the graph in at most one point, the function g is well-defined. By construction, we have $g(f(a)) = a$ for any $a \in \mathrm{Dom}(f)$. Furthermore, if $b \in \mathrm{Dom}(g)$, then the line $y = b$ intersects the graph of f at some point (a, b), so $f(g(b)) = f(a) = b$.

Thus $g = f^{-1}$. □

This leads to:

> **Important:** The **Horizontal Line Test**: given a function f, if every horizontal line (that is, any line given by $y = b$ for some constant $b \in \mathbb{R}$) intersects the graph of f in at most 1 point, then f has an inverse f^{-1}. Conversely, if there is some horizontal line $y = b$ that intersects the graph of f in more than one point, then f does not have an inverse.

We have not proved the converse part of the Horizontal Line Test–we will leave this as an exercise.

The most basic and important graphs in calculus are lines. You should already know how to work with lines in the Cartesian plane. Specifically, the graph of $f(x) = mx + b$ is the line with slope m that passes through the point $(0, b)$. If $m = 0$, this is just the horizontal line $y = b$. Note that a vertical line $x = a$ is not the graph of a function, since it does not pass the Vertical Line Test.

Conversely, given a non-vertical line ℓ in the plane, we can select any two distinct points on the line, (x_1, y_1) and (x_2, y_2). Then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

and thus ℓ is the graph of $y = mx + b$, where $(0, b)$ is the y-intercept of the graph.

The last important graph skill that we'll cover in this section is the ability to identify and draw "related" graphs of a function. The following exercise is typical:

Problem 1.16

Shown at right is the graph of a function f. Sketch the graphs of the following related functions (don't worry about exact accuracy or scale, just try to show the general behavior):

(a) $g(x) = f(x + 1)$

(b) $g(x) = f(x) + 1$

(c) $g(x) = f(-x)$

(d) $g(x) = f(2x)$

(e) $g(x) = 2f(x)$

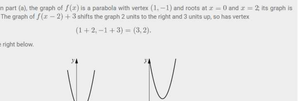
Solution for Problem 1.16: In all of the pictures in this solution, the original graph of f is shown as a dotted curve, and the graph of g is shown as a darker, solid curve.

(a) Note that if (x, y) is on the graph of f, then $(x - 1, y)$ is on the graph of g, since

$$g(x - 1) = f((x - 1) + 1) = f(x) = y.$$

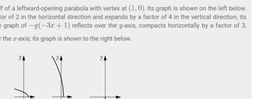
In other words, each point on the graph of g is 1 unit to the left of the corresponding point on the graph of f. Thus, the graph of g is just the graph of f shifted 1 unit to the left, as in picture (a) below:

(b) If (x, y) is on the graph of the f, then $(x, y + 1)$ is on the graph of g, since $g(x) = f(x) + 1 = y + 1$. That is, each point on the graph of g is 1 unit above the corresponding point on the graph of f. Thus, the graph of g is just the graph of f shifted 1 unit upwards, as in picture (b) below.



(c) If (x, y) is on the graph of f, then $(-x, y)$ is on the graph of g, since $g(-x) = f(x) = y$. Thus, every point (x, y) on the graph of f is reflected over the y-axis to the point $(-x, y)$ on the graph of g. Thus, the graph of g is the graph of f reflected over the y-axis, as in picture (c) above.

(d) If (x, y) is on the graph of f, then $(x/2, y)$ is on the graph of g, since $g(x/2) = f(x) = y$. So each point (x, y) on the graph of f gets contracted horizontally by a factor of 2 towards the y-axis to the point $(x/2, y)$ on the graph of g. Thus, the graph of g is the graph of f contracted horizontally by a factor of 2, as in picture (d) below.



(e) If (x, y) is on the graph of f, then $(x, 2y)$ is on the graph of g, since $g(x) = 2f(x) = 2y$. Thus, each point (x, y) on the graph of f gets expanded vertically by a factor of 2 away from the x-axis to the point $(x, 2y)$ on the graph of g. Therefore, the graph of g is the graph of f expanded vertically by a factor of 2, as in picture (e) above.

□

Exercises

1.4.1:

Prove that if f is a function and there is a horizontal line $y = b$ that intersects the graph of f in more than one point, then f cannot have an inverse. (This is the "converse" part of the Horizontal Line Test.)

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Solution: If such a line exists, suppose it intersects the graph of f at points (c, b) and (d, b) with $c \neq d$. Then, since $f(c) = b$, we would require $f^{-1}(b) = c$, but since $f(d) = b$, we would require $f^{-1}(b) = d$. Since $f^{-1}(b)$ must be a unique value, we conclude that f^{-1} cannot exist, and thus f does not have an inverse.

1.4.2:

Suppose that $f(x) = x^2 - 2x$ and $g(x) = \sqrt{1 - x}$.

(a) Find the domain and range of f and g.

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Solution: We see immediately that $\mathrm{Dom}(f) = \mathbb{R}$. To compute the range of f, notice that the graph of f is a parabola opening upward with roots 0 and 2, so its vertex is at $x = 1$, where $f(1) = -1$. Thus the range of f is $[-1, +\infty)$.

The function g is defined for $1 - x \geq 0$, or $1 \geq x$, so its domain is $(-\infty, 1]$. Since the square root can be any nonnegative real number, we have $\mathrm{Rng}(g) = [0, +\infty)$.

(b) Find $(f \circ g)(x)$, $(g \circ f)(x)$, and their domains and ranges.

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Solution: We compute that

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{1 - x}) \\ &= (\sqrt{1 - x})^2 - 2\sqrt{1 - x} \\ &= 1 - x - 2\sqrt{1 - x}.\end{aligned}$$

Since $\mathrm{Dom}(f) = \mathbb{R}$, the function $(f \circ g)$ is defined wherever g is defined, so $\mathrm{Dom}(f \circ g) = (-\infty, 1]$. Furthermore,

$$\begin{aligned}\mathrm{Rng}(f \circ g) &= f(\mathrm{Rng}(g)) \\ &= f([0, +\infty)) \\ &= \mathrm{Rng}(f) \\ &= [-1, +\infty),\end{aligned}$$

since 0 is to the left of the vertex of f.

Also,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2 - 2x) \\ &= \sqrt{1 - x^2 + 2x}.\end{aligned}$$

For this to be defined, we must have $1 - x^2 + 2x \geq 0$ or $x^2 - 2x - 1 \leq 0$. This quadratic has roots $1 \pm \sqrt{2}$, so the domain of $g \circ f$ is $[1 - \sqrt{2}, 1 + \sqrt{2}]$. Also, the quadratic $-x^2 + 2x + 1$ has its maximum at its vertex $(1, 2)$, so the range of $g \circ f$ is $[0, \sqrt{2}]$.

(c) Graph $f(x)$ and $f(x - 2) + 3$.

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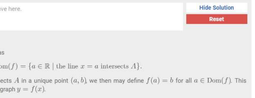
Solution: As discussed in part (a), the graph of $f(x)$ is a parabola with vertex $(1, -1)$ and roots at $x = 0$ and $x = 2$; its graph is shown to the left below. The graph of $f(x - 2) + 3$ shifts the graph 2 units to the right and 3 units up, so has vertex

$$(1 + 2, -1 + 3) = (3, 2).$$

Its graph is shown to the right below.



(d) Graph $g(x)$, $4g(2x)$, and $-g(-3x + 1)$.

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Solution: The graph of $g(x)$ is the top half of a leftward-opening parabola with vertex at $(1, 0)$. Its graph is shown on the left below. The graph of $4g(2x)$ compacts by a factor of 2 in the horizontal direction and expands by a factor of 4 in the vertical direction; its graph is shown in the middle below. The graph of $-g(-3x + 1)$ reflects over the y-axis, compacts horizontally by a factor of 3, shifts left by $\frac{1}{3}$ unit, and then reflects over the x-axis; its graph is shown to the right below.

1.4.3:

Show that the converse of the Vertical Line Test is true: that is, if $A \subset \mathbb{R} \times \mathbb{R}$ satisfies the Vertical Line Test, then A must be the graph of some real-valued function f.

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Solution: We can define the domain of f as

$$\mathrm{Dom}(f) = \{a \in \mathbb{R} \mid \text{the line } x = a \text{ intersects } A\}.$$

Then, since each such line $x = a$ intersects A in a unique point (a, b), we then may define $f(a) = b$ for all $a \in \mathrm{Dom}(f)$. This defines our function f such that A is the graph $y = f(x)$.

1.3 Trigonometric functions

WARNING! Calculators and computers in Section 1.3 are set to **radians** by default. You should already have unpublished experiences with trigonometric functions attempting to "learn" trigonometry. Practice textbook contains a thorough review of trigonometry.

You are already familiar with the trigonometric functions in Section 1.2. If in an acute angle of a right triangle, then:

$$\sin\theta = \frac{\text{Length of side opposite } \theta}{\text{Length of hypotenuse}} = \frac{b}{c}$$

$$\cos\theta = \frac{\text{Length of side adjacent to } \theta}{\text{Length of hypotenuse}} = \frac{a}{c}$$

$$\tan\theta = \frac{\text{Length of side opposite } \theta}{\text{Length of side adjacent to } \theta} = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}$$

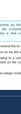
Because any two right triangles with the same angle are similar, the ratios of corresponding sides are equal. Therefore, the trigonometric functions depend only on an angle and are a particular choice of triangle.

In calculus (as with most of higher mathematics), we always measure our angles in **radians**. Radians are a much more natural system of measurement than degrees, which are a holdover from the Babylonian culture of the ancient Middle East.

Definition: Let C be a circle of radius r, and let θ be a central angle of C. Let s be the length of the arc in C that is cut out by θ. Then the measure of θ in **radians** is equal to the length of the arc that is subtended by θ.

In particular, we can compute the radian measure of some common angles:

- A right angle measures $\frac{\pi}{2}$ since it subtends one-quarter of the circle. The circumference of the circle is 2π, so the length of the subtended arc is $\frac{2\pi}{4} = \frac{\pi}{2}$.
- An isosceles right triangle has two acute angles each measure $\frac{\pi}{4}$.



It is easy to remember where corresponding radian measures lie on $[0, 2\pi]$:

$$0 \text{ radians} = 360 \text{ degrees}.$$

Do we can always easily convert between degrees and radians by setting up a ratio.

Problem 1.17

(a) Suppose angle β measures β degrees. What is its measure in radians?

(b) Suppose angle β measures β radians. What is its measure in degrees?

Solution for Problem 1.17

(a) Let the radian measure of β be r. Then we must have the ratio

$$\frac{r}{2\pi} = \frac{\beta}{360}$$

Solving for r, we see that $r = \frac{\beta\pi}{180}$.

(b) Let the degree measure of β be d. Then we must have the ratio

$$\frac{d}{360} = \frac{\beta}{2\pi}$$

Solving for d, we see that $d = \frac{360\beta}{2\pi}$.

WARNING! If you are using a calculator in your study of calculus, make sure that you put it in "radians" mode and not in "degrees" mode.

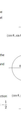
There are two basic right triangles that you should know. They are always at right with their angles labeled in radians. You should already know these triangles from your previous study of trigonometry. The first one is the half of a unit square, cut along the diagonal, and the second is the $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ triangle, that is half of an equilateral triangle of side length 1 unit. We can make the following chart of the trig functions for the acute angles in these triangles:

	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\sin$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\cos$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\tan$	1	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$

Example! You should learn the above table. It should be internalized to the extent that it is as basic as addition. You shouldn't have to think very hard about what $\sin\left(\frac{\pi}{6}\right)$ is. It should be automatic that this is $\frac{1}{2}$.

Using right triangles, we can define our trigonometric functions for any angle θ such that $0 \leq \theta \leq \frac{\pi}{2}$, since any acute angle measures between 0 and $\frac{\pi}{2}$ radians.

Moreover, we want also and we do to be functions with domains of all of $\mathbb{R}$. To extend their definitions beyond acute angles, we will use **unit circle**. The unit circle is the circle of radius 1 centered on the coordinate plane at the origin $(0, 0)$. We construct a right triangle with one leg along the x-axis and the hypotenuse of length 1 with one end at the origin. We place our angle θ between the x-axis and the hypotenuse. We then define the sine and cosine functions the lengths of the sides of the triangle, and we use from the picture that the point on the circle corresponding to the angle θ is $(\cos\theta, \sin\theta)$.



Problem 1.18

(a) Compute $\sin\left(\frac{\pi}{2}\right)$ and $\cos\left(\frac{\pi}{2}\right)$.

(b) Compute $\sin(\pi)$ and $\cos(\pi)$.

(c) Compute $\sin\left(\frac{3\pi}{2}\right)$ and $\cos\left(\frac{3\pi}{2}\right)$.

Solution for Problem 1.18

(a) We draw an angle of $\theta = \frac{\pi}{2}$ right. We see that this puts us in the third quadrant, so both the cosine and the sine will be negative. We can now essentially read the picture that $\sin\left(\frac{\pi}{2}\right) = -1$ and $\cos\left(\frac{\pi}{2}\right) = 0$.

(b) We draw an angle of $\theta = \frac{\pi}{2}$ right. We see that this puts us in the fourth quadrant, so the cosine will be positive and the sine will be negative. We have that $\sin(\pi) = 0$ and $\cos(\pi) = -1$.

(c) We draw an angle of $\theta = \frac{3\pi}{2}$ right ensure that this angle goes in the clockwise direction starting from the positive x-axis, because the angle is negative. We have that $\sin\left(\frac{3\pi}{2}\right) = -1$ and $\cos\left(\frac{3\pi}{2}\right) = 0$.

We define tangent for (although very angle) in terms of sine and cosine. Specifically, for any angle θ such that $\cos\theta \neq 0$ we define

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

It is not $\theta = 0$ does true if θ is undefined.

Problem 1.19

(a) Compute $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$.

(b) Compute $\tan\left(\frac{\pi}{4}\right)$ and $\tan\left(\frac{3\pi}{4}\right)$.

Solution for Problem 1.19

(a) An angle of θ means that we don't know of the position x axis at all. Moreover the point on the unit circle corresponding to an angle of $\theta = \frac{\pi}{4}$ is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. We know that the positive x axis. Thus, $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Also, we have $\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = 1$.

(b) We have $\frac{\pi}{4}$ implies that we know one angle of θ is $\frac{\pi}{4}$. Thus $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan\left(\frac{3\pi}{4}\right) = -1$. Then, we have the point corresponding to an angle of $\frac{3\pi}{4}$ radians is the point $(0, 1)$. Thus $\cos\left(\frac{3\pi}{4}\right) = 0$ and $\sin\left(\frac{3\pi}{4}\right) = 1$. Then, we have $\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)}$ is undefined. So $\tan\left(\frac{3\pi}{4}\right)$ is undefined.

Problem 1.20

(a) Compute $\sin(\pi)$ and $\cos(\pi)$.

(b) Compute $\tan(\pi)$ and $\tan(2\pi)$.

Solution for Problem 1.20

(a) An angle of θ means that we don't know of the position x axis at all. Moreover the point on the unit circle corresponding to an angle of $\theta = \pi$ is $(-1, 0)$. We know that the positive x axis. Thus, $\cos(\pi) = -1$ and $\sin(\pi) = 0$. Also, we have $\tan(\pi) = \frac{\sin(\pi)}{\cos(\pi)} = 0$.

(b) We have $\tan(\pi)$ implies that we know one angle of θ is π. Thus $\tan(\pi) = 0$. The point corresponding to an angle of 2π radians is the point $(0, 1)$. Thus $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$. Then, we have $\tan\left(\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)}$ is undefined. So $\tan\left(\frac{\pi}{2}\right)$ is undefined.

Problem 1.21

Let f be any angle. Show $\sin(\pi + f) = -\sin(\pi - f)$.

Solution for Problem 1.21. Adding π to an angle θ corresponds to making a complete revolution around the unit circle, since the entire circle has angle 2π. So if and $\theta = 2\pi$ give the same point on the unit circle, and thus $\sin(\theta + 2\pi) = \sin\theta$ for any angle θ.

Of course, the same argument shows that $\sin(\theta - 2\pi) = \sin\theta$ and $\cos(\theta + 2\pi) = \cos\theta$. The sum of of situation comes up frequently in mathematics, so we have some terminology for it.

Definition: Let f be a function. We say that f **periodic** if there exists a positive real number λ such that $f(x + \lambda) = f(x)$ for all x. The number λ is called the **period** of f. The smallest such λ if it exists is called the **period** of f.

In light of Problem 1.21 we can say that the sine and cosine functions are periodic with period 2π. On fact, tangent is periodic with a smaller period we will leave this fact as an exercise (I basically this means that the graphs of $y = \sin\theta$ and $y = \cos\theta$ repeat every 2π units. Thus, if we know the graph of $y = \sin\theta$ on $[0, 2\pi]$ we know the graph of $y = \sin\theta$ everywhere. The graph of $\sin\theta$ is a multiple of $\pi/2$ in $\pi/2$. Pitching these and then Connecting the dots" gives us a graph of $\sin\theta$ on $[0, 2\pi]$.

You probably notice that the graph of cosine appears to be the graph of sine shifted $\frac{\pi}{2}$ units to the left. In fact, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ for all x. You'll be asked to explore this relationship in the homework.

There are three other trigonometric functions that are commonly used. They are simply the reciprocals of the functions that we already know.

$$\text{secant:} \quad \sec\theta = \frac{1}{\cos\theta}$$

$$\text{cosecant:} \quad \csc\theta = \frac{1}{\sin\theta}$$

$$\text{cotangent:} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Notice that $\sec\theta = \frac{1}{\cos\theta}$ where both sides are defined. However, we have $\tan\left(\frac{\pi}{2}\right) = 0$ whereas $\sec\left(\frac{\pi}{2}\right)$ is undefined.

WARNING! Note that cosecant is the reciprocal of sine and secant is the reciprocal of cosine.

We also need to discuss the inverses of the trig functions. Of course, they do not have inverses as we defined "inverse function" in Section 1.2 as their graphs pass the horizontal line test. For example, $\sin(0) = \sin(\pi) = 0$.

$$\sin(\theta) = \sin(\pi - \theta) = 0 \text{ for } \theta \in [0, \pi]$$

So we can't positively determine $\sin^{-1}(0)$ because we don't know if we should find $\theta = 0$ that will work and "baffled, sky." by restricting the domain of the sine function to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we can define an inverse function.

Definition: The **inverse sine function**, denoted $\sin^{-1}$ or $\arcsin$, is the function with domain $[-1, 1]$ defined by $\sin^{-1}(x) = \theta$ for all $x \in [-1, 1]$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Thus, $\sin^{-1}$ is the function that takes a number x from $[-1, 1]$ and returns the number $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is that value. Note that this is not technically an inverse of sine, however, it is the inverse of the function we get by restricting the domain of the sine function to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

We can define the inverse cosine function too, but we need to make a slight modification.

Problem 1.22

Define $\cos^{-1}$ **to a similar fashion as we defined** $\sin^{-1}$. **How do the domain and range differ?**

Solution for Problem 1.22. We want $\cos^{-1}$ to have the same domain $[-1, 1]$, but we want the range of cosine is $[-1, 1]$. However the range of $\sin^{-1}$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Since cosine only takes nonnegative values if we restrict the domain to $\theta \in [0, \pi]$ for $\theta \in [0, \pi]$. We need the range of $\cos^{-1}$ to be an interval on which cosine takes all values $[-1, 1]$. The most obvious choice is the interval $[0, \pi]$ so we have our definition:

Definition: The **inverse cosine function**, denoted $\cos^{-1}$ or $\arccos$, is the function with domain $[-1, 1]$ defined by $\cos^{-1}(x) = \theta$ for all $x \in [-1, 1]$ and $\theta \in [0, \pi]$.

WARNING! Most calculus books use the two inverse trig relations interchangeably. In this book, we will frequently switch between $\sin^{-1}$ and $\arcsin$ to get you used to seeing both notations.

Naturally, we also have all inverse tangent function, with the appropriate restrictions to the domain and range.

Definition: The **inverse tangent function**, denoted $\tan^{-1}$ or $\arctan$, is the function with domain $\mathbb{R}$ that satisfies $\tan^{-1}(x) = \theta$ for all $x \in \mathbb{R}$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Exercises

1.5.1.

Determine the domains and ranges of $\sin x$, $\cos x$, and $\tan x$.

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Solution: Recall that $\sin x = \frac{y}{r}$. Therefore the domain of sound is all x for which $\cos x \neq 0$.

$$\text{Dom}(\cos) = \{x \mid \cos x \neq 0\} = \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\right\}.$$

Likewise, since $\cos x = \frac{x}{r}$, we have $\text{Dom}(\sin) = \mathbb{R}$. Finally, the tangent function is defined whenever cosine is nonzero. Therefore

$$\text{Dom}(\tan) = \{\text{Dom}(\sin) \cap \{x \mid \cos x \neq 0\}\} = \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$$

Since $\sin x$ and $\cos x$ have range $[-1, 1]$ we have

$$\text{Range}(\sin) = \text{Range}(\cos) = [-1, -1] = [-1, 1].$$

Finally, $\tan x$ has range $\mathbb{R}$, as $\text{Range}(\tan) = \mathbb{R} = [-\infty, +\infty]$.

1.5.2.

(a) Write $\sin(-\theta)$ in terms of $\sin\theta$.

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Solution: As in part (a), $\sin(-\theta)$ is the coordinate of the point on the unit circle that is the reflection of the point at θ through the x-axis. Reflecting through the x-axis changes the sign of the y-coordinate, so $\sin(-\theta) = -\sin(\theta)$.

(b) Write $\cos(-\theta)$ in terms of $\cos\theta$.

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Solution: As in part (a), $\cos(-\theta)$ is the coordinate of the point on the unit circle that is the reflection of the point at θ through the x-axis. Reflecting through the x-axis does not affect the x-coordinate, so $\cos(-\theta) = \cos(\theta)$.

1.5.3.

Show that $\sin(\theta + \pi) = -\sin\theta$ for all θ. (Each that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so the sign changes θ.

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Solution: For all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the sine function is a counterclockwise rotation of $\frac{\pi}{2}$ so we know the right triangle with one leg on the x-axis and hypotenuse the segment from the origin to the terminal point for each of θ and $\theta + \frac{\pi}{2}$ we get the following figures.

Notice that since $\frac{\pi}{2}$ is a right angle, this rotation gives congruent triangles. The terminal point of θ is $(\cos\theta, \sin\theta)$. By congruence, we can read the lengths of sides of the triangle associated to $\theta + \frac{\pi}{2}$ as $-\sin\theta$ and $\cos\theta$. Therefore the y and x coordinates of the terminal point $(\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2}))$ are $(-\sin\theta, \cos\theta)$.

Next we need to determine the sign of these values. If θ is in the first quadrant then $\theta + \frac{\pi}{2}$ is in the second quadrant. In the first quadrant, the sine and cosine are positive and in the second, only the cosine is negative. Therefore θ is in the first quadrant, then

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta, \quad \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

Considering the cases where θ is a different quadrant shows that the same equalities hold for all θ.

1.5.5.

Find the following quantities in terms of $\sin\theta$ or $\cos\theta$.

(a) $\sin\left(\frac{\pi}{2} - \theta\right)$

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$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \quad \text{and} \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

(b) $\sin(\pi + \theta)$

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$$\sin(\pi + \theta) = \sin\theta \quad \text{and} \quad \cos(\pi + \theta) = -\cos\theta$$

(c) $\cos(\pi + \theta)$

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$$\cos(\pi + \theta) = \cos\theta \quad \text{and} \quad \sin(\pi + \theta) = -\sin\theta$$

1.5.6.

Prove that $\sin\theta$ and $\cos\theta$ are not periodic smaller than 2π. (Hint: Use the unit circle.)

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Solution: Let λ be the angle $\theta \in [0, \pi]$ such that $\cos\theta = 0$. Therefore $\cos\lambda = 0$.

(b) $\sin^{-1}$

Your Submission: Solution

Solution: Let λ be the angle $\theta \in [0, \pi]$ such that $\sin\theta = 0$. Therefore $\sin\lambda = 0$.

(c) $\cos^{-1}$

Your Submission: Solution

Solution: Let λ be the angle $\theta \in [0, \pi]$ such that $\cos\theta = 0$. Therefore $\cos\lambda = 0$.

(d) $\tan^{-1}$

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Solution: Let λ be the angle $\theta \in [0, \pi]$ such that $\tan\theta = 0$. Therefore $\tan\lambda = 0$.

1.5.7.

Compute the following:

(a) $\cos^{-1}\left(\frac{\pi}{2}\right)$

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Solution: $\cos^{-1}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ because $\cos\left(\frac{\pi}{2}\right) = 0$. Therefore $\cos^{-1}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$.

(b) $\sin^{-1}\left(\frac{\pi}{2}\right)$

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Solution: $\sin^{-1}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ because $\sin\left(\frac{\pi}{2}\right) = 0$. Therefore $\sin^{-1}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$.

(c) $\cos^{-1}\left(-\frac{\pi}{2}\right)$

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Solution: $\cos^{-1}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$ because $\cos\left(-\frac{\pi}{2}\right) = 0$. Therefore $\cos^{-1}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$.

(d) $\tan^{-1}\left(-\frac{\pi}{2}\right)$

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Solution: $\tan^{-1}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$ because $\tan\left(-\frac{\pi}{2}\right) = 0$. Therefore $\tan^{-1}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$.

1.5.8.

Find an angle θ such that $\cos\theta = \sin\theta$ and $\sin\theta = \cos\theta$. (Hint: $\sin\theta = \cos\theta$.)

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Solution: We note that the range of $\cos^{-1}$ is $\cos\theta \in [0, \pi]$ while the range of $\sin^{-1}$ is $\sin\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus, for any θ we can have $\cos^{-1}(\cos\theta) = \theta$ but we cannot have $\sin^{-1}(\sin\theta) = \theta$. Equal. Therefore, $\sin^{-1}(\sin\theta) = \theta$ if and only if $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\sin^{-1}(\sin\theta) = \sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta.$$

1.6 Basic trigonometric identities

In this section we'll prove some basic identities for the trig functions. The first identity is the most fundamental and also the most useful.

Problem 1.23

Prove that, for any $\theta \in \mathbb{R}$,

$$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

Solution for Problem 1.23: We recall that, for any $\theta \in \mathbb{R}$, the point $(\cos\theta, \sin\theta)$ is, by definition, on the circle centered at $(0,0)$ with radius 1. But of course any point (x, y) on that circle satisfies the equation $x^2 + y^2 = 1$, and that proves our identity! □

By convention, we usually write powers of trig functions with a notational shorthand: $\sin^2\theta$ denotes $(\sin\theta)^2$ and similarly $\cos^2\theta$ denotes $(\cos\theta)^2$. Thus, our fundamental identity becomes simply:

> **Important:** For any $\theta \in \mathbb{R}$,
>
> $$\sin^2\theta + \cos^2\theta = 1.$$

It's hard to overstate how fundamental this identity is to calculus, and we'll see it reappear time and time again throughout the rest of the book.

Next, we have the angle addition and subtraction formulas. These are a bit difficult to prove (see Section 1.A for some further discussion), so we will just present them without proof:

> **Important:** For any $\alpha, \beta \in \mathbb{R}$,
>
> $$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}$$

While these formulas in their general form are not especially useful for calculus, some specific applications of them are very important. In particular:

Problem 1.24

Find simple formulas for $\sin 2\theta$ and $\cos 2\theta$ in terms of $\sin\theta$ and/or $\cos\theta$.

Solution for Problem 1.24: We can just use our angle-addition formulas above. Specifically,

$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.$$

Similarly,

$$\cos 2\theta = \cos(\theta+\theta) = \cos^2\theta - \sin^2\theta.$$

The latter formula is sometimes simplified using $\sin^2\theta + \cos^2\theta = 1$ to write it only in terms of $\sin$ or of $\cos$:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

□

The formulas from Problem 1.24 are called the **double-angle formulas**:

> **Important:** Double-angle formulas: for any $\theta \in \mathbb{R}$,
>
> $$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta,\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta.\end{aligned}$$

As an exercise, you'll be asked to find formulas for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$. Not surprisingly, these are called the **half-angle formulas**.

The formula in the next problem is also commonly used in calculus:

Problem 1.25

Prove that $\tan^2\theta + 1 = \sec^2\theta$ for all $\theta \in \mathbb{R}$ where both sides are defined.

Solution for Problem 1.25: Our first step is a common trig-problem strategy:

> **Concept:** When dealing with a trigonometric expression, it is often helpful to write everything in terms of sines and cosines.

We write the left side of our desired identity as

$$\tan^2\theta + 1 = \frac{\sin^2\theta}{\cos^2\theta} + 1.$$

Putting this over a common denominator gives us

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}.$$

But the numerator is just 1–remember, this is our most basic trig identity (from Problem 1.23). Thus, our expression is just $\frac{1}{\cos^2\theta}$, which by definition is $\sec^2\theta$. □

Finally, let's see how some of these identities work in a nontrivial trig problem.

Problem 1.26 *Source: AIME*

Given that $t \in \mathbb{R}$ satisfies $(1+\sin t)(1+\cos t) = \frac{5}{4}$, compute $(1-\sin t)(1-\cos t)$.

Solution for Problem 1.26: It's not immediately clear how to start, so we can begin by assigning a variable to what we want to compute.

> **Concept:** In general, when trying to evaluate a complicated expression, set some variable equal to the complicated expression. Then try to combine this new equation with the given data to solve for the variable.

We let $x = (1-\sin t)(1-\cos t)$ be the expression that we want. This gives us a system of equations:

$$\begin{aligned}(1+\sin t)(1+\cos t) &= \frac{5}{4},\\ (1-\sin t)(1-\cos t) &= x.\end{aligned}$$

There doesn't seem to be any good choice other than to multiply out the two left sides.

$$\begin{aligned}1 + \sin t\cos t + \cos t + \sin t &= \frac{5}{4},\\ 1 + \sin t\cos t - \cos t - \sin t &= x.\end{aligned}$$

We notice that all of the left side terms are the same (but some with different signs), so adding and subtracting the equations will make some terms cancel. In particular, adding them gives

$$2 + 2\sin t\cos t = \frac{5}{4} + x, \tag{1}$$

and subtracting them gives

$$2\cos t + 2\sin t = \frac{5}{4} - x. \tag{2}$$

You might recognize the double-angle formula for sine in equation (1), and try to replace $2\sin t\cos t = \sin 2t$. But that doesn't really get us closer to our goal of solving for x. We want to make all the t terms cancel. Instead, a better idea is to square equation (2), because that will create $\sin^2 t$ and $\cos^2 t$ terms that we should be able to eliminate completely. Squaring (2) gives us

$$4\cos^2 t + 8\sin t\cos t + 4\sin^2 t = \frac{25}{16} - \frac{5}{2}x + x^2. \tag{3}$$

This is good, because $4\cos^2 t + 4\sin^2 t = 4$. So (3) becomes

$$4 + 8\sin t\cos t = \frac{25}{16} - \frac{5}{2}x + x^2. \tag{4}$$

Aha! Multiplying (1) by 4 and subtracting from (4) will cancel all the t terms! We get

$$4 - 8 = \left(\frac{25}{16} - \frac{5}{2}x + x^2\right) - 4\left(\frac{5}{4} + x\right).$$

This is just a quadratic in x, which after simplification is

$$x^2 - \frac{13}{2}x + \frac{9}{16} = 0,$$

so by the quadratic formula the solutions to the quadratic are $x = \frac{13}{4} \pm \sqrt{10}$. But which solution do we want?

> **Concept:** Sine and cosine have ranges $[-1, 1]$, and these ranges often put bounds on possible values of expressions defined in terms of sine and cosine.

We recall that we defined $x = (1-\sin t)(1-\cos t)$. Each term on the right side has value in the interval $[0, 2]$, so we know that $x \in [0, 4]$. However, $\frac{13}{4} + \sqrt{10} > 3 + 3 = 6$ is way too big. So we must have $x = \frac{13}{4} - \sqrt{10}$ as our answer. □

Exercises

1.6.1:

Compute $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$.

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Solution: By the double-angle formula for cosine, we have

$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) = 1 - 2\sin^2\left(\frac{\pi}{12}\right).$$

Since $\frac{\pi}{12}$ is in the first quadrant, we are looking for the positive square root of $\sin^2$. Thus

$$\boxed{\sin\frac{\pi}{12} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}}$$

Similarly,

$$\frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) = 2\cos^2\left(\frac{\pi}{12}\right) - 1.$$

Thus, since $\cos\frac{\pi}{12}$ is positive,

$$\boxed{\cos\frac{\pi}{12} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}}$$

Notice that

$$(\sqrt{6} \pm \sqrt{2})^2 = (8 \pm 2\sqrt{12}) = 4(2 \pm \sqrt{3}),$$

so we can also write

$$\boxed{\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}, \quad \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

1.6.2:

Find formulas for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ in terms of $\sin\theta$ and/or $\cos\theta$.

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Solution: We notice that the double-angle identity

$$\cos(2x) = 2\cos^2 x - 1$$

is an expression containing only an angle and its double. Substituting $x = \theta/2$ we get

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}.$$

We take the square root to get:

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}}$$

The sign is determined by locating the quadrant of $\frac{\theta}{2}$. For example, if $-\pi < \theta < \pi$, then $\cos\frac{\theta}{2} > 0$.

An alternative double-angle identity for cosine is

$$\cos(2x) = 1 - 2\sin^2 x.$$

Again we substitute $x = \frac{\theta}{2}$ to find

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

or

$$\boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}},$$

where again the sign is determined by the quadrant of $\frac{\theta}{2}$.

1.6.3:

Find a formula for $\tan(\alpha+\beta)$ in terms of $\tan\alpha$ and $\tan\beta$.

Hint: Write $\tan x$ as $\frac{\sin x}{\cos x}$.

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Solution: We already know the angle-addition formulas for sine and cosine, so we rewrite tangent in terms of these and apply the angle addition identities:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\end{aligned}$$

To get these in terms of tangent, we then divide the numerator and denominator by $\cos\alpha\cos\beta$. Since cosine is nonzero anywhere tangent is defined, we will get an expression that is equivalent to the expression above whenever $\tan\alpha$ and $\tan\beta$ are defined.

$$\tan(\alpha+\beta) = \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \boxed{\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

Reconsidering the steps above, we see that if the denominator is zero, we get that $\cos(\alpha+\beta) = 0$, so $\tan(\alpha+\beta)$ is undefined. Therefore the equation we found is true whenever $\tan\alpha$, $\tan\beta$, and $\tan(\alpha+\beta)$ are all defined.

1.6.4:

Find formulas for $\tan 2\theta$ and $\tan\frac{\theta}{2}$.

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Solution: The double-angle formula for tangent follows from the previous exercise:

$$\tan 2\theta = \tan(\theta+\theta) = \boxed{\frac{2\tan\theta}{1-\tan^2\theta}}$$

To find $\tan\frac{\theta}{2}$ we recall the double-angle formulas for sine and cosine: $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 2\cos^2 x - 1$. Letting $x = \frac{\theta}{2}$ in these formulas and applying some algebra, we have:

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \boxed{\frac{\sin\theta}{\cos\theta+1}}$$

This identity holds when $\cos(\theta) \neq -1$, which is exactly when $\tan\frac{\theta}{2}$ is defined.

Notice that we can use

$$\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)(1+\cos\theta)$$

to simplify the denominator, which gives an alternative version of the half-angle formula for tangent:

$$\begin{aligned}\tan\frac{\theta}{2} &= \frac{\sin\theta}{1+\cos\theta}\\ &= \frac{(\sin\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\\ &= \frac{(\sin\theta)(1-\cos\theta)}{\sin^2\theta}\\ &= \boxed{\frac{1-\cos\theta}{\sin\theta}}\end{aligned}$$

In deriving this identity, we have assumed that $\cos\theta \neq \pm 1$, so the identity holds for those values. In fact the right hand side is undefined for both of these values, but the left hand side is defined (and zero) when $\cos\theta = 1$.

1.7 Exponentials and logarithms

At this point, we have several classes of functions to work with: polynomial functions, square roots, and trigonometric functions. But there's one more very important class of functions that we need.

From your algebra and precalculus courses, you are familiar with exponentials and their inverses, logarithms:

$$a^r = b \quad \Leftrightarrow \quad r = \log_a(b).$$

For calculus, we will need a deeper understanding of these sorts of functions. In order to gain this understanding, let's suppose that we were trying to construct an exponential function f from scratch. What properties should it have? We can try to answer this question by looking at a common exponential function that we should be very familiar with:

$$f(x) = 2^x.$$

What properties does this have?

Problem 1.27

Let $f(x) = 2^x$.

(a) What is $f(0)$? What is $f(1)$?

(b) Write $f(a+b)$ in terms of $f(a)$ and $f(b)$.

(c) Write $f(-a)$ in terms of $f(a)$.

(d) Sketch a graph of f. What are the domain and range of f?

Solution for Problem 1.27:

(a) $f(0) = 2^0 = 1$ and $f(1) = 2^1 = 2$.

(b) This is a basic property of exponentials:

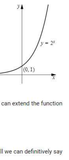
$$f(a+b) = 2^{a+b} = 2^a 2^b = f(a)f(b).$$

(c) We have

$$f(-a) = 2^{-a} = \frac{1}{2^a} = \frac{1}{f(a)} = f(a)^{-1}.$$

We can also use the fact from part (b). We know that $f(a+(-a)) = f(0) = 1$, but on the other hand $f(a+(-a)) = f(a)f(-a)$. So $f(a)f(-a) = 1$, and thus $f(-a) = 1/f(a)$.

(d) We sketch the graph of $y = 2^x$ at right. Note that from part (a), we know that the graph must pass through the points $(0,1)$ and $(1,2)$.



This graph seems to imply that the domain of f is all of $\mathbb{R}$ and that the range of f is the positive real numbers, but are these statements really true? Certainly there's no problem describing 2^x if x is a positive integer:

$$2^n = \underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ times}}.$$

Further, if x is a negative integer we have $2^x = \frac{1}{2^{-x}}$, and we know $2^0 = 1$, so 2^x is well-defined for any integer x. We can extend the function to reciprocals of nonzero integers by defining $2^{\frac{1}{b}}$ (for any nonzero integer n) to be the unique positive real number b such that $b^n = 2$. (The fact that such a number must exist and is unique is nontrivial to prove at this stage, so we will just assume that it exists.) Going further, we can extend the function to all rational numbers by letting

$$2^{\frac{m}{n}} = (2^m)^{\frac{1}{n}},$$

for all integers m and positive integers n. But what, for example, is 2^{π}? How do we define this?

So, at this point, although we may believe that the domain of 2^x is $\mathbb{R}$ and the range is positive real numbers, all we can definitively say is that the domain of 2^x contains $\mathbb{Q}$.

□

Notice that there's nothing special about the "2" in our above discussion—we could just have easily worked with $f(x) = a^x$ for any positive real number a. The issue is that although it's straightforward to define a^x when x is rational, it's not obvious what to do when x is irrational, and we'd like to have a function a^x whose domain is all of $\mathbb{R}$.

Let's focus on what should be the properties of the function a^x. To simplify things a little bit, let's assume that $a > 1$. Then we want to have:

- $f(x) = a^x$ has domain $\mathbb{R}$ and range $(0, +\infty)$.
- $f(x) = a^x$ is **strictly increasing**: if $m < n$ then $a^m < a^n$.
- $a^{m+n} = a^m a^n$ for any real numbers m and n.

Problem 1.28

How do we need to alter the above list if $a = 1$ or $0 < a < 1$?

Solution for Problem 1.28: If $a = 1$, then the function is just the constant function $f(x) = 1^x = 1$. So the range is just the point $\{1\}$, and the function is constant, not increasing.

If $a < 1$, then the function should be **strictly decreasing** instead of strictly increasing: that is, if $m < n$, then $a^m > a^n$. The domain is still $\mathbb{R}$ and the range is still $(0, +\infty)$. This also follows from the fact that $a^x = \frac{1}{a^{-x}} = \left(\frac{1}{a}\right)^x$ for all $x \in \mathbb{R}$, so the function a^x is decreasing if and only if $\left(\frac{1}{a}\right)^x$ is increasing.

In any case, the function should satisfy $a^{m+n} = a^m a^n$:

Using these properties, we can make an abstract definition of an "exponential" function:

> **Definition:** Let f be a strictly increasing, strictly decreasing, or constant function with domain $\mathbb{R}$ that takes positive values. We say that f is **exponential** if $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

We know that $a^0 = 1$ for any positive a, so we hope that our abstract exponential functions have this property too:

Problem 1.29

Suppose that f is an exponential function.

(a) Show that $f(0) = 1$.

(b) Let $a = f(1)$. Show that $f(n) = a^n$ for any integer n.

Solution for Problem 1.29: We use a common technique:

> **Concept:** When dealing with a functional equation, it often helps to plug in simple values to the equation, such as 0 or 1.

(a) Set $y = 0$ in the functional equation $f(x+y) = f(x)f(y)$. This gives us $f(x) = f(x)f(0)$. Since f takes positive values, $f(x)$ is nonzero, so we can divide by it to get $1 = f(0)$.

(b) We prove $f(n) = a^n$ for positive n by induction. Note that $f(1) = a^1 = a$ by definition, and if $f(k) = a^k$ for any positive integer k, then

$$f(k+1) = f(k)f(1) = (a^k)(a) = a^{k+1}.$$

Thus $f(n) = a^n$ for all positive integers n. By part (a), $f(0) = 1 = a^0$. Finally, for negative integers n, we have $f(n)f(-n) = f(0) = 1$, so

$$f(n) = \frac{1}{f(-n)} = \frac{1}{a^{-n}} = a^n.$$

□

We can extend Problem 1.29, by essentially repeating the arguments from Problem 1.27(d), to show that $f(x) = a^x$ for all *rational* x. However, this does not prove that $f(x) = a^x$ for values of x that are irrational. In fact, we haven't even defined a^x when x is irrational. But f, by definition, has domain $\mathbb{R}$, so what we do is define a^x to equal $f(x)$ for the exponential function f such that $f(1) = a$.

There is one more fundamental exponential property that we will need:

$$a^{rs} = (a^r)^s$$

for all $r, s \in \mathbb{R}$. Unfortunately, this is quite difficult to prove using our current notion of exponential, so we will have to take it on faith (for now) that this is true, and defer the proof until Section 5.A.

> **Concept:** The point of this discussion is that, for any positive a, we can consider $f(x) = a^x$ as a function whose domain is all of $\mathbb{R}$, and that this function has all of the nice properties that we expect exponentials to have.
>
> In Chapter 5, we will be able to define very precisely an exponential function, and prove all of these properties. However, we introduce the exponential function now, because we'd really like to be able to use it in examples throughout Chapters 2-4, even if we have to assume certain facts about the function that we cannot prove right now.

There is one very special exponential function that we use throughout calculus. In fact, it is so special that it is usually called the **exponential function**. It is the exponential with the special base e. The number e can be defined as

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

There are several issues with this "definition." First of all, it's not at all clear how to define rigorously the above infinite sum, but we will discuss such matters in Chapter 7. Second, even if we assume that the above number exists—and it does, it is approximately 2.71828—what's so special about e? Unfortunately, we can't say too much right now. You can see a little bit about why e is special in Section 1.A, but the real power of e will have to wait until Chapter 3.

> **Definition:** The **exponential function**, denoted $\exp(x)$, is the exponential with base e:
>
> $$\exp(x) = e^x,$$
>
> for all $x \in \mathbb{R}$.

Again, recall that this means that $\exp$ is the strictly increasing, positive-valued function satisfying $\exp(1) = e$ and $\exp(a+b) = \exp(a)\exp(b)$ for all $a, b \in \mathbb{R}$. We usually write it to look like an exponential, so we would write the above properties as $e^1 = e$ and $e^{a+b} = e^a e^b$.

Right now, this is mostly a tease. We'll see why $\exp$ is a really nice function in Chapter 3, but we won't be able to make the definition rigorous, or to prove facts about the exponential function, until Chapter 5.

Exponential functions have inverses, called **logarithms**. For example,

$$2^4 = 16 \quad \Leftrightarrow \quad \log_2 16 = 4.$$

More generally, if a is any positive real number and $x \in \mathbb{R}$, then

$$a^x = y \quad \Leftrightarrow \quad \log_a y = x.$$

And if we let a be our special number e, then we have

$$e^x = y \quad \Leftrightarrow \quad \log_e y = x.$$

More abstractly, we defined the exponential function to be a strictly increasing function with domain $\mathbb{R}$ and range $(0, +\infty)$. Because it is strictly increasing, it has an inverse (you will explore this in an exercise). This allows us to define our "natural" logarithm function.

> **Definition:** The **natural logarithm** function $\log : (0, +\infty) \to \mathbb{R}$ is the inverse of the exponential function; that is, for any $x \in (0, +\infty)$ and $y \in \mathbb{R}$, we have
>
> $$\log(x) = y \quad \Leftrightarrow \quad x = e^y.$$

Note that we think of $\log$ as being $\log_e$, that is, a logarithm with "base" e.

> **WARNING!!** Most calculus textbooks (and the AP Calculus Examination), and most calculators, use the notation $\ln$ for the natural logarithm function. But most mathematicians use $\log$, thus so shall we.

Let's verify that the $\log$ function has the properties that a good logarithm should have.

Problem 1.30

(a) What is $\log(1)$?

(b) Show that, if $a, b \in (0, +\infty)$, then $\log(ab) = \log(a) + \log(b)$.

(c) If $r \in \mathbb{R}$, then what is $\log(a^r)$?

Solution for Problem 1.30:

(a) Since $\exp(0) = 1$, we have $\log(1) = 0$.

(b) Let $A = \log(a)$ and $B = \log(b)$. Then $\exp(A) = a$ and $\exp(B) = b$, so

$$\exp(A+B) = (\exp A)(\exp B) = ab.$$

Therefore,

$$\log(ab) = A + B = \log(a) + \log(b).$$

(c) Let $A = \log(a)$. Then $\exp(A) = a$, so

$$\exp(rA) = e^{rA} = (e^A)^r = (\exp(A))^r = a^r.$$

Therefore,

$$\log(a^r) = rA = r\log(a).$$

□

So our $\log$ function behaves like a logarithm should.

Exercises

1.7.1:

For practice, simplify the following. (The goal is for these computations to become nearly automatic.)

(a) $\log(e^5)$

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Solution: $\log(e^5) = \boxed{5}$ since the logarithm and exponential are inverses.

(b) $\log(\sqrt{e})$

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Solution: $\log(\sqrt{e}) = \log(e^{\frac{1}{2}}) = \boxed{\frac{1}{2}}$

(c) $e^{3\log 9}$

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Solution: $e^{3\log 9} = e^{\log(9^3)} = e^{\log 9} = \boxed{9}$

(d) $(e^{-(\log 4)})^2$

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Solution: $(e^{-(\log 4)})^2 = (e^{\log(4^{-1})})^2 = (4^{-1})^2 = \boxed{\frac{1}{16}}$

1.7.2:

(a) Write $\log_a(7)$ as an expression in terms of the natural log function.

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Solution: Note that $\log_a(7)$ is the solution to the equation $3^x = 7$. If we take the natural logarithm of both sides we get $y \log 3 = \log 7$. We may divide by $\log 3$ to solve for y, giving $\boxed{\log_a 7 = \frac{\log 7}{\log 3}}$

(b) Assume a and b are positive and $a \neq 1$. How do we write $\log_a b$ in terms of natural log?

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Solution: We start with $a^{\log_a b} = b$, so taking the log of both sides gives

$$\log_a b \log a = \log b.$$

Dividing by $\log a \neq 0$ we get $\boxed{\log_a b = \frac{\log b}{\log a}}$

1.7.3:

Explain why if f is a strictly increasing function, then f must have an inverse function.

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Solution: We know that any function which passes the Horizontal Line Test has an inverse. Let b be any real number. We want to show that the graph of $f(x)$ intersects the graph of $y = b$ at most once. Assume that these graphs intersect (otherwise we're done). Then there is some $a \in \mathbb{R}$ with $f(a) = b$. Since f is strictly increasing, if $x < a$, then $f(x) < f(a)$ and if $x > a$, then $f(x) > f(a)$. These show that if $x \neq a$, then $f(x) \neq b$. Thus the graph of f intersects this line exactly once, as necessary, so the graph passes the Horizontal Line Test, and hence has an inverse.

1.A Relationship between trigonometric functions and exponentials

Two of the types of functions that we looked at in this chapter–trig functions and exponentials–are connected using complex numbers. Although we will not be using complex numbers in our study of calculus in this book, we will briefly discuss how complex numbers connect trig functions to exponentials.

The set of **complex numbers**, denoted $\mathbb{C}$, is the set

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

The **imaginary number** i is defined to have the property that $i^2 = -1$. In the complex number $a + bi$, the real number a is called the **real part** and the real number b is called the **imaginary part**.

Complex numbers are added and multiplied as follows:

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(bi+di)\\ &= (a+c)+(b+d)i,\\ (a+bi)(c+di) &= ac+bci+adi+bdi^2\\ &= (ac-bd)+(ad+bc)i.\end{aligned}$$

It turns out that trig functions are related to exponentials of complex numbers via one of the most important equations of analysis:

Important: **Euler's Formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

For example, letting $\theta = \pi$ in Euler's Formula gives

$$e^{\pi i} = \cos\pi + i\sin\pi = -1 + i(0) = -1.$$

This is often rewritten as

$$e^{\pi i} + 1 = 0,$$

and is considered by many to be the most elegant equation in all of mathematics, as it combines what are arguably the five most fundamental constants in math: 0, 1, π, e, and i.

A less trivial example is

$$e^{\frac{\pi i}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Notice that the left side cubed is

$$(e^{\frac{\pi i}{3}})^3 = e^{\pi i} = -1,$$

and we see that this is true when we cube the right side as well:

$$\begin{aligned}&\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3\\ &= \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}i\right) + 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)^2 + \left(\frac{\sqrt{3}}{2}i\right)^3\\ &= \frac{1}{8} + \frac{3\sqrt{3}}{8}i + \frac{9}{8}i^2 + \frac{3\sqrt{3}}{8}i^3\\ &= \frac{1}{8} + \frac{3\sqrt{3}}{8}i - \frac{9}{8} - \frac{3\sqrt{3}}{8}i\\ &= -1.\end{aligned}$$

One of the many wonderful things about Euler's Formula is that it encapsulates the sine and cosine angle-addition formulas. If $\alpha, \beta \in \mathbb{R}$, then we have

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta),$$

but on the other hand

$$\begin{aligned}e^{i(\alpha+\beta)} &= (e^{i\alpha})(e^{i\beta})\\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).\end{aligned}$$

Comparing the real and imaginary parts of this with the Euler's Formula expansion of $e^{i(\alpha+\beta)}$ from above gives us the angle-addition formulas:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

So if you remember Euler's Formula, you don't have to memorize the angle-addition formulas, since they are very easily reconstituted via the above calculation.

Right now, this is magic. In Chapter 7, we'll see more clearly why Euler's Formula is true.

Challenge Problems

1.43:

(a) If $A \cup B = A$, what is the relationship (if any) between A and B?

Hint: An example might help.

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Solution: We are given that if $x \in A$ or $x \in B$, then $x \in A$. Logically, this means that if $x \in B$, then $x \in A$. Therefore, $\boxed{B \subseteq A}$.

(b) If $A \cap B = A$, what is the relationship (if any) between A and B?

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Solution: We are given that if $x \in A$ and $x \in B$, then $x \in A$. This means that any element of A is also an element of B, so $\boxed{A \subseteq B}$.

1.44:

Let $f(x) = \dfrac{ax+b}{cx+d}$ for some real numbers a, b, c, d with $ad \neq bc$.

(a) Find the domain and range of f.

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Solution: We have $x \in \mathrm{Dom}(f)$ if and only if $cx + d \neq 0$, thus the domain is $\boxed{\mathbb{R} \setminus \left\{\frac{-d}{c}\right\}}$. (This assumes that $c \neq 0$; if $c = 0$, then the domain is $\mathbb{R}$).

If $y \in \mathrm{Rng}(f)$, then letting $y = \dfrac{ax+b}{cx+d}$ and solving for x gives $x = \dfrac{b-dy}{cy-a}$. Thus we must have $cy - a \neq 0$, hence $y \neq \dfrac{a}{c}$. We verify that

$$f(x) = \frac{a(b-dy)+b(cy-a)}{c(b-dy)+d(cy-a)} = \frac{bcy-ady}{bc-ad} = y.$$

Thus $\mathrm{Rng}(f) = \boxed{\mathbb{R} \setminus \left\{\frac{a}{c}\right\}}$.

(b) Find the inverse of f if it exists.

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Solution: As determined in part (a), $f^{-1}(x) = \dfrac{b-dx}{cx-a}$. We have already checked that $f(f^{-1}(x)) = x$; we also have

$$f^{-1}(f(x)) = \frac{b(cx+d)-d(ax+b)}{c(ax+b)-a(cx+d)} = \frac{bcx-adx}{bc-ad} = x,$$

as desired.

(c) Why is the condition that $ad \neq bc$ important? What happens if $ad = bc$?

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Solution: If $ad = bc$, then the domain of f is unchanged. However, we have

$$f(x) = \frac{ax+b}{cx+d} = \frac{ax+b}{cx+d} \cdot \frac{ad}{bc} = \frac{a}{c} \cdot \left(\frac{adx+bd}{bcx+bd}\right) = \frac{a}{c}$$

for all $x \in \mathrm{Dom}(f)$; in other words, f is the constant function with range $\left\{\frac{a}{c}\right\}$, and f does not have an inverse.

1.45:

Determine all θ such that $0 \leq \theta \leq \frac{\pi}{2}$ and $\sin^5\theta + \cos^5\theta = 1$.

Hint: Look first for the "obvious" solutions.

Hint: We'd be much happier working with $\sin^5\theta$ and $\cos^5\theta$. What's a relationship between $\sin^5\theta$ and $\sin^2\theta$?

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Solution: We notice two solutions immediately: $\theta = 0$ and $\theta = \frac{\pi}{2}$. Furthermore, on the interval $(0, \pi/2)$, we know $\sin\theta$ and $\cos\theta$ are between 0 and 1, hence

$$\sin^5\theta < \sin^2\theta,$$
$$\cos^5\theta < \cos^2\theta.$$

Combining these we get

$$\sin^5\theta + \cos^5\theta < \sin^2\theta + \cos^2\theta = 1,$$

so there are no other solutions. Therefore the only solutions are $\boxed{\theta \in \left\{0, \frac{\pi}{2}\right\}}$.

1.46:

Compute $\sin(\arctan 3)$.

Hint: Try drawing a right triangle.

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Solution: Since 3 is positive, $\arctan 3$ is an acute angle, so we consider a right triangle with angle $\theta = \arctan 3$. We construct the right triangle with side opposite θ of length 3 and side adjacent to θ of length 1. The hypotenuse of this triangle has length $\sqrt{10}$. We can read from this triangle

$$\sin(\arctan 3) = \sin\theta = \boxed{\frac{3}{\sqrt{10}}}$$

1.47:

If $\sin 2x = \dfrac{21}{25}$ and $\cos x > \sin x$, then compute $\cos x - \sin x$.

Hint: Since we're starting with $\sin 2x$, the double-angle formula may help.

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Solution: We know the double-angle formula $\sin 2x = 2\sin x \cos x$. It's not obvious how to get from $\cos x - \sin x$ to this value, but we notice that squaring $\cos x - \sin x$ gives us a $2\sin x \cos x$ term:

$$(\cos x - \sin x)^2 = \cos^2 x + \sin^2 x - 2\sin x \cos x.$$

Using $\sin^2 + \cos^2 = 1$ and the double-angle identity, we have.

$$(\cos x - \sin x)^2 = 1 - \sin(2x) = 1 - \frac{21}{25} = \frac{4}{25}.$$

Since we're given $\cos x > \sin x$, we can take the positive square root to find $\cos x - \sin x = \boxed{\frac{2}{5}}$

1.48:

Compute $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3}$.

Hint: Computing the tangent of what you want might be easier.

Hint: Use the tangent double-angle formula to compute $\tan\left(\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3}\right)$

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Solution: Take the tangent and use the tangent angle-sum formula:

$$\begin{aligned}\tan\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right) &= \frac{\tan(\tan^{-1}\frac{1}{2}) + \tan(\tan^{-1}\frac{1}{3})}{1 - \tan(\tan^{-1}\frac{1}{2})\tan(\tan^{-1}\frac{1}{3})} \\ &= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} \\ &= \underset{\text{simplify}}{\frac{5/6}{5/6}} \\ &= 1.\end{aligned}$$

Thus

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1} 1 = \boxed{\frac{\pi}{4}}.$$

1.49:

The following are the **hyperbolic trig functions**:

$$\sinh x = \frac{e^x - e^{-x}}{2},$$
$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

(a) Prove that $\cosh^2 x - \sinh^2 x = 1$ for all $x \in \mathbb{R}$.

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Solution: We compute:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} = 1.\end{aligned}$$

(b) Find a formula for $\sinh(x+y)$ in terms of $\sinh x$, $\sinh y$, $\cosh x$, and $\cosh y$. Do the same for $\cosh(x+y)$.

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Solution: We can write

$$\sinh(x+y) = \frac{e^{x+y} - e^{-(x+y)}}{2}.$$

To try to get this in terms of sinh and cosh of just x and y, we experiment to see what happens when we multiply:

$$\begin{aligned}\sinh x \cosh y &= \frac{(e^x - e^{-x})(e^y + e^{-y})}{4} \\ &= \frac{e^{x+y} - e^{-(x+y)} + e^{x-y} - e^{y-x}}{4}.\end{aligned}$$

This is close, except for the e^{x-y} and e^{y-x} terms. We try swapping x and y:

$$\begin{aligned}\cosh x \sinh y &= \frac{(e^x + e^{-x})(e^y - e^{-y})}{4} \\ &= \frac{e^{x+y} - e^{-(x+y)} - e^{x-y} + e^{y-x}}{4}.\end{aligned}$$

Aha—we add these together and the unwanted terms will cancel:

$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \frac{2(e^{x+y} - e^{-(x+y)})}{4} \\ &= \sinh(x+y).\end{aligned}$$

Similar calculations show that

$$\cosh(x+y) = \sinh x \sinh y + \cosh x \cosh y.$$

(c) Can you suggest why we use the name *hyperbolic* for these functions?

Hint: What's the equation for a hyperbola?

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Solution: We try to do the same thing as with the regular trig functions: think of $\cosh$ as the x-coordinate and $\sinh$ as the y-coordinate of some graph. Part (a) then gives us $x^2 - y^2 = 1$, whose graph is a hyperbola. (We restrict to the $x > 0$ half of the hyperbola since $\cosh$ is always positive.)

1.50★:

There's an interesting phenomenon called **Russell's paradox** that occurs with sets: Let C be the set whose elements are sets that do not contain themselves as an element. Is $C \in C$? Can you explain what's going on?

Hint: There's no "right answer." Just try to explain what's going on as best as you can. Then read the explanation in the Solutions Manual.

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Solution: If $C \in C$, then C contains itself as a member, so by definition $C \notin C$, a contradiction. On the other hand, if $C \notin C$, then C does not contain itself as a member, so $C \in C$, again a contradiction. So we cannot have either $C \in C$ or $C \notin C$.

However, the fundamental property of a "set" S is that for any object x, either $x \in S$ or $x \notin S$; that is, S is determined precisely by its elements. Since we cannot make this determination for the set C and the object C, we have a paradox.

The paradox is that such a set C cannot exist, so our notion of "set" is not broad enough. Mathematicians have developed various generalizations of set theory to deal with such complications.

1.31:

Assume A and B are two sets, their two current have both $A \subset B$ and $B \subset A$

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Solution: If $A \subset B$ and $B \subset A$, then every element of A is an element of B, and every element of B is an element of A. But this means that $A = B$, which contradicts the assertion that the subsets are proper.

1.32:

Thus **symmetric difference** of two sets A and B is

$$A \triangle B = \{x \mid x \text{ is in exactly one of } A \text{ or } B\}.$$

(a) Show that symmetric difference is commutative: $A \triangle B = B \triangle A$

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Solution: If x is in exactly one of A and B, then x is in exactly one of B and A, so symmetric difference is commutative (or commutative, as the same result right).

(b) Show that symmetric difference is associative: $(A \triangle B) \triangle (C = A \triangle (B \triangle C)$

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Solution: We can break elements into cases based on which sets they are in, but this is rather tedious. More simply, we observe that the set on each side of the associative law is defined as the elements in exactly one of A, B, C together with elements in all three sets.

(c) What is $A \triangle \emptyset$?

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Solution: $A \triangle \emptyset = A$ because if an element is in A it is not in the empty set, so it is in exactly one of A and $\emptyset$. There are no elements in $\emptyset$, so we do not have to consider any other cases. Therefore, this is $\boxed{A}$.

(d) Describe, in words, the set $A \triangle (B \triangle C) \triangle D$ and why it is valid to write this without parentheses.

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Solution: The set $A \triangle (B \triangle C) \triangle D$ consists of all elements that are in an odd number of A, B, C, or D. That is, it is the set of elements that are in exactly one, or all three of A, B, C, or D. Because symmetric difference is associative and commutative, it is not necessary to check this for an element in or out of the set. Definition of membership is one good number of sets. This is also why we don't need parentheses when we write $A \cup B \cup C \cup D$. For example, if x is in A but not in B, C, or D, then $x \notin C \cup D$ so $x \notin B \cup C \cup D$ and hence $x \in A \triangle (B \cup C \cup D)$. Similar checks can be made for elements in exactly 0, 1, 3, or 4 of the sets.

1.33:

Suppose $a < b < c < d$ are real numbers. Write each of the following as an interval if possible. If it is not an interval, explain why not.

(a) $(a, b) \cup (b, d)$

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Solution: This is not an interval, since no elements between b and c are in the set.

(b) $(a, b) \cap (c, d)$

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Solution: This is the interval $\boxed{\emptyset}$

(c) $(a, c) \cup (b, d)$

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Solution: This is the interval $\boxed{(a, d)}$

(d) $(a, c) \cap (b, d)$

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Solution: This is the interval $\boxed{(b, c)}$

1.34:

Find the domain and range of $f(x) = \sqrt{2 - x - x^2}$

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Solution: For $f(x)$ to be defined, we must have $2 - x - x^2 \geq 0$. This factors as $(2 + x)(1 - x) \geq 0$. This occurs when $-2 \leq x \leq 1$, so the domain is $\boxed{[-2, 1]}$.

The quadratic has a maximum at its vertex, which is at $x = -\frac{1}{2}$, at which the quadratic equals $2 + \frac{1}{2} - \left(-\frac{1}{2}\right)^2 = \frac{9}{4}$. Since $\sqrt{\frac{9}{4}} = \frac{3}{2}$, the range of f is $\boxed{\left[0, \frac{3}{2}\right]}$

1.35:

Suppose that a function f has domain $(-2, 2)$ and range $[-3, 5]$. Find the domain and range (if possible) of

(a) $f(2x)$

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Solution: The domain is $\boxed{(-1, 1)}$ and the range is $\boxed{[-3, 5]}$.

(b) $f(\sqrt{x})$

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Solution: The domain is $\boxed{[0, 4)}$. We cannot determine the range explicitly: the range of $f(\sqrt{x})$ is equal to the range $f([0, 2))$, but what do outcrop precisely what subset of $[-3, 5]$ this image is.

(c) $f^{-1}(x)$ (assuming it exists)

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Solution: If f^{-1} exists, then the domain of f^{-1} is $\boxed{[-3, 5]}$ and its range is $\boxed{[-2, 2)}$. (The domain and range of the inverse are the range and domain of the original function, respectively.)

1.36:

Evaluate the following. (The goal is for these computations to become nearly automatic.)

(a) $\tan \frac{\pi}{4}$

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Solution: $\tan \frac{\pi}{4} = \boxed{1}$

(b) $\cos \frac{7\pi}{6}$

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Solution: $\cos \frac{7\pi}{6} = \boxed{-\frac{\sqrt{3}}{2}}$

(c) $\sin \frac{5\pi}{4}$

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Solution: $\sin \frac{5\pi}{4} = \boxed{-\frac{\sqrt{2}}{2}}$

(d) $\cos \frac{3\pi}{2}$

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Solution: $\cos \frac{3\pi}{2} = \boxed{0}$

(e) $\sec \frac{2\pi}{3}$

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Solution: $\sec \frac{2\pi}{3} = \frac{1}{\cos \frac{2\pi}{3}} = \frac{1}{-\frac{1}{2}} = \boxed{-2}$

(f) $\sin\left(\tan \frac{\pi}{4}\right)$

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Solution: $\sin\left(\tan \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \boxed{\frac{1}{2}}$

(g) $\tan 21\pi$

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Solution: $\tan 21\pi = \tan 0 = \boxed{0}$

(h) $\sec\left(\frac{7\pi}{3}\right)$

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Solution: $\sec\left(\frac{7\pi}{3}\right) = \frac{1}{\cos \frac{7\pi}{3}} = \frac{1}{\frac{1}{2}} = \boxed{2}$. Since cosine is zero at $\frac{\pi}{2}$ and $\boxed{\text{undefined}}$

(You can verify that these two answers are in fact the same by squaring both.)

We can use either method above, or the fact that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$ to compute

$$\tan \frac{\pi}{12} = \frac{\sin \frac{\pi}{12}}{\cos \frac{\pi}{12}} = \frac{\sqrt{1 - \cos^2 \frac{\pi}{12}}}{\cos \frac{\pi}{12}}.$$

Finally,

$$\begin{aligned} \tan \frac{\pi}{12} &= \frac{\sin \frac{\pi}{12}}{\cos \frac{\pi}{12}} \\ &= \frac{\sqrt{1 - \cos^2 \frac{\pi}{12}}}{\cos \frac{\pi}{12}} \\ &= \frac{\sqrt{1 - \cos^2 \frac{\pi}{12}}}{\sqrt{\cos^2 \frac{\pi}{12}}} \\ &= \sqrt{\frac{1 - \cos^2 \frac{\pi}{12}}{\cos^2 \frac{\pi}{12}}} \\ &= \sqrt{\sec^2 \frac{\pi}{12} - 1} \\ &= \sqrt{2 + 1} \\ &= \boxed{2 + \sqrt{3}} \end{aligned}$$

1.37:

Find the following quantities in terms of $\cos \theta$ or $\sin \theta$:

(a) $\cos(\pi - \theta)$

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Solution:

$$\begin{aligned} \cos(\pi - \theta) &= \cos\left(\frac{\pi}{2} + \left(\frac{\pi}{2} - \theta\right)\right) \\ &= -\sin\left(\frac{\pi}{2} - \theta\right) \\ &= -\cos \theta \\ &= \boxed{-\cos \theta} \end{aligned}$$

(b) $\sin\left(\frac{\pi}{2} - \theta\right)$

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Solution:

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \theta\right) &= \cos(-\theta) \\ &= \boxed{\cos \theta} \end{aligned}$$

(c) $\sin\left(\frac{3\pi}{2} + \theta\right)$

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Solution:

$$\begin{aligned} \sin\left(\frac{3\pi}{2} + \theta\right) &= \cos\left(\frac{\pi}{2} - \left(\frac{3\pi}{2} + \theta\right)\right) \\ &= -\cos(2\pi + \theta) \\ &= \boxed{-\cos \theta} \end{aligned}$$

1.38:

Compute $\sin \frac{5\pi}{12}$, $\cos \frac{5\pi}{12}$, and $\tan \frac{5\pi}{12}$.

Hint: There are two basic ways to do this problem, and they might lead you to two very different-looking answers. Check that those two answers are the same.

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Solution: Using the angle-addition formula,

$$\begin{aligned} \sin \frac{5\pi}{12} &= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) \\ &= \sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3} \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \boxed{\frac{\sqrt{2} + \sqrt{6}}{4}} \end{aligned}$$

Alternatively, using the half-angle formula,

$$\sin \frac{5\pi}{12} = \sqrt{\frac{1 - \cos \frac{\pi}{6}}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \boxed{\frac{\sqrt{2 + \sqrt{3}}}{2}}$$

We can use either method above, or the fact that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$, to compute

$$\cos \frac{5\pi}{12} = \sqrt{\frac{1 + \cos \frac{\pi}{6}}{2}} = \boxed{\frac{\sqrt{2 + \sqrt{3}}}{2}}$$

Finally,

$$\begin{aligned} \tan \frac{5\pi}{12} &= \frac{\sin \frac{5\pi}{12}}{\cos \frac{5\pi}{12}} \\ &= \frac{\sqrt{2 + \sqrt{3}}}{2} \\ &= \frac{\sqrt{2 + \sqrt{3}}}{\sqrt{2 + \sqrt{3}}} \\ &= \frac{\sqrt{(2 + \sqrt{3})^2}}{\sqrt{2 + \sqrt{3}}} \\ &= \frac{2 + \sqrt{3}}{\sqrt{2 + \sqrt{3}}} \\ &= \boxed{2 + \sqrt{3}} \end{aligned}$$

1.39:

Find the smallest positive θ such that $\sin 2\theta = \cos 7\theta$

Hint: Use $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$.

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Solution: Recall the identity $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$. Thus we're looking for the smallest positive solution to

$$\cos\left(\frac{\pi}{2} - 2\theta\right) = \cos 7\theta.$$

More generally, we know that $\cos \alpha = \cos \beta$ if and only if $\alpha = \pm\beta$ is an integer multiple of 2π. Thus, our equation is satisfied if and only if $\frac{\pi}{2} - 2\theta = 2\pi k$ or $\frac{\pi}{2} - 2\theta = 2\pi k$, where k is an integer. Isolating θ we get the possible solutions

$$\begin{aligned} \theta &= \frac{\pi}{4} - \pi k \\ \theta &= \frac{\pi}{4} - \frac{2\pi}{9} k. \end{aligned}$$

Finally, since $\pi > \frac{2\pi}{9}$, the smallest positive solution is thus $\theta = \frac{\pi}{4}$. Checking we see that $\sin \frac{\pi}{2} = \cos \frac{7\pi}{4}$ since $\frac{\pi}{2} = \frac{7\pi}{4} - 2\pi$. Furthermore, for smaller positive θ, one is larger and cosine is smaller, so this must be the smallest positive solution.

1.40:

Find arctan $\left(\tan \frac{47\pi}{7}\right)$

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Solution: We know that

$$\tan \frac{47\pi}{7} = \tan\left(\frac{42\pi}{7} + 5\pi\right) = \tan \frac{5\pi}{7}.$$

Thus, since $-\frac{\pi}{2} < \frac{5\pi}{7} < \frac{\pi}{2}$ we have

$$\arctan\left(\tan \frac{47\pi}{7}\right) = \arctan\left(\tan \frac{5\pi}{7}\right) = \boxed{\frac{5\pi}{7}}$$

1.41:

If $x \in [-1, 1]$, write $\sin \arccos^{-1} x$ more simply in terms of x.

Hint: Recall $\sin^2 \theta + \cos^2 \theta = 1$.

Hint: Be careful about the range of $\arccos^{-1}$.

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Solution: The range of $\arccos^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on which cosine is positive. Hence, letting $\theta = \arccos^{-1} x$, we have

$$\begin{aligned} 1 &= \cos^2 \theta + \sin^2 \theta \\ &= \cos^2(\arccos^{-1} x) + \sin^2(\arccos^{-1} x) \\ &= \cos^2(\arccos^{-1} x) + x^2, \end{aligned}$$

and solving for the cosine term, we get cos(arc^-1 x) = $\boxed{\sqrt{1 - x^2}}$

1.42:

Solve the following equations:

(a) $\log_{\sqrt{2}}(25^x) = 10$

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Solution: Using logarithm properties, we have

$$10 = \log_{\sqrt{2}}(25^x) = 2x \log_2(25) = 2x(2) = 4x.$$

Thus $x = \boxed{\frac{5}{2}}$

(b) $\log_2(x^2) + \log_2(3x) = 16$

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Solution: This equation is the same as $\log_2(x^2 \cdot 3x) = 16$, so $3x^3 = 2^{16}$. Therefore $x = \sqrt[3]{\frac{2^{16}}{3}} = \boxed{\frac{2^{5}\sqrt[3]{2}}{\sqrt[3]{3}}}$

(c) $e^{2x} - 3e^x - 4 = 0$

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Solution: The equation is a quadratic in e^x. We factor

$$(e^x - 4)(e^x + 1) = 0.$$

Since $e^x > 0$ positive, we must have $e^x = 4$, so $x = \boxed{\log 4}$

2.1 Limits

Limit is probably the calculus concept that students initially find the most confusing. Intuitively, a limit describes what happens as a function "approaches" a point in $\mathbb{R}$. We can get some idea of what a limit means by looking at an example.

Problem 2.1

Consider the functions $f(x) = x + 1$ and $g(x) = \frac{x^2-1}{x-1}$.

(a) Determine $\mathrm{Dom}(f)$ and $\mathrm{Dom}(g)$.

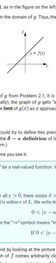
(b) Sketch the graphs of f and g. How do they differ?

Solution for Problem 2.1:

(a) It is clear that $\mathrm{Dom}(f) = \mathbb{R}$. However, $1 \notin \mathrm{Dom}(g)$ since that would make the denominator equal to 0; otherwise, g is defined. So

$$\mathrm{Dom}(g) = \mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, +\infty).$$

(b) The graph of f is the line $y = x + 1$, as in the figure on the left. The graph of g is also this line for all $x \neq 1$, because if $x \neq 1$ then $\frac{x^2-1}{x-1} = x + 1$. However, 1 is not in the domain of g. Thus, there is a "hole" in the graph of g at $(1, 2)$, as in the figure on the right.



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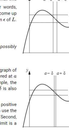
On the other hand, looking at the graph of g from Problem 2.1, it is "obvious" that the graph of g "approaches" the point $(1, 2)$. Slightly more precisely (though still pretty informally), the graph of g gets "arbitrarily close" to the point $(1, 2)$. Mathematically, we write this as $\lim_{x\to 1} g(x) = 2$. This should be read as "the limit of $g(x)$ as x approaches 1 is 2." We could also write it without referring to the name of the function, as $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$.

There are lots of different ways that we could try to define this precisely, and mathematicians over the years tried some different ideas. The idea that stuck is colloquially called the $\boldsymbol{\delta - \epsilon}$ **definition** of limit. (The Greek letters δ (delta) and ϵ (epsilon) are used throughout calculus in this context, so get used to them.)

The definition is a bit confusing the first time you see it:

> **Definition:** Let f be a real-valued function. We say that the **limit** of $f(x)$ as x approaches a is L, or
>
> $$\lim_{x\to a} f(x) = L,$$
>
> if, for all $\epsilon > 0$, there exists $\delta > 0$ such that if x is within δ of a (with $x \neq a$), then $f(x)$ is within of L. We write this more precisely as
>
> $$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon,$$
>
> where the "$\Rightarrow$" symbol means "implies."

This definition is much easier to understand by looking at the picture to the right. Intuitively, we think of $\lim_{x\to a} f(x) = L$ as meaning that the graph of f comes arbitrarily close to the point (a, L). What this means is: suppose someone tells you that you need to have $f(x)$ within of L. In other words, someone gives you a value of ϵ and insists that $|f(x) - L| < \epsilon$. Then, you have to be able to come up with a value of δ such that if x is within δ of a (and $x \neq a$), then $f(x)$ will be within ϵ of L. This means you must choose δ such that



$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

To put it another way, the entire graph of f on the (horizontal) interval $(a - \delta, a + \delta)$ *except possibly* at a itself must be within the (vertical) interval $(L - \epsilon, L + \epsilon)$.

In other words, you are given the width of the vertical region centered at L that we require the graph of the function to be contained in, and you have to determine a width of a horizontal region centered at a so that the graph of the function is contained entirely within the box near (a, L). For example, the picture to the right shows that if δ is chosen small enough, then as long as x is within δ of a (the diagonal dashed lines), the graph of f is also contained in the box.

There are a couple of things slightly tricky about this definition. First, it has to work for any positive value of ϵ, but once it is given, we get to choose the value of δ that works. (Some courses even use the functional notation $\delta(\epsilon)$ to emphasize the fact that the value of δ depends on the value of ϵ.) Second, the value given ϵ only doesn't enter into this at all—in fact, a need not even be in the domain of f. Limit is a concept that describes what happens as x approaches a, not what happens at a.

> **Concept:** The definition of limit is a tricky concept. Don't be discouraged if it doesn't make sense right away. Hopefully, as we progress through the next several examples, the concept will become more clear.

Let's practice with this definition by going back to our original example:

Problem 2.2

Consider the function $g(x) = \frac{x^2-1}{x-1}$. We wish to prove that $\lim_{x\to 1} g(x) = 2$.

(a) If $\epsilon = 0.2$ is given, what value of δ can we choose to satisfy the definition?

(b) If $\epsilon = 0.1$ is given, what value of δ can we choose to satisfy the definition?

(c) Prove that $\lim_{x\to 1} g(x) = 2$.

Solution for Problem 2.2: We first note that $g(x) = x + 1$ for all values of $x \in \mathbb{R} \setminus \{1\}$. Note that the definition of our limit does not concern itself with what happens when $x = 1$, because we only examine values of x such that $0 < |x - 1| < \delta$ for some $\delta > 0$, and in particular we don't have to worry about $x = 1$. Thus, we can treat $g(x)$ and $x + 1$ when trying to prove the condition in the definition of limit, so $g(x) = x + 1$ for all values of x that we will be considering.

> **Concept:** The definition of $\lim_{x\to a} f(x)$ does not concern itself with what happens at $x = a$, only with what happens near $x = a$.

With this idea, let's proceed to the specific calculations:

(a) We need to choose δ such that

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |g(x) - 2| < 0.2.$$

But since $x \neq 1$, the condition on the right side of the implication above is equivalent to

$$|(x + 1) - 2| < 0.2,$$

or $|x - 1| < 0.2$. Thus, letting $\delta = 0.2$ will satisfy the condition. In other words, if x is within 0.2 (that is, if) of 1, then $f(x)$ will be within 0.2 (that is, of 2).

(b) We need to choose δ such that

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |g(x) - 2| < 0.05.$$

But since $x \neq 1$, the condition on the right side of the implication above is equal to

$$|(x + 1) - 2| < 0.05,$$

or $|x - 1| < 0.05$. Thus, letting $\delta = 0.05$ will satisfy the condition.

(c) Our solutions to parts (a) and (b) were exactly the same argument with just the value of ϵ changed, so this argument should work for any value of ϵ. Specifically, we need to choose δ such that

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |g(x) - 2| < \epsilon.$$

But since $x \neq 1$, the condition on the right side of the implication above is equal to

$$|(x + 1) - 2| < \epsilon,$$

or $|x - 1| < \epsilon$. Thus, letting $\delta = \epsilon$ will satisfy the condition.

◇

It just so happened in Problem 2.2 that $\delta = \epsilon$ worked. This will not usually be the case, as in the next example:

Problem 2.3

Determine $\lim_{x\to 0} 3x^2$. Prove your answer is correct using the δ-ϵ definition of limit.

Solution for Problem 2.3: Let $f(x) = 3x^2$. Since the graph of f is "smooth" and has no holes or skips, we expect that $f(x)$ approaches $f(2)$ as x approaches 2. In other words, we strongly suspect that

$$\lim_{x\to 0} 3x^2 = f(2) = 12.$$

(Later in this chapter, we'll make this intuition more precise.) Let's prove this limit using the δ-ϵ definition.

We are given some $\epsilon > 0$, and we need to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \Rightarrow \quad |3x^2 - 12| < \epsilon.$$

The inequality $|3x^2 - 12| < \epsilon$ will be more useful if it is in terms of $x - 2$ rather than x, since the inequality $0 < |x - 2| < \delta$ is in terms of $x - 2$. For simplicity, let $\epsilon = \rho - 2$. Then we wish to find δ such that

$$0 < |x| < \delta \quad \Rightarrow \quad |3x^2 - 12| < \rho - 2.$$

We can simplify this to

$$0 < |x| < \delta \quad \Rightarrow \quad |3x^2 + 12| < \rho.$$

However, we know that

$$|3x^2 + 12| \leq |3x^2| + |12| = 3x^2 + 12 \geq 12.$$

So it suffices to find δ such that

$$0 < |x| < \delta \quad \Rightarrow \quad 3x^2 + 12 \leq |x| < \rho.$$

If $0 < |x| < \delta$, then

$$3x^2 + 12 \leq 3\delta^2 + 12 = 3\delta(\epsilon + \delta).$$

Thus it suffices to choose δ such that

$$3\delta(\epsilon + \delta) < \epsilon.$$

The $\epsilon + \delta$ term is somewhat annoying. We can make it than much simpler by assuming that $\delta \leq 1$.

> **Concept:** All we have to do is find any δ that works for our given ϵ. So we can always make δ smaller than it needs to be.

If we assume that $\delta \leq 1$, then $\epsilon + \delta \leq 5$, and the inequality that we need becomes simply

$$3\delta(\epsilon + \delta) \leq 15\delta < \epsilon.$$

To force this to be true, we select $\delta = \frac{\epsilon}{15}$. (In the unlikely event that $\epsilon > 15$, we can just take $\delta = 1$.) We then conclude that

$$0 < |x| < \delta \quad \Rightarrow \quad |3x^2 + 12| \leq 3\delta(\epsilon + \delta) \leq 15\delta < \epsilon.$$

Thus, for any $\epsilon < 15$, we have found that $\delta = \frac{\epsilon}{15}$ satisfies the δ-ϵ condition:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |3x^2 - 12| < \epsilon,$$

and hence we have established that $\lim_{x\to 0} 3x^2 = 12$.

While performing the sort of calculation that we did in Problem 2.3 is useful to do once or twice in your life, in practice it's not how we usually think about computing limits. We'll have other, more useful techniques to compute limits that we'll develop later in this chapter and in other points in the book.

Again, we stress that the limit of $f(x)$ as x approaches a does not depend at all on $f(a)$. Let's look at an extreme example of this:

Problem 2.4

Consider the function

$$f(x) = \begin{cases} x & \text{if } x \neq 0, \\ 2 & \text{if } x = 0. \end{cases}$$

What is the value of $\lim_{x\to 0} f(x)$?

Solution for Problem 2.4: Looking at the graph of this function, we see that this is just the graph of $y = x$, except the point $(0, 0)$ on the graph is replaced by the point $(0, 2)$. But as the function approaches $x = 0$, it is clear that the function approaches the point $(0, 0)$. Therefore, we should have $\lim_{x\to 0} f(x) = 0$, since the function is "arbitrarily close" to $(0, 0)$ as x approaches 0.

Another way to look at this is to compare f to the function $g(x) = x$. These functions are equal at all of $\mathbb{R}$ except for 0. Thus, since the limit doesn't depend on what happens at $x = 0$, but only depends on what happens near $x = 0$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0.$$

Yet another way to see this is to appeal to the δ-ϵ definition. For any $\epsilon > 0$, we can choose $0 < \delta < \epsilon$, since

$$0 < |x - 0| < \delta \quad \Rightarrow \quad |f(x) - 0| = |x - 0| < \delta < \epsilon.$$

◇

Once again, the important concept to take from Problem 2.4 is:

> **Important:** When determining $\lim_{x\to a} f(x)$, the value of $f(a)$ is irrelevant.

In fact, $f(a)$ doesn't even have to be defined, as in Problem 2.2, where we computed $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$, even though 1 is not in the domain of this function.

It may not immediately be clear that our definition gives us a unique limit. In other words, we need to check the following:

Problem 2.5

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = L'$. Prove that $L = L'$.

Solution for Problem 2.5: Intuitively, if we had two different limits L and L', there's no way that the function could "approach" both points (a, L) and (a, L') simultaneously.

The picture to the right gives us the idea of the proof. We want to pick ϵ small enough so that the shown horizontal bands for the two limits don't overlap. We then pick any x such that $0 < \epsilon < \frac{|L-L'|}{2}$ (which is possible because by assumption $L \neq L'$). The picture clearly shows that no matter how close x is to a, we cannot have $f(x)$ within ϵ of both L and L', because the horizontal bands in the picture don't overlap. A picture, however, is not a proof, so let us prove this rigorously using the δ-ϵ definition of limit.

> **Concept:** A picture may give you intuition, but it is never a substitute for a rigorous proof.

We'll now prove the result rigorously using the definition of limit. We suppose that $L \neq L'$ and try to derive a contradiction. As above, let ϵ be such that $0 < \epsilon < \frac{|L-L'|}{2}$. Since L is a limit, we can find $\delta_1 > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

But since L' is also a limit, we can find δ' such that

$$0 < |x - a| < \delta' \quad \Rightarrow \quad |f(x) - L'| < \epsilon.$$

This means if we pick x close enough to a, then both $|f(x) - L| < \epsilon$ and $|f(x) - L'| < \epsilon$ will be simultaneously satisfied; that is,

$$\begin{aligned} &0 < |x - a| < \min(\delta, \delta') \\ \Rightarrow \quad &|f(x) - L| < \epsilon \text{ and } |f(x) - L'| < \epsilon. \end{aligned}$$

We write the first inequality on the right as $|L - f(x)| < \epsilon$, and add the two inequalities on the right. We get

$$|L - f(x)| + |f(x) - L'| < 2\epsilon.$$

Then, applying the Triangle Inequality (which states that $|y + z| \leq |y| + |z|$ for all $y, z \in \mathbb{R}$) gives us

$$\begin{aligned} |L - L'| &= |(L - f(x)) + (f(x) - L')| \\ &\leq |L - f(x)| + |f(x) - L'| \\ &< 2\epsilon. \end{aligned}$$

So $\frac{|L-L'|}{2} < \epsilon$. But this contradicts our original choice of $\epsilon < \frac{|L-L'|}{2}$, so we have our contradiction, and thus the limit must be unique. ◇

When they are defined, limits have very nice properties. We'll prove one, and then list a bunch of others that are similar (some of which you will prove as exercises).

Problem 2.6

Let f and g be functions and $a \in \mathbb{R}$. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, compute $\lim_{x\to a} (f + g)(x)$.

Solution for Problem 2.6: We should be shocked if the answer were not $L + M$. Let's prove that it is $L + M$. As usual when trying to rigorously prove facts about limits, we appeal to the δ-ϵ definition.

We want to show that if given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |(f + g)(x) - (L + M)| < \epsilon.$$

This suggests constructing the similar inequalities for f and g separately using $\frac{\epsilon}{2}$ and adding them together, as we shall see.

Specifically, because we have $\lim_{x\to a} f(x) = L$, we can choose δ_1 such that

$$0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \frac{\epsilon}{2}.$$

Similarly, because we have $\lim_{x\to a} g(x) = M$, we can choose δ_2 such that

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \frac{\epsilon}{2}.$$

So if we let $\delta = \min(\delta_1, \delta_2)$, then when $0 < |x - a| < \delta$, both $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$ above are satisfied, so we have

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \frac{\epsilon}{2} \text{ and } |g(x) - M| < \frac{\epsilon}{2}.$$

When we add the two inequalities on the right side and apply the Triangle Inequality, we get

$$|(f + g)(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus we have satisfied the δ-ϵ definition for $\lim_{x\to a} f + g)(x) = L + M$. ◇

There are other similar properties; we will leave the proofs as exercises.

> **Important:** Let f and g be functions with $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then:
>
> 1. $\lim_{x\to a} (f + g)(x) = L + M$
> 2. $\lim_{x\to a} (fg)(x) = LM$
> 3. If $c \in \mathbb{R}$, then $\lim_{x\to a} (cf)(x) = cL$.
>
> 4. If $M \neq 0$ then $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = L/M$

Now we see that we can easily determine the limit of many "nice" functions by breaking them up into their constituent parts. For example, revisiting Problem 2.3, we quickly see that:

$$\begin{aligned} \lim_{x\to 0} 3x^2 &= 3\left(\lim_{x\to 0} x^2\right) \\ &= 3\left(\lim_{x\to 0} x\right)^2 \\ &= 3(2)^2 = 12. \end{aligned}$$

Problem 2.7

(a) Suppose that f is a real-valued function such that $f(x) \geq 0$ for all $x \in \mathrm{Dom}(f)$, and suppose that $\lim_{x\to a} f(x) = L$ for some $a \in \mathbb{R}$. Show that $L \geq 0$.

(b) In part (a), now suppose that $f(x) > 0$. Can we conclude that $L > 0$?

(c) Suppose that f and g are real-valued functions such that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Show that

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x),$$

provided both limits are defined.

Solution for Problem 2.7:

(a) To prove this, we proceed by contradiction and assume that $L < 0$. Pick $\epsilon > 0$ sufficiently small so that $L + \epsilon < 0$ (for example, we could take $\epsilon = -L/2$). Then, by the definition of limit, there must exist δ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

But $|f(x) - L| < \epsilon$ means $f(x) \in (L - \epsilon, L + \epsilon)$, which, by our choice of ϵ, means $f(x) < 0$, a contradiction. So we must have $L \geq 0$.

(b) A function that is strictly positive can still approach a value of 0, that is, we may have $\lim_{x\to a} f(x) = 0$ but $f(x) \neq 0$ for all $x \in \mathbb{R}$ and yet have some value of a such that $\lim_{x\to a} f(x) = 0$. A simple example is $f(x) = |x|$ for all $x \in \mathbb{R} \setminus \{0\}$. Note that we are explicitly excluding 0 from the domain of f, so that $f(x) > 0$ for all $x \in \mathrm{Dom}(f)$ that clearly tends to 0 as we only have $L \geq 0$, so we only have $L \geq 0$. We could also construct an example with $\mathrm{Dom}(f) = \mathbb{R}$; for example,

$$f(x) = \begin{cases} |x|, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Note that $f(x) > 0$ for all $x \in \mathbb{R}$, but $\lim_{x\to 0} f(x) = 0$.

(c) Consider the function $g - f$, so that $(g - f)(x) \geq 0$ for all $x \in \mathbb{R}$, and use part (a). That is,

$$0 \leq \lim_{x\to a} (g - f)(x) = \lim_{x\to a} g(x) - \lim_{x\to a} f(x),$$

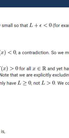
so we conclude that $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

◇

Along the lines of Problem 2.7 is an important result called the **Squeeze Theorem**:

> **Important:** The **Squeeze Theorem**: Let $a \in \mathbb{R}$ and let f, g, h be real-valued functions such that $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing a (but not necessarily at a itself). If
>
> $$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$
>
> then $\lim_{x\to a} g(x) = L$ as well.

We can see what's going on in the picture to the right. The function g is trapped between the function f below and the function h above. Since f and h both approach the point (a, L), and the function g is "squeezed" between them, we conclude that the function g must approach (a, L) as well. Note that the Squeeze Theorem tells us both that the limit must exist and that it must equal L. The proof of the Squeeze Theorem uses the result from Problem 2.7, so we will leave the details of the proof as an exercise.



We'll continue our initial exploration of limits with a classic nontrivial example that is quite important for calculus:

Problem 2.9

Compute $\lim_{x\to 0}\frac{\sin x^2}{x}$.

Solution for Problem 2.9: Of course, the function is undefined at $x = 0$. You can make a guess of the limit by computing (one calculator) $\frac{\sin x^2}{x}$ for values of x close to 0. You can also make a guess by using your graphing calculator to graph $y = \frac{\sin x^2}{x}$ and observing where the graph appears to cross the y-axis. Doing either of these should convince you that the limit should be 0.

We can see geometrically what's going on by looking at the picture to the right. Shown is a unit circle with a central angle θ drawn. By definition, the length of the bold arc of the circle and the length of the vertical line segment is $\sin\theta$. As θ decreases towards 0, the lengths of these two curves grow closer together, so in the bottom picture, it is geometrically "obvious" that the ratio $\sin\theta/\theta$ approaching 1 as the angle θ approaches 0.

To prove rigorously that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, we will use the Squeeze Theorem. Specifically, we are looking for functions $f(\theta)$ and $h(\theta)$ both with limits of 1 at $\theta = 0$ such that:

$$f(\theta) \le \frac{\sin\theta}{\theta} \le h(\theta).$$

To do this, we'll extend our unit circle diagram a bit, and look at the areas of the shaded regions in the pictures below:

The shaded region in the first picture is a triangle with base 1 and height $\sin\theta$, so its area is $(\sin\theta)/2$. The shaded region in the second picture is a sector of a circle of radius 1 and central angle θ. The shaded region in the third picture is a triangle with base 1 and height $\tan\theta$, so its area is $(\tan\theta)/2$. Thus, comparing the areas, we have

$$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2},$$

or more simply,

$$\sin\theta \le \theta \le \tan\theta.$$

Taking reciprocals (which we can do since we are only considering small $\theta > 0$) reverses the direction of the inequalities, so we have

$$\frac{1}{\tan\theta} \le \frac{1}{\theta} \le \frac{1}{\sin\theta}.$$

Then, multiplying by $\sin\theta$ (which is positive), and noting that $\frac{\sin\theta}{\tan\theta} = \cos\theta$, gives us

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

Our calculation only establishes this inequality chain for $0 < \theta < \frac{\pi}{2}$ but since all the terms are the same when we replace θ with $-\theta$, it is true for all nonzero $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Furthermore, our above argument shows that

$$-\theta < \sin\theta < \theta$$

for values of θ close to 0, and hence

$$\cos\theta = \sqrt{1-\sin^2\theta} > \sqrt{1-\theta^2} > 1-\theta^2,$$

so we have

$$1-\theta^2 < \cos\theta \le \frac{\sin\theta}{\theta} < 1$$

for values of θ sufficiently close to 0. Thus, by the Squeeze Theorem, $\lim_{\theta\to 0}\left(1-\theta^2\right) = \lim_{\theta\to 0} 1 = 1$, we conclude that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Often, when considering $\lim_{x\to a} f(x)$ for some function f, it is useful to look only as x approaches a from one side—that is, we only look at the behavior for $x < a$ or for $x > a$. We illustrate the basic idea with the following example:

Problem 2.10

Consider the function

$$f(x) = \begin{cases} x^2 & x \le 0, \\ x+1 & x > 0. \end{cases}$$

(a) Show that $\lim_{x\to 0} f(x)$ does not exist.

(b) What would be a reasonable answer to the question "what is the limit as $f(x)$ approaches $x = 0$ from the left?" What if we replace "from the left" with "from the right"?

Solution for Problem 2.9:

(a) The graph of $y = f(x)$ has a "jump" at $x = 0$. It certainly does not appear that a limit exists at $x = 0$. To prove this, suppose that $\lim_{x\to 0} f(x) = L$, and we will attempt to show a contradiction. If the limit exists, then for any $\varepsilon > 0$, there must exist some $\delta > 0$ such that:

$$0 < |x| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Choose some x such that $0 < x < \delta$. Then, since $|x| = x$ and $|x^2| = x^2 \le x$, we must have

$$|x+1-L| < \varepsilon \quad \text{and} \quad |L+x| = |-x-L| < \varepsilon.$$

Adding these and applying the Triangle Inequality gives

$$|1+2x| < 2\varepsilon \quad \Rightarrow \quad 1 < 2\varepsilon.$$

But this must be true for any value of ε, and clearly it cannot be true for $\varepsilon < \frac{1}{2}$. This is our contradiction, so the limit cannot exist.

(b) When we look at values of x to the left of 0, we'll ignore the values of x to the right of 0, so we see that the graph of the function $f(x)$ approaches the point $(0, 0)$. So it makes sense to say "the limit of $f(x)$ as x approaches 0 from the left is 0."

Similarly, when we look at values of x to the right of 0, and ignore the values of x to the left of 0, we see that the graph of the function $f(x)$ approaches the point $(0, 1)$. So it makes sense to say "the limit of $f(x)$ as x approaches 0 from the right is 1."

0

We'll denote the so-called **one-sided limits** from Problem 2.9 with the notations

$$\lim_{x\to 0^-} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = 1.$$

The superscript "$-$" in 0^- means to consider only values to the left of 0 (that is, less than 0), and the superscript "$+$" in 0^+ means to consider only values to the right of 0 (that is, greater than 0).

To make a formal definition of one-sided limits, we replace the values of x under consideration to those on one side or the other:

Definition: (a) We say that $\lim_{x\to a^-} f(x)$ has a limit L as approaches a from the left, denoted

$$\lim_{x\to a^-} f(x) = L,$$

if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < a - x < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

(b) We say that $f(x)$ has limit L as x approaches a from the right, denoted

$$\lim_{x\to a^+} f(x) = L,$$

if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Note that all we changed are the values of x that we have to consider. When we approach from the left, we only consider x such that $0 < a - x$; that is, $x < a$. When we approach from the right, we only consider x such that $0 < x - a$; that is, $x > a$.

Let's verify that these are smaller conditions than the regular definition of limit.

Problem 2.10

Let f be a real-valued function. Show that

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = L$$

if and only if $\lim_{x\to a^-} f(x) = L$. (In other words, if the two one-sided limits agree at a point a, then the limit at a exists and is equal to the one-sided limits, and vice versa.)

Solution for Problem 2.10: This is primarily a matter of chasing the definitions. Suppose the two one-sided limits each equal L. Then, by definition, for any $\varepsilon > 0$ we can find δ_+ such that:

$$0 < a - x < \delta_+ \quad \Rightarrow \quad |f(x) - L| < \varepsilon,$$

and we can find δ_- such that

$$0 < x - a < \delta_- \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

We can satisfy these simultaneously by letting $\delta = \min\{\delta_+, \delta_-\}$. Then the two statements combine into one:

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

This is the definition of $\lim_{x\to a} f(x) = L$, so we have finished the proof in this direction.

We will leave the proof of the other direction—that the existence of the limit implies the existence of the two one-sided limits—as an exercise. □

Exercises

2.1.1:

Give an example of a function f in which $0 \in \mathrm{Dom}\, f$ but where $\lim_{x\to 0} f(x)$ is undefined.

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Solution: The function from Problem 2.9 will do. $f(0) = 0$ but since the left and right hand limits at zero are different, there is no limit at 0.

2.1.2:

Show, using the ε-δ definition, that for any $a \in \mathbb{R}$, $\lim_{x\to a} x = a$.

Hint: Does the value of a matter?

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Solution: Given ε, we need to find a δ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |x - a| < \varepsilon.$$

Comparing these two, it is clear that choosing $\delta = \varepsilon$ will do the trick. Then

$$\lim_{x\to a} x = a.$$

□

2.1.3:

Prove that if $\lim_{x\to a} f(x) = L$ and $c \in \mathbb{R}$, then $\lim_{x\to a}(cf)(x) = cL$.

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Solution: Let $\varepsilon > 0$ be given. By definition, we need to find a δ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |(cf)(x) - cL| < \varepsilon.$$

If $c = 0$, then this last inequality is just $0 < \varepsilon$, which is always true by definition. If $c \neq 0$, then $|c|(|f(x) - L| < \varepsilon$ is the same as $|f(x) - L| < \frac{\varepsilon}{|c|}$ and because we are given that $\lim_{x\to a} f(x) = L$, there is some δ such that

$$\begin{aligned} &0 < |x - a| < \delta \\ \Rightarrow\ &|f(x) - L| < \frac{\varepsilon}{|c|} \\ \Rightarrow\ &|c||f(x) - L| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{x\to a}(cf)(x) = cL$.

□

2.1.4:

The **greatest integer function**, denoted $f(x) = \lfloor x \rfloor$, is the function in which $f(x)$ equals the greatest integer less than or equal to x. So

$$\lim_{x\to 2^-} \lfloor x \rfloor = 1 \text{ and } \lim_{x\to 2^+} \lfloor x \rfloor = 2.$$

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Solution: We notice first that the greatest integer function is constant on the intervals $[n, n+1)$ between integers. Specifically if $1 \le x < 2$ then $f(x) = 1$ and if $2 \le x < 3$ then $f(x) = 2$.

Let $\varepsilon > 0$ be given, and choose $\delta = 1$. Then

$$\begin{aligned} &0 < |x - a| < \delta \\ \Rightarrow\ &1 \le x < 2 \\ \Rightarrow\ &f(x) = 1 \\ \Rightarrow\ &|f(x) - 1| = |1 - 1| = 0 < \varepsilon. \end{aligned}$$

Therefore $\lim_{x\to 2^-} \lfloor x \rfloor = 1$.

Likewise, we check the limit from the right. Let $\varepsilon > 0$ be given and let $\delta = 1$. Then

$$\begin{aligned} &0 < x - 2 < \delta \\ \Rightarrow\ &2 \le x < 3 \\ \Rightarrow\ &f(x) = 2 \\ \Rightarrow\ &|f(x) - 2| = |2 - 2| = 0 < \varepsilon. \end{aligned}$$

Therefore $\lim_{x\to 2^+} \lfloor x \rfloor = 2$.

□

2.1.5:

Prove that if $\lim_{x\to a} f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(x+h) = L.$$

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Solution: Let's begin by trying to show that if f has a limit at a, then f has a limit from the left. Assume that $\lim_{x\to a} f(x) = L$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

We need to find δ' such that

$$0 < a - x < \delta' \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

However, notice that the δ from above works, since $0 < a - x < \delta$ is more strict than $0 < |x - a| < \delta$, meaning when the first is satisfied then the second is satisfied. So if $\delta' = \delta$, then

$$0 < a - x < \delta \quad \Rightarrow \quad 0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

This shows $\lim_{x\to a^-} f(x) = L$. Furthermore the same logic works to show that f has the same limit from the right:

$$0 < x - a < \delta \quad \Rightarrow \quad 0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon,$$

showing $\lim_{x\to a^+} f(x) = L$.

□

2.1.6★:

Show that

$$\lim_{x\to 0}(f+g)(x) = \left(\lim_{x\to 0} f(x)\right) + \left(\lim_{x\to 0} g(x)\right),$$

assuming all terms are defined. (Caution: this is harder than it looks.)

Hint: Let $\lim_{x\to 0} f(x) = F$ and $\lim_{x\to 0} g(x) = G$. Write

$$(f+g)(x) - FG$$

as

$$(f(x)g(x) - f(x)G) + (f(x)G - FG)$$

and use the Triangle Inequality.

Hint: Find δ so that each of $|f(x)g(x) - f(x)G|$ and $|f(x)G - FG|$ are bounded by $\frac{\varepsilon}{2}$.

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Solution: We begin by labeling $\lim_{x\to 0} f(x) = F$, and $\lim_{x\to 0} g(x) = G$. In general for problems involving limits, it is a good strategy to begin with the quantity we want to show is less than ε and work from there:

$$|f(x)g(x) - FG|.$$

What we want to do now is distinguish the appearances of the functions f and g. We apply a popular algebraic trick of adding and subtracting the function $f(x)G$ and applying the Triangle Inequality:

$$\begin{aligned} &|f(x)g(x) - FG| = |f(x)g(x) - f(x)G + f(x)G - FG| \\ &\qquad = |f(x)(g(x) - G) + (f(x)G - FG)| \\ &\qquad \le |f(x)||g(x) - G| + |f(x) - F||G|. \end{aligned}$$

We applied the Triangle Inequality in the middle to separate the two quantities. This gave us a value which is an upper bound of the quantity we're interested in.

Notice that this term is not guaranteed to vanish. We want we chose might have been zero. It may be possible that we can make the left-hand side arbitrarily small regardless of what G has been chosen. Fortunately what we've written looks very promising, since the quantities $|g(x) - G|$ and $|f(x) - F|$ things we can make small by choosing δ small. We will try to force each of $|f(x)| \cdot |g(x) - G|$ and $|f(x) - F||G|$ to be smaller than $\frac{\varepsilon}{2}$.

First consider the quantity

$$|f(x) - F| \cdot |G|.$$

If $|G| = 0$ then this is always smaller than $\frac{\varepsilon}{2}$. We notice that

$$|f(x) - F| \cdot |G| \le |f(x) - F| \cdot (|G| + 1).$$

Now since $\lim_{x\to 0} f(x) = F$, for any given ε, we can choose δ_1 such that

$$0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - F| < \frac{\varepsilon}{2(|G| + 1)}$$

from which it follows that

$$|f(x) - F| \cdot |G| \le \frac{\varepsilon}{2(|G| + 1)} \cdot |G| \le \frac{\varepsilon}{2}.$$

Next we try to bound

$$|f(x)| \cdot |g(x) - G|.$$

$\frac{\varepsilon}{2}$. This is more difficult since the factor $|f(x)|$ varies based on x. However we notice that when x is near a, the value $f(x)$ should be near F. Specifically another application of the Triangle Inequality gives

$$|f(x)| = |f(x) - F + F| \le |f(x) - F| + |F|.$$

From here we can see, because $\lim_{x\to 0} f(x) = F$, that there must be some δ_2 such that

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |f(x) - F| \le 1.$$

Therefore

$$0 < |x - a| < \delta_2 \quad \Rightarrow \quad |f(x)| \le |F| + 1.$$

Therefore

$$|f(x)| \cdot |g(x) - G| \le (|F| + 1)|g(x) - G|.$$

and we are in the same situation as before once we are faced with bounding $|g(x) - G|(|F| + 1)$. Because $\lim_{x\to 0} g(x) = G$, we can choose δ_3 such that

$$0 < |x - a| < \delta_3 \quad \Rightarrow \quad |g(x) - G| < \frac{\varepsilon}{2(|F| + 1)}.$$

Putting all of this together, we can now construct the proof:

Proof. Let $\varepsilon > 0$ be given. Choose δ_1, δ_2 and δ_3 such that

$$\begin{aligned} &0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - F| < \frac{\varepsilon}{2(|G| + 1)} \\ &0 < |x - a| < \delta_2 \quad \Rightarrow \quad |f(x) - F| \le 1 \\ &0 < |x - a| < \delta_3 \quad \Rightarrow \quad |g(x) - G| < \frac{\varepsilon}{2(|F| + 1)}. \end{aligned}$$

The Triangle Inequality gives

$$|f(x) - F| \le 1 \quad \Rightarrow \quad |f(x)| \le |F| + 1.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then if $0 < |x - a| < \delta$,

$$\begin{aligned} |f(x)g(x) - FG| &= |f(x)(g(x) - G) + (f(x) - F)G| \\ &\le |f(x)||g(x) - G| + |f(x) - F||G| \\ &= |f(x)| \cdot |g(x) - G| + |f(x) - F| \cdot |G| \\ &\le (|F| + 1) \cdot \frac{\varepsilon}{2(|F| + 1)} + \frac{\varepsilon}{2(|G| + 1)} \cdot |G| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

2.1.7★:

Prove the Squeeze Theorem.

Hint: If $\frac{\varepsilon}{2} \le g \le h$ and the limit of f and h is L, consider the inequality

$$f(x) - L \le g(x) - L \le h(x) - L.$$

Hint Use

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Solution: We want to prove that if $f(x) \le g(x) \le h(x)$ for all x near the limit a and if h at a are both L, then the limit of g at a must also be L. We start by considering the quantity we want to bound, namely

$$|g(x) - L|.$$

The one fact we need to use is that $f \le g \le h$. We know that this comes in handy if we subtract L from each term:

$$f(x) - L \le g(x) - L \le h(x) - L.$$

Observe now that if $g(x) \ge L$, then

$$0 \le g(x) - L \le h(x) - L = |h(x) - L|.$$

Also, if $g(x) \le L$, then

$$0 \le L - g(x) \le L - f(x) = |f(x) - L|.$$

In either case, we get that

$$|g(x) - L| \le \max\{|h(x) - L|, |f(x) - L|\}.$$

From here we can construct our proof:

Proof. Let $\varepsilon > 0$ and let δ_1, δ_2. Then choose such that

$$\begin{aligned} &0 < |x - a| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \varepsilon \\ &0 < |x - a| < \delta_2 \quad \Rightarrow \quad |h(x) - L| < \varepsilon. \end{aligned}$$

Now we let $\delta = \min\{\delta_1, \delta_2\}$. From the inequalities $f(x) \le g(x) \le h(x)$ we have

$$|g(x) - L| \le \max\{|h(x) - L|, |f(x) - L|\} < \varepsilon.$$

2.2 Continuity

Intuitively, a function is **continuous** if its graph can be drawn without lifting your pen from the paper. What this means is that the graph has no "jumps" or "holes". Another way to think about continuity is that small changes in the input value should produce small changes in the "jumps" at, if that, it seems unnecessary to stop to carefully "wait for the dust to settle" before the jumps" means. But it's actually pretty easy: we use limits. Continuity means that if the graph of a function approaches a point, then the graph actually hits that point; it doesn't skip it or jump away at the last minute.

Definition: Let f be a real-valued function. We say that f is **continuous** at a point $a \in \mathrm{Dom}(f)$ if:

$$\lim_{x \to a} f(x) = f(a).$$

If f is continuous at every point in its domain, we say that f is **continuous everywhere** (or simply **continuous**).

However, we need to modify this definition slightly if a is on the boundary of $\mathrm{Dom}(f)$. In this case, instead of the limit, we only need use a suitable one-sided limit. For instance, if $\mathrm{Dom}(f) = [a, b)$ then we say that f is continuous at a if $\lim_{x \to a^+} f(x) = f(a)$ since the function isn't defined to the left of a, we just use the one-sided limit as f approaches a from the right. Similarly, we say that f is continuous at b if $\lim_{x \to b^-} f(x) = f(b)$ since we only care about what happens to f approaches b from the left.

We'll start with some simple examples to show that continuity is not quite as mysterious as it may seem.

Problem 2.11

(a) Prove that $f(x) = c$, where $c \in \mathbb{R}$ is a constant, is continuous.

(b) Prove that $f(x) = x$ is continuous.

Solution for Problem 2.11:

(a) Let $a \in \mathbb{R}$. We wish to show that $\lim_{x \to a} f(x) = f(a) = c$. So, using the definition of limit, we let $\epsilon > 0$ given. We need to choose δ so that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

But since $f(x) = f(a) = c$ for all x, the inequality on the right is just $0 < \epsilon$, which is true for any value of δ. So the constant function is continuous.

(b) Let $a \in \mathbb{R}$. We wish to show that $\lim_{x \to a} f(x) = f(a) = a$. Again, we let $\epsilon > 0$ this given, and we need to choose $\delta > 0$ so that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

But the inequality on the right is just $|x - a| < \epsilon$, so choosing $\delta = \epsilon$ does the trick. Thus $\lim_{x \to a} f(x) = a$ and the function is continuous.

One thing to note about continuity is that it is a **local** property. What this means is that, for any $a \in \mathbb{R}$, if two functions f and g are equal on an open neighborhood $(a - \delta, a + \delta)$ for some $\delta > 0$, then f and g are continuous at a both are discontinuous at a. This is because they have the same value at a and the same limit at a (if these exist). If f is an interval, we write $f|_a = g|_a$ to mean that f and g agree on I; that is, $f(x) = g(x)$ for all $x \in I$. For instance $f|_a$ is equal to f restricted to I.

Problem 2.12

Consider the function $f(x) = \frac{1}{x}$ where $\frac{1}{0}$ is the greatest integer less than or equal to x. At what points is f continuous?

Solution for Problem 2.12: The graph of $y = \frac{1}{x}$ is shown at right. Just by observing the graph, we see that it "jumps" at every integer, so we suspect that the function is continuous at all non-integer values and discontinuous at the integers. Let's try to prove this.

2.A Proofs of some continuity results

We prove two important but technical results about continuous functions defined on closed intervals.

> **Important:** The **Intermediate Value Theorem**: If f is a continuous, real-valued function defined on an interval $[a, b]$, with $f(a) \neq f(b)$, and y is a real number between $f(a)$ and $f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = y$.

We prove the theorem in the case that $f(a) < f(b)$. (The proof in the case with $f(a) > f(b)$ is nearly identical.)

Let y be given with $f(a) < y < f(b)$. We want to find some $c \in (a, b)$ such that $f(c) = y$. Define

$$U = \{x \in [a, b] \mid f(x) \leq y\}.$$

In other words, U is the subset of $[a, b]$ that f maps to values less than or equal to y. Since U is a nonempty bounded subset of $\mathbb{R}$ (it's nonempty since $a \in U$), it has a least upper bound. Let $c = \sup U$. We will prove that $f(c) = y$.

First, we show that $c \neq a$. Choose ϵ such that $f(a) < y - \epsilon < y$. (For example, we could take $\epsilon = (y - f(a))/2$, since $f(a) < y$.) Then, since f is continuous at a, we can choose $\delta > 0$ such that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(a) - f(x)| < \epsilon.$$

But then $f(x) < f(a) + \epsilon < y$, so $x \in U$, and thus $[a, a + \delta) \subseteq U$. This means than any upper bound, and in particular the least upper bound, must be at least $a + \delta$; specifically, $c \geq a + \delta$, and in particular $c > a$.

Next, we show that $c \neq b$, using a similar argument. Choose ϵ such that $y < y + \epsilon < f(b)$. (For example, we could take $\epsilon = (f(b) - y)/2$, since $y < f(b)$.) Then, since f is continuous at b, we can choose $\delta > 0$ such that

$$0 < b - x < \delta \quad \Rightarrow \quad |f(x) - f(b)| < \epsilon.$$

But then $f(x) > f(b) - \epsilon > y$, so $x \notin U$, and thus $(b - \delta, b] \cap U = \emptyset$. But this means that $b - \delta$ is an upper bound of U, and hence $c < b$.

Recall our goal is to show that $f(c) = y$, where $c = \sup U$. Let $z = f(c)$; we will show that $z = y$ by showing that both $z < y$ and $z > y$ lead to contradictions.

If $f(c) = z < y$, then pick any ϵ such that $z < z + \epsilon < y$. Since f is continuous, we have $\lim_{x \to c} f(x) = z$, so there is some δ such that

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - z| < \epsilon.$$

Pick δ satisfying the above to be small enough so that $c + \delta < b$. (We can do this since we've already proved that $c < b$.) Let c' be any real number such that $c < c' < c + \delta$. Then $|f(c') - z| < \epsilon$, so $f(c') < z + \epsilon < y$. This means that $c' \in U$, which contradicts the fact that c is an upper bound for U.

On the other hand, if $f(c) = z > y$, then pick any ϵ such that $y < z - \epsilon < z$. Since f is continuous, we have $\lim_{x \to c} f(x) = z$, so there is some δ such that

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - z| < \epsilon.$$

Pick δ small enough so that $a < c - \delta$, and let c' be any real number such that $c - \delta < c' < c$. Then $|f(c') - z| < \epsilon$, giving $f(c') > z - \epsilon > y$. But then c' is an upper bound for U as well, contradicting the fact that c was the least upper bound.

Thus we must have $f(c) = y$, proving the theorem.

> **Important:** The **Boundedness Theorem**: suppose f is continuous on a closed interval $[a, b]$. Then there exists some value $M \in \mathbb{R}$ such that $f(c) \leq M$ for all $c \in [a, b]$. (That is, M is an upper bound for f on the interval $[a, b]$.)

Let's try to isolate the point where f might first become unbounded. In particular, let's start at the point $x = a$ and move slowly rightward, and try to figure out exactly where the function might become unbounded.

Clearly f is bounded on the interval $[a, a]$ (which is just the point a), since $f(a)$ is an upper bound. We move to the right, and keep track of where the function stays bounded. Specifically, let

$$D = \{x \in [a, b] \mid f([a, x]) \text{ is bounded}\}.$$

Clearly D is a bounded set (it's a subset of $[a, b]$), and by our basic properties of $\mathbb{R}$, we know that D must have a least upper bound. Let $d = \sup D$. Note that for any c such that $a \leq c < d$, the set $f([a, c])$ is bounded.

First, we prove that $d > a$. By assumption, f is continuous at a, thus for any $\epsilon > 0$ we can choose $\delta > 0$ such that

$$f((a, a + \delta)) \subseteq (f(a) - \epsilon, f(a) + \epsilon).$$

In particular, this implies that $(a + \delta) \in D$, so $d \geq a + \delta > a$.

We next consider the possibility that $d \neq b$. Since f is continuous at d, we have $\lim_{x \to d} f(x) = f(d)$. Thus, given $\epsilon > 0$, we can find $\delta > 0$ such that

$$0 \leq |x - d| < \delta \quad \Rightarrow \quad |f(x) - f(d)| < \epsilon.$$

Choose δ small enough so that $(d - \delta, d + \delta) \subseteq (a, b)$. Then

$$f([a, d + \delta)) = f([a, d - \delta]) \cup f((d - \delta, d + \delta)).$$

The first term on the right is bounded since $d - \delta < d$, and the second term is bounded since it lies entirely within the interval $(f(d) - \epsilon, f(d) + \epsilon)$. Thus we see that $[a, d + \delta) \subseteq D$, but this contradicts the fact that $d = \sup D$.

Thus, we have shown that $d = b$. Since f is continuous at b, we have $\lim_{x \to b^-} f(x) = f(b)$. Thus, given $\epsilon > 0$ we can find $\delta > 0$ such that

$$0 \leq b - x < \delta \quad \Rightarrow \quad |f(x) - f(b)| < \epsilon.$$

Choose δ small enough so that $a < b - \delta$. Then

$$f([a, b]) = f([a, b - \delta]) \cup f((b - \delta, b]).$$

The first set on the right is bounded since $\sup D = b$, and the second set is bounded since it lies entirely within the interval $(f(b) - \epsilon, f(b) + \epsilon)$. Thus we see that $f([a, b])$ is bounded, as desired.

Challenge Problems

2.22:

Suppose f is a function with domain $\mathbb{R}$ such that $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Prove that f is continuous at 0.

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Solution: We just use the limit definition with $\delta = \epsilon$: if $\epsilon > 0$ is given, then setting $\delta = \epsilon$ gives

$$0 < |x| < \delta \quad \Rightarrow \quad |f(x)| \leq |x| < \delta = \epsilon.$$

So we have $\lim_{x \to 0} f(x) = 0$. Further, $|f(0)| \leq 0$, so $f(0) = 0$. Since $\lim_{x \to 0} f(x) = f(0)$, we conclude that f is continuous at 0. (Alternatively, we can use the Squeeze Theorem on $f(x)$ and squeeze it between $-|x|$ and $|x|$.)

2.23:

Suppose we define $f(x) = \dfrac{\sin(x)}{x}$, but where x is in *degrees*, not radians. Find $\lim_{x \to 0} f(x)$.

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Solution: To avoid confusion, we write $\overset{\circ}{\sin} x$ to denote sine in degrees. We convert this to the sine function in radians.

$$f(x) = \frac{\overset{\circ}{\sin} x}{x} = \frac{\sin\left(\frac{\pi x}{180}\right)}{x} = \frac{\pi}{180} \cdot \frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}}$$

Since we know that $\dfrac{\sin t}{t} \to 1$ as $t \to 0$, we get

$$\lim_{x \to 0} f(x) = \frac{\pi}{180} \lim_{x \to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}} = \frac{\pi}{180}.$$

(Here we have applied the results from 2.20.)

2.24:

(a) Assume that $\lim_{x \to 0} f(x) = L$. Prove that $\lim_{x \to 0} f(x^3) = L$.

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Solution: Let $\epsilon > 0$ be given. Since $\lim_{x \to 0} f(x) = L$, there exists some $\delta' > 0$ such that

$$|x - 0| < \delta' \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Let $\delta' = \delta^3$. Then

$$|x - 0| < \delta \quad \Rightarrow \quad |x^3 - 0| < \delta^3 = \delta' \quad \Rightarrow \quad |f(x^3) - L| < \epsilon.$$

Therefore, $\lim_{x \to 0} f(x^3) = L$.

(b) Find an example where $\lim_{x \to 0} f(x^2)$ exists but $\lim_{x \to 0} f(x)$ does not exist.

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Your Submission: Solution

Solution: Let

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then f has no limit at 0, but $f(x^2) = 0$ for all x, so $\lim_{x \to 0} f(x^2) = 0$.

2.25:

Suppose that f is a continuous function.

(a) Show that if I is an interval with $I \subseteq \mathrm{Dom}(f)$, then $f(I)$ is an interval.

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Solution: By definition, $f(I)$ is an interval if and only if $u, v \in f(I)$ and $u < w < v$ implies $w \in f(I)$. This follows immediately from the Intermediate Value Theorem.

(b) Show that if I is a closed interval with $I \subseteq \mathrm{Dom}(f)$, then $f(I)$ is a closed interval.

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Solution: We know that $f(I)$ is an interval by part (a). By the Boundedness Lemma, f is bounded and the least upper bound and greatest lower bound are each elements of $f(I)$. Therefore $f(I)$ is a closed interval.

(c) Find examples of noncontinuous functions such that (a) and (b) are not true.

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Solution: Let $f(x) = \lfloor x \rfloor$. The set $I = \left[\frac{1}{2}, \frac{3}{2}\right]$ is a closed interval, but $f(I) = \{0, 1\}$ is not an interval.

2.26:

Let f be a continuous function with domain $[0, 1]$ and range $[0, 1]$. Prove that f must have a **fixed point**: that is, there exists some $a \in [0, 1]$ such that $f(a) = a$. Is the result still true if the domain and range are $(0, 1)$?

Hint: Consider the function $g(x) = f(x) - x$.

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Solution: If $f(0) = 0$ or $f(1) = 1$, then we're done. Otherwise, let $g(x) = f(x) - x$. This is still continuous. Since $f(0) > 0$, we have $g(0) > 0$. Since $f(1) < 1$, we have $g(1) < 0$. So by the Intermediate Value Theorem, there is some a such that $g(a) = 0$, meaning that $f(a) = a$.

It is not true on the open interval $(0, 1)$. For example, let $f(x) = x^2$. If we restrict the domain to $(0, 1)$, then the range is also $(0, 1)$, but there is no fixed point.

2.27★:

Let f and g be functions with domain $\mathbb{R}$. Suppose

$$\lim_{x \to a} f(x) = b \quad \text{and} \quad \lim_{x \to b} g(x) = c.$$

Prove or disprove that we must have $\lim_{x \to a} (g \circ f)(x) = c$. (If true, explain why, with a rigorous proof if possible; if false, give an example.)

Hint: The statement is false.

Hint: It can be false if g is not continuous.

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Solution: This is false. Let

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

and let $f(x) = 0$. Then $(g \circ f)(x) = 1$ and

$$\lim_{x \to 0} f(x) = 0,$$
$$\lim_{x \to 0} g(x) = 0,$$
$$\lim_{x \to 0} (g \circ f)(x) = 1.$$

2.28★:

Consider the function

$$f(x) = \begin{cases} 0 \text{ if } x \notin \mathbb{Q}, \\ \frac{1}{q} \text{ if } x = \frac{p}{q} \in \mathbb{Q}, \end{cases}$$

with $p, q \in \mathbb{Z}, q > 0$, and p, q relatively prime. Show that f is continuous at every irrational point and discontinuous at every rational point in $\mathbb{R}$.

Hint: Show that $\lim_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$.

Hint: For a given ϵ, choose a positive integer n such that $\epsilon > \dfrac{1}{n}$. Then describe how to choose δ so that $|f(x)| < \dfrac{1}{n}$ for all $0 < |x - a| < \delta$.

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Solution: We claim that $\lim_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$.

To prove this, let $\epsilon > 0$ be given. Then $\epsilon > \dfrac{1}{n}$ for some positive integer n. Pick δ small enough so that no integer multiple of $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-1}, \frac{1}{n}$ other than possibly a itself, is within δ of a. (We can always do this since there are only finitely many of these multiples in any bounded interval.) Then $f(x)$ takes values less than $\dfrac{1}{n}$ when $x \neq a$ is within δ of a, and hence

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x)| < \frac{1}{n} < \epsilon.$$

Therefore, $\lim_{x \to a} f(x) = 0$.

Thus, f is continuous exactly at the points at which $f(x) = 0$, and these are exactly the irrational numbers.

Review Problems

2.15:

Use the δ-ϵ definition of limit to prove that $\lim_{x\to 2} x^3 = 8$.

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Solution: We want to be able to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x-2| < \delta \quad \Rightarrow \quad |x^3 - 8| < \epsilon.$$

For $\delta > 0$, if $0 < |x-2| < \delta$, then $2 - \delta < x < 2 + \delta$, and we have

$$(2-\delta)^3 < x^3 < (2+\delta)^3.$$

Expanding gives

$$8 - 12\delta + 6\delta^2 - \delta^3 < x^3 < 8 + 12\delta + 6\delta^2 + \delta^3,$$

thus

$$-12\delta + 6\delta^2 - \delta^3 < x^3 - 8 < 12\delta + 6\delta^2 + \delta^3.$$

But note that if $\delta < 1$, then $\delta^3 < \delta^2 < \delta$, and

$$12\delta + 6\delta^2 + \delta^3 < 19\delta,$$
$$-12\delta + 6\delta^2 - \delta^3 > -19\delta,$$

so we have $|x^3 - 8| < 19\delta$. Thus, choosing $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$ will ensure that

$$0 < |x-2| < \delta \quad \Rightarrow \quad |x^3 - 8| < 19\delta \le \epsilon.$$

2.16:

Compute the following:

(a) $\lim_{x\to 1} \frac{x^3-1}{x-1}$

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Solution: If $x \neq 1$, then

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1} = x^2 + x + 1$$

so

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1}(x^2+x+1) = 1^2 + 1 + 1 = \boxed{3}.$$

This is valid because $x^2 + x + 1$ is a polynomial and thus continuous everywhere.

(b) $\lim_{x\to 0} \frac{(x+2)^2-4}{x}$

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Solution: If $x \neq 0$, then

$$\frac{(x+2)^2-4}{x} = \frac{x^2+4x}{x} = x + 4.$$

Therefore,

$$\lim_{x\to 0} \frac{(x+2)^2-4}{x} = \lim_{x\to 0}(x+4) = \boxed{4}.$$

(c) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

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Solution: Factor the denominator as $(1+\sqrt{x})(1-\sqrt{x})$. Then we have $\lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \boxed{\frac{1}{2}}$

(d) $\lim_{x\to 0} \sqrt{x^2}$

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Solution: This is just $0^2 = \boxed{0}$ since $\sqrt{x^2} = |x|$ is continuous.

(e) $\lim_{x\to 1^-} (x - \lfloor x \rfloor)$

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Solution: From the right, $x - \lfloor x \rfloor$ approaches $\boxed{0}$ at each integer.

(f) $\lim_{x\to 1^+} (x - \lfloor x \rfloor)$

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Solution: From the left, $x - \lfloor x \rfloor$ approaches $\boxed{1}$.

2.17:

Find an example of a function $f(x)$ with domain $\mathbb{R}$ for which $\lim_{x\to 0^+} f(x)$ exists but $\lim_{x\to 0^-} f(x)$ does not exist.

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Solution: Here is an example:

$$f(x) = \begin{cases} 1/x & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

In this example, the limit from the left side does not exist, as $\frac{1}{x}$ is unbounded. On the right side, the limit is obviously 0. So $\lim_{x\to 0^+} f(x)$ exists, but $\lim_{x\to 0^-} f(x)$ does not exist, as desired.

2.18:

Suppose that f has domain $\mathbb{R}$ and is continuous. Prove or disprove: if the range of f contains both positive and negative numbers, then it must contain 0.

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Solution: If f is continuous on $\mathbb{R}$ and takes on both positive and negative values, then f must hit zero. Take $a < b$ such that $f(a)$ and $f(b)$ have different signs. Since f is continuous, by the Intermediate Value Theorem $f([a,b])$ contains all values between $f(a)$ and $f(b)$. Since one of these is positive and the other negative, $0 \in f([a,b]) \subseteq f(\mathbb{R})$.

2.19:

Find a function f with domain $\mathbb{R}$ that is continuous on all of $\mathbb{R}$ except -1, 0, and 1.

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Solution: Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \{-1, 0, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(x)$ is 0 except for $x \in \{-1, 0, 1\}$, we have $\lim_{x\to a} f(x) = 0$ for any $a \in \mathbb{R}$. Therefore f is continuous on $\mathbb{R} \setminus \{-1, 0, 1\}$.

2.20:

(a) Let f be a function such that $\lim_{x\to 0} f(x) = L$. Prove that $\lim_{x\to 0} f(cx) = L$ for any nonzero constant c.

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Solution: Let $\epsilon > 0$ be given. There exists δ such that

$$0 < |x| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

This means that

$$0 < |cx| < \delta \quad \Rightarrow \quad |f(cx) - L| < \epsilon,$$

so

$$0 < |x| < \frac{\delta}{|c|} \quad \Rightarrow \quad |f(cx) - L| < \epsilon.$$

Therefore, $\lim_{x\to 0} f(cx) = L$.

(b) Use part (a) to compute $\lim_{x\to 0} \frac{\sin ax}{x}$, where a is a nonzero real number.

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Solution: By part (a), and the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we have $\lim_{x\to 0} \frac{\sin ax}{ax} = 1$. Therefore,

$$\lim_{x\to 0} \frac{\sin ax}{x} = \lim_{x\to 0} a\left(\frac{\sin ax}{ax}\right) = a\left(\lim_{x\to 0} \frac{\sin ax}{ax}\right) = a.$$

(c) Use part (b) to compute $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$, where a and b are nonzero real numbers.

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Solution: Write

$$\frac{\sin ax}{\sin bx} = \frac{\sin ax}{x} \cdot \frac{x}{\sin bx}.$$

The limits of these two terms are a and $\frac{1}{b}$, respectively, so

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}.$$

2.21:

Show that f is continuous if and only if for all $a \in \mathrm{Dom}(f)$ and all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$

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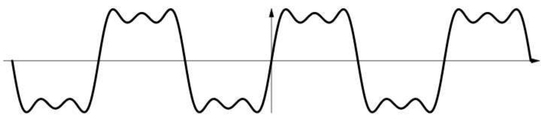
Solution: f is continuous if and only if $\lim_{x\to a} f(x) = f(a)$ for all $a \in \mathrm{Dom}(f)$. This means that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon,$$

which matches the condition in the problem except for the "0 <" portion at the far left. However, if $0 = |x - a|$, then $x = a$ and $f(x) = f(a)$ so $|f(x) - f(a)| = 0 < \epsilon$. Thus, we have

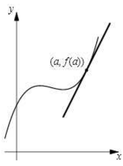
$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \epsilon.$$



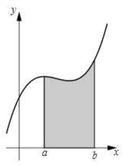
CHAPTER 3 The Derivative

Calculus provides the answers to two natural geometric questions regarding the graph $y = f(x)$ of a function f.

Question 1: Given a real number a in the domain of a function f, what is the slope of the line that is tangent to the graph $y = f(x)$ at the point $(a, f(a))$?



Question 2: Given real numbers $a \leq b$ such that the interval $[a, b]$ is a subset of the domain of a function f, what is the area of the region bounded by the graph $y = f(x)$, the lines $x = a$ and $x = b$, and the x-axis?



We'll defer our discussion of Question 2 until Chapter 5, when we will have some more tools at our disposal. Our immediate goal is to try to answer Question 1. The answer to Question 1 is the fundamental calculus concept known as the **derivative**.

3.1 Intuitive introduction

Our first task is to decide what we mean by "tangent." We can think back to plane geometry: given a circle $\mathcal{C}$ and a point P on $\mathcal{C}$, there is a unique line ℓ through P that does not intersect $\mathcal{C}$ at another point. This is called the **tangent line** to $\mathcal{C}$ at P. Every other line through P will also intersect the circle at some other point; such lines are called **secant lines**.

Let's provisionally use this as our definition of a tangent line to the graph of a function:

Provisional Definition #1:	**Provisional Definition #1**: A **tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$, where a is in the domain of f, is a line ℓ passing through $(a, f(a))$ that does not intersect the graph at any other point.

Let's try this provisional definition with a simple function:

Problem 3.1

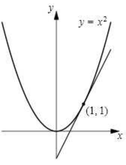
Let $f(x) = x^2$. Find a "tangent line" (under our provisional definition above) to the graph $y = f(x)$ at the point $(1, 1)$.

Solution for Problem 3.1: Any line (other than a vertical line) through $(1, 1)$ will satisfy the equation $(y - 1) = m(x - 1)$, where m is the slope of the line. So $y = mx + (1 - m)$. We want to find m such that the point $(1, 1)$ is the only intersection of $y = x^2$ and $y = mx + (1 - m)$. This means that $(x, y) = (1, 1)$ is the only solution to the system of equations

$$\begin{aligned} y &= x^2, \\ y &= mx + (1 - m). \end{aligned}$$

Eliminating y gives us $x^2 - mx + (m - 1) = 0$, which is a quadratic equation in x. Clearly $x = 1$ is a solution to this equation (as we already know), but it will have another solution if the discriminant is positive. So for $x = 1$ to be the only solution, we must have $m^2 - 4(m - 1) = 0$. This simplifies to $(m - 2)^2 = 0$, so $m = 2$. (Note that for all other values of m, the discriminant is positive, so there are two solutions.) Therefore, the slope of the tangent line is $m = 2$, and the equation of the line is $y = 2x - 1$. □

We can sketch a picture of this, as shown to the right, to convince ourselves that we did the right thing. It does indeed look like the correct tangent line. This all looks fine, but it seems like a lot of trouble to go through just to find a tangent line. Another issue is that there is another line that intersects $y = x^2$ only at the point $(1, 1)$, and that is the line $x = 1$. We didn't get that line in our solution above because its slope is undefined. (Remember our "other than a vertical line" qualifier in the first sentence of our solution to Problem 3.1.)



Moreover, we can see that in any graph $y = f(x)$, the line $x = a$ will intersect the graph in only one point, namely $(a, f(a))$. This is the Vertical Line Test for graphs of functions: there is only one y-value for each possible x-value in the domain.

We can revise our provisional definition to exclude vertical tangent lines:

Provisional Definition #2:	**Provisional Definition #2**: A **tangent line** to the graph of $y = f(x)$ at the point $(a, f(a))$, where a is in the domain of f, is a line ℓ, not parallel to the y-axis, passing through $(a, f(a))$ that does not intersect the graph at any other point.

But there's a more significant flaw, as we'll see with a slightly more complicated example:

Problem 3.2

Find the equation of the tangent line to the graph of $y = x^3 + x$ at the point $(1, 2)$.

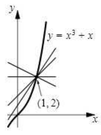
Solution for Problem 3.2: A non-vertical line through $(1, 2)$ has the equation $(y - 2) = m(x - 1)$. So we are looking for a value of m that gives only the solution $(x, y) = (1, 2)$ to the system

$$\begin{aligned} y &= x^3 + x, \\ y &= mx + (2 - m). \end{aligned}$$

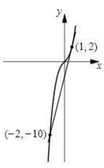
Setting the two equations equal gives $x^3 + (1 - m)x + (m - 2) = 0$. We know that $x = 1$ is always a solution, so we can factor the equation and get $(x - 1)(x^2 + x - (m - 2)) = 0$. We need the cubic to have no other roots, so either the quadratic has the double root $x = 1$, or the quadratic has no roots (since we want $x = 1$ to be the only root of the original cubic). The only monic quadratic with double root $x = 1$ is $x^2 - 2x + 1$, which clearly does not match our quadratic, so the only possibility is for the quadratic to have no root.

Thus, we need the discriminant of the quadratic to be negative. The discriminant is $1 + 4(m - 2)$, which is negative when $m < \frac{7}{4}$. So any line with slope less than $\frac{7}{4}$ through $(1, 2)$ intersects the graph of $y = x^3 + x$ only at the point $(1, 2)$. □

This is a bad state of affairs. We have a whole range of "tangent" lines. Worse, none of these lines really look like good candidates for a tangent line, as they all "cross" the graph, as shown in the diagram to the right. If we think back to our original circle example, the tangent line through a point on a circle just "touches" the circle without "crossing" it. We'd like that property to be true of more general tangents to graphs.



We can see a little bit of what's really going on here if we look at the value of m that makes our quadratic $x^2 + x - (m - 2)$ also have a root of $x = 1$. Plugging in $x = 1$ gives $2 - (m - 2) = 0$, so $m = 4$. If we draw the line with slope 4 through the point $(1, 2)$, which is the line $y = 4x - 2$, then we get the picture at left below. This looks like what we should have as our "tangent line" to $y = x^3 + x$ at $(1, 2)$: it just "touches" the curve at $(1, 2)$, but doesn't cross it. However, this line has the unfortunate property that it intersects the curve at another point, namely $(-2, -10)$, as shown in the picture to the left below. Should we be worried about this?

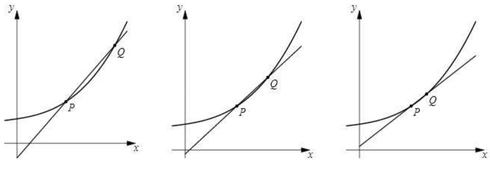


The number of words that appear inside quotation marks in the last two paragraphs is a sign that we're being too vague. So we'll try to define precisely what a tangent line is, but as we've seen, this is not easy.

However, what *is* easy to define is a secant line:

Definition:	A **secant line** of the graph $y = f(x)$ is any line that intersects the graph in at least two distinct points.

So given two points $P = (a, f(a))$ and $Q = (b, f(b))$ on the graph of $y = f(x)$, with $a \neq b$, the line $\overleftrightarrow{PQ}$ is a secant line. Unless the graph of $y = f(x)$ is a line–that is, unless $f(x)$ is a linear polynomial or a constant–we expect that there will be infinitely many secant lines through P. However, if Q is very close to P, then the secant line $\overleftrightarrow{PQ}$ is very close to what we want to be the tangent line at P. In fact, as Q approaches P, then the secant line approaches the tangent line:



"Approaches"... this sounds like a job for a limit!

3.2 Definition of the derivative

Let's take our introductory discussion from the end of the previous section and try to make it rigorous.

Problem 3.2.1

Let $P = (x, f(x))$ be a point on the graph of $y = f(x)$. We want to compute the slope of the tangent line to the curve at the point P.

(a) Let $Q = (h, f(h))$ be another point on the curve (so that $h \neq x$). Compute the slope of the secant line $\overleftrightarrow{PQ}$.

(b) Rewrite part (a) above as $m(h) = \frac{?}{?}$. For some f, f, but that $Q = (x + h, f(x + h))$. Write the slope in terms of x and h.

(c) In terms of x and h, what does it mean for Q to be "approach" P?

(d) Write an expression (using x and h) for the slope of the tangent line at P.

Solution for Problem 3.2

(a) We start with a secant $\overleftrightarrow{PQ}$ where $Q = (h, f(h))$ is some point other than P. The slope of $\overleftrightarrow{PQ}$ is, as usual, the difference in the y-coordinates divided by the difference in the x-coordinates.

$$\text{slope of } \overleftrightarrow{PQ} = \frac{f(h) - f(x)}{h - x}$$

(b) Instead of writing Q in terms of a "new" x-coordinate h let's write it relative to P that is, let's set $h = x + h$ for some nonzero real number h, so that $Q = (x + h, f(x + h))$. Our expression for the slope then becomes:

$$\text{slope of } \overleftrightarrow{PQ} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

(c) As Q approaches P, the number h approaches 0, and the secant line $\overleftrightarrow{PQ}$ "approaches" the tangent line at P. This line, because the secant line, $\overleftrightarrow{PQ}$ approaches the tangent line to not well-defined, but the point about h approaching 0 is, it's a limit.

(d) We can define the tangent line at P as the line through P whose slope is the limit of the slope of the secant $\overleftrightarrow{PQ}$ as Q approaches P. This gives us:

$$\text{slope of tangent line at } P = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

◆

To summarize, we have constructed the following definition (non-provisional at last!) of a tangent line to the graph of a function.

> **Definition:** The **tangent line** to the curve $y = f(x)$ at the point $P = (x, f(x))$ is the line passing through P with slope
>
> $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$
>
> provided that this limit is defined.

The quantity described by the above limit is so fundamental to calculus that it has a name:

> **Definition:** The **derivative** of the function f at a, denoted $f'(a)$ is
>
> $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$
>
> provided this limit exists. If the limit exists, we say that f is **differentiable** at a. We say that f **differentiable** if it is differentiable at every point in its domain.

We see that the derivative $f'(a)$, by definition, is the slope of the tangent line at $(a, f(a))$ provided both are defined.

Note that we could have instead defined

$$f'(a) = \lim_{x \to a} \frac{f(b) - f(a)}{b - a}$$

and this is in fact the exact same quantity as our definition of derivative, but it is usually easier to compute limits when something is approaching 0, as in $\lim_{h \to 0}$. There are other minor variations in the form of the definition of the derivative, you will explore some of these as some exercises.

Let's go back to our earlier example:

Problem 3.4

Let $f(x) = x^2$ that the tangent line to the graph $y = f(x)$ at the point $(1, 1)$.

Solution for Problem 3.4

Using the definition of derivative, we can immediately see that the slope of the tangent line to the graph of $f(x) = x^2 = x$ at $(1, 2)$ is $f'(1) = 2(1)^2 + 1 = 4$ and thus the tangent line is $(1, 2)$ is $y - 4x$.

The derivative of $f(x)$ has many different notations:

$$f'(x) \quad D(f)(x) \quad f(x) \quad \frac{df}{dx}(x) \quad \frac{df}{dx}(x)$$

We'll usually use the first notation. The second notation is called **operator** notation, but we will not use it anywhere in this book. The third notation (the function with a dot over it) is commonly used in physics, but will also not appear elsewhere in this book. The last two notations in the above list are called **differential notation** and are the notations of choice for many of these later in the chapter. The notations can be used both with functions that are explicitly named and functions that are not. For example, if $f(x) = \sin x + x^2$, we could write any of

$$f'(x), \text{ Dsin } x + x^2)', \frac{df}{dx}, \frac{d}{dx}f(x), \frac{d}{dx}(\sin x + x^2)$$

to indicate the derivative of f.

Just because we have a variable chosen at the definition, this doesn't mean that the derivative is always defined. There are basically two ways in which a function can not be differentiable. A function might be undefined for a function f and a real number a in the domain of f in two different ways (circumstances in the real f problem).

Problem 3.5

Suppose f is a function such that f is differentiable at $\frac{1}{2}$ (that means that $f'(a)$ is defined). Compute $\lim_{h \to 0} f(h)$ in terms of $f(x)$ and $f'(x)$ and explain why this means that f must be continuous at x.

Solution for Problem 3.6

We start with $\lim_{h \to 0} f(a + h)$ and do some algebra to get the definition of $f'(a)$ into the picture.

$$\begin{aligned}
\lim_{h \to 0} f(a+h) &= \lim_{h \to 0} f(a+h) - f(a) + f(a) \\
&= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \cdot h + f(a) \right) \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h + \lim_{h \to 0} f(a) \\
&= \lim_{h \to 0} \frac{f(a+h) - a}{h} \cdot 0 + f(a) \\
&= f(a)
\end{aligned}$$

The first term in the last line above is just the definition of $f'(a)$, which exists since we are assuming that f is differentiable at a. Evaluating the other limits in the above expression, we have

$$\lim_{h \to 0} f(a+h) = f'(a) \cdot 0 + f(a) = f(a).$$

So $\lim_{h \to 0} f(a + h) = f(a)$ But this is exactly the same thing as saying that $\lim_{x \to a} f(x) = f(a)$ and hence f is continuous at a.

So Problem 3.5 tells us the following fundamental fact:

> **Concept:** Differentiability implies continuity. That is, $f'(a)$ is defined only if f is continuous at a. If f is not continuous at a then $f'(a)$ is undefined.
>
> **Important:** Continuity does not imply differentiability.

Indeed, the next problem is an example of a continuous function that is not everywhere differentiable:

Problem 3.7

Let $f(x) = |x|$ be the absolute value function.

(a) Where is $f'(x)$ at $x = 0$?

(b) What is $f'(x)$ at $x = 0$?

(c) Attempt to compute $f'(0)$ and verify that is undefined.

(d) Explain, in terms of the graph of f, why $f'(0)$ is undefined.

Solution for Problem 3.7

(a) If $x > 0$ then $f(x) = |x| = x$. So, if we restrict ourselves to h sufficiently close to 0 so that $x + h > 0$ we can write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h) - x}{h}$$
$$= \lim_{h \to 0} \frac{(x+h) - x}{h}$$
$$= \lim_{h \to 0} \frac{h}{h}$$
$$= 1.$$

So $f'(x) = 1$ for all $x > 0$.

(b) If $x < 0$ then $f(x) = |x| = -x$. As in part (a), if we restrict ourselves to h sufficiently close to 0 so that $x + h < 0$, we can write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h) - x}{h}$$
$$= \lim_{h \to 0} \frac{(x+h) - (-x)}{h}$$
$$= \lim_{h \to 0} \frac{-h}{h}$$
$$= -1.$$

So $f'(x) = -1$ for all $x < 0$.

(c) We compute

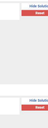
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

But we see a problem with this limit. The function $\frac{|h|}{h}$ is not continuous at 0, since

$$\frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0, \\ -1 & \text{if } h < 0. \end{cases}$$

The graph of $\frac{|h|}{h}$ is shown at right, and we can see that it is not continuous at 0. More specifically, we see that

$$\lim_{h \to 0^+} \frac{|h|}{h} = 1 \quad \text{and} \quad \lim_{h \to 0^-} \frac{|h|}{h} = -1.$$



Since the left-sided limit and the right-sided limit are different, we conclude that $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. Thus, $f'(0)$ is undefined.

(d) We sketch the graph of $f(x) = |x|$ below:

The $x > 0$ portion of the graph suggests that the slope of the tangent line at $x = 0$ should be 1. But the $x < 0$ portion of the graph suggests that the slope of the tangent line at $x = 0$ should be -1. This comes about because there is a sharp corner in the graph at the point $(0, 0)$ so we can't have a uniquely defined tangent line at that point.

◆

> **Concept:** If the graph of $y = f(x)$ has a "sharp corner" at $(a, f(a))$ then the derivative $f'(a)$ is undefined.
>
> **Extreme:** Rather bizarrely, it turns out that there are functions that are continuous everywhere yet $\mathbb{R}$ but are differentiable at no points of $\mathbb{R}$. However, it is very difficult to write down the description of such a function (and virtually impossible to draw its graph).

Problem 3.8

(a) We can try to compute $f'(0)$ using the definition of derivative

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}}.$$

But this limit does not exist: the denominator goes arbitrarily small as $h \to 0$ so the function grows arbitrarily large. (We may say that $\lim_{h \to 0} \frac{1}{\sqrt{h}} = \infty$, as we will explore further in Chapter 6.)

(b) Shown at right is the graph of $y = \sqrt{x}$, with a "tangent line" drawn at the point $(0, 0)$. We see that the tangent line is vertical, and thus has slope ∞. We see that $f'(0) = \infty$ in this case.

(c) If the graph of $y = f(x)$ has a vertical "tangent line" at $(a, f(a))$, then $f'(a)$ is undefined.

> **Concept:** If the graph of $y = f(x)$ has a vertical "tangent line" at $(a, f(a))$, then $f'(a)$ is undefined.
>
> **Extreme:** Rather bizarrely, it turns out that there are functions that are continuous everywhere on $\mathbb{R}$ but are differentiable at no points of $\mathbb{R}$. However, it is very difficult to write down the description of such a function (and virtually impossible to draw its graph).

Problem 3.9

(a) Explain why $f'(0)$ is undefined using the definition of derivative.

(b) Explain why $f'(0)$ is undefined in terms of the graph of f.

Solution for Problem 3.9

(a) We can try to compute $f'(0)$ using the definition of derivative

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}}.$$

But this limit does not exist: the denominator goes arbitrarily small as $h \to 0$ so the function grows arbitrarily large. (We may say that $\lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = \infty$, as we will explore further in Chapter 6.)

(b) Shown at right is the graph of $y = \sqrt[3]{x}$ with a "tangent line" drawn at the point $(0, 0)$. We see that the tangent line is vertical, and thus has slope ∞. We see that $f'(0) = \infty$ in this case.

◆

> **Concept:** If the graph of $y = f(x)$ has a vertical "tangent line" at $(a, f(a))$, then $f'(a)$ is undefined.

We summarize the non-differentiability criteria evident in the previous three problems:

> **Important:** Given a function f and a number $a \in$ Dom f, there are essentially three reasons why the derivative $f'(a)$ might be undefined:
>
> - f is not continuous at a.
> - The graph of f has a sharp corner at $(a, f(a))$.
> - The graph of f has a vertical tangent line at $(a, f(a))$.

These are occasionally more exotic reasons why $f'(a)$ might be undefined, but virtually all of the examples that you will commonly encounter fall into one of the above three categories.

Exercises

3.2.1:

Use the limit definition of derivative to compute the derivatives of the following functions:

(a) $f(x) = x$ where c is any real number

You may type any additional notes you have here.

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Your Submission: Solution

Solution:

$$f'(x) = \lim_{h \to 0} \frac{x - x}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

(b) $f(x) = 2x$

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Your Submission: Solution

Solution:

$$f'(x) = \lim_{h \to 0} \frac{2(x+h) - 2x}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2$$

(c) $f(x) = x^2 - 3x$

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Solution:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \to 0} (2x + h - 3) = 2x - 3$$

(d) $f(x) = (x - 1)^3$

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Your Submission: Solution

Solution:

$$f'(x) = \lim_{h \to 0} \frac{(x-1+h)^3 - (x-1)^3}{h} = \lim_{h \to 0} \frac{3(x-1)^2 h + 3(x-1)h^2 + h^3}{h} = \lim_{h \to 0} 3(x-1)^2 + 3(x-1)h + h^2) = 3(x-1)^2$$

(e) $f(x) = \frac{1}{x}$

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Solution:

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

3.2.2:

Find the equation of the tangent line to the curve $y = x^3 + 3$ at the point $(1, 4)$.

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Solution: Substituting into the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^3 + 3) - (x^3 + 3)}{h} \\
&= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) + 3) - (x^3 + 3)}{h} \\
&= \lim_{h \to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2
\end{aligned}$$

So the slope of the tangent line at $(1, 4)$ is $4(1)^2 = 4$. Therefore, the tangent line is given by the equation $\{y = 3(x - 1) + 4 = 3x\}$

3.2.3:

Find the equation of the tangent line to the graph $f(x) = \frac{1}{x}$ at the point $(2, 3)$.

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Your Submission: Solution

Solution: By the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{(x+h)^2/(2 - x^2)/2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x
\end{aligned}$$

The tangent line has slope $f'(2) = 2$, and so the line has equation $\{y = 2(x - 2) + 2 = 2x - 2\}$

3.2.4:

Find a function f, with domain $\mathbb{R}$, that is continuous on all of $\mathbb{R}$ but that is not differentiable at $x = 0$ and $x = 1$.

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Your Submission: Solution

Solution: An easy example is $f(x) = |x| + |x - 1|$. This function is continuous on all of $\mathbb{R}$, but its graph has "sharp corners" at $x = 0$ and $x = 1$. We may also note that $f'(x) = -2$ if $x \in (-\infty, 0)$, $f'(x) = 0$ if $x \in (0, 1)$, and $f'(x) = 2$ if $x \in (1, +\infty)$, so f is differentiable on all of $\mathbb{R} \setminus \{0, 1\}$.

3.2.5:

Suppose we had defined the derivative as

$$f'(a) = \lim_{h \to 0} \frac{f(a) - f(a - h)}{h}.$$

Is this equivalent to our previous definition? Why or why not?

Hint: Drawing a picture might help to see what's going on.

Hint: Note that $\lim_{h \to 0} g(x) = \lim_{h \to 0} g(-x)$ for any function g (provided either limit is defined).

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Your Submission: Solution

Solution: First, we observe that for any function g,

$$\lim_{h \to 0} g(x) = \lim_{h \to 0} g(-x),$$

provided either limit is defined. This is because replacing x by $-x$ does not change anything in the $\delta\varepsilon$ definition of the limit as $x \to 0$.

Therefore, we can replace h with $-h$ in the original limit definition of derivative, and we get:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h}$$

So the "new" definition of derivative presented in the problem is equivalent to the original definition.

3.3 Basic derivative computations

In this section, we'll begin to explore some of the basic algebraic properties of the derivative. Our goal is to develop a catalog of derivatives of common functions such as polynomials, log functions, and exponentials, and to derive algebraic rules to compute derivatives of more complicated functions in terms of the derivatives of simpler functions. We'will also use differentiation to a very limited extent.

We'll start with a basic algebraic property of the derivative:

Theorem 3.3.1

Let f, g be differentiable functions. Then $(f+g)' = f' + g'$.

Solution for Problem 3.3.1: This property is not far fetch, and since the derivative is defined as a limit, we should be not surprised that this property is true for derivatives. The proof is straightforward, and it follows from the properties of the corresponding rules for limits:

$$(f+g)'(x) = \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}$$
$$= f'(x)+g'(x).$$

□

It is also true that

Theorem 3.3.2

Let f be a differentiable function and $c \in \mathbb{R}$. Then $(cf)' = cf'$.

The proof is very similar to the solution to Problem 3.3, and we'll leave the details as an exercise.

Exercise

The previous results mean that the derivative is **linear**. The derivative of a sum of functions is the sum of the derivatives of the functions, and we can "take out" constant factors in the derivative.

The linear derivatives of an exercise to summarize: Use this fact to build a catalog of common derivatives. So let's start with some simple functions:

Exercise 3.3.3

Let $f(x) = c$ where c is an integer. Find $f'(x)$.

Solution for Problem 3.3: First, let's look at the case where c is a positive integer. We just plug $c^{\prime}$ into our limit definition of derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

We can expand this to c a negative integer. We'll use the fact that $f(x) = c = -(-c)$:

$$f'(x) = \lim_{h\to 0}\frac{(-c)-(-c)}{h} = \lim_{h\to 0}\frac{-(-c)+(-c)}{h} = -\lim_{h\to 0}\frac{c-c}{h} = -0 = 0.$$

Now we want to simplify. Observe that the $c^{\prime}$ terms cancel out. We want to use the fact that we know that we have at least one factor of h, so we can divide by h. So let's the denominator:

$$f'(x) = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Every term except for the first term has a factor of h, so we can take the limit as h approaches 0. Here terms are all 0. Thus, we've just left with the first term, and we get

$$f'(x) = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

We'll state this result using differential notation:

Theorem

If c is a positive integer, then

$$\frac{d}{dx}c = 0.$$

Notice that when $c = 1$, we get $\frac{d}{dx}c = 1$. This makes perfect sense when we think about the graph of $y = x$: the graph is a line with slope 1, so the tangent line to that graph at any point is the line itself, so the derivative is 1.

When $c = 0$, the function is just the constant function $f(x) = x^0 = 1$. We can again use the limit definition of derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1-1}{h}$$
$$= \lim_{h\to 0}\frac{0}{h} = 0.$$

□

Exercise 3.3.4

Find $f'(x)$ for $f(x) = x$ and $f(x) = x^2$.

Solution for Problem 3.3: Let's start with the case where c is a positive integer. We just plug $c^{\prime}$ into our limit definition of derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)-x}{h} = \lim_{h\to 0}\frac{h}{h} = 1.$$

Yes, this makes perfect sense: the graph of $y = x$ is the standard line, which has slope 1 at every point. Now also that this function satisfies our rather formula $f(x) = 1x$.

Now, let's look at $f(x) = x^2$:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h}$$
$$= \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

□

Finally, we consider the function $f(x) = x^{-1}$. There is a pointe integer. We again try to use our limit definition:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^{-1}-x^{-1}}{h}$$
$$= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{\frac{x-(x+h)}{(x+h)x}}{h}$$
$$= \lim_{h\to 0}\frac{\frac{-h}{(x+h)x}}{h} = \lim_{h\to 0}\frac{-h}{(x+h)xh}$$
$$= \lim_{h\to 0}\frac{-1}{(x+h)x} = \frac{-1}{x^2}.$$

□

Since you've mastered the process, there's no need to be so pedantic. Just immediately write:

$$(x^{-1})' = \lim_{h\to 0}\frac{(x+h)^{-1}-x^{-1}}{h} = \lim_{h\to 0}\frac{\frac{-h}{(x+h)x}}{h} = \lim_{h\to 0}\frac{-1}{(x+h)x} = \frac{-1}{x^2}.$$

□

If we now plug in $k = -$ & all of the terms except for the first disappear from the numerator, and the denominator becomes x^{k+1}. So we have:

$$(x^{-1})' = \lim_{h\to 0}\frac{(x+h)^{-1}-x^{-1}}{h} = \lim_{h\to 0}\frac{\frac{-h}{(x+h)x}}{h} = \lim_{h\to 0}\frac{-1}{(x+h)x} = \frac{-1}{x^2}.$$

Notice that this is the same formula that we found for x^2.

Theorem

If x is an integer, then

$$\frac{d}{dx}x^k = kx^{k-1}.$$

We can think of this as "lowering the exponent by one" and "bringing the exponent to the front" as a constant, and then multiplying by the original exponent.

We can now combine the results of the previous two problems to instantly compute the derivative of any polynomial function, such as the following:

Exercise 3.3.5

Compute the derivatives of the following functions

(a) $f(x) = 3x^2 - 5x + 6$

(b) $f(x) = 4x^3 - 3x^2 + 18x$

(c) $f(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

(d) $f(x) = 18x + 9$

Solution for Problem 3.3:

(a) We'll work out all the steps of this one, just so it's clear what we're allowed to do.

$$f'(x) = (x^2 - 5x + 6)'$$
$$= (x^2)' - (5x)' + (6)' \qquad \text{(by Problem 3.3)}$$
$$= (x^2)' - 5(x^1)' + 6(x)' \qquad \text{(by the linear rule of Problem 3.3)}$$
$$= 2x^1 - 5x^0 + 6x^{-1} \qquad \text{(by Problem 3.3)}$$
$$= 2x - 5 + 6x^{-1}.$$

Since you've mastered the process, there's no need to be so pedantic. Just immediately write:

$$(x^2 - 5x + 6)' = 2x - 5 + 6x^{-1} = 2x - 5 + \frac{6}{x}.$$

(b) This time, we'll omit most of the intermediate steps:

$$f'(x) = (4x^3 - 3x^2 + 18x)' = 3(4x^{3-1}) - 2(3x^{2-1}) + 18(1)$$
$$= 12x^2 - 6x^1 + 18.$$

(c) Negative exponents are no problem: we just use the same rule:

$$f'(x) = \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right)' = (x^{-1} + x^{-2} + x^{-3})'$$
$$= -1x^{-2} - 2x^{-3} - 3x^{-4}.$$

(d) Be careful! It's tempting to do this:

Bogus Solution

$$\frac{d}{dx}(18x + 9) = 18(x + 9).$$

But we don't have a right to pull the distribution of things like $(2x + 3)^2$. We can only take derivatives of recurrent terms like x^k, so we need to first expand $(2x + 3)^2$ in terms of recurrent terms, using the Binomial Theorem:

$$(2x + 3)^2 = (2x)^2 + 3(2x)^1(3)^1 + 3(2x)^0 + 3^2$$
$$= 4x^2 + 6x^1 + 6x + 9.$$

$$f'(x) = (4x^2 + 6x + 9)' = 8x + 6x^0 + 6x + 0.$$

Note that this is not equal to

$$18(x + 9)' = 18(x^0 + 1(x + 9)$$
$$= 18(x^0 + 1x + 9)$$

But the derivative is exactly from $18(x + 9)^0$ and we know of this interesting property in the next section.

We know how to take derivatives of sums of functions and of constant multiples of functions. What about derivatives of products of functions?

Exercise 3.3.6

Let f and g be differentiable functions. Find a formula for $(f \cdot g)'$ in terms of f, f', g, g'.

Solution for Problem 3.3: Many people are tempted to say

$$(f \cdot g)' \stackrel{?}{=} f' \cdot g'.$$

Sorry but it's not that simple. In the case of negative of examples, the above formula is not true. For example, let $f(x) = x^2$ and $g(x) = x^3$. Then $(f \cdot g)' = (x^5)' = 5x^4$, but

$$f' \cdot g' = (x^2)' \cdot (x^3)' = (2x)(3x^2) = 6x^3,$$

so clearly $(f \cdot g)' \neq f' \cdot g'$.

What do we do instead? We go back to our limit definition!

$$(f \cdot g)'(x) = \lim_{h\to 0}\frac{(f \cdot g)(x+h)-(f \cdot g)(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

Unfortunately, we have stuck in the way forward. We want to have only one expression some terms that look more like the terms in the definition of derivative, so let's insert a couple of carefully chosen expressions into the numerator:

$$(f \cdot g)'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

This allows us to conveniently group the terms:

$$(f \cdot g)'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$
$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h)-g(x)}{h}+\frac{g(x)(f(x+h)-f(x))}{h}\right)$$

It's even more clear what's going on if we break up it further:

$$(f \cdot g)'(x) = \lim_{h\to 0}f(x+h)\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\frac{f(x+h)-f(x)}{h}$$

Now we see that the limits of the function above are just the derivatives of f and g that are $\lim_{h\to 0}f(x+h) = f(x)$ (continuous f must be continuous, since it is differentiable), we conclude that

$$(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x).$$

□

Theorem (The **Product Rule** for derivatives) **If f and g are differentiable functions, then**

$$(f \cdot g)' = f \cdot g' + g \cdot f'.$$

Or, if we use the Leibniz notation of derivatives, that we'd say for functions u and v,

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Exercise 3.3.7

Let $f(x) = (2x+1)$.

Solution for Problem 3.3: Let $u(x) = 2x + 1$. Again, we'll use the linear definition:

$$f'(x) = \lim_{h\to 0}\frac{(2(x+h)+1)-(2x+1)}{h}$$
$$= \lim_{h\to 0}\frac{2x+2h+1-2x-1}{h}$$
$$= \lim_{h\to 0}\frac{2h}{h} = 2.$$

At first, this looks intractable, but we can instead use product. Or using the new product formula. This gives us:

$$f'(x) = \frac{d}{dx}(u \cdot u) = u\frac{du}{dx} + u\frac{du}{dx} = 2u\frac{du}{dx}.$$

This is easier to work with when we factor out the u and $\frac{du}{dx}$:

$$f'(x) = 2u\frac{du}{dx} = 2(2x+1)\left(\lim_{h\to 0}\frac{2}{h}\right)$$

Fortunately, we know limit $\lim_{h\to 0}\frac{2}{h}$. Recall from Problem 3.3 that $\lim_{h\to 0}\frac{c}{h} = 0$ for c a similar method can be used to show that $\lim_{h\to 0}\frac{mh+b}{h} = m$. Plug in $m = 2$ and $b = 1$ and we get $\lim_{h\to 0}\frac{2h+2}{h} = 2$. Thus, we have:

$$f'(x) = 2(2x+1)\left(\lim_{h\to 0}\frac{2}{h}\right) = 2(2x+1)(2) = 4(2x+1).$$

□

Exercise

Use Product Rule to derivatives of f and g are differentiable functions, then

$$(f \cdot g)' = f \cdot g' + g \cdot f'.$$

Or, if we use the Leibniz notation of derivatives, that we'd say for functions u and v,

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Exercise 3.3.8

Let $f(x) = (2x+1)$.

Solution for Problem 3.3: Let $u(x) = 2x + 1$. Again, we'll use the linear definition:

$$f'(x) = \lim_{h\to 0}\frac{(2(x+h)+1)-(2x+1)}{h}$$
$$= \lim_{h\to 0}\frac{2x+2h+1-2x-1}{h}$$
$$= \lim_{h\to 0}\frac{2h}{h} = 2.$$

At first, this looks intractable, but we can instead use product. Or using the new product formula. This gives us:

$$f'(x) = \frac{d}{dx}(u \cdot u) = u\frac{du}{dx} + u\frac{du}{dx} = 2u\frac{du}{dx}.$$

This is easier to work with when we factor out the u and $\frac{du}{dx}$:

$$f'(x) = 2u\frac{du}{dx} = 2(2x+1)\left(\lim_{h\to 0}\frac{2}{h}\right)$$

Fortunately, we know limit $\lim_{h\to 0}\frac{2}{h}$. Recall from Problem 3.3 that $\lim_{h\to 0}\frac{c}{h} = 0$ for c a similar method can be used to show that $\lim_{h\to 0}\frac{mh+b}{h} = m$. Plug in $m = 2$ and $b = 1$ and we get $\lim_{h\to 0}\frac{2h+2}{h} = 2$. Thus, we have:

$$f'(x) = 2(2x+1)\left(\lim_{h\to 0}\frac{2}{h}\right) = 2(2x+1)(2) = 4(2x+1).$$

□

Now, we can use this rule for derivatives of products and quotients to compute the derivatives of the other trig functions. We will do one as an example:

Exercise 3.3.9

Compute $\frac{d}{dx}\tan x$.

Solution for Problem 3.3: This is a simple matter of writing tangent as a quotient and then applying the Quotient Rule:

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$
$$= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x.$$

□

We can easily compute the derivatives of the remaining trig functions:

Exercise

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \cos x\\ \frac{d}{dx}\cos(x) &= -\sin x\\ \frac{d}{dx}\tan(x) &= \sec^2 x\\ \frac{d}{dx}\sec(x) &= \sec x\tan x\\ \frac{d}{dx}\csc(x) &= -\csc x\cot x\\ \frac{d}{dx}\cot(x) &= -\csc^2 x\end{aligned}$$

We will leave the computations of the derivatives of tan, sec, csc, and cot as exercises.

There are a couple of other basic functions that we will want to our derivative catalog, namely exponentials and logarithms. Unfortunately, we don't have the tools to compute the derivatives of these functions at this point. We'll do that later, and we'll take them in the book as will also give the derivatives.

The derivative limits at a large part of the reason why the number e is special:

Exercise

That is, e^x is its own derivative! In fact, $f(x) = ce^x$ (where c is any constant) are the only functions for which $f' = f$. Although this is hard to prove, it's not too difficult to show that $f(x) = ce^x$ satisfies the differential equation $f' = f$.

The fact that e^x is its own derivative is why $f(x) = e^x$ (where $e

3.4 The Chain Rule for derivatives

In Problem 3.1.10, we saw that

$$\frac{d}{dx}(kx + 3)^2 = 6(kx + 3)^1,$$

although computing this required us first to expand $(2x + 3)^2$ using the Binomial Theorem and then take the derivative of each monomial. We'd like a quicker way to compute such a derivative.

Let's look at another example:

Problem 3.3.1: 7

Our goal is to find an easy way to compute

$$\frac{d}{dx}x^2 = 1x^1$$

without applying the Binomial Theorem.

(a) First compute it the long way by expanding $(x + h)^2$ using the Binomial Theorem and then taking the derivative.

Does your answer from part (a) factor nicely?

(b) Let $f(x) = x^2$, $g(x) = x + h$, and $h(x) = x^2 + h$. Write the function $(x^2 + h)^2$ as a composition of f and g.

Can you diagram/describe/explain the relationship of the composition h to the functions f and g?

Solution to Problem 3.3.1: Before we begin, note the following incorrect solution:

Bogus Solution

$$\frac{d}{dx}(x^2 + h)^2 = 2(x^2 + h)^1.$$

This is not correct, we cannot take $\frac{d}{dx}(x^2 + h)^2$ rule and directly apply it to more complicated expressions related to a power.

(a) Consider the function $h(x) = (x^2 + h)^2$. We can certainly compute its derivative by expanding it, into a polynomial using the Binomial Theorem:

$$h(x) = x^{4} + 2hx^{2} + h^2$$

and hence

$$h'(x) = 2hx^{3} + 4hx^{1} + 2hx^{1} = 4hx^{3} + 2hx.$$

(b) Our answer from part (a) is somewhat ugly, compared to the relatively nice expression that we started with. But if you're observant, you might notice that the above expression factors quite nicely:

$$h'(x) = 2hx(x^2 + 2x^{0}) = 2hx(x^2 + 1) = 2hx(x^2 + 1)^1.$$

That's a lot nicer, and it seems related to our original $h(x) = (x^2 + 1)^2$.

If we set $f(x) = (x^2 + 1)$ and $g(x) = x^2$ then

$$h(x) = f(g(x)).$$

That is, h is the composition of g and f.

(c) At the beginning, if we had been aware and treated $(x^2 + 1)^2$ as if it were x^2, then we might have made the bogus conclusion that the derivative is $2(x^2 + 1)^1$, or we might have guessed that the derivative is a factor of $2x$, as the actual derivative is

$$h'(x) = 2hx(x^2 + 1).$$

Where did the extra factor of $2x$ come from? Hmmm... if we look at what's inside the parentheses, we see the interesting coincidence that

$$\frac{d}{dx}(x^2 + 1) = 2x.$$

That's no coincidence! We can break up $h'(x)$ as follows:

$$h'(x) = 2h(x^2 + 1)^1 = (2x)(2h(x^2 + 1)).$$

We see that the first term is just $f'(x)$. The second term is $g'(x)$ times $f(g(x)) = (x^2 + 1)$ since $g'(x) = 2x^1$ so $g'(x)f(g(x)) = 2(x^2 + 1)^1$.

Based on our work on Problem 3.3, our conjecture is:

Important: If $f(x)$ and $g(x)$ differentiable functions, and $f(g(x))$ exists, then

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

This is known as the **Chain Rule**.

We can get some idea of why the Chain Rule is true by looking at the first definition of the derivative of $(g \circ f)$:

$$(g \circ f)'(x) = \lim_{h \to 0} \frac{g(f(x + h)) - g(f(x))}{h}.$$

We want to rewrite the right hand side so that it will lead to two $f(x + h) - f(x)$ in the numerator somewhere. So let's multiply the above term by $\frac{f(x+h)-f(x)}{f(x+h)-f(x)}$. Of course, this is only valid if $f(x + h) - f(x) \neq 0$ so for the moment assume that $f(x + h) \neq f(x)$ for sufficiently small $|h|$ and $h \neq 0$. Then we have:

$$(g \circ f)'(x) = \lim_{h \to 0} \left[\frac{g(f(x + h)) - g(f(x))}{f(x + h) - f(x)} \cdot \frac{f(x + h) - f(x)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{g(f(x + h)) - g(f(x))}{f(x + h) - f(x)}\right] \cdot \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h}\right]$$

This second term is just the definition of $f'(x)$, so we have:

$$(g \circ f)'(x) = \left[\lim_{h \to 0} \frac{g(f(x + h)) - g(f(x))}{f(x + h) - f(x)}\right] f'(x). \qquad (*)$$

To tell we have left this to establish the Chain Rule is to show that

$$\lim_{h \to 0} \frac{g(f(x + h)) - g(f(x))}{f(x + h) - f(x)} = g'(f(x)). \qquad (*)$$

This seems plausible as $f(x + h)$ approaches $f(x)$ the fraction on the left side of $(*)$ is the slope of the secant line from $(f(x), g(f(x)))$ to $(f(x + h), g(f(x + h)))$ so that we can informally remember the Chain Rule as "Lengthy the slope" where we multiply the "Factors" $f(g(x))$ and $f(x)$.

WARNING: dy/dx and dy/dx are not really fractions. It's just a notation that happens to be very convenient for keeping track of the Chain Rule.

In practice, we usually won't go to all the trouble to write out the full form of $(g \circ f)'$ or anything like that. Once you become adept with the Chain Rule, you'll just write the whole calculation out on one line, like so:

$$(x^2 + 1)^2 = 2(x^2 + 1)^1 x^2 + 1)' = 2(x^2 + 1)^1 (2x) = 4hx(x^2 + 1)^1.$$

Problem 3.3.2: 7

Compute the following derivatives:

(a) $\frac{d}{dx}(x^2 + 3x)^3$

(b) $\frac{d}{dx}\frac{1}{(x^2+1)^2}$

(c) $\frac{d}{dx}(x^2 + 1)^{-2}$

Solution for Problem 3.3.2: All of these are straightforward applications of the Chain Rule.

(a) Let $f(x) = x^3$. So $\exp(x) = x^3$. Then our function $h(x)$ is $f(g(x))$, so by the Chain Rule, we have:

$$\frac{d}{dx}(x^2 + 3x)^3 = f'(g(x))g'(x) = 3(x^2 + 3x)^2(2x + 3).$$

(b) Write the function as $(1 + x^2)^{-2}$ then apply the Chain Rule:

$$((1 + x^2)^{-2})' = -2(1 + x^2)^{-3} \cdot 2x^1 = \frac{4x^1}{(1 + x^2)^3}.$$

(c) We apply the Chain Rule twice:

$$((x^2 + 1)^{-2})' = 2(x^2 + 1)^{-3} \cdot (x^2 + 1)^1$$
$$= 4(x^2 + 1)^{-3} \cdot 2x^1 \cdot (x^2 + 1)^1$$
$$= 4(x^2 + 1)^{-3} \cdot 2x^1 \cdot (x^2 + 1)^1$$
$$= 4(x^2 + 1)^{-3} \cdot 2x^1 \cdot (2x^1 + 0)$$
$$= 4(x^2 + 1)^{-3} \cdot 2x^1 \cdot (2x^1 + 0)$$
$$= 16x^2(x^2 + 1)^{-3}.$$

Problem 3.3.3: 7

Compute the derivatives of the following functions:

(a) $\cos(kx)$

(b) e^{x^2}

(c) $\log(\sin x)$

Solution for Problem 3.3.3:

(a) Write the function as $\cos(x)$, where $x = kx$. Then the derivative is

$$\cos(kx)' = -\sin(kx) \cdot (x)' = -\sin(kx) \cdot (kx)' = -k\sin(kx) = -k\sin(kx).$$

Note the importance of the first step: make sure that the final answer is in terms of the original variable, and does not contain any temporary variables. Students that not connected to help on Chain Rule calculation. Eventually you will get confused at this type of computation, and you won't need all the intermediate steps you can just write

$$\frac{d}{dx}\cos(kx) = -k\sin(kx).$$

(b) Write as e^u where $u = x^2$. Then

$$(e^{x^2} = (e^u)' = e^u \cdot (x^2 = e^{x^2} \cdot 2x^1 = 2xe^{x^2}.$$

(c) We'll consolidate the steps without using u

$$\frac{d}{dx}\log(\sin x) = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x} = \cot x.$$

Do not quantization of the Chain Rule to the formula for the derivative of an inverse function.

Problem 3.3.4: 7

Find $(f^{-1})'(x)$ the derivative of the inverse function of f, and determine where it is defined.

Solution for Problem 3.3.4: Of course, we know that $f(f^{-1}(x)) = x$ as definition, so

$$(f(f^{-1}(x)) = \frac{d}{dx}x = 1.$$

But by the Chain Rule,

$$1 = \frac{d}{dx}(f(f^{-1}(x)) = f'(f^{-1}(x))(f^{-1}(x))'.$$

Thus, assuming that f' is nonzero, we have

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}.$$

Important: The **Inverse Function Rule**: If f is a function with inverse f^{-1}, then

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

wherever $f'(f^{-1}(x)) \neq 0$.

This result opens up a lot more derivative computations for us. A particularly important one (whose importance we will see in Chapter 5):

Problem 3.3.5: 7

Compute $\frac{d}{dx}(\log^{-1} x)$.

Solution for Problem 3.3: By the Inverse Function Rule, and since $\frac{d}{dx}\log(x) = \log x$, we have

$$\frac{d}{dx}(\log^{-1} x) = \frac{1}{\exp(\log^{-1} x)}.$$

But for any $x \in (-1, 1)$, we have $\exp(\log^{-1} x) = \sqrt{1 - x^2}$, where we take the positive square root since $\sin^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus,

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

the importance of which we will also see in Chapter 5.

Problem 3.3.6: 7

Compute $\frac{d}{dx}\sqrt{x}$.

Solution for Problem 3.3: Treat $\sqrt{x}$ as the inverse function of $f(x) = x^2$ where x is a positive real number. Therefore $f'(x) = 2x$, and so we can use the Inverse Function Rule, giving:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Note that if we had written out initial function as $\sqrt{x} = x^{\frac{1}{2}}$ then this is consistent with our exponent derivative rule:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-\frac{1}{2}}.$$

We can use the same general process, which we will leave as an exercise, to show:

Important: For any rational number r,

$$\frac{d}{dx}x^r = rx^{r-1}.$$

In fact, the result is true for any real number r, rational or irrational, as we can see with the following computation:

Problem 3.3.7: 7

Recall that in Chapter 1, we assumed that $x^{\pi} = e^{\pi \log x}$ for any $x \in \mathbb{R}$. Using this, show that $x^{\pi} = \exp(\pi \log x)$ for any $x > 0$ and $\pi \in \mathbb{R}$.

(a) Use this to prove that $\frac{d}{dx}x^{\pi} = \pi x^{\pi-1}$.

Solution for Problem 3.3.7:

(a) We recall that $x = \exp(\log x)$, since $\exp$ and $\log$ are inverses. Thus

$$x^{\pi} = (\exp(\log x))^{\pi} = \exp(\pi \log x).$$

(b) By the Chain Rule, and the fact that $\frac{d}{dx}x^{\pi} = x^{\pi}$ we have

$$\frac{d}{dx}x^{\pi} = \frac{d}{dx}\exp(\pi \log x)$$
$$= \exp(\pi \log x) \cdot (\pi \log x)'$$
$$= \exp(\pi \log x) \cdot \frac{1}{x}$$
$$= x^{\pi} \cdot \frac{1}{x} = x^{\pi-1}.$$

Exercises

3.4.1: 7

Compute the derivatives of the following functions:

(a) $(2x^2 + x)^3$

Hide Solution / Reveal

You may type any additional notes you have here.

Your Submission:

Solution:

$$\frac{d}{dx}(2x^2 + x)^3 = 3(2x^2 + x)^2 \cdot \frac{d}{dx}(2x^2 + x)$$
$$= 3(2x^2 + x)^2(4x^1 + 1)$$
$$= (6x^2 + 3x)(4x^1 + 1)$$

(b) $\sqrt{(x^2)}$

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Preview Solution

You may type any additional notes you have here.

Your Submission:

Solution: Since $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, we have

$$\frac{d}{dx}\sqrt{(x^2)} = \frac{1}{2\sqrt{(x^2)}} \cdot \frac{d}{dx}(x^2)$$
$$= \frac{1}{2\sqrt{(x^2)}} \cdot 2x^1$$
$$= \frac{x^1}{\sqrt{(x^2)}}$$

(c) $\log(x^2 + \sin x)$

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You may type any additional notes you have here.

Your Submission:

Solution: Since $\frac{d}{dx}\log x = \frac{1}{x}$, we have

$$\frac{d}{dx}\log(x^2 + \sin x) = \frac{1}{(x^2 + \sin x)} \cdot \frac{d}{dx}(x^2 + \sin x)$$
$$= \frac{1}{(x^2 + \sin x)} \cdot (2x^1 + \cos x)$$
$$= \frac{2x^1 + \cos x}{x^2 + \sin x}$$

(d) $\sin\sqrt{x}$

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You may type any additional notes you have here.

Your Submission:

Solution:

$$(\sin\sqrt{x})' = (\sin\sqrt{x}) \cdot (\sqrt{x})' = \boxed{\frac{\cos\sqrt{x}}{2\sqrt{x}}}$$

(e) e^{-3x}

Preview Solution

You may type any additional notes you have here.

Your Submission:

Solution:

$$(e^{-3x})' = e^{-3x} \cdot (-3x)' = \boxed{-3e^{-3x}}$$

(f) $\sin^{-1} x$

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Your Submission:

Solution: Since $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, we have

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

(g) $\sqrt{1 + (x^2 + 1)^2}$

Hide Solution / Reveal

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Your Submission:

Solution: We apply the Chain Rule twice:

$$\sqrt{1 + (x^2 + 1)^2} = \frac{1}{2\sqrt{1 + (x^2 + 1)^2}} \cdot \left(1 + (x^2 + 1)^2\right)'$$
$$= \frac{1}{2\sqrt{1 + (x^2 + 1)^2}} \cdot \left(0 + 2(x^2 + 1)^1 \cdot (x^2 + 1)^1\right)$$
$$= \frac{1}{2\sqrt{1 + (x^2 + 1)^2}} \cdot \left(0 + 2(x^2 + 1)^1 \cdot 2x\right)$$
$$= \frac{2(x^2 + 1)^1 \cdot 2x}{2\sqrt{1 + (x^2 + 1)^2}}$$
$$= \boxed{\frac{2x(x^2 + 1)}{\sqrt{1 + (x^2 + 1)^2}}}$$

(h) e^x

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Your Submission:

Solution:

$$(e^x)' = e^x \cdot (x^1)' = e^x \cdot x^0 = \boxed{e^x}$$

3.4.2: 7

Assuming that $\int_0^1 x^2 = \frac{1}{3}$ prove that $\int_0^1 (kx)^2 = \frac{k^2}{3}$.

Hide Solution / Reveal

Your Submission:

Solution: Letting $f(x) = x^2$, and noting $f'(x) = x^1$ we apply the Inverse Function Rule:

$$(kx)^2 = (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2x^1} = \frac{1}{2x^1}.$$

3.4.3: 7

(a) Let n be a nonzero integer. Show that $\frac{d}{dx}x^n = \frac{1}{n}x^{\frac{1}{n}-1}$.

Hide Solution / Reveal

Your Submission:

Solution: Let $f(x) = x^n$ on the domain $(0, \infty)$ and note that $f'(x) = x^{n-1}$. Then by the Inverse Function Rule, noting $f^{-1}(x) = x^{\frac{1}{n}}$, we have

$$\frac{d}{dx}x^{\frac{1}{n}} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{n}x^{-\frac{n-1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

(b) Let p be a nonzero rational number. Use part (a) to show that $\frac{d}{dx}x^p = px^{p-1}$.

Preview Solution

You may type any additional notes you have here.

Your Submission:

Solution: Let $p = \frac{m}{n}$ where m is an integer and n is a positive integer. Then by part (a) and the Chain Rule,

$$\frac{d}{dx}x^p = \frac{d}{dx}x^{\frac{m}{n}} = \frac{1}{n}x^{\frac{m}{n}-1} \cdot m(x^{\frac{m}{n}})^{m-1}$$
$$= \frac{1}{n}x^{\frac{m}{n}-1} \cdot m(x^{\frac{m}{n}})^{m-1}$$
$$= \frac{1}{n}x^{\frac{m}{n}-1} \cdot m(x^{\frac{m}{n}})^{m-1}$$
$$= \frac{1}{n}x^{\frac{m}{n}-1} \cdot m(x^{\frac{m}{n}})^{m-1}$$
$$= \frac{1}{n}x^{\frac{m}{n}-1} \cdot m(x^{\frac{m}{n}})^{m-1}$$
$$= \frac{1}{n}x^{\frac{m}{n}-1} \cdot m(x^{\frac{m}{n}})^{m-1}$$
$$= px^{p-1}.$$

Both cases are defined for all real x.

3.4.4: 7 **Source: OMPT** 7

The derivative of the derivative of f is called the **second derivative** of f, denoted f''. A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$ for some constant c and some positive integer n.

Find f.

Hint: Suppose the leading term of $f(x)$ is cx^n for some constant c and some positive integer n. What are the leading terms of $f'(x)$ and $f''(x)$?

Hide Solution / Reveal

You may type any additional notes you have here.

Your Submission:

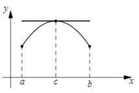
Solution: Suppose that the leading term of $f(x)$ is cx^n, where n is a positive integer. Then the leading term of $f'(x)$ is ncx^{n-1}, and the leading term of $f''(x)$ is $n(n-1)cx^{n-2}$. So we have

$$cx^n = f(x) = f'(x)f''(x) = (ncx^{n-1})(n(n-1)cx^{n-2}) = n^2c^2(n-1)x^{2n-3}.$$

Comparing the leading terms, we have $n = 2(n-1)$, hence $n = 2$. Now, comparing the coefficients, we have $c = 16c^2$, so since c is nonzero, we have $c = \boxed{\frac{1}{16}}$. Now that we determined $f(x) = \frac{1}{16}x^2$, we verifies the condition:

3.5 Rolle's Theorem and the Mean Value Theorem

Consider a continuous function defined on a closed interval $[a, b]$. By our work in Chapter 2, we know that this function must attain a maximum at some point $c \in [a, b]$. What do we know about the derivative at this point? Looking at a sample sketch should give us some idea. We see that the graph tends to "flatten out" at its maximum, and thus the tangent line at the point $(c, f(c))$ is horizontal.

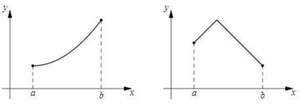


This gives us a conjecture:

Problem 3.22

Let f be a continuous function on a closed interval $[a, b]$, and let $c \in (a, b)$ be a point at which f attains its maximum on $[a, b]$. Show that if f is differentiable at c, then $f'(c) = 0$.

Before we present the solution, note that there are a couple of subtle yet important conditions in this problem. First, f might attain its maximum at one of the endpoints of $[a, b]$, and there's no reason to expect that $f' = 0$ at these points, as the diagram on the left below shows. Second, f might not be differentiable at its maximum, as the diagram on the right below shows.



Solution for Problem 3.22: Let $c \in (a, b)$ be the point where f attains its maximum. We write the definition of the derivative $f'(c)$:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

Assuming that h is small enough so that $c + h \in [a, b]$, what do we know about the numerator of the above fraction? Since $f(c)$ is the maximum value that f attains, we know that $f(c+h) \leq f(c)$; in other words, the numerator is negative or zero. But the denominator is positive for $h > 0$ and negative for $h < 0$. Thus, by considering only positive values of h, we see that $f'(c) \leq 0$, and by considering only negative h, we see that $f'(c) \geq 0$. Thus, the only possibility is $f'(c) = 0$. □

Of course, essentially the same argument works for the minimum value of f on $[a, b]$. We thus have the following theorem:

Important:

Let f be a continuous function on $[a, b]$.

- If f attains its maximum at $c \in (a, b)$, and f is differentiable at c, then $f'(c) = 0$.
- If f attains its minimum at $d \in (a, b)$, and f is differentiable at d, then $f'(d) = 0$.

Once again, remember the important "exceptions" to the above theorem: f might not be differentiable at its minimum or maximum, and the minimum or maximum might occur at one of the endpoints of the interval.

If we change the conditions slightly to remove these exceptions, we get a very useful theorem:

Problem 3.23

Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b), such that $f(a) = f(b)$. Show that there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Solution for Problem 3.23: Suppose $c \in (a, b)$ is such that f attains its maximum on $[a, b]$ at c. Then we can apply Problem 3.22 (since f is differentiable at all points on (a, b)) and conclude that $f'(c) = 0$.

Similarly, suppose $c \in (a, b)$ is such that f attains its minimum on $[a, b]$ at c. Again we can conclude that $f'(c) = 0$.

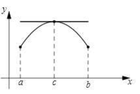
What is the only other possibility? That's if both the minimum and the maximum occur at the endpoints. But since $f(a) = f(b)$, this means that the function is constant on $[a, b]$, and thus $f' = 0$ everywhere!

So in all cases, $f'(c) = 0$ for some $c \in (a, b)$. □

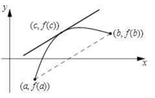
This result is known as:

Important: **Rolle's Theorem**: If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Again, the easiest way to remember Rolle's Theorem is to keep the picture to the right in mind–if a continuous function on $[a, b]$ starts and ends at the same value $f(a) = f(b)$, then it must "flatten out" at some point c with $f'(c) = 0$. It's a very nice theorem, but it's pretty specific: it only works if $f(a) = f(b)$. We'd like a more general version that works on any differentiable function on an interval.



Let's consider an arbitrary function that's continuous on $[a, b]$ and differentiable on (a, b), as shown at right. If we tilt our head and pretend that the x-axis runs through $(a, f(a))$ and $(b, f(b))$, we get the idea that we should be able to find a point $c \in (a, b)$ such that the tangent line through $(c, f(c))$ is parallel to our new "x-axis."



What does this really mean? It means that the tangent line through $(c, f(c))$ is parallel to the line through $(a, f(a))$ and $(b, f(b))$. And, of course, "parallel" means that the two lines have the same slope. If we put it all together, we have that

$$\begin{aligned} f'(c) &= \text{slope of line through } (a, f(a)) \text{ and } (b, f(b)) \\ &= \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

We're now ready to state the theorem, which is one of the most important theorems in calculus.

Important: **The Mean Value Theorem**: If f is a continuous function on the closed interval $[a, b]$, with $a < b$, and f is differentiable on (a, b), then there exists some real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Mean Value Theorem is arguably the most important theoretical result about the derivative. You should think about the Mean Value Theorem until its statement becomes intuitively obvious to you. The proof is a little bit tricky, so we'll defer it to Section 3.B. We'll see more of the significance of the Mean Value Theorem in Chapter 4.

Exercises

3.5.1:

Suppose that f is continuous on a closed interval $[a, b]$ and that $f'(c) = 0$ for all $c \in (a, b)$. What can you conclude about f? Prove your result.

Hint: Use the Mean Value Theorem.

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Solution: The conclusion is that f must be constant. To prove this, let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. Then by the Mean Value Theorem, there is some $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But we know that $f'(c) = 0$ for all $c \in (x_1, x_2)$, so we conclude that $f(x_2) - f(x_1) = 0$, that is, $f(x_1) = f(x_2)$ for all $x_1, x_2 \in [a, b]$, and thus f is constant on $[a, b]$.

3.5.2:

Suppose f and g are differentiable functions such that $f' = g'$. Show that $f - g$ is a constant function.

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Solution: We note that $(f - g)' = f' - g' = 0$, so by the previous exercise, $f - g$ is a constant function.

3.5.3:

Let f be a degree n polynomial with n distinct real roots. Show that f' has $n - 1$ distinct real roots.

Hint: Use Rolle's Theorem.

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Solution: Let the roots of f be $r_1 < r_2 < \cdots < r_n$. Since f is a polynomial, and thus differentiable, then by Rolle's Theorem, there must be some $s_1 \in (r_1, r_2)$ such that $f'(s_1) = 0$. Similarly, for any $1 \leq k < n$, we can apply Rolle's Theorem to find $s_k \in (r_k, r_{k+1})$ such that $f'(s_k) = 0$. Thus, $\{s_1, s_2, \ldots, s_{n-1}\}$ are the $n - 1$ distinct roots of f'.

3.5.4:

Suppose that f is a differentiable function with domain $\mathbb{R}$ such that $f'(x) > 0$ for all $x \in \mathbb{R}$. Show that if $a, b \in \mathbb{R}$ with $a < b$, then $f(a) < f(b)$.

Hint: Use the Mean Value Theorem.

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Solution: By the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(c) > 0$ by assumption, thus

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $a < b$, the denominator is positive, so the numerator must be positive as well, and thus $f(a) < f(b)$.

3.6 Implicit differentiation

Many curves are not defined as the graph of a function, but instead are defined **implicitly** as the graph of an equation involving x and y. For example, consider the circle with center $(0,0)$ and radius 1. This circle is the set of all points (x,y) satisfying the equation $x^2+y^2=1$. Since it fails the Vertical Line Test, we know that this circle is not the graph of any function. This presents an issue with an example like the following:

Problem 3.24

Find the slope of the tangent line to the circle $x^2+y^2=1$ at the point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

Solution for Problem 3.24: Of course, we know from our knowledge of geometry and trigonometry that the slope of this line is -1. However, for the sake of this exercise, let's compute this slope using calculus. Although the entire circle is not the graph of a function, the top half of the circle is the graph of a function, and we can compute the derivative of this function to get the slope of this line.

If we replace y with $f(x)$ in our equation for the circle, we get

$$x^2+(f(x))^2=1,$$

which we can solve to get

$$f(x)=\sqrt{1-x^2},$$

the graph of which is the top half of the circle. If we instead had taken the negative square root, we would have $f(x)=-\sqrt{1-x^2}$, whose graph is the bottom half of the circle.

We can now use the Chain Rule to take the derivative of $f(x)=(1-x^2)^{\frac{1}{2}}$:

$$f'(x)=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)=-\frac{x}{\sqrt{1-x^2}}.$$

We now see that

$$f'\left(\frac{\sqrt{2}}{2}\right)=-\frac{\sqrt{2}/2}{\sqrt{1/2}}=-1,$$

so that the slope of the tangent line to the circle at the point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ is -1, as we expect. ◻

But a simpler way of finding this derivative is to just take the derivative directly of the equation that implicitly defines the circle. Specifically, we start with

$$x^2+(f(x))^2=1,$$

and we take the derivative of both sides:

$$\frac{d}{dx}\left(x^2+(f(x))^2\right)=\frac{d}{dx}(1).$$

Since the derivative is linear, we can break up the left side into a sum of derivatives. We also note that $\frac{d}{dx}(x^2)=2x$ and $\frac{d}{dx}(1)=0$, so we have

$$2x+\frac{d}{dx}\left((f(x))^2\right)=0.$$

How do we take the derivative of that remaining term? We can use the Chain Rule! This gives us

$$2x+2(f(x))(f'(x))=0.$$

Now we solve for the derivative:

$$f'(x)=-\frac{2x}{2f(x)}=-\frac{x}{f(x)}.$$

Now we see, for example, that at the point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ the derivative is $-\frac{\sqrt{2}/2}{\sqrt{2}/2}=-1$, just as before.

More typically, equations that implicitly define functions are written in terms of x and y. Thus, we would write the circle as

$$x^2+y^2=1,$$

and implicitly differentiate to get

$$2x+2yy'=0,$$

from which we could solve to get $y'=-\frac{x}{y}$.

> **Concept:** The main thing to keep in mind when performing implicit differentiation is that taking the derivative (with respect to x) of any term containing a "y" is going to result in a y' term being introduced. In other words, $\frac{d}{dx}y=y'$.

Here is a somewhat more complicated example:

Problem 3.25

Find the equation for the tangent line to the curve $3y^2-6xy+2x^3-5y=2$ at the point $(2,1)$.

Solution for Problem 3.25: First, it's a good idea to verify that $(2,1)$ is on the curve, which we can do by plugging $(x,y)=(2,1)$ into the equation:

$$3(1)^2-6(2)(1)+2(2)^3-5(1)=3-12+16-5=2.$$

Next, we compute the derivative of the equation that implicitly defines the curve. Again, the idea is simple: whenever we have a term with a y in it, we have to apply the Chain Rule to get a y' term. Thus we get

$$6yy'-6xy'-6y+6x^2-5y'=0.$$

Don't forget to differentiate the constant on the right side as well (to get 0). We could solve this for y' in terms of x and y, but in this problem it's simpler to plug in $(x,y)=(2,1)$ right away to get the slope of the tangent at $(2,1)$:

$$6y'-12y'-6+24-5y'=0,$$

so $y'=\frac{18}{11}$. Thus, in point-slope form, the equation of the tangent line is

$$y-1=\frac{18}{11}(x-2).$$

◻

Exercises

3.6.1:

Find the slope of the tangent line to the hyperbola $x^2-y^2=1$ at the point $(2,\sqrt{3})$.

You may type any additional notes you have here.

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Solution: Implicit differentiation gives $2x-2yy'=0$, so $y'=\frac{x}{y}$. Substituting $x=2$ and $y=\sqrt{3}$ gives $y'=\boxed{\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}}$

3.6.2:

Find the equation of the tangent line to the curve $y^3-3xy^2+x^2-xy=6$ at the point $(-1,1)$.

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Solution: Implicit differentiation gives

$$\begin{aligned}&3y^2y'-3x(y^2)'-(3x)'y^2+2x-xy'-x'y\\&=3y^2y'-6xyy'-3y^2+2x-xy'-y\\&=0.\end{aligned}$$

Plugging in $x=-1$ and $y=1$ gives

$$3y'+6y'-3-2+y'-1=0,$$

so $10y'=6$ and thus $y'=\frac{3}{5}$. Thus, the equation of the line is $y-1=\frac{3}{5}(x+1)$, or $\boxed{y=\frac{3}{5}x+\frac{8}{5}}$

3.6.3:

Find $\frac{dy}{dx}$ if $x^2+y=\log(y^2-1)$.

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Solution: Implicit differentiation gives

$$2x+y'=\frac{1}{y^2-1}\cdot\frac{d}{dx}(y^2-1)=\frac{1}{y^2-1}\cdot 2yy'=\frac{2yy'}{y^2-1}.$$

Solving for y' gives

$$y'=\frac{2x}{\frac{2y}{y^2-1}-1}=\boxed{\frac{2x(y^2-1)}{-y^2+2y+1}}$$

3.6.4:

Find the slope of the tangent line to the curve $x\sin(x+y)=y\cos(x-y)$ at the point $\left(0,\frac{\pi}{2}\right)$

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Solution: Implicitly differentiating the equation (using the Product Rule on both sides) gives

$$\begin{aligned}&x\cos(x+y)(1+y')+\sin(x+y)\\&=-y\sin(x-y)(1-y')+y'\cos(x-y).\end{aligned}$$

Substituting $x=0$ and $y=\frac{\pi}{2}$ gives

$$\sin\frac{\pi}{2}=-\frac{\pi}{2}\sin\left(-\frac{\pi}{2}\right)(1-y')+y'\cos\left(-\frac{\pi}{2}\right).$$

Evaluating the trig functions gives

$$1=-\frac{\pi}{2}(-1)(1-y')+y'\cdot 0=\frac{\pi}{2}(1-y'),$$

hence $y'=1-\frac{2}{\pi}$. Thus the slope of the tangent line is $\boxed{1-\frac{2}{\pi}}$

3.6.5:

Assume that $\frac{d}{d\theta}\sin\theta=\cos\theta$ and that $\frac{d}{d\theta}\cos\theta$ is continuous. Use implicit differentiation to prove that $\frac{d}{d\theta}\cos\theta=-\sin\theta$.

Hint: Start with $\sin^2\theta+\cos^2\theta=1$ and differentiate.

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Solution: We take the equation $\sin^2\theta+\cos^2\theta=1$ and differentiate with respect to θ. This gives

$$2\sin\theta\cdot\frac{d}{d\theta}\sin\theta+2\cos\theta\cdot\frac{d}{d\theta}\cos\theta=0.$$

Using $\frac{d}{d\theta}\sin\theta=\cos\theta$, we have

$$\sin\theta\cos\theta=-\cos\theta\left(\frac{d}{d\theta}\cos\theta\right),$$

from which we conclude that $\frac{d}{d\theta}\cos\theta=-\sin\theta$ everywhere that $\cos\theta\neq 0$. But these are isolated points, so by our assumption of continuity, we conclude that $\frac{d}{d\theta}\cos\theta=-\sin\theta$ everywhere.

3.7 Summary of derivative computation

Here is a summary of derivative rules and the catalog of common derivatives that we developed in this chapter. You should learn all of the items below, and practice until computing a derivative is as easy as addition or multiplication.

Definition

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Linearity

$$(f+g)'(x) = f'(x) + g'(x)$$
$$(cf)'(x) = cf'(x)$$

Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Inverse Function Rule

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Monomials

$$\frac{d}{dx}x^n = nx^{n-1}$$

Trig Functions

$$\tfrac{d}{dx}(\sin x) = \cos x \qquad \tfrac{d}{dx}(\cos x) = -\sin x$$

$$\tfrac{d}{dx}(\tan x) = \sec^2 x \qquad \tfrac{d}{dx}(\cot x) = -\csc^2 x$$

$$\tfrac{d}{dx}(\sec x) = \sec x \tan x \qquad \tfrac{d}{dx}(\csc x) = -\csc x \cot x$$

Exponential and Natural Logarithm

$$\tfrac{d}{dx}(e^x) = e^x$$

$$\tfrac{d}{dx}(\log x) = \tfrac{1}{x}$$

Chain Rule

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3.A Proof of the Chain Rule

Problem 3.41

Prove that if f and g are differentiable functions, and if $a \in \mathbb{R}$ is such that $(g \circ f)(a)$ exists, then

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Sidenote: The argument presented below is largely based on the argument in [**Sp**,Chapter 10]. There are other ways to prove this, but I find the argument in [**Sp**] to be the nicest I've seen.

Solution for Problem 3.41: We start as we did in Section 3.4 here, with the limit definition of $(g \circ f)'(a)$:

$$(g \circ f)'(a) = \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h}.$$

We want an $f'(a)$ term to appear on the right side of the above, thus it appears that we'll need to have $f(a+h) - f(a)$ in the numerator. So let's multiply by it. Of course, if we multiply by it, we have to divide by it too:

$$\begin{aligned} &\frac{g(f(a+h)) - g(f(a))}{h} \\ &= \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h}. \end{aligned} \tag{3.1}$$

This is nice: the second fraction in (3.1) has limit $f'(a)$ as $h \to 0$. But we have a problem if some $h \neq 0$ satisfies $f(a+h) - f(a) = 0$. If this occurs, then the expression in (3.1) is not defined at that h, because the denominator of the first term is 0.

To be a little more precise about what the problem is here and how to fix it, suppose we define

$$r(h) = \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)}.$$

This is the first fraction on the right side of (3.1). We'd like to be able to show that $\lim_{h \to 0} r(h) = g'(f(a))$. The problem is that $r(h)$ is undefined wherever $f(a+h) = f(a)$. This is not an issue at $h = 0$, because we're taking a limit, but it is an issue if this happens at any $h \neq 0$. So what we'll do is use some "spackle" and fill in the "holes" in the definition of r that occur where $f(a+h) = f(a)$. Specifically, we extend $r(h)$ to a new function $\rho(h)$ by:

$$\rho(h) = \begin{cases} \dfrac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} & \text{if } f(a+h) \neq f(a), \\ g'(f(a)) & \text{if } f(a+h) = f(a). \end{cases}$$

Now ρ doesn't have any holes, since we've filled the holes with the value $g'(f(a))$.

We now show that $\lim_{h \to 0} \rho(h) = g'(f(a))$. This means that for any $\epsilon > 0$, we need to be able to find $\delta > 0$ such that

$$0 < |h| < \delta \quad \Rightarrow \quad |\rho(h) - g'(f(a))| < \epsilon. \tag{3.2}$$

However, we know by definition that

$$\lim_{k \to 0} \frac{g(f(a)+k) - g(f(a))}{k} = g'(f(a)), \tag{3.3}$$

which means that we can choose $\zeta > 0$ such that

$$0 < |k| < \zeta \quad \Rightarrow \quad \left| \frac{g(f(a)+k) - g(f(a))}{k} - g'(f(a)) \right| < \epsilon, \tag{3.4}$$

and we know that f is continuous at a, so we can choose $\delta > 0$ such that

$$0 < |x| < \delta \quad \Rightarrow \quad |f(a+x) - f(a)| < \zeta. \tag{3.5}$$

We claim that this is the δ that satisfies (3.2). To see why, let h be given such that $0 < |h| < \delta$. If $f(a+h) = f(a)$, then by definition $\rho(h) = g'(f(a))$ (remember, this is the "spackle"), so clearly

$$|\rho(h) - g'(f(a))| = |g'(f(a)) - g'(f(a))| = 0 < \epsilon,$$

as needed. On the other hand, if $f(a+h) \neq f(a)$, then we use (3.5) to get $0 < |f(a+h) - f(a)| < \zeta$. Then, plugging $f(a+h) - f(a)$ in for k in (3.4), we get

$$\left| \frac{g(f(a) + (f(a+h) - f(a))) - g(f(a))}{f(a+h) - f(a)} - g'(f(a)) \right| < \epsilon. \tag{3.6}$$

But that fraction in (3.6) is just $\rho(h)$:

$$\begin{aligned} \frac{g(f(a) + (f(a+h) - f(a))) - g(f(a))}{f(a+h) - f(a)} &= \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \\ &= \rho(h). \end{aligned}$$

Thus, (3.6) becomes

$$|\rho(h) - g'(f(a))| < \epsilon,$$

which is what we needed to establish in (3.2). Therefore, we have shown that $\lim_{h \to 0} \rho(h) = g'(f(a))$.

Now we can prove the Chain Rule. We modify (3.1) by observing that, for all $h \neq 0$, we have

$$\frac{g(f(a+h)) - g(f(a))}{h} = \left(\rho(h) \cdot \frac{f(a+h) - f(a)}{h} \right),$$

because if $f(a+h) \neq f(a)$, this is the same expression that we had in (3.1), and if $f(a+h) = f(a)$, then both sides of the above are 0. Thus, we can now finally complete the computation of the derivative of $g \circ f$:

$$\begin{aligned} (g \circ f)'(a) &= \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h} \\ &= \lim_{h \to 0} \left(\rho(h) \cdot \frac{f(a+h) - f(a)}{h} \right) \\ &= \left(\lim_{h \to 0} \rho(h) \right) \cdot \left(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right) \\ &= g'(f(a))f'(a). \end{aligned}$$

□

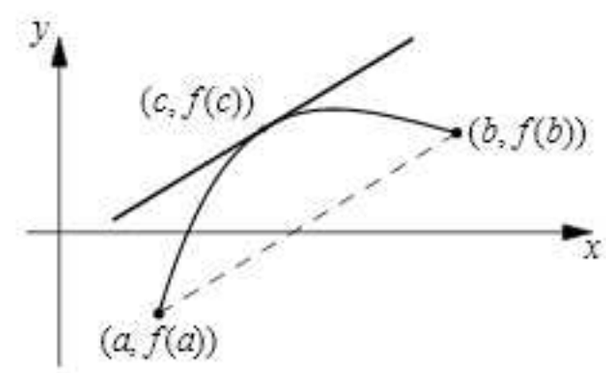
3.B Proof of the Mean Value Theorem

Problem 3.42

The **Mean Value Theorem:** Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Show that there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Solution for Problem 3.42: Recall that the expression $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the secant line between $(a, f(a))$ and $(b, f(b))$, shown by the dashed line in the picture to the right. The Mean Value Theorem asserts that there is a point $(c, f(c))$ on the graph of $y = f(x)$ such that the tangent line at $(c, f(c))$ is parallel to the secant line from $(a, f(a))$ to $(b, f(b))$, as in the picture at right. Also note that if $f(a) = f(b)$, then the Mean Value Theorem asserts that $f'(c) = 0$; in other words, we just have Rolle's Theorem.



We'd like to define a new function g with $g(a) = g(b)$ in order to apply Rolle's Theorem. It's not immediately clear how we might do this. We might as well take $g(a) = f(a)$, and then we will aim for $g(b) = g(a) = f(a)$. A simple approach is to try to have some sort of linear "correction" of the form

$$g(x) = f(x) - (x - a)(\text{something}),$$

noting that

$$g(a) = f(a) + (a - a)(\text{something}) = f(a).$$

Plugging in $x = b$ gives

$$g(b) = f(b) - (b - a)(\text{something}).$$

We want $g(b) = g(a) = f(a)$, so we need the "something" to be $\dfrac{f(b) - f(a)}{b - a}$, which happens to be the slope of the secant line from $(a, f(a))$ to $(b, f(b))$. Therefore, our new function is

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Notice that g is continuous on $[a, b]$ (since it is a linear combination of the continuous functions $f(x)$ and $(x - a)$), and that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so g is differentiable on (a, b). Also we verify that $g(a) = f(a)$ and

$$g(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(a),$$

so $g(a) = g(b)$. Thus, we may apply Rolle's Theorem to g and find a point $c \in (a, b)$ such that $g'(c) = 0$. Finally,

$$f'(c) = g'(c) + \frac{f(b) - f(a)}{b - a} = \frac{f(b) - f(a)}{b - a},$$

as desired. □

Challenge Problems

3.34:

Let a be a positive real number. Find $\frac{d}{dx}a^x$ and $\frac{d}{dx}\log_a(x)$.

Hint: Recall that $a^x = e^{x\log a}$.

You may type any additional notes you have here.

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Solution: Using $a^x = e^{x\log a}$, we see

$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx}e^{x\log a}\\ &= (\log a)e^{x\log a}\\ &= \boxed{a^x\log a}.\end{aligned}$$

Using $\log_a x = \frac{\log x}{\log a}$, we get

$$\frac{d}{dx}\log_a x = \boxed{\frac{1}{x\log a}}.$$

3.35:

Source: HMMT

Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose that the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

Hint: "The tangent lines . . . are parallel" means that $f'(a) = f'(b)$.

Hint: You don't necessarily need to solve for a and b. You just need to find $f(1)$.

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Solution: We have $f'(x) = 3x^2 + a$, so the slopes of the tangent lines at a and b are $3a^2 + a$ and $3b^2 + a$. If these lines are parallel, then these slopes must be equal, so $3a^2 + a = 3b^2 + a$, or $a^2 = b^2$. Since $a \neq b$, this means that $a = -b$, so $a + b = 0$. Therefore, $f(1) = 1 + a + b = \boxed{1}$.

3.36:

Recall that a function f is *even* if $f(-x) = f(x)$ and is *odd* if $f(-x) = -f(x)$. Prove that if f is odd, then f' is even, and that if f is even, then f' is odd.

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Solution: Assume that $f(-x) = f(x)$. Then taking the derivatives of both sides we get $-f'(-x) = f'(x)$, or $f'(-x) = -f'(x)$. Therefore if f is even then f' is odd.

Likewise, taking the derivative of $f(-x) = -f(x)$, we get $f'(-x) = f'(x)$. Therefore if f is odd then f' is even.

3.37:

Source: HMMT

Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

Hint: Notice that

$$f(x) - f(4x) = (f(x) - f(2x)) + (f(2x) - f(4x)).$$

Hint: Let $g(x) = f(x) - f(2x)$ and $h(x) = f(x) - f(4x)$. Write an equation relating $h'(x)$, $g'(x)$, and $g'(2x)$.

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Solution: Define $g(x) = f(x) - f(2x)$. We are given that $g'(1) = 5$ and $g'(2) = 7$. Then note that

$$\begin{aligned}h(x) &= f(x) - f(4x)\\ &= (f(x) - f(2x)) + (f(2x) - f(4x))\\ &= g(x) + g(2x).\end{aligned}$$

So $h'(x) = g'(x) + 2g'(2x)$, and hence $h'(1) = 5 + 2(7) = \boxed{19}$.

3.38:

We write $f^{(k)}(x)$ to mean taking the derivative of $f(x)$ k times. Prove that if $f(x)$ is a degree n polynomial, then $f^{(n)}(x) = c$ where c is a nonzero constant.

Hint: Prove by induction.

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Solution: We prove this by induction. The 0th derivative of a nonzero constant function (a degree 0 polynomial) is a nonzero constant. For the inductive step, assume that the result is true for a degree $n-1$ polynomial (where n is a positive integer). Now, if p is a degree n polynomial, then

$$p^{(n)} = (p')^{(n-1)}.$$

Since p is degree n, the polynomial p' must be degree $n-1$ (notice that if the leading coefficient of p is $a \neq 0$, then the leading coefficient of p' is $na \neq 0$). Applying the inductive assumption, we then have $p^{(n)} = (p')^{(n-1)} = c \neq 0$, so we are done.

3.39:

A polynomial function $f(x)$ has a root r with **multiplicity** m if

$$f(x) = (x - r)^m h(x)$$

for some polynomial $h(x)$ with $h(r) \neq 0$.

(a) Prove that if f has root r with multiplicity m, then f' has root r with multiplicity $m - 1$.

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Solution: Using the Product Rule, we have

$$\begin{aligned}f'(x) &= m(x-r)^{m-1}h(x) + (x-r)^m h'(x)\\ &= (x-r)^{m-1}(mh(x) + (x-r)h'(x)).\end{aligned}$$

Letting $h_1(x) = mh(x) + (x-r)h'(x)$, we see that we can write $f'(x) = (x-r)^{m-1}h_1(x)$, and further

$$\begin{aligned}h_1(r) &= mh(r) + (r-r)h'(r)\\ &= mh(r)\\ &\neq 0.\end{aligned}$$

Thus, f' has root r with multiplicity $r - 1$.

(b) Prove that $f^{(k)}(r) = 0$ for all $1 \leq k < m$.

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Solution: By repeated application of part (a), we have that $f^{(k)}$ has root r with multiplicity $m - k$. Thus, for $1 \leq k < m$, we have $f^{(k)}(r) = 0$.

3.40:

Suppose f and g are differentiable functions.

(a) Show that

$$(fg)'' = f''g + 2f'g' + fg''.$$

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Solution: By the Product Rule,

$$(fg)' = f'g + fg'.$$

We simply apply the Product Rule again:

$$\begin{aligned}(f'g + fg')' &= (f'g)' + (fg')'\\ &= f''g' + f'g' + f'g' + fg''\\ &= f''g + 2f'g' + fg''.\end{aligned}$$

(b)★ Generalize part (a) to show that for any positive integer n,

$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k)}.$$

This is called the **Leibniz Rule**.

Hint: Prove by induction, using the fact that $(fg)^{(n+1)} = ((fg)^{(n)})'$.

Hint: You may find Pascal's Identity useful: $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

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Solution: We prove this by induction. We have the base case for $n = 0$: $fg = fg$. Now assume that the result is true for some nonnegative integer n. Then

$$\begin{aligned}(fg)^{(n+1)} &= ((fg)^{(n)})'\\ &= \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k)}\\ &= \sum_{k=0}^{n}\binom{n}{k}\left(f^{(k+1)}g^{(n-k)} + f^{(k)}g^{(n-k+1)}\right)\\ &= \left[\sum_{k=0}^{n}\binom{n}{k}f^{(k+1)}g^{(n-k)}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k+1)}\right].\end{aligned}$$

Replacing k with $k-1$ in the first sum gives

$$\left[\sum_{k=1}^{n+1}\binom{n}{k-1}f^{(k)}g^{(n-(k-1))}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n+1)-k)}\right].$$

Noting that $\binom{n}{-1} = \binom{n}{n+1} = 0$, we see that both of these sums can be taken from 0 to $n+1$:

$$\sum_{k=0}^{n+1}\left(\binom{n}{k-1} + \binom{n}{k}\right)f^{(k)}g^{((n+1)-k)}.$$

Finally we apply Pascal's identity, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ to get

$$(fg)^{(n+1)} = \sum_{k=0}^{n+1}\binom{n+1}{k}f^{(k)}g^{((n+1)-k)},$$

completing the proof.

Review Problems

3.26:

Compute the derivatives of the following:

(a) $\dfrac{1}{(2x+3)^3}$.

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Your Submission: Solution

Solution: By the Chain Rule,

$$\frac{d}{dx}(2x+3)^{-3} = (-3)(2x+3)^{-4}(2) = \boxed{\frac{-6}{(2x+3)^4}}.$$

(b) $\sqrt{x^3+e^x}$.

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Solution: By the Chain Rule,

$$\begin{aligned}\frac{d}{dx}\sqrt{x^3+e^x} &= \frac{1}{2}(x^3+e^x)^{-\frac{1}{2}}(x^3+e^x)'\\ &= \frac{1}{2}(x^3+e^x)^{-\frac{1}{2}}(3x^2+e^x)\\ &= \boxed{\frac{3x^2+e^x}{2\sqrt{x^3+e^x}}}.\end{aligned}$$

(c) $\log(\sin x \cos x)$

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Solution: Using the Chain Rule and then the Product Rule,

$$\begin{aligned}\frac{d}{dx}\log(\sin x\cos x) &= \frac{\frac{d}{dx}(\sin x\cos x)}{\sin x\cos x}\\ &= \frac{(\sin x)'(-\sin x)+(\cos x)(\cos x)}{\sin x\cos x}\\ &= \boxed{\frac{\cos^2 x-\sin^2 x}{\sin x\cos x}}.\end{aligned}$$

If you are observant, you may notice that the numerator is $\cos 2x$ and the denominator is $\frac{1}{2}\sin 2x$, so the whole thing is $\boxed{2\cot 2x}$.

(d) $\sqrt{1+\sqrt{x+\sqrt{e^x}}}$

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Your Submission: Solution

Solution: Again, we use the Chain Rule repeatedly:

$$\begin{aligned}&\frac{d}{dx}(1+(x+e^{x/2})^{1/2})^{1/2}\\ &= (1/3)(1+(x+e^{x/2})^{1/2})^{-2/3}(1+(x+e^{x/2})^{1/2})'\\ &= (1/3)(1+(x+e^{x/2})^{1/2})^{-2/3}(1/2)(x+e^{x/2})^{-1/2}(x+e^{x/2})'\\ &= (1/6)(1+(x+e^{x/2})^{1/2})^{-2/3}(x+e^{x/2})^{-1/2}(1+e^{x/2})\\ &= \boxed{\frac{2+e^{x/2}}{12(1+\sqrt{x+e^{x/2}})^{2/3}\sqrt{x+e^{x/2}}}}\end{aligned}$$

(e) axe^{-bx} where a and b are real constants

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Your Submission: Solution

Solution: We use the Product Rule, and then the Chain Rule:

$$\begin{aligned}\frac{d}{dx}(axe^{-bx}) &= (ax)'e^{-bx}+(ax)(e^{-bx})'\\ &= ae^{-bx}-abxe^{-bx}\\ &= \boxed{ae^{-bx}(1-bx)}.\end{aligned}$$

(f) e^{-e^x}

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Solution: This requires the Chain Rule:

$$\frac{d}{dx}e^{-e^x} = e^{-e^x}(-e^x)' = \boxed{(-2x)e^{-e^x}}.$$

(g) $\sin^{-1} x^2$

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Solution: Using the derivative of $\sin^{-1}$ and the Chain Rule, we get

$$\frac{d}{dx}\sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}}\cdot(x^2)' = \boxed{\frac{2x}{\sqrt{1-x^4}}}$$

(h) $\sin^3 x$

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Your Submission: Solution

Solution: By the Chain Rule,

$$\frac{d}{dx}\sin^3 x = 3(\sin^2 x)(\sin x)' = \boxed{3\sin^2 x\cos x}.$$

3.27:

Find the equation of the tangent line to the graph $y=(x-1)^3+2$ at the point $(3,10)$.

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Solution: First we find the slope of the tangent line, which is just the derivative $y'=3(x-1)^2$ at $x=3$. Thus we have $y'(3)=3(3-1)^2=12$. Therefore, the line is

$$\boxed{y=12(x-3)+10=12x-26}.$$

3.28:

Let $f(x)=x^2-6x+10$ with domain $[3,+\infty)$. Find $(f^{-1})'(2)$.

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Solution: We solve $2=x^2-6x+10$ to get

$$x^2-6x+8=(x-4)(x-2)=0.$$

So, since $x\geq 3$ we must have $x=4$. Then, noting that $f'(x)=2x-6$, we have, by the Inverse Function Theorem,

$$(f^{-1})'(2)=\frac{1}{f'(f^{-1}(2))}=\frac{1}{f'(4)}=\frac{1}{2(4)-6}=\boxed{\frac{1}{2}}.$$

3.29:

Find the tangent line to the curve $y^3=x^3-3x+1$ at the point $(2,\sqrt{3})$.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Taking the implicit derivative with respect to x of both sides, we get

$$2yy'=3x^2-3.$$

Plugging in $x=2$ and $y=\sqrt{3}$ gives $2\sqrt{3}y'=9$, so $y'=\dfrac{3\sqrt{3}}{2}$. Thus the tangent line is the graph of $\boxed{y=\dfrac{3\sqrt{3}}{2}(x-2)+\sqrt{3}}$.

3.30:

Define

$$f''(a)=\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}.$$

Show that if $f'(a)$ exists, then so does $f''(a)$, and $f''(a)=f'(a)$.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We add $-f(a)+f(a)=0$ to the numerator of the definition of $f''(a)$, and we obtain:

$$\begin{aligned}f''(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)+f(a)-f(a-h)}{2h}\\ &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{2h}+\frac{f(a)-f(a-h)}{2h}\right)\\ &= \lim_{h\to 0}\left(\frac{1}{2}\frac{f(a+h)-f(a)}{h}+\frac{1}{2}\frac{f(a-h)-f(a)}{-h}\right)\\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\frac{1}{2}\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}\\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\frac{1}{2}\lim_{h\to 0}\frac{f(a+(-h))-f(a)}{(-h)}\\ &= \frac{1}{2}f'(a)+\frac{1}{2}f'(a)=f'(a).\end{aligned}$$

It follows that $f''(a)=f'(a)$ for all a where $f'(a)$ exists. Notice that it is possible that $f''(a)$ may exist while $f'(a)$ does not! (The reader is encouraged to find an example of a function f where this occurs.)

3.31:

(a) Show that if f,g,h are differentiable functions, then

$$(fgh)'=f'gh+fg'h+fgh'.$$

Hint: Start by computing $((fg)h)'$ using the Product Rule.

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We apply the product rule twice:

$$(fgh)'=f'(gh)+f(gh)'=f'gh+fg'h+fgh'.$$

(b) Suppose $f_1,f_2,\ldots,f_k$ are differentiable functions. Find a formula for $(f_1f_2\cdots f_k)'$.

Hint: Use induction.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We conjecture that

$$(f_1f_2\cdots f_k)'=(f_1'f_2\cdots f_k)+(f_1f_2'\cdots f_k)+(f_1f_2\cdots f_k'),$$

where there are n summands and each summand is a product that contains the derivative of exactly one of the f_i.

We can prove this by induction on the number of terms. Clearly the base case holds: $f'=f'$. For the inductive step, assume that the result holds for a product of $k-1$ terms. Then we compute:

$$\begin{aligned}(f_1f_2\cdots f_k)' &= f_1(f_2\cdots f_k)+f_k(f_1f_2\cdots f_{k-1})'\\ &= f_1(f_2'\cdots f_k)+f_1(f_2f_3'\cdots f_k)+\cdots+(f_kf_1'\cdots f_{k-1})\\ &= (f_1'f_2\cdots f_k)+(f_1f_2'\cdots f_k)+(f_1f_2\cdots f_k').\end{aligned}$$

3.32:

Find a continuous function f for which the Mean Value Theorem fails. That is, find a continuous function f and $a<b$ with $[a,b]\subseteq \mathrm{Dom}(f)$ such that there does not exist $c\in(a,b)$ satisfying

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Hint: Since we know that the Mean Value Theorem holds for differentiable functions, you should be looking for a function that is not differentiable.

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Solution: Since we are trying to find an example where the Mean Value Theorem fails, we know that f must be non-differentiable at some point in (a,b). So we look for an example that is not differentiable. The simplest example is $|x|$. Indeed, if $f(x)=|x|$, then on the interval $[-1,1]$ there is no point $c\in(-1,1)$ such that $f'(c)=\dfrac{|1|-|-1|}{1-(-1)}=0$.

3.33:

Source: HMMT

Let $f(x)=1+x+x^2+\cdots+x^{100}$. Find $f'(1)$.

Hint: You shouldn't need to write out $f'(x)$ in order to compute $f'(1)$.

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Your Submission: Solution

Solution: The derivative is

$$f'(x)=1+2x+3x^2+\cdots+100x^{99}.$$

So

$$f'(1)=1+2+\cdots+100=\frac{(100)(101)}{2}=\boxed{5050}.$$

4 Applications of the Derivative

In Chapter 3 we introduced the concept of derivative in terms of geometry: the derivative at $x=2$ of a curve $y=f(x)$ is the slope of the tangent line to the graph of f at the point $(2, f(2))$. We also saw how the derivative can be used to describe the velocity of an object moving along a straight line. In this chapter we will see that the derivative can be used to describe a wide variety of other quantities, and we will use the derivative to find the maximum and minimum values of certain quantities, and to sketch graphs of functions. We will also use the derivative to find approximate values of functions, and we will see how the derivative can be used to find the limit of a function when the limit cannot be found by other methods.

4.1 Graphical interpretation of the derivative

Definition: Let f be a function.

- f is **increasing** on an interval I if for all $a, b \in I$ with $a < b$, we have $f(a) < f(b)$.
- f is **decreasing** on an interval I if for all $a, b \in I$ with $a < b$, we have $f(a) > f(b)$.
- f is **constant** on an interval I if for all $a, b \in I$ we have $f(a) = f(b)$.

Intuitively, the graph of an increasing function is "always rising": as we move from left to right, the graph of a decreasing function is "always falling": as we move from left to right, the graph of a constant function is "level": as we move from left to right, the graph of a constant function is "level". We can also describe these properties in terms of the slopes of the tangent lines: if f is increasing, then the slope of the tangent line is always positive, and if f is decreasing, then the slope of the tangent line is always negative, and if f is constant, then the slope of the tangent line is zero.

We can modify our definition slightly:

Definition: Let f be a function.

- f is **strictly increasing** on an interval I if for all $a, b \in I$ with $a < b$, we have $f(a) < f(b)$.
- f is **strictly decreasing** on an interval I if for all $a, b \in I$ with $a < b$, we have $f(a) > f(b)$.

In either case we say that f is **strictly monotonic** on I. If f is the entire domain of f, we say that f is **strictly monotonic**. Similarly, we have the notion of **monotonicity**.

Concept: Monotonicity is a property of functions that we will use in the future. We do not yet have the function to "test" monotonicity. We will develop such a test in Section 4.2.

If the function is differentiable, then the derivative gives us a fast test for monotonicity.

Problem 4.1 ☐ ☐ ☐

Let f be a differentiable function.

(a) **Show that** $f'(x) > 0$ **for all** x **implies that** f **is strictly increasing.**

(b) **Show that** $f'(x) < 0$ **for all** x **implies that** f **is strictly decreasing.**

(c) Replacing the inequality stronger and we know about $f'(x)$ on our Motion Takes Inequalities, tell the key fact that is that when $f'(x) > 0$ for all x implies that f is strictly increasing, and the function is increasing, $f'(x) < 0$ for all x implies that f is strictly decreasing.

$$\begin{array}{l} f'(x) > 0 \text{ for all } x \Rightarrow f \text{ is strictly increasing.} \\ f'(x) < 0 \text{ for all } x \Rightarrow f \text{ is strictly decreasing.} \end{array}$$

(d) Replacing the inequality stronger and we know about $f'(x)$ on our Motion Takes Inequalities, tell the key fact that is that when $f'(x) > 0$ for all x implies that f is strictly increasing, and the function is increasing, $f'(x) < 0$ for all x implies that f is strictly decreasing.

$$\begin{array}{l} f'(x) > 0 \text{ for all } x \Rightarrow f \text{ is strictly increasing.} \\ f'(x) < 0 \text{ for all } x \Rightarrow f \text{ is strictly decreasing.} \end{array}$$

More generally, looking at specific intervals (and not necessary the entire domain of f), we have the following result:

Important: Let f be a continuous differentiable function on an interval I_0.

- If $f'(x) > 0$ for all $x \in I_0$, then f is strictly increasing on I_0.
- If $f'(x) < 0$ for all $x \in I_0$, then f is strictly decreasing on I_0.
- If $f'(x) = 0$ for all $x \in I_0$, then f is strictly decreasing on I_0.
- If $f'(x) = 0$ for all $x \in I_0$, then f is constant on I_0.

This is our first glimpse of how we can interpret the derivative as the rate of change of f. Basically, f' measuring the rate at which $f(x)$ changes as we increase x. If f' is positive, then f is increasing, and the function is increasing. If f' is negative, then f is decreasing, and the function is decreasing.

Concept: Thinking of the derivative as a rate of change is probably the most important application of the derivative. We will see this interpretation throughout this chapter, and throughout the course.

Problem 4.2 ☐ ☐ ☐

Let f be a differentiable function with $f'(x) > 0$. Let x and $x+h$ be arbitrary. What can we conclude about f?

Solution for Problem 4.2: Because $f' > 0$ the function is increasing on $\mathbb{R}$. This means that $f(x) \le f(x+h)$ for all $x, x+h \in \mathbb{R}$. We know that $f'(x) > 0$ for all x implies that f is increasing. Since $f'(x) > 0$ for all x, we know that f is increasing. Since f is increasing, $f(x) \le f(x+h)$ for all x and $x+h$. Since $f'(x) > 0$ for all x, we know that f is increasing. Since f is increasing, $f(x) \le f(x+h)$ for all x and $x+h$.

Let f be a differentiable function with $f'(x) < 0$. Let x and $x+h$ be arbitrary. What can we conclude about f?

Solution for Problem 4.2: Because $f' < 0$ the function is decreasing on $\mathbb{R}$. This means that $f(x) \ge f(x+h)$ for all $x, x+h \in \mathbb{R}$. We know that $f'(x) < 0$ for all x implies that f is decreasing. Since $f'(x) < 0$ for all x, we know that f is decreasing. Since f is decreasing, $f(x) \ge f(x+h)$ for all x and $x+h$. Since $f'(x) < 0$ for all x, we know that f is decreasing. Since f is decreasing, $f(x) \ge f(x+h)$ for all x and $x+h$.

This is the first time we have seen the derivative. Where $f(x)$ is the function, f is strictly increasing, and where $f(x) > 0$, the function f is strictly decreasing.

Next, since the function changes from decreasing to increasing at $x = 2$, we probably want to calculate $f'(2) = 0$ + 10 + 0 = 10. So, the graph passes through $(2, 10)$.

Therefore, the graph increases on $(0, 2)$ (passing through $(1, -1)$), on the way, then decreases from $(2, 10)$ (passing through $(3, 2)$ and $(4, -1)$) on the way, and then increases to ∞.

Putting all this information together, we get this graph:

Interval	$f'(x)$	Behavior
$(-\infty, 0)$	$+$	increasing
$(0, 2)$	$-$	decreasing
$(2, 4)$	$+$	increasing
$(4, \infty)$	$-$	decreasing

Finally, we can use the derivative to get information about the critical points of the curve. By the Intermediate Value Theorem, we know that one root must be between -1 and 0, one between 0 and 2, and one between 2 and 4.

We might ask a question about the critical points: what is the relationship between $f'(x)$ and $f(x)$ in terms of the critical points? We can see that the critical points of $f(x)$ are the **zeroes of the derivative** of the original function f (equivalently, we contribute of f the **first derivative** of f).

Let's continue with the example from Problem 4.3:

Problem 4.4 ☐ ☐ ☐

Let $f(x) = x^3 - 3x^2 + 4$.

(a) **Where is** f **increasing? Where is it decreasing?**

(b) **What does information we also know about the shape of the graph of** f?

Solution for Problem 4.4

(a) We find $f'(x) = 3x^2 - 6x$ so $f'(x) = 3x(x-2)$. This is positive for $x < 0$ and $x > 2$, and negative for $x \in (0, 2)$ when $x < 0$, the derivative is decreasing, and when $x > 2$, the derivative is increasing. We see that this information is our graph:

We notice that f starts positive, but is decreasing, at $x = 0$ we have f switching from positive to negative, and still decreasing, at $x = 2$ we have f switching from decreasing to increasing, now f is still negative, at $x = 4$ f switching from negative to positive, and continues to increase.

(b) We can see the effect of f' on the graph of f.

Interval	$f'(x)$	Behavior
$(-\infty, -1)$	$+$	increasing
$(-1, 0)$	$-$	decreasing
$(0, 2)$	$-$	decreasing
$(2, 4)$	$+$	increasing
$(4, \infty)$	$+$	increasing

We have shown some examples of the tangent lines as various points on the curve. We can see that as we move along the curve from left to right (starting at $-\infty$), the tangent line is decreasing in slope, and is decreasing between $x = 0$. We find that the derivative is decreasing between $x = 0$ and $x = 2$, and increasing between $x = 2$ and $x = 4$.

Another way that we can think of this is as follows: the derivative is the slope of the tangent line, and the slope of the curve is decreasing to the right as you walk along it. In our example, if we walk along the curve from $(-1, -1)$ through $(0, 2)$ and towards $(2, -1)$, the slope of the curve is decreasing. Similarly, if we walk along the curve from $(2, -1)$ through $(4, 2)$ and towards $(4, 2)$, the slope of the curve is increasing.

We say that the portion of the function where $f'(x) > 0$ **concave down**. Similarly, where f is concave down, the graph of f is decreasing (downward) and where f is increasing, the graph is increasing (upward). We can also ask whether the curve is increasing or decreasing, or we can ask whether the function is increasing or is $x = 0$ has decreasing between $x = 0$ and $x = 2$ is the derivative f' is increasing.

Conversely, if f' is decreasing, the function is increasing. This means that the tangent line is increasing in slope, which means that they are rotating counterclockwise. In our example, this occurs when $x > 0$. We add more tangent line segments to our picture to illustrate this behavior:

To summarize some of the definitions and concepts from Problem 4.4:

Problem 4.5 ☐ ☐ ☐

Let f be a twice differentiable function and f'' be continuous.

(a) **Prove that** $f'(x) > 0$ **implies that** f **is concave up.**

(b) **Prove that** $f'(x) < 0$ **implies that** f **is concave down.**

Solution for Problem 4.5

If $f'(x) > 0$ for all x then f is increasing. Since $f'(x) > 0$ for all x the function changes from up to down to from down to up and f is concave up. Since $f'(x) > 0$ for all x then f is increasing. Since $f'(x) > 0$ for all x then f is increasing. Since f is increasing, $f(x) \le f(x+h)$ for all x and $x+h$. Since $f'(x) > 0$ for all x then f is increasing. Since f is increasing, $f(x) \le f(x+h)$ for all x and $x+h$. Since $f'(x) > 0$ for all x then f is increasing. Since f is increasing, $f(x) \le f(x+h)$ for all x and $x+h$.

If $f'(x) < 0$ for all x then f is decreasing. Since $f'(x) < 0$ for all x the function changes from up to down to from down to up and f is concave down. Since $f'(x) < 0$ for all x then f is decreasing. Since $f'(x) < 0$ for all x then f is decreasing. Since f is decreasing, $f(x) \ge f(x+h)$ for all x and $x+h$. Since $f'(x) < 0$ for all x then f is decreasing. Since f is decreasing, $f(x) \ge f(x+h)$ for all x and $x+h$. Since $f'(x) < 0$ for all x then f is decreasing. Since f is decreasing, $f(x) \ge f(x+h)$ for all x and $x+h$.

Since f is a twice differentiable function and f'' is continuous we can use f' using directly from negative to positive (or vice versa) and f is concave up (or down) and f is increasing (or decreasing) and we all know that f is an extremum f is to construct the f.

(a) The concavity doesn't have to change at $x = 0$ and $f'(x) > 0$. There are many examples of this. One simple example is $f(x) = x^4$ at $x = 0$. $f'(x) = 4x^3$ so $f'(x) > 0$ for all $x \ne 0$. Since $f'(x) > 0$ for all x and f is continuous, we know that f is concave up everywhere except at $x = 0$. We know that $f''(0) = 0$ and f has an inflection point at $(0, 0)$, since the function is concave up on both sides of $x = 0$.

(b) $f(x) = -x^4$

There are some other arguments ways to describe concavity:

Problem 4.6 ☐ ☐ ☐

Let f be a twice differentiable function with continuous first and second derivatives, and suppose that $f' > 0$ and $f'' > 0$ on the open interval I containing $x = 0$ and $f(0) = 0$. Let $g(x) = \frac{f(x)}{x}$ for $x \in I$. Show that the graph of g is increasing on I.

Solution for Problem 4.6: Derivative quotient rule gives us $g'(x)$. We know that $f'(x) > 0$ and $f''(x) > 0$ for $x \in I$. Since $f'(x) > 0$ for $x \in I$ we know that f is increasing on I. Since f is increasing on I we know that $f(x) > f(0) = 0$ for $x \in I$. Since $f''(x) > 0$ for $x \in I$ we know that f' is increasing on I. Since f' is increasing on I we know that $f'(x) > f'(0)$ for $x \in I$. Since $f'(x) > 0$ for $x \in I$ we know that f is increasing on I. Since f is increasing on I we know that $f(x) > f(0) = 0$ for $x \in I$. Since $f''(x) > 0$ for $x \in I$ we know that f' is increasing on I. Since f' is increasing on I we know that $f'(x) > f'(0)$ for $x \in I$.

Let's look at the graph of the following function f and its derivative f':

Since f' is positive at $x = 0$ and f' is increasing, we know that f is increasing at $x = 0$. Since f is increasing at $x = 0$ and $f(0) = 0$, we know that f is positive for $x > 0$. Since f' is increasing at $x = 0$ and f' is increasing, we know that f' is increasing for $x > 0$. Since f' is increasing for $x > 0$ and $f'(0) > 0$, we know that f' is positive for $x > 0$. Since f' is positive for $x > 0$ and f is increasing for $x > 0$, we know that f is increasing for $x > 0$.

We'll have the products of these two functions. They are a similar Motion Value Theorem argument as the proof in Problem 4.6.

We can summarize the features of concavity in the following table:

Concave up	Concave down
f' is increasing	f' is decreasing
$f'' > 0$	$f'' < 0$
Tangent line rotates counterclockwise as x increases	Tangent line rotates clockwise as x increases
Tangent line lies below curve	Tangent line lies above curve

Concept: Concave down means the above tangent curve and understand the geometric features of the second derivative. Keeping a basic example the $f(x) = x^2$ is a curve is also very helpful.

Exercises

4.1.1

Sketch the graphs of the following functions. Find the intervals in which the functions are increasing or decreasing, find intervals in which the functions are concave up or down, and any inflection points.

(a) $3x^2 - 4x + 5$

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You may type any additional notes here here.

Your Submission: Solution

Solution: Let $f(x)$ be the given function. We calculate $f'(x) = 6x^2 - 4$ and $f''(x) = 18x$.

We know that f is increasing where $f' > 0$, positive, that occurs when $6x^2 - 4 > 0$ or $|x| > 2/3$. Thus f is increasing on $(-\infty, -2/3)$ and $(2/3, \infty)$. We know that f is decreasing where $f' < 0$, negative, which occurs when $-2/3 < x < 2/3$.

Finally, f is concave up where $f'(x) > 0$, which occurs when $18x > 0$. Thus f is concave up on $(0, \infty)$. Similarly, f is concave down where $f'(x) < 0$, which occurs when $x < 0$. Thus f is concave down on $(-\infty, 0)$. Finally, $f''(x) = 0$ when $x = 0$, and f is the only inflection point.

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(b) $x^3 - 3x^2 + 4$

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Your Submission: Solution

Solution: Let $f(x)$ be the given function. We calculate

$$\begin{aligned} f'(x) &= 3x^2 - 6x \\ f''(x) &= 6x - 6 \end{aligned}$$

$$\begin{aligned} &= 3(x^2 - 2x) \\ &= 3(x^2 - 2x + 1 - 1) \\ &= 3(x^2 - 2x + 1) - 3 \\ &= 3(x - 1)^2 - 3 \end{aligned}$$

In both f and f' the discriminant is always positive, and thus does not affect the sign.

Therefore, f is increasing (decreasing) in $(-\infty, 0)$ and $(2, \infty)$ ($(0, 2)$), and f is decreasing (increasing) in $(-\infty, 0)$ and $(2, \infty)$ ($(0, 2)$). Since f is increasing on $(-\infty, 0)$ and f is decreasing on $(0, 2)$ and f is increasing on $(2, \infty)$ and f is increasing on $(0, 2)$ and f is decreasing on $(2, \infty)$ and f is increasing on $(2, \infty)$. Since f is increasing on $(-\infty, 0)$ and f is decreasing on $(0, 2)$ and f is increasing on $(2, \infty)$ and f is increasing on $(0, 2)$ and f is decreasing on $(2, \infty)$ and f is increasing on $(2, \infty)$. The points of inflection are $-3/3$ and $3/3$.

(c) $x + 4x$

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Your Submission: Solution

Solution: Let $f(x)$ be the given function. We calculate

$$\begin{aligned} f'(x) &= 1 + 4 \\ f''(x) &= 0 \end{aligned}$$

Since f' is positive and f'' is zero, we know that f is increasing on $(-\infty, \infty)$. Since f is increasing on $(-\infty, \infty)$ and f is increasing on $(-\infty, \infty)$ and f is increasing on $(-\infty, \infty)$. Since f' is increasing on $(-\infty, \infty)$ and

4.2 Extrema and optimization

Example: Find the absolute extrema of $f(x) = x^2$ on the interval $[-1, 2]$.

From calc one we already seen in Section 4.1 a continuous on an interval $[a, b]$ has f attains an interval $[a, b]$ has f must attain a minimum and a maximum. Since $f(x) = x^2$ is continuous on $[-1, 2]$ we know that $f(x) = x^2$ has a minimum and a maximum. Since $f(x) = x^2$ is continuous on the interval $[-1, 2]$ which gives us the absolute extrema.

We've identified a continuous function on a closed interval $[a, b]$ has a minimum and a maximum. Now we want to find the largest (top) & the smallest (bottom) of $f(x)$ on $[a, b]$ we are pretty check that the largest (top) & smallest (bottom) of $f(x)$ on $[a, b]$ occurs at the endpoints.

Consider the Problem 4.1 We saw the extremum of the function $f(x)$

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

Assuming the f is small enough so that $x \in [-1, 1]$ & $f(x)$ is not less than $f(1) = 0 = f(-1)$ (Find the $f(x)$ is not less than $f(1) = 0 = f(-1)$ & $f(x)$ is not greater than $f(0) = -1$ (Find the $f(x)$ is not greater than $f(0) = -1$ (Find the $f(x)$ is not greater than $f(0) = -1$ (Find the

4.3 Velocity

In our previous discussions, we've alluded to our primary interpretation of the derivative: as a rate of change. In particular, if $f(t)$ is a function where the variable t represents time, then $f'(t)$ represents the instantaneous rate of change of $f(t)$. As a special case, if $f(t)$ represents a position of an object at time t, then $f'(t)$ represents the object's **velocity**. Further, $f''(t)$ represents the rate of change of velocity, which we call **acceleration**.

Here's an example:

Problem 4.18

A particle's position on the coordinate line is given by $s(t) = 2t^3 - 21t^2 + 60t + 3$, where t is the time in seconds. Describe the position, velocity, and acceleration of the particle.

Solution for Problem 4.18: The rate of change–that is, the derivative–of position is velocity, and the rate of change of velocity is acceleration. We can compute these functions. (For obvious reasons, we typically denote velocity at time t by $v(t)$ and acceleration at time t by $a(t)$.)

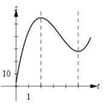
$$\begin{aligned} s(t) &= 2t^3 - 21t^2 + 60t + 3, \\ v(t) &= s'(t) = 6t^2 - 42t + 60, \\ a(t) &= v'(t) = s''(t) = 12t - 42. \end{aligned}$$

We can make a few observations. For instance, the particle starts at $t = 0$ at position $s(0) = 3$ with initial velocity $v(0) = 60$ and initial acceleration $a(0) = -42$. In words, the particle is moving to the right at 60 units/second, but is decelerating at a rate of 42 units/sec/sec.

We also note the inflection point where $a(t) = 0$ at $t = 7/2$. This is the time at which the particle switches from accelerating to the left and starts accelerating to the right.

We can also analyze the velocity. Specifically, we can factor $v(t) = 6(t-2)(t-5)$. So the velocity is negative for $2 < t < 5$ and positive otherwise. This means that the particle is moving to the left between $t = 2$ and $t = 5$, and moving to the right otherwise.

Using this information, we can now sketch a graph of the particle's movement, shown at right. Note that motion upwards on this graph is motion where s is increasing, so the particle is moving to the right; conversely downward motion on the graph corresponds to the particle moving to the left. Between the dashed lines, where $2 < t < 5$, the particle is moving to the left (that is, its position is decreasing). We also note that the "peak" position of the particle during the interval $0 \le t \le 5$ comes at $t = 2$. This is where the particle stops moving to the right and starts moving back to the left. Note that $v(2) = s'(2) = 0$ at this time. □



One very common application is acceleration due to gravity. On Earth, the acceleration due to the force of the planet's gravity is a constant $32\ \mathrm{ft/sec^2}$. (This neglects things like air resistance and the fact that at very high altitudes the Earth's gravity is marginally smaller, but we will for now ignore such details.) A free-falling body's height (in feet) is given by the equation

$$y(t) = -16t^2 + v_0 t + y_0,$$

where t is in seconds, v_0 is the initial velocity in feet/second, and y_0 is the initial height in feet. (In the next chapter, we will see how to derive this formula.) Note that $y(0) = y_0$. Also, we can differentiate $y(t)$ to get functions for velocity and acceleration:

$$\begin{aligned} v(t) &= y'(t) = -32t + v_0, \\ a(t) &= v'(t) = y''(t) = -32. \end{aligned}$$

Notice how these functions make sense: in particular, the velocity starts at v_0 and decreases by 32 every second, and the acceleration is our constant -32.

In metric, the acceleration due to gravity is $-9.8\ \mathrm{m/sec^2}$. The formula for the body's height in meters is thus:

$$y(t) = -4.9t^2 + v_0 t + y_0,$$

giving

$$\begin{aligned} v(t) &= y'(t) = -9.8t + v_0, \\ a(t) &= v'(t) = y''(t) = -9.8. \end{aligned}$$

Here is a typical problem involving gravity:

Problem 4.19

A cannon shoots a cannonball off of the top of a 1000-foot high cliff with an initial upwards velocity of 200 feet per second.

(a) What is the maximum height (above the ground) that the cannonball attains?

(b) How fast is the cannonball traveling downwards when it hits the ground (1000 feet below the top of the cliff)?

Solution for Problem 4.19: We have $y(0) = 1000$ and $v_0 = 200$, so the height of the cannonball is given by

$$y(t) = -16t^2 + 200t + 1000.$$

(a) We know that the maximum height is reached at the time when $v(t) = y'(t) = 0$. This should (by now) make intuitive sense to you. Moreover, this is the only critical point of the function other than its endpoints (at $t = 0$ and at the time at which the cannonball hits the ground), and the heights of the cannonball at the endpoints are lower than the height at the critical point.

We know that the velocity is $v(t) = -32t + 200$, so the velocity is 0 at $t = 200/32$ seconds after the launch. Thus, the maximum height is

$$y(200/32) = -16(200/32)^2 + 200(200/32) + 1000 = 1625 \text{ feet}.$$

(b) The cannonball hits the ground when $y(t) = 0$, so we solve

$$0 = -16t^2 + 200t + 1000.$$

Using the quadratic formula, we find

$$t = \frac{-200 \pm \sqrt{40000 + 4(16)(1000)}}{-32} = \frac{25 \pm 5\sqrt{65}}{4}.$$

Since we only want to consider $t \ge 0$, the cannonball hits the ground $(25 + 5\sqrt{65})/4 \approx 16.328$ seconds after launch. At this time, it is traveling downwards at a velocity of

$$\begin{aligned} v\left(\frac{25+5\sqrt{65}}{4}\right) &= -\frac{32(25+5\sqrt{65})}{4} + 200 \\ &= -40\sqrt{65} \\ &\approx -322.49 \end{aligned}$$

feet per second. (This is about 220 miles per hour).

□

Exercises

4.3.1:

A rock is thrown straight down from the top of a 480-foot high building, with an initial downward velocity of 16 feet/second. When does it hit the ground, and at what speed is it traveling when it hits the ground?

You may type any additional notes you have here.

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Solution: The formula for the height of the rock is given by $h = -16t^2 + v_0 t + h_0$ where v_0 is the initial velocity in feet/second and h_0 is the initial height in feet. Plugging the given data into this equation gives $h = -16t^2 - 16t + 480$; note that the initial velocity is negative, since the rock is thrown downward. Setting $h = 0$, we can simplify:

$$\begin{aligned} & -16t^2 - 16t + 480 = 0 \\ \Rightarrow\ & t^2 + t - 30 = 0 \\ \Rightarrow\ & (t-5)(t+6) = 0. \end{aligned}$$

Since we begin at $t = 0$, the only positive solution is $\boxed{t = 5}$ seconds after it is thrown. The speed then is

$$\begin{aligned} v &= h' \\ &= -32t - 16 \\ &= -32(5) - 16 \\ &= \boxed{-176} \end{aligned}$$

feet per second, or 176 feet per second downwards (assuming no air resistance of course).

4.3.2:

A cannonball is shot at a 30 degree angle from the ground with an initial velocity of 30 meters/second. How far from the cannon is the cannonball when it hits the ground?

You may type any additional notes you have here.

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Solution: When the cannonball is shot, it has a vertical velocity component of $30 \cdot \sin(30°) = 15$ meters per second. Plugging this and an initial height of 0 into our equation for motion under the force of gravity, we get $y = -4.9t^2 + 15t$. Besides $t = 0$, the other solution to $y = 0$ is $t = 15/4.9 = 150/49 \approx 3.06$ seconds later.

The horizontal component is easier to analyze. Initially, it is $30\cos(30°) \approx 25.98$ meters per second. Assuming no air resistance, it remains this for its entire flight. Thus, 3.06 seconds later, it has moved a total of 25.98 - 3.06 $\approx$ $\boxed{79.5}$ meters horizontally.

4.3.3:

A certain road has speed limit $L(x)$ (in miles per hour), where x is the distance from the start of the road in miles. Cars A and B start at mile 0 and are at position $a(t)$ and $b(t)$ at time t (in hours); in particular note that $a(0) = b(0) = 0$.

(a) Write an inequality that states the fact that car A is always going as fast or faster than car B.

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Solution: Speed is the derivative of position, so we have $\boxed{a'(t) \ge b'(t)}$.

(b) Write an equation that states the fact that car B is always going exactly at the speed limit.

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Solution: We can't just write $b'(t) = L(x)$, since one is a function of x and the other is a function of t. The correct equation is $\boxed{b'(t) = L(b(t))}$.

(c) Say (in words) what the equation $a(t+1) = b(t)$ means (for all $t \ge 0$).

You may type any additional notes you have here.

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Your Submission: Solution

Solution: This means that car A is always 1 hour behind car B.

(d) Say (in words) what the equation $a'(t) = L(a(t-1))$ means (for all $t \ge 1$).

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: This means that car A is always traveling at the speed limit of the point where it was one hour ago.

4.4 Tangent line approximation

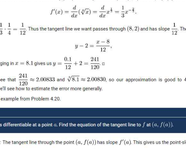
Another application of the derivative comes directly from its definition as the slope of the tangent line to a point on a graph. Suppose f is a function that is continuous and differentiable at $x = a$. We can approximate f near a by its tangent line at $(a, f(a))$. The advantage of this is that the tangent line is a line, and lines are really easy to work with.

Let's see a simple example.

Problem 4.20

Let $f(x) = \sqrt{x}$. Using f' and f', approximate $\sqrt{8.1}$ by constructing the tangent line to f at $(8, 2)$ and finding the point on this line with $x = 8.1$.

Solution for Problem 4.20: We start by sketching $f(x) = \sqrt{x}$ and drawing the tangent line at the point $(8, 2)$:



On the right above, we have zoomed in the graph near the point $(8, 2)$. We see that the graph looks really similar to its tangent line near the point $(8, 2)$. We can use this information to estimate $f(8.1)$. The tangent line to f at $(8, 2)$ has slope $f'(8)$, so we need to compute f':

$$f'(x) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}.$$

So $f'(8) = \frac{1}{2} \cdot 8^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Thus the tangent line we want passes through $(8, 2)$ and has slope $\frac{1}{12}$. The equation of this line is

$$y - 2 = \frac{x - 8}{12},$$

so $y = \frac{x-8}{12} + 2$. Plugging in $x = 8.1$ gives us $y = \frac{0.1}{12} + 2 = \frac{241}{120} \approx$

Using a calculator, we see that $\frac{241}{120} = 2.008\overline{33}$ and $\sqrt{8.1} = 2.00830\ldots$ so our approximation is good to 4 decimal places (in this example). In Chapter 7, we'll see how to estimate the error generally.

Let's try to generalize our example from Problem 4.20.

Problem 4.21

Let f be a function that is differentiable at a point a. Find the equation of the tangent line to f at $(a, f(a))$.

Solution for Problem 4.21: The tangent line through the point $(a, f(a))$ has slope $f'(a)$. This gives us the point-slope form

$$y - f(a) = f'(a)(x - a),$$

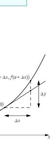
or $y = f(a) + f'(a)(x - a)$. $\square$

We can use the equation from Problem 4.21 as a method for approximating functions near the point $(a, f(a))$.

Important: The **tangent line approximation** of a function f near a is

$$f(x) \approx f(a) + f'(a)(x - a).$$

We can see what's going on by looking at the diagram to the right. We see that the point on the tangent line has y-coordinate $f(a) + f'(a)(x - a)$, and this is a close approximation to the actual value $f(x)$.



The general idea is that we can use a tangent line approximation in situations where $f(a)$ and $f'(a)$ are easy to compute but $f(x)$ may be difficult to compute.

Concept: We can use tangent line approximation to compute the approximate value of a function at a value that is hard to compute (such as $\sqrt{8.1}$ in Problem 4.20), but that is near a value that is easy to compute (such as $\sqrt{8}$ in Problem 4.20).

Intuitively, the closer x is to a, the more accurate the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

will be. We will discuss this more precisely in Chapter 7.

We often write the above formula using differential notation, as follows. We are trying to approximate f near the point $(a, f(a))$. If we look at the tangent line at $(a, f(a))$, we see that the ratio of the change in y (denoted Δy) to the change in x (denoted Δx) is approximately the slope of the tangent line, which is $f'(a)$. This gives us $\Delta y/\Delta x \approx f'(a)$ or $\Delta y \approx f'(a)\Delta x$.

We can also do this computation algebraically. We start with our tangent line approximation

$$f(x) \approx f(a) + f'(a)(x - a).$$

Subtracting $f(a)$ from both sides gives

$$f(x) - f(a) \approx f'(a)(x - a).$$

Note that $f(x) - f(a)$ is the change in the y-coordinate on the graph of the function as we move from $(a, f(a))$ to $(x, f(x))$, so we have $\Delta y = f(x) - f(a)$. Similarly, since $\Delta x = x - a$. Thus:

Important: Near the point $(a, f(a))$, we have $\Delta y \approx f'(a)\Delta x$.

Important: This is probably the most important concept of calculus: the derivative f' represents the rate of change of f.

Replacing a function with its tangent line approximation at a point $x = a$ is also called the **local linearization** of the function at $x = a$ (or simply at a).

Concept: Near $x = a$, a differentiable function and its local linearization at a are almost the same.

The above statement is pretty vague, with the mathematically imprecise words "near" and "almost." We can more precisely state what we mean by the local linearization being "almost" the same as $f(x)$ "near" $x = a$:

Problem 4.22

Let f be a differentiable function with f' continuous. Define $E(x)$ to be the error in the approximation of $f(x)$ using $f(a)$, that is

$$E(x) = f(x) - (f(a) + f'(a)(x - a)).$$

Prove that $\lim_{x \to a} \frac{E(x)}{x - a} = 0$. What (in words) does this mean?

Solution for Problem 4.22: We simply compute:

$$\lim_{x \to a} \frac{E(x)}{x - a} = \lim_{x \to a} \frac{f(x) - (f(a) + f'(a)(x - a))}{x - a}$$
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} - f'(a)$$
$$= f'(a) - f'(a) = 0.$$

In words, this means that $E(x)$ approaches 0 faster than $x - a$ does when $x \to a$. So as x gets near a, the error in the linear approximation approaches 0 faster than x approaches a. This is the formal explanation of what we mean when we say that the linear approximation "close" to $f(x)$ "near" $x = a$. $\square$

You can explore this idea further in Challenge Problem 4.51.

Problem 4.23

Find the local linearization of $f(x) = e^{.001x}$ at $x = 0$, and use it to approximate $e^{.002}$.

Solution for Problem 4.23: We first compute $f'(x) = 2e^{2x}$. So $f'(0) = 2e^{0} = 2$, and thus the tangent line at $x = 0$ has slope 2. Since $f(0) = 1$, the tangent line passes through $(0, 1)$. Therefore the tangent line is $y = 2x + 1$, and this is the local linearization of e^{2x} at $x = 0$.

To approximate $e^{.002} = f(.001)$, we plug in $x = .001$ to the local linearization:

$$f(.001) \approx 2(.001) + 1 = 1.002.$$

The actual value of $e^{.002}$ to 6 decimal places is 1.002002, so this is a very close approximation. $\square$

One nice advantage of the $\Delta y \approx f'(x)\Delta x$ perspective is that sometimes we're only interested in the value of the change away from a point. Specifically, one application of tangent line approximation–also called **linear approximation**–is to estimate the error of an observation quantity when we know that there is an error in our measurement. Let's look at a couple of examples.

Problem 4.24

Paul is trying to measure the height of a building using a sextant. He stands 100 meters away and determines that the angle from the ground where he is standing to the top of the building is 30°, so he computes that the height of the building is $\frac{100}{\sqrt{3}}$ meters. But, if his sextant is only accurate to within 1 degree, then (approximately) by how much might his measurement of the height of the building be off?

Solution for Problem 4.24: We start by determining the function that converts Paul's sextant measurement to a height measurement. But be careful! Don't do this:

Bogus Solution: The height of the building is given by $f(x) = 100\tan(x)$.

This won't work, because the angle given by the sextant is in degrees! In calculus, we always need to have our angles in radians, or else we won't get the correct derivatives. So we must convert the angle to radians first, and we get the height function

$$f(x) = 100\tan\left(\frac{\pi}{180}x\right),$$

where x is the angle in degrees.

To compute how much our height might be off, we use our $\Delta y \approx f'(x)\Delta x$ formula. We are told that $|\Delta x| \leq 1$. We now need to compute the derivative of f:

$$f'(x) = \frac{100\pi}{180}\sec^2\left(\frac{\pi}{180}x\right) = \frac{5\pi}{9}\sec^2\left(\frac{\pi}{180}x\right).$$

Therefore, since we have $x = 30$, we get

$$f'(30) = \frac{5\pi}{9}\sec^2\frac{\pi}{6} = \frac{5\pi}{9} \cdot \frac{4}{3} = \frac{20\pi}{27}.$$

Thus, using $|\Delta x| \leq 1$ we have

$$|\Delta y| \approx |f'(30)\Delta x| \leq f'(30) = \frac{20\pi}{27}.$$

So we estimate that Paul's measurement may be off by approximately $\frac{20\pi}{27}$ meters, which is about 2.327 meters. $\square$

Problem 4.25

A distant planet (assumed to be a perfect sphere) is measured, using a telescope, to have a diameter of 2000 kilometers. However, it is known that the telescope will introduce a measurement error of within ±1%. What is the estimated percentage error in the computed volume of the planet?

Solution for Problem 4.25: We can write the volume as a function of the diameter:

$$V = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{\pi}{6}d^3.$$

Note that we use V for volume and d for diameter, for the obvious reason that it's easier to remember what the variables stand for.

Concept: Don't be too attached to x and y. Use variable names that relate to the quantities that they represent.

The error in the computation of the volume can be approximated by

$$\Delta V \approx V'\Delta d.$$

Note that $V' = \frac{3}{6}d^2 = \frac{d^2}{2}$, so when $d = 2000$ we have $V' = \frac{1}{2}(2000)^2 = 2000000\pi$. We are also given that $|\Delta d| \leq 0.01d = 20$ so

$$|\Delta V| \approx |V'\Delta d| \leq (2000000\pi)(20) = (4 \cdot 10^7)\pi.$$

Also we know that $V = \frac{\pi}{6}d^3 = \frac{\pi}{6}(2000)^3 = \frac{4}{3} \cdot 10^9\pi$. So

$$\frac{|\Delta V|}{V} \leq \frac{(4 \cdot 10^7)\pi}{(4/3) \cdot 10^9\pi} = \frac{3}{100} = 3\%.$$

So the error in the volume measurement is approximately ±3%.

In fact, we didn't need to plug in d. We could have left the expressions in terms of d and simplified algebraically, as follows:

$$\frac{|\Delta V|}{V} \leq \frac{|V'\Delta d|}{V} = \frac{\frac{\pi}{2}d^2|\Delta d|}{\frac{\pi}{6}d^3} = 3\frac{|\Delta d|}{d}.$$

Thus, whatever the relative error in the measurement of d, we see that the relative error in the measurement of V is approximately 3 times as great. Specifically, in our problem, since we are given that $\frac{|\Delta d|}{d} = 1\%$, we conclude that $\frac{|\Delta V|}{V} \approx 3(1\%) = 3\%$. $\square$

Once again, we will explore how to compute the error in a tangent line approximation in Chapter 7, but there's one thing that we can usually easily determine: is the approximation given by the tangent line too small or too large? To do this, we can look at the second derivative.

Problem 4.26

If $f''(x) > 0$ near $x = a$, then explain why the tangent line approximation of $f(a + \epsilon)$ is less than the actual value of $f(a + \epsilon)$ for small values of ϵ.

Solution for Problem 4.26: If $f''(x) > 0$ near $x = a$, then the curve is concave down, meaning that the tangent line lies below the curve, as shown in the picture to the right. So the y-coordinate of a point on the tangent line (with x-coordinate near a) is going be less than the y-coordinate of the corresponding point on the graph of f. Therefore, the tangent line approximation is going to be too low. $\square$

Conversely, if $f''(x) < 0$ near $x = a$, then the curve is concave down, meaning that the tangent line lies above the curve. So in this case, a tangent line approximation is going to be too high.

Exercises

4.4.1:

Use tangent line approximation to approximate the following quantities:

(a) $\sqrt{80}$

You may type any additional notes you have here.

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Solution: We consider the function $f(x) = x^{\frac{1}{4}}$, approximated at $x = 81$. The derivative of $x^{\frac{1}{4}}$ is $\frac{1}{4}x^{-\frac{3}{4}}$, which at 81 is $\frac{1}{4}(81^{-\frac{3}{4}}) = \frac{1}{108}$. Therefore,

$$\sqrt[4]{80} \approx \sqrt[4]{81} + \frac{1}{108}(80 - 81) = 3 - \frac{1}{108} = \boxed{\frac{323}{108}}$$

(b) $\cos 62°$

You may type any additional notes you have here.

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Solution: We convert 62° to radians: $62° = \frac{62\pi}{180} = \frac{\pi}{3} + \frac{\pi}{90}$. We approximate $\cos x$ at $x = \frac{\pi}{3}$ to get

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{90}\right) \approx \cos\frac{\pi}{3} + \frac{\pi}{90}\left(-\sin\frac{\pi}{3}\right)$$
$$= \frac{1}{2} - \frac{\sqrt{3}\pi}{180}$$
$$= \boxed{\frac{90 - \sqrt{3}\pi}{180}}$$
$$\approx 0.4698.$$

(c) $\log_2 17$

You may type any additional notes you have here.

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Solution: We approximate $\log_2 x$ at $x = 16$. Write $\log_2 x = \frac{\log x}{\log 2}$ so that the derivative of $\log_2 x$ is $\frac{1}{x \log 2}$. Therefore,

$$\log_2 17 \approx \log_2 16 + \frac{1}{16\log 2} = \boxed{4 + \frac{1}{16\log 2}} \approx 4.0902.$$

4.4.2:

A cube is measured to have side 30 cm, but the measuring equipment (a cheap ruler) has a measurement error of 2 mm. What is the approximate error in the measured volume of the cube?

You may type any additional notes you have here.

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Solution: If the actual side length is x, the volume is x^3. We know that x is close to 30 cm, so we can approximate the side length near $x = 30$. The derivative of the volume at $x = 30$ is $3x^2 = 3(30)^2 = 2700$ cm^2, so the linear approximation gives an error of $0.2 \cdot 2700 = \boxed{540\text{ cm}^3}$.

4.4.3:

I want to paint a sphere (with diameter 1 meter) with a 2 mm-thick coating of paint. Approximately what volume of paint do I need?

You may type any additional notes you have here.

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Solution: The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius. We need to find the difference between a sphere with radius 500 mm and one with radius 502 mm. So we use the tangent line approximation of the formula for volume at $r = 500$. The derivative is $4\pi r^2$, so the added volume (paint) is approximately

$$4\pi(500)^2 \cdot 2 = \boxed{2000000\pi\text{ mm}^3 = 2000\pi\text{ cm}^3 = 2\pi\text{ liters}}$$

4.4.4:

We have a right circular cylinder whose radius equals its height. I want to measure the height in such a way that the measured volume is estimated to be within 2% of the actual volume. How much error (as a percentage) can I tolerate in the measurement of the height?

You may type any additional notes you have here.

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Solution: We know that a cylinder is πh^3. If the error in the measurement is E, then the measured volume is approximately $\pi h^3 + 3\pi h^2 E$. The error in this measurement is $3\pi h^2 E$, making the percent error $\frac{3\pi h^2 E}{\pi h^3} = \frac{3E}{h}$. Therefore, if we require $\frac{3|E|}{h} \leq \frac{2}{100}$ then we have $\frac{|E|}{h} \leq \frac{2}{300}$ so we can tolerate a $\boxed{\frac{2}{3}\%}$ error in the measurement of the height.

4.4.5★:

Let $f(x) = x^n$ where n is a positive integer. Use the Binomial Theorem to explain why $f(x + \epsilon) \approx f(x) + \epsilon f'(x)$ for small values of ϵ.

You may type any additional notes you have here.

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Solution: We know that $f'(x) = nx^{n-1}$. Therefore, by the tangent line approximation,

$$f(x + \epsilon) \approx f(x) + \epsilon f'(x) = x^n + n\epsilon x^{n-1}.$$

But the Binomial Theorem says that

$$f(x + \epsilon) = (x + \epsilon)^n = x^n + n\epsilon x^{n-1} + \epsilon \cdots + \epsilon^n.$$

We see that the tangent line approximation is just the first two terms of the Binomial Theorem, ignoring the other terms. When ϵ is small, this is not unreasonable, each of those terms includes ϵ^k for some $m \geq 2$.

4.5 Newton's Method

One profound application of tangent line approximation is called **Newton's Method** for approximating a root of a function. This is a numerical method, meaning that its use gives us a numerical approximation of the root. One nice thing (as we'll see) about Newton's Method is that using it repeatedly usually lets us get as close an approximation to a root as we want.

Here's the setup: we have a differentiable function f, and we want to find a root r (so that $f(r) = 0$). We start by making an initial guess r_0. The idea is to guess a value that's easy to calculate. Unless we are psychic, this initial guess will probably be wrong. Then we think about how can we use tangent line approximation to refine our guess.

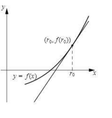
Problem 4.27

Let f be a differentiable function with $r_0 \in \mathrm{Dom}(f)$ such that $f'(r_0) \neq 0$. Find, in terms of r_0, $f(r_0)$, and $f'(r_0)$, the x-intercept of the tangent line to f at r_0.

Solution for Problem 4.27: Using the tangent line approximation at r_0, we get

$$f(x) \approx f(r_0) + f'(r_0)(x - r_0).$$

Let's draw a picture with our initial guess r_0 and the tangent line at $(r_0, f(r_0))$ displayed:



We solve for the x-intercept of the tangent line:

$$0 = f(r_0) + f'(r_0)(x - r_0).$$

This simplifies to

$$x - r_0 = -\frac{f(r_0)}{f'(r_0)},$$

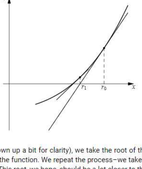
so we have $x = r_0 - \dfrac{f(r_0)}{f'(r_0)}$. □

Of course, unless we were really lucky, this value of x is not actually a root of f; it is only a root of the tangent line approximation. But we hope that this x is a better approximation of a root of f than our initial blind guess r_0 because we believe that the tangent line is a good approximation to f. So let's set

$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)},$$

with the idea that r_1 is a better guess for a root of f than r_0 was.

Now, we repeat the process from Problem 4.27 again, starting at r_1.



Referring to the picture above (which we've blown up a bit for clarity), we take the root of the first tangent line approximation, call that r_1, and use that as our next "guess" of the root of the function. We repeat the process–we take the tangent line approximation to the function at r_1 and find the root of this new tangent line. This root, we hope, should be a lot closer to the actual root of f.

Specifically, we let

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)},$$

and now r_2 should be an even better guess at the root than r_1 (and a lot better than our initial blind guess r_0).

More generally, we can repeat this for as long as we want. At each step, we take our guess r_{n-1} from the previous step, and set

$$r_n = r_{n-1} - \frac{f(r_{n-1})}{f'(r_{n-1})}$$

for all positive integers n. As we perform more and more steps, we should get closer and closer to an actual root.

Let's see a basic example.

Problem 4.28

Approximate $\sqrt[5]{23}$ to five decimal places using Newton's Method.

Solution for Problem 4.28: We might first think of simply using a tangent line approximation. However, trying to use a tangent line approximation using the function $x^{\frac{1}{5}}$ is not going to work very well, because 23 is pretty far from any known values of $x^{\frac{1}{5}}$ that we can use as the point at which we construct our tangent line.

For example, we could try a tangent line approximation using $f(x) = x^{\frac{1}{5}}$ at the point $(32, 2)$. We have $\Delta x = 23 - 32 = -9$ and $f'(x) = \dfrac{1}{5}x^{-4/5}$. So

$$f'(32) = \frac{1}{5} \cdot 32^{(-4/5)} = \frac{1}{5} \cdot \frac{1}{16} = \frac{1}{80}.$$

Thus our approximation is

$$f(23) \approx 2 - 9\left(\frac{1}{80}\right) = \frac{151}{80} = 1.8875.$$

This is not horrible, but because Δx was so large, it doesn't match the actual value of $\sqrt[5]{23}$ very closely. (We do know that our estimate is too high, since $f''(x) = -\left(\dfrac{4}{25}\right)x^{-\frac{9}{5}} < 0$, so f is concave down.)

Let's see how Newton's Method gives a better answer. Instead of working with $f(x) = x^{\frac{1}{5}}$, we note that $\sqrt[5]{23}$ is a root of $g(x) = x^5 - 23$. So we try to estimate the numerical value of the root of g using Newton's Method. First, note that $g'(x) = 5x^4$.

We start with an initial guess of $r_0 = 2$, because the fifth root of 23 is reasonably close to the fifth root of 32, which is 2. Then the first step of Newton's Method gives

$$r_1 = 2 - \frac{g(2)}{g'(2)} = 2 - \frac{9}{80} = 1.8875.$$

Note that this is the same answer as the tangent line approximation gave us. This is not a coincidence: the first step of Newton's Method is just using a tangent line approximation to guess at the root. The advantage of Newton's Method, though, is that we can refine our initial estimate by applying tangent line approximation again and again.

In general, we have

$$r_n = r_{n-1} - \frac{f(r_{n-1})}{f'(r_{n-1})} = r_{n-1} - \frac{r_{n-1}^5 - 23}{5r_{n-1}^4}.$$

So we can continue:

$$\begin{aligned} r_2 &= 1.8875 - \frac{g(1.8875)}{g'(1.8875)} \\ &\approx 1.8875 - \frac{0.95713}{63.46260} \\ &\approx 1.8875 - 0.01508 \\ &\approx 1.87242. \end{aligned}$$

Now our estimate is 1.87242.

We could do as many more iterations as we like; let's do one more:

$$\begin{aligned} r_3 &= 1.87242 - \frac{g(1.87242)}{g'(1.87242)} \\ &\approx 1.87242 - \frac{0.01528}{61.45866} \\ &\approx 1.87242 - 0.00025 \\ &\approx 1.87217. \end{aligned}$$

Now our estimate is 1.87217.

You can put $23^{\frac{1}{5}}$ into your calculator, and you'll see that this is now accurate to 5 decimal places. □

A more typical example of Newton's Method might be to approximate a solution to an equation:

Problem 4.29

Using Newton's Method, estimate the positive value of x such that $\cos x = x$. Perform enough iterations of Newton's Method so that your estimate gives $\cos x$ within 0.001 of x.

Solution for Problem 4.29: What we're really doing is trying to find a root of $f(x) = x - \cos x$. This function has derivative $f'(x) = 1 + \sin x$.

Part of the skill in applying Newton's Method is making a decent initial guess. In this problem, $r_0 = 0$ seems like a simple place to start. We then compute:

$$r_1 = -\frac{f(0)}{f'(0)} = \frac{\cos 0}{1 + \sin 0} = 1.$$

The next step is then

$$r_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1 - \cos 1}{1 + \sin 1}.$$

We can approximate these trig functions to 3 decimal places using a calculator:

$$r_2 \approx 1 - \frac{0.460}{1.841} \approx 0.750.$$

We continue:

$$r_3 = 0.750 - \frac{f(0.750)}{f'(0.750)} = 0.750 - \frac{0.750 - \cos 0.750}{1 + \sin 0.750}.$$

Again, we approximate this to 3 decimal places:

$$r_3 \approx 0.750 - \frac{0.018}{1.682} \approx 0.739.$$

Let's do it one more time:

$$r_4 = 0.739 - \frac{f(0.739)}{f'(0.739)} = 0.739 - \frac{0.739 - \cos 0.739}{1 + \sin 0.739}.$$

Computing again to 3 decimal places:

$$r_4 \approx 0.739 - \frac{0.000}{1.674} \approx 0.739.$$

The "0.000" in the numerator of the above fraction indicates that we are at a point in the process where our answer will not change in the first 3 decimal places, so we can stop now.

Indeed, we check using a calculator that $\cos 0.739 = 0.739142477\ldots$, and these quantities agree in the first three decimal places. □

If we wanted more accuracy, we could continue to run Newton's Method, keeping track of more decimal places.

Newton's Method seems like a good method for finding a root. It is simple enough that you can write a very short computer program to execute it. But it's not foolproof, and it's worth thinking about situations where it might fail to accurately approximate a root. Obviously it will fail if there is no root to be found. For example, try Newton's Method on the function $f(x) = x^2 + 1$ and see what happens! It might also converge to the wrong root if you choose a bad starting point, since many functions have more than one root. Also, we get stuck if we end up at a point with $f' = 0$, because then the tangent line is horizontal, so the formula for Newton's Method would require us to divide by 0.

Exercises

4.5.1:

Use Newton's Method to estimate $\sqrt[5]{2}$. (Whatever starting point you pick, do two iterations, then check against a calculator to see how close you are.)

You may type any additional notes you have here.

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Solution: The number $\sqrt[5]{2}$ is the positive root of the function $x^4 - 2$, whose derivative is $4x^3$. One possible starting point is $x = 1$. Then the tangent line approximation for $x^4 - 2$ is $4(x - 1) - 1 = 4x - 5$, which hits the x-axis at 1.25. The tangent line approximation at $x = 1.25$ is

$$x^4 - 2 \approx (4 \cdot 1.25^3)(x - 1.25) + (1.25^4 - 2),$$

which is zero when

$$\begin{aligned} x &= 1.25 + \frac{2 - 1.25^4}{4 \cdot 1.25^3} \\ &= 1.25 + \frac{512 - 625}{2000} \\ &= 1.25 - .0565 \\ &= 1.1935. \end{aligned}$$

A calculator gives $\sqrt[5]{2} \approx 1.1892$, so this achieves 3 digits of accuracy.

4.5.2★:

A classic example in which Newton's Method fails is $f(x) = \sqrt[3]{x}$ where we start too far away from 0. Compute 3 steps of Newton's Method for this function using the initial guess $r_0 = 1$. Can you explain what is happening?

Hint: You should get the sequence $1, -2, 4, -8, 16, \ldots$.

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Solution: The step in Newton's method for $f(x) = x^{\frac{1}{3}}$ is

$$r_{n+1} = r_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = -2r_n.$$

Starting with $r_0 = 1$, the sequence we find is $1, -2, 4, -8, 16 \ldots$. Of course we know the only root is at $x = 0$.

The issue that we run into is that the slope grows (in magnitude) without bound as x approaches 0. This is forcing the linear approximation to be bad regardless of how close we get to 0. When we draw the tangent line at r_2, it overshoots the root and intersects the x-axis at a point which is on the other side of the y-axis and further from the root than our original point. It seems intuitively clear that if the slope of f between r_n and r_{n+1} remains relatively constant, then the Newton's method approximation should be good. This example gives some credence to that conjecture since it is an example of Newton's method's failing at a point near which the derivative is unbounded.

4.6 Related rates

We have already mentioned several times that the most important interpretation of the derivative of a function is as the rate of change of that function. We can take this a step further. Suppose we have two functions that are somehow dependent on each other. We can use their derivatives to compare their rates of change.

The term **related rates** refers to two quantities that are dependent on each other and that are changing over time. We can use the dependent relationship between the quantities to determine a relationship between their rates of change.

Here is a basic example:

Problem 4.30

A balloon in the shape of a sphere is being inflated at the rate of $2\ \mathrm{cu}^3/\sec$. At the time at which the radius of the balloon is $2\ \mathrm{cu}$, how fast is its radius increasing?

Solution for Problem 4.30: Let r denote the radius of the balloon (in cm) and V be its volume (in cm^3). This already is a little bit deceptive, since both of the quantities are changing over time. We can ask exactly really do is let $r(t)$ denote the radius and $V(t)$ denote the volume, where t and V are functions of t, and where t denotes the time (in seconds). The relationship between the quantities is

$$V(t) = \frac{4}{3}\pi (r(t))^3.$$

What we want is an expression that relates the rates of change of these quantities. For that, we take the derivative of the entire equation with respect to t that, we take

$$\frac{d}{dt}V(t) = \frac{d}{dt}\left(\frac{4}{3}\pi (r(t))^3\right).$$

The left side of this equation is just $\frac{dV}{dt}$. On the right side, we must use the Chain Rule:

$$\frac{d}{dt}\left(\frac{4}{3}\pi (r(t))^3\right) = \frac{4}{3}\pi \frac{d}{dt}(r(t))^3 = 4\pi (r(t))^2 \frac{dr}{dt}.$$

Thus $\frac{dV}{dt} = 4\pi (r(t))^2 \frac{dr}{dt}$. Now we can plug in our given data. The rate of change of the volume is given as a constant $2\ \mathrm{cu}^3/\sec$, so $\frac{dV}{dt} = 2$ Also, at the time t we are interested in, we are given $r(t) = 2$ So we plug these in:

$$2 = 4\pi (2)^2 \frac{dr}{dt}.$$

Thus, we solve for $\frac{dr}{dt}$: $\frac{dr}{dt} = \frac{1}{8\pi}$ So the radius is increasing by $\frac{1}{8\pi}\ \mathrm{cm}/\sec$. ☺

This is the basic premise behind all related rates problems. We have two quantities, both of which are a function of the same independent variable (usually time). If there is an equation relating the two quantities, then we can take the implicitly differentiate that equation with respect to the independent variable to get an equation relating the variables and their derivatives.

In related rates problems, it is customary not to write explicitly the (t) part of the notation. For example, in Problem 4.30, we would typically write the equation relating the quantities as simply $V = \frac{4}{3}\pi r^3$ omitting the mention of the variable t. Then, the derivative of this equation is $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. However, it's important to remember that in this expression, V and r (and r^2) are quantities that depend on t.

Problem 4.31

An observer in 1 kilometer away from a rocket launch pad. At time $t = 0$ the rocket lifts off straight upwards, and at time t seconds has achieved an altitude of $10t^2$ meters. At 5 seconds after takeoff, how fast is the distance between the rocket and the observer increasing?

Solution for Problem 4.31: Even though there is only one moving object—the rocket—there are two quantities that we are interested in that are changing with respect to time: the altitude of the rocket (which we'll call x) and the distance from the rocket to the observer (which we'll call s).

The picture at right shows how these quantities are related. But don't make the following mistake:

Bogus Solution:

$$x^2 = h^2 + 1.$$

We have to be more careful about the units! The observer is 1 kilometer from the launch pad, but the height of the rocket is given in meters. So, if we want to express everything in meters, we have the equation

$$x^2 = h^2 + (1000)^2.$$

Taking the derivative of both sides with respect to t, we get

$$2x\frac{dx}{dt} = 2h\frac{dh}{dt}.$$

We are given $t = 5$. But it doesn't directly appear as a term in the above expression. So we need to figure out what the quantities in the above equation are in terms of t.

When $t = 5$, we have $h = 10t^2 = 250$. This means that

$$s^2 = (250)^2 + (1000)^2 = 1\text{,}272\text{,}032^2,$$

so $x = 250\sqrt{17}$. Also, $\frac{dh}{dt} = 20t = 100$ Thus, we have

$$\begin{aligned} 2x\frac{dx}{dt} &= 2h\frac{dh}{dt} \\ \Rightarrow \quad x\frac{dx}{dt} &= h\frac{dh}{dt} \\ \Rightarrow \quad (250\sqrt{17})\frac{dx}{dt} &= (250)(100) \\ \Rightarrow \quad \frac{dx}{dt} &= \frac{100}{\sqrt{17}} = \frac{100\sqrt{17}}{17}\ \mathrm{m}/\sec. \end{aligned}$$

So the distance between the rocket and the observer is increasing at a rate of $\frac{100\sqrt{17}}{17}\ \mathrm{m}/\sec$. ☺

The next problem is a classic related rates problem:

Problem 4.32

A 10-foot-tall ladder rests against a wall, but the foot of the ladder is slipping away from the wall at a rate of 1 m/sec. When the ladder forms a $60°$ degree angle to the ground, how fast is the top of the ladder sliding down?

Solution for Problem 4.32: We should set this up to have as our two "variables" the functions whose rates we care about. Therefore, we should use w (the distance from the wall) and h (the height of the top of the ladder). We can then write an equation relating these quantities:

$$w^2 + h^2 = 100.$$

But this uses feet as the units on the right side. Since the rate is in m/sec, maybe we should write the whole thing in inches:

$$w^2 + h^2 = 14400.$$

Although it is always good practice to keep our units consistent, it doesn't really matter here, because the next step is to take the derivative with respect to time:

$$2ww' + 2hh' = 0.$$

Now we need to plug in the values of these quantities so that we can solve for h'. We know that $w' = 1(\mathrm{in}/\sec)$. We need to figure out w and h. But the other bit of information that we haven't used yet is the 60° angle. This gives us $w = 5\ (\mathrm{feet})$ and $h = 5\sqrt{3}\ (\mathrm{feet})$. But the rate of change is in inches per second So we need to use $w = 60\ (\mathrm{inches})$ and $h = 60\sqrt{3}\ (\mathrm{inches})$. We then plug these values in to our related rates expression:

$$2(60)(1) + 2(60\sqrt{3})h' = 0.$$

Solving for h' gives $h' = -\frac{1}{\sqrt{3}}$. So the top of the ladder is falling at a rate of $\frac{1}{\sqrt{3}}$ inches per second. ☺

By the way, in Problem 4.32, what does the function $h(t)$ look like in general? We can solve for h' in our related rates equation:

$$h' = -w\frac{w'}{h} = -\frac{w}{h},$$

since $w' = 1$. But we know what the quantity w/h is: it's the cotangent of the angle that the ladder makes with the ground. So we have

$$h' = -\cot\theta,$$

where θ is the angle that the ladder makes with the ground.

This makes "real world" sense. At $\theta = \pi/2$ the ladder is vertical, and $\cot(\pi/2) = 0$, meaning that the top of the ladder is stationary. As θ decreases, cot θ increases, so the top of the ladder moves faster and faster downwards.

Let's look at one more related rates problem that's a bit harder:

Problem 4.33

A conical vessel filler has a hole at the top of radius 4 cm and a hole at the bottom of radius 1 cm, and is 4 cm in height. Water flows out of the bottom at a rate of $2\ \mathrm{cu}^3/\sec$. If the filler begins completely filled at time $t = 0$, how fast is the water level decreasing after 30 seconds?

Solution for Problem 4.33: We start by identifying the relevant quantities that are changing with respect to time. These are the height of the water, which we'll call h, and the volume of water in the filler, which we'll call V. We need to determine how they are related.

We see that the filler is a **frustum**, which is a cone with a smaller cone chopped off. The volume of a frustum is most easily computed as the difference in volume between the two cones. To compute the volumes of the cones, we need their heights. So we will be much better off if our height is measured from the vertex of the imaginary cone, not from the bottom of the filler.

Next, the formula for the volume of a cone is $\frac{1}{3}\pi r^2 h$, where r is the radius and h is the height. So we need to determine how the height is related to the radius. Fortunately, this is easy to do because we are can see from our picture at right. In particular, decreasing the height by 4 causes a decrease of 3 in the radius, so we conclude that $h/r = 4$. and thus the radius at height h is $h/4$. Note that the top of our frustum is at $h = 8$ and $h = 4$), and at the top we have radius $R/2 = 4$, as expected. The bottom of the frustum is at $h = 2$ - $1 = 2$.

So the volume of the water at height h is:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi (1)^2 (2) \\ &= \frac{1}{3}\pi \left(\frac{h}{4}\right)^2 h - \frac{1}{3}\pi (1)^2 (2) \\ &= \frac{1}{3}\pi h^3 - \frac{2}{3}\pi. \end{aligned}$$

We can differentiate this equation to get an equation relating the rates:

$$V' = \frac{1}{4}\pi h^2 h'.$$

We are told that $V' = -2$ as the rate of decrease of volume is constant. But we also need to find the value of h in order to solve for h'. So we need to find the volume at $t = 30$ and compute its height.

We can start by finding the initial volume (at $t = 0$):

$$V(0) = \frac{1}{12}\pi h^3 - \frac{2}{3}\pi.$$

This simplifies to 42π:

When $t = 30$, we will have lost $2 \cdot 30 = 60\ \mathrm{cu}^3$ of water, so the volume will be $V(30) = 42\pi - 60$. We can plug this in to our equation for volume to find h at time $t = 30$:

$$42\pi - 60 = \frac{1}{12}\pi h^3 - \frac{2}{3}\pi.$$

This gives $h^3 = \frac{512\pi - 720}{\pi}$, and thus $h = \sqrt[3]{\frac{512\pi - 720}{\pi}} \approx 6.564$.

Finally, we can go back to our related rates equation:

$$-2 = \frac{1}{4}\pi h^2 h',$$

and plug in the found value of h. This gives:

$$-2 = \frac{1}{4}\pi \left(\frac{512\pi - 720}{\pi}\right)^{\frac{2}{3}} h'.$$

We solve for h' to get our answer:

$$h' = \frac{8}{\sqrt[3]{512\pi - 720}^2} = \frac{8}{\sqrt[3]{604\pi - 90}^2}.$$

This is approximately -0.0591, and thus at time $t = 30$ the water level is decreasing at a rate of $0.0591\ \mathrm{cm}/\sec$. ☺

Exercises

4.6.1:

A plane is flying overhead at an altitude of 10 km and a speed of 200 m/sec. The plane is 20 km away from a camera on the ground, and the plane is flying in a direction that will take it directly over the camera. The camera continuously rotates to keep the plane centered in its lens. When the plane is 15 km away from the camera, how fast is the camera rotating?

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Solution: Let's put everything in kilometers first, and assign some variables. Let the horizontal distance from the camera to the spot on the ground directly below the plane be x, time be t, and the angle the camera makes with the ground be θ. Then trigonometry yields $\tan\theta = 10/x$, or $x = 10\cot\theta$ (in km). We differentiate to get

$$\frac{dx}{dt} = -10\cos^2\frac{d\theta}{dt}.$$

We know that $\frac{dx}{dt} = -0.2\ \mathrm{km}/\mathrm{s}$, where the sign is negative since the plane is approaching the camera. We are considering the moment when $\sin\theta = \frac{10}{15} = \frac{2}{3}$, so $\cos^2\theta = \frac{5}{9}$. Therefore,

$$\frac{d\theta}{dt} = \frac{-2}{-10 \cdot \frac{5}{9}} = \frac{\frac{2\ \mathrm{rad}}{225\ \mathrm{sec}}}{}.$$

4.6.2:

Boyle's Law states (assuming that the temperature is constant) that the pressure and volume of a gas are inversely proportional. Suppose Gas X starts at 1 atm (atmosphere) of pressure and takes up $200\ \mathrm{cm}^3$. If the pressure is increased by 0.1 atm/min, then after 10 minutes, at what rate is the volume decreasing?

Hint: The relevant equation is $PV = k$, where k is some constant.

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Solution: If pressure is P and volume V, then Boyle's Law tells us that $PV = 200\ \mathrm{cm}^3 \cdot \mathrm{atm}$. We differentiate both sides of this equation, yielding

$$P\frac{dV}{dt} + V\frac{dP}{dt} = 0.$$

We know that at the time in question, $P = 1 + 10 \cdot \frac{1}{10} = 2$ atm and V therefore is $100\ \mathrm{cm}^3$. Solving our equation,

$$\frac{dV}{dt} = -\frac{V}{P}\frac{dP}{dt} = -\frac{100\ \mathrm{cm}^3}{2\ \mathrm{atm}} \cdot (0.1\ \mathrm{atm}/\mathrm{min}) = -5\ \mathrm{cm}^3/\mathrm{min}.$$

Thus the volume is decreasing at a rate of $5\ \mathrm{cm}^3/\mathrm{min}$.

4.6.3:

A spherical snowball is 10 cm in diameter, but is melting at the rate of $0.5\ \mathrm{cm}^3$ per second. When the snowball is reduced to half its original volume, at what rate is its diameter decreasing?

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Solution: Let V be the volume of the snowball and x be its diameter. Using the formula for the volume of a sphere, we can write $V = \frac{4}{3}\pi\left(\frac{x}{2}\right)^3 = \frac{\pi}{6}x^3$. Then, differentiating both sides with respect to t, we get

$$\frac{dV}{dt} = \frac{\pi}{2}x^2\frac{dx}{dt}.$$

To find $\frac{dx}{dt}$, we just need $\frac{dV}{dt}$ and x. Since volume is proportional to the cube of diameter, at half its volume the snowball will have a diameter of $\frac{10\ \mathrm{cm}}{\sqrt[3]{2}} = 5\sqrt[3]{4}$ cm. Thus,

$$\frac{dx}{dt} = \frac{dV}{dt} \cdot \frac{2}{\pi x^2} = (0.5\ \mathrm{cm}^3/\mathrm{sec}) \cdot \frac{2}{\pi 50\sqrt[3]{2}\ \mathrm{cm}^2} = \frac{\sqrt[3]{4}\ \mathrm{cm}}{100\pi\ \mathrm{sec}}.$$

4.6.4:

A meteorite has entered the earth's atmosphere and is burning up at a rate that is proportional to the meteorite's surface area. What can you determine about the rate that the meteorite's radius is decreasing?

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Solution: We are given that $\frac{dA}{dt} = k\pi$ is proportional to surface area, which is proportional to r^2 (where r is the radius), so $\frac{dA}{dt} = kr^2$ for some constant k. We assume for now that the meteorite is spherical, so $V = \frac{4}{3}\pi r^3$. Upon differentiating by t, we get $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Eliminating $\frac{dV}{dt}$ from this system we deduce that $\frac{dr}{dt} = \frac{k}{4\pi}$ is constant, so the meteorite's radius is decreasing at a constant rate.

Notice that we needn't have assumed that the meteorite was spherical, but only that the surface area was always proportional to $V^{2/3}$. Letting r denote the square root of surface area allows us to recycle the above computation to prove this more general case.

4.6.5:

Two cars start at an intersection at time $t = 0$ with velocity 5. At $t = 0$, one car accelerates due north at $8\ \mathrm{m}/\sec^2$, and the other car accelerates due east at $8\ \mathrm{m}/\sec^2$. After 6 seconds, by what rate is the distance between the cars increasing?

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Solution: The position of the first car after t seconds is 48^2 north of the starting point, and the position of the second car after t seconds is $\frac{1}{2}t^2$ east of the starting point. So the distance between them is

$$\sqrt{\left(48^2\right)^2 + \left(\frac{1}{2}t^2\right)^2} = t^2\sqrt{576 + \frac{t^2}{4}}.$$

The rate of change of this distance is $t\sqrt{992}$, so at $t = 6$ the distance between the cars is increasing by $\boxed{6\sqrt{992}}$ meters per second.

4.6.6a:

The clock on the wall at my office has an hour hand that's 6 cm long and a minute hand that's 10 cm long. At exactly 2:00, at what rate is the distance between the tips of the two hands changing?

Hint: It may be helpful to measure angles clockwise from the 12:00 position.

Hint: Use the Law of Cosines to determine the distance between the tips of the hands in terms of the angle between the hands.

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Solution: We let $t = 0$ represent 12:00 and measure time in minutes. Also, because we are working with a clock, it makes sense to measure angles clockwise from the 12:00 position. The minute hand sweeps an angle of 2π every 60 minutes, so its angular velocity is $\frac{2\pi}{60} = \frac{\pi}{30}$ radians per minute. At the minute hand at time t is at $\theta = \frac{\pi t}{30}$ radians clockwise from the 12:00 position.

Similarly, the hour hand sweeps an angle of 2π every 720 minutes, so its angular velocity is $\frac{2\pi}{720} = \frac{\pi}{360}$, and thus the position of the hour hand at time t is $\psi = \frac{\pi t}{360}$ radians clockwise from the 12:00 position. The angle between the two hands as a function of t is $\rho\theta - \psi\theta = (\rho - \psi)\theta$.

We are concerned with the time $t = 120$, representing 2:00. By the Law of Cosines, the distance x between the tips of the two hands satisfies

$$\begin{aligned} x^2 &= 6^2 + 10^2 - 2 \cdot 10 \cdot 6\cos\left((\rho - \psi)\theta\right) \\ &= 136 - 120\cos\left((\rho - \psi)\theta\right). \end{aligned}$$

Now we consider $t = 120$. The distance satisfies

$$\begin{aligned} x^2 &= 136 - 120\cos\left(\left(\frac{\pi}{30} - \frac{\pi}{360}\right)(120)\right) \\ &= 136 - 120\cos\left(\left(\frac{11\pi}{360}\right)(120)\right) \\ &= 136 - 120\cos\left(\frac{11\pi}{3}\right) \\ &= 76, \end{aligned}$$

so $x = \sqrt{76} = 2\sqrt{19}$ is the distance between the tips of the two hands at $t = 120$.

We also have

$$x\frac{dx}{dt} = 60\left(\frac{\pi}{30} - \frac{\pi}{360}\right)\sin\left(\left(\frac{\pi}{30} - \frac{\pi}{360}\right)(120)\right)$$

$$= \frac{11\pi}{12}\sin\left(\frac{11\pi}{3}\right)$$

$$= \frac{11\pi\sqrt{3}}{12}.$$

Therefore

$$\frac{dx}{dt} = \frac{\frac{11\pi\sqrt{3}}{12}}{x} = \frac{11\pi\sqrt{3}}{24\sqrt{19}} = \frac{11\pi\sqrt{3}}{24\sqrt{19}} \approx 0.572\ \mathrm{cm}/\sec$$

Thus, the distance between the tips of the hands is decreasing at a rate of $\frac{11\pi\sqrt{3}}{24\sqrt{19}} \approx 0.572\ \mathrm{cm}/\sec$

Challenge Problems

4.45:

Explain and prove the **Racetrack Theorem**: if f and g are differentiable functions, with $f(a) = g(a)$ for some a and $f'(x) > g'(x) > 0$ for all $x \geq a$, then $f(x) > g(x)$ for all $x > a$.

Preview: Solution

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Solution: Define $h(x) = f(x) - g(x)$. Note that $h(a) = 0$ and $h'(x) > 0$ for all $x > a$. So we must have $h(b) > h(a)$ for all $b > a$ (because, if $h(b) = h(a)$ for some $b > a$, then we'd have to have $h'(c) = 0$ for some $b > c > a$, but this would violate Rolle's Theorem). In particular, $h(x) > h(a) = 0$ for all $x > a$, and thus $f(x) > g(x)$. The reason it's called the Racetrack Theorem: think of f and g as representing the positions of two cars on a racetrack. If they start at the same point (at time a), and car f is always going faster than car g, then car f will always be ahead of car g.

4.46:

What is the relationship between the critical points of f and the critical points of f^2?

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Solution: We know that $(f^2)' = 2f f'$, so x is a critical point of f^2 if and only if either $(f')' = 0$, in which case either $f(x) = 0$ or $f'(x) = 0$, or $(f^2)'$ is undefined, which can only happen if f' is undefined. Therefore, the critical points of f^2 are precisely the critical points of f together with the roots of f.

4.47:

Source: HMMT

Determine the real number a having the property that $f(a) = a$ is a relative minimum of

$$f(x) = x^5 - x^3 - x^2 + ax + 1.$$

Hint

You may type any additional notes you have here.

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Solution: a must be a critical point, so since f is differentiable we must have $f'(a) = 0$. This gives us the system of equations

$$\begin{aligned} a &= f(a) = a^5 - a^3 + 1, \\ 0 &= f'(a) = 4a^5 - 3a^2 - a. \end{aligned}$$

The latter equation is $a(4a + 1)(a - 1) = 0$, so its roots are 0, 1, and $-\frac{1}{4}$. Inspection shows that only $a = 1$ satisfies the top equation, so we use the Second Derivative Test to check if this is a minimum. We have $f''(a) = 12a^3 - 6a - 2$, so $f''(1) = 4 > 0$, and indeed $\boxed{a = 1}$ is a local minimum.

4.48:

(a) Sam wishes to cross a circular lake with diameter 1 km. He can row across the water at a rate of 4 km/hour, or he can walk along the shore (carrying the rowboat) at a rate of 6 km/hour. What is the minimum amount of time necessary to cross the lake?

Hint: You can assume that Sam first rows in a straight line and then walks the rest of the way. (Why is this OK to do?)

Hint: Drawing a picture will help a lot.

Hint: Try computing in terms of the angle at which Sam chooses to start rowing (relative to the diameter of the lake).

Preview: Solution

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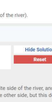
Your Submission: Solution

Solution: We may assume that Sam rows along a chord to some point on the lakeshore, and then walks directly to the end point. (He may in fact walk first and then row, but by the symmetry of the lake it doesn't matter. There is no advantage to mixing up shorter segments of rowing and walking: again by the symmetry, we can rearrange the segments so that all the rowing segments come first, and then it is clear that rowing in a straight line to the rowing endpoint is faster than rowing in multiple segments.)

Suppose that Sam rows at an angle θ below the diameter from his start point to his destination, as shown in the picture at right. The distance that he rows is row θ (in the side adjacent to the angle with measure θ in a right triangle with hypotenuse 1), and the length that he walks is θ (an arc of measure 2θ of a circle of radius $\frac{1}{2}$). So the total amount of time that Sam takes is



$$f(\theta) = \frac{1}{4}\cos\theta + \frac{1}{6}\theta.$$

Note that the domain is $\theta \in \left[0, \frac{\pi}{2}\right]$, where $\theta = 0$ corresponds to rowing the entire distance, and $\theta = \frac{\pi}{2}$ corresponds to walking the entire distance. We then compute

$$f'(\theta) = -\frac{1}{4}\sin\theta + \frac{1}{6},$$

so a critical point is where $\sin\theta = \frac{2}{3}$ or $\theta = \sin^{-1}\frac{2}{3}$. However, using the Second Derivative Test, we see that $f''(\theta) = -\frac{1}{4}\cos\theta < 0$, so any critical point $\theta \in \left(0, \frac{\pi}{2}\right)$ must be a local maximum. We want to minimize the time, so we conclude that the minimum on $\left[0, \frac{\pi}{2}\right]$ must be at one of the endpoints. Rowing the entire distance (where $\theta = 0$) takes $f(0) = \frac{1}{4}$ hours, whereas walking the entire distance takes $f\left(\frac{\pi}{2}\right) = \frac{\pi}{12} > \frac{1}{12} = \frac{1}{4}$ hours, so Sam is best off rowing the whole way, which takes $\boxed{\frac{1}{4}\text{ hours}}$.

(b) Now Sam wishes to cross a river of width 1 km, and reach a point on the opposite bank that is 1 km downstream. Again, he can row across the water at a rate of 4 km/hour (ignore the effect of the river's current), or he walk along the shore (carrying the rowboat) at a rate of 6 km/hour. What is the minimum amount of time necessary to reach his destination?

Hint: As in part (a), you can assume that Sam first rows in a straight line and then walks the rest of the way. (Why is this OK to do?)

Hint: Drawing a picture will help a lot.

Hint: Try computing in terms of the angle at which Sam chooses to start rowing (relative to the shore of the river).

Preview: Solution

You may type any additional notes you have here.

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Solution: We can assume that Sam rows in a straight line from his starting point to a point on the opposite side of the river, and then walks the rest of the way. (He might, in fact, walk on the first side, then row across, and then walk on the other side, but this doesn't alter his total time than if he had just done the rowing first.)

Suppose that Sam rows at an angle θ to the side that he starts on, as shown in the diagram. The distance that he rows is row θ (it is hypotenuse of a right triangle where the side opposite θ has length 1), and the length that he walks is $1 - \cot\theta$. So the total amount of time that Sam takes is

$$f(\theta) = \frac{1}{4}\csc\theta + \frac{1}{6}(1 - \cot\theta).$$

Note that the domain is $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, where $\theta = \frac{\pi}{2}$ corresponds to rowing the entire distance, and $\theta = \frac{\pi}{4}$ corresponds to rowing directly across, then walking the entire shoreline. We then compute

$$\begin{aligned} f'(\theta) &= -\frac{1}{4}\csc\theta\cot\theta + \frac{1}{6}\csc^2\theta = \csc\theta\left(-\frac{1}{4}\cot\theta + \frac{1}{6}\csc\theta\right), \end{aligned}$$

so, since $\csc\theta \neq 0$, a critical point is where $\cot\theta = \frac{\csc\theta}{\csc^2\theta} = \frac{2}{3}$, or $\theta = \cot^{-1}\frac{2}{3}$. Using the Second Derivative Test, we see that for $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$,

$$\begin{aligned} f''(\theta) &= \csc\theta\left(\frac{1}{4}\csc^2\theta - \frac{1}{6}\cot\theta\csc\theta\right) \\ &\quad - (\cot\theta\csc\theta)\left(-\frac{1}{4}\cot\theta + \frac{1}{6}\csc\theta\right) \\ &= \csc\theta\left(\frac{1}{4}\csc^2\theta - \frac{1}{3}\cot\theta\csc\theta + \frac{1}{4}\cot^2\theta\right) \\ &= \frac{1}{4}\csc\theta\left((\csc\theta - \cot\theta)^2 + \frac{2}{3}\csc\theta\cot\theta\right) \\ &> 0, \end{aligned}$$

so any critical point $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ must be the minimum. Thus, the total time is minimized by choosing $\theta = \cot^{-1}\frac{2}{3}$ and the time necessary is

$$\begin{aligned} f\left(\cot^{-1}\frac{2}{3}\right) &= \frac{1}{4}\left(\cot^{-1}\frac{2}{3}\right) + \frac{1}{6}\left(1 - \cot\left(\cot^{-1}\frac{2}{3}\right)\right) \\ &= \frac{1}{4}\left(\frac{\pi}{2}\right) + \frac{1}{6}\left(1 - \frac{2}{3}\right) \\ &= \boxed{\frac{1}{6} + \frac{\sqrt{5}}{12}} \approx 0.353\text{ hours}. \end{aligned}$$

4.49:

The **Rule of 72** states that if an amount of money is invested at $r\%$ interest per year, then it will take approximately $72/r$ years for the money to double. Use linear approximation to explain why this rule is a good approximation for small values of r.

Hint: The key fact is that $100 \log 2$ is reasonably close to 72.

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Solution: If we invest m at $r\%$ interest, then the amount that we have after t years is $(\$m)\left(1 + \frac{r}{100}\right)^t$. So we are looking for the value of t such that

$$\left(1 + \frac{r}{100}\right)^t = 2.$$

Thus $t = \log_{(1+\frac{r}{100})} 2 = \frac{\log 2}{\log(1 + \frac{r}{100})}$.

We can approximate $\log(1 + x)$ by tangent line approximation: let $f(x) = \log x$, so $f'(x) = \frac{1}{x}$. Then

$$\log(1 + x) \approx \log(1) + x = x.$$

Thus we have

$$t = \frac{\log 2}{\log(1 + \frac{r}{100})} \approx \frac{100 \log 2}{r}.$$

Finally, we can approximate $100 \log 2 \approx 69.3$, which is close enough to 72 to not matter too much if r is small. (Accountants and other investment advisors prefer 72 since it's divisible by lots of small numbers: 1, 2, 3, 4, 6, 8, ... Look up "Rule of 72" on the internet to see some examples.)

4.50:

Snell's Law describes how light passes through two mediums in which the speed of light is different. Suppose that light travels from the point A in the positive half-plane (where $y > 0$) to the point B in the negative half-plane (where $y < 0$). Further suppose that light has speed c_1 in the positive half-plane and speed c_2 in the negative half-plane. Find an equation that relates the angles that the light ray hits the x-axis with the speeds of light in the two half-planes, given that light will always travel such that its total travel time is minimized.

Hint: Write a function for the total travel time, then minimize it.

Hint: It may be easier to work in terms of coordinates rather than angles.

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Solution: Let A, B, X, and θ_1 be as in the diagram. Let point A be distance a from the x-axis, point B be distance b from the x-axis, and the horizontal distance between the two be d. Let the distance from the point on the interface nearest A and X be x, where $0 \leq x \leq d$. By Pythagorean Theorem, $AX = \sqrt{a^2 + x^2}$ and $XB = \sqrt{b^2 + (d - x)^2}$. Time is distance divided by rate, so the total time of travel is

$$f(x) = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

Setting $f'(x) = 0$, we have

$$0 = f'(x) = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}.$$

Now we note that the first fraction in this equation is equal to $\frac{\sin\theta_1}{c_1}$, and the second is $\frac{\sin\theta_2}{c_2}$. Therefore, we must have

$$\boxed{c_1 \sin\theta_2 = c_2 \sin\theta_1}.$$

We also note that $f'(x) < 0$ for small x (specifically, to the left of the critical point) and $f'(x) > 0$ for large x (specifically, to the right of the critical point), so by the First Derivative Test, the above critical point does indeed give a global minimum.

4.51★:

Suppose that a function f is twice-differentiable. Recall from Problem 4.22 that we defined $E(x)$ to be the error in the approximation of $f(x)$ using $f(a)$, that is

$$E(x) = f(x) - (f(a) + f'(a)(x - a)).$$

Show that for any $x > a$, we have $|E(x)| \leq M(x - a)^2$, where M is the maximum value of $|f''(x)|$ on the interval $[a, x]$. (In fact, later in the book, we will show that $|E(x)| \leq \frac{1}{2}M(x - a)^2$, but this is difficult with our current technology.)

Hint: Find a bound, in terms of f'', for $\frac{|E(x)|}{x - a}$, then use the Mean Value Theorem to write your expression solely in terms of f'.

Hint: Find a bound, in terms of f', for $\frac{|E(x)|}{(x - a)^2}$, and then use the Mean Value Theorem to write the bound solely in terms of f''.

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Solution: We start with

$$E(x) = f(x) - f(a) - f'(a)(x - a).$$

Divide by $x - a$ to get

$$\frac{|E(x)|}{x - a} = \left|\frac{f(x) - f(a)}{x - a} - f'(a)\right|.$$

By the Mean Value Theorem, there is some $a < c < x$ such that $f'(c) = \frac{f(x) - f(a)}{x - a}$, hence

$$\frac{|E(x)|}{x - a} = |f'(c) - f'(a)|.$$

Note that $(c - a) < (x - a)$, so dividing again gives

$$\frac{|E(x)|}{(x - a)^2} = \frac{|f'(c) - f'(a)|}{x - a} \leq \frac{|f'(c) - f'(a)|}{c - a}$$

and applying the Mean Value Theorem (to f') means that there is some $a < \xi < c$ such that $f''(\xi) = \frac{f'(c) - f'(a)}{c - a}$. Hence

$$\frac{|E(x)|}{(x - a)^2} \leq |f''(\xi)|(c - a) \leq M(c - a)^2,$$

and thus

$$|E(x)| < |f''(\xi)|(x - a)^3 \leq M(x - a)^3.$$

4.52★:

A snowplow can remove snow at a constant rate (m^3/mile). One day, there was no snow on the ground at sunrise, but sometime in the morning it began snowing at a steady rate. At noon, the plow began to remove snow. It had cleared 2 miles of snow between noon and 1 PM, and 1 more mile of snow between 1 PM and 2 PM. At what time did it start snowing?

Hint: The velocity that the snowplow travels is inversely proportional to the amount of time since it started snowing.

Hint: Note that $\frac{dx}{dt} = \frac{k}{t}$ for some constant k (where x is the distance traveled by the plow).

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Solution: Suppose noon is h hours after it started snowing. We're told that the rate of change of the position of the plow (which we can denote as x) is inversely proportional to the amount of snow, which itself is proportional to the time since it started snowing. So we have $\frac{dx}{dt} = \frac{k}{t}$ for some constant k. This means that $x = k \log t + C$ for some constants k and C.

The given information about the position of the plow tells us that

$$k \log(h + 1) + C = 2 + k \log h + C$$

and

$$k \log(h + 2) + C = 3 + k \log h + C.$$

So $k(\log(h + 1) - \log(h)) = 2$ and $k(\log(h + 2) - \log(h)) = 3$, meaning that

$$3 \log\left(\frac{h + 1}{h}\right) = 2 \log\left(\frac{h + 2}{h}\right).$$

Taking the exponential of both sides gives

$$\left(\frac{h + 1}{h}\right)^3 = \left(\frac{h + 2}{h}\right)^2,$$

so $(h + 1)^3 h^2 = (h + 2)^2 h^3$, and thus $(h + 1)^3 = (h + 2)^2 h$, giving $h^3 + h - 1 = 0$. Thus

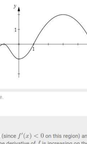
$$h = \frac{-1 \pm \sqrt{5}}{2},$$

and only the positive solution makes sense. So $h = (\sqrt{5} - 1)/2 \approx 0.618$, so it started snowing 0.618 hours before noon, or at about 11:23 a.m..

Review Problems

4.34:

The following is the graph of the derivative f' for some function f. Sketch possible graphs of f and f''. Take into account where the function is increasing, decreasing, its concavity, and any inflection points.



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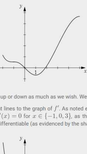
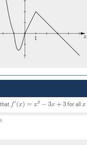
Solution: We see that f is decreasing for $x < 1$ (since $f'(x) < 0$ on this region) and increasing for $x > 1$. We also notice that f is concave up for $x < -1$ and $x \in (0, 3)$, since the derivative of f is increasing on these regions; f is concave down elsewhere. Also, we see that the graph of f is steepest near $x = 3$ (where f' is relatively large), and f has critical points at $x = -1, x = 1$, and $x = 5$, where the graph of f should have horizontal tangent lines. Using all of this information, we can sketch a graph of f, shown below.



Note that this graph can be translated vertically up or down as much as we wish. We have arbitrarily chosen to make $f(0) = 0$.

To graph f'', we graph the slopes of the tangent lines to the graph of f'. As noted earlier, f'' is positive on $x < -1$ and $x \in (0, 3)$ and negative otherwise. We also note that $f''(x) = 0$ for $x \in \{-1, 0, 3\}$, as these are the critical points of f'. Below is one possible graph; note that f'' need not be itself differentiable (as evidenced by the sharp corners in the graph of f'').



4.35:

Suppose that f is a function with domain $\mathbb{R}$ such that $f'(x) = x^2 - 3x = 3$ for all $x \in \mathbb{R}$. Prove that f has an inverse.

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Solution: Note that f' has no real roots, so $f' > 0$ for all x. Thus f is always strictly increasing, and we know that any strictly increasing function has an inverse. Another way to see this is that if we have $a < b$ with $f(a) = f(b)$, then by Rolle's Theorem some c between a and b must have $f'(c) = 0$, but this cannot happen since $f'(c) > 0$ for all c.

4.36:

(a) Show that if f is a quadratic polynomial, then the graph of f has no inflection points.

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Solution: Suppose $f(x) = ax^2 + bx + c$ with $a \neq 0$. Then $f'(x) = 2ax + b$, and $f''(x) = 2a \neq 0$. Thus, since f'' is continuous and $f'' \neq 0$, we conclude that f has no inflection points.

(b) Show that if f is a cubic polynomial, then the graph of f has exactly one inflection point.

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Solution: Suppose $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$. Then $f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$. We see that $f''(x) = 0$ if and only if $x = -\frac{b}{3a}$. Since f'' is positive to one side of $x = -\frac{b}{3a}$ and negative to the other side (which side is which depends on the sign of a), we see that $x = -\frac{b}{3a}$ is the (unique) inflection point of f.

(c) Can you generalize?

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Solution: If f is of degree n, then $f(x) = a_n x^n + \cdots + a_0$, and $f''(x) = n(n-1)a_n x^{n-2} + \cdots$ is of degree $n-2$. Thus, we can say that f has at most $n-2$ inflection points. It is possible that f may have fewer than $n-2$ inflection points; if f'' has fewer than $n-2$ real roots.

4.37:

Describe the graph of $f(x) = x^3 + px + q$ where p and q are constants. Determine where the function is increasing or decreasing, where any local maxima or minima are located, and the concavity and inflection points of the graph.

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Solution: f is differentiable with derivative $f'(x) = 3x^2 + p$. If $p \geq 0$, then f is increasing everywhere, since $3x^2 \geq 0$, and therefore has no local maxima or minima. Otherwise, f is increasing when $|x| > \sqrt{-\frac{p}{3}}$ and decreasing when $x \in \left(-\sqrt{-\frac{p}{3}}, \sqrt{-\frac{p}{3}}\right)$. Furthermore, the critical points are $\pm\sqrt{-\frac{p}{3}}$, and by the First Derivative Test, we see that $x = -\sqrt{-\frac{p}{3}}$ gives a local maximum, and $x = \sqrt{-\frac{p}{3}}$ gives a local minimum.

Also, note that $f''(x) = 6x$ is positive for $x > 0$ and negative for $x < 0$, so f is concave up on $(0, \infty)$, concave down on $(-\infty, 0)$ and has an inflection point at $x = 0$. This also confirms to us, via the Second Derivative Test, that in the case where $p < 0$, we have that $x = \sqrt{-\frac{p}{3}}$ gives a local minimum and $x = -\sqrt{-\frac{p}{3}}$ gives a local maximum.

4.38:

Let $m, n \geq 2$ be integers. Describe (in as much detail as possible) the graph of $f(x) = x^m(1-x)^n$ on the interval $[0, 1]$

Hint: Note $f(0) = f(1) = 0$ and $f(x) > 0$ for all $x \in (0, 1)$. Try to find the maximum on $[0, 1]$.

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Solution: We first note that $f(0) = f(1) = 0$, and $f(x) > 0$ for all $x \in (0, 1)$. So f must have a local maximum on $(0, 1)$. Continuing, we can calculate

$$\begin{aligned} f'(x) &= mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1} \\ &= x^{m-1}(1-x)^{n-1}(m(1-x) - nx) \\ &= x^{m-1}(1-x)^{n-1}(m - (m+n)x). \end{aligned}$$

We see that $f'(x) = 0$ with $x \in (0, 1)$ if and only if $x = \frac{m}{m+n}$. So f increases to a maximum at $x = \frac{m}{m+n}$, then decreases back to 0.

4.39:

A factory is making bikvets. The factory costs \$2500 per day to run, and each bikvet costs \$900 to make. Tina, the factory owner, knows that if she prices the bikvets at \$$(2500 - x)$, then she will sell x of them per day. What should she do to maximize her profit?

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Solution: Suppose she sells x bikvets at \$$(2500 - x)$ apiece. Her overall revenue is \$$(2500 - x)x =$ \$$(2500x - x^2)$. Her cost includes the factory and the bikvets themselves, totaling \$$(2500 + 900x)$. Subtracting this from the revenue yields a profit of \$$(-x^2 + 1600x - 2500)$ per day. This has a maximum either at $x = 0$ (the lower bound of x) or where its derivative is equal to zero, since this is a downward opening parabola. But the derivative of the profit is \$$(-2x + 1600)$ per bikvet, which is zero when $x = 800$ bikvets. If she makes 800 bikvets, she gets a total profit of

$$\begin{aligned} &\$(-800^2 + 1600 \cdot 800 - 2500) = \$(-640000 + 1280000 - 2500) \\ &= \$637500, \end{aligned}$$

which is definitely more than the $-\$2500$ she would make at $x = 0$, and thus is the maximum. Therefore, to maximize her profit, Tina should price the bikvets at \$$(2500 - 800) =$ $\boxed{\$1700}$ and make exactly 800 of them per day.

4.40:

We wish to construct a fenced garden with the shape of a circular sector, with area 300 m^2. What radius r and angle θ should we choose to minimize the amount of fencing required (shown in dark grey in the picture at right)? (Note that we could have $\theta < \pi$, contrary to the diagram; the only restriction is $0 < \theta < 2\pi$.)

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Solution: The area of such a sector is given by $\frac{1}{2}r^2\theta = 300$, so $\theta = 600/r^2$. The perimeter consists of two radii and a portion of the circumference, with total length $L = 2r + \theta r = 2r + \frac{600}{r}$. The restriction that $0 < \theta < 2\pi$ corresponds to the restriction $r > \sqrt{\frac{600}{2\pi}}$. Note that L grows arbitrarily large as $r \to \infty$, so we look for critical points on $r \in \left(\sqrt{\frac{300}{\pi}}, \infty\right)$ where $L'(r) = 0$. We have $L'(r) = 2 - \frac{600}{r^2}$, which is zero where $\boxed{r = \sqrt{300} \text{ meters and } \theta = 2 \text{ radians}}$. This is a minimum since the second derivative $\frac{1200}{r^3}$ is positive.

4.41:

Acceleration due to gravity is actually slightly lower at higher altitudes. In particular, $g = \frac{GM}{r^2}$, where G is a constant (called the gravitational constant), M is the mass of the Earth, and r is the distance from the center of the Earth. If the radius of the Earth is 6400 km (approximately), estimate the percentage decrease in gravity on an airplane cruising at an altitude of 10 km above the surface of the Earth.

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Solution: We need to estimate how much the formula GM/r^2 changes when r goes from 6400 km to 6410 km. Since we are comparing with the initial value, we can ignore the constants G and M. We can approximate using a tangent line approximation at $r = 6400$ km. Since the derivative of $\frac{1}{r^2}$ is $-\frac{2}{r^3}$ at $r = 6400 + \Delta r$, we get a difference of

$$\frac{\left(-\frac{2}{(6400)^3}\right)\Delta r}{\frac{1}{(6400)^2}} = -\frac{\Delta r}{3200} = -\frac{\Delta r}{32}\%.$$

Using $\Delta r = 10$, this gives $\boxed{-\frac{10}{32}\% = -0.312\%}$.

4.42:

The Ferris wheel at the County Fair is 50 meters in diameter and rotates at a rate of 4 revolutions per minute. If Natalie is sitting in one of the cars, at what rate is she gaining altitude when she is passing (on the way up) through a height of 40 meters?

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Solution: Her height above the ground is $h = 30 + 30\sin\theta$, where θ is the angle that her car forms with the center of the wheel (where, as usual, 0 means that she is horizontal to the center). Differentiating gives $\frac{dh}{dt} = 30\cos\theta\frac{d\theta}{dt}$. We know that $\frac{d\theta}{dt} = 8\pi$ rad/min. When she is ascending through 40 meters, we have $h = 40$, so $\sin\theta = \frac{1}{3}$ and thus $\cos\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$. This gives

$$\frac{dh}{dt} = 30 \cdot \frac{2\sqrt{2}}{3} \cdot 8\pi = \boxed{160\sqrt{2}\pi \text{ (in meters/minute)}},$$

or perhaps more meaningful would be $\boxed{\frac{8\sqrt{2}}{3}\pi \text{ (in meters/second)}}$.

4.43:

(Use a calculator for this one.) The temperature of a pie after it is removed from the oven is $70 + 230e^{-kt}$ degrees Fahrenheit, where t is the time in minutes and k is some constant. After 10 minutes, the pie has cooled to 200 degrees.

(a) When will the pie be 150 degrees?

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Solution: We use the given data to get $200 = 70 + 230e^{-10k}$, so $e^{-10k} = 13/23$, and hence $k = -(0.1)\log(13/23) \approx 0.057$. To find when the pie is 150 degrees, we must solve $150 = 70 + 230e^{-kt}$ for t, so $e^{-kt} = 8/23$ and

$$t = -\frac{1}{k}\log(8/23) = \boxed{10\frac{\log(8/23)}{\log(13/23)} \approx 18.51 \text{ minutes}}.$$

(b) At the time in part (a), how fast is the pie cooling?

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Solution: To find the rate of cooling, we differentiate to get $\frac{dT}{dt} = -230ke^{-kt}$ and plugging in $k \approx 0.057$ and $t \approx 18.51$, we get

$$\frac{dT}{dt} \approx -230(0.057)e^{-(0.057)(18.51)} \approx -4.56,$$

so the pie is cooling at $\boxed{\text{approximately 4.56 degrees per minute.}}$

4.44:

(a) Suppose that f and g are directly proportional (so that $f(t) = kg(t)$ for some constant k, for all t). What can you say about the relationship between the rates of change of f and g?

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Solution: We can differentiate to get $\frac{df}{dt} = k\frac{dg}{dt}$. So, the rates of change of f and g are also directly proportional, with the same constant of proportionality.

(b) Suppose that f and g are inversely proportional (so that $f(t)g(t) = k$ for some constant k, for all t). What can you say about the relationship between the rates of change of f and g?

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Solution: We can differentiate to get $\frac{df}{dt}g + f\frac{dg}{dt} = 0$, or $\frac{df/dt}{dg/dt} = -\frac{f}{g}$. This implies that the ratio of the rates of change of f and g is equal to the negative of the ratio of f and g.

Back to Chapter 2, we mentioned two fundamental geometric questions that motivate calculus:

Question 1: Given a tangent line or a secant line of a function f, what is the slope of the line that is tangent to the graph $y = f(x)$ at the point $(a, f(a))$?

Question 2: Given an function f and the area of a region R bounded by the graph of f and the x-axis, what is the area of the region R bounded by the graph $y = f(x)$ from $x = a$ to $x = b$?

5.1 Area under a curve

We already have discussed how to find the slope of a tangent line and secant line, and we discuss the area of a geometric figure. The concept of "area under a curve" is not really clear at first.

For example, what is the area of the region R in the picture below?

Problem 1 (Click to open/close)

$f(x) = \frac{1}{x}$; $a = 1$; $b = 2$

Click the area under f in the interval $[1, 2]$.

Click the area under f in the interval $[1, 3]$.

Click the area under f in the interval $[1, 4]$.

Solution for Problem 1: The graph of $y = \frac{1}{x}$ is a hyperbola. Its graph is shown below. So the region in question is just a rectangle with height 1 and width 1. So the area is 1.

Problem 2 (Click to open/close)

$f(x) = \sqrt{x}$; $a = 0$; $b = 1$

Click the area under f in the interval $[0, 1]$.

Click the area under f in the interval $[0, 2]$.

Click the area under f in the interval $[0, 3]$.

Solution for Problem 2: In this case, there is a difficult formula to find area. But if we write $y = \sqrt{x}$, $x^2 = y$ and square both sides, we get $x = y^2$. So the area under $y = \sqrt{x}$ is the same as the area under $x = y^2$. So the area is $\int_0^1 y^2 \, dy = \frac{1}{3}$.

(1) The picture of the region whose area we want to compute is shown at right. If we draw the rectangle from $(0, 0)$ to $(1, 1)$, we see that the area under $y = \sqrt{x}$ is a right triangle with area 1 and a Toppings piece.

(2) A circle of radius 1. The area is $\frac{\pi}{4}$.

We don't want to restrict ourselves just to positive-valued functions. We want to define area "under" the curve of functions that take on negative values. We want to define area "under" the curve of $f(x) = \sin x$ from $x = 0$ to $x = 2\pi$ to be zero. Intuitively, we can think the portion of the curve that is "underwater" as the area "under" the curve.

Definition: Let $f(x)$ be a closed interval and suppose that $f \geq 0$ continuous on $[a, b]$. Divide the interval $[a, b]$ into n equal pieces $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. Let $x_0 = a$ and $x_n = b$. Let x_i^* be the point in the i-th piece. Let the rectangle R_i be the rectangle with height $f(x_i^*)$ and width $\frac{b-a}{n}$. Then the total area under the curve is $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{b-a}{n} f(x_i^*)$.

Problem 3 (Click to open/close)

$f(x) = \sin x$; $a = 0$; $b = \pi$

Click the area under f in the interval $[0, \pi]$.

Click the area under f in the interval $[0, 2\pi]$.

Click the area under f in the interval $[0, 3\pi]$.

Solution for Problem 3: When we draw the graph of $y = \sin x$, we see that the region under the graph consists of two parts: the region above the x-axis and the region below the x-axis. The area of the first part is 2. The area of the second part is 2. So the area is the sum of the two parts, which is 4. So the area under the curve $y = \sin x$ from $x = 0$ to $x = 2\pi$ is 0. The area under the curve $y = \sin x$ from $x = 0$ to $x = 3\pi$ is 2.

Problem 4 (Click to open/close)

$f(x) = x$; $a = 0$; $b = 1$

Click the area under f in the interval $[0, 1]$.

Click the area under f in the interval $[0, 2]$.

Click the area under f in the interval $[0, 3]$.

Solution for Problem 4: In this case, the area under the curve $y = x$ from $x = 0$ to $x = 1$ is $\frac{1}{2}$. The area under the curve $y = x$ from $x = 0$ to $x = 2$ is 2. The area under the curve $y = x$ from $x = 0$ to $x = 3$ is $\frac{9}{2}$.

This is a rather cumbersome definition, but the picture to the right shows exactly what's going on. We're just listing the graph of $y = f(x)$ in the rectangle $[a, b] \times [0, \infty)$ and considering vertical slices from the very bottom (closest to the x-axis). We then take the area of the region between the slices and the x-axis.

If f is sufficiently nice, then the area of the region is $\int_a^b f(x) \, dx$.

Problem 5 (Click to open/close)

$f(x) = \frac{1}{x}$; $a = 1$; $b = 2$

Click the area under f in the interval $[1, 2]$.

Click the area under f in the interval $[1, 3]$.

Click the area under f in the interval $[1, 4]$.

Solution for Problem 5: Let $f(x) = \frac{1}{x}$. Let $n = 4$. Divide the interval $[1, 2]$ into 4 equal pieces: $[1, \frac{5}{4}], [\frac{5}{4}, \frac{6}{4}], [\frac{6}{4}, \frac{7}{4}], [\frac{7}{4}, 2]$. Let $x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{6}{4}$, $x_3 = \frac{7}{4}$, $x_4 = 2$. Let x_i^* be the point in the i-th piece. Let the rectangle R_i be the rectangle with height $f(x_i^*)$ and width $\frac{b-a}{n}$. Then the total area under the curve is $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{b-a}{n} f(x_i^*)$.

Problem 6 (Click to open/close)

$f(x) = \sin x$; $a = 0$; $b = \pi$

Click the area under f in the interval $[0, \pi]$.

Click the area under f in the interval $[0, 2\pi]$.

Click the area under f in the interval $[0, 3\pi]$.

Solution for Problem 6: When we draw the graph of $y = \sin x$, we see that the region under the graph consists of two parts: the region above the x-axis and the region below the x-axis. The area of the first part is 2. The area of the second part is 2. So the area is the sum of the two parts, which is 4. So the area under the curve $y = \sin x$ from $x = 0$ to $x = 2\pi$ is 0. The area under the curve $y = \sin x$ from $x = 0$ to $x = 3\pi$ is 2.

Problem 7 (Click to open/close)

$f(x) = x$; $a = 0$; $b = 1$

Click the area under f in the interval $[0, 1]$.

Click the area under f in the interval $[0, 2]$.

Click the area under f in the interval $[0, 3]$.

Solution for Problem 7: The region in question is shown in the diagram at right. Unfortunately, we don't know geometrically how to compute the area of a trapezoid bounded by a parabola.

Definition: Actually, the current Greek mathematician Archimedes did know how to compute the area of a trapezoid bounded by a parabola. Archimedes's method was to approximate the area of a parabolic segment by a series of triangles.

Moreover, we know how to compute the area of simple shapes, and the simplest shape to compute the area of is a rectangle. We will approximate the area under a curve by using rectangles. We will approximate the area of a parabolic segment by a series of rectangles.

In a very crude estimate, we can draw a rectangle with opposite corners at $(0, 1)$ and $(1, 0)$. The area of this rectangle is 1. We can improve the estimate by using two rectangles. We can draw a rectangle with opposite corners at $(0, 1)$ and $(\frac{1}{2}, 0)$ and another rectangle with opposite corners at $(\frac{1}{2}, 1)$ and $(1, 0)$. The area of the first rectangle is $\frac{1}{2}$ and the area of the second rectangle is $\frac{1}{2}$. So the area is $\frac{1}{2} + \frac{1}{2} = 1$. We can improve the estimate by using three rectangles. We can draw a rectangle with opposite corners at $(0, 1)$ and $(\frac{1}{3}, 0)$, another rectangle with opposite corners at $(\frac{1}{3}, \frac{8}{9})$ and $(\frac{2}{3}, 0)$, and another rectangle with opposite corners at $(\frac{2}{3}, \frac{4}{9})$ and $(1, 0)$. The area of the first rectangle is $\frac{1}{3}$, the area of the second rectangle is $\frac{8}{9}$, and the area of the third rectangle is $\frac{4}{9}$. So the area is $\frac{1}{3} + \frac{8}{9} + \frac{4}{9} = \frac{11}{9}$. We can improve the estimate by using four rectangles. We can draw a rectangle with opposite corners at $(0, 1)$ and $(\frac{1}{4}, 0)$, another rectangle with opposite corners at $(\frac{1}{4}, \frac{15}{16})$ and $(\frac{1}{2}, 0)$, another rectangle with opposite corners at $(\frac{1}{2}, \frac{9}{16})$ and $(\frac{3}{4}, 0)$, and another rectangle with opposite corners at $(\frac{3}{4}, \frac{7}{16})$ and $(1, 0)$. The area of the first rectangle is $\frac{1}{4}$, the area of the second rectangle is $\frac{15}{16}$, the area of the third rectangle is $\frac{9}{16}$, and the area of the fourth rectangle is $\frac{7}{16}$. So the area is $\frac{1}{4} + \frac{15}{16} + \frac{9}{16} + \frac{7}{16} = \frac{7}{2}$.

(1) We can do the same thing with an arbitrary number of rectangles. For example, to the right we show an arbitrary number of rectangles. We see that the area of the rectangles is a lot closer to the area of the parabolic segment. We see that the area of the rectangles is closer to the area of the parabolic segment the more rectangles we use. We see that the area of the rectangles is closer to the area of the parabolic segment the more rectangles we use.

In the limit, the area is the very positive integer and with dates in rectangles, each with width $\frac{1}{n}$. There is each rectangle will have vertical slice at $x = \frac{i}{n}$ for $i = 1, 2, \ldots, n$. The height of the ith rectangle is $\left(\frac{i}{n}\right)^2 = \frac{i^2}{n^2}$. So the area of the rectangles is $\sum_{i=1}^{n} \frac{1}{n} \cdot \frac{i^2}{n^2} = \sum_{i=1}^{n} \frac{i^2}{n^3}$.

(2) To get a somewhat better estimate, we can draw two rectangles, each of width 0.5 and with the opposite corners at $(0, 1)$ and $(0.5, 0)$ and $(0.5, 0.25)$ and $(1, 0)$. The area of the first rectangle is $0.5 \cdot 1 = 0.5$ and the area of the second rectangle is $0.5 \cdot 0.25 = 0.125$. So the area is $0.5 + 0.125 = 0.625$. We can do the same thing with three rectangles, each of width $\frac{1}{3}$ and with the opposite corners at $(0, 1)$ and $(\frac{1}{3}, 0)$ and $(\frac{1}{3}, \frac{1}{9})$ and $(\frac{2}{3}, 0)$ and $(\frac{2}{3}, \frac{4}{9})$ and $(1, 0)$. The area of the first rectangle is $\frac{1}{3} \cdot 1 = \frac{1}{3}$, the area of the second rectangle is $\frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}$, and the area of the third rectangle is $\frac{1}{3} \cdot \frac{4}{9} = \frac{4}{27}$. So the area is $\frac{1}{3} + \frac{1}{27} + \frac{4}{27} = \frac{14}{27}$.

(3) We can do the same thing with an arbitrary number of rectangles. For example, to the right we show an arbitrary number of rectangles. We see that the area of the rectangles is a lot closer to the area of the parabolic segment. We see that the area of the rectangles is closer to the area of the parabolic segment the more rectangles we use. We see that the area of the rectangles is closer to the area of the parabolic segment the more rectangles we use.

In the limit, the area is the very positive integer and with dates in rectangles, each with width $\frac{1}{n}$. There is each rectangle will have vertical slice at $x = \frac{i}{n}$ for $i = 1, 2, \ldots, n$. The height of the ith rectangle is $\left(\frac{i}{n}\right)^2 = \frac{i^2}{n^2}$. So the area of the rectangles is $\sum_{i=1}^{n} \frac{1}{n} \cdot \frac{i^2}{n^2} = \sum_{i=1}^{n} \frac{i^2}{n^3}$.

(4) The expression that we got in part (1) for the area of rectangles

$$\sum_{i=1}^{n} \frac{i^2}{n^3}$$

is a lower bound for the area under $f(x) = x^2$. So, $\frac{14}{27}$ is a lower bound for the area under $f(x) = x^2$. Similarly, we can find a lower bound for the area under $f(x) = x^2$ using the rectangles in (2).

We can also find an upper bound for the area under $f(x) = x^2$ using the rectangles in part (1). There, we guess that the area under $f(x) = x^2$ is $\frac{1}{3}$.

Now, we can use the formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to give a closed form expression in terms of n:

$$\sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$= \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

This is the lower bound for the area under $f(x) = x^2$. We can see that as n gets very large, the lower bound approaches $\frac{1}{3}$.

(5) The expression that we got in part (2) for the area of rectangles

$$\sum_{i=1}^{n} \frac{i^2}{n^3}$$

is a lower bound for the area under $f(x) = x^2$. So, $\frac{0.625}{0.5}$ is a lower bound for the area under $f(x) = x^2$. Similarly, we can find a lower bound for the area under $f(x) = x^2$ using the rectangles in (3).

We can also find an upper bound for the area under $f(x) = x^2$ using the rectangles in part (2). There, we guess that the area under $f(x) = x^2$ is $\frac{1}{3}$.

Now, we can use the formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to give a closed form expression in terms of n:

$$\sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$= \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

This is the upper bound for the area under $f(x) = x^2$. We can see that as n gets very large, the upper bound approaches $\frac{1}{3}$.

(6) As the rectangles get smaller and smaller, the area under consideration is the area under the curve. So, $\frac{14}{27}$ is a lower bound for the area under $f(x) = x^2$. Similarly, we can find a lower bound for the area under $f(x) = x^2$ using the rectangles in (3).

Similarly, we can find an upper bound for the area under $f(x) = x^2$ using the rectangles in (4). So, $\frac{4}{9}$ is an upper bound for the area under $f(x) = x^2$.

(7) Let $f(x) = x^2$. Let $n = 100$. Let R_i be the area and an upper rectangle $[i, i]$ of the area under $f(x) = x^2$. We can say that we expect that

$$f(1)R_1 \leq \text{Area} \leq f(2)R_1$$

for all positive R_i and R_i and R_i and

5.2 The Fundamental Theorem of Calculus

It would be quite a hassle if every time we wished to compute a definite integral we had to set up Darboux sums or Riemann sums and compute upper and lower bounds. It would be a relatively easy way to compute definite integrals. There is a fundamental theorem of calculus in fact, we call it the **Fundamental Theorem of Calculus**: that relates definite integrals to derivatives in an amazing way. The way we approach this is via the Mean Value Theorem.

Let f be a continuous function on $[a, b]$ and let F be a function whose derivative is f, so that $F' = f$. Such a function is called an **antiderivative** of f. We're interested in computing $\int_a^b f$, so we let $\mathcal{P}$ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For each interval $[x_{i-1}, x_i]$ in the partition, the Mean Value Theorem, applied to the function F, tells us that there is some value $t_i \in [x_{i-1}, x_i]$ such that

$$F(t_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = \frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i}.$$

When we multiply both sides by the denominator, we get:

$$F(x_i)(x_{i+1} - x_i) = F(x_{i+1}) - F(x_i).$$

When we sum this over all the intervals, we get

$$\sum_{i=1}^{n} F(x_i)(x_{i+1} - x_i) = \sum_{i=1}^{n} (F(x_{i+1}) - F(x_i)). \tag{5.5}$$

Let's look at the two sides of equation (5.1) separately:

On the left side of (5.1), we have a Riemann sum. In particular, going back to the notation in the previous section, we note that $h_i \leq f(t_i) \leq H_i$ for all $0 \leq i < n$, and thus the left side of (5.1) lies between $U_f(\mathcal{P})$ and $L_f(\mathcal{P})$:

$$\begin{aligned} U_f(\mathcal{P}) &= \sum_{i=1}^{n} [h_i(x_{i+1} - x_i)] \\ &\leq \sum_{i=1}^{n} f(t_i)(x_{i+1} - x_i) \\ &\leq \sum_{i=1}^{n} [H_i(x_{i+1} - x_i)] \\ &= L_f(\mathcal{P}). \end{aligned} \tag{5.2}$$

On the right side of (5.1), we have a **telescoping sum**, which means that most of the terms cancel out:

$$\begin{aligned} &\sum_{i=1}^{n} (F(x_{i+1}) - F(x_i)) \\ &= (F(x_2) - F(x_1)) + (F(x_3) - F(x_2)) \\ &\quad + (F(x_4) - F(x_3)) + \cdots + (F(x_{n-1}) - F(x_{n-2})) \\ &\quad + (F(x_n) - F(x_{n-1})) \\ &= F(x_n) - F(x_1). \end{aligned}$$

Thus, the right side of (5.1) is just $F(b) - F(a)$. Substituting this expression into the second line of (5.2) gives

$$U_f(\mathcal{P}) \leq F(b) - F(a) \leq L_f(\mathcal{P})$$

for all partitions $\mathcal{P}$ of $[a, b]$. Thus, $F(b) - F(a)$ is an upper bound for the set of all $U_f(\mathcal{P})$ and is a lower bound for the set of all $L_f(\mathcal{P})$, and hence

$$L_f^*(f) \leq F(b) - F(a) \leq U_f^*(f).$$

But if the definite integral $\int_a^b f$ is defined, then $L_f^*(f) = \int_a^b f = U_f^*(f)$. Thus, there is exactly one value that can be placed in the middle of the above inequality for all $\mathcal{P}$, and that value is definitely $\int_a^b f$. Therefore, we conclude that

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

provided that the definite integral is defined. As we mentioned in the last section, if f is continuous on $[a, b]$ then the definite integral $\int_a^b f$ is defined, so we have:

> **Important:** The **Fundamental Theorem of Calculus**, Part I: If f is a continuous function on $[a, b]$ with antiderivative F, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a).$$

Note that we have assumed that f has an antiderivative F. It turns out that every continuous function has an antiderivative; this is Part II of the Fundamental Theorem of Calculus, which we will see in a little bit.

Let's go back to our example from Problem 5.8, and we'll see how much the Fundamental Theorem of Calculus make our lives easier:

Problem 5.9

Compute $\int_0^1 x^2\,dx$.

Solution for Problem 5.9: In order to apply the Fundamental Theorem of Calculus, we need to determine an antiderivative of $f(x) = x^2$. Happily, that's easy: we let $F(x) = \frac{1}{3}x^3$, so that $F'(x) = x^2 = f(x)$. The Fundamental Theorem then tells us immediately that

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3} - 0 = \frac{1}{3}.$$

□

More commonly, the computation from Problem 5.9 is written like this:

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

The vertical bar means to take the function $\left(\frac{x^3}{3}\right)$ evaluated at the first number ($x = 1$) minus the function evaluated at the bottom number ($x = 0$). If there is any question about what variable we are using to evaluate the function, we can list the variable explicitly, such as

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_{x=0}^{x=1}.$$

However, we will usually omit the variable if there is no ambiguity.

Let's look at another basic example of computing a definite integral.

Problem 5.10

Compute the area under a positive half period of the sine function, that is, compute

$$\int_0^\pi \sin x\,dx.$$

Solution for Problem 5.10: An antiderivative of $\sin x$ is $-\cos x$, because

$$\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x.$$

Thus, via the Fundamental Theorem, we have:

$$\int_0^\pi \sin x\,dx = (-\cos x)\Big|_{x=0}^{x=\pi}.$$

Therefore, the integral is equal to

$$(-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 2.$$

□

There is another version of the Fundamental Theorem of Calculus that is also very useful. Let's see how this works with an example.

Problem 5.11

Define a function y by

$$y(x) = \int_x^\pi \cos t\,dt$$

for all $x \in \mathbb{R}$. Prove that $y(x) = \sin x$.

We notice that there are two variables in our equation. In particular, the variable x is one of our limits of integration, so we need a different variable (in this example, t) to represent the variable in the function that we are integrating.

Solution for Problem 5.11: We proceed as usual: since $\sin x$ is an antiderivative of $\cos$, we can simply evaluate the integral as

$$y(x) = \int_x^\pi \cos t\,dt = \sin t\Big|_{t=x}^{t=\pi} = \sin x - \sin \pi = \sin x.$$

Evaluation of the integral eliminated the dummy variable t, and what remains is a function of x. □

More generally, we start with a continuous function f and use the definite integral to define a new function, by selecting a point x_0 in the domain of f and defining

$$y(x) = \int_{x_0}^x f(t)\,dt.$$

Note that we use a new dummy variable t in our integral. If F is an antiderivative of f, then the Fundamental Theorem gives us:

$$y(x) = \int_{x_0}^x f(t)\,dt = F(x) - F(x_0).$$

We now differentiate this with respect to x. Note that since $F(x_0)$ is a constant, we have $\frac{d}{dx}F(x_0) = 0$, and thus we get:

$$y'(x) = \frac{d}{dx}F(x) - \frac{d}{dx}F(x_0) = F(x).$$

Putting this all together gives us:

> **Important:** The **Fundamental Theorem of Calculus**, Part II: Let f be a continuous function and x_0 a number in the domain of f. Define a function y by
>
> $$y(x) = \int_{x_0}^x f(t)\,dt$$
>
> for all x such that $[x_0, x]$ (for $x \geq x_0$) or $[x, x_0]$ (for $x \leq x_0$) is a subset of the domain of f. Then $y'(x) = f(x)$ for all x where both functions are defined.

This is sometimes also written as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Unfortunately, our argument above presupposes the existence of an antiderivative F of f. We want the Fundamental Theorem of Calculus to be stronger: we want it to imply the existence of an antiderivative.

Problem 5.12

Let f be a continuous function and $x_0 \in \mathrm{Dom}(f)$. We wish to prove Part II of the Fundamental Theorem of Calculus without presupposing the existence of an antiderivative of f. Define

$$y(x) = \int_{x_0}^x f(t)\,dt$$

wherever the right side is defined. We will prove that $y'(x) = f(x)$ as follows:

(a) Show that $y(x + h) - y(x) = \int_x^{x+h} f(t)\,dt$ for any h.

(b) Show that

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = f(x).$$

Solution for Problem 5.12:

> **WARNING!** The proof can be confusing, because of the different variables that are used. We are using the variable x as the independent variable of the function y, and the variable t as the dummy variable in the definite integrals.

(a) We know that

$$\int_{x_0}^{x+h} f(t)\,dt = \int_{x_0}^{x} f(t)\,dt + \int_{x}^{x+h} f(t)\,dt,$$

so rearranging gives

$$\int_{x}^{x+h} f(t)\,dt = \int_{x_0}^{x+h} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt = y(x+h) - y(x).$$

(b) By the definition of limit, we must show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |h| < \delta \quad \Rightarrow \quad \left|\frac{y(x+h) - y(x)}{h} - f(x)\right| < \varepsilon.$$

Since f is continuous, we may choose δ such that

$$\begin{aligned} &|t - x| < \delta \\ \Rightarrow\ &|f(t) - f(x)| < \varepsilon \\ \Rightarrow\ &f(x) - \varepsilon < f(t) < f(x) + \varepsilon. \end{aligned}$$

Thus, if $|h| < \delta$, we have

$$\int_x^{x+h} (f(x) - \varepsilon)\,dt < \int_x^{x+h} f(t)\,dt < \int_x^{x+h} (f(x) + \varepsilon)\,dt.$$

But the outer integrals in the above expression are definite integrals of constant functions (that is, they don't depend on t), so they can be explicitly evaluated, and we get

$$h(f(x) - \varepsilon) < \int_x^{x+h} f(t)\,dt < h(f(x) + \varepsilon).$$

We now apply part (a) and divide by h, giving

$$f(x) - \varepsilon < \frac{y(x+h) - y(x)}{h} < f(x) + \varepsilon.$$

This means that

$$0 < |h| < \delta \quad \Rightarrow \quad \left|\frac{y(x+h) - y(x)}{h} - f(x)\right| < \varepsilon,$$

hence, by definition,

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = f(x),$$

as desired.

□

A nice feature of Part II of the Fundamental Theorem is that we no longer have to presuppose that a continuous function f has an antiderivative F. The Fundamental Theorem not only tells us that F must exist, it even tells us how to construct it:

$$F(x) = \int_a^x f(t)\,dt.$$

> **Concept:** A continuous function f has an antiderivative. In fact, it has infinitely many antiderivatives, since if F is an antiderivative, then so is $F + \epsilon$ where $\epsilon \in \mathbb{R}$ is any constant.

Another use of this version of the Fundamental Theorem of Calculus is to rigorously define functions that would otherwise be hard to describe. For example, back in Chapter 1, we gave a vague definition of the exponential and natural logarithm functions. This definition forced us to assume many of the nice properties these functions actually satisfy. However, using a definite integral, we can rigorously define the natural logarithm to be the function whose derivative is $\frac{1}{x}$ by defining for all $x > 0$:

$$\log x = \int_1^x \frac{1}{t}\,dt.$$

Using this definition, we can now prove all of the features of the natural logarithm function and its inverse, the exponential function. We present the details in Section 5.4.

Also, we often use definite integrals to define functions that couldn't really be defined any other way. For example, the following function is quite important in statistics:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

This is called the **error function**. Note that, by definition, we have

$$\frac{d}{dx}\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}.$$

The constant $\frac{2}{\sqrt{\pi}}$ is there to make certain calculations in statistics come out nicely.

Exercises

5.2.1:

Compute the following definite integrals:

(a) $\int_0^1 x^3\,dx$

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Your Submission: Solution

Solution: An antiderivative of x^3 is $\frac{1}{4}x^4$. Thus,

$$\int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_{x=0}^{x=1} = \frac{1}{4}\left(2^4 - (-1)^4\right) = \boxed{\frac{15}{4}}.$$

(b) $\int_0^\pi e^x\,dx$

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Your Submission: Solution

Solution: e^x is its own antiderivative, so

$$\int_0^\pi e^x\,dx = e^x\Big|_{x=0}^{x=\pi} = e^\pi - e^0 = \boxed{e^\pi - 1}.$$

(c) $\int_{\pi/4}^{\pi/3} \sec^2 \theta\,d\theta$

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Your Submission: Solution

Solution: An antiderivative of $\sec^2 \theta$ is $\tan \theta$, so

$$\int_{\pi/4}^{\pi/3} \sec^2 \theta\,d\theta = \tan \theta\Big|_{\theta=\pi/4}^{\theta=\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4} = \boxed{\sqrt{3} - \frac{\sqrt{3}}{3}}.$$

(d) $\int_{-1}^{1} 2x\,dx$

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Your Submission: Solution

Solution: An antiderivative of $2x$ is x^2, hence

$$\int_{-1}^{1} 2x\,dx = x^2\Big|_{x=-1}^{x=1} = (-2)^2 - (-4)^2 = \boxed{-12}.$$

(One could instead note that this integral equals the negative area of a trapezoid with bases 8 and 4 and height 2, so it is $-\frac{1}{2}(8 + 4) = -12$.)

5.2.2:

Suppose f is continuous (with antiderivative F) and $a < c < b$. Explain why

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

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Your Submission: Solution

Solution: We use the Fundamental Theorem of Calculus:

$$\begin{aligned} \int_a^b f &= F(b) - F(a) \\ &= (F(c) - F(a)) + (F(b) - F(c)) \\ &= \int_a^c f + \int_c^b f. \end{aligned}$$

□

5.2.3:

Recall that a function f is called **odd** *if $f(-x) = -f(x)$ for all x.* Show that, if f is continuous and odd, and $[-a, a]$ is in the domain of f then $\int_{-a}^{a} f(x)\,dx = 0$.

Hint: Show that if f is odd, then an antiderivative of f is even.

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Your Submission: Solution

Solution: Let F be an antiderivative of f such that $F(0) = 0$. (For instance, we could define $F(x) = \int_0^x f(t)\,dt$ using the Fundamental Theorem of Calculus.) Then

$$\frac{d}{dx}F(-x) = \frac{dF}{dx}(-x)\frac{d}{dx}(-x) = f(-x)(-1) = -f(-x) = f(x).$$

That is, the function $G(x) = F(-x)$ is also an antiderivative of f. But $F(0) = G(0) = 0$, so since any two antiderivatives must differ by a constant, we have $F(x) = G(x) = F(-x)$ that is, F is an even function. To finish, we compute

$$\int_{-a}^{a} f(x)\,dx = F(a) - F(-a) = F(a) - F(a) = 0.$$

□

5.2.4:

Suppose f is differentiable and g is continuous. Show that

$$\frac{d}{dx}\int_a^{g(x)} g(t)\,dt = g(f(x))f'(x).$$

Hint: Let $G(x) = \int_a^x g(t)\,dt$, and compute $\frac{d}{dx}G(f(x))$ using the Chain Rule.

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Your Submission: Solution

Solution: Let $G(x) = \int_a^x g(t)\,dt$, so that by the Fundamental Theorem of Calculus, $G'(x) = g(x)$. Then by definition,

$$\int_a^{g(x)} g(t)\,dt = G(f(x)),$$

and taking the derivative of both sides and applying the Chain Rule gives

$$\frac{d}{dx}\int_a^{g(x)} g(t)\,dt = \frac{d}{dx}G(f(x)) = G'(f(x))f'(x) = g(f(x))f'(x),$$

as desired.

□

5.3 Integration methods

In order to apply the Fundamental Theorem of Calculus to compute definite integrals of the form $\int_a^b f\,dx$, we need to be able to find an antiderivative of f; that is, we need a function F such that $F' = f$. This process is called **antidifferentiation** (for obvious reasons), and while finding derivatives is pretty easy, finding antiderivatives can be much more difficult, as we will explore in this section.

5.3.1 Definition and basic examples

Recall the definition:

> **Definition:** Let f be a function. A function g is called an **antiderivative** of f if g is differentiable and $g' = f$.

As we saw when discussing the Fundamental Theorem of Calculus, the usual convention is that if a function is denoted by a lowercase letter, then its antiderivative is denoted by the same letter, but in uppercase. For example, F is an antiderivative of f, meaning that $F' = f$.

Notice that we said *an* antiderivative rather than *the* antiderivative. Did we need to be so careful? In other words, does a function have a unique antiderivative?

Problem 5.13

Suppose that F_1 and F_2 are antiderivatives of f. How are F_1 and F_2 related? Must they be equal?

Solution for Problem 5.13: It's not so hard to come up with an example of two different functions that have the same derivative. For example, if $f = 0$ and $g = 1$ (the constant functions 0 and 1, respectively), then $f' = g' = 0$. Furthermore, all constant functions have derivative 0, so any constant function is an antiderivative of the function $f = 0$. Perhaps this generalizes somehow.

Suppose F_1 and F_2 are both antiderivatives of f; that is, $F_1' = F_2' = f$. Then

$$(F_1 - F_2)' = F_1' - F_2' = f - f = 0.$$

So $F_1 - F_2$ is a function whose derivative is 0 everywhere, and we know that the only such functions are constant functions. Thus $F_1 - F_2 = c$ for some constant c. □

> **Important:** If F_1 and F_2 are antiderivatives of the same function f, then $F_1 - F_2 = c$ for some constant $c \in \mathbb{R}$.

So there is no "unique" antiderivative of a function, but the antiderivative is determined up to a constant term.

Antiderivatives are also called **indefinite integrals**, and have a familiar notation: if F is an antiderivative of f, then we write

$$F = \int f = \int f(x)\,dx.$$

Note that this is the same notation as we use for definite integrals, except that an indefinite integral does not have limits of integration. As with definite integrals, if we write a dummy variable (like x) in the function expression of the indefinite integral, then we write dx as a "term" of the integral.

The use of an equal sign in the above "equation" is a little strange, as we know that $\int f$ is determined only up to a constant term. Instead, we write

$$\int f = F + C,$$

where C denotes the arbitrary constant term. (Traditionally, it's almost always written with a capital "C.")

This notation also gives us a convenient shorthand that relates our two types of integrals via the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$$

Pay close attention to the difference between the two types of integrals. The integral on the left, $\int_a^b f(x)\,dx$, is a **definite** integral, and is equal to a *number*, specifically the area under the curve $y = f(x)$ on the interval $[a, b]$. The integral on the right, $\int f(x)\,dx$, is an **indefinite** integral, and is equal to a *function*, specifically the antiderivative of f. (Actually, since it's only determined up to a constant, it's a whole set of functions.) When we then take the difference of this antiderivative evaluated at $x = b$ and $x = a$, we get the definite integral on the left side of the above equation.

Since differentiation is linear–that is, $(f + g)' = f' + g'$ and $(cf)' = cf'$–we expect that antidifferentiation is too.

Problem 5.14

(a) Show that if F and G are antiderivatives of f and g, respectively, then $F + G$ is an antiderivative of $f + g$.

(b) Show that if F is an antiderivative of f and $c \in \mathbb{R}$ is a constant, then cF is an antiderivative of cf.

Solution for Problem 5.14: For both parts, we simply use the corresponding linearity properties of the derivative:

(a) $(F + G)' = F' + G' = f + g$.

(b) $(cF)' = c(F') = cf$.

□

We can express the linearity of antidifferentiation in integral notation:

> **Important:** If f and g are continuous functions and $c \in \mathbb{R}$, then
>
> $$\int (f + g) = \int f + \int g \quad \text{and} \quad \int cf = c\int f.$$

> **WARNING!!** Statements "equating" indefinite integrals are only true up to a constant.

By the Fundamental Theorem, these linearity properties extend to definite integrals too:

> **Important:** If f and g are continuous functions with $[a, b] \subseteq (\mathrm{Dom}(f) \cap \mathrm{Dom}(g))$, and $c \in \mathbb{R}$, then
>
> $$\int_a^b (f + g) = \int_a^b f + \int_a^b g \quad \text{and} \quad \int_a^b cf = c\int_a^b f.$$

5.3.2 Antiderivatives of common functions

We have a nice catalog of functions–polynomials, trig functions, exponentials, etc.–that we know how to differentiate. We'll try to develop a similar catalog of functions that we can antidifferentiate.

We'll start with the class of functions that we typically find easiest to deal with: polynomials.

Problem 5.15

(a) If n is any positive integer, find $\int x^n\,dx$.

(b) Does your formula from part (a) work for $n = 0$? Why or why not?

(c) Does your formula from part (a) work for negative integers? Why or why not?

Solution for Problem 5.15:

(a) We know that when we take the derivative of a monomial, the exponent decreases by 1:

$$\frac{d}{dx}x^m = mx^{m-1}.$$

Therefore, when we take the antiderivative, which is the opposite of taking the derivative, we expect the exponent to increase by 1. We start with

$$\frac{d}{dx}x^{n+1} = (n+1)x^n.$$

But we want something whose derivative is x^n, not $(n+1)x^n$, so we divide by $n+1$:

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n.$$

Therefore,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Don't forget the "$+C$"! Remember that antiderivatives are only determined up to a constant term.

(b) Our derivative formula $(x^m)' = mx^{m-1}$ works for any integer m, whether positive, zero, or negative. Thus, everything that we did is valid for $n = 0$ too. Hence,

$$\int x^0\,dx = \int 1\,dx = x + C.$$

(c) Why can't we do the same thing?

> **Bogus Solution:** Everything that we did in part (a) is valid for any integer, not just positive integers. Therefore,
>
> $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$
>
> for any integer n.

It's correct that the statement

$$\frac{d}{dx}x^{n+1} = (n+1)x^n$$

is true for any integer n. However, it's the last little step from our argument in part (a) that might break: dividing by $n+1$ is only valid if $n \neq -1$, since we can't divide by 0.

> **Concept:** Inadvertently dividing by 0 is one of the most common mistakes in algebra and calculus. Always be on the lookout for it.

Thus, the formula holds for every integer except -1.

□

> **Important:** If n is any integer and $n \neq -1$, then
>
> $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

> **WARNING!!** It's easy for calculus beginners to make computational mistakes while taking the derivatives and antiderivatives of polynomials. To take the derivative of x^n, we *decrease* the exponent by 1, then *multiply* by the *original* exponent:
>
> $$\frac{d}{dx}x^n = nx^{n-1}.$$
>
> To take the antiderivative of x^n, we *increase* the exponent by 1, then *divide* by the *new* exponent:
>
> $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$
>
> Make sure that you see how these two operations are the reverse of each other. Eventually, these operations will be as automatic as addition and subtraction.

This naturally leads us to wonder: what is the antiderivative of $f(x) = x^{-1} = \frac{1}{x}$? Fortunately, we know a function whose derivative is $\frac{1}{x}$. We saw this in Chapter 3:

$$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

But be careful!

> **Bogus Solution:** Therefore we conclude that
>
> $$\int \frac{1}{x}\,dx = \log x + C.$$

The problem is that $\frac{1}{x}$ is defined for all nonzero x, whereas $\log x$ is defined only for positive x. But we can fix this! Notice that

$$\frac{d}{dx}(\log(-x)) = -\frac{1}{-x} = \frac{1}{x}.$$

So for $x < 0$, the function $\log(-x)$ is an antiderivative of $\frac{1}{x}$. We can combine the $x > 0$ and $x < 0$ cases by using absolute value:

$$\int \frac{1}{x}\,dx = \log|x| + C.$$

This now makes sense for all nonzero x. Note that we sometimes "put the dx in the numerator" and write this as $\int \frac{dx}{x} = \log|x| + C$.

> **WARNING!!** Don't forget the absolute value when taking the antiderivative of $\frac{1}{x}$.

Combining our antiderivatives of monomials with our linearity rules from Problem 5.14, we can find the antiderivative of any polynomial.

Problem 5.16

Find the following antiderivatives:

(a) $\int (3x^2 - 7x + 2)\,dx$

(b) $\int \left(\frac{4}{x^{11}} - \frac{2}{x^5}\right)dx$

(c) $\int (5x+2)^3\,dx$

Solution for Problem 5.16:

(a) We use our formula from Problem 5.15, and the fact that antiderivatives are linear:

$$\begin{aligned}\int (3x^2 - 7x + 2)\,dx &= 3\int x^2\,dx - 7\int x\,dx + 2\int 1\,dx \\ &= 3\left(\frac{x^3}{3}\right) - 7\left(\frac{x^2}{2}\right) + 2x + C \\ &= x^3 - \frac{7}{2}x^2 + 2x + C.\end{aligned}$$

Once you've gotten the hang of this, there's no need to write all the intermediate steps. Just do the whole thing in one step:

$$\int (3x^2 - 7x + 2)\,dx = x^3 - \frac{7}{2}x^2 + 2x + C.$$

One thing that might worry you is the constant "$+C$" at the end. Specifically, since we broke up our integral into 3 smaller integrals, why don't we have 3 constants? In other words, shouldn't the answer be the following:

$$\begin{aligned}&\int (3x^2 - 7x + 2)\,dx \\ &= 3\int x^2\,dx - 7\int x\,dx + 2\int 1\,dx \\ &= \left[3\left(\frac{x^3}{3}\right) + C_1\right] - \left[7\left(\frac{x^2}{2}\right) + C_2\right] + [2x + C_3] \\ &= x^3 - \frac{7}{2}x^2 + 2x + (C_1 + C_2 + C_3)?\end{aligned}$$

One way to answer this is that we just say $C = C_1 + C_2 + C_3$ so we don't have to worry about it. A more thoughtful answer is that any antiderivative of $3x^2 - 7x + 2$ is going to differ from $x^3 - \frac{7}{2}x^2 + 2x$ by a constant, and we label that single constant C. All of this is a result of the rather sloppy notation of writing that $\int f(x)\,dx = F(x) + C$, when actually $\int f(x)\,dx$ signifies any function that is an antiderivative of $f(x)$, not one particular function. But the "sloppy" notation is nearly universal, so we'll stick with it.

(b) We can do the same sort of calculation with negative exponents, but extra caution is required as it is a lot easier to make computational mistakes when dealing with negative exponents:

$$\begin{aligned}\int \left(\frac{4}{x^{11}} - \frac{2}{x^5}\right)dx &= 4\int \frac{dx}{x^{11}} - 2\int \frac{dx}{x^5} \\ &= -4\left(\frac{1}{10x^{10}}\right) + 2\left(\frac{1}{4x^4}\right) + C \\ &= -\frac{2}{5x^{10}} + \frac{1}{2x^4} + C.\end{aligned}$$

Again, while you're still getting accustomed to these calculations, be extra careful with the signs and the constants. You might find it easier to rewrite the original integral in terms of negative exponents before proceeding with computing the antiderivatives:

$$\int \left(\frac{4}{x^{11}} - \frac{2}{x^5}\right)dx = \int (4x^{-11} - 2x^{-5})\,dx.$$

Once you've become more comfortable with antidifferentiation, you'll be able to do the whole thing in one line without writing the intermediate steps:

$$\int \left(\frac{4}{x^{11}} - \frac{2}{x^5}\right)dx = -\frac{4}{10x^{10}} + \frac{2}{4x^4} + C = -\frac{2}{5x^{10}} + \frac{1}{2x^4} + C.$$

(c) Just as with the corresponding derivative calculation, it's tempting to take the following "shortcut":

> **Bogus Solution:**
>
> $$\int (5x+2)^3\,dx = \frac{(5x+2)^4}{4} + C.$$

We can see the error of our ways by reversing the process–that is, by taking the derivative. Of course, doing so requires using the Chain Rule.

$$\frac{d}{dx}\frac{(5x+2)^4}{4} = 4\left(\frac{(5x+2)^3}{4}\right)\left(\frac{d}{dx}(5x+2)\right) = 5(5x+2)^3.$$

So we're off by a factor of 5. Indeed, what we've just shown is that

$$\int 5(5x+2)^3\,dx = \frac{(5x+2)^4}{4} + C,$$

and therefore, dividing by 5 gives our answer:

$$\int (5x+2)^3\,dx = \frac{(5x+2)^4}{20} + C.$$

As you might suspect, this leads us to a Chain Rule for antiderivatives; we'll see this in the next subsection.

□

Let's continue building our catalog of antiderivatives of common functions. By now, we should be very comfortable using trigonometric and exponential functions, and we know their derivatives, so we should be able to find their antiderivatives too.

Problem 5.17

(a) Find $\int \sin x\,dx$ and $\int \cos x\,dx$.

(b) Find $\int e^x\,dx$.

Solution for Problem 5.17:

(a) Since we know that

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx}\cos x = -\sin x,$$

we immediately see that

$$\int \cos x\,dx = \sin x + C \qquad \text{and} \qquad \int \sin x\,dx = -\cos x + C.$$

> **WARNING!!** Be very careful about the signs in the above formulas!

(b) Since e^x is the derivative of e^x, it is also an antiderivative of e^x. In other words,

$$\int e^x\,dx = e^x + C.$$

□

5.3.3 The Chain Rule

With differentiation, we've got it pretty easy. Using our catalog of basic functions, such as polynomials, trig functions, and exponentials, together with some basic rules such as the Product and Quotient Rules, the Chain Rule, and the Inverse Function Rule, we can take the derivative of pretty much any "nice" function that we're likely to encounter.

With antidifferentiation, however, we're going to have to work up a little bit of a sweat. Antidifferentiation is straightforward for really simple functions like polynomials, basic trig functions, and exponentials, but once we start getting away from the basics, we quickly find that antidifferentiation is a lot harder than differentiation. There are some general techniques that we can use, but as you'll discover as you work through the progressively-harder examples in this section, antidifferentiation is a bit of an art form, and requires lots of practice.

> **Sidenote:** Every January, the Massachusetts Institute of Technology holds its annual **Integration Bee**, where MIT students compete against the clock and each other to quickly and accurately evaluate integrals. The winner is crowned **Integrator**.

Recall that, in Chapter 3, we learned the Chain Rule for taking derivatives. We can use the Chain Rule in reverse to compute many antiderivatives, such as the following:

Problem 5.18

Compute $\displaystyle\int (12x)(2x^2+1)^2\,dx$.

Solution for Problem 5.18: We have

$$\begin{aligned}\frac{d}{dx}(2x^2+1)^3 &= 3(2x^2+1)^2\cdot\frac{d}{dx}(2x^2+1)\\ &= 3(2x^2+1)^2(4x)\\ &= (12x)(2x^2+1)^2.\end{aligned}$$

Thus

$$\int (12x)(2x^2+1)^2\,dx = (2x^2+1)^3+C.$$

□

Our goal is to develop a systematic way to apply the Chain Rule in reverse. That is, if we start with a function like $(12x)(2x^2+1)^2$, how do we "undo" the Chain Rule in order to compute $\int (12x)(2x^2+1)^2\,dx$?

To see how we do this for antiderivatives, we go back and think about what the Chain Rule really means. One way that we wrote the Chain Rule was:

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}.$$

For example, if $f(x) = (2x^2+1)^3$ as in Problem 5.18, then we can set $u = 2x^2+1$. We then write $f(u) = u^3$, so that $\frac{df}{du} = 3u^2$. Also we have $\frac{du}{dx} = \frac{d}{dx}(2x^2+1) = 4x$. When we put all this together, we get

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx} = 3u^2\cdot 4x = 3(2x^2+1)^2\cdot 4x = (12x)(2x^2+1)^2.$$

Of course, by now you've probably internalized this procedure so that you don't explicitly write the "u" step–you just go straight to the $\frac{df}{dx}$. We did above when we first presented this example. However, when we try to apply the Chain Rule "in reverse" to compute an antiderivative, as we're about to see, we'll usually be explicit about denoting what u is in our Chain Rule calculation.

Let's now look at our example from Problem 5.18:

$$\int (12x)(2x^2+1)^2\,dx$$

We will make a clever substitution so that we can use the Chain Rule in reverse. We substitute $u = 2x^2+1$. If we just did this alone, and nothing else, we'd have

$$\int (12x)u^2\,dx.$$

That's not too good–this integral is an ugly mix of x and u. So not only do we need to substitute u, we also need to substitute for the derivative of u. Specifically, we have

$$\frac{du}{dx} = \frac{d}{dx}(2x^2+1) = 4x,$$

so our integral becomes

$$\int (12x)u^2\,dx = \int 3u^2\left(\frac{du}{dx}\right)dx.$$

We now have the terms from the Chain Rule explicitly shown in our integral, since

$$\frac{d}{dx}(u^3) = 3u^2\left(\frac{du}{dx}\right),$$

so our integral is

$$\int (12x)u^2\,dx = \int 3u^2\left(\frac{du}{dx}\right)dx = \int \frac{d}{dx}(u^3)\,dx = u^3+C,$$

where the final step is simply applying the Fundamental Theorem of Calculus.

We have a notational shortcut that is almost universally used for this sort of Chain Rule calculation. Starting with $\frac{du}{dx} = 4x$, we "multiply by dx" and get $du = 4x\,dx$. When we make this substitution in our integral, we're left with simply

$$\int 3u^2\,du.$$

Note that du is notational shorthand for $\frac{du}{dx}\,dx$. Now it is clear that the antiderivative is just u^3+C.

But we don't want our final answer in terms of u–we want it in terms of x, our original variable. When we undo the substitution, we have our final answer of $(2x^2+1)^3+C$.

More generally, we can use the Chain Rule for antiderivatives whenever we have an integral of the form

$$\int g(f(x))f'(x)\,dx.$$

We then make the substitutions $u = f(x)$ and $du = f'(x)\,dx$ to get a simpler integral of the form

$$\int g(u)\,du.$$

We take the antiderivative of this integral to get a function $G(u)+C$, and finally we rewrite it in terms of x to get our final answer $G(f(x))+C$. Note that reversing all of these steps gives us our usual Chain Rule for derivatives:

> **Concept:** The Chain Rule for antiderivatives–although a bit more cumbersome to use than the Chain Rule for derivatives–is really nothing new! It's just the Chain Rule for derivatives in reverse.

The tricky part is recognizing when our function is of the form $g(f(x))f'(x)$. This can be hard in practice and is part of the reason why computing antiderivatives is so much more difficult than computing derivatives.

Let's see the Chain Rule at work in a few problems.

Problem 5.19

Compute $\displaystyle\int x\sqrt{2x^2-1}\,dx$.

Solution for Problem 5.19: Usually when trying to apply the Chain Rule to an antiderivative, we look for the "ugly" expression in the integral. In this case, the ugly expression is $\sqrt{2x^2-1}$. So our guess is to try $u = 2x^2-1$. Note that we don't set u to be the entire ugly expression; rather, we set u to be something that's inside our ugly expression.

Now we have to get lucky. We need to have du appear somewhere in the integral. In this problem, $du = 4x\,dx$; and sure enough, there's an x term sitting in there, so we multiply and divide by 4 to get

$$\int \frac{1}{6}(4x)\sqrt{2x^2-1}\,dx.$$

Then, after making the substitutions $u = 2x^2-1$ and $du = 4x\,dx$, our integral becomes

$$\int \frac{1}{6}\sqrt{u}\,du.$$

Don't overlook the factor of $\frac{1}{6}$ that we had to throw in there. We have $du = 4x\,dx$, but we only had $x\,dx$ in the integral, so $x\,dx = \frac{1}{6}du$.

An alternative method of making the substitution is to start with our original integral

$$\int x\sqrt{2x^2-1}\,dx$$

and then make the substitution $u = 2x^2-1$, but then take our $du = 4x\,dx$ expression and "solve for dx" to get $dx = \frac{du}{4x}$. We then substitute into the integral to get

$$\int x\sqrt{u}\,\frac{du}{4x},$$

from which the x's cancel and leave

$$\int \frac{1}{6}\sqrt{u}\,du.$$

This method looks a bit uglier, because of the intermediate step of $\int x\sqrt{u}\,\frac{du}{4x}$ which is a nasty mix of x's and u's. But happily the x's cancel, which *we must have* happen in order for this method to work.

Using either method, we are left with an indefinite integral in terms of u that we can evaluate:

$$\int \frac{1}{6}\sqrt{u}\,du = \frac{1}{6}\int u^{\frac{1}{2}}\,du = \frac{1}{6}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)+C = \frac{1}{9}u^{\frac{3}{2}}+C.$$

To finish, we put everything back in terms of x:

$$\frac{1}{9}u^{\frac{3}{2}}+C = \frac{1}{9}(2x^2-1)^{\frac{3}{2}}+C.$$

As a check, you can take the derivative of this last expression and verify that it is equal to the integrand of the integral that we started with.

□

> **Concept:** The general strategy for using the Chain Rule is to look for an expression inside the integral to set equal to u. However, we must also be able to find the expression for du in the integral as well. After substituting for u, and replacing the corresponding dx expression with du, we must have an integral that is entirely in terms of u with no x terms remaining.
>
> **WARNING!!** The most common mistake is not correctly converting from dx to du. We cannot just replace dx with du–we must solve for du in terms of dx, or vice versa, and substitute accordingly.

Problem 5.20

Compute $\displaystyle\int \frac{x-3}{2x^2-12x}\,dx$.

Solution for Problem 5.20: The denominator is the more complicated expression. So we'll try $u = 2x^2-12x$. This gives $du = (4x-12)\,dx$, so the numerator $(x-3)\,dx$ is $du/4$. Therefore, the integral is:

$$\int \frac{x-3}{2x^2-12x}\,dx = \int \frac{1}{4u}\,du.$$

This antiderivative is

$$\frac{1}{4}\log|u|+C = \frac{1}{4}\log|2x^2-12x|+C.$$

□

The next problem involves exponentials:

Problem 5.21

(a) **Compute** $\displaystyle\int e^{3x}\,dx$.

(b) **Compute** $\displaystyle\int xe^{x^2}\,dx$.

(c) **Compute** $\displaystyle\int (e^x+1)^2\,dx$.

Solution for Problem 5.21:

(a) Usually with exponentials, we'll try setting u to be the quantity in the exponent, because we know how to integrate e^u. So for part (a), we let $u = 3x$ and $du = 3\,dx$. Then the integral becomes

$$\int e^{3x}\,dx = \int \frac{1}{3}e^u\,du.$$

The antiderivative is just $\frac{1}{3}e^u+C = \frac{1}{3}e^{3x}+C$. We can immediately see that this is correct, since $\frac{d}{dx}\frac{1}{3}e^{3x} = e^{3x}$.

> **Concept:** Setting $u = cx$ for some constant c is a simple application of the Chain Rule. This will produce $du = c\,dx$, so the effect of substituting u for cx is to divide the integral by c. This application of the Chain Rule is so common that you will probably end up internalizing it and not need to write du out explicitly.

(b) The x^2 in the exponent is hard to deal with, so we let $u = x^2$. This makes $du = 2x\,dx$, and we're in good shape since we have an x term out front. We get

$$\int xe^{x^2}\,dx = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u+C = \frac{1}{2}e^{x^2}+C.$$

(c) This is more tricky. We might try $u = e^x+1$, but then $du = e^x\,dx$, and we don't have an "extra" e^x term to combine with dx and create du. So we can't just apply the Chain Rule in reverse. However, our problem go away simply by expanding the cubic:

$$\begin{aligned}\int (e^x+1)^3\,dx &= \int (e^{3x}+3e^{2x}+3e^x+1)\,dx\\ &= \frac{1}{3}e^{3x}+\frac{3}{2}e^{2x}+3e^x+x+C.\end{aligned}$$

> **Concept:** Don't be afraid to do some algebra to make antiderivatives easier to evaluate.

□

We have to be especially careful when we evaluate a definite integral using a substitution, as we'll see in the next problem:

Problem 5.22

Compute $\displaystyle\int_0^1 xe^{x^2}\,dx$.

Solution for Problem 5.22: To get the antiderivative, we use the substitution $u = x^2$, so $du = 2x\,dx$. But this is where we have to be careful:

> **Bogus Solution:**
>
> $$\int^1 xe^{x^2}\,dx = \int^1 \frac{1}{2}e^u\,du.$$

This is not correct, because the new limits of integration need to be in terms of u, not x, since our integral is now in terms of u. We have two methods by which to proceed.

Method 1: Find the antiderivative in terms of x. We compute the antiderivative of xe^{x^2} in terms of x in the usual way:

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u+C = \frac{1}{2}e^{x^2}+C.$$

We then use this antiderivative to compute the definite integral:

$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}(e^1-e).$$

Method 2: Change the limits of the definite integral to be in terms of u. We can change the interval on which we are computing the definite integral, from an interval on x to an interval on u. We see that when $x = 1$, we have $u = x^2 = 1$, and when $x = 2$, we have $u = x^2 = 4$. Thus,

$$\int_1^4 xe^{x^2}\,dx = \int_1^4 \frac{1}{2}e^u\,du.$$

We now finish the computation doing everything in terms of u. This gives

$$\int_1^4 \frac{1}{2}e^u\,du = \frac{1}{2}e^u\Big|_1^4 = \frac{1}{2}(e^4-e).$$

□

The second method in the above solution is usually easier, since we can then skip the step of converting the antiderivative back in terms of the original variable.

> **Important:** When we do a substitution of variables in a definite integral, we also need to be careful that the limits of integration are in terms of the correct variable.

The Chain Rule is also useful with trigonometric antiderivatives:

Problem 5.23

(a) **Compute** $\int \cos 3\theta\,d\theta$.

(b) **Compute** $\int \sin^3\theta\cos\theta\,d\theta$.

(c) **Compute** $\int \sin^2\theta\cos^2\theta\,d\theta$.

Solution for Problem 5.23:

(a) Once again, the simple substitution $u = c\theta$ is the key. In this case, we substitute $u = 3\theta$ and $du = 3\,d\theta$, and we get:

$$\int \cos 3\theta\,d\theta = \int \frac{1}{3}\cos u\,du = \frac{1}{3}\sin u+C = \frac{1}{3}\sin 3\theta+C.$$

We can quickly check that $\frac{d}{d\theta}\frac{1}{3}\sin 3\theta = \cos 3\theta$, as required.

(b) We don't know how to integrate $\sin^3$. But we do know how to integrate u^3. So let's try $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, and we have:

$$\int \sin^3\theta\cos\theta\,d\theta = \int u^3\,du = \frac{1}{4}u^4+C = \frac{1}{4}\sin^4\theta+C.$$

(c) Why doesn't this work:

> **Bogus Solution:** Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and we have
>
> $$\int \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{3}u^3+C = \frac{1}{3}\sin^3\theta+C.$$

The problem is that we have $\cos^2\theta$ in our original integral, but du only has a single factor of $\cos\theta$.

So instead we'll try some algebraic manipulation. We can write

$$\begin{aligned}\sin^2\theta\cos^2\theta &= (\sin\theta\cos\theta)^2\\ &= \left(\frac{1}{2}\sin(2\theta)\right)^2\\ &= \frac{1}{4}\sin^2(2\theta).\end{aligned}$$

Then, we have

$$\int \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int \sin^2 2\theta\,d\theta.$$

That's still not good enough, because we don't know how to integrate $\sin^2$. We have to do more trig manipulation to reduce the number of powers of trig functions, by noting that $\cos 2x = 1-2\sin^2 x$. So $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, and thus we have

$$\frac{1}{4}\int \sin^2 2\theta\,d\theta = \frac{1}{8}\int (1-\cos 4\theta)\,d\theta.$$

Now we can integrate:

$$\begin{aligned}\frac{1}{8}\int (1-\cos 4\theta)\,d\theta &= \frac{1}{8}\left(\theta-\frac{1}{4}\sin 4\theta\right)+C\\ &= \frac{1}{8}\theta-\frac{1}{32}\sin 4\theta+C.\end{aligned}$$

□

5.3.4 Integration by parts

We recall the Product Rule for derivatives:

$$(fg)' = fg' + f'g.$$

This may lead us to wonder if there is a similar product rule for antiderivatives. The answer is "sort of," as we can see in the following example:

Problem 5.24

Compute $\int xe^x\,dx$.

Solution for Problem 5.24: We can't do much with this using the techniques that we already have. Letting $u = x$ or $u = e^x$ doesn't help at all. But noticing that we have a product gives us an idea.

We know that

$$(xe^x)' = x(e^x)' + (x)'e^x = xe^x + e^x.$$

Rearranging, we have

$$xe^x = (xe^x)' - e^x.$$

Now we can take the antiderivative of both sides:

$$\begin{aligned}\int xe^x\,dx &= \int ((xe^x)' - e^x)\,dx \\ &= \int (xe^x)'\,dx - \int e^x\,dx \\ &= xe^x - e^x + C.\end{aligned}$$

In the last step, notice that we used the "trivial" antidifferentiation rule $\int f' = f + C$. □

"Undoing the Product Rule" as in Problem 5.24 is the general idea of the technique called **integration by parts**. Suppose that we want to find the antiderivative of something of the form uv', where u and v are functions. We can write out the product rule for the derivative of uv:

$$(uv)' = uv' + u'v.$$

Solving for uv' gives

$$uv' = (uv)' - u'v.$$

So the antiderivative of uv' is just the antiderivative of $(uv)' - u'v$. That is,

$$\int uv'\,dx = \int ((uv)' - u'v)\,dx = uv - \int u'v\,dx.$$

Typically, we write the above equation using our usual notational shortcuts $du = u'\,dx$ and $dv = v'\,dx$, giving:

Important: **Integration By Parts**

$$\int u\,dv = uv - \int v\,du$$

Let's revisit Problem 5.24 using this new notation. We set $u = x$ and $dv = e^x\,dx$. Then $du = dx$ (the derivative of u) and $v = e^x$ (the antiderivative of dv). Thus, our integration by parts formula is

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The advantage of this is that we know how to compute the antiderivative on the far right: it's just $e^x + C$. Therefore,

$$\int xe^x\,dx = xe^x - e^x + C = (x-1)e^x + C.$$

This is the general strategy for applying integration by parts: if we have an integral that is difficult to evaluate, but which has a *part* (denoted by dv) that is easier to integrate, then we can try applying integration by parts. This is especially useful if the other part of the integral–the u part–has a simple derivative. Keep this general strategy in the back of your mind as we work through the following examples.

Problem 5.25

(a) Compute $\int x\sin x\,dx$.

(b) Compute $\int x^2e^x\,dx$.

Solution for Problem 5.25:

(a) Often, a tricky step in using integration by parts is deciding which part should be u and which part should be dv. If we make the wrong choice, we can make things worse:

Bogus Solution: Let $u = \sin x$ and $dv = x\,dx$. Then $du = \cos x\,dx$ and $v = \frac{1}{2}x^2$, and integration by parts gives us:

$$\int x\sin x\,dx = \frac{1}{2}x^2\sin x - \int \frac{1}{2}x^2\cos x\,dx.$$

Although our calculation above is correct, we see that we've made our integral more complicated, because now we have to integrate a quadratic term times a trigonometric function. That's because we picked the parts incorrectly.

Concept: Try to pick u and dv so that du and v are simpler, if possible. This usually means we don't want to pick polynomial terms to be in dv, since their antiderivatives are more complicated.

In our problem, we should set $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$, and our integration by parts formula gives us:

$$\begin{aligned}\int x\sin x\,dx &= -x\cos x - \int(-\cos x)\,dx \\ &= -x\cos x + \sin x + C.\end{aligned}$$

(b) We set $u = x^2$ (so that $du = 2x\,dx$) and $dv = e^x\,dx$ (so that $v = e^x$). Then integration by parts gives us:

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx.$$

We know how to do the latter integral: it's integration by parts again! We let $u = 2x$ and $dv = e^x\,dx$, so that $du = 2\,dx$ and $v = e^x$. Thus, the entire computation is

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - \int 2xe^x\,dx \\ &= x^2e^x - \left(2xe^x - \int 2e^x\,dx\right) \\ &= x^2e^x - 2xe^x + 2e^x + C \\ &= (x^2 - 2x + 2)e^x + C.\end{aligned}$$

□

Let's again revisit Problem 5.24, but this time let's make it a definite integral and see how that works:

Problem 5.26

Compute $\int_0^2 xe^x\,dx$.

Solution for Problem 5.26: As before, we apply integration by parts by letting $u = x$ (so that $du = dx$) and $dv = e^x\,dx$ (so that $v = e^x$). Then integration by parts gives

$$\int_0^2 xe^x\,dx = xe^x\Big|_0^2 - \int_0^2 e^x\,dx.$$

Notice that the uv term must now be evaluated at the limits of integration. We continue:

$$\begin{aligned}\int_0^2 xe^x\,dx &= xe^x\Big|_0^2 - \int_0^2 e^x\,dx \\ &= (2e^2 - 0e^0) - e^x\Big|_0^2 \\ &= 2e^2 - (e^2 - e^0) \\ &= e^2 + 1.\end{aligned}$$

□

Problem 5.27

Compute $\int e^x\sin x\,dx$.

Solution for Problem 5.27: It's not clear which part we should set equal to u and which part to dv: they both have "easy" derivatives and antiderivatives. So let's just try one and see what happens.

If we set $u = e^x$ (so that $du = e^x\,dx$) and $dv = \sin x\,dx$ (so that $v = -\cos x$), then we have

$$\int e^x\sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx.$$

This doesn't seem all that helpful, since we still have an integral that we don't know how to compute. But it's very similar to the integral that we started with. So let's try integration by parts again. Let $u = e^x$ and $dv = \cos x\,dx$; then $du = e^x\,dx$ and $v = \sin x$, so we have

$$\begin{aligned}\int e^x\sin x\,dx &= -e^x\cos x + \int e^x\cos x\,dx \\ &= -e^x\cos x + \left(e^x\sin x - \int e^x\sin x\,dx\right) \\ &= e^x(\sin x - \cos x) - \int e^x\sin x\,dx.\end{aligned}$$

We seem to be back where we started, with $\int e^x\sin x\,dx$ on the right side. Are we stuck?

No! If we move the integral on the right side over to the left side, we have

$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x),$$

and hence, dividing by 2, we have our answer:

$$\int e^x\sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + C.$$

(Note the C that we add at the end.) □

Integration by parts also lets us compute the antiderivative of a "simple" function:

Problem 5.28

Compute $\int \log x\,dx$.

Solution for Problem 5.28: This integral seems a little hard to grasp. What are the "parts"? There seems to just be one term: $\log x$. So what can we set to be u and dv? We don't seem to have much choice for u: since we don't know how to integrate $\log x$, we have little choice but to set $u = \log x$, so that $du = \frac{1}{x}\,dx$. What does that leave for dv? All that's left in the integral is dx, so we set $dv = dx$. This makes $v = x$, and our integration by parts calculation gives us:

$$\begin{aligned}\int \log x\,dx &= x\log x - \int x\left(\frac{1}{x}\right)dx \\ &= x\log x - \int 1\,dx \\ &= x\log x - x + C.\end{aligned}$$

As a check, note that

$$\frac{d}{dx}(x\log x - x) = x\left(\frac{1}{x}\right) + 1(\log x) - 1 = 1 + \log x - 1 = \log x.$$

□

Concept: Integration by parts is nothing more than a rearranged version of the Product Rule for derivatives. If you can remember

$$(uv)' = uv' + u'v,$$

you can just rearrange it to

$$uv' = (uv)' - u'v$$

and then integrate this formula to get

$$\int u\,dv = uv - \int v\,du.$$

5.3.5 Substitution methods

The Chain Rule is one example of a **substitution method** for evaluating antiderivatives. There are other more sophisticated sorts of substitutions that we can make that let us evaluate more difficult integrals. All of the substitutions follow the same basic rule:

> **Important:** If we substitute for x in $\int f(x)\,dx$, then we have to substitute for dx too.

The methods that we will discuss in this subsection are essentially "backwards" versions of the Chain Rule substitutions that we studied earlier. Before, we would take an integral of the form

$$\int g(f(x))f'(x)\,dx$$

and make the substitutions $u = f(x)$ and $du = f'(x)\,dx$ to get an integral of the form

$$\int g(u)\,du.$$

However, now we are going to consider substitutions in the other direction. Specifically, we'll start with an integral of the form

$$\int g(x)\,dx$$

and make a substitution of the form $x = f(u)$. This gives $dx = f'(u)\,du$, so after substitution, our integral becomes

$$\int g(f(u))f'(u)\,du.$$

This seems strange—it looks like we've made our integral a lot more complicated! But for the right type of function, this sort of substitution actually makes things work out nicely.

Let's see some examples. Our first example is a substitution that might seem quite surprising!

Problem 5.29

Compute $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Solution for Problem 5.29: We don't have any good way to deal with that ugly denominator. But the expression $\sqrt{1-x^2}$ may remind us of the unit circle, and the unit circle should remind us of trig functions. So we try the substitution $x = \sin\theta$. We note that this substitution makes sense, because the function that we're integrating is only defined for $-1 < x < 1$, and $\sin\theta$ contains the interval $(-1,1)$ in its range.

This means that $dx = \cos\theta\,d\theta$, so we have

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta.$$

Note that the $\cos\theta$ term in the numerator comes from the conversion of dx to $d\theta$.

Now the denominator simplifies nicely! We have

$$\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|,$$

so our integral becomes

$$\int \frac{\cos\theta}{|\cos\theta|}\,d\theta.$$

This looks very nice except for that annoying absolute value sign. Perhaps we can make it go away. . .

We can! Notice that in the substitution $x = \sin\theta$, we can restrict to $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and we get $x \in (-1,1)$. But on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the cosine function is strictly positive. So we can replace $|\cos\theta|$ with just $\cos\theta$. Hence, our integral is simply

$$\int \frac{\cos\theta}{\cos\theta}\,d\theta = \int 1\,d\theta = \theta + C.$$

But we don't want the final answer in terms of θ, so we make the reverse substitution $\theta = \arcsin x$, to get the final answer of

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

Again, note that the domain of $\arcsin$ is $(-1,1)$ and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so all of our substitutions are consistent. □

But wait a minute. What if we had instead made the substitution $x = \cos\theta$ over the interval $\theta \in (0,\pi)$? Then we'd have $dx = -\sin\theta\,d\theta$, and the integral becomes:

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int -\frac{\sin\theta}{\sqrt{1-\cos^2\theta}}\,d\theta \\ &= \int -\frac{\sin\theta}{\sin\theta}\,d\theta \\ &= \int (-1)\,d\theta \\ &= -\theta + C = -\arccos x + C.\end{aligned}$$

How can this be? Is the integral equal to *both* $\arcsin x$ and $-\arccos x$? It can't possibly be, since any two antiderivatives must differ by a constant, right?

There's no problem, because $\arcsin x$ and $-\arccos x$ *do* differ by a constant! Recall that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$. This implies that $\arcsin x = \frac{\pi}{2} - \arccos x$ for any $-1 \le x \le 1$. (Note that there are also some issues with the ranges of $\arcsin$ and $\arccos$, but these also amount to merely adding or subtracting a constant.)

In general, we can use substitutions—trigonometric or otherwise—to make "ugly" terms in integrals come out nicer. A common use is as in the last problem: when we have a term of the form $1 - x^2$, especially $\sqrt{1-x^2}$, we use $x = \sin\theta$ or $x = \cos\theta$ so that we can apply the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

We'll reiterate a subtle point about this substitution: we can only make the substitution $x = \sin\theta$ if x only takes on values between -1 and 1, because those are the only values in the range of $\sin\theta$. But when we make a substitution for x in $\sqrt{1-x^2}$, we must have $1 - x^2 \ge 0$ (in order to take the square root), so we must have $x^2 \le 1$, giving $-1 \le x \le 1$, which makes the trig substitution valid.

The next problem is similar and also quite important:

Problem 5.30

Compute $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$.

Solution for Problem 5.30: This is very similar to Problem 5.29, except we have $1 + x^2$ in the denominator instead of $\sqrt{1-x^2}$. One thing to note is that the domain of x is all of $\mathbb{R}$, so we know that we can't make the substitution $x = \sin\theta$ (since that only takes values between -1 and 1). Fortunately, we do know a trig function that has a range of all the reals: tangent.

We let $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$, and we have

$$\int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta.$$

This may look ugly, but actually we're in great shape. We use the identity $1 + \tan^2\theta = \sec^2\theta$, so our integral becomes

$$\int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \int 1\,d\theta = \theta + C.$$

As always, we want to put this back in terms of $x = \tan\theta$. This means that $\theta = \arctan x$, and we have

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

Finally, we compute the definite integral:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan x\Big|_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}.$$

Note that we could have also written the entire calculation, including the substitution, in terms of a definite integral, as follows:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \int_0^{\pi/4} d\theta = \frac{\pi}{4}.$$

□

The integrals in the previous two problems are fairly common; you should learn them.

> **Important:**
> $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C,$$
> $$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

> **Concept:** More generally, whenever we see a term of the form $1 \pm x^2$, we can consider using a trigonometric substitution.

Here's a more complicated example of this:

Problem 5.31

Compute $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

Solution for Problem 5.31: We have a $\sqrt{1-x^2}$ term, so we think to use the trig substitution $x = \sin\theta$. This gives $dx = \cos\theta\,d\theta$, so the integral becomes

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta \\ &= \int \sin^3\theta\cos^2\theta\,d\theta.\end{aligned}$$

There are a number of ways to continue from here. One method is to use the identity $\sin^2\theta = 1 - \cos^2\theta$ on two of the sine factors, giving

$$\int \sin^3\theta\cos^2\theta\,d\theta = \int (\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta.$$

Now we can make the substitution $u = \cos\theta$ and $du = -\sin\theta\,d\theta$, giving us

$$\begin{aligned}\int (\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta &= \int -(u^2 - u^4)\,du \\ &= \int (u^4 - u^2)\,du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C.\end{aligned}$$

Finally, we use the fact that

$$u = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2},$$

giving the solution:

$$\int x^3\sqrt{1-x^2}\,dx = \frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.$$

Alternatively, noting that we had $u = \sqrt{1-x^2}$ at the end of our calculation above, we could have made this substitution to begin with. Then $u^2 = 1 - x^2$, so $2u\,du = -2x\,dx$, hence $-u\,du = x\,dx$, and we have

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int (1-u^2)u(-u)\,du \\ &= \int (u^4 - u^2)\,du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5}(1-x^2)^{\frac{5}{2}} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.\end{aligned}$$

□

Although trig substitution is by far the most commonly used substitution method, some functions can be integrated with an unusual substitution, such as the following example:

Problem 5.32

Compute $\displaystyle\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx$.

Solution for Problem 5.32: There's no obvious way to attack this integral. Substituting $u = e^x$ might work. This gives $du = e^x\,dx$, and we have

$$\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = \int \frac{u}{\sqrt{1+u}}\,du.$$

Now we can evaluate this by parts. We let $u = u$ (convenient!) and $dv = (1+u)^{-\frac{1}{2}}du$, so that $v = 2(1+u)^{\frac{1}{2}}$. This gives us

$$\begin{aligned}\int \frac{u}{\sqrt{1+u}}\,du &= 2u\sqrt{1+u} - 2\int \sqrt{1+u}\,du \\ &= 2u\sqrt{1+u} - \frac{4}{3}(1+u)^{\frac{3}{2}} + C.\end{aligned}$$

To finish, we replace $u = e^x$ to get

$$\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = 2e^x(1+e^x)^{\frac{1}{2}} - \frac{4}{3}(1+e^x)^{\frac{3}{2}} + C.$$

Alternatively, we could use a more creative substitution on the original integral. Let's get rid of the whole messy denominator at once, by making the substitution $u = \sqrt{1+e^x}$. Now we need to convert $e^{2x}\,dx$ into something in terms of u. To isolate e^{2x}, we "solve" our substitution:

$$u = \sqrt{1+e^x} \quad\Rightarrow\quad e^x = u^2 - 1.$$

Thus, the numerator of our integral will become $(u^2-1)^2$.

We also need to convert dx into something involving du. We continue solving for x, so that we can compute dx:

$$e^x = u^2 - 1 \quad\Rightarrow\quad x = \log(u^2-1) \quad\Rightarrow\quad dx = \frac{2u}{u^2-1}\,du.$$

Now we can write our integral in terms of u:

$$\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = \int \frac{(u^2-1)^2}{u}\,\frac{2u}{u^2-1}\,du.$$

How nice! A lot of things cancel, and we're left with just

$$\int 2(u^2-1)\,du = \frac{2}{3}u^3 - 2u + C,$$

and converting this back into x gives us our answer:

$$\int \frac{e^{2x}}{\sqrt{1+e^x}}\,dx = \frac{2}{3}(1+e^x)^{\frac{3}{2}} - 2(1+e^x)^{\frac{1}{2}} + C.$$

The answers that we got from our two different methods may look different, but if you check them carefully (and rearrange some terms), you will see that they are the same. □

5.3.6 Partial fractions

The technique of **partial fractions** is designed for integrals such as the following:

Problem 5.33

Compute $\int \frac{1}{x^2-1}\,dx$.

Solution for Problem 5.33: We can see that a trig substitution won't help. In particular, since x can be any real number except for ± 1, the only possibilities are $x = \tan\theta$ or $x = \cot\theta$, but the quantities $\tan^2\theta - 1$ and $\cot^2\theta - 1$ are not particularly nice.

However, we notice that we can factor $x^2 - 1 = (x-1)(x+1)$. Surely that must help. Indeed, that factorization lets us rewrite the function in terms of simpler fractions:

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right).$$

This is very helpful, because those simpler fractions have easy antiderivatives. This lets us easily finish the problem:

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \frac{1}{2}\left(\int \frac{1}{x-1}\,dx - \int \frac{1}{x+1}\,dx\right)\\ &= \frac{1}{2}(\log|x-1| - \log|x+1|) + C\\ &= \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.\end{aligned}$$

□

Problem 5.33 is a basic example of the technique of partial fractions. The general idea is that we break up a complicated fraction into simpler pieces that are easy to integrate. Here is a slightly more complicated example:

Problem 5.34

Compute $\int \frac{x+2}{x^2-2x-3}\,dx$

Solution for Problem 5.34: If we were lucky, we could try the Chain Rule substitution $u = x^2 - 2x - 3$. But then $du = (2x-2)\,dx$ doesn't match the term in the numerator, so we're stuck.

However, we notice that $x^2 - 2x - 3$ factors as $(x-3)(x+1)$, so we can try partial fractions. We hope to write

$$\frac{x+2}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1},$$

where A and B are suitable constants.

It is an easy algebra exercise to solve for A and B. One method is to place the right side of the above expression over a common denominator:

$$\frac{x+2}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1} = \frac{A(x+1)+B(x-3)}{x^2-2x-3},$$

so that we must have

$$x + 2 = A(x+1) + B(x-3).$$

We can then match coefficients to get the system of equations $A + B = 1$ and $A - 3B = 2$, which gives $A = \frac{5}{4}$ and $B = -\frac{1}{4}$. Alternatively, we can plug in $x = -1$ to the above equation to get $1 = B(-4)$, hence $B = -\frac{1}{4}$, and we can plug in $x = 3$ to get $5 = A(4)$, so $A = \frac{5}{4}$.

Thus, our integral becomes

$$\int \frac{x+2}{x^2-2x-3}\,dx = \frac{5}{4}\int \frac{1}{x-3}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx.$$

Both of the integrals on the right side of the above equation give us logarithm terms in our final answer:

$$\int \frac{x+2}{x^2-2x-3}\,dx = \frac{5}{4}\log|x-3| - \frac{1}{4}\log|x+1| + C.$$

Some people might prefer to combine the logarithm terms into one expression to get a more compact final answer:

$$\int \frac{x+2}{x^2-2x-3}\,dx = \frac{1}{4}\log\left|\frac{(x-3)^5}{x+1}\right| + C.$$

□

5.3.7 A monster example

We'll finish this long section on antiderivative methods with an example that combines a few different techniques.

Problem 5.35

Compute $\displaystyle\int \frac{3}{x^3-1}\,dx$.

Solution for Problem 5.35: There appears to be no clever substitution that seems to help us. But we have a factorization of the denominator as $(x^3-1)=(x-1)(x^2+x+1)$, so we can try for a partial fraction decomposition:

$$\frac{3}{x^3-1}=\frac{?}{x-1}+\frac{?}{x^2+x+1}.$$

We can try for constant terms in the numerators, but that runs into problems:

Bogus Solution: We wish to have

$$\frac{3}{x^3-1}=\frac{A}{x-1}+\frac{B}{x^2+x+1}.$$

Clearing the denominators gives

$$\begin{aligned}3&=(x^2+x+1)A+(x-1)B\\&=Ax^2+(A+B)x+(A-B).\end{aligned}$$

Comparing quadratic terms gives $A=0$, so we have

$$3=Bx-B.$$

Comparing linear terms gives $B=0$, so $3=0$. Uh-oh.

Instead, we will need a linear term in the numerator over x^2+x+1, otherwise we have no hope of canceling the quadratic factor that will result when we place the right side over a common denominator. Specifically, we will have

$$\frac{3}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1},$$

where A, B, C are constants that we need to find. Placing the right side over a common denominator gives

$$\begin{aligned}\frac{3}{x^3-1}&=\frac{A(x^2+x+1)+(Bx+C)(x-1)}{x^3-1}\\&=\frac{(A+B)x^2+(A-B+C)x+(A-C)}{x^3-1}.\end{aligned}$$

Matching the corresponding coefficients of the numerators gives us the system of linear equations

$$\begin{aligned}A+B&=0,\\A-B+C&=0,\\A-C&=3.\end{aligned}$$

Solving this yields $A=1$, $B=-1$, and $C=-2$.

Thus, our partial fraction decomposition is

$$\int\frac{3}{x^3-1}=\int\left(\frac{1}{x-1}-\frac{x+2}{x^2+x+1}\right)dx.$$

The first term of the integral on the right side above is straightforward:

$$\int\frac{1}{x-1}\,dx=\log|x-1|+C.$$

But how do we compute $\displaystyle\int\frac{x+2}{x^2+x+1}\,dx$? The denominator doesn't factor any further, and trying the Chain Rule with $u=x^2+x+1$ gives $du=(2x+1)\,dx$, which doesn't match the numerator.

But we can force this substitution to match at least partially, by writing

$$\frac{x+2}{x^2+x+1}=\frac{1}{2}\cdot\frac{2x+1}{x^2+x+1}+\frac{3}{2}\cdot\frac{1}{x^2+x+1}.$$

Now the Chain Rule does work on the first summand:

$$\begin{aligned}\int\frac{1}{2}\frac{2x+1}{x^2+x+1}\,dx&=\int\frac{1}{2}\frac{du}{u}\\&=\frac{1}{2}\log|u|+C\\&=\frac{1}{2}\log|x^2+x+1|+C.\end{aligned}$$

The term that remains is

$$\frac{3}{2}\int\frac{1}{x^2+x+1}\,dx.$$

This looks a little bit like an arctangent, but there's that annoying linear term in there. We can make it look more like the derivative of an arctangent if we complete the square of the denominator:

$$\frac{1}{x^2+x+1}=\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}.$$

We evaluate the antiderivative of this via a trig substitution. We want the denominator of the above fraction to look like a constant times $\tan^2\theta+1$, so we let $x=\dfrac{\sqrt{3}}{2}\tan\theta-\dfrac{1}{2}$. Noting that $dx=\dfrac{\sqrt{3}}{2}\sec^2\theta\,d\theta$, we get

$$\int\frac{1}{x^2+x+1}\,dx=\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx=\int\frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}(\tan^2\theta+1)}\,d\theta.$$

But $\tan^2\theta+1=\sec^2\theta$, so this is just

$$\int\frac{2}{\sqrt{3}}\,d\theta=\frac{2}{\sqrt{3}}\theta+C.$$

Undoing the substitution, we see that $\theta=\tan^{-1}\left(\dfrac{2}{\sqrt{3}}\left(x+\dfrac{1}{2}\right)\right)$ so we get

$$\int\frac{1}{x^2+x+1}\,dx=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+C.$$

Putting all of these steps together, we get our answer:

$$\begin{aligned}\int\frac{3}{x^3-1}\,dx&=\int\frac{1}{x-1}\,dx-\int\frac{x+2}{x^2+x+1}\,dx\\&=\log|x-1|-\frac{1}{2}\int\frac{2x+1}{x^2+x+1}\,dx-\frac{3}{2}\int\frac{1}{x^2+x+1}\,dx\\&=\log|x-1|-\frac{1}{2}\log|x^2+x+1|-\frac{3}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx\\&=\log|x-1|-\frac{1}{2}\log|x^2+x+1|\\&\quad-\sqrt{3}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+C.\end{aligned}$$

□

This last problem combined several nontrivial integration techniques. Also worth noting is that we started with the relatively simple function $\dfrac{3}{x^3-1}$ and ended up with a really ugly-looking antiderivative. This is quite common: relatively "nice" functions can have really horrible antiderivatives.

Concept: While it's good to slog through an integral like the one in Problem 5.35 at least once in your life, in practice it's much easier to just have your favorite calculator, computer, or website do it for you. It's just like in elementary school, where at first you learned how to multiply three-digit numbers by hand, but quickly started using a calculator instead. Learning to do the process by hand at first will help you learn the deeper meaning behind it, but there's no reason not to take advantage of the technology that is available.

WARNING!! However, evaluating an integral using technology may not give you an answer in the form that you expect. For example, if you enter the integral from Problem 5.35,

$$\int\frac{3}{x^3-1}\,dx$$

into a TI-Nspire CAS (computer algebra system) calculator, it outputs

$$-\frac{\ln\left(|x^2+x+1|\right)}{2}-\frac{2\cdot\left(\ln\left(|x-1|\right)-\sqrt{3}\cdot\tan^{-1}\left(\frac{\sqrt{3}\cdot(2\cdot x+1)}{3}\right)\right)}{2}.$$

Note that the TI (like most calculators) uses $\ln$ for the natural logarithm, and that the arbitrary constant is omitted, but otherwise this is a correct antiderivative. It is the same as our answer from Problem 5.35, but it is arranged somewhat differently.

5.1.4 Exercises

Exercises

5.1.1.

Evaluate the following integrals.

(a) $\int (x^2 - x) \, dx$

Hint Solution

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: This is just a polynomial, which we know how to integrate:

$$\int (x^2 - x) \, dx = \frac{1}{3}x^3 - \frac{1}{2}x + C$$

(b) $\int \left(5x^4 - x - \frac{1}{x^2} \right) \, dx$

Hint: Write $\frac{1}{x^2}$ as x^{-2}.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: As we saw in part (a), we can integrate this using our usual rules:

$$\int \left(5x^4 - x - \frac{1}{x^2} \right) \, dx = \frac{5}{5}x^5 - \frac{1}{2}x^2 - \frac{1}{-1}x^{-1} + C$$

(c) $\int (\cos x - 3 \sin x) \, dx$

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Hide Solution

Your Submission Solution:

Solution: We know that $\int \cos x \, dx = \sin x + C$ and $\int \sin x \, dx = -\cos x + C$. Combining these,

$$\int (\cos x - 3 \sin x) \, dx = \int \cos x \, dx - 3 \int \sin x \, dx = \sin x + 3 \cos x + C$$

(d) $\int \frac{x^2 + 3\sqrt{x} + 5}{x} \, dx$

Hint: Divide through by the x first.

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Hide Solution

Your Submission Solution:

Solution: First, we distribute the x, so:

$$\int \frac{x^2 + 3\sqrt{x} + 5}{x} \, dx = \int x + 3x^{-1/2} + 5x^{-1} \, dx$$

Then we integrate:

$$\int x + 3x^{-1/2} + 5x^{-1} \, dx = \frac{1}{2}x^2 + 6x^{1/2} + 5 \ln|x| + C$$

(e) $\int e^{-3x} \, dx$

Hint: Substitute $u = -3x$.

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Hide Solution

Your Submission Solution:

Solution: This requires substitution of $u = -3x$. Then $du = -3 \, dx$, so:

$$\int e^{-3x} \, dx = \int e^u \, du = -\frac{1}{3} \int e^u \, du = -\frac{1}{3}e^u + C$$

(f) $\int \cos x \, dx$

Hint: Substitute $u = x^2$.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: This requires the substitution $u = x^2$. Then $du = 2x \, dx$, so:

$$\int \cos(x^2) \, dx = \int \begin{cases} \cos u \, du \\ = \sin u + C \\ = \sin(x^2) + C \end{cases}$$

(g) $\int \sin^2 x \cos x \, dx$

Hint: Substitute $u = \sin x$.

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Hide Solution

Your Submission Solution:

Solution: We integrate by substitution of $u = \sin x$. Then $du = \cos x \, dx$, so:

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$$

(h) $\int \frac{1}{\sqrt{1 - x^2}} \, dx$

Hint: Is something in the integral the derivative of something else in the integral?

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Hide Solution

Your Submission Solution:

Solution: Since $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$, it looks as if we should be substitution $u = \sin x$. Then:

$$\int \frac{1}{\sqrt{1 - x^2}} \, dx = \int \frac{1}{\sqrt{1 - \sin^2 u}} \cos u \, du = \int \frac{1}{\sqrt{\cos^2 u}} \cos u \, du = \int \frac{\cos u}{\cos u} \, du$$

(i) $\int e^{\cos x} \, dx$

Hint: Use integration by parts.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: We integrate by parts. Let $u = e^{\cos x}$ and $v = \int dx = dx$ so that $du = -e^{\cos x} \sin x \, dx$. Then:

$$\int e^{\cos x} \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = \int e^{\cos x} \, dx \\ - \int -e^{\cos x} \sin x \, dx \\ = \int e^{\cos x} \, dx \\ + \int e^{\cos x} \sin x \, dx \end{cases}$$

(j) $\int x e^x \, dx$

Hint: Use integration by parts.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: We integrate by parts. Let $u = x$ and $v = \int e^x \, dx = e^x$ so that $du = dx$ and $dv = e^x \, dx$. Then:

$$\int x e^x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = xe^x - \int e^x \, dx \\ = xe^x \\ - \int e^x \, dx \\ = xe^x \\ - \int e^x \, dx \\ = (x - 1)e^x + C \end{cases}$$

(k) $\int x^2 e^x \, dx$

Hint: Use integration by parts twice.

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Hide Solution

Your Submission Solution:

Solution: We integrate by parts, with $u = x^2$ and $v = \int e^x \, dx = e^x$ so that $du = 2x \, dx$ and $dv = e^x \, dx$. Then:

$$\int x^2 e^x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = x^2 e^x - \int 2x e^x \, dx \end{cases}$$

To calculate the last integral, we use integration by parts again, using $u = x$ and $v = \int e^x \, dx = e^x$ so that $du = dx$ and $dv = e^x \, dx$. We have:

$$\int 2x e^x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = 2xe^x - \int 2e^x \, dx \\ = 2xe^x \\ - \int 2e^x \, dx \\ = 2xe^x \\ - 2e^x + C \end{cases}$$

Substituting this for the integral above, yields:

$$\int x^2 e^x \, dx = \begin{cases} x^2 e^x - \left(2xe^x \right. \\ \left. - 2e^x + C \right) \\ = x^2 e^x \\ - 2xe^x \\ + 2e^x \\ - C \end{cases}$$

(l) $\int \log x \, dx$

Hint: Use integration by parts.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: We integrate by parts, but we need to set u equal to something that is easy to differentiate. Let $u = \log x$ and $v = \int 1 \, dx = x$. Then $du = \frac{1}{x} \, dx$ and $dv = x \, dx$, so we get:

$$\int \log x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = x \log x - \int x \cdot \frac{1}{x} \, dx \\ = x \log x \\ - \int 1 \, dx \\ = x \log x \\ - x + C \end{cases}$$

(m) $\int \frac{1}{x^2 + 1} \, dx$

Hint: Substitution $x = \tan \theta$.

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Hide Solution

Your Submission Solution:

Solution: We make the substitution $x = \tan \theta$. Then $dx = \sec^2 \theta \, d\theta$ and:

$$\int \frac{1}{x^2 + 1} \, dx = \int \frac{1}{\tan^2 \theta + 1} \sec^2 \theta \, d\theta = \int \frac{1}{\sec^2 \theta} \sec^2 \theta \, d\theta = \int 1 \, d\theta = \theta + C$$

(n) $\int \frac{1}{x^2 - 1} \, dx$

Hint: Consider the square in the denominator.

Hint: After completing the square, consider a trig substitution.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: Since $x^2 - 1$ is a 2 factor and metic, we cannot use partial fractions. Instead, we complete the square:

$$\int \frac{1}{x^2 - 1} \, dx = \int \frac{1}{(x - 1)(x + 1)} \, dx$$

We make things cleaner by substituting $u = x - 1$. Then $du = dx$ and we have:

$$\int \frac{1}{(x - 1)(x + 1)} \, dx = \int \frac{1}{u(u + 2)} \, du$$

Now we substitute $u = \frac{2}{\tan \theta}$ and the integral becomes:

$$\int \frac{1}{u(u + 2)} \, du = \int \frac{1}{\frac{2}{\tan \theta}\left(\frac{2}{\tan \theta} + 2 \right)} \, d\left(\frac{2}{\tan \theta} \right)$$

$$= \int \frac{1}{\frac{2}{\tan \theta}\left(\frac{2}{\tan \theta} + 2 \right)} \left(-\frac{2}{\sin^2 \theta} \right) d\theta$$

$$= \int \frac{1}{\sin \theta} \, d\theta$$

$$= \int \csc \theta \, d\theta$$

Removing the substitutions, we have:

$$\theta = \cos^{-1}\left(\frac{2}{u + 2} \right)$$

so our final answer is $\int \frac{1}{x^2 - 1} \, dx = \int \csc \left(\cos^{-1}\left(\frac{2}{x + 1} \right) \right) \, d\left(\frac{2}{x + 1} \right)$

(o) $\int \frac{x^2 + 1}{x^2 - 1} \, dx$

Hint: Consider the square in the denominator.

Hint: After completing the square, consider a trig substitution.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: $x^2 - 1$ is 2factors so $x^2 - 1 = (x - 1)(x + 1)$, so we write the partial fraction decomposition:

$$\frac{x^2 + 1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

Thus,

$$\int \frac{x^2 + 1}{x^2 - 1} \, dx = \begin{cases} \int \left(1 + \frac{2}{x^2 - 1} \right) \, dx \\ = \int 1 \, dx \\ + \int \frac{2}{x^2 - 1} \, dx \\ = x \\ + \int \frac{2}{x^2 - 1} \, dx \end{cases}$$

$$= x + \int \frac{2}{(x - 1)(x + 1)} \, dx$$

Hint: Use partial fractions.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: $x^2 - 1$ is 2factors so $\frac{2}{x^2 - 1}$ = 2 this gives the partial fraction decomposition:

$$\frac{2}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

comparing, we get

$$\int \frac{2}{(x - 1)(x + 1)} \, dx = \begin{cases} \int \left(\frac{1}{x - 1} - \frac{1}{x + 1} \right) \, dx \\ = \int \left(\frac{1}{x - 1} \right) \, dx \\ - \int \left(\frac{1}{x + 1} \right) \, dx \\ = \int \left(\frac{1}{x - 1} \right) \, dx \\ - \int \left(\frac{1}{x + 1} \right) \, dx \\ = \ln|x - 1| \\ - \ln|x + 1| \\ = \ln\left| \frac{x - 1}{x + 1} \right| \end{cases}$$

(p) $\int \frac{x^2 + 1}{x^2 - 1} \, dx$

Hint: Use partial fractions.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: $x^2 - 1$ is 2factors so $\frac{2}{x^2 - 1}$ = 2 this gives the partial fraction decomposition:

$$\frac{2}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

comparing, we get

$$\int \frac{x^2 + 1}{x^2 - 1} \, dx = \begin{cases} \int \left(1 + \frac{2}{x^2 - 1} \right) \, dx \\ = \int 1 \, dx \\ + \int \frac{2}{x^2 - 1} \, dx \\ = x \\ + \int \frac{2}{x^2 - 1} \, dx \\ = x \\ + \int \left(\frac{1}{x - 1} - \frac{1}{x + 1} \right) \, dx \\ = x \\ + \int \left(\frac{1}{x - 1} \right) \, dx \\ - \int \left(\frac{1}{x + 1} \right) \, dx \\ = x \\ + \ln|x - 1| \\ - \ln|x + 1| \\ = x \\ + \ln\left| \frac{x - 1}{x + 1} \right| \end{cases}$$

(q) $\int \frac{x^2 - 2x}{x^2 - 1} \, dx$

Hint: Use partial fractions.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: The denominator factors as:

$$x^2 - x = x(x^2 - 1) = x(x + 1)(x - 1).$$

Thus, we have the partial fraction decomposition:

$$\frac{x^2 - 2x}{x^2 - 1} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}$$

Hence

$$\int \frac{x^2 - 2x}{x^2 - 1} \, dx = \begin{cases} \int \left(\frac{A}{x} + \frac{B}{x - 1} \right. \\ \left. + \frac{C}{x + 1} \right) \, dx \\ = \int \left(\frac{A}{x} \right) \, dx \\ + \int \left(\frac{B}{x - 1} \right) \, dx \\ + \int \left(\frac{C}{x + 1} \right) \, dx \\ = \int \left(\frac{A}{x} \right) \, dx \\ + \int \left(\frac{B}{x - 1} \right) \, dx \\ + \int \left(\frac{C}{x + 1} \right) \, dx \\ = A \ln|x| \\ + B \ln|x - 1| \\ + C \ln|x + 1| \end{cases}$$

taking the integral of each sides yields

$$\int \frac{x^2 - 2x}{x^2 - 1} \, dx = A \ln|x| + B \ln|x - 1| + C \ln|x + 1|$$

(r) $\int \frac{1}{x^2 - 2x} \, dx$

Hint: Substitution $u = x^2 - 2x$.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: Let $u = x^2 - 2x$ and let $du = (x^2 - 2x)' \, dx$ so $du = 2x(x - 1) \, dx$ so:

$$\int \frac{1}{x^2 - 2x} \, dx = \begin{cases} \int \frac{1}{u} \, du \\ = \ln|u| \\ = \ln|x^2 - 2x| \end{cases}$$

(s) $\int \frac{1}{x^2 - 2x} \, dx$

Hint: Substitution $u = x^2 - 2x$.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: Let $u = x^2 - 2x$ and let $du = (x^2 - 2x)' \, dx$ so $du = 2x(x - 1) \, dx$ so:

$$\int \frac{1}{x^2 - 2x} \, dx = \begin{cases} \int \frac{1}{u} \, du \\ = \ln|u| \\ = \ln|x^2 - 2x| \end{cases}$$

(t) $\int \frac{1}{\sqrt{1 - x^2}} \, dx$

Hint: Substitution $x = \sin \theta$.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: Let $x = \sin \theta$ so $dx = \cos \theta \, d\theta$ and we get:

$$\int \frac{1}{\sqrt{1 - x^2}} \, dx = \int \frac{1}{\sqrt{1 - \sin^2 \theta}} \cos \theta \, d\theta = \int \frac{1}{\sqrt{\cos^2 \theta}} \cos \theta \, d\theta = \int \frac{\cos \theta}{\cos \theta} \, d\theta$$

5.1.2.

Let f be a differentiable function. Express $\int f(x) f'(x) \, dx$ in terms of $f(x)$ and $f(0)$.

Hide Solution

Your Submission Solution:

Solution: Let $u = f(x)$ so that $du = f'(x) \, dx$. Then the integral becomes:

$$\int f(x) f'(x) \, dx = \int_{f(0)}^{f(x)} u \, du = \frac{1}{2}\left(f(x)^2 - f(0)^2 \right)$$

5.1.3.

(a) Describe a method for computing $\int \sin^n x \, dx$ and $\int \cos^n x \, dx$ where n is a positive integer.

Hint: The trick is to use $\sin^2 x + \cos^2 x = 1$.

Hint: Use integration by parts (twice) to find an equation relating $\int \sin^n x \, dx$ to $\int \sin^{n-2} x \, dx$.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: Use integration by parts with $u = \sin^{n-1}$, $dv = (\sin x) \, dx$. $\int (\sin x) \, dx = -\cos x$. Then we get the formula:

$$\int \sin^n x \, dx = -\cos^{n-1} \cos x + (n - 1) \int \cos^2 \cos^{n-2} x \, dx$$

Not using $\int \sin^2 x = 1 - \cos^2 x$, this becomes:

$$\int \sin^n x \, dx = -\cos^{n-1} \cos x + (n - 1) \int \sin^{n-2} x \, dx$$

Adding $(n - 1) \int \sin^n$ to both sides gives:

$$n \int \sin^n x \, dx = -\cos^{n-1} \cos x + (n - 1) \int \sin^{n-2} x \, dx$$

and dividing by n gives:

$$\int \sin^n x \, dx = -\frac{1}{n} \cos^{n-1} \cos x + \frac{n - 1}{n} \int \sin^{n-2} x \, dx$$

Applying this repeatedly we can arrive at a formula for any positive integer n for instance:

$$\int \sin^4 x \, dx = \begin{cases} -\frac{1}{4} \cos^{4-1} \cos x \\ + \frac{4 - 1}{4} \int \sin^{4-2} x \, dx \\ = -\frac{1}{4} \cos^3 \cos x \\ + \frac{3}{4} \int \sin^2 x \, dx \\ = -\frac{1}{4} \cos^3 \cos x \\ + \frac{3}{4} \int (1 - \cos^2 x) \, dx \\ = -\frac{1}{4} \cos^3 \cos x \\ + \frac{3}{4} \int 1 \, dx \\ - \frac{3}{4} \int \cos^2 x \, dx \end{cases}$$

Essentially the same computation gives the recursive formula:

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} \sin x + \frac{n - 1}{n} \int \cos^{n-2} x \, dx$$

which can be applied repeatedly to compute $\int \cos^n x \, dx$ for any positive integer n.

(b) Describe a method for computing $\int \sin^m \cos^n x \, dx$ where m and n are positive integers.

Hint: Prefer to use u to be a sine, and du to be $\cos^n x \, dx$ (if possible). If it is impossible, use the trick that eventually substitutes $u = \sin x$ and $du = \cos x \, dx$.

Hint: Prefer to use u to be a cosine, and du to be $\sin^m x \, dx$ (if possible).

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Hide Solution

Your Submission Solution:

Solution: If either m or n is even, then we can use the identities $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$ to derive a recursive substitution. For example:

$$\int \sin^2 \cos^4 x \, dx = \begin{cases} \int \sin^2 \cos^2 x \cos^2 x \, dx \\ = \int \sin^2 \cos^2 x (1 - \sin^2 x) \, dx \\ = \int \sin^2 \cos^2 x \, dx \\ - \int \sin^4 \cos^2 x \, dx \end{cases}$$

If both m and n are even, then we can use the substitutions:

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

to reduce the powers by a factor of 2. Either either repeat or use the "u is a cos" technique described above.

5.1.4.

Evaluate $\int \frac{1}{x^2 - 1} \, dx$.

Hint: Start with partial fraction decomposition.

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Hide Solution

Your Submission Solution:

Solution: We use the partial fraction decomposition:

$$\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

We further decompose:

$$\int \frac{1}{x^2 - 1} \, dx = \begin{cases} \int \left(\frac{1}{2(x - 1)} \right. \\ \left. - \frac{1}{2(x + 1)} \right) \, dx \\ = \int \left(\frac{1}{2(x - 1)} \right) \, dx \\ - \int \left(\frac{1}{2(x + 1)} \right) \, dx \\ = \frac{1}{2} \int \left(\frac{1}{x - 1} \right) \, dx \\ - \frac{1}{2} \int \left(\frac{1}{x + 1} \right) \, dx \\ = \frac{1}{2} \ln|x - 1| \\ - \frac{1}{2} \ln|x + 1| \\ = \frac{1}{2} \ln\left| \frac{x - 1}{x + 1} \right| \end{cases}$$

5.1.5.

Evaluate $\int \sec^3 x \, dx$.

Hint: Use integration by parts.

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Hide Solution

Your Submission Solution:

Solution: Let $u = \sec^{-1}$, $dv = \sec x \, dx$ so $du = \sec x \tan x \, dx$ and $v = \tan x$. Then:

$$\int \sec^3 x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = \sec^{-1} \tan x \\ - \int \sec x \tan^2 x \, dx \end{cases}$$

Then we apply integration by parts again, using $u = \sec x$ and $dv = \tan^2 x \, dx$ so that $du = \sec x \tan x \, dx$ and $v = \int \tan^2 x \, dx = \tan x - x$. We have:

$$\int \sec x \tan^2 x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = \sec^{-1} \tan x \\ - \int \sec x \tan x \, dx \end{cases}$$

Substituting this for the integral above yields:

$$\int \sec^3 x \, dx = \begin{cases} \sec^{-1} \tan x \\ - \left(\sec^{-1} \tan x \right. \\ \left. - \int \sec x \, dx \right) \\ = \int \sec x \, dx \end{cases}$$

Evaluating the last integral, we get:

$$\int \sec^3 x \, dx = \begin{cases} \int \sec x \, dx \\ = \ln|\sec x + \tan x| \\ + C \end{cases}$$

5.1.6.

Evaluate $\int \frac{1}{\sqrt{x^2 + 1}} \, dx$.

Hint: Use a trig substitution.

Type your text here and press enter to preview. Save

Hide Solution

Your Submission Solution:

Solution: Let $u = \sec^{-1} x$, $du = \sec x \tan x \, dx$ and $v = \tan x$. Then:

$$\int \sec^3 x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = \sec^{-1} \tan x \\ - \int \sec x \tan^2 x \, dx \end{cases}$$

Then we apply integration by parts again, using $u = \sec x$ and $dv = \tan^2 x \, dx$ so that $du = \sec x \tan x \, dx$ and $v = \int \tan^2 x \, dx = \tan x - x$. We have:

$$\int \sec x \tan^2 x \, dx = \begin{cases} \int u \, dv \\ = uv - \int v \, du \\ = \sec^{-1} \tan x \\ - \int \sec x \tan x \, dx \end{cases}$$

Substituting this for the integral above, yields:

$$\int \sec^3 x \, dx = \begin{cases} \sec^{-1} \tan x \\ - \left(\sec^{-1} \tan x \right. \\ \left. - \int \sec x \, dx \right) \\ = \int \sec x \, dx \end{cases}$$

Evaluating the last integral, we get:

$$\int \sec^3 x \, dx = \begin{cases} \int \sec x \, dx \\ = \ln|\sec x + \tan x| \\ + C \end{cases}$$

5.4 Applications of the definite integral

The definite integral has many different applications. Although we will present a number of different types of applications, we will not do very many examples of each application. Once you understand the concept, the "examples" are primarily just practice computing antiderivatives, at which you should by now be somewhat proficient.

5.4.1 Areas of regions in the plane

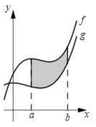
We defined the definite integral in order to compute the area of a very specific region of the plane: the integral $\int_a^b f$ equals the area of the region below the curve $y = f(x)$, above the x-axis, and between the lines $x = a$ and $x = b$. But the definite integral is a flexible instrument, and we can use it to compute areas of other regions of the plane. For example:

Problem 5.36

Let f and g be continuous functions on $[a, b]$, with $f(x) \geq g(x) \geq 0$ for all $x \in [a, b]$. What is the area between the graphs of f and g and the lines $x = a$ and $x = b$?

Solution for Problem 5.36: We can see what's going on by looking at a picture such as the one to the right. The area under f is $\int_a^b f$, and the area under g is $\int_a^b g$. So the area between the two curves is their difference, given by

$$\int_a^b (f - g).$$



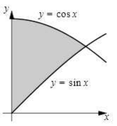
□

Often, we will have to do a little bit of work to determine the proper definite integral, as in the following problem:

Problem 5.37

Find the area of the region in the first quadrant bounded by the graph of $y = \cos x$, the graph of $y = \sin x$, and the y-axis.

Solution for Problem 5.37: We can see from the picture that we want the area between $\cos x$ and $\sin x$. However, we still need to determine the limits of integration. It is clear from the picture that the lower limit of integration is $x = 0$. The upper limit is the smallest positive value of x at which $\sin x = \cos x$. This means that $\tan x = 1$, so $x = \frac{\pi}{4}$.



Thus, the area is given by the definite integral

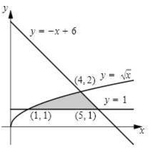
$$\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} = \sqrt{2} - 1.$$

□

Problem 5.38

Find the area of the region bounded by the curves $y = 1$, $y = -x + 6$, and $y = \sqrt{x}$.

Solution for Problem 5.38: We can start by sketching a picture of the region, as shown to the right. In order to work with the region, it's necessary to determine the intersection points of the graphs. Some easy algebra will verify that the intersection points are as labeled in the picture. So our region ranges from $x = 1$ to $x = 5$. However, the shaded region is not the region between two graphs, as in our previous examples.



We can break up our desired region into two smaller regions. We note that for $x \in [1, 4]$, we have the region between $y = \sqrt{x}$ and $y = 1$. Similarly, for $x \in [4, 5]$, we have the region between $y = -x + 6$ and $y = 1$. The sum of these two subregions will give us our entire region.

Thus, the area that we want is

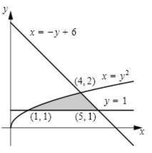
$$\int_1^4 (\sqrt{x} - 1)\, dx + \int_4^5 (-x + 6 - 1)\, dx.$$

The first integral gives us

$$\left(\frac{2}{3}x^{3/2} - x\right)\Big|_1^4 = \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 1\right) = \frac{5}{3}.$$

We could also evaluate the other integral using calculus, but we could instead just observe that it is an isosceles right triangle with legs 1, and thus has area $\frac{1}{2}$. Therefore, the total area is $\frac{5}{3} + \frac{1}{2} = \frac{13}{6}$.

A slightly more elegant solution is to reverse the roles of x and y. That is, we think of the curves as graphs of functions of x in terms of y, as labeled in the picture to the right. We now see that the region that we want is just the region between the graphs of $x = -y + 6$ and $x = y^2$ in the interval from $y = 1$ to $y = 2$. Thus, we have the definite integral



$$\begin{aligned}\int_1^2 (6 - y - y^2)\, dy &= \left(6y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_1^2 \\ &= \left(12 - 2 - \frac{8}{3}\right) - \left(6 - \frac{1}{2} - \frac{1}{3}\right) \\ &= \frac{22}{3} - \frac{31}{6} \\ &= \frac{13}{6},\end{aligned}$$

as before. □

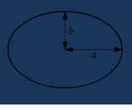
This is an important technique to keep in mind. It is sometimes easier to think of x as a function of y, rather than the usual y as a function of x. Of course, we can only do this if the function is invertible, or if we can restrict the domain suitably to where the function is 1-to-1.

Concept:	When we are working with invertible functions, it may be easier to think of them as functions of y instead of functions of x.

Here's a harder area-computation example where we use a definite integral:

Problem 5.39

Determine (with proof) the area of an ellipse with semimajor axis a and semiminor axis b. (**Semimajor axis** means half the distance in the longer direction, and **semiminor axis** means half the distance in the shorter direction, as in the picture to the right.)



Solution for Problem 5.39: We can make this calculation simplest if we center the ellipse at the origin of the plane. Then, the ellipse passes through the points $(\pm a, 0)$ and $(0, \pm b)$, and is given implicitly by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

However, we can't directly integrate an implicit equation, so we solve for y in terms of x:

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}.$$

This is not a function (since it produces two values of y for each value of $x \in (-a, a)$), but we get the top half of the ellipse if we choose the "+" sign (and the bottom half if we choose the "−" sign). Thus, the area of top half of the ellipse is the area under the curve

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

on the interval $[-a, a]$, and thus the area of the whole ellipse is twice this area. Therefore, the area is equal to

$$2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\, dx.$$

All we have left to do is to compute this integral. The form of this function strongly suggests a trig substitution. Indeed, we can use the substitution $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$. But we also have to change the limits of integration! When $x = -a$, we have $\theta = -\pi/2$, and when $x = a$, we have $\theta = \pi/2$, so the interval $[-a, a]$ in terms of x becomes the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in terms of θ.

Therefore, the area of the ellipse is equal to

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b\sqrt{1 - \frac{a^2\sin^2\theta}{a^2}}\,(a\cos\theta)\, d\theta.$$

The expression under the square root is just $1 - \sin^2\theta = \cos^2\theta$, and for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the cosine function is nonnegative, so the square root term is just $\cos\theta$, and we have

$$2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta.$$

(We have moved the constants a and b to the front of the integral.) At this point, there are a number of different ways that we could finish the problem. One method is to use the trig identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, making our integral

$$ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta.$$

(Note the $\frac{1}{2}$ cancels with the 2 that used to be out front.) This can be broken up into a sum of definite integrals:

$$ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, d\theta + ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta\, d\theta.$$

The first definite integral is just π (since the length of the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is π), so the first term is πab. We could evaluate the second integral using the Fundamental Theorem:

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta\, d\theta &= \frac{1}{2}\sin 2\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{1}{2}(\sin(\pi) - \sin(-\pi)) \\ &= \frac{1}{2}(0 - 0) \\ &= 0.\end{aligned}$$

But we didn't need to compute it explicitly to see that it would be 0. Instead, we notice that the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is an entire period of the function $\cos 2\theta$ (which is periodic with period π). Looking at the graph of $y = \cos 2x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we see that the area under the graph of the part above the x-axis will exactly cancel with the area under the graph of the part below the x-axis, leaving a net area of 0.

Thus, the area of the ellipse is πab. Note that if our ellipse satisfies $a = b = r$, then the ellipse is in fact a circle of radius r, and we get our usual area of πr^2. □

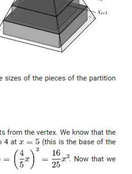
5.4.2 Volumes

Although we defined the definite integral to compute areas, we can go up one dimension and use them to compute certain volumes as well. Recall that, for computing areas, we constructed a Darboux (or Riemann) sum by breaking up our region into a sum of rectangles, where we used a partition of an interval to determine widths and a function to determine heights. For volumes, we'll form a Riemann sum by breaking up a three-dimensional solid into a sum of slices, where we'll again use a partition of an interval to determine the widths of the slices, but where a function will determine the cross-sectional area of each slice.

Let's begin with an example for which you probably already know the answer.

Problem 5.40

Determine the volume of a pyramid of height 5 with a square base of side length 4.



Solution for Problem 5.40: We'll try a 3-dimensional version of the same strategy that we used to find area under a curve. Specifically, we'll approximate the volume of the pyramid by breaking it up into a sum of rectangular 3-dimensional solids (called **cuboids**), as shown to the right. We partition the height interval $[0, 5]$ into

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 5.$$

At distance x_i from the vertex, we will construct a cuboid whose base is a square of area $A(x_i)$ (where A is a function that we haven't yet determined) and whose height is $x_{i+1} - x_i$ as shown in the picture to the right. This cuboid thus has volume $A(x_i)(x_{i+1} - x_i)$. If we sum the volumes of the cuboid over our entire partition, we have

$$\text{Volume} \approx \sum_{i=0}^{n-1} A(x_i)(x_{i+1} - x_i).$$

This is a lower Darboux sum of the function $A(x)$ on the interval $[0, 5]$, so if we take the limit as the sizes of the pieces of the partition become arbitrarily small, we get

$$\text{Volume} = \int_0^5 A(x)\,dx.$$

What remains is to determine the function $A(x)$ that gives the area of the cross-section that is x units from the vertex. We know that the side length of the cross-section varies linearly, from 0 at $x = 0$ (this is the vertex of the pyramid) to 4 at $x = 5$ (this is the base of the pyramid). Thus, the side length of the cross-section at height x is $\frac{4}{5}x$. Therefore, the area is $A(x) = \left(\frac{4}{5}x\right)^2 = \frac{16}{25}x^2$. Now that we know $A(x)$, we can evaluate the definite integral and compute the volume:

$$\begin{aligned}\text{Volume} &= \int_0^5 A(x)\,dx = \int_0^5 \frac{16}{25}x^2\,dx = \frac{16}{75}x^3\Big|_0^5 = \frac{16}{75}(5^3) = \frac{80}{3}.\end{aligned}$$

□

The technique of Problem 5.40 can easily be generalized to an arbitrary pyramid.

Problem 5.41

Compute the volume of a pyramid (or cone) with a base of area b and height h.

Solution for Problem 5.41: Note that we're not assuming anything about the shape of the base. Our only basic assumption is that the cross-sectional area is proportional to the square of the distance from the vertex. In particular, let $A(x)$ denote the area of the cross-sectional slice that is x units from the vertex of the pyramid; we are given that $A(0) = 0$ and $A(h) = b$. We note that each cross section is similar to the base, which means that the cross-sectional area that is x units from the base satisfies the proportionality equation

$$\left(\frac{x}{h}\right)^2 = \frac{A(x)}{A(h)} = \frac{A(x)}{b},$$

because the square of the ratio of lengths gives us the ratio of corresponding areas. Thus

$$A(x) = b\left(\frac{x}{h}\right)^2.$$

We will use these cross-sectional areas to estimate the volume of the pyramid. Just as we did in Problem 5.40, we'll start with a partition of the interval $[0, h]$:

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = h.$$

We break up the pyramid into prisms, where each prism has a height given by the size of the corresponding piece of the partition, so the height of the ith prism (for $0 \le i < n$) is $x_{i+1} - x_i$. The area of the base of this prism is given by $A(x_i)$, the cross-sectional area of the pyramid at height x_i. Thus, the volume of the ith prism is

$$\begin{aligned}\text{Volume of } i^{\text{th}} \text{ prism} &= (\text{Area of base}) \cdot (\text{Height}) \\ &= A(x_i)(x_{i+1} - x_i) \\ &= b\left(\frac{x_i}{h}\right)^2 (x_{i+1} - x_i).\end{aligned}$$

Summing these volumes over our partition gives

$$\begin{aligned}\text{Sum of volumes} &= \sum_{i=0}^{n-1} A(x_i)(x_{i+1} - x_i) \\ &= \sum_{i=0}^{n-1} b\left(\frac{x_i}{h}\right)^2 (x_{i+1} - x_i).\end{aligned}$$

But this sum is exactly a lower Darboux sum for the definite integral of the function $A(x)$ on the interval $[0, h]$. Therefore, as the pieces of the partition get smaller and smaller, the volume approaches the definite integral

$$\int_0^h A(x)\,dx = \int_0^h b\left(\frac{x}{h}\right)^2 dx.$$

We can easily evaluate this integral to get our volume:

$$\int_0^h b\left(\frac{x}{h}\right)^2 dx = \frac{b}{h^2}\int_0^h x^2\,dx = \frac{b}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}bh.$$

□

We can use the method of Problem 5.41 to compute volumes of more general regions. The only essential property that we used was the existence of a function $A(x)$ for the cross-sectional area at x. Thus, we can generalize:

Problem 5.42

Suppose that we have a three-dimensional solid bounded by the planes $x = a$ and $x = b$, such that $A(x)$ is the area of the cross-section at x. What is the volume of the solid?

Solution for Problem 5.42: We can divide the interval $[a, b]$ into a partition

$$a = x_0 < x_1 < \cdots < x_n = b.$$

We'll use this partition to break up our solid into smaller prisms, where the ith prism has its bases in the planes $x = x_i$ and $x = x_{i+1}$. If we choose a point $w_i \in [x_i, x_{i+1}]$ then we can approximate the volume of the piece of the solid lying within the interval $[x_i, x_{i+1}]$ as the volume of the prism with height $x_{i+1} - x_i$ and cross-sectional area $A(w_i)$. Thus, the volume of the solid is approximated by

$$\sum_{i=0}^{n-1} (x_{i+1} - x_i)A(w_i).$$

But this is a Riemann sum! As we let the sizes of the pieces of the partition get smaller and smaller, this Riemann sum approaches a definite integral. Thus, the volume of our region is given by the definite integral

$$\int_a^b A(x)\,dx.$$

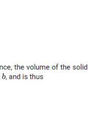
□

> **Important:** The volume of a solid between the planes $x = a$ and $x = b$, with cross-sectional area given by the function $A(x)$, is
>
> $$\int_a^b A(x)\,dx.$$
>
> This should be thought of as "summing the areas" over the interval $[a, b]$

This method of computing a volume is sometimes called **slicing**. We are essentially summing up the areas of cross-sectional slices. This technique is very flexible and can be used in a variety of different volume computations. For example:

Problem 5.43

Find the volume of a sphere of radius r.



Solution for Problem 5.43: We can think of the sphere as sitting with its center at the origin of three-dimension space. As x varies from $-r$ to r, the cross-section of the sphere is a circle with radius $\sqrt{r^2 - x^2}$, and thus has area $\pi(r^2 - x^2)$. Thus, the volume of the sphere is given by the definite integral

$$\begin{aligned}\int_{-r}^{r} \pi(r^2 - x^2)\,dx &= \pi\left(r^2x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} \\ &= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right) \\ &= \frac{4}{3}\pi r^3.\end{aligned}$$

□

There are a couple of common scenarios in which we use the graph of a function to get a three-dimensional solid.

Problem 5.44

Suppose that we start with the graph of $y = f(x)$ between $[a, b]$. We rotate this graph, in three dimensions, around the x-axis to get the surface of a three-dimensional solid. Find the volume of this **solid of revolution**.

Solution for Problem 5.44: The cross-section of the solid at any given value of x is a circle of radius $f(x)$. Thus, the cross-section has area $\pi(f(x))^2$. Therefore, using our general principle of "summing the areas" to get volume, we conclude that the volume is given by the definite integral

$$\pi\int_a^b (f(x))^2\,dx.$$

□

> **Important:** The volume of the **solid of revolution** given by revolving the graph of $y = f(x)$, on the interval $[a, b]$ around the x-axis is
>
> $$\pi\int_a^b (f(x))^2\,dx.$$

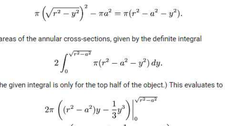
> **Concept:** Most calculus courses teach the formula for the volume of a solid of revolution as a formula to be memorized. However, if you understand this formula as just a special case of the general slicing method, then you won't need to memorize it–it should be obvious to you!

There is another commonly-used type of rotation that results in a solid of revolution:

Problem 5.45

Suppose that we start with the graph of $y = f(x)$ between $[a, b]$ where $0 < a < b$. We rotate the region under graph around the y-axis to get a three-dimensional solid. Find the volume of this solid.

Solution for Problem 5.45: The solid is a bit difficult to visualize–an example is shown in the first image below. The y-axis is the axis of rotation, and passes through the hollow center of the solid. A "cross-section" of this object, for a specific value of x, is the surface of a cylinder. Two such cylinders are in the second image below.



At each value of x, the cylinder has circumference $2\pi x$ and height $f(x)$, so it has surface area $2\pi x f(x)$. Hence, the volume of the solid (as x ranges from a to b) equals the definite integral of the cross-sectional surface areas as x ranges from a to b, and is thus

$$2\pi\int_a^b x f(x)\,dx.$$

□

This method of computing volume is sometimes called the **cylindrical shell** method.

> **Important:** The volume of the **solid of revolution** given by revolving the area under the graph of $y = f(x)$ on the interval $[a, b]$ around the y-axis is
>
> $$2\pi\int_a^b x f(x)\,dx.$$

Problem 5.46

A hole of radius a is drilled through the center of a solid sphere of radius r (with $a < r$). Find the volume of the remaining object.

Solution for Problem 5.46: We can imagine our sphere with the hole cut along the y-axis, as shown in the picture to the right. The top edge of the hole produces a right triangle with hypotenuse r (the radius of the sphere) and leg a (the radius of the hole), so the distance from the center of the sphere to the top edge of the hole is $\sqrt{r^2 - a^2}$. Thus, our integral will sum the areas of the cross-sectional slices of the solid, perpendicular to the direction of the hole, from $-\sqrt{r^2 - a^2}$ to $\sqrt{r^2 - a^2}$ (where this quantity measures the distance of the cross-section from the center of the sphere). We can make the calculation slightly simpler by noticing that the top half of the object has the same volume as the bottom half of the object, so the total volume is just twice the volume from $y = 0$ to $y = \sqrt{r^2 - a^2}$.

To find the cross-sectional area of the slice at height y, we observe that this cross-section is an **annulus**: a circle with a smaller circle removed. The smaller circle has radius a. The radius of the larger circle is the leg of a right triangle with hypotenuse r and other leg y, so its radius is $\sqrt{r^2 - y^2}$. Thus, the area of the annular cross-section at height y is the difference in the areas of the two circles, which is:

$$\pi\left(\sqrt{r^2 - y^2}\right)^2 - \pi a^2 = \pi(r^2 - a^2 - y^2).$$

The total volume is the sum of the areas of the annular cross-sections, given by the definite integral

$$2\int_0^{\sqrt{r^2-a^2}} \pi(r^2 - a^2 - y^2)\,dy.$$

(Don't forget the factor of 2, since the given integral is only for the top half of the object.) This evaluates to

$$\begin{aligned}&2\pi\left((r^2 - a^2)y - \frac{1}{3}y^3\right)\Bigg|_0^{\sqrt{r^2-a^2}} \\ &= 2\pi\left((r^2 - a^2)^{\frac{3}{2}} - \frac{1}{3}(r^2 - a^2)^{\frac{3}{2}}\right) \\ &= \frac{4}{3}\pi(r^2 - a^2)^{\frac{3}{2}}.\end{aligned}$$

As a check, note that when $a = 0$, we get the entire sphere's volume of $\frac{4}{3}\pi r^3$, and when $a = r$, we get 0, as expected. □

> **Concept:** All of these volume methods are essentially the same. They all depend on the idea that to compute volume, we sum the areas of cross-sections along an interval. So you don't need to memorize any of the "special" volume formulas, since they all derive from the same basic principle.

5.4.3 Length of a curve

Let's now look at the problem of computing the length of the graph of $y = f(x)$ on the interval $[a, b]$. If the curve is sufficiently simple, then this is easy. For example:

Problem 5.47

Compute the length of the graph of $y = 2x - 3$ on the interval $[1, 4]$.

Solution for Problem 5.47: This curve is just the line segment from $(1, -1)$ to $(4, 5)$, which has length

$$\sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}.$$

□

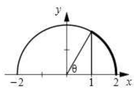
Problem 5.48

Compute the length of the graph of $y = \sqrt{4 - x^2}$ on the interval $[1, 2]$.

Solution for Problem 5.48: The domain of the function is $[-2, 2]$, and the entire graph (for $x \in [-2, 2]$) is a semicircle of radius 2. The circumference of the semicircle is 2π. This might lead to a quick "solution":

> **Bogus Solution:** The interval in question, $[1, 2]$, is $\frac{1}{4}$ of the entire domain, so the length of the curve on $[1, 2]$ is $\frac{1}{4}$ of the total length on $[-2, 2]$, or $\frac{1}{4}(2\pi) = \frac{\pi}{2}$.

This is not correct! Just because $[1, 2]$ is one-quarter of the domain does not mean we get $\frac{1}{4}$ of the total length of the curve. If we look more carefully at the diagram at right, we will see the correct procedure. The section of the circle from $x = 1$ to $x = 2$ is the arc subtended by the angle θ as shown in the diagram. This angle satisfies $\cos\theta = \frac{1}{2}$ (using the triangle shown), so $\theta = \frac{\pi}{3}$. This implies that the arc is $\frac{1}{3}$ of the complete semicircle. Thus, the correct answer is that the length for $x \in [1, 2]$ is $\frac{1}{3}$ of the circumference of the semicircle, or $\frac{2\pi}{3}$. □



In general, it is hard to compute the length of an arbitrary curve. But we know that computing the lengths of line segments is really easy, as in Problem 5.47. Thus, our strategy will be to break up the curve into small pieces, approximate each small piece of the curve with a line segment, measure the lengths of the line segments, and sum these segment lengths.

Problem 5.49

Let f be a differentiable function on $[a, b]$. We wish to determine the length of the curve $y = f(x)$ from $x = a$ to $x = b$.

(a) Partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_n = b.$$

What quantity approximates the length from $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$? Sum these approximate lengths to get an expression that approximates the length of the curve.

(b) We'd like the sum from part (a) to look like a Riemann sum of the form

$$\sum_{i=0}^{n-1} g(w_i)(x_{i+1} - x_i)$$

for some function g and some $w_i \in [x_i, x_{i+1}]$. How can we manipulate our sum to be in this form?

(c) Write a definite integral for the length of $y = f(x)$ from $x = a$ to $x = b$.

Solution for Problem 5.49:

(a) As usual, we'll start by partitioning the interval $[a, b]$ into smaller pieces:

$$a = x_0 < x_1 < \cdots < x_n = b.$$

For each interval $[x_i, x_{i+1}]$ with $0 \le i < n$, a close approximation to the curve is the straight line between $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$. Its length is

$$\sqrt{(f(x_{i+1}) - f(x_i))^2 + (x_{i+1} - x_i)^2}.$$

Thus, our total arc length can be approximated as a sum:

$$\sum_{i=0}^{n-1} \sqrt{(f(x_{i+1}) - f(x_i))^2 + (x_{i+1} - x_i)^2}.$$

(b) Unfortunately, our sum above doesn't exactly look like a Riemann sum, so it's not immediately clear that this quantity becomes a definite integral as we let the partition pieces get smaller and smaller.

Here's the trick: what do we know about the difference of $f(x_{i+1})$ and $f(x_i)$? How can we approximate it or otherwise express it? We can use the Mean Value Theorem! Specifically, there is a point $w_i \in [x_i, x_{i+1}]$ such that

$$f'(w_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i},$$

which gives

$$(x_{i+1} - x_i)f'(w_i) = f(x_{i+1}) - f(x_i).$$

We can substitute this into our sum:

$$\begin{aligned}&\sum_{i=0}^{n-1} \sqrt{(f(x_{i+1}) - f(x_i))^2 + (x_{i+1} - x_i)^2}\\ &= \sum_{i=0}^{n-1} \sqrt{(f'(w_i)(x_{i+1} - x_i))^2 + (x_{i+1} - x_i)^2}.\end{aligned}$$

We factor the $(x_{i+1} - x_i)$ term out of the square root:

$$\sum_{i=0}^{n-1} \left(\sqrt{(f'(w_i))^2 + 1}\right)(x_{i+1} - x_i).$$

Now it looks exactly like a Riemann sum!

(c) The expression above is a Riemann sum of the function $\sqrt{(f'(x))^2 + 1}$ along the interval $[a, b]$. Thus, as the partition widths approach 0, the sum becomes a definite integral. Therefore, the length of the curve $y = f(x)$ along the interval $[a, b]$ is

$$\int_a^b \sqrt{(f'(x))^2 + 1}\, dx.$$

□

> **Important:** The length of the graph of $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by
>
> $$\int_a^b \sqrt{(f'(x))^2 + 1}\, dx.$$

Let's go back to our circle example from Problem 5.48 and verify that this gives what we expect:

Problem 5.50

Use a definite integral to compute the length of the graph of $y = \sqrt{4 - x^2}$ on the interval $[1, 2]$.

Solution for Problem 5.50: Let $f(x) = \sqrt{4 - x^2}$. Then $f'(x) = -\frac{x}{\sqrt{4 - x^2}}$, and the length is given by the integral

$$\int_1^2 \sqrt{\left(-\frac{x}{\sqrt{4 - x^2}}\right)^2 + 1}\, dx.$$

This simplifies to

$$\int_1^2 \sqrt{\frac{x^2}{4 - x^2} + 1}\, dx$$

Writing the integrand with a common denominator, we get:

$$\int_1^2 \sqrt{\frac{4}{4 - x^2}}\, dx = \int_1^2 \sqrt{\frac{1}{1 - \left(\frac{x}{2}\right)^2}}\, dx.$$

The antiderivative of $\frac{1}{\sqrt{1 - u^2}}$ is $\sin^{-1} u$, so our definite integral is

$$\begin{aligned}\int_1^2 \sqrt{\frac{1}{1 - \left(\frac{x}{2}\right)^2}}\, dx &= 2\sin^{-1}\left(\frac{x}{2}\right)\Big|_1^2\\ &= 2\left(\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right)\\ &= 2\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\\ &= \frac{2\pi}{3}.\end{aligned}$$

□

Unfortunately, for most functions f, the integral $\int_a^b \sqrt{(f'(x))^2 + 1}\, dx$ is rather difficult to compute. This indicates that arc length is often a difficult quantity to deal with (although it can be analyzed using the integral approximation techniques that we will study in Section 5.5).

5.4.4 Average value of a function

A key way to think of a definite integral is as a "continuous sum." (That's why the integral sign looks like an elongated letter S.) We have already used this interpretation a number of times–for instance, to compute volume as a sum of cross-sectional areas, or to compute arc length as a sum of lengths of line segments. This general viewpoint of "a definite integral as a sum" gives us some other uses for the definite integral. For example, we know that when we have discrete data–that is, we have n data points for some positive integer n–then the average value is the sum of the values of the data points divided by the number of data points. We can use the definite integral as a sum to extend this to an average of continuous data:

Problem 5.51

Suppose f is a continuous function defined on an interval $[a, b]$ (with $a < b$). What expression could be interpreted as the average value of f on $[a, b]$?

Solution for Problem 5.51: For a continuous function f, we think of the definite integral as performing a "sum" of the values of the function, and we use the length of the interval as the analog of the "number of data points" that we must divide by to get the average. So our candidate for the average value of f on $[a, b]$ is:

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

We can see why this makes sense by thinking in terms of a Riemann sum. As usual, suppose that we partition the interval $[a, b]$ into n pieces:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and for simplicity, suppose that the pieces are equal sized, so that $x_{i+1} - x_i = \dfrac{b-a}{n}$ for all $0 \le i < n$. We can approximate the average value of f by choosing some $w_i \in [x_i, x_{i+1}]$ and then averaging (in the usual discrete sense) the values $f(w_i)$. This gives us

$$\text{Average of } f \approx \frac{1}{n}\sum_{i=0}^{n-1} f(w_i).$$

Let's make this look more like a Riemann sum by multiplying inside the sum by $(x_{i+1} - x_i)$, and dividing outside the sum by the same quantity, which we assumed was equal to $\dfrac{b-a}{n}$ for all i:

$$\text{Average of } f \approx \frac{n}{b-a}\cdot\frac{1}{n}\sum_{i=0}^{n-1}(x_{i+1} - x_i)f(w_i).$$

As the widths of the partition get smaller, the Riemann sum approaches a definite integral, and the n's outside the sum cancel; therefore we are justified in writing

$$\text{Average of } f = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Geometrically, you can think of the average of f as the "average" height of the region under f. That is, taking the area of a rectangle with base $[a, b]$ and height the average value of f gives us:

$$\begin{aligned}\text{Area} &= (b-a)\cdot(\text{Average value of } f)\\ &= (b-a)\cdot\frac{1}{b-a}\int_a^b f(x)\,dx\\ &= \int_a^b f(x)\,dx,\end{aligned}$$

which is the area under f on $[a, b]$. □

In summary, we have

Definition: The **average value** of a continuous function f along an interval $[a, b]$ (with $a < b$) is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Here's a simple application:

Problem 5.52

The temperature at time t (given in hours from 0 to 24 after midnight) in downtown Aopsville is given by $T = 10 - 5\sin\left(\dfrac{t\pi}{12}\right)$ (degrees Celsius). What is the average temperature between noon and midnight?

Solution for Problem 5.52: By definition, the quantity we seek is

$$\frac{1}{12}\int_{12}^{24}\left(10 - 5\sin\left(\frac{t\pi}{12}\right)\right)dt.$$

This gives us

$$\frac{1}{12}\left(10t + \frac{12}{\pi}5\cos\left(\frac{t\pi}{12}\right)\right)\Bigg|_{12}^{24}.$$

This is

$$10 + \frac{5}{\pi}(\cos 2\pi - \cos\pi) = 10 + \frac{10}{\pi} \approx 13.18.$$

This is reasonable: at noon the temperature is 10, at 6:00 it is 15, and at midnight it is 10 again. But the temperature stays closer to 15 for a longer period of time. So an average of 13.18 over the time period makes sense. □

Note that, in Problem 5.52, if we had asked for the average temperature between 6 a.m. and 6 p.m., we wouldn't need to write an integral: the symmetry of the temperature function about $t = 12$ shows clearly that the average is 10.

Closely related to the idea of average value is another interpretation of the definite integral that goes back to the Fundamental Theorem of Calculus. If f is differentiable, then for any $a \le b$ we can write

$$f(b) = f(a) + \int_a^b f'(t)\,dt.$$

That is, the definite integral of f' sums up the accumulated rate of change of f. In particular, when we rearrange this and divide by $b - a$ (assuming that $a < b$), we get:

$$\frac{f(b) - f(a)}{b-a} = \frac{1}{b-a}\int_a^b f'(t)\,dt = \text{average value of } f'(x) \text{ on } [a, b]$$

That is, $\dfrac{f(b) - f(a)}{b-a}$ is equal to the average rate of change of the function on $[a, b]$, as we expect.

Concept: All of the interpretations of the definite integral that we have discussed are interrelated! They all follow from the general idea of the definite integral as a continuous sum.

5.4.5 Exercises

Exercises

5.4.1:

Find the areas of the following regions:

(a) The bounded region between $y = x^2$ and $y = x^3$.

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Your Submission: Solution

Solution: These two curves intersect at $(0, 0)$ and $(1, 1)$. On $[0, 1]$ we have $x^2 > x^3$, so we integrate $x^2 - x^3$ from 0 to 1:

$$\int_0^1 (x^2 - x^3)\,dx = \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

(b) The bounded region between $y = \log x$ and the line segment connecting $(e, 1)$ and $(e^2, 2)$.

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Solution: The area in question is the area under $y = \log x$ on the interval $[e, e^2]$ minus the area of the trapezoid bordered by $x = e$, $x = e^2$, the line segment from $(e, 1)$ to $(e^2, 2)$, and the x-axis. The area under the curve is

$$\begin{aligned}\int_e^{e^2} \log x\,dx &= (x\log x - x)\Big|_e^{e^2} \\ &= (e^2\log e^2 - e^2) - (e\log e - e) \\ &= e^2.\end{aligned}$$

The area of the trapezoid is the average of the lengths of the parallel sides $\frac{1}{2}(1+2)$ times the distance between the parallel sides $(e^2 - e)$, thus its area is $\frac{3}{2}(e^2 - e)$. Therefore, the area of the desired region is

$$e^2 - \frac{3}{2}(e^2 - e) = \boxed{\frac{3}{2}e - \frac{1}{2}e^2}.$$

(c) The region bounded by the y-axis, $y = x^2$, and $y = \cos x$. Express the area in terms of the positive constant α such that $\cos\alpha = \alpha^2$.

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Solution: Note that $(\alpha, \alpha^2) = (\alpha, \cos\alpha)$ is the first intersection point of $y = x^2$ and $y = \cos x$, and have $\cos x > x^2$ for $x < \alpha$. Therefore, our desired integral is

$$\int_0^\alpha (\cos x - x^2)\,dx = \left(\sin x - \frac{1}{3}x^3\right)\Big|_0^\alpha = \sin\alpha - \frac{1}{3}\alpha^3.$$

But $\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \alpha^4}$, so the answer is $\boxed{\sqrt{1-\alpha^4} - \frac{1}{3}\alpha^3}$.

5.4.2:

Find the volumes of the following:

(a) The region enclosed by the surface resulting when the curve $y = x^3$ on $[0, 2]$ is rotated about the x-axis.

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Solution: A cross-section is a circle with radius x^3, so has area $\pi(x^3)^2 = \pi x^6$. Thus the volume is

$$\pi\int_0^2 x^6\,dx = \frac{\pi}{7}x^7\Big|_0^2 = \boxed{\frac{128\pi}{7}}.$$

(b) The solid consisting of the region under the curve $y = x^3$ along $[0, 2]$, rotated about the y-axis.

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Solution: The solid consists of cylindrical shells of surface area $2\pi x f(x) = 2\pi x^4$. Thus the volume is

$$2\pi\int_0^2 x^4\,dx = \frac{2}{5}\pi x^5\Big|_0^2 = \boxed{\frac{64\pi}{5}}.$$

(c) The region enclosed by the surface resulting when the curve $y = \cos x$ on $[0, \pi/2]$ is rotated about the x-axis.

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Solution: A cross-section is a circle with radius $\cos x$, so has area $\pi\cos^2 x$. Thus the volume is

$$\pi\int_0^{\pi/2} \cos^2 x\,dx.$$

We know that $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ so

$$\begin{aligned}\pi\int_0^{\pi/2} \cos^2 x\,dx &= \frac{\pi}{2}\int_0^{\pi/2} (1 + \cos(2x))\,dx \\ &= \frac{\pi}{2}\left(x + \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi/2} \\ &= \frac{\pi}{2}\left(\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(\frac{1}{2}\sin 0\right)\right) \\ &= \boxed{\frac{\pi^2}{4}}.\end{aligned}$$

(d) The region $y^2 \le x^3 - 1$ for $x \in [2, 3]$ rotated about the y-axis.

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Solution: The part of the solid above the x-axis is equal to the area under $f(x) = \sqrt{x^3 - 1}$ rotated about the y-axis. The solid consists of cylindrical shells of surface area $2\pi x f(x) = 2\pi x\sqrt{x^3 - 1}$. Thus, when we multiply by 2π get the total volume both above and below the x-axis, we get a total volume of

$$4\pi\int_2^3 x\sqrt{x^3 - 1}\,x.$$

Substituting $u = x^3 - 1$ and $du = 2x\,dx$ yields

$$\begin{aligned}4\pi\int_2^3 x\sqrt{x^3 - 1}\,dx &= 2\pi\int_2^3 \sqrt{u}\,du \\ &= \frac{4}{3}\pi\,u^{\frac{3}{2}}\Big|_2^8 \\ &= \boxed{\pi\left(\frac{64}{3}\sqrt{2} - 4\sqrt{3}\right)}.\end{aligned}$$

(e) The ellipsoid obtained by rotating the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ about the x-axis.

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Solution: A cross-section perpendicular to the x-axis is a circle with radius $y = b\sqrt{1 - \frac{x^2}{a^2}}$. Integrating from $-a$ to a, we get a volume of

$$\begin{aligned}\int_{-a}^a \pi b^2\left(1 - \frac{x^2}{a^2}\right)dx &= \frac{\pi b^2}{a^2}\int_{-a}^a (a^2 - x^2)\,dx \\ &= \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx,\end{aligned}$$

where the last step follows because $a^2 - x^2$ is symmetric with respect to the y-axis. This integral gives the volume as

$$\frac{2\pi b^2}{a^2}\left(a^2x - \frac{1}{3}x^3\right)\Big|_0^a = \boxed{\frac{4\pi}{3}a^2b}.$$

5.4.3:

Find the lengths of the following curves:

(a) $y = \frac{x^4}{20} + \frac{1}{3x^2}$ between $x = 2$ and $x = 4$.

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Solution: First we find $f'(x) = \frac{x^3}{4} - \frac{1}{x^3}$. Then

$$\begin{aligned}1 + (f'(x))^2 &= 1 + \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6} \\ &= \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} \\ &= \left(\frac{x^3}{4} + \frac{1}{x^3}\right).\end{aligned}$$

Thus the length is given by the definite integral

$$\begin{aligned}\int_2^4 \left(\frac{x^3}{4} + \frac{1}{x^3}\right)dx &= \left(\frac{x^4}{20} - \frac{1}{2x^2}\right)\Big|_2^4 \\ &= \left(\frac{1024}{20} - \frac{1}{32}\right) - \left(\frac{32}{20} - \frac{1}{24}\right) \\ &= \frac{248}{5} - \frac{7}{192} \\ &= \boxed{\frac{47651}{960}}.\end{aligned}$$

(b) $y = 3x^{\frac{2}{3}} - 1$ between $(0, -1)$ and $(4, 23)$.

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Your Submission: Solution

Solution: We compute

$$1 + (f'(x))^2 = 1 + \left(\frac{2}{3}\sqrt{x}\right)^2 = 1 + \frac{81x}{4}.$$

Thus the length is

$$\int_0^4 \sqrt{1 + \frac{81x}{4}}\,dx = \frac{4}{81}\cdot\frac{2}{3}\left(1 + \frac{81x}{4}\right)^{\frac{3}{2}}\Big|_0^4 = \boxed{\frac{8}{243}(82^{3/2} - 1)}.$$

5.4.4★:

Compute the volume of a torus of radius a with cross-sectional radius b, with $a > b$. (The **torus** is the donut-shaped object that results when a circle of radius b rotated around a line in the same plane as the circle, where a is the distance between the line and the center of the circle.)

Hint: Start with the equation of a circle of radius b centered at $(a, 0)$, and rotate it about the y-axis using the cylindrical shell method.

Hint: Use the function $y = \sqrt{b^2 - (x - a)^2}$ for the cross-section of the upper half of the torus.

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Your Submission: Solution

Solution: We can construct the torus by rotating, about the y-axis, the interior of a circle of radius b centered at $(a, 0)$. This circle is given by the equation

$$(x - a)^2 + y^2 = b^2.$$

Hence,

$$y = \sqrt{b^2 - (x - a)^2}$$

gives the upper half of a cross-section of the torus. A cylindrical shell thus has surface area $4\pi x\sqrt{b^2 - (x - a)^2}$ (note the factor of 4 rather than 2 because we need the region both above and below the y-axis.) Thus, the volume is

$$4\pi\int_{a-b}^{a+b} x\sqrt{b^2 - (x - a)^2}\,dx.$$

To compute this integral, substitute $u = x - a$, giving

$$4\pi\int_{-b}^{b} (u + a)\sqrt{b^2 - u^2}\,du.$$

Note that $u\sqrt{b^2 - u^2}$ is an odd function, so $\int_{-b}^{b} u\sqrt{b^2 - u^2}\,du = 0$. This just leaves

$$16\pi\int_{-b}^{b} \sqrt{b^2 - u^2}\,du.$$

The integral is just the area of a semicircle of radius b, so it is $\frac{1}{2}\pi b^2$. Hence the volume of the torus is $\boxed{2\pi^2ab^2}$.

5.4.5★:

(a) Let f be a continuous function defined on $[a, b]$. Show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Hint: This looks a lot like the Intermediate Value Theorem.

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Your Submission: Solution

Solution: This is essentially just the Intermediate Value Theorem. We know that $\frac{1}{b-a}\int_a^b f(x)\,dx$ is the average value of f, so this must lie between the maximum and minimum values of f on $[a, b]$. So by the Intermediate Value Theorem, there is some c such that $f(c)$ equals the average value of f.

(b) Suppose that g is also a continuous function defined on $[a, b]$ whose range is nonnegative. Show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Hint: Bound $\int_a^b f(x)g(x)\,dx$ above and below by appropriate multiples of $\int_a^b g(x)\,dx$.

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Your Submission: Solution

Solution: Let M be the maximum value of f on $[a, b]$ and let m be the minimum value of f on $[a, b]$ (since f is continuous and $[a, b]$ is closed, these values must exist). Then since g is never negative, we have the inequality

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx,$$

and hence $\int_a^b f(x)g(x) = \int_a^b g(x)\,dx$ for some $m \le \xi \le M$. But then, by the Intermediate Value Theorem, we must have $c \in [a, b]$ such that $f(c) = \xi$, and the result follows.

(c) Find an example of continuous functions f and g defined on $[a, b]$ for which the equation in part (b) is not satisfied for any $c \in [a, b]$.

Hint: Look for g whose range contains both positive and negative values.

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Your Submission: Solution

Solution: A simple example is $f(x) = g(x) = x$ on the interval $[-1, 1]$. Then

$$\int_{-1}^1 f(x)g(x)\,dx = \int_{-1}^1 x^2\,dx = \frac{2}{3},$$

but since $\int_{-1}^1 g(x)\,dx = \int_{-1}^1 x\,dx = 0$, there cannot be any value c such that $c\int_{-1}^1 g(x)\,dx = \frac{2}{3}$.

5.5 Approximation techniques

Many integrals, such as

$$\int_0^2 e^{-x^2}\,dx \qquad \text{and} \qquad \int_1^2 \frac{\sin t}{t}\,dt,$$

cannot be explicitly evaluated. For these and other integrals, it is important to have a technique for approximating them via numerical methods. This basically amounts to manually computing an approximation of the area under the curve.

As an example, we will approximate the definite integral

$$\int_{-1}^2 \left(-\frac{1}{5}x^3 + 2\right)\,dx.$$

Of course, we can easily compute this integral explicitly:

$$\begin{aligned}\int_{-1}^2 \left(-\frac{1}{5}x^3 + 2\right)\,dx &= \left(-\frac{1}{15}x^4 + 2x\right)\Big|_{-1}^2\\ &= \left(-\frac{8}{15}+4\right) - \left(-\frac{1}{15}-2\right)\\ &= \frac{27}{5}\\ &= 5.4.\end{aligned}$$

This is the area under the curve $y = -\frac{1}{5}x^3 + 2$ between $x = -1$ and $x = 2$.

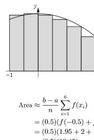
There are a number of approximation techniques using rectangles. As you know by now, we defined the definite integral as a limit of sums of areas of rectangles. We can thus approximate an integral as a sum of areas of rectangles. There are several different ways that we can perform this approximation, based on different ways that we choose the rectangles.

Let's do a bunch of different approximations with $n = 6$ rectangles. (Of course, the more rectangles that we use, the greater the accuracy of the estimate should be.) For simplicity, we'll partition $[a, b]$ into subintervals of equal length, so that each rectangle will have width $\frac{b-a}{n}$. We then get a partition

$$a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 = b,$$

where $x_i = a + i\left(\frac{b-a}{n}\right)$. In our example, $[a, b] = [-1, 2]$ and $n = 6$ so $x_0 = -1, x_1 = -0.5, x_2 = 0$, and so on up to $x_6 = 2$.

We can use the left endpoint of each interval to determine the height:



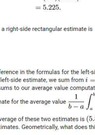
$$\begin{aligned}\text{Area} &\approx \frac{b-a}{n}\sum_{i=0}^{n} f(x_i)\\ &= (0.5)(f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5))\\ &= (0.5)(1.8 + 1.95 + 2 + 1.95 + 1.8 + 1.55)\\ &= (0.5)(11.05)\\ &= 5.525.\end{aligned}$$

The general formula is

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i),$$

but don't memorize this formula: think of it in terms of the rectangles.

Similarly, we can use the right endpoint of each interval to determine the height:



$$\begin{aligned}\text{Area} &\approx \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)\\ &= (0.5)(f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2))\\ &= (0.5)(1.95 + 2 + 1.95 + 1.8 + 1.55 + 1.2)\\ &= (0.5)(10.45)\\ &= 5.225.\end{aligned}$$

The general formula for a right-side rectangular estimate is

$$\frac{b-a}{n}\sum_{i=1}^{n} f(x_i).$$

Notice that the only difference in the formulas for the left-side rectangular estimate and the right-side rectangular estimate is the limits of the summation: for the left-side estimate, we sum from $i = 0$ to $n - 1$, but for the right-side estimate, we sum from $i = 1$ to n. Also note the similarity of these sums to our average value computation from Problem 5.51; in particular, if we divide these sums by $b - a$, we get our usual discrete estimate for the average value $\frac{1}{b-a}\int_a^b f$, as expected.

We also note that the average of these two estimates is $(5.525 + 5.225)/2 = 5.375$, which is a lot closer to the actual value of 5.4 than either of the original estimates. Geometrically, what does this "average" represent?

If we average the left-endpoint and right-endpoint estimates, we get:

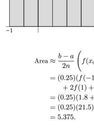
$$\frac{b-a}{n}\sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2},$$

which is sum of the area of trapezoids. This formula is usually written by pulling out the $\frac{1}{2}$ and combining the other terms:

$$\begin{aligned}&\frac{1}{2}\left(\left(\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)\right) + \left(\frac{b-a}{n}\sum_{i=1}^{n} f(x_i)\right)\right)\\ &= \frac{b-a}{2n}\left(f(x_0) + \left(2\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right),\end{aligned}$$

and it is called the **Trapezoid Rule**.

Let's see it explicitly with our example:



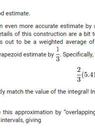
$$\begin{aligned}\text{Area} &\approx \frac{b-a}{2n}\left(f(x_0) + \left(2\sum_{i=1}^{n} f(x_i)\right) + f(x_n)\right)\\ &= (0.5)(f(-1) + 2(f(-0.5) + 2f(0) + 2f(0.5)\\ &\quad + 2f(1) + 2f(1.5) + f(2))\\ &= (0.25)(1.8 + 3.9 + 4 + 3.9 + 3.6 + 3.1 + 1.2)\\ &= (0.25)(21.5)\\ &= 5.375.\end{aligned}$$

This is very close to the actual value of 5.4, and we can see in the picture that the trapezoids very closely fill the entire area under the curve.

There are two other important approximation methods. One of these methods is to use rectangles where the height is determined by the midpoint of each interval. That is,

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right).$$

Here's the picture and computation with our example:



$$\begin{aligned}\text{Area} &\approx \frac{b-a}{n}\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)\\ &= (0.5)(f(-0.75) + f(-0.25) + f(0.25) + f(0.75)\\ &\quad + f(1.25) + f(1.75))\\ &= (0.5)(1.8875 + 1.9875 + 1.9875 + 1.8875\\ &\quad + 1.6875 + 1.3875)\\ &= (0.5)(10.8925)\\ &= 5.4125.\end{aligned}$$

This is also a pretty good estimate.

Finally, we can make an even more accurate estimate by using parabolas, instead of rectangles or trapezoids, to approximate the area under the curve. The details of this construction are a bit technical, so we will defer these details to Section 5.8. The surprising result is that this estimate turns out to be a weighted average of the midpoint estimate and the trapezoid estimate: we weight the midpoint estimate by $\frac{2}{3}$ and the trapezoid estimate by $\frac{1}{3}$. Specifically, in our example, this gives us

$$\frac{2}{3}(5.4125) + \frac{1}{3}(5.375) = 5.4,$$

which happens to exactly match the value of the integral! In general we won't get the exact answer, but this estimate tends to be the most accurate.

In practice, we execute this approximation by "overlapping" the two rules via a partition with twice as many components. So our new partition uses $m = 2n$ intervals, giving

$$a = x_0' < x_1' < x_2' < \cdots < x_m' = b.$$

Note that this new partition just splits all of the intervals in our old partition in half: if i is even, then our "new" x_i' is equal to the "old" $x_{i/2}$, and if i is odd, then our "new" x_i' is the midpoint between the "old" $x_{(i-1)/2}$ and $x_{(i+1)/2}$.

The trapezoid rule with the old partition gives

$$\begin{aligned}&\frac{b-a}{2n}\left(f(x_0) + \left(2\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right)\\ &= \frac{b-a}{2n}\left(f(x_0') + \left(2\sum_{i=1}^{m-1} f(x_{2i}')\right) + f(x_m')\right).\end{aligned}$$

The midpoint rule with the old partition gives

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) = \frac{b-a}{n}\sum_{i=0}^{n-1} f(x_{2i+1}').$$

Weighting the trapezoid rule by $\frac{1}{3}$ and the midpoint rule by $\frac{2}{3}$ and combining gives

$$\frac{b-a}{3n}\left(\frac{1}{2}\left(f(x_0') + \left(2\sum_{i=1}^{m-1} f(x_{2i}')\right) + f(x_m')\right) + 2\sum_{i=0}^{n-1} f(x_{2i+1}')\right).$$

It is more clear what is going when we pull $\frac{1}{2}$ outside:

$$\frac{b-a}{6n}\left(f(x_0') + \left(2\sum_{i=1}^{m-1} f(x_{2i}')\right) + f(x_m') + 4\left(\sum_{i=0}^{n-1} f(x_{2i+1}')\right)\right).$$

This is ugly, but it's clearer if we write it without the summation symbols, write it in terms of m instead of n, and list the terms in order:

$$\begin{aligned}&\frac{b-a}{3m}(f(x_0') + 4f(x_1') + 2f(x_2') + 4f(x_3') + 2f(x_4') + \cdots\\ &\quad + 4f(x_{m-1}') + f(x_m')).\end{aligned}$$

This is called **Simpson's Rule**, and it generally has the smallest error of all the approximation techniques.

Let's use all of these techniques on another example:

Problem 5.53

Compute approximations for the integral $\int_0^2 e^{-x^2}\,dx$ (to 4 decimal places) using $n = 4$ via: (a) left-side rectangles, (b) right-side rectangles, (c) the Trapezoid Rule, (d) midpoint rectangles, and (e) Simpson's Rule (with $m = 2n = 8$).

Solution for Problem 5.53: Let $f(x) = e^{-x^2}$. Here is a chart of the computations:

Method	Computation	Result
Left-side rectangles	$\frac{1}{4}(f(0) + f(0.25) + f(0.5) + f(0.75))$	0.8220
Right-side rectangles	$\frac{1}{4}(f(0.25) + f(0.5) + f(0.75) + f(1))$	0.6640
Trapezoid Rule	$\frac{1}{4}(f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1))$	0.7430
Midpoint rectangles	$\frac{1}{4}(f(0.125) + f(0.375) + f(0.625) + f(0.875))$	0.7487
Simpson's Rule	$\frac{1}{12}(f(0) + 4f(0.125) + 2f(0.25) + 4f(0.375) + 2f(0.5) + 4f(0.625) + 2f(0.75) + 4f(0.875) + f(1))$	0.7468

The Simpson's Rule estimate above is accurate to 4 decimal places. In fact, if we carried out the calculation to 6 decimal places, the Simpson's Rule estimate is 0.748526 and the actual value of the integral is 0.746824. $\square$

Concept: $\mathcal{R}_{\mathbb{C}}$

Problem 5.53 shows the usefulness of the various approximation methods. The integral $\int_0^2 e^{-x^2}\,dx$ is not directly computable: no integration technique will allow us to get an exact answer for this integral. So we are forced to use some sort of approximation method. Many functions that are very useful to mathematics and engineering are in the same boat: their integrals are important, but uncomputable except by numerical approximation.

Exercises

5.5.1:

Compute (to 4 decimal places) the different estimates on the integral $\int_1^2 \frac{\sin t}{t}\,dt$, using $n = 4$ (and $m = 8$ for Simpson's Rule).

You may type any additional notes you have here.

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Solution: Let $f(x) = \frac{\sin x}{x}$. We list the relevant values of f below:

x	$f(x)$
1	0.8415
1.125	0.8020
1.25	0.7592
1.375	0.7134
1.5	0.6650
1.625	0.6145
1.75	0.5623
1.875	0.5088
2	0.4546

We now compute the various estimates:

Method	Computation	Result
Left-side rectangles	$\frac{1}{4}(f(1) + f(1.25) + f(1.5) + f(1.75))$	0.7070
Right-side rectangles	$\frac{1}{4}(f(1.25) + f(1.5) + f(1.75) + f(2))$	0.6103
Trapezoid Rule	$\frac{1}{4}(f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2))$	0.6586
Midpoint rectangles	$\frac{1}{4}(f(1.125) + f(1.375) + f(1.625) + f(1.875))$	0.6597
Simpson's Rule	$\frac{1}{12}(f(1) + 4f(1.125) + 2f(1.25) + 4f(1.375) + 2f(1.5) + 4f(1.625) + 2f(1.75) + 4f(1.875) + f(2))$	0.6593

A computer algebra system computes $\int_1^2 \frac{\sin x}{x}\,dx = 0.65903\ldots$, so the Simpson's Rule estimate is accurate to 4 decimal places.

5.A Formal definitions of log and exp

Throughout the book, we have been forced to have imprecise "definitions" of the exponential and natural logarithm functions. Now that we have defined the definite integral and proved the Fundamental Theorem of Calculus, we are finally able to make rigorous definitions of these functions. We start by defining the natural logarithm function:

Definition: The **natural logarithm** function is the function $\log : (0, +\infty) \to \mathbb{R}$ defined for any $x > 0$ as

$$\log x = \int_1^x \frac{1}{t}\, dt.$$

Note that the Fundamental Theorem of Calculus tells us that $\frac{d}{dx} \log x = \frac{1}{x}$.

We can prove all of the nice properties of the natural logarithm that we were forced to assume earlier in the book. In particular:

Problem 5.76

Let a, b be positive real numbers, and let n be an integer. Show that:

(a) $\log 1 = 0$

(b) $\log ab = \log a + \log b$

(c) $\log \frac{a}{b} = \log a - \log b$

(d) $\log a^n = n \log a$

Solution for Problem 5.76:

(a) We just apply the definition: $\log 1 = \int_1^1 \frac{1}{t}\, dt = 0$, since $\int_1^1 f = 0$ for any function f.

(b) Pretend that b is constant, and we'll differentiate the function $\log ab$ with respect to a, using the Chain Rule:

$$\frac{d}{da} \log ab = \frac{1}{ab} \cdot \frac{d}{da}(ab) = \frac{1}{ab} \cdot b = \frac{1}{a}.$$

Thus, we conclude that $\log a$ and $\log ab$, thought of as functions of a, have the same derivative, and thus must differ by a constant. That is,

$$\log ab = \log a + C$$

for some constant C. But plugging in $a = 1$ gives $\log b = \log 1 + C = C$, so $C = \log b$, and we have reached our conclusion that

$$\log ab = \log a + \log b.$$

(c) We simply rearrange part (b):

$$\log a = \log\left(b \cdot \frac{a}{b}\right) = \log b + \log \frac{a}{b},$$

and then subtracting $\log b$ from both sides gives the desired result.

(d) First, assume $n > 0$. Then the result is an easy consequence of part (b):

$$\begin{aligned} \log a^n &= \log\Big(\underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}\Big) \\ &= \underbrace{(\log a) + (\log a) + \cdots + (\log a)}_{n \text{ times}} \\ &= n \log a. \end{aligned}$$

If $n = 0$, then the result is just $\log a^0 = \log 1 = 0 = 0 \cdot \log a$. Finally, if $n < 0$, we use part (c):

$$\log a^n = \log\left(\frac{1}{a}\right)^{-n} = -n \log \frac{1}{a} = -n(\log 1 - \log a) = n \log a.$$

□

We now would like to be able to take the inverse function of the logarithm and get the exponential function. Since $\frac{1}{t} > 0$ for all $t > 0$, we know that $\log x = \int_1^x \frac{1}{t}\, dt$ is a strictly increasing function, so it has an inverse whose domain is the range of $\log$. By part (c) of Problem 5.76, we have $\log a^n = n \log a$ for any integer n and positive a, and the quantity $n \log a$ is clearly unbounded as n ranges over all the integers (for any $a \neq 1$). Thus, since $\log$ is continuous and has unbounded range, its range must be $\mathbb{R}$, and hence $\log$ must have an inverse with domain $\mathbb{R}$:

Definition: The **exponential** function is the function $\exp : \mathbb{R} \to (0, +\infty)$ defined as the inverse of the natural logarithm function. We usually write $\exp x = e^x$, where $e = \exp 1$.

Using the properties of inverse functions, we can prove the properties corresponding to Problem 5.76 for the exponential function:

Problem 5.77

Let a, b be real numbers and let n be an integer. Show that:

(a) $e^{a+b} = e^a e^b$

(b) $e^{a-b} = \frac{e^a}{e^b}$

(c) $e^{an} = (e^a)^n$

Solution for Problem 5.77: All of these result from taking the exponential of the corresponding fact from Problem 5.76.

(a) Note that

$$\log e^{a+b} = a + b = \log e^a + \log e^b = \log e^a e^b.$$

Taking the exponential of both sides gives the desired result.

(b) We do essentially the same as part (a):

$$\log e^{a-b} = a - b = \log e^a - \log e^b = \log \frac{e^a}{e^b}.$$

Taking the exponential of both sides gives the desired result.

(c) We have

$$\log e^{an} = an = n \log e^a = \log(e^a)^n,$$

and again, taking the exponential of both sides gives the desired result.

□

We can also easily show that $\exp$ is its own derivative:

Problem 5.78

Show that $\frac{d}{dx} e^x = e^x$.

Solution for Problem 5.78: We use the Inverse Function Rule for derivatives:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

where here $f(x) = \log x$, so that $f'(x) = \frac{1}{x}$, and $f^{-1}(x) = e^x$. This gives

$$\frac{d}{dx} e^x = \frac{1}{\frac{1}{e^x}} = e^x,$$

as desired. □

We can further define exponential functions with any positive base:

Definition: Let a be a positive real number. Define the function $a^x : \mathbb{R} \to (0, +\infty)$ by

$$a^x = e^{x \log a}.$$

Note that this definition even works for $a = e$, since $\log e = 1$. We quickly see that

$$a^0 = e^{0 \log a} = e^0 = 1$$

and

$$a^1 = e^{\log a} = a.$$

We can also show that a^x has all the properties that we expect for exponentials:

Problem 5.79

Let a be a positive real number, and $b, c \in \mathbb{R}$. Show that:

(a) $a^{bc} = (a^b)^c$

(b) $a^{b+c} = a^b a^c$

Solution for Problem 5.79: We simply use the definition of a^x and apply the properties that we know about the function e^x.

(a) It's easier if we start from the right side and write, by definition,

$$(a^b)^c = e^{c \log a^b}.$$

But again we can apply the definition and get

$$(a^b)^c = e^{c \log a^b} = e^{c \log e^{b \log a}}.$$

Using the fact that $\log$ and $\exp$ are inverses gives

$$(a^b)^c = e^{c \log a^b} = e^{c \log e^{b \log a}} = e^{c(b \log a)}.$$

And now we just apply the definition one final time:

$$(a^b)^c = e^{c \log a^b} = e^{c \log e^{b \log a}} = e^{c(b \log a)} = e^{bc \log a} = a^{bc}.$$

(b) This is easier than part (a):

$$a^{b+c} = e^{(b+c) \log a} = e^{b \log a + c \log a} = \left(e^{b \log a}\right)\left(e^{c \log a}\right) = a^b a^c.$$

□

We could go on and define $\log_a x$ as the inverse of a^x, and prove its properties, and compute the derivatives of a^x and $\log_a x$, but you get the idea. The point is that the definitions of exponentials and logarithms can be made rigorous and all of their relevant properties can be proved. So we weren't really cheating in Chapters 1-4 when we used these properties.

5.B Simpson's Rule

Simpson's Rule is an approximation technique used to estimate $\int_a^b f$, in which we form a Riemann sum of areas under parabolas. As with the other approximation methods that we studied in Section 5.5, we first choose a positive integer n, and we partition $[a, b]$ into n pieces of size $\frac{b-a}{n}$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where $x_i = a + i\left(\frac{b-a}{n}\right)$.

We will approximate the area under f on each interval $[x_i, x_{i+1}]$ of the partition by constructing the parabola that passes through the same points as the graph of f at each end of the interval and at the midpoint of the interval. That is, for each $0 \le i < n$, we wish to find a quadratic $q_i(x)$ such that

$$\begin{aligned} &q(x_i) = f(x_i), \\ &q\left(\frac{x_i + x_{i+1}}{2}\right) = f\left(\frac{x_i + x_{i+1}}{2}\right), \text{ and} \\ &q(x_{i+1}) = f(x_{i+1}). \end{aligned}$$

Then, since quadratics are easy to integrate, we will approximate the integral by the sum of the integrals of the quadratic pieces:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} q_i(x)\,dx. \tag{5.3}$$

Our next step is to investigate the integrals of these quadratics:

Problem 5.80

Suppose that $q(x)$ is a quadratic function, and $c < d$ are real numbers. Let $m = \frac{c+d}{2}$ be the midpoint of $[c, d]$. Compute $\int_c^d q(x)\,dx$ in terms of c, d, $q(c)$, $q(d)$, and $q(m)$.

Solution for Problem 5.80: Since q is quadratic, we can write it as $q(x) = a_2x^2 + a_1x + a_0$ for some coefficients a_0, a_1, a_2. We can now compute:

$$\begin{aligned} &\int_c^d (a_2x^2 + a_1x + a_0)\,dx \\ &= \left(\frac{1}{3}a_2x^3 + \frac{1}{2}a_1x^2 + a_0x\right)\Big|_c^d \\ &= \frac{1}{3}a_2(d^3 - c^3) + \frac{1}{2}a_1(d^2 - c^2) + a_0(d - c) \\ &= \frac{d-c}{6}\left(2a_2(d^2 + dc + c^2) + 3a_1(d + c) + 6a_0\right). \end{aligned} \tag{5.4}$$

We want to write this expression in terms of $q(c)$, $q(d)$, and $q(m)$. We have

$$\begin{aligned} q(c) &= a_2c^2 + a_1c + a_0, \\ q(m) &= \frac{1}{4}a_2(d^2 + 2dc + c^2) + \frac{1}{2}a_1(d + c) + a_0, \\ q(d) &= a_2d^2 + a_1d + a_0. \end{aligned}$$

In particular, note that

$$\begin{aligned} q(c) + 4q(m) + q(d) &= a_2(c^2 + d^2 + 2dc + c^2 + d^2) \\ &\quad + a_1(c + 2d + 2c + d) + a_0(1 + 4 + 1) \\ &= 2a_2(d^2 + dc + c^2) + 3a_1(d + c) + 6a_0. \end{aligned}$$

This exactly matches the expression in parentheses in (5.4), so we conclude that

$$\begin{aligned} &\int_c^d (a_2x^2 + a_1x + a_0)\,dx \\ &= \frac{d-c}{6}\left(2a_2(d^2 + dc + c^2) + 3a_1(d + c) + a_0\right) \\ &= \frac{d-c}{6}(q(c) + 4q(m) + q(d)). \end{aligned}$$

□

Using the result from Problem 5.80 in equation (5.3), where $c = x_i$, $d = x_{i+1}$, and $m = \left(\frac{x_i + x_{i+1}}{2}\right)$, we get

$$\begin{aligned} &\int_a^b f(x)\,dx \\ &\approx \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} q_i(x)\,dx \\ &= \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{6}\left(f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right) \\ &= \frac{b-a}{6n}\sum_{i=0}^{n-1}\left(f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right). \end{aligned} \tag{5.5}$$

As discussed in Section 5.5, we usually apply this approximation by constructing a new partition with $m = 2n$ pieces

$$a = x'_0 < x'_1 < x'_2 < \cdots < x'_m = b,$$

where the even terms are the terms of the original partition (so that $x'_{2i} = x_i$) and the odd terms are the midpoints of the intervals in the original partition $\left(\text{so that } x'_{2i+1} = \frac{x_i + x_{i+1}}{2}\right)$. Then the Simpson's Rule approximation in (5.5) becomes

$$\begin{aligned} &\frac{b-a}{3m}\big(f(x'_0) + 4f(x'_1) + 2f(x'_2) + 4f(x'_3) + 2f(x'_4) + \cdots \\ &\quad + 4f(x'_{m-1}) + f(x'_m)\big). \end{aligned}$$

Challenge Problems

5.4.1

Use the following integrals and all from the first two rounds of the 2007 MIT Integration Bee (see description in Section 5.1.2 Bees). Try these yourself.

(a) $\int \sqrt{\sin x}\,(1+\cos x)^{3/2}\,dx$

Hint: Try u-substitution $u = 1 + \cos x$ and then integrate by parts.

Hint: Does the integral look like the derivative of a product?

This may have any additional notes you have here.

Hide Solution

Edit

Your Submission: Solution

Solution: A way to help to compute $\int \sqrt{\sin x}\,u^{3/2}\,dx = \int \sqrt{\sin x}\,(1+\cos x)^{3/2}\,dx$ and notice that this is just the first term of the integrand divided by $\frac{5}{2}$. Furthermore, we notice that

$$\int \sqrt{\sin x}\,u^{3/2} = \int \sin x\,(1+\cos x)^{3/2} + \int \cos x\,(1+\cos x)^{3/2}$$

which is exactly the function we are integrating. Therefore,

$$\int \sqrt{\sin x}\,(1+\cos x)^{3/2}\,dx = \frac{2}{5}\left(\int \sin x\,(1+\cos x)^{3/2} + \int \cos x\,(1+\cos x)^{3/2}\right)$$

Note that this is essentially just a rearranged integration by parts:

$$\int u\,dv = uv - \int v\,du$$

where in our example $u = \sin x$ and $dv = (1+\cos x)^{3/2}\,dx$.

(b) $\int \frac{\sin^2 x}{\sqrt{x}}\,dx$

Hint: Rewrite the denominator, but not all the way.

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Your Submission: Solution

Solution: The denominator factors as $x(x^2+1)$ and we have the partial decomposition

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Thus,

$$\int \frac{\sin^2 x}{\sqrt{x}}\,dx = \int \frac{\sin^2 x}{x\sqrt{x}} - \int \frac{\sin^2 x\,(Bx+C)}{\sqrt{x}\,(x^2+1)}$$

(c) $\int \sin(\sqrt{x})\,dx$

Hint: Try the substitution $u = \sqrt{x}$.

This may have any additional notes you have here.

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Your Submission: Solution

Solution: Let $u = \sqrt{x}$ so that $u = x^{1/2}$ and $du = 2x^{1/2}\,dx$. The integral then becomes

$$\int \sin(\sqrt{x})\,dx = \int 2u\,\sin u\,du.$$

This can now be integrated by parts (twice) to get

$$\int 2u^2 \sin u\,du = -2u^2\cos u + 4\int u\cos u\,du$$
$$= -2u^2\cos u + 8\sin u + 8u\cos u + C$$
$$= -2x\cos\sqrt{x} + 8\sin\sqrt{x} + 8\sqrt{x}\cos\sqrt{x} + C.$$

Removing the substitution gives an answer of

$$\boxed{-2x\cos\sqrt{x} + 8\sin\sqrt{x} + 8\sqrt{x}\cos\sqrt{x} + C}$$

5.4.2

(d) $\int \frac{x^2+1}{x^4+1}\,dx$

Hint: What substitution would get rid of the fractional exponents?

Hint: Try the substitution $u = x - \frac{1}{x}$.

This may have any additional notes you have here.

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Your Submission: Solution

Solution: Since we want to deal nicely with both a square root and a cube root of x, we try the substitution $u = x^{1/6}$. Then $du = 6x^5\,dx$, and the integral becomes

$$\int \frac{x^2+1}{x^4+1}\,dx = \int \frac{u^2+1}{u^{24}+1}\,6u^5\,du = 6\int \frac{u^7+u^5}{u^{24}+1}\,du$$

We now split the integral as

$$6\int \frac{u^7}{u^{24}+1}\,du + 6\int \frac{u^5}{u^{24}+1}\,du.$$

Notice that the substitution gives the answer $\frac{3}{2}\ln(u^{12}+1) + \frac{3}{2}\arctan(u^6) + C$.

5.4.7

Source: Le Parisien, Du Pinel, Le Texas (1968, 22 MINUT)

Compute the following definite integrals:

(a) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$

Hint: Use the symmetry of the sine and cosine functions.

Hint: Use the substitution $u = \frac{\pi}{2} - x$ in one does what you end up with relate to the original integral?

This may have any additional notes you have here.

Hide Solution

Edit

Your Submission: Solution

Solution: Note that using the substitution $u = \frac{\pi}{2} - x$ gives

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\cos u + \sin u}\,du$$

Adding these together gives

$$2\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$
$$= \frac{\pi}{2}$$

and thus $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$.

(b) $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx$

Hint: Integrate by parts twice. Is a reasonable thing to try

Hint: Try integration by parts with $u = (1-x)^2$ and $dv = e^{x/2}\,dx$. What do you get?

Hint: Try it probably need to use integration by parts 3 times.

This may have any additional notes you have here.

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Your Submission: Solution

Solution: Integrate by parts with $u = (1-x)^2$, $du = -2(1-x)$, $v = 2e^{x/2}$, $dv = e^{x/2}\,dx$. We get

$$\int_0^1 \left(\frac{1}{2}\right)^2 e^{x/2}(1-x)^2\,dx = \left(\frac{1}{2}\right)^2 \left(2e^{x/2}(1-x)^2\Big|_0^1 + \int_0^1 4e^{x/2}(1-x)\,dx\right)$$

Note that the first term on the right side is 0, and that $\int_0^1 \left(\frac{1}{2}\right)^2 e^{x/2}(1-x)\,dx$ has been

$$\int_0^1 \left(\frac{1}{2}\right)^2 e^{x/2}(1-x)\,dx = \left(\frac{1}{2}\right)^2 \left(2e^{x/2}(1-x)\Big|_0^1 + \int_0^1 2e^{x/2}\,dx\right)$$

We can repeat the process one more time to get

$$\int_0^1 \left(\frac{1}{2}\right)^2 e^{x/2}(1-x)\,dx = \left(\frac{1}{2}\right)^2 \left(2e^{x/2}\Big|_0^1 - 2e^{x/2}\Big|_0^1\right) = \left(\frac{1}{2}\right)^2 \left[\frac{2}{e^{1/2}} - 2\right]$$

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$

Hint: Well this time make the substitution $u = x^2 + 3x^{1/2}$. Is that where are we going to first do?

Hint: Letting $u = x^2 + 3x^{1/2}$, u when integrated is not and and and and and do in the numerator?

This may have any additional notes you have here.

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Your Submission: Solution

Solution: Note that the make the substitution $u = x^2 + 3x^{1/2}$, it is that that we need the substitution $du = (6x^{1/2} + 2x + 1)\,dx$. So we write

$$6x + x + 3x^{1/2} + 2x + 3x^{1/2} + 2x^2 + 6x + 1.$$

So we integrd terminos

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

This is

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

So this integrand is $7\ln|x^2 + 3x^{1/2}| + 3\ln|\frac{x}{2}|$ which you can further simplify if you like to $\boxed{\ln\left|\frac{x^{7/2}}{2}\right|}$

5.4.8

Find a general formula for $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$ where n and k are real numbers.

This may have any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: $I_n = 1 - I_k$ then

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

For $x \neq k$, then we can partial fractions. We claim that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

This can be verified by multiplying by $x - k(x-1)$. Expanding $\cos x$ to the left and

$$x + \frac{1}{2}(x-1)(x-k) = \frac{1}{2}x + \frac{1}{2}$$

on the right. Then we integrate

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx - \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx - \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

5.4.9

Compute the length of the curve $y = \ln(\sin x)$ between the points $\left(\frac{\pi}{6}, \ln\sqrt{\frac{1}{2}}\right)$ and $\left(\frac{\pi}{2}, 0\right)$.

Hint: Note that $\sqrt{1+\cot x} = \sqrt{\frac{1}{1-\cos^2 x}} = \sqrt{\frac{1}{\sin^2 x}} = \csc x$. What big identity can we now use?

Hint: Integrating $\csc x$, a Sub for trig expressions that have $\cos x$ in their derivative.

This may have any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We compute

$$f'(x) = \frac{\cos x}{\sin x} = \cot x$$

and

$$\sqrt{1+(f'(x))^2} = \sqrt{1+\cot^2 x} = \csc x$$

So $L = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc x\,dx$. Now we are ready to go.

To compute this integral, we consider two expressions that we know have $\csc x$ term in their derivative:

$$\csc x = \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} \quad \text{and} \quad \frac{d}{dx}\ln(\csc x - \cot x).$$

In particular, we notice that

$$\frac{d}{dx}\ln(\csc x - \cot x) = -\csc x + \csc x\cot x$$
$$= -\csc x(\csc x - \cot x)$$

Thus, if $f(x) = \csc x + \cot x$, then we have

$$\csc x = -\frac{f'(x)}{f(x)}$$

and hence $-\ln\left(f(x)\right)$ is an antiderivative of $\csc x$. To finish, we compute

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc x\,dx = \left(-\ln\left(\csc x + \cot x\right)\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= -\ln\left(0+1\right) + \ln\left(\frac{1}{2}+\sqrt{3}\right)$$
$$= \boxed{\ln\left(1+\sqrt{3}\right)}$$

5.4.10

The **hyperbolic cosine** function is given by:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Find the length of the graph of $y = \cosh x$ from $x = 0$ to $x = 3$.

Hint (see the identity $1 + \sinh^2 x = \cosh^2 x$.

This may have any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The integral we want is

$$L = \int_0^3 \sqrt{1+(f'(x))^2}\,dx$$

We note that $\cosh^2 x = \sinh^2 x$, where $\sinh x = \frac{e^x - e^{-x}}{2}$. So we are ready to

$$L = \int_0^3 \sqrt{1+\sinh^2 x}\,dx$$

It is easy to check that $1 + \sinh^2 x = \cosh^2 x$.

$$L = \left(\frac{e^x - e^{-x}}{2}\right)\Big|_0^3$$
$$= \frac{e^3 - e^{-3}}{2} - \frac{e^0 - e^{-0}}{2}$$
$$= \frac{e^3 - e^{-3}}{2} - 0$$

so the integral is

$$\int_0^3 \cosh x\,dx = \sinh x\Big|_0^3$$
$$= \sinh 3 - \sinh 0$$
$$= \boxed{\frac{e^3 - e^{-3}}{2}}$$

5.4.11

Find $\int \sec x\,dx$ by **parts fraction** is defined by

$$\sec x = \frac{1}{\cos x}$$

Compute the following (your answer may be expressed in terms of $\sin(x)$):

(a) $\int_0^{\frac{\pi}{2}} \frac{1}{1-x^2}\,dx$

This may have any additional notes you have here.

Hide Solution

Edit

Your Submission: Solution

Solution: Substitution $u = \frac{1}{\sqrt{2}}\sin x$ so that $du = \frac{1}{\sqrt{2}}\cos x\,dx$. Then

$$\int_0^{\frac{\pi}{2}} \frac{1}{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1-\sin^2 x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1-\left(\frac{1}{\sqrt{2}}\sin x\right)^2}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1-\frac{1}{2}\sin^2 x}\,dx$$

(b) $\int_0^{\frac{\pi}{2}} \frac{1}{x^2}\,dx$

Solution: Note $\int_0^{\frac{\pi}{2}} x^{-2}\,dx$ $\int_0^{\frac{\pi}{2}}$ by taking the derivative, we get

$$\int_0^{\frac{\pi}{2}} \frac{1}{x^2}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{x^2}\,dx = \int_0^{\frac{\pi}{2}} x^{-2}\,dx = \boxed{-2x^{-1}}$$

(c) $\int_0^{\frac{\pi}{2}} \frac{1}{x^2+1}\,dx$

Solution: We use the product rule

$$\int_0^{\frac{\pi}{2}} \frac{1}{x^2+1}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{x^2+1}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{x^2+1}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{x^2+1}\,dx$$

(d) $\int \sec(x)\,dx$

Hint: See integration by parts

This may have any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Integrate by parts let $u = \sec(x)$ so $du = \left(\frac{1}{\sqrt{2}}\right)^{-1}\,dx$, and let $v = x$, so $dv = dx$. Then

$$\int \sec(x)\,dx = x\sec(x) - \int x^{-1}\,dx$$
$$= x\sec(x) - \int \frac{1}{x}\,dx$$
$$= \boxed{x\sec(x) - \ln|x| + C}$$

5.7.4

Find the volume of the intersection of two infinite long cylinders of radius r (whose axes intersect in a right angle).

Hint: In $\mathbb{R}_{\geq 0}$ Cropped, let one cylinder be given by $x^2 + z^2 = 1$ and let the other cylinder be given by $y^2 + z^2 = 1$.

Hint: Integrate the cross-sectional area as a function of z.

Hint: The cross sections are squared

This may have any additional notes you have here.

Hide Solution

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Your Submission: Solution

Hint: Cut the integral of a positive function and be 0?

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Your Submission: Solution

Solution: Pie have

$$\int_0^1 f(x)(x^2 - 3x) + f(x)x - x(x^2 - 3x)\,dx.$$

OR

$$\int_0^1 f(x)(x^2 - 3x) + f(x)x - x(x^2 - 3x)\,dx.$$

Pie the function $f(x) = -x^2$ is positive for all $x \geq 0$, so the integral must be positive too. Notice there are no negative terms.

5.7.6

Find the volume of the intersection of two infinite long cylinders of radius r (whose axes intersect in a right angle).

Hint: In $\mathbb{R}_{\geq 0}$ Cropped, let one cylinder be given by $x^2 + z^2 = 1$ and let the other cylinder be given by $y^2 + z^2 = 1$.

Hint: Integrate the cross-sectional area as a function of z.

Hint: The cross sections are squared

This may have any additional notes you have here.

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Your Submission: Solution

Solution: Let cylinder C_1 have the x axis as its axis, and have equation $y^2 + z^2 = 1$. Let cylinder C_2 have the y axis as its axis, and have equation $x^2 + z^2 = 1$. Let $C_1 \cap C_2$ be the intersection of the two cylinders. Let V be the volume of $C_1 \cap C_2$. Let $A(z)$ be the cross-section area at z. The cross section is a square with side length $\sqrt{1-z^2}$. So the cross-section has area $A(z) = 4(1-z^2)$ and then the volume is

$$V = \int_{-1}^{1} A(z)\,dz$$
$$= \int_{-1}^{1} 4(1-z^2)\,dz$$
$$= 4\int_{-1}^{1} 1 - z^2\,dz$$
$$= \boxed{\frac{16}{3}}$$

This perhaps is pumping just about this is free there is no it is the reason!

5.7.8

Find the surface area of the intersection of two infinite long cylinders of radius r (whose axes intersect in a right angle).

Hint: Let a Cropped, let one cylinder be given by $x^2 + z^2 = 1$ and let the other cylinder be given by $y^2 + z^2 = 1$.

Hint: Integrate the cross-sectional area as a function of z.

Hint: The cross sections are squared

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Your Submission: Solution

Solution: Let cylinder C_1 have the x axis as its axis, and have equation $y^2 + z^2 = 1$. Let cylinder C_2 have the y axis as its axis, and have equation $x^2 + z^2 = 1$. Let $C_1 \cap C_2$ be the intersection of the two cylinders. Let V be the volume of $C_1 \cap C_2$. Let $A(z)$ be the cross-section area at z. The cross section is a square with side length $\sqrt{1-z^2}$. So the cross-section has area $A(z) = 4(1-z^2)$ and then the volume is

$$V = \int_{-1}^{1} A(z)\,dz$$
$$= \int_{-1}^{1} 4(1-z^2)\,dz$$
$$= 4\int_{-1}^{1} 1 - z^2\,dz$$
$$= \boxed{\frac{16}{3}}$$

This perhaps is pumping just about this is free there is no it is the reason!

5.7.9

Find the surface area of the intersection of two infinite long cylinders of radius r (whose axes intersect in a right angle).

Hint: Let a Cropped, let one cylinder be given by $x^2 + z^2 = 1$ and let the other cylinder be given by $y^2 + z^2 = 1$.

Hint: Integrate the cross-sectional area as a function of z.

Hint: The cross sections are squared

This may have any additional notes you have here.

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Your Submission: Solution

Solution: Let cylinder C_1 have the x axis as its axis, and have equation $y^2 + z^2 = 1$. Let cylinder C_2 have the y axis as its axis, and have equation $x^2 + z^2 = 1$. Let $C_1 \cap C_2$ be the intersection of the two cylinders. Let V be the volume of $C_1 \cap C_2$. Let $A(z)$ be the cross-section area at z. The cross section is a square with side length $\sqrt{1-z^2}$. So the cross-section has area $A(z) = 4(1-z^2)$ and then the volume is

$$V = \int_{-1}^{1} A(z)\,dz$$
$$= \int_{-1}^{1} 4(1-z^2)\,dz$$
$$= 4\int_{-1}^{1} 1 - z^2\,dz$$
$$= \boxed{\frac{16}{3}}$$

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Review Problems

1.1b ★ 2

Evaluate the following integral:

$$\int \frac{\sin(\ln x)}{x}\,dx$$

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Let $u = \ln x$ and substitute the function before integrating. Multiplying numerator and denominator by x^2 yields $\frac{\sin(\ln x)}{x} = \frac{x\sin(\ln x)}{x^2}$. Therefore, the integral is such

$$\int \frac{\sin(\ln x)}{x}\,dx$$

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Your Explanation

Your Explanation Solution

Question: Write as

$$\int \sin(\ln x)\,\frac{x}{x^2}\,dx$$

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Your Explanation

Your Explanation Solution

Question: Recall that this was easier chunk by long division.

Then

$$\int \frac{\sin(\ln x)}{x}\,dx = \int \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)\,dx$$

$$=

CHAPTER 6

Infinity

Infinity is a concept that you probably have an intuitive feel for. Informally, some of the meanings of infinity are "larger than finite" and "grows without bound" and "arbitrarily large". But what does "infinity" really mean? How can we rigorously work with infinity?

In calculus, we have different meanings for infinity and for the infinity symbol: to depending on the context. Sometimes, to represents a number, sometimes it represents a direction, sometimes it represents a process. Sometimes it represents a quantity that grows arbitrarily negative. In other places, to will be a placeholder for a quantity that is unbounded. For example, the interval $[1, \infty)$ that has no upper bound.

In this chapter, we will explore different notions of to as they occur in calculus.

6.1 Limits towards infinity

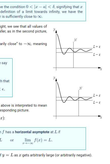
Let's recall our definition of limit. If f is a real-valued function, then we say that

$$\lim_{x \to c} f(x) = L.$$

if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Also recall the graphical representation of this, shown at right. In words, this says that we can get $f(x)$ to be as close L as we want, by making x close enough to c. Graphically, this means that given any pair of horizontal lines, we can find a sufficiently narrow rate of vertical lines so that the graph of f is sandwiched between the vertical lines lies entirely inside the horizontal strip between the given horizontal lines.



How can we modify this concept to define

$$\lim_{x \to \infty} f(x) = L?$$

We want to "make x arbitrarily close to infinity" but what does that mean? Intuitively, we know that x is arbitrarily close to infinity" means "x is arbitrarily large" or "x is arbitrarily negative" or "x is arbitrarily close to infinity means getting larger and larger. This motivates our definition:

Definition: Let f be a real-valued function. We say

$$\lim_{x \to \infty} f(x) = L.$$

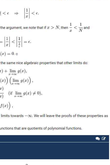
if, for all $\varepsilon > 0$ there exists N such that

$$x > N \implies |f(x) - L| < \varepsilon.$$

In words, this states that we can make $f(x)$ arbitrarily close to L meaning that we can $|f(x) - L| < \varepsilon$ for some given $\varepsilon > 0$ by restricting x to sufficiently close" to ∞ meaning that we have $x > N$ for some large N.

Concept: In the usual limit definition, we have the condition $0 < |x - c| < \delta$, signifying that x is sufficiently close to c. In the definition of a limit towards infinity, we have the condition $x > N$, signifying that x is sufficiently close to ∞.

We can also see what this looks like graphically. In the picture to the right, we see that all values of $x > N$ make the function lie within ε of our limit L. If we make ε smaller, so in the second picture, we can move N to the right to still "squeeze" the function within ε of L.



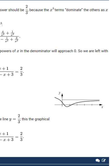
The definition of $\lim_{x \to \infty} f(x) = \infty$ is similar, except now we have a "arbitrarily close" to ∞ - so, meaning that $x > N$ for some N.

Definition: Let f be a real-valued function. We say

$$\lim_{x \to \infty} f(x) = \infty.$$

if, for all $\varepsilon > 0$ there exists N such that

$$x > N \implies |f(x) - L| < \varepsilon.$$



As with limits at positive infinity, the condition $x < N$ in the definition above is interpreted to mean that x is arbitrarily close to $-\infty$ meaning that $x < N$ for some large negative N.

We also use the following terminology that describes the graph $y = f(x)$.

Definition: We say that the graph of a function f has a **horizontal asymptote** at L if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

This means that the graph of f becomes arbitrarily close to the graph of $y = L$ as x gets arbitrarily large (or arbitrarily negative).

Let's practice using the definition of a limit at infinity.

Problem 6.1

Find $\lim_{x \to \infty} \frac{1}{x}$.

Solution for Problem 6.1: Let $f(x) = \frac{1}{x}$. It seems that the limit should be L as the value of $f(x)$ gets closer and closer to L as x gets large and large and how to see proof?

As usual, in limit-definition computations, we start with a given $\varepsilon > 0$ and we need to find a corresponding value of N so that the definition is satisfied. Specifically, we want N such that

$$x > N \implies |f(x) - 0| < \varepsilon \implies \left|\frac{1}{x}\right| < \varepsilon.$$

But $\left|\frac{1}{x}\right| < \varepsilon \iff |x| > \frac{1}{\varepsilon}$, so we can choose $N = \frac{1}{\varepsilon}$. Then, reversing the argument, we note that if $x > N$ then $\left|\frac{1}{x}\right| < \frac{1}{N}$ and

$$|f(x) - 0| = |f(x)| = \left|\frac{1}{x}\right| < \frac{1}{N} = \varepsilon$$

Thus, $x > N$ implies $|f(x) - 0| < \varepsilon$, and hence, by definition, $\lim_{x \to \infty} f(x) = 0$.

One thing that should not surprise you is that limits at infinity satisfy all the same nice algebraic properties that other limits do:

$$\lim_{x \to \infty} (f + g)(x) = \lim_{x \to \infty} f(x) + \lim_{x \to \infty} g(x),$$
$$\lim_{x \to \infty} (fg)(x) = \left(\lim_{x \to \infty} f(x)\right)\left(\lim_{x \to \infty} g(x)\right),$$
$$\lim_{x \to \infty} (f/g)(x) = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)} \quad (\text{if } \lim_{x \to \infty} g(x) \neq 0),$$
$$\lim_{x \to \infty} (cf)(x) = c\left(\lim_{x \to \infty} f(x)\right)$$

provided all of the above limits exist. The same properties also hold for limits towards $-\infty$. We will leave the proofs of these properties as exercises.

Next, we study the behavior approaching infinity of **rational functions**: functions that are quotients of polynomial functions.

Problem 6.2

Find $\lim_{x \to \infty} \frac{2x^2 - 4x + 5}{3x^2 + 2x + 8}$.

Solution for Problem 6.2: You may already intuitively believe the answer should be $\frac{2}{3}$ because the x^2 terms "dominate" the others as x grows arbitrarily large. But how can we more rigorously see this?

Let's call the function $f(x)$ and divide numerator and denominator by x^2.

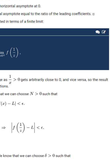
$$f(x) = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{3 + \frac{2}{x} + \frac{8}{x^2}}.$$

Now it is clear what is happening: when $x \to \infty$, all of the terms with powers of x in the denominator will approach 0. So we are left with just $\frac{2}{3}$. Hence

$$\lim_{x \to \infty} f(x) = \frac{2}{3}.$$

We can see the horizontal asymptote at $\frac{2}{3}$ drawing the graph of

$$f(x) = \frac{2x^2 - 4x + 5}{3x^2 + 2x + 8}$$



shown at right. As x grows large, we see that this graph approaches the line $y = \frac{2}{3}$ this the graphical representation of

$$\lim_{x \to \infty} \frac{2x^2 - 4x + 5}{3x^2 + 2x + 8} = \frac{2}{3}.$$

This sort of thing generalizes to any rational function:

Problem 6.3

Describe how to find the limit $\lim_{x \to \infty} \frac{p(x)}{q(x)}$ where $f(x)$ and $g(x)$ are polynomials. How does this limit depend on the degrees of f and g?

Solution for Problem 6.3: For rational calc, let's suppose f has degree m and g has degree n. So we write

$$h(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0}$$

where $a_m \neq 0$ and $b_n \neq 0$.

In keeping with our technique for Problem 6.2, we divide by the highest power of x present in either polynomial.

This leads to 3 cases:

Case 1: $m > n$ If the polynomial in the numerator is higher degree than the polynomial in the denominator, we get, after division by x^m:

$$h(x) = \frac{a_m + \frac{a_{m-1}}{x} + \cdots + \frac{a_0}{x^m}}{b_n x^{n-m} + \frac{b_{n-1}}{x^{m-n+1}} + \cdots + \frac{b_0}{x^m}}.$$

We see that as $x \to \infty$, the numerator approaches a_m which is nonzero and the denominator approaches 0. So the function as a whole becomes arbitrarily large (positive if $\frac{a_m}{b_n}$ is negative if $\frac{a_m}{b_n} < 0$). We say the numerator approaches ∞, while the denominator becomes arbitrarily small. Therefore the limit does not exist. Although in this situation we sometimes write

$$\lim_{x \to \infty} h(x) = \infty \text{ or } -\infty,$$

which we will say more about in Section 6.3.

Case 2: $n = m$ We again divide by the highest power of x, which in this case is x^n:

$$h(x) = \frac{a_m + \frac{a_{m-1}}{x} + \cdots + \frac{a_0}{x^m}}{b_n + \frac{b_{n-1}}{x} + \cdots + \frac{b_0}{x^n}}.$$

Now the numerator approaches 0 while the denominator approaches b_n. So the limit is $0/b_n = 0$.

Case 3: $m < n$ This is the case in our earlier example from Problem 6.2. We divide by x^m:

$$h(x) = \frac{a_m + \frac{a_{m-1}}{x} + \cdots + \frac{a_0}{x^m}}{b_n x^{n-m} + \frac{b_{n-1}}{x^{m-n+1}} + \cdots + \frac{b_0}{x^m}}.$$

As x approaches infinity, all the terms with powers of x in the denominator approach 0. So we have

$$\lim_{x \to \infty} h(x) = \frac{a_m}{b_n}.$$

To summarize:

- If the numerator has higher degree, the rational function grows without bound.
- If the denominator has higher degree, the rational function has a horizontal asymptote at 0.
- If they have the same degree, the rational function has a horizontal asymptote equal to the ratio of the leading coefficients.

Limits are ∞ approaches ∞ are not all that mysterious, and can be restated in terms of a finite limit.

Problem 6.4

Suppose f is a function whose domain includes $[0, \infty)$. Show that

$$\lim_{x \to \infty} f(x) = \lim_{y \to 0^+} f\left(\frac{1}{y}\right).$$

provided either limit is defined.

Solution for Problem 6.4: Intuitively, we see that y grows arbitrarily large as $\frac{1}{y} \to 0$ gets arbitrarily close to 0, and vice versa, so the result seems plausible. The proof is a matter of plowing through the ε-δ definitions.

First, suppose that $\lim_{x \to \infty} f(x) = L$, and let $\varepsilon > 0$ be given. We know that we can choose $N > 0$ such that

$$x > N \implies |f(x) - L| < \varepsilon.$$

But this last statement is exactly the definition of $\lim_{y \to 0^+} f\left(\frac{1}{y}\right) = L$.

Conversely, suppose that $\lim_{y \to 0^+} f\left(\frac{1}{y}\right) = L$, and let $\varepsilon > 0$ be given. We know that we can choose $\delta > 0$ such that

$$0 < |y - 0| < \delta \implies \left|f\left(\frac{1}{y}\right) - L\right| < \varepsilon.$$

Now, we can choose $N = \frac{1}{\delta}$. Then, if $x > N$, we have $\frac{1}{x} < \delta$, so

$$\left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon.$$

This is the definition of $\lim_{x \to \infty} f(x) = L$.

Therefore, the two limits are equal (provided they are defined). $\square$

A statement similar to that of Problem 6.4 exists for limits on $-\infty$ approaches $-\infty$; we will leave this as an exercise.

There is a special case of a function f to which we know that $\lim_{x \to \infty} f(x)$ must exist. If f is an increasing function and $f(x)$ is bounded above, then f cannot continue increasing forever, but must approach some value as x approaches ∞. Not surprisingly, the value that f must approach is the least upper bound of $f(x)$ We can make this statement more precise.

Problem 6.5

Suppose that f is a monotonically increasing function and $f(x)$ is bounded above that is, there exists some M such that $f(x) \leq M$ for all x. Show that $\lim_{x \to \infty} f(x)$ exists and is equal to $\sup \text{Ran}(f)$.

Solution for Problem 6.5: Let $L = \sup \text{Ran}(f)$ we know that L must exist since $\text{Ran}(f)$ is a subset of $\mathbb{R}$ that has an upper bound (namely M), so by the axiom of the real numbers (from Chapter 1), $\text{Ran}(f)$ must have a least upper bound. We wish to show that $\lim_{x \to \infty} f(x) = L$. This means by definition that for any $\varepsilon > 0$ we must find $N > 0$ such that

$$x > N \implies |f(x) - L| < \varepsilon.$$

Because L is an upper bound, we have $f(x) - L \leq 0$ for all $x \in \text{Dom}(f)$ so the above statement is equivalent to

$$x > N \implies |f(x) - L| < \varepsilon.$$

We simply choose any N such that $f(N) > L - \varepsilon$. Such an N must exist, because if it doesn't, then $L - \varepsilon$ is an upper bound for f, contradicting the fact that L is the least upper bound for f. Then, because f is increasing, we have $f(x) \geq f(N) > L - \varepsilon$ for any $x > N$, so we have $\square$

Essentially the same argument shows that if f is monotonically decreasing and is bounded below, then

$$\lim_{x \to \infty} f(x) = \inf \text{Ran}(f).$$

Exercises

6.5.1:

Compute the following limits:

(a) $\lim_{x \to \infty} \frac{2x^2 - x^3 + 5x - 2}{3x^3 + 2x^2 - x + 8}$

Hide Solution

Hint

You may type any additional notes you have here.

Your Submission:

Solution: The limit as $x \to \infty$ of a rational function with equal degree in the numerator and denominator is the ratio of the leading coefficients, which in this problem is $\frac{-1}{3}$. We can also compute this explicitly by dividing both numerator and denominator by x^3, which yields

$$\lim_{x \to \infty} \frac{2x^2 - x^3 + 5x - 2}{3x^3 + 2x^2 - x + 8} = \lim_{x \to \infty} \frac{\frac{2}{x} - 1 + \frac{5}{x^2} - \frac{2}{x^3}}{3 + \frac{2}{x} - \frac{1}{x^2} + \frac{8}{x^3}}$$
$$= \frac{0 + 0 + 0 - 1}{3 + 0 + 0 + 0}$$
$$= \frac{-1}{3}$$

(b) $\lim_{x \to \infty} \frac{2x^2 + x^3 + 5}{x^3 + 2x^2 + 4x + 8}$

Hide Solution

Hint

You may type any additional notes you have here.

Your Submission:

Solution: Since the degree of the numerator is greater than the degree of the numerator, the limit is ∞. We can also compute this explicitly:

$$\lim_{x \to \infty} \frac{2x^2 + x^3 + 5}{x^3 + 2x^2 + 4x + 8} = \lim_{x \to \infty} \frac{\frac{2}{x} + 1 + \frac{5}{x^3}}{1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3}} = \infty$$

(c) $\lim_{x \to \infty} \frac{x^2 + \sin x}{2x^2 + \sin x}$

Hint: Divide by numerator and denominator by x^2.

Hide Solution

Hint

You may type any additional notes you have here.

Your Submission:

Solution: As with a rational function, we divide by the highest power of x:

$$\lim_{x \to \infty} \frac{x^2 + \sin x}{2x^2 + \sin x} = \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x^2}}{2 + \frac{\sin x}{x^2}}.$$

As x gets large, $\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$ by the Squeeze Theorem,

$$0 = \lim_{x \to \infty} \left(-\frac{1}{x^2}\right) \leq \lim_{x \to \infty} \frac{\sin x}{x^2} \leq \lim_{x \to \infty} \frac{1}{x^2} = 0,$$

so $\lim_{x \to \infty} \frac{\sin x}{x^2} = 0$, and thus

$$\lim_{x \to \infty} \frac{x^2 + \sin x}{2x^2 + \sin x} = \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x^2}}{2 + \frac{\sin x}{x^2}} = \frac{1}{2}.$$

6.1.2:

Suppose f is an function with $\lim_{x \to \infty} f = L$ and $\lim_{x \to \infty} g = M$. Prove that

(a) $\lim_{x \to \infty} (f + g) = L + M$

Hide Solution

Hint

You may type any additional notes you have here.

Your Submission:

Solution: Let $\varepsilon > 0$ be given. We know that there exist numbers N_1 and N_2 such that

$$x > N_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$
$$x > N_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

Let N be the larger of these two numbers. Then by the Triangle Inequality, for any $x > N$ we have

$$|(f + g)(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \varepsilon.$$

(b) $\lim_{x \to \infty} (cf) = cL$ for any $c \in \mathbb{R}$

Hide Solution

Hint

You may type any additional notes you have here.

Your Submission:

Solution: If $c = 0$, then the statement is clearly true. Assume $c \neq 0$ and let $\varepsilon > 0$ be given. We know that there exists some N such that

$$x > N \implies |f(x) - L| < \frac{\varepsilon}{|c|}.$$

For this N,

$$x > N \implies |(cf)(x) - cL| = |c| \cdot |f(x) - L| < |c| \frac{\varepsilon}{|c|} = \varepsilon,$$

and hence $\lim_{x \to \infty} (cf) = cL$.

6.1.3:

Show that $\lim_{x \to \infty} f(x) = \lim_{y \to 0^+} f\left(\frac{1}{y}\right)$.

Hint: This is essentially the same argument as Problem 6.4 with some inequalities reversed.

Hide Solution

Hint

You may type any additional notes you have here.

Your Submission:

Solution: First, suppose that $\lim_{x \to \infty} f(x) = L$, and let $\varepsilon > 0$ be given. We know that we can choose $N > 0$ such that

$$x > N \implies |f(x) - L| < \varepsilon.$$

Let $\delta = \frac{1}{N}$. Then

$$-\delta < y < 0 \implies \frac{1}{y} < N \implies |f\left(\frac{1}{y}\right) - L| < \varepsilon.$$

But this last statement is exactly the definition of $\lim_{y \to 0^+} f\left(\frac{1}{y}\right) = L$.

Conversely, suppose that $\lim_{y \to 0^+} f\left(\frac{1}{y}\right) = L$, and let $\varepsilon > 0$ be given. We know that we can choose $\delta > 0$ such that

$$-\delta < y < 0 \implies |f\left(\frac{1}{y}\right) - L| < \varepsilon.$$

Let $N = \frac{1}{\delta}$. Then

$$x > N \implies -\delta < \frac{1}{x} < 0$$
$$\implies |f\left(\frac{1}{x}\right) - L| < \varepsilon$$
$$\implies |f(x) - L| < \varepsilon.$$

This is the definition of $\lim_{x \to \infty} f(x) = L$.

6.1.4:

If $f(x)$ is differentiable and $\lim_{x \to \infty} f(x) = L$, then what can we say about $\lim_{x \to \infty} f'(x)$?

Hint: There are two possibilities.

Hint: Try to construct an example where $\lim_{x \to \infty} f'(x)$ does not exist. Trig functions are usually the best examples of functions that don't have nice asymptotic behavior as $x \to \infty$.

Hint 2: If $\lim_{x \to \infty} f'(x)$ does exist, what must it be?

Hide Solution

Hint

You may type any additional notes you have here.

Your Submission:

Solution: One possibility is that the limit of the derivative might not exist. For example, let $f(x) = \frac{1}{x} \sin x^2$. Note that for all $x \in \mathbb{R}$,

$$-\frac{1}{x} \leq f(x) \leq \frac{1}{x},$$

and as $x \to \infty$, the $-\frac{1}{x}$ and $\frac{1}{x}$ terms approaches 0, but the $2\cos x^2$ term oscillates between -2 and 2, so $\lim_{x \to \infty} f'(x)$ does not exist.

On the contrary, if $\lim_{x \to \infty} f'(x)$ does exist, we expect that it is 0. This makes sense geometrically: the function f has a horizontal asymptote at $y = L$, so the slope approaches the slope of the line $y = L$, which is 0. To prove this, assume that the limit exists but is nonzero. For simplicity, assume that $\lim_{x \to \infty} f'(x) = L > 0$. If the limit is less than 0, we can consider the function $-f$, and the proof is the same.

By definition, for any $\varepsilon > 0$, there exists some N such that

$$x > N \implies |f'(x) - L| < \varepsilon.$$

Choose $\varepsilon = \frac{L}{2}$. Then

$$x > N \implies |f'(x) - L| < \frac{L}{2} \implies \frac{L}{2} < f'(x) < \frac{3L}{2}.$$

Specifically, for sufficiently large x, $f'(x) > \frac{L}{2}$.

Now let $x > N$. We apply the Mean Value Theorem. For every $x > N$, there exists some $\xi \in (N, x)$ such that

$$f'(\xi) = \frac{f(x) - f(N)}{x - N}.$$

Thus

$$x = \lim_{x \to \infty} f'(x) \geq f'(\xi) = \frac{f(x) - f(N)}{x - N} \geq \frac{L}{2}.$$

But the quantity on the right is unbounded, giving a contradiction.

In summary, $\lim_{x \to \infty} f'(x) \neq L$ for any $L > 0$.

6.2 Limits of infinity

In Section 6.1, we discussed what it means for there to be a limit of $f(x)$ as x approaches ∞ or $-\infty$. Here, we discuss the related question of what it means for the value of the function to approach ∞; that is, the meaning of

$$\lim_{x\to a} f(x) = \infty,$$

for some $a \in \mathbb{R}$. Informally, this should mean that as x gets close to a, the value of $f(x)$ approaches ∞, meaning the value of $f(x)$ grows arbitrarily large. But of course we want to define this rigorously.

Definition: Let f be a real-valued function and $a \in \mathbb{R}$. We say

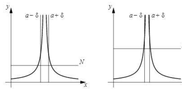
$$\lim_{x\to a} f(x) = \infty$$

if, for all N, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) > N.$$

In words, this means that we can make $f(x)$ become arbitrarily close to ∞–that is, we can require $f(x) > N$ for any given N–by choosing x sufficiently close to a–that is, by choosing δ such that $0 < |x - a| < \delta$.

We can also represent this definition graphically, as in the picture below. Given any value of N, we can find a value of δ so that the entire graph between $x = a - \delta$ and $x = a + \delta$ lies above $y = N$. If we increase N, then we can always find a smaller δ, as shown in the second picture. Everything between the vertical lines at $x = a - \delta$ and $x = a + \delta$ lies above the horizontal line $y = N$.



There's a similar definition for

$$\lim_{x\to a} f(x) = -\infty,$$

but we will leave the statement of this definition as an exercise.

WARNING!! $\lim_{x\to a} f(x) = \infty$ is just a convenient notation. It is important not to think of ∞ as a number. $\lim_{x\to a} f(x) = \infty$ means that $f(x)$ grows arbitrarily large as x approaches a. At no point does $f(x)$ "equal" ∞.

Let's look at a quick example of the definition.

Problem 6.6

Show that $\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty$.

Solution for Problem 6.6: Let $f(x) = \frac{1}{(x-2)^2}$. Intuitively, we see that as x gets close to 2, the denominator is positive but very close to 0, so the value of the $f(x)$ gets very large. To show this rigorously, we have to show that, given any N, we can find a δ satisfying the definition.

So, given N, we need to find δ such that

$$0 < |x - 2| < \delta \quad \Rightarrow \quad \frac{1}{(x-2)^2} > N.$$

Note that if $N \leq 0$, then any δ will work, since $f(x)$ is positive-valued. Thus we can assume that $N > 0$. The condition $\frac{1}{(x-2)^2} > N$ means that $(x-2)^2 < \frac{1}{N}$, so $|x - 2| < \sqrt{\frac{1}{N}} = \frac{1}{\sqrt{N}}$.

So we simply choose $\delta = \frac{1}{\sqrt{N}}$, and we're done. □

We can also define one-sided limits that approach ∞ or $-\infty$, in the obvious way that is analogous to one-sided finite limits:

Definition: Let f be a real-valued function and $a \in R$.

- $\lim_{x\to a^+} f(x) = \infty$ means that for all N, there exists $\delta > 0$ such that
$$0 < x - a < \delta \quad \Rightarrow \quad f(x) > N.$$
- $\lim_{x\to a^-} f(x) = \infty$ means that for all N, there exists $\delta > 0$ such that
$$0 < a - x < \delta \quad \Rightarrow \quad f(x) > N.$$

These one-sided limits are useful for a function such as $f(x) = \frac{1}{x}$, shown to the right. The graph of this function approaches the line $x = 0$, so we want to think of this function as having a vertical asymptote at $x = 0$. But $\lim_{x\to 0} \frac{1}{x}$ is undefined, since the function grows without bound as x approaches 0 from the right, but decreases without bound as x approaches 0 from the left. Instead, we can write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Again, note that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

This example motivates our definition of vertical asymptote:

Definition: We say that the graph of a function f has a **vertical asymptote** at $x = a$ if

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

(Note that $\pm\infty$ means that the limit is ∞ or $-\infty$.)

Geometrically, this means that the graph of f approaches the line $x = a$ as x approaches a from one side.

In Problem 6.3, we determined the horizontal asymptote (if any) of a rational function. Next, we examine the vertical asymptotes of such a function:

Problem 6.7

What are the vertical asymptotes of $h(x) = \frac{f(x)}{g(x)}$**, where** f **and** g **are polynomials?**

Solution for Problem 6.7: Clearly, if a is such that $g(a) \neq 0$, then $x = a$ cannot be a vertical asymptote, since then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} \in \mathbb{R}.$$

We might we tempted to "conclude" that:

Bogus Solution: If r is a root of $g(x)$ (so that $g(r) = 0$), then there will be a vertical asymptote at $x = r$.

But this is not necessarily the case. For example, the function

$$h(x) = \frac{x^2 - 1}{x - 1}$$

does not have a vertical asymptote at $x = 1$, since

$$\lim_{x\to 1} h(x) = \lim_{x\to 1} (x + 1) = 2.$$

It is true that if r is a root of g (the denominator of the rational function) but not of f (the numerator of the function), then $h = f/g$ will have a vertical asymptote at $x = r$. However, if r is a root of the numerator and denominator, then we divide both by the common factor $(x - r)$ and recompute. □

Concept: To find vertical asymptotes of a rational function $\frac{f(x)}{g(x)}$, first cancel out all factors corresponding to common roots of f and g. Then, any remaining roots of g will be vertical asymptotes of $\frac{f}{g}$.

Exercises

6.2.1:

(a) Write a definition of $\lim_{x\to a} f(x) = -\infty$.

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Solution: Let f be a real-valued function and $a \in \mathbb{R}$. We say $\lim_{x\to a} f(x) = -\infty$ if, for all N, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) < N.$$

(b) Write a definition of $\lim_{x\to\infty} f(x) = \infty$.

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Solution: Let f be a real-valued function. We say $\lim_{x\to\infty} f(x) = \infty$ if, for all N, there exists M such that

$$x > M \quad \Rightarrow \quad f(x) > N.$$

6.2.2:

Compute the following limits:

(a) $\lim_{x\to 2^+} \frac{1}{x^2 - 3x + 2}$

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Solution: We factor the denominator:

$$\begin{aligned}\lim_{x\to 2^+} \frac{1}{x^2 - 3x + 2} &= \lim_{x\to 2^+} \frac{1}{(x-2)(x-1)} \\ &= \left(\lim_{x\to 2^+} \frac{1}{x-2}\right)\left(\lim_{x\to 2^+} \frac{1}{x-1}\right) \\ &= \lim_{x\to 2^+} \frac{1}{x-2}.\end{aligned}$$

Since $x - 2 > 0$ as x approaches 2 from the right, the limit is $\boxed{+\infty}$.

(b) $\lim_{x\to 1^-} \frac{x-1}{x^3 - x^2 + x - 1}$

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Solution: Since $x - 1$ is a factor of both the numerator and denominator,

$$\begin{aligned}\lim_{x\to 1^+} \frac{x-1}{x^3 - x^2 + x - 1} &= \lim_{x\to 1^+} \frac{x-1}{(x-1)(x^2+1)} \\ &= \lim_{x\to 1} \frac{1}{x^2+1} \\ &= \boxed{\frac{1}{2}}.\end{aligned}$$

(c) $\lim_{x\to\frac{\pi}{2}^-} \tan x$

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Solution: Since $\tan$ increases without bound on $\left(0, \frac{\pi}{2}\right)$, we suspect that $\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$. We can prove this using the definition of limit. Let $N > 0$ be given and let $\delta = \frac{\pi}{2} - \arctan N$. Note that $\delta \in \left(0, \frac{\pi}{2}\right)$, and hence

$$\begin{aligned}&0 < \frac{\pi}{2} - x < \delta \\ \Rightarrow \quad &\tan\left(\frac{\pi}{2} - x\right) < \tan\delta \\ \Rightarrow \quad &\tan x > \tan\left(\frac{\pi}{2} - \delta\right) = N.\end{aligned}$$

This proves

$$\lim_{x\to\frac{\pi}{2}^-} \tan x = \boxed{+\infty}.$$

6.2.3:

Find a continuous function f such that $\lim_{x\to\infty} f(x)$ is undefined (in particular, this limit is neither a for any $a \in \mathbb{R}$ nor $\pm\infty$).

Hint: Look at trig functions.

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Solution: One example is $f(x) = \cos x$. Note that $\cos 2n\pi x = 1$ and $\cos(2n+1)\pi x = -1$ for all positive integers n. Thus, $\cos$ does not increase nor decrease without bound, so the limit cannot be $\pm\infty$. Also, if $\lim_{x\to\infty} \cos x = a$ for some $a \in \mathbb{R}$, then we must have $|1 - a| < \epsilon$ and $|-1 - a| < \epsilon$ for all $\epsilon > 0$, and clearly this cannot be the case (for instance, choosing $\epsilon = \frac{1}{2}$ and applying the Triangle Inequality gives $2 < 1$, which is not true).

Informally, what is happening is that $\cos x$ oscillates between 1 and -1 infinitely often, so cannot approach a single value as x gets large.

6.3 Rational indeterminate forms and l'Hôpital's Rule

In Problems 4.5 and 6.1, we determined the asymptotics of rational functions—functions that are quotients of polynomials. We'd like to be able to study more generally functions of the form $\frac{f(x)}{g(x)}$ when f and g are arbitrary functions, and in particular describe their behavior as x approaches ∞ or $-\infty$.

For example, you probably already believe that exponentials "dominate" polynomials. This means that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$$

for any polynomial function f. However, why do we believe this? We have an idea that exponentials "grow faster" than polynomials, but how do we make this feel computable?

Let us first examine a related question that is slightly more tractable. Suppose f and g are differentiable functions such that $f(a)=g(a)=0$ that some point a. How can we compute

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

We immediately see the difficulty: although the function $\frac{f(x)}{g(x)}$ is continuous where it is defined, it is not defined at $x=a$, since $g(a)=0$. So we can't just "plug in" $x=a$ to compute the limit.

But the key here is that the functions are not just continuous they are differentiable, and that gives us an additional tool to work with—specifically, we have the tool of tangent line approximation. See if you can work it out from here.

Problem 6.8

Let f and g be differentiable functions at a with $f(a)=g(a)=0$ and $g'(a)\neq 0$. Our goal is to compute

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

Replace $f(x)$ and $g(x)$ with their tangent line approximations at $x=a$, and explain why we can conclude that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

Solution for Problem 6.8: The tangent line approximation of f at $x=a$ is

$$f(x)\approx f(a)+f'(a)(x-a),$$

and similarly the tangent line approximation of g at $x=a$ is

$$g(x)\approx g(a)+g'(a)(x-a).$$

This gives us the approximation

$$\lim_{x\to a}\frac{f(x)}{g(x)}\approx\lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}.$$

But recall that $f(a)=g(a)=0$. So we're left with

$$\lim_{x\to a}\frac{f(x)}{g(x)}\approx\lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)}.$$

If $a\neq a$, we can cancel the $x-a$ terms and we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}\approx\frac{f'(a)}{g'(a)}.$$

The above argument is only informal and is not rigorous, because we approximated the limit with the tangent line approximations of f and g. Specifically, we'd like to get rid of that "$\approx$" symbol and say

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

It turns out this is OK to do. Recall that in Problem 4.22, we defined

$$E_f(x)=f(x)-(f(a)+f'(a)(x-a))$$

to be the error of the linear approximation of f, and we also showed that $\lim_{x\to a}\frac{E_f(x)}{x-a}=0$. Thus, defining $E_g(x)$ similarly as the error of the linear approximation of g, we have

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)}&=\lim_{x\to a}\frac{f(a)+f'(a)(x-a)+E_f(x)}{g(a)+g'(a)(x-a)+E_g(x)}\\&=\lim_{x\to a}\frac{f'(a)+\frac{E_f(x)}{x-a}}{g'(a)+\frac{E_g(x)}{x-a}}\\&=\frac{f'(a)}{g'(a)}\end{aligned}$$

□

Putting this all together gives us:

> **Important:** Basic version of **l'Hôpital's Rule**: If f,g are differentiable functions with $f(a)=g(a)=0$ and $g'(a)\neq 0$ then
>
> $$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

In fact, l'Hôpital's Rule works with $g'(a)=0$ too! But then the quotient of the derivatives becomes a limit too. We have:

> **Important:** More general version of l'Hôpital's Rule: If f,g are differentiable functions with $f(a)=g(a)=0$ then
>
> $$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

In this more general version, if $f'(a)=g'(a)=0$ then we can apply l'Hôpital's Rule again if necessary. We will show the details of the proof in Section 6.A.

Here's a basic example of l'Hôpital's Rule in practice:

Problem 6.9

Compute $\lim_{x\to 0}\frac{\sin x}{x}$.

Solution for Problem 6.9: We've seen this limit before, but having l'Hôpital's Rule in the toolbox makes it easy:

If we let $f(x)=\sin x$ and $g(x)=x$, then $f(0)=g(0)=0$. l'Hôpital's Rule says that we can evaluate $\frac{f'(0)}{g'(0)}$ to find the limit.

We see that $f'(x)=\cos x$, so $f'(0)=1$ and $g'(x)=1$, so $g'(0)=1$. Thus, the limit is $\frac{f'(0)}{g'(0)}=\frac{1}{1}=1$. □

A function $\frac{f(x)}{g(x)}$ where $f(x)=g(x)=0$ is called a **$\frac{0}{0}$ indeterminate form** at $x=a$. We can use l'Hôpital's Rule to compute limits that are $\frac{0}{0}$ indeterminate forms.

Problem 6.10

Compute $\lim_{x\to 1}\frac{x^n-3x-1}{4^x-1}$.

Solution for Problem 6.10: The numerator and denominator are 0 at $x=0$, so this limit is a $\frac{0}{0}$ indeterminate form, and we can use l'Hôpital's Rule by taking the derivative of both the numerator and the denominator:

$$\lim_{x\to 1}\frac{x^n-3x-1}{4^x-1}=\lim_{x\to 1}\frac{nx^{n-1}-3}{4^x\ln 4}.$$

We try to compute the limit by plugging in $t=1$ so that this is still a $\frac{0}{0}$ indeterminate form at $t=0$. No problem, we can use l'Hôpital's Rule again:

$$\lim_{x\to 1}\frac{x^n-3x-1}{4^x-1}=\lim_{x\to 1}\frac{nx^{n-1}-3}{4^x\ln 4}=\lim_{x\to 1}\frac{4x^{n-1}}{4^x\ln 4}.$$

Now we plug in $t=1$ and see that the limit is $\frac{n}{4}$. □

A very nice feature of l'Hôpital's Rule is that it is flexible, so that it can be used in many situations other than $\frac{0}{0}$ indeterminate forms. One such situation is to compute a limit of a quotient of functions towards ∞:

Problem 6.11

Suppose f,g are differentiable functions with

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=0.$$

Use the substitution $t=\frac{1}{x}$ to show that l'Hôpital's Rule applies "at infinity", that is,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)},$$

provided this last limit is defined.

Solution for Problem 6.11: We use the result of Problem 6.4. If we substitute $t=\frac{1}{x}$ then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{t\to 0}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)}.$$

But $\lim_{t\to 0}f\left(\frac{1}{t}\right)=\lim_{x\to\infty}f(x)=0$, and similarly $\lim_{t\to 0}g\left(\frac{1}{t}\right)=\lim_{x\to\infty}g(x)=0$. So l'Hôpital's Rule applies, and we have

$$\lim_{t\to 0}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)}=\lim_{t\to 0}\frac{\frac{d}{dt}f\left(\frac{1}{t}\right)}{\frac{d}{dt}g\left(\frac{1}{t}\right)}=\lim_{t\to 0}\frac{-\frac{1}{t^2}f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2}g'\left(\frac{1}{t}\right)}=\lim_{t\to 0}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}.$$

Putting this all together gives

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{t\to 0}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)}=\lim_{t\to 0}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)},$$

as desired. □

Of course, eventually the same argument works for limits as x approaches $-\infty$, we will leave the details as an exercise.

Yet another flavor of l'Hôpital's Rule deals with infinite limits:

> **Important:** **l'Hôpital's Rule** for infinite limits: If f,g are differentiable functions with
>
> $$\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=\pm\infty,$$
>
> then
>
> $$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$
>
> provided this last limit is defined. This also works as x approaches $\pm\infty$ if
>
> $$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=\pm\infty,$$
>
> then
>
> $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

These statements are considerably harder to prove, so we will leave the proofs to section 6.A. Not surprisingly, a function $\frac{f(x)}{g(x)}$ where $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=\infty$ is called an **$\frac{\infty}{\infty}$ indeterminate form** at $x=a$. We can use l'Hôpital's Rule to compute limits at $\frac{\infty}{\infty}$ indeterminate forms, and in particular we can use it to address the question posed at the beginning of this section:

Problem 6.12

Suppose f is a polynomial function. Compute $\lim_{x\to\infty}\frac{f(x)}{e^x}$.

Solution for Problem 6.12: We know that $\lim_{x\to\infty}f(x)=\pm\infty$ for any non-constant polynomial f and also $\lim_{x\to\infty}e^x=\infty$. Thus $\frac{f(x)}{e^x}$ is an $\frac{\infty}{\infty}$ indeterminate form as x approaches ∞, and we can apply l'Hôpital's Rule, noting that $\frac{d}{dx}e^x=e^x$:

$$\lim_{x\to\infty}\frac{f(x)}{e^x}=\lim_{x\to\infty}\frac{f'(x)}{e^x}.$$

This helps because the degree of f' is one less than the degree of f. If f is non-constant, then this is still an $\frac{\infty}{\infty}$ indeterminate form, and we can continue to apply l'Hôpital's Rule. After n applications of step n^{th}, but the degree of the numerator will decrease by 1 again. Repeating this process will eventually lead to a polynomial with degree 0 in the numerator, which is a constant. Specifically, if deg $f=n$, then

$$\lim_{x\to\infty}\frac{f(x)}{e^x}=\lim_{x\to\infty}\frac{f'(x)}{e^x}=\cdots=\lim_{x\to\infty}\frac{f^{(n)}(x)}{e^x}=\lim_{x\to\infty}\frac{c}{e^x},$$

for some constant c. But this limit is 0, since the denominator grows without bound while the numerator is constant. Thus

$$\lim_{x\to\infty}\frac{f(x)}{e^x}=0.$$

□

Problem 6.12 is a special case of a more general property that we examine for a quotient of functions:

> **Definition:** We say that a function f **dominates** a function f if
>
> $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=0.$$

Informally, this means that $g(x)$ "grows faster" than $f(x)$ as x grows arbitrary large. In our previous examples, we have seen that higher degree polynomials dominate lower degree polynomials, and that exponential functions dominate all polynomials. As an exercise, you will show that all polynomials dominate the ever-slow function $\ln x$.

We often have to be a little careful using l'Hôpital's Rule, as in the following example:

Problem 6.13

Compute $\lim_{x\to\infty}\frac{x+\sin x}{2x+3}$.

Solution for Problem 6.13: This function is an $\frac{\infty}{\infty}$ indeterminate form. So we use l'Hôpital's Rule to compute the limit. We take the derivatives of numerator and denominator:

$$\lim_{x\to\infty}\frac{x+\sin x}{2x+3}=\lim_{x\to\infty}\frac{1+\cos x}{2}.$$

However, this latter limit is undefined, since cosine oscillates and does not approach any value. This might lead to the wrong conclusion:

> **Bogus Solution:** l'Hôpital's Rule tells us that
>
> $$\lim_{x\to\infty}\frac{x+\sin x}{2x+3}=\lim_{x\to\infty}\frac{1+\cos x}{2}.$$
>
> Since this latter limit is undefined, the original limit is also undefined.

No! This just means that we can't apply l'Hôpital's Rule to compute the original limit. Then, how do we calculate the original limit? We do it in the way that we first computed limits at ∞ for rational functions, by dividing by the highest power of x:

$$\lim_{x\to\infty}\frac{x+\sin x}{2x+3}=\lim_{x\to\infty}\frac{1+\frac{\sin x}{x}}{2+\frac{3}{x}}.$$

Now the answer is clear: the two terms with x in the denominator approach 0 as x approaches ∞, and thus the limit is $\frac{1}{2}$. □

Exercises

6.3.1:

Compute the following limits:

(a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

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Solution: l'Hôpital's Rule gives

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\boxed{\frac{1}{2}}.$$

(b) $\lim_{x\to\frac{\pi}{2}}\frac{\sin 3x}{\cos 5x}$.

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Solution: l'Hôpital's Rule gives

$$\lim_{x\to\frac{\pi}{2}}\frac{\sin 3x}{\cos 5x}=\lim_{x\to\frac{\pi}{2}}\frac{3\cos 3x}{-5\sin 5x}=\boxed{-\frac{3}{5}}.$$

(c) $\lim_{x\to\infty}\frac{e^{x^2}}{e^{x^3}}$.

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Solution: l'Hôpital's Rule gives

$$\lim_{x\to\infty}\frac{e^{x^2}}{e^{x^3}}=\lim_{x\to\infty}\frac{2xe^{x^2}}{3x^2e^{x^3}}=\lim_{x\to\infty}\frac{2}{3x}=\boxed{0}.$$

(The last step is simply cancelling the e^{x^2} terms.)

(d) $\lim_{x\to\infty}\frac{\left(\log x\right)^2}{x}$.

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Solution: Since $\log x$ grows (negatively) without bound as $x\to 0^+$, and $x^+\to 1$ is negative and bounded near 0, the limit is $\boxed{\infty}$.

6.3.2:

Show that f is a non-constant polynomial, then f dominates the sine and cosine functions.

Hint: Use the Squeeze Theorem.

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Solution: Note that $\lim_{x\to\infty}|f(x)|=\infty$. So for any non-constant polynomial f, and in particular $f(x)\neq 0$ for sufficiently large x. Thus, for sufficiently large x,

$$0\leq\left|\frac{\sin x}{f(x)}\right|\leq\frac{1}{|f(x)|}.$$

Thus, by the Squeeze Theorem, since $\lim_{x\to\infty}\frac{1}{|f(x)|}=0$, we have $\lim_{x\to\infty}\left|\frac{\sin x}{f(x)}\right|=0$, and thus $\lim_{x\to\infty}\frac{\sin x}{f(x)}=0$. Therefore, $f(x)$ dominates $\sin x$.

The exact same proof works for cosine just replace "sin" with "cos" in the above argument.

6.3.3:

Show that any non-constant polynomial dominates the function $f(x)=\sqrt{x}$.

Hint: $\sqrt{x}=x^{\frac{1}{2}}$.

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Solution: Let

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$$

where $n\geq 1$ and $a_n\neq 0$. Then

$$\lim_{x\to\infty}\frac{\sqrt{x}}{f(x)}=\lim_{x\to\infty}\frac{x^{\frac{1}{2}}}{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}$$

$$=\lim_{x\to\infty}\frac{x^{\frac{1}{2}}}{x^n(a_n+a_{n-1}x^{-1}+\cdots+a_0x^{-n})}$$

$$=0,$$

since $n-\frac{1}{2}>0$ forces the denominator to grow without bound.

6.3.4:

If we apply l'Hôpital's Rule to

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

what happens?

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Solution: We note that as $h\to 0$, the numerator and denominator both approach 0. Therefore l'Hôpital's Rule is applicable.

Treating x as a constant, we have

$$\frac{d}{dh}(f(x+h)-f(x))=f'(x+h)$$

and

$$\frac{d}{dh}(h)=1.$$

Taking the limits as $h\to 0$, we get

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f'(x+h)}{1}=f'(x),$$

assuming that f' is continuous. This just recaptures our original definition of the derivative.

6.3.5:

Show that l'Hôpital's Rule works for $\frac{0}{0}$ indeterminate forms as $x\to-\infty$.

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Solution: If we substitute $u=1/x$, then

$$\lim_{x\to-\infty}\frac{f(x)}{g(x)}=\lim_{u\to 0^-}\frac{f(1/u)}{g(1/u)}.$$

But $\lim_{u\to 0^-}f(1/u)=\lim_{x\to-\infty}f(x)=0$, and similarly $\lim_{u\to 0^-}g(1/u)=\lim_{x\to-\infty}g(x)=0$. So l'Hôpital's Rule applies, and we have

$$\lim_{u\to 0^-}\frac{f(1/u)}{g(1/u)}=\lim_{u\to 0^-}\frac{\frac{d}{du}f(1/u)}{\frac{d}{du}g(1/u)}$$

$$=\lim_{u\to 0^-}\frac{-\frac{1}{u^2}f'(1/u)}{-\frac{1}{u^2}g'(1/u)}$$

$$=\lim_{u\to 0^-}\frac{f'(1/u)}{g'(1/u)}.$$

Undoing the transformation gives

$$\lim_{x\to-\infty}\frac{f(x)}{g(x)}=\lim_{u\to 0^-}\frac{f(1/u)}{g(1/u)}$$

$$=\lim_{u\to 0^-}\frac{f'(1/u)}{g'(1/u)}$$

$$=\lim_{x\to-\infty}\frac{f'(x)}{g'(x)},$$

as desired.

6.4 Exponential indeterminate Forms

As we saw in the last section, l'Hôpital's Rule is a very useful tool for computing limits of quotients of functions that produce indeterminate forms. But l'Hôpital's Rule has even more powerful uses than in the examples we have already seen.

Our motivation is the following classic limit:

Problem 6.14

Our goal is to compute

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

(a) Let $y=\left(1+\frac{1}{x}\right)^x$. Write $\log y$ as a quotient of functions of x.

(b) Use l'Hôpital's Rule to compute $\lim_{x\to\infty}\log y$.

(c) Compute the original limit.

Solution for Problem 6.14: At first glance, this doesn't really look like a limit for which we can use l'Hôpital's Rule. But taking the logarithm is the key to converting it into a more traditional indeterminate form.

> **Concept:** We can often eliminate unpleasant exponential terms by taking logarithms.

(a) We let $y=\left(1+\frac{1}{x}\right)^x$. Then

$$\log y = x\log\left(1+\frac{1}{x}\right).$$

But this can be rewritten as a quotient of functions:

$$\log y=\frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

Now we see that as $x\to\infty$, this limit goes to $\frac{0}{0}$ so we have a $\frac{0}{0}$ indeterminate form.

(b) We can apply l'Hôpital's Rule:

$$\lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}.$$

This looks ugly, but the $-\frac{1}{x^2}$ terms cancel, and we have

$$\lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)}=1.$$

(c) We can undo the logarithm by taking the exponential of both sides of the above equation, and this preserves the limit because the exponential function is continuous. Thus, we have

$$\lim_{x\to\infty}y=e^1=e,$$

and hence

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$$

□

This produces one of the classical formulas for the magic number e:

> **Important:**
>
> $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$$

Problem 6.14 is an example that shows that l'Hôpital's Rule is useful for more than just $\frac{0}{0}$ and $\frac{\infty}{\infty}$ indeterminate forms. Let's look at a more complicated example:

Problem 6.15

Compute $\lim_{x\to 0^+}x^{\sin x}$.

Solution for Problem 6.15: We write $f(x)=x^{\sin x}$. Taking the logarithm of both sides, we get

$$\log f(x)=\log\left(x^{\sin x}\right)=(\sin x)(\log x).$$

We'd like this in a form to which we can apply l'Hôpital's Rule, so write this as

$$\log f(x)=\frac{\log x}{\csc x}.$$

Now it is an $\frac{\infty}{\infty}$ indeterminate form as $x\to 0^+$, so we apply l'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to 0^+}\log f(x)&=\lim_{x\to 0^+}\frac{\log x}{\csc x}\\&=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x}\\&=-\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x}.\end{aligned}$$

This is now a $\frac{0}{0}$ indeterminate form, so we apply l'Hôpital's Rule again:

$$\lim_{x\to 0^+}(\log f(x))=-\lim_{x\to 0^+}\frac{2\sin x\cos x}{\cos x-x\sin x}.$$

At $x=0$, the numerator of the last expression is 0 and the denominator is 1, so the limit is 0. Thus,

$$\lim_{x\to 0^+}\log f(x)=0,$$

and hence $\lim_{x\to 0^+}f(x)=e^0=1$. □

> **Concept:** Converting exponential indeterminate forms into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate forms uses the fact that "exponentials commute with limits," by which we mean
>
> $$e^{\lim f}=\lim e^f$$
>
> for any function f. This works precisely because the exponential function is continuous. Specifically, we start with
>
> $$h(x)=f(x)^{g(x)}$$
>
> and take logs to get
>
> $$\log h(x)=\log\left(f(x)^{g(x)}\right)=g(x)(\log f(x)).$$
>
> If we find that the limit of the rightmost expression above is L, then
>
> $$\lim\log h(x)=L,$$
>
> which gives $\lim h(x)=e^L$.

Exercises

6.4.1:

Source: (c) HMMT

Compute the following limits:

(a) $\lim_{x\to 0^+}x\log x$

You may type any additional notes you have here.

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Solution: We can write $x\log x=\frac{\log x}{\frac{1}{x}}$, and note that both numerator and denominator of this fraction approach $\pm\infty$ as $x\to 0^+$. Thus, by l'Hôpital's Rule,

$$\lim_{x\to 0^+}x\log x=\lim_{x\to 0^+}\frac{\log x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=\boxed{0}.$$

(b) $\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)$

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Solution: We can let $z=\frac{1}{x}$, and we compute:

$$\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)=\lim_{z\to 0}\frac{\sin z}{z}=\boxed{1}.$$

(c) $\lim_{x\to 1}x^{\left(\frac{x}{\sin(1-x)}\right)}$

Hint: It's not as hard as it looks. Start with the usual idea of writing it as $\exp\left(\lim_{x\to 1}\log\left(x^{\frac{x}{\sin(1-x)}}\right)\right)$.

Hint: Bring the exponent out to the front of the logarithm, and then you should have something to which you can apply l'Hôpital's Rule.

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Solution: This limit is just

$$\exp\left(\lim_{x\to 1}\log\left(x^{\frac{x}{\sin(1-x)}}\right)\right).$$

We can write expression inside the limit as $\frac{x\log x}{\sin(1-x)}$. As $x\to 1$, both numerator and denominator approach 0, so we can try l'Hôpital's Rule:

$$\lim_{x\to 1}\frac{x\log x}{\sin(1-x)}=\lim_{x\to 1}\frac{\log x+1}{-\cos(1-x)}=\frac{\log 1+1}{-\cos 0}=-1.$$

Exponentiating yields

$$\lim_{x\to 1}x^{\left(\frac{x}{\sin(1-x)}\right)}=e^{-1}=\boxed{\frac{1}{e}}.$$

6.4.2:

Show that $e=\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$.

You may type any additional notes you have here.

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Solution: Since we have an exponential expression, we try taking a logarithm:

$$\log\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}=\lim_{x\to 0^+}\log(1+x)^{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\log(1+x)}{x}.$$

Now we note that the numerator and denominator of this last fraction both approach 0 as $x\to 0^+$, so we use l'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{\log(1+x)}{x}=\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}=\lim_{x\to 0^+}\frac{1}{1+x}=1.$$

Thus, exponentiating gives the original limit as $e^1=\boxed{e}$.

6.4.3:

Compute $\lim_{x\to 0^+}(1+kx)^{\frac{1}{x}}$, where $k>0$ is a positive constant.

Hint: This looks a lot like the previous problem, so the same technique should work.

Preview: Solution

Solution

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Solution: As in the previous problem, we take the natural logarithm:

$$\log\lim_{x\to 0^+}(1+kx)^{\frac{1}{x}}=\lim_{x\to 0^+}\log(1+kx)^{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\log(1+kx)}{x}.$$

As $x\to 0^+$, this fraction approaches $\frac{\log(1)}{0}=\frac{0}{0}$ so we can use l'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{\log(1+kx)}{x}=\lim_{x\to 0^+}\frac{\frac{k}{1+kx}}{1}=\lim_{x\to 0^+}\frac{k}{1+kx}=k.$$

Therefore, the original limit is $\boxed{e^k}$.

6.4.4:

Suppose f and g are functions such that $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=\infty$. Explain how we can use l'Hôpital's Rule as a tool to compute $\lim_{x\to\infty}(f(x)-g(x))$.

Hint: Try to turn $f-g$ into a $\frac{0}{0}$ indeterminate form.

Hint: Convert $f-g$ to a quotient of fractions involving f and/or g.

Hint: Start with $\frac{f-g}{1}$ and divide numerator and denominator by fg.

You may type any additional notes you have here.

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Solution: We write

$$f-g=\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{fg}}.$$

This is now a $\frac{0}{0}$ indeterminate form at ∞, so we can apply l'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to\infty}(f(x)-g(x))&=\lim_{x\to\infty}\frac{-\frac{g'(x)}{(g(x))^2}+\frac{f'(x)}{(f(x))^2}}{-\frac{f'(x)g(x)+f(x)g'(x)}{(f(x)g(x))^2}}\\&=\lim_{x\to\infty}\frac{f^2(x)g'(x)-g^2(x)f'(x)}{f(x)g'(x)+g(x)f'(x)}.\end{aligned}$$

Of course, there's no guarantee that this limit will exist, but at least it gives us something to try!

So we need to make use of the definition of the integral. To compute the area of the region under the graph of $y = f(x)$ between $x = 0$ and $x = 1$, we divide the interval $[0, 1]$ into n equal subintervals, each of length $\Delta x = \frac{1}{n}$. We then approximate the area of the region under the graph of $y = f(x)$ between $x = 0$ and $x = 1$ by adding up the areas of the rectangles with heights the values of the function at the right hand endpoints of the subintervals, namely $f(\Delta x), f(2\Delta x), f(3\Delta x), \ldots, f(n\Delta x)$. The area of the rectangle corresponding to the subinterval $[k\Delta x, (k+1)\Delta x]$ is $f(k\Delta x)\Delta x$. So the total area of the rectangles is $\sum_{k=1}^{n} f(k\Delta x)\Delta x$.

Since the area of the region between the graph of $y = f(x)$ and the x-axis is the limit of the area of the rectangles as n tends to infinity, we have

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(k\Delta x)\Delta x$$

So it gives sense, we expect that the area under the curve on $[0, 1]$ approaches the area of the entire region as n tends to infinity. Specifically, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta x = \int_0^1 1\,dx = \int_0^1 \left(\frac{1}{1+x} + \frac{x}{1+x}\right) = \ln 2 + 1,$$

The integral is easy to evaluate.

$$\int_0^1 \frac{1}{1+x}\,dx = \int_0^1 \frac{x}{1+x}\,dx = \ln 2 - 1.$$

Thus, when we take the limit, we get that the area of the region is

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(k\Delta x)\Delta x = \int_0^1 \left(\frac{1}{1+x} + \frac{x}{1+x}\right) = \ln 2 + 1.$$

Now the "pseudo" have seen through the region is unbounded. Hence the area. Problem 5: suppose a typical definition:

Definition: Let f be a continuous function and $a \in \mathbb{R}$ such that $[a, \infty) \subset \mathrm{Dom}(f)$. We define the improper integral **converges**. Otherwise, we say that the improper integral **diverges**.

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

provided the limit is defined. If the limit is defined and $a \le c$, we say that the improper integral **converges**. Otherwise, we say that the improper integral **diverges**.

Let's generalize Problem 5.1:

Problem 5.1.1 ✎ ✓

Let $a \in \mathbb{R}$. Use the limit definition. Compute

$$\int_a^\infty \frac{1}{x^2}\,dx$$

Solution for Problem 5.1.1: By definition, we compute the improper integral by writing it into $\lim_{b\to\infty} \int_a^b f(x)\,dx$ and then

$$\lim_{b\to\infty} \int_a^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big|_a^b$$

The integrals

$$\lim_{b\to\infty} \left(\frac{1}{a} - \frac{1}{b}\right)$$

If $a > 1$, then the limit $\frac{1}{a}$ approaches 0 as b approaches ∞. Thus, in this case, the improper integral converges to $\frac{1}{a-1}$.

If $a = 1$, then the limit $\frac{1}{a}$ grows without bound as b approaches ∞. Thus, the improper integral diverges. We might also write

$$\int_1^\infty \frac{1}{x^2}\,dx = \infty.$$

Our original integration was not valid $a = 1$ as we have to do that case separately

$$\int_0^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{x^2}\,dx$$

As Figure towards infinity, this gives without bound as the integral diverges.

In summary:

$$\int_a^\infty \frac{1}{x^2}\,dx = \begin{cases} \frac{1}{a-1}, & \text{if } a > 1, \\ \infty, & \text{if } a \le 1. \end{cases}$$

Problem 5.1.2 ✎ ✓

Suppose f and g are continuous functions on $[a, \infty)$ and $f(x) \le g(x)$ for all $x \ge a$.

(a) Show that $\int_a^\infty f\,dx \le \int_a^\infty g\,dx$ if both converge, then

(b) Show that if both functions are positive, and $\int_a^\infty g\,dx$ converges then $\int_a^\infty f\,dx$ converges.

(c) Show that if both functions are positive, and $\int_a^\infty f\,dx$ diverges, then $\int_a^\infty g\,dx$ diverges.

Solution for Problem 5.1.2:

(a) For part (1), we know $\int_a^b f\,dx \le \int_a^b g\,dx$ for all $a \le b$, so then

$$\int_a^\infty f\,dx \le \int_a^\infty g\,dx.$$

Therefore,

$$\int_a^\infty f\,dx \le \int_a^\infty g\,dx.$$

and since limits preserve non-strict inequalities, we conclude that

$$\int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$$

(b) Define a function

$$F(x) = \int_a^x f(t)\,dt.$$

Note that F is an increasing function since $\int_a^x f(t)\,dt \le \int_a^b f(t)\,dt$ for $x \le b$ and $f(x) \ge 0$ for $x \ge a$. Also, since $f(t) \le g(t)$, $F(x) \le \int_a^x g(t)\,dt \le \int_a^\infty g(t)\,dt$ for $x \ge a$.

$$F(x) = \int_a^x f(t)\,dt \le \int_a^x g(t)\,dt \le \int_a^\infty g(t)\,dt$$

Thus F is increasing and has an upper bound. Hence, F is bounded, and so $\int_a^\infty f(x)\,dx$ converges. (c) As the result of Problem 6.5.6, the book

$$\int_a^\infty f(x)\,dx = \int_a^\infty g(x)\,dx$$

exists, so the integral converges.

This proof the contrapositive statement is just (b), so there is nothing additional to prove.

WARNING! We can only use the comparison tests in parts (b) and (c) of Problem 5.1.2 if both functions are positive. In a small example, if $f(x) = -1$ and $g(x) = 0$, then for any $a \in \mathbb{R}$, $\int_a^\infty f\,dx = \int_a^\infty g\,dx = \infty$ **provided** $\int_a^\infty f\,dx$ **converges**.

Problem 5.1.3 ✎ ✓

Let $a \in \mathbb{R}$. Compute

$$\int_a^\infty \frac{1}{x}\,dx$$

Solution for Problem 5.1.3: Matching the graph will immediately show the issue. We have $\frac{1}{x} > \frac{1}{x^2}$ on $[a, \infty)$. We proceed as in $\int_a^\infty \frac{1}{x^2}\,dx$ and then we find that the limit does not exist without a 0.

We can do essentially the same thing as for the improper integrals with a form of integration of x. We write

$$\int_a^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_a^b \frac{1}{x}\,dx$$

Note the $\frac{1}{x}$ case so only case about the interval $[0, 1]$ and we cannot what happens to the right of 1.

The integral is now easy to compute.

$$\int_a^b \frac{1}{x}\,dx = \ln|x|\Big|_a^b = \ln|b| - \ln|a|.$$

As $x \to 0^+$, this approaches ∞. Hence

$$\int_a^\infty \frac{1}{x}\,dx = \infty.$$

Once again, a seemingly infinite area turns out to be finite.

We can generalize the definition Problem 5.1.3:

Definition: Suppose f is a function continuous on $[a, b]$ such that $\lim_{x\to b^-} f(x) = \infty$. We define the **improper integral**

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

provided the limit is defined. If the limit is defined, we say that the improper integral **converges**. Otherwise, we say that the improper integral **diverges**.

Of course, we can do the same thing if the function is not defined at a or at b. ∞ and $-\infty$ ∞ ∞ and $-\infty$ and writing out the "formal definition".

Exercise: Note that the above definition is consistent with our usual improper integrals. In particular, if f is defined, then by the Fundamental Theorem of Calculus, the function

$$g(x) = \int_a^x f(t)\,dt$$

is differentiable, hence continuous, and thus

$$\int_a^b f(x)\,dx = g(a) = \lim_{x\to a^+} g(x) = \lim_{x\to a^+} \int_a^x f(t)\,dt$$

We know that regular improper integrals, we can break them apart at any point into two separate integrals. Specifically, if $c \in (a, b)$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This is also true we evaluate integrals that are improper at both ends, as long as the integral converges.

Problem 5.1.4 ✎ ✓

Let $a \in \mathbb{R}$. Compute $\int_a^\infty \sin x\,dx$. Use alternative when it diverges.

Solution for Problem 5.1.4: The usual way to do such an improper integral with ends is to split it sometimes is the middle. For example, we can write

$$\int_a^\infty \sin x\,dx = \int_a^0 \sin x\,dx + \int_0^\infty \sin x\,dx$$

(We didn't have to pick $x = 0$ as the point at which to split them, but it seems convenient since x^2 is nicely behaved at $x = 0$.) We already found in Problem 5.1.1 that $\int_0^\infty \sin x\,dx$ diverges, so $\int_a^\infty \sin x\,dx$ diverges.

Caution: We can break up integrals at any point in the interval (a, ∞) ∞ ∞ $\infty

6.A Proof of L'Hôpital's Rule

Before we can rigorously prove l'Hôpital's Rule, we need to have a more general version of the Mean Value Theorem.

Problem 6.35

Let f, g be continuous functions on an interval $[a, b]$ and differentiable on (a, b). Show that there exists $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

Note that if $g(x) = x$, then $g'(c) = 1$ for all c, and the above expression can be rearranged as

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which just the Mean Value Theorem.

Solution for Problem 6.35: Define a function h as

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

Notice that

$$h(a) = h(b) = f(a)g(b) - f(b)g(a),$$

thus by Rolle's Theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. Therefore,

$$0 = h'(c) = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)),$$

which can be rearranged to

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)),$$

as desired. □

The result of Problem 6.35 is called the **Extended Mean Value Theorem**.

> **Important:** **General statement of l'Hôpital's Rule**: In the following statement,
>
> - $\heartsuit$ can be a real number, a "one-sided real number" (for example, $\lim_{x \to 1^+}$), or $\pm\infty$, and
> - $\spadesuit$ can be 0 or $\pm\infty$.
>
> Let f, g be differentiable functions such that
>
> $$\lim_{x \to \heartsuit} f(x) = \lim_{x \to \heartsuit} g(x) = \spadesuit.$$
>
> Then
>
> $$\lim_{x \to \heartsuit} \frac{f(x)}{g(x)} = \lim_{x \to \heartsuit} \frac{f'(x)}{g'(x)},$$
>
> provided this latter limit exists.

We will first rigorously prove the "basic" case of l'Hôpital's Rule.

Problem 6.36

Prove l'Hôpital's Rule where $\spadesuit = 0, \heartsuit = a$ for some $a \in \mathbb{R}$, and

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

for some $L \in \mathbb{R}$.

Note: The modifications to the following proof in the cases that $\heartsuit = a^+$ or $\heartsuit = a^-$ should be clear, and we will not present them.

Solution for Problem 6.36: We alter f (if necessary) by setting $f(a) = 0$; note that since $\lim_{x \to a} f(x) = 0$ by assumption, this makes f continuous at a. Similarly alter g if necessary so that $g(a) = 0$.

We are assuming that $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$. This means that by definition, for any $\epsilon > 0$, we can choose $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon. \tag{6.A.1}$$

Note that this implies that $g'(x) \neq 0$ for all $0 < |x - a| < \delta$. This further implies that $g(x) \neq 0$ for all $0 < |x - a| < \delta$, since if $g(x) = 0 = g(a)$, then by the Mean Value Theorem there is some ξ between x and a such that $g'(\xi) = 0$, a contradiction.

We wish to show that

$$0 < |x - a| < \delta \quad \Rightarrow \quad \left| \frac{f(x)}{g(x)} - L \right| < \epsilon.$$

But $f(a) = g(a) = 0$, so we can write

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right|.$$

Now we apply the Extended Mean Value Theorem: there exists z between x and a such that

$$f'(z)(g(x) - g(a)) = g'(z)(f(x) - f(a)).$$

Furthermore, since we know that $g'(z) \neq 0$ and $g(x) \neq g(a)$, we may divide by $g'(z)(g(x) - g(a))$ to get

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(z)}{g'(z)},$$

and since $0 < |z - a| < |x - a| < \delta$, we have by (6.A.1) that

$$\epsilon > \left| \frac{f'(z)}{g'(z)} - L \right| = \left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right| = \left| \frac{f(x)}{g(x)} - L \right|.$$

Thus, by the δ-ϵ definition of limit, we conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L,$$

as desired. □

Problem 6.37

Modify the solution to Problem 6.36 to prove l'Hôpital's Rule where $\spadesuit = 0, \heartsuit = a$ for some $a \in \mathbb{R}$, and

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \pm\infty.$$

Solution for Problem 6.37: If L is $\pm\infty$, then all the statements in the solution to Problem 6.36 of the form

$$|\text{something } - L| < \epsilon$$

get replaced by statements of the form

$$\begin{aligned} &|\text{something}| > N \text{ (if } L = +\infty) \text{ or} \\ &|\text{something}| < N \text{ (if } L = -\infty), \end{aligned}$$

but all of the same arguments hold. □

Problem 6.38

Use Problem 6.4 to prove l'Hôpital's Rule for the case where $\heartsuit = \pm\infty$ and $\spadesuit = 0$.

Solution for Problem 6.38: By Problems 6.36 and 6.37, we know that l'Hôpital's Rule is true for $\heartsuit = 0$. Thus, we can prove the case where $\heartsuit = \pm\infty$ by replacing $f(x)$ and $g(x)$ by $f(1/x)$ and $g(1/x)$, respectively. In particular:

$$\begin{aligned} \lim_{x \to \pm\infty} \frac{f(x)}{g(x)} &= \lim_{u \to 0^+} \frac{f(1/u)}{g(1/u)} \\ &= \lim_{u \to 0^+} \frac{\frac{d}{du} f(1/u)}{\frac{d}{du} g(1/u)} \\ &= \lim_{u \to 0^+} \frac{(-1/u^2) f'(1/u)}{(-1/u^2) g'(1/u)} \\ &= \lim_{u \to 0^+} \frac{f'(1/u)}{g'(1/u)} \\ &= \lim_{x \to \pm\infty} \frac{f'(x)}{g'(x)}. \end{aligned}$$

□

The previous problems prove l'Hôpital's Rule in all cases where $\spadesuit = 0$. Unfortunately, the case where $\spadesuit = \pm\infty$ is considerably harder. We will prove one specific case of this, and leave it to the reader to extend it to other cases in a similar manner as above.

We first prove the following lemma:

Problem 6.39

Let f be a function such that $\lim_{x \to \infty} f(x) = \infty$, and let $a \in \mathbb{R}$. Show that

$$\lim_{x \to \infty} \frac{f(x) + a}{f(x)} = 1.$$

Solution for Problem 6.39: We simply compute:

$$\lim_{x \to \infty} \frac{f(x) + a}{f(x)} = \lim_{x \to \infty} \left(1 + \frac{a}{f(x)} \right) = 1 + 0 = 1.$$

□

Problem 6.40

Let f, g be differentiable functions such that

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty,$$

and

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$$

for some $L \in \mathbb{R}$. Prove that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L.$$

> **WARNING!!** The proof is quite difficult.

Solution for Problem 6.40: By definition, for any $\epsilon > 0$, there exists some $M > 0$ such that

$$x > M \quad \Rightarrow \quad \left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\epsilon}{4}. \tag{6.A.2}$$

(The reason that we use $\frac{\epsilon}{4}$ rather than ϵ will become more clear later in the proof.) Also, for all $x > M$, there exists, via the Extended Mean Value Theorem, some $z \in (M, x)$ such that

$$\frac{f(x) - f(M)}{g(x) - g(M)} = \frac{f'(z)}{g'(z)}.$$

(We have $g'(z) \neq 0$ and $g(x) \neq g(M)$ for essentially the same reason as in Problem 6.36; we will not repeat the details here.) This, combined with (6.A.2), means that

$$x > M \quad \Rightarrow \quad \left| \frac{f(x) - f(M)}{g(x) - g(M)} - L \right| < \frac{\epsilon}{4}.$$

However, we want to be able to compare $\frac{f(x)}{g(x)}$ to L, so we introduce some algebraic sleight-of-hand. Define a new function

$$j(x) = \frac{f(x)}{f(x) - f(M)} \cdot \frac{g(x) - g(M)}{g(x)},$$

so that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(M)}{g(x) - g(M)} \cdot j(x).$$

By Problem 6.39, we know that

$$\begin{aligned} \lim_{x \to \infty} j(x) &= \lim_{x \to \infty} \left(\frac{f(x)}{f(x) - f(M)} \right) \cdot \lim_{x \to \infty} \left(\frac{g(x) - g(M)}{g(x)} \right) \\ &= 1 \cdot 1 \\ &= 1. \end{aligned}$$

Thus, we can choose $N > M$ sufficiently large so that for all $x > N$,

$$|j(x) - 1| < \min \left\{ 1, \frac{\epsilon}{2|L|} \right\}.$$

Then for all $x > N$, we have $|j(x)| < 2$, and we compute (for $L \neq 0$):

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - L \right| &= \left| \left(\frac{f(x) - f(M)}{g(x) - g(M)} \right) j(x) - L \right| \\ &\leq \left| \left(\frac{f(x) - f(M)}{g(x) - g(M)} \right) j(x) - Lj(x) \right| + |Lj(x) - L| \\ &= |j(x)| \left| \frac{f(x) - f(M)}{g(x) - g(M)} - L \right| + |L| |j(x) - 1| \\ &< 2 \left(\frac{\epsilon}{4} \right) + |L| \left(\frac{\epsilon}{2|L|} \right) \\ &\leq \epsilon. \end{aligned}$$

(If $L = 0$, then we only require N sufficiently large so that $|j(x) - 1| < 1$ for all $x > N$, and in the above computation, the second term on the next-to-last line will be 0.)

In summary, we have shown that for any $\epsilon > 0$, there exists N such that

$$x > N \quad \Rightarrow \quad \left| \frac{f(x)}{g(x)} - L \right| < \epsilon,$$

thus we have established $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$. □

The proof that l'Hôpital's Rule holds as in Problem 6.40 for $L = \pm\infty$ will be left to the reader.

Challenge Problems

6.30:

The **gamma function** is defined for a positive real number z as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx.$$

(a) Compute $\Gamma(1)$.

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Your Submission: Solution

Solution: By definition,

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left. -e^{-x} \right|_{x=0}^{\infty} = \boxed{1}.$$

(b) Show that $\Gamma(z+1) = z\Gamma(z)$ for any $z > 0$.

Hint: Use integration by parts with $u = x^z$.

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Solution: We start with

$$\Gamma(z+1) = \int_0^\infty x^z e^{-x}\, dx.$$

We use integration by parts with $u = x^z, du = zx^{z-1}\, dx$ and $v = -e^{-x}, dv = e^{-x}\, dx$. This gives:

$$\begin{aligned}\Gamma(z+1) &= \left. -x^z e^{-x} \right|_0^\infty + \int_0^\infty zx^{z-1} e^{-x}\, dx \\ &= \left(\lim_{x\to-\infty} -x^z e^{-x} \right) + 0 + z\int_0^\infty x^{z-1} e^{-x}\, dx \\ &= z\Gamma(z),\end{aligned}$$

noting that the limit in the second line above is 0 because e^x dominates x^z as $x \to \infty$ (or you could prove this using l'Hôpital's Rule repeatedly). Note also that part (b) holds for any $z > 0$, even if z is not an integer.

(c) Use (a) and (b) to find a simple formula for $\Gamma(n)$, where n is a positive integer.

Hint: Compute $\Gamma(2), \Gamma(3), \Gamma(4)$ using your formula from part (b). Do you see the pattern?

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Solution: We have

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1, \\ \Gamma(3) &= 2\Gamma(2) = 2, \\ \Gamma(4) &= 3\Gamma(3) = 6,\end{aligned}$$

and so on, and we see (and can easily prove by induction if we so desire) that $\boxed{\Gamma(n) = (n-1)!}$.

6.31:

Suppose a,b,x,y are all positive and $a + b = 1$. Compute $\lim_{t\to0^+} (ax^t + by^t)^{\frac{1}{t}}$.

Preview: Solution

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Solution: Let $z(t) = (ax^t + by^t)^{\frac{1}{t}}$. Taking the log gives

$$\log z(t) = \frac{\log(ax^t + by^t)}{t}.$$

At $t = 0$ this is a $\frac{0}{0}$ indeterminate form (since $\log(a+b) = \log(1) = 0$), so l'Hôpital's Rule gives us

$$\begin{aligned}\lim_{t\to0^+} \log z(t) &= \lim_{t\to0^+} \frac{\log(ax^t + by^t)}{t} \\ &= \lim_{t\to0^+} \frac{\frac{1}{ax^t+by^t} \cdot \frac{d}{dt}(ax^t + by^t)}{1} \\ &= \lim_{t\to0^+} \frac{d}{dt}(ax^t + by^t).\end{aligned}$$

Recall that $\frac{d}{dt} c^t = c^t \log c$, where c is a constant, so the above limit is

$$\lim_{t\to0} (ax^t(\log x) + by^t(\log y)) = a\log x + b\log y = \log(x^a y^b).$$

Hence, upon exponentiation, we get that the original limit is $\boxed{x^a y^b}$.

6.32:

Recall from Section 5.2 **here** the **error function**, defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

It has that weird constant in the front because

$$\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

Using this fact, compute

$$\int_{-\infty}^\infty e^{-\frac{(x-a)^2}{b}}\, dx,$$

where a, b are real numbers with $b \neq 0$.

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Your Submission: Solution

Solution: First, note that since e^{-t^2} is an even function, we have

$$\int_{-\infty}^\infty e^{-t^2}\, dt = 2\int_0^\infty e^{-t^2}\, dt = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

If $b < 0$, then $e^{-\frac{(x-a)^2}{b}} \to \infty$ as $x \to \infty$, so the improper integral diverges. If $b > 0$, we make the substitution $u = \frac{x-a}{\sqrt{b}}$, so $du = \frac{1}{\sqrt{b}}\, dx$, and the limits of integration are the same (since u is linear in x and increasing as x is increasing). Therefore, our integral becomes

$$\sqrt{b}\int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{b} \cdot \sqrt{\pi} = \boxed{\sqrt{b\pi}}.$$

6.33★:

Source: [Sp]

Let f be a continuous function on $[0, 1]$. Evaluate $\lim_{x\to0^+} x\int_x^1 \frac{f(t)}{t}\, dt$.

Hint: Experiment with some simple functions, and take a guess at the answer. This should help suggest a strategy.

Hint: f is continuous on $[0, 1]$ so $|f(x)|$ has an upper bound on $[0, 1]$.

Hint: Compute $\lim_{x\to0^+} x\int_x^1 \frac{1}{t}\, dt$.

Hint: Write the limit as a fraction with the integral in the numerator and $\frac{1}{x}$ in the denominator, and use l'Hôpital's Rule.

Preview: Solution

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Solution: Since f is continuous, there is a maximum value M of $|f(x)|$ on [0,1]. Therefore,

\|\color{rgb}{0.35,0.35,0.35}
\left| \int_x^1 \frac{f(t)}{t}\,dt
\right| \le M \int_x^1 \frac{1}{t}\,dt.

So we compute $\lim_{x\to0^+} x\int_x^1 \frac{1}{t}\, dt$. The integral goes to infinity as x approaches 0, so we rewrite it as:

$$\lim_{x\to0^+} \frac{\int_x^1 \frac{1}{t}\, dt}{\frac{1}{x}},$$

and since this is an $\frac{\infty}{\infty}$ indeterminate form, we apply l'Hôpital's Rule:

$$\lim_{x\to0^+} \frac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to0^+} x = 0.$$

Therefore,

$$\lim_{x\to0^+} \left| x\int_x^1 \frac{f(t)}{t}\, dt \right| \le \lim_{x\to0^+} Mx\int_x^1 \frac{1}{t}\, dt = 0,$$

and thus the limit is $\boxed{0}$.

6.34★:

Source: Putnam

Evaluate $\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}}$.

Hint: How can we introduce some symmetry into the integral?

Hint: The substitution $\color{rgb}{0.35,0.35,0.35}u = x-1$ will make the integral look more symmetric.

Hint: Factor out what you can, then multiply and divide to get rid of terms with negative exponents.

Hint: At some point, the substitution $v = e^u$ will probably help.

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Solution: We'd like to take advantage of some symmetry, so we perform the change of variable $u = x - 1$. Then $du = dx$ and we have

$$\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}} = \int_0^\infty \frac{du}{e^{u+2} + e^{2-u}}.$$

Now we can factor e^2 out of the denominator:

$$\frac{1}{e^2}\int_0^\infty \frac{du}{e^u + e^{-u}}\, du.$$

We next try to get rid of the negative power of e in the denominator by multiplying numerator and denominator by e^u:

$$\frac{1}{e^2}\int_0^\infty \frac{e^u}{(e^u)^2 + 1}\, du.$$

Now the "obvious" substitution is $v = e^u, dv = e^u\, du$, so we have

$$\frac{1}{e^2}\int_1^\infty \frac{dv}{v^2 + 1} = \frac{1}{e^2} \left. \arctan v \right|_1^\infty.$$

This gives our answer:

$$\frac{1}{e^2}(\arctan\infty - \arctan 1) = \frac{1}{e^2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \boxed{\frac{\pi}{4e^2}}.$$

(Of course, "$\arctan\infty$" is an ugly shorthand for $\lim_{x\to\infty} \arctan x$.)

Review Problems

6.23:

Source: (c) Rice

Compute the following:

(a) $\lim_{x\to 1} \frac{x^2-1}{\log x}$.

Preview: Solution

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Your Submission: Solution

Solution: At $x = 1$, both numerator and denominator are 0, so l'Hôpital's Rule gives us

$$\lim_{x\to 1} \frac{x^2-1}{\log x} = \lim_{x\to 1} \frac{2x}{\frac{1}{x}} = \boxed{2}.$$

(b) $\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2}$.

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Your Submission: Solution

Solution: At $x = 0$, both numerator and denominator are 0, so l'Hôpital's Rule gives us

$$\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2} = \lim_{x\to 0} \frac{-2\sin x \cos x}{2x} = \lim_{x\to 0} \left(-\frac{\sin x}{x}\cos x\right).$$

This still evaluates to $\frac{0}{0}$ so we apply l'Hôpital's Rule again:

$$\lim_{x\to 0} \left(-\frac{\sin x}{x}\cos x\right) = \lim_{x\to 0} \left(\frac{-\sin^2 x + \cos^2 x}{1}\right) = \boxed{-1}.$$

(c) $\lim_{x\to 0} \frac{10x^2 - \frac{1}{2}x^3}{e^{5x^2} - 1}$.

Preview: Solution

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Solution: At $x = 0$, both numerator and denominator are 0, so applying l'Hôpital's Rule, and then performing some simplification, gives

$$\begin{aligned}\lim_{x\to 0} \frac{10x^2 - \frac{1}{2}x^3}{e^{5x^2} - 1} &= \lim_{x\to 0} \frac{20x - \frac{3}{2}x^2}{8xe^{5x^2}} \\ &= \lim_{x\to 0} \frac{x(20 - \frac{3}{2}x)}{8xe^{5x^2}} \\ &= \lim_{x\to 0} \frac{20 - \frac{3}{2}x}{8e^{5x^2}} \\ &= \frac{20}{8} \\ &= \boxed{\frac{5}{2}}.\end{aligned}$$

6.24:

Suppose a and b are nonzero real numbers. Find $\lim_{t\to 0} \frac{\sin at}{\sin bt}$ and $\lim_{t\to 0} \frac{\tan at}{\tan bt}$.

Hint: There's no need to use l'Hôpital's Rule for the quotient of tangents.

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Solution: By l'Hôpital's Rule, we have

$$\lim_{t\to 0} \frac{\sin at}{\sin bt} = \lim_{t\to 0} \frac{a\cos at}{b\cos bt} = \boxed{\frac{a}{b}}.$$

Rather than also using l'Hôpital's Rule on the ratio of tangents, we write it as

$$\lim_{t\to 0} \frac{\tan at}{\tan bt} = \lim_{t\to 0} \left(\frac{\sin at}{\sin bt} \cdot \frac{\cos bt}{\cos at}\right) = \frac{a}{b} \cdot 1 = \boxed{\frac{a}{b}}.$$

6.25:

Compute

(a) $\int_1^\infty e^{-2x}\,dx$

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Solution: We evaluate

$$\begin{aligned}\int_1^\infty e^{-2x}\,dx &= \lim_{b\to\infty} \int_1^b e^{-2x}\,dx \\ &= \lim_{b\to\infty} \left(-\frac{1}{2}e^{-2x}\Big|_1^b\right) \\ &= \lim_{b\to\infty} \left(-\frac{1}{2}e^{-2b} + \frac{1}{2}e^{-2}\right) \\ &= \frac{1}{2}e^{-2} \\ &= \boxed{\frac{1}{2e^2}}.\end{aligned}$$

It's also OK to write this without explicitly writing the limit:

$$\int_1^\infty e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_1^\infty = 0 + \frac{1}{2}e^{-2} = \frac{1}{2e^2}.$$

(b) $\int_0^1 \frac{1}{x^2}\,dx$

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Solution: This is improper at the $x = 0$ end. We have

$$\int_0^1 \frac{1}{x^2}\,dx = -\frac{1}{2x^2}\Big|_0^1.$$

But $\lim_{x\to 0^+} \frac{1}{2x^2} = \infty$, so the improper integral $\boxed{diverges}$.

(c) $\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx$

Hint: Complete the square in the denominator.

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Solution: Complete the square of the denominator as $x^2 + 2x + 2 = (x+1)^2 + 1$. Then

$$\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx = \int_{-\infty}^\infty \frac{1}{(x+1)^2+1}$$

Since this integral is evaluated over all of $\mathbb{R}$, we can do the change of variable $u = x + 1$, and the integral is still evaluated over all of $\mathbb{R}$. Thus we have

$$\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx = \int_{-\infty}^\infty \frac{1}{u^2+1}\,du = \tan^{-1} u\Big|_{-\infty}^\infty = \boxed{\pi}$$

6.26:

Source: HMMT

Compute $\int_0^4 \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

Hint: What substitution makes the fractional exponents go away?

Hint: Substitute $x = u^6$.

Hint: After simplification, the integral should no longer be improper.

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Solution: This is an improper integral. Make the substitution $x = u^6$. Then the limits of integration are the same, and $dx = 6u^5\,du$. Therefore, the integral becomes

$$\int_0^4 \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int_0^2 \frac{6u^5}{u^3+u^2}\,du = 6\int_0^2 \frac{u^3}{u+1}\,du.$$

This is no longer improper. We can evaluate this integral by rewriting the function as

$$\frac{u^3}{u+1} = \frac{(u^3+1)-1}{u+1} = u^2 - u + 1 - \frac{1}{u+1},$$

hence we have

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx &= 6\int_0^2 \left(u^2 - u + 1 - \frac{1}{u+1}\right)du \\ &= \left(2u^3 - 3u^2 + 6u - 6\log|u+1|\right)\Big|_0^2 \\ &= \boxed{5 - 6\log 2}.\end{aligned}$$

6.27:

Source: [Sp]

Compute $\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right)$.

Hint: Multiply and divide by the conjugate $\sqrt{x^2+x} + x$.

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Solution: We notice that $\sqrt{x^2+x}$ approaches x as $x \to \infty$, so this "$\infty - \infty$" form may look like it approaches 0, but this is incorrect. Instead, we eliminate the ambiguous term by multiplying and dividing by its conjugate:

$$\sqrt{x^2+x} - x = \frac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x} + x)}{\sqrt{x^2+x} + x} = \frac{x}{\sqrt{x^2+x} + x}.$$

The limit of the last term is

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}} + 1} = \boxed{\frac{1}{2}}.$$

6.28:

Sometimes, "little-o" notation is used to describe the growth rates of functions. Specifically, if f is a function, then $o(f)$ is the set of functions defined as:

$$o(f) = \left\{g \;\middle|\; \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0\right\}.$$

Prove the following:

(a) $o(f)o(g) \subseteq o(fg)$ (This means that if $h_1 \in o(f)$ and $h_2 \in o(g)$, then $h_1h_2 \in o(fg)$)

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Solution: If $h_1 \in o(f)$ and $h_2 \in o(g)$, then

$$\lim_{x\to\infty} \frac{h_1(x)}{f(x)} = \lim_{x\to\infty} \frac{h_2(x)}{g(x)} = 0.$$

Multiply these limits together to get

$$\lim_{x\to\infty} \frac{(h_1h_2)(x)}{(fg)(x)} = 0,$$

so $h_1h_2 \in o(fg)$.

(b) $o(o(f)) \subseteq o(f)$ (This means that if $h_1 \in o(f)$ and $h_2 \in o(h_1)$, then $h_2 \in o(f)$)

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Solution: If $h_1 \in o(f)$ and $h_2 \in o(h_1)$, then

$$\lim_{x\to\infty} \frac{h_1(x)}{f(x)} = \lim_{x\to\infty} \frac{h_2(x)}{h_1(x)} = 0.$$

Multiply these limits together to get

$$\lim_{x\to\infty} \frac{h_2(x)}{f(x)} = 0,$$

so $h_2 \in o(f)$.

(c) If f is a polynomial and $g(x) = e^x$, then $f \in o(g)$.

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Solution: We need $\lim_{x\to\infty} \frac{f(x)}{e^x} = 0$. But if $f(x)$ is degree n, we can simply apply l'Hôpital's Rule n times: the degree of f will decrease by 1 on each application, until we eventually are reduced to

$$\lim_{x\to\infty} \frac{f(x)}{e^x} = \lim_{x\to\infty} \frac{f^{(n)}(x)}{e^x},$$

where $f^{(n)}$ denotes the nth derivative of f. But $f^{(n)}$ is a constant, so the limit is 0, and $f \in o(\exp)$.

6.29:

For any $a \le b$, show that $\lim_{m\to\infty} \int_a^b \sin mx\,dx = 0$.

(In fact, it's true that if f is continuous on $[a, b]$, then $\lim_{m\to\infty} \int_a^b f(x)\sin mx\,dx = 0$, but this is much more difficult to prove.)

Hint: Just compute it.

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Solution: We just compute:

$$\int_a^b \sin mx\,dx = \frac{1}{m}\left(-\cos mx\right)\Big|_a^b = \frac{1}{m}(\cos ma - \cos mb).$$

The difference of the cosine terms is bounded (between -2 and 2), but $\frac{1}{m}$ approaches 0 as m grows large. So the limit of the integral is 0.

7.1 Infinite sequences

Definition: A **sequence** is a list of numbers.

Occasionally this list will be finite, but most often in calculus we deal with infinite sequences. We may write an infinite sequence as a list of numbers separated by commas, with an ellipsis $\ldots$ to indicate that the sequence is infinite, assuming the pattern is clear. For example,

$$1, 2, 4, 8, 16, \ldots$$

is the infinite sequence consisting of all the positive powers of 2.

However, more often we refer to sequences of a certain general form using variables like a_n:

where the **index** n of each term in the sequence is denoted using a subscript. We would denote the entire sequence by $\{a_n\}_{n=1}^{\infty}$. Sequences need not start at $n = 1$. For instance, some sequences might start at $n = 0$ rather than $n = 1$ so we would write such as sequence as $\{a_n\}_{n=0}^{\infty}$.

Note, however, that a sequence is not the same as a set (despite the similar-looking notation), because a key fact of a sequence is that the order matters. For example, the set $\{1, 2\}$ is the same set as $\{2, 1\}$ because the order of the elements in a set doesn't matter, but $\{a_n\}_{n=1}^{\infty}$ is the sequence $1, 2, 4, \ldots$ and $\{a_n\}_{n=1}^{\infty}$ is the sequence $2, 1, 4, \ldots$ and, of course, if the sequence is finite rather than infinite, we replace $\{a_n\}_{n=1}^{\infty}$ with the index of the last term, $\{a_n\}_{n=1}^{N}$.

Sequences are very useful tyes of sequence in calculus because we can use sequences in which the ratio between every two consecutive terms is constant. For example, the sequence

$$3, 6, 12, 24, 48, \ldots$$

is a geometric sequence with first term 3 and common ratio 2.

Naturally, there is a more formal definition:

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is a **geometric sequence** if there exists a real number r such that $a_{n+1} = ra_n$ for all $k \geq 1$. The number r is called the **common ratio** of the geometric sequence.

Problem 7.1.1: 🔖 ↗

Suppose $\{a_n\}_{n=1}^{\infty}$ is a geometric sequence with common ratio r. Write a formula for a_n in terms of r and a_1.

Suppose $\{a_n\}_{n=1}^{\infty}$ is a geometric sequence with common ratio r. When does this sequence converge, and to what value (in terms of r, and a_1)?

WARNING! It is a common mistake to write $a_n = r^n a_1$ for the nth term of the geometric series instead of $a_n = ra_{n-1}$. Although $r^n a_1$ is equal to ra_{n-1} if a_n is a geometric sequence, and as such we get the nth term, so the formula $a_n = r^n a_1$ is correct.

Sidenote: Although not used in calculus, the Fibonacci sequence occurs quite often in discrete mathematics. You can read more about the Fibonacci sequence and its applications in Art of Problem Solving's Intermediate Counting & Probability textbook.

Another way to define an infinite sequence is to define a recursive formula which relates each term in the sequence to previous terms (or possibly the nonnegative integers). Specifically we think of $\{a_n\}_{n=1}^{\infty}$ as a function a such that $a(n) = a_n$ for any positive integer n. This is sometimes called a **discrete function**, because the domain of a is the discrete set of integers.

A natural question that we can ask about an infinite sequence $\{a_n\}_{n=1}^{\infty}$ is whether it **converges** to any number. In other words, we can try to compute $\lim_{n\to\infty} a_n$. This leads to the idea of a limit of a sequence. The limit of a sequence is a number, so we need to define a function and not a function defined on all of $\mathbb{R}$.

Definition: We say that an infinite sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to L, also written

$$\lim_{n\to\infty} a_n = L,$$

if for any $\varepsilon > 0$ there exists a positive integer N such that for any $n > N$ we have $|a_n - L| < \varepsilon$.

WARNING! The sequence might converge and the function doesn't, so be careful! For example, the sequence $a_n = \cos(\pi n)$ converges to 0, but the function $f(x) = \sin(\pi x)$ does not have a limit as $x \to \infty$, even though $f(n) = a_n$ for all positive integers n.

Problem 7.1.2: 🔖 ↗

Let $\{a_n\}_{n=1}^{\infty}$ be a geometric sequence with common ratio r. When does this sequence converge, and to what value (in terms of r, and a_1)?

Solution for Problem 7.1.2: If $r = 0$, then the sequence is $a_1, 0, 0, 0, \ldots$, which clearly converges to 0. If $r \neq 0$, then we have $a_n = r^{n-1}a_1$ and we compute the

$$\lim_{n\to\infty} r^{n-1}a_1 = \lim_{n\to\infty} ar^n.$$

If $a_1 = 0$ then the limit is 0, and the sequence is the constant sequence in which every term is 0. Otherwise, if $a_1 \neq 0$, then the sequence converges if and only if r^n converges. This will converge if and only if $r \leq (-1, 1]$. Note that if $r = 1$ then the sequence is constant, and if $r = 1$, then the sequence converges to 1, also since the sequence does not converge if $r = -1$, since $(-1)^n$ does not converge as $r \to \infty$ but instead the sequence alternates between a_1 and $-a_1$.

Some sequences are a bit trickier to analyze. For example

Problem 7.1.3: 🔖 ↗

Does the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ converge?

Solution for Problem 7.1.3: This sequence consists of the values at positive integers of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$. We computed the limit of this function in Problem 6.14, and thus

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = e.$$

In many instances, we don't really care what value a particular infinite sequence converges to, we just want to know whether it converges or not. In these cases, we can consider the sequence of partial sums of the original sequence.

Problem 7.1.4: 🔖 ↗

Suppose that we have a sequence $\{a_n\}$ that is **increasing**, that is, $a_n \leq a_{n+1}$ for all $k \geq 1$. Show that the sequence converges if and only if it is bounded above, meaning that there is some number M such that $a_n \leq M$ for all positive integers n.

An example of such a sequence is

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots,$$

that is, the sequence given by $a_n = \frac{n-1}{n}$ for all positive integers n. This sequence is increasing, and clearly converges to 1.

Solution for Problem 7.1.4: Provided we can prove the statement above, we will prove that it converges. Thinking back to Chapter 1, what do we know about a nonempty set of real numbers that are bounded above? We know that the set has a least upper bound, called the **supremum** of the set. In our proof, this is one of the defining properties we use.

We claim that $\{a_n\}$ is increasing and bounded, then

$$\lim_{n\to\infty} a_n = \sup\{a_n\}.$$

Note that when we write $\sup\{a_n\}$, we are thinking of the elements of the sequence merely as a set and not as an ordered list.

To prove our claim, let $S = \sup\{a_n\}$ and let $\varepsilon > 0$. Since S is the least upper bound, $S - \varepsilon$ is not an upper bound. Because S does not, then $S - \varepsilon$ is an upper bound for $\{a_n\}$, contradicting the fact that S is the least upper bound for $\{a_n\}$. Then, since the sequence is increasing, we have

$$n > N \quad \Rightarrow \quad S \geq a_n \geq a_N > S - \varepsilon \quad \Rightarrow \quad 0 \leq S - a_n < \varepsilon.$$

Thus, by the definition of convergence, we have $\lim_{n\to\infty} a_n = S$.

We leave the opposite direction of the proof—that is, proving that if an increasing sequence converges, then it has an upper bound—as an exercise.

There is a similar statement for decreasing sequences: if a sequence $\{a_n\}$ is a decreasing sequence, so that $a_{n+1} \leq a_n$ for all $k \geq 1$, then the sequence converges if and only if it has a lower bound, and if this is the case, then the sequence converges to the greatest lower bound $\inf\{a_n\}$. The proof is virtually identical to the solution to Problem 7.4, so we will omit it.

We have some terminology:

Definition:
- A sequence $\{a_n\}_{n=1}^{\infty}$ is called **monotonically increasing** if $a_n \leq a_{n+1}$ for all $k \geq 1$.
- A sequence $\{a_n\}_{n=1}^{\infty}$ is called **monotonically decreasing** if $a_n \geq a_{n+1}$ for all $k \geq 1$.
- A sequence $\{a_n\}_{n=1}^{\infty}$ is called **strictly monotonically increasing** if $a_n < a_{n+1}$ for all $k \geq 1$.
- A sequence $\{a_n\}_{n=1}^{\infty}$ is called **strictly monotonically decreasing** if $a_n > a_{n+1}$ for all $k \geq 1$.
- In any of these cases, $\{a_n\}$ is called **monotonic**.

Thus our conclusion is:

Concept: A bounded monotonic sequence converges.

Let's look at a couple more sequence convergence examples.

Problem 7.1.5: 🔖 ↗

Consider the infinite sequence $a_n = \frac{n^2}{2^n}$. Does this sequence converge, and if so, to what value?

Solution for Problem 7.1.5: It sometimes helps to list a few terms, to get some idea of what's going on:

$$a_1 = \frac{1^2}{2^1} = 1,$$
$$a_2 = \frac{2^2}{2^2} = 1,$$
$$a_3 = \frac{3^2}{2^3} = \frac{9}{8},$$
$$a_4 = \frac{4^2}{2^4} = 1,$$
$$a_5 = \frac{5^2}{2^5} = \frac{25}{32}.$$

It seems that the sequence is strictly monotonically decreasing, and that 0 is the greatest lower bound. So we suspect that the sequence converges to 0. Let's try to prove both of these facts.

First, we want to show that $a_{n+1} < a_n$ for all $n \geq 3$. Since all the terms are positive, this is equivalent to showing that $\frac{a_{n+1}}{a_n} < 1$.

Concept: Examining the ratio of consecutive terms of a sequence is often a useful tool for getting information about the sequence.

Let's write out this expression:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{2n^2}$$
$$= \frac{n^2 + 2n + 1}{2n^2}$$
$$= \frac{\left(n + \frac{1}{n}\right)^2}{\left(n + \frac{1}{n}\right)^2}$$

So the sequence is strictly monotonically decreasing. Since it has a lower bound (namely, 0), we know it converges. We'd like to show that $0 = \inf\{a_n\}$ so that we can conclude that the sequence converges to 0.

Concept: Note the distinction between the previous two sentences. Knowing that a monotonic sequence converges means knowing that it has a bound, but it doesn't tell you what the value is. In order to find the greatest lower bound (for a decreasing sequence) or the least upper bound (for an increasing sequence).

To show that 0 is the greatest lower bound, we need to show that the sequence gets arbitrarily close to 0 as n grows large. To see this, let's write a_n in a slightly different way:

$$a_n = \frac{n^2}{2^n} = \frac{1}{n} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{4}{2} \cdots \frac{n}{2}.$$

The last term is $\frac{n}{2} = 1$, so we have

$$a_n = \frac{1}{n} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{4}{2} \cdots \frac{n-1}{2}.$$

We notice that at least half of these terms are less than or equal to $\frac{3}{2}$. In particular, the terms where the numerator is at most $\frac{3}{2}$, and the rest of the terms are less than 1. Therefore,

$$0 \leq a_n \leq \frac{1}{n} \cdot \left(\frac{3}{2}\right)^{\frac{n}{2}}.$$

Thus

$$0 \leq a_n \leq \frac{1}{(\sqrt{2})^n}$$

for all $n \geq 2$, and taking the limit as $n \to \infty$ and using the Squeeze Theorem, we have

$$0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{1}{(\sqrt{2})^n} = 0.$$

Thus $\inf\{a_n\} = \lim_{n\to\infty} a_n = 0$.

Problem 7.1.6: 🔖 ↗

Determine the sequence $a_n = \sqrt[n]{n}$ converges, and if so, to what value.

Solution for Problem 7.1.6: As in the previous problem, experimenting (perhaps with your calculator) might convince you that this sequence is monotonically decreasing and that 1 is a lower bound. So we might guess that it converges to 1.

We can try computing

$$\lim_{n\to\infty} \sqrt[n]{n}.$$

If this limit exists, then it is also the limit of our sequence (although, recall the warning in this section here—the sequence might converge and the function doesn't). Unfortunately, recalling that the function is $a(x) = x^{\frac{1}{x}}$, determinate from $x = \infty$, we're taking logs and using l'Hôpital's Rule. Specifically, we have that

$$\log x^{\frac{1}{x}} = \frac{\log x}{x}.$$

We see that $\frac{\log x}{x} \to \frac{\infty}{\infty}$, an indeterminate form, so we apply l'Hôpital's Rule, giving

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Taking exponentials, we get $\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$, and hence the original sequence converges to 1.

Exercises

7.1.1:

In each of the following, determine if the sequence $\{a_n\}_{n=1}^{\infty}$ converges, and, if it converges, what it converges to.

(a) $a_n = \frac{n^2 - 1}{n^2 + 2n^2 + 3n + 1}$

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Teacher Solution:

Solution: Just as we did with limits of rational functions, $a_n = \frac{1}{1 + \frac{2}{n} + \frac{3}{n^2} + \frac{1}{n^3}} \to \left[\frac{1}{1}\right] \to \infty$.

(b) $a_n = \frac{3n^2 + 2n^3}{n^3 + 4n^2}$

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Teacher Solution:

Solution Letting $f(x) = \frac{3x^2 + 2x^3}{x^3 + 4x^2}$, we apply l'Hôpital's Rule:

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} f(x)$$
$$= \lim_{x\to\infty} \frac{3x^2 + 2x^3}{x^3 + 4x^2}$$
$$= \lim_{x\to\infty} \frac{6x + 6x^2}{3x^2 + 8x}$$
$$= \lim_{x\to\infty} \frac{6 + 12x}{6x + 8}$$
$$= \lim_{x\to\infty} \frac{12}{6} = 2.$$

(c) $a_n = \frac{n \sin n}{n^2 + 1}$

Preview: Solution

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Teacher Solution:

Hint Use a bounding argument.

(d) $a_n = \sqrt[n]{n^2 + 1}$

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Teacher Solution:

Solution: We can manipulate this via difference of squares:

$$\sqrt{n^2+1} - \sqrt{n^2-1} = \frac{\sqrt{n^2+1} - \sqrt{n^2-1}}{1} \cdot \frac{\sqrt{n^2+1} + \sqrt{n^2-1}}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$
$$= \frac{\sqrt{n^2+1} - \sqrt{n^2-1}}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$
$$= \frac{2}{\sqrt{n^2+1} + \sqrt{n^2-1}}.$$

This is still a bit unpleasant to work with, so we use difference of squares again on the numerator:

$$\frac{\sqrt{n^2+1} - \sqrt{n^2-1}}{\sqrt{n^2+1} + \sqrt{n^2-1}} = \frac{\sqrt{n^2+1} - (n-1)}{\sqrt{n^2+1} + \sqrt{n^2-1}} \cdot \frac{\sqrt{n^2+1} + (n-1)}{\sqrt{n^2+1} + (n-1)}$$
$$= \frac{2n}{(\sqrt{n^2+1} + (n-1))(\sqrt{n^2+1} + (n-1))}.$$

Notice that

$$\sqrt{n^2+1} + (n-1) > n + n = 2n,$$

so

$$\frac{2n}{(\sqrt{n^2+1} + (n-1))} < \frac{2n}{(n+1)(n+1)}$$
$$= \frac{2n}{(n+1)^2}$$
$$= \frac{2n}{n^2+2n+1}$$
$$= \frac{2}{n+2+\frac{1}{n}}.$$

As n gets large, the denominator on the right gets arbitrarily large, so this fraction goes to zero. But this means that $\sqrt{n^2+1} - \sqrt{n^2-1} \to 0$.

(e) $a_n = \sqrt[n]{n^2+1}$

Hint Try taking the log of the sequence.

Preview: Solution

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Teacher Solution:

Solution: $\log a_n = \frac{\log(n^2+1)}{n}$. Then for all $n > 1$

$$0 \leq \log a_n = \frac{\log(n^2+1)}{n} \leq \frac{\log n^2}{n} = \frac{2\log n}{n}.$$

So $\frac{2\log n}{n} \to 0$ as $n \to \infty$, so $\log a_n \to 0$ as $n \to \infty$. Therefore, $a_n \to e^0 = \boxed{1}$.

7.1.2:

Prove that when each term in a geometric sequence is multiplied by a constant k, then the resulting sequence is also geometric.

Preview: Solution

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Teacher Solution:

Solution Suppose the original sequence has common ratio r. Let s and k be two consecutive terms in the original sequence; then we have $k = sr$. These terms become ks and kr in the new sequence, and we have $kr = k(rs) = r(ks)$ in the new sequence. Therefore, the new sequence is also geometric with common ratio r.

7.1.3:

(a) Show that nonzero numbers x, y and z are consecutive terms of a geometric sequence if and only if $y^2 = xz$.

Preview: Solution

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Teacher Solution:

Solution If x, y and z are consecutive terms in a geometric sequence with common ratio r, then $y = rx$ and $z = r^2x$ for some real number r. Then $y^2 = r^2x^2$ and $xz = r^2x^2$, so $y^2 = xz$. Conversely, if $y^2 = xz$, then dividing by xy gives us $\frac{y}{x} = \frac{z}{y}$, which means that x, y and z are consecutive terms of a geometric sequence (with common ratio $\frac{y}{x}$).

(b) Is it true that nonzero numbers x, y and z are consecutive terms of a geometric sequence if and only if $y = \sqrt{xz}$?

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Teacher Solution:

Solution In general, it is not true that if x, y and z are consecutive terms of a geometric sequence, then $y = \sqrt{xz}$. For example, take $x = 1, y = -2$, and $z = 4$. Then x, y and z are in consecutive terms of a geometric sequence with common ratio -2, but $\sqrt{xz} = 2 \neq y$.

7.1.4:

Prove a stronger version of the "only if" direction of Problem 7.6.4 for infinite sequences, then it is bounded.

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Teacher Solution:

Solution Suppose $\{a_n\}$ converges to L. Choose any $\varepsilon > 0$; then by definition of convergence there exists some N such that $L - \varepsilon < a_n < L + \varepsilon$ for all $n > N$. Let $M = \max\{a_1, a_2, \ldots, a_N, L + \varepsilon\}$. It follows, and thus also has an upper bound, specifically M. Conversely, if $\{a_n\}$ is bounded above, then it has a least upper bound, call it L. Then, by Problem 7.6.4, the same argument shows that the sequence has a lower bound the smaller of $L - \varepsilon$ and $\inf\{a_n\}$, so $\{a_n\}$ is bounded below.

7.2 Infinite series

Summing the elements of a sequence results in a **series**. If the sequence is finite, this is pretty easy, although the technical details of the definition may be somewhat more complicated than you are used to:

Definition: The **series** corresponding to the finite sequence $\{a_n\}_{n=1}^N$ is the finite sequence of **partial sums** of the initial terms of the sequence:

$$a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ \ldots,\ a_1+a_2+\cdots+a_N.$$

The final term $a_1+\cdots+a_N = \sum_{n=1}^N a_n$ is the **sum** of the series.

WARNING! The terminology can be somewhat ambiguous. We often refer to the "series" as simply the sum $\sum_{n=1}^N a_n$, even though this is really a number. However, inherent in the definition of "series" is the order of the terms that we are summing, which is why it is more proper to consider a series as a sequence of partial sums. This is not much of a concern with finite series, but as we will soon see it is more of an issue for infinite series.

More generally, sometimes we say that a "series" is the sum of a sequence, and other times we say that a series "is" a sequence. Usually the context makes it clear what we are talking about.

If our sequence is finite, then there's really no issue: we just sum the terms. In particular, if we have a finite sequence $\{a_n\}_{n=m}^N$ then the sum of the corresponding series is

$$a_1+a_2+\cdots+a_N = \sum_{n=1}^N a_n.$$

We see this with a basic example that you probably already know:

Problem 7.7

What is the sum of the series corresponding to the finite geometric sequence with r terms, first term c, and common ratio r?

Solution for Problem 7.7: Let's write the sum explicitly:

$$\text{Sum} = a+ar+r^2a+\cdots+r^{r-1}a$$
$$= a(1+r+r^2+\cdots+r^{r-1}).$$

There are two ways to finish from here. One way is to recall the factorization

$$r^n-1=(r-1)(r^{n-1}+r^{n-2}+\cdots+1).$$

Thus we can write our sum as

$$\frac{a(r^n-1)}{r-1}.$$

Of course, we can't do this if $r=1$ in that case, the sum is just ra.

The other method is to denote the sum as

$$S=a(1+r+r^2+\cdots+r^{n-1}),$$

so that

$$rS=a(r+r^2+r^3+\cdots+r^n).$$

When we subtract, most of the terms cancel:

$$rS-S=a(r^n-1),$$

and dividing by $r-1$ (again assuming $r\neq 1$) gives

$$S=\frac{a(r^n-1)}{r-1},$$

□

Concept: The sum of the series corresponding to the finite geometric sequence with n terms, first term a, and common ratio r is $\frac{a(r^n-1)}{r-1}$ if $r\neq 1$, and ra if $r=1$.

Much more interesting, and more useful to calculus, are sums of terms of infinite sequences, and in particular the "sum" of an entire infinite sequence. Since we can't literally evaluate an infinite sum, we do the most logical alternative: we take the limit of finite sums.

Definition: The **infinite series** corresponding to the infinite sequence $\{a_n\}_{n=1}^\infty$ is the infinite sequence of **partial sums** of the initial terms of the sequence:

$$a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ \ldots.$$

The **sum** of the series is defined the sum of the terms of the series; that is,

$$\sum_{n=1}^\infty a_n = \lim_{N\to\infty}\sum_{n=1}^N a_n.$$

If this limit exists and is equal to some real number S, we say that the series **converges** to S; if the limit is $\pm\infty$ or does not exist, we say that the series **diverges**.

WARNING! The terminology tends to be a bit sloppy. We often refer to the "infinite series" as the "sum" $\sum_{n=1}^\infty a_n$ even though this is really a limit of the partial sums in our series.

Let's look at our basic example:

Problem 7.8

What is the sum of the series corresponding to the infinite geometric sequence with first term a and common ratio r?

Solution for Problem 7.8: If $r=1$, then the geometric sequence is the constant sequence $a, a, a, \ldots$, and thus the corresponding series does not converge unless $a=0$. So let's assume that $r\neq 1$. We need to compute

$$\lim_{N\to\infty}\frac{a(r^N-1)}{r-1}.$$

If $a=0$, then the sum of the series is trivially 0, so let's further assume that a is nonzero.

We can factor out the constants:

$$\frac{a}{r-1}\lim_{N\to\infty}(r^N-1).$$

We now have cases depending on the value of r:

If $|r|<1$, then r^N approaches 0, and the sum becomes $\frac{a}{1-r}$. Note that this works regardless of whether r is positive or negative.

If $r=-1$ or $|r|>1$, then the limit of r^N diverges. So the sum diverges.

Important: Given an infinite geometric sequence with first term $a\neq 0$ and common ratio r, the corresponding series converges to $\frac{a}{1-r}$ if $|r|<1$, and diverges if $|r|\geq 1$.

Let's practice using this infinite series:

Problem 7.9 Source: AMC

The sum of an infinite geometric series with common ratio r such that $|r|<1$ is 15, and the sum of the squares of the terms of this series is 45. What is the first term of the series?

Solution for Problem 7.9: We can sum an infinite geometric series if we know the first term and the common ratio, but in this problem the first term is not given to us. However, we can let the first term be a and see if we can find any helpful equations to work with. The sum of the first series, in terms of a, is

$$\frac{a}{1-r}=15.$$

We can find the sum of the second series in terms of a and r in the same way. We note that the second series is geometric with first term a^2 and common ratio r^2, so we have the sum

$$\frac{a^2}{1-r^2}=45.$$

Thus, we now have two equations involving r and a:

$$\frac{a}{1-r}=15,$$
$$\frac{a^2}{1-r^2}=45.$$

To solve this system of equations most easily, we can multiply through by the denominators:

$$a=15(1-r),$$
$$a^2=45(1-r^2).$$

We notice that the first equation involves expressions that are both factors of the corresponding parts of the second equation. This gives us the idea to divide the second equation by the first equation, giving $a=3(1+r)$.

We now have two linear expressions for a, so we can equate these expressions and solve for r:

$$15-15r=3+3r,$$

giving $12=18r$ and $r=\frac{2}{3}$.

To finish the problem, we need the first term of the original series, which is just a, so

$$a=3+3r=3+2=5.$$

□

Exercises

7.2.1:

Suppose $\{a_n\}$ and $\{b_n\}$ are infinite sequences such that the corresponding infinite series converge:

$$\sum_{n=1}^\infty a_n=A \quad \text{and} \quad \sum_{n=1}^\infty b_n=B.$$

(a) Let $m\in\mathbb{R}$ and let $\{c_n\}$ be the sequence such that $c_n=ma_n$ for all n. Prove that

$$\sum_{n=1}^\infty c_n=mA.$$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: If $m=0$ then $\{c_n\}$ is simply the constant sequence 0, and the series clearly converges to $0=0A=mA$.

If $m\neq 0$ then let $\varepsilon>0$ be given. We must find N such that for any $n>N$,

$$\varepsilon>\left|\sum_{i=1}^n c_n-mA\right|=\left|\sum_{i=1}^n ma_n-mA\right|=m\left|\sum_{i=1}^n a_n-A\right|.$$

But by the convergence of $\sum_{n=1}^\infty a_n$ we know that we can find an N such that for all $n>N$,

$$\frac{\varepsilon}{m}>\left|\sum_{i=1}^n a_n-A\right|,$$

so that N will satisfy the above inequality, and we are done.

(b) Let $\{d_n\}$ be the sequence such that $d_n=a_n+b_n$ for all n. Prove that

$$\sum_{n=1}^\infty d_n=A+B.$$

Hint: You'll need to use the Triangle Inequality.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let $\varepsilon>0$ be given. We must find N such that for any $n>N$,

$$\varepsilon>\left|\sum_{i=1}^n d_n-(A+B)\right|.$$

But by the Triangle Inequality,

$$\left|\sum_{i=1}^n d_n-(A+B)\right|=\left|\sum_{i=1}^n a_n+\sum_{i=1}^n b_n-B\right|$$
$$\leq\left|\sum_{i=1}^n a_n-A\right|+\left|\sum_{i=1}^n b_n-B\right|.$$

By assumption, since $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ converge to A and B, respectively, we can find N sufficiently large so that both $\left|\sum_{i=1}^n a_n-A\right|$ and $\left|\sum_{i=1}^n b_n-B\right|$ are less than $\frac{\varepsilon}{2}$. Thus $\left|\sum_{i=1}^n d_n-(A+B)\right|$ is less than ε, as needed, and we are done.

7.2.2:

Sum each of the following infinite geometric series:

(a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The first term is $\frac{1}{2}$ and the common ratio is $\frac{1}{2}$, so the sum of the series is $\frac{1}{1-\frac{1}{2}}=\boxed{1}$.

(b) $192+144+108+\cdots$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The first term is 192 and the common ratio is $\frac{144}{192}=\frac{3}{4}$, so the sum of the series is $\frac{192}{1-\frac{3}{4}}=\boxed{768}$.

(c) $2-\sqrt{2}+1-\frac{\sqrt{2}}{2}+\frac{1}{2}-\cdots$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The first term is 2 and the common ratio is $\frac{-1}{\sqrt{2}}$, so the sum of the series is

$$\frac{2}{1-\left(\frac{-1}{\sqrt{2}}\right)}=\frac{2}{1+\frac{1}{\sqrt{2}}}$$
$$=\frac{2\sqrt{2}}{1+\sqrt{2}}$$
$$=\frac{2\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})}$$
$$=\boxed{4-2\sqrt{2}}.$$

7.2.3: Source: (c) Putnam

Let $\{a_n\}_{n=1}^\infty$ be an infinite sequence, and suppose that $a_n=b_n-b_{n+1}$ (for all $n\geq 1$) for some sequence $\{b_n\}_{n=1}^\infty$.

(a) Show that, if $\lim_{n\to\infty}b_n=0$, then $\sum_{n=1}^\infty a_n=b_1$ (We call this method of computing the sum **telescoping**.)

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Note that for any k,

$$\sum_{n=1}^k a_n=(b_1-b_2)+(b_2-b_3)+\cdots+(b_k-b_{k+1})=b_1-b_{k+1}.$$

Thus

$$\sum_{n=1}^\infty a_n=\lim_{k\to\infty}(b_1-b_{k+1})=b_1.$$

(b) Compute $\sum_{n=1}^\infty\frac{1}{n^2+n}$.

Hint: Use partial fractions to write the sum as a telescoping sum.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Write each term as $\frac{1}{n^2+n}=\frac{1}{n}-\frac{1}{n+1}$. Then by part (a), we have $\sum_{n=1}^\infty\frac{1}{n^2+n}=\boxed{1}$.

(c) Compute $\sum_{n=1}^\infty\frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)}$.

Hint: Again, try to use partial fractions.

Hint: You might have to make an educated guess, and then check that your guess works.

Hint: Try to get powers of 2 in the numerator.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: In order to use telescoping, we would like to write

$$\frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)}=\frac{C_n}{3^n-2^n}-\frac{C_{n+1}}{3^{n+1}-2^{n+1}}$$

for some sequence $\{c_n\}$. A little experimentation with small values of n will lead to the guess that $c_n=2^n$. Indeed, we verify that

$$\frac{2^n}{3^n-2^n}-\frac{2^{n+1}}{3^{n+1}-2^{n+1}}=\frac{3^{n+1}-2^{n+1}-3^{n+1}+2^{n+1}}{(3^{n+1}-2^{n+1})(3^n-2^n)}$$
$$=\frac{2(3^n)-2(3^n)}{(3^{n+1}-2^{n+1})(3^n-2^n)}$$
$$=\frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)}.$$

So the series is telescoping, and since

$$\lim_{n\to\infty}\frac{2^n}{3^n-2^n}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0,$$

we conclude that

$$\sum_{n=1}^\infty\frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)}=\sum_{n=1}^\infty\left(\frac{2^n}{3^n-2^n}-\frac{2^{n+1}}{3^{n+1}-2^{n+1}}\right)$$
$$=\frac{2^1}{3^1-2^1}$$
$$=\boxed{2}.$$

7.2.4: Source: AIDSME

Find the sum $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots$.

Hint: Break up the series into a sum of two geometric series.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The given series can be expressed as the sum of two infinite geometric series:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots=\left(\frac{1}{2}+\frac{1}{4}+\cdots\right)+\left(\frac{2}{4}+\frac{3}{8}+\cdots\right)$$
$$=\frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)}+\frac{\frac{1}{4}}{1-\left(\frac{1}{2}\right)}$$
$$=\frac{1}{2}+\frac{1}{2}$$
$$=\frac{3}{4}$$
$$=\boxed{\frac{3}{16}}.$$

Problem 7.1.1

Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$

Solution to Problem 7.1.1: Intuitively, we can reason as follows. When taking an infinite sum of terms of a sequence, at first you add terms that are large, but eventually you add terms that are so small that they don't matter anymore. Since the sum of the terms of the sequence is finite, then sequence-wise we can say that these sequences P are convergent.

Now, how do we prove this rigorously in a detailed manner? The proof between $\sum$ and ∞ is that if we can construct the infinite sequence S_n from the finite sum $\sum_{n=1}^{N}$ and show that S_n is convergent, then $\sum_{n=1}^{\infty}$ is convergent.

Let $S_n = \sum_{n=1}^{N} \frac{1}{n^2}$ be the sum of the series. There are three ways that S_n can behave as $N \to \infty$. If S_n increases across N, or S_n has the form

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \frac{1}{16^2} + \frac{1}{17^2} + \frac{1}{18^2} + \frac{1}{19^2} + \frac{1}{20^2} + \frac{1}{21^2} + \frac{1}{22^2} + \frac{1}{23^2} + \frac{1}{24^2} + \frac{1}{25^2} + \frac{1}{26^2} + \frac{1}{27^2} + \frac{1}{28^2} + \frac{1}{29^2} + \frac{1}{30^2} + \frac{1}{31^2} + \frac{1}{32^2} + \frac{1}{33^2} + \frac{1}{34^2} + \frac{1}{35^2} + \frac{1}{36^2} + \frac{1}{37^2} + \frac{1}{38^2} + \frac{1}{39^2} + \frac{1}{40^2} + \frac{1}{41^2} + \frac{1}{42^2} + \frac{1}{43^2} + \frac{1}{44^2} + \frac{1}{45^2} + \frac{1}{46^2} + \frac{1}{47^2} + \frac{1}{48^2} + \frac{1}{49^2} + \frac{1}{50^2} + \frac{1}{51^2} + \frac{1}{52^2} + \frac{1}{53^2} + \frac{1}{54^2} + \frac{1}{55^2} + \frac{1}{56^2} + \frac{1}{57^2} + \frac{1}{58^2} + \frac{1}{59^2} + \frac{1}{60^2} + \frac{1}{61^2} + \frac{1}{62^2} + \frac{1}{63^2} + \frac{1}{64^2} + \frac{1}{65^2} + \frac{1}{66^2} + \frac{1}{67^2} + \frac{1}{68^2} + \frac{1}{69^2} + \frac{1}{70^2} + \frac{1}{71^2} + \frac{1}{72^2} + \frac{1}{73^2} + \frac{1}{74^2} + \frac{1}{75^2} + \frac{1}{76^2} + \frac{1}{77^2} + \frac{1}{78^2} + \frac{1}{79^2} + \frac{1}{80^2} + \frac{1}{81^2} + \frac{1}{82^2} + \frac{1}{83^2} + \frac{1}{84^2} + \frac{1}{85^2} + \frac{1}{86^2} + \frac{1}{87^2} + \frac{1}{88^2} + \frac{1}{89^2} + \frac{1}{90^2} + \frac{1}{91^2} + \frac{1}{92^2} + \frac{1}{93^2} + \frac{1}{94^2} + \frac{1}{95^2} + \frac{1}{96^2} + \frac{1}{97^2} + \frac{1}{98^2} + \frac{1}{99^2} + \frac{1}{100^2}$$

In particular, we can compare the terms for $n = 1, 2, 3$

$$\frac{1}{1^2} = 1 \qquad \frac{1}{2^2} = \frac{1}{4} \qquad \frac{1}{3^2} = \frac{1}{9}$$

Then we get that $\frac{1}{1^2} > \frac{1}{2^2} > \frac{1}{3^2}$. So, this is true for all $N \in \mathbb{N}$ in the sum from some point onwards. $N = 3$ we can find $N > 3$ such that $\frac{1}{N^2} < \frac{1}{3^2}$. Thus, we can say that S_n is a decreasing sequence, $S_n \geq S_{n+1}$ for all $n \in \mathbb{N}$.

Taking the complement of Problem 7.1.1 gives us our lower bound on the sum of the series:

Important — **The Complement Test** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a decreasing sequence and that $\lim_{n\to\infty} \frac{1}{n^2} = 0$. So, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Our contrast will be to show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Another mathematician is greedy so

$$\frac{1}{n^2} \leq \frac{1}{n^2}$$

where $\frac{1}{n^2}$ is the series itself and number. This is called a **partial** $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is the **harmonic series**.

Problem 7.1.2

Prove that harmonic series converges and it is by ratio?

Solution to Problem 7.1.2: We can tell that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31} + \frac{1}{32} + \frac{1}{33} + \frac{1}{34} + \frac{1}{35} + \frac{1}{36} + \frac{1}{37} + \frac{1}{38} + \frac{1}{39} + \frac{1}{40} + \frac{1}{41} + \frac{1}{42} + \frac{1}{43} + \frac{1}{44} + \frac{1}{45} + \frac{1}{46} + \frac{1}{47} + \frac{1}{48} + \frac{1}{49} + \frac{1}{50} + \frac{1}{51} + \frac{1}{52} + \frac{1}{53} + \frac{1}{54} + \frac{1}{55} + \frac{1}{56} + \frac{1}{57} + \frac{1}{58} + \frac{1}{59} + \frac{1}{60} + \frac{1}{61} + \frac{1}{62} + \frac{1}{63} + \frac{1}{64} + \frac{1}{65} + \frac{1}{66} + \frac{1}{67} + \frac{1}{68} + \frac{1}{69} + \frac{1}{70} + \frac{1}{71} + \frac{1}{72} + \frac{1}{73} + \frac{1}{74} + \frac{1}{75} + \frac{1}{76} + \frac{1}{77} + \frac{1}{78} + \frac{1}{79} + \frac{1}{80} + \frac{1}{81} + \frac{1}{82} + \frac{1}{83} + \frac{1}{84} + \frac{1}{85} + \frac{1}{86} + \frac{1}{87} + \frac{1}{88} + \frac{1}{89} + \frac{1}{90} + \frac{1}{91} + \frac{1}{92} + \frac{1}{93} + \frac{1}{94} + \frac{1}{95} + \frac{1}{96} + \frac{1}{97} + \frac{1}{98} + \frac{1}{99} + \frac{1}{100}$$

It is clear that these series have a monotonically increasing sequence and because the growth rate of the sequence is slowing, we may suspect that it could grow to infinity.

However, if changes, and there is a sequence and an upper-bound point. The closed value is the group the terms to group of 2.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{24} + \frac{1}{25} + \frac{1}{26} + \frac{1}{27} + \frac{1}{28} + \frac{1}{29} + \frac{1}{30} + \frac{1}{31} + \frac{1}{32} + \frac{1}{33} + \frac{1}{34} + \frac{1}{35} + \frac{1}{36} + \frac{1}{37} + \frac{1}{38} + \frac{1}{39} + \frac{1}{40} + \frac{1}{41} + \frac{1}{42} + \frac{1}{43} + \frac{1}{44} + \frac{1}{45} + \frac{1}{46} + \frac{1}{47} + \frac{1}{48} + \frac{1}{49} + \frac{1}{50} + \frac{1}{51} + \frac{1}{52} + \frac{1}{53} + \frac{1}{54} + \frac{1}{55} + \frac{1}{56} + \frac{1}{57} + \frac{1}{58} + \frac{1}{59} + \frac{1}{60} + \frac{1}{61} + \frac{1}{62} + \frac{1}{63} + \frac{1}{64} + \frac{1}{65} + \frac{1}{66} + \frac{1}{67} + \frac{1}{68} + \frac{1}{69} + \frac{1}{70} + \frac{1}{71} + \frac{1}{72} + \frac{1}{73} + \frac{1}{74} + \frac{1}{75} + \frac{1}{76} + \frac{1}{77} + \frac{1}{78} + \frac{1}{79} + \frac{1}{80} + \frac{1}{81} + \frac{1}{82} + \frac{1}{83} + \frac{1}{84} + \frac{1}{85} + \frac{1}{86} + \frac{1}{87} + \frac{1}{88} + \frac{1}{89} + \frac{1}{90} + \frac{1}{91} + \frac{1}{92} + \frac{1}{93} + \frac{1}{94} + \frac{1}{95} + \frac{1}{96} + \frac{1}{97} + \frac{1}{98} + \frac{1}{99} + \frac{1}{100}$$

Note that we can continue the process indefinitely, and that each row (after the first) will be greater than $\frac{1}{2}$. Thus, the sum grows without bound, and the series diverges.

The reasoning that we used in the previous solution is a basic example of the **Cauchy Condensation Test** which our solution is that test for convergence of a series of non-negative terms, whose terms are monotonically decreasing. The Cauchy condensation test states that:

Important — **The Cauchy Condensation Test** $\sum_{n=1}^{\infty} \frac{1}{n}$ is a decreasing sequence and that $\lim_{n\to\infty} \frac{1}{n} = 0$. So, $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent. $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.

Problem 7.1.3

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for all $1 \geq p \geq 1$.

Solution to Problem 7.1.3: Intuitively, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is the harmonic series when is the power to $p = 0$ diverges. Also when $1 \geq p \geq 1$, it seems that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Proof

(a) Diverges when $p = 0$

(b) Diverges for p is a rational number $p = \frac{1}{q}$ for some $q \in \mathbb{N}$

(c) Diverges for p is an irrational number

(d) Diverges when $1 \geq p \geq 1$

Solution to Problem 7.1.3: Let $p = 0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 +

7.4 Alternating series

All of the convergence tests that we've developed so far are for series all of whose terms are nonnegative. The tests can be easily modified for series all of whose terms are nonpositive: just multiply all the terms by -1. But trickier are series that have both positive and negative terms.

One easy way to tell if a general series converges is when the same series converges if we take the absolute value of the terms:

> **Definition:** We say that $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.

Absolute convergence is a stronger condition than convergence:

Problem 7.19

Prove that if $\sum a_n$ is absolutely convergent, then $\sum a_n$ converges.

Solution for Problem 7.19: Since all of our convergence tests from Section 7.3 deal with series with nonnegative terms, we write the terms in our given series as the difference of two nonnegative terms:

$$a_n = (a_n + |a_n|) - |a_n|$$

for all n. We know that $\sum |a_n|$ converges by definition, and furthermore we have

$$0 \leq a_n + |a_n| \leq 2|a_n|,$$

so by the Series Comparison Test, $\sum (a_n + |a_n|)$ also converges. Thus, since a sum of convergent series is also a convergent series, we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|,$$

and hence $\sum a_n$ converges. □

For example, we now know that

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

converges, because it absolutely converges–that is,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

converges. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

However, there are series that converge but that do not converge absolutely:

Problem 7.20

Show that the **alternating harmonic series**

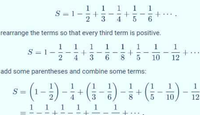
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges.

Solution for Problem 7.20: We can explore this by looking at the partial sums:

$$\begin{aligned}
s_1 &= a_1 = 1 \\
s_2 &= s_1 + a_2 = 1 - \frac{1}{2} = \frac{1}{2} \\
s_3 &= s_2 + a_3 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \\
s_4 &= s_3 + a_4 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12} \\
s_5 &= s_4 + a_5 = \frac{7}{12} + \frac{1}{5} = \frac{47}{60} \\
s_6 &= s_5 + a_6 = \frac{47}{60} - \frac{1}{6} = \frac{37}{60}
\end{aligned}$$

We can represent this graphically on the number line, as in the diagram below. Each arrow is the result of adding the next term.



We notice that the sums of an odd number of terms $\left(1, \frac{5}{6}, \frac{47}{60}, \ldots\right)$ are all greater than the sums of an even number of terms $\left(0, \frac{1}{2}, \frac{7}{12}, \frac{37}{60}, \ldots\right)$. Furthermore, the odd sums are decreasing whereas the even sums are increasing, so since both sequences (the odd sums and the even sums) are bounded and monotonic, they each converge. Finally, because $\lim_{n\to\infty}(s_{2n+1} - s_{2n}) = 0$, the odd sums and the even sums must converge to the same value, which is the value to which the entire series converges. □

The alternating harmonic series actually converges to $\log 2$, but we won't be able to prove this until Section 7.6.

The argument that we used in Problem 7.20 to prove that the alternating harmonic series converges can be generalized:

> **Important:** The **Alternating Series Test**: Let $\sum a_n$ be a series in which:
>
> (a) the terms are alternating: a_n is positive for odd n and negative for even n (or vice versa),
>
> (b) $|a_{n+1}| < |a_n|$ for all n, and
>
> (c) $\lim_{n\to\infty} a_n = 0$.
>
> Then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

We can sketch a proof; it's essentially the same as our argument in Problem 7.20 of the convergence of the alternating harmonic series. Suppose $a_1 > 0$ and $a_2 < 0$ (the proof is essentially the same if they are reversed). We let $s_n = a_1 + a_2 + \cdots + a_n$ be the sum of the first n terms.

Note that $s_1, s_3, s_5, \ldots$ is a strictly decreasing sequence: for example, note that $s_3 = s_1 + a_2 + a_3$ and the conditions imply that $a_2 + a_3 < 0$. Similarly, $s_2, s_4, s_6, \ldots$ is a strictly increasing sequence. Further, each s_i (with i odd) is strictly greater than each s_j (with j even), and $\lim(s_i - s_j) = 0$.

Thus, the odd s's converge downwards and the even s's converge upwards, and where they meet is the limit of the original sequence. They must converge to the same value, since $\lim(s_i - s_j) = 0$.

The alternating harmonic series from Problem 7.20 is an example of a series that is convergent but that is not absolutely convergent. Such a series is called **conditionally convergent**. One thing we have to be very careful about is that if we have a sequence that is convergent but not absolutely convergent, then the order in which we sum the terms of the series is very important! For example, if we make the faulty assumption that we can "rearrange" the terms of the series, then we can arrive at the following bogus conclusion:

> **Bogus Solution:** We start with
>
> $$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$
>
> We rearrange the terms so that every third term is positive.
>
> $$\begin{aligned}
> S &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots. \\
> \text{We add some parentheses and combine some terms:} \\
> S &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots \\
> &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots.
> \end{aligned}$$
>
> But now we have
>
> $$S = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) = \frac{1}{2}S,$$
>
> so $S = 0$.

Clearly the sum is not 0: indeed, we saw in Problem 7.20 that it is between $\frac{1}{2}$ and 1 (in fact, it is $\log 2$). So what we did must be illegal.

> **WARNING!!** We can't arbitrarily rearrange terms in an infinite series. This is why series are formally defined as sequences of partial sums: the order in which we sum the terms of a sequence to get the sum of a series is important!

A remarkable fact is that we can rearrange the terms of a conditionally convergent series to get it to sum to any real number! So we have to be really careful when working with them. We don't have such a problem with absolutely convergent series: any rearrangement of the terms will still converge to the same sum, although this is somewhat difficult to prove.

Exercises

7.4.1:

For each of the following series, determine whether it converges or diverges. If it converges, determine whether it is absolutely convergent, and if possible compute the value to which it converges.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{3^k}$

Preview: Solution

You may type any additional notes you have here.

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Solution: This is a geometric series with ratio $-\frac{1}{3}$. It converges to $\frac{-1}{1-(-\frac{1}{3})} = \boxed{\frac{-1}{4}}$. The absolute value of the terms of the series give a geometric series with ratio $\frac{1}{3}$, which converges, so the original series is absolutely convergent.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$

Preview: Solution

You may type any additional notes you have here.

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Solution: The absolute value of the kth term is $\frac{1}{k^2}$. This is strictly decreasing and converges to 0, so the series converges. (In fact, the series converges to $\frac{\pi}{12}$, but this is difficult to show at present.) The absolute value of the terms of the series give the p-series where $p = 2$, which converges, so the original series is absolutely convergent.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

Preview: Solution

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Solution: The absolute value of the kth term is $\frac{1}{\sqrt{k}}$. This is strictly decreasing and converges to 0, so the series converges. The absolute value of the terms of the series give the p-series where $p = \frac{1}{2}$, which diverges, so the original alternating series is conditionally convergent.

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k \log k}{k}$

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Solution: The absolute value of the kth term is $\frac{\log k}{k}$. This is strictly decreasing (for $k > 1$) and converges to 0, so the series converges. Taking the absolute value of the terms of the series gives the series $\sum \frac{\log k}{k}$, which is strictly greater than the divergent harmonic series (for $k > 1$), so it diverges; that is, the original alternating series is conditionally convergent.

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k^2}$

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Solution: This series is not alternating, since $(-1)^{2k} = (-1^2)^k = 1^k = 1$ for all k. Thus the series is just the p-series where $p = \frac{2}{2}$ and hence is convergent. It is also trivially absolutely convergent since taking the absolute value of the terms of the series gives the same series, since all the terms are positive.

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k^2} k^2}{k^2+1}$

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Solution: Since k and k^2 have the same parity (both are even or both are odd), we have $(-1)^{k^2} = (-1)^k$, so this series is alternating. The absolute value of the kth term is $\frac{k^2}{k^2+1}$, which is strictly decreasing, so the series converges by the Alternating Series Test. However, by the Limit Comparison Test, the absolute value series behaves in the same way as the harmonic series, so it diverges. Thus the original series is conditionally convergent.

7.4.2:

Show that if $\sum_{n=1}^{\infty} a_n$ absolutely converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Preview: Solution

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Solution: If $\sum_{n=1}^{\infty} a_n$ converges, we must have $|a_n| \to 0$ as $n \to \infty$. Thus, by definition, for some $N > 0$, we have $|a_n| < 1$ for all $n > N$, which means that $0 \leq a_n^2 < |a_n| < 1$ for all $n > N$. Thus, by the Series Comparison Test, since $\sum |a_n|$ converges, so does $\sum a_n^2$.

7.5 Taylor polynomials

Let f be a function, and let a be a fixed number in its domain. The tangent line approximation is the tangent line approximation of the graph of f near a — the tangent line to the graph of f at the point $(a, f(a))$ has slope $f'(a)$, so the formula for the linear function $p(x)$ that approximates $f(x)$ near a is

$$p(x) = f(a) + f'(a)(x - a).$$

This is called a **first-order approximation** of f at a — it has $p(a) = f(a)$ and $p'(a) = f'(a)$, so it shares f's value and first derivative at a.

A better one will add relatively simple approximations with such a quadratic polynomial. Of course, we will demand that our approximation be slightly better than the linear approximation—specifically, it should also match the second derivative at a.

Let's look at the special case where $a = 0$. Given that's simpler, and it will guide us to the general case.

Problem 7.21 🔗 ✓

Let f be a twice-differentiable function at $x = 0$. Find the quadratic polynomial q such that $q(0) = f(0)$, $q'(0) = f'(0)$, and $q''(0) = f''(0)$.

Solution for Problem 7.21: We want a quadratic polynomial, so we know that p is of the form

$$p(x) = c_0x^2 + c_1x + c_0.$$

We want to determine the coefficients so that $p(0) = f(0)$, $p'(0) = f'(0)$, and $p''(0) = f''(0)$.

We can just plug in at $x = 0$ to get

$$c_0 = f(0).$$

Then we take the derivative, we have

$$p'(x) = 2c_2x + c_1,$$

and here $c_1 = f'(0)$ for sure.

When we take the second derivative, we have

$$p''(x) = 2c_2,$$

so we have $2c_2 = f''(0)$, hence $c_2 = \frac{f''(0)}{2}$.

Therefore, our quadratic approximation is

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

Quadratic approximations are typically more accurate than linear approximations, and are especially useful in situations when a linear approximation is no good. Like the following example.

Problem 7.22 🔗 ✓

Estimate $\cos 0.1$ using a quadratic approximation at $x = 0$.

Solution for Problem 7.22: We let $f(x) = \cos x$. We could try a linear approximation we have $f'(x) = -\sin x$, so $f'(0) = -\sin 0 = 0$ and our linear approximation is

$$p(x) = f(0) + f'(0)(x - 0) = 1 + 0x = 1.$$

That is, our linear approximation tells us that $\cos x \approx 1$ near $x = 0$. This is interesting, but it doesn't help us to approximate $\cos 0.1$ — where we are willing to settle for $\cos 0.1 \approx 1$.

Instead, noting that $f''(x) = -\cos x$ so that $f''(0) = -\cos 0 = -1$, we can use a quadratic approximation

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 - \frac{1}{2}x^2.$$

This approximation gives us $\cos 0.1 \approx 1 - \frac{(0.1)^2}{2} = .995$.

If you fix this one (adjust, perhaps, so that $\cos 0.1 \approx 0.99500833...$), our approximation is pretty good. We can also sketch the graphs $y = \cos x$ and $y = 1 - \frac{1}{2}x^2$ at right, to see how close they compare. The graph of $\cos x$ is the dashed black curve and the graph of $1 - \frac{1}{2}x^2$ is the solid gray curve. Near $x = 0$, they are very close—so our sketch they are virtually indistinguishable—so this is a pretty good approximation (and much better than the linear approximation $y = 1$).

Let's try to generalize this to an approximation near $x = a$?

Problem 7.23 🔗 ✓

Let f be a twice-differentiable function at $x = a$. Find a quadratic polynomial p such that $p(a) = f(a)$, $p'(a) = f'(a)$, and $p''(a) = f''(a)$.

Solution for Problem 7.23: Recall the formula for the linear approximation at $x = a$:

$$p(x) = f(a) + f'(a)(x - a)$$

The trick was writing the polynomial in terms of $x - a$ rather than x.

So, using this same trick to get a quadratic approximation at $x = a$, we let

$$p(x) = c_0(x - a)^2 + c_1(x - a) + c_0,$$

and solve so that $p(a) = f(a)$, $p'(a) = f'(a)$, and $p''(a) = f''(a)$.

Doing so, we get the same calculations that we did in Problem 7.21. Plugging in $x = a$ and solving $p(a) = f(a)$ gives us

$$c_0 = p(a) = f(a),$$

so $c_0 = f(a)$.

Next, differentiating and plugging in $x = a$ gives us

$$c_1 = p'(a) = f'(a),$$

so $c_1 = \frac{f'(a)}{2}$.

Lastly, differentiating again and plugging in $x = a$ gives us

$$2c_2 = p''(a) = f''(0),$$

so $c_2 = \frac{1}{2}f''(a)$.

Therefore, the quadratic approximation is

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

Note that when $a = 0$ this gives us our approximation near $x = 0$ from Problem 7.21.

Important: The quadratic approximation of a twice-differentiable function f at $x = a$ is

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

Note that p agrees with f in value, first derivative, and second derivative at $x = a$. We say that p is a **second-order approximation** of f at $x = a$.

The more general form of the quadratic approximation is useful when we want to approximate values far from $x = 0$.

Problem 7.24 🔗 ✓

Estimate $\ln 1.5$ using a quadratic estimation.

Solution for Problem 7.24: We set up a quadratic approximation at $a = 1$, call $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$. Thus $f'(1) = 1$ and $f''(1) = -1$, and the approximating quadratic polynomial is

$$p(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2$$
$$= (x - 1) - \frac{1}{2}(x - 1)^2.$$

Plugging in $x = 0.9$ gives

$$p(0.9) = (-.1) - (0.5)(-.1)^2 = -0.095 = 0.1 - \frac{1}{2}(.1)^2 = -0.105.$$

We know $\ln 1.5 = -0.105$. Your calculator will say $\log 0.9 = -0.105036\ldots$, so again this is a pretty good approximation, much better than the linear $\log 0.9 = 0$.

It should seem reasonable that we can make a polynomial approximation even better by adding more terms. Again, to keep things simple, we'll start by examining approximations near 0.

Problem 7.25 🔗 ✓

We want to construct a degree n polynomial

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n.$$

Note that $p^{(0)}(0) = p(0)$ for all $0 \leq n$. (Note that $p^{(0)}(x) = p(x)$ by convention.)

(a) For all $0 \leq k \leq n$, compute $p^{(k)}(0)$ in terms of the coefficients $\{c_k\}$.

(b) Solve for the coefficients so that $p^{(k)}(0) = f^{(k)}(0)$ for all $0 \leq k \leq n$.

Solution for Problem 7.25:

(a) We can certainly compute the kth derivative $p^{(k)}(x)$ as a polynomial in x, but it's a bit messy, and we only need to compute $p^{(k)}(0)$. This means that we only need to determine the constant term of $p^{(k)}(x)$.

We know that taking the derivative of x^m lowers it to mx^{m-1}. Therefore, the kth derivative of a monomial will be a nonzero constant only if the monomial is c_kx^k. Thus, only the c_kx^k term of $p(x)$ will matter in $p^{(k)}(0)$. The smaller degree terms will vanish when we take the kth derivative, and the larger degree terms will still have a factor of x, so they will vanish when we plug in $x = 0$.

So we just need to compute the kth derivative of c_kx^k. The first derivative is kc_kx^{k-1}, the second derivative is $k(k-1)c_kx^{k-2}$, and so on, so we can recover from k down to 1 gets multiplied as a constant. So the kth derivative of c_kx^k is $k!c_k$, hence $p^{(k)}(0) = k!c_k$ for all $0 \leq k \leq n$.

(b) We need $c_k!k! = p^{(k)}(0) = f^{(k)}(0)$, so

$$c_k = \frac{f^{(k)}(0)}{k!}$$

for all $0 \leq k \leq n$. (Note that $0! = 1! = 1$, so this matches our earlier formula.) Hence, the degree n polynomial approximating $f(x)$ will be

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k.$$

It is the unique degree n polynomial with the property that $p^{(k)}(0) = f^{(k)}(0)$ for all $0 \leq k \leq n$. It is the unique degree n polynomial whose value at $x = 0$ and first k derivatives at x agree with those of f.

We can easily do the same calculation at $x = a$ (we will omit it here for brevity, but it is exactly the same calculation as in Problem 7.23), except we replace x with $x - a$ throughout:

Regex Solution: Let f be an n-times-differentiable function. The degree n **Taylor polynomial of f at a** is the any monomial x in the n variables $\{x, x^2, \ldots, x^n\}$

$$p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

It is the unique degree n polynomial with the property that $p^{(k)}(a) = f^{(k)}(a)$ for all $0 \leq k \leq n$. It is the unique degree n polynomial whose value at x and first k derivatives at x agree with those of f.

Concept: Think of Taylor polynomials as a natural extension of our earlier concept of tangent line approximations. Instead of just matching the slope of the tangent line, the degree n Taylor polynomial of f at a is the degree n polynomial that matches f at a.

You will also sometimes see the term **Maclaurin polynomial** to mean the Taylor polynomial at 0, but this term less commonly used.

Of course, we can only have a Taylor polynomial of f at a if all the derivatives that we need are defined.

Let's try a quick example:

Problem 7.26 🔗 ✓

Estimate $\cos 0.2$ using a cubic Taylor polynomial.

Solution for Problem 7.26: Of course, we choose to construct the polynomial at 0. (Always try for 0 unless there's a good reason not to.)

We list the function and its derivatives at 0:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1. \end{aligned}$$

At this point, don't make the following silly, but regrettably common, error:

Regex Solution: The polynomial is

$$p(x) = f(0) + f'(0)x + f''(0)x^2 + f'''(0)x^3 = x - x^3.$$

Each of the coefficients has to be divided by the corresponding factorial. So the polynomial is

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - \frac{1}{6}x^3.$$

To finish, we plug in $x = 0.2$:

$$\sin 0.2 \approx (0.2) - \frac{1}{6}(.008) = \frac{1}{5} - \frac{1}{750} = 0.199667\ldots$$

The calculator says $\sin 0.2 = 0.198669\ldots$, so this approximation is pretty close.

It is reasonable to ask the question: how good are the approximations given by Taylor polynomials? Given a function f, we can estimate the error of a degree n Taylor polynomial of f at a the difference in the most obvious manner, as the amount that the approximation differs from the actual value.

Definition: The **error** in the degree n Taylor polynomial approximation of f at a is

$$E(x) = f(x) - p(x),$$

where $f(x)$ is the actual value of f, and $p(x)$ is the degree n Taylor polynomial at a.

For example, in Problem 7.26, the error in our approximation is

$$E(x) = \sin x - x + \frac{1}{6}x^3,$$

and in particular,

$$E(0.2) = \sin 0.2 - \frac{149}{750}$$

we know that $E(a) = 0$ and that the first n derivatives of E match at a, that is,

$$E(a) = E'(a) = E''(a) = \cdots = E^{(n)}(a) = 0.$$

And since p is a degree n polynomial, its higher derivatives are all 0. So in particular, we have

$$E^{(n+1)}(x) = f^{(n+1)}(x).$$

Suppose we are one-ping n-times differentiable $f(x)$ for some $x > a$. If we know $f^{(n+1)}$ is continuous, then $|f^{(n+1)}(x)|$ has some maximum value M on $[a, x]$ for

$$|E^{(n+1)}(x)| \leq M.$$

What happens if we integrate this $n + 1$ times? We get

$$|E(x)| \leq \frac{M}{(n+1)!}(x - a)^{n+1}.$$

Note that all of the constants of integration disappear while we do this, since all the lower derivatives of E vanish at a.

So our conclusion is:

Important: The absolute value of the error in estimating $f(x)$ by a Taylor polynomial of degree n is about a is at most

$$\frac{M}{(n+1)!}(x - a)^{n+1},$$

where M is the maximum value of $|f^{(n+1)}(x)|$ for x between a and x.

Here's an example:

Problem 7.27 🔗 ✓

Determine the absolute value of the maximum error in the quadratic estimation of $\log 0.9$.

Solution for Problem 7.27: We have the quadratic approximation

$$p(x) = (x - 1) - \frac{1}{2}(x - 1)^2,$$

which gave us

$$p(0.9) = (-.1) - (0.5)(-.1)^2 = -0.095 = 0.1 - \frac{1}{2}(.1)^2,$$

So $\log 0.9 \approx -0.105$.

To determine the error bound, we need to look at $f^{(3)}(x) = \frac{2}{x^3}$ on the interval $[0.9, 1]$. This is maximized at $x = 0.9$, giving

$$f^{(3)}(0.9) = \frac{2}{(0.9)^3} = \frac{2000}{729} = 2.7435\ldots$$

Thus, our error bound is

$$\left|\frac{M}{(n+1)!}(x - a)^{n+1}\right| \approx 0.006027\ldots$$

Note that this is the absolute value of the maximum possible error, not the actual error. Indeed, a calculator will show that the actual error is about 0.000951.

Exercises

7.5.1:

Approximate $\sqrt{0.9}$ using a cubic Taylor polynomial. Determine the bound on the error.

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Your Submission Solution

Solution: We are approximating $f(x) = x^{\frac{1}{2}}$ using the Taylor polynomial at $x = 1$. The relevant derivatives are:

$$\begin{aligned} f(1) &= 1, \\ f'(x) &= \frac{1}{2}x^{-\frac{1}{2}}, \quad f'(1) = \frac{1}{2}, \\ f''(x) &= -\frac{1}{4}x^{-\frac{3}{2}}, \quad f''(1) = -\frac{1}{4}, \\ f'''(x) &= \frac{3}{8}x^{-\frac{5}{2}}, \quad f'''(1) = \frac{3}{8}. \end{aligned}$$

So the cubic Taylor polynomial is

$$p(x) = 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3.$$

(Don't forget to divide by $n!$ in the $(x - 1)^{n+1}$ term) Accordingly, $\sqrt{0.9} \approx p(0.9) = 1 + \frac{1}{2}(-.1) - \frac{1}{8}(-.1)^2 + \frac{1}{16}(-.1)^3 = 0.948375$

The calculator says $\sqrt{0.9} = 0.948683\ldots$, so this approximation is pretty close.

It is reasonable to ask the question: how good are the approximations given by Taylor polynomials? Given a function f, we can estimate the error of a degree n Taylor polynomial of f at a the difference in the most obvious manner, as the amount that the approximation differs from the actual value.

Definition: The **error** in the degree n Taylor polynomial approximation of f at a is

$$E(x) = f(x) - p(x),$$

where $f(x)$ is the actual value of f, and $p(x)$ is the degree n Taylor polynomial at a.

For example, in Problem 7.26, the error in our approximation is

$$E(x) = \sin x - x + \frac{1}{6}x^3,$$

and in particular,

$$E(0.2) = \sin 0.2 - \frac{149}{750}$$

we know that $E(a) = 0$ and that the first n derivatives of E match at a, that is,

$$E(a) = E'(a) = E''(a) = \cdots = E^{(n)}(a) = 0.$$

And since p is a degree n polynomial, its higher derivatives are all 0. So in particular, we have

$$E^{(n+1)}(x) = f^{(n+1)}(x).$$

Suppose we are one-ping n-times differentiable $f(x)$ for some $x > a$. If we know $f^{(n+1)}$ is continuous, then $|f^{(n+1)}(x)|$ has some maximum value M on $[a, x]$ for

$$|E^{(n+1)}(x)| \leq M.$$

What happens if we integrate this $n + 1$ times? We get

$$|E(x)| \leq \frac{M}{(n+1)!}(x - a)^{n+1}.$$

Note that all of the constants of integration disappear while we do this, since all the lower derivatives of E vanish at a.

So our conclusion is:

Important: The absolute value of the error in estimating $f(x)$ by a Taylor polynomial of degree n about a is at most

$$\frac{M}{(n+1)!}(x - a)^{n+1},$$

where M is the maximum value of $|f^{(n+1)}(x)|$ for x between a and x.

Here's an example:

Problem 7.27 🔗 ✓

Determine the absolute value of the maximum error in the quadratic estimation of $\log 0.9$.

Solution for Problem 7.27: We have the quadratic approximation

$$p(x) = (x - 1) - \frac{1}{2}(x - 1)^2,$$

which gave us

$$p(0.9) = (-.1) - (0.5)(-.1)^2 = -0.095 = 0.1 - \frac{1}{2}(.1)^2,$$

So $\log 0.9 \approx -0.105$.

To determine the error bound, we need to look at $f^{(3)}(x) = \frac{2}{x^3}$ on the interval $[0.9, 1]$. This is maximized at $x = 0.9$, giving

$$f^{(3)}(0.9) = \frac{2}{(0.9)^3} = \frac{2000}{729} = 2.7435\ldots$$

Thus, our error bound is

$$\left|\frac{M}{(n+1)!}(x - a)^{n+1}\right| \approx 0.006027\ldots$$

Note that this is the absolute value of the maximum possible error, not the actual error. Indeed, a calculator will show that the actual error is about 0.000951.

7.5.2:

What is the estimate of $(1 + x)^{\frac{1}{3}}$ by a cubic Taylor polynomial, where x is a positive integer?

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Your Submission Solution

Solution: We can write the cubic Taylor polynomial of $f(x) = (1 + x)^{\frac{1}{3}}$ about $x = 0$. The relevant derivatives are:

$$\begin{aligned} f(0) &= 1, \\ f'(x) &= \frac{1}{3}(1 + x)^{-\frac{2}{3}}, \quad f'(0) = \frac{1}{3}, \\ f''(x) &= \frac{2}{9}(1 + x)^{-\frac{5}{3}}, \quad f''(0) = \frac{2}{9}, \\ f'''(x) &= -\frac{10}{27}(1 + x)^{-\frac{8}{3}}, \quad f'''(0) = -\frac{10}{27}. \end{aligned}$$

So the polynomial is

$$p(x) = 1 + \frac{1}{3}x + \frac{\frac{2}{9}(x - 1)}{2!}x^2 + \frac{\frac{-10}{27}(x - 1)}{3!}x^3,$$

and hence

$$(1 + x)^{\frac{1}{3}} \approx 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3.$$

7.5.3:

Estimate $\sqrt[3]{9}$ to 4 decimal places using a Taylor polynomial about 8.

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Your Submission Solution

Solution: If $f(x) = x^{\frac{1}{3}}$, then all derivatives are also equal to $x^{\frac{1}{3}}$, so $f^{(n)}(0) = 1$ for all n. Therefore, the Taylor polynomial is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

where we can carry the polynomial to as many terms as we need. For 4 decimal places, we find $f(1.7) = 0.00983$ and $f(1.8) = 0.00983$ and $f(1.9) = 0.00983$ and $f(2) = 0.00983$ ... and should not affect the first 4 decimal places. So we can use a quartic polynomial, and we get

$$x^{\frac{1}{3}} \approx 1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}$$

which is 1.09375 to 4 decimal places. (And indeed, a calculator will confirm that this is accurate to 4 decimal places.)

7.5.4:

Let $p(x) = x^3 + 5x + 1$ be a quadratic polynomial. Find the degree 2 Taylor polynomial at p.

(a) $x = 0$

Preview Solution

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Your Submission Solution

Solution: Since $p(x) = 3x^2 + 5$ and $p''(x) = 6x$, we have $p(0) = 1$, $p'(0) = 5$ and $p''(0) = 3x$. Thus the Taylor polynomial is

$$c + bx + \frac{a}{2}x^2 = 1 + 5x + \frac{3}{2}x^2.$$

Of course, this is exactly the same as the original quadratic $p(x)$.

(b) $x = 1$

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Your Submission Solution

Solution: Since $p'(x) = 3x^2 + 5$ and $p''(x) = 6x$, we have $p(1) = a + b + c$, $p'(1) = 2a + b$, and $p''(1) = 2a$, thus the Taylor polynomial is

$$a + b + c + (2a + b)(x - 1) + \frac{2a}{2}(x - 1)^2$$
$$= a + b + c + 2ax + bx - 2a - b + ax^2 - 2ax + a$$
$$= ax^2 + bx + c.$$

Again, this is just $p(x)$.

(c) $x = \pi$ where π is a root of p

Hint: All of these parts are actually the same – why?

Hint: It doesn't really matter that π is a root.

Preview Solution

You may type any additional notes you have here.

Your Submission Solution

Solution: Since $p'(x) = 3x^2 + 5$ and $p''(x) = 6x$, we have $p(\pi) = a\pi^2 + b\pi + c$, $p'(\pi) = 2a\pi + b$, and $p''(\pi) = 2a$, thus the Taylor polynomial is

$$a\pi^2 + b\pi + c + (2a\pi + b)(x - \pi) + \frac{2a}{2}(x - \pi)^2.$$

Expanding gives

$$ax^2 + bx + c + 2ax\pi - 2a\pi^2 + bx - b\pi + ax^2 - 2ax\pi + a\pi^2$$
$$= ax^2 + bx + c.$$

which is just $p(x)$ as before. In fact, we never used the fact that π is a root, so we will get $p(x)$ as our quadratic Taylor polynomial at any value of x.

7.6 Taylor series

If f(x) has infinitely differentiable functions, you can approximate it by polynomials with arbitrarily high degree. We can "factor the form" of a polynomial in terms of its derivatives at a point a in the following way:

In general, a **power series** is an infinite series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ for some sequence of coefficients $\{c_n\}$. Then, plug into x.

$a = 0$ giving a power series of the form $\sum_{n=0}^{\infty} c_n x^n$ that the coefficients for the coefficients of a Taylor polynomial, then we get:

Definition The Taylor series of f at infinitely differentiable function f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Important For the case of the whole interval, the series converges. This is also called the **Maclaurin series** of f.

Concept A function f is analytic at a if the Taylor series converges to $f(x)$ for all x in some interval. Analytic functions are infinitely differentiable functions that are equal to their Taylor series.

Not just sin(x) what arguably is the most important example:

Example 7.6.1

Find the Taylor series of $f(x) = \sin x$ at $a = 0$ and determine the what interval of f converges.

Solution by Professor Colin: We know that derivatives of $\sin^{(n)}$ are periodic $\sin^{(4)} = \sin$. So we have $f^{(n)}(0) = \sin^{(n)} 0$. Hence the Taylor series is:

$$\sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!}(x-0)^n$$

To determine the radius of the series this converges on, we use the ratio test. The series is usually the most useful test for power series. Let $a_n = \frac{\sin^{(n)}(0)}{n!}(x-0)^n$.

Put any value of x, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. Thus the Taylor Series. Taylor series converges for all $x \in \mathbb{R}$.

In Problem 7.3.6, we showed $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges. Now we can find what exact it converges to. We'll do this now that

$$\sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!}(x-0)^n = \sin x$$

but how do we know that this is true, and that the power series doesn't converge to some other value?

Wait we do know a lot:

$$\sin^{(n)} = \begin{cases} \sin & \text{if } n \equiv 0 \pmod 4 \\ \cos & \text{if } n \equiv 1 \pmod 4 \\ -\sin & \text{if } n \equiv 2 \pmod 4 \\ -\cos & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

In the degree 0, Taylor polynomial approximation, and derivatives, the approximation is gives:

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| \leq \frac{|x-a|^{n+1}}{(n+1)!}$$

where R_n is the remainder, c is a number between a and x, and $f^{(n+1)}$ is the derivative. The derivative is half the exponential function, so $M = e^x$ for $x \in [-r, r]$. So the exponential function is bad. Of course exponent is nice, and present in Problem 7.3.6 that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges. So the exponential function is bad. The derivative of the exponential function is a converging Taylor series and $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$$

That is, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to $\sin x$. So, and those we can say that e^x equals the sum of its power series.

In particular, we have a proof of a fact that has been used in the past:

Concept A function f is analytic at a if the Taylor series converges to $f(x)$ for all x in some interval. Analytic functions are infinitely differentiable functions that are equal to their Taylor series.

Example 7.6.2

Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, that the derivatives $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is self-derivative, and that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a power series converges at $x = 1$ that is convergence absolutely at $x = 1$.

Solution by Professor Colin: Let $a_n = \frac{x^n}{n!}$. Put any value of x, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. Thus the Taylor series converges at $x = 0$. Thus f has an open interval of convergence $(-\infty, \infty)$. Therefore, f is analytic. Now, we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, and we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely at $x = 1$.

Example 7.6.3

Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, that the derivatives $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is self-derivative, and that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a power series converges at $x = 1$ that is convergence absolutely at $x = 1$.

Solution by Professor Colin: Let $a_n = \frac{x^n}{n!}$. Put any value of x, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. Thus the Taylor series converges at $x = 0$. Thus f has an open interval of convergence $(-\infty, \infty)$. Therefore, f is analytic. Now, we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, and we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely at $x = 1$.

Example 7.6.4

Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, that the derivatives $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is self-derivative, and that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a power series converges at $x = 1$ that is convergence absolutely at $x = 1$.

Solution by Professor Colin: Let $a_n = \frac{x^n}{n!}$. Put any value of x, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. Thus the Taylor series converges at $x = 0$. Thus f has an open interval of convergence $(-\infty, \infty)$. Therefore, f is analytic. Now, we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, and we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely at $x = 1$.

Example 7.6.5

Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, that the derivatives $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is self-derivative, and that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a power series converges at $x = 1$ that is convergence absolutely at $x = 1$.

Solution by Professor Colin: Let $a_n = \frac{x^n}{n!}$. Put any value of x, we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. Thus the Taylor series converges at $x = 0$. Thus f has an open interval of convergence $(-\infty, \infty)$. Therefore, f is analytic. Now, we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, and we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely at $x = 1$.

Exercises

7.6.1.

Find the Taylor series, and their radius of convergence, for the following functions:

(a) e^{-x} at $a = 0$

Hint: Write down the Taylor series for e^x, then in the Taylor series for e^{-x} instead!

Your Solution/Solution Sketch: Hide Solution

Solution. By plug $-x$ into the Taylor's polynomial for e^x we compute:

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

This corresponds to $f(x)$ at a by the Taylor Series.

Hide Solution

(b) $\sin x^2$ about $a = 0$.

Write Taylor's polynomial derivatives: start with the Taylor polynomial for $\sin x$. Then, plug x^2 into the Taylor's polynomial for $\sin x$. Then, plug x^2 into the Taylor's polynomial for $\sin x$.

Your Solution/Solution Sketch: Hide Solution

Solution. Replacing x with x^2 in the Taylor's polynomial for $\sin x$. Then, plug x^2 into the Taylor's polynomial for $\sin x$.

$$\sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!}(x^2-0)^n$$

This corresponds to $f(x)$ at a by the Taylor Series.

Hide Solution

(c) $\sin x$ about $a = 0$.

Write the non-multiple Taylor series.

Your Solution/Solution Sketch: Hide Solution

Solution. Replacing x with x^2 in the Taylor's polynomial for $\sin x$. Then, plug x^2 into the Taylor's polynomial for $\sin x$.

$$\sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!}(x-0)^n$$

This corresponds to $f(x)$ at a by the Taylor Series.

Hide Solution

(d) $\sqrt{1+x}$ about $a = 0$.

Write the $\sqrt{1+x}$ about $a = 0$.

Your Solution/Solution Sketch: Hide Solution

Solution. Let $f(x) = \sqrt{1+x}$. Then:

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-1/2} \\ f''(x) &= -\frac{1}{4}(1+x)^{-3/2} \\ f'''(x) &= \frac{3}{8}(1+x)^{-5/2} \end{aligned}$$

and so on. Thus, $f^{(n)}(0) = \frac{(-1)^{n-1}(2n-1)!!}{2^n}$ and therefore the series is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-1)!!}{2^n n!}(x-0)^n$$

Let a_n be the coefficient $(x-0)^n$ in this power series. Note that:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n-1}(2n-1)!!}{2^n n!}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n-1}(2n-1)!!}{2^n n!}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n-1}(2n-1)!!}{2^n n!}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n-1}(2n-1)!!}{2^n n!}\right|

7.A A strange formula for π

WARNING!! This section will use a lot of "magic" results that we have not proved, or even really rigorously defined.

Problem 7.51

Show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

Sidenote: The argument below was first shown by the mathematician Leonhard Euler in the 18th century.

Solution for Problem 7.51: We start with the power series for sine:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots ,$$

and we divide by x:

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots .$$

As we know, this converges for all $x \in \mathbb{R}$.

We can make the substitution $y = x^2$ to define

$$g(y) = 1 - \frac{y}{3!} + \frac{y^2}{5!} - \cdots .$$

This still converges for all $y \in \mathbb{R}$.

We now examine the zeroes of this power series; that is, we find the values of y such that $g(y) = 0$. The zeroes of $f(x)$ are the integer multiples of π, so the zeroes of $g(y)$ are the squares of integer multiples of π. That is, the zeroes of $g(y)$ are exactly the elements of the set $\{\pi^2, 4\pi^2, 9\pi^2, \ldots\}$.

This means–and here we are taking a lot of liberty and using some manipulations that we have not proved–that we can write

$$1 - \frac{y}{3!} + \frac{y^2}{5!} - \cdots = g(y) = \left(1 - \frac{y}{\pi^2}\right)\left(1 - \frac{y}{4\pi^2}\right)\left(1 - \frac{y}{9\pi^2}\right)\cdots .$$

We now examine the linear (or y) terms on both sides of this equation. On the left side, it is clearly $-\frac{1}{6}$, but on the right side, it is the sum of the y-coefficients of the individual factors, giving

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right).$$

These must be equal, so we have

$$-\frac{1}{6} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right).$$

Finally, we multiply by $-\pi^2$, and we get

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

□

Challenge Problems

7.44:

Source: Putnam

Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.

Hint: Try to group the terms of the series is a useful way.

Hint: Group the terms by powers of 2.

You may type any additional notes you have here.

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Solution: Let $b_k = \sum_{n=2^k+1}^{2^{k+1}} a_n$ for all $k \geq 1$. Note that $b_1 = a_3 > 0$, and the given inequality implies that $b_k \geq b_{k-1}$ for all $k > 1$. For example,

$$\begin{aligned} b_2 &= a_4 + a_5 + a_6 + a_7 \\ &\leq (a_4 + a_8) + (a_{10} + a_{11}) + (a_{12} + a_{13}) + (a_{14} + a_{15}) \\ &= b_3. \end{aligned}$$

Thus the b_k's do not converge to 0 but rather increase, so $\sum_{k=1}^{\infty} b_k$ diverges. But $\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} b_k$ so the series $\sum_{n=1}^{\infty} a_n$ diverges as well.

7.45:

Define $a_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \log n$.

(a) Use appropriate definite integral(s) to prove that $0 < a_{n+1} < a_n$ for all $n > 1$.

Hint: Compare the sum of the first n terms to $\int_{1}^{n+1} \frac{1}{x}\,dx$.

Hint: For the other inequality, show that $\log(n+1) - \log(n) > \frac{1}{n+1}$.

Preview: Solution

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Solution: Let $f(x) = \frac{1}{x}$. Then since $f(x)$ is decreasing, the Riemann sum

$$f(1) + f(2) + \cdots + f(n) = \sum_{k=1}^{n} \frac{1}{k}$$

is strictly larger than the definite integral

$$\int_{1}^{n+1} f(x)\,dx = \log(n+1) > \log n.$$

Thus all the terms of $\{a_n\}$ are positive.

For the other inequality, we must prove that

$$0 < a_n - a_{n+1} = \log(n+1) - \log(n) - \frac{1}{n+1}.$$

But

$$\log(n+1) - \log(n) = \int_{n}^{n+1} \frac{1}{x}\,dx > \int_{n}^{n+1} \frac{1}{n+1}\,dx = \frac{1}{n+1},$$

so the inequality holds.

(b) Use (a) to prove that $\lim_{n\to\infty} a_n$ exists.

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Solution: Since $\{a_n\}$ is monotone decreasing and bounded below (by 0), it must converge, and in fact it converges to $\inf\{a_n\}$.

Note: $\lim_{n\to\infty} a_n$ is called **Euler's Constant** and is usually denoted γ. It is an important number in analysis, but not much is known about it: indeed it is not known whether γ is rational or irrational, although it has been computed to 2 billion decimal places.

7.46:

Source: HMMT

Define a sequence $\{a_n\}$ by

$$a_n = \left(\frac{1}{\sqrt{n^2-0^2}} + \frac{1}{\sqrt{n^2-1^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}}\right).$$

Compute $\lim_{n\to\infty} a_n$.

Hint: Try to make the sum look like a Riemann sum.

Hint: Factor out $\frac{1}{n}$. Now what does the sum look like?

Hint: How is our sum related to $\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$?

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Solution: This looks more like a Riemann sum if we factor out $\frac{1}{n}$:

$$\begin{aligned} a_n &= \frac{1}{n}\left(\frac{1}{\sqrt{1-\left(\frac{0}{n}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}} + \cdots + \frac{1}{\sqrt{1-\left(\frac{n-1}{n}\right)^2}}\right) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \frac{1}{\sqrt{1-\left(\frac{k}{n}\right)^2}}. \end{aligned}$$

Now it is a Riemann sum! Taking the limit as $n \to \infty$ gives

$$\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx.$$

(The terms of the Riemann sum are the areas of boxes of width $\frac{1}{n}$ and height $f\left(\frac{k}{n}\right)$ where $f(x) = \frac{1}{\sqrt{1-x^2}}$.) So to evaluate the limit, we just need to compute this integral. Fortunately, this is easy:

$$\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x\Big|_{0}^{1} = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2}.$$

7.47:

The **Fibonacci numbers** are defined as $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots.$$

(a) Prove that the radius of convergence of f is at least $\frac{1}{2}$.

Hint: Use the Ratio Test.

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Solution: The sequence $\{a_n\}$ is clearly nondecreasing, and so $a_n = a_{n-1} + a_{n-2} \leq 2a_{n-1}$. Now we apply the Ratio Test to $f(x)$:

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| \leq |2x|.$$

So if $|x| < \frac{1}{2}$ the series converges, as desired.

(b) Prove that if $|x| < \frac{1}{2}$ then $f(x) = \frac{1}{1-x-x^2}$.

Hint: Write out a few terms of $(1-x-x^2)f(x)$ to see what's going on.

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Solution: Note that

$$\begin{aligned} f(x) &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \cdots \\ xf(x) &= \quad\;\; x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots \\ x^2 f(x) &= \qquad\quad x^2 + x^3 + 2x^4 + 3x^5 + \cdots. \end{aligned}$$

If we subtract the second and third equations from the first, all the terms with x to a positive power will vanish by the recursive formula $a_n = a_{n-1} + a_{n-2}$. Therefore $(1-x-x^2)f(x) = 1$ and $f(x) = \frac{1}{1-x-x^2}$. Since we showed the series converges for $|x| < \frac{1}{2}$ in (a), the series is identically equal to $\frac{1}{1-x-x^2}$ for $|x| < \frac{1}{2}$ as desired.

7.48:

The **Binomial Theorem** works for non-integer exponents too! Specifically, we can write

$$(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \binom{p}{3}x^3 + \cdots,$$

even when p is not a positive integer. Explain this equation and prove it. What are the necessary conditions (if any) on x and p?

Hint: Compute the Taylor series of $(1+x)^p$.

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Solution: For a nonnegative integer k and real p, we say

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

Then the Binomial Theorem expression makes sense as the Taylor series of $(1+x)^p$ about $x = 0$. Notice that if $f(x) = (1+x)^p$, then

$$f^{(n)}(x) = p(p-1)\cdots(p-k+1)(1+x)^{p-k}.$$

Putting in $x = 0$ and dividing by $k!$ gives $\binom{p}{k}$ as desired.

By using the Ratio Test, we can verify that if p is not a nonnegative integer, then the expression converges only when $|x| < 1$. Note that $a_n = (p)(p-1)\cdots(p-k+1)$ is an unbounded sequence for any p that is not a nonnegative integer.

7.49★:

Source: Putnam

Let $n \geq 2$ be an integer, and for any real number $0 \leq \alpha \leq 1$, let $C(\alpha)$ be the coefficient of x^n in the power series expansion of $(1+x)^{\alpha}$. Prove that

$$\int_{0}^{1} \left(C(-t-1)\left(\sum_{k=1}^{n} \frac{1}{t+k}\right)\right) dt = (-1)^n n.$$

Hint: Try it first for $n = 2$ to get a better idea of what's happening.

Hint: Write the sum as the derivative of some function.

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Solution: Let's first try it for $n = 2$. In this case, $C(-t-1)$ is the coefficient of x^2 in $(1+x)^{-t-1}$, which is

$$\frac{(-t-1)(-t-2)}{2} = \frac{(t+1)(t+2)}{2}.$$

So the integral is

$$\int_{0}^{1} \left(\frac{(t+1)(t+2)}{2}\left(\frac{1}{t+1} + \frac{1}{t+2}\right)\right) dt.$$

This easily simplifies:

$$\frac{1}{2}\int_{0}^{1} ((t+2) + (t+1))\,dt.$$

We can evaluate this directly very easily, but the "trick" here is to rewrite it as

$$\frac{1}{2}\int_{0}^{1} \frac{d}{dt}((t+2)(t+1))\,dt = \frac{1}{2}\left((t+2)(t+1)\right)\Big|_{0}^{1} = \frac{1}{2}(6-2) = 2.$$

Let's now see the general case. We compute

$$\begin{aligned} C(-t-1) &= \frac{(-t-1)(-t-2)(-t-3)\cdots(-t-n)}{n!} \\ &= (-1)^n \frac{(t+1)(t+2)\cdots(t+n)}{n!}. \end{aligned}$$

So the integral becomes

$$\int_{0}^{1} (-1)^n \frac{(t+1)(t+2)\cdots(t+n)}{n!}\left(\frac{1}{t+1} + \cdots + \frac{1}{t+n}\right) dt.$$

Rewrite the integrand as

$$\frac{(-1)^n}{n!}\int_{0}^{1} \frac{d}{dt}((t+1)(t+2)\cdots(t+n))\,dt.$$

Evaluating this, we get

$$\begin{aligned} \frac{(-1)^n}{n!}((t+1)(t+2)\cdots(t+n))\Big|_{0}^{1} &= \frac{(-1)^n}{n!}((n+1)!-n!) \\ &= (-1)^n((n+1)-1) \\ &= (-1)^n n. \end{aligned}$$

7.50★:

Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Show that, for all positive integers n, there exists a polynomial p_n such that

$$f^{(n)}(x) = \begin{cases} p_n\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Hint: Prove it by induction.

Hint: For $x \neq 0$, you will be able to compute the derivative just using the Product Formula. But at $x = 0$, you'll have to use the limit definition of derivative.

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Solution: We proceed by induction. Note that the base case $n = 0$ works with $p_0 = 1$.

For the inductive step, let n be a positive integer and assume that $f^{(n-1)}$ is of the specified form. For nonzero x, we have

$$\begin{aligned} f^{(n)}(x) &= \frac{d}{dx} f^{(n-1)}(x) \\ &= \frac{d}{dx}\left(p_{n-1}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}\right) \\ &= \frac{1}{x^2}p'_{n-1}\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} + p_{n-1}\left(\frac{1}{x}\right)\frac{2}{x^3}e^{-\frac{1}{x^2}} \\ &= \left(-\frac{1}{x^2}p'_{n-1}\left(\frac{1}{x}\right) + \frac{2}{x^3}p_{n-1}\left(\frac{1}{x}\right)\right)e^{-\frac{1}{x^2}}. \end{aligned}$$

We define p_n by

$$p_n(y) = -y^2 p'_{n-1}(y) + 2y^3 p_{n-1}(y),$$

and we have

$$p_n\left(\frac{1}{x}\right) = -\frac{1}{x^2}p'_{n-1}\left(\frac{1}{x}\right) + \frac{2}{x^3}p_{n-1}\left(\frac{1}{x}\right),$$

giving the result.

At $x = 0$ we must apply the definition of derivative:

$$f^{(n)}(0) = \lim_{h\to 0} \frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h} = \lim_{h\to 0} p_{n-1}\left(\frac{1}{h}\right)e^{-\frac{1}{h^2}}.$$

This looks like a good place for the substitution $t = \frac{1}{h}$. However in this case we need to consider both the limit at ∞ and at $-\infty$:

$$\lim_{t\to\infty} p_{n-1}(t)e^{-t^2} = \lim_{t\to\infty} \frac{p_{n-1}(t)}{e^{t^2}} = 0,$$

where the limit is zero because exponentials dominate polynomials. Likewise the limit at $-\infty$ is zero, so $\lim_{h\to 0} p_{n-1}\left(\frac{1}{h}\right)f(h) = 0$ for any polynomial p. Thus $f^{(n)}(0) = 0$, as required.

(b) Using part (a), compute the Taylor series of f at 0, and show that for all nonzero $x \in \mathbb{R}$, the Taylor series converges to something other than $f(x)$.

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Solution: The Taylor series for f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} 0 = 0.$$

This Taylor series converges for all x; however, since $f(x)$ is always nonzero when $x \neq 0$, the Taylor series converges to the wrong value for every $x \neq 0$.

Review Problems

7.36:

Define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_1 = \sqrt{2}$ and $a_n = \sqrt{2a_{n-1}}$ for all $n > 1$. Determine $\lim_{n\to\infty} a_n$.

Hint: Try letting $b_n = \log_2 a_n$.

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Solution: Let $b_n = \log_2 a_n$ for all n. Then $b_1 = \frac{1}{2}$ and $b_n = \frac{1 + b_{n-1}}{2}$ for all n. We claim that $b_n = 1 - \frac{1}{2^n}$ for all n, which we can show by induction. For the base case, $b_1 = 1 - \frac{1}{2} = \frac{1}{2}$ as desired, and for the inductive step, if $b_{n-1} = 1 - \frac{1}{2^{n-1}}$, then

$$b_n = \frac{1}{2}(1 + b_{n-1}) = \frac{1}{2}\left(1 + 1 - \frac{1}{2^{n-1}}\right) = 1 - \frac{1}{2^n},$$

as desired. Therefore, we have $b_n \to 1$, so by the continuity of $f(x) = \log_2 x$, we conclude that $a_n \to 2^1 = \boxed{2}$.

7.37:

For each of the following series, prove that it converges, and if possible, determine what it converges to:

(a) $\sum_{k=1}^{\infty} \frac{2k-1}{3k^2+1}$.

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Solution: This series has positive terms for $k > 1$, and we can compare it to $\sum_{k=1}^{\infty} \frac{1}{k^2}$ using the Limit Comparison Test. We have

$$\lim_{k\to\infty} \frac{\frac{2k-1}{3k^2+1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{2k^3 - k^2}{3k^3 + 1} = \frac{2}{3}.$$

Since this is nonzero, we conclude that since the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does our given series.

(b) $\sum_{k=0}^{\infty} \frac{3^k + 4^k}{7^k}$.

Preview: Solution

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Solution: This is the sum of two geometric series, one with ratio $\frac{3}{7}$ and one with ratio $\frac{4}{7}$. Thus the series converges to

$$\frac{1}{1-\frac{3}{7}} + \frac{1}{1-\frac{4}{7}} = \frac{7}{4} + \frac{7}{3} = \frac{49}{12}.$$

(c) $\sum_{k=1}^{\infty} \left(\frac{1-2^k}{2^k-1}\right)$.

Preview: Solution

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Your Submission: Solution

Solution: This is the sum of two geometric series, one with ratio $\frac{3}{4}$ and one with ratio $\frac{4}{7}$. Thus the series converges to

$$\frac{1}{1-\frac{3}{4}} + \frac{1}{1-\frac{4}{7}} = \frac{4}{1} + \frac{7}{3} = \frac{25}{12}.$$

(d) $\sum_{k=1}^{\infty} \frac{2^k}{2^k - 1}$.

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Your Submission: Solution

Solution: This is the sum of two geometric series, one with ratio $\frac{3}{4}$ and one with ratio $\frac{4}{7}$. Thus the series converges to

$$\frac{1}{1-\frac{3}{4}} + \frac{1}{1-\frac{4}{7}} = \frac{4}{1} + \frac{7}{3} = \frac{25}{12}.$$

(e) $\sum_{k=1}^{\infty} \frac{(-2)^k}{2^k - 1}$.

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Solution: This is an alternating series. The absolute value of the kth term is $\frac{2^k}{2^k - 1}$, which is decreasing for $k > 1$ and converges to 0, so the series converges conditionally. Furthermore, via the Limit Comparison Test with the geometric series with ratio $\frac{2}{3}$, we see that the series is absolutely convergent.

(f) $\sum_{k=1}^{\infty} \frac{k^2 3^k}{4^k}$.

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Solution: We apply the Ratio Test:

$$\lim_{k\to\infty} \frac{|\frac{(k+1)^2 3^{k+1}}{4^{k+1}}|}{|\frac{k^2 3^k}{4^k}|} = \lim_{k\to\infty} \frac{3(k+1)^2}{4^1(k+1)} = 0.$$

Thus, the series converges.

(g) $\sum_{k=1}^{\infty} \frac{1}{k^r}$.

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Solution: This is a p-series with $p = r > 1$, so it converges.

7.38:

Find the first four terms of the Taylor series about $x = 0$ of the following functions:

(a) e^{-x^2}

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Solution: We plug in $-x^2$ to the Taylor series of e^x:

$$e^{-x^2} = \boxed{1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots}.$$

(b) $\cos x^2$

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Solution: We plug in x^2 to the Taylor series of $\cos x$:

$$\cos x^2 = \boxed{1 - \frac{x^4}{2!} + \frac{x^{12}}{4!} - \frac{x^{16}}{6!} + \cdots}.$$

(c) $e^{x^2 \cos x}$

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Solution: The Taylor series for e^{x^2} is

$$1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots.$$

We get the Taylor series for $e^{x^2 \cos x}$ by plugging in the power series for $\sin x$ into the Taylor series of e^u:

$$1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) + \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^2}{2!} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^3 / 3! + \cdots.$$

We can simplify this and collect terms:

$$1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots\right) + \left(x^2 - \frac{x^4}{3} + \cdots\right) + \left(\frac{x^3}{6} - \frac{x^5}{6} + \cdots\right),$$

which gives

$$1 + x + \frac{x^2}{2} - \frac{x^4}{3} + \cdots.$$

(The higher degree terms won't matter as we only need to compute the first four terms.) Now we multiply e^{x^2} by $e^{\cos x}$:

$$e^{x^2 \cos x} = \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots\right)\left(1 + x + \frac{x^2}{2} - \frac{x^4}{6} + \cdots\right)$$
$$= \boxed{1 + x + \frac{3}{2}x^2 + x^3 + \cdots}.$$

You could also solve this by taking the various derivatives of $e^{x^2 \cos x}$.

(d) $\frac{1}{\sqrt{1-4x^2}}$

Preview: Solution

You may type any additional notes you have here.

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Solution: We can start with the Taylor series of $\frac{1}{\sqrt{1+y}} = (1+y)^{-\frac{1}{2}}$:

$$(1+y)^{-\frac{1}{2}} = 1 - \frac{1}{2}y + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}y^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}y^3 + \cdots$$
$$= 1 - \frac{1}{2}y + \frac{3}{8}y^2 - \frac{5}{16}y^3 + \cdots,$$

and then substitute $y = -4x^2$:

$$\frac{1}{\sqrt{1-4x^2}} = \boxed{1 + 2x^2 + 6x^4 + 20x^6 + \cdots}.$$

As a check, we can plug in $x = \frac{1}{4}$ and see if it's close:

$$\frac{1}{\sqrt{1-4\left(\frac{1}{4}\right)^2}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}} \approx 1.1547,$$

and

$$1 + 2\left(\frac{1}{4}\right)^2 + 6\left(\frac{1}{4}\right)^4 + 20\left(\frac{1}{4}\right)^6$$
$$\approx 1 + 0.125 + 0.0234 + 0.0049$$
$$\approx 1.1533,$$

so that's pretty close. (The next term is $70x^8$, which adds 0.0011 to our approximation, giving 1.1544, and now it's very close.)

7.39:

Compute $\sum_{n=0}^{\infty} \frac{2^{n-1}}{n!}$.

Hint: What function does this look like the Taylor series of?

You may type any additional notes you have here.

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Solution: We note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Substituting $x = 2$ gives

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!},$$

thus

$$\sum_{n=0}^{\infty} \frac{2^{n-1}}{n!} = \boxed{\frac{1}{2}e^2}.$$

7.40:

Estimate $\int_0^1 \sin x^2\, dx$ to 3 decimal places using power series.

Hint: Write the power series for $\sin x^2$, then integrate term-by-term. Be a little careful about the resulting constant of integration.

Preview: Solution

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Solution: The power series expansion for $\sin x^2$ is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots.$$

Integrating, this becomes

$$g(x) = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots,$$

for some C. By the Fundamental Theorem of Calculus it suffices to find $g(1) - g(0)$. When we do this, C will cancel, and

$$g(1) - g(0) = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \cdots.$$

The fourth term is far less than 0.001, so we take the first three to get $\boxed{0.310}$.

7.41:

Let p and q be positive real numbers. Compute $\lim_{n\to\infty} \sqrt{p^n + q^n}$.

Hint: Think about it a bit first before diving into computations. What do you expect the answer to be?

Preview: Solution

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Your Submission: Solution

Solution: Suppose $p > q$. Then we have

$$\lim_{n\to\infty} \sqrt{p^n + q^n} = \lim_{n\to\infty} p\sqrt{1 + \left(\frac{q}{p}\right)^n} = p \lim_{n\to\infty} \sqrt{1 + \left(\frac{q}{p}\right)^n} = p,$$

since $\lim_{n\to\infty} \sqrt{1 + x^n} = 1$ if $0 \le x < 1$. Similarly, if $p < q$, then the limit is q. If $p = q$, then we have

$$\lim_{n\to\infty} \sqrt{2p^n} = p \lim_{n\to\infty} \sqrt{2} = p.$$

This can all be summarized as

$$\lim_{n\to\infty} \sqrt{p^n + q^n} = \boxed{\max\{p, q\}},$$

which makes intuitive sense, as the larger term "dominates" the smaller term.

7.42:

Source: HMMT

Let $x > 0$ be a real number. Prove that $\sum_{n=1}^{\infty} \frac{n!}{(xn)^n}$ converges if $x > \frac{1}{e}$ and diverges if $x < \frac{1}{e}$.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We can use the Ratio Test. We compute the limit of $\frac{a_{n+1}}{a_n}$, where a_n is the nth term of the series:

$$L = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(x(n+1))^{n+1}}}{\frac{n!}{(xn)^n}} = \lim_{n\to\infty} \frac{(n+1)(x^n)}{x(n+1)^{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left(\frac{n}{n+1}\right)^n.$$

But the last term of this limit is

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}.$$

Thus we get $L = \frac{1}{xe}$. The series converges if $L < 1$, which occurs if $x > \frac{1}{e}$. It diverges if $L > 1$, which occurs if $x < \frac{1}{e}$.

7.43:

Source: HMMT

Define a function f via the power series

$$f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{2^n}.$$

for $-1 \le x \le 1$. Compute $\sqrt{e\int_0^1 f(x)\,dx}$.

Hint: It's not as hard as it looks. Just logically work through it step-by-step.

Your Submission: Solution

Solution: We note that $f(x)$ is a geometric series with ratio $\frac{x}{2}$ so

$$f(x) = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x},$$

with convergence $\theta - 2 < x < 2$. The interval $[0, 1]$ is within the radius of convergence, thus

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{2}{2-x}\,dx$$
$$= -2\log(2-x)\Big|_0^1$$
$$= -2(\log 1 - \log 2)$$
$$= 2\log 2.$$

Finally,

$$\sqrt{e\int_0^1 f(x)\,dx} = \sqrt{e2\log 2} = e^{\log 2} = \boxed{2}.$$

Plane Curves

8.1 Parametric curves

Let's go back to our example of the circle, but now let's consider the parametrization of the circle given by the two sets of the circle as the set of all points (x, y) such that

$$x = \cos\theta, \quad y = \sin\theta,$$

where $\theta \in \mathbb{R}$. We further restrict that $0 \leq \theta < 2\pi$ because we don't want to take a unique point on the circle.

Now, we can think of the parametrization θ as a function of the variable t and consider the curve whose coordinates of the x-coordinate and the y-coordinate of the circle are given by the two functions $x(t)$ and $y(t)$ respectively.

Definition: A **parametric curve** C is a set of points in the coordinate plane satisfying the two functions $x(t)$ and $y(t)$ where t is the variable, $t \in I$, and I is an interval (finite or infinite). I is called the **domain** of C.

$$C = \{(x(t), y(t)) \mid t \in I\}$$

The functions x and y are called **parametric functions**, and the pair (x, y) is called a **parametrization** of C.

Usually, if x is chosen arbitrarily, then the value of y is fixed. We call x corresponding to the endpoints of I are the **endpoints** of the parametric curve. We call the parametric curve closed if the endpoints of I are equal.

In our circle example above, we have $x(t) = \cos t$ and $y(t) = \sin t$ where t ranges over the interval $[0, 2\pi)$. We could have used a different interval, say $[-\pi, \pi)$, or $[0, 4\pi)$, or $[0, 2\pi]$, or $[0, 2\pi)$ with $t \in \mathbb{Z}$, or $[0, 2\pi)$ with $t \in \mathbb{Q}$, or $[0, 2\pi)$ with $t \in \mathbb{R} \setminus \mathbb{Q}$, or $[0, 2\pi)$ with $t \in \mathbb{C}$, or $[0, 2\pi)$ with $t \in \mathbb{R}^2$, or $[0, 2\pi)$ with $t \in \mathbb{R}^3$, or $[0, 2\pi)$ with $t \in \mathbb{R}^n$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$ and $t \in \mathbb{Q}$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$ and $t \in \mathbb{R} \setminus \mathbb{Q}$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$ and $t \in \mathbb{C}$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$ and $t \in \mathbb{R}^2$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$ and $t \in \mathbb{R}^3$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$ and $t \in \mathbb{R}^n$, or $[0, 2\pi)$ with $t \in \mathbb{R}^\infty$ and $t \in \mathbb{R}^\infty$, or $[0, 2\pi)$ with $t \in 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8.2 Polar coordinates

You have likely seen polar coordinates before. Polar coordinates provide an alternative system for describing points in the plane, in terms of their positions relative to the origin.

Definition: A point P in the plane can be described by polar coordinates (r, θ) where r is the distance from P to the origin, and θ is the angle between the positive x-axis and the ray $\overrightarrow{OP}$, where θ is the origin, measured in radians, anticlockwise. A given point P can have many different polar coordinates, for example the point P in the diagram is described by the coordinates (r, θ) and $(r, \theta + 2\pi)$.

The first task is converting between our usual "rectangular" coordinates and polar coordinates.

Problem 8.11

(a) If P has given rectangular coordinates (x, y), what are the polar coordinates (r, θ) in terms of x and y?

(b) If P has given polar coordinates (r, θ) what are the rectangular coordinates (x, y) in terms of r and θ?

Solution for Problem 8.11:

(a) The picture accompanying the definition of polar coordinates makes it pretty clear what we need to do. We just use basic trees about right triangles. We have $r^2 = x^2 + y^2$, so $r = \sqrt{x^2 + y^2}$ (as $r \geq 0$). Also, $\tan(\theta) = \frac{y}{x}$, so $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. However, this only works for points in the first quadrant, as in the diagram to the right. Also, if $\tan(\theta) = \frac{y}{x}$ we have a point in the second quadrant, unless our angle θ is the case, we still have $\tan\left(\frac{y}{x}\right) \neq \tan(\theta)$ but we don't have $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ since the range of $\tan^{-1}\left(\frac{y}{x}\right)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ for the second quadrant.

The formula $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ gives us a point in the second quadrant, but we need to add π to get the right angle.

For example, the point $(-1, -1)$ in polar coordinates is given by $r = \sqrt{2}$ and $\theta = \frac{5\pi}{4}$. So $\tan^{-1}\left(\frac{y}{x}\right)$ We need to use our trigonometric knowledge of where the sine and cosine are positive and to add or subtract a factor of π. Another problem is if $x = 0$. Then $\frac{y}{x}$ is undefined, so we cannot compute $\tan^{-1}\left(\frac{y}{x}\right)$. However, know that we must have $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ depending on whether y is positive or negative.

(b) The other direction going from polar to rectangular is much easier:

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$

and there are no exceptions to this rule.

It will be convenient for us to allow for polar coordinates with negative r. We think of this as a distance "in the negative direction" of the ray that forms angle θ with the positive x-axis. So, we have the same polar coordinates $(r, \theta) = (-r, \theta + \pi)$ for any r and θ.

One perhaps unfortunate feature of polar coordinates is that any point has infinitely many different representations. For example, the point $(x, y) = (1, 0)$ in rectangular coordinates can be represented in polar coordinates as any of the following:

$$(1, 0), (1, 2\pi), (1, 4\pi), (1, -2\pi), (1, -4\pi), (-1, \pi), (-1, 3\pi),$$
$$(-1, -\pi), (-1, -3\pi), \ldots$$

Our main use of polar coordinates will be to describe the graph of curves of the form $r = f(\theta)$, as in the following example

Problem 8.12

Describe the graph of $r = \cos\theta$. Where is it positive/ negative?

Solution for Problem 8.12: We can actually try to study this graph by plugging in a few common values of θ and determining what the corresponding points are on the graph.

For example:

- Setting $\theta = \frac{\pi}{2}$ gives $r = \cos\frac{\pi}{2} = 0$, so the (rectangular) point $(0, 0)$ lies on the graph.
- Setting $\theta = \frac{\pi}{4}$ gives $r = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so the (rectangular) point $(0, \frac{1}{\sqrt{2}})$ lies on the graph.
- Setting $\theta = \pi$ gives $r = \cos\pi = -1$ so the (rectangular) point $(0, 0)$ lies on the graph.
- Setting $\theta = \frac{3\pi}{2}$ gives $r = \cos\frac{3\pi}{2} = 0$, so the (rectangular) point $(0, 0)$ lies on the graph again.

This is not much information.

Let's look a little more closely at the shape of this graph by considering what happens as θ varies from 0 to 2π. As θ increases from 0 to $\frac{\pi}{2}$ the radius r increases from 0 to 1. Then as θ increases $\frac{\pi}{2}$ to π, the radius decreases from a back to 0. We may also note as θ ranges from 0 to 2π, we get exactly the same graph, since $\cos(\theta) = \cos(-\theta)$.

To determine exactly what the graph looks like, we can convert to rectangular coordinates. Noting that $r = \cos\theta$, we have $\frac{r}{1} = \cos\theta$, so we equations $r = \cos\theta$ and $\frac{r}{1} = \frac{x}{r}$ or $r^2 = x$. (We need to be a little careful, as this doesn't work when $r = 0$, but this point is just the origin, which we know is in the graph.) But we also have $r^2 = x^2 + y^2$, so we get the equation

$$x^2 + y^2 = x.$$

To examine the graph of this, we complete the square:

$$x^2 + \left(\frac{1}{2}\right)^2 = \frac{x^2}{2}.$$

Therefore, the graph is a circle centered at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$. Here is a picture

Problem 8.13

Sketch the graph of $r = 2\cos\theta$. Where is it positive/ negative? Write it in rectangular equation in x and y.

Solution for Problem 8.13: As in the previous problem, we can analyze what happens to r as we vary θ.

- As θ ranges from 0 to $\frac{\pi}{2}$, the radius increases from 0 to 2 and then decreases back to 0.
- As θ ranges from $\frac{\pi}{2}$ to π, the radius goes from 0 to -2 and then back to 0.
- As θ ranges from π to $\frac{3\pi}{2}$, the radius goes from 0 to 2 and then back to 0.
- And so on...

Sketching this gives us a lovely **rose**, as shown to the right. Note that each range of θ between multiples of $\frac{\pi}{2}$ gives a leaf of the rose, but the leaves overlap in parts. For example, the leaf for $\theta \in \left[0, \frac{\pi}{2}\right]$ overlaps with the leaf for $\theta \in \left[\pi, \frac{3\pi}{2}\right]$.

To get a rectangular equation, we can use the polar angle formula:

$$\sin 2\theta = 2\sin\theta = 2\sin^2\theta.$$

So we have

$$r = 2\left(\frac{y}{r}\right)^2.$$

We multiply by r and substitute $r^2 = x^2 + y^2$ to get:

$$x^2 + y^2 = 2y - 2\frac{y^2}{x^2}.$$

To clean this up, we can clear the denominator:

$$(x^2 + y^2)^2 + 2y^2 - 4y^3 y = 0.$$

More generally, the graph of $r = a\sin n\theta$ where n is a positive integer, is called a **rose**. As we saw in Problem 8.13, a circle is a special case of a rose. Roses are good examples of curves that are ideal for polar coordinates: the rectangular equation for these curves are usually somewhat complicated, but the polar equation is very simple and more importantly gives us an idea of the nature of the curves.

When exploring these sorts of graphs, you may want to be a little careful about missing "hidden" points. For example:

Problem 8.14

Is the polar point $\left(-\frac{1}{2}, \frac{1}{2}\right)$ on the graph of $r = \sin 2\theta$

Solution for Problem 8.14: You might quickly say "no".

Bogus Solution: Plugging $\theta = \frac{\pi}{4}$ gives $r = \sin\pi = -1$, so the point $\left(-1, \frac{\pi}{4}\right)$ is on the graph, not $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

However, $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is on the graph! Remember that any point has infinitely many different polar representations. Since $\theta = \frac{3\pi}{4}$ gives $r = -\frac{1}{2}$, we have $\left(-\frac{1}{2}, \frac{3\pi}{4}\right)$ as the graph. Also this is the same point as $\left(-\frac{1}{2}, \frac{7\pi}{4}\right)$ so the point $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is on the graph.

WARNING! Points can be represented in polar coordinates in infinitely many ways.

The polar curve $r = f(\theta)$ can be viewed as a special case of a parametric curve. Since $x = r\cos\theta$ and $y = r\sin\theta$, our curve is parametrized by

$$(r\cos\theta, r\sin\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta).$$

So the parameter θ. This allows us to we apply our parametric curve tools from Section 8.1 to study polar curves. For instance:

Problem 8.15

Let $r = f(\theta)$ be the polar equation of a curve, and let $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$. What is the slope of the tangent line to this curve at any given value of θ?

Solution for Problem 8.15: Treating our polar curve as a parametric curve, we can use the usual idea of

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Thus, we evaluate $dx/d\theta$ and $dy/d\theta$ in terms of r and θ.

We start with:

$$y = r\sin\theta = f(\theta)\sin\theta.$$

Then we use the product rule:

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta.$$

Since $r = f(\theta)$, we will write $r' = f'(\theta)$, and thus

$$\frac{dy}{d\theta} = r\cos\theta + r'\sin\theta.$$

Similarly,

$$x = r\cos\theta = f(\theta)\cos\theta,$$

so upon differentiating, we get

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta.$$

Thus

$$\frac{dx}{d\theta} = -r\sin\theta + r'\cos\theta.$$

Putting this all together, we have

$$\text{Slope of tangent at } r = f(\theta) = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}.$$

There's really no point of memorizing this formula. It doesn't come up that often, and it's easy to rederive when you need it. The key thing to remember is:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Problem 8.16

Find the slope of the tangent line to $r = 2\cos 2\theta$ at $\theta = \frac{\pi}{6}$

Solution for Problem 8.16: First, we should get a rough idea of what the graph looks like. The cos 2θ form has one leaf centered at $\frac{\pi}{2}$ between zeroes, so the graph is a lemniscate (the zeroes do not overlap), as shown to the right. We have also shown the ray corresponding to $\theta = \pi/6$. At this point, you might pause to try the wrong conclusion.

Bogus Solution: We want the slope of the tangent at the point where the ray hits the curve.

However, it's not that point at all! We plug in $\theta = \frac{\pi}{6}$ and we get $r = 2\cos\frac{2\pi}{6} = 1$. The resulting point is in the 3rd quadrant, as shown below:

In rectangular coordinates, it's the point $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Now, to compute the slope of the tangent line, we just use the formula from Problem 8.15:

$$\frac{dy}{dx} = \frac{2\cos 2\theta\sin\theta + 4\sin 2\theta\cos\theta}{-2\cos 2\theta\cos\theta + 4\sin 2\theta\sin\theta}.$$

We then plug in $\theta = \frac{\pi}{6}$:

$$\frac{dy}{dx} = \frac{2\cos\frac{\pi}{3}\sin\frac{\pi}{6} + 4\sin\frac{\pi}{3}\cos\frac{\pi}{6}}{-2\cos\frac{\pi}{3}\cos\frac{\pi}{6} + 4\sin\frac{\pi}{3}\sin\frac{\pi}{6}}.$$

To finish, we substitute into $\frac{2\cos\frac{\pi}{3}}{-2\cos\frac{\pi}{3}} = -\frac{1}{2}$ and $\frac{4\sin\frac{\pi}{3}}{4\sin\frac{\pi}{3}} = 1$ to get:

$$\frac{dy}{dx} = \frac{-\frac{1}{2}\sin\frac{\pi}{6} + \cos\frac{\pi}{6}}{-\frac{1}{2}\cos\frac{\pi}{6} + \sin\frac{\pi}{6}} = \frac{\frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}}{\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}} = \frac{\frac{1}{4} + \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{4} + \frac{1}{2}}.$$

So the slope is $\frac{1+2\sqrt{3}}{\sqrt{3}+2}$ and the tangent line is shown in the picture below:

As we've seen, finding tangent lines to polar curves is a bit messy. This is not a big surprise, since lines themselves are not all that nice in polar coordinates, as we will see in the next problem.

Problem 8.17

Find the equation, in polar coordinates of the line through the rectangular point (x_0, y_0) with slope m.

Solution for Problem 8.17: There's really no easy way to do this except to do it first in rectangular coordinates and then convert to polar coordinates. The rectangular equation for the line is

$$y - y_0 = m(x - x_0).$$

We convert this to polar coordinates:

$$r\sin\theta - y_0 = m(r\cos\theta - x_0).$$

To write this in the usual form of a polar curve, we solve for r:

$$r(\sin\theta - m\cos\theta) = y_0 - mx_0,$$

giving

$$r = \frac{y_0 - mx_0}{\sin\theta - m\cos\theta}.$$

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giving

$$r = \frac{y_0 - mx_0}{\sin\theta - m\cos\theta}.$$

These are other ways to describe lines in polar coordinates, some of them will be explored further in the exercises.

Exercises

8.2.1:

For each of the following equations in polar coordinates, describe the graph, and find the slope of the tangent line at the point where $\theta = \frac{\pi}{6}$.

(a) $r = \cos\theta + 3\sin\theta$ where a and b are positive real numbers, with the slope of the tangent line at the point where $\theta = \frac{\pi}{6}$.

Hint: It may be easiest to first convert to rectangular coordinates.

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Solution Note that $r = 0$ when $0 = \cos\theta + 3\sin\theta$ or $-\frac{1}{3} = \tan\theta$. So the origin of the plane is on the graph.

If $\theta \neq 0$, then multiply by r to get $r^2 = r\cos\theta + 3r\sin\theta$. Then, converting to rectangular coordinates, we have $x^2 + y^2 = ax + by$, which can be rearranged (after completing the square in both x and y) as

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}.$$

The graph of this is a circle centered at $\left(\frac{a}{2}, \frac{b}{2}\right)$ with radius $\frac{\sqrt{a^2+b^2}}{2}$. (Note that this graph includes the origin.)

For the slope of the tangent line at $\theta = \frac{\pi}{6}$ it is easiest to implicitly differentiate the rectangular equation, noting that $\theta = \frac{\pi}{6}$ gives the rectangular point $(x, y) = (\frac{a}{2}, \frac{b}{2})$. Differentiating gives

$$2\left(x - \frac{a}{2}\right) + \left(y - \frac{b}{2}\right)\frac{dy}{dx} = b.$$

Plugging in $x = a$ and $y = \frac{b}{2}$ gives $a + \frac{b}{2}\frac{dy}{dx} = b$, so $\frac{dy}{dx} = \frac{2b-2a}{b}$.

(b) $r = a + b\sin\theta$ where a and b are positive real numbers, with the slope of the tangent line at the point where $\theta = \frac{\pi}{6}$. (These curves are called limaçons.)

Hint: The nature of the graph will depend on the sign of $a - b$.

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Your Submission Solution

Solution Note that, as Processes from Ex 21, the graph passes through the following points (in polar coordinates):

$$(a, 0), (a, \frac{\pi}{2}), (a, \pi), (a, \frac{3\pi}{2}), (a + b, \frac{\pi}{2}), (a, 2\pi).$$

The nature of the graph will vary depending on whether $a - b$ is positive, zero, or negative. We sketch three graphs below (all using $a = 2$). The first graph uses $b = 3$ so $a - b$ is negative for some values of θ. The second uses $b = 2$ and is a cardioid. The third uses $b = 1$ so $a - b$ is positive throughout. The curves are collectively known as **limaçons**.

For the slope of the tangent line, we note that $r = r\sin\theta = (a + b\sin\theta)\sin\theta$ so

$$\frac{dy}{d\theta} = (a + b\sin\theta)\cos\theta + (b\cos\theta)(\sin\theta)$$
$$= a\cos\theta + b\cos^2\theta - \sin^2\theta,$$
$$= a\cos\theta + b\cos 2\theta.$$

Similarly, $y = r\cos\theta = (a + b\sin\theta)\cos\theta$, so

$$\frac{dx}{d\theta} = (a + b\sin\theta)(\cos\theta) + (\cos\theta)(b\cos\theta)$$
$$= a\cos\theta + 2b\cos\theta\cos\theta$$
$$= a\cos\theta + b\cos 2\theta.$$

Thus,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta + b\cos 2\theta}{-a\sin\theta + b\cos 2\theta}.$$

and at $\theta = \frac{\pi}{6}$ this slope equals

$$\frac{a\cos\frac{\pi}{6} + b\cos\pi}{-a\sin\frac{\pi}{6} + b\cos\pi} = \frac{0}{-a - b} = \frac{a}{b}.$$

Note that this answer is consistent with our pictures above: all of the graphs cross the positive y-axis at a point at which the graph has slope $\frac{a}{b}$. In the case that $a = b$, the tangent line at the positive y-intercept is horizontal.

Let $r = 1 - \sin 2\theta$, with the slope of the tangent line at the origin.

Hint: Draw some points for easy values of θ and connect the dots.

Hint: To get the slope of the tangent line, you'll have to take a limit. You may also need Hospital's Rule.

Pencil Solution

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Your Submission Solution

Solution The curve passes through the following points (in polar coordinates):

$$(1, 0), (0, \frac{\pi}{4}), (1, \frac{\pi}{2}), (2, \frac{3\pi}{4}), (1, \pi).$$

Also note that $\sin 2\theta \leq 1$, so the curve lies in the x-y plane in the quadrant where $\sin 2\theta$ is positive.

The graph of the curve is shown at right. To graph the correct behavior near the curve at the origin, it is useful to know the slope of the tangent line at the origin, which occurs when $\theta = \frac{\pi}{4}$.

We have $x = r\cos\theta = (1 - \sin 2\theta)\cos\theta$, so

$$\frac{dx}{d\theta} = (1 - \sin 2\theta)(-\sin\theta) + (-2\cos 2\theta)(\cos\theta).$$

Thus, the slope of the tangent line at the point corresponding to some value of θ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 - \sin 2\theta)\cos\theta + (-2\cos 2\theta)(\sin\theta)}{(1 - \sin 2\theta)(-\sin\theta) + (-2\cos 2\theta)(\cos\theta)}.$$

However at $\theta = \frac{\pi}{4}$, we have $\frac{dx}{d\theta} = 0$ so the above expression for $\frac{dy}{dx}$ is undefined, and instead we must compute

$$\lim_{\theta\to\frac{\pi}{4}} \frac{dy}{dx}$$

which we can compute using Hospital's Rule. We compute

$$\frac{d}{d\theta}\left[(1 - \sin 2\theta)\cos\theta + (-2\cos 2\theta)(\sin\theta)\right]$$
$$= (1 - \sin 2\theta)(-\sin\theta) + (-2\cos 2\theta)(\cos\theta)$$
$$+ (-2\cos 2\theta)(\cos\theta) + (4\sin 2\theta)(\sin\theta)$$
$$= (1 - \sin 2\theta)(-\sin\theta) + 4\cos 2\theta\cos\theta + 8\sin 2\theta\sin\theta,$$

and

$$\frac{d}{d\theta}\left[(1 - \sin 2\theta)(-\sin\theta) + (-2\cos 2\theta)(\cos\theta)\right]$$
$$= (1 - \sin 2\theta)\cos\theta + (-2\cos 2\theta)(\sin\theta)$$
$$+ (-2\cos 2\theta)\cos\theta + (4\sin 2\theta)\sin\theta$$
$$+ (-4\cos 2\theta)(-\sin\theta) + 8\sin 2\theta\cos\theta.$$

Then

$$\frac{dy/d\theta}{dx/d\theta} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d^2y}{d\theta^2}}{\frac{d^2x}{d\theta^2}}$$
$$= \frac{8\sin 2\theta}{8\cos 2\theta}$$
$$= \tan 2\theta$$
$$= 1,$$

so the slope of the tangent line at the origin is $\frac{1}{1}$.

8.2.2:

Find the equation in polar coordinates of the line with slope m that passes through the point (r_0, θ_0).

Hint: Applying the sine angle addition formula at some stage may make your answer look much nicer.

Pencil Solution

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Your Submission Solution

Solution Using the fact that (r_0, θ_0) has rectangular coordinates $(r_0\cos\theta_0, r_0\sin\theta_0)$, we can use the formula in the text to get:

$$r = \frac{r_0\sin\theta_0 - m\cos\theta_0}{\sin\theta - m\cos\theta}.$$

This can be more naturally expressed by letting $\psi = \arctan m$. Then our equation becomes

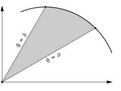
$$r = \frac{r_0\cos\psi\sin\theta_0 - \cos\psi\cos\theta_0}{\cos\psi\sin\theta - \cos\psi\cos\theta}$$

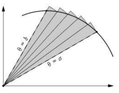
and then applying the angle subtraction formula for sine, we have

$$r = r_0\frac{\sin(\theta_0 - \psi)}{\sin(\theta - \psi)}.$$

8.3 Areas in polar coordinates

Suppose $r = f(\theta)$ is a curve given in polar coordinates. We would like to compute the area of the "sector" between the curve and the rays given by the starting and ending angles $\theta = a$ and $\theta = b$, as in the picture to the right. To compute this area, we can set up an appropriate Riemann sum of smaller regions whose areas we know how to compute. When we compute the area under a rectangular curve given by $y = f(x)$, we divide the region into rectangles. However, rectangles are not a natural object to work with in polar coordinates, so instead we divide our region into a number of small circular sectors:





Those regions may look like triangles, but they actually have circular arcs at the far end, so that they are sectors of a circle. Fortunately, areas of sectors are easy to compute:

Problem 8.18

If a sector has radius r and angle $\Delta\theta$, then what is its area?

Solution for Problem 8.18: A circle of radius r has area πr^2, and a sector of angle $\Delta\theta$ covers a proportion of $\frac{\Delta\theta}{2\pi}$ of the entire circle. So the area of the sector is $\frac{1}{2}r^2(\Delta\theta)$. (Note that if $\Delta\theta = 2\pi$, then we get the whole circle of area πr^2.) ◻

Given that we can compute the areas of the sectors, we can sum up the sectors to get an approximation of the area of the region:

$$\sum_i \frac{1}{2}(r_i)^2(\Delta\theta_i),$$

where r_i is the radius of the ith sector, and $\Delta\theta_i$ is the measure of the angle of that sector. This is a Riemann sum, so when we take the limit of this as the number of sectors grows, we get a definite integral.

Summing up, we see the following:

Important:

The area of the region between $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ is

$$\frac{1}{2}\int_a^b r^2\,d\theta = \frac{1}{2}\int_a^b (f(\theta))^2\,d\theta.$$

WARNING!!

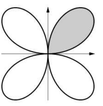
The most common mistake is to use simply $\int_a^b f(\theta)\,d\theta$ for the area. To avoid this error, remember that the area of a circle of radius r is πr^2. This circle is given by the polar curve $r = 1$, and we can compute its area via the integral

$$\frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}(2\pi r^2) = \pi r^2.$$

A typical example is the following:

Problem 8.19

Find the area of one leaf of the rose given by $r = 3\sin 2\theta$.



Solution for Problem 8.19: As we've seen before, this is a 4-leaf rose. One leaf of the rose ranges from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Thus the definite integral for the area is

$$\frac{1}{2}\int_0^{\pi/2} 9\sin^2 2\theta\,d\theta = \frac{9}{2}\int_0^{\pi/2} \sin^2 2\theta\,d\theta.$$

Now use the substitution $u = 2\theta$ to get

$$\frac{9}{4}\int_0^{\pi} \sin^2 u\,du.$$

There are lots of ways to finish from here. For instance, we can now use the trig identity $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$ to get

$$\frac{9}{8}\int_0^{\pi} (1 - \cos 2u)\,du.$$

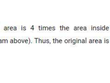
Note that the latter term integrates to 0 since the integral covers an entire period of $\cos 2u$. Therefore, the area is just $\frac{9}{8}\pi$. ◻

A slightly more complicated example is the following:

Problem 8.20

Find the area of the region that is the intersection of the interiors of the graphs of $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$.

These curves are called **cardioids**.

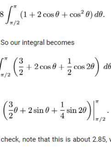


Solution for Problem 8.20: We can sketch a picture, shown at right. The curve on the right is $r = 2(1 + \cos\theta)$ and the curve on the left is $r = 2(1 - \cos\theta)$. These curves intersect when

$$2(1 + \cos\theta) = 2(1 - \cos\theta),$$

so $\cos\theta = 0$, which corresponds to points on the y-axis.

One way to compute the area is to note that the graphs are reflections of each other across the y-axis, so the total area is just twice the area inside $r = 2(1 + \cos\theta)$ between $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, as shown below:



Even easier than that, because of the symmetry of the graph across the x-axis, the original area is 4 times the area inside $r = 2(1 + \cos\theta)$ between $\theta = \frac{\pi}{2}$ and $\theta = \pi$ (which is the top half of the shaded region in the diagram above). Thus, the original area is given by the definite integral

$$4 \cdot \frac{1}{2}\int_{\pi/2}^{\pi} r^2\,d\theta = 2\int_{\pi/2}^{\pi} (2(1 + \cos\theta))^2\,d\theta.$$

Expanding the square in the integrand gives

$$8\int_{\pi/2}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta.$$

Now we use the fact that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$. So our integral becomes

$$8\int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)\,d\theta.$$

This gives

$$8\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_{\pi/2}^{\pi}.$$

Hence, the area is $8\left(\frac{3\pi}{4} - 2\right) = 6\pi - 16$. As a check, note that this is about 2.85, which is plausible for a region that fills much of a rectangle with an approximate "height" of 4 and an approximate "width" of about 1. ◻

Exercises

8.3.1:

Find the area between the curves $r = \theta$ and $r = 2\theta$ for $0 \le \theta \le \pi$.

Preview: Solution

You may type any additional notes you have here.

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Solution: The graphs and the desired region are shown at right. The area inside the outer spiral is $\frac{1}{2}\int_0^{\pi}(2\theta)^2\,d\theta$. The area inside the inner spiral is $\frac{1}{2}\int_0^{\pi}\theta^2\,d\theta$. Thus, the desired area is the difference:

$$\frac{1}{2}\int_0^{\pi} 3\theta^2\,d\theta = \frac{1}{2}\theta^3\Big|_0^{\pi} = \boxed{\frac{1}{2}\pi^3}.$$

8.3.2:

Find the area of region that is outside the curve $r = 1 - \cos\theta$ but inside the curve $r = 1$.

Preview: Solution

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Solution: The graphs of the cardioid and the circle are shown at right, with the desired region shaded. The cardioid crosses $r = 1$ at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$. Thus we want

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} ((1)^2 - (1 - \cos\theta)^2)\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (2\cos\theta - \cos^2\theta)\,d\theta.$$

Since the regions from $-\frac{\pi}{2}$ to 0 and from 0 to $\frac{\pi}{2}$ have equal area, this area is just

$$\begin{aligned}\int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\,d\theta &= \int_0^{\pi/2}\left(2\cos\theta - \frac{1}{2}(1 + \cos 2\theta)\right)\,d\theta \\ &= \left(2\sin\theta - \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right)\Bigg|_0^{\pi/2} \\ &= \boxed{2 - \frac{\pi}{4}}.\end{aligned}$$

8.3.3★:

Find the shaded area, shown at right, inside the limaçon given by the graph of $r = 1 + 2\sin\theta$.

Hint: If we take the area inside the entire limaçon for $\theta \in [0, 2\pi]$, how many times is the small white inner loop region counted?

Hint: Note the limaçon passes through the origin when $\sin\theta = -\frac{1}{2}$ and forms the boundary of the inner white region when $\sin\theta < -\frac{1}{2}$.

Hint: You'll need to express this area as a difference of two integrals.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The outer boundary of the desired region is the part of the limaçon where $r \ge 0$, which occurs when $2\sin\theta \ge -1$, or $\sin\theta \ge -\frac{1}{2}$. This occurs for $-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$. Thus, the entire area inside the limaçon–including both the shaded and unshaded regions–is given by

$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6} (1 + 2\sin\theta)^2\,d\theta.$$

To exclude the unshaded region, we subtract its area, which is the region from $7\pi/6$ to $11\pi/6$, giving the area of the shaded region as

$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6} (1 + 2\sin\theta)^2\,d\theta - \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2\,d\theta.$$

Noting that the combined intervals of the integrals is $[-\pi/6, 11\pi/6]$, which is a full period for sine, gives us the idea to use a clever trick to make the computation a bit simpler. We note that for any function f and $a < c < b$,

$$\int_a^b f - \int_c^b f = \int_a^c f + \int_c^b f - 2\int_c^b f = \int_a^b f - 2\int_c^b f.$$

Using this makes our area equal to

$$\frac{1}{2}\int_{-\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2\,d\theta - \int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2\,d\theta.$$

The advantage of this is that, by the periodicity of the function $1 + 2\sin\theta$, the first integral is the same as the integral from 0 to 2π. Thus our area is

$$\frac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta)^2\,d\theta - \int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2\,d\theta.$$

We now note that

$$\begin{aligned}(1 + 2\sin\theta)^2 &= 1 + 4\sin\theta + 4\sin^2\theta \\ &= 1 + 4\sin\theta + (2 - 2\cos 2\theta) \\ &= 3 + 4\sin\theta - 2\cos 2\theta,\end{aligned}$$

which has antiderivative $3\theta - 4\cos\theta - \sin 2\theta$. Thus, our area evaluates to

$$\frac{1}{2}\left(3\theta - 4\cos\theta - \sin 2\theta\right)\Big|_0^{2\pi} - \left(3\theta - 4\cos\theta - \sin 2\theta\right)\Big|_{7\pi/6}^{11\pi/6}.$$

The first term is just 3π. In the second term, we have

$$\begin{aligned}&(3\theta - 4\cos\theta - \sin 2\theta)\Big|_{7\pi/6}^{11\pi/6} \\ &= 3\left(\frac{11\pi}{6} - \frac{7\pi}{6}\right) - 4\left(\cos\frac{11\pi}{6} - \cos\frac{7\pi}{6}\right) - \left(\sin\frac{11\pi}{3} - \sin\frac{7\pi}{3}\right) \\ &= 2\pi - 4\sqrt{3} - (-\sqrt{3}) = 2\pi - 3\sqrt{3}.\end{aligned}$$

Thus, the area is $3\pi - (2\pi - 3\sqrt{3}) = \boxed{\pi + 3\sqrt{3}}$.

Challenge Problems

8.27:

Suppose a planet revolves around a star at distance R and the planet has a moon that revolves around the planet at distance r, with $0 < r < R$. (Assume that all motion is in the same plane and that all orbits are circular.) At time $t = 0$, the planet is at $(R, 0)$ and the moon is at $(R + r, 0)$. The planet takes 1 year to revolve around the star and the moon takes $\frac{1}{4}$ year to revolve around the planet.

(a) Write parametric equations for the position of the moon at time t (in years). (Assume all movement is counterclockwise.) As a check, at time $t = \frac{1}{2}$ the position of the moon should be $(r - R, 0)$ and at $t = 1$ it should be back to $(R + r, 0)$.

Preview: Solution

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Solution: The planet moves with parameterization $(R\cos 2\pi t, R\sin 2\pi t)$. The moon moves around the planet with parameterization $(r\cos 8\pi t, r\sin 8\pi t)$ relative to the center of the planet. (Note that as t goes from 0 to 1, the planet revolves once around the star, and the moon revolves 4 times around the planet.) Adding these, we get parametric equations for the moon relative to the star:

$$\boxed{(R\cos 2\pi t + r\cos 8\pi t, R\sin 2\pi t + r\sin 8\pi t).}$$

(b) Sketch some graphs of the moon's path. Do this for different values of $\frac{r}{R}$ and try to determine the conditions on $\frac{r}{R}$ that affect the shape of the graph.

Hint: The three different behaviors are $\frac{r}{R} < \frac{1}{4}$, $\frac{r}{R} = \frac{1}{4}$, and $\frac{r}{R} > \frac{1}{4}$.

Preview: Solution

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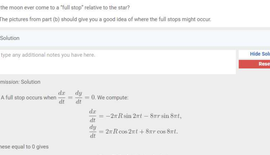
Your Submission: Solution

Solution: There are different behaviors depending on the ratio $\frac{r}{R}$. Below, we plot three different values:



The thin circle is the path of the planet, and the bold curve is the path of the moon. Note that if $\frac{r}{R} < \frac{1}{4}$, the path of the moon loops on itself, and if $\frac{r}{R} > \frac{1}{4}$, then the moon stays close to the path of the planet and doesn't loop. If $\frac{r}{R} = \frac{1}{4}$, then we get sharp corners, as we'll see in part (c).

(c) Does the moon ever come to a "full stop" relative to the star?

Hint: The pictures from part (b) should give you a good idea of where the full stops might occur.

Preview: Solution

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Your Submission: Solution

Solution: A full stop occurs when $\frac{dx}{dt} = \frac{dy}{dt} = 0$. We compute:

$$\begin{aligned} \frac{dx}{dt} &= -2\pi R\sin 2\pi t - 8\pi r\sin 8\pi t, \\ \frac{dy}{dt} &= 2\pi R\cos 2\pi t + 8\pi r\cos 8\pi t. \end{aligned}$$

Setting these equal to 0 gives

$$\begin{aligned} 0 &= R\sin 2\pi t + 4r\sin 8\pi t, \\ 0 &= R\cos 2\pi t + 4r\cos 8\pi t. \end{aligned}$$

One way to solve this system is to square both equations and add them:

$$0 = R^2 + 4r\left(\sin 2\pi t\sin 8\pi t + \cos 2\pi t\cos 8\pi t\right).$$

The trig expression in the parentheses is the cosine angle subtraction expression of $\cos(6\pi t - 2\pi t)$, so the equation simplifies to

$$0 = R^2 + 16r^2 + 8Rr\cos 6\pi t,$$

hence

$$\cos 6\pi t = \frac{R^2 + 16r^2}{8Rr} = \frac{R}{8r} - \frac{2r}{R}.$$

Let $c = \frac{r}{R} < 1$, and we have

$$\cos 6\pi t = -\left(2c + \frac{1}{8c}\right).$$

This will have a solution if and only if the right side is between -1 and 1, so we must have

$$2c + \frac{1}{8c} \leq 1.$$

Multiplying by c and rearranging gives $2c^2 - c + \frac{1}{8} \leq 0$, or $\frac{1}{8}(4c - 1)^2 \leq 0$. Clearly this occurs if and only if $c = \frac{1}{4}$.

In summary: if $c = \frac{r}{R} < \frac{1}{4}$, then the planet comes to a full stop when $\cos 6\pi t = -\left(2c + \frac{1}{8c}\right) = -1$; this happens 3 times a year (at $t = \frac{1}{6}$, $t = \frac{1}{2}$, and $t = \frac{5}{6}$) and these times correspond to the sharp corners in the graph for the case $\frac{r}{R} = \frac{1}{4}$ shown above.

8.28:

Describe the graph of $r = \frac{\ell}{1 + e\cos\theta}$ where $e > 1$ and ℓ is a positive constant.

Hint: You could try to sketch an example first.

Hint: Try converting to rectangular coordinates.

Preview: Solution

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Solution: We have $r + er\cos\theta = \ell$, so $\sqrt{x^2 + y^2} + ex = \ell$. Thus

$$x^2 + y^2 = (\ell - ex)^2 = \ell^2 - 2\ell x + e^2x^2.$$

If $e > 1$, this can be written in the form

$$y^2 - a(x - b)^2 = c^2$$

for some positive constant a and constants b and c. So the graph is a hyperbola. More specifically, it is a hyperbola with one focus at $(0, 0)$.

8.29:

Show that the length of a parametric curve is independent of the choice of parameterization in the following sense:

Suppose that $(u(t), v(t))$ is a parameterization of a curve C for $t \in [a, b]$. Let f be a differentiable function whose domain includes $[c, d]$ such that:

- $f(c) = a$ and $f(d) = b$,
- $f([c, d]) = [a, b]$ and
- $f'(t) \geq 0$ for all $t \in [c, d]$.

Show that C is the curve given by the parameterization $(u(f(t)), v(f(t)))$ for $t \in [c, d]$ and that the length of C computed using the parameterization $(u(f(t)), v(f(t)))$ equals the length of C computed using the parameterization $(u(t), v(t))$.

Hint: First, show that the new parameterization gives the same curve (as a set) as the original parameterization.

Hint: This is just an (admittedly complicated) application of the Chain Rule.

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Solution: To show that both parameterizations give the same curve, start with

$$C = \{(u(t), v(t)) \mid t \in [a, b]\}$$

and the "new" curve

$$C_{new} = \{(u(f(t)), v(f(t))) \mid r \in [c, d]\}.$$

(We use a different variable for C_{new} for clarity.) We wish to show that $C = C_{new}$ if $(u(t), v(t)) \in C$, then $t \in [a, b]$ so $t = f(r)$ for some $r \in [c, d]$ and hence

$$(u(t), v(t)) = (u(f(r)), v(f(r))) \in C_{new}.$$

Thus $C \subseteq C_{new}$. Conversely, if $(u(f(r)), v(f(r))) \in C_{new}$ for some $r \in [c, d]$ then $t = f(r) \in [a, b]$ and hence

$$(u(f(r)), v(f(r))) = (u(t), v(t)) \in C.$$

Thus $C_{new} \subseteq C$, and therefore $C_{new} = C$. So both parameterizations give the same curve.

The length of the curve C computed using the new parameterization is

$$\int_c^d \sqrt{\left(\frac{d}{dr}(u(f(r)))\right)^2 + \left(\frac{d}{dr}(v(f(r)))\right)^2}\, dr.$$

By the Chain Rule, this is equal to

$$\int_c^d \sqrt{(u'(f(r))f'(r))^2 + (v'(f(r))f'(r))^2}\, dr.$$

Since $f'(r) \geq 0$, we can pull $(f'(r))^2$ outside of the square root, giving:

$$\int_c^d f'(r)\sqrt{(u'(f(r)))^2 + (v'(f(r)))^2}\, dr.$$

Now with the substitution $t = f(r)$ and $dt = f'(r)\,dr$, we have

$$\int_a^b \sqrt{(u'(t))^2 + (v'(t))^2}\, dt,$$

which equals the length of C computed using the original parameterization.

8.30:

Consider the rose $r = \cos n\theta$ where n is a positive integer.

(a) Describe the graph of the rose (in terms of n). (You will have to distinguish between the cases where n is even and where n is odd.)

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Solution: If n is odd, we have n petals, but if n is even, we have $2n$ petals. That is because $\cos n\theta$ will have n periods from $\theta = 0$ to $\theta = 2\pi$. But in the even n case, each period will produce 2 petals (one where $r \geq 0$ and one where $r \leq 0$), whereas in the odd n case, the $r \leq 0$ petal will exactly overlap with an $r \geq 0$ petal from a different period. This is because $\cos(n(\theta + \pi)) = \cos n\theta$ when n is even, but $\cos(n(\theta + \pi)) = -\cos n\theta$ when n is odd.

(b) Compute the area of one petal of the rose.

Preview: Solution

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Solution: The area of the first petal is twice the area inside the curve from $\theta = 0$ to $\theta = \frac{\pi}{2n}$. So the integral that gives the area we want is

$$2 \cdot \frac{1}{2}\int_0^{\pi/2n} \cos^2 n\theta\, d\theta.$$

We then compute:

$$\begin{aligned} \int_0^{\pi/2n} \cos^2 n\theta\, d\theta &= \frac{1}{2}\int_0^{\pi/2n} (1 + \cos 2n\theta)\, d\theta \\ &= \frac{\pi}{4n} + \frac{1}{2}\int_0^{\pi/2n} \cos 2n\theta\, d\theta \\ &= \frac{\pi}{4n} + \frac{1}{4n}\sin 2n\theta\Big|_0^{\pi/2n} \\ &= \frac{\pi}{4n} + \frac{1}{4n}(\sin\pi - \sin 0) \\ &= \boxed{\frac{\pi}{4n}}. \end{aligned}$$

(c) ★ The **width** of the rose is the length of the longest segment, parallel to the y-axis, that can be inscribed in the first petal of the rose (that is, the petal that contains part of the positive x-axis). The example for $n = 2$ is shown below (the width is the length of the dark black segment):

Find the width of the rose $r = \cos 2\theta$.

Hint: Note that you're trying to maximize $y = r\sin\theta$ for $\theta \in \left[0, \frac{\pi}{4}\right]$.

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Solution: Note that we have $y = r\sin\theta = \cos 2\theta\sin\theta$. We need to maximize this for the domain $\theta \in \left[0, \frac{\pi}{4}\right]$. This is easier to work with if we write the whole thing in terms of sine:

$$y = \cos 2\theta\sin\theta = (1 - 2\sin^2\theta)\sin\theta = \sin\theta - 2\sin^3\theta.$$

Now we can maximize by differentiating and finding the critical points:

$$\frac{dy}{d\theta} = \cos\theta - 6\sin^2\theta\cos\theta.$$

Since $\cos\theta \neq 0$ in the interval that we are considering, we see that the critical point occurs where $\sin^2\theta = \frac{1}{6}$ or $\sin\theta = \frac{1}{\sqrt{6}}$. Note that this critical point must be a local maximum, as the endpoints of the interval give $y = 0$ as local minimums. So our width is

$$2y = 2\left(\frac{1}{\sqrt{6}} - 2\frac{1}{6\sqrt{6}}\right) = \boxed{\frac{2\sqrt{6}}{9}}.$$

(d) ★ Determine what you can about the width of the rose $r = \cos n\theta$. (You will not be able to find an explicit formula for $n > 2$, but see if you can find a relatively simple equation for the value of θ such that $(\cos n\theta, \theta)$ and $(\cos n\theta, -\theta)$ are the endpoints, in polar coordinates, of the required line segment.)

Hint: The best you can do for most n is to get an equation relating $\cos(n - 1)\theta$ and $\cos(n + 1)\theta$. This equation can be solved for $n = 3$, but I wouldn't try it without using a calculator or computer.

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Solution: We can do similar to part (c): we wish to maximize $y = \cos n\theta\sin\theta$ on the interval $\left[0, \frac{\pi}{2n}\right]$. We differentiate

$$\frac{dy}{d\theta} = \cos n\theta\cos\theta - n\sin n\theta\sin\theta.$$

We set this equal to 0 and solve for θ to get the value of θ that maximizes y. So the value of θ that we want is the solution to

$$0 = \cos n\theta\cos\theta - n\sin n\theta\sin\theta$$

for $\theta \in \left[0, \frac{\pi}{2n}\right]$. We can make this a little bit simpler by using trigonometric product-to-sum identities:

$$\begin{aligned} 0 &= \cos n\theta\cos\theta - n\sin n\theta\sin\theta \\ &= \frac{1}{2}(\cos(n+1)\theta + \cos(n-1)\theta) \\ &\quad - \frac{n}{2}(\cos(n-1)\theta - \cos(n+1)\theta) \\ &= \frac{n+1}{2}\cos(n+1)\theta - \frac{n-1}{2}\cos(n-1)\theta. \end{aligned}$$

Thus, we want the value of $\theta \in \left[0, \frac{\pi}{2n}\right]$ such that

$$(n-1)\cos(n-1)\theta = (n+1)\cos(n+1)\theta.$$

In part (c), where $n = 2$, this amounts to solving $\cos\theta = 3\cos 3\theta$, and you can verify that $\theta = \sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$ is indeed the solution. However, for $n > 2$, this equation is difficult to solve. The $n = 3$ case is a bit tractable: we have $2\cos 2\theta = 4\cos 4\theta$. Letting $z = \cos 2\theta$ and using the cosine double-angle formula, this is equivalent to $2z = 4(2z^2 - 1)$, which has positive root $z = \frac{1 + \sqrt{33}}{8}$. Thus,

$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{1 + \sqrt{33}}{8}\right),$$

and we could use this value of θ to get the width $2y = 2\cos 3\theta\sin\theta$, but don't try this last step at home. If $n > 3$, then the computation becomes virtually impossible without a computer algebra system.

Review Problems

8.21:

Write parametric equations to describe the curves traced by the following motions:

(a) A particle tracing a circle with center $(0,0)$ and radius 2, starting at $(2,0)$ at time $t=0$, moving counterclockwise with constant speed 1.

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Solution: Constant speed 1 means that we make a revolution in time 2π, so the parameterization is $\boxed{(2\cos t, 2\sin t)}$.

(b) A cannonball fired from $(0,0)$ with initial velocity $100\mathrm{m}/\mathrm{sec}$, shot at an angle $\frac{\pi}{3}$ above the ground, with gravity $g=-9.8\mathrm{m}/\mathrm{sec}^2$.

Preview: Solution

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Solution: The horizontal movement of the cannonball is with constant velocity $100\cos(\pi/3)=50$, so the horizontal position at time t is $50t$. The vertical movement is

$$100t\sin(\pi/3)+\frac{1}{2}gt^2=50\sqrt{3}t-4.9t^2.$$

So the parameterization is

$$\boxed{(50t, 50\sqrt{3}t-4.9t^2)}.$$

(c) A point starting at the bottom edge of a bicycle wheel (with radius 30 cm) that is rotating at 1 revolution per second, where the bicycle is moving forward at the rate implied by the rotation of the wheel. (For example, using meters as units, $(0,0)$ is the starting point, and after 0.5 seconds the point is at $(0.3\pi, 0.6)$.)

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Solution: The wheel itself is moving forward at a rate of 0.6π meters per second (as it travels one circumference each second). So the center of the wheel is given by the parameterization $(0.6\pi t, 0.3)$.

The point on the edge, relative to the center of the wheel, is given by the parameterization

$$(-0.3\sin 2\pi t, -0.3\cos 2\pi t).$$

(Note how this parameterization takes into account the clockwise motion, the starting position of $(0,-0.3)$, and the fact that the period of rotation is 1 second.)

Thus, the overall position of the point is the sum of these two parameterizations:

$$\boxed{(0.6\pi t-0.3\sin 2\pi t, 0.3-0.3\cos 2\pi t)}.$$

(d) ★ A particle tracing a circle with center $(0,0)$ and radius 2, starting at $(2,0)$ at time $t=0$, moving clockwise with speed $\sqrt{t}$.

Hint: Look for a parameterization similar to part (a), but where the speed is appropriate.

Preview: Solution

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Solution: The parameterization will be of the form $(2\cos f(t), -2\sin f(t))$ for some increasing function f. The speed at time t is

$$\begin{aligned}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} &= \sqrt{(-2f'(t)\sin f(t))^2+(-2f'(t)\cos f(t))^2}\\ &= 2f'(t)\sqrt{\sin^2 f(t)+\cos^2 f(t)}\\ &= 2f'(t).\end{aligned}$$

So $2f'(t)=\sqrt{t}$, hence $f(t)=\frac{1}{3}\sqrt{t^3}$ for $t\geq 0$. (Note $f(0)=0$ as required to get the proper starting point.) Thus the parameterization is

$$\boxed{\left(2\cos\left(\frac{1}{3}\sqrt{t^3}\right), -2\sin\left(\frac{1}{3}\sqrt{t^3}\right)\right)}.$$

8.22:

Imagine a string (of negligible thickness) unwinding from a fixed circular bobbin of radius 10, so that the string that is unwound is always tangent to the bobbin, as in the picture to the right. (Three positions of the string are shown in gray.) Assume that the string unwinds at the constant rate of one full loop of string per second. Determine parametric equations for the curve traced out by the end of the string, as shown by the dark curve at right.

Hint: Write the direction of the string as a function of t.

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Solution: The point of tangency of the string moves around the circle at 2π radians per second. First, we compute the position of the point of tangency of the string with the bobbin. Because this is simply a revolution around a circle of radius 10, the parameterization of the point of tangency is $(10\cos 2\pi t, 10\sin 2\pi t)$.

Now we compute the position of the end of the string relative to the point of tangency with the bobbin. After t seconds have passed, the point of tangency has moved a length of $20\pi t$ around the bobbin. Thus, a length of $20\pi t$ of string has been unwound. The direction that the string points is tangent to the bobbin. By computing the derivatives of the components in $(10\cos 2\pi t, 10\sin 2\pi t)$, we realize that the direction the string points in is the direction of $(\sin 2\pi t, -\cos 2\pi t)$. Thus, the position of the end of the string relative to the point of tangency with the bobbin is parameterized by $(20\pi t\sin 2\pi t, -20\pi t\cos 2\pi t)$.

Adding these two together gives the parameterization

$$\boxed{(10\cos 2\pi t+20\pi t\sin 2\pi t, 10\sin 2\pi t-20\pi t\cos 2\pi t)}.$$

8.23:

Let $\mathcal{C}$ be a circle of radius 3 centered at $(0,0)$. Let $\mathcal{D}$ be a circle of radius 1 centered at $(4,0)$, and let $P=(3,0)$ be the point on $\mathcal{D}$ that is tangent to $\mathcal{C}$. Find parametric equations for the **epicycloid** that is produced when we trace the path of P as $\mathcal{D}$ rolls around $\mathcal{C}$, so that the center of $\mathcal{D}$ moves counterclockwise around $\mathcal{C}$ at a rate of 1 revolution per second.

Hint: Start by drawing a picture.

Hint: When does P first again touch $\mathcal{C}$?

Preview: Solution

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Solution: The starting position is shown in black in the picture to the right. There are two components to the motion of P: the rotation of $\mathcal{D}$ around $\mathcal{C}$, and the rotation of P around $\mathcal{D}$.

The center of $\mathcal{D}$ rotates around $\mathcal{C}$ along a path that is a circle of radius 4 centered at $(0,0)$. Since it starts at $(4,0)$ and takes 1 unit of time to make 1 counterclockwise rotation, the parameterization of the motion of the center of $\mathcal{D}$ is given by $(4\cos 2\pi t, 4\sin 2\pi t)$.

The point P rotates counterclockwise around $\mathcal{D}$. Looking at the picture, since the circumference of $\mathcal{C}$ is 3 times that of $\mathcal{D}$, we see that P will again be tangent to $\mathcal{C}$ after $\mathcal{D}$ has moved $\frac{1}{3}$ of the way around $\mathcal{C}$, as shown by the grey circle in the upper-left of the picture. During the time it takes for $\mathcal{D}$ to get from its starting position (in black) to this new position (in grey), we see that P has rotated a total of $\frac{4}{3}$ of the way around $\mathcal{D}$. Thus P makes $\frac{4}{3}$ rotations around $\mathcal{D}$ in $\frac{1}{3}$ second, and hence makes 1 rotation in $\frac{1}{4}$ second, or 4 rotations in 1 second. Thus, the parameterization of P relative to the center of $\mathcal{D}$ is given by $(-\cos 8\pi t, -\sin 8\pi t)$. (The minus signs in this parameterization are due to the fact that P starts at $(-1,0)$ relative to the center of $\mathcal{D}$.)

Adding the two parameterizations, we get

$$\boxed{(4\cos 2\pi t-\cos 8\pi t, 4\sin 2\pi t-\sin 8\pi t)}$$

as a parameterization of the epicycloid. The epicycloid is shown by the dotted line in the picture.

8.24:

The **Archimedes spiral** is the graph of $r=\theta$ for $\theta\geq 0$.

(a) Sketch the spiral.

Preview: Solution

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Solution: Shown at right is the graph for θ from 0 to 2π.

(b) Find the slope of the tangent line to the spiral at θ.

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Solution: Note that $x=r\cos\theta=\theta\cos\theta$, so $\frac{dx}{d\theta}=-\theta\sin\theta+\cos\theta$. Similarly, $y=r\sin\theta=\theta\sin\theta$, so $\frac{dy}{d\theta}=\theta\cos\theta+\sin\theta$. Thus,

$$\frac{dy}{dx}=\frac{\theta\cos\theta+\sin\theta}{-\theta\sin\theta+\cos\theta}.$$

(c) Find the area of region bounded by the spiral and to the left of the y-axis when $0\leq\theta\leq 2\pi$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We must integrate between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Thus we have

$$\frac{1}{2}\int_{\pi/2}^{3\pi/2}\theta^2\,d\theta=\frac{1}{6}\theta^3\Big|_{\pi/2}^{3\pi/2}=\frac{1}{6}\left(\frac{27\pi^3}{8}-\frac{\pi^3}{8}\right)=\boxed{\frac{13\pi^3}{24}}.$$

8.25:

Find the area of the region that is in the interior of the two cardioids $r=1+\sin\theta$ and $r=1+\cos\theta$. (The region is shown at right.)

Hint: Use the fact that the picture is symmetric across the line $y=x$.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We see that the picture is symmetric about the line $x=y$, as shown. We can also see this algebraically: reflecting across $x=y$ is the same as replacing θ with $\frac{\pi}{2}-\theta$, and using the identity $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ establishes that the two cardioids are reflections of each other.

Thus, the area is just twice the area of shaded region below the line $y=x$. This half of the overall region is the region inside the cardioid $r=1+\sin\theta$ from $\theta=-\frac{3\pi}{4}$ to $\theta=\frac{\pi}{4}$. Hence, the total area is

$$\begin{aligned}&2\cdot\frac{1}{2}\int_{-3\pi/4}^{\pi/4}(1+\sin\theta)^2\,d\theta\\ &=\int_{-3\pi/4}^{\pi/4}(1+2\sin\theta+\sin^2\theta)\,d\theta\\ &=\int_{-3\pi/4}^{\pi/4}\left(1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right)\,d\theta\\ &=\int_{-3\pi/4}^{\pi/4}\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)\,d\theta\\ &=\left(\frac{3}{2}\theta-2\cos\theta-\frac{1}{4}\sin 2\theta\right)\Big|_{-3\pi/4}^{\pi/4}\\ &=\frac{3}{2}\pi-2\left(\cos\frac{\pi}{4}-\cos\left(-\frac{3\pi}{4}\right)\right)-\frac{1}{4}\left(\sin\frac{\pi}{2}-\sin\left(-\frac{3\pi}{2}\right)\right)\\ &=\boxed{\frac{3}{2}\pi-2\sqrt{2}}.\end{aligned}$$

8.26:

Prove that the graph of the polar equation

$$A\cos\theta+B\sin\theta+\frac{C}{r}=0,$$

where A, B, C are nonzero real numbers, is a line. What is the geometric interpretation of the constants A, B, and C? And what happens if one or more of them is zero?

Hint: Multiplying by r may make it easier to convert to rectangular coordinates.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Note that r is nonzero, so we may multiply the equation by r to get

$$Ar\cos\theta+Br\sin\theta+C=0.$$

This converts to rectangular form to get $Ax+By+C=0$, which is the equation for a line. The slope is $-\frac{A}{B}$ and the y-intercept is $\left(0,-\frac{C}{B}\right)$. If $A=0$ then the line is the horizontal line $y=-\frac{C}{B}$ and if $B=0$ then the line is the vertical line $x=-\frac{C}{A}$. If both A and B are zero then the graph is empty if C is nonzero, and if they're all zero then the equation is $0=0$, which is content-free.

9.1 Definitions and basic examples

A **differential equation** is an equation that relates a function, the function's derivative(s), and an independent variable. This is not a totally new idea: we've already seen some simple types of differential equations. For example,

$$y' = 2x$$

is a differential equation that relates two variables x and y. Usually, in differential equations, we have a dependent variable (often y) that is understood to be a function of x or for whatever our independent variable is. The context will usually make clear which variable represents the function and which represents the independent variable.

As we know, to solve the differential equation $y' = 2x$, we just compute the antiderivative of f':

$$y = \int 2x\,dx = x^2 + C.$$

Of course, this only determines y up to a constant C.

Differential equations involve both y and y' terms. For example:

Problem 9.1.1

Find a solution to the differential equation $y' = y$.

Solution for Problem 9.1.1: In words, this equation asks: what function is equal to its derivative? One function that you probably immediately think of is $y = e^x$, but that's not the only answer. The complete answer is $y = ce^x$, where c is a constant (possibly 0). A bit later, we will see how to show that these are no other solutions.

Often, a differential equation will come with an **initial condition**. This is a condition of the form $x = x_0$, $y = y_0$ for some constants x_0 and y_0. This means that we are looking for the function $y(x)$ that is the graph of our solution—we sometimes think of this as the "initial" point of our solution. Assuming that we are thinking of y as a function of x, we may write the initial condition as $y(x_0) = y_0$. This is a shorthand for $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$, etc. When writing differential equations, we generally are a little sloppy and use y to represent both a function and a dependent variable.

Regardless of how we write the initial condition, we plug it into our solution in order to solve for the unknown constant. Continuing our basic example:

Problem 9.1.2

Find the solution to $y' = y$ where $y(0) = 2$.

Solution for Problem 9.1.2: From Problem 9.1, we know that solutions to $y' = y$ are of the form $y = ce^x$ for some constant c. We plug in $x = 0$ and $y = 2$ to get $2 = ce^0 = c$ so $c = 2$. Thus, the solution is $y = 2e^x$.

A **first-order** differential equation involves y' but no higher derivatives of y. The most basic form of a first-order differential equation is

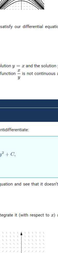
$$y' = f(x, y)$$

where $f(x, y)$ is some function involving x and y.

Many differential equations cannot be solved exactly. However, we have other tools that we can use to analyze them. One geometric tool that we use to study first-order differential equations is called **slope field**. Using a slope field involves thinking geometrically about what $y(x, y)$ really means. Since one interpretation of y' is as the slope of the tangent line to the graph of y, the equation $y' = f(x, y)$ means that the slope of a tangent line to the graph of y at the point (x, y) is $f(x, y)$.

A slope field is what results when we plot many tangent lines at many points in the plane. For instance, shown to the right is the slope field for the differential equation $y' = \frac{x}{y}$. At each point (x, y) we draw a small line segment with slope $\frac{x}{y}$. For example, at $(3, 2)$, we have a segment with slope $\frac{3}{2}$. At $(0, 1)$, we have a segment with slope $\frac{0}{1} = 0$ (that is, horizontal). At $(2, 0)$, we have segments with "infinite" slope; that is, vertical segments.

The key fact is: any curve that is the graph of a solution to $y' = f(x, y)$ must match these segments in slope. This allows us to sketch likely solution curves by following the "trails" formed by the slope field. For instance, in the differential equation $y' = \frac{x}{y}$ whose slope field is shown at right, the slope field suggests that the solution curves might be circles, as in the diagram below:



Specifically, it looks like the solutions to $y' = \frac{x}{y}$ are circles centered at the origin.

Problem 9.1.3

Show that the functions given implicitly by the equation $x^2 + y^2 = c$, where c is some positive constant, are solutions to the differential equation $y' = -\frac{x}{y}$.

Solution for Problem 9.1.3: We know that the functions given implicitly by the equation $x^2 + y^2 = c$, where c is some positive constant, are solutions to the differential equation $y' = -\frac{x}{y}$.

$$2x + 2yy' = 0,$$

and solving for y' we see that indeed $y' = -\frac{x}{y}$ (where this is defined).

An important hard theorem of analysis states that if f is continuous (although we haven't really defined what "continuous" means for a multivariable function), then the first-order differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$ has a unique solution. This is not too surprising when we think about it in terms of slope fields: if f is continuous, then we are given the slope field from the slope field tells us what the solution curve through (x_0, y_0) must look like. We simply trace our curve so that, at each point, the direction of the curve is given by the slope field.

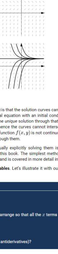
Thus, in Problem 9.3, since there is exactly one curve of the form $x^2 + y^2 = c$ passing through any point of the plane, this theorem about uniqueness of solutions tells us that these indeed are the only solutions to this differential equation.

Using slope fields is a fancy game of connect-the-dots. Here's another example:

Problem 9.1.4

Use slope fields to guess at the solutions to $y' = \frac{y^2}{x^2}$. Then, confirm that your guesses satisfy the differential equation.

Solution for Problem 9.1.4: Note that this is almost the same function as we had in Problem 9.3, except without the minus sign. We sketch our slope field at each point (x, y) we draw a little line segment with slope $\frac{y^2}{x^2}$. We've drawn it zoomed "out" than in Problem 9.3—meaning we've drawn more little slopes—so that we can better see the behavior. Notice the qualitative features of this slope field. Near the x-axis and away from the y-axis, we have slopes near 0. Near the y-axis and away from the x-axis, we have large slopes, and slopes are near vertical. On the other hand, near the y-axis and away from the x-axis, we have slopes near 0. In general, when $|x|$ is relatively small and $|y|$ is relatively large, so $\left|\frac{y}{x}\right|$ is small, and the slopes are near horizontal.



We connect the dots—that is, we try to draw curves along the directions given by the slope field—and we get something like the picture to the right. These curves look a lot like hyperbolas. How can we check?

We suspect the solutions are $y^2 - x^2 = c$ for some constant c. To check that these satisfy our differential equation, we implicitly differentiate:

$$2yy' - 2x = 0.$$

Solving for y' gives $y' = \frac{x}{y}$ as we want.

Our diagram appears to have a problem at $(0, 0)$ in that there are two lines crossing the solution $y = x$ and the solution $y = -x$. This is an apparent violation of the uniqueness of solutions through a given point. However, the function $\frac{y^2}{x^2}$ is not continuous at $(0, 0)$, so this result does not violate the theorem about uniqueness of solutions.

Let's do one more slope field example that looks slightly different than the previous two:

Problem 9.1.5

Using a slope field, analyze the solutions to $y' = 2 - y$.

Solution for Problem 9.1.5: This may look like a "trivial" differential equation that we can just antidifferentiate:

Bogus Solution: We antidifferentiate $y' = 2 - y$ to get

$$y = \int (2 - y)\,dy = 2y - \frac{1}{2}y^2 + C,$$

so solving gives $\frac{1}{2}y^2 - y = C$.

This is incorrect. Indeed, we can plug the above "solution" into the original differential equation and see that it doesn't work. What we should have written is

$$y = \int (2 - y)\,dx.$$

However, the last line is a function of y integrated as a function of x, so we cannot integrate it (with respect to x) unless we know beforehand how to express y in terms of x.

We draw the slope field at right: at each point (x, y) we draw a little line with slope $2 - y$. Note that the slopes only depend on y and not on x. We notice that the slopes seem to "funnel" the graph towards $y = 2$. We can further analyze this phenomenon by looking at the sign of y': if $y > 2$, then $y' < 0$ so the solutions look like they are going right below. Similarly, this makes sense: if y is far from 2, then $y' = 2 - y$ gives a negative slope. If y is very close to 2, the rate of change y' gets small, so the y-coordinate of the curve moves very slowly towards 2.

You might have a guess as to what equations have these curves as their graphs. The "funnel" behavior of y towards 2 at $x \to \infty$ is our clue. The correct answer is that these curves are the graphs of $y = 2 - ce^{-x}$ for some constant c. To see that $y = 2 - ce^{-x}$ solves $y' = 2 - y$, we compute y':

$$y' = ce^{-x} = 2 - (2 + ce^{-x}) = 2 - y.$$

To check that our curve is correct, we compute $y' = ce^{-x}$, and indeed we see that

$$y' = ce^{-x} = 2 - (2 + ce^{-x}) = 2 - y.$$

Important fact to note about drawing solutions to a slope field of $y' = f(x, y)$ is that the solution curves can never intersect. This is due to the theorem about the uniqueness of the solution of a first-order differential equation with an initial condition. Drawing a curve through a point of a slope field is equivalent to specifying an initial condition. Thus, if two curves could intersect, we would have two uniqueness of such a solution, there can only be one curve through any given point, and hence the curves cannot intersect. Except (and there is often an "except" in calculus), there may be multiple solutions at points where the function $f(x, y)$ is not continuous. Such points (for example, the point $(0, 0)$ in Problem 9.3) may have multiple solution curves passing through them.

Besides slope fields, another way that we can analyze differential equations without actually explicitly solving them is by numerically estimating solutions. This is a very useful skill, since that is largely beyond the scope of this book. The simplest method of estimating solutions is to calculate repeated tangent line approximations; this is called **Euler's Method** and is covered in more detail in Section 9.4.

Generally, the easiest method of solving differential equations is called **separation of variables**. Let's illustrate it with our basic example from Problem 9.1:

Problem 9.1.6

Our goal is to solve $y' = \frac{x}{y}$.

(a) Write the equation as $\frac{dy}{dx} = \frac{x}{y}$. Pretend that the "$\frac{d}{dx}$" and "$\frac{dy}{dx}$" are variables, and rearrange so that the x terms are on one side and the y terms are on the other side.

(b) Compute the antiderivatives of both sides.

(c) Write the solution. Why do we need only one constant and not two (since we took two antiderivatives)?

Solution for Problem 9.6:

(a) We write the differential equation as

$$\frac{dy}{dx} = \frac{x}{y}.$$

We move all the x terms to one side and all the y terms to the other side. This includes "multiplying by dx":

$$y\,dy = x\,dx.$$

(b) We can antidifferentiate both sides:

$$\int y\,dy = \int x\,dx.$$

This gives

$$\frac{1}{2}y^2 = \frac{x^2}{2} + C.$$

(c) Cleaning it up a little gives $x^2 + y^2 = 2C = c$ as expected. Notice that we don't need a constant on both sides: each antiderivative is determined up to an arbitrary constant, so their difference is also determined up to an arbitrary constant.

This general method is called **separation of variables**. It requires that our differential equation be **separable**, meaning that we can split all the x terms on one side and all the y terms on the other side, with "$\frac{dy}{dx}$" on the side with the y terms and "dx" on the side with the x terms. To be more specific:

Definition: A two-variable function $f(x, y)$ is called **separable** if we can write

$$f(x, y) = g(x)h(y)$$

where $g(x)$ is a function that depends solely on x, and $h(y)$ is a function that depends solely on y.

Important: We can use **separation of variables** to solve the differential equation $y' = f(x, y)$ if f is separable. We write $\frac{dy}{dx} = f(x, y) = g(x)h(y)$ and rearrange the equation as

$$\frac{dy}{h(y)} = g(x)\,dx.$$

We then antidifferentiate both sides: the left side in terms of y and the right side in terms of x.

Note that we will always get a constant from our antidifferentiation. If we have an initial condition, we can plug it in and solve for the constant to get the unique solution.

WARNING: Even though we are treating dx and dy as variable terms, they are really just notational conveniences. What we really have when we write

$$\frac{dy}{h(y)} = g(x)\,dx$$

is instead the equation

$$\int \frac{1}{h(y)}\,\frac{dy}{dx}\,dx = \int g(x).$$

Then when we integrate this equation with respect to x, the left side becomes

$$\int \frac{1}{h(y)}\,\frac{dy}{dx}\,dx = \int \frac{1}{h(y)}\,dy,$$

where the last equality is an application of the Chain Rule. In other words, when we "multiply by dx," we are really just setting up our equation to properly apply the Chain Rule when antidifferentiating.

We can go back and revisit our basic example of a differential equation from Problem 9.1:

Problem 9.7

Solve the differential equation $y' = y$.

Solution for Problem 9.7: We write this equation as $\frac{dy}{dx} = y$ and use separation of variables to write as

$$\frac{dy}{y} = dx.$$

Integrating both sides gives

$$\int \frac{dy}{y} = \int dx,$$

giving $\log|y| = x + C$. Exponentiating then gives $|y| = e^{x+C} = ce^x$, where $c = e^C$. Note that $c > 0$ in this expression.

The absolute value sign is a bit annoying. We can get rid of it by absorbing any factor of -1 into the constant c. So we are OK with writing $y = ce^x$ for some nonzero expression $|y| = ce^x$ the constant c had to be positive. When we take away the absolute value sign, we allow c to be negative (or zero).

Thus, any solution of $y' = y$ is of the form $y = ce^x$ for some constant $c \in \mathbb{R}$. $\square$

Here's another example:

Problem 9.8

Solve the differential equation $x' = y - 2xy$ with $y(1) = 2$.

Solution for Problem 9.8: We write the equation as

$$\frac{dx}{dy} = y(1 - 2x).$$

We can now integrate both sides:

$$\int \frac{dx}{y} = (1 - 2x)\,dx.$$

This gives us

$$\log|y| = x - x^2 + C,$$

so we can exponentiate both sides to get

$$|y| = e^{x - x^2 + C} = ce^{x - x^2}.$$

(Note that we've adjusted the constant by letting $c = e^C$.)

Again, we can get rid of the absolute value sign by absorbing any factor of -1 into the constant c, giving $y = ce^{x - x^2}$. (Recall in our previous expression $|y| = ce^{x - x^2}$ the constant c had to be positive. When we take away the absolute value sign, we allow c to be negative (or zero).)

Finally, since we are given an initial condition, we can solve for c. We plug in $x = 1$ and $y = 2$:

$$2 = ce^{1 - 1} = ce^0 = c.$$

So $c = 2$ and our solution is $y = 2e^{x - x^2}$. $\square$

We can graph this over a large field to double-check that our answer looks correct. The slope field and the graph of $y = 2e^{x - x^2}$ are shown at right. We see the expected behavior of the graph. As x gets large, the function tends to 0 too (to peak at the value of y that maximizes $y - x^2$ which is $x = \frac{1}{2}$). The slope field is decreasing near the x-axis, since y gets close to 0, the slopes get close to horizontal. The slope field does is not for sure, but it looks like it does tend to 0 as $x \to \infty$.

One of the important examples of a differential equation is $y' = ky$, where k is some constant.

In English, this means that the quantity changes at a rate proportional to the quantity itself. Another way to write this is $\frac{y'}{y} = k$. This interpretation means that y is growing at a constant **relative rate**.

There are many "real-world" phenomena that are modeled with the differential equation $y' = ky$. Some examples are:

- population growth
- spread of disease
- radioactive decay
- temperature change
- continuously compounded interest
- ...or indeed any situation which is properly modeled with a constant relative growth rate. We'll look at some of these examples a bit more closely in a moment.

First, let's solve the equation:

Problem 9.9

Find the solutions to $y' = ky$, where k is a constant.

Solution for Problem 9.9: We can solve this by separation of variables. Arrange the equation as

$$\frac{dy}{y} = k\,dx,$$

and integrate to get $\log|y| = kx + C$. Exponentiating gives

$$|y| = ce^{kx} \quad \text{or} \quad y = ce^{kx},$$

where $c = e^C > 0$. Allowing y to be 0 or negative lets us remove the absolute value sign and have any $c \in \mathbb{R}$, so the solutions are $y = ce^{kx}$ for any constant c.

As a check, we compute

$$y' = \frac{d}{dx}(ce^{kx}) = k(ce^{kx}) = ky,$$

which is valid for any $c \in \mathbb{R}$. $\square$

Not surprisingly, this is called the **exponential growth** differential equation:

Important: The solutions to $y' = ky$, where k is a constant, are $y = ce^{kx}$ for any constant c.

Before we look at specific examples, let's examine how the solutions behave. If $k > 0$, then the quantity y grows without bound since $\lim e^{kx} = \infty$. On the other hand, if $k < 0$, then the quantity y tends towards 0, since $\lim e^{-x} = 0$. Finally, if $k = 0$, then the quantity is a constant: the differential equation is just $y' = 0$, and our solutions are $y = c$ for a constant c.

We can see this differential equation in action in some problems.

Problem 9.10

A radioactive substance decays at a rate proportional to the amount of the substance present. Suppose we start with 100 g of Aljenium, and after 1 day we have 99 g of the radioactive material. After 1 hour, we have 75 g of Aljenium remaining. How long will it take until we have only 10 g of Aljenium?

Solution for Problem 9.10: The first sentence of the problem describes our differential equation. It tells us that we have the equation $y' = ky$ for some $k < 0$. We know that k is negative because the quantity is decreasing over time. The solution to this equation is $y = ce^{kx}$ where k is some other constant. (Note that we use x because it is the independent variable in time.)

We don't know k or c, so we might be in a little bit of trouble. However, fear not, for we are given two conditions: at $t = 0$, we know $y = 100$ g and at $t = 1$ we know $y = 75$. Plugging in the first condition gives $100 = ce^0 = c$, so $c = 100$. Now we have $y = 100e^{kx}$, and we can plug in the second condition, giving $75 = 100e^k$. Thus $k = \log(3/4) \approx -0.2877$.

Hence our solution is that the amount of material at time t is $y = 100e^{(\log(3/4))t}$. We could leave it like this, or we could write it as:

$$y = 100e^{(-0.2877)t}.$$

But the first form is preferred to finish the problem. We want to know when the amount of material will be 10 g. So, we plug in $y = 10$ and we need to solve for t:

$$10 = 100e^{(\log(3/4))t}.$$

This gives

$$\log(1/10) = (\log(3/4))t,$$

and thus

$$t = \frac{\log(1/10)}{\log(3/4)} = \frac{-2.3026}{-0.2877} \approx 8.0039.$$

Thus, it will take slightly over 8 hours for our sample to decay to 10 g. $\square$

Another basic example is temperature change: heating and cooling. **Newton's Law of Heating** states that the temperature of a body changes at a rate proportional to the difference between the body's temperature and the ambient temperature. Here's an example:

Problem 9.11

On a PC's desk at a temperature of 70° F, the room has its coffee in the microwave to a temperature of 170°. After 1 minute, the coffee has cooled to 110°. Since it is picky and will only drink his coffee if it is at least 120°, how long does his lunar (after removing the coffee from the microwave) drink take to be ready to drink?

Solution for Problem 9.11 You know that the rate of change of the coffee temperature is proportional to the difference between the ambient temperature and the temperature of the coffee and the temperature of the room.

Hence the differential equation is our usual $y' = k_{\text{a}}y$ with the two conditions $y(0) = 130$ and $y(1) = 100$. As we know, the solution is $y = ce^{kt}$ for some constants c and k (with $k < 0$ since y tends to 0).

Plugging in $y = 130$ and $t = 0$ gives $130 = ce^{k\cdot 0} = c$; so we have $c = 130$ and thus our solution is now $y = 130e^{kt}$. Then, plugging in $y = 100$ and $t = 1$ gives $100 = 130e^{k}$, so $k = \log(100/130)$. Thus the solution is

$$y = 130e^{t\log(100/130)}.$$

We want to know when $y = 90$ so we plug it in and solve for t. This gives

$$90 = 130e^{t\log(100/130)},$$

from which we get

$$t = \log(90/130)/\log(100/130) \approx 3.64.$$

Thus, Eric has only about 3.64 minutes to drink up (the should be less pitchy.) ☺

These are just some of the real-world problems that can be simply modeled by the exponential growth equation $y' = ky$. You will see more examples later.

However, in many cases, the $y' = ky$ model for population growth is not quite accurate. For instance, most populations have external factors that limit the population from growing exponentially. Instead, we need to slow down the growth rate as the population gives larger. For these situations, we can modify our differential equation slightly:

$$y' = ky - ay^2.$$

This is called the **logistic growth equation**. Note that $y = 0$ is a solution to $y' = ky - ay^2$ if y' is 0; then we can write the logistic equation as

$$\frac{y'}{y} = k - ay.$$

Written in this form, we note that $\frac{y'}{y}$ is the relative growth rate; that is, the growth rate as a function of the total population. In the regular exponential growth model, $\frac{y'}{y}$ is constant. The new $-ay$ term in the logistic growth equation is the "limiting" factor that slows the growth as the population gets larger.

Before solving the logistic equation, we might inquire about an **equilibrium**. This is a value c such that $y = c$ is a solution.

Problem 9.12

Does the logistic equation $y' = ky - ay^2$ have any equilibrium solutions? If so, what?

Solution for Problem 9.12 An equilibrium solution $y = c$ has $y' = 0$. We plug these in to the equation and get $0 = kc - ac^2 \Rightarrow c(k - ac)$. Thus $c = 0$ and $c = \frac{k}{a}$ are the equilibrium solutions. ☺

When studying the logistic equation, we often let $L = \frac{k}{a}$ (equal the equilibrium), and rewrite the equation in terms of L. The equation then becomes:

$$\frac{y'}{y} = k\left(1 - \frac{y}{L}\right).$$

Problem 9.13

Solve the logistic differential equation $y' = ky - ay^2$ where k and a are positive constants and y is a function of t. (Write your answer in terms of $L = k/a$.) Assuming that $y > 0$ and $y < L$, what happens to y as $t \to \infty$?

Solution for Problem 9.13 We'll write the equation in terms of L as

$$\frac{y'}{y} = k\left(1 - \frac{y}{L}\right).$$

Note that $y = \frac{L}{2}$ is a solution: the right side of the above equation is zero, and the left side is also 0 since $y' = 0$. If $y \neq L$, then we can separate the variables:

$$\frac{dy}{y(1 - \frac{y}{L})} = k\,dt.$$

We want to integrate both sides of this, but first we can divide both sides by L to get the left side in a nicer form:

$$\int \frac{dy}{y(1 - \frac{y}{L})} = \int \frac{L}{y}\,dy.$$

The left side is now the reciprocal of a factorable quadratic, so we can use partial fractions on the left side. Note that

$$\frac{1}{y(1 - \frac{y}{L})} = \frac{1}{y}\left(\frac{L}{L - y}\right) = \frac{L}{y} + \frac{L}{L - y}.$$

Our equation becomes

$$\int \frac{L}{y} + \frac{L}{L - y}\,dy = \int \frac{L}{y}\,dt.$$

Now the $\int L/y$ terms cancel, and after integrating we have

$$\log|y| - \log|L - y| = kt + C.$$

We consolidate the logs, and this becomes

$$\log\left|\frac{y}{L - y}\right| = kt + C.$$

Next, we exponentiate, giving

$$\left|\frac{y}{L - y}\right| = e^{kt+C} = ce^{kt},$$

where $c = e^C \geq 0$. Allowing c to be 0 or negative allows us to eliminate the absolute value, leaving

$$\frac{y}{L - y} = ce^{kt}.$$

If $c = 0$, then we get the solution $y = 0$ (we consider this a solution because it is a solution to the original equation $y' = ky - ay^2$). If $c \neq 0$, then solving for y, we conclude that

$$y = \frac{L}{1 + \frac{1}{ce^{kt}}},$$

where $c = e^{-C}$ is a nonzero constant. Finally, note that letting $c = 0$ recovers our equilibrium solution $y = L$.

As $t \to \infty$, the exponential term in the denominator approaches 0. Therefore, y approaches L, the limiting population, as we expect. ☺

We'll force this section with a better differential equation modeling problem.

Problem 9.14

Sunshine Monday morning, it starts snowing in Los Angeles (it is a constant rate, at those times, a snowplow starts to plow Hollywood Boulevard. It's 1 A.M., and plows in a straight line at a rate inversely proportional to the amount of snow on the ground. The plow covers equally twice as much ground between noon and 1 p.m. as it does between 1 p.m. and 2 p.m. (and it is snowing at the same constant rate throughout). What time did it start snowing?

Solution for Problem 9.14 It's not clear immediately how to approach this problem, so we begin by trying to model the problem and bring all of the data that we know.

We can model this situation by letting t be our time variable (in hours), so that $t = 0$ corresponds to noon. Suppose that S is the time that it starts snowing; note that $S < 0$. We assume that the ground at time t is then $c(t - S)$, where c is the snow rate.

Let the position of the snowplow along Hollywood Boulevard be given by $x(t)$ with $x(0) = 0$. The fact that the distance covered in the second hour is one-half the distance covered in the first hour is represented by the equation

$$x(2) - x(1) = \frac{1}{2}(x(1) - x(0)) = \frac{1}{2}x(1),$$

which simplifies to $x(2) = \frac{3}{2}x(1)$. We also have the stated fact that the plow moves at a rate inversely proportional to the amount of snow on the ground, so this gives the differential equation

$$\frac{dx}{dt} = \frac{k}{c(t - S)},$$

where $k > 0$ is some constant. Integrating this gives

$$x(t) = \frac{k}{c}\log|t - S| + C_1$$

where C_1 is some constant. Since we are only concerned with $t \geq 0$, we always have $t - S > 0$, and we can eliminate the absolute value signs. Next, plugging in $t = 0$ and using $x(0) = 0$ yields

$$0 = \frac{k}{c}\log(-S) + C_1$$

so $C = -\frac{k}{c}\log(-S)$, and our equation becomes

$$x(t) = \frac{k}{c}\log(t - S) - \log(-S) = \frac{k}{c}\log\left(\frac{t - S}{-S}\right).$$

Our last bit of data that we haven't used yet is $x(2) = \frac{3}{2}x(1)$, so we plug that in:

$$\frac{k}{c}\log\left(\frac{2 - S}{-S}\right) = x(2) = \frac{3}{2}x(1) = \frac{3}{2}\frac{k}{c}\log\left(\frac{1 - S}{-S}\right).$$

The constant terms $\frac{k}{c}$ cancel, and changing the denominators we have

$$2\log\left(\frac{2 - S}{-S}\right) = 3\log\left(\frac{1 - S}{-S}\right).$$

The constants outside the logarithms can be brought inside as exponents, so after exponentiating both sides we have

$$\left(\frac{2 - S}{-S}\right)^2 = \left(\frac{1 - S}{-S}\right)^3.$$

This simplifies to $2(2 - S)^2 = (1 - S)^3 \Rightarrow 2^3 - 2 = 1 + 3S$ by the binomial theorem. Thus, $S = \frac{3 + 6}{3} = 3$.

Since $2^3 < 9$, we choose the negative root, giving $2 = \frac{1 - \sqrt{9}}{3} = -0.618$, and we conclude that it started snowing approximately 0.618 hours before noon, or at about 11:23 a.m. ☺

Exercises

9.1.1:

In 29-34 use Euler half-step and some solutions curves for the following equations. Find the explicit solutions if you can.

(a) $y' = x(y - 1)$

Preview Solution

You may type any additional notes here.

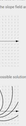
Hide Solution

Good

Bad

Your Submission Solution

Solution: Using separation of variables, we write $\frac{dy}{y-1} = x\,dx$. Integrating both sides gives $\log|y - 1| = \frac{x^2}{2} + C$; so $y = \left(\frac{x^2}{2} + C\right)$ for some constant C.



(b) $y' = x + y$

Hint: Find solve $y' = y$; then worry about the "extra" x term.

Preview Solution

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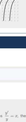
Your Submission Solution

Answer: If we ignore the x term, we get the equation $y' = y$, which we know has the solution $y = ce^x$ for a real constant c. Now we need to add some extra terms to this so that y' has an extra x. We can guess that these extra terms will have the form $a_1x + a_0$ for some real a_1, a_0. Let $y = ce^x + ax + b$, then substituting into $y' = y + x$ gives

$$ce^x + a_1 = ce^x + (a_1 + b)x + a_0.$$

Equating corresponding coefficients gives $a_1 = 1 = b$ and $a_1 = a_0$, so $a_0 = a_1 = 1$. Thus, we find that $\boxed{y = ce^x + x - 1}$ is our solution.



(c) $y' = \frac{x + y}{x - y}$

Hint: This is pretty hard to solve. You can try the substitution $y = xz$.

Preview Solution

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Your Submission Solution

Solution: Make the substitution $y = xz$, so $y' = z + xz'$. Then the equation becomes

$$z + xz' = \frac{1 + z}{1 - z}$$

giving

$$xz' = \frac{1 + z}{1 - z} - \frac{1 + z^2}{1 - z}$$

Now using separation of variables, this becomes

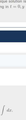
$$\frac{dz}{1 + z^2} = \frac{dx}{x}.$$

This can be solved, but it's fairly ugly, and then we have to solve the substitution $xz = \frac{y}{x}$. It turns out (and you can check this yourself if you like) that the solution curves are implicitly given by

$$2\tan^{-1}\left(\frac{y}{x}\right) = \log(x^2 + y^2) + C,$$

where C is a constant. Also, this equation does not really give us a good idea of what the solution curves look like.

Rather than explicitly solve the differential equation, we can just draw the slope field and sketch some curves. First, the slope field:



The solutions look like they spiral towards (0, 0). We can draw some possible solution curves, resulting in the picture below.

(d) $y' = 5 + y^2$

Preview Solution

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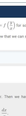
Your Submission Solution

Solution: Using separation of variables, we have $\frac{dy}{5 + y^2} = dx$. Integrating both sides gives $\arctan y = x + C$. Thus $\boxed{y = \tan(x + C)}$ is the general solution.



9.1.2:

Solve the following differential equations:

(a) $y' = xy$

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Your Submission Solution

Solution: We use separation of variables. Rewrite the equation as $\frac{y'}{y} = x$, then $\frac{dy}{y} = x\,dx$. Integrating both sides gives $\log|y| = \frac{x^2}{2} + C$ for some constant C. Then $\boxed{y = ce^{x^2/2}}$ for some real c. We can verify that any solution of this form works.

(b) $y' = xe^x \sin(y)$

Preview Solution

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Your Submission Solution

Solution: We use separation of variables. Rewrite the equation as $y'\csc^{-1} = x\,\text{Then}\,e^{-x}dy = x\,dx$. Integrating both sides, we get $-e^{-x} = \frac{x^2}{2} + C$. We have $y = 0$ at $x = 0$ so this gives $-1 = C$. So $e^{-x} = -\frac{x^2}{2} - 1$. Thus

$$y = \log\left(\frac{2}{x^2 + 2}\right) = -\log\left(1 - \frac{x^2}{2}\right).$$

Note that as x approaches $\sqrt{2}$ or $-\sqrt{2}$, the value of y goes to infinity, so the function is only defined for $x \in (-\sqrt{2}, \sqrt{2})$.

(c) $y' = y^2 \sin x$

Preview Solution

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Your Submission Solution

Solution: Rewrite the equation as $\frac{y'}{y^2} = \sin x$. Then $\frac{dy}{y^2} = \sin x\,dx$ and $-\frac{1}{y} = -\cos x + C$ for a real constant C. Then $\boxed{y = \frac{1}{\cos x - C}}$ is the general solution. We can verify that any solution of this form works.

9.1.3:

Solve $y' = 2\sin x\,y(0) = 1$ and $y'(0) = 0$.

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Your Submission Solution

Solution: We solve the first differentiation. First, let $w = y'$. Then $w' = 2\sin x$ with $y(0) = 0$. The general solution $w = 2\sin x + C_1$ for some constant C_1. Plugging in $t = 0$, $w = 0$ gives $C_1 = 0$, so the unique solution is $w = 2\sin x$. Now we need to solve $y' = 2\sin x$. Integrating, taking the antiderivative gives $y = C + C_2$. Putting in $t = 0$ $y = 1$ gives $C_2 = 1$, so the unique solution is $\boxed{y = 2 + 1}$.

9.1.4:

Solve $y' = 2y - y^2$ where $y(0) = 1$.

Preview Solution

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Your Submission Solution

Solution: We separate variables and integrate:

$$\int \frac{dy}{2y - y^2} = \int dx.$$

The left side decomposes into partial fractions

$$\frac{1}{2}\left(\int \frac{dy}{y} + \int \frac{dy}{2 - y}\right) = \int dx.$$

Integrating gives

$$\frac{1}{2}\left(\log|y| - \log|2 - y|\right) = x + C,$$

and simplifying gives

$$\log\left|\frac{y}{2 - y}\right| = 2x + C.$$

Taking the exponential gives

$$\frac{y}{2 - y} = ce^{2x},$$

where c could be negative. Plugging in $y(0) = 1$ gives $\frac{1}{1} = ce^0 \Rightarrow c = 1$, so $c = 1$. Hence

$$\frac{y}{2 - y} = e^{2x},$$

and solving for y gives $\boxed{y = \frac{2e^{2x}}{1 + e^{2x}}}$.

9.1.5:

The milk is removed from the refrigerator at a temperature of 2 degrees Celsius and placed in a glass in a room that's feel at a constant temperature of 22 degrees. After 1 minute, the milk has warmed to 3 degrees. When will the milk warm to 10 degrees?

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Your Submission Solution

Solution: Let y denote the difference in temperature between the milk and the room. Then the milk's temperature is governed by the differential equation $y' = k_{\text{a}}y$, which has solution $y = Ce^{kt}$. We are given that $y(\text{minutes})(0) = 18, y(1) = 16$ and $y(2) = 12$. Putting in $t = 0$, $y = 18$ gives $C = 18$, so $y = 18e^{kt}$. Putting in $t = 1$, $y = 16$ gives $16 = 18e^k$. Then $k = \log\left(\frac{8}{9}\right)$. Putting this in and simplifying gives $y = 18e^{t\log(8/9)} = 18(\frac{8}{9})^{t/1}$. We want to know what t is when $y = -10$.

Using logarithms, we can determine that this $t = \log_{\frac{8}{9}}(-1/9 \approx 3.22 \text{ minutes})$ after the milk is removed from the refrigerator.

9.1.6:

A **homogeneous** differential equation is an equation of the form $y' = f\left(\frac{y}{x}\right)$ for some function f. For example $y' = \frac{x^2 - y^2}{xy}$ is homogeneous, since the right side can be written as $\frac{1}{y}\left(\frac{x}{y} - \frac{y}{x}\right)$. Show that we can solve any homogeneous equation by making the substitution $y = xz$.

Preview Solution

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Your Submission Solution

Solution: Rewrite the substitution as $z = xz$, so that $y' = z + xz'$. Then we have $z + xz' = f(z)$, which we can write as $xz' = f(z) - z$. Now we separate the variables as

$$\frac{dz}{f(z) - z} = \frac{dx}{x}.$$

Finally, we can antidifferentiate both sides and solve.

9.2 Second-order linear differential equations

A type of differential equation that comes up very frequently (especially in physics) is a **2nd order linear homogeneous equation with constant coefficients**. That's a mouthful—what that means is that an equation of the form

$$y'' + ay' + by = 0,$$

where a and b are real numbers. We can decode some of the adjectives:

- *2nd order means* y'' appears in the equation, but not y''' or any other higher derivatives of y
- *linear means* y'', y', and y appear only by themselves as linear terms: nothing like $y''y$ is allowed.
- *homogeneous* means there are no terms that involve only the independent variable.
- *constant coefficients* means that a and b are real numbers and not functions of the independent variable (such as x or t).

To begin to study this type of differential equation, let's first take the middle term out

Problem 9.15:

Consider the differential equation $y'' + by = 0$, where y is a function of x.

(a) If $b = 0$, what are the solutions?

(b) If $b < 0$, what are the solutions? (Try $b = -1$ first if you're stuck.)

(c) If $b > 0$, what are the solutions? (Try $b = 1$ at first if you're stuck.)

Solution for Problem 9.15: The nature of the solutions depends on whether b is positive, negative or 0. However, we expect that in all cases, solutions to second-order differential equations will depend on two arbitrary constants, as opposed to a single constant that we saw in solutions to first-order equations. One reason to expect this is that we may have to antidifferentiate twice to solve a second-order equation, and each antiderivative will introduce an arbitrary constant. To see why we get two constants to fix we work through each case below:

(a) Naturally, $b = 0$ is the easiest. In this case, the equation is simply $y'' = 0$. We integrate this once to get $y' = c$ for some constant c, and then we integrate this again to get $y = cx + d$ for some new constant d. More typically, we call the constants c_1 and c_2, so that the solutions of $y'' = 0$ are all of the form

$$y = c_1 x + c_2,$$

where c_1 and c_2 are constants. Once again, note that our solution contains two constants, since we are solving a second-order equation.

(b) Next is the case where $b < 0$. It may be easier to determine the solutions if we write it as $y'' = -by$. This looks very similar to the exponential growth equation $y' = ky$, except that we now have y'' on the left side instead of y'. In a similar manner, we can search for an exponential solution of the form $y = e^{rx}$ for some constant r. Then we have $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$, so to satisfy $y'' = -by$ we must have $\lambda^2 = -b$, or $\lambda = \pm\sqrt{-b}$. (Now we see why we must have $b < 0$; otherwise, the square root does not exist. We treat the case of $b > 0$ in part (c) below.) For simplicity, let $k = \sqrt{-b}$, so that the original differential equation is $y'' = -k^2 y = 0$ and thus we have found solutions $y = e^{\pm kx}$.

Because differentiation is linear, any linear combination of our two solutions e^{kx} and e^{-kx} will be a solution to the differential equation $y'' = k^2 y = 0$. To verify this, we simply compute, where c_1 and c_2 are arbitrary constants:

$$\begin{aligned} y &= c_1 e^{kx} + c_2 e^{-kx} \\ y' &= k c_1 e^{kx} - k c_2 e^{-kx} \\ y'' &= k^2 c_1 e^{kx} + k^2 c_2 e^{-kx} = k^2 y. \end{aligned}$$

(c) The method from part (b) doesn't work if $b > 0$ because $\sqrt{-b}$ is not a real number. Instead, for this case, we can look first at $b = 1$ to get an idea. If $b = 1$, then we are looking for solutions to $y'' + y = 0$. Happily, we already know two functions that are solutions to this equation: $y = \sin x$ and $y = \cos x$. Moreover, we can prove that any solution is of the form $y = c_1 \sin x + c_2 \cos x$ for constants c_1 and c_2, although we will leave the details of this as a Challenge Problem.

To extend this argument to other values of b, we can simply repeat the same logic as in part (b). Each differentiation of y must introduce a factor of $\omega = \pm\sqrt{b}$, so that differentiating twice will introduce a factor of $\omega^2 = b$. Thus, the solutions of $y'' + \omega^2 y = 0$ are of the form

$$y = c_1 \sin \omega x + c_2 \cos \omega x,$$

where c_1 and c_2 are constants.

□

Let's summarize the solutions to Problem 9.15.

Important: In all of the following, k and ω are positive constants, and c_1 and c_2 are arbitrary constants.

- Solutions to $y'' = 0$ are of the form $y = c_1 x + c_2$.
- Solutions to $y'' - k^2 y = 0$ are of the form $y = c_1 e^{kx} + c_2 e^{-kx}$.
- Solutions to $y'' + \omega^2 y = 0$ are of the form $y = c_1 \sin \omega x + c_2 \cos \omega x$.

Now let's turn our attention to the more general form of the equation:

$$y'' + ay' + by = 0,$$

where a and b are constants. Since exponentials figure prominently in solutions to many of the differential equations that we have already seen, let's start our investigation there.

Problem 9.16:

Suppose that $y = e^{rx}$ is a solution to $y'' + ay' + by = 0$. Where is r an unknown. What equation does r satisfy?

Solution for Problem 9.16: We can immediately compute $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. We plug these in to the differential equation and get

$$r^2 e^{rx} + are^{rx} + be^{rx} = 0.$$

Dividing by e^{rx} (which is never 0) leaves $r^2 + ar + b = 0$ □

Thus, we get an exponential solution of $y'' + ay' + by = 0$ of the form $y = e^{rx}$ if and only if r is a root of the **characteristic polynomial** $\lambda^2 + a\lambda + b$. More generally, if this quadratic has two distinct real roots r_1 and r_2 then every solution to the differential equation is of the form

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

for some constants c_1 and c_2.

WARNING! We have not actually proved that every solution to the above differential equation is of the form $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. We have shown that these are solutions, but we haven't proved that there are no other solutions. In fact, those are all of the solutions, although this is difficult to prove with the tools we currently have.

If the characteristic polynomial $\lambda^2 + a\lambda + b$ has a double root r, then every solution to the differential equation is of the form

$$y = (c_1 x + c_2)e^{rx},$$

although we will leave it as an exercise to check that these are in fact solutions.

The remaining case is if the roots of the characteristic polynomial are not real. Not surprisingly, in light of part (c) of Problem 9.15, we get trigonometric terms:

Problem 9.17:

Consider the differential equation $y'' + ay' + by = 0$, and suppose that the roots of the characteristic polynomial $\lambda^2 + a\lambda + b$ are $r \pm i\omega$ where $r, \omega \in \mathbb{R}$. Show that $y = e^{rx} \sin \omega x$ is a solution to the differential equation.

Solution for Problem 9.17: Let $y = e^{rx} \sin \omega x$. We first compute y' and y'':

$$\begin{aligned} y' &= \omega e^{rx} \cos \omega x + re^{rx} \sin \omega x \\ y'' &= e^{rx}(r\sin \omega x + \omega \cos \omega x), \end{aligned}$$

and

$$\begin{aligned} y'' &= e^{rx}(r\cos \omega x - \omega^2 \sin \omega x) + re^{rx}(r\sin \omega x + \omega \cos \omega x) \\ &= e^{rx}((r^2 - \omega^2)\sin \omega x + 2r\omega \cos \omega x). \end{aligned}$$

This gives

$$\begin{aligned} y'' + ay' + by &= e^{rx}((r^2 - \omega^2 + ar + b)\sin \omega x + (2r\omega + a\omega)\cos \omega x). \end{aligned}$$

But we also know that the sum of the roots of $\lambda^2 + a\lambda + b = 0$ is $-a$, and the product of the roots is b. Therefore, $2r = -a$ and $r^2 + \omega^2 = b$, which gives

$$\begin{aligned} y'' + ay' + by &= e^{rx}((2r^2 - b) + ar + b)\sin \omega x + (-a\omega + a\omega)\cos \omega x) \\ &= e^{rx}((-ar + ar)\sin \omega x) \\ &= 0, \end{aligned}$$

as desired. □

A similar computation shows that $y = e^{rx} \cos \omega x$ is also a solution to the differential equation (we will leave the details of this computation as an exercise). Further, any linear combination of solutions is also a solution. Thus, we see that solutions to $y'' + ay' + by$, where the characteristic polynomial has complex roots $r \pm i\omega$, are

$$y = e^{rx}(c_1 \sin \omega x + c_2 \cos \omega x)$$

for constants c_1 and c_2.

We can now summarize the solutions to second-order linear differential equations with constant coefficients:

Important: Consider the differential equation

$$y'' + ay' + by = 0,$$

with characteristic equation $\lambda^2 + a\lambda + b = 0$. In all cases, c_1 and c_2 are constants.

- If the characteristic equation has distinct real roots r_1 and r_2, then every solution is of the form

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

- If the characteristic equation has a double root r, then every solution is of the form

$$y = e^{rx}(c_1 x + c_2).$$

- If the characteristic equation has complex roots $r \pm i\omega$, then every solution is of the form

$$y = e^{rx}(c_1 \sin \omega x + c_2 \cos \omega x).$$

A common example of this sort of differential equation is the motion of a spring.

Problem 9.18:

A spring obeys **Hooke's Law**, which states that the force exerted by the spring is proportional to the amount of displacement from the equilibrium position, and in the opposite direction (that is, towards equilibrium). Newton's Second Law of Motion states that $F = ma$, where F is force, m is mass, and a is acceleration. Show that the motion of a spring, with no resistance or other force present, is given by

$$x = A\sin(\theta + \omega t)$$

for suitable constants A, θ and ω.

Solution for Problem 9.18: We can write Hooke's Law as $F = -kx$, where F is force, x is displacement, and $k > 0$ is the constant of proportionality. We are also given Newton's Second Law $F = ma$, and we know that acceleration is the second derivative of displacement.

Putting this all together, we have the equation

$$x'' = a = \frac{F}{M} = -\frac{k}{m}x.$$

If we relabel the constant factor by letting $\omega = \sqrt{\frac{k}{m}}$, then we can rewrite this more conveniently as

$$x'' = -\omega^2 x.$$

As we have just seen, the solutions to this differential equation are of the form

$$x = c_1 \cos \omega t + c_2 \sin \omega t.$$

This does not quite look like what we want, which is $A\sin(\theta + \omega t)$. But we can expand our desired answer using the sine angle-addition formula:

$$A\sin(\theta + \omega t) = A(\sin\theta\cos \omega t + \cos\theta \sin \omega t).$$

This does look like the solution to our differential equation, provided that we can solve $c_1 = A\sin\theta$ and $c_2 = A\cos\theta$ for some θ. We must have $A = \sqrt{c_1^2 + c_2^2}$, so we divide both sides of our solution by A:

$$\frac{x}{A} = \frac{c_1}{A}\cos \omega t + \frac{c_2}{A}\sin \omega t.$$

Now we can choose θ so that $\sin\theta = \frac{c_1}{A}$ and $\cos\theta = \frac{c_2}{A}$, noting that $\sin^2\theta + \cos^2\theta = 1$. Then we have

$$\frac{x}{A} = \sin\theta\cos \omega t + \cos\theta \sin \omega t.$$

The right side is exactly the sine angle-sum formula, and we can finish:

$$x = A\sin(\theta + \omega t),$$

as desired. □

Exercises

9.2.1:

Solve the following differential equations:

(a) $y'' - 4y' + 3y = 0$ where $y(0) = 1$ and $y'(0) = 2$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The characteristic equation

$$r^2 - 4r + 3 = (r-3)(r-1) = 0$$

has roots 1 and 3, so the general solution is of the form

$$y = c_1 e^x + c_2 e^{3x}.$$

Plugging in $y(0) = 1$ gives $1 = c_1 + c_2$ and plugging in $y'(0) = 2$ gives $c_1 + 3c_2 = 2$. Thus $c_1 = c_2 = \frac{1}{2}$ and the solution is

$$\boxed{y = \frac{1}{2}e^x + \frac{1}{2}e^{3x}}$$

(b) $y'' + 6y' + 9y = 0$ where $y(0) = 0$ and $y'(0) = 1$.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The characteristic equation $r^2 + 6r + 9 = (r+3)^2 = 0$ has a double root -3, so the general solution is of the form

$$y = c_1 e^{-3x} + c_2 xe^{-3x}.$$

Plugging in $y(0) = 0$ gives $c_1 = 0$. Then $y' = c_2(e^{-3x} - 3xe^{-3x})$ so $y'(0) = 1$ gives $c_2 = 1$. Thus the solution is $\boxed{y = xe^{-3x}}$

(c) $y'' - 4y' + 13y = 0$ where $y(0) = 2$ and $y'(0) = -1$.

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Solution: The characteristic equation $r^2 - 4r + 13 = 0$ has roots $2 \pm 3i$, so the general solution is of the form

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x).$$

Plugging in $y(0) = 2$ gives $c_1 = 2$. Differentiating gives

$$y' = e^{2x}((2c_1 + 3c_2)\cos 3x + (-3c_1 + 2c_2)\sin 3x),$$

so plugging in $y'(0) = -1$ gives $2c_1 + 3c_2 = -1$, hence $c_2 = -\frac{5}{3}$. Thus the solution is

$$\boxed{y = e^{2x}\left(2\cos 3x - \frac{5}{3}\sin 3x\right)}$$

9.2.2:

Show that, if $\lambda^2 + a\lambda + b = 0$ has complex roots $\lambda = r \pm i\epsilon$, then $e^{rx}\cos \epsilon x$ is a solution to the differential equation $y'' + ay' + by = 0$.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let $y = e^{rx}\cos \epsilon x$. We then compute:

$$\begin{aligned} y' &= re^{rx}\cos \epsilon x - \epsilon e^{rx}\sin \epsilon x \\ &= e^{rx}(r\cos \epsilon x - \epsilon \sin \epsilon x), \end{aligned}$$

and

$$\begin{aligned} y'' &= re^{rx}(r\cos \epsilon x - \epsilon \sin \epsilon x) + e^{rx}(-r\epsilon \sin \epsilon x - \epsilon^2 \cos \epsilon x) \\ &= e^{rx}((r^2 - \epsilon^2)\cos \epsilon x - 2r\epsilon \sin \epsilon x). \end{aligned}$$

This gives

$$\begin{aligned} y'' + ay' + by &= e^{rx}((r^2 - \epsilon^2) + ar + b)(\cos \epsilon x) + (-2r\epsilon - a\epsilon)(\sin \epsilon x)). \end{aligned}$$

But we know that $2r = -a$ and $r^2 + \epsilon^2 = b$ because $r \pm i\epsilon$ are roots of $\lambda^2 + a\lambda + b$, so both coefficients of the trig functions in the above expression are 0, and hence $y'' + ay' + by = 0$, as desired.

9.2.3:

Show that if $\lambda^2 + a\lambda + b = 0$ has a double root r, then $e^{rx}(c_1x + c_2)$ is a solution to the differential equation $y'' + ay' + by = 0$.

Hint: Use the facts that r is a double root, that $a = -2r$ and $b = r^2$.

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Your Submission: Solution

Solution: Let $y = e^{rx}(c_1x + c_2)$. We compute:

$$\begin{aligned} y &= re^{rx}(c_1x + c_2) + e^{rx}(c_1) \\ &= e^{rx}(rc_1x + c_1 + rc_2), \end{aligned}$$

and

$$\begin{aligned} y' &= re^{rx}(rc_1x + c_1 + rc_2) + e^{rx}(rc_1) \\ &= e^{rx}(r^2c_1x + 2rc_1 + r^2c_2). \end{aligned}$$

This gives

$$\begin{aligned} y'' + ay' + by &= e^{rx}((r^2 + ar + b)c_1x + (2rc_1 + r^2c_2 + a(c_1 + rc_2) + bc_2)). \end{aligned}$$

Since r is a root of the characteristic polynomial, we know $r^2 + ar + b = 0$, so it suffices to show that

$$2rc_1 + r^2c_2 + a(c_1 + rc_2) + bc_2 = 0.$$

But we also know that $a = -2r$ and $b = r^2$, since r is a double root of the characteristic polynomial, so substituting these gives

$$\begin{aligned} &2rc_1 + r^2c_2 + a(c_1 + rc_2) + bc_2 \\ &= 2rc_1 + r^2c_2 - 2r(c_1 + rc_2) + r^2c_2 \\ &= 0, \end{aligned}$$

as desired.

9.2.4a:

Solve the differential equation $y'' + 2y' + 2y = e^{-3x}$.

Hint: First, solve $y'' + 2y' + 2y = 0$.

Hint: Then, find a function of the form $y = ae^{-3x}$ that is a solution to the equation, for some constant a.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: First, we solve the homogeneous equation $y'' + 2y' + 2y = 0$. The characteristic equation $r^2 + 2r + 2$ has roots $r = -1 \pm i$, so its general solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x).$$

Next, we look for a solution to the original equation of the form $y = ae^{-3x}$ for some constant a. Noting that $y' = -3ae^{-3x}$ and $y'' = 9ae^{-3x}$, we must have

$$\begin{aligned} e^{-3x} &= y'' + 2y' + 2y \\ &= 9ae^{-3x} + 2(-3ae^{-3x}) + 2ae^{-3x} \\ &= 3ae^{-3x}, \end{aligned}$$

so $a = \frac{1}{3}$. Thus, the general solution is

$$\boxed{y = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{1}{3}e^{-3x}}$$

9.A Euler's Method

As mentioned in Section 9.1, **Euler's Method** is a method for approximating solutions to differential equations of the form $y' = f(x, y)$. Euler's Method is basically just repeated tangent line approximation, as we shall see.

Let's go back to our first example from Problem 9.3 of $y' = -\frac{x}{y}$ with initial condition $y(0) = 1$. We know the solution to this is the circle $x^2 + y^2 = 1$. When $x = 0.3$, we have

$$y(0.3) = \sqrt{1 - (0.3)^2} = \sqrt{0.91} \approx 0.954.$$

Let's now see how we can estimate $y(0.3)$ using only the differential equation $y' = -\frac{x}{y}$.

Problem 9.35

Suppose a function y satisfies $y' = -\frac{x}{y}$ and $y(0) = 1$. We wish to estimate $y(0.3)$.

(a) Using $y(0) = 1$ and the fact that $y'(0) = -\frac{0}{1} = 0$, estimate $y(0.1)$ via a tangent line approximation.

(b) Using the value of $y(0.1)$ that you found in part (a), compute an estimate for $y'(0.1)$.

(c) Using the values from parts (a) and (b), estimate $y(0.2)$ via a tangent line approximation.

(d) Repeat steps (b) and (c) to estimate $y(0.3)$.

Solution for Problem 9.35: Our method is basically to perform repeated tangent-line approximations. We can decide how accurate to make the approximation by choosing how many "steps" to do. Since we want $y(0.3)$ and we're starting with $y(0)$, we'll try steps of 0.1, so we will end up doing three successive tangent-line approximations: first $y(0.1)$, then $y(0.2)$, and finally $y(0.3)$.

(a) Our first step is to apply our usual tangent line approximation, starting at the point $(0, 1)$ with slope $y' = -0/1 = 0$. As you recall, the expression for the tangent line approximation at a is

$$y(x) \approx y(a) + y'(a)(x - a).$$

(If you don't "recall" this, please go back to Chapter 4 and review it.) Since $y'(0) = 0$, our tangent line is just the horizontal line $y = 1$, and thus our first approximation is $y(0.1) = 1$. If we insist on applying the formula, we get

$$y(0.1) \approx y(0) + y'(0)(0.1) = 1 + 0(0.1) = 1.$$

(b) Now we're at the point $(0.1, 1)$. We use the differential equation to get the slope for our next tangent line approximation, by substituting $x = 0.1$ and $y = 1$ to get

$$y'(0.1) = -x/y = -0.1/1 = -0.1.$$

(c) Now we have $y(0.1)$ and $y'(0.1)$, so we can use these to perform a tangent line approximation of $y(0.2)$:

$$y(0.2) \approx y(0.1) + y'(0.1)(0.1) \approx 1 + (-0.1)(0.1) = 0.99.$$

(d) Now we're at the point $(0.2, 0.99)$, so we substitute $x = 0.2$ and $y = 0.99$ into the differential equation to get

$$y'(0.2) = -x/y \approx -0.2/0.99 \approx -0.202.$$

Thus, our final tangent line approximation is

$$y(0.3) \approx y(0.2) + y'(0.2)(0.1) \approx 0.99 + (-0.202)(0.1) \approx 0.970.$$

So our final answer is $y(0.3) \approx 0.970$.

□

We recall the exact answer was approximately 0.954, so our estimate is not that great, but it is considerably better than a 1-step tangent line approximation starting at $(1, 0)$ would give us–that would give us $y(0.3) \approx 1$, since $y'(0) = 0$ would be the slope of our tangent line at $y = 0$. Not surprisingly, performing more steps (for example, increasing x by 0.05 between estimates instead of 0.1) would give a better approximation.

Concept: Euler's Method is just repeated application of tangent line approximation.

Let's see a slightly different example, where we don't even have a differential equation to start with.

Problem 9.36

Estimate $e^{0.3}$ using a 3-step Euler's Method approximation starting from $e^0 = 1$.

Solution for Problem 9.36: To put this more in the context of Problem 9.35, we use the fact that $y = e^x$ is a solution to the differential equation $y' = y$.

We start at the point $(0, 1)$. The first tangent line approximation uses $y'(0) = 1$. So we have

$$y(0.1) \approx 1 + 1(0.1) = 1.1.$$

Now we're at the point $(0.1, 1.1)$. We have $y'(0.1) = y \approx 1.1$, so we have

$$y(0.2) \approx y(0.1) + (0.1)y'(0.1) \approx 1.1 + (0.1)(1.1) = 1.21.$$

Now we're at the point $(0.2, 1.21)$. We have $y'(0.2) = y \approx 1.21$. So we have

$$y(0.3) \approx y(0.2) + (0.1)y'(0.2) \approx 1.21 + (0.1)(1.21) = 1.331.$$

Thus, we conclude that $e^{0.3} \approx 1.331$. □

Your calculator will say $e^{0.3} = 1.349859\ldots$, so this is not an especially great approximation. We can make the estimate of $e^{0.3}$ more accurate by doing more steps to get from 0 to 0.3. Here's the calculation for 10 steps: in each step, x increases by 0.03:

x	y	y'	new y
0.000000	1.000000	1.000000	1.030000
0.030000	1.030000	1.030000	1.060900
0.060000	1.060900	1.060900	1.092727
0.090000	1.092727	1.092727	1.125509
0.120000	1.125509	1.125509	1.159274
0.150000	1.159274	1.159274	1.194052
0.180000	1.194052	1.194052	1.229874
0.210000	1.229874	1.229874	1.266770
0.240000	1.266770	1.266770	1.304773
0.270000	1.304773	1.304773	1.343916

This gives $e^{0.3} \approx 1.3439\ldots$, which is better but still not too good. If we did 100 steps (which I won't torture you by showing, but is a nice computer programming exercise if you are so inclined), we get $e^{0.3} \approx 1.3493\ldots$ which is within 0.0005 of the true value.

Sidenote: Finding numerical solutions to differential equations is a vast area of mathematics, because differential equations appear very frequently in science and engineering applications, but are often quite difficult to solve explicitly. There are many more sophisticated methods for getting numerical solutions to differential equations, some of which you may learn when you take a differential equations or numerical methods course.

Challenge Problems

9.27: Source: Putnam ✎ ✓

The component calculus minister is to deliver that the product rule for derivatives says that $(fg)' = f'g' \text{ if } f(x) = x^2$. Unfortunately, this analysis leads one to expect interest $\{4, 5\}$ and a nonzero continuous function g defined on $\{4, 5\}$ such that this wrong product rule is true for $x \in \{4, 5\}$.

Preview Solution

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Solution: If the wrong "product rule" works, then we must have $f'g' = (fg)' = fg + fg'$. Plugging in $f = x^2$ and $g' = 2x^2$ gives

$$2x^2 g' = 3x^2 g + x^2 g'.$$

Dividing by x^2 leaves

$$2x^2 = 3xg + g'.$$

This can be separated as

$$\frac{dg}{dx} = \frac{2x}{g} - 3x,$$

so

$$\log|g| = \int \frac{2x}{g} - 3x \, dx$$

$$= \int \left(\frac{2}{g} - 3\right) x \, dx$$

$$= \frac{x^2}{2}(g - 3) + C.$$

Exponentiating both sides gives $\boxed{g(x) = ce^{x^2/2} + 3}$ on any interval with $x > \frac{3}{2}$.

9.28: ✎ ✓

(a) Explain how the equation $y' = ay^2 - by$, where a and b are positive real numbers, models a population with a birth rate represented by a and a death rate represented by b.

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Solution: Population growth is proportional to the number of couples in the population (since it takes 2 people to reproduce), so this is the right term in the growth rate. Death, however, is proportional to the number of people, so this is the $-by$ term in the growth rate.

(b) Solve the equation $y' = ay^2 - by$, where a and b are positive real numbers.

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Solution: Separation of variables gives

$$\int \frac{dy}{ay^2 - by} = \int dx.$$

Factoring out $\frac{1}{b}$ lets us use partial fractions:

$$\int \frac{dy}{ay - b} = \int \left(\frac{a/b}{y - \frac{b}{a}} - \frac{1}{y}\right) dy.$$

This is

$$\frac{1}{b}\left(-\log|y| + \log\left|y - \frac{b}{a}\right|\right) = \frac{1}{b}\log\left|\frac{b/a}{y}\right| = \frac{1}{b}\log\left|\frac{a}{y}\right|.$$

Thus, we have the equation

$$\frac{1}{b}\log\left|\frac{a}{y}\right| = x + C,$$

hence exponentiating gives

$$y = \frac{a}{ce^{bx}},$$

and solving gives

$$\boxed{y = \frac{a}{1 + ce^{bx}}}.$$

(c) Further suppose that $\frac{a}{b}(0) = m$ (this gives the behavior of the solutions depend on whether $m > 0$, $m = 0$, or $m = 0$.)? Does this equation make sense?

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Solution: Plugging $m > 0$ and $y = m$ gives $m = \frac{a}{m(1 + ce^{0})}$ so $1 = \frac{a}{m^2(1 + ce^{0})}$ and hence $c = 1 - \frac{a}{m^2}$. Thus our equation becomes

$$y = \frac{a}{1 + \left(1 - \frac{a}{m^2}\right)e^{bx}}.$$

If $m > 0$, then $\frac{a}{m^2} < 1$, and $0 < c < 1$. But note that some value of x (specifically, the value of x for which $e^{bx} = 1/c$) is the x for which y is equal to m. So this equation does make sense (it blows up to infinite that this is equal model).

If $m = 0$, then $c = 0$ so we have a constant population $y = 0/a$.

If $m < 0$, then $c > 1$ and the population approaches 0 as $t \to \infty$. People are not being born fast enough to make up for the people who are dying.

This analysis leads us to believe that the differential equation is too simplistic to model population: the model only really "works" when the population is already exactly at the equilibrium size.

9.29: ✎ ✓

A first-order linear differential equation is a differential equation of the form

$$y' + a(x)y = g(x),$$

where y and y are functions of x.

(a) Solve $y' - 2y = e^{-x^2}$ by multiplying both sides by e^{-x^2} and integrating.

Preview Solution

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Your Submission: Solution

Solution: We take our original equation $y' - 2y = e^{-x^2}$ and multiply through by e^{-x^2}, giving

$$e^{-x^2}y' - 2e^{-x^2}y = e^{-2x^2}.$$

It seems like we've made things more complicated by doing this, but in fact we've actually made things simpler! The left side is now simply $(e^{-x^2}y)'$ and our differential equation has become

$$(e^{-x^2}y)' = e^{-2x^2}.$$

Now we can integrate both sides with respect to x, giving

$$e^{-x^2}y = \int e^{-2x^2} + C.$$

To finish, we multiply by e^{x^2} to solve for y, and the solution is

$$\boxed{y = e^{x^2}\int e^{-2x^2} + Ce^{x^2}}$$

(b) Find a function $h(x)$, in terms of $g(x)$, such that $(y)' = (y' + ay)h$.

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Solution: We can generalize the example from part (a) to develop a method to solve first-order linear differential equations. We start with

$$y' + a(x)y = g(x).$$

We want to multiply by some function $h(x)$ so that the left side becomes $(y)'$. Therefore, $h(x)$ must satisfy

$$h(x)y' + a(x)h(x)y = (h(x)y)' = h(x)y' + h'(x)y.$$

Comparing the y terms of the above equation, we see that this requires $h'(x) = h(x)a(x)$. This new equation for h is a separable differential equation: in particular, $\frac{h'(x)}{h(x)} = a(x)$ integrating gives $\log|h(x)| = \int a(x)\,dx$, and then exponentiation gives

$$\boxed{h(x) = e^{\int a(x)\,dx}}$$

(c) Multiply $y' + ay = g$ on both sides by the function h from part (b), and solve the equation.

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Solution: Where $h(x)$ is as in part (b) above, we have the equation

$$(y)h' = y'h + ay'h = y'h + ayh = h(y' + ay) = hg,$$

so integrating gives $yh = \int hg$ and hence the solution is

$$y = \frac{1}{h}\int hg = \frac{1}{e^{\int a(x)\,dx}}\int \left(\int e^{\int a(x)\,dx} g(x)\right) dx$$

Note the function h is called an **integrating factor** for the linear differential equation.

9.30: ✎ ✓

We prove that all solutions of $y' = y$ + 9 are of the form $y = ce_x + e_x$ xxx for some constants c and e_x, as follows:

(a) Show that f is a function such that $f'(x) = f(x) + 9$ for all x, then the function

$$g(x) = f(x)e^{\int -1\,dx}$$

is a constant function.

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Solution: We differentiate g

$$g'(x) = \frac{d}{dx}\left(f(x)e^{\int -1\,dx}\right)$$

$$= f(x)e^{\int -1\,dx}(-1) + f'(x)e^{\int -1\,dx}$$

$$= f(x)e^{\int -1\,dx}(-1) + f(x)e^{\int -1\,dx}$$

So $g' = 0$, hence g is constant.

(b) Using part (a), show that if f is a function such that $f'(x) = f(x) + 9$ and $f(0) = f(0) = 0$, then f is the constant function 9.

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Your Submission: Solution

Solution: We note that g is constant and

$$g(0) = f(0)e^{\int -1\,dx} = 0 \cdot 0 = 0.$$

Hence $g = 0$. But the only way that the sum of two squares can be 0 is if each square is itself 0. In particular, we must have $f = 0$.

(c) Prove that f is a function such that $f'(x) = f(x) + 9$, then

$$f(x) = f'(0)e_x + f(0)\,xxx.$$

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Your Submission: Solution

Solution: We know from part (b) that $f(x) = f(0)\,xxx + f(0)\,xxx$ is a solution. If h is another solution with $h(0) = f(0)$ and $h'(0) = f'(0)$, then $f - h = 0$ by the proof in part (b), hence $h = f$. Thus, the functions shown are the only solutions.

9.31: Source: USAMO ✎ ✓

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(x) = f(-x)$ for all x and $f(0) = 1$ find $f(1)$.

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Your Submission: Solution

Solution: Differentiate the given $f'(x) = f(-x)$ to get

$$f'(x) = -f(-x) \Rightarrow f'(x) = -f(-x) \Rightarrow f'(x) = -f(x).$$

Thus $f(x) = c\cos(x) + b\sin(x)$ for some constants c and b. Plugging in $f(0) = 1$ gives $c = 1$, so the function is $f(x) = \cos(x) + b\sin(x)$. To find b, we must solve for b since

$$f(1) = f(0) \Rightarrow -\sin(b) = b\cos(0) \Rightarrow b.$$

However, we can compute

$$f'(x) = -\sin(x) \Rightarrow b\cos(x)$$

and

$$f'(1) = -\sin(1 - x) \Rightarrow b\cos(1 - x).$$

Rewriting the latter in terms of $\cos(x)$ and $\sin(x)$ gives

$$f'(1 - x) = \cos\,1\sin\,x + \sin\,1\sin\,x + b\sin\,1\cos\,x - b\cos\,1\sin\,x$$

$$= \cos\,1 + b\sin\,1\cos\,x + (\sin\,1 - b\cos\,1)\sin\,x.$$

Thus, since $f'(x) = f'(1 - x)$ we have the system

$$\begin{aligned} b &= \cos\,1 + b\sin\,1, \\ 1 &= \sin\,1 - b\cos\,1. \end{aligned}$$

These have the common solution

$$f(1) = b = \frac{\cos\,1}{\cos\,1} = \frac{1 + \sin\,1}{\cos\,1}.$$

Note this can also be written as $b = \sec\,1 + \tan\,1$.

9.32: ✎ ✓

(This problem is closely based on a problem from 2K48).

Let $f(x)$ be the function $\sin x$, but Romeo is more fickle. The more Juliet loves him, the more he hates her, and the more she hates him, the more he loves her. On the other hand, Juliet is more sensible: the more Romeo loves her, the more she loves him, and the more he hates her, the more she hates him.

(a) Explain why a reasonable model for their love for each other is

$$\frac{d}{dt} = ky, \quad \frac{dr}{dt} = -kj.$$

where j and r denote their love (if positive) or hate (if negative) for each other, and where k is a positive constant.

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Your Submission: Solution

Solution: j should grow proportionally with r, and r should grow proportionally with $-j$.

(b) Solve this system for r and j. Your answer will depend on k and two other arbitrary constants.

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Your Submission: Solution

Solution: Take the derivative of the first equation to get

$$j'' = kr' = -k^2 j.$$

This gives the solution

$$j = c_1\cos(kt) + c_2\sin(kt).$$

Differentiating gives

$$r = -\frac{1}{k}j' = -c_1\sin(kt) + c_2\cos(kt).$$

(c) Suppose at time $t = 0$ Romeo is fully in love with Juliet (so that $r(0) = 1$), but Juliet is indifferent to Romeo (so that $j(0) = 0$). Write equations for the solution in terms of k.

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Your Submission: Solution

Solution: If we have $r(0) = 1$, then plugging in $t = 0$ to the equation for r gives $1 = c_2$ if we have $j(0) = 0$, then plugging in $t = 0$ to the equation for j gives $0 = c_1$. Thus the solution is

$$j = \sin(kt),$$

$$r = \cos(kt).$$

(d) Sketch the solution from (c), and explain in words what is going on.

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Your Submission: Solution

Solution: The solution is a circle of radius 1, where x is along the x-axis and j is along the y-axis, and the motion is counterclockwise. In the first quarter, they are both in love with each other, although Romeo's love is fading while Juliet's is rising. In the second quarter, Juliet's love for Romeo fades below Romeo's hatred increases. Then in the third quarter, Juliet starts hating Romeo more and more (since she hates him so much), and Romeo's hatred for Juliet fades. In the fourth quarter, Juliet's hatred fades and Romeo (since she hates him so much) and Romeo's hatred increases. Then in the final quarter, Juliet starts hating Romeo more and more (since she hates him so much), and this makes Juliet's hatred decrease. And the cycle continues forever!

(e)★ (If you want to explore further) How would things change if we had

$$\frac{d}{dt} = ky, \quad \frac{dr}{dt} = -kj,$$

where k and k are positive constants?

Hide Solution

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Your Submission: Solution

Solution: You should get an ellipse instead of a circle, but it shows the same basic phenomenon. The details are left for the reader to explore.

9.33★: Source: Putnam ✎ ✓

Find all real-valued continuously differentiable functions f with domain $\mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$(f(x))^2 = 2000 + \int_0^x \left((f(t))^2 + (f'(t))^2\right) dt.$$

(Hint: Differentiating both sides of the equation seems like a reasonable first step.

Hint: Can you factor the equation that results after differentiation?

Hide Solution

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Your Submission: Solution

Solution: Differentiating both sides:

$$2f(x)f'(x) = (f(x))^2 + (f'(x))^2.$$

This can be factored as $(f(x) - f'(x))^2 = 0$. This means that $f(x) = cx^c$ for some constant c. Plugging this back into the original equation gives

$$x^2c^2 = 2000 + \int_0^x 2x^2c^2\,dx$$

$$= 2000 + x^3c^2$$

$$\Rightarrow -2000 = x^3c^2 - x^2c^2$$

so $c^2 = 2000$ and $c = \pm\sqrt{2000}$. Therefore the functions are $\boxed{f(x) = \pm x\sqrt{2000}}$.

9.34★: ✎ ✓

In the proof of existence, infinitely differentiable functions, f with domain $\mathbb{R}$ such that

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

for all $x, y \in \mathbb{R}$. (Warning: this is quite hard. First just play with the equation a bit, to see if you can discover any facts about such a function. There has to be at least one such function, and it is polynomial. Then try to find some nonpolynomial solutions). Then try to prove that you've found them all.)

Hint: First try all the ideas in the remarks accompanying the problem.

Hint: Plugging in now values for x and/or y is a good strategy.

Hint: Try substituting $x = 0$.

Hint: At some point you'll want to differentiate both sides. Differentiating multiple times might help too.

Hint: The hard part will be proof that all the possible functions. If you're stuck, you might be able to rearrange-and-repeat the solution once you know the answer.

Hint: Some functions that work are $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, $\frac{1}{x^4}$, $\frac{1}{x^5}$, $\frac{1}{x^6}$, $\frac{1}{x^7}$, $\frac{1}{x^8}$, $\frac{1}{x^9}$, $\frac{1}{x^{10}}$, $\frac{1}{x^{11}}$, $\frac{1}{x^{12}}$, $\frac{1}{x^{13}}$, $\frac{1}{x^{14}}$, $\frac{1}{x^{15}}$, $\frac{1}{x^{16}}$, $\frac{1}{x^{17}}$, $\frac{1}{x^{18}}$, $\frac{1}{x^{19}}$, $\frac{1}{x^{20}}$, $\frac{1}{x^{21}}$, $\frac{1}{x^{22}}$, $\frac{1}{x^{23}}$, $\frac{1}{x^{24}}$, $\frac{1}{x^{25}}$, $\frac{1}{x^{26}}$, $\frac{1}{x^{27}}$, $\frac{1}{x^{28}}$, $\frac{1}{x^{29}}$, $\frac{1}{x^{30}}$, $\frac{1}{x^{31}}$, $\frac{1}{x^{32}}$, $\frac{1}{x^{33}}$, $\frac{1}{x^{34}}$, $\frac{1}{x^{35}}$, $\frac{1}{x^{36}}$, $\frac{1}{x^{37}}$, $\frac{1}{x^{38}}$, $\frac{1}{x^{39}}$, $\frac{1}{x^{40}}$, $\frac{1}{x^{41}}$, $\frac{1}{x^{42}}$, $\frac{1}{x^{43}}$, $\frac{1}{x^{44}}$, $\frac{1}{x^{45}}$, $\frac{1}{x^{46}}$, $\frac{1}{x^{47}}$, $\frac{1}{x^{48}}$, $\frac{1}{x^{49}}$,

Review Problems

9.19:

Solve the differential equation $y' = 1 + y^2$ with initial condition $y(0) = 0$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Separate as $\dfrac{dy}{1+y^2} = dx$, so integration gives $\tan^{-1} y = x + C$. Plugging in $x = y = 0$ gives $C = 0$, so we have $\boxed{y = \tan x}$.

9.20:

Solve the differential equation $(x^2+1)y' = (xy)^2$ with initial condition $y(1) = 2$.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Expand as $(x^2+1)y' = x^2y^2$, and separate variables to give $\dfrac{dy}{y^2} = \dfrac{x^2}{x^2+1}\,dx$. Integrating both sides gives

$$-\frac{1}{y} = \int \frac{x^2}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C,$$

hence $y = -\dfrac{1}{x - \tan^{-1} x + C}$ Plugging in $y(1) = 2$ gives

$$2 = -\frac{1}{1 - \frac{\pi}{4} + C},$$

so $1 - \dfrac{\pi}{4} + C = -\dfrac{1}{2}$ hence $C = -\dfrac{3}{2} + \dfrac{\pi}{4}$. Thus, the solution is

$$\boxed{y = -\frac{1}{x - \tan^{-1} x - \frac{3}{2} + \frac{\pi}{4}}}.$$

This can be made to look slightly nicer by multiplying numerator and denominator by -4:

$$y = \frac{4}{6 - \pi - 4x + 4\tan^{-1} x}.$$

9.21:

Solve the equation $y' = 3y - y^2$ with initial condition $y(0) = 5$, and determine $\lim_{t\to\infty} y(t)$.

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Your Submission: Solution

Solution: Separate variables to give $\dfrac{dy}{3y - y^2} = dt$. The integral of the left side is

$$\begin{aligned}\int \frac{dy}{3y - y^2} &= \frac{1}{3}\int \left(\frac{1}{y} + \frac{1}{3-y}\right) dy \\ &= \frac{1}{3}\left(\log|y| - \log|3-y|\right) \\ &= \frac{1}{3}\log\left|\frac{y}{3-y}\right| \\ &= \frac{1}{3}\log\left|\frac{3}{3-y} - 1\right|.\end{aligned}$$

Thus we have

$$\frac{1}{3}\log\left|\frac{3}{3-y} - 1\right| = t + C,$$

and exponentiating gives

$$\left|\frac{3}{3-y} - 1\right| = ce^{3t}.$$

Thus,

$$\frac{3}{3-y} = ce^{3t} + 1,$$

and solving for y yields

$$y = 3 - \frac{3}{ce^{3t} + 1}.$$

Plugging in $t = 0$ gives $5 = 3 - \dfrac{3}{c+1}$, so $c = -\dfrac{5}{2}$, and the solution is

$$\boxed{y = 3 - \frac{3}{1 - \frac{5}{2}e^{3t}} = 3 + \frac{6}{5e^{3t} - 2}}.$$

As $t \to \infty$, the fraction $\dfrac{6}{2 - 5e^{3t}}$ approaches 0, so $\boxed{\lim_{t\to\infty} y(t) = 3}$.

9.22:

Solve the equation $2y'' + (y')^2 = -1$.

Hint: Substitute $v = y'$, solve for v, then integrate.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let $v = y'$, so the equation becomes $2v' + v^2 = -1$. This separates to $-2\dfrac{dv}{1+v^2} = dx$, and hence we integrate to get $-2\arctan v = x + C$, or $v = \tan\left(-\dfrac{x+C}{2}\right)$ Therefore

$$y = \int v\,dx = \boxed{-2\log\left|\sec\left(-\frac{x+C}{2}\right)\right| + D}.$$

We could write this a bit more simply as

$$\begin{aligned}y &= 2\log\left|\left(\sec\left(-\frac{x+C}{2}\right)\right)^{-1}\right| + D \\ &= 2\log\left|\cos\left(-\frac{x+C}{2}\right)\right| + D \\ &= 2\log\left|\cos\left(\frac{x}{2} + C_0\right)\right| + D.\end{aligned}$$

9.23:

Solve $y'' = y'$ with initial conditions $y(0) = 1$ and $y(1) = 2$.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let $z = y'$, so that $z' = z$. This is the basic exponential growth equation, so $y' = z = ce^x$. Integrating again gives $y = ce^x + d$. Plugging in the initial conditions gives the equations $1 = c + d$ and $2 = ce + d$, so $c(e-1) = 1$, hence $c = \dfrac{1}{e-1}$ and $d = 1 - \dfrac{1}{e-1}$. Thus the solution is $\boxed{y = 1 + \dfrac{e^x - 1}{e-1}}$.

9.24:

Motion of an object through a fluid can be modeled by $y' = -ky^2$ where y represents velocity and k is a positive constant. Suppose an object starts at time $t = 0$ sec with velocity 10 m/sec and $k = 0.5$. Determine when the object slows to 1 m/sec.

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Your Submission: Solution

Solution: First, we find the explicit solution using separation of variables. We can write $\dfrac{dy}{y^2} = -0.5\,dt$. Integrating both sides gives $-\dfrac{1}{y} = -0.5t + C$ for a constant C. Therefore $y = \dfrac{1}{0.5t - C}$. We use the initial condition to find C. Putting in $t = 0$ and $y = 10$ gives $10 = \dfrac{1}{0 - C}$, so $C = -0.1$. Thus $y = \dfrac{1}{0.5t + 0.1}$. Putting in $y = 1$ and solving for t gives us that $t = 1.8$, so the object slows to 1 m/sec after $\boxed{1.8 \text{ seconds}}$.

9.25:

We can model the rate that a rumor spreads among a population with the differential equation $\dfrac{dy}{dt} = ky(1-y)$, where $0 \le y \le 1$ is the fraction of the population that knows the rumor, and $k > 0$ is a constant.

(a) Explain in words why this differential equation is a reasonable model.

Hint: The rumor spreads when sometime who knows the rumor tells it to someone who doesn't already know it.

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Your Submission: Solution

Solution: A rumor spreads when a person who knows the rumor comes into contact with a person that doesn't know the rumor, and gossip ensues. Thus, the rate of spread of the rumor is proportional to the quantity

$$\begin{aligned}&(\text{population that knows the rumor}) \\ &\times (\text{population that doesn't know the rumor}),\end{aligned}$$

which is $y(1-y)$. Letting k be the constant of proportionality, we get $y' = ky(1-y)$.

(b) Solve the equation.

Preview: Solution

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Your Submission: Solution

Solution: Separating variables and integrating gives

$$\int \frac{dy}{y(1-y)} = \int k\,dt = kt + C.$$

To evaluate the integral on the left side, we use the partial fraction decomposition

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y},$$

which then gives

$$\log|y| - \log|1-y| = kt + C.$$

The left side of the above is $\log\left|\dfrac{y}{1-y}\right|$ so exponentiating gives

$$\frac{y}{1-y} = c_0e^{kt},$$

which is more useful when we take reciprocals, giving $\dfrac{1}{y} - 1 = ce^{-kt}$ (where $c = \dfrac{1}{c_0}$). Thus, solving for y, we have

$$\boxed{y = \frac{1}{1 + ce^{-kt}}}.$$

for some positive constant c.

(c) If 10% of the population knows the rumor at noon on Sunday, and 20% of the population knows the rumor at noon on Monday, then approximately when will 90% of the population know the rumor?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We measure time in days starting from noon on Sunday. At $t = 0$, we have $0.1 = y = \dfrac{1}{1+c}$ so $c = 9$ and our equation is

$$y = \frac{1}{1 + 9e^{-kt}}.$$

At $t = 1$, we have $0.2 = y = \dfrac{1}{1 + 9e^{-k}}$, so $9e^{-k} = 4$, hence $k = -\log(4/9) \approx 0.8109$. To find when $y = 0.9$, we solve

$$0.9 = \frac{1}{1 + 9e^{-\log(4/9)t}},$$

giving $e^{-\log(4/9)t} = 1/81$, or $\log(4/9)t = \log(1/81)$, and finally

$$t = \frac{\log(1/81)}{\log(4/9)} \approx 5.419 \text{ days}.$$

Noting that 0.419 days is about 10 hours and 3 minutes, we conclude that 90% of the population will know the rumor at approximately $\boxed{\text{10:03 PM on Friday}}$.

9.26:

Solve the differential equation $(\sin x)y' + (\cos x)y = \tan x$.

Hint: Does the left side of the equation look like the derivative of a product?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The key observation is to note that the left side is the derivative of a product–the equation can we rewritten as

$$((\sin x)y)' = \tan x.$$

Therefore we can just integrate:

$$(\sin x)y = \int \tan x\,dx.$$

To integrate the right side, you can look it up, or write it as

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

and substitute $u = \cos x$, $du = -\sin x\,dx$. This makes our integral

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\log|u| + C = -\log|\cos x| + C.$$

We usually write this as $\log|\sec x| + C$. Thus, we have that

$$(\sin x)y = \log|\sec x| + C,$$

and dividing by $\sin x$ solves for y:

$$\boxed{y = (\csc x)(\log|\sec x| + C)}.$$

Epilogue

Now what?

Congratulations on completing the book! You should now have a good understanding of the fundamentals of single-variable calculus. What do you do now?

Most likely, you are now at the stage of your career where the mathematics that you "have to" study is dictated by what subject you are going to pursue. For many paths of study, you may not "need" any more mathematics! But assuming that most readers of this book will be pursuing careers in mathematics, science, engineering, or some other technically-demanding field, there is probably more mathematics in your future.

One very good option is to study **physics**. You may have already seen some physics, but physics without calculus is like language without verbs: it's hard to do much physics without some basic calculus. For example, one of the most fundamental equations in physics is $F = ma$, but since a (acceleration) is the second derivative of position, what is $F = ma$ if not a differential equation? Virtually all major concepts in physics hinge on calculus–in fact, it's largely why much of calculus was invented!

Another option is to study **statistics**. You've likely already seen some discrete statistics or probability, where you were analyzing finite sets of data or situations with finitely many discrete outcomes. Now that you know calculus, you can explore the much wider arena of probability and statistics of events with continuous outcomes. For example, many events can be modeled by a **normal distribution**, where an event with mean μ and variance σ^2 is modeled with the **probability density function**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Using this function, the probability that $a \le x \le b$ is $\int_a^b f(x)\,dx$. (Note that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ by Problem 6.32, reflecting the fact that the probability of *any* outcome should be 1.)

Within pure mathematics, there are a number of possibilities for the next course after single-variable calculus. The most obvious choice is to continue with **multivariable calculus** or **vector calculus** (these two are essentially the same material with slightly different nuances). This subject extends the tools of calculus that we developed in this book to functions of two or more variables. It also generalizes many of the techniques from Chapter 8 to work with curves and surfaces in 3-dimensional space.

You could also embark on a course of study of **differential equations**. Our work in Chapter 9 barely scratched the surface of this very deep subject.

A third direction is to study **linear algebra**. Linear algebra, at its core, is the study of vectors and matrices, but it is much more than that. Also, many argue that having some background with linear algebra is preferred before beginning multivariable calculus or differential equations, because the tools of linear algebra are necessary in both of these subjects.

Finally, we will repeat what we said in the Preface: we strongly recommend that students study **discrete mathematics**, which broadly includes combinatorics, probability, logic, and number theory. These subjects are often overlooked–especially in high-school curricula–but are vitally important and are broadly applicable to many "real world" problems.

The overall point is that once you've completed a first course in calculus, mathematics is no longer "do A, then B, then C" like it might have been throughout high school. There are many different branches of math to explore, most of them interrelated. Don't be in a rush to pigeonhole yourself–explore lots of different areas of math, and most importantly, have fun!

Explanation of top-of-chapter diagrams

The diagrams at the top of each chapter are graphs of **Fourier series approximations**. Specifically, the picture at the top of Chapter n (where $1 \le n \le 9$) is the graph of

$$y = \frac{2}{\pi} \sum_{k=1}^{n} \frac{\sin((2k-1)\pi x)}{2k-1}.$$

As $n \to \infty$, this function approaches the **square wave** function:

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } \lfloor x \rfloor \text{ is even,} \\ -\frac{1}{2} & \text{if } \lfloor x \rfloor \text{ is odd.} \end{cases}$$

The graph of this function is shown at the top of the previous page.

The nth Fourier series approximation of a function $f(x)$ with period $2L$ is

$$F_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L} \right),$$

where the coefficients are given by:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}\,dx, \quad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}\,dx.$$

If we extend this to an infinite series, we get the Fourier series for $f(x)$. Part of the reason that the series converges is that the integrals above approach 0 as $k \to \infty$, due to the result from Problem 6.29.

Just as Taylor polynomials compute the best polynomial approximations of a function at a point, Fourier series compute the best approximations of a function as a sum of sine and cosine functions. But whereas Taylor series give an approximation of a function at a single point, Fourier series are instead global in nature, considering the entire function at once, and thus are used to study the global properties of the function.

Fourier series are useful for studying functions which are periodic in nature. Such functions arise in acoustics, cryptography, and signal processing. Fourier series are also essential for studying quantum mechanics and differential equations, and Fourier analysis even makes multiplication of large numbers easier.

References

Before studying calculus, you should be ready. If you need to brush up on your precalculus topics, a good source is:

[Ru] R. Rusczyk, *Precalculus*, AoPS Incorporated, 2009.

Covers trigonometry, complex numbers, and an introduction to vectors and matrices. Lots of hard problems, including many problems from advanced US high-school math contests such as the AIME and USAMO.

Next are a couple of "mainstream" calculus textbooks. Most widely-used college calculus textbooks (including the two listed below) come in many varieties: single-variable (like this book) vs. multi-variable (typically the next calculus course after this book) vs. both, and "early transcendentals" (where $\exp$ and $\log$ are introduced early, as in this book) vs. "late transcendentals" (where $\exp$ and $\log$ are not introduced until after differentiation and integration are defined).

[H-H] D. Hughes-Hallett et al., *Calculus* (6th Edition), John Wiley & Sons, 2012.

An above-average "mainstream" calculus textbook. The "Single Variable" edition is the version most closely aligned with this textbook. Light on rigor in spots, but a stronger emphasis than most books on using calculus to model "real-world" problems.

[St] J. Stewart, *Calculus* (7th edition), Brooks Cole, 2011.

By far the most widely-used college calculus textbook–reportedly sells more than all other calculus textbooks combined. A competent book designed for typical first-year college calculus courses, with the usual advantages (lots of routine exercises and word problems) and drawbacks (not much rigor in places and not a lot of very challenging problems). The main version (combining single-variable and multivariable calculus, with late transcendentals) is 1368 pages long, and at press time retails for over $\$250$.

Some more advanced calculus texts:

[Sp] M. Spivak, *Calculus* (Third Edition), Publish or Perish, Inc., 1994.

A very thorough, very rigorous treatment of calculus. Spivak covers many topics much more thoroughly than we do, and proves *everything*. If you want to read something more rigorous than this book, that really covers all the nuts-and-bolts of calculus, then you want to read Spivak's book. His detailed construction of the real numbers is particularly illuminating. A new 4th edition was published in 2008.

[Ap] T. M. Apostol, *Calculus, Volume 1: One-Variable Calculus with an Introduction to Linear Algebra* (2nd Edition), John Wiley & Sons, 1967.

Another excellent rigorous treatment of calculus. A bit unconventional in that definite integrals are covered before differentiation. Also, as the title implies, has extensive coverage of beginning linear algebra, which leads into an introduction of vector calculus. (Volume 2 of this textbook covers multivariable calculus.)

[MacC] C. R. MacCluer, *Honors Calculus*, Princeton University Press, 2006.

A very compact (only 168 pages!) treatment of single-variable calculus. Generally assumes that students have seen calculus before, and fills in much of the rigor. Leaves a lot of results as "exercises" for the reader. In particular, the book has a very mature treatment of continuity, using topology rather than the δ-ϵ construction.

There are a number of essentially identical test-prep books published by various test-prep companies. These books are recommended for students who would like more practice with routine exercises before taking calculus placement tests for the purpose of potential college credit. These books may also contain test-taking tips. While such books are largely interchangeable, one of the more popular choices is:

[Bar] S. O. Hockett and D. Bock, *Barron's AP Calculus*, Barron's Educational Services, Inc., 2008.

Includes four full-length practice exams, a brief review of the topics on the exam, and test-taking tips (including tips on calculator usage).

A definitive source for recent Putnam Competition problems is:

[KPV] K. S. Kedlaya, B. Poonen, and R. Vakil, *The William Lowell Putnam Competition Problems & Solutions: 1985–2000*, Mathematical Association of America, 2002.

All the problems from the Putnam during the period 1985–2000. Contains hints and very detailed solutions, including areas for further study. Warning: the majority of the problems on the Putnam involve branches of mathematics beyond calculus, and Putnam problems as a whole are *very* difficult.