Vectors

Definition 1

An *inner product* on a vector space is such that

1. $\langle u, v\rangle = \langle v, u\rangle$
2. $\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$
3. $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$
4. $\langle u, u\rangle \geq 0$, and $= 0 \iff u = 0$

Definition 2

A *norm* is $\|v\| = \sqrt{\langle v, v\rangle}$.

Definition 3

A *line* $l(t) : \mathbb{R} \to \mathbb{R}^n$ is $l(t) = P + td$, where $P, d \in \mathbb{R}^n$. It may also be given by $l(t) = (1-t)Q + tP$, where $P, Q \in \mathbb{R}^n$, and $t \in [0, 1]$.

Definition 4

A *plane* $p(s, t) : \mathbb{R}^2 \to \mathbb{R}^3$ is $p(s, t) = P + sd_1 + td_2$, where $d_1, d_2 \in \mathbb{R}^3$.

It may also be given in point-normal form, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $\langle a, b, c\rangle \in \mathbb{R}^3$ is normal to the plane, and $(x_0, y_0, z_0) \in \mathbb{R}^3$ lies on the plane.

Definition 5

A *linear transformation* $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is such that $\lambda(\vec{0}) = \vec{0}$, and $\lambda(\vec{a} + \vec{b}) = \lambda(\vec{a}) + \lambda(\vec{b})$.

Alternatively, write $\lambda(x_1, ..., x_n) = x_1\vec{d_1} + ... + x_n\vec{d_n}$, where $\vec{d_i} \in \mathbb{R}^m$.

Definition 6

An *affine* transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation plus a point $P \in \mathbb{R}^m$.

Definition 7

The *projection* of v onto u, denoted $\text{proj}_u(v)$, is given by $(u \cdot v)\frac{u}{\|u\|}$.

Definition 8

The *tangent line* of $r : \mathbb{R} \to \mathbb{R}^n$ at $a \in \mathbb{R}$ is an affine transformation λ satisfying

$$\lim_{t \to a} \frac{\|r(t) - \lambda(t)\|}{|t - a|} = 0$$

$$\lambda(t) = r(a) + (t - a)\vec{d} \quad \vec{d} \neq 0$$

Prop 1 (Inequalities)

$\|v + u\| \leq \|v\| + \|u\|$ ($\triangle$) and

$|\langle u, v\rangle| \leq \|u\| \, \|v\|$ (Cauchy-Schwartz).

Prop 2 (Cross/Dot Products)

$(u \times v) \cdot u = 0$

$\|u \times v\|$ is the area of the parallelogram bounded by u, v.

$\|u \times v\| = \|u\| \, \|v\| \sin(\theta)$

$u \cdot v = \|u\| \, \|v\| \cos(\theta)$

Prop 3 (Distances)

(a) The distance between a point R and a plane may be given by $\|\text{proj}_{\vec{n}}(P - R)\|$, where $P, \vec{n}$ are as in point-normal form.

(b) The distance between skew lines may be given by projecting [a third line which contains points of both] onto [the normal vector of the skew lines].

(c) The distance between R and PQ may be given by $\|PR - \text{proj}_{PQ}(R)\|$.

Differentiation

Definition 1

(a) $r : \mathbb{R} \to \mathbb{R}^n$ is *differentiable* at $\vec{a}$ if $\exists$ a linear transformation $\lambda : \mathbb{R} \to \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{\|r(\vec{a}+h) - r(\vec{a}) - \lambda(\vec{h})\|}{|h|} = 0$$

(b) Similarly, $F : \mathbb{R}^2 \to \mathbb{R}$ is *differentiable* at $\vec{a}$ if $\exists$ a linear transformation $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(h,k)\to(0,0)} \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\|\langle h, k\rangle\|}$$

(c) Generally, $F : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $\vec{a}$ if $\exists$ a linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{\|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})\|}{\|\vec{h}\|}$$

We denote the derivative $\lambda(\vec{a})$ as $DF_{\vec{a}}$

Definition 2

The *arc length* of a curve $r(t) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

The arc length parameterization is some $t = \alpha(s)$ such that $\|r'(\alpha(s))\| = 1$.

Definition 3

The *Jacobian* matrix of $F : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$F' = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix}$$

Note that, where $m > 1$, each element is a column vector.

Definition 4

A *level surface* is $F : \mathbb{R}^n \to 0$.

Definition 5

The *gradient* of $F : \mathbb{R}^n \to \mathbb{R}$ is $\left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle$ (i.e. its Jacobian) and denoted by ∇F.

Definition 6

$\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $\vec{a}$ if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon$.

Definition 7

$F : \mathbb{R}^n \to \mathbb{R}$ is called C^k*-continuous* (or k-times continuously differentiable) at $\vec{a}$ if all k^{th}-order partial derivatives exist near $\vec{a}$ and are continuous at $\vec{a}$.

Definition 8

(a) The k^{th} *partial derivative* of $F : \mathbb{R}^n \to \mathbb{R}$ at $\vec{a}$, denoted by $\frac{\partial F}{\partial x_k}(\vec{a})$ is given by

$$\lim_{t\to 0} \frac{F(\vec{a} + te_k) - F(\vec{a})}{t}$$

where e_k is the standard basis vector, e.g. $\vec{i}$ for $k = 1$.

(b) The *directional derivative* of $F : \mathbb{R}^n \to \mathbb{R}$ along $\vec{h}$, denoted by $\partial_{\vec{h}} F(\vec{a})$, is given by

$$\lim_{t\to 0} \frac{F(\vec{a} + t\vec{h}) - F(\vec{a})}{t}$$

(c) The j^{th} *iterated directional derivative* of $F : \mathbb{R}^n \to \mathbb{R}$ at $\vec{a}$ along $\vec{h}$, denoted by $\partial^j_{\vec{h}} F(\vec{a})$, is given by $g^{(j)}(0)$, where $g(t) = F(\vec{a} + t\vec{h})$. Note that $g'(0) = \partial_{\vec{h}} F(\vec{a})$.

Prop 1 (Differentiability and Partials)

(a) If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then all partial derivatives of F exist at $\vec{a}$.

(b) If all partial derivatives of $F : \mathbb{R}^n \to \mathbb{R}$ exist near $\vec{a}$ and are continuous at $\vec{a}$, then F is differentiable.

(c) If $F : \mathbb{R}^n \to \mathbb{R}$ is k-times continuously differentiable, then all $(k-1)^{th}$-order partial derivative are continuously differentiable.

Prop 2 (Chain Rule)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{a} \in \mathbb{R}^n$ and $f(\vec{a}) \in \mathbb{R}^m$, respectively. Then

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l$$

is differentiable. Furthermore, $Dh_{\vec{a}} = Dg_{f(\vec{a})} \circ Df_{\vec{a}}$, and $h'(\vec{a}) = g'(f(\vec{a}))f'(\vec{a})$.

Prop 3 (Iterated Partials)

(a) If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then

$$\partial_{\vec{h}}(\vec{a}) = \vec{h} \cdot \nabla F = \sum_{i=1}^{n} h_i \frac{\partial F}{\partial x_i}(\vec{a})$$

(b) If $F : \mathbb{R}^n \to \mathbb{R}$ if k-times continuously differentiable, then

$$\partial^j_{\vec{h}}(\vec{a}) = (\vec{h} \cdot \nabla)^j F(\vec{a})$$

Prop 4 (Mixed Partials)

If $F : \mathbb{R}^2 \to \mathbb{R}$ is C^2-continuous, then $\partial_x \partial_y F = \partial_y \partial_x F$.

Prop 5 (Taylor)

If $F : \mathbb{R}^n \to \mathbb{R}$ is C^k continuous, let

$$\alpha_j(\vec{h}) = \frac{1}{j!}\partial^j_{\vec{h}}(\vec{a}) = \frac{1}{j!}(\vec{h} \cdot \nabla)^j F(\vec{a})$$

Then $F(\vec{a}) + \sum_{i=1}^{k} \alpha_i(\vec{x} - \vec{a})$ is the best degree k approximation of F near $\vec{a}$.

Integration

Definition 1

Let $\mathcal{B} = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Then $F : \mathbb{R}^n \to \mathbb{R}$ is *integrable* over $\mathcal{B}$ if

$$\lim_{m \to \infty} \sum_{i=1}^{m} F(c_j^m)\frac{\text{vol}(\mathcal{B})}{m^n} \text{ exists}$$

and is equivalent up to choices of c_j^m, where $c_j^m \in [a_j^k, b_j^k]$, $k \in [1, m]$ (slice of size $\frac{1}{m}$).

Definition 2

A set $S \subseteq \mathbb{R}^n$ has *zero measure* if $\forall \varepsilon > 0$ we can choose a countable set of open balls such that $S \subseteq \cup B(x_i, \varepsilon_i)$, where $\sum \text{vol} B(x_i, \varepsilon_i) < \varepsilon$.

Prop 1 (Integrability)

(a) If $F : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathcal{B}$, then F is integrable on $\mathcal{B}$.

(b) The set of discontinuities of F in $\mathcal{B}$ has zero measure if and only if F is integrable.

Prop 2 (Fubini)

Let $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$. If $F : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathcal{B}$, then

$$\int_{\mathcal{B}} F = \int_{x_n=a_n}^{x_n=b_n} \cdots \left(\int_{x_1=a_1}^{x_1=b_1} F(x_1, \ldots, x_n) dx_1 \right) \cdots dx_n$$

The order of integration doesn't matter.

Prop 3 (MVT)

$\int_a^b g(x) = g(c)(b-a)$ for some $c \in [a, b]$