ALGEBRA 3 NOTES

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In Algebra 3, we will study abstract algebraic structures. Chiefly among them, we have *groups*, which are useful in representing symmetries, *rings & fields*, which help us think about number systems, and *vector spaces & modules*, which encode physical space.

AXIOMS AND FIRST PROPERTIES

A *group* is a set G endowed with a binary composition $G \times G \to G$ such that the following axioms hold:

1. $\exists e \in G$, an identity element, such that $e * a = a * e = a \; \forall a \in G$.
2. $\forall a \in G, \exists a' \in G$ such that $a * a' = a' * a = e$.
3. $a * (b * c) = (a * b) * c \; \forall a, b, c \in G$.

If $a * b = b * a \; \forall a, b \in G$, we call G *commutative*.

Why do we care about groups? If X is an object, we call a *symmetry* of X a function $X \to X$ which preserves the structure of the object.

e.g. a polygon, graphs, tilings, "crystal," "molecules," rings, vector spaces, metric spaces, manifolds

The collection of symmetries, $\text{Aut}(X) = \{f : X \to X\}$, we can structure as a group: let $* = \circ$, $e = \text{Id}$, and $f \in \text{Aut}(X)$ (note that, by axiom 2, these must be bijective).

A note on notation: for non-commutative groups, we write $a * b = ab$, $e = 1$ or $\mathbb{1}$, $a' = a^{-1}$, and $a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}}$. This is called *multiplicative notation*. For commutative rings, we write $a * b = a + b$, $e = 0$ or $\mathbb{0}$, $a' = -a$, and $na = \underbrace{a + \ldots + a}_{n \text{ times}}$.

The following are some examples of groups generated by sets:

1. If X is a set with no operations, $\text{Aut}(X)$ is the set of all bijections $f : X \to X$. One calls this the *permutation group*, or, if $|X| = n < \infty$, the *symmetric group*, and we write $\text{Aut}(X) = S_n$.
2. If V is a vector space over $\mathbb{F}$, $\text{Aut}(V) = \{T : V \to V\}$, the set of vector space isomorphism. If $\dim(V) = n$, recall that we assocate V with $\mathbb{F}^n$, whose set of isomorphism is given by $GL_n(\mathbb{F})$, the collection of $n \times n$ invertible matrices. This is called the *linear group*.
3. If R is a ring, then $(R, +, \mathbb{0})$ is a commutative group. Furthermore, $(R^\times, \times, \mathbb{1})$ is a non-commutative group, where $R^\times := R \setminus \{\text{non-invertible elements of } R\}$.
4. If V is Euclidean space endowed with a dot product, where $\mathbb{F} = \mathbb{R}$, with $\dim(V) < \infty$, $\text{Aut}(V) = O(V)$ is called the *orthogonal group of V*. In particular, $O(V) = \{T : V \to V : T(u) \cdot T(v) = u \cdot v\}$.

5. If X is a geometric figure (e.g. a polygon), we write $\mathrm{Aut}(X) = D_n$, where $|\mathrm{Aut}(X)| = n$, and call this the *dihedral group*.

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A *homomorphism* from groups $G_1 \to G_2$ is a function $\varphi : G_1 \to G_2$ satisfying $\varphi(ab) = \varphi(a)\varphi(b)$, where $a, b \in G_1$.

Prop. 1.1 $\varphi(\mathbb{1}_{G_1}) = \mathbb{1}_{G_2}$ and $\varphi(a^{-1}) = \varphi(a)^{-1}\ \forall a \in G_1$.

Proof.

$\varphi(\mathbb{1}_{G_1}) = \varphi(\mathbb{1}_{G_1}^2) = \varphi(\mathbb{1}_{G_1})^2 \implies \varphi(\mathbb{1}_{G_1}) = \varphi(\mathbb{1}_{G_1}^{-1})\varphi(\mathbb{1}_{G_1}) = \mathbb{1}_{G_2}$.
$\varphi(a^{-1})\varphi(a) = \varphi(a^{-1}a) = \varphi(\mathbb{1}_{G_1}) = \mathbb{1}_{G_2} \implies \varphi(a_{-1}) = \varphi(a)^{-1}$. □

A homomorphism which is bijective is called an *isomorphism*. If there exists an isomorphism between two groups G_1 and G_2, we call them *isomorphic*, and write $G_1 \cong G_2$. One can thus call $\mathrm{Aut}(G)$ the set of isomorphisms from $G \to G$.

As an example, take $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, ..., n-1\}$. Note that $\varphi : G \to G$ is determined entirely by $\varphi(1)$, since $\varphi(i) = \varphi(\underbrace{1 + ... + 1}_{i \text{ times}}) = \underbrace{\varphi(1) + ... + \varphi(1)}_{i \text{ times}}$. How can we find an element of $\mathrm{Aut}(G)$? Clearly, not all mappings $\varphi(1)$ are bijective: take n to be even and $\varphi(1) = 2$. Then $\varphi(2) = 4$, $\varphi(3) = 6$, ..., $\varphi(n/2) = 0$, so φ is not surjective. We know then that $\varphi(G) = \varphi(1)\mathbb{Z} \mod n$, and would like $\varphi(G) = G$. If $\varphi(1)$ and n are co-prime, then we can write $k\varphi(1) + ln = k\varphi = 1$, so every element can be reached.

We can construct a group isomorphism $\eta : \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^\times$ which sends $\varphi \to \varphi(1)$. Clearly $\eta(\varphi_{t_1} \circ \varphi_{t_2}) = \varphi_{t_1} \circ \varphi_{t_2}(1) = \varphi_{t_1}(t_2) = t_1 t_2 = \eta(\varphi_{t_1})\eta(\varphi_{t_2})$, so η is a homomorphism. It is also bijective: given $\varphi(1)$, we can deduce a mapping for each element.

For a group G and an object X, define an *action* to be a function from $G \times X \to X$ such that

1. $\mathbb{1} \times x = x$
2. $(g_1 g_2)x = g_1(g_2 x)$
3. $m_g : x \to gx$ is an automorphism of X.

You can derive (3): if $gx_1 = gx_2$, one can take the group inverse to conclude $x_1 = x_2$. Similarly, given $x \in X$, we know $m_g(g^{-1}x) = x$.

for $x \in X$, $g_1, g_2 \in G$.

Prop. 1.2 Given an action of G on X, the assignment $g \to m_g$ is a homomorphism between $G \to \mathrm{Aut}(X)$.

Proof.

$m_{g_1 g_2}(x) = g_1 g_2 x = g_1(g_2 x) = g_1 m_{g_2}(x) = m_{g_1}(m_{g_2}(x)) = m_{g_1} \circ m_{g_2}(x)$ □