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# DISCRETE MATHEMATICS

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# I Preliminaries and Trees

## DEFINITIONS

*Graph theory* is the study of pairwise relations between objects, e.g. computer networks, interpersonal relationships, airport flights. Graphs will help us simplify and abstract networks .

A *graph*  $G$  is comprised of a set of vertices, denoted  $V(G)$ , where  $|V(G)| < \infty$ , a set of edges, denoted  $E(G)$ , where every edge is associated with two vertices.

At least in this course

We say that an edge *joins* the two vertices it's associated with. Similarly, an edge is *incident* to a vertex which is an end of it. Conversely, a vertex is incident to an edge if it is an end of it.

Two vertices are *adjacent* or *neighbors* if they are joined by an edge, and a vertex has *degree* edges incident to it.

The *null graph* is the graph such that  $V(G) = \emptyset$ . The *complete graph* on  $n$  vertices, denoted  $K_n$ , is such that  $|V(K_n)| = n$  and  $|E(K_n)|$  is maximal.

For a graph of  $n$  vertices, the maximal number of edges it may have is  $\binom{n}{2}$ .

PROP 1.1

Suppose every vertex is connected to every other vertex. Then  $\sum_{v \in V(G)} \deg(v) = n(n-1) \implies |E(G)| = \frac{n(n-1)}{2} = \binom{n}{2}$ .  $\square$

PROOF.

A graph of  $n$  vertices, where  $v_i$  is only adjacent to  $v_{i-1}$  and  $v_{i+1}$ , is called a *path* and is sometimes denoted  $P_n$ .  $v_1$  and  $v_n$  are called the ends of  $P_n$ .

For  $n \geq 3$ , a *cycle*  $C_n$  is a graph with  $V(G) = \{v_1, \dots, v_n\}$  and  $E(G) = \{v_1 v_2, v_2 v_3, \dots, v_{n-1} v_n, v_n v_1\}$ .

An *adjacency matrix* is a matrix contained all vertices on both axis. Pairwise adjacency is denoted by a 1 entry, and 0 otherwise. For example, the following is an adjacency table for a 4 element cycle:

	$v_1$	$v_2$	$v_3$	$v_4$
$v_1$	$\times$	1	0	1
$v_2$	1	$\times$	1	0
$v_3$	0	1	$\times$	1
$v_4$	1	0	1	$\times$

Similarly, an *incidence matrix* has rows in  $V(G)$  and columns in  $E(G)$ , and marks with 1 pairs which are incident to each other. The following is the incidence

matrix for a 4 element cycle:

	$v_1$	$v_2$	$v_3$	$v_4$
$e_1$	1	1	0	0
$e_2$	0	1	1	0
$e_3$	0	0	1	1
$e_4$	1	0	0	1

PROP 1.2

For a graph  $G$ , we always have  $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$ .

PROOF.

Every edge has two vertices incident to it. Thus,  $\sum \deg(v)$  will be the number of times an edge is incident to a vertex, i.e. the number of edges  $\times 2$ .  $\square$

$H$  is a *subgraph* of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .

We cannot do the same for " $G \setminus H$ ," since we may delete vertices and keep their incident edges!

For two graphs  $G, H$ , the union  $G \cup H$  is a graph such that  $V(G \cup H) = V(G) \cup V(H)$  and  $E(G \cup H) = E(G) \cup E(H)$ . We similarly define the intersection  $G \cap H$  to be such that  $V(G \cap H) = V(G) \cap V(H)$  and  $E(G \cap H) = E(G) \cap E(H)$ .

PROP 1.3

There are  $2^{\binom{n}{2}}$  graphs with  $n$  vertices.

PROOF.

We know the maximal number of edges of this graph is  $\binom{n}{2}$ . Then, for each edge, one may make a binary choice whether to include it or not  $\therefore$  the number of graphs is  $2^{\binom{n}{2}}$ .  $\square$

We can now ask: how many graphs are there with  $n$  vertices up to isomorphism?

An *isomorphism* between  $H$  and  $G$  is a bijection  $\varphi : V(G) \rightarrow V(H)$  such that  $uv \in E(G) \implies \varphi(u)\varphi(v) \in E(H)$ .

## CONNECTIVITY

A *walk* in  $G$  with ends  $u_0$  and  $u_k$  is a sequence  $(u_0, u_1, \dots, u_k)$  such that  $u_i \in V(G)$  and  $u_i u_{i+1} \in E(G)$ . The length of this walk is  $k$ .

$u$  and  $v$  are called *connected* if there exists a walk in  $G$  with ends  $u$  and  $v$  OR, equivalently, there exists a path  $P \subseteq G$  with ends  $u$  and  $v$ .

PROP 1.4

$\exists$  a walk in  $G$  with ends  $u$  and  $v \iff \exists$  a path  $P \subseteq G$  with ends  $u$  and  $v$ .

PROOF.

( $\Leftarrow$ ) Let  $P \subseteq G$  be a path with ends  $u$  and  $v$ . Then  $V(P)$  can be numbered  $u = v_0, v_1, \dots, v_k = v$ , where  $v_i v_{i+1} \in E(P)$ . Then  $(v_0, \dots, v_k)$  is a walk in  $G$ .

( $\Rightarrow$ ) Let there exist a walk  $(u = v_0, \dots, v_k = v)$  with  $v_i v_{i+1} \in E(G)$ . WLOG suppose this is the walk of minimal length. If  $v_i \neq v_j$ , i.e. are pairwise distinct, then we already have a path. Suppose otherwise, and let  $v_i = v_j$ . Then

$(v_0, \dots, v_i, v_{j+1}, \dots, v_k)$  is a *smaller* walk with ends  $u$  and  $v$ , which establishes the contradiction  $\nmid$ .  $\square$

A graph  $G$  is called *connected* if  $\forall u, v \in V(G)$ ,  $u$  and  $v$  are connected.

A *partition* of  $V(G)$  is  $(X_1, \dots, X_k)$  such that  $\cup_{i=1}^k X_i = V(G)$  and  $X_i \cap X_j = \emptyset \forall i \neq j$ .

A graph  $G$  is not connected  $\iff \exists$  a partition  $(X, Y)$  of  $V(G)$  such that no edge of  $G$  is incident to one vertex in  $X$  and one in  $Y$ . PROP 1.5

( $\Leftarrow$ ) Suppose  $G$  were connected. Then choose  $u \in X, v \in Y$  such that there exists a walk  $(u = u_0, \dots, u_k = v)$ . Let  $u_i$  be minimal over  $i$  such that  $u_i \in Y$ . Then  $u_{i-1} \in X$ , and  $u_{i-1}u_i \in E(G) \nmid$ . PROOF.

( $\Rightarrow$ ) Let  $u, v \in V(G)$  be such that there is no walk from  $u$  to  $v$ . Let  $X$  be the set of all  $w \in V(G)$  such that  $\exists$  a walk with ends  $u$  and  $w$ . Similarly, let  $Y = V(G) \setminus X$ . Clearly  $V(G) = X \cup Y$ ,  $X \cap Y = \emptyset$ , and  $(X, Y)$  is a partition. Suppose there exists an edge from a vertex in  $X$  to a vertex in  $Y$ , i.e.  $x \in X, y \in Y$ . Then we have the walk  $(u, \dots, w, \dots, x, y)$ . But  $y \notin X \nmid$ .  $\square$

Let  $G$  be a graph.  $H \subseteq G$  is called a *connected component* of  $G$  if  $H$  is a maximal connected subgraph of  $G$ , i.e. if  $\exists H \subseteq H' \subseteq G$  with  $H'$  connected, then  $H = H'$ .

Sometimes we just say “component.”  
PROP 1.6

If  $H_1, H_2$  are connected graphs, and  $V(H_1) \cap V(H_2) \neq \emptyset$ , then  $H_1 \cup H_2$  is also connected.

Let  $u \in H_1, v \in H_1 \cap H_2, w \in H_2$ . Then  $(u, \dots, v)$  and  $(v, \dots, w)$  are both walks, and thus  $(u, \dots, v, \dots, w)$  is a walk.  $\square$  PROOF.

Every  $v \in V(G)$  is a member of a unique connected component  $H \subseteq G$ . PROP 1.7

$\{v\}$  is connected. If there does not exist  $H \supseteq \{v\}$  also connected, then we are done. Otherwise, we may choose the maximal such connected superset. PROOF.

Suppose  $v \in H_1$  and  $H_2$ , two connected components. Then by Prop 1.6,  $H_1 \cup H_2$  is connected. But since  $H_1 \cup H_2 \supseteq H_1, H_2$ , this violates maximality. We conclude that  $H_1 = H_2$ .  $\square$

Let  $G$  be a graph, and let  $H \subseteq G$  be a non-null and connected subgraph. Then  $H$  is a connected component of  $G \iff \forall e \in E(G)$  with an end in  $V(H)$ , we have  $e \in E(H)$ . PROP 1.8

For ( $\Rightarrow$ ), let  $e = uv$ , with  $u \in V(H)$ . If  $v \in V(H)$ , then we are done. Otherwise, suppose  $e \notin E(H)$ . We know  $v$  is a member of a unique connected component. But adding  $e$  to  $H$  would yield a further connected graph: take the graphs of  $\{uv\}$  and  $H$ . Both are clearly connected, so  $H \cup \{uv\}$  is connected. PROOF.

(  $\Leftarrow$  )

Proof idea.

Obtained from  $G$  by deleting  
 $e$

For  $e \in E(G)$ ,  $G \setminus e$  is a graph such that  $V(G \setminus e) = V(G)$  and  $E(G \setminus e) = E(G) \setminus \{e\}$ .

Similarly, for  $v \in V(G)$ ,  $G \setminus v$  is a graph such that  $V(G \setminus v) = V(G) \setminus \{v\}$  and  $E(G \setminus v) = E(G) \setminus \{e : e \text{ incident to } v\}$ .

Let  $\text{comp}(G) = \#$  of connected components of  $G$ .

PROP 1.9

$\text{comp}(G) = 1 \iff G$  is connected.

PROOF.

(  $\implies$  ) direction is trivial. For (  $\Leftarrow$  ), if  $G$  is connected, then there cannot exist a more maximal connected subgraph, e.g.  $G$  is a connected component. Since every vertex belongs to a unique connected component, and this must be  $G$ ,  $\text{comp}(G) = 1$ .  $\square$

Let  $e = \{u, v\} \in E(G)$ . Define a *cut-edge* to be an edge which is not part of any cycle.

PROP 1.10

Exactly one of the following holds:

1.  **$e$  is a cut-edge:**  $\text{comp}(G \setminus e) = \text{comp}(G) + 1$ , and  $u, v$  belong to different components of  $G \setminus e$ .
2.  **$e$  is not a cut-edge:**  $\text{comp}(G \setminus e) = \text{comp}(G)$ , and  $u, v$  belong to the same component.

PROOF.

Let  $e$  be a cut-edge. Let  $H_1, \dots, H_k$  be the connected components of  $G \setminus e$ . If  $u, v$  belong to  $H_i$ , then  $\exists$  a path  $P \subseteq H_i$  with ends  $u$  and  $v$ . Adding  $e$ , this is a cycle  $\nmid$ .

WLOG, assume that  $u, v$  belong to  $V(H_1), V(H_2)$ , respectively. Then let  $H'$  be obtained by  $H_1 \cup H_2$  by adding  $e$ . We claim that  $H', H_2, \dots, H_k$  are all components of  $G$ . By Prop 1.8, we only need to check the connectivity of  $H'$ , and this holds by Prop 1.6. Since there do not exist any vertices *not* in  $V(H_i) : i \geq 2$  or  $V(H')$ , these are all the components of  $G$ . Thus,  $\text{comp}(G) + 1 = \text{comp}(G \setminus e)$ .  $\square$

## TREES AND FORESTS

A *forest* is a graph with no cycles, i.e. every edge is a cut-edge.

A *tree* is a non-null connected forest.

PROP 1.11

Let  $F$  be a non-null forest. Then  $\text{comp}(F) = |V(F)| - |E(F)|$ .

PROOF.

We'll show by induction on  $|E(F)|$ . If  $n = 0$  then all vertices are their own connected components. Let  $|E(F)| = n$ , and assume  $\text{comp}(F) = |V(F)| - |E(F)|$ . Let  $e \in E(F)$ . Since  $F$  is a forest,  $e$  is a cut-edge, and thus  $\text{comp}(G \setminus e) = \text{comp}(G) + 1 = |V(F)| - |E(F)| + 1 = |V(F)| - (|E(F)| - 1) = |V(F)| - |E(F \setminus e)| = |V(F \setminus e)| - |E(F \setminus e)|$ .  $\square$

A *leaf* is a vertex with degree 1.

Let  $T$  be a tree with  $|V(T)| \geq 2$ . let  $X = \{\text{leaves of } T\}$ ,  $Y = \{v \in V(G) : \deg(v) \geq 3\}$ . Then  $|X| \geq |Y| + 2$ .

PROP 1.12

Thus, trees have  $\geq 2$  leaves!

By Prop 1.1, we have

PROOF.

$$\begin{aligned}
 \sum_{v \in V(T)} \deg(v) &= 2|E(T)| \stackrel{1.11}{=} 2(|V(T)| - \text{comp}(G)) \stackrel{1.9}{=} 2(|V(T)| - 1) \\
 &\Rightarrow \sum_{v \in V(T)} (\deg(v) - 2) = 2(|V(T)| - 1) - 2|V(T)| = -2 \\
 &= \underbrace{\sum_{v \in X} (\deg(v) - 2)}_{=-|X|} + \underbrace{\sum_{v \in Y} (\deg(v) - 2)}_{\geq |Y|} + \underbrace{\sum_{v \in V(T) - X - Y} (\deg(v) - 2)}_{=0} \\
 &\Rightarrow -2 \geq -|X| + |Y| \Rightarrow |X| \geq |Y| + 2 \quad \square
 \end{aligned}$$

A note for the following few proofs: if  $w$  is a leaf, then any path which exists in  $T$  (with ends not  $w$ ) exists in  $T \setminus w$ .

Let  $T$  be a tree with 2 leaves,  $u$  and  $v$ . Then  $T$  is a path with ends  $u$  and  $v$ .

PROP 1.13

Let  $P \subseteq T$  be a path with ends  $u$  and  $v$ . By Prop 1.12,  $\deg_T(w) = 2 \forall w \in V(P) \setminus \{u, v\}$ . Moreover,  $\deg_T(w) = \deg_P(w)$ , so no vertex in  $V(P)$  is incident to an edge in  $E(T) \setminus E(P)$ . Then, by Prop 1.8,  $P$  is a connected component. But  $T$  is connected, so  $T = P$ .  $\square$

PROOF.

Let  $T$  be a tree and  $v \in V(T)$  be a leaf. Then  $T \setminus v$  is a tree.

PROP 1.14

$T \setminus v$  is non-null, since  $v$  has a neighbor.  $T \setminus v$  has no cycles, since  $T$  has no cycles, and  $T \setminus V$  is connected. We know there exists a path between any two vertices in  $V(T) \setminus \{v\}$ . Such a path still exists.  $\square$

PROOF.

If  $G$  is a graph,  $v \in V(G)$  a leaf, and  $G \setminus v$  a tree, then  $G$  is a tree.

PROP 1.15

$G$  is non-null, since  $G \setminus v$  is non-null. We know that  $v$  belongs to no cycles, since it is a leaf, so any cycles apparent in  $G$  would exist in  $G \setminus v$ . Thus,  $G$  has no cycles. For connectedness, let  $H$  be the graph containing  $v$ , its

PROOF.

incident edge, and that edge's other vertex  $v'$ .  $H$  is connected, as is  $G \setminus v$ , and  $G \setminus v \cap H \neq \emptyset$ , so  $G \setminus v \cup H = G$  is connected by Prop 1.6.  $\square$

PROP 1.16

Let  $T$  be a tree,  $u, v \in V(T)$ . Then  $T$  contains a unique path with ends  $u$  and  $v$ .

PROOF.

We'll show by induction on  $|V(T)|$ . This clearly holds for  $|V(T)| = 1$ . Let  $|V(T)| \geq 2$ . Suppose  $T$  contains a leaf  $w \in V(T) \setminus \{u, v\}$ . Then  $T \setminus w$  is a tree by Prop 1.14. By our induction hypothesis,  $T \setminus w$  contains a unique path with ends  $u$  and  $v$ . By connectedness,  $\exists$  a path with ends  $u, v$  in  $T$ . But this path must exist in  $T \setminus w$ , whose uniqueness follows.

If no such leaf exists, then  $T$  has exactly 2 leaves ( $u$  and  $v$ ). Thus, by Prop 1.13,  $T$  is a path with ends  $u$  and  $v$ , and thus the only path in  $T$ .  $\square$

## SPANNING TREES

Let  $G$  be a graph. A subgraph  $T \subseteq G$  is called a *spanning tree* of  $G$  if  $T$  is a tree and  $V(T) = V(G)$ .

PROP 1.17

Let  $G$  be connected and non-null. Let  $H \subseteq G$ , chosen minimal such that  $V(H) = V(G)$  and  $H$  is connected. Then  $H$  is a spanning tree of  $G$ .

PROOF.

We only need to check that  $T$  is non-null and contains no cycles. The first is automatic, since  $V(T) = V(G)$ , and  $G$  is non-null. If  $H$  has a cycle, then let  $e$  be an edge in the cycle.  $H \setminus e$  is connected by Prop 1.9 and Prop 1.10. But this contradicts minimality, so  $T$  contains no cycles.  $\square$

PROP 1.18

Let  $G$  be a connected non-null graph. Let  $H \subseteq G$  be maximal such that  $H$  contains no cycles. Then  $H$  is a spanning tree of  $G$ .

PROOF.

We need to show that  $V(H) = V(G)$  and  $H$  is connected (it is non-null, since at least a singleton of  $G$  contains no cycles; it contains no cycles by construction). If  $\exists v \in V(G) \setminus V(H)$ , adding  $v$  such that  $\deg(v) = 0$  would maintain  $H$  having no cycles, thus contradicting maximality.

Suppose  $H$  is not connected. Then by Prop 1.5 there exists a partition  $H = X \cup Y$  such that no edge has a vertex in both  $X$  and  $Y$ . However, such an edge must exist in  $G$ , say  $e \in E(G)$ , so we may add this edge to  $H$  to produce  $H'$ . Observe that  $H'$  contains no cycles, since  $e$  belongs to no cycles in  $H$ . But this contradicts maximality, so  $H$  must contain no cycles.  $\square$

Let  $T$  be a spanning tree of  $G$ . Let  $f \in E(G) \setminus E(T)$ . Then  $T$  with  $f$  has one cycle (by Prop 1.16). This is called the *fundamental cycle* of  $f$  with respect to  $T$ , and denoted  $FC(T, f)$ .



Let  $T$  be a spanning tree of  $G$ ,  $f \in E(G) \setminus E(T)$ . Let  $C = \text{FC}(T, f)$ ,  $e \in E(C)$ . Then  $(T + f) \setminus \{e\}$  is a spanning tree. PROP 1.19

Let  $T' = (T + f) \setminus \{e\}$ .  $T + f$  is connected, and since  $e$  is not a cut-edge,  $(T + f) \setminus \{e\} = T'$  is also connected.  $C$  is a unique cycle in  $T + f$ , so  $T'$  contains no cycles. Thus,  $T'$  is a tree.  $V(T') = V(T) = V(G)$ , since  $T$  is a spanning tree, so we conclude that  $T'$  is a spanning tree. PROOF. □