ASSIGNMENT 1

MATH 251

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QUESTION 1

Define $V := (0, \infty) \subset \mathbb{R}$ with the operations $r \oplus s = rs$ and $\lambda \otimes r = r^\lambda$, where rs and r^λ are defined as usual, and $\mathbb{0}_V = 1$. This is a vector space over $\mathbb{R}$.

We are given that V is abelian. Let $\alpha, \beta \in \mathbb{R}$ and $r, s \in V$. Then

1. Where $\mathbb{R}$ is a field, $\mathbb{1}_\mathbb{R} = 1$. Then for any $r \in V$, $\mathbb{1}_\mathbb{R} \otimes r = r^1 = r$
2. $\alpha \otimes (\beta \otimes r) = \alpha \otimes (r^\beta) = (r^\beta)^\alpha = r^{\beta\alpha} = (\alpha\beta) \otimes r$.
3. $(\alpha + \beta) \otimes r = r^{\alpha+\beta}$. Similarly, $\alpha \otimes r \oplus \beta \otimes r = r^\alpha r^\beta = r^{\alpha+\beta}$.
4. $\alpha \otimes (r \oplus s) = \alpha \otimes (rs) = (rs)^\alpha$. Similarly, $\alpha \otimes r \oplus \alpha \otimes s = r^\alpha s^\alpha = (rs)^\alpha$

QUESTION 2

Part (a): $M_{n\times m}(\mathbb{F})$ is an abelian group, where $(a_{ij}) \oplus (b_{ij}) = (a_{ij} + b_{ij})$ for all $(a_{ij}), (b_{ij}) \in M_{n\times m}(\mathbb{F})$, and $+$ is defined as usual over $\mathbb{F}$.

Group axioms: Define the neutral element $\mathbb{0}_M$ to be the matrix of $\mathbb{0}_\mathbb{F}$ entries. Choose $(a_{ij}) \in M_{n\times m}(\mathbb{F})$. Then $(a_{ij}) \oplus \mathbb{0}_M = (a_{ij} + \mathbb{0}_\mathbb{F}) = (a_{ij})$.

Furthermore, we have the inverse element $(-a_{ij})$ for (a_{ij}), the matrix of additive inverses in $\mathbb{F}$ for each coordinate a_{ij}.

$M_{n\times m}(\mathbb{F})$ is associative: $[(a_{ij}) \oplus (b_{ij})] \oplus (c_{ij}) = (a_{ij} + b_{ij} + c_{ij}) = (a_{ij}) \oplus [(b_{ij}) \oplus (c_{ij})]$; and commutative: $(a_{ij} \oplus b_{ij}) = (a_{ij} + b_{ij}) = (b_{ij}) \oplus (a_{ij})$.

For notational reasons, [ ] will be used for associativity. ( ) are all sequences.

$M_{n\times m}(\mathbb{F})$ is a vector space over $\mathbb{F}$, where scalar multiplication is defined $\lambda \otimes (a_{ij}) = (\lambda a_{ij})$, and λa_{ij} is defined as usual over $\mathbb{F}$. Let $\alpha, \beta \in \mathbb{F}$. We check the axioms:

1. $\mathbb{1}_\mathbb{F} \otimes (a_{ij}) = (\mathbb{1}_\mathbb{F} a_{ij}) = (a_{ij})$, since $a_{ij} \in \mathbb{F}$.
2. $\alpha \otimes [\beta \otimes (a_{ij})] = \alpha \otimes (\beta a_{ij}) = (\alpha\beta a_{ij}) = [\alpha\beta](a_{ij})$
3. $[\alpha + \beta] \otimes (a_{ij}) = ([\alpha + \beta]a_{ij})$. Similarly, $\alpha \otimes (a_{ij}) \oplus \beta \otimes (a_{ij}) = (\alpha a_{ij}) \oplus (\beta a_{ij}) = (\alpha a_{ij} + \beta a_{ij}) = ([\alpha + \beta]a_{ij})$
4. $\alpha \otimes [(a_{ij}) \oplus (b_{ij})] = \alpha \otimes (a_{ij} + b_{ij}) = (\alpha[a_{ij} + b_{ij}]) = (\alpha a_{ij} + \alpha b_{ij}) = (\alpha a_{ij}) \oplus (\alpha b_{ij}) = \alpha \otimes (a_{ij}) \oplus \alpha \otimes (b_{ij})$

Dropping the $\oplus$, $\otimes$ notation now.

Part (b): Consider $M_n(\mathbb{F})$ and $\mathrm{Sym}_n(\mathbb{F})$, the set of matrices for which $(a_{ij}) = (a_{ji})$. This is a subspace of $M_n(\mathbb{F})$. To check:

1. As above, $\mathbb{0}_M$ is the $n \times n$ matrix of $\mathbb{0}_\mathbb{F}$ entries. This is clearly symmetric, so $\mathbb{0}_M \in \mathrm{Sym}_n(\mathbb{F})$
2. Suppose (a_{ij}) and (b_{ij}) are in $\mathrm{Sym}_n(\mathbb{F})$. Then $(a_{ij} + b_{ij}) = (a_{ij}) + (b_{ij}) = (a_{ji}) + (b_{ji}) = (a_{ji} + b_{ji})$, so $(a_{ij}) + (b_{ij})$ is symmetric, i.e. $x, y \in \mathrm{Sym}_n(\mathbb{F}) \implies x + y \in \mathrm{Sym}_n(\mathbb{F})$
3. Suppose $(a_{ij}) \in \mathrm{Sym}_n(\mathbb{F})$, and let $\alpha \in \mathbb{F}$. Then $\alpha(a_{ij}) = (\alpha a_{ij})$. Similarly, $\alpha(a_{ij}) = \alpha(a_{ji}) = (\alpha a_{ji})$, so especially $(\alpha a_{ij}) = (\alpha a_{ji})$, and $\alpha(a_{ij}) \in \mathrm{Sym}_n(\mathbb{F})$.

$\implies \mathrm{Sym}_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$ is a subspace.

QUESTION 3

(a) Consider $U := \{(x, y, z) : x, y, z > 0\}$. For any $a, b > 0$, we know that $a + b > 0$, and thus if $(x, y, z), (x', y', z') \in U$, their sum $(x + x', y + y', z + z') \in U$. Thus, U is closed under addition.

This is not closed under scalar multiplication: take $\lambda = 0$, for example. $0 \cdot (x, y, z) = (0, 0, 0)$, and clearly 0 is not positive, so $0(x, y, z) \notin U$

(b) Consider $V := \{(x, y, z) : x, y, z \in \mathbb{N}\}$. This is closed under addition, since the naturals are closed under addition. However, $\lambda(x, y, z) \notin V$, where λ is irrational, for example, so it is not closed under scalar multiplication.

(c) Consider $W := \{(0, 0, z) : z \in \mathbb{R}\}$. This is closed under addition: $(0, 0, z) + (0, 0, z') = (0, 0, z + z') \in W$. This is also closed under scalar multiplication: $\lambda \cdot (0, 0, z) = (0, 0, \lambda z)$. Lastly, $\mathbb{0}_{\mathbb{R}^3} \in W$, so $W \subseteq \mathbb{R}^3$ is a subspace. It is proper and nontrivial.

Proper: (1,0, 0) cannot be represented
Non-trivial: (0, 0, 1) exists

Question 4

It has been shown in class that $X + Y$ is a subspace. Also, $X + Y \supseteq X \cup Y$, as any element in $X \cup Y$ can be represented as $x + \mathbb{0}_V$ or $\mathbb{0}_V + y$.

Suppose that $U \supseteq X \cup Y$ is a subspace. Let $v \in X + Y$. Then $v = x + y$ for $x \in X$ and $y \in Y$. We know that $x, y \in U$, since $x, y \in X \cup Y \subseteq U$. Since U is a subspace, it is closed under addition, and so $x + y \in U$, or $v \in U$, and we conclude that $U \supseteq X + Y$.

QUESTION 5

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$ be a subspace.

($\Longrightarrow$) Let S be linearly dependent. Then there exists at least one $v_j \in S$ such that $v_j \in \text{Span}(S \setminus v_j)$, so $v_j = b_1 v_1 + ... + b_{j-1}v_{j-1} + b_{j+1}v_{j+1} ... + b_n v_n$, where b_i not all zero and $\{v_i\}_{i\in[n]} = S$

Let $w \in \text{Span}(S)$. Then we can write $w = a_1 v_1 + ... + a_n v_n$, where a_i not all zero. Substituting our expression for v_j in the linear combination for w yields

$$w = a_1 v_1 + .. + a_{j-1}v_{j-1} + (b_1 v_1 + ... + b_{j-1}v_{j-1} + b_{j+1}v_{j+1} ... + b_n v_n) + a_{j+1}v_{j+1} + ... + a_n v_n$$

This is a linear combination that does not contain v_j, so we conclude that $w \in \text{Span}(S \setminus v_j) \implies \text{Span}(S) \subseteq \text{Span}(S \setminus v_j)$, where clearly $S \setminus v_j \subsetneq S$. Furthermore, $\text{Span}(S \setminus v_j) \subseteq \text{Span}(S)$ automatically, so $\text{Span}(S) = \text{Span}(S \setminus v_j)$

($\Longleftarrow$) Suppose there exists a proper subset $S' \subsetneq S$ such that $\text{Span}(S) = \text{Span}(S')$. Let $w \in \text{Span}(S) = \text{Span}(S')$ be defined as $v_1 + ... + v_n$. Then $w = \sum_{i\in I} a_i v_i$, where $I \subsetneq [n]$, and WLOG, all a_i are non-zero.

To clarfiy: one "throws out" any zero coefficients in w's represntation. Indices $i : a_i \neq 0$ then make up the set I, which we know to be non-empty.

$$\implies v_1 + ... + v_n = \sum_{i\in I} a_i v_i \implies v_1 + ... + v_n - \sum_{i\in I} a_i v_i = 0$$

Since $S' \subsetneq S$, there exists some vectors in $\{v_1, ..., v_n\}$ which are *not* in $\{v_i : i \in I\}$. In particular, these vectors have the coefficient 1 in front. Thus, we've found a non-trivial linear combination = 0, and S is linearly dependent.

QUESTION 6

Let V be a v.s. over a field $\mathbb{F}$ for which $2 \neq 0$.

$(\implies)$ Suppose $x, y \in V$ are independent. Consider $a(x+y) + b(x-y) = 0$, and assume that at least one of $a, b \neq 0$. Rearranging, we get $(a+b)x + (a-b)y = 0$. Since x, y are independent, we know $a + b = 0$ and $a - b = 0$. This implies $b = -b \implies 2 = 0$, which is a contradiction. Thus, $a = b = 0$.

$\implies x + y, x - y$ are independent.

$(\impliedby)$ Suppose $x + y, x - y$ are independent. As we did above, we can rearrange $ax + by = 0$, which we assume to be non-trivial.

$$ax + by = (x+y)\frac{a+b}{2} + (x-y)\frac{a-b}{2} = 0$$

This is non dividing by 0, since our assumption is that $2 \neq 0$!

Thus, since $x+y, x-y$ are independent, we need $\frac{a+b}{2} = 0$ and $\frac{a-b}{2} = 0$. This implies $a + b = 0$ and $a - b = 0 \implies b = -b \implies 2 = 0 \lightning$.

$\implies x, y$ are independent.

QUESTION 7

Let $S \subseteq \mathbb{F}[t]$ be a possibly infinite set containing non-zero polynomials with pairwise different degrees. This is linearly independent. Consider an arbitrary, finite subset $S_0 \subset S$. If S_0 is linearly independent, so is S. Take $f \in S_0$ to have degree n, and assume $f \in \mathrm{Span}(S_0 \setminus f)$. Since $S_0 \subset S$ contains functions of distinct degrees, one writes $f = \sum\limits_{i \in I} a_i t^i$, where $n \notin I$.

However, this is impossible: the summation would have degree $\max(i \in I)$, which is especially $\neq n$, while f has degree n.

$\implies f \notin \mathrm{Span}(S_0 \setminus v)$, so S_0, and thus S, is linearly independent.

QUESTION 8

Consider the set $S \subseteq \mathrm{Sym}_n(\mathbb{F})$ of matrices with all zero entries except at a two particular locations, ij and ji, where a 1 is placed. If $i = j$, i.e. lie on the diagonal, only one 1 will be placed. This set is $S = \{(z_{ij} + z_{ji}) : ij \in I\} \cup \{z_{ij} : ij \in D\}$, where I is the set of coordinates in the upper triangle ($i \neq j$), and D is the set of coordinates on the diagonal ($i = j$).

Notice also that all elements of S are symmetric: $(z_{ij} + z_{ji}) = (z_{ji} + z_{ij})$, where $i \neq j$, and $z_{ij} = z_{ii} = z_{ji}$ where $i = j$. The following is a rough enumeration of these elements:

$$S = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right\}$$

This is independent. Write an arbitrary linear combination in S as follows:

$$\sum_{ij \in I} \lambda_{ij}(z_{ij} + z_{ji}) + \sum_{ij \in D} \lambda_{ij}(z_{ij}) = \mathbb{0}_{\mathrm{Sym}_n(\mathbb{F})}$$

This is exactly the following matrix:

$$\begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} & \cdots & \lambda_{n1} \\ \lambda_{21} & \lambda_{22} & \lambda_{32} & \cdots & \lambda_{n2} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \cdots & \lambda_{n3} \\ \vdots & \vdots & \vdots & \ddots & \\ \lambda_{n1} & \lambda_{n2} & \lambda_{n3} & \cdots & \lambda_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\implies \lambda_{ij} = 0$ for all i, j, so S is linearly independent. Now consider a nonzero symmetric matrix $(a_{ij}) \in \mathrm{Sym}_n(\mathbb{F}) \setminus S$, and let $\varphi(ij)$ be a function returning the value of (a_{ij}) at each coordinate. Then we have:

$$(a_{ij}) - \sum_{ij \in I} \varphi(ij)(z_{ij} + z_{ji}) - \sum_{ij \in D} \varphi(ij)(z_{ij}) = \mathbb{0}_{\mathrm{Sym}_n(\mathbb{F})}$$

This is a nontrivial linear combination, so $S \cup \{(a_{ij})\}$ is linearly dependent.

$\implies$ S is maximally independent, and thus a basis for $\mathrm{Sym}_n(\mathbb{F})$.

QUESTION 9

Consider t and e^t in the vector space of continuous functions, $C[\mathbb{R}]$, over the field $\mathbb{R}$. Let $at + be^t = 0$, and assume WLOG that one of $a, b \neq 0$. Choose $t = 0$. Then $b = 0$. We update our assumption to find $at = 0 \implies a = 0$, since t is arbitrary. $a = b = 0$, so t, e^t are linearly independent.

t and e^t do not form a basis. Consider $t^2 = at + be^t$. At $t = 0$ we find that $b = 0$. We update our assumption to $t^2 = at$. Choosing $t = 1$ yields $a = 1$, and we are left with $t^2 = t$, which is clearly not true in generality. Thus, we've found $f \in C[\mathbb{R}]$ such that $f \notin \text{Span}(t, e^t)$, so $\{t, e^2\}$ does not span $C[\mathbb{R}]$, and is not a basis.

t^2 is continuous