DISCRETE MATH NOTES

NICHOLAS HAYEK

Lectures by Prof. Sergie Norin

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I Graphs

DEFINITIONS

Graph theory is the study of pairwise relations between objects, e.g. computer networks, interpersonal relationships, airport flights. Graphs will help us simplify and abstract networks .

A *graph* G is comprised of a set of vertices, denoted $V(G)$, where $|V(G)| < \infty$, a set of edges, denoted $E(G)$, where every edge is associated with two vertices. At least in this course

We say that an edge *joins* the two vertices it's associated with. Similarly, an edge is *incident* to a vertex which is an end of it. Conversely, a vertex is incident to an edge if it is an end of it.

Two vertices are *adjacent* or *neighbors* if they are joined by an edge, and a vertex has *degree* edges incident to it.

The *null graph* is the graph such that $V(G) = \varnothing$. The *complete graph* on n vertices, denoted K_n, is such that $|V(K_n)| = n$ and $|E(K_n)|$ is maximal.

PROP 1.1 For a graph of n vertices, the maximal number of edges it may have is $\binom{n}{2}$.

PROOF. Suppose every vertex is connected to every other vertex. Then $\sum_{v \in V(G)} \deg(v) = n(n-1) \implies |E(G)| = \frac{n(n-1)}{2} = \binom{n}{2}$. □

A graph of n vertices, where v_i is only adjacent to v_{i-1} and v_{i+1}, is called a *path* and is sometimes denoted P_n. v_1 and v_n are called the ends of P_n.

For $n \geq 3$, a *cycle* C_n is a graph with $V(G) = \{v_1, ..., v_n\}$ and $E(G) = \{v_1 v_2, v_2 v_3, ..., v_{n-1} v_n, v_n v_1\}$.

An *adjacency matrix* is a matrix contained all vertices on both axis. Pairwise adjacency is denoted by a 1 entry, and 0 otherwise. For example, the following is an adjacency table for a 4 element cycle:

	v_1	v_2	v_3	v_4
v_1	×	1	0	1
v_2	1	×	1	0
v_3	0	1	×	1
v_4	1	0	1	×

Similarly, an *incidence* matrix has rows in $V(G)$ and columns in $E(G)$, and marks with 1 pairs which are incident to eachother. The following is the incidence matrix for a 4 element cycle:

	v_1	v_2	v_3	v_4
e_1	1	1	0	0
e_2	0	1	1	0
e_3	0	0	1	1
e_4	1	0	0	1

Prop 1.1 For a graph G, we always have $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

Proof. Every edge has two vertices incident to it. Thus, $\sum \deg(v)$ will be the number of times an edge is incident to a vertex, i.e. the number of edges × 2. □

H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

For two graphs G, H, the union $G \cup H$ is a graph such that $V(G \cup H) = V(G) \cup V(G)$ and $E(G \cup H) = E(G) \cup E(H)$. We similarly define the intersection $G \cap H$ to be such that $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$.

We cannot do the same for "$G \setminus H$," since we may delete vertices and keep their incident edges!

Prop 1.3 There are $2^{\binom{n}{2}}$ graphs with n vertices.

Proof. We know the maximal number of edges of this graph is $\binom{n}{2}$. Then, for each edge, one may make a binary choice whether to include it or not $\therefore$ the number of graphs is $2^{\binom{n}{2}}$. □

An *isomorphism* between H and G is a bijection $\varphi : V(G) \to V(H)$ such that $uv \in E(G) \implies \varphi(u)\varphi(v) \in E(H)$..

We can now ask: how many graphs are there with n vertices up to isomorphism?

Connectivity

A *walk* in G with ends u_0 and u_k is a sequence $(u_0, u_1, ..., u_k)$ such that $u_i \in V(G)$ and $u_i u_{i+1} \in E(G)$. The length of this walk is k.

u and v are called *connected* if there exists a walk in G with ends u and v OR, equivalently, there exists a path $P \subseteq G$ with ends u and v.

Prop 1.4 $\exists$ a walk in G with ends u and $v \iff \exists$ a path $P \subseteq G$ with ends u and v.

Proof. ($\Longleftarrow$) Let $P \subseteq G$ be a path with ends u and v. Then $V(P)$ can be numbered $u = v_0, v_1, ..., v_k = v$, where $v_i v_{i+1} \in E(P)$. Then $(v_0, ..., v_k)$ is a walk in G.

($\implies$) Let there exist a walk $(u = v_0, ..., v_k = v)$ with $v_i v_{i+1} \in E(G)$. WLOG suppose this is the walk of minimal length. If $v_i \neq v_j$, i.e. are pairwise distinct, then we already have a path. Suppose otherwise, and let $v_i = v_j$. Then

$(v_0, ..., v_i, v_{j+1}, ..., v_k)$ is a *smaller* walk with ends u and v, which establishes the contradiction $\lightning$. $\square$

A graph G is called *connected* if $\forall u, v \in V(G)$, u and v are connected.

A *partition* of $V(G)$ is $(X_1, ..., X_k)$ such that $\cup_{i=1}^{k} X_i = V(G)$ and $X_i \cap X_j = \emptyset \; \forall i \neq j$.

Prop 1.5

A graph G is not connected $\iff$ $\exists$ a partition (X, Y) of $V(G)$ such that no edge of G is incident to one vertex in X and one in Y.

Proof.

$(\impliedby)$ Suppose G were connected. Then choose $u \in X, v \in Y$ such that there exists a walk $(u = u_0, ..., u_k = v)$. Let u_i be minimal over i such that $u_i \in Y$. Then $u_{i-1} \in X$, and $u_{i-1}u_i \in E(G)$ $\lightning$.

$(\implies)$ Let $u, v \in V(G)$ be such that there is no walk from u to v. Let X be the set of all $w \in V(G)$ such that $\exists$ a walk with ends u and w. Similarly, let $Y = V(G) \setminus X$. Clearly $V(G) = X \cup Y$, $X \cap Y = \emptyset$, and (X, Y) is a partition. Suppose there exists an edge from a vertex in X to a vertex in Y, i.e. $x \in X$, $y \in Y$. Then we have the walk $(u, ..., w, ..., x, y)$. But $y \notin X$ $\lightning$. $\square$

Let G be a graph. $H \subseteq G$ is called a *connected component* of G if H is a maximal connected subgraph of G, i.e. if $\exists H \subseteq H' \subseteq G$ with H' connected, then $H = H'$.

Prop 1.6

If H_1, H_2 are connected graphs, and $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is also connected.

Proof.

Let $u \in H_1, v \in H_1 \cap H_2, w \in H_2$. Then $(u, ..., v)$ and $(v, ..., w)$ are both walks, and thus $(v, ..., v, ..., w)$ is a walk. $\square$

Prop 1.7

Every $v \in V(G)$ is a member of a unique connected component $H \subseteq G$.

Proof.

$\{v\}$ is connected. If there does not exist $H \supseteq \{v\}$ also connected, then we are done. Otherwise, we may choose the maximal such connected superset.

Suppose $v \in H_1$ and H_2, two connected components. Then by Prop 1.6, $H_1 \cup H_2$ is connected. But since $H_1 \cup H_2 \supseteq H_1, H_2$, this violates maximality. We conclude that $H_1 = H_2$. $\square$