Stochastic Processes

MATH 447

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CONTENTS

We assume working knowledge of probability and no knowledge of measure theory (though a grasp of analysis is essential). See these MATH 356 notes, also taught by Louigi!

I Markov Chains

Conditional expectations will be important in this course. Recall $\mathbb{E}[X|Y = y_0]$, where X, Y are random variables. If Y is continuous, writing $\mathbb{E}[X|Y = y_0] = \frac{\mathbb{P}(X, Y = y_0)}{\mathbb{P}(Y = y_0)}$, will not work. Instead, we consider the slice of the joint density function $f(x, y)$ at $y = y_0$. The result is a one dimensional function $g(x)$ which may not have probability 1. Hence, we divide by $\int g(x)$ to make it into a density function:

$$\mathbb{E}[X|Y = y_0] = \int_{\mathbb{R}} \frac{f(x, y_0)}{\int_{\mathbb{R}} f(x, y_0)dx} xdx$$

DEF 1.1 We frequently write $f_{X|Y}(x) = f(x, y) / \int_{\mathbb{R}} f(x, y)dx$, and call this the *conditional density* of X given Y. For fixed y, then, $\mathbb{E}[X|Y = y] = \mathbb{E}[Z]$, where $Z \sim f_{X|Y}$.

INTRODUCTION

Before providing definitions, we give some examples of stochastic processes:

Eg. 1.1 A simple random walk: $S_{i+1} = S_i + X_i$, where $X_i \sim \text{Ber}(p)$ and $S_0 = 0$. We might ask: does S_i ever return to 0, i.e.

$$\mathbb{P}(\exists i > 0 : S_i = 0)$$

Eg. 1.2 A branching process: as in asexual reproduction, we have an initial node. Each node n has a number of children X_n, where $\frac{X_n}{2} \sim \text{Ber}(p)$. We denote Z_i to be the number of inididuals in the i-th generation. We might ask: does Z_i ever have no children, i.e.

$$\mathbb{P}(\exists i > 0 : Z_i = 0)$$

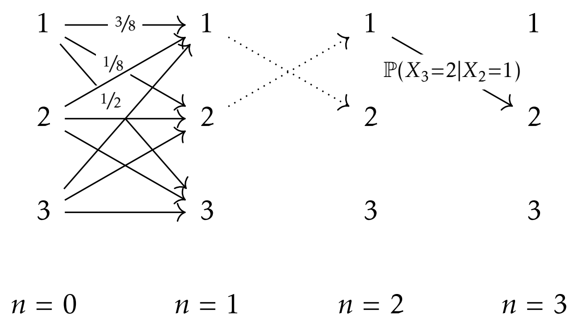
Eg. 1.3 Choose k independent random points in the square $[0, \sqrt{k}]^2$. On average, then, there is 1 point within any unit square $U \subseteq [0, \sqrt{k}]^2$.

DEF 1.2 Given a finite or countable set V, a *Markov chain* with *state space* V is a sequence $X_n : n \geq 0$
DEF 1.3 of random variables, with $X_n \in V$, such that:

$$\mathbb{P}(\underbrace{X_{n+1} = v_{n+1}}_{\text{future}} | \underbrace{X_0 = v_0, ..., X_{n-1} = v_{n-1}}_{\text{past}}, \underbrace{X_n = v_n}_{\text{present}}) = \mathbb{P}(X_{n+1} = v_{n+1} | X_n = v_n)$$

In other words, the future only depends on the past via the present. This is called the
DEF 1.4 *Markov property*.

Sometimes we allow Markov chains to be only finitely large (i.e. $0 \leq n \leq m$). For instance, we limit ourselves to one weekend of gambling in Las Vegas. A graphical example would look something like:



By repeated Bayes' Law, we observe PROP 1.1

$$
\begin{aligned}
&\mathbb{P}(X_1 = v_1, ..., X_n = v_n | X_0 = v_0) \\
=&\mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_0 = v_0, X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_0 = v_0, ..., X_{n-1} = v_{n-1}) \\
=&\mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_{n-1} = v_{n-1}) \quad \text{by Markov property}
\end{aligned}
$$

TIME-HOMOGENEOUS MARKOV CHAINS

We often write THMC

We say that a Markov chain is *time-homogeneous* if, for all $u, v \in V$ and $n \geq 0$ DEF 1.5

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u)$$

In other words, the chain's behavior is described entirely by $\mathbb{P}(X_1 = v | X_0 = u)$ for each $(v, u) \in V \times V$. In this case, we can describe such probabilities in a *transition matrix* P: DEF 1.6

$$P = \left(p_{u,v}\right)_{(u,v)\in V^2} = \left(\mathbb{P}(X_1 = v | X_0 = u)\right)_{(u,v)\in V^2}$$

Eg. 1.4 Recall the game Snakes and Ladders. A 6×6 grid is indexed $1, ..., 36$. Players start at the 1 cell. The game ends when a player reaches the 36 cell. A die roll dictates how many spots one advances. The are some directed edges between cells (increasing: "ladders", decreasing: "snakes"). One must follow these edges when one lands at its tail. Suppose a ladder exists from 11 to 27. Then

$$\mathbb{P}(X_{11} = 27 | X_{10} = 6, X_9 = 3) = \frac{1}{6} = \mathbb{P}(X_{11} = 27 | X_{10} = 6) = \mathbb{P}(X_2 = 27 | X_1 = 6)$$

We see that Snakes and Ladders is naturally modeled as a time-homogeneous Markov chain.

Eg. 1.5 Sampling without replacement is *not* a Markov chain. If we sample from

$|X| = 10$, we have

$$\mathbb{P}(X_3 = a \,|X_2 = b) = 1/9$$
$$\mathbb{P}(X_3 = a \,|X_2 = b, X_1 = c) = 1/8$$
$$\mathbb{P}(X_3 = a \,|X_2 = b, X_1 = a) = 0$$

so we do not satisfy the Markov property.

Eg. 1.6 Returning to the Snakes and Ladders example, consider $S \subseteq V$. Let $T_S = \inf\{n \geq 0 : X_n \in S\}$, which we call the "*hitting time*" of S. We may ask... DEF 1.7

- What is the average number of rounds to finite? We can write this as $\mathbb{E}[T_{\{36\}}|X_0 = 1]$.
- What is the probability of landing on 18 or 19 before the game ends? We can write this as $\mathbb{P}(T_{\{18,19\}} < T_{\{36\}}|X_0 = 1)$.
- What is the average number of visits to 6 before the game ends? We can write this as
$$\mathbb{E}[\#\{n \in [T_{\{36\}}] : X_n = 6\}|X_0 = 1]$$
- What is the expected proportion of time spent on state 5 before the game ends?
- If we allow two players, what is the probability that player 1 wins? Is this still a Markov chain?

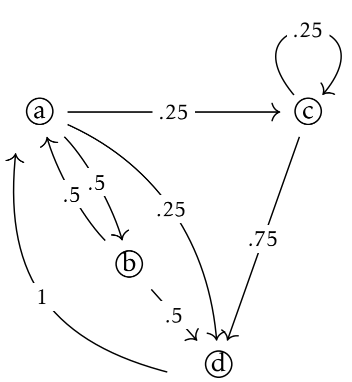
DEF 1.8 A matrix $P = \left(p_{u,v}\right)_{(u,v)\in V^2}$ is called a *stochastic matrix* if every row sums to 1, i.e.

$$\forall\ u \in V, \sum_{v \in V} p_{u,v} = 1$$

Note that any stochastic matrix is the transition matrix of some time-homogeneous Markov chain with state space V and transition probabilities

$$\mathbb{P}(X_{n+1} = v|X_n = u) = \mathbb{P}(X_1 = v|X_0 = u) = p_{u,v}$$

A directed graph, together with its stochastic matrix, can visualize any THMC:



$$\longleftrightarrow \begin{bmatrix} 0 & .5 & .25 & .25 \\ .5 & 0 & 0 & .5 \\ 0 & 0 & .25 & .75 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Eg. 1.7 Random walks on an undirected weighted graph, where edge weights dictate the proportional probability of transitioning between two states, are a special class of THMCs. In particular, given a graph $G = (V, E)$ with weights $w(e) > 0 : e \in E$, we set

$$p_{u,v} = \frac{w(\{u, v\})}{\sum_{z \in N(u)} w(\{u, z\})}$$

If there are no edges $u \leftrightarrow v$, we write $p_{u,v} = 0$.

Not every THMC can be represented by a random walk on an undirected weighted graph. In particular, see the directed graph listed above, or any transition matrix which is not symmetric.

As a concrete example, we can consider a random walk on the number line $\mathbb{Z}$, where, if $w(k, k+1) = \alpha$, $w(k-1, k) = \frac{\alpha}{2}$.

$$\cdots \overset{\frac{1}{16}}{\text{——}} -3 \overset{\frac{1}{8}}{\text{——}} -2 \overset{\frac{1}{4}}{\text{——}} -1 \overset{\frac{1}{2}}{\text{——}} 0 \overset{1}{\text{——}} 1 \overset{2}{\text{——}} 2 \overset{4}{\text{——}} 3 \overset{8}{\text{——}} \cdots$$

Multi-Step Transition Probabilities

Given a THMC $X = X_n : n \geq 0$ with a transition matrix P, we write

$$\begin{aligned}
\mathbb{P}(X_2 = v | X_0 = u) &= \sum_{w \in V} \mathbb{P}(X_2 = v, X_1 = w | X_0 = u) \\
&= \sum_{w \in V} \mathbb{P}(X_1 = w | X_0 = u) \mathbb{P}(X_2 = v | X_1 = w, \cancel{X_0 = u}) \quad \text{by Markov property} \\
&= \sum_{w \in V} p_{u,w} p_{w,v} = (P^2)_{u,v} \quad \text{or write } P^2_{u,v}
\end{aligned}$$

Hence, to determine a two-step transition probability, and by extension an n-step transition probability from u to w, we consider $P^n_{u,v}$. **PROP 1.2**

PROOF.

See Prop 1.1 to expand probabilities, using Bayes', as needed. We get that

$$\begin{aligned}
\mathbb{P}(X_n = v | X_0 = u) &= \sum_{v_1, \ldots, v_{n-1} \in V} \mathbb{P}(X_1 = v_1, \ldots, X_{n-1} = v_{n-1}, X_n = v | X_0 = u) \\
&= \sum_{v_1, \ldots, v_{n-1} \in V} p_{u,v_1} p(v_1, v_2) \cdots p(v_{n-1}, v) = (P^n)_{u,v} \qquad \square
\end{aligned}$$

Thus, if P is a stochastic matrix, then so is P^n. **PROP 1.3**

PROOF.

$\sum_{v \in V} P^n_{u,v} = \sum_{v \in V} \mathbb{P}(X_n = v | X_0 = u) = 1.$ $\square$

Theorem 1.1 Markov Property

If $X_n : n \geq 0$ is a THMC with state space V, then for all $u_0, ..., u_{n-1}, u, v \in V$,

$$\mathbb{P}(X_{n+m} = v | X_0 = u_0, ..., X_{n-1} = u_{n-1}, X_n = u) = \mathbb{P}(X_{n+m} = v | X_n = u) = P^m_{u,v}$$

PROOF. One shows this by combining the Markov property with Prop 1.2 via induction. □

Somewhat nonsensically, we *also* call this the Markov property. When talking about THMCs, this will be the default notion.

DEF 1.9 We say that a Markov chain has an *initial distribution* $\alpha = (\alpha_v : v \in V)$ if $\mathbb{P}(X_0 = v) = \alpha_v$ for each $v \in V$. If this is the case, we often write α as a subscript of our state probabilities. For instance,

$$\mathbb{P}_\alpha(X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u, X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u)\mathbb{P}_\alpha(X_n = v | X_0 = u) = \sum_{u \in V} \alpha_u P^n_{u,v}$$

PROP 1.4 For any event E depending only on $X_0, ..., X_n$, with $\mathbb{P}(X_n = u, E) > 0$, we have

$$\mathbb{P}(X_{n+m} = v | X_n = u, E) = P^m_{u,v}$$

PROOF. For any such event E, we can determine whether E occurs exactly when we know the realized values u_i of X_i for $i = 1, ... n-1$. Hence, we may write $\mathcal{S}$ to be the set of tuples $(u_0, ..., u_{n-1})$ that guarantee E. It follows that

$$\mathbb{P}(X_n = u, E) = \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(\mathbf{x} = \mathbf{s}, X_n = u)$$

Similarly, we have

$$\begin{aligned}
\mathbb{P}(X_{n+m} = v, X_n = u, E) &= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v, X_n = u, \mathbf{x} = \mathbf{s}) \\
&= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v | X_n = u, \mathbf{x} = \mathbf{s}) \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) \\
&= P^m_{u,v} \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) = P^m_{u,v} \mathbb{P}(X_n = u, E)
\end{aligned}$$

Divide and use Bayes, and the result follows. □

PROP 1.5 If X is a THMC with transition matrix P, then, for all $k \geq 1$, $X_{kn} : n \geq 0$ is a THMC with transition matrix P^k.

PROOF.

For any $n \neq 0$, any sequence $u_0, \ldots, u_{n+1} \in V$ satisfies

$$\mathbb{P}(X_{(n+1)k} = u_{n+1} | X_0 = u_0, X_k = u_1, \ldots, X_{nk} = u_n) = P^k_{u_n, u_{n+1}} \qquad \square$$

Theorem 1.2 Chapman-Kolmogorov

For any Markov chain X with state space V, any $m, n \geq 0$, and $u, v \in V$,

$$\mathbb{P}(X_{m+n} = v | X_0 = u) = \sum_{w \in V} \mathbb{P}(X_n = w | X_0 = u) \mathbb{P}(X_{m+n} = v | X_n = w)$$

If the X is time homogeneous, then this is $P^{n+m}_{u,v}$, which agrees with Prop 1.2.

LONG TERM BEHAVIOR

Recall from probability the *law of large numbers*: if $Y_n : n \geq 1$ are IID with common mean μ, then $\frac{S_n}{n} \to \mu$ in probability, where $S_n = \sum_{i=1}^n Y_i$, i.e. $\forall \varepsilon > 0$, DEF 1.10

$$\limsup_{n \to \infty} \mathbb{P}\left(\left| \frac{S_n}{n} - \mu \right| > \varepsilon \right) = 0$$

If $Y_i \in \mathbb{Z}$ then, for $k, \ell, u_i \in \mathbb{Z}$ and $i = 1, \ldots, n-1$,

$$\begin{aligned}\mathbb{P}(S_{n+1} = \ell | S_n = k, S_i = u_i \ \forall i) &= \mathbb{P}(Y_{n+1} = \ell - k | S_n = k, S_i = u_i \ \forall i) \\ &= \mathbb{P}(Y_{n+1} = \ell - k | Y_1 = u_1 - u_0, Y_2 = u_2 - u_1, \ldots, Y_n = k - u_{n-1}) \\ &= \mathbb{P}(Y_{n+1} | \ell - k) = \mathbb{P}(Y_1 = \ell - k) = P_{k,\ell}\end{aligned}$$

where $S_n : n \geq 0$ has transition matrix P, noting that it may be viewed as a THMC.

From now on, we denote by $\mathbb{P}_v(E)$ the probability $\mathbb{P}(E|v)$.

Eg. 1.8 A general two-state chain, with states A and B, can be described by

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Let $q_n = \mathbb{P}_A(X_n = A) = \mathbb{P}(X_n = A | X_0 = A)$. Then

$$\begin{aligned}q_{n+1} &= \mathbb{P}_A(X_{n+1} = A, X_n = A) + \mathbb{P}_A(X_{n+1} = A, X_n = B) \\ &= \mathbb{P}_A(X_{n+1} = A | X_n = A)\mathbb{P}_A(X_n = A) + \mathbb{P}_A(X_{n+1} = A | X_n = B)\mathbb{P}_A(X_n = B) \\ &= (1-\alpha)q_n + \beta(1-q_n) = \beta + (1-\alpha-\beta)q_n\end{aligned}$$

This recurrence has a unique solution. In particular, one can find

$$q_n = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^n \frac{\alpha}{\alpha+\beta}$$

It follows that $q_n \to \frac{\beta}{\alpha+\beta}$, and hence $\mathbb{P}_A(X_n = B) = 1 - q_n \to \frac{\alpha}{\alpha+\beta}$. Likewise:

$$\mathbb{P}_B(X_n = B) = \frac{\alpha}{\alpha+\beta} + (1_\alpha - \beta)^n \frac{\beta}{\alpha+\beta}$$

So $\mathbb{P}_B(X_n = B) \to \frac{\alpha}{\alpha+\beta}$.

Let $\pi := (\pi_A, \pi_B) := \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$ be the distribution of our initial state X_0, Then

$$\mathbb{P}_\pi(X_1 = A) = \pi_A \mathbb{P}_A(X_1 = A) + \pi_B \mathbb{P}_B(X_1 = A) = \pi_A$$

and, similarly, $\mathbb{P}_\pi(X_1 = B) = \pi_B$. Hence, if X_0 has initial distribution π, then X_1 also has distribution π. By induction, X_n has distribution π $\forall n \geq 0$.

When we say $X = \text{Markov}(P)$, we mean that X is a THMC with transition matrix P.

DEF 1.11 A probability distribution π is called *stationary* if $\pi P = \pi$. Similarly, a probability
DEF 1.12 distribution λ is called a *limiting distribution* if, for each $u, v \in V$

$$(P^n)_{u,v} \to \lambda_v \text{ as } n \to \infty$$

In other words, $\mathbb{P}_u(X_n = v) \to \lambda_v$. Note that, for any initial distribution α, we have $\alpha P^n \to \lambda$, i.e. $(\alpha P^n)_v \to \lambda_v$, where λ is limiting.

PROP 1.6 If λ is a limiting distribution for P, then λ is stationary for P.

PROOF. Fix any initial distribution α, we have

$$\lambda = \lim_{n\to\infty}(\alpha P^n) = \lim_{n\to\infty}(\alpha P^{n-1}P) = (\lim_{n\to\infty} \alpha P^{n-1})P = \lambda P \qquad \square$$

Stationary distributions need not be unique, but limiting distributions are (as the limit $\lim_{n\to\infty} \alpha P^n$ is well-defined). In general, then, stationary distributions need not be limiting distributions.

When $n = 0$, $P^n = I$, which encapsulates the idea that, at timestep 0, we will be at our initial positions.

DEF 1.13 A stochastic matrix P is called *regular* if $\exists n \geq 1$ such that $P^n > 0$ on all entries.

Theorem 1.3 Fundamental Theorem of Markov Chains

Every finite, regular stochastic matrix P has a limiting distribution π.

Incorporating some of the formulations above, this is equivalent to saying: *For a regular stochastic matrix, there exists a unique distribution $\pi = (\pi_v : v \in V)$ such that $\pi P = \pi$ and $\mathbb{P}_u(X_n = v) \to \pi_v$ $\forall u, v \in V$.*

In this case, there is a unique stationary distribution, and it is the unique limiting distribution.

A stationary distribution always exists!

Let $\rho = \langle 1, \ldots, 1\rangle$. Then note that $P\rho = \rho$, since the sum of any row in P must be 1. Hence, P has eigenvalue 1. It follows that it has a left eigenvector, i.e. $\pi : \pi P = \pi$. This is exactly a stationary distribution, as long as we scale suitably such that π is a distribution.

However, then process of scaling into a distribution is non-trivial. Since π may have negative coordinates, and hence $\sum \pi_i = 0$, we must consider instead $|\pi|$, i.e. prove it is also an eigenvalue.

This is true, but requires the fact that P is stochastic

Periodicity of States

For $u, v \in V$, we say that v is *accessible* from u if $\exists n \geq 0$ such that $(P^n)_{u,v} > 0$. Equivalently, in the directed graph generated by P, there is a directed path from u to v. When v is accessible from u, we write $u \to v$. DEF 1.14

States u and v *communicate* if $u \to v$ and $v \to u$. When u and v communicate, we write $u \leftrightarrow v$. Observe that communication is a equivalence relation. Hence, the state space V can be written as a disjoint union of mutually-communicating states, called a *communication class*. Note that, in the directed graph generated by P, these correspond to the strongly connected components. DEF 1.15 DEF 1.16

Clearly, if P is regular, then it is irreducible

We say that P is *irreducible* if there is only one communication class. DEF 1.17

$u \to v \iff \mathbb{P}_u(T_{\{v\}} < \infty) > 0$. **PROP 1.7**

The *period* of a state $u \in V$ is DEF 1.18

$$d(u) := \gcd(n > 0 : P^n_{u,u} > 0)$$

If $d(u) = 1$, we call u *aperiodic*. By extension, P is aperiodic if $d(u) = 1 \; \forall u \in V$, and X is aperiodic if $X = \text{Markov}(P)$ for P aperiodic. DEF 1.19

If $u \leftrightarrow v$, then $d(u) = d(v)$. **PROP 1.8**

PROOF. Let $I = \{n > 0 : P^n_{u,u} > 0\}$, and similarly J for v. Hence, $d(u) = \gcd(I)$ and $d(v) = \gcd(J)$. Let $a, b > 0$ such that $P^a_{u,v} > 0$ and $P^b_{v,u} > 0$. Then

$$P^{a+b}_{u,u} \geq P^a_{u,v} P^b_{v,u} > 0$$

$\implies a + b \in I$, so $d(u) | a + b$. Now, if $n \in J$, then

$$P^{a+b+n}_{u,u} \geq P^a_{u,v} P^n_{v,v} P^b_{v,u} > 0$$

$\implies a + b + n \in I$, so $d(u) | n + a + b$. But, by the previous line, $d(u) | n$. Since $n \in J$ is arbitrary, we can write $d(u) | \gcd(J) = d(v)$.

Symmetrically, we could conclude that $d(v) | d(u)$, so indeed $d(v) = d(u)$. □

Let $I = \{n > 0 : P^n_{u,u} > 0\}$. If $\gcd(I) = 1$, then $\exists a, b \in I$ such that $\gcd(a, b) = 1$. **PROP 1.9**

PROOF. This is not true for any I (and thus relies not only on number theory). Let $\ell, m \in I$, with $\ell < m$. Let $k = m - \ell$. If $k = 1$, then $\gcd(\ell, m) = 1$. Otherwise, since $\gcd(I) = 1$, there is an $n \in I$ with $k \nmid n$. We then write $n = qk + r$, with $r \in [k - 1]$. Then $m' \in (q + 1)m \in I$, since $P^{(q+1)m}_{u,u} \geq (P^m_{u,u})^{q+1}$. Symmetrically, we can argue $\ell' = (q + 1)\ell \in I$.

Similarly, $\ell^* := \ell' + n \in I$, since $P^{\ell'+n}_{u,u} \geq P^{\ell'}_{u,u} P^n_{u,u}$. We have

$$\begin{aligned} m' - \ell^* &= (q+1)m - (q+1)\ell - n = (q+1)(m-\ell) - n \\ &= (q+1)k - n = k - r \in [k-1] \end{aligned}$$

TODO... □

Theorem 1.4 Postage Stamp Lemma

If P is irreducible and aperiodic, then $\forall u, v \in V, \exists N$ such that $P^n_{u,v} > 0 \; \forall n \geq N$.

Before proving this, we note that, for $a, b \geq 1$ with $\gcd(a, b) = 1$, then for any $q \geq ab$, we can write $q = ja + kb$ for integers $j, k \geq 0$.

PROOF. Fix $u, v \in V$. Since P is aperiodic, there are $a, b \geq 1$ with $P^a_{u,u}, P^b_{u,u} > 0$ and $\gcd(a, b) = 1$, by Prop 1.9. Since P is irreducible, there is some $m > 0$ with $P^m_{u,v} > 0$. Thus, let $N = m + ab$. For any $n \geq N$, let $q = n - m$. We have that $q \geq ab$, so we can find $j, k \geq 0$ with $q = ja + kb$. Then

$$P^n_{u,v} = P^{q+m}_{u,v} = P^{ja+kb+m}_{u,v} \geq P^{ja}_{u,u} P^{kb}_{u,u} P^m_{u,v} \geq (P^a_{u,u})^j (P^b_{u,u})^k P^m_{u,v}$$

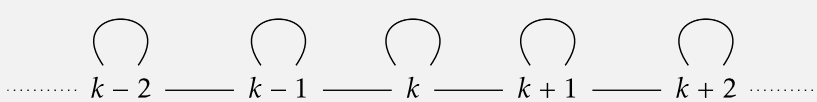
All are positive, so $P^n_{u,v} > 0$, as desired. □

Theorem 1.5 Characterization of Regular Markov Chains

Let $P = (p_{u,v})_{u,v \in V}$ be a stochastic matrix, where $|V| < \infty$. Then

$$P \text{ is regular} \iff P \text{ is irreducible and aperiodic}$$

PROOF. We first note why finiteness is necessary. Consider:



with all edges having weight 1. This graph is clearly aperiodic and irreducible, but not regular.

($\implies$) We start with the "easy" direction. If P is regular, then $\exists n > 0$ s.t. $P^n_{u,v} > 0$ for all $u, v \in V$. Then, for all $u, v \in V$, we have $u \to v$ and $v \to u$. Hence, P is irreducible. Now, if P is irreducible, then for all $u \in V$, there is some $v \in V$ such that $P_{v,u} > 0$ (think about this in graph theoretic terms). Then, let $n > 0$ be such that $P^n_{u,u}$ is positive. We have

$$P^{n+1}_{u,u} \geq P^n_{u,v} P_{u,v} > 0$$

So, with $I = \{m > 0 : P^m_{u,u} > 0\}$, $d(u) = \gcd(I) \leq \gcd(n, n+1) = 1$. It follows that

$d(u) = 1$, so u is aperiodic (and hence P is aperiodic).

($\Longleftarrow$) By Thm 1.4, for each $u, v \in V$, there exists $N : P^n_{u,v} > 0\ \forall n \geq N$. Let N^* be the maximum value of N determined over all pairs $(u, v) \in V^2$. Then, for $n \geq N^*$ and all $u, v \in V$, $P^n_{u,v} > 0$. It follows that all entries of P^n are positive, and we are done. $\square$

Finding Stationary Distributions

Recall that $x = (x_v : v \in V)$ is a stationary distribution if $xP = x$. Let V be finite. Then, for a stationary distribution x, we have

$$\begin{aligned} x_1 p_{1,1} + \cdots + x_n p_{n,1} &= x_1 \\ x_1 p_{1,2} + \cdots + x_n p_{n,2} &= x_2 \\ &\vdots \\ x_1 p_{1,n} + \cdots + x_n p_{n,n} &= x_n \end{aligned}$$

We have n equations, n unknowns, and a homogeneous system, so there is not a unique solution. If

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

We can compute $x = \langle t, 2t, 2t \rangle$. But, noting that x is a probability distribution, and hence $5t = 1$, this yields $x = \langle 1/5, 2/5, 2/5 \rangle$. We'll consider some special cases.

UNDIRECTED GRAPHS

This is distinct from Example 1.7

Let $G = (V, E)$ be undirected. Then we define a THMC by

$$P_{u,v} = \begin{cases} \frac{1}{\deg(u)} & \{u, v\} \in E \\ 0 & \text{o.w.} \end{cases}$$

Let $x = (\deg(v) : v \in V)$. We have

$$\begin{aligned} (xP)_v &= \sum_{u \in V} \deg(u) P_{u,v} = \sum_{u \in N(v)} \deg(u) \cdot \frac{1}{\deg(u)} \\ &= \deg(v) \end{aligned}$$

Hence, $xP = x$. Recalling that $\sum_{v \in V} \deg(v) = 2|E|$, we conclude that

$$\left(\frac{\deg(v)}{2|E|} : v \in V \right)$$

is a stationary distribution.

UNDIRECTED WEIGHTED GRAPHS

Let $G = (V, E)$ be undirected. Then, we define a THMC by

This is not distinct from Example 1.7

$$P_{u,v} = \begin{cases} \frac{w(\{u,v\})}{\sum_{z \in N(u)} w(\{u,z\})} & v \in N(u) \\ 0 & \text{o.w.} \end{cases}$$

Let $x = (\sum_{e:e \ni v} w(e) : v \in V)$. Then we can compute $xP = x$, and similar to above,

$$x = \left(\frac{\sum_{e:e \ni v} w(e)}{2 \sum_{e \in E} w(e)} : v \in V \right)$$

is a stationary distribution.

Transience and Recurrence

Recall $T_S = \inf\{n \geq 0 : X_n \in S\}$, the "hitting time" of S. We let $R_S = \inf\{n > 0 : X_n \in S\}$.
DEF 1.20 Note that if $T_S > 0$, $T_S = R_S$. Otherwise, R_S gives the first "*return time*" to the set S.

DEF 1.21 A state $v \in V$ is called *recurrent* if $\mathbb{P}_v(R_{\{v\}} < \infty) = 1$. If all states of v are recurrent, we may
DEF 1.22 P and $X = \text{Markov}(P)$ recurrent. Otherwise, we call v *transient*, and similarly extend the notion to the transition matrix and chain when all state are transient.

DEF 1.23 For a given state $v \in V$, we call $L_v = |\{n \geq 0 : X_n = v\}|$ the *local time* of v. This notion is not probabilistic: we simply consider a realized walk on the chain (or a part of the chain). Note that, if $v = X_j$ and v is recurrent, then $L_v = \infty$.

PROP 1.10 Let $X = \text{Markov}(P)$. For any state $v \in V$ and $k > 1$,

$$\mathbb{P}_v(L_v > k) = \mathbb{P}_v(L_v > 1)^k$$

Intuitively, if $L_v > k$ when $X_0 = v$, then $L_v > k - 1$ when $X_{i_1} = v$, where i_1 is the first time we return to v.

PROOF. Using the law of total probability:

$$\begin{aligned} \mathbb{P}_v(L_v > k) = \mathbb{E}\left[\mathbb{P}_v(L_v > k | R_v)\right] &= \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t)\mathbb{P}_v(L_v > k | R_v = t) \\ &= \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t)\mathbb{P}_v(L_v > k | R_v = t, X_t = v) \\ &= \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t)\mathbb{P}_v(L_v > k - 1) \\ &= \mathbb{P}_v(L_v > k - 1) \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t) \\ &= \mathbb{P}_v(L_v > k - 1)\mathbb{P}_v(R_v < \infty) = \mathbb{P}_v(L_v > k - 1)\mathbb{P}_v(L_v > 1) \end{aligned}$$

As $R_v = t \iff R_v = t \wedge X_t = v$

The result follows by induction. $\square$

PROP 1.11

$$\mathbb{P}_v(L_v = \infty) = \begin{cases} 1 & v \text{ recurrent} \\ 0 & v \text{ transient} \end{cases}$$

PROOF. This follows directly from Prop 1.10 + monotonicity of probability. □

PROP 1.12

$$\sum_{n=0}^{\infty} P^n_{v,v} = \begin{cases} \infty & v \text{ recurrent} \\ \frac{1}{1-\mathbb{P}_v(R_{\{v\}}<\infty)} & v \text{ transient} \end{cases}$$

PROOF. This follows from linearity of expectation, and the fact that, for a non-negative integer variable Z,

$$\mathbb{E}[Z] = \sum_{k=0}^{\infty} \mathbb{P}(Z > k)$$

In particular... [TODO] □

PROP 1.13 If $u \leftrightarrow v$, then u is transient $\iff$ v is transient.

PROOF. Fix $a, b \geq 0$ with $P^a_{u,v}, P^b_{v,u} > 0$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} P^n_{v,v} \geq \sum_{n=0}^{\infty} P^{a+b+n}_{v,v} &= \sum_{n=0}^{\infty} P^b_{v,u} P^n_{u,u} P^a_{u,v} \\ &= P^b_{v,u} P^a_{u,v} \sum_{n=0}^{\infty} P^n_{u,u} \end{aligned}$$

Thus, if v is transient, then $\sum_{n=0}^{\infty} P^n_{v,v} < \infty$, so it must be that $\sum_{n=0}^{\infty} P^n_{u,u} < \infty$, i.e. u is transient. The argument is identical in reverse. □

Eg. 1.9 If $u \leftrightarrow v$ and u is recurrent, then $\mathbb{P}_u(T_{\{v\}} < \infty) = 1$.

Eg. 1.10 The following chain is completely transient:

$$\begin{tikzcd} \cdots \arrow[r, bend left] & -2 \arrow[l, bend left] \arrow[r, bend left, "2/3"] & -1 \arrow[l, bend left, "1/3"] \arrow[r, bend left, "2/3"] & 0 \arrow[l, bend left, "1/3"] \arrow[r, bend left, "2/3"] & 1 \arrow[l, bend left, "1/3"] \arrow[r, bend left, "2/3"] & 2 \arrow[l, bend left, "1/3"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \end{tikzcd}$$

In fact, we could replace 2/3 by p, and 1/3 by $1 - p$. In this case, the chain is irreducible. To see that it is transient, we have

$$P^{2n}_{0,0} = \binom{2n}{n} p^n (1-p)^n$$

Then

$$\sum_{n=0}^{\infty} P_{0,0}^{2n} < \sum_{n=0}^{\infty} 2^{2n} p^n (1-p)^n = \sum_{n=0}^{\infty} (4p(1-p))^n < \infty \quad \text{if } p \neq \frac{1}{2}$$

By Prop 1.12. Notice that $\sum_{n=0}^{\infty} P_{0,0}^{n} = \sum_{n=0}^{\infty} P_{0,0}^{2n}$, since it is only possible to return on even-length cycles.

We conclude that the chain is transient when $p \neq \frac{1}{2}$.

FACT Stirling's Formula provides

$$m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}$$

in that

$$\lim_{m\to\infty} \frac{m!}{\left(\frac{m}{e}\right)^m \sqrt{2\pi m}} = 1$$

This fact implies

$$e\left(\frac{n}{e}\right)^n \leq n! \leq \frac{e(n+1)}{4}\left(\frac{n+1}{e}\right)^n$$

These facts, though out of the scope of this course, can be derived from a careful treatment of Reimann sums

Eg. 1.11 We return to the previous example, letting $p = \frac{1}{2}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} P_{0,0}^{2n} &= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}[p(1-p)]^n \sim \sum_{n=0}^{\infty} \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi(2n)}}{\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right]^2}[p(1-p)]^n \\ &= \sum_{n=0}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty\end{aligned}$$

We conclude that the chain is recurrent when $p = \frac{1}{2}$.

PROP 1.14 If V is finite, then there is at least one recurrent state.

PROOF. Fix an initial distribution $\alpha = (\alpha_v : v \in V)$. Then $\mathbb{P}_\alpha\left(\sum_{v\in V} L_v = \infty\right) = 1$. We conclude that there is at least one state $v \in V$ with $\mathbb{P}_\alpha(L_v = \infty) > 0$. But also:

$$\begin{aligned}\mathbb{P}_\alpha(L_v = \infty) = \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty, T_v = n) &= \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty | T_v = n)\mathbb{P}_\alpha(T_v = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty | T_v = n, X_n = v)\mathbb{P}_\alpha(T_v = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty | X_n = v)\mathbb{P}_\alpha(T_v = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_v(L_v = \infty)\mathbb{P}_\alpha(T_v = n)\end{aligned}$$

So $\mathbb{P}_v(L_v = \infty) > 0 \implies \mathbb{P}_v(L_v = \infty) = 1$, by Prop 1.11. □

Finite, irreducible chains are recurrent. **PROP 1.15**

PROOF. Since the chain is finite it has at least one recurrent state, by Prop 1.14. Then all states must be recurrent, since the chain is irreducible, by Prop 1.13. □

Canonical Decompositions

Fix a transition matrix P and list the communication classes of V as

$$D_1, D_2, \ldots \quad \text{(transient)} \qquad C_1, C_2, \ldots \quad \text{(recurrent)}$$

Note that we can split the chain up in this way by Prop 1.13. Set $D = \cup_{i \geq 0} D_i$. Then the *canonical decomposition* of the chain is defined to be DEF 1.24

$$D \sqcup C_1 \sqcup C_2 \sqcup \cdots$$

We say that a communication class C is *closed* if, for any $u \in C, v \notin C, p_{u,v} = 0$. Intuitively, if $X_0 \in C$, or we enter C at some later time, we will never leave C. DEF 1.25

If C is a recurrent communication class, then C is closed. **PROP 1.16**

PROOF. Fix $u \in C, v \notin C$. Suppose $v \mapsto u$. If $p_{u,v} > 0$, then $v \mapsto v$, so $v \in C$ ↯. Suppose $v \not\mapsto u$. Then $\mathbb{P}_u(R_u = \infty) \geq \mathbb{P}_u(X_1 = v) = p_{u,v}$. But $\mathbb{P}_u(R_u = \infty) = 0$, since u is recurrent. It follows that $p_{u,v} = 0$. □

The converse of Prop 1.16 is not true in generality, but it is in the finite case:

Finite, closed communication classes are recurrent. **PROP 1.17**

PROOF. From any starting state in C, we must visit some state $u \in C$ infinitely often, as $|C| < \infty$ and $X_t \in C \ \forall t$. But recurrence is a class property by Prop 1.13. Hence, all of C is recurrent. □

When our communication classes are closed, we have

$$\begin{tikzcd} & & D \arrow[dll, Rightarrow, bend right] \arrow[d, Rightarrow] \arrow[drr, Rightarrow, bend left] & & & \\ C_1 & & C_2 & & C_3 & \cdots \end{tikzcd}$$

Proof of Fundamental Theorem of Markov Chains

Recall Thm 1.3:

Every finite, regular stochastic matrix P has a limiting distribution π.

We will prove this in two steps. First, we will find some stationary distribution. Then, we will prove that this is a limiting distribution.

Theorem 1.6 Existence Theorem

Let P be irreducible and recurrent. Let $(X_n : n \geq 0) = \text{Markov}(P)$. Fix $u \in V$, which we call a reference vertex, and, for any $v \in V$, define

$$\gamma_v = \mathbb{E}_u[|\{0 \leq n < R_u : X_n = v\}|]$$

Let $\gamma = (\gamma_v : v \in V)$. Then $\gamma P = \gamma$, and $0 < \gamma_v < \infty\ \forall v \in V$.

By $\mathbb{E}_u$, we mean the expectation, under the assumption that $X_0 = u$

PROOF. Observe that $\gamma_u = 1$. Write

$$\gamma_v = \mathbb{E}_u\left[\sum_{n=0}^{R_u-1} \mathbb{1}_{X_n=v}\right] = \mathbb{E}_u\left[\sum_{n=1}^{R_u} \mathbb{1}_{X_n=v}\right] = \mathbb{E}_u\left[\sum_{n=1}^{\infty} \mathbb{1}_{X_n=v}\mathbb{1}_{n\leq R_u}\right]$$

We utilize a second indicator variable in order to use linearity of expectation (otherwise, our sum would index over a random variable, which is not valid). Then

$$\gamma_v = \sum_{n=1}^{\infty} \mathbb{E}_u[\mathbb{1}_{X_n=v}\mathbb{1}_{n\leq R_u}] = \sum_{n=1}^{\infty} \mathbb{P}_u(X_n = v, n \leq R_u)$$

Now, using the law of total probability,

$$\begin{aligned}
\mathbb{P}_u(X_n = v, n \leq R_u) &= \sum_{w\in V} \mathbb{P}_u(X_{n-1} = w, X_n = v, n \leq R_u) \\
&= \sum_{w\in W} \mathbb{P}_u(X_n = v | X_{n-1} = w, n \leq R_u)\mathbb{P}_u(X_{n-1} = w, n \leq R_u) \\
&= \sum_{w\in V} \mathbb{P}_u(X_{n-1} = w, n \leq R_u)P_{w,v}
\end{aligned}$$

So

$$\begin{aligned}\gamma_v &= \sum_{n=1}^{\infty}\sum_{w\in W} P_{w,v}\mathbb{P}_u(X_{n-1}=w, n\le R_u) = \sum_{w\in V} P_{w,v}\left(\sum_{n=1}^{\infty}\mathbb{P}_u(X_{n-1}=w, n-1<R_u)\right)\\ &= \sum_{w\in V} P_{w,v}\sum_{n=0}^{\infty}\mathbb{P}_u(X_n=w, n<R_u) = \sum_{w\in V} P_{w,v}\sum_{n=0}^{\infty}\mathbb{E}_u[\mathbb{1}_{X_n=w}\mathbb{1}_{n<R_u}]\\ &= \sum_{w\in V} P_{w,v}\mathbb{E}_u\left[\sum_{n=0}^{R_u-1}\mathbb{1}_{X_n=w}\right] = \sum_{w\in V} P_{w,v}\gamma_w = (\gamma P)_v\end{aligned}$$

$\implies \gamma P = \gamma$. Furthermore, $\gamma_v > 0$, since $u \mapsto v$. Letting $n : P^n_{v,u} > 0$, we have $\gamma_u = (\gamma P^n)_u \ge \gamma_v P^n_{v,u}$. Noting that $\gamma_u = 1$, this shows $\gamma_v < \infty$. □

PROP 1.18 As a corollary, under the same conditions, if $\mathbb{E}_u[R_u]$ is finite, $\pi = (\pi_v : v \in V)$, where

$$\pi_v = \frac{\gamma_v}{\sum_{w\in V}\gamma_w}$$

is a stationary distribution.

PROOF. Observe

$$\sum_{w\in V}\gamma_W = \sum_{w\in V}\mathbb{E}_i\left[\sum_{n=0}^{R_n^{-1}}\mathbb{1}_{X_n=w}\right] = \mathbb{E}_u\left[\sum_{n=0}^{R_n^{-1}}\sum_{w\in V}\mathbb{1}_{X_n=w}\right] = \mathbb{E}_u[R_u]$$

So we require that $\mathbb{E}_u[R_u] < \infty$, then a stationary distribution exists. □

PROP 1.19 An an exercise, if u is recurrent and $u \mapsto v$, then $\mathbb{P}_u(T_v < \infty) = 1\ \forall v \in V$. π, as defined above, is a limiting distribution.

PROOF. Consider two independent copies $X_n, Y_n = \text{Markov}(P)$. Then (X_n, Y_n) is a Markov chain with transition matrix $Q = (q_{(v,w),(x,y)})_{(v,w),(x,y)\in V\times V}$. In particular,

$$q_{(v,w),(x,y)} = p_{v,x}p_{w,y}$$

Fix some state $u \in V$. Let α be the initial distribution that has $X_0 = u$ and $Y_0 \sim \pi$.

$$\alpha_{(x,y)} = \begin{cases}\pi(y) & x = u\\ 0 & \text{o.w.}\end{cases}$$

Remark that, as π is stationary, $Y_n \sim \pi\ \forall n \ge 0$. For any $u \in V$, we'd like that $P^n_{u,v} \to \pi(v)$.

Let $M = \inf\{n \ge 0 : X_n = Y_n\}$ be the first meeting time of the X_n and Y_n chains. If P is finite and regular, then Q is finite and regular. Then

$$\mathbb{P}_\alpha(M < \infty) \geq \mathbb{P}_\alpha(T_{(v,v)} < \infty) = \sum_{(x,y)\in V\times V} \alpha(x,y)\mathbb{P}_{(x,y)}(T_{v,v} < \infty) = 1$$

It follows that $\lim_{n\to\infty} \mathbb{P}_\alpha(M > n) = 0$. We claim that $\mathbb{P}_\alpha(X_n = v, M \leq n) = \mathbb{P}_\alpha(Y_n = v, M \leq n)\ \forall n \geq 0$. Assuming this, then

$$\begin{aligned}
\mathbb{P}_u(X_n = v) = \mathbb{P}_\alpha(X_n = v) &= \mathbb{P}_\alpha(X_n = v, M \leq n) + \mathbb{P}_\alpha(X_n = v, M > n)\\
\mathbb{P}_\pi(Y_n = v) = \mathbb{P}_\alpha(Y_n = v) &= \mathbb{P}_\alpha(Y_n = v, M \leq n) + \mathbb{P}_\alpha(Y_n = v, M > n)\\
\implies \mathbb{P}_u(X_n = v) - \pi(v) &= \mathbb{P}_\alpha(X_n = v, M > n) - \mathbb{P}_\alpha(Y_n = v, M > n)\\
\implies |\mathbb{P}_u(X_n = v) - \pi(v)| &= |\mathbb{P}_\alpha(X_n = v, M > n) - \mathbb{P}_\alpha(Y_n = v, M > n)| \leq \mathbb{P}_\alpha(M > n)
\end{aligned}$$

But $P_\alpha(M > n) \to 0$, so it must be that $\mathbb{P}_u(X_n = v) \to \pi(v)$.

We still must show the claim, however.

$$\begin{aligned}
\mathbb{P}_\alpha(X_n = v, M \leq n) &= \sum_{k=0}^{n}\sum_{w\in V} \mathbb{P}_\alpha(X_n = v, M = k, X_k = w)\\
&= \sum_{k=0}^{n}\sum_{w\in V} \mathbb{P}_\alpha(M = k, X_k = w)\mathbb{P}_\alpha(X_n = v | X_k = w, M = k)\\
&\overset{\text{MP}}{=} \sum_{k=0}^{n}\sum_{w\in V} \mathbb{P}_\alpha(M = k, X_k = w)P^{n-k}_{w,v}\\
&\textit{and now we reverse!}\\
&= \sum_{k=0}^{n}\sum_{w\in V} \mathbb{P}_\alpha(M = k, Y_k = w)P^{n-k}_{w,v}\\
&\overset{\text{MP}}{=} \sum_{k=0}^{n}\sum_{w\in V} \mathbb{P}_\alpha(M = k, Y_k = w)\mathbb{P}_\alpha(X_n = v | Y_k = w, M = k)\\
&= \mathbb{P}_\alpha(Y_n = v, M \leq n) \qquad \square
\end{aligned}$$

Hitting Times and Absorbing States

DEF 1.26 We call a state $v \in V$ *absorbing* if $\mathbb{P}_v(X_1 = v) = 1$.

Eg. 1.12 Consider a game where we continually bet \$1 with winning probability p, until we earn \$$k$ or run out of money. Modeling the state space as our balance, we have absorbing states at \$0 and \$$k$.

Observe that each absorbing state forms a recurrent communication class of size 1. As a consequence, we have a variant of the canonical form, where $A \subseteq V$ are absorbing states, and we write $V = A \sqcup (V \setminus A)$. Schematically,

$$\left[\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right]$$

INDEX OF DEFINITIONS