DISCRETE MATHEMATICS

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I Preliminaries and Trees

DEFINITIONS

Graph theory is the study of pairwise relations between objects, e.g. computer networks, interpersonal relationships, airport flights. Graphs will help us simplify and abstract networks .

A *graph* G is comprised of a set of vertices, denoted $V(G)$, where $|V(G)| < \infty$, a set of edges, denoted $E(G)$, where every edge is associated with two vertices. At least in this course

We say that an edge *joins* the two vertices it's associated with. Similarly, an edge is *incident* to a vertex which is an end of it. Conversely, a vertex is incident to an edge if it is an end of it.

Two vertices are *adjacent* or *neighbors* if they are joined by an edge, and a vertex has *degree* edges incident to it.

The *null graph* is the graph such that $V(G) = \varnothing$. The *complete graph* on n vertices, denoted K_n, is such that $|V(K_n)| = n$ and $|E(K_n)|$ is maximal.

PROP 1.1 For a graph of n vertices, the maximal number of edges it may have is $\binom{n}{2}$.

PROOF. Suppose every vertex is connected to every other vertex. Then $\sum_{v \in V(G)} \deg(v) = n(n-1) \implies |E(G)| = \frac{n(n-1)}{2} = \binom{n}{2}$. $\square$

A graph of n vertices, where v_i is only adjacent to v_{i-1} and v_{i+1}, is called a *path* and is sometimes denoted P_n. v_1 and v_n are called the ends of P_n.

For $n \geq 3$, a *cycle* C_n is a graph with $V(G) = \{v_1, ..., v_n\}$ and $E(G) = \{v_1 v_2, v_2 v_3, ..., v_{n-1} v_n, v_n v_1\}$.

An *adjacency matrix* is a matrix contained all vertices on both axis. Pairwise adjacency is denoted by a 1 entry, and 0 otherwise. For example, the following is an adjacency table for a 4 element cycle:

	v_1	v_2	v_3	v_4
v_1	$\times$	1	0	1
v_2	1	$\times$	1	0
v_3	0	1	$\times$	1
v_4	1	0	1	$\times$

Similarly, an *incidence* matrix has rows in $V(G)$ and columns in $E(G)$, and marks with 1 pairs which are incident to eachother. The following is the incidence matrix for a 4 element cycle:

	v_1	v_2	v_3	v_4
e_1	1	1	0	0
e_2	0	1	1	0
e_3	0	0	1	1
e_4	1	0	0	1

Prop 1.2

For a graph G, we always have $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

Proof.

Every edge has two vertices incident to it. Thus, $\sum \deg(v)$ will be the number of times an edge is incident to a vertex, i.e. the number of edges $\times$ 2. $\square$

H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

We cannot do the same for "$G \setminus H$," since we may delete vertices and keep their incident edges!

For two graphs G, H, the union $G \cup H$ is a graph such that $V(G \cup H) = V(G) \cup V(G)$ and $E(G \cup H) = E(G) \cup E(H)$. We similarly define the intersection $G \cap H$ to be such that $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$.

Prop 1.3

There are $2^{\binom{n}{2}}$ graphs with n vertices.

Proof.

We know the maximal number of edges of this graph is $\binom{n}{2}$. Then, for each edge, one may make a binary choice whether to include it or not $\therefore$ the number of graphs is $2^{\binom{n}{2}}$. $\square$

We can now ask: how many graphs are there with n vertices up to isomorphism?

An *isomorphism* between H and G is a bijection $\varphi : V(G) \to V(H)$ such that $uv \in E(G) \implies \varphi(u)\varphi(v) \in E(H)$..

Connectivity

A *walk* in G with ends u_0 and u_k is a sequence $(u_0, u_1, ..., u_k)$ such that $u_i \in V(G)$ and $u_i u_{i+1} \in E(G)$. The length of this walk is k.

u and v are called *connected* if there exists a walk in G with ends u and v OR, equivalently, there exists a path $P \subseteq G$ with ends u and v.

Prop 1.4

$\exists$ a walk in G with ends u and $v \iff \exists$ a path $P \subseteq G$ with ends u and v.

Proof.

($\impliedby$) Let $P \subseteq G$ be a path with ends u and v. Then $V(P)$ can be numbered $u = v_0, v_1, ..., v_k = v$, where $v_i v_{i+1} \in E(P)$. Then $(v_0, ..., v_k)$ is a walk in G.

($\implies$) Let there exist a walk $(u = v_0, ..., v_k = v)$ with $v_i v_{i+1} \in E(G)$. WLOG suppose this is the walk of minimal length. If $v_i \neq v_j$, i.e. are pairwise distinct, then we already have a path. Suppose otherwise, and let $v_i = v_j$. Then

$(v_0, ..., v_i, v_{j+1}, ..., v_k)$ is a *smaller* walk with ends u and v, which establishes the contradiction $\lightning$. □

A graph G is called *connected* if $\forall u, v \in V(G)$, u and v are connected.

A *partition* of $V(G)$ is $(X_1, ..., X_k)$ such that $\cup_{i=1}^{k} X_i = V(G)$ and $X_i \cap X_j = \varnothing \; \forall i \neq j$.

A graph G is not connected $\iff$ $\exists$ a partition (X, Y) of $V(G)$ such that no edge of G is incident to one vertex in X and one in Y. PROP 1.5

PROOF. ($\Longleftarrow$) Suppose G were connected. Then choose $u \in X, v \in Y$ such that there exists a walk $(u = u_0, ..., u_k = v)$. Let u_i be minimal over i such that $u_i \in Y$. Then $u_{i-1} \in X$, and $u_{i-1}u_i \in E(G)$ $\lightning$.

($\Longrightarrow$) Let $u, v \in V(G)$ be such that there is no walk from u to v. Let X be the set of all $w \in V(G)$ such that $\exists$ a walk with ends u and w. Similarly, let $Y = V(G) \setminus X$. Clearly $V(G) = X \cup Y$, $X \cap Y = \varnothing$, and (X, Y) is a partition. Suppose there exists an edge from a vertex in X to a vertex in Y, i.e. $x \in X$, $y \in Y$. Then we have the walk $(u, ..., w, ..., x, y)$. But $y \notin X$ $\lightning$. □

Let G be a graph. $H \subseteq G$ is called a *connected component* of G if H is a maximal connected subgraph of G, i.e. if $\exists H \subseteq H' \subseteq G$ with H' connected, then $H = H'$.

Sometimes we just say "component."

If H_1, H_2 are connected graphs, and $V(H_1) \cap V(H_2) \neq \varnothing$, then $H_1 \cup H_2$ is also connected. PROP 1.6

PROOF. Let $u \in H_1, v \in H_1 \cap H_2, w \in H_2$. Then $(u, ..., v)$ and $(v, ..., w)$ are both walks, and thus $(v, ..., v, ..., w)$ is a walk. □

Every $v \in V(G)$ is a member of a unique connected component $H \subseteq G$. PROP 1.7

PROOF. $\{v\}$ is connected. If there does not exist $H \supseteq \{v\}$ also connected, then we are done. Otherwise, we may choose the maximal such connected superset.

Suppose $v \in H_1$ and H_2, two connected components. Then by Prop 1.6, $H_1 \cup H_2$ is connected. But since $H_1 \cup H_2 \supseteq H_1, H_2$, this violates maximality. We conclude that $H_1 = H_2$. □

Let G be a graph, and let $H \subseteq G$ be a non-null and connected subgraph. Then H is a connected component of G $\iff$ $\forall e \in E(G)$ with an end in $V(H)$, we have $e \in E(H)$. PROP 1.8

PROOF. For ($\Longrightarrow$), let $e = uv$, with $u \in V(H)$. If $v \in V(H)$, then we are done. Otherwise, suppose $e \notin E(H)$. We know v is a member of a unique connected component. But adding e to H would yield a further connected graph: take the graphs of $\{uv\}$ and H. Both are clearly connected, so $H \cup \{uv\}$ is connected.

($\Longleftarrow$) Proof idea.

Obtained from G by deleting e

For $e \in E(G)$, $G \setminus e$ is a graph such that $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.

Similarly, for $v \in V(G)$, $G \setminus v$ is a graph such that $V(G \setminus v) = V(G) \setminus \{v\}$ and $E(G \setminus v) = E(G) \setminus \{e : e \text{ incident to } v\}$.

Let comp(G) = # of connected components of G.

Prop 1.9

comp$(G) = 1 \iff G$ is connected.

Proof.

($\Longrightarrow$) direction is trivial. For ($\Longleftarrow$), if G is connected, then there cannot exist a more maximal connected subgraph, e.g. G is a connected component. Since every vertex belongs to a unique connected component, and this must be G, comp$(G) = 1$. □

Let $e = \{u, v\} \in E(G)$. Define a *cut-edge* to be an edge which is not part of any cycle.

Prop 1.10

Exactly one of the following holds:

1. ***e* is a cut-edge**: comp$(G \setminus e)$ = comp$(G) + 1$, and u, v belong to different components of $G \setminus e$.

2. ***e* is not a cut-edge**: comp$(G \setminus e)$ = comp(G), and u, v belong to the same component.

Proof.

Let e be a cut-edge. Let $H_1, ..., H_k$ be the connected components of $G \setminus e$. If u, v belong to H_i, then $\exists$ a path $P \subseteq H_i$ with ends u and v. Adding e, this is a cycle ↯.

WLOG, assume that u, v belong to $V(H_1), V(H_2)$, respectively. Then let H' be obtained by $H_1 \cup H_2$ by adding e. We claim that $H', H_2, ..., H_k$ are all components of G. By Prop 1.8, we only need to check the connectivity of H', and this holds by Prop 1.6. Since there do not exist any vertices *not* in $V(H_i) : i \geq 2$ or $V(H')$, these are all the components of G. Thus, comp$(G) + 1$ = comp$(G \setminus e)$. □

Trees and Forests

A *forest* is a graph with no cycles, i.e. every edge is a cut-edge.

A *tree* is a non-null connected forest.

Prop 1.11

Let F be a non-null forest. Then comp$(F) = |V(F)| - |E(F)|$.

Proof.

We'll show by induction on $|E(F)|$. If $n = 0$ then all vertices are their own connected components. Let $|E(F)| = n$, and assume $\text{comp}(F) = |V(F)| - |E(F)|$. Let $e \in E(F)$. Since F is a forest, e is a cut-edge, and thus $\text{comp}(G \setminus e) = \text{comp}(G) + 1 = |V(F)| - |E(F)| + 1 = |V(F)| - (|E(F)| - 1) = |V(F)| - |E(F \setminus e)| = |V(F \setminus e)| - |E(F \setminus e)|$. □

A *leaf* is a vertex with degree 1.

Let T be a tree with $|V(T)| \geq 2$. let $X = \{\text{leaves of } T\}$, $Y = \{v \in V(G) : \deg(v) \geq 3\}$. Then $|X| \geq |Y| + 2$.

PROP 1.12

Thus, trees have ≥ 2 leaves!

PROOF.

By Prop 1.1, we have

$$\sum_{v \in V(T)} \deg(v) = 2|E(T)| \overset{1.11}{=} 2(|V(T)| - \text{comp}(G)) \overset{1.9}{=} 2(|V(T)| - 1)$$

$$\implies \sum_{v \in V(T)} (\deg(v) - 2) = 2(|V(T)| - 1) - 2|V(T)| = -2$$

$$= \underbrace{\sum_{v \in X}(\deg(v) - 2)}_{=-|X|} + \underbrace{\sum_{v \in Y}(\deg(v) - 2)}_{\geq |Y|} + \underbrace{\sum_{v \in V(T) - X - Y}(\deg(v) - 2)}_{=0}$$

$$\implies -2 \geq -|X| + |Y| \implies |X| \geq |Y| + 2 \quad \square$$

A note for the following few proofs: if w is a leaf, then any path which exists in T (with ends not w) exists in $T \setminus w$.

Let T be a tree with 2 leaves, u and v. Then T is a path with ends u and v.

PROP 1.13

PROOF.

Let $P \subseteq T$ be a path with ends u and v. By Prop 1.12, $\deg_T(w) = 2 \ \forall w \in V(P) \setminus \{u, v\}$. Moreover, $\deg_T(w) = \deg_P(w)$, so no vertex in $V(P)$ is incident to an edge in $E(T) \setminus E(P)$. Then, by Prop 1.8, P is a connected component. But T is connected, so $T = P$. □

Let T be a tree and $v \in V(T)$ be a leaf. Then $T \setminus v$ is a tree.

PROP 1.14

PROOF.

$T \setminus v$ is non-null, since v has a neighbor. $T \setminus v$ has no cycles, since T has no cycles, and $T \setminus V$ is connected. We know there exists a path between any two vertices in $V(T) \setminus \{v\}$. Such a path still exists. □

If G is a graph, $v \in V(G)$ a leaf, and $G \setminus v$ a tree, then G is a tree.

PROP 1.15

PROOF.

G is non-null, since $G \setminus v$ is non-null. We know that v belongs to no cycles, since it is a leaf, so any cycles apparent in G would exist in $G \setminus v$. Thus, G has no cycles. For connectedness, let H be the graph containing v, its incident edge, and that edge's other vertex v'. H is connected, as is $G \setminus v$, and $G \setminus v \cap H \neq \emptyset$, so $G \setminus v \cup H = G$ is connected by Prop 1.6. □

PROP 1.16 Let T be a tree, $u, v \in V(T)$. Then T contains a unique path with ends u and v.

PROOF. We'll show by induction on $|V(T)|$. This clearly holds for $|V(T)| = 1$. Let $|V(T)| \geq 2$. Suppose T contains a leaf $w \in V(T) \setminus \{u, w\}$. Then $T \setminus w$ is a tree by Prop 1.14. By our induction hypothesis, $T \setminus w$ contains a unique path with ends u and v. By connectedness, $\exists$ a path with ends u, v in T. But this path must exist in $T \setminus w$, whose uniqueness follows.

If no such leaf exists, then T has exactly 2 leaves (u and v). Thus, by Prop 1.13, T is a path with ends u and v, and thus the only path in T. □

SPANNING TREES

Let G be a graph. A subgraph $T \subseteq G$ is called a *spanning tree* of G if T is a tree and $V(T) = V(G)$.

PROP 1.17 Let G be connected and non-null. Let $H \subseteq G$, chosen minimal such that $V(H) = V(G)$ and H is connected. Then H is a spanning tree of G.

PROOF. We only need to check that T is non-null and contains no cycles. The first is automatic, since $V(T) = V(G)$, and G is non-null. If H has a cycle, then let e be an edge in the cycle. $H \setminus e$ is connected by Prop 1.9 and Prop 1.10. But this contradicts minimality, so T contains no cycles. □

PROP 1.18 Let G be a connected non-null graph. Let $H \subseteq G$ be maximal such that H contains no cycles. Then H is a spanning tree of G.

PROOF. We need to show that $V(H) = V(G)$ and H is connected (it is non-null, since at least a singleton of G contains no cycles; it contains no cycles by construction). If $\exists v \in V(G) \setminus V(H)$, adding v such that $\deg(v) = 0$ would maintain H having no cycles, thus contradicting maximality.

Suppose H is not connected. Then by Prop 1.5 there exists a partition $H = X \cup Y$ such that no edge has a vertex in both X and Y. However, such an edge must exist in G, say $e \in E(G)$, so we may add this edge to H to produce H'. Observe that H' contains no cycles, since e belongs to no cycles in H. But this contradicts maximality, so H must contain no cycles. □

Let T be a spanning tree of G. Let $f \in E(G) \setminus E(T)$. Then T with f has one cycle (by Prop 1.16). This is called the *fundamental cycle* of f with respect to T, and denoted $\mathrm{FC}(T, f)$.

Prop 1.19

Let T be a spanning tree of G, $f \in E(G) \setminus E(T)$. Let $C = \mathrm{FC}(T, f)$, $e \in E(C)$. Then $(T + f) \setminus \{e\}$ is a spanning tree.

Proof.

Let $T' = (T + f) \setminus \{e\}$. $T + f$ is connected, and since e is not a cut-edge, $(T + f) \setminus \{e\} = T'$ is also connected. C is a unique cycle in $T + f$, so T' contains no cycles. Thus, T' is a tree. $V(T') = V(T) = V(G)$, since T is a spanning tree, so we conclude that T' is a spanning tree. $\square$