

# Vectors

## Definition 1

An *inner product* on a vector space is such that

1.  $\langle u, v \rangle = \langle v, u \rangle$
2.  $\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$
3.  $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
4.  $\langle u, u \rangle \geq 0$ , and  $\langle u, u \rangle = 0 \iff u = 0$

## Definition 2

A *norm* is  $\|v\| = \sqrt{\langle v, v \rangle}$ .

## Definition 3

A *line*  $l(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  is  $l(t) = P + td$ , where  $P, d \in \mathbb{R}^n$ . It may also be given by  $l(t) = (1-t)Q + tP$ , where  $P, Q \in \mathbb{R}^n$ , and  $t \in [0, 1]$ .

## Definition 4

A *plane*  $p(s, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is  $p(s, t) = P + sd_1 + td_2$ , where  $d_1, d_2 \in \mathbb{R}^3$ .

It may also be given in point-normal form,  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ , where  $\langle a, b, c \rangle \in \mathbb{R}^3$  is normal to the plane, and  $(x_0, y_0, z_0) \in \mathbb{R}^3$  lies on the plane.

## Definition 5

A *linear transformation*  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is such that  $\lambda(\vec{0}) = \vec{0}$ , and  $\lambda(\vec{a} + \vec{b}) = \lambda(\vec{a}) + \lambda(\vec{b})$ .

Alternatively, write  $\lambda(x_1, \dots, x_n) = x_1 \vec{d}_1 + \dots + x_n \vec{d}_n$ , where  $\vec{d}_i \in \mathbb{R}^m$ .

## Definition 6

An *affine transformation*  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation plus a point  $P \in \mathbb{R}^m$ .

## Definition 7

The *projection* of  $v$  onto  $u$ , denoted  $\text{proj}_u(v)$ , is given by  $(u \cdot v) \frac{u}{\|u\|^2}$ .

## Definition 8

The *tangent line* of  $r : \mathbb{R} \rightarrow \mathbb{R}^n$  at  $a \in \mathbb{R}$  is an affine transformation  $\lambda$  satisfying

$$\lim_{t \rightarrow a} \frac{\|r(t) - \lambda(t)\|}{|t - a|} = 0$$

$$\lambda(t) = r(a) + (t - a)\vec{d} \quad \vec{d} \neq 0$$

## Prop 1 (Inequalities)

$\|v + u\| \leq \|v\| + \|u\|$  ( $\Delta$ ) and

$|\langle u, v \rangle| \leq \|u\| \|v\|$  (Cauchy-Schwartz).

## Prop 2 (Cross/Dot Products)

$$(u \times v) \cdot u = 0$$

$\|u \times v\|$  is the area of the parallelogram bounded by  $u, v$ .

$$\|u \times v\| = \|u\| \|v\| \sin(\theta)$$

$$u \cdot v = \|u\| \|v\| \cos(\theta)$$

## Prop 3 (Distances)

- (a) The distance between a point  $R$  and a plane may be given by  $\|\text{proj}_{\vec{n}}(P - R)\|$ , where  $P, \vec{n}$  are as in point-normal form.
- (b) The distance between skew lines may be given by projecting [a third line which contains points of both] onto [the normal vector of the skew lines].
- (c) The distance between  $R$  and  $PQ$  may be given by  $\|PR - \text{proj}_{PQ}(R)\|$ .

# Differentiation

## Definition 1

- (a)  $r : \mathbb{R} \rightarrow \mathbb{R}^n$  is *differentiable* at  $\vec{a}$  if  $\exists$  a linear transformation  $\lambda : \mathbb{R} \rightarrow \mathbb{R}^n$  such that

$$\lim_{h \rightarrow 0} \frac{\|r(\vec{a} + h) - r(\vec{a}) - \lambda(\vec{h})\|}{|h|} = 0$$

- (b) Similarly,  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  is *differentiable* at  $\vec{a}$  if  $\exists$  a linear transformation  $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|(h, k)\|} = 0$$

- (c) Generally,  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is *differentiable* at  $\vec{a}$  if  $\exists$  a linear transformation  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{\|F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})\|}{\|\vec{h}\|} = 0$$

We denote the derivative  $\lambda(\vec{a})$  as  $DF_{\vec{a}}$

## Definition 2

The *arc length* of a curve  $r(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  is given by

$$s = \int_a^b \|r'(t)\| dt$$

The arc length parameterization is some  $t = \alpha(s)$  such that  $\|r'(\alpha(s))\| = 1$ .

## Definition 3

The *Jacobian* matrix of  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is given by

$$F' = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix}$$

Note that, where  $m > 1$ , each element is a column vector.

## Definition 4

A *level surface* is  $F : \mathbb{R}^n \rightarrow 0$ .

## Definition 5

The *gradient* of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $\left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle$  (i.e. its Jacobian) and denoted by  $\nabla F$ .

## Definition 6

$\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is *continuous* at  $\vec{a}$  if,  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  such that  $\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon$ .

## Definition 7

$F : \mathbb{R}^n \rightarrow \mathbb{R}$  is called  *$C^k$ -continuous* (or  *$k$ -times continuously differentiable*) at  $\vec{a}$  if all  $k^{th}$ -order partial derivatives exist near  $\vec{a}$  and are continuous at  $\vec{a}$ .

## Definition 8

- (a) The  $k^{th}$  *partial derivative* of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  at  $\vec{a}$ , denoted by  $\frac{\partial F}{\partial x_k}(\vec{a})$  is given by

$$\lim_{t \rightarrow 0} \frac{F(\vec{a} + te_k) - F(\vec{a})}{t}$$

where  $e_k$  is the standard basis vector, e.g.  $\vec{i}$  for  $k = 1$ .

- (b) The *directional derivative* of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  along  $\vec{h}$ , denoted by  $\partial_{\vec{h}} F(\vec{a})$ , is given by

$$\lim_{t \rightarrow 0} \frac{F(\vec{a} + t\vec{h}) - F(\vec{a})}{t}$$

- (c) The  $j^{th}$  *iterated directional derivative* of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  at  $\vec{a}$  along  $\vec{h}$ , denoted by  $\partial_{\vec{h}}^j F(\vec{a})$ , is given by  $g^{(j)}(0)$ , where  $g(t) = F(\vec{a} + t\vec{h})$ . Note that  $g'(0) = \partial_{\vec{h}} F(\vec{a})$ .

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## Prop 1 (Differentiability and Partialials)

- (a) If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $\vec{a}$ , then all partial derivatives of  $F$  exist at  $\vec{a}$ .

- (b) If all partial derivatives of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  exist near  $\vec{a}$  and are continuous at  $\vec{a}$ , then  $F$  is differentiable.
- (c) If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $k$ -times continuously differentiable, then all  $(k-1)^{th}$ -order partial derivative are continuously differentiable.

### Prop 2 (Chain Rule)

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^l$  be differentiable at  $\vec{a} \in \mathbb{R}^n$  and  $f(\vec{a}) \in \mathbb{R}^m$ , respectively. Then

$$h = g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^l$$

is differentiable. Furthermore,  $Dh_{\vec{a}} = Dg_{f(\vec{a})} \circ Df_{\vec{a}}$ , and  $h'(\vec{a}) = g'(f(\vec{a}))f'(\vec{a})$ .

### Prop 3 (Iterated Partialials)

- (a) If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $\vec{a}$ , then

$$\partial_{\vec{h}}(\vec{a}) = \vec{h} \cdot \nabla F = \sum_{i=1}^n h_i \frac{\partial F}{\partial x_i}(\vec{a})$$

- (b) If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $k$ -times continuously differentiable, then

$$\partial_{\vec{h}}^j(\vec{a}) = (\vec{h} \cdot \nabla)^j F(\vec{a})$$

### Prop 4 (Mixed Partialials)

If  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $C^2$ -continuous, then  $\partial_x \partial_y F = \partial_y \partial_x F$ .

### Prop 5 (Taylor)

If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^k$  continuous, let

$$\alpha_j(\vec{h}) = \frac{1}{j!} \partial_{\vec{h}}^j(\vec{a}) = \frac{1}{j!} (\vec{h} \cdot \nabla)^j F(\vec{a})$$

Then  $F(\vec{a}) + \sum_{i=1}^k \alpha_i(\vec{x} - \vec{a})$  is the best degree  $k$  approximation of  $F$  near  $\vec{a}$ .

## Integration

### Definition 1

Let  $\mathcal{B} = [a_1, b_1] \times \dots \times [a_n, b_n]$ . Then  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is *integrable* over  $\mathcal{B}$  if

$$\lim_{m \rightarrow \infty} \sum_{i=1}^m F(c_j^m) \frac{\text{vol}(\mathcal{B})}{m^n} \text{ exists}$$

and is equivalent up to choices of  $c_j^m$ , where  $c_j^m \in [a_j^k, b_j^k]$ ,  $k \in [1, m]$  (slice of size  $\frac{1}{m}$ ).

### Definition 2

A set  $S \subseteq \mathbb{R}^n$  has *zero measure* if  $\forall \varepsilon > 0$  we can choose a countable set of open balls such that  $S \subseteq \cup B(x_i, \varepsilon_i)$ , where  $\sum \text{vol} B(x_i, \varepsilon_i) < \varepsilon$ .

### Prop 1 (Integrability)

- (a) If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous on  $\mathcal{B}$ , then  $F$  is integrable on  $\mathcal{B}$ .
- (b) The set of discontinuities of  $F$  in  $\mathcal{B}$  has zero measure if and only if  $F$  is integrable.

### Prop 2 (Fubini)

Let  $\mathcal{B} = [a_1, b_1] \times \dots \times [a_n, b_n]$ . If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous on  $\mathcal{B}$ , then

$$\int_{\mathcal{B}} F = \int_{x_n=a_n}^{x_n=b_n} \dots \left( \int_{x_1=a_1}^{x_1=b_1} F(x_1, \dots, x_n) dx_1 \right) \dots dx_n$$

The order of integration doesn't matter.

### Prop 3 (MVT)

$$\int_a^b g(x) = g(c)(b-a) \text{ for some } c \in [a, b]$$