WRITTEN ASSIGNMENT 1

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Based on lectures by Prof. Eyal Goren

CONTENTS

Submitting Questions 1, 2, 3, 4, 5, 6

QUESTION 1

Assume the principle of induction holds. Consider a non-empty subset of $\mathbb{N}$ with 1 element. Trivially, that element is the least element of $\mathbb{N}$.

$n \to n+1$ step:

Consider a nonempty subset $A \in \mathbb{N}$ with $n+1$ elements. We may split this set into two disjoint sets, $A_1 \cup A_{i \geq 2}$. The singleton A_1 contains one element a_1, and $A_{i \geq 2}$, which contains n elements, has a least element a_2 by our assumption. The least element of A, then, is $\min\{a_1, a_2\}$ $\square$

Question 2

Let $C_1 = 1, C_2 = 2, C_3 = 3$. Conjecture that $C_4 = 2+3 = 5$ and $C_5 = 3+5 = 8$ (it can be shown by counting). We have the recurrence relation $C_n = C_{n-1} + C_{n-2}$. Before proving the discrete equation for C_n, note the following polynomial:

$$\phi^2 - \phi - 1 = 0 \implies \phi = \frac{1 \pm \sqrt{5}}{2}$$

Base cases: $n = 1, 2$:

$$\begin{aligned} C_1 &= \tfrac{1}{\sqrt{5}}\left[\left(\tfrac{1+\sqrt{5}}{2}\right)^2 - \left(\tfrac{1-\sqrt{5}}{2}\right)^2\right] \\ &= \tfrac{1}{\sqrt{5}}\left[\left(1 + \tfrac{1+\sqrt{5}}{2}\right) - \left(1 + \tfrac{1-\sqrt{5}}{2}\right)\right] \\ &= \tfrac{1}{\sqrt{5}}\left(\tfrac{2\sqrt{5}}{2}\right) = 1 \end{aligned}$$

$$\begin{aligned} C_2 &= \tfrac{1}{\sqrt{5}}\left[\left(\tfrac{1+\sqrt{5}}{2}\right)^3 - \left(\tfrac{1-\sqrt{5}}{2}\right)^3\right] \\ &= \tfrac{1}{\sqrt{5}}\left[\tfrac{1+\sqrt{5}}{2} + \left(\tfrac{1+\sqrt{5}}{2}\right)^2 - \tfrac{1-\sqrt{5}}{2} + \left(\tfrac{1-\sqrt{5}}{2}\right)^2\right] \\ &= \tfrac{1}{\sqrt{5}}\left[\sqrt{5} + \left(\tfrac{1+\sqrt{5}}{2}\right)^2 - \left(\tfrac{1-\sqrt{5}}{2}\right)^2\right] \\ &= \tfrac{1}{\sqrt{5}}\left[\sqrt{5} + \sqrt{5}\right] = 2 \end{aligned}$$

Note that step (2) for C_1 uses the polynomial identity, and step (3) for C_2 uses the result from C_1.

Take C_n and C_{n+1} to be true. We'll show $C_{n,\, n+1} \to C_{n+2}$

$$\begin{aligned} C_{n+2} &= C_{n+1} + C_n \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left[\frac{1+\sqrt{5}}{2} + \left(\frac{1+\sqrt{5}}{2}\right)^2\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\left[\frac{1-\sqrt{5}}{2} + \left(\frac{1-\sqrt{5}}{2}\right)^2\right]\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{1-\sqrt{5}}{2}\right)^3\right] \quad \text{from our polynomial} \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+3}\right] = C_{n+2} \end{aligned}$$

QUESTION 3

($\implies$) Let $f : A \to B$ be a bijective function. We can then construct $f^{-1} : B \to A$, where $f^{-1}(b) = a$ if $f(a) = b$.

- Since f is injective, $f^{-1}(b)$ has exactly one solution
- Since f is surjective, the entire domain B of f^{-1} is well defined

$$\implies f^{-1} \text{ is a valid function}$$

- $f(a)$ has exactly one solution, so f^{-1} is injective
- The entire domain of f is defined, so $\forall a \in A \exists b : f(a) = b$, meaning, $\forall a \in A \exists b : f^{-1}(b) = a$, or f^{-1} is surjective.

$$\implies f^{-1} : B \to A \text{ is bijective}$$

$f(f^{-1}(B)) = f(A) = B \implies f \circ f^{-1} = 1_B$

$f^{-1}(f(A)) = f^{-1}(B) = A \implies f^{-1} \circ f = 1_A$

Thus, we've found a suitable g such that the above conditions are satisfied.

($\impliedby$) Let $g : B \to A$ exist such that $f \circ g = 1_B$ and $g \circ f = 1_A$, with $f : A \to B$.

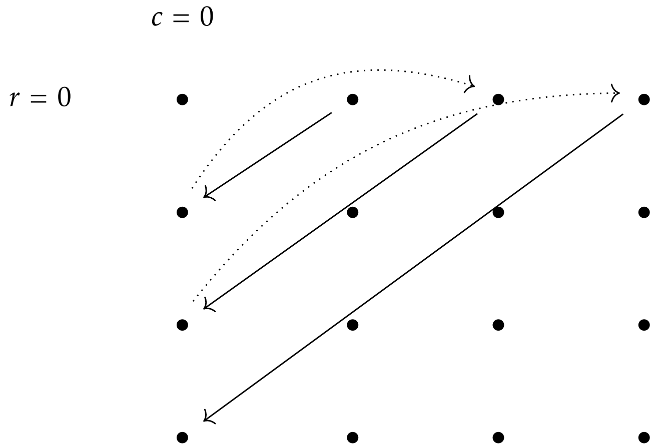
1. $f(g(B)) = B \implies f(C) = B$, where $C \subseteq A$. But we have that f maps all of A, so in fact $f(A) = B$. This implies that $\forall a \in A$ there exists some $c \in C$ such that $f(c) = f(a) = b \in B$. Let $a \neq c$. Then we have $f(g(b) = c) = b$ and $f(g(b) = a) = b$, which cannot happen, since g is well-defined $\lightning$. Thus, $a = c$, $f(A) = B$, and f injective.
2. $g(f(A)) = A \implies g(D) = A$, where $D \subseteq B$. But g maps all of B, and thus $f(A) = B \implies f$ is surjective.

This proves (a). For (b), let $g : B \to A$ be only injective such that $g(B) = C$, a strict subset of A. By our construction, $f(C) = B$ exactly, meaning $\forall a \in A \setminus C \ \exists c \in C : f(a) = f(c) = b$ and $a \neq c$, for some $b \in B$, since f must map all of A. Thus, f cannot be injective, but is surjective. $\square$

c.f. Pigeonhole Principle

QUESTION 4

Consider an n by n grid of lattice points, and denote the first column as $c = 0$ and the first row as $r = 0$. Then, the n^{th} diagonal (connecting $(0, n)$ and $(n, 0)$) will contain $n + 1$ lattice points.



Consider each lattice point in $r = 0$. If we count the total number of lattice points as shown in the diagram, with $(0, 0)$ being the 0^{th} point, we can express a recursive formula for a unique #, a_n, of the point $(0, n)$:

This last identity is well known. We have:

$$\begin{aligned} 2a_n &= 2(1 + 2 + ... + n) \\ &= (1 + 2 + ... + n) \\ &\quad +(n + [n-1] + ... + 1) \\ &= n(n+1) \end{aligned}$$

$$\begin{aligned} a_n &= a_{n-1} + n \quad \text{with } a_0 = 0 \\ &= a_{n-2} + (n-1) + n \\ &\vdots \\ &= a_0 + 1 + 2 + ... + (n-1) + n \\ &= \tfrac{n^2+n}{2} \end{aligned}$$

From here, we can define a general form for the entire grid, $N(m, n)$:

$N(0, c) = a_c$ using the explicit formula derived

$N(1, c) = a_{c+1} + 1$, since our counting procedure "pulls down" from the $c + 1^{th}$ column directly above.

$N(2, c)$ pulls down from $N(1, c)$, and so $N(2, c) = N(1, [c + 1]) + 1 = a_{c+2} + 2$

$N(r, c) = a_{c+r} + r$

$$\implies N(r, c) = \frac{(c + r)^2 + (c + r)}{2} + r$$

Real proof not asked for, though

Our counting procedure ensures that each lattice point is assigned a unique natural number, with no repetitions, and so $N(r, c)$ is bijective.

QUESTION 5

Let $|A_1| = |A_2|$ and $|B_1| = |B_2|$.

Then, $\exists f : A_1 \to A_2$ and $g : B_1 \to B_2$, both bijective.

Let $h(a_1, b_1) = \langle f(a_1), g(b_1)\rangle \;\; \forall (a_1, b_1) \in A_1 \times B_1$ and $\forall \langle f(a_1), g(b_1)\rangle \in A_2 \times B_2$.

Since f and g are injective, we have:

$$f(a_1) = f(a_1') \quad \implies a_1 = a_1'$$

$$g(b_1) = g(b_1') \quad \implies b_1 = b_1'$$

Thus, if $\langle f(a_1), g(b_1)\rangle = \langle f(a_1'), g(b_1')\rangle$, then $a_1 = a_1'$ and $b_1 = b_1' \implies h$ is injective.

Since f and g are surjective, we have:

$$\forall a_2 \in A_2 \, \exists a_1 \quad : f(a_1) = a_2$$

$$\forall b_2 \in B_2 \, \exists b_1 \quad : f(b_1) = b_2$$

And so, $\forall (a_2, b_2) \in A_2 \times B_2 \; \exists (a_1, b_1) \in A_1 \times B_1$ such that

$$\langle f(a_1), g(b_1)\rangle = (a_2, b_2) \implies h \text{ is surjective}$$

h is a bijection between $A_1 \times B_1$ and $A_2 \times B_2$, so

$$|A_1 \times B_1| = |A_2 \times B_2|$$

QUESTION 6

We'll construct two injections to show $|\mathbb{N}| = |\mathbb{Q}|$

From class, note that $|\mathbb{N}| = |\mathbb{Z}|$ and $|\mathbb{Z}| = |\mathbb{Z} \times \mathbb{Z}|$

$\implies |\mathbb{N}| = |\mathbb{Z} \times \mathbb{Z}|$

Let $q \in \mathbb{Q}$ be such that $q = \frac{a}{b}$ in its most reduced form (i.e. for any $a, b \in \mathbb{N}$, a and b have no common factors $\neq 1$). Further, if q is negative, let a be negative and b be positive.

Then, an ordered pair (a, b) defines a unique $q \in \mathbb{Q}$ such that, if $(a, b) = (a', b')$, then $\frac{a}{b} = \frac{a'}{b'} = q$. Clearly, the function $q \to (a, b)$, for all $q \in \mathbb{Q}$, is injective. Thus, $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}| \implies |\mathbb{Q}| \leq |\mathbb{N}|$.

Let $\mathbb{N} \to \mathbb{Q}$ be such that $n \to \frac{1}{n}$, where $\frac{1}{n} \forall n \in \mathbb{N}$ is a defined subset of $\mathbb{Q}$.

$\frac{1}{n} = \frac{1}{n'} \implies 1 = \frac{n}{n'} \implies n' = n \implies$ our function is injective. $|\mathbb{N}| \leq |\mathbb{Q}|$.

By Cantor-Bernstein, we have $|\mathbb{N}| = |\mathbb{Q}|$ $\square$