Quantitative Risk Management

MATH 510

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CONTENTS

This course is based on the ETH course and textbook of the same name, by McNeil, Fray, and Embrechts. In order, we will focus on: stochastic volatility, extreme-value theory, multivariate models, risk aggregation, and backtesting.

Cover: index simulation following the stochastic volatility process

We will assume good working knowledge of probability and statistics. In addition, we remind ourselves of the *survivor function* $\overline{F}(x) := \mathbb{P}(X > x) = 1 - F(x)$, as well as the α-*quantile*, where $\alpha \in (0, 1)$, defined to be

DEF 0.1

DEF 0.2

$$q_\alpha = F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$$

In this course, we would like to quantify one's exposure to bad consequences. The likelihood of loss or less-than-expected gains is called *risk*. The following are types of risk:

DEF 0.3

Risk is necessary: taking this course incurs a risk of a poor grade

Risk	Description
Credit Risk	Odds a debtor defaults on payment
Market Risk	Exposure to price fluctuations of bonds, stocks, or derivatives
Operational Risk	Risk relating to circumstantial adverse events (e.g. institutional fraud)
Liquidity Risk	Risk of damage from not having sufficient assets to pay off debts
Model Risk	Risk associated with financial model inaccuracies; closely related to operational risk
Underwriting Risk	Odds that an insured makes a claim on their policy

The above types of risk interact with each-other. Quantitative risk management aims to model these interactions and hedge against risk.

I Introduction

DEF 1.1

DEF 1.2

A *portfolio* is a collection of assets or liabilities. Denote by V_t the value of the portfolio at time t. We denote by Δt a *time horizon*, i.e. a duration of time. Assuming that V_t is known, that the composition of the portfolio remains constant over the time horizon, and that there are no payments made, we denote by V_{t+1} the value of the portfolio at time $t + \Delta t$.

Portfolios may be stocks, bonds, derivatives, risky loans, or insurance contracts.

Denote by $\Delta V_{t+1} = V_{t+1} - V_t$. It is a random variable which follows a *profit-and-loss distribution*, or *P&L*. Since we care about managing risk in this course, we will primarily consider *loss*, defined as

DEF 1.3

DEF 1.4

Hence, V_{t+k} is the value of the portfolio at time $t + k\Delta t$

$$L_{t+1} = \begin{cases} V_t - V_{t+1} & \text{short horizon} \\ V_t - \frac{V_{t+1}}{1+r_{t,1}} & \text{long horizon} \end{cases}$$

DEF 1.5

We say that L_{t+1} follows the *loss distribution*. By convention, we consider losses positive and profits negative. The loss distribution is typically right-skewed and has a fat right tail: the probability of heavy profits is small, but the probability of heavy losses is comparatively likelier.

Typically, V_t is a function of multiple *risk factors*, usually denoted by $\mathbf{Z}_t = \langle Z_{t,1}, ..., Z_{t,d} \rangle$.   DEF 1.6
Thus, we can view V_t as a function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ with $V_t = f(t, \mathbf{Z}_t)$.

At time t, $\mathbf{Z}_t$ is observable, so we denote the observed value as $\mathbf{z}_t$. Then, $f(t, \mathbf{z}_t)$ is called
the *realized value* of V_t at time t.   DEF 1.7

Eg. 1.1 Consider d stocks ,and let λ_i denote the number of shares in stock i at time t.
Let $S_{t,i}$ denote the price of the stock i. Frequently, we consider the log prices
to be risk factors, i.e.

$$Z_{t,i} = \log S_{t,i} : i \in [d]$$

The value of the portfolio is then

$$V_t = \sum_{i=1}^{d} \lambda_i S_{t,i} = \sum_{i=1}^{d} \lambda_i e^{Z_{t,i}}$$

Risk factors may change over the time horizon. Hence, we consider *risk factor changes*, i.e.   DEF 1.8
$\mathbf{X}_{t+1} = \mathbf{Z}_{t+1} - \mathbf{Z}_t$. We can rearrange our loss function, then:

$$L_{t+1} = -[f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) - f(t, \mathbf{Z}_t)]$$

While modeling loss, $\mathbf{Z}_t = \mathbf{z}_t$ is known, so we are only forced to consider $\mathbf{X}_{t+1}$ as a random
variable. When writing loss as a function of observed risk factor changes, we write
$L_{t+1} = l_{[t]}(\mathbf{x})$, and call it the *loss operator*. While estimating loss, we must consider the   DEF 1.9
multivariate distribution of $\mathbf{X}_{t+1}$, distinguishing between

1. The conditional distribution of risk factor changes, given all information up to time
   t. The resulting loss distribution is called the *conditional loss disrtibution*.   DEF 1.10

2. The stationary distribution if risk factor changes. The resulting loss distribution is
   called the *unconditional loss distribution*.   DEF 1.11

In either case, we may use Taylor approximations (assuming sufficient smoothness of f)
to obtain the *linearized loss* as follows   DEF 1.12

$$L_{t+1}^{\Delta} = -f_t(t, \mathbf{z}_t) - \sum_{i=1}^{d} f_{z_i}(t, \mathbf{z}_t) X_{t+1,i}$$

where $f_\xi = \frac{\partial f(t, \mathbf{z}_t)}{\partial \xi}$.

Eg. 1.2 Following Example 1.1, we compute

$$\mathbf{X}_{t+1} = \left\langle \log \frac{S_{t+1,1}}{S_{t,1}}, ..., \log \frac{S_{t+1,d}}{S_{t,d}} \right\rangle$$

and also rearrange

$$L_{t+1} = V_t - V_{t+1} = -\sum_{i=1}^{d} \lambda_i e^{Z_{t,i}}(e^{X_{t+1,i}} - 1) = -V_t \sum_{i=1}^{d} w_{t,i}(e^{X_{t+1,i}} - 1)$$

DEF 1.13   where $w_{t,i} = \frac{\lambda_i S_{t,i}}{V_i}$ is the *relative weight* of stock i at time t.

INDEX OF DEFINITIONS