ASSIGNMENT 5

MATH 251

NICHOLAS HAYEK

QUESTION 1

The following picture will be useful:

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1k} & c_{11} & c_{12} & \cdots & c_{1l} \\ a_{21} & a_{22} & \cdots & a_{2k} & c_{21} & c_{22} & \cdots & c_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & c_{k1} & c_{k2} & \cdots & c_{kl} \\ \hline 0 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1l} \\ 0 & 0 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & b_{l1} & b_{l2} & \cdots & b_{ll} \end{array}\right]$$

We express the determinant of this matrix as follows, where $M = (m_{ij})$:

$$\det(M) = \sum_{\pi \in S_{k+l}} m_{1\pi(1)} \cdot \ldots \cdot m_{k\pi(k)} m_{(k+1)\pi(k+1)} \cdot \ldots \cdot m_{(k+l)\pi(k+l)} \mathrm{sgn}(\pi)$$

However, if $\pi(i) \in [1,k]$ for some $i \in [k+1, k+l]$, then $m_{i\pi(i)} = 0$, and so $m_{1\pi(1)} \cdot \ldots \cdot m_{i\pi(i)} \cdot \ldots \cdot m_{(k+l)\pi(k+l)} = 0$. Thus, it is sufficient to sum over the set $\{\pi \in S_{k+l} : \pi(i) \in [k+1, k+l] \ \forall i \in [k+1, k+l]\}$. But, if $\pi([k+1, k+l]) = [k+1, k+l]$, then $\pi(i) \in [1,k] \ \forall i \in [1,k]$, since it is a bijection.

We conclude that $m_{i\pi(i)}$ must only define elements either in the A sub-matrix, since π sends $[1,k] \to [1,k]$, or the B sub-matrix, since $[k+1, k+l] \to [k+1, k+l]$. Furthermore, since we are concerned with all permutations π that act, disjointedly, from $[1,k]$ and $[k+1, k+l]$ onto themselves, we can split π piecewise into two permutations $\pi_1, \in S_k$ and $\pi_2 \in S_l$. Our notation adapts as follows:

$$\det(M) = \sum_{\pi_1 \in S_k, \pi_2 \in S_l} a_{1\pi_1(1)} \cdot \ldots \cdot a_{k\pi_1(k)} \cdot b_{1\pi_2(1)} \cdot \ldots \cdot b_{l\pi_2(l)} \mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2)$$

Note, if π has i inversions, then its constituent permutations π_1, π_2 have, together, i inversions. Thus, $\#\pi_1 + \#\pi_2 = \#\pi$, so $\mathrm{sgn}(\pi) = (-1)^{\#\pi_1}(-1)^{\#\pi_2} = \mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2)$.

$$\begin{aligned} \det(M) &= \sum_{\pi_1 \in S_k, \pi_2 \in S_l} a_{1\pi_1(1)} \cdot \ldots \cdot a_{k\pi_1(k)} \cdot b_{1\pi_2(1)} \cdot \ldots \cdot b_{l\pi_2(l)} \mathrm{sgn}(\pi_1)\mathrm{sgn}(\pi_2) \\ &= \sum_{\pi_1 \in S_k} a_{1\pi_1(1)} \cdot \ldots \cdot a_{k\pi_1(k)} \mathrm{sgn}(\pi_1) \sum_{\pi_2 \in S_l} b_{1\pi_2(1)} \cdot \ldots \cdot b_{l\pi_2(l)} \mathrm{sgn}(\pi_2) \\ &= \det(A)\det(B) \end{aligned}$$

Question 2

Let $A \in M_n(\mathbb{F})$, and $t \in \mathbb{F}$. We can express $\det(tI_n - A)$ as

$$\sum_{\pi \in S_n} a_{1\pi(1)} \cdot \ldots \cdot a_{n\pi(n)} \mathrm{sgn}(\pi)$$

Since $a_{i\pi(i)}$ is either a constant or a term like $(t - a_{ii})$, we can be certain that $\det(A)$ is a polynomial. Only one π will yield a summand of degree n, and that is precisely the permutation which yields $a_{1\pi(1)} \cdot \ldots \cdot a_{n\pi(n)} = (t - a_{11}) \cdot \ldots \cdot (t - a_{nn})$, i.e $\pi(i) = i$. Since this permutation has no inversions, $\mathrm{sgn}(\pi) = 1$, and we can further expand:

$$\begin{aligned}(t - a_{11}) \cdot \ldots \cdot (t - a_{nn}) &= t^n + t^{n-1}(-a_{11} - \ldots - a_{nn}) + t^{n-2} \sum_{i,j \leq n} a_{ii}a_{jj} + \ldots + \prod_{i=1}^{n}(-a_{ii}) \\ &= t^n - t^{n-1}\mathrm{tr}(A) + \text{stuff}\end{aligned}$$

To extract the constant term of $\det(tI_n - A)$, we can simply evaluate at $t = 0$, since it is a polynomial:

$$\det(0I_n - A) = \det(-A) = (-1)^n \det(A)$$

Where, by multilinearity, we have

$$\det(-A) = \det\begin{pmatrix} -A_{(1)} \\ \vdots \\ -A_{(n)} \end{pmatrix} = -\det\begin{pmatrix} A_{(1)} \\ -A_{(2)} \\ \vdots \\ -A_{(n)} \end{pmatrix} = \det\begin{pmatrix} A_{(1)} \\ A_{(2)} \\ -A_{(3)} \\ \vdots \\ -A_{(n)} \end{pmatrix} = \ldots = (-1)^n \det\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(n)} \end{pmatrix} = (-1)^n \det(A)$$

Combining with our results above, we have

$$\det(tI_n - A) = t^n - t^{n-1}\mathrm{tr}(A) + \ldots + (-1)^n \det(A)$$

Note: while we know immediately that we've caught all constant and t^n terms, if we went hunting for a t^{n-1} term using some other permutation π, then $\pi(i) = i \ \forall i \in J$, where $|J| = n - 1$. But then, for this last element $k \notin J$, the bijectivity of π dictates that $\pi(k) = k$ as well. Thus, $-\mathrm{tr}(A)t^{n-1}$ are *all* $n-1$ degree terms.

QUESTION 3

Part (a): Let $V = V_1 \oplus ... \oplus V_n$ and let $\beta_i \subseteq V_i$ be a basis for V_i. Then $\beta_1 \cup ... \cup \beta_n$ is linearly independent: let $\beta_i = \{v_{ij} : j \in J_i\}$ and $a_{ij} \in \mathbb{F}$. We write

$$\sum_{\substack{1 \leq i \leq n \\ j \in J_i}} a_{ij} v_{ij} = 0$$

If all $a_{ij} v_{ij} \neq 0$ belong to only one "sub-basis," $\text{Span}(\beta_m)$, then we violate the independence of β_m. Suppose otherwise: that is, there are vectors from at least two "sub-bases." Without loss of generality, suppose $\sum_{j \in J_1} a_{1j} v_{1j}$ is nontrivial. Then:

$$\sum_{\substack{2 \leq i \leq n \\ j \in J_i}} a_{ij} v_{ij} = -\sum_{j \in J_1} a_{1j} v_{1j}$$

Since β_1 forms a basis, the RHS is a unique vector $v_1 \in V_1$, and likewise the LHS are unique vectors $v_2 + ... + v_n$ (at least one non-zero), where $v_i \in V_i$. Thus, $v_1 \in V_1 \cap (V_2 + ... + V_n)$, which is a contradiction $\implies \beta_1 \cup ... \cup \beta_n$ is independent.

$\beta_1 \cup ... \cup \beta_n$ is also spanning: any vector $v \in V$ can be written as $v = \sum_{i=1}^{n} v_i$, where $v_i \in V_i$. Each v_i has a unique representation in β_i, say $\sum_{j \in J_i} a_{ij} v_{ij}$, so we write

$$v = \sum_{\substack{1 \leq i \leq n \\ j \in J_i}} a_{ij} v_{ij} \quad a_{ij} \in \mathbb{F}$$

$\implies v \in \text{Span}(\beta_1 \cup ... \cup \beta_n) \implies \beta_1 \cup ... \cup \beta_n$ is a basis.

Part (b): Since $\beta_i \subseteq V_i$, and $V_i \cap V_j = \{0\}$ for any $i \neq j$, we conclude that β_i contain distinct elements. Thus, $|\beta_1 \cup ... \cup \beta_n| = |\beta_1| + ... + |\beta_n| \implies \dim(V) = \dim(V_1) + ... + \dim(V_n)$, as $\beta_1 \cup ... \cup \beta_n$ is a basis for V by (a).

QUESTION 4

Part (a): Fix $\lambda \in \mathbb{F}$, and let $T(v) = \lambda v$ for some $v \in V$. Then $T(v)\lambda^{-1} = v$. Since T is invertible, we write $T^{-1}(T(v)\lambda^{-1}) = T^{-1}(v) \implies \lambda^{-1}v = T^{-1}(v)$ by linearity of T^{-1} and noting that $T^{-1}(T(v)) = I(v) = v$. Thus, λ^{-1} is an eigenvalue for T^{-1}. This also shows that $\mathrm{Eig}_T(\lambda) \subseteq \mathrm{Eig}_{T^{-1}}(\lambda^{-1})$.

Now let $v \in \mathrm{Eig}_{T^{-1}}(\lambda^{-1})$. Then $T^{-1}(v) = \lambda^{-1}v \implies \lambda T^{-1}(v) = v \implies T(\lambda T^{-1}(v)) = \lambda v = T(v)$ by linearity of T. Thus, $v \in \mathrm{Eig}_T(\lambda) \implies \mathrm{Eig}_{T^{-1}}(\lambda^{-1}) \subseteq \mathrm{Eig}_T(\lambda) \implies \mathrm{Eig}_{T^{-1}}(\lambda^{-1}) = \mathrm{Eig}_T(\lambda)$ as desired.

Part (b): $T : V \to V$ is diagonalizable $\iff \exists \beta := \{v_1, ..., v_n\} \subseteq V$ which is a basis for V, where v_i are eigenvectors of T. From (a), we know that v_i is an eigenvector of $T \iff v_i$ is an eigenvector of T^{-1}. Namely, $T(v_i) = \lambda_i v_i \iff T^{-1}(v_i) = \lambda_i^{-1} v_i$. Thus, β is a basis of eigenvectors for $T \iff \beta$ is a basis of eigenvectors for T^{-1}, so T is diagonalizable $\iff T^{-1}$ is.

QUESTION 5

For $M_2(\mathbb{F})$, no matter the field, we can use the standard basis

$$\text{St} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad 1 := \mathbb{1}_{\mathbb{F}}$$

Let $T : M_2(\mathbb{F}) \to M_2(\mathbb{F})$ be $T(A) = A^t$. Then

$$[T]_{\text{St}} = \begin{bmatrix} | & | & | & | \\ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^t\right]_{\text{St}} & \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^t\right]_{\text{St}} & \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^t\right]_{\text{St}} & \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^t\right]_{\text{St}} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $[tI]_{\text{St}} = [tI_4]$, we have the following form for $p_T(t)$:

$$\det(tI_4 - [T]_{\text{St}}) = \det \begin{pmatrix} t-1 & 0 & 0 & 0 \\ 0 & t & -1 & 0 \\ 0 & -1 & t & 0 \\ 0 & 0 & 0 & t-1 \end{pmatrix} = \sum_{\pi \in S_4} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} a_{4\pi(4)} \text{sgn}(\pi)$$

We know that $\pi(1) = 1$ and $\pi(4) = 4$, else the product will evaluate to 0. Thus, we only need to consider the remaining options: $\{\pi(2) = 3, \pi(3) = 2\}$ and $\{\pi(2) = 2, \pi(3) = 3\}$. The former has sgn $= -1$, since it flips $2 \leftrightarrow 3$, and the latter has sgn $= 1$, since it is the identity. Thus,

$$\det([T]_{\text{St}_n}) = (t-1)^2 t^2 - (t-1)^2 = (t-1)^2(t^2-1) = (t-1)^3(t+1)$$

T thus has eigenvalues $\lambda = 1$ with multiplicity 3 and $\lambda = -1$ with multiplicity 1 in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $\text{char}(\mathbb{F}) = 2$, then $-1 = 1$, so $p_T(t) = (t-1)^4$, i.e. T has one eigenvalue, 1, with multiplicity 4.

For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the eigenvectors of 1 are those which satisfy $A \in \ker(I - T)$, i.e. $T(A) = A^t = I(A) = A$, which are precisely the symmetric matrices in $M_2(\mathbb{F})$, and

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis for $\text{Sym}_2(\mathbb{F})$, as in HW1, so $m_g(1) = 3$. If $A \in \ker(-I - T)$, then $T(A) = -I(A) \implies A^t = -A$. Since $a^t_{11} = a_{11}$ and $a^t_{22} = a_{22}$, we require that $a_{11} = a_{22} = 0$.

Since the corner elements swap, $a_{12} = -a_{21}$, and so all eigenvectors must be

$$\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This has dimension 1, so $m_g(-1) = 1$. Since $m_g(1) + m_g(-1) = 4$, T is diagonalizable in both $\mathbb{R}$ and $\mathbb{C}$. We simply take these aforementioned eigenvectors as a basis:

$$\beta = \{M_1, M_2, M_3, M_4\} := \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \implies [T]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

which is diagonal. To show β is a basis, we show it is spanning:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{12} + a_{21}}{2} M_1 + a_{22} M_2 + a_{11} M_3 + \frac{a_{12} - a_{21}}{2} M_4$$

As $\dim(\beta) = 4 = \dim(\mathrm{St})$, β is a basis for $M_2(\mathbb{F})$, $\mathbb{F} = \mathbb{R} \vee \mathbb{C}$.

Recall, for $\mathrm{char}(\mathbb{F}) = 2$, we found that $p_T(t) = (t-1)^4$. Then $\dim(\ker(I - T)) = 4 - \mathrm{rank}(I - T)$, and $[I - T]$ in the standard basis is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Type III}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \mathrm{rank}([I - T]) = 1$$

Thus, $(I - T) = 4 - 1 = 3$, so $m_g(1) = 3$, as before. But $m_a(1) = 4$, so by our main criterion, T cannot be diagonalizable over $M_2(\mathbb{F})$ since $m_a(1) \neq m_g(1)$.

QUESTION 6

Part (a): Recall that $p_A(s) = \det(sI_n - A) = \det\left((sI_n - A)^t\right)$. But $(sI_n - A)^t = sI_n - A^t$, since $(sI_n)^t = sI_n$. Thus, $\det\left((sI_n - A)^t\right) = \det(sI_n - A^t) = p_{A^t}(s)$, and we conclude that $p_A(s) = p_{A^t}(s)$. The roots of A and A^t's characteristic polynomials are thus identical, and these are precisely the eigenvalues of A and A^t.

Part (b): Fix an eigenvalue λ of both A and A^t. Then $(\lambda I_n - A)^t = \lambda I_n - A^t$, since $(\lambda I_n)^t = \lambda I_n$, as observed before. Let $m_g^A(\lambda)$ and $m_g^{A^t}(\lambda)$ represent the multiplicities of λ with respect to A and A^t, respectively. Then:

$$m_g^A(\lambda) = \text{null}(\lambda I_n - A) = n - \text{rank}(\lambda I_n - A) = n - \text{rank}((\lambda I_n - A)^t)$$

since $\text{rank}(M) = \text{rank}(M^t)\ \forall M \in M_n(\mathbb{F})$. Then

$$\ldots = n - \text{rank}(\lambda I_n - A^t) = m_g^{A^t}(\lambda)$$

Thus, $\dim(\text{Eig}_A(\lambda)) = \dim(\text{Eig}_{A^t}(\lambda))$ as desired.

Part (c): Let Λ be the set of all distinct eigenvalues of A, and thus of A^t, by part (a). Then, A is diagonalizable $\iff \sum_{\lambda\in\Lambda} m_g^A(\lambda) = n \iff \sum_{\lambda\in\Lambda} m_g^{A^t}(\lambda) = n \iff A^t$ is diagonalizable, by part (b), as desired.

There is a small amount of nuance: A is diagonalizable $\iff$ L_A is diagonalizable, but their eigenvalues/vectors remain the same, so the argument in (c) extends to the language of transformations immediately.

QUESTION 7

Part (a): Since $\lambda_1, ..., \lambda_k$ are all the eigenvalues of A, and we know their multiplicities to be $m_a(\lambda_i) = m_i$, we can write

$$p_A(t) = \det(tI_n - A) = (t-\lambda_1)^{m_1} \cdot ... \cdot (t-\lambda_k)^{m_k} q(t) \quad q(t) \in \mathbb{F}[t]_n$$

Note, here, that $q(t)$ does not have roots in $\mathbb{F}$, or else we would find some eigenvalue not λ_i. Since B is upper triangular, we write

$$p_B(t) = \det(tI_n - B) = (t - b_{11}) \cdot ... \cdot (t - b_{nn})$$

Let $A = Q^{-1}BQ$, where $Q \in GL_n(\mathbb{F})$. Then $Q^{-1}(tI_n - A)Q = (tQ^{-1}I_n - Q^{-1}A)Q = tQ^{-1}I_nQ - Q^{-1}AQ = tQ^{-1}Q - B = tI_n - B$. Since the determinant is conjugation-invariant, $\det(tI_n - A) = \det(tI_n - B)$, i.e.

$$(t-\lambda_1)^{m_1} \cdot ... \cdot (t-\lambda_k)^{m_k} q(t) = (t - b_{11}) \cdot ... \cdot (t - b_{nn}) = p_B(t)$$

Since the RHS has n roots, so does the LHS. But recall that $q(t)$ cannot have roots, so we write

$$(t-\lambda_1)^{m_1} \cdot ... \cdot (t-\lambda_k)^{m_k} = (t - b_{11}) \cdot ... \cdot (t - b_{nn})$$

In particular, $q(t) = 1$. Thus, we can take the first m_1 terms of the RHS to be $(t - \lambda_1)$, the next m_2 terms to be $(t-\lambda_2)$, etc. In other words, $b_{(m_{i-1}+1)(m_{i-1}+1)} = ... = b_{(m_{i-1}+m_i)(m_{i-1}+m_i)} = \lambda_i$, as desired.

Part (b): Recall that $\det(B) = \det(A)$ for $A \sim B$. Since B is upper triangular, we can use (a) to conclude $\det(B) = \lambda_1^{m_1} \cdot ... \cdot \lambda_k^{m_k} = \det(A)$. Furthermore, since there are n b_{ii} terms, $\sum_{i=1}^k m_i = n = \sum_{i=1}^k m_a(\lambda_i)$ again by (a), so $p_A(t)$ splits. Lastly, since trace is conjugation-invariant, $\mathrm{tr}(A) = \mathrm{tr}(B) = \sum_{i=1}^k m_i\lambda_i$ by (a).

"Lemmas" to consider:

$\det(B)$, for an upper triangular matrix $B \in M_n(\mathbb{F})$, is $\prod_{i=1}^n b_{ii}$:

$$\det(B) = \sum_{\pi \in S_n} b_{1\pi(1)} \cdot ... \cdot b_{n\pi(n)} \mathrm{sgn}(\pi)$$

But since $B_{(n)} = (0, ..., 0, b_{nn})$, we know that $\pi(n) = n$ (else the summand would $= 0$). Since $B_{(n-1)} = (0, ..., 0, b_{(n-1)(n-1)}, b_{nn})$, we know $\pi(n-1) = n-1 \vee n$. But $\pi(n) = n$, so $\pi(n-1) = n-1$. Thus, by induction, $\pi = \mathrm{Id}$ is the only $\pi \in S_n$ such that $b_{1\pi(1)} \cdot ... \cdot b_{n\pi(n)} \neq 0$, i.e. $\det(B) = b_{11} \cdot ... \cdot b_{nn}$, where we note that $\mathrm{sgn}(\mathrm{Id}) = 1$.

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For $A, B, C \in M_n(\mathbb{F})$, we have $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$:

$$(L_A \circ (L_B + L_C))(v) = L_A((L_B + L_C)(v)) = L_A(L_B(v) + L_C(v)) = L_A \circ L_B(v) + L_A \circ L_C(v)$$

Thus, $L_A \circ (L_B + L_C) = L_A \circ L_B + L_A \circ L_C \implies A(B+C) = AB + AC$. The statement $(B+C)A = BA + CA$ follows similarly.

QUESTION 8

Part (a): $w \in W = \text{Span}(\{T^i(v) : i \in \mathbb{N}\})$, and formalize $T^0 := I$. Consider:

$$T(w) = T\left(\sum_{i\in I} a_i T^i(v)\right) = \sum_{i\in I} a_i T^{i+1}(v) \quad \text{by linearity}$$

Thus, $T(w) \in W$, and we conclude that $T(W) \subseteq W$, i.e. $T_W : W \to W$.

Part (b): Consider the largest linearly independent subspace of the form $W' = \{v, ..., T^l(v)\}$. Such a set exists, or else $\dim(W) = \infty$. If $l > k-1$, then we could extend W' to a basis for W, i.e. $\dim(W) > k$, which we know to be false.

Suppose now $l < k-1$. Then $T^{l+1}(v) \in \text{Span}(\{v, ..., T^l(v)\})$. Let this be the base case, and assume $T^{l+m}(v) \in \text{Span}(W')$, i.e. $T^{l+m}(v) = a_0 v + ... + a_l T^l(v)$. Then $T^{l+m+1}(v) = T(T^{l+m}(v)) = T(a_0 v + ... + a_l T^l(v)) = a_0 T(v) + ... + a_l T^{l+1}(v) \in \text{Span}(W')$, since $T^{l+1}(v) \in \text{Span}(W')$. Thus, $T^{l+i}(v) \in \text{Span}(W')\ \forall i \in \mathbb{N}$

$\implies \{T^i(v) : i \in \mathbb{N}\} \subseteq \text{Span}(W')$, so especially $\text{Span}(\{T^i(v) : i \in \mathbb{N}\}) = W \subseteq \text{Span}(W')$. But since $W' = \{v, ..., T^l(v)\}$ is linearly independent, this is a basis, so $\dim(W) < k$, which is a contradiction.

$\implies l = k-1$. As $\dim(W) = k$, W' independent, and $|W'| = k$, we know that $W' = \{v, ..., T^{k-1}(v)\}$ is a basis for W.

Part (c): Consider $[T_W]_\beta$, where $\beta = \{v, ..., T^{k-1}(v)\}$:

$$\begin{bmatrix} | & & | \\ [T(v)]_\beta & \cdots & [T(T^{k-1}(v))]_\beta \\ | & & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ 0 & 0 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{k-2} \\ 0 & 0 & \cdots & 1 & a_{k-1} \end{bmatrix} \implies tI_n - [T_W]_\beta = \begin{bmatrix} t & 0 & \cdots & 0 & -a_0 \\ -1 & t & \cdots & 0 & -a_1 \\ 0 & -1 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t & -a_{k-2} \\ 0 & 0 & \cdots & -1 & t - a_{k-1} \end{bmatrix}$$

And recall that $p_{T_W}(t) = \det(tI_n - [T_W]_\beta) = \sum_{\pi\in S_k} m_{1\pi(1)} \cdots m_{k\pi(k)} \text{sgn}(\pi)$, where $[T_W]_\beta = (m_{ij})$. For some $\pi \in S_n$, fix $\pi(j) = k$, where $j \neq 1$ or k. Then, in order for $\prod_{i=1}^{k} m_{i\pi(i)} \neq 0$, we need $\pi(1) = 1$. But then $\pi(2) = 2$, since 1 and k are already mapped, and all other $m_{2i} = 0$. We deduce the same for all $i = 1, 2, ..., j-1$.

Now, since $\pi(j) = k$, $\pi(k) \neq k$. Thus, $\pi(k) = k-1$, since all other $m_{ki} = 0$. Similarly, the nonzero elements of m_{ij} are precisely $m_{ii}, m_{i(i-1)}$, and m_{ik}. But $\pi(i) \neq k$, and $\pi(i) \neq i$, since $\pi(i+1) = i \implies \pi(i) = i-1$. Thus, we get the following characterization of π s.t. $\prod_{i=1}^{k} m_{i\pi(i)} \neq 0$ and $j \neq 1, k$:

$$\begin{cases} \pi(1) & = 1 \\ \pi(2) & = 2 \\ & \vdots \\ \pi(j) & = k \\ \pi(j+1) & = j \\ & \vdots \\ \pi(k-1) & = k-2 \\ \pi(k) & = k-1 \end{cases} \implies m_{1\pi(1)} \cdot .. \cdot m_{k\pi(k)} = -t^{j-1} a_{j-1} (-1)^{k-j} \qquad \underbrace{(-1)^{k-j}}_{\text{inversions } (j,j'),j'>j}$$

Edge cases: $\pi(1) = k \implies \pi(k) = k-1$, and inductively $\pi(i) = i-1$ using the same arguments as above. In this case, $m_{1\pi(1)} \cdot .. \cdot m_{k\pi(k)} = -a_0(-1)^{k-1}(-1)^{k-1}$, where every $(1,i) : i > 1$, is an inversion, i.e. $\text{sgn}(\pi) = k-1$. Lastly, if we suppose $\pi(k) = k$, then $\pi(1) = 1$, and we conclude $\pi(i) = i$ as before. This case yields a $t^{k-1}(t - a_{k-1})$ term, with no inversions. The full picture is thus

$$p_{T_W}(t) = t^k - t^{k-1} a_{k-1} + \left(\sum_{j=2}^{k-1} -t^{j-1} a_{j-1} \right) - a_0 = t^k - a_{n-1} t^{k-1} - ... - a_1 t - a_0$$

as desired.

Part (d): We write $p_{T_W}(T_W(v)) = T_W^k(v) - a_{k-1} T_W^{k-1}(v) - ... - a_1 T_W(v) - a_0 I(v)$. But $T_W^k(v) = a_{k-1} T_W^{k-1}(v) + ... + a_1 T_W(v) + a_0 v$ by assumption, so

$$p_{T_W}(T_W(v)) = a_{k-1} T_W^{k-1}(v) + ... + a_0 v - a_{k-1} T_W^{k-1}(v) - ... - a_0 I(v) = 0$$

$\implies p_{T_W}(T_W(v)) = \mathbb{0}_W$.