

Vectors

Definition 1

An *inner product* on a vector space is such that

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u + w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$
3. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
4. $\langle u, u \rangle \geq 0$, and $\langle u, u \rangle = 0 \iff u = 0$

Definition 2

A *norm* is $\|v\| = \sqrt{\langle v, v \rangle}$.

Definition 3

A *line* $l(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ is $l(t) = P + td$, where $P, d \in \mathbb{R}^n$. It may also be given by $l(t) = (1-t)Q + tP$, where $P, Q \in \mathbb{R}^n$, and $t \in [0, 1]$.

Definition 4

A *plane* $p(s, t) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is $p(s, t) = P + sd_1 + td_2$, where $d_1, d_2 \in \mathbb{R}^3$. It may also be given in point-normal form, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $\langle a, b, c \rangle \in \mathbb{R}^3$ is normal to the plane, and $(x_0, y_0, z_0) \in \mathbb{R}^3$ lies on the plane.

Definition 5

A *linear transformation* $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is such that $\lambda(\vec{0}) = \vec{0}$, and $\lambda(\vec{a} + \vec{b}) = \lambda(\vec{a}) + \lambda(\vec{b})$. Alternatively, write $\lambda(x_1, \dots, x_n) = x_1 \vec{d}_1 + \dots + x_n \vec{d}_n$, where $\vec{d}_i \in \mathbb{R}^m$.

Definition 6

An *affine transformation* $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation plus a point $P \in \mathbb{R}^m$.

Definition 7

The *projection* of v onto u , denoted $\text{proj}_u(v)$, is given by $(u \cdot v) \frac{u}{\|u\|}$.

Definition 8

The *tangent line* of $r : \mathbb{R} \rightarrow \mathbb{R}^n$ at $a \in \mathbb{R}$ is an affine transformation λ satisfying

$$\lim_{t \rightarrow a} \frac{\|r(t) - \lambda(t)\|}{|t - a|} = 0$$

$$\lambda(t) = r(a) + (t - a)\vec{d} \quad \vec{d} \neq 0$$

Prop 1

$\|v + u\| \leq \|v\| + \|u\|$ (triangle inequality) and $|\langle u, v \rangle| \leq \|u\| \|v\|$ (Cauchy-Schwartz).

Prop 2

$$(u \times v) \cdot u = 0$$

$\|u \times v\|$ is the area of the parallelogram bounded by u, v .

$$\|u \times v\| = \|u\| \|v\| \sin(\theta)$$

$$u \cdot v = \|u\| \|v\| \cos(\theta)$$

Prop 3

The distance between a point R and a plane is given by $\|\text{proj}_{\vec{n}}(P - R)\|$, where P, \vec{n} are as in point-normal form.

Differentiation

Definition 1

- (a) $r : \mathbb{R} \rightarrow \mathbb{R}^n$ is *differentiable* at \vec{a} if \exists a linear transformation $\lambda : \mathbb{R} \rightarrow \mathbb{R}^n$ such that

$$\lim_{h \rightarrow 0} \frac{\|r(\vec{a} + h) - r(\vec{a}) - \lambda(\vec{h})\|}{|h|} = 0$$

- (b) Similarly, $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is *differentiable* at \vec{a} if \exists a linear transformation $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{|F(a + h, b + k) - F(a, b) - \lambda(h, k)|}{\|(h, k)\|}$$

- (c) Generally, $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *differentiable* at \vec{a} if \exists a linear transformation $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{\|F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})\|}{\|\vec{h}\|}$$

We denote the derivative $\lambda(\vec{a})$ as $DF_{\vec{a}}$

Definition 2

The *arc length* of a curve $r(t) : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

The arc length parameterization is some $t = \alpha(s)$ such that $\|r'(\alpha(s))\| = 1$.

Definition 3

The *Jacobian* matrix of $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is given by

$$F' = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix}$$

Note that, where $m > 1$, each element is a column vector.

Definition 4

A *level surface* is $F : \mathbb{R}^n \rightarrow 0$.

Definition 5

The *gradient* of $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is $\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \rangle$ (i.e. its Jacobian) and denoted by ∇F .

Definition 6

$\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *continuous* at \vec{a} if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon$.

Definition 7

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is called C^k -*continuous* (or k -times continuously differentiable) at \vec{a} if all k^{th} -order partial derivatives exist near \vec{a} and are continuous at \vec{a} .

Definition 8

- (a) The k^{th} *partial derivative* of $F : \mathbb{R}^n \rightarrow \mathbb{R}$ at \vec{a} , denoted by $\frac{\partial F}{\partial x_k}(\vec{a})$ is given by

$$\lim_{t \rightarrow 0} \frac{F(\vec{a} + t e_k) - F(\vec{a})}{t}$$

where e_k is the standard basis vector, e.g. \vec{i} for $k = 1$.

- (b) The *directional derivative* of $F : \mathbb{R}^n \rightarrow \mathbb{R}$ along \vec{h} , denoted by $\partial_{\vec{h}} F(\vec{a})$, is given by

$$\lim_{t \rightarrow 0} \frac{F(\vec{a} + t \vec{h}) - F(\vec{a})}{t}$$

- (c) The j^{th} *iterated directional derivative* of $F : \mathbb{R}^n \rightarrow \mathbb{R}$ at \vec{a} along \vec{h} , denoted by $\partial_{\vec{h}}^j F(\vec{a})$, is given by $g^{(j)}(0)$, where $g(t) = F(\vec{a} + t \vec{h})$. Note that $g'(0) = \partial_{\vec{h}} F(\vec{a})$.

Prop 1 (Differentiability and Partial)

- (a) If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at \vec{a} , then all partial derivatives of F exist at \vec{a} .
- (b) If all partial derivatives of $F : \mathbb{R}^n \rightarrow \mathbb{R}$ exist near \vec{a} and are continuous at \vec{a} , then F is differentiable.
- (c) If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is k -times continuously differentiable, then all $(k - 1)^{th}$ -order partial derivative are continuously differentiable.

Prop 2 (Chain Rule)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \rightarrow \mathbb{R}^l$ be differentiable at $\vec{a} \in \mathbb{R}^n$ and $f(\vec{a}) \in \mathbb{R}^m$, respectively. Then

$$h = g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^l$$

is differentiable. Furthermore, $Dh_{\vec{a}} = Dg_{f(\vec{a})} \circ Df_{\vec{a}}$, and $h'(\vec{a}) = g'(f(\vec{a}))f'(\vec{a})$.

Prop 3 (Iterated Partial)

- (a) If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at \vec{a} , then

$$\partial_{\vec{h}}(\vec{a}) = \vec{h} \cdot \nabla F = \sum_{i=1}^n h_i \frac{\partial F}{\partial x_i}(\vec{a})$$

- (b) If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is k -times continuously differentiable, then

$$\partial_{\vec{h}}^j(\vec{a}) = (\vec{h} \cdot \nabla)^j F(\vec{a})$$

Prop 4 (Mixed Partial)

If $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^2 -continuous, then $\partial_x \partial_y F = \partial_y \partial_x F$.

Prop 5 (Taylor)

If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^k continuous, let

$$\alpha_j(\vec{h}) = \frac{1}{j!} \partial_{\vec{h}}^j(\vec{a}) = \frac{1}{j!} (\vec{h} \cdot \nabla)^j F(\vec{a})$$

Then $F(\vec{a}) + \sum_{i=1}^k \alpha_i(\vec{x} - \vec{a})$ is the best degree k approximation of F near \vec{a} .

- (a) If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous on \mathcal{B} , then F is integrable on \mathcal{B} .

- (b) The set of discontinuities of F in \mathcal{B} has zero measure if and only if F is integrable.

Prop 2 (Fubini)

Let $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$. If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous on \mathcal{B} , then

$$\int_{\mathcal{B}} F = \int_{x_n=a_n}^{x_n=b_n} \cdots \left(\int_{x_1=a_1}^{x_1=b_1} F(x_1, \dots, x_n) dx_1 \right) \cdots dx_n$$

The order of integration doesn't matter.

Prop 3 (MVT)

$$\int_a^b g(x) = g(c)(b-a) \text{ for some } c \in [a, b]$$

Integration

Definition 1

Let $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is *integrable* over \mathcal{B} if

$$\lim_{m \rightarrow \infty} \sum_{i=1}^m F(c_j^m) \frac{\text{vol}(\mathcal{B})}{m^n} \text{ exists}$$

and is equivalent up to choices of c_j^m , where $c_j^m \in [a_j^k, b_j^k]$, $k \in [1, m]$ (slice of size $\frac{1}{m}$).

Definition 2

A set $S \subseteq \mathbb{R}^n$ has *zero measure* if $\forall \varepsilon > 0$ we can choose a countable set of open balls such that $S \subseteq \cup B(x_i, \varepsilon_i)$, where $\sum \text{vol}(B(x_i, \varepsilon_i)) < \varepsilon$.

Prop 1 (Integrability)