

Stochastic Processes

MATH 447

Nicholas Hayek

Taught by Prof. Louigi Addario-Berry

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We assume working knowledge of probability and no knowledge of measure theory (though a grasp of analysis is essential). See these [MATH 356 notes](#), also taught by Louigi!

I Markov Chains

Before providing definitions, we give some examples of stochastic processes:

Eg. 1.1 A simple random walk: $S_{i+1} = S_i + X_i$, where $X_i \sim \text{Ber}(p)$ and $S_0 = 0$. We might ask: does S_i ever return to 0, i.e.

$$\mathbb{P}(\exists i > 0 : S_i = 0)$$

Eg. 1.2 A branching process: as in asexual reproduction, we have an initial node. Each node n has a number of children X_n , where $\frac{X_n}{2} \sim \text{Ber}(p)$. We denote Z_i to be the number of individuals in the i -th generation. We might ask: does Z_i ever have no children, i.e.

$$\mathbb{P}(\exists i > 0 : Z_i = 0)$$

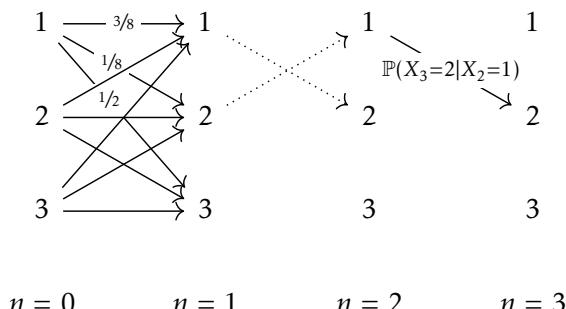
Eg. 1.3 Choose k independent random points in the square $[0, \sqrt{k}]^2$. On average, then, there is 1 point within any unit square $U \subseteq [0, \sqrt{k}]^2$.

DEF 1.1 Given a finite or countable set V , a *Markov chain* with *state space* V is a sequence $X_n : n \geq 0$ of random variables, with $X_n \in V$, such that:

$$\mathbb{P}(\underbrace{X_{n+1} = v_{n+1}}_{\text{future}} \mid \underbrace{X_0 = v_0, \dots, X_{n-1} = v_{n-1}}_{\text{past}}, \underbrace{X_n = v_n}_{\text{present}}) = \mathbb{P}(X_{n+1} = v_{n+1} \mid X_n = v_n)$$

In other words, the future only depends on the past via the present. This is called the *Markov property*.

Sometimes we allow Markov chains to be only finitely large (i.e. $0 \leq n \leq m$). For instance, we limit ourselves to one weekend of gambling in Las Vegas. A graphical example would look something like:



By repeated Bayes' Law, we observe

$$\begin{aligned} & \mathbb{P}(X_1 = v_1, \dots, X_n = v_n | X_0 = v_0) \\ &= \mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_0 = v_0, X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_0 = v_0, \dots, X_{n-1} = v_{n-1}) \\ &= \mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_{n-1} = v_{n-1}) \quad \text{by Markov property} \end{aligned}$$

Time-Homogeneous Markov Chains

We often write
THMC

We say that a Markov chain is *time-homogeneous* if, for all $u, v \in V$ and $n \geq 0$

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u)$$

In other words, the chain's behavior is described entirely by $\mathbb{P}(X_1 = v | X_0 = u)$ for each $(v, u) \in V \times V$. In this case, we can describe such probabilities in a *transition matrix* P :

$$P = (p_{u,v})_{(u,v) \in V^2} = (\mathbb{P}(X_1 = v | X_0 = u))_{(u,v) \in V^2}$$

DEF 1.4

DEF 1.5

Eg. 1.4 Recall the game Snakes and Ladders. A 6×6 grid is indexed $1, \dots, 36$. Players start at the 1 cell. The game ends when a player reaches the 36 cell. A die roll dictates how many spots one advances. There are some directed edges between cells (increasing: "ladders", decreasing: "snakes"). One must follow these edges when one lands at its tail. Suppose a ladder exists from 11 to 27. Then

$$\mathbb{P}(X_{11} = 27 | X_{10} = 6, X_9 = 3) = \frac{1}{6} = \mathbb{P}(X_{11} = 27 | X_{10} = 6) = \mathbb{P}(X_2 = 27 | X_1 = 6)$$

We see that Snakes and Ladders is naturally modeled as a time-homogeneous Markov chain.

Eg. 1.5 Sampling without replacement is *not* a Markov chain. If we sample from $|X| = 10$, we have

$$\begin{aligned} \mathbb{P}(X_3 = a | X_2 = b) &= 1/9 \\ \mathbb{P}(X_3 = a | X_2 = b, X_1 = c) &= 1/8 \\ \mathbb{P}(X_3 = a | X_2 = b, X_1 = a) &= 0 \end{aligned}$$

so we do not satisfy the Markov property.

Eg. 1.6 Returning to the Snakes and Ladders example, consider $S \subseteq V$. Let $T_S = \inf\{n \geq 0 : X_n \in S\}$. We may ask...

- What is the average number of rounds to finite? We can write this as $\mathbb{E}[T_{\{36\}} | X_0 = 1]$.
- What is the probability of landing on 18 or 19 before the game ends? We can write this as $\mathbb{P}(T_{\{18,19\}} < T_{\{36\}} | X_0 = 1)$.
- What is the average number of visits to 6 before the game ends? We can write this as

$$\mathbb{E}[\#\{n \in [T_{\{36\}}] : X_n = 6\} | X_0 = 1]$$

- What is the expected proportion of time spent on state 5 before the game ends?
- If we allow two players, what is the probability that player 1 wins? Is this still a Markov chain?

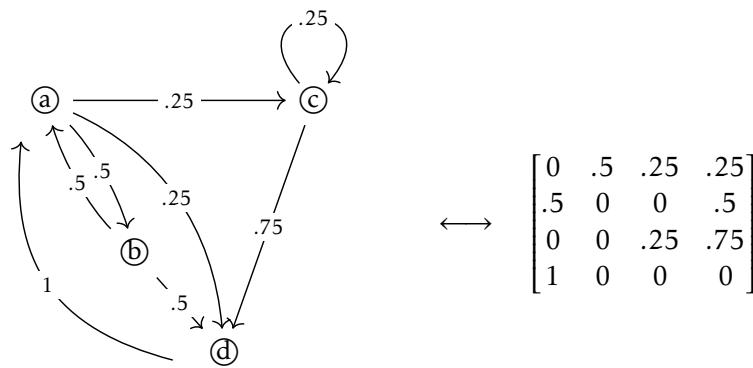
DEF 1.6 A matrix $P = (p_{u,v})_{(u,v) \in V^2}$ is called a *stochastic matrix* if every row sums to 1, i.e.

$$\forall u \in V, \sum_{v \in V} p_{u,v} = 1$$

Note that any stochastic matrix is the transition matrix of some time-homogeneous Markov chain with state space V and transition probabilities

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u) = p_{u,v}$$

A directed graph, together with its stochastic matrix, can visualize any THMC:



Eg. 1.7 Random walks on an undirected weighted graph, where edge weights dictate the proportional probability of transitioning between two states, are a special class of THMCs. In particular, given a graph $G = (V, E)$ with weights $w(e) > 0 : e \in E$, we set

$$p_{u,v} = \frac{w(\{u, v\})}{\sum_{z \in N(u)} w(\{u, z\})}$$

If there are no edges $u \leftrightarrow v$, we write $p_{u,v} = 0$.

Not every THMC can be represented by a random walk on an undirected weighted graph. In particular, see the directed graph listed above, or any transition matrix which is not symmetric.

Multi-Step Transition Probabilities

Given a THMC $X = X_n : n \geq 0$ with a transition matrix P , we write

$$\begin{aligned}\mathbb{P}(X_2 = v | X_0 = u) &= \sum_{w \in V} \mathbb{P}(X_2 = v, X_1 = w | X_0 = u) \\ &= \sum_{w \in V} \mathbb{P}(X_1 = w | X_0 = u) \mathbb{P}(X_2 = v | X_1 = w, X_0 = u) \quad \text{by Markov property} \\ &= \sum_{w \in V} p_{u,w} p_{w,v} = (P^2)_{u,v} \quad \text{or write } P_{u,v}^2\end{aligned}$$

Hence, to determine a two-step transition probability, and by induction an n -step transition probability from u to w , we consider $P_{u,v}^n$. PROP 1.2

See [Prop 1.1](#) to expand probabilities, using Bayes', as needed. We get that

$$\begin{aligned}\mathbb{P}(X_n = v | X_0 = u) &= \sum_{v_1, \dots, v_{n-1} \in V} \mathbb{P}(X_1 = v_1, \dots, X_{n-1} = v_{n-1}, X_n = v | X_0 = u) \\ &= \sum_{v_1, \dots, v_{n-1} \in V} p_{u,v_1} p(v_1, v_2) \cdots p(v_{n-1}, v) = (P^n)_{u,v} \quad \square\end{aligned}$$

Thus, if P is a stochastic matrix, then so is P^n , as $\sum_{v \in V} P_{u,v}^n = \sum_{v \in V} \mathbb{P}(X_n = v | X_0 = u) = 1$.

PROOF.

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