

Stochastic Processes

MATH 447

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We assume working knowledge of probability and no knowledge of measure theory (though a grasp of analysis is essential). See these [MATH 356 notes](#), also taught by Louigi!

I Introduction

Before providing definitions, we give some examples of stochastic processes:

Eg. 1.1 A simple random walk: $S_{i+1} = S_i + X_i$, where $X_i \sim \text{Ber}(p)$ and $S_0 = 0$. We might ask: does S_i ever return to 0, i.e.

$$\mathbb{P}(\exists i > 0 : S_i = 0)$$

Eg. 1.2 A branching process: as in asexual reproduction, we have an initial node. Each node n has a number of children X_n , where $\frac{X_n}{2} \sim \text{Ber}(p)$. We denote Z_i to be the number of individuals in the i -th generation. We might ask: does Z_i ever have no children, i.e.

$$\mathbb{P}(\exists i > 0 : Z_i = 0)$$

Eg. 1.3 Choose k independent random points in the square $[0, \sqrt{k}]^2$. On average, then, there is 1 point within any unit square $U \subseteq [0, \sqrt{k}]^2$.

MARKOV CHAINS

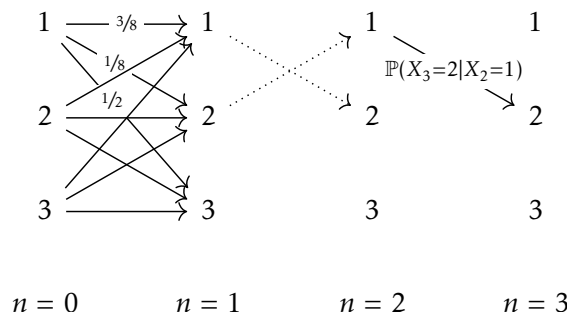
DEF 1.1 Given a finite or countable set V , a *Markov chain* with *state space* V is a sequence $X_n : n \geq 0$ of random variables, with $X_n \in V$, such that:

$$\underbrace{\mathbb{P}(X_{n+1} = v_{n+1})}_{\text{future}} \mid \underbrace{X_0 = v_0, \dots, X_{n-1} = v_{n-1}}_{\text{past}}, \underbrace{X_n = v_n}_{\text{present}} = \mathbb{P}(X_{n+1} = v_{n+1} \mid X_n = v_n)$$

In other words, the future only depends on the past via the present. This is called the *Markov property*.

DEF 1.3

Sometimes we allow Markov chains to be only finitely large (i.e. $0 \leq n \leq m$). For instance, we limit ourselves to one weekend of gambling in Las Vegas. A graphical example would look something like:



By repeated Bayes' Law, we observe

PROP 1.1

$$\begin{aligned} & \mathbb{P}(X_1 = v_1, \dots, X_n = v_n | X_0 = v_0) \\ &= \mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_0 = v_0, X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_0 = v_0, \dots, X_{n-1} = v_{n-1}) \\ &= \mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_{n-1} = v_{n-1}) \quad \text{by Markov property} \end{aligned}$$

We say that a Markov chain is *time-homogeneous* if, for all $u, v \in V$ and $n \geq 0$

DEF 1.4

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u)$$

In other words, the chain's behavior is described entirely by $\mathbb{P}(X_1 = v | X_0 = u)$ for each $(v, u) \in V \times V$. In this case, we can describe such probabilities in a *transition matrix* P :

DEF 1.5

$$P = (p_{u,v})_{(u,v) \in V^2} = (\mathbb{P}(X_1 = v | X_0 = u))_{(u,v) \in V^2}$$

Fig. 1.4 Recall the game Snakes and Ladders. A 6×6 grid is indexed $1, \dots, 36$. Players start at the 1 cell. The game ends when a player reaches the 36 cell. A die roll dictates how many spots one advances. There are some directed edges between cells (increasing: "ladders", decreasing: "snakes"). One must follow these edges when one lands at its tail. Suppose a ladder exists from 11 to 27. Then

$$\mathbb{P}(X_{11} = 27 | X_{10} = 6, X_9 = 3) = \frac{1}{6} = \mathbb{P}(X_{11} = 27 | X_{10} = 6) = \mathbb{P}(X_2 = 27 | X_1 = 6)$$

We see that Snakes and Ladders is naturally modeled as a time-homogeneous Markov chain.

Fig. 1.5 Sampling without replacement is *not* a Markov chain. If we sample from $|X| = 10$, we have

$$\begin{aligned} \mathbb{P}(X_3 = a | X_2 = b) &= 1/9 \\ \mathbb{P}(X_3 = a | X_2 = b, X_1 = c) &= 1/8 \\ \mathbb{P}(X_3 = a | X_2 = b, X_1 = a) &= 0 \end{aligned}$$

so we do not satisfy the Markov property.

Fig. 1.6 Returning to the Snakes and Ladders example, consider $S \subseteq V$ (called a *state space*). Let $T_S = \inf\{n \geq 0 : X_n \in S\}$. We may ask...

DEF 1.6

- What is the average number of rounds to finite? We can write this as $\mathbb{E}[T_{\{36\}} | X_0 = 1]$.
- What is the probability of landing on 18 or 19 before the game ends? We can write this as $\mathbb{P}(T_{\{18,19\}} < T_{\{36\}} | X_0 = 1)$.
- What is the average number of visits to 6 before the game ends? We can write this as

$$\mathbb{E}[\#\{n \in [T_{\{36\}}] : X_n = 6\} | X_0 = 1]$$

- What is the expected proportion of time spent on state 5 before the game ends?
- If we allow two players, what is the probability that player 1 wins? Is this still a Markov chain?

DEF 1.7 A matrix $P = (p_{u,v})_{(u,v) \in V^2}$ is called *stochastic* if every row sums to 1, i.e. $\forall u \in V, \sum_{v \in V} p_{u,v} = 1$.

Any stochastic matrix is the transition matrix of some time-homogeneous Markov chain with state space V and transition probabilities

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u) = p_{u,v}$$

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