ASSIGNMENT 3

MATH 251

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QUESTION 1

Part (a): Consider $\delta : \mathbb{F}[t]_{n+1} \to \mathbb{F}[t]_n$ which takes $p(t) \to p'(t)$. This function is surjective, but not injective. For the latter, consider a counterexample: for t and $t+1 \in \mathbb{F}[t]_{n+1}$, we have $\delta(t) = 1$ and $\delta(t+1) = 1$. To show δ is surjective, consider an arbitrary element in $\mathbb{F}[t]_n$:

$$p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n \qquad a_i \in \mathbb{F}$$

Then $\delta(a_0 t + a_1 \frac{t^2}{2} + ... + a_n \frac{t^{n+1}}{n+1}) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n = p(t)$. Note that the constants $\frac{1}{i}$ and i can be thought of as $\underbrace{\mathbb{1} + ... + \mathbb{1}}_{i \text{ times}}$ and inverses thereof in the field.

Note that, since $\mathbb{F}$ is characteristic 0 or $> n+1$, none of these coefficients $= 0$, so we don't run into any trouble. The same should be remarked for parts (b) and (c).

The kernel of δ is $\mathbb{F}$: clearly, $\delta(a) = 0 \ \forall a \in \mathbb{F}$. Suppose $p(t) \notin \mathbb{F}$, and WLOG write $p(t) = a_i t^i$ where $1 \leq i \leq n$ and $a_i \in \mathbb{F} \neq 0$. Then $\delta(a_i t^i) = a_i i t^{i-1}$, which is non-zero (e.g. take $t = 1$).

"WLOG": for a full-fledged sum, the argument remains the same, as all $a_i i t^{i-1} \neq 0$

Since $\dim(\mathbb{F}) = 1$, $\text{null}(\delta) = 1$

Part (b): Consider $\iota : \mathbb{F}[t]_n \to \mathbb{F}[t]_{n+1}$ which takes $p(t) \to \int_0^t p(t)$. This function is injective, but not surjective: for the latter, take $1 \in \mathbb{F}[t]_{n+1}$. No polynomial on $\mathbb{F}[t]_n$ integrates to this over $[0, t]$: take $p(t) = a_0 + a_1 t + ... + a_n t_n$. Then

$$\iota[p(t)] = a_0 t + a_1 \frac{t^2}{2} + ... + a_n \frac{t^{n+1}}{n+1} \neq 1$$

To overdo it, the LHS has degree at least one, and the RHS has degree 0.

To show injectivity, suppose $\iota(p(t)) = \iota(q(t))$. We then have

$$a_0 t + a_1 \frac{t^2}{2} + ... + a_n \frac{t^{n+1}}{n+1} = b_0 t + b_1 \frac{t^2}{2} + ... + b_n \frac{t^{n+1}}{n+1}$$

Taking the derivative of both sides (which we can do, as δ is a linear transformation), we get $a_0 + ... + a_n t^n = b_0 + ... + b_n t^n \implies p(t) = q(t)$

The image of ι is $\{a_1 t + ... + a_n t_n + a_{n+1} t^{n+1} : a_i \in \mathbb{F}\}$. Suppose $q(t) \in \text{Im}(\iota)$ were not in this set. WLOG, we can just write $q(t) = a + bt^{n+2}$ for some $a, b \in \mathbb{F}$. Any polynomial of degree $\leq n$ will integrate to a polynomial of degree $\leq n+1$, so we arrive at a contradiction, and $\text{Im}(\iota) \subseteq \{a_1 t + ... + a_n t_n + a_{n+1} t^{n+1}\}$.

"WLOG": note that no function integrates to a constant, by definition.

For $p(t) = a_1 t + ... + a_n t^n + a_{n+1} t^{n+1}$, we have that $\iota(a_1 + 2a_2 t + ... + (n+1) a_{n+1} t^n) = p(t)$, so $p(t) \in \text{Im}(\iota)$, and so $\text{Im}(\iota) = \{a_1 t + ... + a_n t^n + a_{n+1} t^{n+1}\}$. This has dimension $n+1$, so $\text{rank}(\iota) = n+1$.

$\mathbb{F}[t]_{n+1}$ has a basis $\{1, t, t^2, ..., t^{n+1}\}$, as shown in class, so $\{t, t^2, ..., t^{n+1}\}$ is linearly independent. Clearly it is spanning for $\text{Im}(\iota)$, so $\{t, t^2, ..., t^{n+1}\}$ is a basis for $\text{Im}(\iota)$ with $n+1$ elements.

Part (c): Let $p(t) := a_0 + a_1 t + ... + a_n t_n$. Then

$$\begin{aligned}
\delta \circ \iota(p(t)) &= \delta \circ \iota(a_0 + a_1 t + ... + a_n t^n) \\
&= \delta \left[a_0 t + \frac{a_1 t^2}{2} + ... + a_n \frac{t^{n+1}}{n+1} \right] \\
&= a_0 + a_1 t + ... + a_n t^n = p(t) \\
&\implies \delta \circ \iota = I_{\mathbb{F}[t]_n}
\end{aligned}$$

Thus ι is a right inverse for δ. Now let $q(t) = a_0 + ... + a_{n+1} t^{n+1}$. Then:

$$\begin{aligned}
\iota \circ \delta(q(t)) &= \iota \circ \delta(a_0 + a_1 t + ... + a_{n+1} t^{n+1}) \\
&= \iota(a_1 + 2a_2 t + a_{n+1}(n+1)t^n) \\
&= a_1 t + a_2 t^2 + ... + a_{n+1} t^{n+1} \neq q(t) \\
&\implies \iota \circ \delta \neq I_{\mathbb{F}[t]_{n+1}}
\end{aligned}$$

so ι is not a left inverse for δ.

QUESTION 2

Part (a): Consider $T : (W_0, W_1) \to V$ which sends $(w_0, w_1) \to w_0 + w_1$. The kernel of this set is $\{(w_0, w_1) \in W_0 \times W_1 : w_0 + w_1 = 0\}$, just by definition.

Consider the transformation $S : W_0 \cap W_1 \to \ker(T)$ which sends $w \to (w, -w)$. This is an isomorphism, and so $W_0 \cap W_1$ and $\ker(T)$ are isomorphic:

OK Definition $(w, -w)$ is indeed $\in \ker(T)$, since $w + (-w) = 0$, $w \in W_0$ and W_1.

Linear $S(aw + w') = (aw + w', -aw - w') = (aw, -aw) + (w', -w') = a(w, -w) + (w', -w') = aS(w) + S(w')$

Injective Let $S(w) = S(w')$. Then $(w, -w) = (w', -w')$. It follows that $w = w'$

Surjective For any $(w_0, w_1) \in \ker(T)$, we have $w_0 + w_1 = 0$, so $(w_0, w_1) = (w_0, -w_0) = S(w_0)$. Note also that $w_0 \in W_1 \cap W_0$: since $w_1 \in W_1$, we have $-w_1 = w_0 \in W_1$ by closure.

Part (b): To show $1 \iff 2 \iff 3$:

$1 \implies 2$ Consider elements $w_0, w_2 \in W_0$ and $w_1, w_3 \in W_1$. The sum $w_0 + w_1$ is equal to a vector $v \in V$, and by assumption this is a unique representation. Then $T[(w_0, w_1)] = T[(w_2, w_3)] \implies v = w_2 + v_3$. But our assumption stipulates $w_2 = w_0, w_3 = w_1$ by uniqueness, so $(w_0, w_1) = (w_2, w_3)$, and T is injective.

$2 \implies 1$ Let $w_0 + w_1$ be some representation of $v \in V$. Suppose another existed, and write $v = w_2 + w_3$ for $w_2 \in W_0, w_3 \in W_1$. Then $v = T[(w_0, w_1)] = T[(w_2, w_3)]$. By injectivity, $(w_0, w_1) = (w_2, w_3)$, so $w_0 = w_2$ and $w_1 = w_3$, and we conclude that this representation is unique.

$2 \iff 3$ We have $V = W_1 + W_2$, so it remains to show $W_1 \cap W_2 = \{0\} \iff T$ injective. But T is injective $\iff \ker(T) = \{0\}$. And from part (a), we have that $\ker(T)$ and $W_0 \cap W_1$ are isomorphic, so $\ker(T) = \{0\} \iff W_0 \cap W_1 = \{0\}$ as well.

Part (c): By dimension theorem, we have

$$
\begin{aligned}
\dim(W_1 \times W_2) &= \dim(\ker(T)) + \dim(\mathrm{Im}(T)) \\
&= \dim(W_1 \cap W_2) + \dim(V) \\
\implies \dim(V) &= \dim(W_1 \times W_2) - \dim(W_1 \cap W_2) \\
\dim(V) &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)
\end{aligned}
$$

Note: $\mathrm{Im}(T) = W_0 + W_1 = V$

c.f. HW2, Q2

Question 3

Part (a): Consider $P_0 : V \to V$, which sends $v \to w_0$, where $v = w_0 + w_1$ is the unique representation of v for $w_0 \in W_0, w_1 \in W_1$. This is linear: let $v = w_0 + w_1$ and $v' = w_0' + w_1'$. Then:

i.e. is the unique representation of $av + v'$

$$\begin{aligned} P_0(av + v') &= P_0(aw_0 + aw_1 + w_0' + w_1') \\ &= aw_0 + w_0' \quad \text{since } \underbrace{(aw_0 + w_0')}_{\in W_0} + \underbrace{(aw_1 + w_1')}_{\in W_1} = av + v' \end{aligned}$$

which is just $aP_0(v) + P_0(v')$, hence P_0 is linear. Furthermore, $P_0^2 = P_0$:

$$P_0^2(v) = P_0(P_0(v)) = P_0(w_0) = w_0 \quad \text{since } w_0 \text{ is its own representation}$$

As $P_0(v) = w_0$, we conclude $P_0^2 = P_0$. Lastly, we have

$$\begin{aligned} \ker(P_0) &= \{v \in V : P_0(v) = 0\} \\ &= \{w_0 + w_1 : w_0 = 0, w_1 \in W_1\} = W_1 \\ &\quad (\text{note that } V = W_1 + W_2) \end{aligned} \qquad \begin{aligned} \operatorname{Im}(P_0) &= \{P_0(v) : v \in V\} \\ &= \{P_0(w_0 + w_1) : w_0 \in W_0, w_1 \in W_1\} \\ &= \{w_0 : w_0 \in W_0\} = W_0 \end{aligned}$$

Part (b): Let $T : V \to V$ be s.t. $T^2 = T$. For $v \in V$, write $v = v - T(v) + T(v)$. We'll show that $v - T(v) \in \ker(T)$: let $v - T(v) = w$ for some $w \in V$. Then $T(v) - T^2(v) = T(w) \implies T(v) - T(v) = 0 = T(w)$, since $T^2 = T$, and thus $w = v - T(v) \in \ker(T)$. Clearly $T(v) \in \operatorname{Im}(T)$, so $V \subseteq \ker(T) + \operatorname{Im}(T)$. The $\supseteq$ direction is trivial, since $\ker(T) \subseteq V$ and $\operatorname{Im}(T) \subseteq V$, and so $V = \ker(T) + \operatorname{Im}(T)$.

It remains to show that $\ker(T) \cap \operatorname{Im}(T) = \{0\}$. Suppose a non-zero element v were in this intersection, and write $T(v) = 0$ and $v = T(w)$ for some $w \in V$. Then $T(v) = T^2(w) \implies T(v) = T(w) \implies 0 = T(w)$, so $w \in \ker(T)$. But we have $v = T(w)$, so $v = 0$ $\lightning$, and $\ker(T) \cap \operatorname{Im}(T) = \{0\} \implies V = \ker(T) \oplus \operatorname{Im}(T)$

The projection onto $\operatorname{Im}(T)$ along $\ker(T)$ is precisely $P : V \to V$ which sends $v \to w$, where $w \in \operatorname{Im}(T)$, and $v = w + y$ for some $y \in \ker(T)$. For *our* T, write $v = w + y$. Then $T(v) = T(w + y) = T(w) + T(y) = T(w)$, so T sends $v \to T(w)$.

Mental gymnastics: $w \in \operatorname{Im}(T)$, so $T(w) = T \circ T(v')$ for some *other* $v' \in V$. Thus, T really sends $v \to T^2(v') = T(v')$. But we said $w = T(v')$, so T sends $v \to w$.

$\implies T = P$, as defined above.

Part(c): Let $(x, y) \in \mathbb{R}^2$. Then $(x, y) = (0, y - x) + (x, x)$. Thus, for $V := \mathbb{R}^2$, the projection onto the y-axis along $\{(t, t) : t \in \mathbb{R}\}$ is the function $P : \mathbb{R}^2 \to \mathbb{R}^2$ which sends $(x, y) \to (0, y - x)$. *Fun little animation here.*

QUESTION 4

Consider the set $\tau = \{T_{v,w} : v \in \beta, w \in \gamma\}$, where β, γ are basis of V and W, respectively, and $\beta := \{v_1, ..., v_n\}$ is finite. This is a basis for $\text{Hom}(V, W)$.

Independence To consider a truly arbitrary subset of τ, we need to represent all $T_{v_i,\times}$ and, for $T_{v_i,\times}$, any number of $\times = w_i$. Thus, we form the following combination:

$$\star \quad a_{11}T_{v_1,w_1} + ... + a_{1k}T_{v_1,w_k} + ... + a_{nl}T_{v_n,w_l} + ... + a_{nm}T_{v_n,w_m} = \mathbb{0}$$

where $\mathbb{0}$ is the transformation that sends $v \to \mathbb{0}_W$.

This must hold for all $v_i \in \beta$, so we can evaluate the combination at $v = v_1$. Since $T_{v_1,w}(v_1) = w$ and $T_{v_j,w}(v_j) = 0$ for $i \neq j$, $w \in \gamma$, we have

$$a_{11}w_1 + ... + a_{1k}w_k = 0 \implies a_{11} = ... = a_{1k} = 0$$

since $w_i \in \gamma$ are members of a basis. Similarly, evaluating $\star$ at any v_j will imply that $a_{jk} = 0$, $w_k \in \gamma$. These are all our coefficients, so $\star$ is a trivial combination, and since any subset of τ is linearly independent, so is τ.

Spanning Consider a transformation $T : V \to W$, which sends $v_i \to w_i$ for $w_i \in W$. Remember $v_i \in \beta$.

$$\begin{aligned} T(v) &= T(a_1v_1 + ... + a_nv_n) \quad \text{for constants } a_i \in \mathbb{F} \\ &= a_1T(v_1) + ... + a_nT(v_n) = a_1w_1 + ... + a_nw_n \\ &= T_{v_1,w_1}(v) + ... + T_{v_n,w_n}(v) \quad \spadesuit \end{aligned}$$

where T_{v_i,w_i} sends $v_i \to w_i$ and $v_j \to 0$ for $j \neq i$. For this last step, see that

$$\begin{aligned} T_{v_i,w_i}(v) &= T_{v_i,w_i}(a_1v_1 + ... + a_nv_n) \\ &= a_1T_{v_i,w_i}(v_1) + ... + a_iT_{v_i,w_i}(v_i) + ... + a_nT_{v_i,w_i}(v_n) = a_iw_i \end{aligned}$$

Thus, it only remains to show that $T_{v_i,w_i} \in \text{Span}(\tau)$, but

This is not immediate, since $w_i \notin \gamma$ necessarily.

$$\begin{aligned} T_{v_i,w_i}(v) = a_iw_i &= a_i[b_1w_1^* + ... + b_nw_n^*] \quad w_i^* \in \gamma, b_i \in \mathbb{F} \\ &= a_i\left[\frac{b_1}{a_1}T_{v_1,w_1^*}(v) + ... + \frac{b_n}{a_n}T_{v_n,w_n^*}(v)\right] \end{aligned}$$

where $w_i^* \in \gamma$. The second line requires the following justification:

$$T_{v_1,w_1^*}(v) = T_{v_1,w_1^*}(a_1v_1 + ... + a_nv_n) = a_1w_1^*$$

Since $w_i^* \in \gamma$, $T_{v_i,w_i^*} \in \tau$, so $T_{v_i,w_i} \in \text{Span}(\tau)$. Thus, $\spadesuit$, i.e. $T(v)$, $\in \text{Span}(\tau)$. Clearly $\text{Span}(\tau) \subseteq \text{Hom}(V, W)$, so $\text{Span}(\tau) = \text{Hom}(V, W)$, and τ is a basis.

QUESTION 5

We need to show $L_{E_{ji}} = T_{v_i,w_j}$, where $v_i \in \beta$, $w_i \in \gamma$, the standard bases for $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively. If this is true, we conclude that

$$\{L_{E_{ij}} : i \in [1,m], j \in [1,n]\} = \{L_{E_{ji}} : j \in [1,m], i \in [1,n]\}$$
$$= \{T_{v_i,w_j} : i \in [1,n], j \in [1,m]\} = \{T_{v,w} : v \in \beta, w \in \gamma\}$$

as desired. Consider $L_{E_{ji}}$. This is the transformation that sends $v \to E_{ji} \cdot v$, where v is represented as a column vector $\langle a_1, ..., a_n \rangle$, $a_i \in \mathbb{F}$:

$$L_{E_{ji}}(v) = E_{ji} \cdot v = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 1_{ji} & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 \cancel{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + ... + a_i \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1_j \\ 0 \end{bmatrix}}_{=w_j} + ... + a_n \cancel{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = a_i w_j$$

And this is precisely $T_{v_i,w_j}(v)$, which expands to

$$T_{v_i,w_j}(v) = T_{v_i,w_j}(a_1 v_1 + ... + a_n v_n) = a_1 T_{v_i,w_j}(v_1) + ... + a_i T_{v_i,w_j}(v_i) + ... + a_n T_{v_i,w_j}(v_n) = a_i w_j$$

QUESTION 6

Linearity: We need to first show $[T_1+T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$ and $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for $a \in \mathbb{F}$, $T \in \mathrm{Hom}(V, W)$. Let $\beta = \{v_1, ..., v_n\}$ and $\gamma = \{w_1, ..., w_m\}$

$$[T_1+T_2]_\beta^\gamma = \begin{bmatrix} | & | & & | \\ [(T_1+T_2)(v_1)]_\gamma & [(T_1+T_2)(v_2)]_\gamma & \cdots & [(T_1+T_2)(v_n)]_\gamma \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ [T_1(v_1)]_\gamma + [T_2(v_1)]_\gamma & [T_1(v_2)]_\gamma + [T_2(v_2)]_\gamma & \cdots & [T_1(v_n)]_\gamma + [T_2(v_n)]_\gamma \\ | & | & & | \end{bmatrix} \quad \text{by linearity of } I_\gamma$$

$$= \begin{bmatrix} | & | & & | \\ [T_1(v_1)]_\gamma & [T_1(v_2)]_\gamma & \cdots & [T_1(v_n)]_\gamma \\ | & | & & | \end{bmatrix} + \begin{bmatrix} | & | & & | \\ [T_2(v_1)]_\gamma & [T_2(v_2)]_\gamma & \cdots & [T_2(v_n)]_\gamma \\ | & | & & | \end{bmatrix}$$

$$= [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$$

$$[aT]_\beta^\gamma = \begin{bmatrix} | & | & & | \\ [aT(v_1)]_\gamma & [aT(v_2)]_\gamma & \cdots & [aT(v_n)]_\gamma \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ a[T(v_1)]_\gamma & a[T(v_2)]_\gamma & \cdots & a[T(v_n)]_\gamma \\ | & | & & | \end{bmatrix} \quad \text{by linearity of } I_\gamma$$

$$= a\begin{bmatrix} | & | & & | \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & | & & | \end{bmatrix} = a[T]_\beta^\gamma$$

Inverse: If an inverse exists for $T \to [T]_\beta^\gamma$, then both are bijective. Since linearity has been shown, this is sufficient to show isomorphism. Consider $A \to I_\gamma^{-1} \circ L_A \circ I_\beta$.

We show the mapping $T \to [T]_\beta^\gamma \to I_\gamma^{-1} \circ L_{[T]_\beta^\gamma} \circ I_\beta$ is the identity on $\mathrm{Hom}(V, W)$, i.e. takes $T \to T$. However, we've previously seen that $I_\gamma \circ T = L_{[T]_\beta^\gamma} \circ I_\beta$, so $T = I_\gamma^{-1} \circ L_{[T]_\beta^\gamma} \circ I_\beta$, as desired.

We can take the inverse and apply it, since I_γ is an isormophism

We now need to show that $A \to I_\gamma^{-1} \circ L_A \circ I_\beta \to [I_\gamma^{-1} \circ L_A \circ I_\beta]_\beta^\gamma$ is the identity on $M_{m\times n}(\mathbb{F})$, i.e. takes $A \to A$:

$$[I_\gamma^{-1} \circ L_A \circ I_\beta]_\beta^\gamma = \begin{bmatrix} | & | & & | \\ [I_\gamma^{-1} \circ L_A \circ I_\beta(v_1)]_\gamma & [I_\gamma^{-1} \circ L_A \circ I_\beta(v_2)]_\gamma & \cdots & [I_\gamma^{-1} \circ L_A \circ I_\beta(v_n)]_\gamma \\ | & | & & | \end{bmatrix}$$

We have $[I_\gamma^{-1} \circ L_A \circ I_\beta(v_i)]_\gamma = [I_\gamma^{-1} \circ L_A([v_i]_\beta)]_\gamma = \left[I_\gamma^{-1}\left(A \cdot [v_i]_\beta\right)\right]_\gamma$, but $v_i \in \beta$, so $[v_i]_\beta = \langle 0, ..., 0, 1, 0, ..., 0\rangle$, where 1 is in the i^{th} position. Then $A \cdot [v_i]_\beta = A^{(i)}$. Finally, I_γ^{-1} is an isomorphism, so $v \to I_\gamma^{-1}(v) \to [I_\gamma^{-1}(v)]_\gamma = \mathrm{Id}$. Combining:

$\implies \left[I_\gamma^{-1}\left(A \cdot [v_i]_\beta\right)\right]_\gamma = A^{(i)}$. We return to the matrix form to find

$$[I_\gamma^{-1} \circ L_A \circ I_\beta]_\beta^\gamma = \begin{bmatrix} | & | & & | \\ A^{(1)} & A^{(2)} & \cdots & A^{(n)} \\ | & | & & | \end{bmatrix} = A$$

and we're done.

QUESTION 7

$$[T]_\alpha^\beta = \begin{bmatrix} | & | & | & | \\ [T(E_{11})]_\beta & [T(E_{12})]_\beta & [T(E_{21})]_\beta & [T(E_{22})]_\beta \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | & | \\ [1]_\beta & [1+t^2]_\beta & [0]_\beta & [2t]_\beta \\ | & | & | & | \end{bmatrix}$$

We now write

$$1 = 1(1) \qquad 1 + t^2 = 1(1) + 1(t^2) \qquad 0 = 0 \quad \text{and} \quad 2t = 2(t)$$

where $(\cdot)$ are our basis vectors. Then $[1]_\beta = \langle 1, 0, 0\rangle$, $[1 + t^2]_\beta = \langle 1, 0, 1\rangle$, $[0] = \langle 0, 0, 0\rangle$, and $[2t]_\beta = \langle 0, 2, 0\rangle$. Thus:

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Implicit in this calculation is the assumption that β is ordered exactly as $\{1, t, t^2\}$ and α as $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ (the final result would change otherwise to some permutation of these columns [order of α] and rows [order of β]).

Question 8

Part (a):

I show $T(\mathrm{Im}(T^k)) = \mathrm{Im}(T^{k+1})$ at the start of part (b), case $l = 1$.

Let $x \in \ker(T^{k+1})$. Then $T^{k+1}(x) = 0 \implies T^k(T(x)) = 0$, so $T(x) \in \ker(T^k)$, i.e. $x \in T^{-1}(\ker(T^k))$, so $\ker(T^{k+1}) \subseteq T^{-1}(\ker(T^k))$.

Now let $x \in T^{-1}(\ker(T^k))$. Then $T(x) \in \ker(T^k)$, i.e. $T^k(T(x)) = 0 \implies T^{k+1}(x) = 0$, so $x \in \ker(T^{k+1})$, and $T^{-1}(\ker(T^k)) \subseteq \ker(T^{k+1}) \implies T^{-1}(\ker(T^k)) = \ker(T^{k+1})$.

Part (b): Note that $T^l(\mathrm{Im}(T^k)) = \mathrm{Im}(T^{k+l})$:

$$T^l(\mathrm{Im}(T^k)) = \{T^l(x) : x \in \mathrm{Im}(T^k)\} = \{T^l(x) : x = T^k(y), y \in V\}$$
$$= \{T^l(T^k(y)) : y \in V\} = \{T^{k+l}(y) : y \in V\} = \mathrm{Im}(T^{k+l})$$

Then, assuming $\mathrm{Im}(T^k) = \mathrm{Im}(T^{k+1})$ for some $k \in \mathbb{N}$, we show $\mathrm{Im}(T^k) = T^l\mathrm{Im}(T^k)$ by induction on l, which shows $\mathrm{Im}(T^k) = \mathrm{Im}(T^{k+l})$ from above.

$$l = 1 \qquad T(\mathrm{Im}(T^k)) = \mathrm{Im}(T^{k+1}) = \mathrm{Im}(T^k)$$
$$l \to l+1 \qquad T^{l+1}(\mathrm{Im}(T^k)) = T \circ T^l(\mathrm{Im}(T^k)) = T(\mathrm{Im}(T^k)) = \mathrm{Im}(T^k)$$

Assume now $\ker(T^k) = \ker(T^{k+1})$ for some k. We'll show $\ker(T^k) = \ker(T^{k+l})$ by induction on l:

$l = 1 \quad \ker(T^k) = \ker(T^{k+1})$

$l \to l+1 \quad$ We know $\ker(T^k) \subseteq \ker(T^{k+l+1})$. Let $x \in \ker(T^{k+l+1})$. Then $T^{k+l}(T(x)) = 0$, so $T(x) \in \ker(T^{k+l})$. By ind. hyp., we have $T(x) \in \ker(T^k)$, so $x \in T^{-1}(\ker(T^k))$. From part (a), this means $x \in \ker(T^{k+1}) \implies x \in \ker(T^k)$ by assumption.

Part (c): Suppose $\mathrm{Im}(T^k) = \mathrm{Im}(T^{k+1})$. Then by dimension theorem, we have

$\dim(\mathrm{Im}(T^k)) + \dim(\ker(T^k)) = \dim(V) = \dim(\mathrm{Im}(T^{k+1})) = \dim(\ker(T^{k+1}))$

$\implies \dim(\ker(T^k)) = \dim(\ker(T^{k+1})) \implies \ker(T^k) = \ker(T^{k+1})$, using the fact that $\ker(T^k) \subseteq \ker(T^{k+1})$.

Similarly, if $\dim(\ker(T^k)) = \dim(\ker(T^{k+1}))$, then $\dim(\mathrm{Im}(T^k)) = \dim(\mathrm{Im}(T^{k+1}))$.

$\implies$ Since $\mathrm{Im}(T^{k+1}) \subseteq \mathrm{Im}(T^k)$, this means $\mathrm{Im}(T^k) = \mathrm{Im}(T^{k+1})$.

Part (d): Suppose $\ker(T^k) \neq \ker(T^{k+1}) \ \ \forall k \leq n$. Since $\ker(T^k) \subseteq \ker(T^{k+1})$, this means

$$\ker(T) \subsetneq \ker(T^2) \subsetneq ... \subsetneq \ker(T^n)$$

By monotonicity, this means

$$0 < \dim(\ker(T)) < \dim(\ker(T^2)) < ... < \dim(\ker(T^n)) \leq n$$

then we can notate below the minimum dimensions of each:

$$0 < \dim(\underbrace{\ker(T)}_{\geq 1}) < \dim(\underbrace{\ker(T^2)}_{\geq 2}) < ... < \dim(\underbrace{\ker(T^n)}_{\geq n}) \leq n$$

We conclude that $\dim(\ker(T^n)) = n$, but $\dim(\ker(T^n)) \leq \dim(\ker(T^{n+1})) \leq n$ by necessity, so $\dim(\ker(T^n)) = \dim(\ker(T^{n+1}))$, and we arrive at a contradiction. $\implies \exists k \leq n : \ker(T^k) = \ker(T^{k+1}) \implies \operatorname{Im}(T^k) = \operatorname{Im}(T^{k+1})$ by (c).

Note that, if $\dim(\ker(T)) = 0$, that means $\ker(T) = \{0\}$, so T is injective, and thus surjective. Then, $\ker(T^2)$ is
$\{x : T(T(x)) = 0\}$
$= \{y : T(y) = 0, y = \operatorname{Im}(T)\}$
$= \{y : T(y) = 0, y \in V\}$
$= \ker(T)$
which also establishes the contradiction

Question 9

Suppose $\exists k \in \mathbb{N}$ such that $T^k = 0$, i.e. $\mathrm{Im}(T^k) = \{0\}$. We can assume $k > n$, else $\mathrm{Im}(T^n) \subseteq \ldots \subseteq \mathrm{Im}(T^k) = \{0\} \implies \mathrm{Im}(T^n) = \{0\}$, i.e. $T^n = 0$.

From (8d), we know $\exists l \leq n$ such that $\mathrm{Im}(T^l) = \mathrm{Im}(T^{l+1})$, and from (8b) this means $\mathrm{Im}(T^l) = \mathrm{Im}(T^{l+l'})$ for any $l' \in \mathbb{N}$. Since $k > n \geq l$, let $l' := k - l$.

Then $\mathrm{Im}(T^l) = \mathrm{Im}(T^{l+(k-l)}) = \mathrm{Im}(T^k) = \{0\}$. Thus $\mathrm{Im}(T^l) = \{0\}$, or $T^l = 0$.

$l \leq n$, so by our first remarks, $T^n = 0$ $\square$