ANALYSIS 4 NOTES

NICHOLAS HAYEK

Lectures by Prof. Henri Darmon

CONTENTS

I Measure

MOTIVATION

In Analysis 3, we will formalize the concept of measure and study integration further. As motivation, consider the lower and upper Riemann integral:

$$\overline{\int_a^b} f(x)dx := \inf\left\{\sum_{i=1}^{n} \sup f_{[x_{i-1},x_i]}(x_i - x_{i-1})\right\}$$

$$\underline{\int_a^b} f(x)dx := \sup\left\{\sum_{i=1}^{n} \inf f_{[x_{i-1},x_i]}(x_i - x_{i-1})\right\}$$

where $a = x_0 < x_1 < ... < x_n = b$. Recall that f is called Riemann integrable if $\overline{\int_a^b} f = \underline{\int_a^b} f$, and we write instead $\int_a^b f$. Note that not all functions are integrable in this sense.

Consider $f : [0,1] \to \mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q} \cap [0,1]$ and 0 otherwise. Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are both dense in $\mathbb{R}$, and in particular $[0,1]$, we conclude that $\overline{\int_a^b} f = 1$ and $\underline{\int_a^b} f = 0$. Thus, f is not Riemann integrable.

We introduce the Lebesgue integral as an alternative. Let $A_i := \{x \in [a,b] : y_i \leq f(x) < y_{i+1}\}$, where the y_i's are increasing. See that now $\sum y_i |A_i| \approx \int_a^b f$. The following questions arise from this:

1. What *is* the "size" of A_i?
2. What sets *can* we measure?

σ-ALGEBRAS

Let X be a non-empty set, and let $\mathcal{F}$ be a collection of subsets of X. We call $\mathcal{F}$ a *σ-algebra* of subsets of X if the following hold:

1. $X \in \mathcal{F}$.
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ ("closed under compliments")
3. If $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, then $\cup_{n=1}^{\infty} \in \mathcal{F}$ ("closed under countable unions").

We can derive the following from these axioms: PROP. 1.1

1. $\varnothing \in \mathcal{F}$

2. If $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, then $\cap_{i=1}^{\infty} \in \mathcal{F}$
3. If $A_1, ..., A_N \in \mathcal{F}$, then $\cap A_i$ and $\cup A_i \in \mathcal{F}$

$A \triangle B := (A \setminus B) \cup (B \setminus A)$

4. If $A, B \in \mathcal{F}$, then $A \setminus B$, $B \setminus A$, and $A \triangle B \in \mathcal{F}$

For a set X, consider $\mathcal{F} = 2^X := \{A : A \subseteq X\}$, the powerset of X. This is the largest σ-algebra of X. The smallest one can construct is $\mathcal{F} = \{\varnothing, X\}$. If we'd like to include a particular subset of X, say A, we can write $\mathcal{F} = \{\varnothing, X, A, A^c\}$.

Let X be a space and $\mathcal{C}$ be a collection of subsets of X. The *σ-algebra generated by $\mathcal{C}$*, denoted by $\sigma(\mathcal{C})$, is defined by the following:

1. $\sigma(\mathcal{C})$ is a σ-algebra with $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. If $\mathcal{F}$ is a σ-algebra with $\mathcal{C} \subseteq \mathcal{F}$, then $\mathcal{F} \supseteq \sigma(\mathcal{C})$.

We also say that $\sigma(\mathcal{C})$ is the "σ-algebra generated by $\mathcal{C}$"

In other words, $\sigma(\mathcal{C})$ is the smallest σ-algebra which contains $\mathcal{C}$. From the example above, we can write $\sigma(A) = \{\varnothing, X, A, A^c\}$.

Prop 1.2

We can state the following about σ-algebras generated by $\mathcal{C}$:

1. $\sigma(\mathcal{C}) = \cap\{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-algebra}, \mathcal{C} \subseteq \mathcal{F}\}$
2. If $\mathcal{C}$ is a σ-algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$
3. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$.