VECTOR CALCULUS NOTES

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CONTENTS

I Curves and Surfaces

PRODUCTS ON VECTOR SPACES

Recall the definition of the *inner product* over a vector space V: DEF 1.1

1. $\langle u, v\rangle = \overline{\langle v, u\rangle} = \langle v, u\rangle$ in $\mathbb{R}$ (where we'll be in this class)
2. $\langle au + bw, v\rangle = a\langle u, v\rangle + v\langle w, v\rangle$
3. $\langle u, u\rangle \geq 0$, and $= 0 \iff u = \mathbb{0}$

From this, we define the *norm* of $u \in V$ to be $\|u\| := \sqrt{\langle u, u\rangle}$. This is well-defined, since $\langle u, u\rangle \geq 0$. DEF 1.2

$$\forall u, v \in V, |\langle u, v\rangle| \leq \|u\|\|v\|$$

PROP 1.1 Cauchy-Schwartz Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

PROP 1.2 Triangle Inequality

The *cross product* of $u, v \in \mathbb{R}$, with respect to $\mathbb{R}^3$, is the determinate of the following "matrix": DEF 1.3

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where $u = \langle u_1, u_2, u_3\rangle$ and $v = \langle v_1, v_2, v_3\rangle$. We observe the following two properties of the cross product in $\mathbb{R}^3$: PROP 1.3

1. $(u \times v) \cdot u = 0$
2. $\|u \times v\| = \|u\|\|v\|\sin(\theta)$, where θ is the angle found between u and v. A conceptualization of this property is that "u-cross-v is equal to the area created by the parallelogram bounded by u and v."

LINES

Define a *line* $l(t) \in \mathbb{R}^n$ to be a function from $\mathbb{R} \to \mathbb{R}^n$, with the primary form $l(t) = P + td$, with $P, d \in \mathbb{R}^n$, $t \in \mathbb{R}$. We call P the "point vector" and d the "direction vector" An alternate form, with two points $P, Q \in \mathbb{R}^n$, would be $l(t) = (1-t)P + tQ$, where $l(t)$ lies along the path between P and Q for $t \in [0, 1]$. DEF 1.4

Distance between a point and line Using this definition, how an we find the shortest path between a point R and a line $l(t)$, which lies between P and Q?

Idea 1 We know the desired vector $w = PR\sin(\theta)$, the angle between PR and PQ. To find this value, note that $\|PR \times PQ\| = \|PR\|\|PQ\|\sin(\theta)$.

Idea 2 We can project R onto PQ, and then subtract this projection from PR.

Idea 3 We can minimize a distance function between R and a point on l, i.e. $l(t)$. Thus, we take $\min_{t\in\mathbb{R}} \|R - l(t)\| = \alpha$, and then take $Rl(\alpha)$ to be the shortest path.

Idea 4 We can find when $(R - l(t)) \cdot d = 0$.

Distance between 2 lines Consider two lines, l_1 and l_2, which do not intersect but are not necessarily parallel. What is the minimal distance between l_1 and l_2?

Sometimes called "skew lines"

Idea 0 Conceptualize this problem as finding the distance between the parallel planes defined by $\{l_1, l_2\}$.

Idea 1 We can minimize $\|l_1(t) - l_2(s)\|$ (really, one should minimize the square to make one's life easier).

Idea 2 Pick any two points, say $l_1(T)$ and $l_2(S)$, and project $l_1(T)l_2(S)$ onto $l_1 \times l_2$.

Idea 3 Minimize $\text{dist}(l_1(t), l_2)$ for fixed t.

Idea 4 Find t and s such that $[l_1(t) - l_2(s)] \cdot \vec{d_1} = 0$ and $[l_1(t) - l_2(s)] \cdot \vec{d_2} = 0$

PROP 1.4

$\|u \times v\| = \|u\|\|v\| \sin(\theta) =$ Area of parallelogram defined by u and v.

PLANES

DEF 1.5

A plane $r(s,t)$ is a function $[0,1]^2 \to \mathbb{R}^3$ defined by $d_1, d_2 \in \mathbb{R}^3$, two vectors, and $P \in \mathbb{R}^3$, a point. In particular, $r(s,t) = P + s\vec{d_1} + t\vec{d_2}$. This is called the *parametric form*.

DEF 1.6

The *point-normal* form is a function $\mathbb{R}^2 \to \mathbb{R}^3$ is given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $\vec{n} = \langle a, b, c \rangle$ is a vector normal to the plane, and $P = \langle x_0, y_0, z_0 \rangle$ is a point lying on the plane.

Distance between a point R and a plane r

Idea 1 Minimize $\|R - r(s,t)\|$ (or the square)

Idea 2 $\|\text{proj}_{\vec{n}}(P - R)\|$, where $\vec{n}$ and P are as given in the point-normal form.

TRANSFORMATIONS AND PARAMETERIZATIONS

The following table give general examples of linear transformations $\lambda : \mathbb{R}^n \to \mathbb{R}^m$.

Dimension	Linear	Affine
$n = 0$	$\lambda(0) = 0$	$\lambda(0) = P$
$n = 1$	$\lambda(t) = t\vec{d}$	$\lambda(t) = P + t\vec{d}$
$n = 2$	$\lambda(t,s) = t\vec{d_1} + s\vec{d_2}$	$\lambda(t,s) = P + t\vec{d_1} + s\vec{d_2}$
$n = 3$	$\lambda(t,s,r) = t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$	$\lambda(t,s,r) = P + t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$

We also define the following important curves in $\mathbb{R}^2$:

Type	Explicit Form	Parametric Form
Ellipse	$x^2 + y^2 = 1$	$r(t) = \left\langle t, \sqrt{1-t^2}\right\rangle_{t\in[-1,1]} = \langle \cos(t), \sin(t)\rangle_{t\in[-\pi,\pi]}$
Hyperbola	$x^2 - y^2 = 1$	$r(t) = \left\langle \sqrt{1+t^2}, t\right\rangle_{t\in\mathbb{R}} = \langle \cosh(t), \sinh(t)\rangle_{t\in\mathbb{R}}$
Parabola	$x = y^2$	
Double Cone	$x^2 = y^2$	
Any Function	$y = F(x)$	$r(t) = \langle t, F(t)\rangle$

Define a *path* in $\mathbb{R}^m$ to be a continuous function $r : \mathbb{R} \to \mathbb{R}^m$, e.g. $[a, b] \to \mathbb{R}^m$. DEF 1.7

Define a *curve* in $\mathbb{R}^m$ to be the image of a path (i.e. a set of points in $\mathbb{R}^m$). Recall the statement "paths parameterize curves." DEF 1.8

For example, the unit circle $x^2 + y^2 = 1$ is parameterized by the path $r : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = \langle \cos(t), \sin(t)\rangle$.

Define the *tangent* line of $\vec{r}$ at $a \in \mathbb{R}$ to be an affine transformation $l : \mathbb{R} \to \mathbb{R}^m$ satisfying the following:

1. $l(t) = r(a) + (t-a)\vec{d} : \vec{d} \neq 0$
2. $\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} = 0$

♠ *Examples* ♣ E.G. 1.1

We'll now find the derivative of the unit circle at a point $a \in \mathbb{R}$: we have $r(a) = \langle \cos(a), \sin(a)\rangle$. Thus:

$$l(t) = \langle \cos(t), \sin(t)\rangle + (t-a)\langle d_1, d_2\rangle$$

Where $\langle d_1, d_2\rangle \neq 0$. Consider now the limit in question 2:

$$\begin{aligned}
\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} &= \lim \frac{1}{|t-a|}\sqrt{(\cos(t)-\cos(a)-(t-a)d_1)^2 + (\sin(t)-\sin(a)-(t-a)d_2)^2} \\
&= \lim \sqrt{\left(\frac{\cos(t)-\cos(a)}{t-a} - d_1\right)^2 + \left(\frac{\sin(t)-\sin(a)}{t-a} - d_2\right)^2} \\
&\underset{t\to a}{=} \sqrt{(-\sin(a)-d_1)^2 + (\cos(a)-d_2)^2} = 0 \\
&\iff d_1 = -\sin(a) \wedge d_2 = \cos(a) \\
&\implies l(t) = \langle -\sin(a), \cos(a)\rangle \quad \square
\end{aligned}$$

DIFFERENTIATION AND CONTINUITY

Frequently, $l(t)$ is referred to as the "velocity vector" of $r(t)$, and is notated as $r'(t)$. Notice that $r'(t)$ is equivalent to the component-wise derivative of the coordinates of $r(t)$ w.r.t. t. Formally:

DEF 1.9 Given $\vec{r}: \mathbb{R} \to \mathbb{R}^n$, the *derivative* of $\vec{r}$ at $a \in \mathbb{R}$ is a linear transformation $\vec{\lambda}: \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\lim_{t\to a} \frac{\|r(t) - r(a) - \lambda(t-a)\|}{|t-a|} = 0 \quad \text{or equivalently} \quad \lim_{h\to 0} \frac{\|r(a+h) - r(a) - \lambda(h)\|}{|h|} = 0$$

It is denoted $D\vec{r}_a$, and represented by the $n \times 1$ matrix $r'(a)$. One may now rewrite the tangent line in the form $l(t) = r(a) + \lambda(t-a)$.

DEF 1.10 The *arc length* of a curve $r(t)$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

DEF 1.11 An *arc length parameterization* of $r(t)$ is some $t = \alpha(s)$ such that $r(\alpha(s))$ has a unit velocity vector, i.e. $\|r'(\alpha(s))\| = 1$. Alternatively, one could find an expression for arc length, and then parameterize $r(t)$ in terms of its arc length. The resultant will be equivalent.

DEF 1.12 $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $\vec{a}$ if, for any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon \;\forall \vec{x} \in \mathbb{R}^n$$

E.G. 1.2 ♠ *Examples* ♣

We'll do an arc length parameterization of a semicircle of radius 1 with its center at the origin, i.e. $y = \sqrt{1-x^2}$. We get the natural parameterization $r(t) = \left\langle t, \sqrt{1-t^2} \right\rangle$, where $t \in [-1, 1]$. We'd like to find a change of parameters $t = \alpha(s)$ such that $\|r(\alpha(s))\| = 1$ and $\alpha' \geq 0$.

$$\begin{aligned} r(\alpha(s)) &= \left\langle \alpha(s), \sqrt{1-\alpha(s)^2} \right\rangle \\ r'(\alpha(s)) &= \left\langle \alpha'(s), \frac{1}{2}(1-\alpha(s)^2)^{-\frac{1}{2}} \cdot (-2\alpha(s)\alpha'(s)) \right\rangle \\ &= \alpha'(s) \left\langle 1, \frac{-\alpha(s)}{\sqrt{1-\alpha(s)^2}} \right\rangle \\ \text{Then } 1 = \|r'(\alpha(s))\| &= \alpha'(s) \sqrt{1 + \frac{\alpha(s)^2}{1-\alpha(s)^2}} \\ &= \frac{\alpha'(s)}{\sqrt{1-\alpha(s)^2}} \end{aligned}$$

Integrating with respect to s, we get $s = \arcsin(\alpha(s)) = \arcsin(t)$. Thus, $t = \sin(s)$, and $s \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, and we yield the parameterization $\langle \sin(s), \cos(s) \rangle : s \in [\frac{-\pi}{2}, \frac{\pi}{2}]$.

SURFACES

We note the following quadric surfaces:

Type	Explicit Form
Ellipsoid	$x^2 + y^2 + z^2 = 1$
Elliptic Hyperboloid	$x^2 + y^2 - z^2 = 1$
Elliptic Paraboloids	$x^2 + y^2 - z^2 = -1$
Hyperbolic Paraboloids	$x = y^2 - z^2$
Double Cones	$x^2 = y^2 + z^2$

A surface $F(x, y)$ is called *differentiable* at (a, b) if there exists some linear transformation $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that DEF 1.13

$$\lim_{(h,k)\to(0,0)} \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\| \langle h, k \rangle \|}$$

One may represent $\lambda(h,k) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = uh + vk$

♠ *Examples* ♣ E.G. 1.3

Let $F(x, y) = xy$. We consider F at (a, b). Then

$$\begin{aligned} 0 \le \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\| \langle h, k \rangle \|} &= \frac{|(a+h)(b+k) - ab - (uk + vk)|}{\| \langle h, k \rangle \|} \\ &= \frac{|bh + ak + hk - uh - vk|}{\| \langle h, k \rangle \|} = \frac{|(b-u)h + (a-v)k + hk|}{\| \langle h, k \rangle \|} \\ &\le \frac{|b-u||h|}{|h|} + \frac{|a-v||k|}{|k|} + \frac{|h||k|}{|h|} \quad \text{since } |h|, |k| \le \| \langle h, k \rangle \| \\ &= |b-u| + |a-v| + |k| \to |b-u| + |a-v| \\ &= 0 \quad \text{when } b = u, a = v \end{aligned}$$

Thus, the desired limit is always $\ge$ and ≤ 0, so especially it is 0. Our derivative at (a, b) is then $\lambda(x, y) = bx + ay$.

One may also find these coefficients as the partial derivative of F, i.e.

$$\nabla F(a,b) = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle \Bigg|_{(a,b)}$$

DEF 1.14

This is called the *gradient*. Similarly, $\alpha(x,y) = F(a,b) + \lambda(x-a, y-b)$ is called the *affine approximation* at (a,b).

PROP 1.5

Note that the converse is *false* (as a counterexample, see $F = \sqrt{|xy|}$)

If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then all partial derivatives of F at $\vec{a}$ exist. Furthermore, $\lambda(\vec{a}) = F'(\vec{a}) = \left[\partial_1 F \cdots \partial_n F\right]\Big|_{\vec{a}}$.

1.1 Partial Converse

If all partial derivatives of $F : \mathbb{R}^n \to \mathbb{R}$ exist near $\vec{a}$ and are continuous at $\vec{a}$, then F is differentiable at $\vec{a}$.

PROOF FOR $n = 2$.

Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a linear transformation defined by $\left[\partial_1 F \cdots \partial_n F\right]\Big|_{\vec{a}}$. Then

$$\lambda(\vec{h}) = \sum_{i=1}^{n} \partial_i F(\vec{a}) h_i$$

Let $n = 2$. Then

$$\begin{aligned}
|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})| &= |F(a_1+h_1, a_2+h_2) - F(a_1+h_1, a_2) + F(a_1+h_1, a_2) \\
&\quad - F(a_1,a_2) - \partial_1 F(\vec{a})h_1 - \partial_2 F(\vec{a})h_2| \\
&\leq |F(a_1+h_1, a_2+h_2) - F(a_1+h_1, a_2) - \partial_2 F(\vec{a})h_2| \\
&\quad + |F(a_1+h_1, a_2) - F(a_1,a_2) - \partial_1 F(\vec{a})h_1| \\
&= |\partial_2 F(\vec{c})h_2 - \partial_2 F(\vec{a})h_2| + |\partial_1 F(\vec{d})h_1 - \partial_1 F(\vec{a})h_1| \\
&\qquad \text{by mean value thm.} \\
&= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})||h_2| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})||h_1| \\
\frac{|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})|}{\|\vec{h}\|} &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})|\frac{|h_2|}{\|\vec{h}\|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})|\frac{|h_1|}{\|\vec{h}\|} \\
&\leq |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})|\frac{|h_2|}{|h_2|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})|\frac{|h_1|}{|h_1|} \\
&\qquad \text{since } |h_i| < \|\vec{h}\| \\
&= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})|
\end{aligned}$$

Then, as $\vec{h} \to 0$, $\vec{c}, \vec{d} \to \vec{a}$. Since F, is continuous, we know $F(\vec{c}) \to F(\vec{a})$ and similarly for $F(\vec{d})$. Thus,

$$|\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \to |\partial_2 F(\vec{a}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{a}) - \partial_1 F(\vec{a})| = 0$$

And we conclude that the limit, as $\leq$ and ≥ 0, is 0. □

DEF 1.15

$F : \mathbb{R}^n \to \mathbb{R}$ is called C^1 continuous (or *continuously differentiable*) at $\vec{a}$ if all partial exists near $\vec{a}$ and are continuous at $\vec{a}$.

Note that the converse to our partial converse is *not* true: i.e. if F is differentiable at $\vec{a}$, it is not necessarily continuously differentiable at $\vec{a}$. Some counter examples include $F(x, y) = |y|$ and $F(x) = x^2 \sin(\frac{1}{x})$ s.t. $x \neq 0$ and 0 otherwise.

We have an alternative and equivalent definition of differentiability. Let E be continuous and $= 0$ at 0. Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a linear transformation. Then PROP 1.6

$$F(\vec{a}+\vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + ||\vec{h}||E(\vec{h}) \qquad \forall h$$

implies differentiability.

♠ *Examples* ♣ E.G. 1.4

In our previous example, we prove (laboriously) that $F(x, y) = xy$ is differentiable for all (a, b). We can now use Thm 1.1 to show this result: the partial derivatives $F_x = y$ and $F_y = x$ exist and are continuous $\forall x, y \in \mathbb{R}$, so F is differentiable $\forall x, y \in \mathbb{R}$.

1.2 Characterization of the Derivative

Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$. The derivative at $\vec{a}$ exists if:

1. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\lim_{\vec{h}\to\vec{0}} \frac{||F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})||}{||\vec{h}||} = 0$$

2. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ and a function E such that

$$F(\vec{a}+\vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + ||\vec{h}||E(\vec{h})$$

   and $E(0) = 0$ is continuous at 0.

Such a λ is unique when found, and is called the derivative. We denote it by $D\vec{F}_{\vec{a}}$.

This follows from Def 1.12 and Thm 1.1. □ PROOF.

We may represent the partial derivatives of $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m = \langle F_1, ..., F_m\rangle$ using a *Jacobian* matrix, denoted $F'(\vec{a})$, and defined as follows: DEF 1.16

$$\left[TBD \right]$$

PROP 1.7
Chain Rule

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{a} \in \mathbb{R}^n$. Let $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$. Then

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l \quad \text{is differentiable at } \vec{a}$$

and $D\vec{h}_{\vec{a}} = D\vec{g}_{\vec{b}} \circ D\vec{f}_{\vec{a}}$. Furthermore, their Jacobians obey

$$h'(a) = g'(b)f'(a)$$

(matrix multiplication)

E.G. 1.5

♠ *Examples* ♣

1. Consider $f(x,y) = \langle x+y, x-y \rangle$ and $g(x,y) = \frac{1}{4}x^2 - \frac{1}{4}y^2$. Then $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}$ is given by
$$\frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$$
Let $\vec{a} = \langle a_1, a_2 \rangle$. Then $f(a) = b = \langle a_1 + a_2, a_1 - a_2 \rangle$. What about the Jacobian of f?
$$f'(a) = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix}\Bigg|_{(a_1,a_2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
Similarly, for g we have
$$g'(b) = \begin{bmatrix} \partial_1 g & \partial_2 g \end{bmatrix}\Big|_{(a_1+a_2, a_1-a_2)} = \begin{bmatrix} \frac{1}{2}(a_1+a_2) - \frac{1}{2}(a_1-a_2) \end{bmatrix}$$
Then, by the chain rule, we multiple these two matrices to yield
$$\begin{bmatrix} \frac{1}{2}(a_1+a_2) - \frac{1}{2}(a_1-a_2) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$
One can (less) manually find that $h = g \circ f$ is xy, and conclude the same.

2. Let S be a surface in R^3 given by $F(x,y,z) = 0$ (this is called a "level surface," e.g. $xy - z = 0$). Let $P = (a,b,c)$ be a point on F, and let C be a curve in S containing P, parameterized by $r(t)$.

   Denote $r(t) = \langle x(t), y(t), z(t) \rangle$. Then $g = F \circ r = F(x(t), y(t), z(t)) = 0$. By chain rule, we have $0 = g'(t_0) = F'(P) \cdot r'(t_0)$, where we choose t_0 such that $r(t_0) = \langle a, b, c \rangle$. Then, we observe that
$$0 = \nabla F(P) \bullet \vec{v}(t_0) \implies \nabla F(P) \perp \vec{v}(t_0)$$
   Where $\vec{v} = r'$ is the velocity vector of r. By considering all curves that satisfy our construction $C \subset S$, we yield the tangent plane of S at P with normal vector $\vec{n} = \nabla F(P)$. In particular, the point-normal form of the tangent plane of a surface F at $P = (a,b,c)$ is given by
$$\boxed{\partial_x F(P)(x-a) + \partial_y F(P)(y-b) + \partial_z F(P)(z-c) = 0}$$

3. Generally, we can consider $S^{n-1} \subset \mathbb{R}^n$ of $F : \mathbb{R}^n \to \mathbb{R}$. (This is called a *hypersurface*). Suppose this is differentiable at $P \in S$. Let $C \subset S$ be a curve in S through P, parameterized by $r : \mathbb{R} \to \mathbb{R}^n$ and differentiable at t_0 with $r(t_0) = P$.

   Then, by the chain rule, $v(t_0) \perp \nabla F(P)$. If $v(t_0) \neq 0$, then the tangent line to C at P has derivative $r(t_0)$. If $\nabla F(P) \neq 0$, then the tangent hyperplane to S at P has a normal vector $n = \nabla F(P)$.

DEF 1.17

Let $\mathbb{R}^n \to \mathbb{R}, \vec{a}, \vec{h} \in \mathbb{R}^n$. Let $l(t) = a + th$. Then the *directional derivative* of F along h at a, denoted $\partial_{\vec{h}} F(\vec{a})$, is given by

$$\lim_{t \to 0} \frac{F(a+th) - F(a)}{t}$$

Then, if F is differentiable at a, we have the more useful form

Thus, if $h = e_1$, then $\partial_{e_1} F(\vec{a}) = \partial_1 F(\vec{a})$.

$$\partial_{\vec{h}} F(\vec{a}) = \vec{h} \cdot \nabla F(\vec{a}) = \sum_{i=1}^{n} h_i \partial_i F(\vec{a})$$

PROP 1.8
Mean Value Thm.

Let $F : \mathbb{R}^n \to R$ be differentiable, and let $a, h \in \mathbb{R}^n$, with $h \neq 0$. Then

$$F(a+h) - F(a) = \partial_{\vec{h}} F(c_h) = h \nabla F(c_h) \quad c_h \in [a, a+h]$$

Note that, since a, h are vectors, by $c_h \in [a, a+h]$ we mean that c_h lies along the line segment connecting a and $a+h$.

We now restate the chain rule:

1.3 Chain Rule

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{a}$. Let $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{b} = F(\vec{a})$. Then

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l$$

is differentiable at $\vec{a}$ and $h'(\vec{a}) = g'(\vec{b}) \circ f'(\vec{a})$.

PROOF.

Let λ be the derivative of f. Let $\vec{t}, \vec{s}$ be arbitrary. Then we have

$$f(\vec{a} + \vec{t}) - f(\vec{a}) = \lambda(\vec{t}) + \|\vec{t}\| \varepsilon_1(\vec{t})$$

where $\varepsilon_1 : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $\vec{0}@\vec{0}$. Similarly, for g:

$$g(\vec{b} + \vec{s}) - g(\vec{b}) = \mu(\vec{s}) + \|\vec{s}\| \varepsilon_2(\vec{s})$$

where μ is the derivative of g, and ε_2 is as above. Our goal is to write $h = g \circ f$

in the same manner. Let $\nu = \mu \circ \lambda$. Then

$$\begin{aligned}
h(\vec{a}+\vec{t}) - h(\vec{a}) &= g(f(\vec{a}+\vec{t})) - g(f(\vec{a})) \\
&= g(f(\vec{a}) + \underbrace{\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})}_{:=\vec{s}}) - g(f(\vec{a})) \\
&= \mu(\vec{s}) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \mu(\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \mu(\lambda(\vec{t})) + \|\vec{t}\|\mu(\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \nu(\vec{t}) + \|\vec{t}\|\underbrace{\left(\mu(\varepsilon_1(\vec{t})) + \frac{\|\vec{s}\|}{\|\vec{t}\|}\varepsilon_2(\vec{s})\right)}_{=\varepsilon_3(\vec{t})} \quad \text{if } \vec{t} \neq 0
\end{aligned}$$

$$\begin{aligned}
\vec{t} \neq 0 \implies 0 \leq \|\varepsilon_3(\vec{t})\| &\leq \|\mu(\varepsilon_1(\vec{t}))\| + \frac{\|\lambda(\vec{t})\| + \|\vec{t}\|\|\varepsilon_1(\vec{t})\|}{\|\vec{t}\|}\|\varepsilon_2(\vec{s})\| \\
&\leq M\|\varepsilon_1(\vec{t})\| + (L + \|\varepsilon_1(\vec{t})\|)\|\varepsilon_2(\vec{s})\| \\
&(\text{where} \lambda(\vec{t}) \leq L\|\vec{x}\| \text{ and } \mu(\vec{x})) \leq M\|\vec{x}\|) \\
&\implies \lim_{\vec{t}\to 0} \varepsilon_3(\vec{t}) = 0 \quad \square
\end{aligned}$$

DEF 1.18
Iterated Partial Derivatives

Suppose $g = \partial_i f$ is defined near $\vec{a} \in \mathbb{R}^n$, where $F : \mathbb{R}^n \to \mathbb{R}$. Then if $\partial_j g$ exists at $\vec{a}$, we call it a 2^{nd} order partial derivative of f at $\vec{a}$. We denote this $\partial_j \partial_i f(\vec{a})$, where $i, j \in [1, n]$.

1.4 Mixed Partials are Equal

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $\vec{a} = \langle a_1, a_2 \rangle$. Let $\partial_1 f, \partial_2 \partial_1 f$ exist near $\vec{a}$, with $\partial_2 \partial_1 f$ continuous at $\vec{a}$. Suppose further that $\partial f(x, a_2)$ is defined near $x = a_1$.

$\implies \partial_1 \partial_2 f$ is defined at $\vec{a}$ and $\partial_1 \partial_2 f(\vec{a}) = \partial_2 \partial_1 f(\vec{a})$.

PROOF.

$$\partial_1 \partial_2 f(\vec{a}) = \lim_{h_1 \to 0} \underbrace{\frac{\partial_2 f(a_1 + h_2) - \partial_2 f(a_1, a_2)}{h_1}}_{\beta(h_1):\mathbb{R}_{\neq 0}\to\mathbb{R}}$$

$$\begin{aligned}
\implies \beta(h_1) &= \frac{\lim_{h_2\to 0} \frac{f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2)}{h_2} - \lim_{h_2 \to 0} \frac{f(a_1, a_2+h_2) - f(a_1, a_2)}{h_2}}{h_1} \\
&= \lim_{h_2 \to 0} \underbrace{\frac{1}{h_2} \frac{(f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2)) - (f(a_1, a_2 + h_2) - f(a_1, a_2))}{h_1}}_{\alpha(h_1, h_2):\mathbb{R}^2_{\neq 0}\to\mathbb{R}}
\end{aligned}$$

Now, for a break...

If $\lim_{\vec{h}\to\vec{0}} \alpha(\vec{h})$ exists, then $\lim_{h_1\to 0} \beta(h_1)$ exists, where $\beta(h_1) = \lim_{\vec{h}\setminus h_1\to 0} \alpha(h_1, (\vec{h}\setminus h_1))$. Furthermore, we conclude PROP 1.9

$$\lim_{h_1\to 0} \beta(h_1) = \lim_{\vec{h}\to\vec{0}} \alpha(\vec{h})$$

PROOF (CONTINUED). Now, it's enough to show that $\lim_{\vec{h}\to\vec{0}} \alpha(\vec{h}) = \partial_2\partial_1 f(\vec{a})$. By the Mean Value Thm, we have

$$\begin{aligned}\alpha(\vec{h}) &= \frac{1}{h_2}\left(\partial_1 f(c_1, a_2 + h_2) - \partial_1 f(c_1, a_2)\right)\\ &= \partial_2\partial_1 f(c_1, c_2) : c_2 \in [a_2, a_2 + h]\end{aligned}$$

Let $\vec{c} = \langle c_1, c_2\rangle$. Then as $\vec{h} \to \vec{0}$, we have $\vec{c} \to \vec{a}$. Thus

$$\lim_{\vec{h}\to\vec{0}} = \lim_{\vec{c}\to\vec{a}} \partial_2\partial_1 f(\vec{c}) = \partial_2\partial_1\vec{a} \qquad \square$$

$f : \mathbb{R}^n \to \mathbb{R}$ is *k-times continuously differentiable* at $\vec{a}$ if all k^{th}-order partial derivatives exist near $\vec{a}$ and are continuous at $\vec{a}$. DEF 1.19

We say that f is k-times continuously differentiable *near* $\vec{a}$ if it is continuously differentiable at $\vec{a}$ and all k-th order partial derivatives are continuous near $\vec{a}$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable at $\vec{a}$, then all mixed partial derivatives are equal at $\vec{a}$. PROP 1.10

If f is k-time continuously differentiable at $\vec{a}$, then the $(k-1)$-order partial derivatives are continuously differentiable (hence differentiable and continuous) at $\vec{a}$. PROP 1.11

is the following a proof? proposition?

PROOF. Let $\vec{h} \in \mathbb{R}^n, \vec{l} : \mathbb{R} \to \mathbb{R}^n$ given by $\vec{l}(t) = \vec{a} + t\vec{h}$. Set $g := f \circ \vec{l} : \mathbb{R} \to \mathbb{R}$, i.e. $g(t) = f(\vec{a} + t\vec{h})$.

Then let f be k-times continuously differentiable at $\vec{a}$. Then g is k-times differentiable at 0, and we have

$$\partial^i_{\vec{h}} f(\vec{a}) = g^{(i)}(0) \underset{\text{CR}}{=} (\vec{h}\cdot\nabla)^i f\Big|_{\vec{a}}$$

For example, with $n = 2$, we have

$$\partial^2_{\vec{h}} = (\vec{h}\cdot\nabla)(\vec{h}\cdot\nabla) = (h_1\partial_1 + h_2\partial_2)(h_1\partial_1 + h_2\partial_2)$$

$\square$

1.5 Multivariable Taylor's Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be k-times continuously differentiable near $\vec{a}$ with $\vec{a} \in \mathbb{R}^n$. Let $\alpha_j : \mathbb{R}^n \to \mathbb{R}$ be a degree j homogeneous polynomial, i.e. all non-zero terms have the same degree.

Let $E : \mathbb{R}^n \to \mathbb{R}$ be such that

$$\begin{cases} \bullet\ f(\vec{a}+\vec{h}) - f(\vec{a}) = \alpha_1(\vec{h}) + \ldots + \overbrace{\alpha_k(\vec{h}) + \underbrace{\|h\|^k E(\vec{h})}_{R_k(\vec{h})}}^{R_{k-1}(\vec{h})} \ \forall \vec{h} \\ \\ \bullet\ E(\vec{0}) = 0 \end{cases}$$

To find such an E, we can take

$$E(\vec{h}) = \begin{cases} \frac{1}{\|\vec{h}\|^k}(f(\vec{a}+\vec{h}) - f(\vec{a}) - \alpha_1(\vec{h}) - \ldots - \alpha_k(\vec{h})) & \vec{h} \neq 0 \\ \vec{0} \quad \vec{h} = 0 \end{cases}$$

Then Taylor's Theorem states:

$$E \text{ continuous at } \vec{0} \iff \alpha_j(\vec{h}) = \frac{1}{j!}\partial^j_{\vec{h}} f(\vec{a}) = \frac{1}{j!}(\vec{h} \bullet \nabla)^j f(\vec{a}) \quad \forall j \in [1,k]$$

If E is continuous at $\vec{a}$ and $\vec{h} \neq \vec{0}$ is near $\vec{0}$, then:

$$R_{k-1}(\vec{h}) = \frac{1}{k!}\partial^k_{\vec{h}} f(\vec{c}_h)$$

where $\vec{c} \in [\vec{a}, \vec{a}+\vec{h}]$.

MIDTERM REVIEW

Recall that the directional derivative is defined as follows

$$\partial_{\vec{h}} f(\vec{a}) := \lim_{t\to 0} \frac{f(\vec{a}+t\vec{h}) - f(\vec{a})}{t} = \lim_{t\to 0} \frac{g(t)-g(0)}{t} = g'(0) \qquad g(t) := f(\vec{a}+t\vec{h})$$

An *iterated directional derivative*, denoted $\partial^i_{\vec{h}} f(\vec{a})$, is then

$$g^{(i)}(0)$$

If f is i-times continuously differentiable at $\vec{a}$, then we can write

$$\partial^i_{\vec{h}}(\vec{a}) = (\vec{h} \cdot \nabla)^i f(\vec{a})$$

II Integration

RIEMANN INTEGRATION

On Hypercubes

DEF 2.1 Let $\mathcal{B}$ be a box in $\mathbb{R}^n$. Choose $F : \mathbb{R}^n \to \mathbb{R}$ which is bounded on the box. Then, informally, F is *integrable* if the limit of its Riemann summation is equivalent across all orderings of tagged partitions.

PROP 2.1 By the extreme value theorem, if F is continuous on $\mathcal{B}$, then F is bounded on $\mathcal{B}$.

2.1 Integrability Criterion

If F is continuous on $\mathcal{B}$, then F is integrable over $\mathcal{B}$.

2.2 Fubini

Let $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathcal{B}$. Then

$$\int_{\mathcal{B}} F\,dV^n = \int_{x_n=a_n}^{x_n=b_n} \cdots \left(\int_{x_1=a_1}^{x_1=b_1} F(x_1, \ldots, x_n)\,dx_1 \right) \cdots dx_n$$

Furthermore, the order of integration doesn't matter.

PROP 2.2 $\int_a^b g(x)dx = g(c)(b-a)$ where $a < c < b$.

PROOF. $\frac{G(b)-G(a)}{b-a} = G'(c) = g(c)$ by the mean value theorem and the FTC. $\square$

2.3

The set of discontinuities of F in $\mathcal{B}$ has zero measure $\iff$ F is integrable over $\mathcal{B}$.

Note that this theorem is not useful in MATH 248, and its proof is out of the scope of this course.

Point-Set Topology

DEF 2.2 A set $S \subseteq \mathbb{R}^n$ has *zero measure* if $\forall \varepsilon > 0$ we can choose a set of open balls such that

$S \subseteq \bigcup B(x_i, \varepsilon_i)$ where $\sum \text{vol}(B(x_i, \varepsilon_i)) < \varepsilon$.

In general, hypersurfaces in $\mathbb{R}^n$ have zero measure. Thus, if $F : \mathbb{R}^n \to \mathbb{R}$ is continuous except on a hypersurface, F is still integrable.

$\vec{p} \in \text{Int}(S)$ is called an *interior point* of S if $\exists \varepsilon > 0$ such that $B(\vec{p}, \varepsilon) \subseteq S$. DEF 2.3

1. If $S \subseteq \mathbb{R}^n$ has zero measure and $S' \subseteq S$, then S' has zero measure. PROP 2.3
2. If $S \subseteq \mathbb{R}^n$ has zero measure, then S has no interior points.

Let $S \subseteq \mathbb{R}^n$. Then DEF 2.4

1. $\text{Int}(S)$, the *interior of* S, is the set of all interior points of S
2. S is called *open* if $S = \text{Int}(S)$.
3. S^c, the *compliment of* S, is $\mathbb{R}^n \setminus S$.
4. $p \in S^c$ is called an *exterior point* of S if $\exists \varepsilon > 0$ with $B(p, \varepsilon) \subseteq S^c$.
5. $\text{Ext}(S)$, the *exterior* of S, is the set of all exterior points of S.
6. S is *closed* if $S^c = \text{Ext}(S)$.
7. $p \in \mathbb{R}^n$ is called a *boundary point* of S if $p \notin \text{Int}(S) \wedge p \notin \text{Ext}(S)$.
8. The *boundary* of S, denoted ∂S, is the set of all boundary points of S.
9. S is *bounded* if $\exists \mathcal{B}$ with $S \subseteq \mathcal{B} \subsetneq \mathbb{R}^n$.

S is closed $\iff$ S^c is open $\iff$ S contains its boundary. PROP 2.4

On Arbitrary $\mathbb{R}^n$ Subsets

Let $\mathscr{D} \subseteq \mathbb{R}^n$ be closed and bounded. Let $f : \mathscr{D} \to \mathbb{R}^n$ be some function. $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{f}(x) = \begin{cases} f(x) & x \in \mathscr{D} \\ 0 & \text{o.w.} \end{cases}$$

is called the *trivial extension of* f.

f is integrable over $\mathscr{D}$ if its trivial extension is integrable over a box $\mathcal{B} \supseteq \mathscr{D}$. PROP 2.5

2.4

Let $\mathscr{D} \subseteq \mathbb{R}^n$ be closed and bounded, with a boundary that has zero measure. Then, if $f : \mathscr{D} \to \mathbb{R}$ is continuous on $\mathscr{D}$, then f is integrable.

PROOF.

If f is continuous on $\mathscr{D}$, then $\hat{f}$ is continuous on both $\mathrm{Int}(\mathscr{D})$ and $\mathrm{Ext}(\mathscr{D})$ (for any point in either of these sets, we can find epsilon balls centered at the point and contained in the set—within these intervals $\hat{f} = f$). Thus, since $\mathscr{D} = \mathrm{Int}(\mathscr{D}) \cup \mathrm{Ext}(\mathscr{D}) \cup \partial D$, the set of discontinuities of $\hat{f}$ has at most measure 0. Hence, $\hat{f}$ is integrable over any box containing $\mathscr{D}$, and hence f is integrable over $\mathscr{D}$ by Prop 2.5. $\square$

DEF 2.5 $\mathscr{D} \subseteq \mathbb{R}^2$ is called *y-simple* if, for $a, b \in \mathbb{R}$ and $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ continuous, we may write

$$\mathscr{D} = \begin{cases} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{cases}$$

Similarly, $\mathscr{D}$ is *x-simple* if

$$\mathscr{D} = \begin{cases} a \leq y \leq b \\ g_1(y) \leq x \leq g_2(y) \end{cases}$$

Note that, since $x \in [a, b]$ is closed (hence compact), $g_1(x)$ and $g_2(x)$ are bounded. We reason similarly for x-simple domains.

DEF 2.6 $\mathscr{D} \subseteq \mathbb{R}^2$ is *elementary* if it is y- or x-simple. It is *simple* if it is both.

2.5 Fubini

If $\mathscr{D} \subseteq \mathbb{R}^n$ is elementary and $f : \mathscr{D} \to \mathbb{R}$ is continuous, then

- $\mathscr{D}$ is y-simple $\implies \iint\limits_{\mathscr{D}} f\,dA = \int\limits_{x=a}^{x=b} \int\limits_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$
- $\mathscr{D}$ is x-simple $\implies \iint\limits_{\mathscr{D}} f\,dA = \int\limits_{y=a}^{y=b} \int\limits_{x=g_1(y)}^{x=g_2(y)} f(x,y)\,dx\,dy$

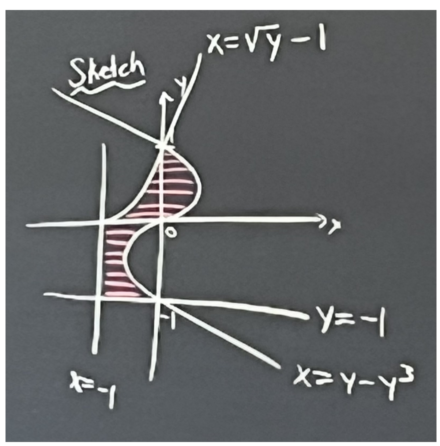
E.G. 2.1 ♠ *Examples* ♣

1. Consider $\iint_{\mathscr{D}} (1+2y)\,dA$, where $\mathscr{D}$ is bounded by $y = 2x^2$ and $y = 1 + x^2$. We first find the intersection between these two curves: $2x^2 = 1 + x^2 \implies x = \pm 1$.

Then, by Thm 2.5 ($\mathscr{D}$ is y-simple), we write

$$\begin{aligned}\iint\limits_{\mathscr{D}}(1+2y)dA &= \int\limits_{x=-1}^{x=1}\int\limits_{2x^2}^{1+x^2}(1+2y)dydx = \int\limits_{-1}^{1} y+y^2\Big|_{2x^2}^{1+x^2} \\ &= \int\limits_{-1}^{1}(1+x^2)+(1+x^2)^2-2x^2-4x^4 \\ &= \int\limits_{-1}^{1} 1+x^2+1+x^4+2x^2-2x^2-4x^4 \\ &= \int\limits_{-1}^{1} -3x^4+x^2+2 = \frac{-3}{5}x^5+\frac{1}{3}x^3+2x\Big|_{-1}^{1} = 2\frac{-3}{5}+2\frac{1}{3}+4 \\ &= 2\left(\frac{-9}{15}+\frac{5}{15}+\frac{30}{15}\right) = \frac{52}{15}\end{aligned}$$

2. Consider $\iint_{\mathscr{D}} y dA$, where $\mathscr{D}$ is bounded by $x = y - y^3$, $x = \sqrt{y} - 1$, $x = -1$, and $y = -1$ (OOF). By Thm 2.5 (y-simple):



We split this up into two x-simple graphs, one in $y \in [-1, 0]$, and one in $y \in [0, 1]$. Then we have $\iint\limits_{\mathscr{D}} = I_1 + I_2$, with

$$I_1 = \int\limits_0^1 \int\limits_{\sqrt{y}-1}^{y-y^3} y dxdy \qquad I_2 = \int\limits_{-1}^{1}\int\limits_{-1}^{y-y^3} y dxdy$$

Computing this integral a hassle. Try it yourself.

3. We may also flip the bounds of integration using Thm 2.5. For example, consider $\int_0^3 \int_y^3 \sin(x^2)\,dx\,dy$. This is a non-elementary integral to evaluate in x. But observe that our bounds are equivalent to $y \in [0, x]$ and $x \in [0, 3]$, so we may re-write this as $\int_0^3 \int_0^x \sin(x^2)\,dy\,dx$.

We pick up an x, not, after integrating WRT y, so this is easy to evaluate!

DEF 2.7

A set $S \subseteq \mathbb{R}^n$ is called *path-connected* if, for every $a, b \in S$, there exists a continuous mapping containing a and b (i.e., there exists a path between them).

DEF 2.8

In $\mathscr{D} \subseteq \mathbb{R}^n$, we call $\mathscr{D}$ *elementary* if it is closed, bounded, and both its interior and boundary are path-connected.

This is distinct from elementary-ness of $\mathscr{D} \subseteq \mathbb{R}^2$, which we characterized by y and x simple-ness.

DEF 2.9

Let $\mathscr{D}, \mathscr{D}^*$ be elementary subsets of $\mathbb{R}^n$. Let $T : \mathscr{D}^* \to \mathscr{D}$. We call T *onto* , or *surjective*, if the whole of $\mathscr{D}$ is mapped to, i.e. $\forall d^* \in \mathscr{D} \exists d \in \mathscr{D} : T(d) = d'$.

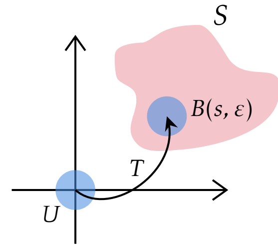
DEF 2.10

Using the same notation, we call T *one-to-one*, or *injective*, if no two points share a mapping, i.e. $\forall d_1^*, d_2^* \in \mathscr{D}^*$, we have $T(d_1^*) = T(d_2^*) \implies d_1^* = d_2^*$.

DEF 2.11

$S \subseteq \mathbb{R}^n$ is a *hypersurface* if, $\forall s \in S$, $\exists \varepsilon > 0$, an open set $\vec{0} \in U$, and a function $T : U \to B(s, \varepsilon)$ such that

- T is injective on $\text{Int}(\mathscr{D}^*)$ and also surjective
- $T(U \cap \{s = \langle x_1, ..., x_n \rangle : x_n = 0\}) = S \cap B(s, \varepsilon)$



2.6 Change of Variables

Let $T : \mathscr{D}^* \to \mathscr{D}$ be continuously differentiable on $\text{Int}(\mathscr{D}^*)$ (i.e. all partial derivatives exist and are continuous on $\text{Int}(\mathscr{D}^*)$). Let T' be the Jacobian induced by T. Let $F^* = F \circ T$.

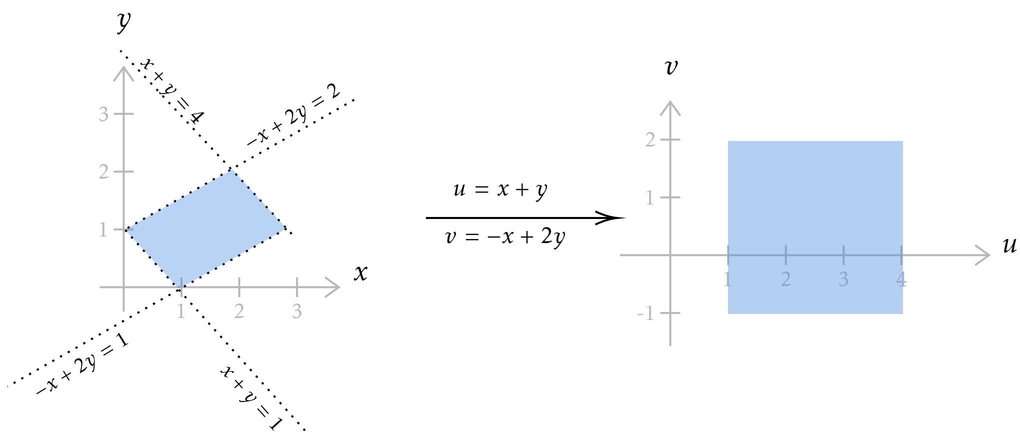
If $F : \mathscr{D} \to \mathbb{R}$ is integrable over $\mathscr{D}$, then F^* is integrable over $\mathscr{D}^*$ and

$$\int_{\mathscr{D}} F\,dV = \int_{\mathscr{D}^*} F^* |\det(T)|\,dV$$

For example, in $n = 2$ polar coordinates, $\int\limits_{\mathscr{D}} F dA = \int\limits_{\mathscr{D}^*} F^* r dA$. For this, see that the relevant Jacobian is

$$T' = \begin{bmatrix} \partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \implies |\det(T')| = |r| = r$$

Consider the area of the following parallelogram:



Then, $x = \frac{2u-v}{3}$ and $y = \frac{u+v}{3}$. Hence, we compute our Jacobian and conclude that $\det(T') = \frac{1}{3}$. However, we may also compute the determinate of the *inverse*'s Jacobian, i.e. $u = x + y$ and $v = -x + 2y$, which will yield 3, and invert the result.

Hence, since the area of the left rectangle is 9, we get an area of 3 for the parallelogram.

2.7 Mean Value Theorem in $\mathbb{R}^n$

Let $F : \mathscr{D} \to \mathbb{R}$ be integrable over an elementary region $\mathscr{D} \subseteq \mathbb{R}^n$. Let $\overline{F} := \int\limits_{\mathscr{D}} F dV \frac{1}{\mathrm{vol}(\mathscr{D})}$ be the mean value of F. Then

$$\exists c \in \mathscr{D} : F(c) = \overline{F}$$

Let $\delta : \mathscr{D} \to \mathbb{R}_+$ be a density function (which is integrable). Then define $\mathrm{mass}(\mathscr{D}) = \int\limits_{\mathscr{D}} \delta dV$. Then the center of mass $x \in \mathscr{D}$ is given by

$$x_i = \frac{\int\limits_{\mathscr{D}} x_i \delta dV}{\mathrm{mass}(\mathscr{D})}$$