
ALGEBRA 3 NOTES

NICHOLAS HAYEK

Lectures by Prof. Henri Darmon

CONTENTS

I Groups	1
Axioms and First Properties	1

I Groups

8/28/24

In Algebra 3, we will study abstract algebraic structures. Chiefly among them, we have *groups*, which are useful in representing symmetries, *rings & fields*, which help us think about number systems, and *vector spaces & modules*, which encode physical space.

AXIOMS AND FIRST PROPERTIES

A *group* is a set G endowed with a binary composition $G \times G \rightarrow G$ such that the following axioms hold:

1. $\exists e \in G$, an identity element, such that $e * a = a * e = a \forall a \in G$.
2. $\forall a \in G, \exists a' \in G$ such that $a * a' = a' * a = e$.
3. $a * (b * c) = (a * b) * c \forall a, b, c \in G$.

If $a * b = b * a \forall a, b \in G$, we call G *commutative*.

Why do we care about groups? If X is an object, we call a *symmetry* of X a function $X \rightarrow X$ which preserves the structure of the object.

e.g. a polygon, graphs, tilings, "crystal," "molecules," rings, vector spaces, metric spaces, manifolds

The collection of symmetries, $\text{Aut}(X) = \{f : X \rightarrow X\}$, we can structure as a group: let $*$ be composition, $e = \text{Id}$, and $f \in \text{Aut}(X)$ (note that, by axiom 2, these must be bijective).

A note on notation: for non-commutative groups, we write $a * b = ab$, $e = 1$ or $\mathbb{1}$, $a' = a^{-1}$, and $a^n = \underbrace{a \cdot \dots \cdot a}_{n \text{ times}}$. This is called *multiplicative notation*. For commutative rings, we write $a * b = a + b$, $e = 0$ or $\mathbb{0}$, $a' = -a$, and $na = \underbrace{a + \dots + a}_{n \text{ times}}$.

The following are some examples of groups generated by sets:

1. If X is a set with no operations, $\text{Aut}(X)$ is the set of all bijections $f : X \rightarrow X$. One calls this the *permutation group*, or, if $|X| = n < \infty$, the *symmetric group*, and we write $\text{Aut}(X) = S_n$.
2. If V is a vector space over \mathbb{F} , $\text{Aut}(V) = \{T : V \rightarrow V\}$, the set of vector space isomorphism. If $\dim(V) = n$, recall that we associate V with \mathbb{F}^n , whose set of isomorphism is given by $GL_n(\mathbb{F})$, the collection of $n \times n$ invertible matrices. This is called the *linear group*.
3. If R is a ring, then $(R, +, \mathbb{0})$ is a commutative group. Furthermore, $(R^\times, \times, \mathbb{1})$ is a non-commutative group, where $R^\times := R \setminus \{\text{non-invertible elements of } R\}$.
4. If V is Euclidean space endowed with a dot product, where $\mathbb{F} = \mathbb{R}$, with $\dim(V) < \infty$, $\text{Aut}(V) = O(V)$ is called the *orthogonal group of V* . In particular, $O(V) = \{T : V \rightarrow V : T(u) \cdot T(v) = u \cdot v\}$.

5. If X is a geometric figure (e.g. a polygon), we write $\text{Aut}(X) = D_n$, where $|\text{Aut}(X)| = n$, and call this the *dihedral group*.

A *homomorphism* from groups $G_1 \rightarrow G_2$ is a function $\varphi : G_1 \rightarrow G_2$ satisfying $\varphi(ab) = \varphi(a)\varphi(b)$, where $a, b \in G_1$.

$$\varphi(1_{G_1}) = 1_{G_2} \text{ and } \varphi(a^{-1}) = \varphi(a)^{-1} \forall a \in G_1.$$

PROP. 1.1

PROOF.

$$\begin{aligned} \varphi(1_{G_1}) &= \varphi(1_{G_1}^2) = \varphi(1_{G_1})^2 \implies \varphi(1_{G_1}) = \varphi(1_{G_1}^{-1})\varphi(1_{G_1}) = 1_{G_2}. \\ \varphi(a^{-1})\varphi(a) &= \varphi(a^{-1}a) = \varphi(1_{G_1}) = 1_{G_2} \implies \varphi(a^{-1}) = \varphi(a)^{-1}. \end{aligned}$$

□

A homomorphism which is bijective is called an *isomorphism*. If there exists an isomorphism between two groups G_1 and G_2 , we call them *isomorphic*, and write $G_1 \cong G_2$. One can thus call $\text{Aut}(G)$ the set of isomorphisms from $G \rightarrow G$.

As an example, take $G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. Note that $\varphi : G \rightarrow G$ is determined entirely by $\varphi(1)$, since $\varphi(i) = \underbrace{\varphi(1 + \dots + 1)}_{i \text{ times}} = \underbrace{\varphi(1) + \dots + \varphi(1)}_{i \text{ times}}$. How can we find

an element of $\text{Aut}(G)$? Clearly, not all mappings $\varphi(1)$ are bijective: take n to be even and $\varphi(1) = 2$. Then $\varphi(2) = 4, \varphi(3) = 6, \dots, \varphi(n/2) = 0$, so φ is not surjective. We know then that $\varphi(G) = \varphi(1)\mathbb{Z} \pmod n$, and would like $\varphi(G) = G$. If $\varphi(1)$ and n are co-prime, then we can write $k\varphi(1) + ln = k\varphi = 1$, so every element can be reached.

We can construct a group isomorphism $\eta : \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ which sends $\varphi \rightarrow \varphi(1)$. Clearly $\eta(\varphi_{t_1} \circ \varphi_{t_2}) = \varphi_{t_1} \circ \varphi_{t_2}(1) = \varphi_{t_1}(t_2) = t_1 t_2 = \eta(\varphi_{t_1})\eta(\varphi_{t_2})$, so η is a homomorphism. It is also bijective: given $\varphi(1)$, we can deduce a mapping for each element.

For a group G and an object X , define an *action* to be a function from $G \times X \rightarrow X$ such that

1. $1 \times x = x$
2. $(g_1 g_2)x = g_1(g_2 x)$

for $x \in X, g_1, g_2 \in G$. One can create from this the automorphism $m_g : x \rightarrow gx$ of X : if $gx_1 = gx_2$, one can take the group inverse to conclude $x_1 = x_2$. Similarly, given $x \in X$, we know $m_g(g^{-1}x) = x$.

PROP. 1.2

Given an action of G on X , the assignment $g \rightarrow m_g$ is a homomorphism between $G \rightarrow \text{Aut}(X)$.

PROOF.

$$m_{g_1 g_2}(x) = g_1 g_2 x = g_1(g_2 x) = g_1 m_{g_2}(x) = m_{g_1}(m_{g_2}(x)) = m_{g_1} \circ m_{g_2}(x)$$

□

9/4/24

In fact, given a homomorphism of this form, one can extract the group action.

A G -set is a set X endowed with a group action of G . If $\forall x, y \in X, \exists g \in G : gx = y$, we say that this G -set is *transitive*. Finally, a transitive G -set of a subset of X (“ G -subset of X ”) is called an *orbit* of G on X .

Every G -set is a disjoint union of orbits.

PROP 1.3

We define a relation on X as follows: $x \underset{G}{\sim} y$ if $\exists g : gx = y$. This is an equivalence relation:

PROOF.

1. Take $g = 1$. Then $1x = x$, so $x \underset{G}{\sim} x$.
2. If $gx = y$, then $g^{-1}y = x$, so $x \underset{G}{\sim} y \implies y \underset{G}{\sim} x$.
3. If $gx = y$ and $hy = z$, then $hgx = z$, so $x \underset{G}{\sim} y \wedge y \underset{G}{\sim} z \implies x \underset{G}{\sim} z$.

From prior theory, we know that equivalence classes of an equivalence relation on X form a partition of X . However, by definition, the equivalence classes of the above relation are exactly the orbits of the G -set on X . \square

We denote the set of equivalence classes defined in the proof above X/G .

Examples:

1. Let $X = \{\clubsuit\}$, G be a group, and $g\clubsuit = \clubsuit$. This is a group action. The homomorphism $m : G \rightarrow \text{Aut}(X) = S_1$ sends g to the identity.
2. Let $X = G$, G be a group, and $gx = gx$ (group action on the LHS, left-multiplication on the RHS). We have the homomorphism $m : G \rightarrow \text{Aut}(G)$ such that $m(g)(x) = gx = gx$. This is an injective function, since we can always take the group inverse, i.e. $m(h)(x) = m(g)(x) \implies g = h$. Thus, $G \cong m(G) \subseteq \text{Aut}(G)$.
3. Let $X = G$ as before, but let $gx = xg^{-1}$. We can check that this is a group action: (1) $1 * x = x1^{-1} = x1 = x$ and (2) $g * (h * x) = (h * x)g^{-1} = xh^{-1}g^{-1}$, where $(gh) * x = x(gh)^{-1} = xh^{-1}g^{-1} \implies g * (h * x) = (gh) * x$.
4. Letting $X = G \times G$, we can form a group action from both left- and right-multiplication: $(g, h) * x = gxh^{-1}$. One can check its validity.

1.1 Cayley

Every group G is isomorphic of a group of permutations (i.e. a subgroup of

a symmetric group). If G is finite, then G is isomorphic to S_n , where $n = |G|$.

If X_1 and X_2 are G -sets, then an *isomorphism* from X_1 to X_2 is a bijection $\varphi : X_1 \rightarrow X_2$ such that $\varphi(gx) = g\varphi(x) \forall x \in X_1, g \in G$.

SOME STUFF