Stochastic Processes

MATH 447

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We assume working knowledge of probability and no knowledge of measure theory (though a grasp of analysis is essential). See these MATH 356 notes, also taught by Louigi!

I Introduction

PROBABILITY REVIEW

Essential to studying stochastic processes are continuous conditional expectations.

Recall $\mathbb{E}[X|Y = y_0]$, where X, Y are random variables. If Y is continuous, writing $\mathbb{E}[X|Y = y_0] = \frac{\mathbb{P}(X,Y=y_0)}{\mathbb{P}(Y=y_0)}$, will not work. Instead, we consider the slice of the joint density function $f(x,y)$ at $y = y_0$. The result is a one dimensional function $g(x)$ which may not have probability 1. Hence, we divide by $\int g(x)$ to make it into a density function:

$$\mathbb{E}[X|Y = y_0] = \int_{\mathbb{R}} \frac{x}{\int_{\mathbb{R}} f(x, y_0)dx} f(x, y_0)dx$$

DEF 1.1 We frequently write $f_{X|Y}(x) = \frac{f(x,y)}{\int_{\mathbb{R}} f(x,y)dx}$, and call this the *conditional density* of X given Y. For fixed y, then, $\mathbb{E}[X|Y = y] = \mathbb{E}[Z]$, where $Z \sim f_{X|Y}$

Before providing definitions, we give some examples of stochastic processes:

Eg. 1.1 A simple random walk: $S_{i+1} = S_i + X_i$, where $X_i \sim \text{Ber}(p)$ and $S_0 = 0$. We might ask: does S_i ever return to 0, i.e.

$$\mathbb{P}(\exists i > 0 : S_i = 0)$$

Eg. 1.2 A branching process: as in asexual reproduction, we have an initial node. Each node n has a number of children X_n, where $\frac{X_n}{2} \sim \text{Ber}(p)$. We denote Z_i to be the number of inididuals in the i-th generation. We might ask: does Z_i ever have no children, i.e.

$$\mathbb{P}(\exists i > 0 : Z_i = 0)$$

Eg. 1.3 Choose k independent random points in the square $[0, \sqrt{k}]^2$. On average, then, there is 1 point within any unit square $U \subseteq [0, \sqrt{k}]^2$.