ANALYSIS 3 NOTES

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CONTENTS

I Measure

MOTIVATION

In Analysis 3, we will formalize the concept of measure and study integration further. As motivation, consider the lower and upper Riemann integral: 8/28/23

$$\overline{\int_a^b} f(x)dx := \inf\left\{\sum_{i=1}^{n} \sup f_{[x_{i-1},x_i]}(x_i - x_{i-1})\right\}$$

$$\underline{\int_a^b} f(x)dx := \sup\left\{\sum_{i=1}^{n} \inf f_{[x_{i-1},x_i]}(x_i - x_{i-1})\right\}$$

where $a = x_0 < x_1 < \ldots < x_n = b$. Recall that f is called Riemann integrable if $\overline{\int}_a^b f = \underline{\int}_a^b f$, and we write instead $\int_a^b f$. Note that not all functions are integrable in this sense. For example:

Consider $f : [0,1] \to \mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q} \cap [0,1]$ and 0 otherwise. Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are both dense in $\mathbb{R}$, and in particular $[0,1]$, we conclude that $\overline{\int}_a^b f = 1$ and $\underline{\int}_a^b f = 0$. Thus, f is not Riemann integrable.

We introduce the Lebesgue integral as an alternative. Let $A_i := \{x \in [a,b] : y_i \le f(x) < y_{i+1}\}$, where the y_i's are increasing. See that now $\sum y_i|A_i| \approx \int_a^b f$. The following questions arise from this:

1. What *is* the "size" of A_i?
2. What sets *can* we measure?

σ-ALGEBRAS

Let X be a non-empty set, and let $\mathcal{F}$ be a collection of subsets of X. We call $\mathcal{F}$ a *σ-algebra* of subsets of X if the following hold:

1. $X \in \mathcal{F}$.
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ ("closed under compliments")
3. If $\{A_n : n \ge 1\} \subseteq \mathcal{F}$, then $\cup_{n=1}^{\infty} \in \mathcal{F}$ ("closed under countable unions").

We can derive the following from these axioms: Prop. 1.1

1. $\varnothing \in \mathcal{F}$

2. If $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, then $\cap_{i=1}^{\infty} \in \mathcal{F}$
3. If $A_1, ..., A_N \in \mathcal{F}$, then $\cap A_i$ and $\cup A_i \in \mathcal{F}$

$A \triangle B := (A \setminus B) \cup (B \setminus A)$

4. If $A, B \in \mathcal{F}$, then $A \setminus B$, $B \setminus A$, and $A \triangle B \in \mathcal{F}$

For a set X, consider $\mathcal{F} = 2^X := \{A : A \subseteq X\}$, the powerset of X. This is the largest σ-algebra of X. The smallest one can construct is $\mathcal{F} = \{\varnothing, X\}$. If we'd like to include a particular subset of X, say A, we can write $\mathcal{F} = \{\varnothing, X, A, A^c\}$.

Let X be a space and $\mathcal{C}$ be a collection of subsets of X. The *σ-algebra generated by $\mathcal{C}$*, denoted by $\sigma(\mathcal{C})$, is defined by the following:

1. $\sigma(\mathcal{C})$ is a σ-algebra with $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. If $\mathcal{F}$ is a σ-algebra with $\mathcal{C} \subseteq \mathcal{F}$, then $\mathcal{F} \supseteq \sigma(\mathcal{C})$.

We also say that $\sigma(\mathcal{C})$ is the "σ-algebra generated by $\mathcal{C}$"

In other words, $\sigma(\mathcal{C})$ is the smallest σ-algebra which contains $\mathcal{C}$. From the example above, we can write $\sigma(A) = \{\varnothing, X, A, A^c\}$.

Prop 1.2

We can state the following about σ-algebras generated by $\mathcal{C}$:

1. $\sigma(\mathcal{C}) = \cap\{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-algebra}, \mathcal{C} \subseteq \mathcal{F}\}$
2. If $\mathcal{C}$ is a σ-algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$
3. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$.

Proofs.

1. Let $\mathcal{D}$ be some σ-algebra containing $\mathcal{C}$, and let $\{\mathcal{F}_i\}$ denote all σ-algebras containing $\mathcal{C}$. Then $\cap_{i=1}^{\infty}\{\mathcal{F}_i\} \subseteq \mathcal{D}$, since $\mathcal{D} \in \{\mathcal{F}_i\}$. We also have to show that $\cap_{i=1}^{\infty}\{\mathcal{F}_i\}$ is a σ-algebra. Clearly $X \in \cap_{i=1}^{\infty}\{\mathcal{F}_i\}$, since it must be in all $\mathcal{F}_i$. Now, let $A \in \cap_{i=1}^{\infty}\{\mathcal{F}_i\}$. Then $A \in \mathcal{F}_i \ \forall i$, so $A^c \in \mathcal{F}_i \ \forall i$. Thus, $A^c \in \cap_{i=1}^{\infty}\{\mathcal{F}_i\}$. Similarly, suppose $\{A_n\} \subseteq \cap_{i=1}^{\infty}\{\mathcal{F}_i\}$. Then $\{A_n\} \subseteq \mathcal{F}_i \ \forall i$, and therefore $\cup_{n=1}^{\infty}\{A_n\} \in \mathcal{F}_i \ \forall i$, so we conclude $\cup_{n=1}^{\infty}\{A_n\} \in \cap_{i=1}^{\infty}\{\mathcal{F}_i\}$. Hence, $\{F_i\}$ is a σ-algebra.
2. Suppose otherwise. Then $\exists$ a σ-algebra containing fewer subsets than $\mathcal{C}$, and yet containing at least all subsets of $\mathcal{C}$. This cannot be.
3. Note that $\{\mathcal{F} : \mathcal{C}_1 \subseteq \mathcal{F}\} \supseteq \{\mathcal{F} : \mathcal{C}_2 \subseteq \mathcal{F}\}$, since $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Thus, $\cap\{\mathcal{F} : \mathcal{C}_1 \subseteq \mathcal{F}\} \subseteq \cap\{\mathcal{F} : \mathcal{C}_2 \subseteq \mathcal{F}\}$, so $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$. $\square$

Measurable Spaces

8/30/23

A *Borel σ-algebra*, denoted $\mathscr{B}_{\mathbb{R}}$, is the σ-algebra generated by all the open subsets of $\mathbb{R}$.

Prop. 1.3

Recall that, for any open $G \subseteq \mathbb{R}$, we can write $G = \cup_{n=1}^{\infty} I_n$, where I_n are finite, disjoint, open intervals.

Proof.

Let G be open. Consider any $x \in G \cap \mathbb{Q}$. G is open $\implies \exists$ an open $x \in I \subseteq G$. Choose the largest such interval (i.e. the union of all intervals containing x). One may associate any rational number in G with an interval of this kind.

Furthermore, for $y \in G \cap \mathbb{Q}^c$, $\exists$ a neighborhood which necessarily contains a rational number (by density), and is therefore contained within an I. Note now: the set of I's are countable, since they are generated by elements of $\mathbb{Q}$; the set of I's are pairwise disjoint, since, otherwise, the union of intersecting sets would constitute a larger-than-maximal set containing x. □

The generation of $\mathscr{B}_{\mathbb{R}}$ is *not* unique, so while $\sigma\{(a,b) : a, b \in \mathbb{R}, a < b\}$ is obvious, there exist other equivalent descriptions:

$$\begin{aligned}
\mathscr{B}_{\mathbb{R}} &= \sigma\{(a,b] : a, b \in \mathbb{R}, a < b\} \\
&= \sigma\{[a,b] : a, b \in \mathbb{R}, a < b\} \\
&= \sigma\{[a,b) : a, b \in \mathbb{R}, a < b\} \\
&= \sigma\{(-\infty, c) : c \in \mathbb{R}\} \\
&= \sigma\{(c, \infty) : c \in \mathbb{R}\}
\end{aligned}$$

As proof of $\mathscr{B}_{\mathbb{R}} = \sigma(\{[a,b) : a < b\})$, it is sufficient to show that (a,b) is in this set, and similarly that $[a,b)$ is in $\sigma\{(a,b) : a < b\}$. For the first, we can write $(a,b) = \cup_{n=1}^{\infty}[a + \frac{1}{n}, b)$, and for the second, $[a,b) = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b)$

If $A \in \mathscr{B}_{\mathbb{R}}$, we call A a *Borel set*. All intervals on $\mathbb{R}$ are Borel, and any set produced by a countable series of set operations (union, intersection, compliment, difference) is also Borel. Lastly, note that $\{x\}$ are Borel for $x \in \mathbb{R}$, so therefore all countable or finite sets in $\mathbb{R}$ are Borel.

e.g. the Cantor set

Take $[x-1, x] \cap [x, x+1]$.

Given a space X and a σ-algebra $\mathcal{F}$ of subsets of X, we call $(X, \mathcal{F})$ a *measurable space*.

e.g. $(\mathscr{B}_{\mathbb{R}}, \mathbb{R})$

Given a measurable space $(X, \mathcal{F})$, define $\mu : \mathcal{F} \to [0, \infty]$. μ is called a *measure* if the following hold:

1. $\mu(\emptyset) = 0$
2. If $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, where $A_i \cap A_j = \emptyset$, then $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Should feel similar to some of Kolmogorov's axioms...

We classify a few types of measures:

1. If $\mu(X) < \infty$, we call μ a *finite measure*.
2. If $\mu(X) = 1$, we call μ a *probability measure*.
3. If $\exists\{A_n : n \geq 1\} \subseteq \mathcal{F}$ with $\cup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty\ \forall n$, we call μ a *σ-finite measure*.

Finally, we call $(X, \mathcal{F}, \mu)$ a *measure space*.

Examples:

$\mu : \mathscr{B}_{\mathbb{R}} \to [0, \infty]$ with $\mu(A) = \begin{cases} |A| \text{ if } A \text{ if finite} \\ \infty \text{ o.w.} \end{cases}$ is called the "counting measure"

Fix $x \in \mathbb{R}$. $\mu : \mathscr{B}_{\mathbb{R}} \to [0, \infty]$ with $\mu(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ o.w.} \end{cases}$ is the "Dirac measure"

1.1 Properties of Measure

Let $(X, \mathcal{F}, \mu)$ be a measure space. Then:

(i) If $A_1, ..., A_N \in \mathcal{F}$ are disjoint, then $\mu(A_1 \cup ... \cup A_N) = \sum_{i=1}^{N} \mu(A_i)$

(ii) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(iii) If $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, then $\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$. This also holds for finite collections.

PROOF.

(i) Let $A_i = \varnothing \ \forall i > N$. The result follows from axiom 1.

(ii) Write $B = A \cup (B \setminus A) \implies \mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.

(iii) Set $B_1 = A_1$ and $B_n = A_n \setminus (\cup_{i=1}^{n-1} A_i)$ for $n \geq 2$. Then B_n are pairwise disjoint $\forall n$, so we can write $\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$. Lastly, notice that $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n$ to conclude that $\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$. □

If $A \in \mathcal{F}$ with $\mu(A) = 0$, we call A a *null set*. Note that the union of null sets is a null set.

PROP 1.4 (Continuity from Below)

Given $\{A_n : n \geq 1\} \subseteq \mathcal{F}$ with $A_n \subseteq A_{n+1}$, $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$.

PROOF.

We call $\{A_n\}$ with $A_n \subseteq A_{n+1}$ *increasing*, and write $A_n \uparrow$. Set $B_1 = A_1$, and $B_n = A_n \setminus A_{n-1} \ \forall n \geq 2$. Then $\{B_n : n \geq 1\} \subseteq \mathcal{F}$ are disjoint, and $\cup_{n=1}^{\infty} B_n =$

$\cup_{n=1}^{\infty} A_n$. Similarly, $N \geq 1$, $\cup_{n=1}^{N} B_n = A_N$. Combining, we have

$$\begin{aligned} \mu(\cup_{n=1}^{\infty} A_n) = \mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) = \lim_{N\to\infty} \left(\cup_{n=1}^{N} B_n\right) \\ = \lim_{N\to\infty} \mu(A_n) \qquad \square \end{aligned}$$

Prop 1.5 (Continuity from Above) or, $\exists j : \mu(A_j) < \infty$

We have the similar property that, given $\{A_n : n \geq 1\} \subseteq \mathcal{F}$, where $A_n \supseteq A_{n+1}$, $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$ IF the measure of this intersection is finite.

This assumption is indeed necessary: take the counting measure over $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$, and let $A_n := \{n, n+1, n+2, ...\}$. Clearly $A_n \downarrow$, and $\mu(A_n) = \infty \; \forall n \geq 1$. We find, though, that $\cap_{n=1}^{\infty} A_n = \varnothing$, and so $\mu(\cap_{n=1}^{\infty} A_n) = 0$.