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# VECTOR CALCULUS NOTES

NICHOLAS HAYEK

*Lectures by Prof. Jean Pierre Mutanguha*

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# I Vector Fields

## PRODUCTS ON VECTOR SPACES

Recall the definition of the *inner product* over a vector space  $V$ :

1.  $\langle u, v \rangle = \overline{\langle v, u \rangle} = \langle v, u \rangle$  in  $\mathbb{R}$  (where we'll be in this class)
2.  $\langle au + bw, v \rangle = a \langle u, v \rangle + b \langle w, v \rangle$
3.  $\langle u, u \rangle \geq 0$ , and  $= 0 \iff u = \mathbf{0}$

From this, we define the *norm* of  $u \in V$  to be  $\|u\| := \sqrt{\langle u, u \rangle}$ . This is well-defined, since  $\langle u, u \rangle \geq 0$ .

$$\forall u, v \in V, |\langle u, v \rangle| \leq \|u\| \|v\|$$

PROP 1.1  
Cauchy-Schwartz Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

PROP 1.2  
Triangle Inequality

The *cross product* of  $u, v \in \mathbb{R}^3$ , with respect to  $\mathbb{R}^3$ , is the determinate of the following “matrix”:

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where  $u = \langle u_1, u_2, u_3 \rangle$  and  $v = \langle v_1, v_2, v_3 \rangle$ . We observe the following two properties of the cross product in  $\mathbb{R}^3$ :

PROP 1.3

1.  $(u \times v) \cdot u = 0$
2.  $\|u \times v\| = \|u\| \|v\| \sin(\theta)$ , where  $\theta$  is the angle found between  $u$  and  $v$ . A conceptualization of this property is that “ $u$ -cross- $v$  is equal to the area created by the parallelogram bounded by  $u$  and  $v$ .”

## LINES

Define a *line*  $l(t) \in \mathbb{R}^n$  to be a function from  $\mathbb{R} \rightarrow \mathbb{R}^n$ , with the primary form  $l(t) = P + td$ , with  $P, d \in \mathbb{R}^n$ ,  $t \in \mathbb{R}$ . We call  $P$  the “point vector” and  $d$  the “direction vector”. An alternate form, with two points  $P, Q \in \mathbb{R}^n$ , would be  $l(t) = (1-t)P + tQ$ , where  $l(t)$  lies along the path between  $P$  and  $Q$  for  $t \in [0, 1]$ .

**Distance between a point and line** Using this definition, how can we find the shortest path between a point  $R$  and a line  $l(t)$ , which lies between  $P$  and  $Q$ ?

*Idea 1* We know the desired vector  $w = PR \sin(\theta)$ , the angle between  $PR$  and  $PQ$ . To find this value, note that  $\|PR \times PQ\| = \|PR\| \|PQ\| \sin(\theta)$ .

*Idea 2* We can project  $R$  onto  $PQ$ , and then subtract this projection from  $PR$ .

*Idea 3* We can minimize a distance function between  $R$  and a point on  $l$ , i.e.  $l(t)$ . Thus, we take  $\min_{t \in \mathbb{R}} \|R - l(t)\| = \alpha$ , and then take  $Rl(\alpha)$  to be the shortest path.

*Idea 4* We can find when  $(R - l(t)) \cdot d = 0$ .

**Distance between 2 lines** Consider two lines,  $l_1$  and  $l_2$ , which do not intersect but are not necessarily parallel. What is the minimal distance between  $l_1$  and  $l_2$ ?

Sometimes called “skew lines”

*Idea 0* Conceptualize this problem as finding the distance between the parallel planes defined by  $\{l_1, l_2\}$ .

*Idea 1* We can minimize  $\|l_1(t) - l_2(s)\|$  (really, one should minimize the square to make one’s life easier).

*Idea 2* Pick any two points, say  $l_1(T)$  and  $l_2(S)$ , and project  $l_1(T)l_2(S)$  onto  $l_1 \times l_2$ .

*Idea 3* Minimize  $\text{dist}(l_1(t), l_2)$  for fixed  $t$ .

*Idea 4* Find  $t$  and  $s$  such that  $[l_1(t) - l_2(s)] \cdot \vec{d}_1 = 0$  and  $[l_1(t) - l_2(s)] \cdot \vec{d}_2 = 0$

PROP 1.4

$\|u \times v\| = \|u\| \|v\| \sin(\theta) = \text{Area of parallelogram defined by } u \text{ and } v.$

## PLANES

A plane  $r(s, t)$  is a function  $[0, 1]^2 \rightarrow \mathbb{R}^3$  defined by  $d_1, d_2 \in \mathbb{R}^3$ , two vectors, and  $P \in \mathbb{R}^3$ , a point. In particular,  $r(s, t) = P + s\vec{d}_1 + t\vec{d}_2$ . This is called the *parametric form*.

The *point-normal* form is a function  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$  is given by  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ , where  $\vec{n} = \langle a, b, c \rangle$  is a vector normal to the plane, and  $P = \langle x_0, y_0, z_0 \rangle$  is a point lying on the plane.

**Distance between a point  $R$  and a plane  $r$**

*Idea 1* Minimize  $\|R - r(s, t)\|$  (or the square)

*Idea 2*  $\|\text{proj}_{\vec{n}}(P - R)\|$ , where  $\vec{n}$  and  $P$  are as given in the point-normal form.

## TRANSFORMATIONS AND PARAMETERIZATIONS

The following table give general examples of linear transformations  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

Dimension	Linear	Affine
$n = 0$	$\lambda(0) = 0$	$\lambda(0) = P$
$n = 1$	$\lambda(t) = t\vec{d}$	$\lambda(t) = P + t\vec{d}$
$n = 2$	$\lambda(t, s) = t\vec{d}_1 + s\vec{d}_2$	$\lambda(t, s) = P + t\vec{d}_1 + s\vec{d}_2$
$n = 3$	$\lambda(t, s, r) = t\vec{d}_1 + s\vec{d}_2 + r\vec{d}_3$	$\lambda(t, s, r) = P + t\vec{d}_1 + s\vec{d}_2 + r\vec{d}_3$

We also define the following important curves in  $\mathbb{R}^2$ :

Type	Explicit Form	Parametric Form
Ellipse	$x^2 + y^2 = 1$	$r(t) = \langle t, \sqrt{1-t^2} \rangle_{t \in [-1,1]} = \langle \cos(t), \sin(t) \rangle_{t \in [-\pi, \pi]}$
Hyperbola	$x^2 - y^2 = 1$	$r(t) = \langle \sqrt{1+t^2}, t \rangle_{t \in \mathbb{R}} = \langle \cosh(t), \sinh(t) \rangle_{t \in \mathbb{R}}$
Parabola	$x = y^2$	
Double Cone	$x^2 = y^2$	
Any Function	$y = F(x)$	$r(t) = \langle t, F(t) \rangle$

Define a *path* in  $\mathbb{R}^m$  to be a continuous function  $r : \mathbb{R} \rightarrow \mathbb{R}^m$ , e.g.  $[a, b] \rightarrow \mathbb{R}^m$ .

Define a *curve* in  $\mathbb{R}^m$  to be the image of a path (i.e. a set of points in  $\mathbb{R}^m$ ). Recall the statement “paths parameterize curves.”

For example, the unit circle  $x^2 + y^2 = 1$  is parameterized by the path  $r : \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $r(t) = \langle \cos(t), \sin(t) \rangle$ .

Define the *tangent* line of  $\vec{r}$  at  $a \in \mathbb{R}$  to be an affine transformation  $l : \mathbb{R} \rightarrow \mathbb{R}^m$  satisfying the following:

1.  $l(t) = r(a) + (t - a)\vec{d} : \vec{d} \neq 0$
2.  $\lim_{t \rightarrow a} \frac{\|r(t) - l(t)\|}{|t - a|} = 0$

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♠ Examples ♣

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We'll now find the derivative of the unit circle at a point  $a \in \mathbb{R}$ : we have  $r(a) = \langle \cos(a), \sin(a) \rangle$ . Thus:

$$l(t) = \langle \cos(t), \sin(t) \rangle + (t - a)\langle d_1, d_2 \rangle$$

Where  $\langle d_1, d_2 \rangle \neq 0$ . Consider now the limit in question 2:

$$\begin{aligned}
 \lim_{t \rightarrow a} \frac{\|r(t) - l(t)\|}{|t - a|} &= \lim_{t \rightarrow a} \frac{1}{|t - a|} \sqrt{(\cos(t) - \cos(a) - (t - a)d_1)^2 + (\sin(t) - \sin(a) - (t - a)d_2)^2} \\
 &= \lim_{t \rightarrow a} \sqrt{\left(\frac{\cos(t) - \cos(a)}{t - a} - d_1\right)^2 + \left(\frac{\sin(t) - \sin(a)}{t - a} - d_2\right)^2} \\
 &\stackrel{t \rightarrow a}{=} \sqrt{(-\sin(a) - d_1)^2 + (\cos(a) - d_2)^2} = 0 \\
 &\iff d_1 = -\sin(a) \wedge d_2 = \cos(a) \\
 &\implies l(t) = \langle -\sin(a), \cos(a) \rangle \quad \square
 \end{aligned}$$


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Frequently,  $l(t)$  is referred to as the “velocity vector” of  $r(t)$ , and is notated as  $r'(t)$ . Notice that  $r'(t)$  is equivalent to the component-wise derivative of the coordinates of  $r(t)$  w.r.t.  $t$ . Formally:

Given  $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^n$ , the *derivative* of  $\vec{r}$  at  $a \in \mathbb{R}$  is a linear transformation  $\vec{\lambda}: \mathbb{R} \rightarrow \mathbb{R}^n$  satisfying

$$\lim_{t \rightarrow a} \frac{\|r(t) - r(a) - \lambda(t - a)\|}{|t - a|} = 0 \quad \text{or equivalently} \quad \lim_{h \rightarrow 0} \frac{\|r(a + h) - r(a) - \lambda(h)\|}{|h|} = 0$$

It is denoted  $D\vec{r}_a$ , and represented by the  $n \times 1$  matrix  $r'(a)$ . One may now rewrite the tangent line in the form  $l(t) = r(a) + \lambda(t - a)$ .

The *arc length* of a curve  $r(t)$  is given by

$$s = \int_a^b \|r'(t)\| dt$$

An *arc length parameterization* of  $r(t)$  is some  $t = \alpha(s)$  such that  $r(\alpha(s))$  has a unit velocity vector, i.e.  $\|r'(\alpha(s))\| = 1$ . Alternatively, one could find an expression for arc length, and then parameterize  $r(t)$  in terms of its arc length. The resultant will be equivalent.

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♠ Examples ♣

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We'll do an arc length parameterization of a semicircle of radius 1 with its center at the origin, i.e.  $y = \sqrt{1 - x^2}$ . We get the natural parameterization  $r(t) = \langle t, \sqrt{1 - t^2} \rangle$ , where  $t \in [-1, 1]$ . We'd like to find a change of parameters  $t = \alpha(s)$  such that  $\|r(\alpha(s))\| = 1$  and  $\alpha' \geq 0$ .

$$\begin{aligned} r(\alpha(s)) &= \left\langle \alpha(s), \sqrt{1 - \alpha(s)^2} \right\rangle \\ r'(\alpha(s)) &= \left\langle \alpha'(s), \frac{1}{2}(1 - \alpha(s)^2)^{-\frac{1}{2}} \cdot (-2\alpha(s)\alpha'(s)) \right\rangle \\ &= \alpha'(s) \left\langle 1, \frac{-\alpha(s)}{\sqrt{1 - \alpha(s)^2}} \right\rangle \end{aligned}$$

$$\begin{aligned} \text{Then } 1 = \|r'(\alpha(s))\| &= \alpha'(s) \sqrt{1 + \frac{\alpha(s)^2}{1 - \alpha(s)^2}} \\ &= \frac{\alpha'(s)}{\sqrt{1 - \alpha(s)^2}} \end{aligned}$$

Integrating with respect to  $s$ , we get  $s = \arcsin(\alpha(s)) = \arcsin(t)$ . Thus,  $t = \sin(s)$ , and  $s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , and we yield the parameterization  $\langle \sin(s), \cos(s) \rangle : s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ .

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## SURFACES

We note the following quadric surfaces:

Type	Explicit Form
Ellipsoid	$x^2 + y^2 + z^2 = 1$
Elliptic Hyperboloid	$x^2 + y^2 - z^2 = 1$
Elliptic Paraboloids	$x^2 + y^2 - z^2 = -1$
Hyperbolic Paraboloids	$x = y^2 - z^2$
Double Cones	$x^2 = y^2 + z^2$

A surface  $F(x, y)$  is called differentiable at  $(a, b)$  if there exists some linear transformation  $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k \rangle\|}$$

One may represent  $\lambda(h, k) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = uh + vk$

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♠ Examples ♣

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Let  $F(x, y) = xy$ . We consider  $F$  at  $(a, b)$ . Then

$$\begin{aligned} 0 \leq \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k \rangle\|} &= \frac{|(a+h)(b+k) - ab - (uh + vk)|}{\|\langle h, k \rangle\|} \\ &= \frac{|bh + ak + hk - uh - vk|}{\|\langle h, k \rangle\|} = \frac{|(b-u)h + (a-v)k + hk|}{\|\langle h, k \rangle\|} \\ &\leq \frac{|b-u||h|}{|h|} + \frac{|a-v||k|}{|k|} + \frac{|h||k|}{|h|} \quad \text{since } |h|, |k| \leq \|\langle h, k \rangle\| \\ &= |b-u| + |a-v| + |k| \rightarrow |b-u| + |a-v| \\ &= 0 \quad \text{when } b = u, a = v \end{aligned}$$

Thus, the desired limit is always  $\geq$  and  $\leq 0$ , so especially it is 0. Our derivative at  $(a, b)$  is then  $\lambda(x, y) = bx + ay$ .

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One may also find these coefficients as the partial derivative of  $F$ , i.e.

$$\nabla F(a, b) = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle \Big|_{(a,b)}$$

This is called the *gradient*. Similarly,  $\alpha(x, y) = F(a, b) + \lambda(x - a, y - b)$  is called the *affine approximation* at  $(a, b)$ .

If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $\vec{a}$ , then all partial derivatives of  $F$  at  $\vec{a}$  exist. Furthermore,  $\lambda(\vec{a}) = F'(\vec{a}) = \left[ \partial_1 F \cdots \partial_n F \right] \Big|_{\vec{a}}$ .

PROP 1.5

Note that the converse is *false* (as a counterexample, see  $F = \sqrt{|xy|}$ )

### 1.1 Partial Converse

If all partial derivatives of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  exist near  $\vec{a}$  and are continuous at  $\vec{a}$ , then  $F$  is differentiable at  $\vec{a}$ .

PROOF FOR  $n = 2$ .

Let  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  be a linear transformation defined by  $\left[ \partial_1 F \cdots \partial_n F \right] \Big|_{\vec{a}}$ . Then

$$\lambda(\vec{h}) = \sum_{i=1}^n \partial_i F(\vec{a}) h_i$$

Let  $n = 2$ . Then

$$\begin{aligned} |F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})| &= |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) + F(a_1 + h_1, a_2) \\ &\quad - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1 - \partial_2 F(\vec{a}) h_2| \\ &\leq |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) - \partial_2 F(\vec{a}) h_2| \\ &\quad + |F(a_1 + h_1, a_2) - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1| \\ &= |\partial_2 F(\vec{c}) h_2 - \partial_2 F(\vec{a}) h_2| + |\partial_1 F(\vec{d}) h_1 - \partial_1 F(\vec{a}) h_1| \\ &\quad \text{by mean value thm.} \\ &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| |h_2| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| |h_1| \\ \frac{|F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})|}{\|\vec{h}\|} &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{\|\vec{h}\|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{\|\vec{h}\|} \\ &\leq |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{|h_2|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{|h_1|} \\ &\quad \text{since } |h_i| < \|\vec{h}\| \\ &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \end{aligned}$$

Then, as  $\vec{h} \rightarrow 0$ ,  $\vec{c}, \vec{d} \rightarrow \vec{a}$ . Since  $F$ , is continuous, we know  $F(\vec{c}) \rightarrow F(\vec{a})$  and similarly for  $F(\vec{d})$ . Thus,

$$|\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \rightarrow |\partial_2 F(\vec{a}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{a}) - \partial_1 F(\vec{a})| = 0$$

And we conclude that the limit, as  $\leq$  and  $\geq 0$ , is 0.  $\square$

$F : \mathbb{R}^n \rightarrow \mathbb{R}$  is called  $C^1$  continuous (or *continuously differentiable*) if all partial exists and are continuous everywhere.