Higher Algebra 2

MATH 571

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CONTENTS

I Review

"Go and be somethingological directly"

Dickens, *Hard Times*

The first third of this course will be review from MATH 457. In particular, we will cover representation theory, with an emphasis on induced representations and Fourier analysis. The remainder of the course will cover homological algebra, beginning from notions of exact sequences, as well as flat, injective, and projective modules.

We require a basic understanding of module tensors, categories, and functors. The official prerequisite for this course is MATH 570 (which covers category theory and commutative algebra), but these notes will be written from the point of view of someone (me) who has not studied these topics.

COMMUTATOR SUBGROUPS

DEF 1.1 Let G be a group. The *commutator* of $a, b \in G$, denoted by $[a, b]$, is the element $aba^{-1}b^{-1}$. Clearly, $[a, b] = 1 \iff a$ and b commute. Let $G' \subseteq G$ be generated by all finite multiplications of commutators, i.e.

$$G' = \langle [a, b] : a, b \in G \rangle$$

DEF 1.2 G' is called the *commutator subgroup* of G.

PROP 1.1 The commutator subgroup of G is normal.

PROOF. Note that $g[a, b]g^{-1} = gaba^{-1}b^{-1}g^{-1} = [gag^{-1}, gbg^{-1}]$. Then,

$$g[a_1, b_1] \cdots [a_N, b_N]g^{-1} = g[a_1, b_1]g^{-1} \cdot g[a_2, b_2]g^{-1} \cdot \ldots \cdot g[a_N, b_N]g^{-1} \in G' \quad \square$$

PROP 1.2 If $H \triangleleft G$, then G/H is abelian $\iff G' \subseteq H$.

PROOF. Suppose G/H is abelian. Consider $aba^{-1}b^{-1} = [a, b] \in G'$. Then

$$aba^{-1}b^{-1}H = aH \cdot bH \cdot a^{-1}H \cdot b^{-1}H = aa^{-1}H \cdot bb^{-1}H = H$$

Hence, $[a, b] \in H$, so $G' \subseteq H$. Conversely, suppose $G' \subseteq H$. Then

$$a^{-1}b^{-1}abH = H \implies abH = baH$$

so G/H is abelian. $\square$

PROP 1.3 G/G' is the largest abelian subgroup of the form G/H for $H \triangleleft G$. In other words, G' is the smallest normal subgroup of G such that G/G' is abelian.

PROOF. Suppose G/H is abelian. Then $G' \subseteq H$ by Prop 1.2. Thus, $|G/G'| \geq |G/H|$. □

DEF 1.3 $G^{ab} := G/G'$ is called the *abelianization* of G.

Theorem 1.1 Unique Factoring Over Abelianizations

Let $\varphi : G \to A$ be a homomorphism into an abelian group. Then φ factors uniquely into $\varphi = \psi \circ \pi$, where $\pi : G \twoheadrightarrow G^{ab}$ is the natural quotient and $\psi : G^{ab} \to A$.

PROOF. Recall the homomorphism theorem, of which the isomorphism theorem is a special case. Let $\varphi : G \to H$. Let $N \subseteq \ker(\varphi)$ be a normal subgroup of G. Then $\varphi = \psi \circ \pi$, where $\pi : G \twoheadrightarrow G/N$ is the natural quotient and $\psi : G/N \to H$ is a homomorphism (surjective into $\mathrm{Im}(\varphi)$). Moreover, this decomposition is unique.

We apply this directly to the theorem above. Since A is abelian, so is $\mathrm{Im}(\varphi)$. But $\mathrm{Im}(\varphi) \cong G/\ker(\varphi)$. By Prop 1.2, it follows that $G' \subseteq \ker(\varphi)$. Since G' is normal, the homomorphism theorem applies. □

TENSOR PRODUCTS OF MODULES

1/7/26 Let $\mathbf{Mod_R}$ and $\mathbf{{}_RMod}$ denote the categories of left and right modules over a ring R, respectively. Recall that, for an R-module M, $r \in R$, and $m \in M$, left modules act by $(r, m) \mapsto rm$ and right modules act by $(r, m) \mapsto mr$.

DEF 1.4 If a module is both a left and right module, and obeys all respective module axioms, we call it a *bimodule*, and write $\mathbf{{}_SMod_R}$ for the category of bimodules.

DEF 1.5 If $A \in \mathbf{Mod_R}$ and $B \in \mathbf{{}_RMod}$, an *R-biadditive* map is a function

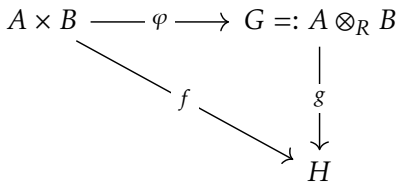
$$f : A \times B \to G$$

where H is a abelian. Additionally, we require that

- $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$
- $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$
- $f(ar, b) = f(a, rb)$

As H is a group, we do not impose any scaling qualities for f with respect to R.

We would like to construct an abelian group G and associated R-biadditive function φ such that, for any R-biadditive function f, there is a unique group homomorphism g with



DEF 1.6 commuting. If such a pair (G, φ) exists, we say it satisfies the *universal property*.

Construction

We will construct a group G which satisfies the universal property, as above

Consider $H = \mathbb{Z} \cdot (A \times B)$, the $\mathbb{Z}$-module, and hence free abelian group. In other words,

$$H \ni h = \oplus_{(a,b)\in A\times B} k_{(a,b)} \cdot (a,b) \quad \text{where} \quad k_{(a,b)} \in \mathbb{Z}$$

Furthermore, consider the subgroup $N < H$ by

$$N = \{(a_1 + a_2, b) - (a_1, b) - (a_2, b)\} \cup \{(a, b_1 + b_2) - (a, b_1) - (a, b_2)\} \cup \{(ar, b) - (a, rb)\}$$

under $a, a_i \in A$, $b, b_i \in B$, and $r \in R$.

One shows manually that this is a group

DEF 1.7 Define $A \otimes_R B := H/N$, and call this the *tensor product* of A and B over R.

Let $\varphi : A \times B \to A \otimes_R B$ be the natural map formed by viewing (a, b) as an element of the $\mathbb{Z}$-module H, and modding out by N as above.

Immediately, we see that the subgroup N ensures that φ is biadditive.

DEF 1.8 We denote the image of $(a, b) : a \in A, b \in B$ under φ by $a \otimes b$, and call the result a *tensor*.

Immediately, by the properties of φ, we have

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b \qquad a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 \qquad ar \otimes b = a \otimes rb$$

for $a \in A, b \in B, r \in R$.

DEF 1.9 An element that can be written as $a \otimes b$ for $a \in A, b \in B$ is called a *pure tensor*.

In practice, pure tensors are rare. Our elements will typically take the form of H, i.e. $\sum_{i\in I} a_i \otimes b_i$ for $a_i \in A, b_i \in B$. Note that coefficients in the formal sum are not necessary, as one can verify that $n(a_i \otimes b_i) = na_i \otimes b_i = a_i \otimes nb_i$ for $n \in \mathbb{Z}$, using additive notation for the groups underlying the modules A and B.

PROP 1.4 $(\varphi, A \otimes_R B)$ has the universal property.

DEF 1.10 $V^* = \mathrm{Hom}_k(V, k)$ is called the *dual vector space*. Recall that $\dim_k(V^*) = \dim_k(V)$.

Theorem 1.2 Properties of the Tensor Product

1. $\mathrm{Hom}_k(V, W) \cong V^* \otimes_k W$, V, W are finite dimensional vector spaces over k.
2. $\dim(V \otimes_k W) = \dim_k(V) \cdot \dim_k(W)$
3. If $f \in \mathrm{Hom}_R(A, A')$, $g \in \mathrm{Hom}_R(B, B')$, then
$$f \otimes g : A \otimes_R B \to A' \otimes_R B' \text{ given by } (a \otimes b) \mapsto f(a) \otimes f(b)$$
is a homomorphism.
4. If $A \cong A'$ and $B \cong B'$, then $A \otimes_R B \cong A' \otimes_R B'$
5. $A \otimes_R R = A$ and $R \otimes_R B = B$
6. $(\oplus_{i\in I} A_i) \otimes_R B \cong \oplus_{i\in I}(A_i \otimes_R B)$ and $A \otimes (\oplus_{i\in I} B_i) = \oplus_{i\in I} A \oplus_R B_i$
7. If R is commutative, then $A \otimes_R B = B \otimes_R A$.
8. If R is commutative, then $A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$.

REPRESENTATIONS OF FINITE GROUPS

DEF 1.11 A *linear representation* of a finite group G is a vector space V over a field $\mathbb{F}$ equipped with a group action
$$G \times V \to V$$
that respects the vector space, i.e. $m_g : V \to V$ with $m_g(v) = gv$ is a linear transformation. We make the following assumptions unless otherwise stated:

1. G is finite.
2. V is finite dimensional.
3. $\mathbb{F}$ is algebraically closed and of characteristic 0. We write $\mathbb{F} = \mathbb{C}$.

Since V is a G-set, $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ which sends $g \mapsto m_g$ is a homomorphism.

Relatedly, if $\dim(V) < \infty$, then $\rho : G \mapsto \mathrm{Aut}_{\mathbb{C}}(V) = \mathrm{GL}_n(\mathbb{C})$.

DEF 1.12 The *group ring* $\mathbb{C}[G]$ is a (typically) non-commutative ring consisting of all finite linear combinations $\{\sum_{g\in G} \lambda_g g : \lambda_g \in \mathbb{C}\}$, with $1 \cdot \mathbb{1}_G = \mathbb{1}_{\mathbb{C}[G]}$. It's endowed with the multiplication rule
$$\left(\sum_{g\in G} \alpha_g g\right)\left(\sum_{h\in G} \beta_h h\right) = \sum_{(g,h)\in G\times G} \alpha_g \beta_h (gh)$$

PROP 1.5 We can view representations as a module over the group ring $\mathbb{C}[G]$.

PROOF. Let V be a $\mathbb{C}[G]$-module. Consider $g \in G \subseteq \mathbb{C}[G]$, $\lambda \mathbb{1}_G \in \mathbb{C}[G]$, and $v_1, v_2 \in V$. Since V is a $\mathbb{C}[G]$-module,

$$g(v_1 + v_2) = gv_1 + gv_2 \qquad (gh)v_1 = g(hv_1)$$

Then: $(g\lambda\mathbb{1}_G)v_1 = (\lambda(g\mathbb{1}_G))v_1 = (\lambda g)v_1$. But also, $(g\lambda\mathbb{1}_G)v_1 = g(\lambda\mathbb{1}_G v_1) = g(\lambda v_1)$. Hence, the map $v \mapsto gv$ is a linear transformation on V over $\mathbb{C}$. □

We will frequently return to this view when module theory is more convenient.

DEF 1.13 **Eg. 1.1** Consider $\rho : G \to \{\mathbb{1}\}$, the *trivial representation*, which maps $\rho(g)(v) = v$. We will denote the trivial representation simply by $\mathbb{1}$, subject to context.

DEF 1.14 **Eg. 1.2** We call $\rho^{\text{reg}} : h \mapsto \left[\sum_{g\in G} \alpha_g g \mapsto \sum_{g\in G} \alpha_g hg\right]$ the *regular representation*, with $G \circlearrowleft \mathbb{C}[G]$ by left multiplication.

Over $\mathbb{C}$, $\mathbb{C}[G]$ has basis $\{g_1, ..., g_n\}$, where $n = |G|$. Then $\chi(h) = \{g_i \in G : hg_i = g_i\}$. If $h = 1$, then $\chi(h) = |G|$. Otherwise, it is impossible for $hg_i = g_i$.

Generally, recall that the trace counts the number of basis vectors which are fixed by a transformation

We conclude that

$$\chi_{\text{reg}}(g) = \begin{cases} |G| & g = 1 \\ 0 & \text{o.w.} \end{cases}$$

Examples

RESTRICTED AND INDUCED REPRESENTATIONS

Let $H < G$ be a subgroup. Then we consider a functor between the categories of representations of G and H,

$$\text{Res}^G_H : \mathbf{Rep}(G) \to \mathbf{Rep}(H) : \rho \mapsto \rho|_H = \text{Res}^G_H(\rho)$$

DEF 1.15 called the *restricted representation* of G to H. Analogously, this sends a $\mathbb{C}[G]$-module V to the submodule W defined over $\mathbb{C}[H]$.

Similarly, we consider a functor

$$\text{Ind}^G_H : \mathbf{Rep}(H) \to \mathbf{Rep}(G) : V \mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

DEF 1.16 called the *induced representation* of H to G, where we view V as a $\mathbb{C}[G]$-module. Observe that $\dim_{\mathbb{C}[H]}(\mathbb{C}[G]) = [G : H]$, so $\dim(\text{Ind}^G_H) = [G : H]\dim(V)$.

Eg. 1.3 Consider $H = \{\mathbb{1}\}$ with the trivial representation on $V = \mathbb{C}$. Then $\text{Ind}^G_H(\mathbb{C}) = \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[G]$, i.e. the regular representation.

DUAL REPRESENTATIONS

Let ρ, V be a representation of G. Recall the dual, $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, the set of linear transformations from $V \to C$. Given an endomorphism $T : V \to V$, we call

$$T^t : V^* \to V^* : (T^t\varphi)(v) := \varphi(Tv)$$

the *transpose*. If $\beta = \{v_1, ..., v_n\}$ is a basis for V, then we construct the *dual basis* $\beta^* = \{\varphi_1, ..., \varphi_n\}$ for V^*, where $\varphi_i(v_j) = \delta_{ij}$. In the dual basis, we have DEF 1.17 DEF 1.18

$$[T^t]_{\beta^*} = [T]_\beta^t \implies \mathrm{tr}(T) = \mathrm{tr}(T^t)$$ **PROP 1.6**

PROOF. See MATH 251 notes. □

When $T = \rho(g) : V \to V$, we also observe

$$(\rho(gh)^t\varphi)(v) = (\rho(h)^t\rho(g)^t\varphi)(v) \implies \rho(gh)^t = \rho(h)^t\rho(g)^t$$

Given a representation ρ, $\rho^* : G \to \mathrm{GL}(V^*)$ by $g \mapsto \rho(g^{-1})^t$ is called the *dual representation*. DEF 1.19

$\chi_{\rho^*} = \overline{\chi_\rho}$ **PROP 1.7**

PROOF. If $g \in G$ has order n, then $\rho(g)$ has order $m|n$, since $\rho(g)^n = \rho(g^n) = \rho(\mathbb{1}) = I$. Hence, in a certain basis,

$$\rho(g) = \begin{pmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{pmatrix} \quad \text{where} \quad \xi_i^m = 1$$

If ξ is a root of unity, $\xi\overline{\xi} = 1$ (try viewing this geometrically)

It follows that

$$\rho(g^{-1}) = \begin{pmatrix} \xi_1^{-1} & & & \\ & \ddots & & \\ & & & \xi_n^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\xi_1} & & & \\ & \ddots & & \\ & & & \overline{\xi_n} \end{pmatrix}$$

Thus, $\mathrm{tr}(\rho^*(g)) = \mathrm{tr}(\rho(g^{-1})^t) = \mathrm{tr}(\rho(g^{-1})) = \overline{\mathrm{tr}(\rho(g))}$, using Prop 1.2. □

1-DIM REPRESENTATIONS

A *1-dim representation* (ρ, V) is a representation with $\dim(V) = 1$. In this case, as V is a $\mathbb{C}$-vector space and $\rho(g) \in \mathrm{GL}(V)$, we write $V = \mathbb{C}^\times$. Also observe that $\chi_\rho = \rho$. DEF 1.20

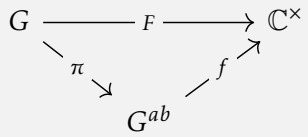
$G^* = \mathrm{Hom}(G, \mathbb{C}^\times)$, as groups, is called the *group of multiplicative characters*. DEF 1.21

If G is a finite, abelian group, then every irreducible representation has dimension 1. **PROP 1.8**

PROOF. See MATH 457. □

PROP 1.9 $(G^{ab})^* \cong G^*$

PROOF. If $f \in (G^{ab})^*$ is a homomorphism $f : G^{ab} \to \mathbb{C}^\times$, then $f \circ \pi : G \twoheadrightarrow G/N \to \mathbb{C}^\times$ is also a homomorphism. Conversely, any $F : G \to \mathbb{C}^\times$ must factor uniquely into $f \circ \pi$ by Thm 1.1, where $f : G^{ab} \to C^\times$. See the following diagram:



TENSOR REPRESENTATIONS

If ρ is a finite representation of G and τ is a 1-dim representation, we can generate a new representation

$$\rho \otimes \tau : G \to \mathrm{GL}(V \otimes_{\mathbb{C}} \mathbb{C}) \cong \mathrm{GL}(V) : g \mapsto \tau(g) \otimes \rho(g)$$

Note that $\tau(g) \in \mathbb{C}^\times$, so $\chi_{\rho\otimes\tau} = \tau\chi_\rho$.

In generality, given two representations ρ_1, ρ_2, we generate the tensor product representation $\rho_1 \otimes \rho_2$ over $V_1 \otimes_{\mathbb{C}} V_2$, with dimension $\dim(V_1)\dim(V_2)$ and trace $\chi_{\rho_1}\chi_{\rho_2}$.

Irreducible Representations

DEF 1.22 Let (ρ, V) be a representation. It is called an *irreducible representation* if there are no G-stable, nontrivial subspaces of V (i.e. no nontrivial subrepresentations). In the language of modules, irreducible representations are simple $\mathbb{C}[G]$-modules.

Theorem 1.3 Semi-Simplicity of Representations

Every finite dimensional, non-zero representation of G is a direct sum of irreducible representations.

This is "Maschke's Theorem" without uniqueness

PROOF. Pick any Hermitian inner product $\langle \cdot, \cdot \rangle$ on V. Define

$$\langle u, v \rangle^* = \frac{1}{|G|} \sum_{g \in G} \langle gu, gv \rangle$$

It can be easily verified that $\langle \cdot, \cdot \rangle^*$ is an inner product which is G-equivariant. If $W \subseteq V$ is a subrepresentation, then set $W^\perp = \{u : \langle u, v \rangle = 0 \ \forall v \in W\}$, i.e. the orthogonal complement of W with respect to $\langle \cdot, \cdot \rangle^*$. It follows that $V = W \oplus W^\perp$, with $W^\perp$ being G-stable by the G-equivariance of the inner product.

We then argue by induction to yield a direct sum of irreducible representations. See MATH 457 for more details on semi-simplicity. □

Based on this proof, we see that $\rho(g)$ is unitary. One necessary and sufficient condition for a transformation to be unitary is the existence of an inner product $\langle\cdot,\cdot\rangle^*$ with $\langle Tv, Tw\rangle^* = \langle v, w\rangle^*$. Unitary matrices are interesting for the following reasons:

- $\overline{\rho(g)}^t = \rho(g)^{-1} = \rho(g^{-1})$
- $\rho(g)$ is diagonalizable. With $g^n = 1$, we must be able to write $\rho(g)$ with roots of unity on the diagonal and zeros otherwise. In particular, the i-th diagonal element is ξ_i, where $\xi_i^{m_i} = 1, m_i | n$.

SCHUR'S LEMMA

In this section, we will build up the intuition necessary for proving Schur's Lemma.

Let (ρ, V) be a representation. Let $V^G = \{v : \rho(g)(v) = v : \ \forall g \in G\}$ be the space of *invariant vectors*. Notice that V^G is a subrepresentation of V equivalent to $\underbrace{\mathbb{1} \oplus \cdots \oplus \mathbb{1}}_{\dim(V^G) \text{ times}}$. 1/9/26 DEF 1.23

Recall that $\mathbb{1}$ denotes the trivial representation.

$\dim(V^G) = \frac{1}{|G|}\sum_{g\in G} \chi_\rho(g)$ **PROP 1.10**

PROOF.

Let $\pi : v \mapsto \frac{1}{|G|}\sum_{g\in G}\rho(g)(v)$. Writing $\rho(h)\pi = \frac{1}{|G|}\sum_{g\in G}\rho(hg) = \frac{1}{|G|}\sum_{g\in G}\rho(g)$ verifies that $\mathrm{Im}(\pi) \subseteq V^G$. It is also easy to verify that $\pi|_{V^G} = \mathrm{Id}_{V^G}$. Hence, we may write $V = \ker(\pi) \oplus V^G$. It follows that, in some basis,

$$\pi = \begin{pmatrix} 0 & 0 \\ 0 & I_{\dim(V^G)} \end{pmatrix}$$

and thus $\mathrm{tr}(\pi) = \dim(V^G) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)$. □

Let $(\rho_1, V_1), (\rho_2, V_2)$ be two representations. Consider

$$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) = \{T : V_1 \to V_2 \text{ with } T \ \mathbb{C}\text{-linear}\}$$

This is a $\mathbb{C}$-vector space of dimension $\dim(V_1)\dim(V_2)$. Similarly, we consider

$$Hom_G(V_1, V_2) = \{T : V_1 \to V_2 \text{ with } T\rho_1(g) = \rho_2(g)T\}$$ DEF 1.24

It is often more natural to think of $\mathrm{Hom}_G(V_1, V_2)$ as transformations which satisfy $T(gv) = gT(v)\ \forall v \in V_1$, noting the distinct actions of g on V_1 and V_2, respectively.

Over the vector space $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, $\rho : g \mapsto [T \mapsto \rho_2(g^{-1})T\rho_1(g)]$ is a G-representation. **PROP 1.11**

PROOF.

Clearly $\rho_2(g^{-1})T\rho_1(g) \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$. Also note

$$\rho_2((gh)^{-1})T\rho_1(gh) = \rho_2(h^{-1})\rho_2(g^{-1})T\rho_1(g)\rho_2(h)$$

so $\rho(gh) = \rho(g)\rho(h)$. □

PROP 1.12 $\mathrm{Hom}_G(V_1, V_2) = (\mathrm{Hom}_{\mathbb{C}}(V_1, V_2))^G$

PROOF.

Let $T \in (\mathrm{Hom}_{\mathbb{C}}(V_1, V_2))^G$. Let $g \in G$. Then

$$gT = T \implies \rho_2(g^{-1})T\rho_1(g) = T \implies T\rho_1(g) = \rho_2(g)T \qquad \square$$

PROP 1.13 $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \cong V_1^* \otimes V_2$ as vector spaces and as G-representations.

PROP 1.14 $\dim(\mathrm{Hom}_G(V_1, V_2)) = \frac{1}{|G|}\sum_{g\in G} \chi_1(g)\overline{\chi_2(g)}$

PROOF.

$$\begin{aligned}\dim(\mathrm{Hom}_G(V_1, V_2)) &= \dim(\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)^G) = \dim((V_1^* \otimes V_2)^G) \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_1^*\otimes\rho_2} = \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_1^*}\chi_{\rho_2} = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_{\rho_1}}\chi_{\rho_2} \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_1}\overline{\chi_{\rho_2}}\end{aligned}$$

In the last step, we use the fact that the dimension is always real, so $\overline{\dim} = \dim$. □

Eg. 1.4 Let $G = S_n$, $V = \mathbb{C}^n$, and let ρ be the standard representation (i.e. permuting indices). Then $V^G = \{\langle x, \ldots, x\rangle : x \in \mathbb{C}\}$. This implies that

$$1 = \dim(V^G) = \frac{1}{|G|}\sum_{\sigma\in S_n} \chi_\rho(\sigma) \qquad \text{Prop 1.10}$$

But the trace of $\rho(\sigma)$ is exactly the number of fixed points of σ. To see this, note that σ permutes $i \to j$ if the i^{th} row is equal to e_j. Hence

$$1 = \frac{1}{n!}\sum_{\sigma\in S_n} \#\text{FP of } \sigma$$

On average, then, a random permutation has 1 fixed point.

Theorem 1.4 Schur's Lemma

Let $(\rho, V), (\tau, W)$ be irreducible representations of G. Then

$$\mathrm{Hom}_G(V, W) \cong \begin{cases}\mathbb{C} & \rho \cong \tau \\ 0 & \rho \ncong \tau\end{cases}$$

PROOF.

We claim that any nonzero $T \in \mathrm{Hom}_G(V, W)$ is an isomorphism.

Equivelantly, the kernel and image of a homomorphism of modules are submodules

$\ker(T)$ and $\mathrm{Im}(T)$ are subrepresentations of ρ and τ, respectively. Since both ρ and τ are irreducible, it follows that $\ker(T) = V \vee 0$, and $\mathrm{Im}(T) = W \vee 0$.

$\ker(T) = 0$, since T is nonzero. $\mathrm{Im}(T) \neq 0$ for the same reason, so $\mathrm{Im}(T) = W$. It follows that T is an isomorphism. Immediately, $\mathrm{Hom}_G(V, W) = 0$ when $\rho \neq \tau$.

Suppose $\rho \cong \tau$. We can write $\mathrm{Hom}_G(V, W) = \mathrm{End}_G(V)$. Let $T \in \mathrm{End}_G(V)$ be nonzero. Let λ be some eigenvalue of T with corresponding eigenspace $U_\lambda \subseteq V$. Then

$$T(gu) = g(Tu) = g(\lambda u) = \lambda gu \ \forall u \in U, g \in G$$

It follows that $gu \in U_\lambda$. Hence, U_λ is a G-stable subspace, and hence a subrepresentation. By irreducibly, $U_\lambda = V$, so $T = \lambda$.

By this argument, we can map $T \mapsto \lambda_T = \mathrm{tr}(T)/\dim(V) \in \mathbb{C}$. The converse map $\lambda \mapsto \lambda I$ completes the proof. The well-structuredness of this map derives from the fact that

$$\frac{\mathrm{tr}(T+G)}{\dim(V)} = \frac{\mathrm{tr}(T)}{\dim(V)} + \frac{\mathrm{tr}(G)}{\dim(V)}$$

□

Class Functions

A function $f : G \to \mathbb{C}$ is called a *class function* if $f(hgh^{-1}) = f(g)$. In other words, f is constant on each conjugacy class of G. DEF 1.25

We will denote by $h(G)$ the number of conjugacy classes of G. Similarly,

$$\mathit{Class(G)} = \{f : f \text{ is a class function on } G\}$$ DEF 1.26

Its basis consists of functions that are the identity on each class, and zero elsewhere

Note that $\mathrm{Class}(G)$ is a $\mathbb{C}$-vector space with dimension $h(G)$.

Eg. 1.5 If G is abelian, then $h(G) = |G|$.

Eg. 1.6 $h(S_n)$ is the number of permutations of n.

We can endow $\mathrm{Class}(G)$ with the inner product

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}$$

Note also that χ_ρ is a class function. For the following theorems and propositions, we will denote by $(p_1, V_1), \ldots, (p_n, V_n)$ the irreducible representations of G, along with their dimensions d_i and characters λ_i for $i \in [n]$.

Theorem 1.5 Irreducible Characters Form Orthonormal Basis of Class(G)

Up to isomorphism, the irreducible characters of G form an orthonormal basis for $\mathrm{Class}(G)$. We conclude that $\#h(G) = \#$irreducible representations of G.

PROOF.

Let σ, τ be irreducible representations. We do not distinguish between σ, τ and their associated vector spaces. Then, by Schur's Lemma (Thm 1.4) and Prop 1.14,

$$\delta_{\sigma,\tau} = \dim(\mathrm{Hom}_{\mathbb{C}}(\sigma,\tau)^G) = \frac{1}{|G|}\sum_{g\in G}\chi_\sigma\overline{\chi_\tau} = \langle\chi_\sigma,\chi_\tau\rangle$$

From this, we conclude that the irreducible characters are orthonormal in $\mathrm{Class}(G)$. It remains to show that they are a basis. From linear algebra (see MATH 251), we recall the criterion

$$\langle\chi_i,\beta\rangle = 0\ \forall i\in[n] \implies \beta\in\mathrm{Class}(G)\equiv 0$$

This ensures that Fourier coefficients always exist using the irreducible characters provided, which establishes spanning-ness. Let $\alpha\in\mathrm{Class}(G): G\to\mathbb{C}$. Consider

$$A_\rho = \sum_{g\in G}\alpha(g)\rho(g)\in\mathrm{End}_{\mathbb{C}}(V)$$

We claim that $A_p\in\mathrm{End}_G(V)$. Write

$$\begin{aligned}\rho(h)A_\rho\rho(h^{-1}) &= \sum_{g\in G}\alpha(g)\rho(hgh^{-1}) = \sum_{g\in G}\alpha(hgh^{-1})\rho(hgh^{-1})\\ &= \sum_{g\in G}\alpha(g)\rho(g) = A_\rho\end{aligned}$$

We claim that, if $\alpha=\overline{\beta}$, with ρ irreducible, then $A_\rho = 0$. Schur's Lemma gives the map

$$\mathrm{End}_G(V)\to\mathbb{C}: T\mapsto\frac{\mathrm{tr}(T)}{\dim(\rho)}$$

Which we apply to A_ρ

$$A_\rho\mapsto\frac{\mathrm{tr}(\sum_{g\in G}\alpha(g)\rho(g))}{\dim(\rho)} = \frac{|G|}{\dim(\rho)}\frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\beta(g) = \frac{|G|}{\dim(\rho)}\left\langle\chi_\rho,\beta\right\rangle = 0$$

This holds for any irreducible representation, so, in particular, $A_{\rho_i}\ \forall i\in[n]$. It must also hold for the regular representation. Hence, $A_{\rho^{\mathrm{reg}}} = 0$ on $\mathrm{End}_G(\mathbb{C}[G])$. Consider $\mathbb{1}_G$. Then we must have

$$\sum_{g\in G}\alpha(g)\rho^{\mathrm{reg}}(g)(\mathbb{1}_G) = \sum_{g\in G}\alpha(g)[g] = 0$$

Since $[g]: g\in G$ is a basis for $\mathbb{C}[G]$, it must be that $\alpha(g)=0\ \forall g\in G$. As $\alpha=\overline{\beta}$, the result follows. □

Theorem 1.6 Mascke's Theorem

If (ρ, V) is a representation of G, then it has a unique decomposition

$$\rho \cong \rho_1^{a_1} \oplus \cdots \oplus \rho_n^{a_n} V\psi$$

where $a_i = \langle \lambda_\rho, \lambda_i \rangle$.

PROOF.

Letting a_i be as in Thm 1.3, we know $\chi_\rho = \sum_{i=1}^n a_i \lambda_i$. It remains to show that a_i are unique. But we can compute

$$\langle \chi_\rho, \chi_i \rangle = \sum_{j=1}^n a_i \underbrace{\langle \chi_j, \chi_i \rangle}_{\delta_{ij} \text{ by Thm 1.5}} = a_i$$

PROP 1.15

$\rho \cong \tau \iff \chi_\rho = \chi_\tau$

PROOF.

We only need to consider the ($\Longleftarrow$) direction. In this case, we write

$$\langle \chi_\rho, \chi_i \rangle = \langle \chi_\tau, \chi_i \rangle \ \forall i$$

But these are the multiplicities of the irreducible characters in ρ and τ, so $\rho \cong \tau$. □

PROP 1.16

Let ρ^{reg} be the regular representation of G on $\mathbb{C}[G]$. We have

$$\rho^{\text{reg}} \cong \rho_1^{d_1} \oplus \cdots \oplus \rho_n^{d_n}$$

Consequently, $|G| = \sum_{i=1}^n d_i^2$.

PROOF.

$$\langle \chi^{\text{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi^{\text{reg}}(g) \overline{\chi_i(g)} = \frac{1}{|G|} |G| \chi_i(1) = d_i$$

□

PROP 1.17

ρ is irreducible $\iff \|\chi_\rho\|^2 = 1$. Similarly, ρ is the direct sum of two irreducible representations $\iff \|\chi_\rho\|^2 = 2$.

PROOF.

We know $\|\chi_\rho\|^2 = \sum_{i=1}^h a_i^2$, where a_i is the multiplicity of the i-th irreducible repre-

sentation in ρ's decomposition. Recall

$$\begin{aligned}\|\chi_\rho\|^2 &= \left\langle \chi_\rho, \chi_\rho \right\rangle = \langle a_1\chi_1 + \ldots + a_h\chi_h, a_1\chi_1 + \ldots + a_h\chi_h \rangle \\ &= \sum_{i=1}^{h} \langle a_i\chi_i, a_1\chi_1 + \ldots + a_h\chi_h \rangle \\ &= \sum_{i=1}^{h} a_i^2 \langle \chi_i, \chi_i \rangle = \sum_{i=1}^{h} a_i^2\end{aligned}$$

It follows that $\|\chi_\rho\|^2 = 1$ if and only if exactly one of $a_i^2 = 1$, i.e. χ_ρ is irreducible. If $\|\chi_\rho\|^2 = 2$, we must have some $i \neq j$ with $a_i^2 = a_j^2 = 1$, and so $\rho = \rho_i \otimes \rho_j$, where ρ_i, ρ_j are irreducible. □

Eg. 1.7 Consider $S_n : n \geq 2$. We consider ρ^{std}, the natural action of S_n on a set of n elements (e.g. permuting the indices of $v \in \mathbb{C}^n$). Recall Example 1.4, where we derived

$$1 = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{\text{std}}(\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} \#\text{FP of } \sigma$$

But also

$$\|\chi^{\text{std}}\|^2 = \frac{1}{n!} \sum_{\sigma \in S_n} (\chi^{\text{std}}(\sigma))^2 = \frac{1}{n!} \sum_{\sigma \in S_n} (\#\text{FP of } \sigma)^2$$

To analyze this equation, we define an action of S_n on $[n]^2$, which sends $\sigma(i,j) = (\sigma(i), \sigma(j))$. We observe exactly 2 orbits under this action: $\{(i,i) : i \in [n]\}$ and $\{(i,j) : i \neq j \in [n]\}$. By Burnside's Lemma,

$$2 = \frac{1}{n!} \sum_{\sigma \in S_n} (\#\text{FP of } \sigma \text{ on } [n]^2)$$

Observe that, σ has a fixed point $(k, \ell) \in F$ on $[n]^2$ if and only if it is fixed on each coordinate of each fixed point. In this way, we have a $n - to - n^2$ mapping, and conclude that $\|\chi^{\text{std}}\|^2 = 2$.

Note that the trivial representation is a G-stable subrepresentation of ρ^{std}. By "subtracting" $\mathbb{1}$ from ρ^{std} we can recover the other irreducible representation implied by the computation above, which we denote by $\rho^{\text{std},0}$. In particular

$$\rho^{\text{std}} = \mathbb{1} \oplus \rho^{\text{std},0}$$

PROP 1.18 Every irreducible representation of an abelian group is one dimensional.

PROOF. As G is abelian, $h = |G|$. Then, $|G| = \sum_{i=1}^{|G|} d_i^2$, from which we conclude $d_i = 1 \; \forall i$. □

Character Tables

We'll fire off a few propositions that follow immediately from the work we've done on class functions. $\sum_{i=1}^{n} d_i \chi_i = \chi_{\text{reg}}$ (recall Def 1.14)

PROP 1.19

PROOF. Follows immediately from Prop 1.19. □

Let χ be irreducible. Let $C_1, ..., C_n$ be conjugacy classes. Then $\sum_{i=1}^{n} \chi(C_i)|C_i| = \begin{cases} 0 & \chi \neq \mathbb{1} \\ |G| & \chi = \mathbb{1} \end{cases}$.

PROP 1.20

By $\chi(C_i)$, we mean the representation evaluated on any element in C_i.

Recall that the number of conjugacy classes and irreducible representations are the same, i.e. n

PROOF. Note that $\chi_{\mathbb{1}}(g) = 1 \ \forall g \in G$. Hence,

$$\sum_{i=1}^{n} \chi(C_i)|C_i| = \sum_{g \in G} \chi(g)\overline{\chi_{\mathbb{1}}(g)} = |G| \langle \chi, \chi_{\mathbb{1}} \rangle$$

□

The number of 1-dim irreducible representations is equal to $|G^{ab}|$.

PROP 1.21

PROOF. Observe that $G^* = \text{Hom}(G, C^\times) \cong \text{Hom}(G^{ab}, C^\times)$ by Thm 1.1. But, in homework, we proved $\text{Hom}(G^{ab}, C^\times) \cong G^{ab}$ (in particular, for any finite, abelian group). □

The inner product of character table rows, weighted by class size, is 0, unless the rows are equal, in which case it is $|G|$. Similarly, the inner product of character table columns is 0, unless the rows are equal, in which case it is $\frac{|G|}{|C_i|}$.

PROP 1.22

PROOF. For rows: $\sum_{i=1}^{n} \chi_i(C_k)\overline{\chi_j(C_k)}|C_k| = \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = |G| \left\langle \chi_i, \chi_j \right\rangle$.

For columns, see MATH 457. □

If $\dim(\chi_i) = 1$, then $\forall j \in [n]$, $\chi_i \chi_j$ is also an irreducible character. We call this *twisting*.

PROP 1.23

DEF 1.27

PROOF. $\chi_i \chi_j$ refers to the character $\chi_{\rho_i \otimes \rho_j}$. We use the criterion outlined in Prop 1.20.

$$\|\chi_i \chi_j\|^2 = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_i(g)}\chi_j(g)\overline{\chi_j(g)} = \|\chi_j\|^2 = 1$$

Observing that $\chi_i(g) \in \mathbb{C}^\times$ is a root of unity, and therefore $\overline{\chi_i(g)} = \chi_i(g)^{-1} = \chi_i(g^{-1})$. We conclude that $\chi_i \chi_j$ is irreducible. □

We define $\ker(\chi) = \{g : \chi(g) = \chi(1)\}$. Recall $\chi(1) = \dim(V)$.

$\ker(\chi_\rho) = \ker(\rho)$, where ρ is not necessarily irreducible.

PROP 1.24

PROOF. With $g^n = 1$, recall that we can write

$$\rho(g) = \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_d \end{pmatrix} \quad \xi_i^n = 1$$

Then, $\chi(g) = d \iff \rho(g) = I_d \iff g \in \ker(\rho)$. □

PROP 1.25 Let ρ be a representation with a decomposition into irreducible characters $\chi_\rho = \sum_{i\in I} a_i \chi_i$, $a_i > 0$. Then $\ker(\chi_\rho) = \cap_{i\in I} \ker(\chi_i)$.

PROOF. $g \in \ker(\chi) \iff \rho(g) = I_{\dim(\rho)}\ \forall i \in I \iff \rho_i(g) = I_{d_i} \iff g \in \ker(\chi_i)\ \forall i \in I$. For the middle if-and-only-if, note that, as a direct sum of irreducible representations, we may write $\rho(g)$'s matrix as a_i diagonally-adjacent block matrices of $\rho_i(g)$, for each $i \in I$. □

PROP 1.26 For any $N \triangleleft G$, we have $N = \ker(\chi)$ for some representation χ.

PROOF. Let σ be the composition of maps

$$G \overset{\pi}{\twoheadrightarrow} G/N \overset{\rho_{\text{reg}}}{\hookrightarrow} \mathrm{GL}\left(\mathbb{C}\left[G/N\right]\right)$$

Then σ is a representation, and $\ker(\pi) = N$. But $\rho_{\text{reg}}(g)$ is faithful, i.e only the identity when $g = 1$. We conclude that $\ker(\sigma) = N$, so $\ker(\chi_\sigma) = N$. □

Theorem 1.7 Normal Subgroups and Characters

Let $N_i = \ker(\chi_i) : i \in [n]$. Then, for any $I \subseteq [n]$, we have

$$N_I := \bigcap_{i\in I} N_i$$

is a normal subgroup of G. Furthermore, for any $N \triangleleft G$, there is some index set $I \subseteq [n]$ for which $N = N_I$.

PROOF. Since $N_i = \ker(\rho_i)$, we know that $N_i \triangleleft G$. Thus, $N_I = \cap_{i\in I} N_i$ is also normal. Finally, if $N \triangleleft G$, then by Prop 1.29, $N = \ker(\chi)$ for some representation χ. By Prop 1.28, we can write $N = \cap_{i\in I} \ker(\chi_i)$, where $\rho = \oplus_{i\in I} \rho^{a_i}$. □

Recall that $\rho_i : G \to \mathrm{GL}(V_i)$ is a homomorphism

Induced Representations

Let $H < G$, and let (ρ, χ, V) be a representation of H. Recall the induced representation, Def 1.16, $\mathrm{Ind}_H^G(\rho) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. We wish to study its character, denoted by $\mathrm{Ind}_H^G(\chi)$.

Let $g_1, ..., g_d$ be the coset representations for H, where $d = [G : H]$. In particular, we can write $G = \sqcup_{i\in[d]} g_i H$, and thus $\mathbb{C}[G] = \oplus_{i\in[d]}[g_i]\mathbb{C}[H]$. Then

$$\mathrm{Ind}_H^G(\rho) = \left(\oplus_{i\in[d]}[g_i]\mathbb{C}[H]\right) \otimes_{\mathbb{C}[H]} V = \oplus_{i\in[d]} \left([g_i]\mathbb{C}[H] \otimes_{\mathbb{C}[H]} V\right) = \oplus_{i\in[d]}([g_i] \otimes_{\mathbb{C}[H]} V)$$

Given $g \in G$, how does $\mathrm{Ind}_H^G(\rho)$ act? We first write $gg_i = g_j h$ for some unique coset representative g_j and $h \in H$. Then, $\forall v \in V$,

$$g([g_i] \otimes v) = (g \otimes 1)([g_i] \otimes v) = [gg_i] \otimes v = [g_j h] \otimes v$$

From here on out, we will drop the $[\cdot]$ notation. Now, viewing ρ as a $\mathbb{C}[H]$-module, and using the balancing property of tensor products, $g_j h \otimes v = g_j \otimes hv$, where $hv = \rho(h)(v)$:

$$g(g_i \otimes v) = g_j \otimes \rho(h)(v)$$

At this point, we let $\hbar(g, i)$ be element $h \in H$ that satisfies $gg_i = g_j h = g_j \hbar(g, i)$. Similarly, we let $\delta(g, i)$ be $g_j \in G$ such that $gg_i = \delta(g, i)\hbar(g, i)$.

In $\mathrm{Ind}_H^G(\rho)$, we have

$$g(g_1 \otimes v_{i_1}, ..., g_d \otimes v_{i_d}) = (\delta(g, 1) \otimes \rho(\hbar(g, 1))(v_1), ..., \delta(g, d) \otimes \rho(\hbar(g, d))(v_d))$$

$\delta(g, i) : i \in [d]$ permutes the basis vectors $g_1, ..., g_d$. Hence, $\mathrm{Ind}_H^G(\rho)(g)$, in some suitable basis, can be thought of as a set of block matrices $\{\rho(\hbar(g, i)) : i \in [d]\}$, positioned accordingly in columns $i \in [d]$. However, to account for the permutation $\delta(g, i)$, the block matrix $\rho(\hbar(g, i))$ is placed in the $\delta(g, i)$-th row, or rather the k-th column, where $g_k = \delta(g, i)$.

This block contributes to the trace if and only if $\delta(g, i)$ corresponds to the i-th basis vector, i.e. $\iff gg_i = g_i\hbar(g, i) \iff g_i^{-1} gg_i \in H$. In this case, the trace contributed is equal to $\chi(\hbar(g, i)) = \chi(g_i^{-1} gg_i)$. In short, then, we have the following result:

Theorem 1.8 Character of the Induced Representation

$$\mathrm{Ind}_H^G(\chi)(g) = \sum_{g_i : g_i^{-1} gg_i \in H} \chi(g_i^{-1} gg_i)$$

PROOF.

See above discussion. □

This lends itself to some simplifications. We adopt the notation $\dot{\chi}(g) = \begin{cases} \chi(g) & g \in H \\ 0 & \text{o.w.} \end{cases}$.

$$\mathrm{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{b\in G} \dot{\chi}(b^{-1} gb)$$

PROP 1.27

PROOF. $\mathrm{Ind}_H^G(\chi)(g) = \sum_{i\in[d]} \dot\chi(g_i^{-1}gg_i)$. Let $h \in H$. Then $(g_ih)^{-1}g(g_ih) = h^{-1}(g_i^{-1}gg_i)h$. But χ is a class function, so $\chi(h^{-1}(g_i^{-1}gg_i)h) = \chi(g_i^{-1}gg_i)$. As g_i are coset representatives,

$$\frac{1}{|H|}\sum_{b\in g_iH} \dot\chi(b^{-1}gb) = \dot\chi(g_i^{-1}gg_i)$$

Then, summing over cosets gives the result. □

PROP 1.28 When $H \triangleleft G$,

$$\mathrm{Ind}_H^G(\chi)(g) = \begin{cases} \frac{1}{|H|}\sum_{b\in G}\chi(b^{-1}gb) & g \in H \\ 0 & g \notin H \end{cases}$$

PROOF. Since H is normal, $b^{-1}gb \in H \iff g \in H$. It follows that $\dot\chi(b^{-1}gb) = \chi(b^{-1}gb)$ when $g \in H$. On the other hand, we have $g \notin H \iff b^{-1}gb \notin H$, so $\dot\chi(b^{-1}gb) = 0$ when $g \notin H$. □

Theorem 1.9 Frobenius Reciprocity

Let $H < G$. Denote by $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_G$ the usual inner products on Class(H) and Class(G), respectively. Let η and γ be representations of H and G, respectively. Then

$$\left\langle \mathrm{Ind}_H^G(\eta), \gamma \right\rangle_G = \left\langle \eta, \mathrm{Res}_H^G(\gamma) \right\rangle_H$$

PROOF.

$$\begin{aligned}
\left\langle \mathrm{Ind}_H^G(\eta), \gamma \right\rangle_G &= \frac{1}{|G|}\sum_{g\in G} \mathrm{Ind}(\eta)(g)\overline{\gamma(g)} = \frac{1}{|G|}\sum_{g\in G}\sum_{b\in G}\frac{1}{|H|}\dot\eta(b^{-1}gb)\overline{\gamma(g)} \\
&= \frac{1}{|G|\cdot|H|}\sum_{g,b:b^{-1}gb=t\in H} \eta(t)\overline{\gamma(btb^{-1})} \\
&= \frac{1}{|G|\cdot|H|}\sum_{t\in H}\sum_{b\in G}\eta(t)\overline{\gamma(btb^{-1})} = \frac{1}{|G|}\sum_{t\in H}\eta(t)\overline{\gamma(t)} \\
&= \left\langle \eta, \mathrm{Res}_H^G(\gamma) \right\rangle_H \quad \square
\end{aligned}$$

Eg. 1.8 Let $H = \{1\}$ and $\eta = \chi_{\mathrm{triv}}$. Then $\mathrm{Ind}_H^G(\eta) = \chi_{\mathrm{reg}}$. Let χ be irreducible on G. By the theorem above,

$$\langle \chi_{\mathrm{reg}}, \chi\rangle_G = \langle \chi_{\mathrm{triv}}, \sigma\rangle_H = \dim(\chi)$$

where σ is a $\dim(\chi)$-identity matrix on $H = \{1\}$. At the same time, we know

that $\langle \chi_{\text{reg}}, \chi \rangle_G$ is the multiplicity of χ in χ_{reg}. But this is exactly consistent with what we found.

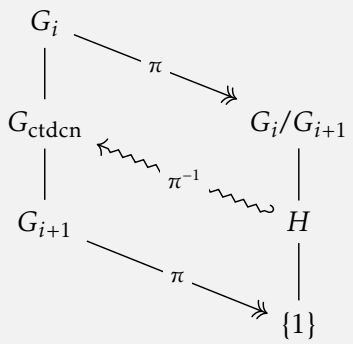
Supersolvable Groups

We say that G is *solvable* if there exists a chain DEF 1.28

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_N = \{1\}$$

with $G_{i+1} \triangleleft G_i$ and G_i/G_{i+1} abelian. We may assume that G_i/G_{i+1} is cyclic of prime order.

PROOF. Refine the chain until no normal subgroups can be inserted. In other words, if there exists $H : G_{i+1} \triangleleft H \triangleleft G_i$, insert this into the chain. Once this is complete, it must be that G_i/G_{i+1} is simple. Suppose not, and let $H \triangleleft G_i/G_{i+1}$ Consider the following:



The preimage under a homomorphism of a normal subgroup is also normal. This contradicts the refinement of the chain. It is well-known that the only simple abelian groups are prime and cyclic. Hence, G_i/G_{i+1} are assumed to be so. □

We say that G is *supersolvable* if there exists a chain DEF 1.29

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_N = \{1\}$$

with $G_i \triangleleft G$ and G_i/G_{i+1} cyclic. As before, we may assume that G_i/G_{i+1} has prime order.

The following are properties of supersolvable groups: **PROP 1.29**

1. If G is supersolvable, then so is $H < G$
2. If G is supersolvable, and $G \twoheadrightarrow H$ is surjective, then so is H
3. If G is a p-group, then it is supersolvable
4. If G_1, G_2 are both supersolvable, then so is $G_1 \oplus G_2$.

Theorem 1.10 Blichfeldt's Lemma

Let G be a finite, non-abelian supersolvable group. Then there exists $N \triangleleft G$, with N abelian but $N \not\subseteq Z(G)$.

PROOF.

Since G is non-abelian, we know $G_0 = G \not\subseteq Z(G)$. But it is true that $G_N = \{1\} \subseteq Z(G)$. Hence, there must be some G_i with $G_i \not\subseteq Z(G)$ but $G_{i-1} \subseteq Z(G)$. It only remains to show that $N = G_i$ is abelian.

Since G_i/G_{i+1} is cyclic, we can generate it with $\langle x \rangle$ for some $x \in N$. Then, any element in G_i can be written as $x^a y$ for some integer a and $y \in G_{i+1}$.

$$x^{a_1} y_1 x^{a_2} y_2 \overset{G_{i-1} \subseteq Z(G)}{=} x^{a_1} x^{a_2} y_1 y_2 = x^{a_2} y_2 x^{a_1} y_1$$

as desired. □

Theorem 1.11 Blichfeldt's Theorem

Let G be a finite, supersolvable group. Let (ρ, V) be some irreducible representation of G. Then $p \cong \mathrm{Ind}_H^G(\psi)$ for some subgroup $H < G$ and 1-dim representation ψ of H.

PROOF.

If G is abelian, we have no work to do. $\mathrm{Ind}_G^G(\rho) = \rho$ clearly, and for all representations of G, $\rho \in G^*$ is 1-dim. Hence, assume G is non-abelian. We also assume that ρ is faithful, i.e. has a trivial kernel, as quotients of supersolvable groups are supersolvable. In particular, ρ is faithful on $G/\ker(\rho)$.

We proceed by induction on $|G|$. Let N be as in Thm 1.10. Then V can be viewed as a representation of N via $\mathrm{Res}_N^G(\rho)$. As N is abelian, its irreducible characters are 1-dim. Hence, $V \cong \oplus_{\psi \in N^*} V_\psi$, where $V_\psi = \{v \in V : \rho(n)(v) = \psi(v)\ \forall n \in N\}$.

For any $\psi \in N^*$ and $g \in G$, we define $\psi^g : N \to \mathbb{C}^\times : \psi^g(n) = \psi(g^{-1}ng)$. We claim that $\rho(g)$ is a map $V_\psi \to V_{\psi^g}$. Let $v \in V_\psi$. Then

$$\rho(n)(\rho(g)(v)) = \rho(ng)(v) = \rho(g)(\rho(g^{-1}ng)(v)) = \cdots$$

But $\rho(g^{-1}ng)(v) = \psi(g^{-1}ng)(v) = \psi^g(n)(v)$, by assumption. As this is a scalar, we can pull it out:

$$\cdots = \psi^g(n)\rho(g)(v)$$

Hence, $\rho(g)(v) \in V_{\psi^g}$, as claimed. Since we have easy access to the inverse $\rho(g^{-1})$, it follows that $V_\psi \cong V_{\psi^g}$. Pick any $\psi \in N^*$ such that $V^\psi \neq \{0\}$. Let $S = \{\psi^g : g \in G\} \subseteq N^*$.

$$\bigoplus_{\chi \in S} V^\chi \subseteq V \implies \bigoplus_{\chi \in S} V^\chi = V$$

by irreducibility. Then, $\dim(V) = \#S \dim(V^\psi)$. In particular, if $H = \{g \in G : \psi^g = \psi\}$, then $\#S = [G : H]$. We claim that $\mathrm{Ind}_H^G(V^\psi) \cong \rho$.

BLAH! □

Fourier Transforms

Let G be a finite group. Let $C(G,\mathbb{C})$ denote the space of functions (with no particular structure) $f : G \to \mathbb{C}$. We can view this as a $\mathbb{C}$-vector space equipped with addition and scalar multiplication:

$$(f+g)(s) = f(s) + f(g) \qquad f(\alpha s) = \alpha f(s)$$

Under this view, $C(G,\mathbb{C})$ has a basis $\{\delta_s : s \in G\}$, where $\delta_s(g) = \begin{cases} 1 & g = s \\ 0 & \text{o.w.} \end{cases}$.

We can also view $C(G,\mathbb{C})$ through the group ring $\mathbb{C}[G]$. In particular, by defining the *convolution* as follows DEF 1.30

$$(f * g)(s) = \sum_{t \in G} f(st^{-1})g(t)$$

we see that $C(G,\mathbb{C}) \cong \mathbb{C}[G]$ by associating

$$\sum_{s\in G} a_s[s] \mapsto f : f(s) = a_s \qquad f \mapsto \sum_{s\in G} f(s)[s]$$

We note the addition maps in the usual way:

$$\sum_{s\in G} a_s[s] + \sum_{s\in G} b_s[s] \quad \leftrightarrows \quad f+g : f(s) = a_s, g(s) = b_s$$

And multiplication maps via convolutions:

$$\left(\sum_{s\in G} a_s[s]\right)\left(\sum_{s\in G} b_s[s]\right) = \sum_{s,t\in G\times G} a_s b_t[st] = \sum_{s,t\in G\times G} a_{st^{-1}} b_t[s] \quad \leftrightarrows \quad h : h(s) = \sum_{t\in G} a_{st^{-1}} b_t = (f*g)(s)$$

where $f \leftrightarrow \sum_{s\in G} a_s[s]$ and $g \leftrightarrow \sum_{s\in G} b_s[s]$.

Theorem 1.12 Properties of Group Convolutions

1. $(f * g) * h = f * (g * h)$
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(g_1 + g_2) * f = g_1 * f + g_2 * f$
4. $\delta_g * \delta_h = \delta_{gh}$
5. The representation ρ of G on $C(G,\mathbb{C})$ given by $(bf)(x) = f(b^{-1}x)$ is equivalent to the regular representation ρ_{reg} on $\mathbb{C}[G]$.

PROOF.

Each of these is established via inheritance from the $\mathbb{C}[G]$ view (in particular, points 1, 2, and 3 are immediate).

For 4, see that $\delta_g \mapsto [g]$ in $\mathbb{C}[G]$, so we conclude that $\delta_g * \delta_h \mapsto [gh] \leftarrow\!\shortmid \delta_{gh}$.

For 5,

$$bf \leftrightarrow \sum_{s\in G}(bf)(s)[s] = \sum_{s\in G} f(b^{-1}s)[s] = \sum_{s\in G} f(s)[bs] = [b]\sum_{s\in G} f(s)[s]$$

which is exactly $\rho(b)\left(\sum_{s\in G} f(s)[s]\right)$. □

DEF 1.31 For $f \in C(G,\mathbb{C})$, we define the *Fourier transform*, denoted $\hat{f}$, to be a function from representations (ρ, V) to their endomorphism group $\mathrm{End}(V)$, with

$$\hat{f}(\rho) = \sum_{s\in G} f(s)\rho(s) \in \mathrm{End}(V)$$

Theorem 1.13 Properties of Fourier Transforms

Let $f, g \in C(G,\mathbb{C})$. Let (ρ, V) be a representation. Then

1. $\hat{f} + \hat{g} = \widehat{f+g}$ and $\widehat{\alpha f} = \alpha\hat{f}$.
2. $\widehat{\delta_s}(\rho) = \rho(s)$
3. $\widehat{f * g} = \hat{f} \circ \hat{g}$
4. Let $U \in C(G,\mathbb{C})$ be the uniform probability distribution on G. Then $\hat{U}(\rho)$ is a projection from $V \to V^G$. We conclude that $\hat{U}(\rho_{\text{triv}}) = 1 \in \mathbb{C}^\times$, and $\hat{U}(p_i) = 0$ for any irreducible ρ_i.

PROOF.

1, 2 can be left as an exercise.

For 3, write

$$\begin{aligned}
\widehat{f*g}(\rho) &= \sum_{s\in G}\left(\sum_{t\in G} f(st^{-1})g(t)\right)\rho(s) \\
&= \sum_{s\in G}\sum_{t\in G} f(st^{-1})g(t)\rho(s) = \sum_{s\in G}\sum_{t\in G} f(s)g(t)\rho(st) \\
&= \sum_{s\in G}\sum_{t\in G} f(s)\rho(s)g(t)\rho(t) = \left(\sum_{s\in G} f(s)\rho(s)\right)\left(\sum_{t\in G} g(t)\rho(t)\right) \\
&= (\hat{f}\circ\hat{g})(\rho)
\end{aligned}$$

For 4, we see that

$$g\hat{U}(\rho)(v) = g\sum_{s\in G} U(s)\rho(s)(v) = U(\mathbb{1})\sum_{s\in G}\rho(gs)(v) = U(\mathbb{1})\sum_{s\in G}\rho(s)(v) = \hat{U}(\rho)(v)$$

Noting that $U(g)$ is constant over all $g \in G$. Thus, $\mathrm{Im}(\hat{U}(\rho)) \subseteq V^G$. Showing $\hat{U}(\rho)(\hat{U}(\rho)(v)) = \hat{U}(\rho)(v)$ is left as an exercise.

Observe that V^G under $\rho_{\text{triv}} = V$, and thus $\hat{U}(\rho_{\text{triv}})$ acts as the identity. However, $\text{Im}(\hat{U}(\rho_i))$, for any irreducible ρ_i, forms a non-trivial G-stable subspace V^G. Thus, $\hat{U}(\rho_i)$ must be 0. □

Note that, if f is a probability distribution on G, we can view $\hat{f}(\rho) = \mathbb{E}[\rho]$, viewing ρ as a random variable which takes on values $\rho(g) : g \in G$.

Theorem 1.14 Fourier Inversion and Plancherel

Fourier Inversion Formula

$$f(s) = \frac{1}{|G|} \sum_{i=1}^{n} d_i \text{tr}\left(\rho_i(s^{-1})\hat{f}(\rho_i)\right)$$

Plancherel's Formula

$$\sum_{s\in G} f_1(s^{-1}) f_2(s) = \frac{1}{|G|} \sum_{i=1}^{n} d_i \text{tr}\left(\hat{f}_1(\rho_i)\hat{f}_2(\rho_i)\right)$$

PROOF.

Note that both sides of the Fourier inversion are linear in f. Similarly, both sides of Plancherel's formula are bi-linear in (f_1, f_2). Thus, it is enough to prove Fourier inversion on $f = \delta_g$, and similarly on $f = \delta_g, g = \delta_h$ for Plancherel. Thus:

Fourier Inversion Formula Recalling that $\widehat{\delta_g}(\rho) = \rho(g)$,

$$\begin{aligned}\frac{1}{|G|} \sum_{i=1}^{n} d_i \text{tr}\left(\rho_i(s^{-1})\widehat{\delta_g}(\rho_i)\right) &= \frac{1}{|G|} \sum_{i=1}^{n} d_i \text{tr}\left(\rho_i(s^{-1})\rho_i(g)\right) \\ &= \frac{1}{|G|} \sum_{i=1}^{n} d_i \chi_i(s^{-1}g) = \frac{1}{|G|}\chi_{\text{reg}}(s^{-1}g) \\ &= \begin{cases} 1 & s = g \\ 0 & \text{o.w.} \end{cases} = \delta_g\end{aligned}$$

Plancherel's Formula

$$\begin{aligned}\frac{1}{|G|} \sum_{i=1}^{n} d_i \text{tr}\left(\rho_i(g)\rho_i(h)\right) &= \frac{1}{|G|}\chi_{\text{reg}}(gh) = \begin{cases} 1 & g = h^{-1} \\ 0 & \text{o.w.} \end{cases} \\ &= \sum_{s\in G} \delta_g(s^{-1})\delta_h(s)\end{aligned}$$

□

PROP 1.30

$$\langle f_1, f_2 \rangle = \frac{1}{|G|^2} \sum_{i=1}^{n} d_i \text{tr}\left(\hat{f}_1(\rho_i)\overline{\hat{f}_2(\rho_i)}^t\right)$$

PROOF.

To help with notation, we write $k(s) = \overline{f_2(s^{-1})}$. We have

$$\langle f_1, f_2 \rangle_G = \frac{1}{|G|} \sum_{s \in G} f_1(s) k(s^{-1}) = \frac{1}{|G|} \sum_{s \in G} f_1(s^{-1}) k(s)$$

Applying Plancherel gives

$$\frac{1}{|G|^2} \sum_{i=1}^{n} d_i \operatorname{tr}\left(\hat{f}_1(\rho_i)\hat{k}(\rho_i)\right)$$

It thus remains to show that $\hat{k}(\rho_i) = \overline{\hat{f}_2(\rho_i)}^t = \hat{f}_2(\rho_i)^* \ \forall i \in [n]$. Let ρ be arbitrary. In some basis, we know that a $\rho(s)$ is unitary, i.e. $\rho(s)\rho(s)^* = \mathrm{Id} \implies \rho(s)^* = \rho(s^{-1})$. With this in mind,

$$\begin{aligned}\hat{k}(\rho) &= \sum_{s \in G} k(s)\rho(s) = \sum_{s \in G} \overline{f_2(s^{-1})}\rho(s) = \sum_{s \in G} \overline{f_2(s)}\rho(s^{-1}) \\ &= \sum_{s \in G} \overline{f_2(s)}\rho(s)^* = \overline{\sum_{s \in G} f_2(s)\rho(s)}^t = \hat{f}_2(\rho)^*\end{aligned}$$

where we note that $f_2(s)^t = f_2(s)$, as it is a scalar. □

We make some final miscellaneous remarks about the Fourier transform:

When G is abelian, we only have abstract isomorphisms $G \cong G^*$. However, a canonical isomorphism can be formed from $G \cong G^{**}$ by $g \mapsto (\chi \mapsto \chi(g))$. We denote by $[g] : G^* \to \mathbb{C}^\times$ the function $\chi \mapsto \chi(g)$, i.e. $G \cong G^{**}$ by $g \mapsto [g]$.

When G is abelian, $C(G, \mathbb{C}) = \mathrm{Class}(G)$, which we know to be an inner product space with the usual inner product. As discussed earlier, $f = \sum_{g \in G} f(g)\delta_g$. Let $N = |G|$. Rewriting this in terms of the inner product gives

$$\left\langle f, \sqrt{N}\delta_g \right\rangle = \sqrt{N} f(g) \implies f = \sum_{g \in G} \left\langle f, \sqrt{N}\delta_g \right\rangle \sqrt{N}\delta_g$$

It follows that $\{\sqrt{N}\delta_g : g \in G\}$ is an orthonormal basis for $C(G, \mathbb{C}) = \mathrm{Class}(G)$.

PROP 1.31 When $G = \mathbb{Z}/N\mathbb{Z}$, with $f \in C(G, \mathbb{C})$, we have

$$f(b) = \frac{1}{N} \sum_{a=0}^{n-1} \hat{f}(a)\xi_N^{-ab} \qquad \|f\|^2 = \frac{1}{N}\|\hat{f}\|^2$$

where $\xi_N = e^{\frac{2\pi i}{N}}$.

PROOF.

By Fourier inversion, we get

$$f(b) = \frac{1}{N}\sum_{a=0}^{n-1} \operatorname{tr}\left(\hat{f}(\chi_a)\chi_a(b^{-1})\right) = \frac{1}{N}\sum_{a=0}^{n-1} \hat{f}(\chi_a)\chi_a(b^{-1})$$

where we remove trace due to G-representations being 1-dim. In particular, we parameterize G's representations by $\chi_a : a \in G$, where

$$\chi_a(b) = e^{\frac{2ab\pi i}{N}} = \xi_N^{ab}$$

Using this parameterization, we view $\hat{f} : G \to \mathbb{C}^\times$. Furthermore, in $\mathbb{Z}/N\mathbb{Z}$, we have $\chi_a(b^{-1}) = \chi_a(-b) = \overline{\chi_a(b)}$. The first result follows. By Prop 1.33,

$$\langle f, f\rangle = \frac{1}{N^2}\sum_{a=0}^{n-1} \hat{f}(\chi_a)\overline{\hat{f}(\chi_a)}^t = \frac{1}{N^2}\sum_{a=0}^{n-1} \hat{f}(a)\overline{\hat{f}(a)} = \frac{1}{N}\langle \hat{f}, \hat{f}\rangle$$

From which the second result follows. □

Random Walks on Cyclic Groups

We can use the theory of finite representations to analyze random walks on $\mathbb{Z}/N\mathbb{Z}$, as Markov processes. Much of this work was pioneered by Diaconis and Shahshahani, who picked much of the "low hanging fruit," i.e. processes on the finite groups whose representations are well-understood (supersolvable, abelian, symmetric, dihedral). Further study has been limited.

Let G be a finite group, with irreducible representations $\rho_1, ..., \rho_n$ of dimensions $d_1, ..., d_n$. Define probability distributions P and Q on G. We define the *total variation norm* on probability measures to be DEF 1.32

$$\|P - Q\|_{\max} = \frac{1}{2}\sum_{s\in G} |P(s) - Q(s)|$$

Apply Kolmogorov's axioms yields $\|P-Q\|_{\max} = \max_{A\subseteq G} |P(A)-Q(A)|$. This norm provides a worst-case measure for the convergence of P to a distribution Q.

Theorem 1.15 Diaconis-Shahshahani Lemma

Let $U \sim U(G)$ be the uniform distribution on G. Let P be an arbitrary probability distribution on G. Then

$$\|P - U\|_{\max}^2 \le \frac{1}{4}\sum_{i=2}^{n} d_i \operatorname{tr}\left(\hat{P}(\rho_i)\hat{P}(\rho_i)^*\right)$$

PROOF.

Utilizing $(\sum a_i b_i)^2 \le (\sum a_i^2)(\sum b_i^2)$, with $b_i = 1$,

$$4\|P-U\|^2 = \left(\sum_{s\in G} |P(s)-U(s)|\right)^2 \le \#G \sum_{s\in G} (P(s)-U(s))^2$$

As $P(s) - U(s) \in \mathbb{R}$, we have $4\|P-U\|^2 \le \#G^2 \cdot \langle P-U, P-U\rangle$. Applying Plancherel and Prop 1.33 gives

$$4\|P-U\|^2 \le \sum_{i=1}^{n} d_i \mathrm{tr}\left(\widehat{P-U}(\rho_i)\ \widehat{P-U}(\rho_i)^*\right)$$

We've noted in Thm 1.13 that $\hat{U}(\rho_i) = 0$ except on the trivial representation. Also note that $\hat{P}(\rho_{\text{triv}}) = 1$ (this dissapears $\rho_{\text{triv}} = \rho_1$ in the sum). Since the Fourier transform is linear on $C(G, \mathbb{C})$, we conclude that

$$4\|P-U\|^2 \le \sum_{i=2}^{n} d_i \mathrm{tr}\left(\widehat{P}(\rho_i)\ \widehat{P}(\rho_i)^*\right)$$

as desired. □

Using the Diaconis-Shahshahani lemma, we will consider a worked example on $G = \mathbb{Z}/N\mathbb{Z}$. To maintain consistent multiplicative notions, we take on the view

$$G = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in G\right\} = \mathbb{Z}/N\mathbb{Z}$$

Suppose $X \sim P$ and $Y \sim Q$. We ask $\mathbb{P}(XY = s)$. This is exactly

$$\sum_{t\in G} P(st^{-1})Q(t) = (P * Q)(s)$$

Inductively, if $X_1, ..., X_n \overset{\text{iid}}{\sim} P$, we conclude that $\mathbb{P}(\prod_{i=1}^n X_i = s) = P^{*n}(s)$. We use these notions to model the process $Y_i = Y_{i-1} + b_i : b_i \in \mathbb{Z}/N\mathbb{Z}$, with $Y_0 = 0$.

Using G, this is

$$Y_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad Y_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix} Y_{i-1} : b_i \in \mathbb{Z}/N\mathbb{Z}$$

Our primary question is the following:

Given some distribution P on $[n]$, do random walks on $\mathbb{Z}/N\mathbb{Z}$ via P converge to the uniform distribution U on G. In other words, $\lim_{n\to\infty} Y_n = \lim_{n\to\infty} P^{*n} \sim U$?

Eg. 1.9 Let $P \sim \text{Ber}(\frac{1}{2})$ on $\{-1, 1\}$, with $G = \mathbb{Z}/N\mathbb{Z}$. Then the random walk on $\mathbb{Z}/N\mathbb{Z}$ is approximately uniform for n large enough.

We can write $P = \frac{1}{2}(\delta_1 + \delta_{-1})$. Then

$$\hat{P}(j) = \frac{1}{2}(\chi_1(j) + \chi_{-1}(j)) = \frac{1}{2}\left(e^{\frac{2\pi ij}{N}} + e^{-\frac{2\pi ij}{N}}\right) = \cos\left(\frac{2\pi j}{N}\right)$$

where χ are the irreducible 1-dim representations parameterized by $-1, 1$. As convolutions on $C(G, \mathbb{C})$ acts as composition on $\mathbb{C}[G]$ (see Thm 1.13), we have

$$\widehat{P^{*n}}(j) = \cos^n\left(\frac{2\pi j}{N}\right)$$

Applying the Diaconis-Shahshahani lemma gives

$$\|P^{*n} - U\|_{\max}^2 \leq \frac{1}{4}\sum_{j=2}^{n} \widehat{P^{*n}}(j) = \frac{1}{4}\sum_{j=2}^{n} \cos^{2n}\left(\frac{2\pi j}{N}\right) \overset{(!)}{\leq} e^{\frac{-\pi^2 n}{2N^2}}$$

for $n \geq N^2$ and $N \geq 7$ is odd. The (!) bound is shown by Shahshahani in a brute-force way. Note, for an even cyclic group $\mathbb{Z}/N\mathbb{Z}$, this process cannot converge: when we take n even steps, we will land on an even number mod N, and when we take n odd steps, we will land on an odd number mod N.

After N^2 steps we are close to the uniform distribution:

$$\|P^{*n} - U\|^2 \leq e^{\frac{-\pi^2}{2}} \approx 0.007$$

When P is uniform on $\{-1, 0, 1\}$, one can show that $P^{*n} \approx U$ after N^2 steps as well. Funnily, if we consider distribution above, with $X_n = 3X_{n-1} + b_n$, Diaconis-Shahshahani showed that we converge to U in only $\log(N)$ steps!

This is as good as you can do, as we require $\log(N)$ steps to simply observe all N integers via a branching process.

II Homological Algebra

An *abelian category* is one in which $\mathrm{Mor}(A, B)$ forms an abelian group via summation, where A, B are any two objects in the category. We also define a zero element for this category. A functor F between two abelian categories $\underline{C}, \underline{D}$ is called *additive* if the following hold: DEF 2.1 DEF 2.2

1. $F(f + g) = F(f) + F(g)$ for any two morphisms on $\underline{C}$
2. $F(A \oplus B) \cong F(A) \oplus F(B)$, where $A \in \mathrm{Ob}(\underline{C})$, $B \in \mathrm{Ob}(\underline{D})$.
3. $F(\mathbb{0}_{\underline{C}}) = \mathbb{0}_{\underline{D}}$

$F(f + g) = F(f) + F(g)$ for any two morphisms on $\mathbf{X}$, and $F(A \oplus B) \cong F(A) \oplus F(B)$, where $A \in \mathrm{Ob}(\mathbf{X})$, $B \in \mathrm{Ob}(\mathbf{Y})$. In these notes, we primarily consider the abelian categories ${}_{\mathbf{R}}\mathbf{Mod}$ and $\mathbf{Mod}_{\mathbf{R}}$. When we say "$R$-modules," we mean either left or right $\mathbb{R}$-modules.

EXACT SEQUENCES

A sequence of morphisms

$$A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots A_2 \xrightarrow{f_2} A_1$$

DEF 2.3 is called an *exact sequence* if the image of a map is equal to the kernel of the subsequent map, i.e. $\mathrm{Im}(f_i) = \ker(f_{i+1})$. Recall that A_i can be thought of as modules and f_i as module homomorphisms.

DEF 2.4 A *short exact sequence* (written SES) is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

In particular, it must be that $\ker(f) = \{0\}$, the module constructed from the trivial group and R. For similar reasons, we conclude that $\mathrm{Im}(g) = C$, and thus

$$B/\ker(g) \cong \mathrm{Im}(g) \implies B/\mathrm{Im}(f) \cong C$$

DEF 2.5 We call an infinite sequence an *R-complex*, and write

$$(A_\bullet, f_\bullet) = \cdots \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \cdots$$

DEF 2.6 A *morphism of R-complexes* $(A_\bullet, f_\bullet) \to (B_\bullet, g_\bullet)$ is a collection of module homomorphisms $h_n : A_n \to B_n$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & A_n & \xrightarrow{f_n} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n-1}} & & \\
\cdots & \longrightarrow & B_n & \xrightarrow[g_n]{} & B_{n-1} & \longrightarrow & \cdots
\end{array}$$

We can extend this diagram as follows

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \ker(h_n) & \xrightarrow{f_n} & \ker(h_{n-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_n & \xrightarrow{f_n} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n-1}} & & \\
\cdots & \longrightarrow & B_n & \xrightarrow{g_n} & B_{n-1} & \longrightarrow & \cdots
\end{array}$$

Note that R-complexes themselves form an abelian category, with the zero object being $\cdots \to 0 \to 0 \to \cdots$, and $(A_\bullet, f_\bullet) \oplus (B_\bullet, g_\bullet) = (A_\bullet \oplus B_\bullet, (f_\bullet, g_\bullet))$, i.e.

$$(A_\bullet \oplus B_\bullet, (f_\bullet, g_\bullet)) = \cdots \to A_n \oplus B_n \xrightarrow{(f_n, g_n)} A_{n-1} \oplus B_{n-1} \xrightarrow{(f_{n-1}, g_{n-1})} A_{n-2} \oplus B_{n-2} \cdots$$

DEF 2.7 An additive covariant functor $F : \underline{C} \to \underline{D}$ is called *left-exact* if

$$0 \to A \to B \to C \text{ exact} \implies 0 \to F(A) \to F(B) \to F(C) \text{ exact}$$

and *right-exact* if DEF 2.8

$$A \to B \to C \to 0 \text{ exact} \implies F(A) \to F(B) \to F(C) \to 0 \text{ exact}$$

Similarly, an additive contravariant functor is called left-exact if

$$A \to B \to C \to 0 \text{ exact} \implies 0 \to F(C) \to F(B) \to F(A) \text{ exact}$$

and right-exact if

$$0 \to A \to B \to C \text{ exact} \implies F(C) \to F(B) \to F(A) \to 0 \text{ exact}$$

Unsurprisingly, F is called an *exact functor* if it is both left- and right-exact. Equivalently, DEF 2.9
we can ask

$$0 \to A \to B \to C \to 0 \text{ exact} \implies 0 \to F(A) \to F(B) \to F(C) \to 0 \text{ exact}$$

with arrows flipped on the RHS in the contravariant case. In other words, short exact sequences induce short exact sequence.

Theorem 2.1 Module Tensors are Right-Exact

For $M \in \mathbf{Mod_R}$, the functor into abelian groups

$$M \otimes_R (-) : N \mapsto M \otimes_R N$$

is additive and right-exact. When working with $M \in {}_{\mathbf{R}}\mathbf{Mod}$, the map $(\cdot) \otimes_R M$ is also additive and right-exact. Note that both functors are covariant.

PROOF.

Fix $M \in \mathbf{Mod_R}$. Let $A \xrightarrow{f} B$. Then we know that $M \otimes A \xrightarrow{\mathrm{Id}\otimes f} M \otimes B$, that $M \otimes (\cdot)$ is additive, and that $\mathrm{Id} \otimes (f_1 + f_2) = \mathrm{Id} \otimes f_1 + \mathrm{Id} \otimes f_2$.

Given some exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, we must show that

$$M \otimes A \xrightarrow{1\otimes\alpha} M \otimes B \xrightarrow{1\otimes\beta} M \otimes C \longrightarrow 0$$

is also exact.

To start, we have that $1 \otimes \beta$ is surjective: given $\sum_{i\in I} m_i \otimes c_i$, we know $\exists b_i \in B$ with $\beta(b_i) = c_i$, as β is surjective. With this notation, $(1 \otimes \beta)(\sum_{i\in I} m_i \otimes b_i) = \sum_{i\in I} m_i \otimes c_i$.

We now must show that $\mathrm{Im}(1 \otimes \alpha) = \ker(1 \otimes \beta)$. We have containment, since $(1 \otimes \beta) \circ (1 \otimes \alpha) = 1 \otimes (\beta \circ \alpha)$. We know that $\mathrm{Im}(\alpha) = \ker(\beta)$, so this is $1 \otimes 0 = 0$, as maps.

For the converse, let $E = \mathrm{Im}(1 \otimes \alpha)$. Since $E \subseteq \ker(1 \otimes \beta)$ (one can show it is normal in $M \otimes B$), by the homomorphism theorem there exists some $\overline{\beta}$ such that $1 \otimes \beta$ is the composition of maps:

$$M \otimes B \twoheadrightarrow (M \otimes B)/E \xrightarrow{\overline{\beta}} \mathrm{Im}(1 \otimes \beta) = M \otimes C$$

In particular, there is a surjective map $\overline{\beta} : (M \otimes B)/E \twoheadrightarrow M \otimes C$, so, by the isomorphism

theorem, we just need to show that this is an isomorphism to conclude $E \cong \ker(1 \oplus \beta)$. One can do this by constructing a left inverse $\overline{f}$ for $\overline{\beta}$. □

DEF 2.10 $M \in \mathbf{Mod_R}$ is called a *flat module* if $M \otimes_R (-)$ is an exact functor. Similarly, $M \in {}_\mathbf{R}\mathbf{Mod}$ is called flat if $(-) \otimes_R M$ is an exact functor.

PROP 2.1 $M \in \mathbf{Mod_R}$ is a flat module if and only if $A \hookrightarrow B \implies M \otimes_R A \hookrightarrow M \otimes_R B$. Similarly, $M \in {}_\mathbf{R}\mathbf{Mod}$ is flat if and only if $A \hookrightarrow B \implies A \otimes_R M \hookrightarrow A \otimes_R M$.

PROOF. We'll consider $M \in \mathbf{Mod_R}$.

Let $A \hookrightarrow B$. Consider a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} B/A \to 0$.

By Thm 2.1, we have that $M \otimes A \xrightarrow{\mathrm{Id}\otimes f} M \otimes B \xrightarrow{\mathrm{Id}\otimes g} B/A \to 0$ is exact. We then have exactness on the complete SES if and only if $\ker(f) = 0 \implies \ker(\mathrm{Id} \otimes f) = 0$. □

Recall the notion of a free module from MATH 456. We say that an R-module M is free if either of the following equivalent statements hold:

- $M \cong \bigoplus_{i\in I} R$. Hence, $m = (r_i)_{i\in I}$, which is zero except at finitely many points. (Recall that finite support is baked into infinite direct sums).
- There exists some set $(m_i)_{i\in I}$ such that every $m \in M$ is uniquely represented by $\sum_{i\in I} \alpha_i m_i$, where $\alpha_i = 0$ except at finitely many points.

PROP 2.2 Free R-modules are flat.

PROOF. Let $M = R^n \in {}_\mathbf{R}\mathbf{Mod}$. Consider $A \xhookrightarrow{f} B$. Then

$$\begin{array}{ccc} A \otimes_R R^n & \xrightarrow{f\otimes \mathrm{Id}} & B \otimes_R R^n \\ \| & & \| \\ \bigoplus_{i=1}^n A \otimes_R R & & \bigoplus_{i=1}^n B \otimes_R R \\ \| & & \| \\ \bigoplus_{i=1}^n A & \xrightarrow[(f,\ldots,f)]{} & \bigoplus_{i=1}^n B \end{array}$$

Since f is injective, so is $(f, \ldots, f)$, and thus $f \otimes \mathrm{Id}$. □

Eg. 2.1 Some modules are verifiably *not* flat. For instance, take $R = \mathbb{Z}$ and $M = \mathbb{Z}/n\mathbb{Z} \in \mathbf{Mod}_\mathbb{Z}$. Consider $[2] : \mathbb{Z} \to \mathbb{Z} : x \mapsto 2x$. Then

$$\begin{array}{ccc} \mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{[2]\otimes \mathrm{Id}} & \mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \\ \| & & \| \\ \mathbb{Z}/2\mathbb{Z} & \xrightarrow[[2]]{} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

But $[2]$ is the zero map on $\mathbb{Z}/2\mathbb{Z}$, which is not injective. A shorter, non-diagrammatical justification would be

$$([2] \otimes \mathrm{Id})(n \otimes i) = 2n \otimes i = n \otimes 2i = n \otimes 0 = 0$$

Recall $\mathrm{Hom}_R(A, B)$, the abelian group formed by all R-module homomorphisms $\varphi : A \to B$, with point-wise addition being its operation. PROP 2.3

Theorem 2.2 Module Homomorphisms are Left-Exact

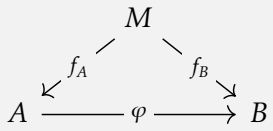
Let M be an R-module (left or right). Then

$$\mathrm{Hom}_R(M, -) : N \mapsto \mathrm{Hom}_R(M, N)$$

is a left-exact covariant functor. Similarly, $\mathrm{Hom}_R(-, M)$ is a left-exact contravariant functor.

PROOF.

For justification of the co/contra-variant claim, let $f_A \in \mathrm{Hom}_R(M, A)$, $f_B \in \mathrm{Hom}_R(M, B)$, and $\varphi : A \to B$. Then, $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B)$ by $f_A \mapsto \varphi \circ f_A$



Similarly, reversing the arrows of f_A and f_B show contravariantness of the $\mathrm{Hom}_R(-, M)$ map. Showing the additiveness of these maps is left as exercise.

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ be exact. We'll assume we're working with left R-modules, and consider $\mathrm{Hom}_R(M, -)$ (the 3 combinations of cases are identically proven). We must show that

$$0 \to \mathrm{Hom}_R(M, A) \xrightarrow{\alpha} \mathrm{Hom}_R(M, B) \xrightarrow{\beta} \mathrm{Hom}_R(M, C)$$

is exact. We start by showing that α is injective. Note, by the above discussion, that α acts as a pre-composition of f. Thus, $\ker(\alpha) = \{f_A : M \to A : f \circ f_A = \mathbb{0}\}$, where $\mathbb{0}(m) = 0 \ \forall m \in M$. But f is injective, so $f \circ f_A = \mathbb{0} \implies f_A = \mathbb{0}$, and so $\ker(\alpha)$ trivial.

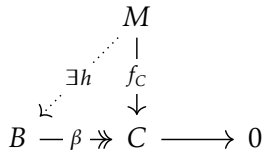
Next, we show that $\mathrm{Im}(\alpha) = \ker(\beta)$. Suppose $f_B \in \ker(\beta)$, i.e. $g \circ f_B = \mathbb{0} \implies g \circ f_B(m) = 0 \ \forall m \in M$, so $f_B(m) \in \ker(g) = \mathrm{Im}(f)$. Hence, we can write $f_B(m) = f(a)$ for some $a \in A$. Now, let $\overline{f_A} : m \mapsto a$, choosing a as described. One can show that this is indeed a module homomorphism. Then $f_B = f \circ \overline{f_A}$, and we conclude $f_B \in \mathrm{Im}(\alpha)$.

Conversely, suppose $f_B \in \mathrm{Im}(\alpha)$. Then $f_B = f \circ f_A$ for some $f_A : M \to A$. We see that $f_B(m) \in \mathrm{Im}(f) = \ker(g)$, so $g \circ f_B(m) = 0 \ \forall m \implies f_B \in \ker(\beta)$. And we are done. □

In the context of right-exact module tensor functors, we described flat modules as those for which we can "complete" exactness, i.e. find left-exactness under this functor. Similarly, we will describe modules for which we can "complete" the exactness of module homomorphism functors.

DEF 2.11 $M \in {}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$ is called a *projective module* if $\mathrm{Hom}_R(M, -)$ is exact.

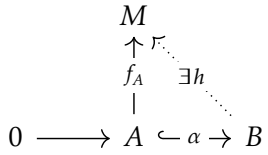
PROP 2.4 An R-module is projective if and only if $B \twoheadrightarrow C \implies \mathrm{Hom}_R(M, B) \twoheadrightarrow \mathrm{Hom}_R(M, C)$. In a picture:



PROOF. Let $\beta : B \twoheadrightarrow C$. Consider a short exact sequence $0 \to \ker(\beta) \to B \to C \to 0$. The map $C \to 0$ suggests $\mathrm{Im}(g) = C$ by exactness. By the previous theorem, $\mathrm{Hom}_R(M, -)$ is exact when $\mathrm{Im}(\beta) = \mathrm{Hom}_R(M, C)$, i.e. $\mathrm{Hom}_R(M, B) \twoheadrightarrow \mathrm{Hom}_R(M, C)$. □

DEF 2.12 Similarly, $M \in {}_R\mathbf{Mod}$ or $\mathbf{Mod}_R$ is called an *injective module* if $\mathrm{Hom}_R(-, M)$ is exact.

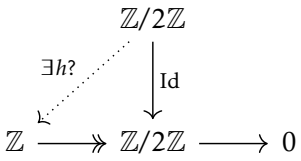
PROP 2.5 An R-module is injective if and only if $A \hookrightarrow B \implies \mathrm{Hom}_R(B, M) \twoheadrightarrow \mathrm{Hom}_R(A, M)$. In a picture:



A proof of the above statement, like Prop 2.4 and Prop 2.1, comes down to stating left-exactness, and considering the missing link. Note that $0 \to A \to B$ can be extended to a short exact sequence $0 \to A \to B \to B/A \to 0$, as we did for flat modules.

Eg. 2.2 Take $R = \mathbb{Z}$. Then $\mathbb{Z}$ is a projective module. A function $f_C : \mathbb{Z} \to C$ is determined exactly by $f_C(1) \in C$. By surjectivity of β, $\exists b \in B$ with $\beta(b) = f_C(1)$. Then, setting $h(1) = b$, $(\beta \circ h)(1) = f_C(1)$, and so $\beta \circ h = f_C$.

Eg. 2.3 Over $R = \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ is *not* a projective module. Consider the following



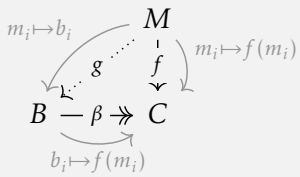
A surjective homormophism from $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is reasonable (e.g. map to the parity), but any homomorphism $h : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is necessarily the 0 map, so the identity cannot be made up of a composition with h.

Projective Modules

We'll take a closer look at the theory of projective modules.

PROP 2.6 Free R-modules are projective.

PROOF. Let M have a basis $(m_i)_{i\in I}$. Let $f : M \to C$. It is determined exactly by the mappings $f(m_i) : i \in I$. Since β is surjective, $\exists b_i \in B$ with $b_i = f(m_i)$. Thus, let $g(m_i) = b_i$. Then $(\beta \circ g)(m_i) = \beta(b_i) = f(m_i)$. In a picture:



□

The ideas expressed so far about projective modules extend easily to abelian categories. We say that a category $\underline{C}$ has *enough projectives* if, $\forall A \in \underline{C}$, there exists an exact complex DEF 2.13

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

where $P_i \in \underline{C}$ are all projective objects. We call this a *projective resolution* for A. DEF 2.14

For any R-module M, there exists a free module F, with $F \twoheadrightarrow M$. **PROP 2.7**

PROOF. We construct $F = \{\sum_{m\in M} r_m[m] : r_m \in R : r_m = 0 \text{ except finitely many}\} = \oplus_{m\in M} R$. $\sum_{m\in M} r_m[m] \mapsto \sum_{m\in M} r_m m$ provides a surjective map. □

The category of R-modules (left or right) has enough projectives. **PROP 2.8**

PROOF. We proceed by induction on the length n of the complex. An exact sequence $F_0 \xrightarrow{\varepsilon} M \to 0$, i.e. a surjection $F_0 \twoheadrightarrow M$, is given by Prop 2.7. Suppose now that

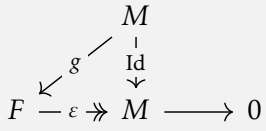
$$F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_0 \varepsilon M \to 0$$

is exact. Let $N = \ker(f_n)$. Construct F_{n+1} and $F_{n+1} \twoheadrightarrow N$ by Prop 2.7. □

Theorem 2.3 Characterization of Projective Modules

An R-module M is projective if and only if M is a direct summand of a free module. In other words, there exists some R-module Q such that $Q \oplus M$ is free.

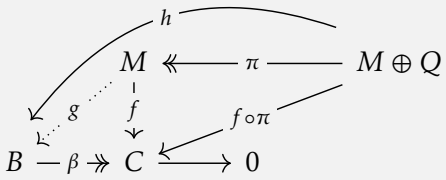
PROOF. ($\implies$) Let M be projective. Then there exists some $F \xrightarrow{\varepsilon} M$, where F is free. We have



Thus, there is some g with $\varepsilon \circ g = \mathrm{Id}$. We can conclude $F \cong \mathrm{Im}(g) \oplus \ker(\varepsilon)$, either via

the splitting lemma (not proven in these notes) or directly. $M/\ker(g) \cong \text{Im}(g)$, but $\ker(g) = \{0\}$, since $g(m) = 0 \implies (\varepsilon \circ g)(m) = 0$. Thus, $M \cong \text{Im}(g)$.

($\impliedby$) Conversely, let $M \oplus Q$ be free. Let $\pi : (m, q) \mapsto m$ be the projection from $M \oplus Q$ onto M. Since $M \oplus Q$ is free, it is projective. Thus, $\exists h : \beta \circ h = f \circ \pi$.



Now, let $g(m) = h(m, q)\ \forall m \in M$, where $q \in Q$ is any fixed element. Then $(\beta \circ g)(m) = (\beta \circ h)(m, q) = (f \circ \pi)(m, q) = f(m)$, as desired. □

We get some useful corollaries from this.

PROP 2.9 Let P_1, P_2 be projective R-modules. Then:

1. $P_1 \oplus P_2$ is projective.
2. If R is commutative (such that $P_1 \otimes_R P_2$ makes sense), $P_1 \otimes_R P_2$ is projective.

PROOF.

Let Q_i be such that $P_i \oplus Q_i = F_i$ is free, for $i = 1, 2$.

For (1), consider

$$(P_1 \oplus P_2) \oplus (Q_1 \oplus Q_2) \cong (P_1 \oplus Q_1) \oplus (P_2 \oplus Q_2) = F_1 \oplus F_2$$

which is free. $P_1 \oplus P_2$ thus satisfies Thm 2.3.

For (2), consider

$$F_1 \otimes_R F_2 = (P_1 \oplus Q_1) \otimes_R (P_2 \oplus Q_2) \cong (P_1 \otimes_R P_2) \oplus (P_1 \otimes_R Q_2 \oplus Q_1 \otimes_r P_2 \oplus Q_1 \otimes_R Q_2)$$

But also, we can show that $F_1 \otimes_R F_2$ is free:

$$F_1 \otimes_R F_2 \cong (\oplus_{i \in I} R) \otimes_R (\oplus_{j \in J} R) \cong \bigoplus_{i,j \in I \times J} R \otimes_R R \cong \bigoplus_{i,j \in I \times J} R \qquad \square$$

Injective Modules

We turn our attention to the mirrored case of injective modules. Recall that exactness of $\text{Hom}_R(M, -)$ established projectiveness, whereas exactness of $\text{Hom}_R(-, M)$ esablishes injectiveness.

PROP 2.10 We call a module M *divisible* if $\forall r \neq 0 \in R$ and $a \in M$, $\exists b \in M : rb = a$.

Theorem 2.4 Characeterization of Projective $\mathbb{Z}$-Modules

A $\mathbb{Z}$-module I is injective if and only if I is divisible.