DISCRETE MATHEMATICS

NICHOLAS HAYEK

Lectures by Prof. Sergey Norin

CONTENTS

I Preliminaries

DEFINITIONS

Graph theory is the study of pairwise relations between objects, e.g. computer networks, interpersonal relationships, airport flights. Graphs will help us simplify and abstract networks . DEF 1.1

A *graph* G is comprised of a set of vertices, denoted $V(G)$, where $|V(G)| < \infty$, a set of edges, denoted $E(G)$, where every edge is associated with two vertices. DEF 1.2 At least in this course

We say that an edge *joins* the two vertices it's associated with. Similarly, an edge is *incident* to a vertex which is an end of it. Conversely, a vertex is incident to an edge if it is an end of it. DEF 1.3

Two vertices are *adjacent* or *neighbors* if they are joined by an edge, and a vertex has *degree* edges incident to it. DEF 1.4

The *null graph* is the graph such that $V(G) = \varnothing$. The *complete graph* on n vertices, denoted K_n, is such that $|V(K_n)| = n$ and $|E(K_n)|$ is maximal. DEF 1.5

For a graph of n vertices, the maximal number of edges it may have is $\binom{n}{2}$. PROP 1.1

PROOF. Suppose every vertex is connected to every other vertex. Then $\sum_{v \in V(G)} \deg(v) = n(n-1) \implies |E(G)| = \frac{n(n-1)}{2} = \binom{n}{2}$. $\square$

A graph of n vertices, where v_i is only adjacent to v_{i-1} and v_{i+1}, is called a *path* and is sometimes denoted P_n. v_1 and v_n are called the ends of P_n. DEF 1.6

For $n \geq 3$, a *cycle* C_n is a graph with $V(G) = \{v_1, ..., v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. DEF 1.7

An *adjacency matrix* is a matrix contained all vertices on both axis. Pairwise adjacency is denoted by a 1 entry, and 0 otherwise. For example, the following is an adjacency table for a 4 element cycle: DEF 1.8

	v_1	v_2	v_3	v_4
v_1	$\times$	1	0	1
v_2	1	$\times$	1	0
v_3	0	1	$\times$	1
v_4	1	0	1	$\times$

Similarly, an *incidence* matrix has rows in $V(G)$ and columns in $E(G)$, and marks DEF 1.9

with 1 pairs which are incident to eachother. The following is the incidence matrix for a 4 element cycle:

	v_1	v_2	v_3	v_4
e_1	1	1	0	0
e_2	0	1	1	0
e_3	0	0	1	1
e_4	1	0	0	1

PROP 1.2 For a graph G, we always have $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

PROOF. Every edge has two vertices incident to it. Thus, $\sum \deg(v)$ will be the number of times an edge is incident to a vertex, i.e. the number of edges $\times$ 2. □

DEF 1.10 H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

We cannot do the same for "$G \setminus H$," since we may delete vertices and keep their incident edges!

For two graphs G, H, the union $G \cup H$ is a graph such that $V(G \cup H) = V(G) \cup V(G)$ and $E(G \cup H) = E(G) \cup E(H)$. We similarly define the intersection $G \cap H$ to be such that $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$.

PROP 1.3 There are $2^{\binom{n}{2}}$ graphs with n vertices.

PROOF. We know the maximal number of edges of this graph is $\binom{n}{2}$. Then, for each edge, one may make a binary choice whether to include it or not $\therefore$ the number of graphs is $2^{\binom{n}{2}}$. □

DEF 1.11 An *isomorphism* between H and G is a bijection $\varphi : V(G) \to V(H)$ such that $uv \in E(G) \implies \varphi(u)\varphi(v) \in E(H)$..

We can now ask: how many graphs are there with n vertices up to isomorphism?

CONNECTIVITY

DEF 1.12 A *walk* in G with ends u_0 and u_k is a sequence $(u_0, u_1, ..., u_k)$ such that $u_i \in V(G)$ and $u_i u_{i+1} \in E(G)$. The length of this walk is k.

u and v are called *connected* if there exists a walk in G with ends u and v OR, equivalently, there exists a path $P \subseteq G$ with ends u and v.

PROP 1.4 $\exists$ a walk in G with ends u and $v \iff \exists$ a path $P \subseteq G$ with ends u and v.

PROOF. ($\impliedby$) Let $P \subseteq G$ be a path with ends u and v. Then $V(P)$ can be numbered $u = v_0, v_1, ..., v_k = v$, where $v_i v_{i+1} \in E(P)$. Then $(v_0, ..., v_k)$ is a walk in G.

($\implies$) Let there exist a walk $(u = v_0, ..., v_k = v)$ with $v_i v_{i+1} \in E(G)$. WLOG suppose this is the walk of minimal length. If $v_i \neq v_j$, i.e. are pairwise distinct, then we already have a path. Suppose otherwise, and let $v_i = v_j$. Then $(v_0, ..., v_i, v_{j+1}, ..., v_k)$ is a *smaller* walk with ends u and v, which establishes the contradiction $\lightning$. $\square$

A graph G is called *connected* if $\forall u, v \in V(G)$, u and v are connected. DEF 1.13

A *partition* of $V(G)$ is $(X_1, ..., X_k)$ such that $\cup_{i=1}^{k} X_i = V(G)$ and $X_i \cap X_j = \varnothing \; \forall i \neq j$. DEF 1.14

A graph G is not connected $\iff \exists$ a partition (X, Y) of $V(G)$ such that no edge of G is incident to one vertex in X and one in Y. PROP 1.5

PROOF. ($\Longleftarrow$) Suppose G were connected. Then choose $u \in X, v \in Y$ such that there exists a walk $(u = u_0, ..., u_k = v)$. Let u_i be minimal over i such that $u_i \in Y$. Then $u_{i-1} \in X$, and $u_{i-1}u_i \in E(G)$ $\lightning$.

($\Longrightarrow$) Let $u, v \in V(G)$ be such that there is no walk from u to v. Let X be the set of all $w \in V(G)$ such that $\exists$ a walk with ends u and w. Similarly, let $Y = V(G) \setminus X$. Clearly $V(G) = X \cup Y$, $X \cap Y = \varnothing$, and (X, Y) is a partition. Suppose there exists an edge from a vertex in X to a vertex in Y, i.e. $x \in X$, $y \in Y$. Then we have the walk $(u, ..., w, ..., x, y)$. But $y \notin X$ $\lightning$. $\square$

Let G be a graph. $H \subseteq G$ is called a *connected component* of G if H is a maximal connected subgraph of G, i.e. if $\exists H \subseteq H' \subseteq G$ with H' connected, then $H = H'$. DEF 1.15

Sometimes we just say "component."

If H_1, H_2 are connected graphs, and $V(H_1) \cap V(H_2) \neq \varnothing$, then $H_1 \cup H_2$ is also connected. PROP 1.6

PROOF. Let $u \in H_1, v \in H_1 \cap H_2, w \in H_2$. Then $(u, ..., v)$ and $(v, ..., w)$ are both walks, and thus $(v, ..., v, ..., w)$ is a walk. $\square$

Every $v \in V(G)$ is a member of a unique connected component $H \subseteq G$. PROP 1.7

PROOF. $\{v\}$ is connected. If there does not exist $H \supseteq \{v\}$ also connected, then we are done. Otherwise, we may choose the maximal such connected superset.

Suppose $v \in H_1$ and H_2, two connected components. Then by Prop 1.6, $H_1 \cup H_2$ is connected. But since $H_1 \cup H_2 \supseteq H_1, H_2$, this violates maximality. We conclude that $H_1 = H_2$. $\square$

Let G be a graph, and let $H \subseteq G$ be a non-null and connected subgraph. Then H is a connected component of $G \iff \forall e \in E(G)$ with an end in $V(H)$, we have $e \in E(H)$. PROP 1.8

$\Longrightarrow$ PROOF. For ($\Longrightarrow$), let $e = uv$, with $u \in V(H)$. If $v \in V(H)$, then we are done. Otherwise, suppose $e \notin E(H)$. We know v is a member of a unique connected component. But adding e to H would yield a further connected graph: take the graphs of $\{uv\}$ and H. Both are clearly connected, so $H \cup \{uv\}$ is connected.

($\Longleftarrow$) $\square$

Obtained from G by deleting e

For $e \in E(G)$, $G \setminus e$ is a graph such that $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.

Similarly, for $v \in V(G)$, $G \setminus v$ is a graph such that $V(G \setminus v) = V(G) \setminus \{v\}$ and $E(G \setminus v) = E(G) \setminus \{e : e \text{ incident to } v\}$.

Let $\text{comp}(G) = \#$ of connected components of G.

PROP 1.9

$\text{comp}(G) = 1 \iff G$ is connected.

PROOF.

($\Longrightarrow$) direction is trivial. For ($\Longleftarrow$), if G is connected, then there cannot exist a more maximal connected subgraph, e.g. G is a connected component. Since every vertex belongs to a unique connected component, and this must be G, $\text{comp}(G) = 1$. $\square$

DEF 1.16

Let $e = \{u, v\} \in E(G)$. Define a *cut-edge* to be an edge which is not part of any cycle.

PROP 1.10

Exactly one of the following holds:

1. ***e* is a cut-edge**: $\text{comp}(G \setminus e) = \text{comp}(G) + 1$, and u, v belong to different components of $G \setminus e$.

2. ***e* is not a cut-edge**: $\text{comp}(G \setminus e) = \text{comp}(G)$, and u, v belong to the same component.

PROOF.

Let e be a cut-edge. Let $H_1, ..., H_k$ be the connected components of $G \setminus e$. If u, v belong to H_i, then $\exists$ a path $P \subseteq H_i$ with ends u and v. Adding e, this is a cycle $\lightning$.

WLOG, assume that u, v belong to $V(H_1), V(H_2)$, respectively. Then let H' be obtained by $H_1 \cup H_2$ by adding e. We claim that $H', H_2, ..., H_k$ are all components of G. By Prop 1.8, we only need to check the connectivity of H', and this holds by Prop 1.6. Since there do not exist any vertices *not* in $V(H_i) : i \geq 2$ or $V(H')$, these are all the components of G. Thus, $\text{comp}(G) + 1 = \text{comp}(G \setminus e)$. $\square$

II Trees and Forests

A *forest* is a graph with no cycles, i.e. every edge is a cut-edge. DEF 2.1

A *tree* is a non-null connected forest. DEF 2.2

Let F be a non-null forest. Then $\mathrm{comp}(F) = |V(F)| - |E(F)|$. PROP 2.1

PROOF. We'll show by induction on $|E(F)|$. If $n = 0$ then all vertices are their own connected components. Let $|E(F)| = n$, and assume $\mathrm{comp}(F) = |V(F)| - |E(F)|$. Let $e \in E(F)$. Since F is a forest, e is a cut-edge, and thus $\mathrm{comp}(G \setminus e) = \mathrm{comp}(G) + 1 = |V(F)| - |E(F)| + 1 = |V(F)| - (|E(F)| - 1) = |V(F)| - |E(F \setminus e)| = |V(F \setminus e)| - |E(F \setminus e)|$. □

A *leaf* is a vertex with degree 1. DEF 2.3

Let T be a tree with $|V(T)| \geq 2$. let $X = \{\text{leaves of } T\}$, $Y = \{v \in V(G) : \deg(v) \geq 3\}$. Then $|X| \geq |Y| + 2$. PROP 2.2

Thus, trees have ≥ 2 leaves!

PROOF. By Prop 1.1, we have

$$\sum_{v \in V(T)} \deg(v) = 2|E(T)| \overset{1.11}{=} 2(|V(T)| - \mathrm{comp}(G)) \overset{1.9}{=} 2(|V(T)| - 1)$$

$$\implies \sum_{v \in V(T)} (\deg(v) - 2) = 2(|V(T)| - 1) - 2|V(T)| = -2$$

$$= \underbrace{\sum_{v \in X}(\deg(v) - 2)}_{=-|X|} + \underbrace{\sum_{v \in Y}(\deg(v) - 2)}_{\geq |Y|} + \underbrace{\sum_{v \in V(T) - X - Y}(\deg(v) - 2)}_{=0}$$

$$\implies -2 \geq -|X| + |Y| \implies |X| \geq |Y| + 2 \quad \square$$

A note for the following few proofs: if w is a leaf, then any path which exists in T (with ends not w) exists in $T \setminus w$.

Let T be a tree with 2 leaves, u and v. Then T is a path with ends u and v. PROP 2.3

PROOF. Let $P \subseteq T$ be a path with ends u and v. By Prop 2.2, $\deg_T(w) = 2\ \forall w \in V(P) \setminus \{u, v\}$. Moreover, $\deg_T(w) = \deg_P(w)$, so no vertex in $V(P)$ is incident to an edge in $E(T) \setminus E(P)$. Then, by Prop 1.8, P is a connected component. But T is connected, so $T = P$. □

Let T be a tree and $v \in V(T)$ be a leaf. Then $T \setminus v$ is a tree. PROP 2.4

PROOF.

$T \setminus v$ is non-null, since v has a neighbor. $T \setminus v$ has no cycles, since T has no cycles, and $T \setminus v$ is connected: we know there exists a path between any two vertices in $V(T) \setminus \{v\}$. Such a path still exists. □

PROP 2.5 If G is a graph, $v \in V(G)$ a leaf, and $G \setminus v$ a tree, then G is a tree.

PROOF. G is non-null, since $G \setminus v$ is non-null. We know that v belongs to no cycles, since it is a leaf, so any cycles apparent in G would exist in $G \setminus v$. Thus, G has no cycles. For connectedness, let H be the graph containing v, its incident edge, and that edge's other vertex v'. H is connected, as is $G \setminus v$, and $G \setminus v \cap H \neq \emptyset$, so $G \setminus v \cup H = G$ is connected by Prop 1.6. □

PROP 2.6 Let T be a tree, $u, v \in V(T)$. Then T contains a unique path with ends u and v.

PROOF. We'll show by induction on $|V(T)|$. This clearly holds for $|V(T)| = 1$. Let $|V(T)| \geq 2$. Suppose T contains a leaf $w \in V(T) \setminus \{u, w\}$. Then $T \setminus w$ is a tree by Prop 2.4. By our induction hypothesis, $T \setminus w$ contains a unique path with ends u and v. By connectedness, $\exists$ a path with ends u, v in T. But this path must exist in $T \setminus w$, whose uniqueness follows.

If no such leaf exists, then T has exactly 2 leaves (u and v). Thus, by Prop 2.3, T is a path with ends u and v, and thus the only path in T. □

SPANNING TREES

DEF 2.4 Let G be a graph. A subgraph $T \subseteq G$ is called a *spanning tree* of G if T is a tree and $V(T) = V(G)$.

PROP 2.7 Let G be connected and non-null. Let $H \subseteq G$, chosen minimal such that $V(H) = V(G)$ and H is connected. Then H is a spanning tree of G.

PROOF. We only need to check that T is non-null and contains no cycles. The first is automatic, since $V(T) = V(G)$, and G is non-null. If H has a cycle, then let e be an edge in the cycle. $H \setminus e$ is connected by Prop 1.9 and Prop 1.10. But this contradicts minimality, so T contains no cycles. □

PROP 2.8 Let G be a connected non-null graph. Let $H \subseteq G$ be maximal such that H contains no cycles. Then H is a spanning tree of G.

PROOF. We need to show that $V(H) = V(G)$ and H is connected (it is non-null, since at least a singleton of G contains no cycles; it contains no cycles by construction). If $\exists v \in V(G) \setminus V(H)$, adding v such that $\deg(v) = 0$ would maintain H having no cycles, thus contradicting maximality.

Suppose H is not connected. Then by Prop 1.5 there exists a partition $H = X \cup Y$ such that no edge has a vertex in both X and Y. However, such an edge must exist in G, say $e \in E(G)$, so we may add this edge to H to produce H'. Observe that H' contains no cycles, since e belongs to no cycles in H. But this contradicts maximality, so H must contain no cycles. □

DEF 2.5

Let T be a spanning tree of G. Let $f \in E(G) \setminus E(T)$. Then T with f has one cycle (by Prop 2.6). This is called the *fundamental cycle* of f with respect to T, and denoted $\mathrm{FC}(T, f)$.

PROP 2.9

Let T be a spanning tree of G, $f \in E(G) \setminus E(T)$. Let $C = \mathrm{FC}(T, f)$, $e \in E(C)$. Then $(T + f) \setminus \{e\}$ is a spanning tree.

PROOF.

Let $T' = (T + f) \setminus \{e\}$. $T + f$ is connected, and since e is not a cut-edge, $(T + f) \setminus \{e\} = T'$ is also connected. C is a unique cycle in $T + f$, so T' contains no cycles. Thus, T' is a tree. $V(T') = V(T) = V(G)$, since T is a spanning tree, so we conclude that T' is a spanning tree. □

DEF 2.6

Let G be a non-null, connected tree. Let $w : E(G) \to \mathbb{R}_+$ by a real valued function on the edges of G. The *minimal spanning tree* of G w.r.t. w, denoted $\mathrm{MST}(G, w)$, is a spanning tree T such that $w(T) = \sum_{e \in E(T)} w(e)$ is minimal.

2.1 Minimality of MST Edges

Let G be connected and non-null. Let $w : E(G) \to \mathbb{R}_+$. Let $T = \mathrm{MST}(G, w)$ and $E(T) = \{e_1, ..., e_k\}$, where we order

$$w(e_1) \leq w(e_2) \leq ... \leq w(e_k)$$

Then $\forall 1 \leq i \leq k$, e_i is an edge of minimum weight subject to the following constraints:

- $e_i \notin \{e_1, ..., e_{i-1}\}$
- $\{e_1, ..., e_i\}$, as an edge set, does not contain any cycles.

In particular, this theorem states that *for any* $f \in E(G) - \{e_1, ..., e_{i-1}\}$ with $\{e_1, ..., e_{i-1}, f\}$ not containing cycles, $w(f) > f(e_i)$.

PROOF.

Suppose otherwise. Then for at least one i, we can choose $f \in E(G) - \{e_1, ..., e_{i-1}\}$ such that $\{e_1, ..., e_{i-1}, f\}$ contains no cycles and $w(f) < w(e_i)$.

Then $f \notin E(T)$, otherwise $f = e_j$ for some $j \geq i$. But $j < i$, since we have an ordering on w. Let $C = \mathrm{FC}(T, f)$, the unique cycle in $T + f$. There is some $j \geq i$ such that $e_j \in E(C)$, since all vertices $< i$ must not contain cycles. Then $w(e_j) > w(f)$.

Let $T' = (T + f) - e_j$. Then by Prop 2.9, T' is still a spanning tree. Let $w(G)$ be the sum of weights of edges of G. Then $w(T') - w(T) = w(f) - w(e_j) < 0$, implying that T is not minimal $\lightning$. □

DEF 2.7 **Kruskal's Algorithm**

| Input

A connected, non-null graph G, and $w : E(G) \to \mathbb{R}_+$

| Output

A graph T such that $V(T) = V(T)$, with $E(T) = \{e_1, ..., e_{|V(G)|-1}\}$

| $n \to n + 1$

Let $e_i \in E(G)$ be chosen such that $w(e_i)$ is minimum subject to

- $e_i \notin \{e_1, ..., e_{i-1}\}$
- $\{e_1, ..., e_i\}$, as an edge set, does not contain any cycles.

for $1 \le i \le |V(G)| - 1$

2.2 Kruskal's Algorithm Outputs an MST

PROOF. Suppose $w : E(G) \to \mathbb{R}_+$ is injective. Then all edges have different weights. Then Thm 2.1 implies that Kruskal's outputs an MST which is unique. If w is not injective, the proof is out of the scope of this course. □

2.3 Spanning Trees of K_n

The complete graph K^n has exactly n^{n-2} spanning trees.

PROOF. The proof for this will require proving multiple statements. Let $\mathcal{T}_k$ be the set of spanning, rooted forests in K^n with k components. Then $\mathcal{T}_1$ is the set of rooted spanning trees in K_n. Since we may choose n roots, $\frac{|\mathcal{T}_1|}{n}$ equals the number of spanning trees in K^n. Thus, we need to show $|\mathcal{T}_1| = n^{n-1}$.

Claim 1 $|\mathcal{T}_n| = 1$

If a spanning, rooted forest has n components, then it is exactly the graph of no edges and the vertex set $V(K^n)$ (each being its own component).

Claim 2 $n(k-1)|\mathcal{T}_k| = (n-k+1)|\mathcal{T}_{k-1}|$

Call a forest F with $k-1$ the *parent* of a forest F' with k components if

$F' = F \setminus e$ for some $e \in E(F)$. Naturally, we call F' a *child* of F under these conditions. We will thus count (parent, child) combinations. Every $F \in \mathcal{T}_{k-1}$ has $|E(F)|$ children, since every edge is a cutedge. This is $|V(F)| - \text{comp}(F) = n - (k-1) = n - k + 1$ by Prop 2.1.

For every $F' \in \mathcal{T}_k$, we can obtain a parent by adding an edge from any vertex to the root of a component not containing this vertex. Thus, every child has $n(k-1)$ parents. Thus, there are $n(k-1)|\mathcal{T}_k|$ parent-child combinations, and also $(n-k+1)|\mathcal{T}_{k-1}|$ such combinations. Thus, we conclude $n(k-1)|\mathcal{T}_k| = (n-k+1)|\mathcal{T}_{k-1}|$.

Claim 3 $|\mathcal{T}_k| = \binom{n}{k} k n^{n-1-k}$

We just solve the recursion. We'll show by induction on $n-k$. If $n - k = 0 \implies n = k$, we have $|\mathcal{T}_n| = \binom{n}{n} n n^{n-1-n} = 1$, which is true by Claim 1.

Letting $n - k \to n - k + 1 = n - (k-1)$, we are having $k \to k-1$. By Claim 2, then,

$$|\mathcal{T}_{k-1}| = \frac{n(k-1)}{n-k+1}|\mathcal{T}_k| \overset{\text{hyp.}}{=} \frac{n(k-1)}{n-k+1}\binom{n}{k} k n^{n-1-k} \overset{?}{=} \binom{n}{k-1}(k-1)n^{n-1-(k-1)}$$
$$= \frac{k}{n-k+1}\binom{n}{k}(k-1)n^{n-1-(k-1)}$$

Note that $\frac{k}{n-k+1}\binom{n}{k} = \frac{kn!}{(n-k+1)k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!(n-k+1)} = \frac{n!}{(k-1)!(n-k+1)!} = \binom{n}{k-1}$, so the last statement above evaluates to

$$\binom{n}{k-1}(k-1)n^{n-1-(k-1)} \qquad \text{as desired.}$$

Claim 4 $|\mathcal{T}_1| = n^{n-1}$

We plug in from above to find $|\mathcal{T}_1| = \frac{n}{1} 1 n^{n-1-1} = n^{n-2+1} = n^{n-1}$. □

EULER'S THEOREM & HAMILTONIAN CYCLES

Recall that a *walk* in G is a sequence $(v_0, ..., v_k) : v_i \in V(G)$, perhaps with repetition, such that $v_i v_{i+1} \in E(G)\ \forall i \leq k-1$. (See Def 1.12).

A walk *uses* an edge e if $e = v_i v_{i+1}$ and v_i, v_{i+1} is contained in the walk. DEF 2.8

A *trail* is a walk that uses every edge at most once. DEF 2.9

A *Euler trail* in G is a trail that uses every edge in $E(G)$ DEF 2.10

A *Euler tour* in G is a closed Euler trail (i.e. $v_0 = v_k$). DEF 2.11

PROP 2.10 Let G be a graph with $E(G) \neq \emptyset$. If G has no leaves, then G has a cycle.

PROOF. Suppose not. Then G is a forest by Def 2.1. Let C be a component with at least one edge. Then C is a tree with $|V(C)| \geq 2$. Thus, C has a leaf by Prop 2.2. $\lightning$. $\square$

PROP 2.11 Let G be a graph with all degrees even. Then $\exists$ cycles $C_1, ..., C_k \subseteq G$ such that $(E(C_1), ..., E(C_k))$ is a partition of $E(G)$.

PROOF. Informally, by the previous proposition we may create one cycle from the outset. Removing the edge set of this cycle from G leaves the graph with still all even edges. We'll show by induction on the edge set. The base case, $|E(G)| = 0$, holds trivially.

Let $n \to n+1$. We know there exists a cycle $C_1 \subseteq G$. Let $G' = G - C_1$. Then all degrees in G' are even. By (strong) induction, G' contains a partition $(E(C_2), ..., E(C_k))$. Since $E(G) = E(G') \cup E(C_1)$, the partition $(E(C_1), ..., E(C_k))$ satisfies the proposition. $\square$

2.4 Euler's Theorem

Let G be connected with all even degrees. Then $\exists$ a Euler tour in G.

PROOF. Let $W = (v_0, ..., v_k)$ be a closed trail of maximal length in G. WLOG suppose W doesn't use every edge. Let $H \subseteq G$ be such that $V(H) = V(G)$ and $E(H) = G \setminus \{\text{edges used by } W\}$. Then all vertices of H have an even degree, so by Prop 2.11, $\exists$ cycles $C_1, ..., C_k$ which partition $E(H)$.

Let $H' \subseteq G$ be the subgraph consisting of the edges and vertices of W. H' is not a component of G (since G is connected), so by Prop 1.8 $\exists e \in E(G) - E(H')$ with an end in $V(H')$, i.e. an end in the walk. Thus, e is part of H, and is part of a cycle. Thus, we may append this cycle to the walk W, creating a larger closed trail, violating maximality $\lightning$. $\square$

PROP 2.12 Let G be a connected graph with ≤ 2 vertices of odd degree. Then G contains a Euler trail.

PROOF. By the handshaking lemma (not seen here, but easy to see), there must be an even number of odd degree vertices in any graph. Thus, the case of one odd vertex is invalid.

Let $\deg(u), \deg(v)$ be odd, and all other vertices even. Let $G' = G + w$, where w is an added edge which is adjacent (joins) u and v. Then G' has all degrees even, and by Thm 2.4 G' has a Euler tour. WLOG we can let it begin and end at w. Thus, removing w, we get a Euler trail in G. $\square$

A *Hamiltonian cycle* is a cycle $C \subseteq G$ s.t. $V(C) = V(G)$. DEF 2.12

Let G be a graph, and $X \subseteq V(G)$ with $X \neq \emptyset$. If $|X| < \text{comp}(G \setminus X)$, then G has no Hamiltonian cycles. PROP 2.13

PROOF. Let C be Hamiltonian cycle. Then

$$\text{comp}(C \setminus X) \geq \text{comp}(G \setminus X) > |X|$$

Observe now that $C \setminus X$ is a forest. Then, from theory, we know that

$$\text{comp}(C \setminus X) = |V(C \setminus X)| - |E(C \setminus X)| \leq |V(C)| - |V(X)| - (|E(C)| - 2|X|) = |X|$$

where we note that $|V(C)| = |E(C)|$, since C is a cycle. This is a contradiction. $\square$

As it turns out, there is no effiicent algorithm to decide if G has a Hamiltonian cycle.

2.5 Dirac-Pósa

Let G be a graph on $n \geq$ vertices. If $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices $u, v \in V(G)$, then G has a Hamiltonian cycle.

PROOF. We'll show by induction on $\binom{n}{2} - |E(G)|$. If $|E(G)| = \binom{n}{2}$, then G is complete, and clearly contains a Hamiltonian cycle.

Let $|E(G)| < \binom{n}{2}$. Let $u, v \in V(G)$ be non-adjacent. Let $G' = G + uv$. By induction hypothesis, $\exists$ a Hamiltonian cycle $C \subseteq G'$. If $uv \notin E(C)$, then C is a Hamiltonian cycle in G. Otherwise, let $uv \in E(C)$. Notate

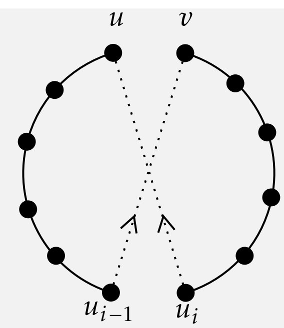
$$V(C) = \{u = u_1, u_2, ..., u_n = v\}$$

Let $A = \{i : uu_i \in E(G)\}$ and $B = \{i : vu_{i-1} \in E(G)\}$. Then $|A| + |B| = \deg(u) + \deg(v) \geq n$.

But we have $n - 1$ such vertices (u, v are non-adjacent, so this takes away a possibility). Thus, $A \cap B \neq \emptyset$, so $\exists i : uu_i, vu_{i-1} \in E(G)$. Then

$$\{u = u_1, ..., u_{i-1}, u_n = v, u_{n-1}, ..., u_i, u_1 = u\}$$

is a Hamiltonian cycle. $\square$



PROP 2.14 Let G be a graph on $n \geq 3$ vertices. Then if $\deg(v) \geq \frac{n}{2}$ $\forall v \in V(G)$ OR $|E(G)| \geq \binom{n}{2} - n - 3$, then G has a Hamiltonian cycle.

PROOF. If $\deg(v) \geq \frac{n}{2}$ $\forall v \in V(G)$, then $\deg(u) + \deg(v) \geq n$ $\forall u, v \in V(G)$, so by Thm 2.5 G has a Hamiltonian cycle.

For the second condition, I was getting a cookie and didn't listen. □

BIPARTITE GRAPHS

DEF 2.13 A *bipartition* of a graph G is a partition (A, B) of $V(G)$ such that every edge of G has one end in A and one end in B. Refer to Def 1.14.

DEF 2.14 A graph G is *bipartite* if it admits a bipartition.

PROP 2.15 Trees are bipartite.

PROOF. We'll show by induction on $n = |V(T)|$. For $|V(T)| = 1$, we have a bipartition $(\{v\}, \varnothing)$.

Let $|V(T)| = n$. Let $v \in V(T)$ be a leaf with a neighbor u. By induction hypothesis, $(T \setminus v)$ is bipartite. Let (A, B) be a bipartition. Assume WLOG that $u \in A$. Then $(A, B \cup \{v\})$ is a bipartition of T. □

2.6 Characterization of Bipartite Graphs

Let G be a graph. Then the following are equivalent:

1. G is bipartite.
2. G contains no closed walk of odd length
3. G contains no odd cycles.

PROOF.

(1 $\implies$ 2). Let (A, B) be a bipartition of G. Let $(v_0, ..., v_k)$ be a walk in G. WLOG let $v_0 \in A$. Then $v_i \in A \iff i$ is even. Thus, if $v_0 = v_k$, k must be even, so the walk must have even length.

(2 $\implies$ 3)If G had a cycle of odd length, it would be a closed walk of odd length.

(3 $\implies$ 1). As bipartitions of components may be combined to form a larger bipartition, it suffices to show this for a connected, non-null graph.

Let T be a spanning tree of G. Then $\exists$ a bipartition (A, B) of $V(T)$ by Prop 2.15. We'll show this is a bipartition of G as well. Let $f \in E(G) - E(T)$. Let $v_0, ..., v_k$ be the vertices of $\mathrm{FC}(T, f)$, with ends on f. Assume WLOG that $v_0 \in A$.

The fundamental cycle $\mathrm{FC}(T, f)$ has even length by assumption, so v_k must be odd (observe the cycle v_0, v_1, v_2, v_3 for reference). Thus, $v_k \in B$, so f has one end in A, and one in B. This may be reasoned for all $f \in E(G) - E(T)$. The bipartition holds for $E(T)$. Thus, it holds for all $e \in E(G)$. $\square$

MATCHINGS IN BIPARTITE GRAPHS

A *matching* M in G is a set of edges such that no vertex in $V(G)$ is incident to more than one edge in M. DEF 2.15

The *matching number* of G, denoted $\nu(G)$, is the maximum size $|M|$ for matchings M in G. DEF 2.16

$\nu(G) \leq \lfloor \frac{V(G)}{2} \rfloor$. PROP 2.16

PROOF. The maximal matching will use every vertex. $\square$

A *vertex cover* in G is a set $X \subseteq V(G)$ such that every edge in $E(G)$ has at least one end in X. DEF 2.17

If M is a matching in G and X is a vertex cover, then $|M| \leq |X|$. PROP 2.17

PROOF. If $|X|$ is a vertex cover, then every edge in M has an end in X. But no vertex $x \in X$ can belong to more than one edge in M, so we have an injection between M and X, i.e. $|M| \leq |X|$. $\square$

Let $\tau(G)$ be the minimum size of a vertex cover in G. DEF 2.18

$\nu(G) \leq \tau(G)$. PROP 2.18

PROOF.

Immediately from Prop 2.17. □

PROP 2.19 If X is a vertex cover of G, then $\sum_{v \in X} \deg(v) \geq E(G) = \frac{1}{2} \sum_{v \in V(G)} \deg(v)$.

PROP 2.20 If G is a graph, and Y is a set of pairwise non-adjacent vertices, then $V(G) \setminus Y$ is a vertex cover.

PROOF. Suppose otherwise. Then $\exists uv \in E(G)$ such that uv is not incident to any $V(G) \setminus Y$. Thus, $u, v \in Y$. But then u, v are adjacent. □

Note that the previous two propositions were not shown in class, but I have high confidence they're true, and they might be useful; take them with a grain of salt.

PROP 2.21 For a graph G, $\nu(G) \leq \tau(G) \leq 2\nu(G)$.

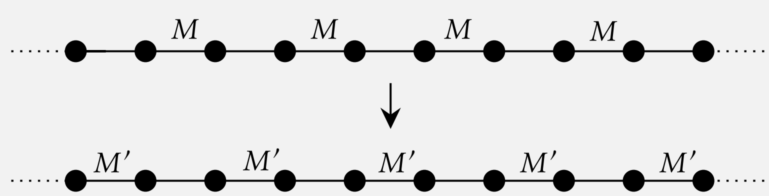
PROOF. It remains to show that $\tau(G) \leq 2\nu(G)$. Let M be a matching with $|M| = \nu(G)$. We want to find a vertex cover X with $|X| \leq 2|M|$. Let X be the set of ends of edges of M. Then $|X| = 2|M|$. Furthermore, X is a vertex cover. Otherwise, $\exists e \in E(G)$ with no end in X. Then $M \cup \{e\}$ is a matching, violating maximality. □

DEF 2.19 Let M be a matching in G. A path $P \subseteq G$ is M-*alternating* if edges in P alternate between edges of M and $E(G) - M$, i.e. every internal vertex of P is incident to an edge in $E(P) \cap M$.

DEF 2.20 An M-alternating path $P \subseteq G$ is M-*augmenting* if $|V(P)| \geq 2$ and the ends of P are not incident to edges of M.

PROP 2.22 If G contains an M-augmenting path, then M is not maximum.

PROOF. Let P be M-augmenting. Let $P = n$. Then $E(P) \cap M = \frac{n-1}{2}$. We may choose a matching $M' = E(P) - (E(P) \cap M)$. Then $E(P) \cap M' = \frac{n+1}{2}$. Then $M' \cup [(E(G) - E(P)) \cap M]$, i.e. M' with the edges of M not in P, is a larger matching. □



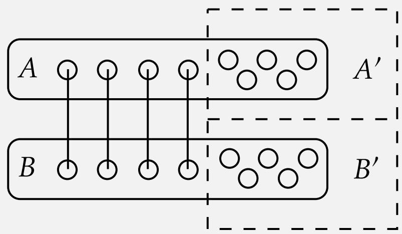
2.7 König

For any bipartite graph G, $\nu(G) = \tau(G)$.

PROOF.

It suffices to show $\tau(G) \leq \nu(G)$. Thus, given a matching M with $\nu(G) = |M|$, we look for a vertex cover $X : |X| = |M|$.

Let (A, B) be a bipartition of G. Let A', B' be vertices not incident to edges in M in A and B, respectively:



Let $Z \subseteq V(G)$ be such that for all $z \in Z$, $\exists$ an M-alternating path in G with one end in v and another in A'. Then we can conclude the following:

1. $A' \subseteq Z$.
2. $Z \cap B' = \varnothing$ (i.e. $\nexists$ an M-augmenting path).
3. Every edge in M with one end in Z has both ends in Z.
4. Every edge with one end in $Z \cap A$ has a second end in $Z \cap B$.

Thus, let $X = (Z \cap B) \cup (A \setminus Z)$. Then $|X| \geq |M|$, since every vertex of X is incident to an edge of M (see (1) and (2)). Every edge of M has exactly one end in X, so $|M| \geq |X|$, and then $|X| = |M|$. Lastly, X is a vertex cover, by (4). □

We say that a matching M *covers* $X \subseteq V(G)$ if every vertex in X is an end of some edge in M. DEF 2.21

We say that a matching is *perfect* if it covers $V(G)$. DEF 2.22

A matching M is perfect $\iff |M| = \frac{|V(G)|}{2}$. PROP 2.23

A graph G is *d-regular* if $\deg(v) = d \ \forall v \in V(G)$. DEF 2.23

2.8 Criterion for Perfect Matchings

Let G be a d-regular bipartite graph for $d \geq 1$. Then G has a perfect matching.

PROOF. If a bipartite graph G contains a perfect matching, then for a bipartition (A, B), $|A| = |B|$.

$d|B| = |E(G)| = d|A|$, so $|A| = |B|$, since every edge has exactly one end in A and one end in B. We wish to show that $\nu(G) \geq |A| = |B|$, since clearly $\nu(G) \geq |A|$. By König, it suffices to show that $\tau(G) \geq |A|$, i.e. for every vertex cover X of G, $|X| \geq |A|$.

As X is a vertex cover, we have

$$d|X| = \sum_{x \in X} \deg(v) \geq E(G)d|A| \implies |X| \geq |A| \quad \square$$

DEF 2.24 Let $N(S)$ denote the set of all vertices in G with at least one neighbor in $S \subseteq V(G)$.

2.9 Hall

Let G be a bipartite graph with a bipartition (A, B). Then G has a matching M covering A if an only if

$$|N(S)| \geq |S| \; \forall S \subseteq A$$

Sometimes we call the qualifier "Hall's condition."

PROOF. ($\implies$) If M is a matching which covers A, then M matches every vertex of $S \subseteq A$ to a vertex in $N(S)$. Thus $|N(S)| \geq |S|$.

($\impliedby$) We want to show that $\nu(G) \geq |A|$, since automatically $\nu(G) \leq |A|$. By König, it suffices to show $\tau(A) \geq |A|$, i.e. $|X| \geq |A|$ for any vertex cover X.

Let $S = A - X$. By Hall's condition, $|B \cap X| \geq \overbrace{|N(S)|}^{\subseteq B \cap X} \geq |S| = |A - X|$. Thus, $|A \cap X| - |B \cap X| \geq |A \cap X| - |A - X| \implies |X| \geq |A|$. $\square$

MENGER'S THEOREM & SEPARATIONS

Let G be a graph, and let $s, t \in V(G)$. We wish to consider when there exists a path in G with ends s and t. If such a path does not exist, then we can conclude that s and t are members of different components. Abstractly, there exists a partition (A, B) of $V(G)$, where $s \in A, t \in B$, such that no edge of G has one end in A and another in B.

Let s, t be non-adjacent, and suppose there exists at least one path between them. How might we guarantee that s cannot be "disconnected" from t by deleting $X \subseteq V(G)$ with $|X| < k$, $s, t \notin X$? The existence of disjoint paths $P_1, ..., P_k$ from s to t would suffice.

DEF 2.25 A *separation* of G is a pair (A, B) such that $A \cup B = V(G)$ and no edge of G has one end in $A - B$ and the other in $B - A$.

DEF 2.26 The *order* of a separation (A, B) is $|A \cap B|$.

PROP 2.24

Let $s, t \in V(G)$. Then either there exists a path with ends s and t in G, or there exists a separation of G with $s \in A$, $t \in B$ of order 0.

PROOF.

This will follow from Thm 2.10, with $k = 1$. □

2.10 Menger

Let $s, t \in V(G)$ be distinct and non-adjacent. Let $k \geq 1$. Then exactly one of the following holds:

1. There exists pairwise disjoint paths $P_1, \ldots, P_k$ with ends s and t.
2. There exists a separation (A, B) of G with order $< k$ such that $s \in A - B$, $t \in B - A$.

PROOF.

If (A, B) is a separation as in (2), then every path P from s to t contains a vertex in $A \cap B$. Thus, if (1) holds, then $P_1, \ldots, P_k$ use k distinct vertices in $A \cap B$, contradicting $|A \cap B| < k$. Thus, (1) and (2) are at least mutually exclusive.

We will assume Thm 2.11 holds, and conclude that Menger holds.

Let Q be the set of neighbors of s and R be the set of neighbors of t. Then either (1) or (2) of the theorem below holds, applied to $G \setminus s \setminus t$.

Suppose (1) of 2.11 holds. Then adding s and t to the ends of each disjoint path, we get that (1) of Menger holds. Suppose (2) of 2.11 holds. Then a separation $(A \cup \{s\}, B \cup \{t\})$ satisfies (2) of Menger. □

2.11 Generalized Menger

Let $Q, R \subseteq V(G)$. Let $k \geq 1$. Then exactly one of the following holds:

1. There exists pairwise disjoint paths $P_1, \ldots, P_k$, each from Q to R.
2. There exists a separation (A, B) of G of order $< k$ such that $Q \subseteq A$ and $R \subseteq B$.

DEF 2.27

For $X \subseteq V(G)$, let $V[X]$, the *subgraph of G induced by X*, have the vertices of X and the edges of G with both ends in X.

EXERCISE CAUTION

PROOF. We only need to show that one of (1), (2) hold. By induction on $|V(G)|+|E(G)|$.

$|V(G)|+|E(G)| = 0 \implies G = \varnothing$. We have the order 0 separation $(\varnothing, \varnothing)$.

Case 1: There exists a separation (A', B') of order exactly k s.t. $Q \subseteq A', R \subseteq B'$, and $A', B' \neq V(G)$. By induction hypothesis applied to $G[A']$, Q,and $A' \cap B'$, either

1. $\exists P_1', ..., P_k'$ in $G[A']$ from Q to $A' \cap B'$, pairwise disjoint.
2. $\exists$ a separation (A'', B'') of $G[A']$ such that $Q \subseteq A''$ and $A' \cap B' \subseteq B''$ of order $< k$.

Then $(A'', B' \cup B'')$ is a separation of G satisfying (2): observe that $Q \subseteq A''$ by definition, and $R \subseteq B' \cup B''$, since $R \subseteq B'$. Furthermore,

$$|A'' \cap (B' \cup B'')| = |\underbrace{(\underbrace{A''}_{\subseteq A'} \cap \underbrace{B'}_{\subseteq B''})}_{\subseteq A'' \cap B''} \cup (A'' \cap B'')| = |A'' \cap B''| < k$$

Similarly, by applying the induction hypothesis to $G[B']$, $A' \cap B'$, R, we may assume there exists pairwise disjoint paths $P_1'', ..., P_k''$ from $A' \cap B'$ to R. By renumbering, we may assume that P_i' and P_i'' share an end in $A' \cap B'$, and then paths $P_1' \cup P_1'', ..., P_k' \cup P_k''$ satisfy (1).

Case 2: $Q \cap R \neq \varnothing$. Let $v \in Q \cap R$. We apply induction hypothesis to $G - v$, $R - v$, $Q - v$, and $k - 1$. If (1) holds in $G - v$, then adding a path P_k with $V(P_k) = \{v\}$, we get k paths in G.

If (2) holds in $G - v$, then let (A', B') be a separation with $Q - v \subseteq A'$, $R - v \subseteq B'$. Then (2) holds for G with the separation

$$(A, B) = (A' \cup v, B' \cup v)$$

Case 3: $k = 1$. If there exists a component C of G such that $V(C) \cap Q \neq \varnothing$, $V(C) \cap R \neq \varnothing$, then (1) holds.

Otherwise, let A be the union of vertex sets of components that contain a vertex of Q. Let $B = V(G) - A$. Then (A, B) is a separation of order 0.

Case 4: Cases 1, 2, 3 do not hold. Let $e \in E(G)$. Apply induction hypothesis to $G \setminus Q, R$. We may assume that there exists a separation (A', B') of $G \setminus e$ with $Q \subseteq A', R \subseteq B'$. WLOG e has ends in $u \in A' - B'$ and $v \in B' - A'$ (otherwise, we are done).

Consider a separation $(A', B' \cup u)$. If it has order $< k$, then we are done.

If it has order $= k$, then Case 1 holds, unless $B' \cup u = V(G)$. Similarly, considering $(A' \cup v, B')$, we may assume $A' \cup v = V(G)$. So $|V(G)| \leq |A \cap B| + 2 \leq k + 2$. Then $|Q| + |R| = |V(G)|$, since Case 2 doesn't hold. So we may assume $|Q| \leq \frac{k+1}{2} < k$. Then, $(Q, V(G))$ is a separation that satisfies (2). □

PROP 2.25

Menger (Thm 2.11) $\Longrightarrow$ König (Thm 2.7)

PROOF.

Let G be a bipartite graph with a bipartition (Q, R). Let $k = \nu(G) + 1$. Then (1) of Menger doesn't hold, since this would imply the existence of a matching of size k. Thus, $\exists$ a separation (A, B) of G of order $\leq \nu(G)$ such that $Q \subseteq A, R \subseteq B$. Then $A \cap B$ is a vertex cover, so $\tau(G) \leq \nu(G)$. (Recall, by Prop 2.18, that we only need to show this direction.) □

DEF 2.28

Let $k \geq 1$ and let G be a graph with $|V(G)| \geq k + 1$. We say that G is k-connected if $G \setminus X$ is connected for all $X \subseteq V(G)$ such that $|X| \leq k - 1$.

E.G. 2.1

♠ *Examples* ♣

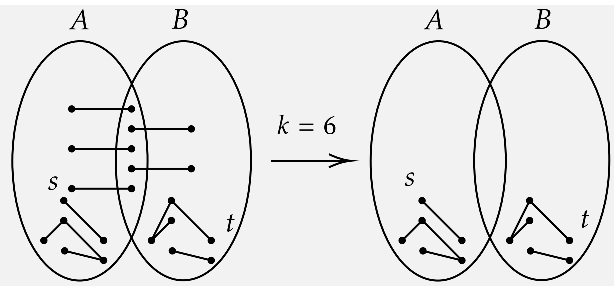
G is 1-connected $\iff$ G is connected and $V(G) \geq 2$. Trees on ≥ 2 vertices are 1-connected, but not 2-connected. Cycles are 2-connected, but not 3 connected.

2.12 Paths in k-Connected Graphs

Let G be a k-connected graph. Let $s, t \in V(G)$. Then there exists paths $P_1, ..., P_k$ in G, each with ends s and t, and otherwise pairwise disjoint.

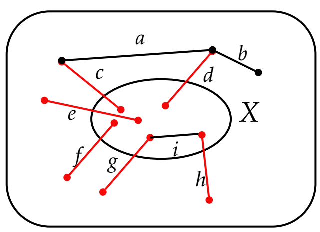
PROOF.

Recall Menger's (Thm 2.10): if $s, t \in V(G)$ are non-adjacent, then either $\exists$ paths as described above, or $\exists$ a separation (A, B) with $s \in A, t \in B$, and $|A \cap B| < k$. However, then $G \setminus (A \cap B)$ is no longer connected. But G is k-connected, so we have a contradiction. Hence, such a separation can't exist, and so the path case holds.



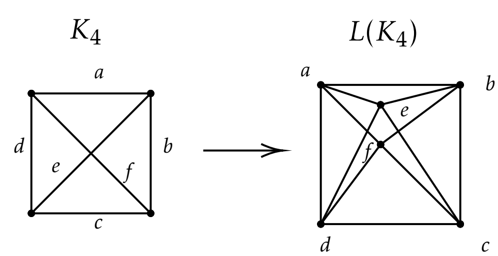
Now suppose that s, t are adjacent. We get $P_k := st$ (the edge connecting them) for free. We'll apply Menger's to $G \setminus st$, i.e. $\exists P_1, \ldots, P_{k-1}$ from $s \leftrightarrow t$, pairwise non-adjacent, or $\exists$ a separation of $G \setminus st$ with $|A \cap B| < k - 1$. Then $G \setminus ((A \cap B) \cup \{s\})$ is disconnected (unless $A - B = s$). Similarly, we find that $B - A = t$ as well. But then $|V(G)| \leq |A \cap B| + 2 \leq k$. This also violates k-connectivity (in particular, the condition that $|V(G)| \geq k + 1$). Thus, $P_1, \ldots, P_{k-1}, P_k$ are paths from $s \leftrightarrow t$. Note that $P_i \in G \setminus st$ for $i \leq k - 1$, so since $P_k = st$, these are all disjoint. $\square$

DEF 2.29 The *cut* associated with $X \subseteq V(G)$, denoted by $\delta(X)$, is the set of edges of G with exactly one end in X.



$G \longrightarrow \delta(X) = \{c, d, e, f, g, h\}$

DEF 2.30 For a graph G, the *line graph*, denoted $L(G)$, is a graph such that $V(L(G)) = E(G)$, and $e, f \in V(L(G))$ adjacent in $V(L(G))$ if and only if they share an end in G.



2.13 Edge Menger

Let $s, t \in V(G)$ be non-adjacent. Then either $\exists$ edge-disjoint paths $P_1, \ldots, P_k$ or $\exists X \subseteq V(G)$ with $s \in X, t \notin X$, and $|\delta(X)| < k$.

PROOF.

Note that (1) and (2) cannot both hold. Suppose (1) holds. Consider a path P_i from s to t. Let $s \in X$ and $t \notin X$. Let v_l be the minimal vertex not in X. Then $v_{l-1}v_l \in \delta(X)$. Since P_i are all pairwise disjoint, we have at least $|\delta(X)| \geq k$, which is a contradiction.

Thus, we need to show that either (1) or (2) holds. Let $G' = L(G)$. Let $Q \subseteq V(G') = E(G)$ be the set of all edges with an end being s. Similarly, let $R \subseteq V(G')$ be the set of edges with an end being t. By Thm 2.11, we first consider the possibility that $\exists$ vertex disjoint paths $P'_1, ..., P'_k \subseteq G'$ with ends in Q and R.

Then $V(P'_i)$ contains $E(P_i)$ for some path P_i from $s \leftrightarrow t$, so in particular we have an edge-disjoint path in G from $s \leftrightarrow t$.

Suppose now that the second condition in Thm 2.11 holds, i.e. $\exists$ a separation (A, B) of G' with $Q \subseteq A$, $R \subseteq B$, $|A \cap B| < k$, and $A \cup B = V(G') = E(G)$. No edge in $A - B$ shares an end with an edge in $B - A$. Let X be the vertices $v \in V(G) \setminus \{t\}$ such that all edges incident to v are in A. Then $s \in X, t \notin X$, and for all $v \notin X$, we have that the edges incident to v are in B. Hence, $\delta(X) \subseteq A \cap B$, so $|\delta(X)| < k$ as desired. □

DIRECTED GRAPHS & FLOWS

A *directed graph*, or *digraph*, D is a graph where, for each edge $e \in E(D)$, one of its ends is designated *tail*, and one end is designated *head*. Then, e is said to be *directed* from its tail to its head. DEF 2.31

A *directed path* P from u to v in a digraph D is a path from u to v in which, for every $v_{i-2}v_{i-1}, v_iv_{i+1} \in E(P)$, v_{i-1} is a head, and v_i is a tail. DEF 2.32

For a digraph D and $X \subseteq V(D)$, $\delta^+(X)$ denotes the vertices in $\delta(X)$ with its tail in X. Similarly, $\delta^-(X)$ denotes the vertices in $\delta(X)$ with its head in X. Note that $\delta^+(X) = \delta^-(V(G) - X)$, and similarly $\delta^-(X) = \delta^+(V(G) - X)$. DEF 2.33

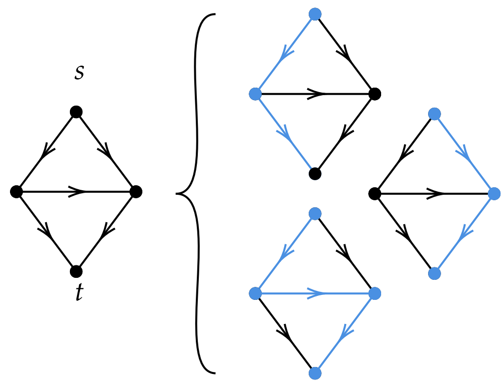
Let D be a digraph, and $s, t \in V(D)$. Then $\nexists$ a directed path in D from $s \to t \iff \exists X \subseteq V(G)$ s.t. $s \in X, t \notin X$, and $\delta^+(X) = \varnothing$. PROP 2.26

PROOF.

($\Longleftarrow$) Suppose there existed a directed path $P \subseteq D$ from $s \to t$. Consider the last vertex $v \in V(P)$ s.t. $v \in X$. Then the edge of the path with a tail in v is in $\delta^+(X)$. Hence, $\delta^+(X) \neq \varnothing \implies \lightning$.

($\Longrightarrow$) Let X be all $v \in V(D)$ s.t. $\exists$ a directed path from s to v. Then $s \in X, t \notin X$ by assumption. If $vw \in \delta^+(X)$ for some $w \notin X$, then we may construct a directed path consisting of the path $s \to v$, and stitching on this edge to w. Hence $w \in X \implies \lightning$. Hence $\delta^+(X) = \varnothing$. □

Consider the following directed paths from s to t:



Typically, we call s the "source" and t the "sink." Let $\delta^+(v)$ for $v \in V(D)$ denote all edges whose tail is v. We then define flow in the following way:

DEF 2.34 An (s,t)–*flow* on a digraph D is a function $\varphi : E(D) \to \mathbb{R}_+$ such that

$$\sum_{e \in \delta^+(v)} \varphi(e) = \sum_{e \in \delta^-(v)} \varphi(e) \quad \forall v \in V(D) - \{s,t\}$$

where s is the source and t is the sink.

DEF 2.35 The *value* of an (s,t)-flow φ is $\sum\limits_{e \in \delta^+(s)} \varphi(s) - \sum\limits_{e \in \delta^-(s)} \varphi(e)$.

PROP 2.27 Let φ be an (s,t)-flow on a digraph D with value k. Then $\forall X \subseteq V(D)$ such that $s \in X, t \notin X$, we have

$$\sum_{e \in \delta^+(X)} \varphi(e) - \sum_{e \in \delta^-(X)} \varphi(e) = k$$

PROOF. By flow conservation,

$$\begin{aligned}
k &= \sum_{e \in \delta^+(s)} \varphi(s) - \sum_{e \in \delta^-(s)} \varphi(e) \\
&= \sum_{v \in X} \underbrace{\left(\sum_{e \in \delta^+(v)} \varphi(e) - \sum_{e \in \delta^-(v)} \varphi(e) \right)}_{0 \text{ if } v \neq s} \\
&= \sum_{e \in E(D)} \varphi(e)\,(t(e) - h(e)) = \left(\sum_{e \in E(D)} \varphi(e) t(e) \right) - \left(\sum_{e \in E(D)} \varphi(e) h(e) \right) \\
&= \sum_{e \in \delta^+(X)} \varphi(e) - \sum_{e \in \delta^-(X)} \varphi(e)
\end{aligned}$$

Where

$$t(e) = \begin{cases} 1 & \text{tail of } e \text{ in } X \\ 0 & \text{o.w.} \end{cases} \qquad h(e) = \begin{cases} 1 & \text{head of } e \text{ in } X \\ 0 & \text{o.w.} \end{cases}$$

□

What is the maximal value of an (s,t)-flow? The answer is uninteresting: if there exists a path from $s \to t$, we can assign any amount of flow to each of these edges, and 0 otherwise, and maintain conservation. Hence, if there exists such a path, we may have ∞ flow. If a path does *not* exist, then we invoke Prop 2.26, which says $\delta^+(X) = \emptyset$ for any $X \subseteq V(G), s \in X, t \notin X$, to conclude that $k = -\sum_{e\in\delta^-(X)} \varphi(e)$. Since φ is non-negative, k is negative, and at most 0 (take $\varphi \equiv 0$).

DEF 2.36 A *capacity function* on a digraph D is a function $c : E(D) \to \mathbb{Z}_+$. An (s,t)-flow φ is *c-admissible* if $\varphi(e) \leq c(e) \forall e \in E(D)$.

DEF 2.37 A (not necessarily directed) path $P \subseteq D$ from $s \leftrightarrow t$ is *φ-augmenting* path for an (s,t)-flow $\varphi : E(D) \to \mathbb{Z}_+$ if:

1. $\varphi(e) \leq c(e) - 1$ if $e \in E(D)$ from tail to head.
2. $\varphi(e) \geq 1$ if $e \in E(P)$ from head to tail.

DEF 2.38 φ is called *integral* if its co-domain is the integers.

PROP 2.28 Let φ be an integral c-admissible (s,t)-flow of value k. If $\exists$ a φ-augmenting path D from $s \leftrightarrow t$, then $\exists$ a c-admissible (s,t)-flow in D of value $k+1$.

DEF 2.39 ψ is an (s,t) *pseudo-flow* if it satisfies flow conservation (but not necessarily non-negativity).

PROOF.

$$\psi(e) = \begin{cases} 1 & e \in E(P) \text{ is head to tail} \\ -1 & e \in E(P) \text{ is tail to head} \\ 0 & e \notin E(P) \end{cases}$$

Then let $\varphi' = \varphi + \psi$. ψ is then a "pseudo-flow," since ψ satisfies flow conservation. φ' is also a pseudo-flow. But $\varphi' \geq 0$, since, if $\psi(e) = -1$, then $\varphi(e) \geq 0$, so $\varphi'(e) \geq 0$. φ' is also c-admissible, since, if $\psi(e) = 1$, then $\varphi(e) + 1 \leq c(e)$, so $\varphi'(e) = \varphi(e) + 1 \leq c(e)$.

Also, the value the φ' is the value of ψ + the value of $\varphi \implies$ the value of φ' is $k+1$. □

2.14 Max Flow-Min Cut (or Ford-Fulkerson)

Let D be digraph, $s, t \in V(D)$ distinct. Let $c : E(D) \to \mathbb{Z}_+$. Then the maximal value of an integral c-admissible (s,t)-flow is equal to the minimum $\sum e \in \delta^+(X)c(e)$, over all $X \subseteq V(D), s \in X, t \notin X$.