DISCRETE MATHEMATICS

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CONTENTS

I   Preliminaries

DEFINITIONS

Graph theory is the study of pairwise relations between objects, e.g. computer networks, interpersonal relationships, airport flights. Graphs will help us simplify and abstract networks .

<div style="text-align:right">DEF 1.1</div>

A *graph G* is comprised of a set of vertices, denoted $V(G)$, where $|V(G)| < \infty$, a set of edges, denoted $E(G)$, where every edge is associated with two vertices.

<div style="text-align:right">DEF 1.2
At least in this course</div>

We say that an edge *joins* the two vertices it's associated with. Similarly, an edge is *incident* to a vertex which is an end of it. Conversely, a vertex is incident to an edge if it is an end of it.

<div style="text-align:right">DEF 1.3</div>

Two vertices are *adjacent* or *neighbors* if they are joined by an edge, and a vertex has *degree* edges incident to it.

<div style="text-align:right">DEF 1.4</div>

The *null graph* is the graph such that $V(G) = \varnothing$. The *complete graph* on n vertices, denoted K_n, is such that $|V(K_n)| = n$ and $|E(K_n)|$ is maximal.

<div style="text-align:right">DEF 1.5</div>

For a graph of n vertices, the maximal number of edges it may have is $\binom{n}{2}$.

<div style="text-align:right">PROP 1.1</div>

> Suppose every vertex is connected to every other vertex. Then $\sum_{v \in V(G)} \deg(v) = n(n-1) \implies |E(G)| = \frac{n(n-1)}{2} = \binom{n}{2}$. $\qquad\square$

<div style="text-align:right">PROOF.</div>

A graph of n vertices, where v_i is only adjacent to v_{i-1} and v_{i+1}, is called a *path* and is sometimes denoted P_n. v_1 and v_n are called the ends of P_n.

<div style="text-align:right">DEF 1.6</div>

For $n \geq 3$, a *cycle C_n* is a graph with $V(G) = \{v_1, ..., v_n\}$ and $E(G) = \{v_1 v_2, v_2 v_3, ..., v_{n-1} v_n, v_n v_1\}$.

<div style="text-align:right">DEF 1.7</div>

An *adjacency matrix* is a matrix contained all vertices on both axis. Pairwise adjacency is denoted by a 1 entry, and 0 otherwise. For example, the following is an adjacency table for a 4 element cycle:

<div style="text-align:right">DEF 1.8</div>

	v_1	v_2	v_3	v_4
v_1	$\times$	1	0	1
v_2	1	$\times$	1	0
v_3	0	1	$\times$	1
v_4	1	0	1	$\times$

Similarly, an *incidence* matrix has rows in $V(G)$ and columns in $E(G)$, and marks

<div style="text-align:right">DEF 1.9</div>

with 1 pairs which are incident to eachother. The following is the incidence matrix for a 4 element cycle:

	v_1	v_2	v_3	v_4
e_1	1	1	0	0
e_2	0	1	1	0
e_3	0	0	1	1
e_4	1	0	0	1

PROP 1.2

For a graph G, we always have $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

PROOF.

> Every edge has two vertices incident to it. Thus, $\sum \deg(v)$ will be the number of times an edge is incident to a vertex, i.e. the number of edges $\times$ 2.   □

DEF 1.10

H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

We cannot do the same for "$G \setminus H$," since we may delete vertices and keep their incident edges!

For two graphs G, H, the union $G \cup H$ is a graph such that $V(G \cup H) = V(G) \cup V(G)$ and $E(G \cup H) = E(G) \cup E(H)$. We similarly define the intersection $G \cap H$ to be such that $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$.

PROP 1.3

There are $2^{\binom{n}{2}}$ graphs with n vertices.

PROOF.

> We know the maximal number of edges of this graph is $\binom{n}{2}$. Then, for each edge, one may make a binary choice whether to include it or not $\therefore$ the number of graphs is $2^{\binom{n}{2}}$.   □

DEF 1.11

We can now ask: how many graphs are there with n vertices up to isomorphism?

An *isomorphism* between H and G is a bijection $\varphi : V(G) \rightarrow V(H)$ such that $uv \in E(G) \implies \varphi(u)\varphi(v) \in E(H)$..

CONNECTIVITY

DEF 1.12

A *walk* in G with ends u_0 and u_k is a sequence $(u_0, u_1, ..., u_k)$ such that $u_i \in V(G)$ and $u_i u_{i+1} \in E(G)$. The length of this walk is k.

u and v are called *connected* if there exists a walk in G with ends u and v OR, equivalently, there exists a path $P \subseteq G$ with ends u and v.

PROP 1.4

$\exists$ a walk in G with ends u and v $\iff$ $\exists$ a path $P \subseteq G$ with ends u and v.

PROOF.

> ($\impliedby$) Let $P \subseteq G$ be a path with ends u and v. Then $V(P)$ can be numbered $u = v_0, v_1, ..., v_k = v$, where $v_i v_{i+1} \in E(P)$. Then $(v_0, ..., v_k)$ is a walk in G.
>
> [8pt]($\implies$) Let there exist a walk $(u = v_0, ..., v_k = v)$ with $v_i v_{i+1} \in E(G)$. WLOG suppose this is the walk of minimal length. If $v_i \neq v_j$, i.e. are pairwise distinct, then we already have a path. Suppose otherwise, and let $v_i = v_j$. Then $(v_0, ..., v_i, v_{j+1}, ..., v_k)$ is a *smaller* walk with ends u and v, which

establishes the contradiction ⨎.   □

A graph G is called *connected* if $\forall u, v \in V(G)$, u and v are connected.   DEF 1.13

A *partition* of $V(G)$ is $(X_1, ..., X_k)$ such that $\cup_{i=1}^{k} X_i = V(G)$ and $X_i \cap X_j = \varnothing$ $\forall i \neq j$.   DEF 1.14

A graph G is not connected $\iff$ $\exists$ a partition (X, Y) of $V(G)$ such that no edge   PROP 1.5
of G is incident to one vertex in X and one in Y.

> ($\impliedby$) Suppose G were connected. Then choose $u \in X, v \in Y$ such that there   PROOF.
> exists a walk $(u = u_0, ..., u_k = v)$. Let u_i be minimal over i such that $u_i \in Y$.
> Then $u_{i-1} \in X$, and $u_{i-1} u_i \in E(G)$ ⨎.
>
> [8pt]($\implies$) Let $u, v \in V(G)$ be such that there is no walk from u to v. Let X
> be the set of all $w \in V(G)$ such that $\exists$ a walk with ends u and w. Similarly,
> let $Y = V(G) \setminus X$. Clearly $V(G) = X \cup Y$, $X \cap Y = \varnothing$, and (X, Y) is a partition.
> Suppose there exists an edge from a vertex in X to a vertex in Y, i.e. $x \in X$,
> $y \in Y$. Then we have the walk $(u, ..., w, ..., x, y)$. But $y \notin X$ ⨎.   □

Let G be a graph. $H \subseteq G$ is called a *connected component* of G if H is a maximal   DEF 1.15
connected subgraph of G, i.e. if $\exists H \subseteq H' \subseteq G$ with H' connected, then $H = H'$.   Sometimes we just say "com-
ponent."

If H_1, H_2 are connected graphs, and $V(H_1) \cap V(H_2) \neq \varnothing$, then $H_1 \cup H_2$ is also   PROP 1.6
connected.

> Let $u \in H_1, v \in H_1 \cap H_2, w \in H_2$. Then $(u, ..., v)$ and $(v, ..., w)$ are both walks,   PROOF.
> and thus $(v, ..., v, ..., w)$ is a walk.   □

Every $v \in V(G)$ is a member of a unique connected component $H \subseteq G$.   PROP 1.7

> $\{v\}$ is connected. If there does not exist $H \supseteq \{v\}$ also connected, then we are   PROOF.
> done. Otherwise, we may choose the maximal such connected superset.
>
> [8pt]Suppose $v \in H_1$ and H_2, two connected components. Then by Prop 1.6,
> $H_1 \cup H_2$ is connected. But since $H_1 \cup H_2 \supseteq H_1, H_2$, this violates maximality.
> We conclude that $H_1 = H_2$.   □

Let G be a graph, and let $H \subseteq G$ be a non-null and connected subgraph. Then H PROP 1.8
is a connected component of G $\iff$ $\forall e \in E(G)$ with an end in $V(H)$, we have
$e \in E(H)$.

> For ($\implies$), let $e = uv$, with $u \in V(H)$. If $v \in V(H)$, then we are done.   PROOF.
> Otherwise, suppose $e \notin E(H)$. We know v is a member of a unique connected
> component. But adding e to H would yield a further connected graph: take
> the graphs of $\{uv\}$ and H. Both are clearly connected, so $H \cup \{uv\}$ is connected.

[8pt]($\Longleftarrow$)   Proof idea.

Obtained from G by deleting e

For $e \in E(G)$, $G \setminus e$ is a graph such that $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.

Similarly, for $v \in V(G)$, $G \setminus v$ is a graph such that $V(G \setminus v) = V(G) \setminus \{v\}$ and $E(G \setminus v) = E(G) \setminus \{e : e \text{ incident to } v\}$.

Let $\text{comp}(G) = \#$ of connected components of G.

PROP 1.9

$\text{comp}(G) = 1 \iff G$ is connected.

PROOF.

($\Longrightarrow$) direction is trivial. For ($\Longleftarrow$), if G is connected, then there cannot exist a more maximal connected subgraph, e.g. G is a connected component. Since every vertex belongs to a unique connected component, and this must be G, $\text{comp}(G) = 1$. $\qquad\square$

DEF 1.16

Let $e = \{u, v\} \in E(G)$. Define a *cut-edge* to be an edge which is not part of any cycle.

PROP 1.10

Exactly one of the following holds:

1. *e is a cut-edge*: $\text{comp}(G \setminus e) = \text{comp}(G) + 1$, and u, v belong to different components of $G \setminus e$.

2. *e is not a cut-edge*: $\text{comp}(G \setminus e) = \text{comp}(G)$, and u, v belong to the same component.

PROOF.

Let e be a cut-edge. Let $H_1, ..., H_k$ be the connected components of $G \setminus e$. If u, v belong to H_i, then $\exists$ a path $P \subseteq H_i$ with ends u and v. Adding e, this is a cycle $\frac{\ell}{2}$.

[8pt] WLOG, assume that u, v belong to $V(H_1)$, $V(H_2)$, respectively. Then let H' be obtained by $H_1 \cup H_2$ by adding e. We claim that $H', H_2, ..., H_k$ are all components of G. By Prop 1.8, we only need to check the connectivity of H', and this holds by $\overline{\text{Prop 1.6}}$. Since there do not exist any vertices *not* in $V(H_i) : i \geq 2$ or $\overline{V(H')}$, these are all the components of G. Thus, $\text{comp}(G) + 1 = \text{comp}(G \setminus e)$. $\qquad\square$

II   Trees and Forests

A *forest* is a graph with no cycles, i.e. every edge is a cut-edge.   DEF 2.1

A *tree* is a non-null connected forest.   DEF 2.2

Let F be a non-null forest. Then $\text{comp}(F) = |V(F)| - |E(F)|$.   PROP 2.1

> We'll show by induction on $|E(F)|$. If $n = 0$ then all vertices are their own   PROOF.
> connected components. Let $|E(F)| = n$, and assume $\text{comp}(F) = |V(F)| - |E(F)|$.
> Let $e \in E(F)$. Since F is a forest, e is a cut-edge, and thus $\text{comp}(G \setminus e) =$
> $\text{comp}(G) + 1 = |V(F)| - |E(F)| + 1 = |V(F)| - (|E(F)| - 1) = |V(F)| - |E(F \setminus e)| =$
> $|V(F \setminus e)| - |E(F \setminus e)|$.   □

A *leaf* is a vertex with degree 1.   DEF 2.3

Let T be a tree with $|V(T)| \geq 2$. let $X = \{$leaves of $T\}$, $Y = \{v \in V(G) : \deg(v) \geq 3\}$.   PROP 2.2
Then $|X| \geq |Y| + 2$.   Thus, trees have ≥ 2 leaves!

> By Prop 1.1, we have   PROOF.
>
> $$\sum_{v \in V(T)} \deg(v) = 2|E(T)| \overset{1.11}{=} 2(|V(T)| - \text{comp}(G)) \overset{1.9}{=} 2(|V(T)| - 1)$$
>
> $$\implies \sum_{v \in V(T)} (\deg(v) - 2) = 2(|V(T)| - 1) - 2|V(T)| = -2$$
>
> $$= \underbrace{\sum_{v \in X} (\deg(v) - 2)}_{= -|X|} + \underbrace{\sum_{v \in Y} (\deg(v) - 2)}_{\geq |Y|} + \underbrace{\sum_{v \in V(T) - X - Y} (\deg(v) - 2)}_{= 0}$$
>
> $$\implies -2 \geq -|X| + |Y| \implies |X| \geq |Y| + 2 \quad □$$

*A note for the following few proofs: if w is a leaf, then any path which exists in T (with
ends not w) exists in $T \setminus w$.*

Let T be a tree with 2 leaves, u and v. Then T is a path with ends u and v.   PROP 2.3

> Let $P \subseteq T$ be a path with ends u and v. By Prop 1.12, $\deg_T(w) = 2 \; \forall w \in$ PROOF.
> $V(P) \setminus \{u, v\}$. Moreover, $\deg_T(w) = \deg_P(w)$, so no vertex in $V(P)$ is incident
> to an edge in $E(T) \setminus E(P)$. Then, by Prop 1.8, P is a connected component.
> But T is connected, so $T = P$.   □

Let T be a tree and $v \in V(T)$ be a leaf. Then $T \setminus v$ is a tree.   PROP 2.4

                                                                                     PROOF.

$T \setminus v$ is non-null, since v has a neighbor. $T \setminus v$ has no cycles, since T has no cycles, and $T \setminus V$ is connected. We know there exists a path between any two vertices in $V(T) \setminus \{v\}$. Such a path still exists. □

PROP 2.5

If G is a graph, $v \in V(G)$ a leaf, and $G \setminus v$ a tree, then G is a tree.

PROOF.

G is non-null, since $G \setminus v$ is non-null. We know that v belongs to no cycles, since it is a leaf, so any cycles apparent in G would exist in $G \setminus v$. Thus, G has no cycles. For connectedness, let H be the graph containing v, its incident edge, and that edge's other vertex v'. H is connected, as is $G \setminus v$, and $G \setminus v \cap H \neq \varnothing$, so $G \setminus v \cup H = G$ is connected by Prop 1.6. □

PROP 2.6

Let T be a tree, $u, v \in V(T)$. Then T contains a unique path with ends u and v.

PROOF.

We'll show by induction on $|V(T)|$. This clearly holds for $|V(T)| = 1$. Let $|V(T)| \geq 2$. Suppose T contains a leaf $w \in V(T) \setminus \{u, w\}$. Then $T \setminus w$ is a tree by Prop 1.14. By our induction hypothesis, $T \setminus w$ contains a unique path with ends u and v. By connectedness, $\exists$ a path with ends u, v in T. But this path must exist in $T \setminus w$, whose uniqueness follows.

[8pt]If no such leaf exists, then T has exactly 2 leaves (u and v). Thus, by Prop 1.13, T is a path with ends u and v, and thus the only path in T. □

SPANNING TREES

DEF 2.4

Let G be a graph. A subgraph $T \subseteq G$ is called a *spanning tree* of G if T is a tree and $V(T) = V(G)$.

PROP 2.7

Let G be connected and non-null. Let $H \subseteq G$, chosen minimal such that $V(H) = V(G)$ and H is connected. Then H is a spanning tree of G.

PROOF.

We only need to check that T is non-null and contains no cycles. The first is automatic, since $V(T) = V(G)$, and G is non-null. If H has a cycle, then let e be an edge in the cycle. $H \setminus e$ is connected by Prop 1.9 and Prop 1.10. But this contradicts minimality, so T contains no cycles. □

PROP 2.8

Let G be a connected non-null graph. Let $H \subseteq G$ be maximal such that H contains no cycles. Then H is a spanning tree of G.

PROOF.

We need to show that $V(H) = V(G)$ and H is connected (it is non-null, since at least a singleton of G contains no cycles; it contains no cycles by construction). If $\exists v \in V(G) \setminus V(H)$, adding v such that $\deg(v) = 0$ would maintain H having no cycles, thus contradicting maximality.

[8pt]Suppose H is not connected. Then by Prop 1.5 there exists a partition $H = X \cup Y$ such that no edge has a vertex in both X and Y. However, such an edge must exist in G, say $e \in E(G)$, so we may add this edge to H to produce H'. Observe that H' contains no cycles, since e belongs to no cycles in H. But this contradicts maximality, so H must contain no cycles. ☐

Let T be a spanning tree of G. Let $f \in E(G) \setminus E(T)$. Then T with f has one cycle (by Prop 2.6). This is called the *fundamental cycle* of f with respect to T, and denoted $\overline{FC}(T, f)$. DEF 2.5

Let T be a spanning tree of G, $f \in E(G) \setminus E(T)$. Let $C = FC(T, f)$, $e \in E(C)$. Then $(T + f) \setminus \{e\}$ is a spanning tree. PROP 2.9

Let $T' = (T + f) \setminus \{e\}$. $T + f$ is connected, and since e is not a cut-edge, $(T + f) \setminus \{e\} = T'$ is also connected. C is a unique cycle in $T + f$, so T' contains no cycles. Thus, T' is a tree. $V(T') = V(T) = V(G)$, since T is a spanning tree, so we conclude that T' is a spanning tree. ☐ PROOF.

Let G be a non-null, connected tree. Let $w : E(G) \to \mathbb{R}_+$ by a real valued function on the edges of G. The *minimal spanning tree* of G w.r.t. w, denoted $MST(G, w)$, is a spanning tree T such that $w(T) = \sum_{e \in E(T)} w(e)$ is minimal. DEF 2.6

2.1   Minimality of MST Edges

Let G be connected and non-null. Let $w : E(G) \to \mathbb{R}_+$. Let $T = MST(G, w)$ and $E(T) = \{e_1, ..., e_k\}$, where we order

$$w(e_1) \le w(e_2) \le ... \le w(e_k)$$

Then $\forall 1 \le i \le k$, e_i is an edge of minimum weight subject to the following constraints:

- $e_i \notin \{e_1, ..., e_{i-1}\}$

- $\{e_1, ..., e_i\}$, as an edge set, does not contain any cycles.

In particular, this theorem states that *for any $f \in E(G) - \{e_1, ..., e_{i-1}\}$ with $\{e_1, ..., e_{i-1}, f\}$ not containing cycles, $w(f) > f(e_i)$.*

Suppose otherwise. Then for at least one i, we can choose $f \in E(G) - \{e_1, ..., e_{i-1}\}$ such that $\{e_1, ..., e_{i-1}, f\}$ contains no cycles and $w(f) < w(e_i)$. PROOF.

Then $f \notin E(T)$, otherwise $f = e_j$ for some $j \ge i$. But $j < i$, since we have an ordering on w. Let $C = FC(T, f)$, the unique cycle in $T + f$. There is some $j \ge i$ such that $e_j \in E(C)$, since all vertices $< i$ must not contain cycles. Then $w(e_j) > w(f)$.

Let $T' = (T + f) - e_j$. Then by Prop 2.9, T' is still a spanning tree. Let $w(G)$ be the sum of weights of edges of G. Then $w(T') - w(T) = w(f) - w(e_j) < 0$, implying that T is not minimal ↯. □

DEF 2.7

Kruskal's Algorithm

| Input

A connected, non-null graph G, and $w : E(G) \to \mathbb{R}_+$

| Output

A graph T such that $V(T) = V(T)$, with $E(T) = \{e_1, ..., e_{|V(G)|-1}\}$

| $n \to n + 1$

Let $e_i \in E(G)$ be chosen such that $w(e_i)$ is minimum subject to

- $e_i \notin \{e_1, ..., e_{i-1}\}$

- $\{e_1, ..., e_i\}$, as an edge set, does not contain any cycles.

for $1 \le i \le |V(G)| - 1$

2.2 Kruskal's Algorithm Outputs an MST

PROOF.

Suppose $w : E(G) \to \mathbb{R}_+$ is injective. Then all edges have different weights. Then Thm 2.1 implies that Kruskal's outputs an MST which is unique. If w is not injective, the proof is out of the scope of this course. □

2.3 Spanning Trees of K_n

The complete graph K^n has exactly n^{n-2} spanning trees.

PROOF.

The proof for this will require proving multiple statements. Let $\mathcal{T}_k$ be the set of spanning, rooted forests in K^n with k components. Then $\mathcal{T}_1$ is the set of rooted spanning trees in K_n. Since we may choose n roots, $\frac{|\mathcal{T}_1|}{n}$ equals the number of spanning trees in K^n. Thus, we need to show $|\mathcal{T}_1| = n^{n-1}$.

Claim 1 $|\mathcal{T}_n| = 1$

If a spanning, rooted forest has n components, then it is exactly the graph of no edges and the vertex set $V(K^n)$ (each being its own component).

Claim 2 $n(k-1)|\mathcal{T}_k| = (n-k+1)|\mathcal{T}_{k-1}|$

Call a forest F with $k - 1$ the *parent* of a forest F' with k components if

$F' = F \setminus e$ for some $e \in E(F)$. Naturally, we call F' a *child* of F under these conditions. We will thus count (parent, child) combinations. Every $F \in \mathcal{T}_{k-1}$ has $|E(F)|$ children, since every edge is a cutedge. This is $|V(F)| - \text{comp}(F) = n - (k - 1) = n - k + 1$ by <u>Prop 2.1</u>.

For every $F' \in \mathcal{T}_k$, we can obtain a parent by adding an edge from any vertex to the root of a component not containing this vertex. Thus, every child has $n(k - 1)$ parents. Thus, there are $n(k - 1)|\mathcal{T}_k|$ parent-child combinations, and also $(n - k + 1)|\mathcal{T}_{k-1}|$ such combinations. Thus, we conclude $n(k - 1)|\mathcal{T}_k| = (n - k + 1)|\mathcal{T}_{k-1}|$.

Claim 3 $|\mathcal{T}_k| = \binom{n}{k} k n^{n-1-k}$

We just solve the recursion. We'll show by induction on $n - k$. If $n - k = 0 \implies n = k$, we have $|\mathcal{T}_n| = \binom{n}{n} n n^{n-1-n} = 1$, which is true by Claim 1.

Letting $n - k \to n - k + 1 = n - (k - 1)$, we are having $k \to k - 1$. By Claim 2, then,

$$|\mathcal{T}_{k-1}| = \frac{n(k-1)}{n-k+1}|\mathcal{T}_k| \stackrel{\text{hyp.}}{=} \frac{n(k-1)}{n-k+1}\binom{n}{k}kn^{n-1-k} \stackrel{?}{=} \binom{n}{k-1}(k-1)n^{n-1-(k-1)}$$

$$= \frac{k}{n-k+1}\binom{n}{k}(k-1)n^{n-1-(k-1)}$$

Note that $\frac{k}{n-k+1}\binom{n}{k} = \frac{kn!}{(n-k+1)k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!(n-k+1)} = \frac{n!}{(k-1)!(n-k+1)!} = \binom{n}{k-1}$, so the last statement above evaluates to

$$\binom{n}{k-1}(k-1)n^{n-1-(k-1)} \qquad \text{as desired.}$$

Claim 4 $|\mathcal{T}_1| = n^{n-1}$

We plug in from above to find $|\mathcal{T}_1| = \frac{n}{1} 1 n^{n-1-1} = n^{n-2+1} = n^{n-1}$. $\qquad\qquad \square$

EULER'S THEOREM & HAMILTONIAN CYCLES

Recall that a *walk* in G is a sequence $(v_0, ..., v_k) : v_i \in V(G)$, perhaps with repetition, such that $v_i v_{i+1} \in E(G) \; \forall i \leq k - 1$. (See <u>Def 1.12</u>).

A walk *uses* an edge e if $e = v_i v_{i+1}$ and v_i, v_{i+1} is contained in the walk. DEF 2.8

A *trail* is a walk that uses every edge at most one. DEF 2.9

A *Euler trail* in G is a trail that uses every edge in $E(G)$ DEF 2.10

A *Euler tour* in G is a closed Euler trail (i.e. $v_0 = v_k$). DEF 2.11

PROP 2.10

Let G be a graph with $E(G) \neq \emptyset$. If G has no leaves, then G has a cycle.

PROOF.

Suppose not. Then G is a forest by Def 2.1. Let C be a component with at least one edge. Then C is a tree with $|V(C)| \geq 2$. Thus, C has a leaf by Prop 2.2. ↯. □

PROP 2.11

Let G be a graph with all degrees even. Then $\exists$ cycles $C_1, ..., C_k \subseteq G$ such that $(E(C_1), ..., E(C_k))$ is a partition of $E(G)$.

PROOF.

Informally, by the previous proposition we may create one cycle from the outset. Removing the edge set of this cycle from G leaves the graph with still all even edges. We'll show by induction on the edge set. The base case, $|E(G)| = 0$, holds trivially.

Let $n \to n + 1$. We know there exists a cycle $C_1 \subseteq G$. Let $G' = G - C_1$. Then all degrees in G' are even. By (strong) induction, G' contains a partition $(E(C_2), ..., E(C_k))$. Since $E(G) = E(G') \cup E(C_1)$, the partition $(E(C_1), ..., E(C_k))$ satisfies the proposition. □

2.4 Euler's Theorem

Let G be connected with all even degrees. Then $\exists$ a Euler tour in G.

PROOF.

Let $W = (v_0, ..., v_k)$ be a closed trail of maximal length in G. WLOG suppose W doesn't use every edge. Let $H \subseteq G$ be such that $V(H) = V(G)$ and $E(H) = E(G) \setminus \{\text{edges not used by } W\}$. Then all vertices of H have an even degree, so by Prop 2.11, $\exists$ cycles $C_1, ..., C_k$ which partition $E(H)$.

Let $H' \subseteq G$ be the subgraph consisting of the edges and vertices of W. H' is not a component of G (since G is connected). so by Prop 1.8 $\exists e \in E(G) - E(H')$ with an end in $V(H')$, i.e. an end in the walk. Thus, e is part of H, and is part of a cycle. Thus, we may append this cycle to the walk W, creating a larger closed trail, violating maximality ↯. □

PROP 2.12

Let G be a connected graph with ≤ 2 vertices of odd degree. Then G contains a Euler trail.

PROOF.

By the handshaking lemma (not seen here, but easy to see), there must be an even number of odd degree vertices in any graph. Thus, the case of one odd vertex is invalid.

Let $\deg(u), \deg(v)$ be odd, and all other vertices even. Let $G' = G + w$, where w is an added edge which is adjacent (joins) u and v. Then G' has all degrees

even, and by <u>Thm 2.4</u> G' has a Euler tour. WLOG we can let it begin and end at w. Thus, removing w, we get a Euler trail in G. □

A *Hamiltonian cycle* is a cycle $C \subseteq G$ s.t. $V(C) = V(G)$.

Let G be a graph, and $X \subseteq V(G)$ with $X \neq \varnothing$. If $|X| < \text{comp}(G \setminus X)$, then G has no Hamiltonian cycles.

DEF 2.12

As it turns out, there is no ef-fiicent algorithm to decide if G has a Hamiltonian cycle.

PROP 2.13

Let C be Hamiltonian cycle. Then

$$\text{comp}(C \setminus X) \geq \text{comp}(G \setminus X) > |X|$$

PROOF.

Observe now that $C \setminus X$ is a forest. Then, from theory, we know that

$$\text{comp}(C \setminus X) = |V(C \setminus X)| - |E(C \setminus X)| \leq |V(C)| - |V(X)| - (|E(C)| - 2|X|) = |X|$$

where we note that $|V(C)| = |E(C)|$, since C is a cycle. This is a contradiction. □

2.5 Dirac-Pósa

Let G be a graph on $n \geq$ vertices. If $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices $u, v \in V(G)$, then G has a Hamiltonian cycle.

We'll show by induction on $\binom{n}{2} - |E(G)|$. If $|E(G)| = \binom{n}{2}$, then G is complete, and clearly contains a Hamiltonian cycle.

PROOF.

Let $|E(G)| < \binom{n}{2}$. Let $u, v \in V(G)$ be non-adjacent. Let $G' = G + uv$. By induction hypothesis, $\exists$ a Hamiltonian cycle $C \subseteq G'$. If $uv \notin E(C)$, then C is a Hamiltonian cycle in G. Otherwise, let $uv \in E(C)$. Notate
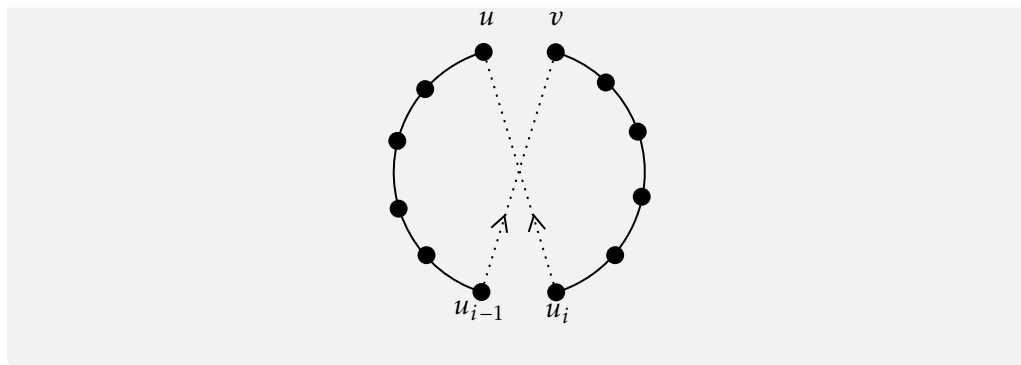
$$V(C) = \{u = u_1, u_2, ..., u_n = v\}$$

Let $A = \{i : uu_i \in E(G)\}$ and $B = \{i : vu_{i-1} \in E(G)\}$. Then $|A| + |B| = \deg(u) + \deg(v) \geq n$.

But we have $n - 1$ such vertices (u, v are non-adjacent, so this takes away a possibility). Thus, $A \cap B \neq \varnothing$, so $\exists i : uu_i, vu_{i-1} \in E(G)$. Then

$$\{u = u_1, ..., u_{i-1}, u_n = v, u_{n-1}, ..., u_i, u_1 = u\}$$

is a Hamiltonian cycle. □

PROP 2.14   Let G be a graph on $n \geq 3$ vertices. Then if $\deg(v) \geq \frac{n}{2}$ $\forall v \in V(G)$ OR $|E(G)| \geq \binom{n}{2} - n - 3$, then G has a Hamiltonian cycle.

PROOF.

> If $\deg(v) \geq \frac{n}{2}$ $\forall v \in V(G)$, then $\deg(u) + \deg(v) \geq n$ $\forall u, v \in V(G)$, so by Thm 2.5 G has a Hamiltonian cycle.
>
> For the second condition, I was getting a cookie and didn't listen. $\qquad\square$

BIPARTITE GRAPHS

DEF 2.13   A *bipartition* of a graph G is a partition (A, B) of $V(G)$ such that every edge of G has one end in A and one end in B. Refer to Def 1.14.

DEF 2.14   A graph G is *bipartite* if it admits a bipartition.

PROP 2.15   Trees are bipartite.

PROOF.

> We'll show by induction on $n = |V(T)|$. For $|V(T)| = 1$, we have a bipartition $(\{v\}, \varnothing)$.
>
> Let $|V(T)| = n$. Let $v \in V(T)$ be a leaf with a neighbor u. By induction hypothesis, $(T \setminus v)$ is bipartite. Let (A, B) be a bipartition. Assume WLOG that $u \in A$. Then $(A, B \cup \{v\})$ is a bipartition of T. $\qquad\square$

> **2.6   Characterization of Bipartite Graphs**
>
> Let G be a graph. Then the following are equivalent:
>
> 1. G is bipartite.
>
> 2. G contains no closed walk of odd length
>
> 3. G contains no odd cycles.

PROOF.

(1 $\implies$ 2). Let (A, B) be a bipartition of G. Let $(v_0, ..., v_k)$ be a walk in G. WLOG let $v_0 \in A$. Then $v_i \in A \iff i$ is even. Thus, if $v_0 = v_k$, k must be even, so the walk must have even length.

(2 $\implies$ 3)If G had a cycle of odd length, it would be a closed walk of odd length.

(3 $\implies$ 1). As bipartitions of components may be combined to form a larger bipartition, it suffices to show this for a connected, non-null graph.

Let T be a spanning tree of G. Then $\exists$ a bipartition (A, B) of $V(T)$ by Prop 2.15. We'll show this is a bipartition of G as well. Let $f \in E(G) - E(T)$. Let $\overline{v_0, ..., v_k}$ be a path with ends on f. Assume WLOG that $v_0 \in A$.

The fundamental cycle $FC(T, f)$ has even length by assumption, so v_k must be odd (observe the cycle v_0, v_1, v_2, v_3 for reference). Thus, $v_k \in B$, so f has one end in A, and one in B. This may be reasoned for all $f \in E(G) - E(T)$. The bipartition holds for $E(T)$. Thus, it holds for all $e \in E(G)$. $\qquad\square$

MATCHINGS IN BIPARTITE GRAPHS

A *matching M* in G is a set of edges such that no vertex in $V(G)$ is incident to more than one edge in M.

<div align="right">DEF 2.15</div>

The *matching number* of G, denoted $\nu(G)$, is the maximum size $|M|$ for matchings M in G.

<div align="right">DEF 2.16</div>

$\nu(G) \leq \lfloor \frac{V(G)}{2} \rfloor$.

<div align="right">PROP 2.16</div>

This follows from the pigeonhole principle. $\qquad\square$

<div align="right">PROOF.</div>

A *vertex cover* in G is a set $X \subseteq V(G)$ such that every edge in $E(G)$ has at least one end in X.

<div align="right">DEF 2.17</div>

If M is a matching in G and X is a vertex cover, then $|M| \leq |X|$.

<div align="right">PROP 2.17</div>

If $|X|$ is a vertex cover, then every edge in M has an end in X. But no vertex $x \in X$ can belong to more than one edge in M, so $|M| \leq |X|$. $\qquad\square$

<div align="right">PROOF.</div>

Let $\tau(G)$ be the minimum size of a vertex cover in G.

<div align="right">DEF 2.18</div>

$\nu(G) \leq \tau(G)$.

<div align="right">PROP 2.18</div>

Immediately from Prop 2.17. $\qquad\square$

<div align="right">PROOF.</div>

For a graph G, $\nu(G) \leq \tau(G) \leq 2\nu(G)$.

<div align="right">PROP 2.19</div>

<div align="right">PROOF.</div>

It remains to show that $\tau(G) \leq 2\nu(G)$. Let M be a matching with $|M| = \nu(G)$. We want to find a vertex cover X with $|X| \leq 2|M|$. Let X be the set of ends of edges of M. Then $|X| = 2|M|$. Furthermore, X is a vertex cover. Otherwise, $\exists e \in E(G)$ with no end in X. Then $M \cup \{e\}$ is a matching, violating maximality.

□

DEF 2.19   Let M be a matching in G. A path $P \subseteq G$ is *M-alternating* if edges in P alternate between edges of M and $E(G) - M$, i.e. every internal vertex of P is incident to an edge in $E(P) \cap M$.

DEF 2.20   An M-alternating path $P \subseteq G$ is *M-augmenting* if $|V(P)| \leq 2$ and the ends of P are not incident to edges of M.

PROP 2.20   If G contains an M-augmenting path, then M is not maximum.