ASSIGNMENT 2

MATH 356

QUESTION 1

For part (a), we want $\mathbb{P}(A|Y)$, where A denotes the event that we pick the first bin, B the second bin, and Y a yellow ball. We have the following:

$\mathbb{P}(A|Y) = \frac{\mathbb{P}(AY)}{\mathbb{P}(Y)}$

$\mathbb{P}(Y) = \mathbb{P}(Y|A)\mathbb{P}(A) + \mathbb{P}(Y|A^c)\mathbb{P}(A^c)$

$$\implies \mathbb{P}(A|Y) = \frac{\mathbb{P}(AY)}{\mathbb{P}(Y|A)\mathbb{P}(A) + \mathbb{P}(Y|A^c)\mathbb{P}(A^c)} \qquad \star$$

with the following probabilities:

$$\mathbb{P}(AY) = \frac{4}{10}\mathbb{P}(A) \quad \mathbb{P}(Y|A) = \frac{4}{10} \quad \mathbb{P}(Y|A^c) = \frac{4}{7}$$

since A^c is just the event we pick the other bin, and $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

Thus we have:

$$\mathbb{P}(A|Y) = \frac{\frac{4}{10}\mathbb{P}(A)}{\frac{4}{10}\mathbb{P}(A) + \frac{4}{7}[1 - \mathbb{P}(A)]}$$

For part (b), we use the same equation from $\star$, but notate with Y_Y (the event one picks yellow twice from a bin, without replacement) instead of Y.

$$\implies \mathbb{P}(A|Y_Y) = \frac{\mathbb{P}(AY_Y)}{\mathbb{P}(Y_Y|A)\mathbb{P}(A) + \mathbb{P}(Y_Y|A^c)\mathbb{P}(A^c)}$$

with the following probabilities:

$$\mathbb{P}(AY_Y) = \frac{16}{100}\mathbb{P}(A) \quad \mathbb{P}(Y_Y|A) = \frac{16}{100} \quad \mathbb{P}(Y_Y|A^c) = \frac{16}{49}$$

Thus we have:

$$\mathbb{P}(A|Y_Y) = \frac{\frac{16}{100}\mathbb{P}(A)}{\frac{16}{100}\mathbb{P}(A) + \frac{16}{49}[1 - \mathbb{P}(A)]}$$

since, without replacement, picking a yellow ball twice is equal to the square of the probability of picking it once.

NOTE: I took the probability of choosing a bin at random to be arbitrary, as though it were it's own random variable $X \sim \text{Ber}(p) := \{A, B\}$, for bins A and B. If we take this "random pick" to have $X \sim \text{Ber}(\frac{1}{2})$, then the probabilities are as follows:

For part (a), $\mathbb{P}(A|Y) = \frac{1/5}{1/5 + 2/7} = \frac{7}{17}$ For part(b) $\mathbb{P}(A|Y_Y) = \frac{2/25}{2/25 + 8/49} \approx \frac{1}{3}$

QUESTION 2

To find $\mathbb{P}$(Best applicant hired), we split the probability into cases:

$$\begin{aligned}
\mathbb{P}(\text{Best applicant hired}) = \quad & \mathbb{P}(\text{Best applicant hired in first interview after rejects}) \\
+ \quad & \mathbb{P}(\text{Best applicant hired in second interview after rejects}) \\
\vdots \quad & \\
+ \quad & \mathbb{P}(\text{Best applicant hired in } N-k^{th} \text{ interview after rejects}) \\
= \quad & \sum_{i=1}^{N-k} \mathbb{P}(\text{Best applicant hired in } i^{th} \text{ interview after rejects})
\end{aligned}$$

By conditional probability, we have

$$\sum_{i=1}^{N-k} \mathbb{P}(A \text{ hired in interview } i) = \sum_{i=1}^{N-k} \mathbb{P}(\text{Given that } A_i = A,\ A_i \text{ is hired })\mathbb{P}(A_i = A)$$

where A_i is the i^{th} applicant interviewed after the round of rejects, and A is the best candidate. Immediately, we know that the probability of any candidate being the best, $\mathbb{P}(A_i = A)$, is $\frac{1}{N}$. If A_i is given to be the best, they are hired only if the best candidate thus far was insta-rejected (otherwise, that candidate would have been selected, not A_i). Thus, we have $\mathbb{P}(A_i \text{ hired } | A_i = A) = \frac{k}{k+i-1}$.

$$\mathbb{P}(A \text{ is hired}) = \sum_{i=1}^{N-k} \mathbb{P}(A \text{ hired in interview } i) = \sum_{i=1}^{N-k} \frac{k}{N(k+i-1)}$$

Parts (b) and (c): We'll need to consider subtler points to find "optimal" k's. One could differentiate our series representation in order to maximize $\mathbb{P}$, but the following leads nowhere when N is finite:

$$\begin{aligned}
\mathbb{P}'(k^*) \quad & = \frac{d}{dk^*}\left[\frac{k^*}{N}\sum_{i=1}^{N-k^*}\frac{1}{k^*+i-1}\right] \\
& \implies \sum_{i=1}^{N-k^*}\frac{1}{k^*+i-1} - k^*\sum_{i=1}^{N-k^*}\frac{1}{(k^*+i-1)^2} = 0 \implies \sum_{i=k^*}^{N-1}\frac{1}{i} - \frac{k^*}{i^2} = 0 \quad \star \\
& = \frac{1}{k^*+1} - \frac{k^*}{(k^*+1)^2} + \frac{1}{k^*+2} - \frac{k^*}{(k^*+2)^2} + \dots + \frac{1}{N-1} - \frac{k^*}{(N-1)^2} \\
& = \frac{1}{k^*+1}(1 - \underbrace{\frac{k^*}{k^*+1}}_{<1}) + \frac{1}{k^*+2}(1 - \underbrace{\frac{k^*}{k^*+2}}_{<1}) + \dots + \frac{1}{N-1}(1 - \underbrace{\frac{k^*}{k^*+1}}_{<1}) \\
& = 0 \textit{ only} \text{ when } k^* \text{ and } N \text{ are very large, if at all.}
\end{aligned}$$

Thus, consider the case where N is large, and construct an integral analogous to our discrete formula for $\mathbb{P}$.

Let $m = \frac{k^*}{N}$, the ratio between k and N for a particular setup. We'll try to maximize $\mathbb{P}$ in terms of this ratio:

$$\sum_{i=1}^{N-k} \frac{k}{N(k+i-1)} = m \sum_{i=1}^{N-k} \frac{1}{k+i-1}$$

Additionally, let $x = \frac{k+i-1}{N}$. Then we can have $dx = \frac{1}{N}$, since N is small.

$$m \sum_{i=1}^{N-k} \frac{1}{k+i-1} = m \sum_{i=1}^{N-k} \frac{1}{x} dx = m \int_D \frac{1}{x} dx$$

To find a suitable domain D, set $x_A = m \implies \frac{k+i-1}{N} = \frac{k}{N} \implies i = 1$, which is the desired lower index.

For an upper bound, have $x_B = 1 - dx = \frac{k+i-1}{N} \implies N - Ndx = k + i - 1 = N - 1 \implies i = N - k$, as desired.

$$m \int_m^{1-\varepsilon} \frac{1}{x} dx = m \ln(x) \Big|_m^{1-\varepsilon} = m \ln(1-\varepsilon) - m \ln(m) = -m \ln(m)$$

Thus, for a ratio $\frac{k^*}{N}$, we can approximate the associated probability as $-m\ln(m)$. To maximize, we take the derivative and set equal to 0:

is automatic, since
N to be large to be-

$$-m\frac{1}{m} - \ln(m) = -1 - \ln(m) = 0 \implies m = \boxed{e^{-1} = \frac{k^*}{N} \text{ as } N \to \infty}$$

Thus, for any given N, the ideal k^* is $\lfloor Ne^{-1} \rfloor$. Using $\star$, one could also express k^* using harmonic numbers:

$$k^* = \frac{H_{N-1} - H_{k-1}}{H_{N-1}^2 - H_{k-1}^2}$$

where $H_A^B = \sum_{i=1}^{A} \frac{1}{i^B}$

Question 3

Note that the probability of flipping heads-then-tails is the same as flipping tails-then-heads: $p(1-p) = (1-p)p$. Also note that, in a sequence of infinite coin tosses, the probability of achieving a *fixed* $\omega = \{\omega_1, \omega_2, ...\}$ is exactly 0 (analogous to this setup is the probability of pinning a dart at *exactly* a given radius, or choosing $c \in \mathbb{R}$ in the interval $a < b$; one could think of an infinite sequence of coin tosses as a binary expansion of a particular real number).

$\implies$ in a given infinite sequence of coin tosses, one is guaranteed to encounter both combinations HT and TH, each of which is equally likely. Thus, the odds that one comes across HT first is $\frac{1}{2}$, and the odds one comes across TH first is $\frac{1}{2}$.

Or alternativly, the
ity that the *second*
like tosses is HT =
or third, or fourth,
biased coin there
potential for infini
fair coins!

QUESTION 4

Let $y(X) = u(X) + u(X)$. Since both u and v are well defined from $\mathbb{R} \to \mathbb{R}$, y is too.

Consider the case where X is discrete. We have:

$$\begin{aligned}
\mathbb{E}[u(X) + v(X)] &= \mathbb{E}[y(X)] \\
&= \sum_{x \in S} y(x)\mathbb{P}(X = x) \\
&= \sum_{x \in S} [u(x) + v(x)]\mathbb{P}(X = x) \\
&= \sum_{x \in S} u(x)\mathbb{P}(X = x) + \sum_{x \in S} v(x)\mathbb{P}(X = x) \\
&= \mathbb{E}[u(X)] + \mathbb{E}[v(X)]
\end{aligned}$$

Now let X be a continuous r.v. with PDF $f : \Omega \to \mathbb{R}$.

$$\begin{aligned}
\mathbb{E}[u(X) + v(X)] &= \mathbb{E}[y(X)] \\
&= \int_{\mathbb{R}} y(X)f(X) \\
&= \int_{\mathbb{R}} [u(X) + v(X)]f(X) \\
&= \int_{\mathbb{R}} u(X)f(X) + v(X)f(X) \\
&= \int_{\mathbb{R}} u(X)f(X) + \int_{\mathbb{R}} v(X)f(X) \\
&= \mathbb{E}[u(X)] + \mathbb{E}[v(X)]
\end{aligned}$$

following linearity of integration.

Question 5

From the previous homework, recall the probability that *exactly* k people receive the correct hat:

$$\mathbb{P}(X = k) = \sum_{j=0}^{N-k} \frac{(-1)^j}{j!k!}$$

Note that X is a discrete random variable, with $S := \mathbb{N} \cup 0$ being the possible values it can take. We have that

Since the question unclear about it, I a ering both infinite hatchecks.

$$\mathbb{E}X = \sum_{k \geq 0} k\mathbb{P}(X = k) = \sum_{k \geq 0} k \sum_{j \geq 0} \frac{(-1)^j}{j!k!}$$

Again from the previous homework, the second summation is equivalent to $\frac{e^{-1}}{k!}$

$$\mathbb{E}X = e^{-1} \sum_{k \geq 0} \frac{k}{k!}$$

Consider the series expansion of $\frac{k}{k!}$ for $k \geq 0$: $1 + \frac{2}{2!} + \frac{3}{3!} + \ldots$

$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = e^1$ by Taylor series

$\implies \mathbb{E}X = e^{-1}e = 1$, so we can expect, interestingly, 1 person to receive their hat in an infinitely huge room of hat-checkers. Lucky guy.

Alternatively, if we take $X = k$ to be the event that k people get their hats in a *finite* scenario, then we can decompose X in much the same way we do $B \sim \text{Bin}(n, p)$, with X_i "heads" if person i gets his hat, "tales" if he doesn't. Denote these events with a 1 or 0. Then $X = X_1 + X_2 + \ldots + X_N$ and $\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \ldots + \mathbb{E}X_N$ by linearity of expectation. For any X_i, we have $\mathbb{E}X_i = \sum_{k=0 \text{ and } 1} k\mathbb{P}(X_i = k) = 0 + 1\mathbb{P}(X_i \text{ gets hat}) = \frac{1}{N}$. Thus $\mathbb{E}X = N(\frac{1}{N}) = 1$ once again. $\square$