ASSIGNMENT 4

MATH 356

QUESTION 1

Part (a): Let $f_X = \frac{1}{2}e^{-|x|}$. Then we have

$$M(t) = \int_{\mathbb{R}} e^{tx}(\frac{1}{2}e^{-|x|})dx$$
$$= \frac{1}{2}\left(\int_{-\infty}^{0} e^{tx+x} + \int_{0}^{\infty} e^{tx-x}\right) = \frac{1}{2}\left(\int_{-\infty}^{0} e^{x(t+1)} + \int_{0}^{\infty} e^{x(t-1)}\right)$$
$$= \frac{1}{2}\left(\frac{1}{t+1}e^{x(t+1)}\Big|_{-\infty}^{0} + \frac{1}{t-1}e^{x(t-1)}\Big|_{0}^{\infty}\right)$$

The first term is finite exactly when $t > -1$, and the second term is finite when $t < 1$. Thus, we require that $|t| < 1$. When this is true, we have:

$$\frac{1}{2}\left(\frac{1}{t+1}e^{x(t+1)}\Big|_{-\infty}^{0} + \frac{1}{t-1}e^{x(t-1)}\Big|_{0}^{\infty}\right) = \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right) = \frac{1}{2}\left(\frac{-2}{t^2-1}\right) = \frac{1}{1-t^2}$$

Thus, $M_X(t) = \frac{1}{1-t^2}$ for $|t| < 1$, and ∞ otherwise.

For the first term: if $t = -1$, $\frac{1}{t+1}$ diverges, and if $t < -1$, then $(t+1)$ is negative, and thus $e^{x(t+1)}$ will evaluate to infinity at the lower bound.

For the second term: let $t = 1$. Then $\frac{1}{t-1}$ is divergent. And if $t > 1$, we have that $(t-1)$ is positive, and thus $e^{x(t-1)}$ will evaluate to infinity at our upper bound.

Part (b): Suppose that $|t| < 1$. The derivatives of $\frac{1}{1-t^2}$ will get hairy quick, so consider its Taylor expansion, or, rather that of $\frac{1}{1-t}$.

Since the derivatives of $\frac{1}{1-t}$ are:

$$f' = \frac{1}{(1-t)^2} \quad f'' = \frac{2!}{(1-t)^3} \quad f''' = \frac{3!}{(1-t)^4} \quad \dots \quad f^{(n)} = \frac{n!}{(1-t)^{n+1}}$$

we have that $f^{(n)}(0) = n!$ Thus, $\frac{1}{1-t}$ expanded about 0 is the series

$$1 + t + t^2 + t^3 + \dots = \sum_{n=0}^{\infty} t^n$$

and with $t \to t^2$ we have that $\frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}$. Evaluating at $t = 0$, notice that, when the exponential of t is 0, we yield the one and only non-zero term in the series. The k-th derivative of this final Taylor series is as follows:

$$f^{(k)} = \sum_{n=0}^{\infty} 2n(2n-1)(2n-2)\dots(2n-k+1)t^{2n-k}$$

...as we "pull down" the power of each subsequent derivative. For the t term to be non-zero when evaluated at 0, we require that its power be 0, or $n = \frac{k}{2}$, which only occurs at even derivatives. In fact, if k is odd, then t^{2n-k} will be odd, and thus evaluate to 0 for all powers. Alternatively, the $\frac{k}{2}$-th term will simply be $k(k-1)(k-2)...1 = k!$

In short, we have that $M^{(k)}(0)$ is non-zero for *even* k, where $M^{(k)}(0) = k!$. Otherwise $M^{(k)}(0) = 0$.

We can finally express the n-th moment of X as

$$\mathbb{E}[X^n] = M^{(n)}(0) = \begin{cases} n! & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

QUESTION 2

Part (a): Let $X \sim \mathcal{N}(0,1)$, and let $Y := e^X$. When $y \leq 0$, $\mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = 0$, since e^X does not span $\mathbb{R}^-$. For a given $y > 0$, the set of all $\{x : e^x = y\}$ is just $\ln(y)$, as e^x is a one-to-one mapping. Further, note that $g'(x) = e^{\ln(y)} = y$ is zero nowhere when $y > 0$, so we can apply the following formula:

$$f_Y(y) = \sum_{x:e^x=y} \frac{f_X}{|g'(x)|} = \frac{f_X}{|g'(x)|}\bigg|_{x=\ln(y)} = \frac{e^{\frac{-\ln^2(y)}{2}}}{\sqrt{2\pi}y}$$

since $|g'| = |y| = y$ and $f_X = \varphi(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\bigg|_{x=\ln(y)} = \frac{e^{\frac{-\ln^2(y)}{2}}}{\sqrt{2\pi}}$.

$$\text{In total, we have} \quad f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{e^{\frac{-\ln^2(y)}{2}}}{\sqrt{2\pi}y} & y > 0 \end{cases}$$

Part (b): We've seen from lecture that $M_X(t) = e^{\frac{t^2}{2}}$ for $X \sim \mathcal{N}(0,1)$. In other words, $\mathbb{E}\left[e^{tX}\right] = e^{\frac{t^2}{2}}$.

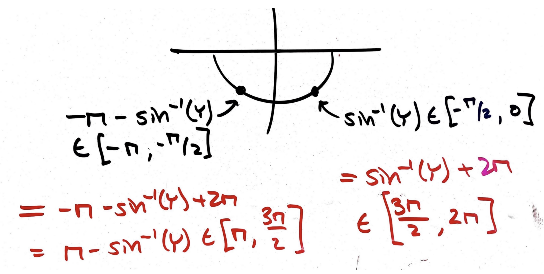
We have that $\mathbb{E}[Y^n]$ is the n-th moment of Y, and this is $\mathbb{E}[(e^X)^n] = \mathbb{E}[e^{nX}] = e^{\frac{n^2}{2}}$ from above.

$$\implies \mathbb{E}[Y^n] = e^{\frac{n^2}{2}}$$

Question 3

Let $X \sim \text{Unif}[-\pi, 2\pi]$ and $Y = \sin(X)$. Define the function $\sin^{-1}(y)$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, as usual. We have that $f_X = \frac{1}{2\pi-(-\pi)} = \frac{1}{3\pi}$. Recall:

$$\star \;\; f_Y = \sum \frac{f_X(x)}{|g'(x)|} \text{ over all } x \text{ such that } g(x) = y \;\text{ and }\; g'(x) \neq 0$$



When $|y| > 1$, we have immediately that $\sin(x) \neq y$ for *any* x, and thus $f_Y = 0$ for these values. Suppose now that $y \in [-1, 0]$. On the interval $[-\pi, 2\pi]$, one sees that $\sin^{-1}(y) = -\pi - \sin^{-1}(y) = \pi - \sin^{-1}(y) = \sin^{-1}(y) + 2\pi$.

Thus, when $y \in [-1, 0]$, there exist 4 equivalent solutions to $g(x) = y$ on the interval $x \in [-\pi, 2\pi]$. From $\star$, we have

$$f_Y = \frac{1}{3\pi}\left(\frac{1}{|\cos(\sin^{-1}(y))|} + \frac{1}{|\cos(-\pi-\sin^{-1}(y))|} + \frac{1}{|\cos(\pi-\sin^{-1}(y))|} + \frac{1}{|\cos(\sin^{-1}(y))+2\pi|}\right) = \frac{4}{3\pi\sqrt{1-y^2}}$$

The following calculations led to this answer: first, we know that all the phase shifts given for $\sin^{-1}$ are equivalent, so the expression is really $\frac{4}{3\pi}\left(\frac{1}{|\cos(\sin^{-1}(y))|}\right)$. Then, for a unit triangle, $\cos(\sin^{-1}(y))$ is $\frac{a}{1}$, where $\sin^{-1}(y)$ is the angle θ at which $\sin(\theta) = \frac{y}{1}$, i.e. the angle adjacent to a. Thus, since $a^2 + y^2 = 1$, we have $a = \cos(\sin^{-1}(y)) = \sqrt{1-y^2}$. We conclude that $f_Y = \frac{4}{3\pi\sqrt{1-y^2}}$ for $y \in [-1, 0]$.

Now consider $y \in (0, 1]$. We have that, for $\sin^{-1}(y) : \mathbb{R}^+ \to (0, \frac{\pi}{2}]$, $\sin^{-1}(y) = \pi - \sin^{-1}(y)$, where the latter term has range $[\frac{\pi}{2}, \pi)$. Notice, however, that either of these expressions $\pm 2\pi$ exits our allowed domain of $[-\pi, 2\pi]$. Thus, we have the only two $x : g(x) = y$, and by $\star$:

The "picture" is sin reflected over the positive y-axis.

$$f_Y = \frac{1}{3\pi}\left(\frac{1}{|\cos(\sin^{-1}(y))|} + \frac{1}{|\cos(\pi - \sin^{-1}(y))|}\right) = \frac{2}{3\pi\sqrt{1-y^2}} \quad \text{as above}$$

Thus, the complete PDF of Y is the following:

$$f_Y(y) = \begin{cases} \frac{4}{3\pi\sqrt{1-y^2}} & y \in [-1, 0] \\ \frac{2}{3\pi\sqrt{1-y^2}} & y \in (0, 1] \\ 0 & \text{otherwise} \end{cases}$$

QUESTION 4

Consider the standard deviation $\sigma = \sqrt{\text{Var}(X)}$. This is defined in $\mathbb{R}$ for all random variables. Thus, $\text{Var}(X) \geq 0$, or $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 \geq 0 \implies \mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$ for all variables X.

Let $M_X(1) = 3$ and $M_X(2) = 4$. Then $\mathbb{E}\left[e^X\right] = 3$ and $\mathbb{E}\left[e^{2X}\right] = 4$. Define $Y := e^X$. This is well defined for any variable X.

We then have that $\mathbb{E}[Y] = 3 \implies (\mathbb{E}[Y])^2 = 9$ and $\mathbb{E}[Y^2] = 4$.

Thus, $\mathbb{E}[Y^2] < (\mathbb{E}[Y])^2 \lightning$, and we are done.

Question 5

Let $\mathbb{P}(X \leq x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$

Considering, alone, the "continuous part" of F_X, we have $f_X = \frac{d}{dx}x = 1$ for $0 \leq x < \frac{1}{2}$. For $x < 0$, it's clear to see that $\rho_X(x) = 0$, the CMF of the "discrete part."

F_X implies that the sum of probabilities over $x \in \left[0, \frac{1}{2}\right]$ is 1. The continuous part contributes $\int\limits_0^{1/2} dx = \frac{1}{2}$, and thus $\mathbb{P}(X = \frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$.

Consider $\mathbb{E}[e^{tX}]$. For x defined on $[0, \frac{1}{2})$, this is

$$\int\limits_0^{1/2} e^{tx} dx = \frac{1}{t} e^{tx}\Big|_0^{1/2} = \frac{e^{t/2} - 1}{t}$$

When $x = \frac{1}{2}$, we have $\mathbb{E}[e^{tX}] = \sum\limits_{x=1/2} e^{tx}\mathbb{P}(X = x) = \frac{e^{t/2}}{2}$

Over the entire domain of X, then, we have

$$\mathbb{E}\left[e^{tX}\right] = \frac{e^{t/2} - 1}{t} + \frac{e^{t/2}}{2}$$

QUESTION 6

Consider $X \sim \text{Geom}(p)$ and $Y \sim \text{Poi}(\lambda)$. We have

$$\mathbb{P}(X = k) = (1-p)^{k-1} p \ \forall k \geq 1 \quad \text{and} \quad \mathbb{P}(Y = k) = \frac{e^{-\lambda}\lambda^k}{k!} \ \forall k \geq 0$$

Now let $Y \to Y+1$ be an affine transformation. $f_{Y+1} = (1) f_Y(k-1) = \frac{e^{-\lambda}\lambda^{k-1}}{k-1!} \ \forall k \geq 1$. Then we have that $\mathbb{P}(Y+1 = X)$ is the sum of probabilities that $\mathbb{P}(X = k)$ and $\mathbb{P}(Y+1 = k)$ for all valid k, as X and Y are independent.

$$\begin{aligned}\mathbb{P}(Y+1 = X) &= \sum_{k\geq 1} \frac{(1-p)^{k-1} p e^{-\lambda} \lambda^{k-1}}{(k-1)!} \\ &= pe^{-\lambda} \sum_{k\geq 1} \frac{(\lambda - \lambda p)^{k-1}}{(k-1)!} = pe^{-\lambda} \sum_{k\geq 0} \frac{(\lambda - \lambda p)^k}{k!} \\ &= e^{\lambda - p\lambda} p e^{-\lambda} = e^{-p\lambda} p\end{aligned}$$

Question 7

Suppose we sample with replacement from an urn of 4 green, 3 yellow, and 2 white balls. Let $N := \{\#\text{ of draws until one sees a green or yellow}\}$. Since we're sampling without replacement, the probability that we pick a green or yellow is constant $\frac{7}{9}$. Thus, N is a geometric r.v. with

$$\mathbb{P}(N = k_1) = \left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right)$$

After one stops picking, there is $\mathbb{P} = \frac{4}{7}$ that the ball is green and $\mathbb{P} = \frac{3}{7}$ the ball is yellow. Thus, Y as defined in the question has probability

$$\mathbb{P}(Y = k_2) = \begin{cases} \frac{4}{7} & k_2 = 1 \\ \frac{3}{7} & k_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Combining, the joint probability function is

$$\rho_{N,Y}(k_1, k_2) = \begin{cases} \left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right)\left(\frac{4}{7}\right) & k_2 = 1 \\ \left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right)\left(\frac{3}{7}\right) & k_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

For N, the marginal probability function is given by summing all possible values of Y, i.e., $Y = \{1, 2\}$:

$$\rho_N(k_1) = \left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right)\left(\frac{4}{7}\right) + \left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right)\left(\frac{3}{7}\right) = \left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right)$$

For Y, its MPF is given by summing over possible values of N, i.e. $\mathbb{N}$. Note that, as in its joint function, we need to separate this into cases for k_2:

$$\rho_Y(k_2) = \begin{cases} \frac{4}{7}\sum\limits_{k_1 \geq 1}\left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right) & k_2 = 1 \\ \frac{3}{7}\sum\limits_{k_1 \geq 1}\left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right) & k_2 = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{4}{7} & k_2 = 1 \\ \frac{3}{7} & k_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Since $\sum\limits_{k_1 \geq 1}\left(\frac{2}{9}\right)^{k_1-1}\left(\frac{7}{9}\right) = \frac{7}{9}\left(\frac{1}{1-\frac{2}{9}}\right) = 1$ by geometric series.

Now consider the event $\{N = k \text{ and } Y = k\}$. From above, we can express $\mathbb{P}(N = k, Y = k) = \rho_{N,Y}(k,k)$.

When $k \leq 0$ and $k > 2$, this probability is 0, since one cannot choose a ball before the game starts, and Y is not defined for $k > 2$. Thus, $\rho_{N,Y}(k,k) = \rho_N(k)\rho_Y(k) = 0$ for these values.

When $k = 1$, we have $\rho_{N,Y}(1,1) = \left(\frac{2}{9}\right)^0\left(\frac{7}{9}\right)\left(\frac{4}{7}\right) = \left(\frac{7}{9}\right)\left(\frac{4}{7}\right) = \rho_N(1)\rho_Y(1) = \left(\frac{7}{9}\right)\left(\frac{4}{7}\right)$ using the expressions defined above.

When $k = 2$, we have $\rho_{N,Y}(2,2) = \left(\frac{2}{9}\right)\left(\frac{7}{9}\right)\left(\frac{3}{7}\right) = \left(\frac{14}{81}\right)\left(\frac{3}{7}\right) = \rho_N(2)\rho_Y(2) = \left(\frac{14}{81}\right)\left(\frac{3}{7}\right)$ using the expressions defined above.

Thus, over all integers k, $\rho_{N,Y}(k,k) = \rho_N(k)\rho_Y(k)$, and thus

$$\mathbb{P}(N = k, Y = k) = \mathbb{P}(N = k)\mathbb{P}(Y = k)$$

so N and Y are independent.

And yes, the PMF of Y makes sense, since we are essentially flipping a coin with bias $p = \frac{3}{7}$.