VECTOR CALCULUS NOTES

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CONTENTS

I Curves and Surfaces

PRODUCTS ON VECTOR SPACES

Recall the definition of the *inner product* over a vector space V: DEF 1.1

1. $\langle u, v\rangle = \overline{\langle v, u\rangle} = \langle v, u\rangle$ in $\mathbb{R}$ (where we'll be in this class)
2. $\langle au + bw, v\rangle = a\langle u, v\rangle + v\langle w, v\rangle$
3. $\langle u, u\rangle \geq 0$, and $= 0 \iff u = \mathbb{0}$

From this, we define the *norm* of $u \in V$ to be $\|u\| := \sqrt{\langle u, u\rangle}$. This is well-defined, since $\langle u, u\rangle \geq 0$. DEF 1.2

$$\forall u, v \in V, |\langle u, v\rangle| \leq \|u\|\|v\|$$

PROP 1.1 Cauchy-Schwartz Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

PROP 1.2 Triangle Inequality

The *cross product* of $u, v \in \mathbb{R}$, with respect to $\mathbb{R}^3$, is the determinate of the following "matrix": DEF 1.3

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where $u = \langle u_1, u_2, u_3\rangle$ and $v = \langle v_1, v_2, v_3\rangle$. We observe the following two properties of the cross product in $\mathbb{R}^3$: PROP 1.3

1. $(u \times v) \cdot u = 0$
2. $\|u \times v\| = \|u\|\|v\|\sin(\theta)$, where θ is the angle found between u and v. A conceptualization of this property is that "u-cross-v is equal to the area created by the parallelogram bounded by u and v."

LINES

Define a *line* $l(t) \in \mathbb{R}^n$ to be a function from $\mathbb{R} \to \mathbb{R}^n$, with the primary form $l(t) = P + td$, with $P, d \in \mathbb{R}^n$, $t \in \mathbb{R}$. We call P the "point vector" and d the "direction vector" An alternate form, with two points $P, Q \in \mathbb{R}^n$, would be $l(t) = (1-t)P + tQ$, where $l(t)$ lies along the path between P and Q for $t \in [0, 1]$. DEF 1.4

Distance between a point and line Using this definition, how an we find the shortest path between a point R and a line $l(t)$, which lies between P and Q?

Idea 1 We know the desired vector $w = PR\sin(\theta)$, the angle between PR and PQ. To find this value, note that $\|PR \times PQ\| = \|PR\|\|PQ\|\sin(\theta)$.

Idea 2 We can project R onto PQ, and then subtract this projection from PR.

Idea 3 We can minimize a distance function between R and a point on l, i.e. $l(t)$. Thus, we take $\min_{t\in\mathbb{R}} \|R - l(t)\| = \alpha$, and then take $Rl(\alpha)$ to be the shortest path.

Idea 4 We can find when $(R - l(t)) \cdot d = 0$.

Distance between 2 lines Consider two lines, l_1 and l_2, which do not intersect but are not necessarily parallel. What is the minimal distance between l_1 and l_2?

Sometimes called "skew lines"

Idea 0 Conceptualize this problem as finding the distance between the parallel planes defined by $\{l_1, l_2\}$.

Idea 1 We can minimize $\|l_1(t) - l_2(s)\|$ (really, one should minimize the square to make one's life easier).

Idea 2 Pick any two points, say $l_1(T)$ and $l_2(S)$, and project $l_1(T)l_2(S)$ onto $l_1 \times l_2$.

Idea 3 Minimize $\mathrm{dist}(l_1(t), l_2)$ for fixed t.

Idea 4 Find t and s such that $[l_1(t) - l_2(s)] \cdot \vec{d_1} = 0$ and $[l_1(t) - l_2(s)] \cdot \vec{d_2} = 0$

PROP 1.4 $\|u \times v\| = \|u\|\|v\|\sin(\theta) =$ Area of parallelogram defined by u and v.

PLANES

DEF 1.5 A plane $r(s,t)$ is a function $[0,1]^2 \to \mathbb{R}^3$ defined by $d_1, d_2 \in \mathbb{R}^3$, two vectors, and $P \in \mathbb{R}^3$, a point. In particular, $r(s,t) = P + s\vec{d_1} + t\vec{d_2}$. This is called the *parametric form*.

DEF 1.6 The *point-normal* form is a function $\mathbb{R}^2 \to \mathbb{R}^3$ is given by $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$, where $\vec{n} = \langle a, b, c\rangle$ is a vector normal to the plane, and $P = \langle x_0, y_0, z_0\rangle$ is a point lying on the plane.

Distance between a point R and a plane r

Idea 1 Minimize $\|R - r(s,t)\|$ (or the square)

Idea 2 $\|\mathrm{proj}_{\vec{n}}(P - R)\|$, where $\vec{n}$ and P are as given in the point-normal form.

TRANSFORMATIONS AND PARAMETERIZATIONS

The following table give general examples of linear transformations $\lambda : \mathbb{R}^n \to \mathbb{R}^m$.

Dimension	Linear	Affine
$n = 0$	$\lambda(0) = 0$	$\lambda(0) = P$
$n = 1$	$\lambda(t) = t\vec{d}$	$\lambda(t) = P + t\vec{d}$
$n = 2$	$\lambda(t,s) = t\vec{d_1} + s\vec{d_2}$	$\lambda(t,s) = P + t\vec{d_1} + s\vec{d_2}$
$n = 3$	$\lambda(t,s,r) = t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$	$\lambda(t,s,r) = P + t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$

We also define the following important curves in $\mathbb{R}^2$:

Type	Explicit Form	Parametric Form
Ellipse	$x^2 + y^2 = 1$	$r(t) = \left\langle t, \sqrt{1-t^2} \right\rangle_{t\in[-1,1]} = \langle \cos(t), \sin(t) \rangle_{t\in[-\pi,\pi]}$
Hyperbola	$x^2 - y^2 = 1$	$r(t) = \left\langle \sqrt{1+t^2}, t \right\rangle_{t\in\mathbb{R}} = \langle \cosh(t), \sinh(t) \rangle_{t\in\mathbb{R}}$
Parabola	$x = y^2$	
Double Cone	$x^2 = y^2$	
Any Function	$y = F(x)$	$r(t) = \langle t, F(t) \rangle$

Define a *path* in $\mathbb{R}^m$ to be a continuous function $r : \mathbb{R} \to \mathbb{R}^m$, e.g. $[a, b] \to \mathbb{R}^m$. DEF 1.7

Define a *curve* in $\mathbb{R}^m$ to be the image of a path (i.e. a set of points in $\mathbb{R}^m$). Recall the statement "paths parameterize curves." DEF 1.8

For example, the unit circle $x^2 + y^2 = 1$ is parameterized by the path $r : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = \langle \cos(t), \sin(t) \rangle$.

Define the *tangent* line of $\vec{r}$ at $a \in \mathbb{R}$ to be an affine transformation $l : \mathbb{R} \to \mathbb{R}^m$ satisfying the following:

1. $l(t) = r(a) + (t-a)\vec{d} : \vec{d} \neq 0$
2. $\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} = 0$

♠ *Examples* ♣

We'll now find the derivative of the unit circle at a point $a \in \mathbb{R}$: we have $r(a) = \langle \cos(a), \sin(a) \rangle$. Thus:

$$l(t) = \langle \cos(t), \sin(t) \rangle + (t-a) \langle d_1, d_2 \rangle$$

Where $\langle d_1, d_2 \rangle \neq 0$. Consider now the limit in question 2:

$$\begin{aligned}
\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} &= \lim \frac{1}{|t-a|} \sqrt{(\cos(t) - \cos(a) - (t-a)d_1)^2 + (\sin(t) - \sin(a) - (t-a)d_2)^2} \\
&= \lim \sqrt{\left(\frac{\cos(t)-\cos(a)}{t-a} - d_1\right)^2 + \left(\frac{\sin(t)-\sin(a)}{t-a} - d_2\right)^2} \\
&\underset{t\to a}{=} \sqrt{(-\sin(a) - d_1)^2 + (\cos(a) - d_2)^2} = 0 \\
&\iff d_1 = -\sin(a) \land d_2 = \cos(a) \\
&\implies l(t) = \langle -\sin(a), \cos(a) \rangle \quad \square
\end{aligned}$$

Frequently, $l(t)$ is referred to as the "velocity vector" of $r(t)$, and is notated as $r'(t)$. Notice that $r'(t)$ is equivalent to the component-wise derivative of the coordinates of $r(t)$ w.r.t. t. Formally:

DEF 1.9 Given $\vec{r}:\mathbb{R}\to\mathbb{R}^n$, the *derivative* of $\vec{r}$ at $a\in\mathbb{R}$ is a linear transformation $\vec{\lambda}:\mathbb{R}\to\mathbb{R}^n$ satisfying

$$\lim_{t\to a}\frac{\|r(t)-r(a)-\lambda(t-a)\|}{|t-a|}=0 \quad \text{or equivalently} \quad \lim_{h\to 0}\frac{\|r(a+h)-r(a)-\lambda(a)\|}{|h|}=0$$

It is denoted $D\vec{r}_a$, and represented by the $n\times 1$ matrix $r'(a)$. One may now rewrite the tangent line in the form $l(t)=r(a)+\lambda(t-a)$.

DEF 1.10 The *arc length* of a curve $r(t)$ is given by

$$s=\int_a^b \|r'(t)\|dt$$

DEF 1.11 An *arc length parameterization* of $r(t)$ is some $t=\alpha(s)$ such that $r(\alpha(s))$ has a unit velocity vector, i.e. $\|r'(\alpha(s))\|=1$. Alternatively, one could find an expression for arc length, and then parameterize $r(t)$ in terms of its arc length. The resultant will be equivalent.

♠ *Examples* ♣

We'll do an arc length parameterization of a semicircle of radius 1 with its center at the origin, i.e. $y=\sqrt{1-x^2}$. We get the natural parameterization $r(t)=\left\langle t,\sqrt{1-t^2}\right\rangle$, where $t\in[-1,1]$. We'd like to find a change of parameters $t=\alpha(s)$ such that $\|r(\alpha(s))\|=1$ and $\alpha'\geq 0$.

$$\begin{aligned}
r(\alpha(s)) &= \left\langle \alpha(s),\sqrt{1-\alpha(s)^2}\right\rangle\\
r'(\alpha(s)) &= \left\langle \alpha'(s),\frac{1}{2}(1-\alpha(s)^2)^{-\frac{1}{2}}\cdot(-2\alpha(s)\alpha'(s))\right\rangle\\
&= \alpha'(s)\left\langle 1,\frac{-\alpha(s)}{\sqrt{1-\alpha(s)^2}}\right\rangle\\
\text{Then } 1=\|r'(\alpha(s))\| &= \alpha'(s)\sqrt{1+\frac{\alpha(s)^2}{1-\alpha(s)^2}}\\
&= \frac{\alpha'(s)}{\sqrt{1-\alpha(s)^2}}
\end{aligned}$$

Integrating with respect to s, we get $s=\arcsin(\alpha(s))=\arcsin(t)$. Thus, $t=\sin(s)$, and $s\in[\frac{-\pi}{2},\frac{\pi}{2}]$, and we yield the parameterization $\langle\sin(s),\cos(s)\rangle : s\in[\frac{-\pi}{2},\frac{\pi}{2}]$.

SURFACES

We note the following quadric surfaces:

Type	Explicit Form
Ellipsoid	$x^2 + y^2 + z^2 = 1$
Elliptic Hyperboloid	$x^2 + y^2 - z^2 = 1$
Elliptic Paraboloids	$x^2 + y^2 - z^2 = -1$
Hyperbolic Paraboloids	$x = y^2 - z^2$
Double Cones	$x^2 = y^2 + z^2$

A surface $F(x, y)$ is called *differentiable* at (a, b) if there exists some linear transformation $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that DEF 1.12

$$\lim_{(h,k)\to(0,0)} \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k\rangle\|}$$

One may represent $\lambda(h, k) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = uh + vk$

♠ *Examples* ♣

Let $F(x, y) = xy$. We consider F at (a, b). Then

$$\begin{aligned}
0 \le \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k\rangle\|} &= \frac{|(a+h)(b+k) - ab - (uk + vk)|}{\|\langle h, k\rangle\|} \\
&= \frac{|bh + ak + hk - uh - vk|}{\|\langle h, k\rangle\|} = \frac{|(b-u)h + (a-v)k + hk|}{\|\langle h, k\rangle\|} \\
&\le \frac{|b-u||h|}{|h|} + \frac{|a-v||k|}{|k|} + \frac{|h||k|}{|h|} \quad \text{since } |h|, |k| \le \|\langle h, k\rangle\| \\
&= |b-u| + |a-v| + |k| \to |b-u| + |a-v| \\
&= 0 \quad \text{when } b = u, a = v
\end{aligned}$$

Thus, the desired limit is always $\geq$ and ≤ 0, so especially it is 0. Our derivative at (a, b) is then $\lambda(x, y) = bx + ay$.

One may also find these coefficients as the partial derivative of F, i.e.

$$\nabla F(a, b) = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle\Bigg|_{(a,b)}$$

This is called the *gradient*. Similarly, $\alpha(x, y) = F(a, b) + \lambda(x - a, y - b)$ is called the *affine approximation* at (a, b). DEF 1.13

If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then all partial derivatives of F at $\vec{a}$ exist. Furthermore, $\lambda(\vec{a}) = F'(\vec{a}) = \begin{bmatrix} \partial_1 F \cdots \partial_n F \end{bmatrix}\Big|_{\vec{a}}$. PROP 1.5

Note that the converse is *false* (as a counterexample, see $F = \sqrt{|xy|}$)

1.1 Partial Converse

If all partial derivatives of $F : \mathbb{R}^n \to \mathbb{R}$ exist near $\vec{a}$ and are continuous at $\vec{a}$, then F is differentiable at $\vec{a}$.

PROOF FOR $n = 2$.

Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a linear transformation defined by $\left[\partial_1 F \cdots \partial_n F\right]\Big|_{\vec{a}}$. Then

$$\lambda(\vec{h}) = \sum_{i=1}^{n} \partial_i F(\vec{a}) h_i$$

Let $n = 2$. Then

$$\begin{aligned}
|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})| &= |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) + F(a_1 + h_1, a_2) \\
&\quad - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1 - \partial_2 F(\vec{a}) h_2| \\
&\leq |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) - \partial_2 F(\vec{a}) h_2| \\
&\quad + |F(a_1 + h_1, a_2) - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1| \\
&= |\partial_2 F(\vec{c}) h_2 - \partial_2 F(\vec{a}) h_2| + |\partial_1 F(\vec{d}) h_1 - \partial_1 F(\vec{a}) h_1| \\
&\quad \text{by mean value thm.} \\
&= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})||h_2| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})||h_1| \\
\frac{|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})|}{||\vec{h}||} &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{||\vec{h}||} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{||\vec{h}||} \\
&\leq |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{|h_2|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{|h_1|} \\
&\quad \text{since } |h_i| < ||\vec{h}|| \\
&= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})|
\end{aligned}$$

Then, as $\vec{h} \to 0$, $\vec{c}, \vec{d} \to \vec{a}$. Since F, is continuous, we know $F(\vec{c}) \to F(\vec{a})$ and similarly for $F(\vec{d})$. Thus,

$$|\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \to |\partial_2 F(\vec{a}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{a}) - \partial_1 F(\vec{a})| = 0$$

And we conclude that the limit, as $\leq$ and ≥ 0, is 0. □

DEF 1.14

$F : \mathbb{R}^n \to \mathbb{R}$ is called C^1 continuous (or *continuously differentiable*) at $\vec{a}$ if all partial exists near $\vec{a}$ and are continuous at $\vec{a}$.

Note that the converse to our partial converse is *not* true: i.e. if F is differentiable at $\vec{a}$, it is not necessarily continuously differentiable at $\vec{a}$. Some counter examples include $F(x, y) = |y|$ and $F(x) = x^2 \sin(\frac{1}{x})$ s.t. $x \neq 0$ and 0 otherwise.

We have an alternative and equivalent definition of differentiability. Let E be continuous and $= 0$ at 0. Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a linear transformation. Then PROP 1.6

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + ||\vec{h}||E(\vec{h}) \qquad \forall h$$

implies differentiability.

♠ *Examples* ♣

In our previous example, we prove (laboriously) that $F(x, y) = xy$ is differentiable for all (a, b). We can now use Thm 1.1 to show this result: the partial derivatives $F_x = y$ and $F_y = x$ exist and are continuous $\forall x, y \in \mathbb{R}$, so F is differentiable $\forall x, y \in \mathbb{R}$.

1.2 Characterization of the Derivative

Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$. The derivative at $\vec{a}$ exists if:

1. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\lim_{\vec{h} \to \vec{0}} \frac{||F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})||}{||\vec{h}||} = 0$$

2. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ and a function E such that

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + ||\vec{h}||E(\vec{h})$$

and $E(0) = 0$ is continuous at 0.

Such a λ is unique when found, and is called the derivative. We denote it by $D\vec{F}_{\vec{a}}$.

This follows from Def 1.12 and Thm 1.1. □ PROOF.

We may represent the partial derivatives of $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m = \langle F_1, ..., F_m \rangle$ using a *Jacobian* matrix, denoted $F'(\vec{a})$, and defined as follows: DEF 1.15

$$\left[TBD \right]$$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{a} \in \mathbb{R}^n$. Let $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$. Then PROP 1.7 Chain Rule

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l \quad \text{is differentiable at } \vec{a}$$

and $D\vec{h}_{\vec{a}} = D\vec{g}_{\vec{b}} \circ D\vec{f}_{\vec{a}}$. Furthermore, their Jacobians obey

$$h'(a) = g'(b)f'(a)$$

(matrix multiplication)

♠ *Examples* ♣

1. Consider $f(x, y) = \langle x + y, x - y\rangle$ and $g(x, y) = \frac{1}{4}x^2 - \frac{1}{4}y^2$. Then $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}$ is given by
$$\frac{1}{4}(x + y)^2 - \frac{1}{4}(x - y)^2$$
Let $\vec{a} = \langle a_1, a_2\rangle$. Then $f(a) = b = \langle a_1 + a_2, a_1 - a_2\rangle$. What about the Jacobian of f?
$$f'(a) = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix}\Bigg|_{(a_1,a_2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
Similarly, for g we have
$$g'(b) = \begin{bmatrix} \partial_1 g & \partial_2 g \end{bmatrix}\Big|_{(a_1+a_2,a_1-a_2)} = \begin{bmatrix} \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1 - a_2) \end{bmatrix}$$
Then, by the chain rule, we multiple these two matrices to yield
$$\begin{bmatrix} \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1 - a_2) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$
One can (less) manually find that $h = g \circ f$ is xy, and conclude the same.

2. Let S be a surface in R^3 given by $F(x, y, z) = 0$ (this is called a "level surface," e.g. $xy - z = 0$). Let $P = (a, b, c)$ be a point on F, and let C be a curve in S containing P, parameterized by $r(t)$.

   Denote $r(t) = \langle x(t), y(t), z(t)\rangle$. Then $g = F \circ r = F(x(t), y(t), z(t)) = 0$. By chain rule, we have $0 = g'(t_0) = F'(P) \cdot r'(t_0)$, where we choose t_0 such that $r(t_0) = \langle a, b, c\rangle$. Then, we observe that
$$0 = \nabla F(P) \bullet \vec{v}(t_0) \implies \nabla F(P) \perp \vec{v}(t_0)$$
Where $\vec{v} = r'$ is the velocity vector of r. By considering all curves that satisfy our construction $C \subset S$, we yield the tangent plane of S at P with normal vector $\vec{n} = \nabla F(P)$. In particular, the point-normal form of the tangent plane of a surface F at $P = (a, b, c)$ is given by
$$\boxed{\partial_x F(P)(x - a) + \partial_y F(P)(y - b) + \partial_z F(P)(z - c) = 0}$$

Let $\mathbb{R}^n \to \mathbb{R}, \vec{a}, \vec{h} \in \mathbb{R}^n$. Let $l(t) = a + th$. Then the *directional derivative* of F along h at a is given by $(F \circ l)'(0)$. The chain rule dictates that DEF 1.16

$$\begin{aligned}(F \circ l)'(0) &= F'(a)l'(0) \\ &= \nabla F(a) \cdot h\end{aligned}$$

which is a more useful form.