# VECTOR CALCULUS NOTES

NICHOLAS HAYEK

*Lectures by Prof. Jean Pierre Mutanguha*

CONTENTS

# I Vector Fields

## PRODUCTS ON VECTOR SPACES

Recall the definition of the *inner product* over a vector space $V$:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle} = \langle v, u \rangle$ in $\mathbb{R}$ (where we'll be in this class)
2. $\langle au + bw, v \rangle = a \langle u, v \rangle + v \langle w, v \rangle$
3. $\langle u, u \rangle \geq 0$, and $= 0 \iff u = \mathbb{0}$

From this, we define the *norm* of $u \in V$ to be $\|u\| := \sqrt{\langle u, u \rangle}$. This is well-defined, since $\langle u, u \rangle \geq 0$.

$$\forall u, v \in V, |\langle u, v \rangle| \leq \|u\|\|v\|$$

Prop 1.1
Cauchy-Schwartz Inequality

Prop 1.2
Triangle Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

The *cross product* of $u, v \in \mathbb{R}$, with respect to $\mathbb{R}^3$, is the determinate of the following "matrix":

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$. We observe the following two properties of the cross product in $\mathbb{R}^3$:

Prop 1.3

1. $(u \times v) \cdot u = 0$
2. $\|u \times v\| = \|u\|\|v\| \sin(\theta)$, where $\theta$ is the angle found between $u$ and $v$. A conceptualization of this property is that "$u$-cross-$v$ is equal to the area created by the parallelogram bounded by $u$ and $v$."

## LINES

Define a *line* $l(t) \in \mathbb{R}^n$ to be a function from $\mathbb{R} \to \mathbb{R}^n$, with the primary form $l(t) = P + td$, with $P, d \in \mathbb{R}^n$, $t \in \mathbb{R}$. We call $P$ the "point vector" and $d$ the "direction vector" An alternate form, with two points $P, Q \in \mathbb{R}^n$, would be $l(t) = (1-t)P + tQ$, where $l(t)$ lies along the path between $P$ and $Q$ for $t \in [0, 1]$.

---

**Distance between a point and line** Using this definition, how an we find the shortest path between a point $R$ and a line $l(t)$, which lies between $P$ and $Q$?

*Idea 1* We know the desired vector $w = PR \sin(\theta)$, the angle between $PR$ and $PQ$. To find this value, note that $\|PR \times PQ\| = \|PR\|\|PQ\| \sin(\theta)$.

*Idea 2* We can project $R$ onto $PQ$, and then subtract this projection from $PR$.

*Idea 3* We can minimize a distance function between $R$ and a point on $l$, i.e. $l(t)$. Thus, we take $\min_{t\in\mathbb{R}} \|R - l(t)\| = \alpha$, and then take $Rl(\alpha)$ to be the shortest path.

*Idea 4* We can find when $(R - l(t)) \cdot d = 0$.

**Distance between 2 lines** Consider two lines, $l_1$ and $l_2$, which do not intersect but are not necessarily parallel. What is the minimal distance between $l_1$ and $l_2$?

Sometimes called "skew lines"

*Idea 0* Conceptualize this problem as finding the distance between the parallel planes defined by $\{l_1, l_2\}$.

*Idea 1* We can minimize $\|l_1(t) - l_2(s)\|$ (really, one should minimize the square to make one's life easier).

*Idea 2* Pick any two points, say $l_1(T)$ and $l_2(S)$, and project $l_1(T)l_2(S)$ onto $l_1 \times l_2$.

*Idea 3* Minimize $\mathrm{dist}(l_1(t), l_2)$ for fixed $t$.

*Idea 4* Find $t$ and $s$ such that $[l_1(t) - l_2(s)] \cdot \vec{d_1} = 0$ and $[l_1(t) - l_2(s)] \cdot \vec{d_2} = 0$

---

PROP 1.4 $\|u \times v\| = \|u\|\|v\| \sin(\theta) =$ Area of parallelogram defined by $u$ and $v$.

## PLANES

A plane $r(s,t)$ is a function $[0,1]^2 \to \mathbb{R}^3$ defined by $d_1, d_2 \in \mathbb{R}^3$, two vectors, and $P \in \mathbb{R}^3$, a point. In particular, $r(s,t) = P + s\vec{d_1} + t\vec{d_2}$. This is called the *parametric form*.

The *point-normal* form is a function $\mathbb{R}^2 \to \mathbb{R}^3$ is given by $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$, where $\vec{n} = \langle a, b, c\rangle$ is a vector normal to the plane, and $P = \langle x_0, y_0, z_0\rangle$ is a point lying on the plane.

---

**Distance between a point $R$ and a plane $r$**

*Idea 1* Minimize $\|R - r(s,t)\|$ (or the square)

*Idea 2* $\|\mathrm{proj}_{\vec{n}}(P - R)\|$, where $\vec{n}$ and $P$ are as given in the point-normal form.

---

## TRANSFORMATIONS AND PARAMETERIZATIONS

The following table give general examples of linear transformations $\lambda : \mathbb{R}^n \to \mathbb{R}^m$.

| Dimension | Linear | Affine |
|---|---|---|
| $n = 0$ | $\lambda(0) = 0$ | $\lambda(0) = P$ |
| $n = 1$ | $\lambda(t) = t\vec{d}$ | $\lambda(t) = P + t\vec{d}$ |
| $n = 2$ | $\lambda(t,s) = t\vec{d_1} + s\vec{d_2}$ | $\lambda(t,s) = P + t\vec{d_1} + s\vec{d_2}$ |
| $n = 3$ | $\lambda(t,s,r) = t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$ | $\lambda(t,s,r) = P + t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$ |

We also define the following important curves in $\mathbb{R}^2$:

| Type | Explicit Form | Parametric Form |
| --- | --- | --- |
| Ellipse | $x^2 + y^2 = 1$ | $r(t) = \left\langle t, \sqrt{1-t^2} \right\rangle_{t\in[-1,1]} = \langle \cos(t), \sin(t)\rangle_{t\in[-\pi,\pi]}$ |
| Hyperbola | $x^2 - y^2 = 1$ | $r(t) = \left\langle \sqrt{1+t^2}, t \right\rangle_{t\in\mathbb{R}} = \langle \cosh(t), \sinh(t)\rangle_{t\in\mathbb{R}}$ |
| Parabola | $x = y^2$ | |
| Double Cone | $x^2 = y^2$ | |
| Any Function | $y = F(x)$ | $r(t) = \langle t, F(t)\rangle$ |

Define a *path* in $\mathbb{R}^m$ to be a continuous function $r : \mathbb{R} \to \mathbb{R}^m$, e.g. $[a, b] \to \mathbb{R}^m$.

Define a *curve* in $\mathbb{R}^m$ to be the image of a path (i.e. a set of points in $\mathbb{R}^m$). Recall the statement "paths parameterize curves."

For example, the unit circle $x^2 + y^2 = 1$ is parameterized by the path $r : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = \langle \cos(t), \sin(t)\rangle$.

Define the *tangent* line of $\vec{r}$ at $a \in \mathbb{R}$ to be an affine transformation $l : \mathbb{R} \to \mathbb{R}^m$ satisfying the following:

1. $l(t) = r(a) + (t-a)\vec{d} : \vec{d} \neq 0$
2. $\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} = 0$

♠ *Examples* ♣

We'll now find the derivative of the unit circle at a point $a \in \mathbb{R}$: we have $r(a) = \langle \cos(a), \sin(a)\rangle$. Thus:

$$l(t) = \langle \cos(t), \sin(t)\rangle + (t-a)\langle d_1, d_2\rangle$$

Where $\langle d_1, d_2\rangle \neq 0$. Consider now the limit in question 2:

$$\begin{aligned}
\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} &= \lim \frac{1}{|t-a|}\sqrt{(\cos(t)-\cos(a)-(t-a)d_1)^2 + (\sin(t)-\sin(a)-(t-a)d_2)^2} \\
&= \lim \sqrt{\left(\frac{\cos(t)-\cos(a)}{t-a} - d_1\right)^2 + \left(\frac{\sin(t)-\sin(a)}{t-a} - d_2\right)^2} \\
&\underset{t\to a}{=} \sqrt{(-\sin(a)-d_1)^2 + (\cos(a)-d_2)^2} = 0 \\
&\iff d_1 = -\sin(a) \wedge d_2 = \cos(a) \\
&\implies l(t) = \langle -\sin(a), \cos(a)\rangle \quad \square
\end{aligned}$$

Frequently, $l(t)$ is referred to as the "velocity vector" of $r(t)$, and is notated as $r'(t)$. Notice that $r'(t)$ is equivalent to the component-wise derivative of the coordinates of $r(t)$ w.r.t. $t$. Formally:

Given $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, the *derivative* of $\vec{r}$ at $a \in \mathbb{R}$ is a linear transformation $\vec{\lambda} : \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\lim_{t \to a} \frac{\|r(t) - r(a) - \lambda(t-a)\|}{|t-a|} = 0 \quad \text{or equivalently} \quad \lim_{h \to 0} \frac{\|r(a+h) - r(a) - \lambda(a)\|}{|h|} = 0$$

It is denoted $D\vec{r}_a$, and represented by the $n \times 1$ matrix $r'(a)$. One may now rewrite the tangent line in the form $l(t) = r(a) + \lambda(t-a)$.

The *arc length* of a curve $r(t)$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

## SURFACES

We note the following quadric surfaces:

| Type | Explicit Form |
|---|---|
| Ellipsoid | $x^2 + y^2 + z^2 = 1$ |
| Elliptic Hyperboloid | $x^2 + y^2 - z^2 = 1$ |
| Elliptic Paraboloids | $x^2 + y^2 - z^2 = -1$ |
| Hyperbolic Paraboloids | $x = y^2 - z^2$ |
| Double Cones | $x^2 = y^2 + z^2$ |