

# Stochastic Processes

MATH 447

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## Index of Definitions

We assume working knowledge of probability and no knowledge of measure theory (though a grasp of analysis is essential). See these [MATH 356 notes](#), also taught by Louigi!

# I Markov Chains

Conditional expectations will be important in this course. Recall  $\mathbb{E}[X|Y = y_0]$ , where  $X, Y$  are random variables. If  $Y$  is continuous, writing  $\mathbb{E}[X|Y = y_0] = \frac{\mathbb{P}(X, Y=y_0)}{\mathbb{P}(Y=y_0)}$ , will not work. Instead, we consider the slice of the joint density function  $f(x, y)$  at  $y = y_0$ . The result is a one dimensional function  $g(x)$  which may not have probability 1. Hence, we divide by  $\int g(x)$  to make it into a density function:

$$\mathbb{E}[X|Y = y_0] = \int_{\mathbb{R}} \frac{f(x, y_0)}{\int_{\mathbb{R}} f(x, y_0) dx} x dx$$

DEF 1.1 We frequently write  $f_{X|Y}(x) = f(x, y) / \int_{\mathbb{R}} f(x, y) dx$ , and call this the *conditional density* of  $X$  given  $Y$ . For fixed  $y$ , then,  $\mathbb{E}[X|Y = y] = \mathbb{E}[Z]$ , where  $Z \sim f_{X|Y}$ .

## INTRODUCTION

Before providing definitions, we give some examples of stochastic processes:

**Eg. 1.1** A simple random walk:  $S_{i+1} = S_i + X_i$ , where  $X_i \sim \text{Ber}(p)$  and  $S_0 = 0$ . We might ask: does  $S_i$  ever return to 0, i.e.

$$\mathbb{P}(\exists i > 0 : S_i = 0)$$

**Eg. 1.2** A branching process: as in asexual reproduction, we have an initial node. Each node  $n$  has a number of children  $X_n$ , where  $\frac{X_n}{2} \sim \text{Ber}(p)$ . We denote  $Z_i$  to be the number of individuals in the  $i$ -th generation. We might ask: does  $Z_i$  ever have no children, i.e.

$$\mathbb{P}(\exists i > 0 : Z_i = 0)$$

**Eg. 1.3** Choose  $k$  independent random points in the square  $[0, \sqrt{k}]^2$ . On average, then, there is 1 point within any unit square  $U \subseteq [0, \sqrt{k}]^2$ .

DEF 1.2 Given a finite or countable set  $V$ , a *Markov chain* with *state space*  $V$  is a sequence  $X_n : n \geq 0$  of random variables, with  $X_n \in V$ , such that:

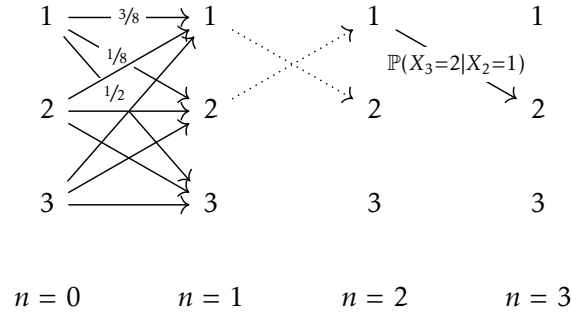
DEF 1.3

$$\underbrace{\mathbb{P}(X_{n+1} = v_{n+1})}_{\text{future}} \mid \underbrace{X_0 = v_0, \dots, X_{n-1} = v_{n-1}}_{\text{past}}, \underbrace{X_n = v_n}_{\text{present}} = \mathbb{P}(X_{n+1} = v_{n+1} \mid X_n = v_n)$$

In other words, the future only depends on the past via the present. This is called the

DEF 1.4 *Markov property*.

Sometimes we allow Markov chains to be only finitely large (i.e.  $0 \leq n \leq m$ ). For instance, we limit ourselves to one weekend of gambling in Las Vegas. A graphical example would look something like:



By repeated Bayes' Law, we observe

PROP 1.1

$$\begin{aligned} & \mathbb{P}(X_1 = v_1, \dots, X_n = v_n | X_0 = v_0) \\ &= \mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_0 = v_0, X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_0 = v_0, \dots, X_{n-1} = v_{n-1}) \\ &= \mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_{n-1} = v_{n-1}) \quad \text{by Markov property} \end{aligned}$$

### TIME-HOMOGENEOUS MARKOV CHAINS

We often write  
THMC

We say that a Markov chain is *time-homogeneous* if, for all  $u, v \in V$  and  $n \geq 0$

DEF 1.5

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u)$$

In other words, the chain's behavior is described entirely by  $\mathbb{P}(X_1 = v | X_0 = u)$  for each  $(v, u) \in V \times V$ . In this case, we can describe such probabilities in a *transition matrix*  $P$ :

DEF 1.6

$$P = (p_{u,v})_{(u,v) \in V^2} = (\mathbb{P}(X_1 = v | X_0 = u))_{(u,v) \in V^2}$$

**Fig. 1.4** Recall the game Snakes and Ladders. A  $6 \times 6$  grid is indexed  $1, \dots, 36$ . Players start at the 1 cell. The game ends when a player reaches the 36 cell. A die roll dictates how many spots one advances. There are some directed edges between cells (increasing: "ladders", decreasing: "snakes"). One must follow these edges when one lands at its tail. Suppose a ladder exists from 11 to 27. Then

$$\mathbb{P}(X_{11} = 27 | X_{10} = 6, X_9 = 3) = \frac{1}{6} = \mathbb{P}(X_{11} = 27 | X_{10} = 6) = \mathbb{P}(X_2 = 27 | X_1 = 6)$$

We see that Snakes and Ladders is naturally modeled as a time-homogeneous Markov chain.

**Fig. 1.5** Sampling without replacement is *not* a Markov chain. If we sample from

$|X| = 10$ , we have

$$\mathbb{P}(X_3 = a | X_2 = b) = 1/9$$

$$\mathbb{P}(X_3 = a | X_2 = b, X_1 = c) = 1/8$$

$$\mathbb{P}(X_3 = a | X_2 = b, X_1 = a) = 0$$

so we do not satisfy the Markov property.

**Eg. 1.6** Returning to the Snakes and Ladders example, consider  $S \subseteq V$ . Let  $T_S = \inf\{n \geq 0 : X_n \in S\}$ , which we call the "*hitting time*" of  $S$ . We may ask...

DEF 1.7

- What is the average number of rounds to finite? We can write this as  $\mathbb{E}[T_{\{36\}} | X_0 = 1]$ .
- What is the probability of landing on 18 or 19 before the game ends? We can write this as  $\mathbb{P}(T_{\{18,19\}} < T_{\{36\}} | X_0 = 1)$ .
- What is the average number of visits to 6 before the game ends? We can write this as

$$\mathbb{E}[\#\{n \in [T_{\{36\}}] : X_n = 6\} | X_0 = 1]$$

- What is the expected proportion of time spent on state 5 before the game ends?
- If we allow two players, what is the probability that player 1 wins? Is this still a Markov chain?

DEF 1.8

A matrix  $P = (p_{u,v})_{(u,v) \in V^2}$  is called a *stochastic matrix* if every row sums to 1, i.e.

$$\forall u \in V, \sum_{v \in V} p_{u,v} = 1$$

Note that any stochastic matrix is the transition matrix of some time-homogeneous Markov chain with state space  $V$  and transition probabilities

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u) = p_{u,v}$$

A directed graph, together with its stochastic matrix, can visualize any THMC:



**Eg. 1.7** Random walks on an undirected weighted graph, where edge weights dictate the proportional probability of transitioning between two states, are a special class of THMCs. In particular, given a graph  $G = (V, E)$  with weights  $w(e) > 0 : e \in E$ , we set

$$p_{u,v} = \frac{w(\{u, v\})}{\sum_{z \in N(u)} w(\{u, z\})}$$

If there are no edges  $u \leftrightarrow v$ , we write  $p_{u,v} = 0$ .

Not every THMC can be represented by a random walk on an undirected weighted graph. In particular, see the directed graph listed above, or any transition matrix which is not symmetric.

As a concrete example, we can consider a random walk on the number line  $\mathbb{Z}$ , where, if  $w(k, k+1) = \alpha$ ,  $w(k-1, k) = \frac{\alpha}{2}$ .

$$\dots \frac{1}{16} \quad -3 \quad \frac{1}{8} \quad -2 \quad \frac{1}{4} \quad -1 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 1 \quad \frac{2}{2} \quad 2 \quad \frac{4}{4} \quad 3 \quad \frac{8}{8} \quad \dots$$

### Multi-Step Transition Probabilities

Given a THMC  $X = X_n : n \geq 0$  with a transition matrix  $P$ , we write

$$\begin{aligned} \mathbb{P}(X_2 = v | X_0 = u) &= \sum_{w \in V} \mathbb{P}(X_2 = v, X_1 = w | X_0 = u) \\ &= \sum_{w \in V} \mathbb{P}(X_1 = w | X_0 = u) \mathbb{P}(X_2 = v | X_1 = w, \cancel{X_0 = u}) \quad \text{by Markov property} \\ &= \sum_{w \in V} p_{u,w} p_{w,v} = (P^2)_{u,v} \quad \text{or write } P_{u,v}^2 \end{aligned}$$

Hence, to determine a two-step transition probability, and by extension an  $n$ -step transition probability from  $u$  to  $w$ , we consider  $P_{u,v}^n$ . PROP 1.2

See [Prop 1.1](#) to expand probabilities, using Bayes', as needed. We get that

PROOF.

$$\begin{aligned} \mathbb{P}(X_n = v | X_0 = u) &= \sum_{v_1, \dots, v_{n-1} \in V} \mathbb{P}(X_1 = v_1, \dots, X_{n-1} = v_{n-1}, X_n = v | X_0 = u) \\ &= \sum_{v_1, \dots, v_{n-1} \in V} p_{u,v_1} p(v_1, v_2) \cdots p(v_{n-1}, v) = (P^n)_{u,v} \quad \square \end{aligned}$$

Thus, if  $P$  is a stochastic matrix, then so is  $P^n$ . PROP 1.3

$$\sum_{v \in V} P_{u,v}^n = \sum_{v \in V} \mathbb{P}(X_n = v | X_0 = u) = 1. \quad \square$$

PROOF.

**Theorem 1.1 Markov Property**

If  $X_n : n \geq 0$  is a THMC with state space  $V$ , then for all  $u_0, \dots, u_{n-1}, u, v \in V$ ,

$$\mathbb{P}(X_{n+m} = v | X_0 = u_0, \dots, X_{n-1} = u_{n-1}, X_n = u) = \mathbb{P}(X_{n+m} = v | X_n = u) = P_{u,v}^m$$

PROOF.

One shows this by combining the Markov property with [Prop 1.2](#) via induction.  $\square$

Somewhat nonsensically, we *also* call this the Markov property. When talking about THMCs, this will be the default notion.

DEF 1.9

We say that a Markov chain has an *initial distribution*  $\alpha = (\alpha_v : v \in V)$  if  $\mathbb{P}(X_0 = v) = \alpha_v$  for each  $v \in V$ . If this is the case, we often write  $\alpha$  as a subscript of our state probabilities. For instance,

$$\mathbb{P}_\alpha(X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u, X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u) \mathbb{P}_\alpha(X_n = v | X_0 = u) = \sum_{u \in V} \alpha_u P_{u,v}^n$$

PROP 1.4

For any event  $E$  depending only on  $X_0, \dots, X_n$ , with  $\mathbb{P}(X_n = u, E) > 0$ , we have

$$\mathbb{P}(X_{n+m} = v | X_n = u, E) = P_{u,v}^m$$

PROOF.

For any such event  $E$ , we can determine whether  $E$  occurs exactly when we know the realized values  $u_i$  of  $X_i$  for  $i = 1, \dots, n-1$ . Hence, we may write  $\mathcal{S}$  to be the set of tuples  $(u_0, \dots, u_{n-1})$  that guarantee  $E$ . It follows that

$$\mathbb{P}(X_n = u, E) = \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(\mathbf{x} = \mathbf{s}, X_n = u)$$

Similarly, we have

$$\begin{aligned} \mathbb{P}(X_{n+m} = v, X_n = u, E) &= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v, X_n = u, \mathbf{x} = \mathbf{s}) \\ &= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v | X_n = u, \mathbf{x} = \mathbf{s}) \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) \\ &= P_{u,v}^m \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) = P_{u,v}^m \mathbb{P}(X_n = u, E) \end{aligned}$$

Divide and use Bayes, and the result follows.  $\square$

PROP 1.5

If  $X$  is a THMC with transition matrix  $P$ , then, for all  $k \geq 1$ ,  $X_{kn} : n \geq 0$  is a THMC with transition matrix  $P^k$ .

For any  $n \neq 0$ , any sequence  $u_0, \dots, u_{n+1} \in V$  satisfies

$$\mathbb{P}(X_{(n+1)k} = u_{n+1} | X_0 = u_0, X_k = u_1, \dots, X_{nk} = u_n) = P_{u_n, u_{n+1}}^k \quad \square$$

PROOF.

### Theorem 1.2 Chapman-Kolmogorov

For any Markov chain  $X$  with state space  $V$ , any  $m, n \geq 0$ , and  $u, v \in V$ ,

$$\mathbb{P}(X_{m+n} = v | X_0 = u) = \sum_{w \in V} \mathbb{P}(X_n = w | X_0 = u) \mathbb{P}(X_{m+n} = v | X_n = w)$$

If the  $X$  is time homogeneous, then this is  $P_{u,v}^{n+m}$ , which agrees with [Prop 1.2](#).

### Long Term Behavior

Recall from probability the *law of large numbers*: if  $Y_n : n \geq 1$  are IID with common mean  $\mu$ , then  $\frac{S_n}{n} \rightarrow \mu$  in probability, where  $S_n = \sum_{i=1}^n Y_i$ , i.e.  $\forall \varepsilon > 0$ , DEF 1.10

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = 0$$

If  $Y_i \in \mathbb{Z}$  then, for  $k, \ell, u_i \in \mathbb{Z}$  and  $i = 1, \dots, n-1$ ,

$$\begin{aligned} \mathbb{P}(S_{n+1} = \ell | S_n = k, S_i = u_i \forall i) &= \mathbb{P}(Y_{n+1} = \ell - k | S_n = k, S_i = u_i \forall i) \\ &= \mathbb{P}(Y_{n+1} = \ell - k | Y_1 = u_1 - u_0, Y_2 = u_2 - u_1, \dots, Y_n = k - u_{n-1}) \\ &= \mathbb{P}(Y_{n+1} | \ell - k) = \mathbb{P}(Y_1 = \ell - k) = P_{k, \ell} \end{aligned}$$

where  $S_n : n \geq 0$  has transition matrix  $P$ , noting that it may be viewed as a THMC.

From now on, we denote by  $\mathbb{P}_v(E)$  the probability  $\mathbb{P}(E|v)$ .

**Eg. 1.8** A general two-state chain, with states  $A$  and  $B$ , can be described by

$$P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}$$

Let  $q_n = \mathbb{P}_A(X_n = A) = \mathbb{P}(X_n = A | X_0 = A)$ . Then

$$\begin{aligned} q_{n+1} &= \mathbb{P}_A(X_{n+1} = A, X_n = A) + \mathbb{P}_A(X_{n+1} = A, X_n = B) \\ &= \mathbb{P}_A(X_{n+1} = A | X_n = A) \mathbb{P}_A(X_n = A) + \mathbb{P}_A(X_{n+1} = A | X_n = B) \mathbb{P}_A(X_n = B) \\ &= (1 - \alpha)q_n + \beta(1 - q_n) = \beta + (1 - \alpha - \beta)q_n \end{aligned}$$

This recurrence has a unique solution. In particular, one can find

$$q_n = \frac{\beta}{\alpha + \beta} + (1 - \alpha - \beta)^n \frac{\alpha}{\alpha + \beta}$$



It follows that  $q_n \rightarrow \frac{\beta}{\alpha+\beta}$ , and hence  $\mathbb{P}_A(X_n = B) = 1 - q_n \rightarrow \frac{\alpha}{\alpha+\beta}$ . Likewise:

$$\mathbb{P}_B(X_n = B) = \frac{\alpha}{\alpha + \beta} + (1 - \beta)^n \frac{\beta}{\alpha + \beta}$$

So  $\mathbb{P}_B(X_n = B) \rightarrow \frac{\alpha}{\alpha+\beta}$ .

Let  $\pi := (\pi_A, \pi_B) := \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$  be the distribution of our initial state  $X_0$ , Then

$$\mathbb{P}_\pi(X_1 = A) = \pi_A \mathbb{P}_A(X_1 = A) + \pi_B \mathbb{P}_B(X_1 = A) = \pi_A$$

and, similarly,  $\mathbb{P}_\pi(X_1 = B) = \pi_B$ . Hence, if  $X_0$  has initial distribution  $\pi$ , then  $X_1$  also has distribution  $\pi$ . By induction,  $X_n$  has distribution  $\pi \forall n \geq 0$ .

When we say  $X = \text{Markov}(P)$ , we mean that  $X$  is a THMC with transition matrix  $P$ .

DEF 1.11 A probability distribution  $\pi$  is called **stationary** if  $\pi P = \pi$ . Similarly, a probability  
DEF 1.12 distribution  $\lambda$  is called a **limiting distribution** if, for each  $u, v \in V$

$$(P^n)_{u,v} \rightarrow \lambda_v \text{ as } n \rightarrow \infty$$

In other words,  $\mathbb{P}_u(X_n = v) \rightarrow \lambda_v$ . Note that, for any initial distribution  $\alpha$ , we have  $\alpha P^n \rightarrow \lambda$ , i.e.  $(\alpha P^n)_v \rightarrow \lambda_v$ , where  $\lambda$  is limiting.

PROP 1.6 If  $\lambda$  is a limiting distribution for  $P$ , then  $\lambda$  is stationary for  $P$ .

PROOF.

Fix any initial distribution  $\alpha$ , we have

$$\lambda = \lim_{n \rightarrow \infty} (\alpha P^n) = \lim_{n \rightarrow \infty} (\alpha P^{n-1} P) = \left( \lim_{n \rightarrow \infty} \alpha P^{n-1} \right) P = \lambda P \quad \square$$

Stationary distributions need not be unique, but limiting distributions are (as the limit  $\lim_{n \rightarrow \infty} \alpha P^n$  is well-defined). In general, then, stationary distributions need not be limiting distributions.

DEF 1.13 A stochastic matrix  $P$  is called **regular** if  $\exists n \geq 1$  such that  $P^n > 0$  on all entries.

### Theorem 1.3 Fundamental Theorem of Markov Chains

Every finite, regular stochastic matrix  $P$  has a limiting distribution  $\pi$ .

Incorporating some of the formulations above, this is equivalent to saying: *For a regular stochastic matrix, there exists a unique distribution  $\pi = (\pi_v : v \in V)$  such that  $\pi P = \pi$  and  $\mathbb{P}_u(X_n = v) \rightarrow \pi_v \forall u, v \in V$ .*

**A stationary distribution always exists!**

Let  $\rho = \langle 1, \dots, 1 \rangle$ . Then note that  $P\rho = \rho$ , since the sum of any row in  $P$  must be 1. Hence,  $P$  has eigenvalue 1. It follows that it has a left eigenvector, i.e.  $\pi : \pi P = \pi$ . This is exactly a stationary distribution, as long as we scale suitably such that  $\pi$  is a distribution.

When  $n = 0$ ,  $P^n = I$ , which encapsulates the idea that, at timestep 0, we will be at our initial positions.

In this case, there is a unique stationary distribution, and it is the unique limiting distribution.

This is true, but  
requires the fact that  
 $P$  is stochastic

However, then process of scaling into a distribution is non-trivial. Since  $\pi$  may have negative coordinates, and hence  $\sum \pi_i = 0$ , we must consider instead  $|\pi|$ , i.e. prove it is also an eigenvalue.

### Periodicity of States

For  $u, v \in V$ , we say that  $v$  is *accessible* from  $u$  if  $\exists n \geq 0$  such that  $(P^n)_{u,v} > 0$ . Equivalently, in the directed graph generated by  $P$ , there is a directed path from  $u$  to  $v$ . When  $v$  is accessible from  $u$ , we write  $u \rightarrow v$ .

DEF 1.14

States  $u$  and  $v$  *communicate* if  $u \rightarrow v$  and  $v \rightarrow u$ . When  $u$  and  $v$  communicate, we write  $u \leftrightarrow v$ . Observe that communication is an equivalence relation. Hence, the state space  $V$  can be written as a disjoint union of mutually-communicating states, called a *communication class*. Note that, in the directed graph generated by  $P$ , these correspond to the strongly connected components.

DEF 1.15

DEF 1.16

Clearly, if  $P$  is  
regular, then it is  
irreducible

We say that  $P$  is *irreducible* if there is only one communication class.

DEF 1.17

$$u \rightarrow v \iff \mathbb{P}_u(T_{\{v\}} < \infty) > 0.$$

PROP 1.7

The *period* of a state  $u \in V$  is

DEF 1.18

$$d(u) := \gcd\{n > 0 : P_{u,u}^n > 0\}$$

If  $d(u) = 1$ , we call  $u$  *aperiodic*. By extension,  $P$  is aperiodic if  $d(u) = 1 \forall u \in V$ , and  $X$  is aperiodic if  $X = \text{Markov}(P)$  for  $P$  aperiodic.

DEF 1.19

If  $u \leftrightarrow v$ , then  $d(u) = d(v)$ .

PROP 1.8

Let  $I = \{n > 0 : P_{u,u}^n > 0\}$ , and similarly  $J$  for  $v$ . Hence,  $d(u) = \gcd(I)$  and  $d(v) = \gcd(J)$ . Let  $a, b > 0$  such that  $P_{u,v}^a > 0$  and  $P_{v,u}^b > 0$ . Then

PROOF.

$$P_{u,u}^{a+b} \geq P_{u,v}^a P_{v,u}^b > 0$$

$\implies a + b \in I$ , so  $d(u) | a + b$ . Now, if  $n \in J$ , then

$$P_{u,u}^{a+b+n} \geq P_{u,v}^a P_{v,v}^n P_{v,u}^b > 0$$

$\implies a + b + n \in I$ , so  $d(u) | n + a + b$ . But, by the previous line,  $d(u) | n$ . Since  $n \in J$  is arbitrary, we can write  $d(u) | \gcd(J) = d(v)$ .

Symmetrically, we could conclude that  $d(v) | d(u)$ , so indeed  $d(v) = d(u)$ .  $\square$

Let  $I = \{n > 0 : P_{u,u}^n > 0\}$ . If  $\gcd(I) = 1$ , then  $\exists a, b \in I$  such that  $\gcd(a, b) = 1$ .

PROP 1.9

This is not true for any  $I$  (and thus relies not only on number theory). Let  $\ell, m \in I$ , with  $\ell < m$ . Let  $k = m - \ell$ . If  $k = 1$ , then  $\gcd(\ell, m) = 1$ . Otherwise, since  $\gcd(I) = 1$ , there is an  $n \in I$  with  $k \nmid n$ . We then write  $n = qk + r$ , with  $r \in [k - 1]$ . Then  $m' \in (q + 1)m \in I$ , since  $P_{u,u}^{(q+1)m} \geq (P_{u,u}^m)^{q+1}$ . Symmetrically, we can argue  $\ell' = (q + 1)\ell \in I$ .

PROOF.

Similarly,  $\ell^* := \ell' + n \in I$ , since  $P_{u,u}^{\ell'+n} \geq P_{u,u}^{\ell'} P_{u,u}^n$ . We have

$$\begin{aligned} m' - \ell^* &= (q+1)m - (q+1)\ell - n = (q+1)(m - \ell) - n \\ &= (q+1)k - n = k - r \in [k-1] \end{aligned}$$

TODO...

□

### Theorem 1.4 Postage Stamp Lemma

If  $P$  is irreducible and aperiodic, then  $\forall u, v \in V, \exists N$  such that  $P_{u,v}^n > 0 \forall n \geq N$ .

Before proving this, we note that, for  $a, b \geq 1$  with  $\gcd(a, b) = 1$ , then for any  $q \geq ab$ , we can write  $q = ja + kb$  for integers  $j, k \geq 0$ .

PROOF.

Fix  $u, v \in V$ . Since  $P$  is aperiodic, there are  $a, b \geq 1$  with  $P_{u,u}^a, P_{u,u}^b > 0$  and  $\gcd(a, b) = 1$ , by [Prop 1.9](#). Since  $P$  is irreducible, there is some  $m > 0$  with  $P_{u,v}^m > 0$ . Thus, let  $N = m + ab$ . For any  $n \geq N$ , let  $q = n - m$ . We have that  $q \geq ab$ , so we can find  $j, k \geq 0$  with  $q = ja + kb$ . Then

$$P_{u,v}^n = P_{u,v}^{q+m} = P_{u,v}^{ja+kb+m} \geq P_{u,u}^{ja} P_{u,u}^{kb} P_{u,v}^m \geq (P_{u,u}^a)^j (P_{u,u}^b)^k P_{u,v}^m$$

All are positive, so  $P_{u,v}^n > 0$ , as desired.

□

### Theorem 1.5 Characterization of Regular Markov Chain

Let  $P = (p_{u,v})_{u,v \in V}$  be a stochastic matrix, where  $|V| < \infty$ . Then

$$P \text{ is regular} \iff P \text{ is irreducible and aperiodic}$$

PROOF.

We first note why finiteness is necessary. Consider:



with all edges having weight 1. This graph is clearly aperiodic and irreducible, but not regular.

( $\implies$ ) We start with the "easy" direction. If  $P$  is regular, then  $\exists n > 0$  s.t.  $P_{u,v}^n > 0$  for all  $u, v \in V$ . Then, for all  $u, v \in V$ , we have  $u \rightarrow v$  and  $v \rightarrow u$ . Hence,  $P$  is irreducible. Now, if  $P$  is irreducible, then for all  $u \in V$ , there is some  $v \in V$  such that  $P_{v,u} > 0$  (think about this in graph theoretic terms). Then, let  $n > 0$  be such that  $P_{u,u}^n$  is positive. We have

$$P_{u,u}^{n+1} \geq P_{u,v}^n P_{u,v} > 0$$

So, with  $I = \{m > 0 : P_{u,u}^m > 0\}$ ,  $d(u) = \gcd(I) \leq \gcd(n, n+1) = 1$ . It follows that

$d(u) = 1$ , so  $u$  is aperiodic (and hence  $P$  is aperiodic).

( $\Leftarrow$ ) By [Thm 1.4](#), for each  $u, v \in V$ , there exists  $N : P_{u,v}^n > 0 \forall n \geq N$ . Let  $N^*$  be the maximum value of  $N$  determined over all pairs  $(u, v) \in V^2$ . Then, for  $n \geq N^*$  and all  $u, v \in V$ ,  $P_{u,v}^n > 0$ . It follows that all entries of  $P^n$  are positive, and we are done.  $\square$

### Finding Stationary Distributions

Recall that  $x = (x_v : v \in V)$  is a stationary distribution if  $xP = x$ . Let  $V$  be finite. Then, for a stationary distribution  $x$ , we have

$$\begin{aligned} x_1 p_{1,1} + \cdots + x_n p_{n,1} &= x_1 \\ x_1 p_{1,2} + \cdots + x_n p_{n,2} &= x_2 \\ &\vdots \\ x_1 p_{1,n} + \cdots + x_n p_{n,n} &= x_n \end{aligned}$$

We have  $n$  equations,  $n$  unknowns, and a homogeneous system, so there is not a unique solution. If

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

We can compute  $x = \langle t, 2t, 2t \rangle$ . But, noting that  $x$  is a probability distribution, and hence  $5t = 1$ , this yields  $x = \langle 1/5, 2/5, 2/5 \rangle$ . We'll consider some special cases.

### UNDIRECTED GRAPHS

This is distinct from [Example 1.7](#)

Let  $G = (V, E)$  be undirected. Then we define a THMC by

$$P_{u,v} = \begin{cases} \frac{1}{\deg(u)} & \{u, v\} \in E \\ 0 & \text{o.w.} \end{cases}$$

Let  $x = (\deg(v) : v \in V)$ . We have

$$\begin{aligned} (xP)_v &= \sum_{u \in V} \deg(u) P_{u,v} = \sum_{u \in N(v)} \deg(u) \cdot \frac{1}{\deg(u)} \\ &= \deg(v) \end{aligned}$$

Hence,  $xP = x$ . Recalling that  $\sum_{v \in V} \deg(v) = 2|E|$ , we conclude that

$$\left( \frac{\deg(v)}{2|E|} : v \in V \right)$$

is a stationary distribution.

## UNDIRECTED WEIGHTED GRAPHS

Let  $G = (V, E)$  be undirected. Then, we define a THMC by

This is not distinct from [Example 1.7](#)

$$P_{u,v} = \begin{cases} \frac{w(\{u,v\})}{\sum_{z \in N(u)} w(\{u,z\})} & v \in N(u) \\ 0 & \text{o.w.} \end{cases}$$

Let  $x = (\sum_{e: e \ni v} w(e) : v \in V)$ . Then we can compute  $xP = x$ , and similar to above,

$$x = \left( \frac{\sum_{e: e \ni v} w(e)}{2 \sum_{e \in E} w(e)} : v \in V \right)$$

is a stationary distribution.

*Transience and Recurrence*

Recall  $T_S = \inf\{n \geq 0 : X_n \in S\}$ , the "hitting time" of  $S$ . We let  $R_S = \inf\{n > 0 : X_n \in S\}$ . Note that if  $T_S > 0$ ,  $T_S = R_S$ . Otherwise,  $R_S$  gives the first "return time" to the set  $S$ .

DEF 1.20

A state  $v \in V$  is called **recurrent** if  $\mathbb{P}_v(R_{\{v\}} < \infty) = 1$ . If all states of  $v$  are recurrent, we may  $P$  and  $X = \text{Markov}(P)$  recurrent. Otherwise, we call  $v$  **transient**, and similarly extend the notion to the transition matrix and chain when all state are transient.

DEF 1.21

DEF 1.22

DEF 1.23

For a given state  $v \in V$ , we call  $L_v = |\{n \geq 0 : X_n = v\}|$  the **local time** of  $v$ . This notion is not probabilistic: we simply consider a realized walk on the chain (or a part of the chain). Note that, if  $v = X_j$  and  $v$  is recurrent, then  $L_v = \infty$ .

PROP 1.10 Let  $X = \text{Markov}(P)$ . For any state  $v \in V$  and  $k > 1$ ,

$$\mathbb{P}_v(L_v > k) = \mathbb{P}_v(L_v > 1)^k$$

Intuitively, if  $L_v > k$  when  $X_0 = v$ , then  $L_v > k - 1$  when  $X_{i_1} = v$ , where  $i_1$  is the first time we return to  $v$ .

PROOF.

Using the law of total probability:

$$\begin{aligned} \mathbb{P}_v(L_v > k) &= \mathbb{E}[\mathbb{P}_v(L_v > k | R_v)] = \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t) \mathbb{P}_v(L_v > k | R_v = t) \\ &= \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t) \mathbb{P}_v(L_v > k | R_v = t, X_t = v) \\ &= \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t) \mathbb{P}_v(L_v > k - 1) \\ &= \mathbb{P}_v(L_v > k - 1) \sum_{t=1}^{\infty} \mathbb{P}_v(R_v = t) \\ &= \mathbb{P}_v(L_v > k - 1) \mathbb{P}_v(R_v < \infty) = \mathbb{P}_v(L_v > k - 1) \mathbb{P}_v(L_v > 1) \end{aligned}$$

As  $R_v = t \iff R_v = t \wedge X_t = v$

The result follows by induction. □

PROP 1.11

$$\mathbb{P}_v(L_v = \infty) = \begin{cases} 1 & v \text{ recurrent} \\ 0 & v \text{ transient} \end{cases}$$

This follows directly from [Prop 1.10](#) + monotonicity of probability.  $\square$

PROOF.

PROP 1.12

$$\sum_{n=0}^{\infty} P_{v,v}^n = \begin{cases} \infty & v \text{ recurrent} \\ \frac{1}{1 - \mathbb{P}_v(R_{\{v\}} < \infty)} & v \text{ transient} \end{cases}$$

This follows from linearity of expectation, and the fact that, for a non-negative integer variable  $Z$ ,

$$\mathbb{E}[Z] = \sum_{k=0}^{\infty} \mathbb{P}(Z > k)$$

In particular... [TODO]  $\square$

PROOF.

If  $u \leftrightarrow v$ , then  $u$  is transient  $\iff v$  is transient.

PROP 1.13

Fix  $a, b \geq 0$  with  $P_{u,v}^a, P_{v,u}^b > 0$ . Then

PROOF.

$$\begin{aligned} \sum_{n=0}^{\infty} P_{v,v}^n &\geq \sum_{n=0}^{\infty} P_{v,v}^{a+b+n} = \sum_{n=0}^{\infty} P_{v,u}^b P_{u,u}^n P_{u,v}^a \\ &= P_{v,u}^b P_{u,v}^a \sum_{n=0}^{\infty} P_{u,u}^n \end{aligned}$$

Thus, if  $v$  is transient, then  $\sum_{n=0}^{\infty} P_{v,v}^n < \infty$ , so it must be that  $\sum_{n=0}^{\infty} P_{u,u}^n < \infty$ , i.e.  $u$  is transient. The argument is identical in reverse.  $\square$

**Eg. 1.9** If  $u \leftrightarrow v$  and  $u$  is recurrent, then  $\mathbb{P}_u(T_{\{v\}} < \infty) = 1$ .

**Eg. 1.10** The following chain is completely transient:



In fact, we could replace  $2/3$  by  $p$ , and  $1/3$  by  $1 - p$ . In this case, the chain is irreducible. To see that it is transient, we have

$$P_{0,0}^{2n} = \binom{2n}{n} p^n (1-p)^n$$

Then

$$\sum_{n=0}^{\infty} P_{0,0}^{2n} < \sum_{n=0}^{\infty} 2^{2n} p^n (1-p)^n = \sum_{n=0}^{\infty} (4p(1-p))^n < \infty \quad \text{if } p \neq \frac{1}{2}$$

By [Prop 1.12](#). Notice that  $\sum_{n=0}^{\infty} P_{0,0}^n = \sum_{n=0}^{\infty} P_{0,0}^{2n}$ , since it is only possible to return on even-length cycles.

We conclude that the chain is transient when  $p \neq \frac{1}{2}$ .

**FACT** Stirling's Formula provides

$$m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}$$

in that

$$\lim_{m \rightarrow \infty} \frac{m!}{\left(\frac{m}{e}\right)^m \sqrt{2\pi m}} = 1$$

This fact implies

$$e \left(\frac{n}{e}\right)^n \leq n! \leq \frac{e(n+1)}{4} \left(\frac{n+1}{e}\right)^n$$

These facts, though out of the scope of this course, can be derived from a careful treatment of Riemann sums

**Fig. 1.11** We return to the previous example, letting  $p = \frac{1}{2}$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} P_{0,0}^{2n} &= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} [p(1-p)]^n \sim \sum_{n=0}^{\infty} \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi(2n)}}{\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right]^2} [p(1-p)]^n \\ &= \sum_{n=0}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty \end{aligned}$$

We conclude that the chain is recurrent when  $p = \frac{1}{2}$ .

**PROP 1.14** If  $V$  is finite, then there is at least one recurrent state.

**PROOF.**

Fix an initial distribution  $\alpha = (\alpha_v : v \in V)$ . Then  $\mathbb{P}_{\alpha}(\sum_{v \in V} L_v = \infty) = 1$ . We conclude

that there is at least one state  $v \in V$  with  $\mathbb{P}_\alpha(L_v = \infty) > 0$ . But also:

$$\begin{aligned} \mathbb{P}_\alpha(L_v = \infty) &= \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty, T_v = n) = \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty | T_v = n) \mathbb{P}_\alpha(T_v = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty | T_v = n, X_n = v) \mathbb{P}_\alpha(T_v = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_\alpha(L_v = \infty | X_n = v) \mathbb{P}_\alpha(T_v = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}_v(L_v = \infty) \mathbb{P}_\alpha(T_v = n) \end{aligned}$$

So  $\mathbb{P}_v(L_v = \infty) > 0 \implies \mathbb{P}_v(L_v = \infty) = 1$ , by [Prop 1.11](#).  $\square$

Finite, irreducible chains are recurrent.

**PROP 1.15**

Since the chain is finite it has at least one recurrent state, by [Prop 1.14](#). Then all states must be recurrent, since the chain is irreducible, by [Prop 1.13](#).  $\square$

PROOF.

### Canonical Decompositions

Fix a transition matrix  $P$  and list the communication classes of  $V$  as

$$D_1, D_2, \dots \quad (\text{transient}) \quad C_1, C_2, \dots \quad (\text{recurrent})$$

Note that we can split the chain up in this way by [Prop 1.13](#). Set  $D = \cup_{i \geq 0} D_i$ . Then the *canonical decomposition* of the chain is defined to be

DEF 1.24

$$D \sqcup C_1 \sqcup C_2 \sqcup \dots$$

We say that a communication class  $C$  is *closed* if, for any  $u \in C, v \notin C, p_{u,v} = 0$ . Intuitively, if  $X_0 \in C$ , or we enter  $C$  at some later time, we will never leave  $C$ .

DEF 1.25

If  $C$  is a recurrent communication class, then  $C$  is closed.

**PROP 1.16**

Fix  $u \in C, v \notin C$ . Suppose  $v \mapsto u$ . If  $p_{u,v} > 0$ , then  $v \mapsto v$ , so  $v \in C$ . Suppose  $v \not\mapsto u$ . Then  $\mathbb{P}_u(R_u = \infty) \geq \mathbb{P}_u(X_1 = v) = p_{u,v}$ . But  $\mathbb{P}_u(R_u = \infty) = 0$ , since  $u$  is recurrent. It follows that  $p_{u,v} = 0$ .  $\square$

PROOF.

The converse of [Prop 1.16](#) is not true in generality, but it is in the finite case:

Finite, closed communication classes are recurrent.

**PROP 1.17**

From any starting state in  $C$ , we must visit some state  $u \in C$  infinitely often, as  $|C| < \infty$  and  $X_t \in C \forall t$ . But recurrence is a class property by [Prop 1.13](#). Hence, all of  $C$  is recurrent.  $\square$

PROOF.



When our communication classes are closed, we have



### *Proof of Fundamental Theorem of Markov Chains*

Recall [Thm 1.3](#):

**Every finite, regular stochastic matrix  $P$  has a limiting distribution  $\pi$ .**

We will prove this in two steps. First, we will find some stationary distribution. Then, we will prove that this is a limiting distribution.

#### **Theorem 1.6 Existence Theorem**

Let  $P$  be irreducible and recurrent. Let  $(X_n : n \geq 0) = \text{Markov}(P)$ . Fix  $u \in V$ , which we call a reference vertex, and, for any  $v \in V$ , define

$$\gamma_v = \mathbb{E}_u[|\{0 \leq n < R_u : X_n = v\}|]$$

Let  $\gamma = (\gamma_v : v \in V)$ . Then  $\gamma P = \gamma$ , and  $0 < \gamma_v < \infty \forall v \in V$ .

By  $\mathbb{E}_u$ , we mean the expectation, under the assumption that  $X_0 = u$

PROOF.

Observe that  $\gamma_u = 1$ . Write

$$\gamma_v = \mathbb{E}_u \left[ \sum_{n=0}^{R_u-1} \mathbb{1}_{X_n=v} \right] = \mathbb{E}_u \left[ \sum_{n=1}^{R_u} \mathbb{1}_{X_n=v} \right] = \mathbb{E}_u \left[ \sum_{n=1}^{\infty} \mathbb{1}_{X_n=v} \mathbb{1}_{n \leq R_u} \right]$$

We utilize a second indicator variable in order to use linearity of expectation (otherwise, our sum would index over a random variable, which is not valid). Then

$$\gamma_v = \sum_{n=1}^{\infty} \mathbb{E}_u[\mathbb{1}_{X_n=v} \mathbb{1}_{n \leq R_u}] = \sum_{n=1}^{\infty} \mathbb{P}_u(X_n = v, n \leq R_u)$$

Now, using the law of total probability,

$$\begin{aligned} \mathbb{P}_u(X_n = v, n \leq R_u) &= \sum_{w \in V} \mathbb{P}_u(X_{n-1} = w, X_n = v, n \leq R_u) \\ &= \sum_{w \in V} \mathbb{P}_u(X_n = v | X_{n-1} = w, n \leq R_u) \mathbb{P}_u(X_{n-1} = w, n \leq R_u) \\ &= \sum_{w \in V} \mathbb{P}_u(X_{n-1} = w, n \leq R_u) P_{w,v} \end{aligned}$$

So

$$\begin{aligned}
 \gamma_v &= \sum_{n=1}^{\infty} \sum_{w \in W} P_{w,v} \mathbb{P}_u(X_{n-1} = w, n \leq R_u) = \sum_{w \in V} P_{w,v} \left( \sum_{n=1}^{\infty} \mathbb{P}_u(X_{n-1} = w, n-1 < R_u) \right) \\
 &= \sum_{w \in V} P_{w,v} \sum_{n=0}^{\infty} \mathbb{P}_u(X_n = w, n < R_u) = \sum_{w \in V} P_{w,v} \sum_{n=0}^{\infty} \mathbb{E}_u[\mathbb{1}_{X_n=w} \mathbb{1}_{n < R_u}] \\
 &= \sum_{w \in V} P_{w,v} \mathbb{E}_u \left[ \sum_{n=0}^{R_u-1} \mathbb{1}_{X_n=w} \right] = \sum_{w \in V} P_{w,v} \gamma_w = (\gamma P)_v
 \end{aligned}$$

$\implies \gamma P = \gamma$ . Furthermore,  $\gamma_v > 0$ , since  $u \mapsto v$ . Letting  $n : P_{v,u}^n > 0$ , we have  $\gamma_u = (\gamma P^n)_u \geq \gamma_v P_{v,u}^n$ . Noting that  $\gamma_u = 1$ , this shows  $\gamma_v < \infty$ .  $\square$

As a corollary, under the same conditions, if  $\mathbb{E}_u[R_u]$  is finite,  $\pi = (\pi_v : v \in V)$ , where

**PROP 1.18**

$$\pi_v = \frac{\gamma_v}{\sum_{w \in V} \gamma_w}$$

is a stationary distribution.

Observe

**PROOF.**

$$\sum_{w \in V} \gamma_w = \sum_{w \in V} \mathbb{E}_i \left[ \sum_{n=0}^{R_i-1} \mathbb{1}_{X_n=w} \right] = \mathbb{E}_i \left[ \sum_{n=0}^{R_i-1} \sum_{w \in V} \mathbb{1}_{X_n=w} \right] = \mathbb{E}_i[R_i]$$

So we require that  $\mathbb{E}_u[R_u] < \infty$ , then a stationary distribution exists.  $\square$

An exercise, if  $u$  is recurrent and  $u \mapsto v$ , then  $\mathbb{P}_u(T_v < \infty) = 1 \ \forall v \in V$ .  $\pi$ , as defined above, is a limiting distribution.

**PROP 1.19**

Consider two independent copies  $X_n, Y_n = \text{Markov}(P)$ . Then  $(X_n, Y_n)$  is a Markov chain with transition matrix  $Q = (q_{(v,w),(x,y)})_{(v,w),(x,y) \in V \times V}$ . In particular,

**PROOF.**

$$q_{(v,w),(x,y)} = p_{v,x} p_{w,y}$$

Fix some state  $u \in V$ . Let  $\alpha$  be the initial distribution that has  $X_0 = u$  and  $Y_0 \sim \pi$ .

$$\alpha_{(x,y)} = \begin{cases} \pi(y) & x = u \\ 0 & \text{o.w.} \end{cases}$$

Remark that, as  $\pi$  is stationary,  $Y_n \sim \pi \ \forall n \geq 0$ . For any  $u \in V$ , we'd like that  $P_{u,v}^n \rightarrow \pi(v)$ .

Let  $M = \inf\{n \geq 0 : X_n = Y_n\}$  be the first meeting time of the  $X_n$  and  $Y_n$  chains. If  $P$  is

finite and regular, then  $Q$  is finite and regular. Then

$$\mathbb{P}_\alpha(M < \infty) \geq \mathbb{P}_\alpha(T_{(v,v)} < \infty) = \sum_{(x,y) \in V \times V} \alpha(x,y) \mathbb{P}_{(x,y)}(T_{v,v} < \infty) = 1$$

We claim that  $\mathbb{P}_\alpha(X_n = v, M \leq n) = \mathbb{P}_\alpha(Y_n = v, M \leq n) \forall n \geq 0$ . Assuming this, then, we are done:

$$\begin{aligned} \mathbb{P}_u(X_n = v) &= \mathbb{P}_\alpha(X_n = v) = \mathbb{P}_\alpha(X_n = v, M \leq n) + \mathbb{P}_\alpha(X_n = v, M > n) \\ \mathbb{P}_\pi(Y_n = v) &= \mathbb{P}_\alpha(Y_n = v) = \mathbb{P}_\alpha(Y_n = v, M \leq n) + \mathbb{P}_\alpha(Y_n = v, M > n) \\ \implies \mathbb{P}_u(X_n = v) - \pi(v) &= \mathbb{P}_\alpha(X_n = v, M > n) - \mathbb{P}_\alpha(Y_n = v, M > n) \\ \implies |\mathbb{P}_u(X_n = v) - \pi(v)| &= |\mathbb{P}_\alpha(X_n = v, M > n) - \mathbb{P}_\alpha(Y_n = v, M > n)| \end{aligned}$$

□

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