
ASSIGNMENT 3

MATH 251

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QUESTION 1

Part (a): Consider $\delta : \mathbb{F}[t]_{n+1} \rightarrow \mathbb{F}[t]_n$ which takes $p(t) \rightarrow p'(t)$. This function is surjective, but not injective: for the latter, consider a counterexample: for t and $t + 1 \in \mathbb{F}[t]_{n+1}$, we have $\delta(t) = 1$ and $\delta(t + 1) = 1$. To show δ is surjective, consider an arbitrary element in $\mathbb{F}[t]_n$:

$$p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

Then $\delta(a_0 t + a_1 \frac{t^2}{2} + \dots + a_n \frac{t^{n+1}}{n+1}) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n = p(t)$, where our input is clearly in $\mathbb{F}[t]_{n+1}$. Note the interpretation of constants $\frac{1}{i}$ and i to simply be $\underbrace{1 + \dots + 1}_{i \text{ times}}$ and their inverses in the field.

The kernel of δ is \mathbb{F} : clearly, $\delta(p) = 0 \forall p \in \mathbb{F}$. Suppose $p(t) \notin \mathbb{F}$, and WLOG write $p(t) = at^i$ where $1 \leq i \leq n$ and $a \in \mathbb{F} \neq 0$. Then $\delta(at^i) = ait^{i-1}$, which is non-zero.

Since $\dim(\mathbb{F}) = 1$, $(\delta) = 1$

Part (b): Consider $\iota : \mathbb{F}[t]_n \rightarrow \mathbb{F}[t]_{n+1}$ which takes $p(t) \rightarrow \int_0^t p(t)$. This function is injective, but not surjective: for the latter, take $1 \in \mathbb{F}[t]_{n+1}$. No polynomial on $\mathbb{F}[t]_n$ integrates to this over $[0, t]$: take $p(t) = a_0 + a_1 t + \dots + a_n t^n$. Then

$$\iota[p(t)] = a_0 t + a_1 \frac{t^2}{2} + \dots + a_n \frac{t^{n+1}}{n+1} \neq 1$$

Suppose $\iota(p(t)) = \iota(q(t))$. Since polynomials of degree ≥ 1 (as all anti-derivatives of $p(t) \in \mathbb{F}[t]_n$ are) evaluate to 0 at 0, we have

$$a_0 t + a_1 \frac{t^2}{2} + \dots + a_n \frac{t^{n+1}}{n+1} = b_0 t + b_1 \frac{t^2}{2} + \dots + b_n \frac{t^{n+1}}{n+1}$$

Taking the derivative of both sides (which we can do, as δ is a linear transformation), we get

$$a_0 + \dots + a_n t^n = b_0 + \dots + b_n t^n \implies p(t) = q(t)$$

The image of ι is $\{a_1 t + \dots + a_n t^n + a_{n+1} t^{n+1} : a_i \in \mathbb{F}\}$. Suppose $q(t) \in \text{Im}(\iota)$ were not in this set. WLOG, we can just write $q(t) = a + bt^{n+2}$ for some $a, b \in \mathbb{F}$. Any polynomial of degree $\leq n$ will integrate to a polynomial of degree $\leq n + 1$, so we arrive at a contradiction, and $\text{Im}(\iota) \subseteq \{a_1 t + \dots + a_n t^n + a_{n+1} t^{n+1}\}$.

For $p(t) = a_1 t + \dots + a_n t^n + a_{n+1} t^{n+1}$, we have that $\iota(a_1 + 2a_2 t + \dots + (n+1)a_{n+1} t^n) = p(t)$, so $p(t) \in \text{Im}(\iota)$, and so $\text{Im}(\iota) = \{a_1 t + \dots + a_n t^n + a_{n+1} t^{n+1}\}$. This has dimension $n + 1$, so $\text{rank}(\iota) = n + 1$.

To overdo it, the LHS has degree at least one, and the RHS has degree 0. In fact, this shows that no $p(t)$ can integrate to a constant.

Note that, since \mathbb{F} is characteristic 0 or $> n + 1$, none of these coefficients = 0, so we don't run into any trouble.

$\mathbb{F}[t]_{n+1}$ has a basis $\{1, t, t^2, \dots, t^{n+1}\}$, as shown in class, so $\{t, t^2, \dots, t^{n+1}\}$ is linearly independent. Clearly it is spanning for $\text{Im}(\iota)$, so $\{t, t^2, \dots, t^{n+1}\}$ is a basis for $\text{Im}(\iota)$ with $n + 1$ elements.

Part (c): Let $p(t) := a_0 + a_1 t + \dots + a_n t^n$. Then

$$\begin{aligned}
 \delta \circ \iota(p(t)) &= \delta \circ \iota(a_0 + a_1 t + \dots + a_n t^n) \\
 &= \delta \left[a_0 t + \frac{a_1 t^2}{2} + \dots + a_n \frac{t^{n+1}}{n+1} \right] \\
 &= a_0 + a_1 t + \dots + a_n t^n = p(t) \\
 &\implies \delta \circ \iota = I_{\mathbb{F}[t]_n}
 \end{aligned}$$

Thus ι is a right inverse for δ . Now let $q(t) = a_0 + \dots + a_{n+1} t^{n+1}$. Then:

$$\begin{aligned}
 \iota \circ \delta(q(t)) &= \iota \circ \delta(a_0 + a_1 t + \dots + a_{n+1} t^{n+1}) \\
 &= \iota(a_1 + 2a_2 t + a_{n+1}(n+1)t^n) \\
 &= a_1 t + a_2 t^2 + \dots + a_{n+1} t^{n+1} \neq q(t) \\
 &\implies \iota \circ \delta \neq I_{\mathbb{F}[t]_{n+1}}
 \end{aligned}$$

so ι is not a left inverse for δ .

QUESTION 2

Part (a): Consider $T : (W_0, W_1) \rightarrow V$ which sends $(w_0, w_1) \rightarrow w_0 + w_1$. The kernel of this set is $\{(w_0, w_1) \in W_0 \times W_1 : w_0 + w_1 = 0\}$, just by definition.

Consider the transformation $S : W_0 \cap W_1 \rightarrow \ker(T)$ which sends $w \rightarrow (w, -w)$. This is an isomorphism:

OK Definition $(w, -w)$ is indeed $\in \ker(T)$, since $w + (-w) = 0$, $w \in W_0$ and W_1 .

$$\begin{aligned} \text{Linear } S(aw + a'w') &= (aw + a'w', -aw - a'w') = (aw, -aw) + (a'w', -a'w') = a(w, -w) + \\ &a'(w', -w') = aS(w) + a'S(w') \end{aligned}$$

Injective Let $S(w) = S(w')$. Then $(w, -w) = (w', -w')$. It follows that $w = w'$

Surjective For any $(w_0, w_1) \in \ker(T)$, we have $w_0 + w_1 = 0$, so $(w_0, w_1) = (w_0, -w_0) = S(w_0)$. Note also that $w_0 \in W_1 \cap W_0$: since $w_1 \in W_1$, we have $-w_1 = w_0 \in W_1$ by closure.

Part (b): To show $1 \iff 2 \iff 3$:

- $1 \implies 2$ Let $v = w_0 + w_1$ be a unique representation of $v \in V$, where $w_0 \in W_0, w_1 \in W_1$. Then $T[(w_0, w_1)] = T[(w_2, w_3)] \implies v = w_2 + w_3$, with $w_2 \in W_0, w_3 \in W_1$. But our assumption stipulates $w_2 = w_0, w_3 = w_1$, so $(w_0, w_1) = (w_2, w_3)$.
- $2 \implies 1$ Let $v = w_0 + w_1$ be some representation of v . Suppose another existed, and write $v = w_2 + w_3$ for $w_2 \in W_0, w_3 \in W_1$. Then $T(v) = T[(w_0, w_1)] = T[(w_2, w_3)]$. By injectivity, $(w_0, w_1) = (w_2, w_3)$, so $w_0 = w_2$ and $w_1 = w_3$, so this representation is unique.
- $2 \iff 3$ We have $V = W_1 + W_2$, so it remains to show $W_1 \cap W_2 = \{0\} \iff T$ injective. But T is injective $\iff \ker(T) = \{0\}$. And from part (a), we have that $\ker(T)$ and $W_0 \cap W_1$ are isomorphic, so $\ker(T) = \{0\} \iff W_0 \cap W_1 = \{0\}$ as well.

Part (c): By dimension theorem, we have

$$\dim(W_1 \times W_2) = \dim(\ker(T)) + \dim(\text{Im}(T))$$

Note: $\text{Im}(T) = W_0 + W_1 = V$

$$= \dim(W_1 \cap W_2) + \dim(V)$$

$$\implies \dim(V) = \dim(W_1 \times W_2) - \dim(W_1 \cap W_2)$$

$$\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

c.f. HW2, Q2

QUESTION 3

Part (a): Consider $P_0 : V \rightarrow V$, which sends $v \rightarrow w_0$, where $v = w_0 + w_1$ is a unique representation of v for $w_0 \in W_0, w_1 \in W_1$. This is linear: let $v = w_0 + w_1$ and $v' = w'_0 + w'_1$. Then:

$$\begin{aligned} P_0(av + a'v') &= P_0(aw_0 + aw_1 + a'w'_0 + a'w'_1) \\ &= aw_0 + a'w'_0 \quad \text{since } \underbrace{(aw_0 + a'w'_0)}_{\in W_0} + \underbrace{(aw_1 + a'w'_1)}_{\in W_1} = v \end{aligned}$$

which is just $aP_0(v) + a'P_0(v')$, hence P_0 is linear. Furthermore, $P_0^2 = P_0$:

$$P_0^2(v) = P_0(P_0(v)) = P_0(w_0) = w_0 \quad \text{since } w_0 \text{ is its own representation}$$

As $P_0(v) = w_0$, we conclude $P_0^2 = P_0$. Lastly, we have

$$\begin{aligned} \ker(P_0) &= \{v \in V : P_0(v) = 0\} & \text{Im}(P_0) &= \{P_0(v) : v \in V\} \\ &= \{w_0 + w_1 : w_0 = 0\} = W_1 & &= \{P_0(w_0 + w_1) : w_0 \in W_0, w_1 \in W_1\} \\ & & &= \{w_0 : w_0 \in W_0\} = W_0 \end{aligned}$$

Part (b): Let $T : V \rightarrow V$ be s.t. $T^2 = T$. For $v \in V$, write $v = v - T(v) + T(v)$. We'll show that $v - T(v) \in \ker(T)$: let $v - T(v) = w$ for some $w \in V$. Then $T(v) - T^2(v) = T(w) \implies T(v) - T(v) = 0 = T(w)$, since $T^2 = T$, and thus $w = v - T(v) \in \ker(T)$. Clearly $T(v) \in \text{Im}(T)$, so $V = \ker(T) + \text{Im}(T)$.

It remains to show that $\ker(T) \cap \text{Im}(T) = \{0\}$. Suppose a non-zero element v were in this intersection, and write $T(v) = 0$ and $v = T(w)$ for some $w \in V$. Then $T(v) = T^2(w) \implies T(v) = T(w) \implies 0 = T(w)$, so $w \in \ker(T)$. But we have $v = T(w)$, so $v = 0$ \nmid , and $\ker(T) \cap \text{Im}(T) = \{0\}$.

$$\implies V = \ker(T) \oplus \text{Im}(T)$$

The projection onto $\text{Im}(T)$ along $\ker(T)$ is precisely $P : V \rightarrow V$ which sends $v \rightarrow w$, where $w \in \text{Im}(T)$, and $v = w + y$ for some $y \in \ker(T)$. Well, if $v = w + y$, then $T(v) = T(w + y) = T(w) + T(y) = T(w)$, so T sends $v \rightarrow T(w)$.

Mental gymnastics: $w \in \text{Im}(T)$, so $T(w) = T \circ T(v')$ for some *other* $v' \in V$. Thus, T really sends $v \rightarrow T^2(v') = T(v')$. But we said $w = T(v')$, so T sends $v \rightarrow w$.

$$\implies T = P, \text{ as defined above.}$$

Part(c): Let $(x, y) \in \mathbb{R}^2$. Then $(x, y) = (0, y - x) + (x, x)$. Thus, for $V := \mathbb{R}^2$, the projection onto the y -axis along $\{(t, t) : t \in \mathbb{R}\}$ is the function $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which sends $(x, y) \rightarrow (0, y - x)$. *Fun little animation here.*

QUESTION 4

Consider the set $\tau = \{T_{v,w} : v \in \beta, w \in \gamma\}$, where β, γ are basis of V and W , respectively, and $\beta := \{v_1, \dots, v_n\}$ is finite. This is a basis for $\text{Hom}(V, W)$.

Independence To consider a truly arbitrary subset of τ , we need to represent all $T_{v_i, \times}$ and, for $T_{v_i, \times}$, any number of $\times = w_i$. Thus, we form the following combination:

$$\star \quad a_{11}T_{v_1, w_1} + \dots + a_{1k}T_{v_1, w_k} + \dots + a_{nl}T_{v_n, w_l} + \dots + a_{nm}T_{v_n, w_m} = \mathbb{O}$$

where \mathbb{O} is the transformation that sends $v \rightarrow \mathbb{O}_W$.

This must hold for all $v_i \in \beta$, so we can evaluate the combination at $v = v_1$. Since $T_{v_1, w}(w) = w$ and $T_{v_i}(w) = 0$ for $i \neq j$, $w \in \gamma$, we have

$$a_{11}w_1 + \dots + a_{1k}w_k = 0 \implies a_{11} = \dots = a_{1k} = 0$$

since $w_i \in \gamma$ are members of a basis. Similarly, evaluating \star at any v_j will imply that $a_{v_j, w} = 0$, $w \in \gamma$. These are all our coefficients, so \star is a trivial combination, and τ is linearly independent.

Spanning Consider a transformation $T : V \rightarrow W$, which sends $v_i \rightarrow w_i$ for $w_i \in W$.

$$\begin{aligned} T(v) &= T(a_1 v_1 + \dots + a_n v_n) \quad \text{for constants } a_i \in \mathbb{F} \\ &= a_1 T(v_1) + \dots + a_n T(v_n) = a_1 w_1 + \dots + a_n w_n \\ &= T_{v_1, w_1}(v) + \dots + T_{v_n, w_n}(v) \quad \spadesuit \end{aligned}$$

where T_{v_i, w_i} sends $v_i \rightarrow w_i$ and $v_j \rightarrow 0$ for $j \neq i$. For this last step, see that

$$\begin{aligned} T_{v_i, w_i}(v) &= T_{v_i, w_i}(a_1 v_1 + \dots + a_n v_n) \\ &= a_1 T_{v_i, w_i}(v_1) + \dots + a_i T_{v_i, w_i}(v_i) + \dots + a_n T_{v_i, w_i}(v_n) = a_i w_i \end{aligned}$$

Thus, it only remains to show that $T_{v_i, w_i} \in \text{Span}(\tau)$, but

$$\begin{aligned} T_{v_i, w_i}(v) &= a_i w_i = a_i [b_1 w_1^* + \dots + b_n w_n^*] \quad w_i^* \in \gamma, b_i \in \mathbb{F} \\ &= a_i \left[\frac{b_1}{a_1} T_{v_1, w_1^*}(v) + \dots + \frac{b_n}{a_n} T_{v_n, w_n^*}(v) \right] \end{aligned}$$

where $w_i^* \in \gamma$. The second line requires the following justification:

$$T_{v_1, w_1^*}(v) = T_{v_1, w_1^*}(a_1 v_1 + \dots + a_n v_n) = a_1 w_1^*$$

Since $w_i^* \in \gamma$, $T_{v_i, w_i^*} \in \tau$, so $T_{v_i, w_i} \in \text{Span}(\tau)$. Thus, \spadesuit , i.e. $T(v)$, $\in \text{Span}(\tau)$. Clearly $\text{Span}(\tau) \subseteq \text{Hom}(V, W)$, so $\text{Span}(\tau) = \text{Hom}(V, W)$, and τ is a basis.

QUESTION 5

We need to show $L_{E_{ji}} = T_{v_i, w_j}$, where $v_i \in \beta$, $w_i \in \gamma$, the standard bases for \mathbb{F}^n and \mathbb{F}^m , respectively. Then

$$\begin{aligned} \{L_{E_{ji}} : i \in [1, m], j \in [1, n]\} &= \{L_{E_{ji}} : j \in [1, m], i \in [1, n]\} \\ &= \{T_{v_i, w_j} : i \in [1, n], j \in [1, m]\} = \{T_{v, w} : v \in \beta, w \in \gamma\} \end{aligned}$$

as desired. Consider $L_{E_{ji}}$. This is the transformation that sends $v \rightarrow E_{ji} \cdot v$, where v is represented as a column vector $\langle a_1, \dots, a_n \rangle$, $a_i \in \mathbb{F}$:

$$L_{E_{ji}}(v) = E_{ji} \cdot v = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & 1_{ji} & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + a_i \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1_j \\ 0 \end{bmatrix}}_{=w_j} + \dots + a_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a_i w_j$$

And this is precisely $T_{v_i, w_j}(v)$, which expands to

$$T_{v_i, w_j}(v) = T_{v_i, w_j}(a_1 v_1 + \dots + a_n v_n) = a_1 T_{v_i, w_j}(v_1) + \dots + a_i T_{v_i, w_j}(v_i) + \dots + a_n T_{v_i, w_j}(v_n) = a_i w_j$$

QUESTION 6

Linearity: We need to first show $[T_1 + T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$ and $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for $a \in \mathbb{F}$, $T \in \text{Hom}(V, W)$. Let $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$

$$\begin{aligned}
[T_1 + T_2]_\beta^\gamma &= \begin{bmatrix} | & | & | & | \\ [(T_1 + T_2)(v_1)]_\gamma & [(T_1 + T_2)(v_2)]_\gamma & \cdots & [(T_1 + T_2)(v_n)]_\gamma \\ | & | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & | & | \\ [T_1(v_1)]_\gamma + [T_2(v_1)]_\gamma & [T_1(v_2)]_\gamma + [T_2(v_2)]_\gamma & \cdots & [T_1(v_n)]_\gamma + [T_2(v_n)]_\gamma \\ | & | & | & | \end{bmatrix} \quad \text{by linearity of } I_\gamma \\
&= \begin{bmatrix} | & | & | & | \\ [T_1(v_1)]_\gamma & [T_1(v_2)]_\gamma & \cdots & [T_1(v_n)]_\gamma \\ | & | & | & | \end{bmatrix} + \begin{bmatrix} | & | & | & | \\ [T_2(v_1)]_\gamma & [T_2(v_2)]_\gamma & \cdots & [T_2(v_n)]_\gamma \\ | & | & | & | \end{bmatrix} \\
&= [T_1]_\beta^\gamma + [T_2]_\beta^\gamma \\
[aT]_\beta^\gamma &= \begin{bmatrix} | & | & | & | \\ [aT(v_1)]_\gamma & [aT(v_2)]_\gamma & \cdots & [aT(v_n)]_\gamma \\ | & | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & | & | \\ a[T(v_1)]_\gamma & a[T(v_2)]_\gamma & \cdots & a[T(v_n)]_\gamma \\ | & | & | & | \end{bmatrix} \quad \text{by linearity of } I_\gamma \\
&= a \begin{bmatrix} | & | & | & | \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & | & | & | \end{bmatrix} = a[T]_\beta^\gamma
\end{aligned}$$

Inverse: If an inverse exists for $T \rightarrow [T]_\beta^\gamma$, then it is bijective. Since linearity has been shown, this is sufficient to show isomorphism. Consider $A \rightarrow I_\gamma^{-1} \circ L_A \circ I_\beta$.

We show the mapping $T \rightarrow [T]_\beta^\gamma \rightarrow I_\gamma^{-1} \circ L_{[T]_\beta^\gamma} \circ I_\beta$ is the identity on $\text{Hom}(V, W)$, i.e. takes $T \rightarrow T$. However, we've previously seen that $I_\gamma \circ T = L_{[T]_\beta^\gamma} \circ I_\beta$, so

$T = I_\gamma^{-1} \circ L_{[T]_\beta^\gamma} \circ I_\beta$, as desired.

We now need to show that $A \rightarrow I_\gamma^{-1} \circ L_A \circ I_\beta \rightarrow [I_\gamma^{-1} \circ L_A \circ I_\beta]_\beta^\gamma$ is the identity on $M_{m \times n}(\mathbb{F})$, i.e. takes $A \rightarrow A$:

$$[I_\gamma^{-1} \circ L_A \circ I_\beta]_\beta^\gamma = \left[\begin{array}{c|c|c|c} & & & \\ \hline [I_\gamma^{-1} \circ L_A \circ I_\beta(v_1)]_\gamma & [I_\gamma^{-1} \circ L_A \circ I_\beta(v_2)]_\gamma & \cdots & [I_\gamma^{-1} \circ L_A \circ I_\beta(v_n)]_\gamma \\ \hline & & & \end{array} \right]$$

We have $[I_\gamma^{-1} \circ L_A \circ I_\beta(v_i)]_\gamma = [I_\gamma^{-1} \circ L_A([v_i]_\beta)]_\gamma = [I_\gamma^{-1}(A \cdot [v_i]_\beta)]_\gamma$, but $v_i \in \beta$, so $[v_i]_\beta = \langle 0, \dots, 0, 1, 0, \dots, 0 \rangle$, where 1 is in the i^{th} position. Then $A \cdot [v_i]_\beta = A^{(i)}$. Finally, I_γ^{-1} is an isomorphism, so $v \rightarrow I_\gamma^{-1}(v) \rightarrow [I_\gamma^{-1}(v)]_\gamma = \text{Id}$. Combining:

$\Rightarrow [I_\gamma^{-1}(A \cdot [v_i]_\beta)]_\gamma = A^{(i)}$. We return to the matrix form to find

$$[I_\gamma^{-1} \circ L_A \circ I_\beta]_\beta^\gamma = \left[\begin{array}{c|c|c|c} & & & \\ \hline A^{(1)} & A^{(2)} & \cdots & A^{(n)} \\ \hline & & & \end{array} \right] = A$$

and we're done.

We can take the inverse and apply it, since I_γ is an isomorphism

QUESTION 7

$$\begin{aligned}
[T]_{\alpha}^{\beta} &= \begin{bmatrix} | & | & | & | \\ [T(E_{11})]_{\beta} & [T(E_{12})]_{\beta} & [T(E_{21})]_{\beta} & [T(E_{22})]_{\beta} \\ | & | & | & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & | & | \\ [1]_{\beta} & [1+t^2]_{\beta} & [0]_{\beta} & [2t]_{\beta} \\ | & | & | & | \end{bmatrix}
\end{aligned}$$

We now write

$$1 = 1(1) \quad 1 + t^2 = 1(1) + 1(t^2) \quad 0 = 0 \quad \text{and} \quad 2t = 2(t)$$

where (\cdot) are our basis vectors. Then $[1]_{\beta} = \langle 1, 0, 0 \rangle$, $[1 + t^2]_{\beta} = \langle 1, 0, 1 \rangle$, $[0] = \langle 0, 0, 0 \rangle$, and $[2t]_{\beta} = \langle 0, 2, 0 \rangle$. Thus:

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Implicit in this calculation is the assumption that β is ordered as $\{1, t, t^2\}$ and α as $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ (the final result would change otherwise).

QUESTION 8

Part (a): Let $x \in \text{Im}(T^k)$. Then $x = T^k(y)$ for some $y \in V \implies T(x) = T(T^k(y)) = T^{k+1}(y)$. This means $x \in T^{-1}(\text{Im}(T^{k+1}))$, so $\text{Im}(T^k) \subseteq T^{-1}(\text{Im}(T^{k+1}))$.

Now let $x \in T^{-1}(\text{Im}(T^{k+1}))$. Then $T(x) \in \text{Im}(T^{k+1})$, which means $T(x) = T^{k+1}(y)$ for some $y \in V$, or $T(x) = T^k(T(y))$. We have $T^k(T(y)) \in \text{Im}(T)$, or $T^k(x) \in \text{Im}(T)$

Let $x \in \ker(T^{k+1})$. Then $T^{k+1}(x) = 0 \implies T^k(T(x)) = 0$, so $T(x) \in \ker(T^k)$, i.e. $x \in T^{-1}(\ker(T^k))$, so $\ker(T^{k+1}) \subseteq T^{-1}(\ker(T^k))$.

Now let $x \in T^{-1}(\ker(T^k))$. Then $T(x) \in \ker(T^k)$, i.e. $T^k(T(x)) = 0 \implies T^{k+1}(x) = 0$, so $x \in \ker(T^{k+1})$, and $T^{-1}(\ker(T^k)) \subseteq \ker(T^{k+1}) \implies T^{-1}(\ker(T^k)) = \ker(T^{k+1})$.

Part (b): Note that $T^l(\text{Im}(T^k)) = \text{Im}(T^{k+l})$:

$$\begin{aligned} T^l(\text{Im}(T^k)) &= \{T^l(x) : x \in \text{Im}(T^k)\} = \{T^l(x) : x = T^k(y), y \in V\} \\ &= \{T^l(T^k(y)) : y \in V\} = \{T^{k+l}(y) : y \in V\} = \text{Im}(T^{k+l}) \end{aligned}$$

Then, assuming $\text{Im}(T^k) = \text{Im}(T^{k+1})$ for some $k \in \mathbb{N}$, we show $\text{Im}(T^k) = T^l \text{Im}(T^k)$ by induction on l , which shows $\text{Im}(T^k) = \text{Im}(T^{k+l})$ from above.

$$\begin{aligned} l = 1 & \quad \text{Im}(T^k) = \text{Im}(T^{k+1}) = T(\text{Im}(T^k)) \\ l \rightarrow l+1 & \quad T^{l+1}(\text{Im}(T^k)) = T \circ T^l(\text{Im}(T^k)) = T(\text{Im}(T^k)) = \text{Im}(T^k) \end{aligned}$$

Assume now $\ker(T^k) = \ker(T^{k+1})$ for some k . We'll show $\ker(T^k) = \ker(T^{k+l})$ by induction on l :

$$l = 1 \quad \ker(T^k) = \ker(T^{k+1})$$

$l \rightarrow l+1$ We know $\ker(T^k) \subseteq \ker(T^{k+l+1})$. Let $x \in \ker(T^{k+l+1})$. Then $T^{k+l}(T(x)) = 0$, so $T(x) \in \ker(T^{k+l})$. By ind. hyp., we have $T(x) \in \ker(T^k)$, so $x \in T^{-1}(\ker(T^k))$. From part (a), this means $x \in \ker(T^{k+1}) \implies x \in \ker(T^k)$ by assumption.

Part (c): Suppose $\text{Im}(T^k) = \text{Im}(T^{k+1})$. Then by dimension theorem, we have

$$\begin{aligned} \dim(\text{Im}(T^k)) + \dim(\ker(T^k)) &= \dim(V) = \dim(\text{Im}(T^{k+1})) + \dim(\ker(T^{k+1})) \\ \implies \dim(\ker(T^k)) &= \dim(\ker(T^{k+1})) \implies \ker(T^k) = \ker(T^{k+1}), \text{ using the fact} \\ &\text{that } \ker(T^k) \subseteq \ker(T^{k+1}). \end{aligned}$$

Similarly, if $\dim(\ker(T^k)) = \dim(\ker(T^{k+1}))$, then $\dim(\text{Im}(T^k)) = \dim(\text{Im}(T^{k+1}))$.

$$\implies \text{Since } \text{Im}(T^{k+1}) \subseteq \text{Im}(T^k), \text{ this means } \text{Im}(T^k) = \text{Im}(T^{k+1}).$$

Part (d): Suppose $\ker(T^k) \neq \ker(T^{k+1}) \quad \forall k \leq n$. Since $\ker(T^k) \subseteq \ker(T^{k+1})$, this means

$$\ker(T) \subsetneq \ker(T^2) \subsetneq \dots \subsetneq \ker(T^n)$$

By monotonicity, this means

$$0 < \dim(\ker(T)) < \dim(\ker(T^2)) < \dots < \dim(\ker(T^n)) \leq n$$

then we can notate below the minimum dimensions of each:

$$0 < \underset{\geq 1}{\dim(\ker(T))} < \underset{\geq 2}{\dim(\ker(T^2))} < \dots < \underset{\geq n}{\dim(\ker(T^n))} \leq n$$

We conclude that $\dim(\ker(T^n)) = n$, but $\dim(\ker(T^n)) \leq \dim(\ker(T^{n+1})) \leq n$ by necessity, so $\dim(\ker(T^n)) = \dim(\ker(T^{n+1}))$, and we arrive at a contradiction.
 $\implies \exists k \leq n : \ker(T^k) = \ker(T^{k+1}) \implies \text{Im}(T^k) = \text{Im}(T^{k+1})$ by (c).

Note that, if $\dim(\ker(T)) = 0$, that means $\ker(T) = \{0\}$, so T is injective, and thus surjective. Then, $\ker(T^2)$ is
 $\{x : T(T(x)) = 0\}$
 $= \{y : T(y) = 0, y = T(x)\}$
 $= \{y : T(y) = 0, y \in V\}$
 $= \ker(T)$
 which is a contradiction

QUESTION 9

Suppose $\exists k \in \mathbb{N}$ such that $T^k = 0$, i.e. $\text{Im}(T^k) = \{0\}$. We can assume $k > n$, else $\text{Im}(T^n) \subseteq \dots \subseteq \text{Im}(T^k) = \{0\} \implies \text{Im}(T^n) = \{0\}$, i.e. $T^n = 0$.

From (8d), we know $\exists l \leq n$ such that $\text{Im}(T^l) = \text{Im}(T^{l+1})$, and from (8b) this means $\text{Im}(T^l) = \text{Im}(T^{l+l'})$ for any $l' \in \mathbb{N}$. Since $k > n$, let $l' := k - l$.

Then $\text{Im}(T^l) = \text{Im}(T^{l+(k-l)}) = \text{Im}(T^k) = \{0\}$. Thus $\text{Im}(T^l) = \{0\}$, or $T^l = 0$.

$l \leq n$, so by our first remarks, $T^n = 0$

□