

# Geometry & Topology 1

MATH576

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## CONTENTS

<b>I</b>	<b>Topologies</b>	<b>1</b>
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# I Topologies

We will study continuous mappings between objects in  $\mathbb{R}^n$ . These, in contrast with isogenies (rotations, reflections, translations), will allow us to identify circles with squares, doughnuts with mugs, etc. A guiding question: how do we describe deformation mathematically?

A *metric* on a space  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  satisfying

DEF 1.1

- (1)  $d(x, y) \geq 0$ , and  $d(x, y) = 0 \iff x = y$
- (2)  $d(x, y) = d(y, x)$
- (3)  $d(x, y) \leq d(x, z) + d(y, z)$

A *metric space*  $(X, d)$  is a space with a specific metric.

The *ball centered at  $x$  of radius  $\varepsilon$* , denoted  $B(x, \varepsilon)$ , is the set

DEF 1.2  
or  $B_d(x, \varepsilon)$   
w.r.t. a metric  
 $d$

$$\{y \in X : d(x, y) < \varepsilon\} \quad \text{with} \quad \varepsilon > 0$$

$U \subseteq X$  is called *open* if, for each  $x \in X$ , there exists  $\varepsilon > 0$  with  $B(x, \varepsilon) \subseteq U$ .

DEF 1.3

Denote by  $\mathcal{T}$  the family of all open sets in  $X$ .

DEF 1.4

Given  $(X, d)$ :

PROP 1.1

- (1)  $\emptyset, X \in \mathcal{T}$
- (2) For any (possibly infinite) collection of sets in  $\mathcal{T}$ , their union is in  $\mathcal{T}$
- (3) For any finite collection of sets in  $\mathcal{T}$ , their intersection is in  $\mathcal{T}$

(1) follows by definition. For (2), let  $U$  be some union of sets in  $\mathcal{T}$ , and  $x \in U$ . Then  $x$  belongs to some open set  $U_i$  comprising the union. Then  $B(x, \varepsilon_i) \subseteq U_i \subseteq U$  for some  $\varepsilon_i$ .

PROOF.

For (3), let  $\{U_1, \dots, U_n\} \subseteq \mathcal{T}$ . Pick  $x \in \bigcup_{i \in [n]} U_i =: U$ . It belongs to all sets  $U_i : i \in [n]$ . Since these sets are open, there exists some  $\varepsilon_i : B(x, \varepsilon_i) \subseteq U_i$ . Let  $\varepsilon := \min\{\varepsilon_i\}$ . Then  $B(x, \varepsilon) \subseteq U_i$  for each  $i$ , and hence  $B(x, \varepsilon) \subseteq U$ .  $\square$

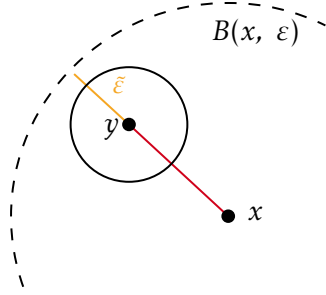
A *topology* on  $(X, d)$  is a collection  $\mathcal{T}$  of subsets of  $X$  satisfying [Prop 1.1](#). Elements of  $\mathcal{T}$  are called *open sets*.

DEF 1.5

If  $\mathcal{T} \subseteq \mathcal{T}'$ , we say that  $\mathcal{T}$  is *coarser* than  $\mathcal{T}'$ . Conversely,  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$ .

**Eg. 1.1.1**  $B(x, \varepsilon)$  is open. Let  $y$  be some internal point. Take  $\tilde{\varepsilon} := \varepsilon - d(x, y)$ . Then if  $z \in B(y, \tilde{\varepsilon})$ , we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon \implies z \in B(x, \varepsilon)$$



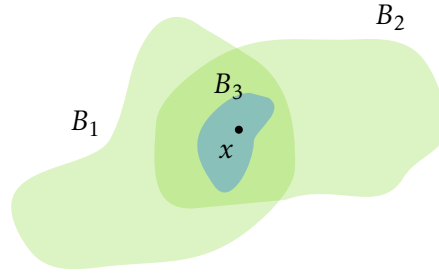
**Eg. 1.1.2** If  $X = \{a, b, c\}$ , we can consider  $\mathcal{T} = \mathcal{P}(X)$ , i.e. the powerset of  $X$ . This is a topology. We can also arrive at this by considering the metric  $d(x, y) = 1, x \neq y$ . The topology  $\mathcal{T}' = \{\emptyset, \{a, b\}, \{b, c\}, \{b\}, X\}$  has a less obvious metric-theoretical explanation.

**Eg. 1.1.3** Condition 3 of [Prop 1.1](#) fails for infinite collections (e.g. balls of radius approaching 0).

**DEF 1.6** The pair  $(\mathcal{T}, X)$  is called a *topological space*.

**DEF 1.7** A *basis*  $\mathcal{B}$  on a topology  $(X, \mathcal{T})$  is a collection of subsets of  $X$  satisfying

- (1) For each  $x \in X$ , there exists  $B \in \mathcal{B}$  with  $x \in B$
- (2) If  $x \in B_1, B_2 \in \mathcal{B}$ , then there exists  $B_3 \in \mathcal{B}$  with  $x \in B_3 \subseteq (B_1 \cap B_2)$ .



**DEF 1.8** The *topology generated by*  $\mathcal{B}$  is the collection of sets

$$\{U : \forall x \in U, \exists B \in \mathcal{B} : x \in B \subseteq U\}$$

**Eg. 1.2.1**  $\mathcal{B} = \{B(x, \varepsilon), x \in X, \varepsilon > 0\}$  is a basis. Clearly condition (1) of [Def 1.7](#) is satisfied. For condition (2), suppose  $x \in B(x_i, \varepsilon_i)$  for  $i = 1, 2$ . Take  $\varepsilon_3 := \min\{\varepsilon_i - d(x, x_i)\}$ . Then  $B(x, \varepsilon_3)$  is contained in both balls  $B(x_i, \varepsilon_i)$ .

**PROP 1.2** The topology generated by a basis  $\mathcal{B}$  is a topology.

Let  $\mathcal{T}$  be generated by  $\mathcal{B}$ . We will go through each condition in [Prop 1.1](#):

PROOF.

- (1) Condition (1) of [Def 1.7](#) provides for  $\emptyset, X \in \mathcal{T}$
- (2) Let  $U_\alpha : \alpha \in J$  be a collection in  $\mathcal{T}$ , for some index  $J$ . Let  $U := \cup_{\alpha \in J} U_\alpha$ . Given  $x \in U$ , we know  $x \in U_\alpha$  for some  $\alpha \in J$ . Hence, since  $U_\alpha \in \mathcal{T}$ , there exists  $B \in \mathcal{B}$  with  $x \in B \subseteq U_\alpha \subseteq U$ .
- (3) It suffices to show  $U_1 \cap U_2 \in \mathcal{T}$ , since we only consider finite collections. Fix  $x \in U_1 \cap U_2$ . There exists  $B_1, B_2 \in \mathcal{B}$  with  $x \in B_1 \subseteq U_1$  and  $x \in B_2 \subseteq U_2$ . We know there exists, then,  $B_3 \in \mathcal{B}$  with  $x \in B_3 \subseteq (B_1 \cap B_2) \subseteq (U_1 \cap U_2)$ .  $\square$

Let  $\mathcal{B}$  be a basis for  $X$  generating  $\mathcal{T}$ . Then  $\mathcal{T}$  is the collection of all possible unions of elements in  $\mathcal{B}$ . PROP 1.3

Let  $\mathcal{T}'$  denote all possible unions of elements in  $\mathcal{B}$ . Then clearly  $\mathcal{T}' \subseteq \mathcal{T}$ , since  $\mathcal{B} \subseteq \mathcal{T}$ .  
Conversely, let  $U \in \mathcal{T}$ . Then each  $x \in U$  belongs to some  $B_x \subseteq U \in \mathcal{B}$ . Hence,  $U \subseteq \cup_{x \in U} B_x$ . But each  $B_x \subseteq U$ , so  $U$  is exactly this union, and hence  $U \in \mathcal{T}'$ .  $\square$

PROOF.

Let  $X, Y$  be topological spaces. Then the *product topology on  $X \times Y$*  is the topology generated by the basis DEF 1.9

$$\{U \times V : U \text{ open in } X, V \text{ open in } Y\}$$

Indeed, this is a proper basis. For (1) of [Def 1.7](#), if  $(x, y) \in X \times Y$ , we take  $U = X$  and  $V = Y$ . For (2), if  $(x, y) \in U_1 \times V_1, U_2 \times V_2$ , then  $U_1 \cap U_2 \in \mathcal{T}_X$  and  $V_1 \cap V_2 \in \mathcal{T}_Y$  are open. Hence,  $x \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$ .

Let  $Y \subseteq (X, \mathcal{T})$ . Then the collection  $\{Y \cap U : U \in \mathcal{T}\}$  is called the *subspace topology* with respect to  $Y$ . DEF 1.10

Let  $\mathcal{S}$  be a collection of subsets of  $X$ . Let  $\mathcal{B}$  be the collection of all finite intersections of elements in  $\mathcal{S}$ . Then  $\mathcal{B}$  is a basis in  $X$ , as in [Def 1.7](#). PROP 1.4

$\mathcal{S}$ , as above, is called a *sub-basis* of the topology in  $X$  generated by  $\mathcal{B}$ .

DEF 1.11

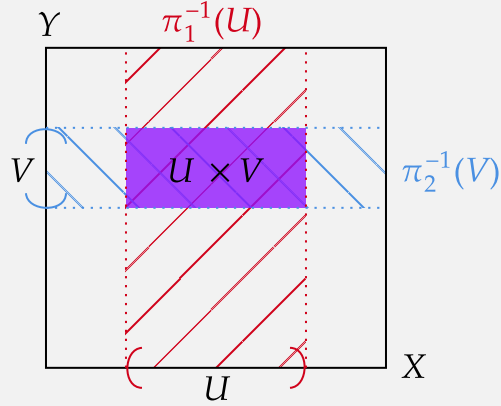
**Eg. 1.3.1** Let  $\pi_1 : X \times Y \rightarrow X, \pi_2 : X \times Y \rightarrow Y$  be the coordinate projections onto the  $X, Y$  subspaces of  $X \times Y$ . Then

$$\{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ open in } Y\} =: \mathcal{S}$$

is a sub-basis of the product topology.

PROOF.

The intersection of any two elements in  $\mathcal{S}$  may be written as  $U \times V$ ,  $U \times Y$ , or  $X \times V$  for open subsets  $U, V$  of  $X, Y$ , respectively.



**Eg. 1.3.2** Let  $X = \prod_{\alpha \in J} X_\alpha$ , with  $\pi_\alpha : X \rightarrow X_\alpha$  the coordinate projection. Then the product topology on  $X$  will have a sub-basis

$$\mathcal{S} = \{\pi_\alpha^{-1}(U) : U \text{ open in } X_\alpha : \alpha \in J\}$$

with corresponding basis

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \text{ open in } X_\alpha \right\}$$

where  $U_\alpha = X_\alpha$  except at finitely many indices.

**DEF 1.12** A subset  $A \subseteq X$  is called *closed* if  $X \setminus A$  is open.

**PROP 1.5** Let  $X$  be a topological space. Then

1.  $X, \emptyset$  are closed
2. Finite unions of closed sets are closed
3. Infinite intersections of closed sets are closed

PROOF.

We'll take a look at (2). Let  $A_1, A_2$  be closed. Then  $(A_1 \cup A_2)^C = A_1^C \cap A_2^C$  by De Morgan's. Then, This is open, since  $A_i^C$  is open, and hence  $A_1 \cup A_2$  is closed.  $\square$

**PROP 1.6** If  $X$  is a metric space, then  $A \subseteq X$  is closed  $\iff$  for each  $x_n \rightarrow x$  (i.e.  $d(x_n, x) \rightarrow 0$ ), with  $x_i \in A$ , then  $x \in A$ .

( $\implies$ ) Suppose  $x \notin A$ , and hence  $x \in X \setminus A$ , which is open. Hence,  $B(x, \varepsilon) \subseteq X \setminus A$  for some  $\varepsilon > 0$ . Choose  $x_n : d(x_n, x) < \varepsilon$ . Then  $x_n \in B(x, \varepsilon) \subseteq X \setminus A$ , which is a contradiction.

PROOF.

( $\impliedby$ ) Let  $x \in X \setminus A$ . Fix some  $n$ . We wish to show that there exists  $B(x, \frac{1}{n}) \subseteq X \setminus A$ . If not, then  $\exists x_n \in B(x, \frac{1}{n})$ . Repeating this, we have a sequence  $x_n \rightarrow x$  with  $x_n \in A$  and  $x \notin A$ , a contradiction.  $\square$

Let  $A$  be a subset of a topological space  $X$ . The *interior* of  $A$ , denoted  $\text{Int}(A)$ , is the union of all open sets contained in  $A$ . Hence, it is open. An alternative characterization of  $\text{Int}(A)$  is: the largest open subset of  $A$ .

DEF 1.13

Conversely, for  $A \subseteq X$ , the *closure* of  $A$ , denoted  $\overline{A}$ , is the intersection of all closed sets containing  $A$ . Hence, it is closed. An alternative characterization of  $\overline{A}$  is: the smallest closed set containing  $A$ .

DEF 1.14

We say  $A$  is *dense* in  $X$  if  $\overline{A} = X$ .

DEF 1.15

For  $x \in X$ ,  $U$  is called a *neighborhood* of  $x$  if  $x \in \text{Int}(U)$ .

DEF 1.16

**Eg. 1.4.1**  $[a, b] \subseteq \mathbb{R}$  is closed, as the complement,  $(-\infty, a) \cup (b, \infty)$ , is open.

**Eg. 1.4.2** "Closed balls," i.e.  $A = \{y : d(x, y) \leq \varepsilon\}$  for  $x \in X, \varepsilon > 0$ , are closed. For  $y \in X \setminus A$ , choose  $\tilde{\varepsilon} = d(x, y) - \varepsilon$  to show openness.

**Eg. 1.4.3**  $\text{Int}(\mathbb{Q}) = \emptyset$  and  $\overline{\mathbb{Q}} = \mathbb{R}$ , i.e.  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

**Eg. 1.4.4**  $X \setminus \text{Int}(A) = \overline{X \setminus A}$

**Eg. 1.4.5**  $U = \{y : d(x, y) \leq \varepsilon\}$  is a neighborhood of  $x$ , since  $B(x, \varepsilon) \subseteq U$ . In general, any open set containing  $x$  is a neighborhood of  $x$ .

Let  $A \subseteq X$  be a topological space. Then  $x \in \overline{A} \iff$  each neighborhood  $U$  of  $x$  intersects  $A$  (i.e.  $U \cap A \neq \emptyset$ ).

PROP 1.7

We'll prove the contrapositive, i.e.  $x \notin \overline{A} \iff$  there exists a neighborhood  $U$  with  $U \cap A = \emptyset$ .

PROOF.

( $\implies$ ) Take  $U = X \setminus \overline{A}$ , since  $x \in X \setminus \overline{A}$ .  $U$  is open, and hence a neighborhood. Since  $A \subseteq \overline{A}$ ,  $U \cap A = \emptyset$ .

( $\impliedby$ ) Let  $U$  be a neighborhood of  $x$  with  $U \cap A = \emptyset$ . Then  $X \setminus \text{Int}(U)$  is closed and contains  $A$ . Hence,  $\overline{A} \subseteq X \setminus \text{Int}(U)$ . But  $x \in \text{Int}(U)$ , so  $x \notin \overline{A}$ .  $\square$

Let  $X, Y$  be topological spaces. A function  $f : X \rightarrow Y$  is *continuous at*  $x \in X$  if, for each neighborhood  $A$  of  $f(x)$ , the preimage  $f^{-1}(A)$  is a neighborhood of  $x$ .

DEF 1.17

If  $f : X \rightarrow Y$  is continuous for all  $x \in X$ , we simply call  $f$  *continuous*.

DEF 1.18

**PROP 1.8**  $f$  is continuous  $\iff$  for each open set  $U \subseteq Y$ ,  $f^{-1}(U)$  is open in  $X$ .

**1.1 boop**

test!