Honours Discrete Mathematics

MATH 350

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CONTENTS

I Definitions

Graph theory is the study of pairwise relations between objects, e.g. computer networks, interpersonal relationships, airport flights. Graphs will help us simplify and abstract networks . DEF 1.1

A *graph* G is comprised of a set of vertices, denoted $V(G)$, where $|V(G)| < \infty$, a set of edges, denoted $E(G)$, where every edge is associated with two vertices. DEF 1.2 At least in this course

We say that an edge *joins* the two vertices it's associated with. Similarly, an edge is *incident* to a vertex which is an end of it. Conversely, a vertex is incident to an edge if it is an end of it. DEF 1.3

Two vertices are *adjacent* or *neighbors* if they are joined by an edge, the *degree* of a vertex is its number of neighbors. DEF 1.4

The *null graph* is the graph such that $V(G) = \varnothing$. The *complete graph* on n vertices, denoted K_n, is such that $|V(K_n)| = n$ and $|E(K_n)|$ is maximal. DEF 1.5

For a graph of n vertices, the maximal number of edges it may have is $\binom{n}{2}$. **PROP 1.1**

PROOF. Suppose every vertex is connected to every other vertex. Then $\sum_{v \in V(G)} \deg(v) = n(n-1) \implies |E(G)| = \frac{n(n-1)}{2} = \binom{n}{2}$. □

A graph of n vertices, where v_i is only adjacent to v_{i-1} and v_{i+1}, is called a *path* and is sometimes denoted by P_n. v_1 and v_n are called the *ends* of P_n. DEF 1.6

For $n \geq 3$, a *cycle* C_n is a graph with $V(G) = \{v_1, ..., v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. DEF 1.7

An *adjacency matrix* is a matrix which specifies if any two vertices are adjacent. Pairwise adjacency is denoted by a 1 entry, and 0 otherwise. For example, the following is an adjacency table for a 4 element cycle: DEF 1.8

	v_1	v_2	v_3	v_4
v_1	×	1	0	1
v_2	1	×	1	0
v_3	0	1	×	1
v_4	1	0	1	×

Similarly, an *incidence* matrix has rows in $V(G)$ and columns in $E(G)$, and marks with "1" pairs that are incident to eachother. The following is the incidence matrix for a 4 element cycle: DEF 1.9

	v_1	v_2	v_3	v_4
e_1	1	1	0	0
e_2	0	1	1	0
e_3	0	0	1	1
e_4	1	0	0	1

PROP 1.2 For a graph G, we always have $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

PROOF. Every edge has two vertices incident to it. Thus, $\sum \deg(v)$ will be the number of times an edge is incident to a vertex, i.e. the number of edges $\times$ 2. $\square$

DEF 1.10 H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

For two graphs G, H, the union $G \cup H$ is a graph such that $V(G \cup H) = V(G) \cup V(G)$ and $E(G \cup H) = E(G) \cup E(H)$. We similarly define the intersection $G \cap H$ to be such that $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$.

We cannot do the same for "$G \setminus H$," since we may delete vertices and keep their incident edges!

PROP 1.3 There are $2^{\binom{n}{2}}$ graphs with n vertices.

PROOF. We know the maximal number of edges of this graph is $\binom{n}{2}$. Then, for each edge, one may make a binary choice whether to include it or not $\therefore$ the number of graphs is $2^{\binom{n}{2}}$. $\square$

DEF 1.11 An *isomorphism* between H and G is a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G) \implies \phi(u)\phi(v) \in E(H)$.

We can now ask: how many graphs are there with n vertices up to isomorphism?

II Connectivity

DEF 2.1 A *walk* in G with ends u_0 and u_k is a sequence $(u_0, u_1, ..., u_k)$ such that $u_i \in V(G)$ and $u_i u_{i+1} \in E(G)$. The length of this walk is k.

u and v are called *connected* if there exists a walk in G with ends u and v OR, equivalently, there exists a path $P \subseteq G$ with ends u and v.

PROP 2.1 $\exists$ a walk in G with ends u and $v \iff \exists$ a path $P \subseteq G$ with ends u and v.

PROOF. ($\impliedby$) Let $P \subseteq G$ be a path with ends u and v. Then $V(P)$ can be numbered $u = v_0, v_1, ..., v_k = v$, where $v_i v_{i+1} \in E(P)$. Then $(v_0, ..., v_k)$ is a walk in G.

($\implies$) Let there exist a walk $(u = v_0, ..., v_k = v)$ with $v_i v_{i+1} \in E(G)$. WLOG suppose this is the walk of minimal length. If $v_i \neq v_j$, i.e. are pairwise distinct, then we already have a path. Suppose otherwise, and let $v_i = v_j$. Then $(v_0, ..., v_i, v_{j+1}, ..., v_k)$ is a *smaller* walk with ends u and v, which establishes the contradiction $\lightning$. $\square$

DEF 2.2 A graph G is called *connected* if $\forall u, v \in V(G)$, u and v are connected.

DEF 2.3 A *partition* of $V(G)$ is $(X_1, ..., X_k)$ such that $\cup_{i=1}^k X_i = V(G)$ and $X_i \cap X_j = \varnothing \; \forall i \neq j$.

PROP 2.2 A graph G is not connected $\iff \exists$ a partition (X, Y) of $V(G)$ such that no edge of G is incident to one vertex in X and one in Y.

PROOF.

($\Longleftarrow$) Suppose G were connected. Then choose $u \in X, v \in Y$ such that there exists a walk $(u = u_0, ..., u_k = v)$. Let u_i be minimal over i such that $u_i \in Y$. Then $u_{i-1} \in X$, and $u_{i-1}u_i \in E(G)$ $\lightning$.

($\Longrightarrow$) Let $u, v \in V(G)$ be such that there is no walk from u to v. Let X be the set of all $w \in V(G)$ such that $\exists$ a walk with ends u and w. Similarly, let $Y = V(G) \setminus X$. Clearly $V(G) = X \cup Y$, $X \cap Y = \varnothing$, and (X, Y) is a partition. Suppose there exists an edge from a vertex in X to a vertex in Y, i.e. $x \in X$, $y \in Y$. Then we have the walk $(u, ..., w, ..., x, y)$. But $y \notin X$ $\lightning$. $\square$

DEF 2.4

Let G be a graph. $H \subseteq G$ is called a *connected component* of G if H is a maximal connected subgraph of G, i.e. if $\exists H \subseteq H' \subseteq G$ with H' connected, then $H = H'$.

Sometimes we just say "component."

PROP 2.3

If H_1, H_2 are connected graphs, and $V(H_1) \cap V(H_2) \neq \varnothing$, then $H_1 \cup H_2$ is also connected.

PROOF.

Let $u \in H_1, v \in H_1 \cap H_2, w \in H_2$. Then $(u, ..., v)$ and $(v, ..., w)$ are both walks, and thus $(v, ..., v, ..., w)$ is a walk. $\square$

PROP 2.4

Every $v \in V(G)$ is a member of a unique connected component $H \subseteq G$.

PROOF.

$\{v\}$ is connected. If there does not exist $H \supseteq \{v\}$ also connected, then we are done. Otherwise, we may choose the maximal such connected superset.

Suppose $v \in H_1$ and H_2, two connected components. Then by Prop 2.3, $H_1 \cup H_2$ is connected. But since $H_1 \cup H_2 \supseteq H_1, H_2$, this violates maximality. We conclude that $H_1 = H_2$. $\square$

PROP 2.5

Let G be a graph, and let $H \subseteq G$ be a non-null and connected subgraph. Then H is a connected component of $G \iff \forall e \in E(G)$ with an end in $V(H)$, we have $e \in E(H)$.

PROOF.

For ($\Longrightarrow$), let $e = uv$, with $u \in V(H)$. If $v \in V(H)$, then we are done. Otherwise, suppose $e \notin E(H)$. We know v is a member of a unique connected component. But adding e to H would yield a further connected graph: take the graphs of $\{uv\}$ and H. Both are clearly connected, so $H \cup \{uv\}$ is connected.

($\Longleftarrow$) Let H' be a connected graph with $H' \supseteq H$. We may assume $H' \neq H$. Let $w \in V(H') \setminus V(H)$. Let $v \in V(H)$. Then, by connectedness of H', $\exists P$ with ends w and v. Enumerate:

$$V(P) = \{w = u_1, ..., u_k = v\}$$

Let u_i be maximal such that $u_i \in V(H') \setminus V(H)$. Then $u_{i+1} \in V(H)$, and $u_i u_{i+1} \in E(G)$. But then, by assumption, $u_{i+1} \in V(H)$, violating maximality of $u_i \implies \lightning$. $\square$

DEF 2.5

For $e \in E(G)$, $G \setminus e$ is a graph such that $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$. We say that $G \setminus e$ is "obtained from G by deleting e."

Similarly, for $v \in V(G)$, $G \setminus v$ is a graph such that $V(G \setminus v) = V(G) \setminus \{v\}$ and $E(G \setminus v) = E(G) \setminus \{e : e \text{ incident to } v\}$. We may also write $G - v$.

$\text{comp}(G)$ denotes the number of connected components of G.

PROP 2.6 $\text{comp}(G) = 1 \iff G$ is connected.

PROOF. ($\Longrightarrow$) Let H be G's one component. Let $v \notin H$. Then v is a member of a unique component, by Prop 2.4, and this cannot be H. Hence, $\text{comp}(G) > 1 \implies \lightning \implies v \in H$, so $H = G$.

($\Longleftarrow$) Let G be connected. Then $G' \supseteq G \implies G = G'$ tautologically, so G is a connected component. By uniqueness outlined in Prop 2.4, it must be the only one $\implies \text{comp}(G) = 1$. □

DEF 2.6 Let $e = uv \in E(G)$. e is called a *cut-edge* if it is not part of any cycle.

PROP 2.7 Let $e = uv \in E(G)$. Exactly one of the following holds:

1. ***e* is a cut-edge**: $\text{comp}(G \setminus e) = \text{comp}(G) + 1$, and u, v belong to different components of $G \setminus e$.

2. ***e* is not a cut-edge**: $\text{comp}(G \setminus e) = \text{comp}(G)$, and u, v belong to the same component.

PROOF. Let e be a cut-edge. Let $H_1, ..., H_k$ be the connected components of $G \setminus e$. If u, v belong to H_i, then $\exists$ a path $P \subseteq H_i$ with ends u and v. Adding e, this is a cycle $\lightning$.

WLOG, assume that u, v belong to $V(H_1), V(H_2)$, respectively. Then let H' be obtained by $H_1 \cup H_2$ by adding e. We claim that $H', H_2, ..., H_k$ are all components of G. By Prop 2.5, we only need to check the connectivity of H', and this holds by Prop 2.3. Since there do not exist any vertices *not* in $V(H_i) : i \geq 2$ or $V(H')$, these are all the components of G. Thus, $\text{comp}(G) + 1 = \text{comp}(G \setminus e)$.

Let e be part of a cycle C. Then $C \setminus e$ is a path with ends u and v. Hence, since u and v are connected, they belong to the same component in $G \setminus e$ (i.e. the component they belong to in G). □

III Trees and Forests

DEF 3.1 A *forest* is a graph with no cycles, i.e. every edge is a cut-edge.

DEF 3.2 A *tree* is a non-null connected forest.

PROP 3.1 Let F be a non-null forest. Then $\text{comp}(F) = |V(F)| - |E(F)|$.

PROOF. We'll show by induction on $|E(F)|$. If $n = 0$ then all vertices are their own connected components. Let $|E(F)| = n$, and assume $\text{comp}(F) = |V(F)| - |E(F)|$. Let $e \in E(F)$. Since F is a forest, e is a cut-edge, and thus $\text{comp}(G \setminus e) = \text{comp}(G) + 1 = |V(F)| - |E(F)| + 1 = |V(F)| - (|E(F)| - 1) = |V(F)| - |E(F \setminus e)| = |V(F \setminus e)| - |E(F \setminus e)|$. □

DEF 3.3 A *leaf* is a vertex with degree 1.

PROP 3.2 Let T be a tree with $|V(T)| \geq 2$. let $X = \{\text{leaves of } T\}$, $Y = \{v \in V(G) : \deg(v) \geq 3\}$. Then $|X| \geq |Y| + 2$.

Thus, trees have ≥ 2 leaves!

PROOF. By Prop 1.1, we have

$$\sum_{v \in V(T)} \deg(v) = 2|E(T)| \overset{3.1}{=} 2(|V(T)| - \text{comp}(G)) \overset{2.6}{=} 2(|V(T)| - 1)$$

$$\implies \sum_{v \in V(T)} (\deg(v) - 2) = 2(|V(T)| - 1) - 2|V(T)| = -2$$

$$= \underbrace{\sum_{v \in X} (\deg(v) - 2)}_{=-|X|} + \underbrace{\sum_{v \in Y} (\deg(v) - 2)}_{\geq |Y|} + \underbrace{\sum_{v \in V(T) - X - Y} (\deg(v) - 2)}_{=0}$$

$$\implies -2 \geq -|X| + |Y| \implies |X| \geq |Y| + 2 \quad \square$$

A note for the following few proofs: if w is a leaf, then any path which exists in T (with ends not w) exists in $T \setminus w$.

PROP 3.3 Let T be a tree with 2 leaves, u and v. Then T is a path with ends u and v.

PROOF. Let $P \subseteq T$ be a path with ends u and v. By Prop 3.2, $\deg_T(w) = 2 \; \forall w \in V(P) \setminus \{u, v\}$. Moreover, $\deg_T(w) = \deg_P(w)$, so no vertex in $V(P)$ is incident to an edge in $E(T) \setminus E(P)$. Then, by Prop 2.5, P is a connected component. But T is connected, so $T = P$. □

PROP 3.4 Let T be a tree and $v \in V(T)$ be a leaf. Then $T \setminus v$ is a tree.

PROOF. $T \setminus v$ is non-null, since v has a neighbor. $T \setminus v$ has no cycles, since T has no cycles, and $T \setminus v$ is connected: we know there exists a path between any two vertices in $V(T) \setminus \{v\}$. Such a path still exists. □

PROP 3.5 If G is a graph, $v \in V(G)$ a leaf, and $G \setminus v$ a tree, then G is a tree.

PROOF. G is non-null, since $G \setminus v$ is non-null. We know that v belongs to no cycles, since it is a leaf, so any cycles apparent in G would exist in $G \setminus v$. Thus, G has no cycles. For connectedness, let H be the graph containing v, its incident edge, and that edge's other vertex v'. H is connected, as is $G \setminus v$, and $G \setminus v \cap H \neq \emptyset$, so

$G \setminus v \cup H = G$ is connected by Prop 3.3. □

PROP 3.6 Let T be a tree, $u, v \in V(T)$. Then T contains a unique path with ends u and v.

PROOF. We'll show by induction on $|V(T)|$. This clearly holds for $|V(T)| = 1$. Let $|V(T)| \geq 2$. Suppose T contains a leaf $w \in V(T) \setminus \{u, w\}$. Then $T \setminus w$ is a tree by Prop 3.4. By our induction hypothesis, $T \setminus w$ contains a unique path with ends u and v. By connectedness, $\exists$ a path with ends u, v in T. But this path must exist in $T \setminus w$, whose uniqueness follows.

If no such leaf exists, then T has exactly 2 leaves (u and v). Thus, by Prop 3.3, T is a path with ends u and v, and thus the only path in T. □

IV Spanning Trees

DEF 4.1 Let G be a graph. A subgraph $T \subseteq G$ is called a *spanning tree* of G if T is a tree and $V(T) = V(G)$.

PROP 4.1 Let G be connected and non-null. Let $H \subseteq G$, chosen minimal such that $V(H) = V(G)$ and H is connected. Then H is a spanning tree of G.

PROOF. We only need to check that T is non-null and contains no cycles. The first is automatic, since $V(T) = V(G)$, and G is non-null. If H has a cycle, then let e be an edge in the cycle. $H \setminus e$ is connected by Prop 2.6 and Prop 2.7. But this contradicts minimality, so T contains no cycles. □

PROP 4.2 Let G be a connected non-null graph. Let $H \subseteq G$ be maximal such that H contains no cycles. Then H is a spanning tree of G.

PROOF. We need to show that $V(H) = V(G)$ and H is connected (it is non-null, since at least a singleton of G contains no cycles; it contains no cycles by construction). If $\exists v \in V(G) \setminus V(H)$, adding v such that $\deg(v) = 0$ would maintain H having no cycles, thus contradicting maximality.

Suppose H is not connected. Then by Prop 2.2 there exists a partition $H = X \cup Y$ such that no edge has a vertex in both X and Y. However, such an edge must exist in G, say $e \in E(G)$, so we may add this edge to H to produce H'. Observe that H' contains no cycles, since e belongs to no cycles in H. But this contradicts maximality, so H must contain no cycles. □

DEF 4.2 Let T be a spanning tree of G. Let $f \in E(G) \setminus E(T)$. Then T with f has one cycle (by Prop 3.6). This is called the *fundamental cycle* of f with respect to T, and denoted $\mathrm{FC}(T, f)$.

PROP 4.3 Let T be a spanning tree of G, $f \in E(G) \setminus E(T)$. Let $C = \mathrm{FC}(T, f), e \in E(C)$. Then $(T + f) \setminus \{e\}$ is a spanning tree.

PROOF. Let $T' = (T+f)\setminus\{e\}$. $T+f$ is connected, and since e is not a cut-edge, $(T+f)\setminus\{e\} = T'$ is also connected. C is a unique cycle in $T+f$, so T' contains no cycles. Thus, T' is a tree. $V(T') = V(T) = V(G)$, since T is a spanning tree, so we conclude that T' is a spanning tree. □

DEF 4.3 Let G be a non-null, connected tree. Let $w : E(G) \to \mathbb{R}_+$ by a real valued function on the edges of G. The *minimal spanning tree* of G w.r.t. w, denoted $\mathrm{MST}(G, w)$, is a spanning tree T such that $w(T) = \sum_{e\in E(T)} w(e)$ is minimal.

4.4 Minimality of MST Edges

Let G be connected and non-null. Let $w : E(G) \to \mathbb{R}_+$. Let $T = \mathrm{MST}(G, w)$ and $E(T) = \{e_1, ..., e_k\}$, where we order

$$w(e_1) \leq w(e_2) \leq ... \leq w(e_k)$$

Then $\forall 1 \leq i \leq k$, e_i is an edge of minimum weight subject to the following constraints:

- $e_i \notin \{e_1, ..., e_{i-1}\}$
- $\{e_1, ..., e_i\}$, as an edge set, does not contain any cycles.

In particular, this theorem states that *for any* $f \in E(G) - \{e_1, ..., e_{i-1}\}$ with $\{e_1, ..., e_{i-1}, f\}$ not containing cycles, $w(f) > w(e_i)$.

PROOF. Suppose otherwise. Then for at least one i, we can choose $f \in E(G) - \{e_1, ..., e_{i-1}\}$ such that $\{e_1, ..., e_{i-1}, f\}$ contains no cycles and $w(f) < w(e_i)$.

Then $f \notin E(T)$, otherwise $f = e_j$ for some $j \geq i$. But $j < i$, since we have an ordering on w. Let $C = \mathrm{FC}(T, f)$, the unique cycle in $T + f$. There is some $j \geq i$ such that $e_j \in E(C)$, since all vertices $< i$ must not contain cycles. Then $w(e_j) > w(f)$.

Let $T' = (T+f) - e_j$. Then by Prop 4.3, T' is still a spanning tree. Let $w(G)$ be the sum of weights of edges of G. Then $w(T') - w(T) = w(f) - w(e_j) < 0$, implying that T is not minimal $\lightning$. □

DEF 4.4 **Kruskal's Algorithm**

| Input

A connected, non-null graph G, and $w : E(G) \to \mathbb{R}_+$

| Output

A graph T such that $V(T) = V(T)$, with $E(T) = \{e_1, ..., e_{|V(G)|-1}\}$

| $n \to n+1$

Let $e_i \in E(G)$ be chosen such that $w(e_i)$ is minimum subject to

- $e_i \notin \{e_1, ..., e_{i-1}\}$
- $\{e_1, ..., e_i\}$, as an edge set, does not contain any cycles.

for $1 \leq i \leq |V(G)| - 1$

4.5 Kruskal's Algorithm Outputs an MST

PROOF. Suppose $w : E(G) \to \mathbb{R}_+$ is injective. Then all edges have different weights. Then Thm 4.4 implies that Kruskal's outputs an MST which is unique. If w is not injective, the proof is out of the scope of this course. □

4.6 Spanning Trees of K_n

The complete graph K^n has exactly n^{n-2} spanning trees.

PROOF. The proof for this will require proving multiple statements. Let $\mathcal{T}_k$ be the set of spanning, rooted forests in K^n with k components. Then $\mathcal{T}_1$ is the set of rooted spanning trees in K_n. Since we may choose n roots, $\frac{|\mathcal{T}_1|}{n}$ equals the number of spanning trees in K^n. Thus, we need to show $|\mathcal{T}_1| = n^{n-1}$.

Claim 1 $|\mathcal{T}_n| = 1$

If a spanning, rooted forest has n components, then it is exactly the graph of no edges and the vertex set $V(K^n)$ (each being its own component).

Claim 2 $n(k-1)|\mathcal{T}_k| = (n-k+1)|\mathcal{T}_{k-1}|$

Call a forest F with $k-1$ the *parent* of a forest F' with k components if $F' = F \setminus e$ for some $e \in E(F)$. Naturally, we call F' a *child* of F under these conditions. We will thus count (parent, child) combinations. Every $F \in \mathcal{T}_{k-1}$ has $|E(F)|$ children, since every edge is a cutedge. This is $|V(F)| - \text{comp}(F) = n - (k-1) = n - k + 1$ by Prop 3.1.

For every $F' \in \mathcal{T}_k$, we can obtain a parent by adding an edge from any vertex to the root of a component not containing this vertex. Thus, every child has $n(k-1)$ parents. Thus, there are $n(k-1)|\mathcal{T}_k|$ parent-child combinations, and also $(n-k+1)|\mathcal{T}_{k-1}|$ such combinations. Thus, we conclude $n(k-1)|\mathcal{T}_k| = (n-k+1)|\mathcal{T}_{k-1}|$.

Claim 3 $|\mathcal{T}_k| = \binom{n}{k}kn^{n-1-k}$

We just solve the recursion. We'll show by induction on $n-k$. If $n - k = 0 \implies n = k$, we have $|\mathcal{T}_n| = \binom{n}{n}nn^{n-1-n} = 1$, which is true by Claim 1.

Letting $n-k \to n-k+1 = n-(k-1)$, we are having $k \to k-1$. By Claim 2, then,

$$|\mathcal{T}_{k-1}| = \frac{n(k-1)}{n-k+1}|\mathcal{T}_k| \overset{\text{hyp.}}{=} \frac{n(k-1)}{n-k+1}\binom{n}{k}kn^{n-1-k} \overset{?}{=} \binom{n}{k-1}(k-1)n^{n-1-(k-1)}$$
$$= \frac{k}{n-k+1}\binom{n}{k}(k-1)n^{n-1-(k-1)}$$

Note that $\frac{k}{n-k+1}\binom{n}{k} = \frac{kn!}{(n-k+1)k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!(n-k+1)} = \frac{n!}{(k-1)!(n-k+1)!} = \binom{n}{k-1}$, so the last statement above evaluates to

$$\binom{n}{k-1}(k-1)n^{n-1-(k-1)} \qquad \text{as desired.}$$

Claim 4 $|\mathcal{T}_1| = n^{n-1}$

We plug in from above to find $|\mathcal{T}_1| = \frac{n}{1}1n^{n-1-1} = n^{n-2+1} = n^{n-1}$. □

V Euler's Thm & Hamiltonian Cycles

Recall that a *walk* in G is a sequence $(v_0, \dots, v_k) : v_i \in V(G)$, perhaps with repetition, such that $v_i v_{i+1} \in E(G)\ \forall i \leq k-1$. (See Def 2.1).

A walk *uses* an edge e if $e = v_i v_{i+1}$ and v_i, v_{i+1} is contained in the walk. DEF 5.1

A *trail* is a walk that uses every edge at most once. DEF 5.2

A *Euler trail* in G is a trail that uses every edge in $E(G)$ DEF 5.3

A *Euler tour* in G is a closed Euler trail (i.e. $v_0 = v_k$). DEF 5.4

Let G be a graph with $E(G) \neq \emptyset$. If G has no leaves, then G has a cycle. **PROP 5.1**

PROOF. Suppose not. Then G is a forest by Def 3.1. Let C be a component with at least one edge. Then C is a tree with $|V(C)| \geq 2$. Thus, C has a leaf by Prop 3.2 $\implies \lightning$. □

Let G be a graph with all degrees even. Then $\exists$ cycles $C_1, \dots, C_k \subseteq G$ such that $(E(C_1), \dots, E(C_k))$ is a partition of $E(G)$. **PROP 5.2**

PROOF. Informally, by the previous proposition we may create one cycle from the outset. Removing the edge set of this cycle from G leaves the graph with still all even edges. We'll show by induction on the edge set. The base case, $|E(G)| = 0$, holds trivially.

Let $n \to n+1$. We know there exists a cycle $C_1 \subseteq G$. Let $G' = G - C_1$. Then all degrees in G' are even. By (strong) induction, G' contains a partition $(E(C_2), ..., E(C_k))$. Since $E(G) = E(G') \cup E(C_1)$, the partition $(E(C_1), ..., E(C_k))$ satisfies the proposition. $\square$

5.3 Euler's Theorem

Let G be connected with all even degrees. Then $\exists$ a Euler tour in G.

PROOF. Let $W = (v_0, ..., v_k)$ be a closed trail of maximal length in G. WLOG suppose W doesn't use every edge. Let $H \subseteq G$ be such that $V(H) = V(G)$ and $E(H) = G \setminus \{\text{edges used by } W\}$. Then all vertices of H have an even degree, so by Prop 5.2, $\exists$ cycles $C_1, ..., C_k$ which partition $E(H)$.

Let $H' \subseteq G$ be the subgraph consisting of the edges and vertices of W. H' is not a component of G (since G is connected), so by Prop 2.5 $\exists e \in E(G) - E(H')$ with an end in $V(H')$, i.e. an end in the walk. Thus, e is part of H, and is part of a cycle. Thus, we may append this cycle to the walk W, creating a larger closed trail, violating maximality $\lightning$. $\square$

PROP 5.4 Let G be a connected graph with ≤ 2 vertices of odd degree. Then G contains a Euler trail.

PROOF. By the handshaking lemma (not seen here, but easy to see), there must be an even number of odd degree vertices in any graph. Thus, the case of one odd vertex is invalid.

Let $\deg(u), \deg(v)$ be odd, and all other vertices even. Let $G' = G + w$, where w is an added edge which is adjacent (joins) u and v. Then G' has all degrees even, and by Thm 5.3 G' has a Euler tour. WLOG we can let it begin and end at w. Thus, removing w, we get a Euler trail in G. $\square$

DEF 5.5 A *Hamiltonian cycle* is a cycle $C \subseteq G$ s.t. $V(C) = V(G)$.

PROP 5.5 Let G be a graph, and $X \subseteq V(G)$ with $X \neq \emptyset$. If $|X| < \text{comp}(G \setminus X)$, then G has no Hamiltonian cycles.

PROOF. Let C be Hamiltonian cycle. Then

$$\text{comp}(C \setminus X) \geq \text{comp}(G \setminus X) > |X|$$

Observe now that $C \setminus X$ is a forest. Then, from theory, we know that

$$\text{comp}(C \setminus X) = |V(C \setminus X)| - |E(C \setminus X)| \leq |V(C)| - |V(X)| - (|E(C)| - 2|X|) = |X|$$

where we note that $|V(C)| = |E(C)|$, since C is a cycle. This is a contradiction. $\square$

As it turns out, there is no effiicent algorithm to decide if G has a Hamiltonian cycle.

5.6 Dirac-Pósa

Let G be a graph on n vertices. If $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices $u, v \in V(G)$, then G has a Hamiltonian cycle.

PROOF.

We'll show by induction on $\binom{n}{2} - |E(G)|$. If $|E(G)| = \binom{n}{2}$, then G is complete, and clearly contains a Hamiltonian cycle.

Let $|E(G)| < \binom{n}{2}$. Let $u, v \in V(G)$ be non-adjacent. Let $G' = G + uv$. By induction hypothesis, $\exists$ a Hamiltonian cycle $C \subseteq G'$. If $uv \notin E(C)$, then C is a Hamiltonian cycle in G. Otherwise, let $uv \in E(C)$. Notate

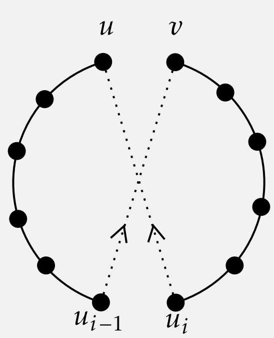
$$V(C) = \{u = u_1, u_2, ..., u_n = v\}$$

Let $A = \{i : uu_i \in E(G)\}$ and $B = \{i : vu_{i-1} \in E(G)\}$. Then $|A| + |B| = \deg(u) + \deg(v) \geq n$.

But we have $n - 1$ such vertices (u, v are non-adjacent, so this takes away a possibility). Thus, $A \cap B \neq \varnothing$, so $\exists i : uu_i, vu_{i-1} \in E(G)$. Then

$$\{u = u_1, ..., u_{i-1}, u_n = v, u_{n-1}, ..., u_i, u_1 = u\}$$

is a Hamiltonian cycle. $\square$



PROP 5.7 Let G be a graph on $n \geq 3$ vertices. Then if $\deg(v) \geq \frac{n}{2}$ $\forall v \in V(G)$ OR $|E(G)| \geq \binom{n}{2} - n - 3$, then G has a Hamiltonian cycle.

PROOF.

If $\deg(v) \geq \frac{n}{2}$ $\forall v \in V(G)$, then $\deg(u) + \deg(v) \geq n$ $\forall u, v \in V(G)$, so by Thm 2.5 G has a Hamiltonian cycle.

For the second condition, I was getting a cookie and didn't listen. $\square$

VI Bipartite Graphs

DEF 6.1 A *bipartition* of a graph G is a partition (A, B) of $V(G)$ such that every edge of G has one end in A and one end in B. Refer to Def 1.14.

DEF 6.2 A graph G is *bipartite* if it admits a bipartition.

PROP 6.1 Trees are bipartite.

PROOF. We'll show by induction on $n = |V(T)|$. For $|V(T)| = 1$, we have a bipartition $(\{v\}, \varnothing)$.

Let $|V(T)| = n$. Let $v \in V(T)$ be a leaf with a neighbor u. By induction hypothesis, $(T \setminus v)$ is bipartite. Let (A, B) be a bipartition. Assume WLOG that $u \in A$. Then $(A, B \cup \{v\})$ is a bipartition of T. □

6.2 Characterization of Bipartite Graphs

Let G be a graph. Then the following are equivalent:

1. G is bipartite.
2. G contains no closed walk of odd length
3. G contains no odd cycles.

PROOF. $(1 \implies 2)$. Let (A, B) be a bipartition of G. Let $(v_0, \ldots, v_k)$ be a walk in G. WLOG let $v_0 \in A$. Then $v_i \in A \iff i$ is even. Thus, if $v_0 = v_k$, k must be even, so the walk must have even length.

$(2 \implies 3)$If G had a cycle of odd length, it would be a closed walk of odd length.

$(3 \implies 1)$. As bipartitions of components may be combined to form a larger bipartition, it suffices to show this for a connected, non-null graph.

Let T be a spanning tree of G. Then $\exists$ a bipartition (A, B) of $V(T)$ by Prop 2.15. We'll show this is a bipartition of G as well. Let $f \in E(G) - E(T)$. Let $v_0, \ldots, v_k$ be the vertices of $\mathrm{FC}(T, f)$, with ends on f. Assume WLOG that $v_0 \in A$.

The fundamental cycle $\mathrm{FC}(T, f)$ has even length by assumption, so v_k must be odd (observe the cycle v_0, v_1, v_2, v_3 for reference). Thus, $v_k \in B$, so f has one end in A, and one in B. This may be reasoned for all $f \in E(G) - E(T)$. The bipartition holds for $E(T)$. Thus, it holds for all $e \in E(G)$. □

VII Matchings in Bipartite Graphs

A *matching* M in G is a set of edges such that no vertex in $V(G)$ is incident to more than one edge in M. DEF 7.1

The *matching number* of G, denoted $\nu(G)$, is the maximum size $|M|$ for matchings M in G. DEF 7.2

$\nu(G) \leq \lfloor \frac{V(G)}{2} \rfloor$. **PROP 7.1**

PROOF. The maximal matching will use every vertex. □

A *vertex cover* in G is a set $X \subseteq V(G)$ such that every edge in $E(G)$ has at least one end in X. DEF 7.3

If M is a matching in G and X is a vertex cover, then $|M| \leq |X|$. **PROP 7.2**

PROOF. If $|X|$ is a vertex cover, then every edge in M has an end in X. But no vertex $x \in X$ can belong to more than one edge in M, so we have an injection between M and X, i.e. $|M| \leq |X|$. □

Let $\tau(G)$ be the minimum size of a vertex cover in G. DEF 7.4

$\nu(G) \leq \tau(G)$. **PROP 7.3**

PROOF. Immediately from Prop 2.17. □

If X is a vertex cover of G, then $\sum_{v \in X} \deg(v) \geq E(G) = \frac{1}{2} \sum_{v \in V(G)} \deg(v)$. **PROP 7.4**

If G is a graph, and Y is a set of pairwise non-adjacent vertices, then $V(G) \setminus Y$ is a vertex cover. **PROP 7.5**

PROOF. Suppose otherwise. Then $\exists uv \in E(G)$ such that uv is not incident to any $V(G) \setminus Y$. Thus, $u, v \in Y$. But then u, v are adjacent. □

Note that the previous two propositions were not shown in class, but I have high confidence they're true, and they might be useful; take them with a grain of salt.

For a graph G, $\nu(G) \leq \tau(G) \leq 2\nu(G)$. **PROP 7.6**

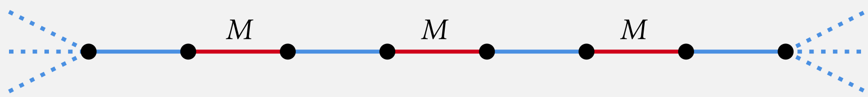
PROOF. It remains to show that $\tau(G) \leq 2\nu(G)$. Let M be a matching with $|M| = \nu(G)$. We want to find a vertex cover X with $|X| \leq 2|M|$. Let X be the set of ends of edges of M. Then $|X| = 2|M|$. Furthermore, X is a vertex cover. Otherwise, $\exists e \in E(G)$ with no end in X. Then $M \cup \{e\}$ is a matching, violating maximality. □

DEF 7.5 Let M be a matching in G. A path $P \subseteq G$ is M-*alternating* if edges in P alternate between edges of M and $E(G) - M$, i.e. every internal vertex of P is incident to an edge in $E(P) \cap M$.

DEF 7.6 An M-alternating path $P \subseteq G$ is M-*augmenting* if $|V(P)| \geq 2$ and the ends of P are not incident to edges of M.

PROP 7.7 If G contains an M-augmenting path, then M is not maximum.

PROOF. Let P be M-augmenting. Let $M_1 = M \cap P$, $M_2 = (E(G) \setminus M) \cap P$, and $M_3 = M \setminus M_1$. Then $|M_2| > |M_1|$, and $M_3 \cup M_2$ is a larger matching (note that $M_3 \cup M_1 = M$).



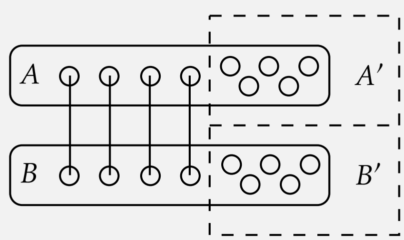
$\square$

7.8 König

For any bipartite graph G, $\nu(G) = \tau(G)$.

PROOF. It suffices to show $\tau(G) \leq \nu(G)$. Thus, given a matching M with $\nu(G) = |M|$, we look for a vertex cover $X : |X| = |M|$.

Let (A, B) be a bipartition of G. Let A', B' be vertices not incident to edges in M in A and B, respectively:



Let $Z \subseteq V(G)$ be such that for all $z \in Z$, $\exists$ an M-alternating path in G with one end in v and another in A'. Then we can conclude the following:

1. $A' \subseteq Z$.
2. $Z \cap B' = \varnothing$ (i.e. $\nexists$ an M-augmenting path).
3. Every edge in M with one end in Z has both ends in Z.
4. Every edge with one end in $Z \cap A$ has a second end in $Z \cap B$.

Thus, let $X = (Z \cap B) \cup (A \setminus Z)$. Then $|X| \geq |M|$, since every vertex of X is incident to an edge of M (see (1) and (2)). Every edge of M has exactly one end in X, so $|M| \geq |X|$, and then $|X| = |M|$. Lastly, X is a vertex cover, by (4). $\square$

We say that a matching M *covers* $X \subseteq V(G)$ if every vertex in X is an end of some edge in M. DEF 7.7

We say that a matching is *perfect* if it covers $V(G)$. DEF 7.8

A matching M is perfect $\iff |M| = \frac{|V(G)|}{2}$. **PROP 7.9**

A graph G is *d-regular* if $\deg(v) = d \ \forall v \in V(G)$. DEF 7.9

7.10 Criterion for Perfect Matchings

Let G be a d-regular bipartite graph for $d \geq 1$. Then G has a perfect matching.

PROOF.

If a bipartite graph G contains a perfect matching, then for a bipartition (A, B), $|A| = |B|$.

$d|B| = |E(G)| = d|A|$, so $|A| = |B|$, since every edge has exactly one end in A and one end in B. We wish to show that $\nu(G) \geq |A| = |B|$, since $\nu(G) \leq |A|$. By König, it suffices to show that $\tau(G) \geq |A|$, i.e. for every vertex cover X of G, $|X| \geq |A|$.

As X is a vertex cover, we have

$$d|X| = \sum_{x \in X} \deg(v) \geq E(G) = d|A| \implies |X| \geq |A| \quad \square$$

Let $N(S)$ denote the set of all vertices in G with at least one neighbor in $S \subseteq V(G)$. DEF 7.10

7.11 Hall

Let G be a bipartite graph with a bipartition (A, B). Then G has a matching M covering A if an only if

$$|N(S)| \geq |S| \ \forall S \subseteq A$$

PROOF.

Sometimes we call the qualifier "Hall's condition."

($\implies$) If M is a matching which covers A, then M matches every vertex of $S \subseteq A$ to a vertex in $N(S)$. Thus $|N(S)| \geq |S|$.

($\impliedby$) We want to show that $\nu(G) \geq |A|$, since automatically $\nu(G) \leq |A|$. By König, it suffices to show $\tau(A) \geq |A|$, i.e. $|X| \geq |A|$ for any vertex cover X.

Let $S = A - X$. By Hall's condition, $|B \cap X| \geq \overbrace{|N(S)|}^{\subseteq B \cap X} \geq |S| = |A - X|$. Thus, $|A \cap X| - |B \cap X| \geq |A \cap X| - |A - X| \implies |X| \geq |A|$. $\square$

VIII Menger's Theorem & Separations

Let G be a graph, and let $s, t \in V(G)$. We wish to consider when there exists a path in G with ends s and t. If such a path does not exist, then we can conclude that s and t are members of different components. Abstractly, there exists a partition (A, B) of $V(G)$, where $s \in A, t \in B$, such that no edge of G has one end in A and another in B.

Let s, t be non-adjacent, and suppose there exists at least one path between them. How might we guarantee that s cannot be "disconnected" from t by deleting $X \subseteq V(G)$ with $|X| < k$, $s, t \notin X$? The existence of disjoint paths $P_1, ..., P_k$ from s to t would suffice.

DEF 8.1 A *separation* of G is a pair (A, B) such that $A \cup B = V(G)$ and no edge of G has one end in $A - B$ and the other in $B - A$.

DEF 8.2 The *order* of a separation (A, B) is $|A \cap B|$.

PROP 8.1 Let $s, t \in V(G)$. Then either there exists a path with ends s and t in G, or there exists a separation of G with $s \in A, t \in B$ of order 0.

PROOF. This will follow from Thm 2.10, with $k = 1$. □

8.2 Menger

Let $s, t \in V(G)$ be distinct and non-adjacent. Let $k \geq 1$. Then exactly one of the following holds:

1. There exists pairwise disjoint paths $P_1, ..., P_k$ with ends s and t.
2. There exists a separation (A, B) of G with order $< k$ such that $s \in A - B, t \in B - A$.

PROOF. If (A, B) is a separation as in (2), then every path P from s to t contains a vertex in $A \cap B$. Thus, if (1) holds, then $P_1, ..., P_k$ use k distinct vertices in $A \cap B$, contradicting $|A \cap B| < k$. Thus, (1) and (2) are at least mutually exclusive.

We will assume Thm 2.11 holds, and conclude that Menger holds.

Let Q be the set of neighbors of s and R be the set of neighbors of t. Then either (1) or (2) of the theorem below holds, applied to $G \setminus s \setminus t$.

Suppose (1) of 2.11 holds. Then adding s and t to the ends of each disjoint path, we get that (1) of Menger holds. Suppose (2) of 2.11 holds. Then a separation $(A \cup \{s\}, B \cup \{t\})$ satisfies (2) of Menger. □

8.3 Generalized Menger

Let $Q, R \subseteq V(G)$. Let $k \geq 1$. Then exactly one of the following holds:

1. There exists pairwise disjoint paths $P_1, \ldots, P_k$, each from Q to R.
2. There exists a separation (A, B) of G of order $< k$ such that $Q \subseteq A$ and $R \subseteq B$.

For $X \subseteq V(G)$, let $V[X]$, the *subgraph of G induced by X*, have the vertices of X and the edges of G with both ends in X. DEF 8.3

EXERCISE CAUTION

PROOF. We only need to show that one of (1), (2) hold. By induction on $|V(G)| + |E(G)|$.

$|V(G)| + |E(G)| = 0 \implies G = \emptyset$. We have the order 0 separation $(\emptyset, \emptyset)$.

Case 1: There exists a separation (A', B') of order exactly k s.t. $Q \subseteq A'$, $R \subseteq B'$, and $A', B' \neq V(G)$. By induction hypothesis applied to $G[A']$, Q,and $A' \cap B'$, either

1. $\exists P_1', \ldots, P_k'$ in $G[A']$ from Q to $A' \cap B'$, pairwise disjoint.
2. $\exists$ a separation (A'', B'') of $G[A']$ such that $Q \subseteq A''$ and $A' \cap B' \subseteq B''$ of order $< k$.

Then $(A'', B' \cup B'')$ is a separation of G satisfying (2): observe that $Q \subseteq A''$ by definition, and $R \subseteq B' \cup B''$, since $R \subseteq B'$. Furthermore,

$$|A'' \cap (B' \cup B'')| = |\underbrace{(\underbrace{A''}_{\subseteq A'} \cap \underbrace{B'}_{\subseteq B''})}_{\subseteq A'' \cap B''} \cup (A'' \cap B'')| = |A'' \cap B''| < k$$

Similarly, by applying the induction hypothesis to $G[B'], A' \cap B', R$, we may assume there exists pairwise disjoint paths $P_1'', \ldots, P_k''$ from $A' \cap B'$ to R. By renumbering, we may assume that P_i' and P_i'' share an end in $A' \cap B'$, and then paths $P_1' \cup P_1'', \ldots, P_k' \cup P_k''$ satisfy (1).

Case 2: $Q \cap R \neq \emptyset$. Let $v \in Q \cap R$. We apply induction hypothesis to $G - v$, $R - v$, $Q - v$, and $k - 1$. If (1) holds in $G - v$, then adding a path P_k with $V(P_k) = \{v\}$, we get k paths in G.

If (2) holds in $G - v$, then let (A', B') be a separation with $Q - v \subseteq A'$, $R - v \subseteq B'$. Then (2) holds for G with the separation

$$(A, B) = (A' \cup v, B' \cup v)$$

Case 3: $k = 1$. If there exists a component C of G such that $V(C) \cap Q \neq \emptyset$, $V(C) \cap R \neq \emptyset$, then (1) holds.

Otherwise, let A be the union of vertex sets of components that contain a vertex of Q. Let $B = V(G) - A$. Then (A, B) is a separation of order 0.

Case 4: Cases 1, 2, 3 do not hold. Let $e \in E(G)$. Apply induction hypothesis to $G \setminus Q, R$. We may assume that there exists a separation (A', B') of $G \setminus e$ with

$Q \subseteq A'$, $R \subseteq B'$. WLOG e has ends in $u \in A' - B'$ and $v \in B' - A'$ (otherwise, we are done).

Consider a separation $(A', B' \cup u)$. If it has order $< k$, then we are done.

If it has order $= k$, then Case 1 holds, unless $B' \cup u = V(G)$. Similarly, considering $(A' \cup v, B')$, we may assume $A' \cup v = V(G)$. So $|V(G)| \leq |A \cap B| + 2 \leq k + 2$. Then $|Q| + |R| = |V(G)|$, since Case 2 doesn't hold. So we may assume $|Q| \leq \frac{k+1}{2} < k$. Then, $(Q, V(G))$ is a separation that satisfies (2). $\square$

PROP 8.4 Menger (Thm 2.11) $\implies$ König (Thm 2.7)

PROOF. Let G be a bipartite graph with a bipartition (Q, R). Let $k = \nu(G) + 1$. Then (1) of Menger doesn't hold, since this would imply the existence of a matching of size k. Thus, $\exists$ a separation (A, B) of G of order $\leq \nu(G)$ such that $Q \subseteq A$, $R \subseteq B$. Then $A \cap B$ is a vertex cover, so $\tau(G) \leq \nu(G)$. (Recall, by Prop 2.18, that we only need to show this direction.) $\square$

DEF 8.4 Let $k \geq 1$ and let G be a graph with $|V(G)| \geq k + 1$. We say that G is k-connected if $G \setminus X$ is connected for all $X \subseteq V(G)$ such that $|X| \leq k - 1$.

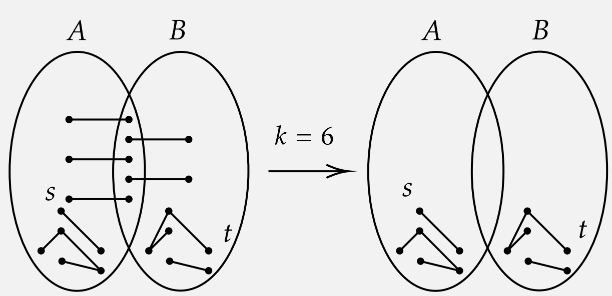
E.G. 8.1 ♠ *Examples* ♣

G is 1-connected $\iff$ G is connected and $V(G) \geq 2$. Trees on ≥ 2 vertices are 1-connected, but not 2-connected. Cycles are 2-connected, but not 3 connected.

8.5 Paths in k-Connected Graphs

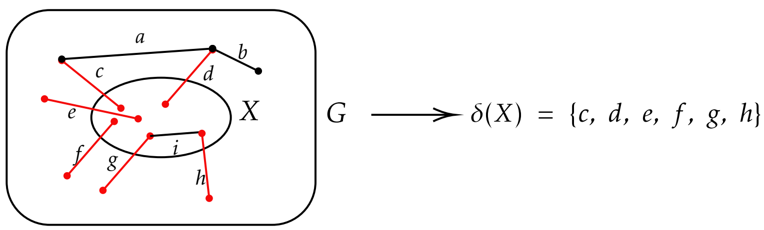
Let G be a k-connected graph. Let $s, t \in V(G)$. Then there exists paths $P_1, ..., P_k$ in G, each with ends s and t, and otherwise pairwise disjoint.

PROOF. Recall Menger's (Thm 2.10): if $s, t \in V(G)$ are non-adjacent, then either $\exists$ paths as described above, or $\exists$ a separation (A, B) with $s \in A$, $t \in B$, and $|A \cap B| < k$. However, then $G \setminus (A \cap B)$ is no longer connected. But G is k-connected, so we have a contradiction. Hence, such a separation can't exist, and so the path case holds.

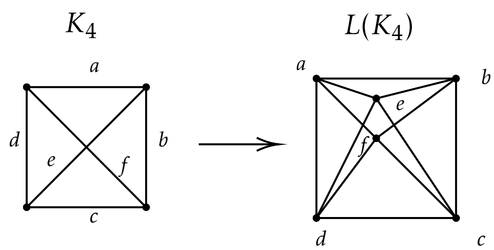


Now suppose that s, t are adjacent. We get $P_k := st$ (the edge connecting them) for free. We'll apply Menger's to $G \setminus st$, i.e. $\exists P_1, ..., P_{k-1}$ from $s \leftrightarrow t$, pairwise non-adjacent, or $\exists$ a separation of $G \setminus st$ with $|A \cap B| < k-1$. Then $G \setminus ((A \cap B) \cup \{s\})$ is disconnected (unless $A - B = s$). Similarly, we find that $B - A = t$ as well. But then $|V(G)| \leq |A \cap B| + 2 \leq k$. This also violates k-connectivity (in particular, the condition that $|V(G)| \geq k+1$). Thus, $P_1, ..., P_{k-1}, P_k$ are paths from $s \leftrightarrow t$. Note that $P_i \in G \setminus st$ for $i \leq k-1$, so since $P_k = st$, these are all disjoint. □

DEF 8.5 The *cut* associated with $X \subseteq V(G)$, denoted by $\delta(X)$, is the set of edges of G with exactly one end in X.



DEF 8.6 For a graph G, the *line graph*, denoted $L(G)$, is a graph such that $V(L(G)) = E(G)$, and $e, f \in V(L(G))$ adjacent in $V(L(G))$ if and only if they share an end in G.



8.6 Edge Menger

Let $s, t \in V(G)$ be non-adjacent. Then either $\exists$ edge-disjoint paths $P_1, ..., P_k$ or $\exists X \subseteq V(G)$ with $s \in X, t \notin X$, and $|\delta(X)| < k$.

PROOF. Note that (1) and (2) cannot both hold. Suppose (1) holds. Consider a path P_i from s to t. Let $s \in X$ and $t \notin X$. Let v_l be the minimal vertex not in X. Then $v_{l-1}v_l \in \delta(X)$. Since P_i are all pairwise disjoint, we have at least $|\delta(X)| \geq k$, which is a contradiction.

Thus, we need to show that either (1) or (2) holds. Let $G' = L(G)$. Let $Q \subseteq V(G') = E(G)$ be the set of all edges with an end being s. Similarly, let $R \subseteq V(G')$ be the set of edges with an end being t. By Thm 2.11, we first consider the possibility that $\exists$ vertex disjoint paths $P'_1, ..., P'_k \subseteq G'$ with ends in Q and R.

Then $V(P'_i)$ contains $E(P_i)$ for some path P_i from $s \leftrightarrow t$, so in particular we have an edge-disjoint path in G from $s \leftrightarrow t$.

Suppose now that the second condition in Thm 2.11 holds, i.e. $\exists$ a separation (A, B) of G' with $Q \subseteq A$, $R \subseteq B$, $|A \cap B| < k$, and $A \cup B = V(G') = E(G)$. No edge in $A - B$ shares an end with an edge in $B - A$. Let X be the vertices $v \in V(G) \setminus \{t\}$ such that all edges incident to v are in A. Then $s \in X$, $t \notin X$, and for all $v \notin X$, we have that the edges incident to v are in B. Hence, $\delta(X) \subseteq A \cap B$, so $|\delta(X)| < k$ as desired. □

IX Directed Graphs & Flows

DEF 9.1 A *directed graph*, or *digraph*, D is a graph where, for each edge $e \in E(D)$, one of its ends is designated *tail*, and one end is designated *head*. Then, e is said to be *directed* from its tail to its head.

DEF 9.2 A *directed path* P from u to v in a digraph D is a path from u to v in which, for every $v_{i-2}v_{i-1}, v_iv_{i+1} \in E(P)$, v_{i-1} is a head, and v_i is a tail.

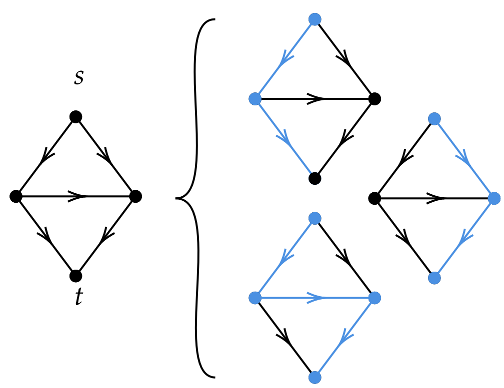
DEF 9.3 For a digraph D and $X \subseteq V(D)$, $\delta^+(X)$ denotes the vertices in $\delta(X)$ with its tail in X. Similarly, $\delta^-(X)$ denotes the vertices in $\delta(X)$ with its head in X. Note that $\delta^+(X) = \delta^-(V(G) - X)$, and similarly $\delta^-(X) = \delta^+(V(G) - X)$.

PROP 9.1 Let D be a digraph, and $s, t \in V(D)$. Then $\nexists$ a directed path in D from $s \to t \iff \exists X \subseteq V(G)$ s.t. $s \in X$, $t \notin X$, and $\delta^+(X) = \varnothing$.

PROOF. ($\impliedby$) Suppose there existed a directed path $P \subseteq D$ from $s \to t$. Consider the last vertex $v \in V(P)$ s.t. $v \in X$. Then the edge of the path with a tail in v is in $\delta^+(X)$. Hence, $\delta^+(X) \neq \varnothing \implies \lightning$.

($\implies$) Let X be all $v \in V(D)$ s.t. $\exists$ a directed path from s to v. Then $s \in X$, $t \notin X$ by assumption. If $vw \in \delta^+(X)$ for some $w \notin X$, then we may construct a directed path consisting of the path $s \to v$, and stitching on this edge to w. Hence $w \in X \implies \lightning$. Hence $\delta^+(X) = \varnothing$. □

Consider the following directed paths from s to t:



Typically, we call s the "source" and t the "sink." Let $\delta^+(v)$ for $v \in V(D)$ denote all edges whose tail is v. We then define flow in the following way:

DEF 9.4 An (s,t)–*flow* on a digraph D is a function $\phi : E(D) \to \mathbb{R}_+$ such that

$$\sum_{e\in\delta^+(v)} \phi(e) = \sum_{e\in\delta^-(v)} \phi(e) \quad \forall v \in V(D) - \{s,t\}$$

where s is the source and t is the sink.

DEF 9.5 The *value* of an (s,t)-flow ϕ is $\sum_{e\in\delta^+(s)} \phi(s) - \sum_{e\in\delta^-(s)} \phi(e)$.

PROP 9.2 Let ϕ be an (s,t)-flow on a digraph D with value k. Then $\forall X \subseteq V(D)$ such that $s \in X, t \notin X$, we have

$$\sum_{e\in\delta^+(X)} \phi(e) - \sum_{e\in\delta^-(X)} \phi(e) = k$$

PROOF. By flow conservation,

$$\begin{aligned}
k &= \sum_{e\in\delta^+(s)} \phi(s) - \sum_{e\in\delta^-(s)} \phi(e) \\
&= \sum_{v\in X} \underbrace{\left(\sum_{e\in\delta^+(v)} \phi(e) - \sum_{e\in\delta^-(v)} \phi(e) \right)}_{0 \text{ if } v \neq s} \\
&= \sum_{e\in E(D)} \phi(e)\,(t(e) - h(e)) = \left(\sum_{e\in E(D)} \phi(e)t(e) \right) - \left(\sum_{e\in E(D)} \phi(e)h(e) \right) \\
&= \sum_{e\in\delta^+(X)} \phi(e) - \sum_{e\in\delta^-(X)} \phi(e)
\end{aligned}$$

Where

$$t(e) = \begin{cases} 1 & \text{tail of } e \text{ in } X \\ 0 & \text{o.w.} \end{cases} \qquad h(e) = \begin{cases} 1 & \text{head of } e \text{ in } X \\ 0 & \text{o.w.} \end{cases}$$

□

What is the maximal value of an (s,t)-flow? The answer is uninteresting: if there exists a path from $s \to t$, we can assign any amount of flow to each of these edges, and 0 otherwise, and maintain conservation. Hence, if there exists such a path, we may have ∞ flow. If a path does *not* exist, then we invoke Prop 2.26, which says $\delta^+(X) = \varnothing$ for any $X \subseteq V(G), s \in X, t \notin X$, to conclude that $k = -\sum_{e\in\delta^-(X)} \phi(e)$. Since ϕ is non-negative, k is negative, and at most 0 (take $\phi \equiv 0$).

DEF 9.6 A *capacity function* on a digraph D is a function $c : E(D) \to \mathbb{Z}_+$. An (s,t)-flow ϕ is *c-admissible* if $\phi(e) \leq c(e) \forall e \in E(D)$.

DEF 9.7 A (not necessarily directed) path $P \subseteq D$ from $s \leftrightarrow t$ is ϕ*-augmenting* path for an (s,t)-flow $\phi : E(D) \to \mathbb{Z}_+$ if:

1. $\phi(e) \leq c(e) - 1$ if $e \in E(D)$ from tail to head.

2. $\phi(e) \geq 1$ if $e \in E(P)$ from head to tail.

ϕ is called *integral* if its co-domain is the integers. DEF 9.8

Let ϕ be an integral c-admissible (s,t)-flow of value k. If $\exists$ a ϕ-augmenting path D from $s \leftrightarrow t$, then $\exists$ a c-admissible (s,t)-flow in D of value $k+1$. **PROP 9.3**

ψ is an (s,t) *pseudo-flow* if it satisfies flow conservation (but not necessarily non-negativity). DEF 9.9

PROOF.

$$\psi(e) = \begin{cases} 1 & e \in E(P) \text{ is head to tail} \\ -1 & e \in E(P) \text{ is tail to head} \\ 0 & e \notin E(P) \end{cases}$$

Then let $\phi' = \phi + \psi$. ψ is then a "pseudo-flow," since ψ satisfies flow conservation. ϕ' is also a pseudo-flow. But $\phi' \geq 0$, since, if $\psi(e) = -1$, then $\phi(e) \geq 0$, so $\phi'(e) \geq 0$. ϕ' is also c-admissible, since, if $\psi(e) = 1$, then $\phi(e) + 1 \leq c(e)$, so $\phi'(e) = \phi(e) + 1 \leq c(e)$.

Also, the value the ϕ' is the value of ψ + the value of $\phi \implies$ the value of ϕ' is $k+1$. □

9.4 Max Flow-Min Cut (or Ford-Fulkerson)

Let D be digraph, $s,t \in V(D)$ distinct. Let $c : E(D) \to \mathbb{Z}_+$. Then the maximal value of an integral c-admissible (s,t)-flow is equal to the minimum $\sum_{e \in \delta^+(X)} c(e)$, over all $X \subseteq V(D), s \in X, t \notin X$.

PROOF.

Let $\phi : E(D) \to \mathbb{R}_+$ be a c-admissible (s,t)-flow of maximum value, k. Let $X \subseteq V(D)$ be such that $s \in X$, $t \notin X$. By Prop 2.27,

$$k = \sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^{-1}(X)} \phi(e) \leq \sum_{e \in \delta^+(X)} \phi(e) \leq \sum_{e \in \delta^+(X)} c(e)$$

Now let X be the set of $v \in V(D)$ such that there exists a ϕ-augmenting path in D from $s \leftrightarrow v$ (recall: not necessarily directed). Then $s \in X$ and $t \notin X$, since, if $t \in X$, then our (s,t)-flow is not maximal, by Prop 2.28. Then

$$k = \sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^{-1}(X)} \phi(e)$$

Now, if any edge $e \in \delta^+(X)$ had $\phi(e) \leq c(e) - 1$, then we could extend the augmenting path to the head h of e, hence deriving a contradiction $h \notin, \in X$. Hence,

$\phi(e) \geq c(e)$, so $\phi(e) = c(e)$. Similarly, we conclude that $\phi(e) = 0$ for any $e \in \delta^-(X)$. Thus,

$$k = \sum_{e\in\delta^+(X)} c(e) - \sum_{e\in\delta^{-1}(X)} 0 = \sum_{e\in\delta^+(X)} c(e)$$

Thus, minimizing over X yields $k \geq \sum_{e\in\delta^+(X)} c(e)$ as desired. □

X Ramsey's Theorem

Recall that $\nu(G)$ denotes the maximum size of a matching $M \subseteq E(G)$, where M is such that no two edges in M share an end (alternatively, no vertex is incident to two edges in M). Recall also that $\tau(G)$ denotes the minimum size of a vertex cover $X \subseteq V(G)$, where X is such that every edge has an end in X. These motivate the following definitions:

DEF 10.1 $X \subseteq V(G)$ is *independent* if no edge of G has both ends in X. $\alpha(G)$ denotes the maximum size of an independent set in G.

DEF 10.2 $L \subseteq E(G)$ is an *edge cover* of G if every vertex in G is an end of some edge in L. $\rho(G)$ denotes the minimum size of an edge cover in G.

Remark that this is only well-defined when every vertex is incident to at least one edge

	ν	τ	α	ρ
P_n	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	$\lceil \frac{n}{2} \rceil$
C_n	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$
K_n	$\lfloor \frac{n}{2} \rfloor$	$n-1$	1	$\lceil \frac{n}{2} \rceil$

Notice how, in these elementary examples, $\alpha(G) + \tau(G) = |V(G)|$. This holds in generality:

PROP 10.1 For a graph G with $|V(G)| = n$, $\alpha(G) + \tau(G) = n$.

PROOF. Remark that $X \subseteq V(G)$ is a vertex cover $\iff$ $V(G) \setminus X$ is independent.

$\alpha(G) + \tau(G) \geq n$: We need to find one independent set larger than $n - |X|$, where $|X| = \tau(G)$ is a vertex cover. Take $V(G) \setminus X$. Then $\alpha(G) \geq |V(G)| - |X| = n - \tau(G)$.

$\alpha(G) + \tau(G) \leq n$: We need to find one vertex cover smaller than $n - |X|$, where $|X| = \alpha(G)$ is an independent set. Take $V(G) \setminus X$. Then $\tau(G) \leq |V(G)| - |X| = n - \alpha(G)$. □

PROP 10.2 Let G admit an edge cover, with $|V(G)| = n$. Then $\nu(G) + \rho(G) = n$.

PROOF.

$\nu(G) + \rho(G) \leq n$: Let M be a matching in G with $|M| = \nu(G)$. Let L be obtained from M by adding an edge to every vertex not already covered by the ends of M. Then $\rho(G) \leq |L| = |M| + (n - 2|M|) = n - |M|$, since each edge M covers 2 vertices (distinct from those covered by any other edge).

$\nu(G) + \rho(G) \geq n$: Let L be an edge cover with $|L| = \rho(G)$. It suffices to consider H with $V(G) = V(H)$ and $E(H) = L$. For this, not that if M is a matching in H, then it is a matching in G.

Since L is minimal, every edge has a degree 1 end. Otherwise, we may delete such an edge and maintain covering. Hence, no cycles exist in H, so H is a forest. Let M consist of one edge per component of H. Then $\nu(G) \geq |M| = \text{comp}(H) = |V(H)| - |E(H)| = n - |L|$. □

10.3 Gallai's Equations

Let G be a graph with $|V(G)| = n$. Then

1. $\alpha(G) + \tau(G) = n$
2. $\nu(G) + \rho(G) = n$ if G admits an edge covering

PROOF. Prop 2.29 and Prop 2.30 □

PROP 10.4 Let G be bipartite with no degree 0 vertices. Then $\alpha(G) = \rho(G)$.

PROOF. By König and Thm 2.15, $\alpha(G) + \cancel{\tau(G)} = |V(G)| = \cancel{\nu(G)} + \rho(G) \implies \alpha(G) = \rho(G)$. □

DEF 10.3 A *clique* in G is a collection of vertices $X \subseteq V(G)$ such that every two vertices in X are adjacent. $\omega(G)$ denotes the maximal size of a clique in G.

PROP 10.5 A clique in G is independent in G^c, and vice versa.

$V(G^c) = V(G)$ and two vertices in G are adjacent $\iff$ they are not adjacent in G^c

	$\omega(G)$
P_n	2
C_n	2 if $n \neq 3$
K_n	n

DEF 10.4 Let $R(s,t)$, the *Ramsey number*, denote the minimum integer N such that every graph on N vertices has an independent set of size s or a clique of size t.

PROP 10.6 $R(s,t) = R(t,s)$.

PROOF. Follows from Prop 2.32 □

10.7 Ramsey's Theorem

$R(s,t)$ exists for all positive integers s and t, and $R(s,t) \leq R(s,t-1) + R(s-1,t)$ for $s,t \geq 2$.

PROOF. We prove the second part of this statement by induction on $s+t$. The base case, $R(s,1) = R(1,t) = 1$, holds by observation. Suppose $R(s-1,t)$ and $R(s,t-1)$ exist for $s,t \geq 2$. Let $N = R(s-1,t) + R(s,t-1)$, and G a graph with N vertices. We wish to show that G contains both a independent set of size s *or* a clique of size t. Let $v \in V(G)$.

Case 1: $\deg(v) \geq R(s,t-1)$. Let A be the neighbors of v. Then $A \subseteq V(G)$ contains either an independent set of size s or a clique X of size $t-1$. Then $X \cup v$ is a clique of size t.

Case 2:$\deg(v) \geq R(s,t-1)$. Let B be the non-neighbors of v. Then $|B| = N - \deg(v) - 1 \geq N - R(s,t-1) = R(s-1,t)$. Hence, $B \subseteq V(G)$ contains an independent set X of size $s-1$ or a clique of size t. But then $X \cup v$ is an independent set of size s. □

PROP 10.8

$$R(s,t) \leq \binom{s+t-2}{t-1} \quad \forall s,t \geq 2$$

PROOF. We show by induction on $s+t$.

Base case: $R(s,1)$ and $R(t,1) = 1 = \binom{s-1}{0} = \binom{t-1}{t-1}$.

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \\ &\leq \binom{s+t-3}{t-1} + \binom{s+t-3}{t-2} = \binom{s+t-2}{t-1} \end{aligned}$$

as desired. □

From this, we observe that $R(s,s) \leq \binom{2(s-1)}{s-1} = 4^s$.

PROP 10.9 If N, s are positive integers such that $\binom{N}{s} 2^{1-\binom{s}{2}} < 1$, then there is a graph on N vertices that has no independent set or clique of size s, i.e. $R(s,s) > N$.

PROOF. Let V be a vertex set of size N. We will consider subgraphs of K_N with $V(K_N) = V$. Let $F \subseteq E(K_N)$. Denote by G_F the graph with $V(G_F) = V$ and $E(G_F) = F$. Note that there are $2^{\binom{N}{2}}$ graphs G_F. Let $X \subseteq V$ with $|X| = s$. Then X is independent set in exactly $2^{\binom{N}{2} - \binom{s}{2}}$ graphs G_F. Since there are $\binom{N}{s}$ ways to construct X, there are at most $\binom{N}{s} 2^{\binom{N}{2} - \binom{s}{2}}$ graphs G_F with independent sets of size s. We conclude identically for cliques of size s.

If $2^{\binom{N}{2}} > 2\binom{N}{s}2^{\binom{N}{2}-\binom{s}{2}}$, then by Pigeonhole, there exists some graph G_F with neither an independent set or clique of size s. □

10.10 Erdos

For $s \geq 2$, $R(s,s) \geq 2^{\frac{s}{2}}$

PROOF. By the previous proposition, it suffices to show that, for $N < 2^{\frac{s}{2}}$, we have $\binom{N}{s}2^{1-\binom{s}{2}} < 1$. We expand a little:

$$\binom{N}{s}2^{1-\binom{s}{2}} < \frac{N^2}{s!}2^{1-\binom{s}{2}} < \frac{2^{\frac{s^2}{2}}}{s!}2^{1-\frac{s(s-1)}{2}} = \frac{2^{\frac{s}{2}+1}}{s!}$$

Hence, it suffices to show $2^{\frac{s}{2}+1} < s!$. For $s = 3$, we have $2^5 < 3^6$. One can show by induction easily. Note that this is not true for $s = 2$, but we can manually see that $R(2,2) = 2 \geq 2^{\frac{2}{2}} = 2$. □

DEF 10.5 Let $R_k(s_1, ..., s_k)$ be the minimum N such that in every coloring of edges in K_N by colors $\{1, ..., k\}$, one can find K_{s_i} with all edges colored by i for some i.

10.11 Ramsey Coloring Theorem

$R_k(s_1, ..., s_k)$ exists for all k and choices of $s_1, ..., s_k$.

PROOF. We show by induction on k. For $k = 2$, we have $R_2(s_1, s_2) = R(s_1, s_2)$ (i.e. the ordinary Ramsey number), and thus the result holds by Ramsey's Theorem.

We show that $R_k(s_1, ..., s_k) \leq R_{k-1}(R_2(s_1, s_2), s_3, ..., s_k) =: N$. We have that N exists by assumption. Let $\star$ be a merged color with $s_\star = R_2(s_1, s_2)$. Then either K_{s_i} is completely covered with i in K_n (for $i \geq 3$), or $K_{R_2(s_1,s_2)}$ is colored completely with $\star$. But this suffices. □

XI Vertex Coloring

DEF 11.1 Let G be a graph. Let S be a set of "colors," with $|S| = k$. We say that $c : V(G) \to S$ is a *k-coloring* of G if $c(u) \neq c(v)$ for any two adjacent $u, v \in V(G)$.

DEF 11.2 The *chromatic number* of G, denoted $\chi(G)$, is the minimum k such that G admits a k-coloring.

For example, G is 1-colorable $\iff$ there exist no edges in G. G is 2-colorable $\iff$ G is bipartite.

DEF 11.3 Let $c : V(G) \to S$ be some k-coloring. Then $\forall i \in S$, the set of all vertices colored by i is called the *color class* of i.

PROP 11.1 Let G be a graph. Then $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \lceil \frac{|V(G)|}{\alpha(G)} \rceil$. Recall from Def 10.1 and Def 10.3 these numbers.

PROOF. The first result is almost automatic. Given a clique and a coloring, every vertex must be colored with pairwise different colors (since, otherwise, we'd have two adjacent vertices colored the same). Hence, we must at least always use as many colors as the maximum size of a clique, i.e. $\chi(G) \geq \omega(G)$.

For the second result, let $\chi(G) = k$. It suffices to show that $k \geq \frac{|V(G)|}{\alpha(G)}$, i.e. $k\alpha(G) \geq |V(G)|$. Note that every color class is independent (if an internal edge existed, we'd find two adjacent vertices of the same color). Hence, if $V_1, ..., V_k$ are the color classes of G, we have

$$|V(G)| = |V_1| + ... + |V_k| \leq k\alpha(G)$$

□

DEF 11.4 **Greedy Coloring Algorithm**

| Input

A graph G and an ordering of vertices $(v_1, ..., v_n)$, $v_i \in V(G)$.

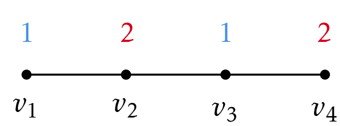
| Output

A k-coloring $c : V(G) \to \{1, ..., k\}$

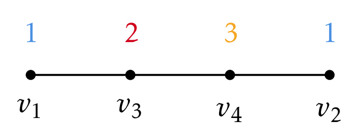
| $i \to i + 1$

Let $c(v_i)$ be the minimal positive integer not already assigned to one of its neighbors among $(v_1, ..., v_{i-1})$.

Note that the ordering we provide is essential. Consider the following 2 and 3 colorings that result from different orderings:



or



DEF 11.5 A graph G is *k-degenerate* if every subgraph of G contains a vertex of degree $\leq k$ (measured in the subgraph).

For example, G is 0-degenerate $\iff$ it has no edges $\iff$ it is 1-colorable. G is 1-degenerate $\iff$ it is a forest ($\implies$ it is 2-colorable).

PROP 11.2 Let G be k-degenerate. Then $\chi(G) \leq k + 1$.

PROOF.

It suffices to provide an ordering $(v_1, ..., v_n)$ such that v_i has at most k neighbors among $v, ..., v_{i-1}$. Then, applying the greedy algorithm above, we'll always use $k+1$ colors. Hence, $\chi(G) \leq k+1$.

We'll construct this ordering backwards. Suppose $v_n, ..., v_{n-i+1}$ satisfy our conditions. Consider $G' = G \setminus \{v_n, ..., v_{n-i+1}\}$. As G is k-degenerate, G' has a vertex of degree $\leq k$. Choose this vertex to be v_{n-i}. Then, v_{n-i} has at most k neighbors in $\{v_1, ..., v_{n-i-1}\}$, as desired. □

DEF 11.6

Let $\Delta(G)$ denote the maximum degree of a vertex in G.

Observe that G is $\Delta(G)$-degenerate. Hence, we get

PROP 11.3

$\chi(G) \leq \Delta(G) + 1$

11.4 Brooks

Let G be connected, non-null, and not either a complete graph or an odd cycle. Then

$$\chi(G) \leq \Delta(G)$$

CONJECTURE (REED)

$$\chi(G) \leq \left\lceil \frac{\omega(G) + (\Delta(G)+1)}{2} \right\rceil$$

PROOF IDEA

Equivalently, we show, for any integer $k \geq 1$, that if G is connected with $\Delta(G) \leq k$, then G admits a k-coloring unless $G = K_{k+1}$ or $k = 2$ and G is an odd cycle.

Case 0: $k = 1, 2$. For $k = 1$, we have the connected non-null graphs of maximal degree 1, i.e. K_2 and a singleton vertex. The former case is handled by K_{k+1}, and the latter can clearly by 1-colored. For $k = 2$, if G is bipartite, then it is 2-colorable, as desired. If G is not bipartite, then it contains an odd cycle by Thm 6.2. By connectedness, and the fact that the maximal degree of G is 2, we conclude that G itself is an odd cycle.

We proceed by induction on $|V(G)|$, assuming, by the case above, that $k \geq 3$ (otherwise arbitrary). If $|V(G)| = 0$, then we run into non-nullity case, so Brooks holds. Hence, we continue with a strong induction hypothesis.

Case 1: G is not 2-connected. Then $\exists$ a separation (A, B) of G such that $|A|, |B| < |V(G)|$ with $|A \cap B| = 1$ by Menger. Recall the notation $G[X]$ for $X \subseteq V(G)$, which is the graph induced by the vertices X and all its internal edges. We apply our induction hypothesis to $G[A]$ and $G[B]$, i.e. $\exists k$-colorings $c_A : A \to S$ and $c_B : B \to S$ with $|S| = k$.

Almost: neither $G[A]$ nor $G[B]$ are K_{k+1}. WLOG suppose A is. Then $\exists v \in A \cap B$ with k neighbors in A. But v must have a neighbor in B to maintain connectedness.

Then $\deg(v) \leq k+1$, contradicting $\Delta(G) \leq k$. By permuting colors as needed, we can ensure that c_A and c_B agree on $A \cap B$. This hence creates a k coloring

$$c(u) = \begin{cases} c_A(u) : u \in A \\ c_B(u) : u \in B \end{cases}$$

Case 2: G is 2-connected, but not 3-connected.

Just as above, but a little more careful.

Case 3: G is 3-connected.

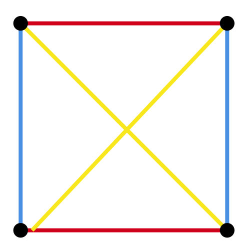
Constructive k-coloring using greedy algorithm. □

XII Edge Coloring

DEF 12.1 A function $c : E(G) \to S$, with $|S| = k$, is called a *k-edge coloring* if $c(e) \neq e(f)$ for any $e, f \in E(G)$ which share an end.

The *edge chromatic number* $\chi'(G)$ is the minimum k such that G admits a k-edge coloring.

Consider the following edge coloring of K_4:



This shows that $\chi'(K_4) \leq 3$. But we cannot color with 2 colors, so $\chi'(K_4) = 3$.

PROP 12.1 Let G be a graph with at least one edge. Then $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$.

PROOF. Recall that $L(G)$ has vertex set $E(G)$, and two edges of G are adjacent in $L(G)$ if they share an end. Hence, a k-edge coloring of G is a k-coloring of $L(G)$ (and vice versa). In particular, $\chi'(G) = \chi(L(G))$.

Since $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$, $\omega(L(G)) \leq \chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1$.

What does the maximal clique look like in $L(G)$? This is the maximal number of edges incident to a single vertex. Hence, $\Delta(G) \leq \omega(L(G))$. What is vertex degree in $L(G)$? We observe that $\deg_{L(G)}(e) = \deg(u) + \deg(v) - 2$, where $e = uv$. Hence, $\Delta(L(G)) \leq 2\Delta(G) - 2$. We re-write the inequality above, then:

$$\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1 \quad \square$$

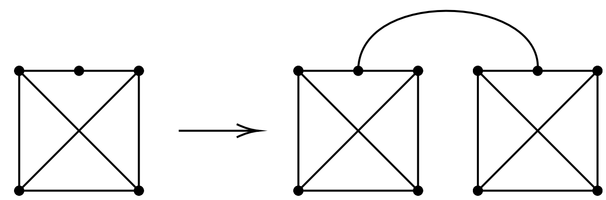
$\Delta(G) = \omega(L(G))$ unless $G = K_3$

Note here that $2 = \Delta(G) < \chi'(G) = 3$ for odd cycles. However, this is an uncommon exception. We can show $\chi'(G) = \Delta(G)$ for bipartite G, and $\chi'(G) \leq 3\left\lceil \frac{\Delta(G)}{2} \right\rceil$ in generality. Recall that G is k-regular if $\deg(v) = k$ for any $v \in V(G)$. (cf. Def 7.9)

In particular,

PROP 12.2 Let $k \geq 1$ and G be a graph with $\Delta(G) \leq k$. Then there exists a k-regular graph H with $G \subseteq H$. Moreover, if G is bipartite, then so is H.

Consider an example for an *almost* 3-regular graph as inspiration for a proof:



PROOF.

We will show by induction on $k - L$ such that, if $L \leq \deg(v) \leq k\ \forall v \in V(G)$, then G is a subgraph of a k-regular H.

If $k = L$, then we have a k-regular $H = G$.

Let G' be an isomorphic copy of G with $V(G')$ disjoint from $V(G)$. Denote the copy of v as v'. Let G'' be obtained from $G \cup G'$ by adding an edge vv' for every v with $\deg_G(v) = \deg_{G'}(v') < k$. Then $L + 1 \leq \deg_{G''}(u) \leq k$ for every $u \in V(G'')$. So, by induction hypothesis, G'' is a subgraph of a k-regular graph H. Hence, $G \subseteq G'' \subseteq H$.

Note that, if G is bipartite with bipartition (A, B) (with (A', B') corresponding to G'), then G'' is bipartite with a bipartition $(A \cup B', B \cup A')$. Induction then proves the final statement. $\square$

12.3 König

Let G be bipartite. Then $\chi'(G) = \Delta(G)$

PROOF.

We've shown that $\chi'(G) \geq \Delta(G)$ by Prop 12.1. Hence, it suffices to show that $\chi'(G) \leq \Delta(G)$. But, by Prop 12.2, we only need to $\chi'(G) \leq k$ for every k-regular, bipartite G. We'll show this by induction on k.

For $k = 0$, this is trivial.

Let $k \geq 1$. We know that k-regular bipartite G has a perfect matching, say M. Then $G \setminus M$ is $(k-1)$-regular. By induction hypothesis, $\chi'(G \setminus M) \leq k - 1$. We use another color k to color M, and get a k-edge coloring of G. $\square$

To show $\chi'(G) \leq 3\left\lceil \frac{\Delta(G)}{2} \right\rceil$, we can show that $\Delta(G) \leq 2k \implies \chi'(G) \leq 3k$. Even easier, we show that if G is $(2k)$-regular, then $\chi'(G) \leq 3k$.

DEF 12.2 $F \subseteq E(G)$ is called a *2-factor* if every vertex of G is incident to exactly 2 edges in F. Similarly, F is *1-factor* if it is a perfect matching.

PROP 12.4 Let G be a $(2k)$-regular graph. Then $E(G)$ can be partitioned into k 2-factors.

PROOF. If $k = 1$, then the lemma is trivial. As all degrees of G are even, there exists a partition of $E(G)$ into edge sets of cycles. Direct all edges of G so that every cycle in the partition is oriented in one direction of traversal. Then every vertex is a head of exactly k edges and a tail of exactly k edges.

Let H be a bipartite graph with bipartition (A, B), where A contains a copy v_1 of $v\ \forall v \in V(G)$, and, similarly, B contains a copy v_2 of $v\ \forall v \in V(G)$. For every edge $uv \in E(G)$ directed from $u \to v$, add an edge u_1v_2 to H.

Then H is a k-regular bipartite graph. $E(H)$ can be partitioned into k perfect matchings, $M_1, \ldots, M_k$.

The matchings M_i correspond to the desired 2-factors in G. □

12.5 Shannon

$$\chi'(G) \leq 3\left\lceil \frac{\Delta(G)}{2} \right\rceil$$

PROOF. Let $k = \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Then $\Delta(G) \leq 2k$. By Prop 12.2, we may assume that G is $2k$-regular. Then by Prop 12.4, $E(G)$ may be partitioned into k 2-factors, $F_1, \ldots, F_k$. Each F_i is an edge set of a union of cycles, so it can be colored using 3 colors, which gives a $(3k)$-edge coloring of G. Hence $\chi'(G) \leq 3k$. □

12.6 Vizing

$$\chi'(G) \leq \Delta(G) + 1$$

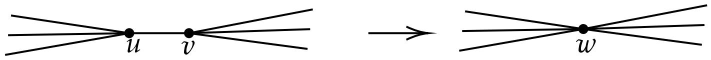
We won't prove this in class (too long and technical). In fact, $\chi'(G)$ is *either* $\Delta(G)$ or $\Delta(G) + 1$.

XIII Graph Minors & Hadwiger's

H is a subgraph of G if H can be obtained from G be repeatedly deleting vertices and/or edges. Hence

1. Every graph is a subgraph of itself.
2. If J is a subgraph of H, and H is a subgraph of G, then J is a subgraph of G.

DEF 13.1 Let $e \in E(G)$ with ends u, v. A graph G' obtained from G by *contracting* e is produced by deleting u, v, and replacing them by w, such that $N(w) = N(u) \cup N(v)$ in G'.

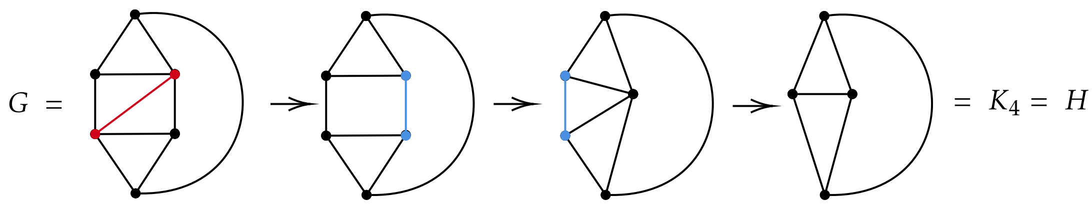


DEF 13.2 H is a *minor* of G if H can be obtained from G by deleting vertices and/or edges, and/or contracting edges.

1. Every graph is a minor of itself.
2. If J is a minor of H, and H is a minor of G, then J is a minor of G.

If G has no K_2, then certainty G is edgeless (which is the same as when G has no K_2 subgraph). These are $\iff$ statements. This implies that G has a 1-coloring.

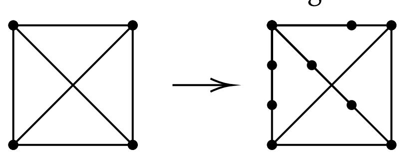
If G has no K_3 minor, then G has no cycles. Otherwise, we may isolate a cycle, and contract each edge inductively. In fact, G has no K_3 minor $\iff$ it has no cycles. This implies that G is bipartite.



CONJECTURE (HADWINGER) For every positive integer t, if G has no K_{t+1} minor, then $\chi(G) \leq t$.

For $t = 1, 2$, this conjecture holds easily. For $t = 3$, i.e. no K_4 minor $\implies \chi(G) \leq 3$, it is provable and not too difficult. For $t = 4$, i.e. no $K_5 \implies \chi(G) \leq 4$, we would show the famous Four Color Theorem, as planar graphs have no K_5 minor. This theorem was proven by computer (Appel, Haken), and is 10,000 pages in length. The $t = 5$ case was proven in 1993 *assuming* $t = 4$ holds (Robertson, Seymour, Thomas), and is not computer assisted, but is 80 pages of human reasoning. $t \geq 6$ is open.

DEF 13.3 A *subdivision of a graph H* is obtained from H by replacing edges of H by internally vertex disjoint paths with the same ends as the original edges.



Observe that H is a minor of any subdivision of H, by repeated contraction of new edges. If G contains a subdivision of H as a subgraph, then G contains H as a minor. Hence, for the $t = 3$ case of Hadwiger's, we may prove that no K_4 subdivision $\implies \chi(G) \leq 3$. Even simpler, we may prove instead that G is 2-degenerate, by Prop 11.2.

PROP 13.1 Let G be a 3-connected graph. Then G contains a subdivision of K_4 as a subgraph. Hence, G contains a K_4 minor.

PROOF. Let $s, t \in V(G)$ be distinct vertices. Then, there exists vertex disjoint paths $P, Q, \& R$ from $s \to t$. (cf Thm 8.5). Let L be the shortest path in $G - s - t$ with ends on two different paths among $P, Q, \& R$. Such a choice is possible: $G - s - t$ is connected, since G is 3-connected, and at least two of the paths P, Q, R contain internal vertices.

Then L is disjoint from $P, Q, \& R$ except for its ends. Suppose $p \in P, r \in R$ are ends of L, and suppose for a contradiction that L has an internal vertex $q \in (P \cup Q \cup R) - \{s, t\}$. If $q \notin V(R)$, then a subpath of L from $q \to r$ contradicts the choice of L as minimal. Otherwise, the subpath of L from $q \to p$ contradicts it.

Hence, $P \cup Q \cup R \cup L$ is a subdivision of K_4 in G. □

PROP 13.2 Let G be a graph with no K_4 minor. Let X be a clique in G s.t. $|X| \leq 2$ and $X \neq V(G)$. Then there exists $v \in V(G) - X$ such that $\deg(v) \leq 2$

PROOF. We proceed by induction on $|V(G)|$. For $|V(G)| \leq 3$, every $v \in V(G)$ is s.t $\deg(V) \leq 2$.

By Prop 13.1, G is not 3-connected. Hence, there exists a separation (A, B) of G s.t. $A, B \neq V(G)$ of order ≤ 2. Choose such a separation with $|A \cap B|$ minimum. WLOG we assume $X \subseteq A$.

Let $Y = A \cap B$ and let G' be obtained from $G[B]$ by adding an edge between two vertices of Y if needed. Then, by induction hypothesis on G', there exists $v \in V(G') - Y = B - A \supseteq V(G) - X$ s.t. $\deg_G(v) \leq 2$, but $\deg_G(v) = \deg_{G'}(v) \leq 2$, so v is as needed.

Note that G' is a minor of G, and so has no K_4 minor, as $G[A]$ contains a path P with the same ends as the edge we added (if we did), by minimality of $|A \cap B|$. We could then contract P. □

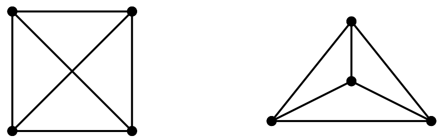
13.3 Hadwiger's Conjecture for $t = 3$

Let G be a graph with no K_4 minor. Then $\chi(G) \leq 3$

PROOF. By Thm 11.2, it is enough to show that G is 2-degenerate. That is, every non-null subgraph H of G contains a vertex v of degree ≤ 2. But H has no K_4 minor (since G doesn't), so it has such a vertex by Prop 13.2. □

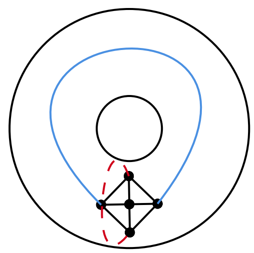
XIV Planar Graphs

DEF 14.1 A *planar drawing* of a graph G in the plane is a representation of G such that its vertices are distinct points in the plane and its edges are curves joining its ends, such that no two curves intersect eachother. The picture on the left is not a drawing of K_4, while the one on the right is.

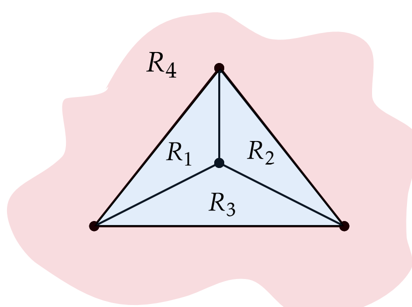


A graph G is *planar* if it admits a planar drawing. DEF 14.2

We may also be interested in representing graphs on *locally* planar objects, like a torus. K_5 is not planar in $\mathbb{R}^2$, but one can draw it on a torus, like so:



A drawing of a graph separates the plane into *regions*, where two points (not in the drawing) are in the same region if they can be joined by a curve disjoint from the drawing.



A *curve* from $p \to q$ for $p, q \in \mathbb{R}^2$ is a continuous map $\phi : [0,1] \to \mathbb{R}^2$ such that $\phi(0) = p$ and $\phi(1) = q$. A curve is *simple* if ϕ is injective on $[0,1)$ and *closed* if $\phi(0) = \phi(1)$. DEF 14.3

14.1 Jordan Curve Theorem

Every simple closed separates the plane into two regions.

Deleting an edge decreases the number of regions by one if the regions on the two sides of the edge are different. It remains the same if the regions are the same.

Let G be a graph drawn in the plane OBSERVATION 14.1

1. If $e \in E(G)$ belong to a cycle, then the regions on two sides of e are different.
2. If one of the ends of e is a leaf, then the regions on both sides of e are the same.

A priori, we don't know if this is well-defined

DEF 14.4 Let $\text{Reg}(G)$ denote the number of regions in the drawing of a planar graph G..

14.2 Euler's Formula

Let G be a non-null planar graph. Then

$$|V(G)| - |E(G)| + \text{Reg}(G) = 1 + \text{comp}(G)$$

In particular, if G is connected and non-null, then $|V(G)| - |E(G)| + \text{Reg}(G) = 2$. Often, one write $V - E + F = 2$.

PROOF. We'll show by induction on $|E(G)|$. If $|E(G)| = 0$, then $|V(G)| + \text{Reg}(G) = |V(G)| + 1$. But $|V(G)| = \text{comp}(G)$ if G is edgeless. Hence, let $|E(G)| \geq 1$.

Case 1: There exists $e \in E(G)$ that belongs to a cycle. Then by the observation above, $\text{Reg}(G \setminus e) = \text{Reg}(G) - 1$, while $|V(G \setminus e)| = |V(G)|$ and $|E(G \setminus e)| = |E(G)| - 1$. Lastly $\text{comp}(G) = \text{comp}(G \setminus e)$ by Prop 2.7. Hence,

$$\begin{aligned} &|V(G \setminus e)| - |E(G \setminus e)| + \text{Reg}(G \setminus e) = 1 + \text{comp}(G \setminus e) \\ \Longrightarrow\ &|V(G)| - |E(G)| + \text{Reg}(G) = 1 + \text{comp}(G) \end{aligned}$$

Case 2: There does not exist such an edge e. Then G is a forest, so in particular there exists an edge f incident to a leaf. We perform a similar cancellation as above with $G \setminus f$. □

DEF 14.5 The *length* of a region R in a planar drawing of G is the number of edges of G on the boundary of the region. Some observations about length:

The sum of lengths of all regions generated by G is always $2|E(G)|$.

PROP 14.3 If G is a planar connected graph with $|V(G)| \geq 3$, then $\text{length}(R) \geq 3$ for all regions R. If $\text{length}(R) = 3$, then the boundary of R is a 3 cycle.

14.4 Size of Planar Graphs

Let G be a planar graph with $|V(G)| \geq 3$. Then $|E(G)| \leq 3|V(G)| - 6$. Moreover, if G has no K_3 subgraphs, then $|E(G)| \leq 2|V(G)| - 4$.

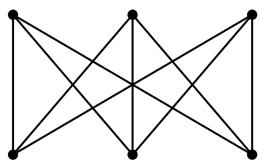
PROOF. Let G be a graph drawn in the plane. By adding edges (if needed), we may assume that G is connected. Then

$$2|E(G)| = \sum_{R=\text{region}} \text{length}(R) \geq 3\text{Reg}(G)$$

Then, by Euler's formula, we conclude $|E(G)| \le 3|V(G)| - 6$. For the second statement, if there are no cycles of length 3, then $\sum_{R=\text{region}} \text{length}(R) \ge 4\text{Reg}(G)$, and we substitute into Euler's formula to yield the result. □

As a corollary, we see that K_5 is not planar, since $|E(K_5)| = 10$ and $|V(G)| = 5$, so $10 \ge 9 \implies \lightning$. In fact, K_5 is minimally planar (draw and see, or take a look at the torus figure).

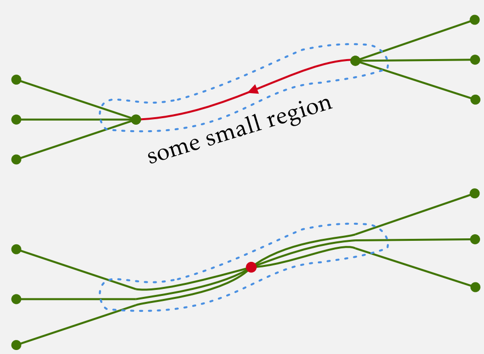
Recall also that bipartite graphs have no K_3 subgraphs, so they satisfy the stricter bound of Thm 14.4. Hence, $K_{3,3}$ (two size 3 bipartitions, connect all of one to all of the other) is non-planar (see picture). This too, like K_5, is minimally non-planar.

$K_{3,3} =$ 

OBSERVATION 14.2 Let H be a subdivision of G. Then H is planar if and only if G is planar. Hence, subdivisions of K_5 and $K_{3,3}$ are minimally non-planar.

We wish to prove the following for the sake of a future theorem: a graph G is planar if and only if it contains neither K_5 or $K_{3,3}$ as a minor.

PROOF. ($\implies$) If H is a minor of G, and G is planar, then H is planar. Clearly, deleting edges or vertices will leave G planar. Contracting edges is more complicated, but see the proof by drawing:



Hence, if G is planar, then it contains no K_5 or $K_{3,3}$ minor.

($\impliedby$) We will first show the following proposition: □

PROP 14.5 Let G be a planar graph with $|V(G)| \ge 3$. Then $\sum_{v \in V(G)} (6 - \deg(v)) \ge 12$. In particular, every non-null planar graph contains a vertex v with $\deg(v) \le 5$.

PROOF. By Thm 14.4,

$$\sum_{v\in V(G)} \deg(v) = 2|E(G)| \leq 6|V(G)| - 12 \implies \sum_{v\in V(G)} (6-\deg(v)) \geq 12$$

□

PROP 14.6 Every planar graph is 5-degenerate, and therefore 6-colorable.

PROOF. Prop 14.5 □

XV Kuratowski's Theorem

Our goal is to show that every non-planar graph G has a K_5 or $K_{3,3}$ minor. A proof outline is as follows:

PROOF OUTLINE. We'll show by induction on $|V(G)|$. Let G' be obtained from G by contracting an edge. If G' is not planar, then we are done (it will contain K_5 or $K_{3,3}$ as a minor by induction hypothesis). Hence, assume G' is planar. Let w be the vertex which was contracted from $u, v \in V(G)$. We pop out into the following lemma for help: □

PROP 15.1 Let G be a 2-connected graph drawn in the plane. Then the boundary of every region in the drawing is a cycle.

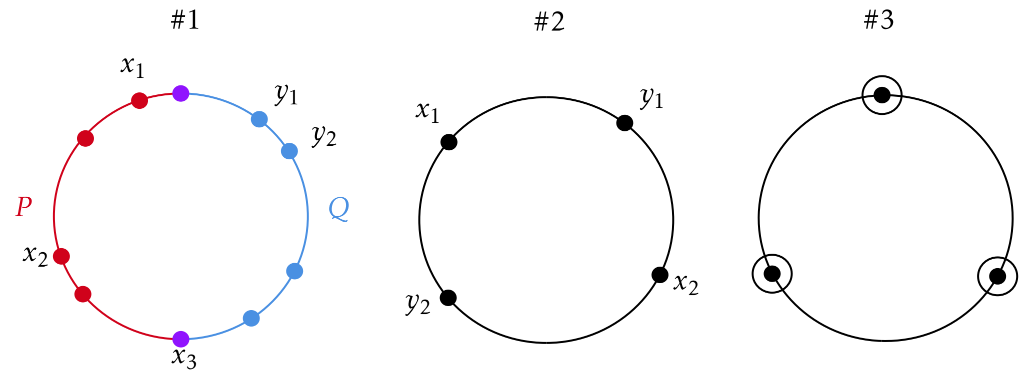
PROOF. Here's the idea: by "walking along" the boundary of a region R, we find a cycle C in G such that C belongs to the boundary of R. All vertices of C, except possible one $v \in V(C)$, have no neighbors in the interior of the region R' bounded by C. If $R' \neq R$, then a subgraph G' of G is drawn strictly inside R'. Deleting v then disconnects G' from $C \setminus v \implies \lightning$. □

PROP 15.2 Let C be a cycle and let $X, Y \subseteq C$. Then one of the following holds:

1. There exist distinct vertices $z_1, z_2 \in V(C)$ and two paths $P, Q \subseteq C$ with ends z_1, z_2 such that $P \cup Q = C$ and $X \subseteq V(P)$, $Y \subseteq V(Q)$.

2. There exist distinct $x_1, y_1, x_2, y_2 \in V(C)$ occurring in this order, where $x_1, x_2 \in X$, $y_1, y_2 \in Y$.

3. $X = Y$ and $|X| = |Y| = 3$.



PROOF.

Case 1: $X = Y$. If $|X| = |Y| \leq 2$, then (1) holds, since we can just have P, Q with ends on X and Y. If $|X| = |Y| = 3$, then (3) holds. If $|X| = |Y| \geq 4$, then outcome two holds by choosing two from each X and Y.

Case 2: WLOG take $x_1 \in X - Y$. Let y_1, y_2 be the closest vertices of Y to x_1 (in either direction along C). Let Q be a path from y_1 to y_2 not containing x_1. Then $Y \subseteq V(Q)$. If $X \cap V(Q) \subseteq \{y_1, y_2\}$, then (1) holds. Otherwise, $\exists x_2 \in (X \cap V(Q)) \setminus \{y_1, y_2\}$, so (2) holds. □

15.3 Kuratowski-Wagner

A graph G is planar if and only if it contains neither K_5 or $K_{3,3}$ as a minor.

PROOF.

By induction on $|V(G)| + |E(G)|$. One only needs to prove the ($\Longleftarrow$) direction, since we did the converse at the end of Chapter 14.

Let G' be obtained from G be contracting an edge e with ends u and v. If G' is not planar, then the theorem follows by induction hypothesis. Hence, assume G' is planar. Let w be the vertex which results from contracting e. Let $G'' = G' \setminus w = G \setminus u \setminus v$.

Case 1: G'' is 2–connected. Then by Prop 15.1, the boundary of the region of G'' that contained w is a cycle C.

Let X and Y be the sets of neighbors of u and v, respectively, in C. Then $X, Y \subseteq V(G)$. Then, by Prop 15.2, we consider a few outcomes:

(ii) There exist distinct $x_1, y_1, x_2, y_2 \in V(C)$, appearing in this order, where $x_i \in X, y_i \in Y$.

Then C together with edges ux_1, ux_2, vy_1, vy_2 is a subdivision of $K_{3,3}$ in G.

(iii) There exist distinct $x_1, x_2, x_3 \in V(C) \cap X \cap Y$.

Similarly, C together with $ux_1, ux_2, ux_3, vx_1, vx_2, vx_3$ is a K_5 subdivision in

G.

(i) There exists paths $P, Q \subseteq C$ with common ends x_1, x_2, and otherwise disjoint, such that $P \cup Q = C$ and $X \subseteq V(P)$, $Y \subseteq V(Q)$.

Let S be a curve in R joining x_1 and x_2, dividing it into two regions R_1 (with boundary $P \cup S$) and R_2 (with boundary $Q \cup S$). Drawing u in R_1 "close" to S and v in R_2 "close" to S, and joining them to their neighbors and eachother, we yield a planar drawing of $G \implies \lightning$.

Case 2: G is not connected, i.e. $G = G_1 \cup G_2$ s.t. $V(G_1) \cap V(G_2) = \varnothing$ and G_1, G_2 are non-null. If G_1, G_2 are planar, then so is G, so we may assume WLOG that G_1 is non-planar. Then, by induction hypothesis, G_1 has a K_5 or $K_{3,3}$ minor, and hence G does.

Case 3: G is not 2-connected. Then $G = G_2 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v\}$ for distinct, non-null G_1, G_2. We are done (as in case 2) if we can show G_1, G_2 planar $\implies G_1 \cup G_2$ planar.

We first make the following observation: let R be a region in a planar drawing of a graph H. The H can be redrawn such that the boundary of R is the boundary of an infinite region. (For proof of this, we draw H on the sphere and project it stereographically from a point inside R). Hence, for any path P that is part of the boundary of R, we can ensure that P is drawn on some line L and the rest of the drawing is a half plane bounded by L. Hence, we "glue" together G_1 and G_2.

Case 4: G is 2-connected, but not 3-connected, i.e. $G = G_1 \cup G_2$ such that $V(G_1) \cap V(G_2) = \{u, v\}$, where G_1, G_2 are non-null. As before, we just need to show that G_1, G_2 planar $\implies G$ planar.

Let G_1' and G_2' be obtained from G_1 and G_2, respectively, by adding an edge uv (if needed). If G_1' and G_2' are planar, so is G (by the arguments above). Hence, WLOG we may assume that G_1' is non-planar.

G_1' is a minor of G, so applying the induction hypothesis to G_1', we get a $K_{3,3}$ or K_5 minor of G.

Case 5: G is 3-connected, but for every $uv \in E(G)$, $G \setminus u \setminus v$ is no 2-connected. We write $G = G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = \{u, v\}$ and G_1, G_2 non-null.

Let G_1' be obtained from G_1 by adding a vertex $x_2 \in V(G_2)$ adjacent to u, v, w and deleting the edge uv. Let G_2' be obtained similarly. Then G_1', G_2' are minors of G.

If G_1', G_2' are planar, then G is planar. Hence, we assume WLOG that G_1' is non-planar. *BLAH* □

15.4 Kuratowski

A graph G is non-planar if and only if G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.

PROOF.

We only need to show the ($\Longrightarrow$) direction, and proceed by induction on $|V(G)| + |E(G)|$. By Thm 15.3, G contains either a K_5 or $K_{3,3}$ minor. Hence, $\exists G'$, obtained from G by performing a valid minor operation, which also contains K_5 or $K_{3,3}$ as a minor. By induction hypothesis, G' then contains a subdivision H of K_5 or $K_{3,3}$.

If the operation was a deletion of an edge or vertex, then G' is a subgraph of G, so G contains H. Hence, assume we contracted an edge u, v.

If either u or v are incident to only one edge of H, then extending the corresponding path in the subdivision yields a subdivision in G. Otherwise, the new vertex has degree four in H, so H is a subdivision of K_5, so G has a subdivision of $K_{3,3}$.

□

XVI Coloring Planar Graphs

Six Color Theorem Let G be planar and non-null. We want to show $\chi(G) \leq 6$. By Prop 14.5, there exists some vertex v with $\deg(v) \leq 5$. We'll show the result by induction on $|V(G)|$. $G \setminus v$ is 6-colorable. Since v has at most 5 neighbors, assign a color not used by one of its neighbors. Hence, G is 6-colorable.

16.1 Five Color Theorem

Let G be planar and non-null. Then $\chi(G) \leq 5$.

PROOF.

We show by induction on $|V(G)|$. The non-null graph is trivial (and so is any graph with $|V(G)| \leq 5$).

Let $|V(G)| \geq 1$. Let v be such that $\deg(v) \leq 5$, as in Prop 14.5. If $\deg(v) \leq 4$, then we apply induction hypothesis to $G \setminus v$, and extend the coloring to G with a color not used by any of its neighbors.

Hence, assume $\deg(v) = 5$ exactly. There exist neighbors u, w that are non-adjacent, or else we'd get a K_5 (even K_6) subgraph of G (but G is planar). Then let G' be obtained from G by deleting edges incident to v except uv and wv, and then contracting uv and wv. There is a 5 coloring $c : V(G') \to \{1, 2, 3, 4, 5\}$.

We extend c to $V(G) - v$ by setting $c(u) = c(w)$ to be the color of the new vertex of G'. Hence, we can extend the coloring once more to $V(G)$ by choosing a color not used by any of its neighbors (since $c(u) = c(w)$, its neighbors are colored by only 4 colors). □

This proof does *not* imply the (true) statement: $\chi(G) \leq 5$ for G with no K_6 minor. We don't necessarily know there exists $v : \deg(v) \leq 5$. There might always be $v : \deg(v) \leq 6$, but this is not well-known (it is a key point in showing Hadwiger's conjecture).

DEF 16.1 Consider a "maximal" planar graph G with $|V(G)| \geq 3$, in the sense that adding any edge makes it non-planar. Then G is 2-connected. Hence, by Prop 15.1, the boundary of every region of G is a cycle. By maximality, the boundary of every region is a length 3 cycle (any more, and we could add an edge). G is then called a *triangulation*.

DEF 16.2 Let G and G^* be two graphs drawn in the plane. Then G^* is called a *dual* of G if (1) every region of G contains exactly one vertex of G^*, and (2) every edge of G crosses exactly one edge of G^*, and vice verse, and (3) the drawings are otherwise disjoint.

If G is not well-connected (e.g. a tree), the dual may not be a simple graph.

16.2 Tait (1884)

Let G be a planar triangulation with $|V(G)| \geq 4$. Let G^* be a dual of G. Then $\chi'(G^*) = 3 \iff \chi(G) \leq 4$

This proves that the four color theorem is equivalent to the statement: *every planar, 3-regular, 2-connected graph is 3-edge-colorable.*

PROOF. ($\Longleftarrow$) Let $\chi'(G) \leq 4$, and let $c : V(G) \to \{1, 2, 3, 4\}$. However, we may write $c : V(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $e \in E(G^*)$. It crosses an edge of G, say with ends u and v. Define $c^*(e) = c(u) + c(v)$.

We claim that c^* is a 3-edge-coloring of G^*. Since $c(u) \neq c(v)$, it cannot be that $c(u) + c(v) = (0, 0)$. Hence, c^* maps $E(G)$ into 3 colors, $\{(0, 1), (1, 0), (1, 1)\}$.

The edges incident to a vertex in G^*, corresponding to a region in G with vertices u, v, w, receive the following 3 colors

$$c(u) + c(v) \quad c(v) + c(w) \quad c(w) + c(u)$$

If any of these colors were equivalent, then, since one summand is always shared, we will get the equivalence of colors of two vertices among u, v, w, which is a contradiction of the valid coloring c. Hence, c^* is a 3-edge-coloring.

($\Longrightarrow$) Let $\chi(G) \leq 3$, and let $c^* : E(G^*) \to \{(0, 1), (1, 0), (1, 1)\}$. Let H_1 be the subgraph of G^* with $V(H_1) = V(G^*)$ and edges consisting of those colored by $(1, 0)$ or $(1, 1)$.

Then the regions of H_1 may be 2-colored such that regions sharing a boundary receive different colors. Let $c_1 : V(G) \to \{0, 1\}$ be a coloring of $V(G)$ according to the regions they belong to.

Similarly, let H_2 be a subgraph of G^* consisting of edges with color $(0,1)$ or $(1,1)$. The regions of H_2 may be 2-colored, as before, so let c_2 be a coloring of $V(G)$ according to the regions they belong to.

Then $c(v) = (c_1(v), c_2(v))$ is a 4-coloring of G, with $c : V(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. Every edge of G crosses exactly one edge of G^*, which belongs to either H_1 or H_2, making the colors of the ends different. $\square$

Hard segue: Let $\hat{i}, \hat{j}, \hat{k}$ be the standard basis of $\mathbb{R}^3$. One can define the *cross product* $x : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by $i \times j = k$, $j \times k = i$, and $k \times i = j$. x is then anti-symmetric and bilinear, i.e.

$$u \times v = -v \times u \quad u \times u = 0$$

But $(i \times j) \times j = k \times j = -i$ and $i \times (j \times j) = 0$, so the cross product is especially not associative. This motivates the following theorem:

16.3 Kaufman

For any two bracketing (i.e. placings of parentheses) of the product $u_1 \times \cdots \times u_m$, there exists an assignment $u_i \in \{i, j, k\}$ such that the resulting products are equal and non-zero.

For example, suppose we consider $(u_1 \times u_2) \times u_3$ and $u_1 \times (u_2 \times u_3)$. We may write $(i \times j) \times i = k \times i = i \times -k = i \times (j \times i)$.

Thm 16.3 is, in fact, equivalent to the four color theorem (proving this is technical, but not conceptually impossible).