Stochastic Processes

MATH 447

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CONTENTS

I Markov Chains **3**

Time-Homogeneous Markov Chains
Multi-Step Transition Probabilities
Long Term Behavior
Classification of States

Index of Definitions

We assume working knowledge of probability and no knowledge of measure theory (though a grasp of analysis is essential). See these MATH 356 notes, also taught by Louigi!

I Markov Chains

Before providing definitions, we give some examples of stochastic processes:

Eg. 1.1 A simple random walk: $S_{i+1} = S_i + X_i$, where $X_i \sim \text{Ber}(p)$ and $S_0 = 0$. We might ask: does S_i ever return to 0, i.e.

$$\mathbb{P}(\exists i > 0 : S_i = 0)$$

Eg. 1.2 A branching process: as in asexual reproduction, we have an initial node. Each node n has a number of children X_n, where $\frac{X_n}{2} \sim \text{Ber}(p)$. We denote Z_i to be the number of inididuals in the i-th generation. We might ask: does Z_i ever have no children, i.e.

$$\mathbb{P}(\exists i > 0 : Z_i = 0)$$

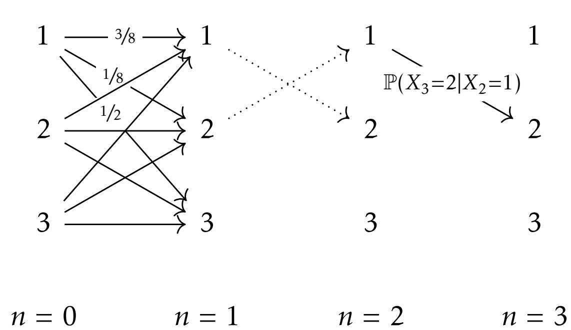
Eg. 1.3 Choose k independent random points in the square $[0, \sqrt{k}]^2$. On average, then, there is 1 point within any unit square $U \subseteq [0, \sqrt{k}]^2$.

DEF 1.1 DEF 1.2 Given a finite or countable set V, a *Markov chain* with *state space* V is a sequence $X_n : n \geq 0$ of random variables, with $X_n \in V$, such that:

$$\mathbb{P}(\underbrace{X_{n+1} = v_{n+1}}_{\text{future}} | \underbrace{X_0 = v_0, \ldots, X_{n-1} = v_{n-1}}_{\text{past}}, \underbrace{X_n = v_n}_{\text{present}}) = \mathbb{P}(X_{n+1} = v_{n+1} | X_n = v_n)$$

In other words, the future only depends on the past via the present. This is called the
DEF 1.3 *Markov property*.

Sometimes we allow Markov chains to be only finitely large (i.e. $0 \leq n \leq m$). For instance, we limit ourselves to one weekend of gambling in Las Vegas. A graphical example would look something like:



By repeated Bayes' Law, we observe **PROP 1.1**

$$\begin{aligned}
&\mathbb{P}(X_1 = v_1, ..., X_n = v_n | X_0 = v_0) \\
=&\mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_0 = v_0, X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_0 = v_0, ..., X_{n-1} = v_{n-1}) \\
=&\mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_{n-1} = v_{n-1}) \quad \text{by Markov property}
\end{aligned}$$

Time-Homogeneous Markov Chains

We often write THMC

We say that a Markov chain is *time-homogeneous* if, for all $u, v \in V$ and $n \geq 0$ DEF 1.4

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u)$$

In other words, the chain's behavior is described entirely by $\mathbb{P}(X_1 = v | X_0 = u)$ for each $(v, u) \in V \times V$. In this case, we can describe such probabilities in a *transition matrix* P: DEF 1.5

$$P = \left(p_{u,v}\right)_{(u,v)\in V^2} = \left(\mathbb{P}(X_1 = v | X_0 = u)\right)_{(u,v)\in V^2}$$

Eg. 1.4 Recall the game Snakes and Ladders. A 6×6 grid is indexed $1, ..., 36$. Players start at the 1 cell. The game ends when a player reaches the 36 cell. A die roll dictates how many spots one advances. The are some directed edges between cells (increasing: "ladders", decreasing: "snakes"). One must follow these edges when one lands at its tail. Suppose a ladder exists from 11 to 27. Then

$$\mathbb{P}(X_{11} = 27 | X_{10} = 6, X_9 = 3) = \frac{1}{6} = \mathbb{P}(X_{11} = 27 | X_{10} = 6) = \mathbb{P}(X_2 = 27 | X_1 = 6)$$

We see that Snakes and Ladders is naturally modeled as a time-homogeneous Markov chain.

Eg. 1.5 Sampling without replacement is *not* a Markov chain. If we sample from $|X| = 10$, we have

$$\begin{aligned}
&\mathbb{P}(X_3 = a \,| X_2 = b) = 1/9 \\
&\mathbb{P}(X_3 = a \,| X_2 = b, X_1 = c) = 1/8 \\
&\mathbb{P}(X_3 = a \,| X_2 = b, X_1 = a) = 0
\end{aligned}$$

so we do not satisfy the Markov property.

Eg. 1.6 Returning to the Snakes and Ladders example, consider $S \subseteq V$. Let $T_S = \inf\{n \geq 0 : X_n \in S\}$. We may ask...

- What is the average number of rounds to finite? We can write this as $\mathbb{E}[T_{\{36\}} | X_0 = 1]$.
- What is the probability of landing on 18 or 19 before the game ends? We can write this as $\mathbb{P}(T_{\{18,19\}} < T_{\{36\}} | X_0 = 1)$.
- What is the average number of visits to 6 before the game ends? We can write this as
$$\mathbb{E}[\#\{n \in [T_{\{36\}}] : X_n = 6\} | X_0 = 1]$$

- What is the expected proportion of time spent on state 5 before the game ends?
- If we allow two players, what is the probability that player 1 wins? Is this still a Markov chain?

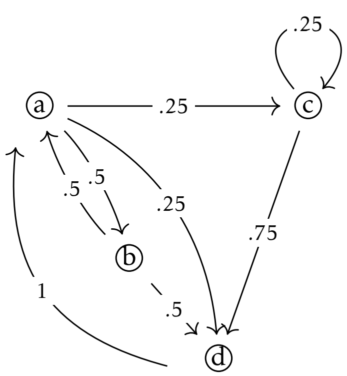
DEF 1.6 A matrix $P = \left(p_{u,v}\right)_{(u,v)\in V^2}$ is called a *stochastic matrix* if every row sums to 1, i.e.

$$\forall\ u \in V, \sum_{v \in V} p_{u,v} = 1$$

Note that any stochastic matrix is the transition matrix of some time-homogeneous Markov chain with state space V and transition probabilities

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u) = p_{u,v}$$

A directed graph, together with its stochastic matrix, can visualize any THMC:



$$\longleftrightarrow \begin{bmatrix} 0 & .5 & .25 & .25 \\ .5 & 0 & 0 & .5 \\ 0 & 0 & .25 & .75 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Eg. 1.7 Random walks on an undirected weighted graph, where edge weights dictate the proportional probability of transitioning between two states, are a special class of THMCs. In particular, given a graph $G = (V, E)$ with weights $w(e) > 0 : e \in E$, we set

$$p_{u,v} = \frac{w(\{u, v\})}{\sum_{z \in N(u)} w(\{u, z\})}$$

If there are no edges $u \leftrightarrow v$, we write $p_{u,v} = 0$.

Not every THMC can be represented by a random walk on an undirected weighted graph. In particular, see the directed graph listed above, or any transition matrix which is not symmetric.

As a concrete example, we can consider a random walk on the number line $\mathbb{Z}$, where, if $w(k, k+1) = \alpha$, $w(k-1, k) = \frac{\alpha}{2}$.

$$\cdots \overset{\frac{1}{16}}{\text{---}} -3 \overset{\frac{1}{8}}{\text{---}} -2 \overset{\frac{1}{4}}{\text{---}} -1 \overset{\frac{1}{2}}{\text{---}} 0 \overset{1}{\text{---}} 1 \overset{2}{\text{---}} 2 \overset{4}{\text{---}} 3 \overset{8}{\text{---}} \cdots$$

Multi-Step Transition Probabilities

Given a THMC $X = X_n : n \geq 0$ with a transition matrix P, we write

$$\begin{aligned}\mathbb{P}(X_2 = v|X_0 = u) &= \sum_{w \in V} \mathbb{P}(X_2 = v, X_1 = w|X_0 = u) \\ &= \sum_{w \in V} \mathbb{P}(X_1 = w|X_0 = u)\mathbb{P}(X_2 = v|X_1 = w, \cancel{X_0 = u}) \quad \text{by Markov property} \\ &= \sum_{w \in V} p_{u,w}p_{w,v} = (P^2)_{u,v} \quad \text{or write } P^2_{u,v}\end{aligned}$$

Hence, to determine a two-step transition probability, and by extension an n-step transition probability from u to w, we consider $P^n_{u,v}$. PROP 1.2

PROOF. See Prop 1.1 to expand probabilities, using Bayes', as needed. We get that

$$\begin{aligned}\mathbb{P}(X_n = v|X_0 = u) &= \sum_{v_1,\ldots,v_{n-1} \in V} \mathbb{P}(X_1 = v_1, \ldots, X_{n-1} = v_{n-1}, X_n = v|X_0 = u) \\ &= \sum_{v_1,\ldots,v_{n-1} \in V} p_{u,v_1}p(v_1, v_2)\cdots p(v_{n-1}, v) = (P^n)_{u,v} \qquad \square\end{aligned}$$

Thus, if P is a stochastic matrix, then so is P^n. PROP 1.3

PROOF. $\sum_{v \in V} P^n_{u,v} = \sum_{v \in V} \mathbb{P}(X_n = v|X_0 = u) = 1.$ $\square$

Theorem 1.1 Markov Property

If $X_n : n \geq 0$ is a THMC with state space V, then for all $u_0, \ldots, u_{n-1}, u, v \in V$,

$$\mathbb{P}(X_{n+m} = v|X_0 = u_0, \ldots, X_{n-1} = u_{n-1}, X_n = u) = \mathbb{P}(X_{n+m} = v|X_n = u) = P^m_{u,v}$$

PROOF. One shows this by combining the Markov property with Prop 1.2 via induction. $\square$

Somewhat nonsensically, we *also* call this the Markov property. When talking about THMCs, this will be the default notion.

We say that a Markov chain has an *initial distribution* $\alpha = (\alpha_v : v \in V)$ if $\mathbb{P}(X_0 = v) = \alpha_v$ for each $v \in V$. If this is the case, we often write α as a subscript of our state probabilities. For instance, DEF 1.7

$$\mathbb{P}_\alpha(X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u, X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u)\mathbb{P}_\alpha(X_n = v|X_0 = u) = \sum_{u \in V} \alpha_u P^n_{u,v}$$

For any event E depending only on $X_0, \ldots, X_n$, with $\mathbb{P}(X_n = u, E) > 0$, we have PROP 1.4

$$\mathbb{P}(X_{n+m} = v|X_n = u, E) = P^m_{u,v}$$

PROOF. For any such event E, we can determine whether E occurs exactly when we know the realized values u_i of X_i for $i = 1, \dots n-1$. Hence, we may write $\mathcal{S}$ to be the set of tuples $(u_0, \dots, u_{n-1})$ that guarantee E. It follows that

$$\mathbb{P}(X_n = u, E) = \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(\mathbf{x} = \mathbf{s}, X_n = u)$$

Similarly, we have

$$\begin{aligned} \mathbb{P}(X_{n+m} = v, X_n = u, E) &= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v, X_n = u, \mathbf{x} = \mathbf{s}) \\ &= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v | X_n = u, \mathbf{x} = \mathbf{s}) \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) \\ &= P_{u,v}^m \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) = P_{u,v}^m \mathbb{P}(X_n = u, E) \end{aligned}$$

Divide and use Bayes, and the result follows. □

PROP 1.5 If X is a THMC with transition matrix P, then, for all $k \geq 1$, $X_{kn} : n \geq 0$ is a THMC with transition matrix P^k.

PROOF. For any $n \neq 0$, any sequence $u_0, \dots, u_{n+1} \in V$ satisfies

$$\mathbb{P}(X_{(n+1)k} = u_{n+1} | X_0 = u_0, X_k = u_1, \dots, X_{nk} = u_n) = P_{u_n, u_{n+1}}^k \quad \square$$

Theorem 1.2 Chapman-Kolmogorov

For any Markov chain X with state space V, any $m, n \geq 0$, and $u, v \in V$,

$$\mathbb{P}(X_{m+n} = v | X_0 = u) = \sum_{w \in V} \mathbb{P}(X_n = w | X_0 = u) \mathbb{P}(X_{m+n} = v | X_n = w)$$

If the X is time homogeneous, then this is $P_{u,v}^{n+m}$, which agrees with Prop 1.2.

Long Term Behavior

DEF 1.8 Recall from probability the *law of large numbers*: if $Y_n : n \geq 1$ are IID with common mean μ, then $\frac{S_n}{n} \to \mu$ in probability, where $S_n = \sum_{i=1}^n Y_i$, i.e. $\forall \varepsilon > 0$,

$$\limsup_{n \to \infty} \mathbb{P}\left(\left| \frac{S_n}{n} - \mu \right| > \varepsilon \right) = 0$$

If $Y_i \in \mathbb{Z}$ then, for $k, \ell, u_i \in \mathbb{Z}$ and $i = 1, \dots, n-1$,

$$\begin{aligned} \mathbb{P}(S_{n+1} = \ell | S_n = k, S_i = u_i \ \forall i) &= \mathbb{P}(Y_{n+1} = \ell - k | S_n = k, S_i = u_i \ \forall i) \\ &= \mathbb{P}(Y_{n+1} = \ell - k | Y_1 = u_1 - u_0, Y_2 = u_2 - u_1, \dots, Y_n = k - u_{n-1}) \\ &= \mathbb{P}(Y_{n+1} | \ell - k) = \mathbb{P}(Y_1 = \ell - k) = P_{k,\ell} \end{aligned}$$

where $S_n : n \geq 0$ has transition matrix P, noting that it may be viewed as a THMC.

From now on, we denote by $\mathbb{P}_v(E)$ the probability $\mathbb{P}(E|v)$.

Eg. 1.8 A general two-state chain, with states A and B, can be described by

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Let $q_n = \mathbb{P}_A(X_n = A) = \mathbb{P}(X_n = A|X_0 = A)$. Then

$$\begin{aligned} q_{n+1} &= \mathbb{P}_A(X_{n+1} = A, X_n = A) + \mathbb{P}_A(X_{n+1} = A, X_n = B) \\ &= \mathbb{P}_A(X_{n+1} = A|X_n = A)\mathbb{P}_A(X_n = A) + \mathbb{P}_A(X_{n+1} = A|X_n = B)\mathbb{P}_A(X_n = B) \\ &= (1-\alpha)q_n + \beta(1-q_n) = \beta + (1-\alpha-\beta)q_n \end{aligned}$$

This recurrence has a unique solution. In particular, one can find

$$q_n = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^n \frac{\alpha}{\alpha+\beta}$$

It follows that $q_n \to \frac{\beta}{\alpha+\beta}$, and hence $\mathbb{P}_A(X_n = B) = 1 - q_n \to \frac{\alpha}{\alpha+\beta}$. Likewise:

$$\mathbb{P}_B(X_n = B) = \frac{\alpha}{\alpha+\beta} + (1_\alpha - \beta)^n \frac{\beta}{\alpha+\beta}$$

So $\mathbb{P}_B(X_n = B) \to \frac{\alpha}{\alpha+\beta}$.

Let $\pi := (\pi_A, \pi_B) := \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$ be the distribution of our initial state X_0, Then

$$\mathbb{P}_\pi(X_1 = A) = \pi_A \mathbb{P}_A(X_1 = A) + \pi_B \mathbb{P}_B(X_1 = A) = \pi_A$$

and, similarly, $\mathbb{P}_\pi(X_1 = B) = \pi_B$. Hence, if X_0 has initial distribution π, then X_1 also has distribution π. By induction, X_n has distribution π $\forall n \geq 0$.

When we say $X = \text{Markov}(P)$, we mean that X is a THMC with transition matrix P.

A probability distribution π is called *stationary* if $\pi P = \pi$. Similarly, a probability DEF 1.9
distribution λ is called a *limiting distribution* if, for each $u, v \in V$ DEF 1.10

$$(P^n)_{u,v} \to \lambda_v \text{ as } n \to \infty$$

In other words, $\mathbb{P}_u(X_n = v) \to \lambda_v$. Note that, for any initial distribution α, we have $\alpha P^n \to \lambda$, i.e. $(\alpha P^n)_v \to \lambda_v$, where λ is limiting.

If λ is a limiting distribution for P, then λ is stationary for P. PROP 1.6

PROOF. Fix any initial distribution α, we have

$$\lambda = \lim_{n\to\infty}(\alpha P^n) = \lim_{n\to\infty}(\alpha P^{n-1}P) = (\lim_{n\to\infty}\alpha P^{n-1})P = \lambda P \qquad \square$$

Stationary distributions need not be unique, but limiting distributions are (as the limit $\lim_{n\to\infty}\alpha P^n$ is well-defined). In general, then, stationary distributions need not be limiting distributions.

DEF 1.11 A stochastic matrix P is called *regular* if $\exists n \geq 1$ such that all entries of P^n are positive.

When $n = 0$, $P^n = I$, which encapsulates the idea that, at timestep 0, we will be at our initial positions.

Theorem 1.3 Fundamental Theorem of Markov Chains

Every regular stochastic matrix P has a limiting distribution π.

Incorporating some of the formulations above, this is equivalent to saying: *For a regular stochastic matrix, there exists a unique distribution $\pi = (\pi_v : v \in V)$ such that $\pi P = \pi$ and $\mathbb{P}_u(X_n = v) \to \pi_v\ \forall u, v \in V$.*

In this case, there is a unique stationary distribution, and it is the unique limiting distribution.

Let $\rho = \langle 1, ..., 1\rangle$. Then note that $P\rho = \rho$, since the sum of any row in P must be 1. Hence, P has eigenvalue 1. It follows that it has a left eigenvector, i.e. $\pi : \pi P = \pi$. This is exactly a stationary distribution (as long as we scale suitably such that π is a distribution).

Classification of States

DEF 1.12 For $u, v \in V$, we say that v is *accessible* from u if $\exists n \geq 0$ such that $(P^n)_{u,v} > 0$. Equivalently, in the directed graph generated by P, there is a directed path from u to v. When v is accessible from u, we write $u \to v$.

DEF 1.13 States u and v *communicate* if $u \to v$ and $v \to u$. When u and v communicate, we write $u \leftrightarrow v$. Observe that communication is a equivalence relation. Hence, the state space V can be written as a disjoint union of mutually-communicating states, called a

DEF 1.14 *communication class*. Note that, in the directed graph generated by P, these correspond to the strongly connected components.

DEF 1.15 We say that P is *irreducible* if there is only one communication class.

Clearly, if P is regular, then it is irreducible

PROP 1.7 $u \to v \iff \mathbb{P}_u(T_{\{v\}} < \infty) > 0$.

DEF 1.16 The *period* of a state $u \in V$ is

$$d(u) := \gcd(n > 0 : P^n_{u,u} > 0)$$

DEF 1.17 If $d(u) = 1$, we call u *aperiodic*. By extension, P is aperiodic if $d(u) = 1\ \forall u \in V$, and X is

PROP 1.8 aperiodic if $X = \text{Markov}(P)$ for P aperiodic. If $u \leftrightarrow v$, then $d(u) = d(v)$.

PROOF. Let $I = \{n > 0 : P^n_{u,u} > 0\}$, and similarly J for v. Hence, $d(u) = \gcd(I)$ and $d(v) = \gcd(J)$.

Let $a, b > 0$ such that $P^a_{u,v} > 0$ and $P^b_{v,u} > 0$. Then

$$P^{a+b}_{u,u} \geq P^a_{u,v} P^b_{v,u} > 0$$

$\implies a + b \in I$, so $d(u) | a + b$. Now, if $n \in J$, then

$$P^{a+b+n}_{u,u} \geq P^a_{u,v} P^n_{v,v} P^b_{v,u} > 0$$

$\implies a + b + n \in I$, so $d(u) | n + a + b$. But, by the previous line, $d(u)|n$. Since $n \in J$ is arbitrary, we can write $d(u) | \gcd(J) = d(v)$.

Symmetrically, we could conclude that $d(v)|d(u)$, so indeed $d(v) = d(u)$. □

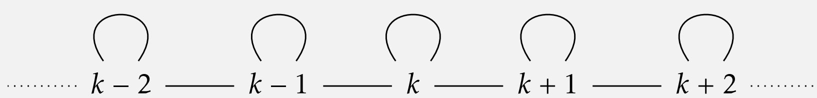
Theorem 1.4 theorem name

Let $P = (p_{u,v})_{u,v \in V}$ be a stochastic matrix, where $|V| < \infty$. Then

$$P \text{ is regular} \iff P \text{ is irreducible and aperiodic}$$

PROOF.

We first note why finiteness is necessary. Consider:



with all edges having weight 1. This graph is clearly aperiodic and irreducible, but not regular.

($\implies$) If P is regular, then $\exists n > 0$s.t. $P^n_{u,v} > 0$ for all $u, v \in V$. Then, for all $u, v \in V$, we have $u \to v$ and $v \to u$. Hence, P is irreducible.

($\impliedby$) If P is irreducible, then for all $u \in V$, there is some $v \in V$ such that $P_{v,u} > 0$ (think about this in graph theoretic term). Then, let $n > 0$ be such that $P^n_{u,u}$ is positive. We have

$$P^{n+1}_{u,u} \geq P^n_{u,v} P_{u,v} > 0$$

So, with $I = \{m > 0 : P^m_{u,u} > 0\}$, $d(u) = \gcd(I) \leq \gcd(n, n+1) = 1$. □

We now have the tools to prove Thm 1.3, the fundamental theorem of Markov chains.

PF. OF 1.3

Statement: every regular stochastic matrix P has a limiting distribution.

Let $P = (p_{u,v})_{u,v \in V}$ be a stochastic □

INDEX OF DEFINITIONS

accessible 1.12
aperiodic 1.17

communicate 1.13
communication class 1.14

initial distribution 1.7
irreducible 1.15

law of large numbers 1.8
limiting distribution 1.10

Markov chain 1.1
Markov property 1.3

period 1.16

regular 1.11

state space 1.2
stationary 1.9
stochastic matrix 1.6

time-homogeneous 1.4
transition matrix 1.5