
VECTOR CALCULUS NOTES

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I Vector Fields

PRODUCTS ON VECTOR SPACES

Recall the definition of the *inner product* over a vector space V :

1. $\langle u, v \rangle = \overline{\langle v, u \rangle} = \langle v, u \rangle$ in \mathbb{R} (where we'll be in this class)
2. $\langle au + bw, v \rangle = a \langle u, v \rangle + b \langle w, v \rangle$
3. $\langle u, u \rangle \geq 0$, and $= 0 \iff u = \mathbf{0}$

From this, we define the *norm* of $u \in V$ to be $\|u\| := \sqrt{\langle u, u \rangle}$. This is well-defined, since $\langle u, u \rangle \geq 0$.

$$\forall u, v \in V, |\langle u, v \rangle| \leq \|u\| \|v\|$$

PROP 1.1
Cauchy-Schwartz Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

PROP 1.2
Triangle Inequality

The *cross product* of $u, v \in \mathbb{R}^3$, with respect to \mathbb{R}^3 , is the determinate of the following “matrix”:

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$. We observe the following two properties of the cross product in \mathbb{R}^3 :

PROP 1.3

1. $(u \times v) \cdot u = 0$
2. $\|u \times v\| = \|u\| \|v\| \sin(\theta)$, where θ is the angle found between u and v . A conceptualization of this property is that “ u -cross- v is equal to the area created by the parallelogram bounded by u and v .”

LINES

Define a *line* $l(t) \in \mathbb{R}^n$ to be a function from $\mathbb{R} \rightarrow \mathbb{R}^n$, with the primary form $l(t) = P + td$, with $P, d \in \mathbb{R}^n$, $t \in \mathbb{R}$. We call P the “point vector” and d the “direction vector”. An alternate form, with two points $P, Q \in \mathbb{R}^n$, would be $l(t) = (1-t)P + tQ$, where $l(t)$ lies along the path between P and Q for $t \in [0, 1]$.

Distance between a point and line Using this definition, how can we find the shortest path between a point R and a line $l(t)$, which lies between P and Q ?

Idea 1 We know the desired vector $w = PR \sin(\theta)$, the angle between PR and PQ . To find this value, note that $\|PR \times PQ\| = \|PR\| \|PQ\| \sin(\theta)$.

Idea 2 We can project R onto PQ , and then subtract this projection from PR .

Idea 3 We can minimize a distance function between R and a point on l , i.e. $l(t)$. Thus, we take $\min_{t \in \mathbb{R}} \|R - l(t)\| = \alpha$, and then take $RI(\alpha)$ to be the shortest path.

Idea 4 We can find when $(R - l(t)) \cdot d = 0$.

Sometimes called “skew lines”

Distance between 2 lines Consider two lines, l_1 and l_2 , which do not intersect but are not necessarily parallel. What is the minimal distance between l_1 and l_2 ?

Idea 0 Conceptualize this problem as finding the distance between the parallel planes defined by $\{l_1, l_2\}$.

Idea 1 We can minimize $\|l_1(t) - l_2(s)\|$ (really, one should minimize the square to make one’s life easier).

Idea 2 Pick any two points, say $l_1(T)$ and $l_2(S)$, and project $l_1(T)l_2(S)$ onto $l_1 \times l_2$.

Idea 3 Minimize $\text{dist}(l_1(t), l_2)$ for fixed t .

Idea 4 Find t and s such that $[l_1(t) - l_2(s)] \cdot \vec{d}_1 = 0$ and $[l_1(t) - l_2(s)] \cdot \vec{d}_2 = 0$

PROP 1.4

$\|u \times v\| = \|u\| \|v\| \sin(\theta) = \text{Area of parallelogram defined by } u \text{ and } v.$

PLANES

A plane $r(s, t)$ is a function $[0, 1]^2 \rightarrow \mathbb{R}^3$ defined by $d_1, d_2 \in \mathbb{R}^3$, two vectors, and $P \in \mathbb{R}^3$, a point. In particular, $r(s, t) = P + s\vec{d}_1 + t\vec{d}_2$. This is called the *parametric form*.

The *point-normal* form is a function $\mathbb{R}^2 \rightarrow \mathbb{R}^3$ is given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $\vec{n} = \langle a, b, c \rangle$ is a vector normal to the plane, and $P = \langle x_0, y_0, z_0 \rangle$ is a point lying on the plane.

Distance between a point R and a plane r

Idea 1 Minimize $\|R - r(s, t)\|$ (or the square)

Idea 2 $\|\text{proj}_{\vec{n}}(P - R)\|$, where \vec{n} and P are as given in the point-normal form.

TRANSFORMATIONS AND PARAMETERIZATIONS

The following table give general examples of linear transformations $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Dimension	Linear	Affine
$n = 0$	$\lambda(0) = 0$	$\lambda(0) = P$
$n = 1$	$\lambda(t) = t\vec{d}$	$\lambda(t) = P + t\vec{d}$
$n = 2$	$\lambda(t, s) = t\vec{d}_1 + s\vec{d}_2$	$\lambda(t, s) = P + t\vec{d}_1 + s\vec{d}_2$
$n = 3$	$\lambda(t, s, r) = t\vec{d}_1 + s\vec{d}_2 + r\vec{d}_3$	$\lambda(t, s, r) = P + t\vec{d}_1 + s\vec{d}_2 + r\vec{d}_3$

We also define the following important curves in \mathbb{R}^2 :

Type	Explicit Form	Parametric Form
Ellipse	$x^2 + y^2 = 1$	$r(t) = \langle t, \sqrt{1-t^2} \rangle_{t \in [-1,1]} = \langle \cos(t), \sin(t) \rangle_{t \in [-\pi, \pi]}$
Hyperbola	$x^2 - y^2 = 1$	$r(t) = \langle \sqrt{1+t^2}, t \rangle_{t \in \mathbb{R}} = \langle \cosh(t), \sinh(t) \rangle_{t \in \mathbb{R}}$
Parabola	$x = y^2$	
Double Cone	$x^2 = y^2$	
Any Function	$y = F(x)$	$r(t) = \langle t, F(t) \rangle$

Define a *path* in \mathbb{R}^m to be a continuous function $r : \mathbb{R} \rightarrow \mathbb{R}^m$, e.g. $[a, b] \rightarrow \mathbb{R}^m$.

Define a *curve* in \mathbb{R}^m to be the image of a path (i.e. a set of points in \mathbb{R}^m). Recall the statement “paths parameterize curves.”

For example, the unit circle $x^2 + y^2 = 1$ is parameterized by the path $r : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $r(t) = \langle \cos(t), \sin(t) \rangle$.

Define the *tangent* line of \vec{r} at $a \in \mathbb{R}$ to be an affine transformation $l : \mathbb{R} \rightarrow \mathbb{R}^m$ satisfying the following:

1. $l(t) = r(a) + (t - a)\vec{d} : \vec{d} \neq 0$
2. $\lim_{t \rightarrow a} \frac{\|r(t) - l(t)\|}{|t - a|} = 0$

♠ Examples ♣

We'll now find the derivative of the unit circle at a point $a \in \mathbb{R}$: we have $r(a) = \langle \cos(a), \sin(a) \rangle$. Thus:

$$l(t) = \langle \cos(t), \sin(t) \rangle + (t - a)\langle d_1, d_2 \rangle$$

Where $\langle d_1, d_2 \rangle \neq 0$. Consider now the limit in question 2:

$$\begin{aligned}
 \lim_{t \rightarrow a} \frac{\|r(t) - l(t)\|}{|t - a|} &= \lim_{t \rightarrow a} \frac{1}{|t - a|} \sqrt{(\cos(t) - \cos(a) - (t - a)d_1)^2 + (\sin(t) - \sin(a) - (t - a)d_2)^2} \\
 &= \lim_{t \rightarrow a} \sqrt{\left(\frac{\cos(t) - \cos(a)}{t - a} - d_1\right)^2 + \left(\frac{\sin(t) - \sin(a)}{t - a} - d_2\right)^2} \\
 &\stackrel{=}{=} \sqrt{(-\sin(a) - d_1)^2 + (\cos(a) - d_2)^2} = 0 \\
 &\iff d_1 = -\sin(a) \wedge d_2 = \cos(a) \\
 &\implies l(t) = \langle -\sin(a), \cos(a) \rangle \quad \square
 \end{aligned}$$

Frequently, $l(t)$ is referred to as the “velocity vector” of $r(t)$, and is notated as $r'(t)$. Notice that $r'(t)$ is equivalent to the component-wise derivative of the coordinates of $r(t)$ w.r.t. t . Formally:

Given $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^n$, the *derivative* of \vec{r} at $a \in \mathbb{R}$ is a linear transformation $\vec{\lambda}: \mathbb{R} \rightarrow \mathbb{R}^n$ satisfying

$$\lim_{t \rightarrow a} \frac{\|r(t) - r(a) - \lambda(t - a)\|}{|t - a|} = 0 \quad \text{or equivalently} \quad \lim_{h \rightarrow 0} \frac{\|r(a + h) - r(a) - \lambda(h)\|}{|h|} = 0$$

It is denoted $D\vec{r}_a$, and represented by the $n \times 1$ matrix $r'(a)$. One may now rewrite the tangent line in the form $l(t) = r(a) + \lambda(t - a)$.

The *arc length* of a curve $r(t)$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

An *arc length parameterization* of $r(t)$ is some $t = \alpha(s)$ such that $r(\alpha(s))$ has a unit velocity vector, i.e. $\|r'(\alpha(s))\| = 1$. Alternatively, one could find an expression for arc length, and then parameterize $r(t)$ in terms of its arc length. The resultant will be equivalent.

♠ Examples ♣

We'll do an arc length parameterization of a semicircle of radius 1 with its center at the origin, i.e. $y = \sqrt{1 - x^2}$. We get the natural parameterization $r(t) = \langle t, \sqrt{1 - t^2} \rangle$, where $t \in [-1, 1]$. We'd like to find a change of parameters $t = \alpha(s)$ such that $\|r(\alpha(s))\| = 1$ and $\alpha' \geq 0$.

$$\begin{aligned} r(\alpha(s)) &= \left\langle \alpha(s), \sqrt{1 - \alpha(s)^2} \right\rangle \\ r'(\alpha(s)) &= \left\langle \alpha'(s), \frac{1}{2}(1 - \alpha(s)^2)^{-\frac{1}{2}} \cdot (-2\alpha(s)\alpha'(s)) \right\rangle \\ &= \alpha'(s) \left\langle 1, \frac{-\alpha(s)}{\sqrt{1 - \alpha(s)^2}} \right\rangle \end{aligned}$$

$$\begin{aligned} \text{Then } 1 = \|r'(\alpha(s))\| &= \alpha'(s) \sqrt{1 + \frac{\alpha(s)^2}{1 - \alpha(s)^2}} \\ &= \frac{\alpha'(s)}{\sqrt{1 - \alpha(s)^2}} \end{aligned}$$

Integrating with respect to s , we get $s = \arcsin(\alpha(s)) = \arcsin(t)$. Thus, $t = \sin(s)$, and $s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and we yield the parameterization $\langle \sin(s), \cos(s) \rangle : s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

SURFACES

We note the following quadric surfaces:

Type	Explicit Form
Ellipsoid	$x^2 + y^2 + z^2 = 1$
Elliptic Hyperboloid	$x^2 + y^2 - z^2 = 1$
Elliptic Paraboloids	$x^2 + y^2 - z^2 = -1$
Hyperbolic Paraboloids	$x = y^2 - z^2$
Double Cones	$x^2 = y^2 + z^2$

A surface $F(x, y)$ is called differentiable at (a, b) if there exists some linear transformation $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k \rangle\|}$$

One may represent $\lambda(h, k) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = uh + vk$

♠ Examples ♣

Let $F(x, y) = xy$. We consider F at (a, b) . Then

$$\begin{aligned} 0 \leq \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k \rangle\|} &= \frac{|(a+h)(b+k) - ab - (uh + vk)|}{\|\langle h, k \rangle\|} \\ &= \frac{|bh + ak + hk - uh - vk|}{\|\langle h, k \rangle\|} = \frac{|(b-u)h + (a-v)k + hk|}{\|\langle h, k \rangle\|} \\ &\leq \frac{|b-u||h|}{|h|} + \frac{|a-v||k|}{|k|} + \frac{|h||k|}{|h|} \quad \text{since } |h|, |k| \leq \|\langle h, k \rangle\| \\ &= |b-u| + |a-v| + |k| \rightarrow |b-u| + |a-v| \\ &= 0 \quad \text{when } b = u, a = v \end{aligned}$$

Thus, the desired limit is always \geq and ≤ 0 , so especially it is 0. Our derivative at (a, b) is then $\lambda(x, y) = bx + ay$.

One may also find these coefficients as the partial derivative of F , i.e.

$$\nabla F(a, b) = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle \Big|_{(a,b)}$$

This is called the *gradient*. Similarly, $\alpha(x, y) = F(a, b) + \lambda(x - a, y - b)$ is called the *affine approximation* at (a, b) .

If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at \vec{a} , then all partial derivatives of F at \vec{a} exist. Furthermore, $\lambda(\vec{a}) = F'(\vec{a}) = \left[\partial_1 F \cdots \partial_n F \right] \Big|_{\vec{a}}$.

PROP 1.5

Note that the converse is *false* (as a counterexample, see $F = \sqrt{|xy|}$)

1.1 Partial Converse

If all partial derivatives of $F : \mathbb{R}^n \rightarrow \mathbb{R}$ exist near \vec{a} and are continuous at \vec{a} , then F is differentiable at \vec{a} .

PROOF FOR $n = 2$.

Let $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ be a linear transformation defined by $\left[\partial_1 F \cdots \partial_n F \right] \Big|_{\vec{a}}$. Then

$$\lambda(\vec{h}) = \sum_{i=1}^n \partial_i F(\vec{a}) h_i$$

Let $n = 2$. Then

$$\begin{aligned} |F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})| &= |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) + F(a_1 + h_1, a_2) \\ &\quad - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1 - \partial_2 F(\vec{a}) h_2| \\ &\leq |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) - \partial_2 F(\vec{a}) h_2| \\ &\quad + |F(a_1 + h_1, a_2) - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1| \\ &= |\partial_2 F(\vec{c}) h_2 - \partial_2 F(\vec{a}) h_2| + |\partial_1 F(\vec{d}) h_1 - \partial_1 F(\vec{a}) h_1| \\ &\quad \text{by mean value thm.} \\ &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| |h_2| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| |h_1| \\ \frac{|F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})|}{\|\vec{h}\|} &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{\|\vec{h}\|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{\|\vec{h}\|} \\ &\leq |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{|h_2|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{|h_1|} \\ &\quad \text{since } |h_i| < \|\vec{h}\| \\ &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \end{aligned}$$

Then, as $\vec{h} \rightarrow 0$, $\vec{c}, \vec{d} \rightarrow \vec{a}$. Since F , is continuous, we know $F(\vec{c}) \rightarrow F(\vec{a})$ and similarly for $F(\vec{d})$. Thus,

$$|\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \rightarrow |\partial_2 F(\vec{a}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{a}) - \partial_1 F(\vec{a})| = 0$$

And we conclude that the limit, as \leq and ≥ 0 , is 0. \square

$F : \mathbb{R}^n \rightarrow \mathbb{R}$ is called C^1 continuous (or *continuously differentiable*) if all partial exists and are continuous everywhere.