Quantitative Risk Management

MATH 510

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CONTENTS

This course is based on the ETH course and textbook of the same name, by McNeil, Fray, and Embrechts. The present document is based on this textbook and Prof. Neshlehov's lectures. In order, we will focus on: stochastic volatility, extreme-value theory, multivariate models, risk aggregation, and backtesting.

Cover: index simulation following the stochastic volatility process

DEF 0.1
DEF 0.2

We will assume good working knowledge of probability and statistics. In addition, we remind ourselves of the *survivor function* $\overline{F}(x) := \mathbb{P}(X > x) = 1 - F(x)$, as well as the *$\alpha$-quantile*, where $\alpha \in (0, 1)$, defined to be

$$q_\alpha = F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$$

DEF 0.3

In this course, we would like to quantify one's exposure to bad consequences. The likelihood of loss or less-than-expected gains is called *risk*. The following are types of risk:

Risk is necessary: taking this course incurs a risk of a poor grade

Risk	Description
Credit Risk	Odds a debtor defaults on payment
Market Risk	Exposure to price fluctuations of bonds, stocks, or derivatives
Operational Risk	Risk relating to circumstantial adverse events (e.g. institutional fraud)
Liquidity Risk	Risk of damage from not having sufficient assets to pay off debts
Model Risk	Risk associated with financial model inaccuracies; closely related to operational risk
Underwriting Risk	Odds that an insured makes a claim on their policy

The above types of risk interact with each-other. Quantitative risk management aims to model these interactions and hedge against risk.

I   Loss

RISK FACTORS

DEF 1.1
DEF 1.2

A *portfolio* is a collection of assets or liabilities. Denote by V_t the value of the portfolio at time t. We denote by Δt a *time horizon*, i.e. a duration of time. Assuming that V_t is known, that the composition of the portfolio remains constant over the time horizon, and that there are no payments made, we denote by V_{t+1} the value of the portfolio at time $t + \Delta t$.

Portfolios may include stocks, bonds, derivatives, risky loans, or insurance contracts, for example.

DEF 1.3

We write $\Delta V_{t+1} = V_{t+1} - V_t$. This is a random variable which takes a negative value on losses and a positive value on profits. Since we care about managing risk in this course, we prefer that losses are positively indicated. Define *loss* as

Hence, V_{t+k} is the value of the portfolio at time $t + k\Delta t$

$$L_{t+1} = -\Delta V_{t+1} = \begin{cases} V_t - V_{t+1} & \Delta t \text{ is a short horizon} \\ V_t - \frac{V_{t+1}}{1+r_{t,1}} & \Delta t \text{ is a long horizon} \end{cases}$$

Note that, for long time horizons, we must account for the time value of money. If $r_{t,1}$ is the risk-free interest rate, applied over Δt, then $\alpha(1 + r_{t,1})$ at time $t + 1$ is the equivalent

in value to α at time t. Hence, V_{t+1} must be adjusted to $V_{t+1}/(1 + r_{r,1})$ to keep L_{t+1}'s time-value of money consistent. Working in "$t + 1$ dollars," one could equivalently write $L_{t+1} = V_t(1 + r_{t,1}) - V_{t+1}$ for long time intervals.

V_t is a function of multiple *risk factors*, denoted by $\mathbf{Z}_t = \langle Z_{t,1}, ..., Z_{t,d} \rangle$. Thus, we write $V_t = f(t, \mathbf{Z}_t)$ with $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, called the *risk function*. When $\mathbf{Z}_t$ is known, we write $\mathbf{Z}_t = \mathbf{z}_t$. In this case, $f(t, \mathbf{z}_t)$ is called the *realized value* of V_t. **DEF 1.4** **DEF 1.5** **DEF 1.6**

> **Eg. 1.1** Consider a portfolio of d stocks, and let λ_i denote the number of shares in stock i at time t. Let $S_{t,i}$ denote the price of the stock i. Write
>
> $$Z_{t,i} = \log S_{t,i} : i \in [d]$$
>
> The value of the portfolio is then
>
> $$V_t = \sum_{i=1}^{d} \lambda_i S_{t,i} = \sum_{i=1}^{d} \lambda_i e^{Z_{t,i}}$$
>
> This formula is called the *value with log prices*. The purpose of writing $\mathbf{Z}_t = \log(\mathbf{S}_t)$ versus $\mathbf{Z}_t = \mathbf{S}_t$ is purely numerical. **DEF 1.7**

Risk factors may change over the time horizon. We call time-series changes in risk factors *risk factor changes*, denoted by $\mathbf{X}_{t+1}$. In particular, **DEF 1.8**

$$\mathbf{X}_{t+1} = \mathbf{Z}_{t+1} - \mathbf{Z}_t$$

Rearranging Def 1.5 gives

$$L_{t+1} = f(t, \mathbf{Z}_t) - f(t + 1, \mathbf{Z}_t + \mathbf{X}_{t+1})$$

When modeling loss, we assume that $\mathbf{z}_t$ is observable. Thus, we are forced only to consider $\mathbf{X}_{t+1}$ as a random variable. If $\mathbf{x}_{t+1}$ is observable, then L_{t+1} is called the *realized loss*. Realized loss is notated by $l_{[t]}(\mathbf{x}) = L_{t+1}$, especially when we wish to parameterize $\mathbf{x}_{t+1}$. For simplicity, we drop the subscript, i.e. $\mathbf{x} = \mathbf{x}_{t+1}$. We call this function the *loss operator*. **DEF 1.9** **DEF 1.10**

We say that L_{t+1} follows a *loss distribution*. The loss distribution is typically right-skewed and has a fat right tail: the probability of heavy profits is small, but it is easy to incur heavy losses. **DEF 1.11**

We may use a linearization to obtain the *linearized loss*, denoted $L_{t+1}^{\Delta} \approx L_{t+1}$. **DEF 1.12**

The linearized loss, assuming differentiability of f along t and $\mathbf{Z}_t$, is given by **PROP 1.1**

$$L_{t+1}^{\Delta} = -f_t(t, \mathbf{z}_t) - \sum_{i=1}^{d} f_{z_{t,i}}(t, \mathbf{z}_t) X_{t+1,i}$$

PROOF.

Recall $L_{t+1} = f(t, \mathbf{z}_t) - f(t+1, \mathbf{z}_{t+1})$. We will approximate $f(t+1, \mathbf{z}_{t+1})$:

$$f(t+1, \mathbf{z}_{t+1}) \approx f(t, \mathbf{z}_t) + \nabla f(t, \mathbf{z}_t) \cdot (\langle t+1, \mathbf{z}_{t+1}\rangle - \langle t, \mathbf{z}_t\rangle)$$

This is

$$f(t, \mathbf{z}_t) + f_t(t, \mathbf{z}_t) \cdot 1 + \sum_{i=1}^{d} f_{z_{t,i}}(t, \mathbf{z}_t) X_{t+1,i}$$

Applying to L_{t+1} gives

$$L_{t+1}^{\Delta} = f(t, \mathbf{z}_t) - \left[f(t, \mathbf{z}_t) + f_t(t, \mathbf{z}_t) + \sum_{i=1}^{d} f_{z_{t,i}}(t, \mathbf{z}_t) X_{t+1,i} \right] = -f_t(t, \mathbf{z}_t) - \sum_{i=1}^{d} f_{z_{t,i}}(t, \mathbf{z}_t) X_{t+1,i}$$

as desired.   □

Linearized loss is convenient and easy to understand, but the assumption of its accuracy is a heavy one. An easy improvement would be to consider higher-order Taylor approximations, but the estimation of the necessary derivatives may not be numerically stable.

Eg. 1.2 Following Example 1.1, we compute

$$\mathbf{X}_{t+1} = \mathbf{Z}_{t+1} - \mathbf{Z}_t = \left\langle \log \frac{S_{t+1,1}}{S_{t,1}}, ..., \log \frac{S_{t+1,d}}{S_{t,d}} \right\rangle$$

DEF 1.13

which, when $\mathbf{Z}_t = \log(\mathbf{S}_t)$, we call *log returns*. Rearranging L_{t+1}:

$$L_{t+1} = \sum_{i=1}^{d} \lambda_i e^{Z_{t,i}} - \sum_{i=1}^{d} \lambda_i e^{X_{t+1,i} - Z_{t,i}} = -\sum_{i=1}^{d} \lambda_i e^{Z_{t,i}}(e^{X_{t+1,i}} - 1) = -V_t \sum_{i=1}^{d} w_{t,i}(e^{X_{t+1,i}} - 1)$$

DEF 1.14

where $w_{t,i} = \frac{\lambda_i S_{t,i}}{V_t}$ is the *relative weight* of stock i at time t. In the language of operators:

$$l_{[t]}(\mathbf{x}) = -V_t \sum_{i=1}^{d} w_{t,i}(e^{x_i} - 1)$$

Following Def 1.12,

$$L_{t+1}^{\Delta} = -\sum_{i=1}^{d} \lambda_i e^{Z_{t,i}} X_{t+1,i} = -V_t \sum_{i=1}^{d} w_{t,i} X_{t+1,i}$$

Thus, if we assume that $\mathbf{X}_{t+1} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ follows the multivariate normal distribution,

$$\mathbb{E}[L_{t+1}^{\Delta}] = -V_t(\boldsymbol{w_t} \cdot \boldsymbol{\mu})$$

with $w_t = \langle w_{t,1}, ..., w_{t,d} \rangle$. Similarly,

$$\text{Var}(L_{t+1}^{\Delta}) = V_t^2 \sum_{i=1}^{d} \text{Var}(w_{t+1} X_{t+1,i}) = V_t^2 (w_t^T \Sigma w_t)$$

Note that Σ is a covariance matrix, so $w_t^T \Sigma w_t$ is indeed a scalar, as expected.

Eg. 1.3 Consider a portfolio with one standard European call on a stock S with maturity T and exercise price K. The value of European options is modeled by the Black-Scholes equation:

$$V_t = C^{BS}(t, S_t, r_t, \sigma_t)$$

where S_t and σ_t is the price and volatility of the underlying stock, respectively, and r_t is the risk-free interest rate. We write $\mathbf{Z}_t = \langle \log(S_t), r_t, \sigma_t \rangle$. The Black-Scholes equation satisfies

$$\frac{\partial C^{BS}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C^{BS}}{\partial S^2} + rS \frac{\partial C^{BS}}{\partial S} - rC^{BS} = 0$$

Differentiating C^{BS} gives "the Greeks"

$$C_t^{BS}(\text{theta}) \quad C_r^{BS}(\text{rho}) \quad C_\sigma^{BS}(\text{vega}) \quad C_S^{BS}(\text{delta})$$

with $C_z^{BS} = S_t \times C_S^{BS}$. One can compute the linearized loss to be

$$L_{t+1}^{\Delta} = -C_t^{BS} - S_t C_S^{BS} X_{t+1,1} - C_r^{BS} X_{t+1,2} - C_\sigma^{BS} X_{t+1,3}$$

Note that

$$\mathbf{X}_{t+1} = \left\langle \log\left(\frac{S_{t+1}}{S_t}\right), r_{t+1} - r_t, \sigma_{t+1} - \sigma_t \right\rangle$$

Stylized Loan Portfolio

This is an example of a similar flavor to Example 1.2, but deserves a section of its own.

We will consider a stylized loan portfolio. Let the time horizon Δt be one year (with this horizon, we should account for the time value of money).

We lend to m obligors. For each obligor i, let k_i denote the amount to be paid at the end of the present time period, consisting of the loan principle plus interest. Since this payment is made at time $t + 1$, $e_i = \frac{k_i}{1+r_{t,1}}$, called the exposure, is the present-day money value of this payment, where $r_{t,1}$ is the risk-free interest rate.

There is a possibility that an obligor defaults. Let $Y_{t,i}$ be an indicator variable, called the default state, that detects whether an obligor defaults by time t. We will assume $Y_{t,i} = 0$ and $\mathbb{E}[Y_{t+1,i}] = p_i$. In case of default, the lender may recover a portion of the loan, i.e. $(1 - \delta_i)k_i$, where $\delta_i \in (0, 1]$. The expected shortfall associated with obligor i is the difference between [the present-dollar value of the loan payment at time $t + 1$] and [the

present-dollar expected loan payment at time $t + 1$]

$$\frac{k_i}{1 + r_{t,i}} - \underbrace{\left[p_i(1 - \delta_i)\frac{k_i}{1 + r_{t,i}} + (1 - p_i)\frac{k_i}{1 + r_{t,i}} \right]}_{e_i(1 - p_i\delta_i)} = p_i\delta_i\frac{k_i}{1 + r_{t,i}} = p_i\delta_i e_i$$

The value of the loan itself, then, is the [present-dollar loan payment's value] minus the [expected shortfall], or $e_i - p_i\delta_i e_i$. Note that this is exactly [the present-dollar expected loan payment at time $t + 1$], which we already computed. The value of whole the portfolio is given by

$$V_t = \sum_{i=1}^{m} e_i - p_i\delta_i e_i = \sum_{i=1}^{m} e_i(1 - p_i\delta_i)$$

At time $t + 1$, we assume that all obligors have paid off the principle. Hence, V_{t+1} encapsulates the actualized payoff made at time $t + 1$. However, if the loan payments continued into a second year, we would *add on* to V_{t+1} an estimation of the expected loan payment at time $t + 2$. In the former case, we have

$$V_{t+1} = \sum_{i=1}^{m} Y_{t,i}(1 - \delta_i)k_i + (1 - Y_{t,i})k_i = k_i(1 - Y_{t,i}\delta_i)$$

We adjust for the time value of money in the next calculation:

$$L_{t+1} = V_t - \frac{V_{t+1}}{1 + r_{t,1}} = \sum_{i=1}^{m} e_i(1 - \delta_i p_i) - e_i(1 - Y_{t,i}\delta_i) = \sum_{i=1}^{m} e_i\delta_i(Y_{t,i} - p_i)$$

Determining Loss Distributions

In order to determine the loss distribution (Def 1.11), one must model risk factor changes $\mathbf{X}_{t+1}$, given a known mapping $f(t, \mathbf{Z}_t) = V_t$. We distinguish between:

DEF 1.15

1. The conditional distribution of risk factor changes, as a function of all information up to time t. We call the resulting loss distribution is called the *conditional loss disrtibution*. Note that an estimation of risk factor changes which relies on historical data is *not* necessarily a conditional distribution.

DEF 1.16

2. The stationary distribution of risk factor changes. The resulting loss distribution is called the *unconditional loss distribution*.

VARIANCE-COVARIANCE METHOD

We will first consider an analytical method for the unconditional distribution of L_{t+1}. Assume that $\mathbf{X}_{t+1}$ has a multivariate normal distribution, and write $\mathbf{X}_{t+1} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$. Suppose also that the linearized loss, written as

$$L_{t+1}^{\Delta} = -(c_t + \mathbf{b}_t^T \mathbf{X}_{t+1})$$

This is not new: c_t denotes the partial of the realized value with respect to t, and $\mathbf{b}_t$ denotes the partials with respect to $\mathbf{z}_t$.

is sufficiently accurate, i.e. $L_{t+1}^{\Delta} = L_{t+1}$. Consequently,

$$L_{t+1}^{\Delta} \sim \mathcal{N}(-c_t - \mathbf{b}_t^T \boldsymbol{\mu}, \mathbf{b}_t^T \Sigma \mathbf{b}_t)$$

In this case, the mean vector $\boldsymbol{\mu}$ and covariance matrix Σ are estimated from past data. Hence, inference about the loss distribution are made using these estimates:

$$L_{t+1}^{\Delta} \sim \mathcal{N}(-c_t - \mathbf{b}_t^T \hat{\boldsymbol{\mu}}, \mathbf{b}_t^T \hat{\Sigma} \mathbf{b}_t)$$

The estimation of L_{t+1} in this manner is called the *Variance-Covariance Method*. Of its advantages: it is easy to implement and understand, and it is a closed solution, which eases compute requirements. However, we reply on heavy assumptions. In particular, linearized loss is crude, and the normality assumption may seriously underestimate the tail of the loss distribution. **DEF 1.17**

Granted, we can use the t-distribution to achieve heavy tailed-ness

HISTORICAL SIMULATION METHOD

This also estimates the *unconditional* loss distribution

Alternatively, we use an empirical distribution based on historical data $\mathbf{X}_{t-n+1}, ..., \mathbf{X}_t$. To do so, we construct the historically simulated loss data:

$$\tilde{L}_s = f(t, \mathbf{z}_t) - f(t+1, \mathbf{z}_t + \mathbf{X}_s) : s \in [t-n+1, t]$$

Then, we average these estimates to yield a final estimation of the loss distribution:

$$\mathbb{P}(L_{t+1} \leq x) \approx \frac{1}{n} \sum_{s=1}^{n} \mathbb{1}(\tilde{L}_{t-n+s} \leq x)$$

Past performance does not indicate future gains. The historical simulation method is akin to driving by only looking through the rear view mirror

If $\mathbf{X}_s$ are IID, the convergence of the empirical distribution to the true distribution is ensured. (Real-life risk factor changes are not IID.) Note that this method can only estimate the unconditional loss distribution.

MONTE CARLO METHOD

We create a model for risk factor changes, based on data $\mathbf{X}_{t-n+1}, ..., \mathbf{X}_t$, to simulate m new data points $\tilde{\mathbf{X}}_{t+1}^{(1)}, ..., \tilde{\mathbf{X}}_{t+1}^{(m)}$. Using this data, we construct simulated future loss data:

$$\tilde{L}_s = f(t, \mathbf{z}_t) - f(t+1, \mathbf{z}_t + \mathbf{X}_{t+1}^{(m)}) : m \in [M]$$

As before, we average these estimates into a final loss distribution

$$\mathbb{P}(L_{t+1} \leq x) \approx \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}(\tilde{L}_{t+1}^{(m)} \leq x)$$

The Monte Carlo Method is akin to driving by using the rear view mirror to view a front-facing funny mirror. Slightly better.

This method can potential circumvent a lack of historical data with a better model, more simulations with the model (i.e. $M \ggg 0$), or more models. However, this method is potentially expensive, and it relies on training a suitably accurate model.

RISK MEASURES

A *risk measure* is a function $\rho : \mathcal{L} \to \mathbb{R}$, where $\mathcal{L}$ is the space of risk functions. **DEF 1.18**

INDEX OF DEFINITIONS