ASSIGNMENT 6

MATH 356

QUESTION 1

Part (1): Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{\pi(1+x^2)}$. To show this is a probability density function, see that

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} = \frac{1}{\pi} \arcsin(x)|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{\pi}\pi = 1$$

Part (2): Let X_1, X_2 be positive, independent RVS with PDFs f_1 and f_2, respectively. Consider the variable $Y := \frac{X_1}{X_2}$. Then

$$F_Y(y) = \mathbb{P}\left(\frac{x_1}{x_2} \leq y\right) = \mathbb{P}(x_1 \leq x_2 y) = \int_0^{\infty} \int_0^{x_2 y} f_1(x_1) f_2(x_2) dx_1 dx_2$$

Consider now the substitution $x_1 = x_2 u \implies dx_1 = x_2 du$. Plugging in, we get

Since X_1, X_2 are independent, their joint density is $\int_a^b \int_c^d f_1(x_1) f_2(x_2) dx_1 dx_2$.

$$F_Y(y) = \int_0^{\infty} \int_{0}^{x_2 y} f_1(x_2 u) f_2(x_2) x_2 du dx_2$$

To correct the lower bound on the inner integral, we have $x_2 u = 0 \implies u = 0$, since $x_2 > 0$. Similarly, for the upper bound, $x_2 u = x_2 y \implies u = y$.

$$F_Y(y) = \int_0^{\infty} \int_0^{y} f_1(x_2 u) f_2(x_2) x_2 du dx_2 \implies f_Y(y) = \frac{d}{dy} \int_0^{y} \int_0^{\infty} f_1(x_2 u) f_2(x_2) x_2 dx_2 du$$

$$= \int_0^{\infty} f_1(x_2 y) f_2(x_2) x_2 dx_2 \cdot \frac{d}{dy}[y] = \int_0^{\infty} f_1(xy) f_2(x) x dx$$

by the fund. theorem of calculus, and subbing in x for x_2 (purely notation).

Checking intuition: looking at $f(x)$ for Cauchy variables one sees $f(x) = f(-x)$

$\mathbb{P}(X_1 \in [-b,-a]) = \int\limits_{-b}^{-a} f(x)dx$

$= \int\limits_{-b}^{-a} f(x) = \int\limits_{b}^{a} f(-x)(-dx)$, subbing in $x \to -x$. This is $\int\limits_{a}^{b} f(-x)dx = \int\limits_{a}^{b} f(x)dx$

Part (3): Consider $|X_1|$ and $|X_2|$, where X_1, X_2 are IID Cauchy variables with density $f(x) = \frac{1}{\pi(1+x^2)}$. We have that $\mathbb{P}(|X_1| \in [a,b]) = \mathbb{P}(X_1 \in [a,b]) + \mathbb{P}(X_1 \in [-b,-a]) = 2\mathbb{P}(X_1 \in [a,b])$, since f is symmetric.

$f_{|X_1|} = f_{|X_2|} = 2f(x)$. We can now apply our equation from part 2 as follows:

$$f_Z(z) = \frac{4}{\pi^2} \underbrace{\int\limits_0^\infty \frac{x}{(1+x^2)(1+(xz)^2)} dx}_{\star}$$

Some partial fractions...

$$\frac{x}{(1+x^2)(1+(xz)^2)} = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{1+(xz)^2} \implies x = (Ax+B)(1+(xz)^2) + (Cx+D)(1+x^2)$$

Plugging in $x = 0$ yields $0 = B + D \implies B = -D$. We can also decompose:

$x = Ax + Az^2x^3 + Bz^2x^2 + \cancel{B} + Cx + Cx^3 + \cancel{D} + Dx^2$. Thus:

$x|\ \ 1 = A + C$

$x^2|\ \ 0 = Bz^2 + D = B(z^2-1) \implies B = 0$, so also $D = 0$.

$x^3|\ \ 0 = Az^2 + C = Az^2 + 1 - A \implies 1 = A(1-z^2) \implies A = \frac{1}{1-z^2}$.

Since $C = 1 - A$, $C = 1 - \frac{1}{1-z^2} = \frac{1-z^2-1}{1-z^2} = \frac{z^2}{z^2-1}$

Thus, $\int\limits_0^\infty \frac{x}{(1+x^2)(1+(xz)^2)} = \int\limits_0^\infty \left(\frac{1}{1-z^2}\right)\frac{x}{1+x^2} + \left(\frac{z^2}{z^2-1}\right)\frac{x}{1+x^2z^2}$

$$= \frac{\ln(1+x^2)}{2(1-z^2)} + \frac{\ln(1+(xz)^2)z^2}{2z^2(z^2-1)}\Bigg|_0^\infty = \frac{\ln(1+x^2) - \ln(1+(xz)^2)}{2-2z^2}\Bigg|_0^\infty = \frac{1}{2-2z^2}\ln\left(\frac{1+x^2}{1+(xz)^2}\right)\Bigg|_0^\infty \star$$

$$\overset{\text{Aside}}{\implies} \lim_{x\to\infty} \ln\left(\frac{1+x^2}{1+(xz)^2}\right) = \lim_{x\to\infty} \ln\left(\frac{z^2+x^2z^2}{1+(xz)^2}\frac{1}{z^2}\right) = \lim_{x\to\infty} \ln\left(\left[\cancelto{0}{\frac{z^2}{1+x^2z^2}} + \cancelto{1}{\frac{x^2z^2}{1+x^2z^2}}\right]\frac{1}{z^2}\right)$$

$$= \ln\left(\frac{1}{z^2}\right) = -2\ln(z)$$

$$\implies \star = \frac{\ln(z)}{z^2-1} - \cancel{\frac{1}{2-2z^2}\ln\left(\frac{1+x^2}{1+(xz)^2}\right)\Bigg|_0} = \frac{\ln(z)}{z^2-1}$$

Finally, we have

$$f_Z(z) = \frac{4}{\pi^2} \cdot \star = \frac{4}{\pi^2}\frac{\ln(z)}{z^2-1}$$

QUESTION 2

Part (1): Since X_1, X_2 are IID, they are exchangeable, so $\mathbb{P}\left(\frac{|X_1|}{|X_2|} \le 1\right) = \mathbb{P}\left(\frac{|X_2|}{|X_1|} \le 1\right) = \mathbb{P}\left(\frac{|X_1|}{|X_2|} \ge 1\right)$. Since $\mathbb{P}(Z \le 1) + \mathbb{P}(Z \ge 1) = 1$, we have $\mathbb{P}(Z \le 1) = \frac{1}{2}$.

Part (2):

$$\int_0^1 \frac{-\ln(y)}{1-y^2} dy = -\int_0^1 \ln(y) \sum_{k\ge0} y^{2k} = -\ln(y) \sum_{k\ge0} \frac{y^{2k+1}}{2k+1}\Bigg|_0^1 + \int_0^1 \sum_{k\ge0} \frac{y^{2k}}{2k+1}$$

where the last equality is by $uv - \int v du$:

$u := \ln(y) \implies du = \frac{1}{y}$ and $dv := \sum_{k\ge0} y^{2k} \implies v = \sum_{k\ge0} \frac{y^{2k+1}}{2k+1}$

$$-\ln(y) \sum_{k\ge0} \frac{y^{2k+1}}{2k+1}\Bigg|_0^1 + \int_0^1 \sum_{k\ge0} \frac{y^{2k}}{2k+1} = (0-0) + \int_0^1 \sum_{k\ge0} \frac{y^{2k}}{2k+1} = \sum_{k\ge0} \frac{y^{2k+1}}{(2k+1)^2}\Bigg|_0^1 = \sum_{k\ge0} \frac{1}{(2k+1)^2}$$

This completes the proof. To show that $\int_0^1 \frac{-\ln(y)}{1-y^2} dy = \frac{\pi^2}{8}$, note that part (1) implies that $\int_0^1 f_Z = \frac{1}{2}$, as Z is always positive. Rearranging, we get:

$$\int_0^1 f_Z = \int_0^1 \frac{4\ln(z)}{\pi^2(z^2-1)} = \frac{1}{2} \implies \int_0^1 \frac{\ln(z)}{z^2-1} = \int_0^1 \frac{-\ln(z)}{1-z^2} = \frac{\pi^2}{8}$$

Part (3): $\sum_{k\ge0} \frac{1}{(2k+1)^2}$ sums the inverse squares of odd integers, $\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \ldots$. However, we can get to this by subtracting *even* inverse squares from that of all numbers, and so

$$\sum_{k\ge0} \frac{1}{(2k+1)^2} = \sum_{k\ge1} \frac{1}{k^2} - \sum_{k\ge1} \frac{1}{(2k)^2} = \frac{3}{4} \sum_{k\ge1} \frac{1}{k^2} \implies \sum_{k\ge1} \frac{1}{k^2} = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}$$

Question 3

We want to study the joint PMF $\rho_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4)$. This probability is given by the number of successful 5-card hands divided by the total number of distinct 5-card hands one could draw. The latter is just $\binom{52}{5}$.

A successful hand would contain $W := x_1 + x_2 + x_3 + x_4$ aces, as given, and thus we count combinations to fill the remaining $5 - W$ slots. However, we are now choosing from a revised deck containing no aces, as our aces (W) have already been fixed, and any more would be extraneous and break our conditions. This deck has 48 cards. Thus:

$$\rho(x_1, x_2, x_3, x_4) = \frac{\binom{48}{5-W}}{\binom{52}{5}} = \frac{\binom{48}{5-x_1-x_2-x_3-x_4}}{\binom{52}{5}}$$

i.e. $x_1 + x_2 + x_3 + x_4 = x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} + x_{\sigma(4)}$

See that W is symmetric, so ρ is symmetric. We conclude that X_i are exchangeable.

QUESTION 4

Part (1): Let X be the number of distinct rolls, and denote R_i as the i^{th} roll. If $\mathbb{1}_{R_i}$ is the indicator that R_i is distinct from all previous rolls, we have that $X = \mathbb{1}_{R_1} + \mathbb{1}_{R_2} + \mathbb{1}_{R_3} + \mathbb{1}_{R_4}$. The probabilities of $\mathbb{1}_{R_i}$ are as follows:

$$\mathbb{P}\text{ of:} \quad \mathbb{1}_{R_1} = 1 \quad \mathbb{1}_{R_2} = \frac{5}{6} \quad \mathbb{1}_{R_3} = \left(\frac{5}{6}\right)^2 \quad \mathbb{1}_{R_4} = \left(\frac{5}{6}\right)^3$$

$$\begin{aligned} \implies \mathbb{E}X &= \mathbb{E}[\mathbb{1}_{R_1} + \mathbb{1}_{R_2} + \mathbb{1}_{R_3} + \mathbb{1}_{R_4}] \\ &= \mathbb{E}[\mathbb{1}_{R_1}] + \mathbb{E}[\mathbb{1}_{R_2}] + \mathbb{E}[\mathbb{1}_{R_3}] + \mathbb{E}[\mathbb{1}_{R_4}] \\ &= \mathbb{P}[\mathbb{1}_{R_1}] + \mathbb{P}[\mathbb{1}_{R_2}] + \mathbb{P}[\mathbb{1}_{R_3}] + \mathbb{P}[\mathbb{1}_{R_4}] = 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 \end{aligned}$$

Part (2): Let $\mathbb{1}_i$ be the indicator that the number i is contained in our sequence of rolls. We have that $X = \mathbb{1}_1 + \ldots + \mathbb{1}_6$. This will count distinct occurrences only, and negate any duplicates.

We needed a new expression for X, since squaring our current formula will lead to $\mathbb{1}_{R_i \cap R_j}$ terms, which, since R_i and R_j are not independent or exchangeable variables, will lead to madness.

$\mathbb{E}[X^2] = \mathbb{E}[(\mathbb{1}_1 + \ldots + \mathbb{1}_6)^2]$. Squaring indicators does not affect their behavior, so $\mathbb{1}_i^2 = \mathbb{1}_i$. Evaluating $\mathbb{E}[\mathbb{1}_i \mathbb{1}_j]$ is a little trickier.

This is $\mathbb{P}(i, j \in \text{ roll seq.}) = 1 - \mathbb{P}(\text{one of } i, j \text{ not rolled}) = 1 - [\mathbb{P}(i \text{ not rolled}) + \mathbb{P}(j \text{ not rolled}) - \mathbb{P}(i, j \text{ not rolled})]$. The probability that i is not rolled is $(5/6)^2$, as is the probability that j is not rolled. The probability neither i nor j are rolled is $(4/6)^4$. Thus, $\mathbb{E}[\mathbb{1}_i \mathbb{1}_j] = 1 - 2(5/6)^4 + (2/3)^4$.

The event one rolls an i and the event one rolls a j are exchangeable, as they are both Bernoulli with probability $\frac{1}{6}$ and independent

$$\mathbb{E}[X^2] = \mathbb{E}[\mathbb{1}_1 + \ldots + \mathbb{1}_6] + \mathbb{E}\left[\sum_{\substack{i \neq j: \\ i,j \leq 6}} \mathbb{1}_i \mathbb{1}_j\right] = \mathbb{E}X + 2\binom{6}{2}\left(1 - 2\left(\frac{5}{6}\right)^4 + \left(\frac{2}{3}\right)^4\right)$$

From part (1), $\mathbb{E}X = \sum_{i=0}^{3} \left(\frac{5}{6}\right)^i$. Putting together, we get

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \sum_{i=0}^{3}\left(\frac{5}{6}\right)^i + 2\binom{6}{2}\left(1 - 2\left(\frac{5}{6}\right)^4 + \left(\frac{2}{3}\right)^4\right) - \left[\sum_{i=0}^{3}\left(\frac{5}{6}\right)^i\right]^2$$

QUESTION 5

Consider $\overline{X_n} := \frac{X_1+...+X_n}{n}$. Assume X_i are IID, and the following:

$\mathbb{E}[X_1] = 0$, $\mathbb{E}[X_1^2] = a$, $\mathbb{E}[X_1^3] = b$ and $\mathbb{E}[X_1^4] = c$. We have

$$\mathbb{E}[\overline{X_n}^4] = \mathbb{E}\left[\left(\frac{X_1 + ... + X_n}{n}\right)^4\right] = \frac{1}{n^4}\mathbb{E}\left[(X_1 + ... + X_n)^4\right] = \frac{1}{n^4}\mathbb{E}\left[\sum_{i,j,k,l\in[n]} X_i X_j X_k X_l\right]$$

When the summation above is expanded, every term will be in one of the following forms: (X_i^4), $(X_i^3 X_j)$, $(X_i^2 X_j^2)$, $(X_i^2 X_j X_k)$, and $(X_i X_j X_k X_l)$, where i, j, k, l are all distinct. The expectations of all of these *except* (X_i^4) and $(X_i^2 X_j^2)$ go to 0 (by independence, and since $\mathbb{E}[X_i] = 0$). By linearity these terms can be removed:

We do now know what I is at the moment, but there is certainly only one X_i^4 term for each $1 \le i \le n$.

$$\mathbb{E}\left[\sum_{i,j,k,l\in[n]} X_i X_j X_k X_l\right] = \mathbb{E}\left[\sum_{i\neq j\in I} X_i^2 X_j^2\right] + \mathbb{E}\left[\sum_{i\in[n]} X_i^4\right]$$

Since X_i are all exchangeable and independent:

$$\mathbb{E}\left[\sum_{i\neq j\in I} X_i^2 X_j^2\right] = N\mathbb{E}[X_i^2 X_j^2] = N\mathbb{E}[X_i^2]\mathbb{E}[X_j^2] = N(\mathbb{E}[X_i^2])^2 = Na^2$$

and

$$\mathbb{E}\left[\sum_{i\in[n]} X_i^4\right] = n\mathbb{E}[X_i^4] = n\mathbb{E}[X_1^4] = nc$$

where N is the number of terms that look like $(X_i^2 X_j^2, i \neq j)$ in the sum.

$X_i^2 X_j^2$ will be generated in 6 ways by $(X_1 + ... + X_n)^4$, for any unique pair of $i \neq j$:

$$X_i X_j X_i X_j \quad X_j X_i X_j X_i \quad X_i X_j^2 X_i \quad X_j X_i^2 X_j \quad X_i^2 X_j^2 \quad X_j^2 X_i^2$$

Since there are $\binom{n}{2}$ pairs of $i \neq j$, $N = 6\binom{n}{2}$.

$$\implies \mathbb{E}[\overline{X_n}^4] = \frac{1}{n^4}\left(6a^2\binom{n}{2} + nc\right) = \frac{1}{n^4}\left(\frac{6a^2 n!}{2!(n-2)!} + nc\right) = \frac{3a^2(n-1) + c}{n^3}$$

QUESTION 6

Consider the event $\{W_1 = 1, ..., W_{n-1} = a_{n-1}\}$. Then $T_2 - T_1 = a_1, ..., T_n - T_{n-1} = a_{n-1}$. Since $T_1 = 1$, the recursion becomes

$$T_1 = 1, T_2 = a_1 + 1, ..., T_n = 1 + ... + a_{n-1}$$

A way of calculating the probability of our initial event is counting the number of ways toys may appear and still satisfy the time constraints $T_1 = 1$, $T_2 = a_1 + 1$, etc., and then dividing over all ways in the sample space toys may appear, which, since we stop opening boxes at T_n, is $n^{T_n} = n^{1+a_1+...+a_{n-1}}$.

At T_1, there are n toys that might've appeared. $\star$

Our counting "blocks" begin at T_2. Let $2 \leq i \leq n$. Consider the time T_i. Since the last new toy was picked (not inclusive), we have opened $T_i - T_{i-1} - 1$ boxes with toys already in our collection. This collection contains $i - 1$ distinct elements prior to T_i. Thus, there are $(i-1)^{T_i - T_{i-1} - 1} = (i-1)^{a_{i-1}-1}$ possible ways toys could have been opened in between T_{i-1} and T_i. At the time T_i, we choose a new toy. Since there are $i - 1$ toys *not new*, there are $n - i + 1$ possibilities for the new one. Thus, we have the total ways toys appear in our block of time after T_{i-1}, up to and including T_i, and so for the entirety of $i \geq 2$:

$$\prod_{i=2}^{n}(i-1)^{a_{i-1}-1}(n-i+1) \overset{\text{adding back } \star}{\Longrightarrow} n\prod_{i=2}^{n}(i-1)^{a_{i-1}-1}(n-i+1) \overset{n \to n+1}{=} n\prod_{i=1}^{n-1} i^{a_i - 1}(n-i)$$

The probability of $\{W_1 = 1, ..., W_{n-1} = a_{n-1}\}$ is then

$$\frac{n\prod_{i=1}^{n-1} i^{a_i-1}(n-i)}{n^{1+a_1+...+a_{n-1}}} = \left(\prod_{i=1}^{n-1}\frac{1}{n^{a_i}}\right)\left(\prod_{i=1}^{n-1} i^{a_i-1}(n-i)\right) = \prod_{i=1}^{n-1}\frac{i^{a_i-1}}{n^{a_i}}(n-i) = \prod_{i=1}^{n-1}\left(\frac{i}{n}\right)^{a_i-1}\frac{n-i}{n}$$

This last term is just $\prod_{i=1}^{n-1}\mathbb{P}(W_i = a_i)$, as W_i are all $\sim \text{Geom}(\frac{n-i}{n})$ See that $1 - \frac{n-i}{n} = \frac{n-n+i}{n} = \frac{i}{n}$

$\implies \mathbb{P}(W_1 = 1, ..., W_{n-1} = a_{n-1}) = \prod_{i=1}^{n-1}\mathbb{P}(W_i = a_i)$, and we are done. $\square$