VECTOR CALCULUS NOTES

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CONTENTS

I Curves and Surfaces

VECTOR SPACES

Recall the definition of the *inner product* over a vector space V: DEF 1.1

1. $\langle u, v\rangle = \overline{\langle v, u\rangle} = \langle v, u\rangle$ in $\mathbb{R}$ (where we'll be in this class)
2. $\langle au + bw, v\rangle = a\langle u, v\rangle + v\langle w, v\rangle$
3. $\langle u, u\rangle \geq 0$, and $= 0 \iff u = \mathbb{0}$

From this, we define the *norm* of $u \in V$ to be $\|u\| := \sqrt{\langle u, u\rangle}$. This is well-defined, since $\langle u, u\rangle \geq 0$. DEF 1.2

$$\forall u, v \in V, |\langle u, v\rangle| \leq \|u\|\|v\|$$

PROP 1.1 Cauchy-Schwartz Inequality

PROP 1.2 Triangle Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

The *cross product* of $u, v \in \mathbb{R}$, with respect to $\mathbb{R}^3$, is the determinate of the following: DEF 1.3

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where $u = \langle u_1, u_2, u_3\rangle$ and $v = \langle v_1, v_2, v_3\rangle$. We observe the following two properties of the cross product in $\mathbb{R}^3$: PROP 1.3

1. $(u \times v) \cdot u = 0$
2. $\|u \times v\| = \|u\|\|v\| \sin(\theta)$, where θ is the minimal angle found between u and v. A conceptualization of this property is that "u-cross-v is equal to the area created by the parallelogram bounded by u and v."

Inner products are not just abstractly useful: by defining a norm on continuous functions in $C[0,1]$, with $\langle f, g\rangle = \int_{0,1} f(x)g(x)dx$, we yield inequalities that are otherwise nontrivial via analysis:

$$\left|\int_0^1 f(x)g(x)dx\right| \leq \left(\int_0^1 f(x)^2 dx\right)^2 + \left(\int_0^1 g(x)^2 dx\right)^2$$

$$\left(\int_0^1 (f(x) \pm g(x))^2 dx\right)^{\frac{1}{2}} \leq \left(\int_0^1 f(x)^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 g(x)^2 dx\right)^{\frac{1}{2}}$$

LINES AND PLANES

DEF 1.4 Define a *line* $l(t) \in \mathbb{R}^n$ to be a function from $\mathbb{R} \to \mathbb{R}^n$ of the form $l(t) = P + td$, with $P, d \in \mathbb{R}^n$, $t \in \mathbb{R}$. We call P the "point vector" and d the "direction vector" An alternate form, with two points $P, Q \in \mathbb{R}^n$, would be $l(t) = (1-t)P + tQ$, where $l(t)$ lies along the path between P and Q for $t \in [0, 1]$.

DEF 1.5 $\text{proj}_u(v)$, the *projection* of v onto u, is given by

$$(u \cdot v)\frac{v}{\|v\|^2}$$

Distance between a point and line Using this definition, how an we find the shortest path between a point R and a line $l(t)$, which lies between P and Q?

Idea 1 We know the desired vector $w = PR\sin(\theta)$, the angle between PR and PQ. To find this value, note that $\|PR \times PQ\| = \|PR\|\|PQ\|\sin(\theta)$.

Idea 2 We can project R onto PQ, and then subtract this projection from PR.

Idea 3 We can minimize a distance function between R and a point on l, i.e. $l(t)$. Thus, we take $\min_{t\in\mathbb{R}} \|R - l(t)\| = \alpha$, and then take $Rl(\alpha)$ to be the shortest path.

Idea 4 We can find when $(R - l(t)) \cdot d = 0$.

Distance between 2 lines Consider two lines, l_1 and l_2, which do not intersect but are not necessarily parallel. What is the minimal distance between l_1 and l_2?

Sometimes called "skew lines"

Idea 0 Conceptualize this problem as finding the distance between the parallel planes defined by $\{l_1, l_2\}$.

Idea 1 We can minimize $\|l_1(t) - l_2(s)\|$ (really, one should minimize the square to make one's life easier).

Idea 2 Pick any two points, say $l_1(T)$ and $l_2(S)$, and project $l_1(T)l_2(S)$ onto $l_1 \times l_2$.

Idea 3 Minimize $\text{dist}(l_1(t), l_2)$ for fixed t.

Idea 4 Find t and s such that $[l_1(t) - l_2(s)] \cdot \vec{d_1} = 0$ and $[l_1(t) - l_2(s)] \cdot \vec{d_2} = 0$

Idea 5 For lines l_1, l_2 with direction vectors d_1, d_2, let $n = d_1 \times d_2$. Then calculate $\|\text{proj}_n(l_1(x_1) - l_2(x_2))\|$, where we may choose any two points $l_1(x_1)$ and $l_2(x_2)$ arbitrarily.

PROP 1.4 $\|u \times v\| = \|u\|\|v\|\sin(\theta)$ gives the area of the parallelogram bounded by u and v.

DEF 1.6 A plane $r(s, t)$ is a function $[0, 1]^2 \to \mathbb{R}^3$ defined by $d_1, d_2 \in \mathbb{R}^3$, two vectors lying on the plane, and $P \in \mathbb{R}^3$, a point. In particular, $r(s,t) = P + s\vec{d_1} + t\vec{d_2}$. This is called the *parametric form*.

The *point-normal* form of a plane is a function $\mathbb{R}^2 \to \mathbb{R}^3$ given by $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, where $\vec{n} = \langle a, b, c \rangle$ is a vector normal to the plane, and $P = \langle x_0, y_0, z_0 \rangle$ is a point lying on the plane. DEF 1.7

Distance between a point R and a plane r

Idea 1 Minimize $\|R - r(s,t)\|$ (or the square)

Idea 2 $\|\text{proj}_{\vec{n}}(P-R)\|$, where $\vec{n}$ and P are as given in the point-normal form.

TRANSFORMATIONS AND PARAMETERIZATIONS

The following table give general examples of linear transformations $\lambda : \mathbb{R}^n \to \mathbb{R}^m$.

Dimension	Linear	Affine
$n = 0$	$\lambda(0) = 0$	$\lambda(0) = P$
$n = 1$	$\lambda(t) = t\vec{d}$	$\lambda(t) = P + t\vec{d}$
$n = 2$	$\lambda(t,s) = t\vec{d_1} + s\vec{d_2}$	$\lambda(t,s) = P + t\vec{d_1} + s\vec{d_2}$
$n = 3$	$\lambda(t,s,r) = t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$	$\lambda(t,s,r) = P + t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$

We also define the following important curves in $\mathbb{R}^2$:

Type	Explicit Form	Parametric Form
Ellipse	$x^2 + y^2 = 1$	$r(t) = \left\langle t, \sqrt{1-t^2} \right\rangle_{t\in[-1,1]} = \langle \cos(t), \sin(t) \rangle_{t\in[-\pi,\pi]}$
Hyperbola	$x^2 - y^2 = 1$	$r(t) = \left\langle \sqrt{1+t^2}, t \right\rangle_{t\in\mathbb{R}} = \langle \cosh(t), \sinh(t) \rangle_{t\in\mathbb{R}}$
Parabola	$x = y^2$	
Double Cone	$x^2 = y^2$	
Any Function	$y = F(x)$	$r(t) = \langle t, F(t) \rangle$

Define a *path* in $\mathbb{R}^m$ to be a continuous function $r : \mathbb{R} \to \mathbb{R}^m$, e.g. $[a,b] \to \mathbb{R}^m$. DEF 1.8

Define a *curve* in $\mathbb{R}^m$ to be the image of a path (i.e. a set of points in $\mathbb{R}^m$). Remember always the phrase "paths parameterize curves." For example, the unit circle curve is parameterized by the path $r : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = \langle \cos(t), \sin(t) \rangle$. DEF 1.9

Define the *tangent line* of $\vec{r}$ at $a \in \mathbb{R}$ to be an affine transformation $l : \mathbb{R} \to \mathbb{R}^m$ satisfying the following:

$$l(t) = r(a) + (t-a)\vec{d} : \vec{d} \neq 0 \quad \text{and} \quad \lim_{t\to a} \frac{\|r(t) - l(t)\|}{|t-a|} = 0$$

E.G. 1.1

♠ *Examples* ♣

Consider the tangent to the unit circle at a point $a \in \mathbb{R}$: we have $r(a) = \langle \cos(a), \sin(a) \rangle$:

$$l(t) = \langle \cos(t), \sin(t) \rangle + (t-a)\langle d_1, d_2 \rangle$$

Where $\langle d_1, d_2 \rangle \neq 0$. Consider now the limit:

$$\begin{aligned}
\lim_{t \to a} \frac{\|r(t) - l(t)\|}{|t-a|} &= \lim \frac{1}{|t-a|}\sqrt{(\cos(t) - \cos(a) - (t-a)d_1)^2 + (\sin(t) - \sin(a) - (t-a)d_2)^2} \\
&= \lim \sqrt{\left(\frac{\cos(t) - \cos(a)}{t-a} - d_1\right)^2 + \left(\frac{\sin(t) - \sin(a)}{t-a} - d_2\right)^2} \\
&\underset{t \to a}{=} \sqrt{(-\sin(a) - d_1)^2 + (\cos(a) - d_2)^2} = 0 \\
&\iff (d_1 = -\sin(a)) \wedge (d_2 = \cos(a)) \\
&\implies l(t) = \langle -\sin(a), \cos(a) \rangle \quad \square
\end{aligned}$$

DIFFERENTIATION AND CONTINUITY

DEF 1.10 Given $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, the *derivative* of $\vec{r}$ at $a \in \mathbb{R}$ is a linear transformation $\vec{\lambda} : \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\lim_{t \to a} \frac{\|r(t) - r(a) - \lambda(t-a)\|}{|t-a|} = 0 \quad \text{or equivalently} \quad \lim_{h \to 0} \frac{\|r(a+h) - r(a) - \lambda(h)\|}{|h|} = 0$$

It is denoted $D\vec{r}_a$, and represented by the $n \times 1$ matrix $r'(a)$. One may now rewrite the tangent line in the form $l(t) = r(a) + \lambda(t-a)$.

DEF 1.11 The *arc length* of a curve $r(t)$ in $t \in [a,b]$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

DEF 1.12 An *arc length parameterization* of $r(t)$ is some $t = \alpha(s)$ such that $r(\alpha(s))$ has a unit velocity vector, i.e. $\|r'(\alpha(s))\| = 1$. Alternatively, one could find an expression for arc length, and then parameterize $r(t)$ in terms of its arc length. The resultant will be equivalent.

DEF 1.13 $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $\vec{a}$ if, for any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon \ \forall \vec{x} \in \mathbb{R}^n$$

E.G. 1.2

♠ *Examples* ♣

We'll do an arc length parameterization of a semicircle of radius 1 with its center at the origin, i.e. $y = \sqrt{1-x^2}$. We get the natural parameterization $r(t) = \left\langle t, \sqrt{1-t^2} \right\rangle$, where $t \in [-1, 1]$. We'd like to find a change of parameters $t = \alpha(s)$ such that $\|r(\alpha(s))\| = 1$ and $\alpha' \geq 0$.

$$\begin{aligned}
r(\alpha(s)) &= \left\langle \alpha(s), \sqrt{1-\alpha(s)^2} \right\rangle \\
r'(\alpha(s)) &= \left\langle \alpha'(s), \frac{1}{2}(1-\alpha(s)^2)^{-\frac{1}{2}} \cdot (-2\alpha(s)\alpha'(s)) \right\rangle \\
&= \alpha'(s) \left\langle 1, \frac{-\alpha(s)}{\sqrt{1-\alpha(s)^2}} \right\rangle \\
\text{Then } 1 = \|r'(\alpha(s))\| &= \alpha'(s)\sqrt{1 + \frac{\alpha(s)^2}{1-\alpha(s)^2}} \\
&= \frac{\alpha'(s)}{\sqrt{1-\alpha(s)^2}}
\end{aligned}$$

Integrating with respect to s, we get $s = \arcsin(\alpha(s)) = \arcsin(t)$. Thus, $t = \sin(s)$, and $s \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, and we yield the parameterization $\langle \sin(s), \cos(s) \rangle : s \in [\frac{-\pi}{2}, \frac{\pi}{2}]$.

SURFACES

We note the following quadric surfaces:

Type	Explicit Form
Ellipsoid	$x^2 + y^2 + z^2 = 1$
Elliptic Hyperboloid	$x^2 + y^2 - z^2 = 1$
Elliptic Paraboloids	$x^2 + y^2 - z^2 = -1$
Hyperbolic Paraboloids	$x = y^2 - z^2$
Double Cones	$x^2 = y^2 + z^2$

A surface $F(x, y)$ is called *differentiable* at (a, b) if there exists some linear transformation $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that DEF 1.14

$$\lim_{(h,k)\to(0,0)} \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\| \langle h, k \rangle \|} = 0 \quad \text{or alternatively}$$

$$\lim_{(x,y)\to(a,b)} \frac{|F(x,y) - F(a,b) - \lambda(x-a, y-b)|}{\| \langle x, y \rangle - \langle a, b \rangle} = 0$$

$\lambda : \mathbb{R}^2 \to \mathbb{R}$, as above, is called the *derivative* of $F(x, y)$ at (a, b), and is denoted by DEF 1.15

$D_{F_{(a,b)}}$. It is a linear transformation, and may be represented by multiplication by a 1×2 matrix $[u, v]$ for $u, v \in \mathbb{R}$.

E.G. 1.3

♠ *Examples* ♣

Let $F(x, y) = xy$. We consider F at (a, b). Then

$$\begin{aligned} 0 \leq \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\|\langle h, k\rangle\|} &= \frac{|(a+h)(b+k) - ab - (uk + vk)|}{\|\langle h, k\rangle\|} \\ &= \frac{|bh + ak + hk - uh - vk|}{\|\langle h, k\rangle\|} = \frac{|(b-u)h + (a-v)k + hk|}{\|\langle h, k\rangle\|} \\ &\leq \frac{|b-u||h|}{|h|} + \frac{|a-v||k|}{|k|} + \frac{|h||k|}{|h|} \quad \text{since } |h|, |k| \leq \|\langle h, k\rangle\| \\ &= |b-u| + |a-v| + |k| \to |b-u| + |a-v| \\ &= 0 \quad \text{when } b = u, a = v \end{aligned}$$

Thus, the desired limit is always $\geq$ and ≤ 0, so especially it is 0. Our derivative at (a, b) is then $\lambda(x, y) = bx + ay$.

One may also find these coefficients as the partial derivative of F at (a, b), i.e.

$$\nabla F(a,b) = \left.\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle\right|_{(a,b)}$$

DEF 1.16

This is called the *gradient*. Similarly, $\alpha(x, y) = F(a, b) + \lambda(x - a, y - b)$ is called the *affine approximation* of F at (a, b), and is analogous to the tangent line of a curve r at a.

1.1 Characterization of the Derivative

Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$. The derivative of F at $\vec{a}$, λ, exists and is unique if:

1. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\lim_{\vec{h} \to \vec{0}} \frac{\|F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})\|}{\|\vec{h}\|} = 0$$

2. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ and a function E such that

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + \|\vec{h}\| E(\vec{h})$$

and $E(0) = 0$ is continuous at 0.

PROP 1.5

If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then all partial derivatives of F at $\vec{a}$ exist. Furthermore, $\lambda(\vec{a}) = \left[\partial_1 F \cdots \partial_n F\right]\Big|_{\vec{a}}$.

Note that the full converse is *false* (as a counterexample, see that the partial derivative of $F = \sqrt{|xy|}$ exist at $(0, 0)$, but it is not differentiable there)

1.2 Partial Converse

If all partial derivatives of $F : \mathbb{R}^n \to \mathbb{R}$ exist near $\vec{a}$ and are continuous at $\vec{a}$, then F is differentiable at $\vec{a}$.

$F : \mathbb{R}^n \to \mathbb{R}$ is called *continuously differentiable* at $\vec{a}$ if all partial derivatives of F exist near $\vec{a}$ and are continuous at $\vec{a}$. We also say that F is C^1 *continuous*. DEF 1.17

If $F : \mathbb{R}^n \to \mathbb{R}$ is C^1 continuous at a, then it is differentiable at a. PROP 1.6

PROOF. This is a restating of Thm 1.2 using Def 1.17 □

Note that the converse to our partial converse is *not* true: i.e. if F is differentiable at $\vec{a}$, it is not necessarily continuously differentiable at $\vec{a}$. Some counter examples include $F(x, y) = |y|$ and $\{F(x) = x^2 \sin(\frac{1}{x})$ s.t. $x \neq 0$ and 0 otherwise$\}$.

♠ *Examples* ♣ E.G. 1.4

In Example 1.3, we prove (laboriously) that $F(x, y) = xy$ is differentiable for all (a, b). We can now use Thm 1.2 to show this result: the partial derivatives $F_x = y$ and $F_y = x$ exist and are continuous $\forall x, y \in \mathbb{R}$, so F is differentiable $\forall x, y \in \mathbb{R}$.

We may represent the partial derivatives of $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m = \langle F_1, ..., F_m \rangle$ at a using the *Jacobian* matrix, denoted $F'(\vec{a})$ or J_a, and defined as follows: DEF 1.18

$$F'(a) = J_a = \begin{bmatrix} \dfrac{\partial F}{\partial x_1} & \cdots & \dfrac{\partial F}{\partial x_n} \end{bmatrix}\Bigg|_a = \begin{bmatrix} \nabla^\mathsf{T} F_1 \\ \vdots \\ \nabla^\mathsf{T} F_m \end{bmatrix}\Bigg|_a = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix}\Bigg|_a$$

1.3 Chain Rule

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{a}$. Let $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{b} = f(\vec{a})$. Then

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l \quad \text{is differentiable at } \vec{a} \text{ and } D\vec{h}_{\vec{a}} = D\vec{g}_{\vec{b}} \circ D\vec{f}_{\vec{a}}$$

Furthermore, their Jacobians obey $[h'(a)] = [g'(b)][f'(a)]$

PROOF.

Let λ be the derivative of f. Let $\vec{t}, \vec{s}$ be arbitrary. Then we have

$$f(\vec{a}+\vec{t}) - f(\vec{a}) = \lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})$$

where $\varepsilon_1 : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $\vec{0}@\vec{0}$. Similarly, for g:

$$g(\vec{b}+\vec{s}) - g(\vec{b}) = \mu(\vec{s}) + \|\vec{s}\|\varepsilon_2(\vec{s})$$

where μ is the derivative of g, and ε_2 is as above. Our goal is to write $h = g \circ f$ in the same manner. Let $\nu = \mu \circ \lambda$. Then

$$\begin{aligned}
h(\vec{a}+\vec{t}) - h(\vec{a}) &= g(f(\vec{a}+\vec{t})) - g(f(\vec{a})) \\
&= g(f(\vec{a}) + \underbrace{\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})}_{:=\vec{s}}) - g(f(\vec{a})) \\
&= \mu(\vec{s}) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \mu(\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \mu(\lambda(\vec{t})) + \|\vec{t}\|\mu(\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \nu(\vec{t}) + \|\vec{t}\|\underbrace{\left(\mu(\varepsilon_1(\vec{t})) + \frac{\|\vec{s}\|}{\|\vec{t}\|}\varepsilon_2(\vec{s})\right)}_{=\varepsilon_3(\vec{t})} \quad \text{if } \vec{t} \neq 0
\end{aligned}$$

$$\begin{aligned}
\vec{t} \neq 0 \implies 0 &\leq \|\varepsilon_3(\vec{t})\| \leq \|\mu(\varepsilon_1(\vec{t}))\| + \frac{\|\lambda(\vec{t})\| + \|\vec{t}\|\|\varepsilon_1(\vec{t})\|}{\|\vec{t}\|}\|\varepsilon_2(\vec{s})\| \\
&\leq M\|\varepsilon_1(\vec{t})\| + (L + \|\varepsilon_1(\vec{t})\|)\|\varepsilon_2(\vec{s})\| \\
&\text{(where} \lambda(\vec{t}) \leq L\|\vec{x}\| \text{ and } \mu(\vec{x})) \leq M\|\vec{x}\|) \\
\implies &\lim_{\vec{t}\to 0} \varepsilon_3(\vec{t}) = 0 \quad \square
\end{aligned}$$

E.G. 1.5

♠ *Examples* ♣

1. Consider $f(x,y) = \langle x+y, x-y\rangle$ and $g(x,y) = \frac{1}{4}x^2 - \frac{1}{4}y^2$. Then $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$$

Let $\vec{a} = \langle a_1, a_2\rangle$. Then $f(a) = b = \langle a_1 + a_2, a_1 - a_2\rangle$. What about the Jacobian of f?

$$f'(a) = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix}\Bigg|_{(a_1,a_2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Similarly, for g we have

$$g'(b) = \begin{bmatrix} \partial_1 g & \partial_2 g \end{bmatrix}\Big|_{(a_1+a_2, a_1-a_2)} = \begin{bmatrix} \frac{1}{2}(a_1+a_2) & -\frac{1}{2}(a_1-a_2) \end{bmatrix}$$

Then, by the chain rule, we multiple these two matrices to yield

$$\begin{bmatrix} \frac{1}{2}(a_1+a_2) & -\frac{1}{2}(a_1-a_2) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$

One can (less) manually find that $h = g \circ f$ is xy, and conclude the same.

2. Let S be a surface in R^3 given by $F(x, y, z) = 0$ (this is called a "level surface," e.g. $xy - z = 0$). Let $P = (a, b, c)$ be a point on F, and let C be a curve in S containing P, parameterized by $r(t)$.

   Denote $r(t) = \langle x(t), y(t), z(t) \rangle$. Then $g = F \circ r = F(x(t), y(t), z(t)) = 0$. By chain rule, we have $0 = g'(t_0) = F'(P) \cdot r'(t_0)$, where we choose t_0 such that $r(t_0) = \langle a, b, c \rangle$. Then, we observe that

   $$0 = \nabla F(P) \bullet \vec{v}(t_0) \implies \nabla F(P) \perp \vec{v}(t_0)$$

   Where $\vec{v} = r'$ is the velocity vector of r. By considering all curves that satisfy our construction $C \subset S$, we yield the tangent plane of S at P with normal vector $\vec{n} = \nabla F(P)$. In particular, the point-normal form of the tangent plane of a surface F at $P = (a, b, c)$ is given by

   $$\boxed{\partial_x F(P)(x-a) + \partial_y F(P)(y-b) + \partial_z F(P)(z-c) = 0}$$

3. Generally, we can consider $S^{n-1} \subset \mathbb{R}^n$ of $F : \mathbb{R}^n \to \mathbb{R}$. (This is called a *hypersurface*). Suppose this is differentiable at $P \in S$. Let $C \subset S$ be a curve in S through P, parameterized by $r : \mathbb{R} \to \mathbb{R}^n$ and differentiable at t_0 with $r(t_0) = P$.

   Then, by the chain rule, $v(t_0) \perp \nabla F(P)$. If $v(t_0) \neq 0$, then the tangent line to C at P has derivative $r(t_0)$. If $\nabla F(P) \neq 0$, then the tangent hyperplane to S at P has a normal vector $n = \nabla F(P)$.

Let $\mathbb{R}^n \to \mathbb{R}$, $\vec{a}, \vec{h} \in \mathbb{R}^n$. Then the *directional derivative* of F along $\vec{h}$ at $\vec{a}$ is given by DEF 1.19

$$\partial_{\vec{h}} F(\vec{a}) = \lim_{t \to 0} \frac{F(\vec{a} + t\vec{h}) - F(\vec{a})}{t}$$

For $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{a} \in \mathbb{R}^n$, $\partial_i F(\vec{a}) = \partial_{e_i} F(\vec{a})$ is the *partial derivative* of F at $\vec{a}$ DEF 1.20

along the i^{th} direction. In particular, for $n \leq 3$, $\partial_x = \partial_{\hat{i}}$, $\partial_y = \partial_{\hat{j}}$, and $\partial_z = \partial_{\hat{j}}$.

PROP 1.7 Then, if $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a} \in \mathbb{R}^n$, then

$$\partial_{\vec{h}} F(\vec{a}) = \vec{h} \cdot \nabla F(\vec{a}) = \sum_{i=1}^{n} h_i \partial_i F(\vec{a})$$

PROP 1.8 Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $\vec{a}, \vec{h} \in \mathbb{R}^n$. By Def 1.19, we have

$$\partial_{\vec{h}} f(\vec{a}) := \lim_{t \to 0} \frac{f(\vec{a} + t\vec{h}) - f(\vec{a})}{t} = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = g'(0) \qquad g(t) := f(\vec{a} + t\vec{h})$$

The *iterated directional derivative* on these parameters, denoted $\partial^i_{\vec{h}} f(\vec{a})$, is $g^{(i)}(0)$.

PROP 1.9 If f is i-times continuously differentiable at $\vec{a}$, then we can write

$$\partial^i_{\vec{h}}(\vec{a}) = (\vec{h} \cdot \nabla)^i f(\vec{a})$$

PROP 1.10
Mean Value Thm. Let $F : \mathbb{R}^n \to \mathbb{R}$ be differentiable, and let $\vec{a}, \vec{h} \in \mathbb{R}^n$, with $\vec{h} \neq 0$. Then

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \partial_{\vec{h}} F(c_{\vec{h}}) = \vec{h} \nabla F(c_{\vec{h}}) \quad \text{for some} \quad c_{\vec{h}} \in [\vec{a}, \vec{a} + \vec{h}]$$

1.4 Mixed Partials are Equal

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $\vec{a} = \langle a_1, a_2 \rangle$. Let $\partial_1 f, \partial_2 \partial_1 f$ be defined near $\vec{a}$, let $\partial_2 \partial_1 f$ be continuous at $\vec{a}$, and let $\partial_2 f(\cdot, a_2)$ be defined near $\vec{a}$.

$\implies \partial_1 \partial_2 f$ is defined at $\vec{a}$ and $\partial_1 \partial_2 f(\vec{a}) = \partial_2 \partial_1 f(\vec{a})$.

PROP 1.11 If $f : \mathbb{R}^2 \to \mathbb{R}$ is C^2 continuous near $\vec{a}$, then $\partial_1 \partial_2 f = \partial_2 \partial_1 f$ at $\vec{a}$.

DEF 1.21 $f : \mathbb{R}^n \to \mathbb{R}$ is *k-times continuously differentiable* at $\vec{a}$ if all k^{th}-order partial derivatives exist near $\vec{a}$ and are continuous at $\vec{a}$. We also say that f is C^k *continuous*.

PROP 1.12 If f is C^k continuous at $\vec{a}$, then its $(k-1)^{th}$ order partial derivatives are C^1 continuous at $\vec{a}$.

1.5 Multivariable Taylor's Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^k continuous near some $\vec{a} \in \mathbb{R}^n$. For $j \in [1, k]$, let $\alpha_j(\vec{h})$ be defined by

$$\alpha_j(\vec{h}) = \frac{1}{j!} \partial^j_{\vec{h}} f(\vec{a}) = \frac{1}{j!} (\vec{h} \cdot \nabla)^j f(\vec{a}) \quad \forall$$

Let $p(\vec{h}) = \alpha_1(\vec{h}) + \ldots + \alpha_k(\vec{h})$. Then $G : \mathbb{R}^n \to \mathbb{R}$ by $G(\vec{x}) = f(\vec{a}) + p(\vec{x} - \vec{a})$ is the best degree k approximation of f at $\vec{a}$.

II Integration

RIEMANN INTEGRATION

Let $\mathcal{B}$ be a box in $\mathbb{R}^n$. Choose $F : \mathbb{R}^n \to \mathbb{R}$ which is bounded on the box. Then, informally, F is *integrable* if the limit of its Riemann summation is equivalent across all orderings of tagged partitions. DEF 2.1

By the extreme value theorem, if F is continuous on $\mathcal{B}$, then F is bounded on $\mathcal{B}$. PROP 2.1

2.1 Integrability Criterion on Boxes

If F is continuous on $\mathcal{B}$, then F is integrable over $\mathcal{B}$.

2.2 Fubini

Let $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathcal{B}$. Then

$$\int\limits_{\mathcal{B}} F dV^n = \int\limits_{x_n=a_n}^{x_n=b_n} \cdots \left(\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, \ldots, x_n) dx_1 \right) \cdots dx_n$$

Furthermore, the order of integration doesn't matter.

$\int\limits_a^b g(x)dx = g(c)(b-a)$ where $a < c < b$. PROP 2.2

PROOF. $\frac{G(b)-G(a)}{b-a} = G'(c) = g(c)$ by the mean value theorem and the FTC. □

Point-Set Topology

A set $S \subseteq \mathbb{R}^n$ has *zero measure* if $\forall \varepsilon > 0$ we can choose a set of open balls such that $S \subseteq \bigcup B(x_i, \varepsilon_i)$ where $\sum \mathrm{vol}(B(x_i, \varepsilon_i)) < \varepsilon$. DEF 2.2

In general, hypersurfaces in $\mathbb{R}^n$ have zero measure. Thus, if $F : \mathbb{R}^n \to \mathbb{R}$ is continuous except on a hypersurface, F is still integrable.

$\vec{p} \in \mathrm{Int}(S)$ is called an *interior point* of S if $\exists \varepsilon > 0$ such that $B(\vec{p}, \varepsilon) \subseteq S$. DEF 2.3

PROP 2.3

1. If $S \subseteq \mathbb{R}^n$ has zero measure and $S' \subseteq S$, then S' has zero measure.
2. If $S \subseteq \mathbb{R}^n$ has zero measure, then S has no interior points.

Let $S \subseteq \mathbb{R}^n$. Then DEF 2.4

1. $\text{Int}(S)$, the *interior of* S, is the set of all interior points of S
2. S is called *open* if $S = \text{Int}(S)$.
3. S^c, the *compliment of* S, is $\mathbb{R}^n \setminus S$.
4. $p \in S^c$ is called an *exterior point* of S if $\exists \varepsilon > 0$ with $B(p, \varepsilon) \subseteq S^c$.
5. $\text{Ext}(S)$, the *exterior* of S, is the set of all exterior points of S.
6. S is *closed* if $S^c = \text{Ext}(S)$.
7. $p \in \mathbb{R}^n$ is called a *boundary point* of S if $p \notin \text{Int}(S) \wedge p \notin \text{Ext}(S)$.
8. The *boundary* of S, denoted ∂S, is the set of all boundary points of S.
9. S is *bounded* if $\exists \mathcal{B}$ with $S \subseteq \mathcal{B} \subsetneq \mathbb{R}^n$.

PROP 2.4 S is closed $\iff$ S^c is open $\iff$ S contains its boundary.

2.3 Integrable $\iff$ Trivial Discontinuities

The set of discontinuities of F in $\mathcal{B}$ has zero measure $\iff$ F is integrable over $\mathcal{B}$.

DEF 2.5 Let $\mathcal{D} \subseteq \mathbb{R}^n$ be closed and bounded. Let $f : \mathcal{D} \to \mathbb{R}^n$ be some function. $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{f}(x) = \begin{cases} f(x) & x \in \mathcal{D} \\ 0 & \text{o.w.} \end{cases}$$

is called the *trivial extension of* f.

PROP 2.5 f is integrable over $\mathcal{D}$ if its trivial extension is integrable over a box $\mathcal{B} \supseteq \mathcal{D}$.

2.4 Integrability Criterion on Sets

Let $\mathcal{D} \subseteq \mathbb{R}^n$ be closed and bounded, with a boundary that has zero measure. Then, if $f : \mathcal{D} \to \mathbb{R}$ is continuous on $\mathcal{D}$, then f is integrable.

PROOF. If f is continuous on $\mathcal{D}$, then $\hat{f}$ is continuous on both $\text{Int}(\mathcal{D})$ and $\text{Ext}(\mathcal{D})$ (for any point in either of these sets, we can find epsilon balls centered at the point and contained in the set—within these intervals $\hat{f} = f$). Thus, since $\mathcal{D} = \text{Int}(\mathcal{D}) \cup \text{Ext}(\mathcal{D}) \cup \partial D$, the set of discontinuities of $\hat{f}$ has at most measure 0. Hence, $\hat{f}$ is integrable over any box containing $\mathcal{D}$, and hence f is integrable over $\mathcal{D}$ by Prop 2.5. $\square$

$\mathcal{D} \subseteq \mathbb{R}^2$ is called *y-simple* if, for $a, b \in \mathbb{R}$ and $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ continuous, we may write DEF 2.6

$$\mathcal{D} = \begin{cases} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{cases}$$

Similarly, $\mathcal{D}$ is *x-simple* if

$$\mathcal{D} = \begin{cases} a \leq y \leq b \\ g_1(y) \leq x \leq g_2(y) \end{cases}$$

Note that, since $x \in [a, b]$ is closed (hence compact), $g_1(x)$ and $g_2(x)$ are bounded. We reason similarly for x-simple domains.

$\mathcal{D} \subseteq \mathbb{R}^2$ is *elementary* if it is y- or x-simple. It is *simple* if it is both. DEF 2.7

2.5 Fubini

If $\mathcal{D} \subseteq \mathbb{R}^n$ is elementary and $f : \mathcal{D} \to \mathbb{R}$ is continuous, then

- $\mathcal{D}$ is y-simple $\implies \iint\limits_{\mathcal{D}} f \, dA = \int\limits_{x=a}^{x=b} \int\limits_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy dx$
- $\mathcal{D}$ is x-simple $\implies \iint\limits_{\mathcal{D}} f \, dA = \int\limits_{y=a}^{y=b} \int\limits_{x=g_1(y)}^{x=g_2(y)} f(x, y) dx dy$

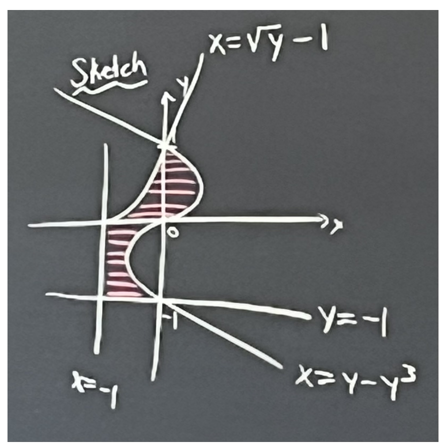
♠ *Examples* ♣ E.G. 2.1

1. Consider $\iint_{\mathcal{D}}(1 + 2y) dA$, where $\mathcal{D}$ is bounded by $y = 2x^2$ and $y = 1 + x^2$. We first find the intersection between these two curves: $2x^2 = 1 + x^2 \implies x = \pm 1$.

Then, by Thm 2.5 ($\mathcal{D}$ is y-simple), we write

$$\iint\limits_{\mathcal{D}} (1+2y)\,dA = \int_{x=-1}^{x=1}\int_{2x^2}^{1+x^2} (1+2y)\,dy\,dx = \int_{-1}^{1} y + y^2\Big|_{2x^2}^{1+x^2}$$
$$= \int_{-1}^{1} (1+x^2) + (1+x^2)^2 - 2x^2 - 4x^4$$
$$= \int_{-1}^{1} 1 + x^2 + 1 + x^4 + 2x^2 - 2x^2 - 4x^4$$
$$= \int_{-1}^{1} -3x^4 + x^2 + 2 = \frac{-3}{5}x^5 + \frac{1}{3}x^3 + 2x\Big|_{-1}^{1} = 2\frac{-3}{5} + 2\frac{1}{3} + 4$$
$$= 2\left(\frac{-9}{15} + \frac{5}{15} + \frac{30}{15}\right) = \frac{52}{15}$$

2. Consider $\iint_{\mathcal{D}} y\,dA$, where $\mathcal{D}$ is bounded by $x = y - y^3$, $x = \sqrt{y} - 1$, $x = -1$, and $y = -1$ (OOF). By Thm 2.5 (y-simple):



We split this up into two x-simple graphs, one in $y \in [-1, 0]$, and one in $y \in [0, 1]$. Then we have $\iint\limits_{\mathcal{D}} = I_1 + I_2$, with

$$I_1 = \int_{0}^{1}\int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy \qquad I_2 = \int_{-1}^{1}\int_{-1}^{y-y^3} y\,dx\,dy$$

Computing this integral a hassle. Try it yourself.

3. We may also flip the bounds of integration using Thm 2.5. For example, consider $\int_0^3 \int_y^3 \sin(x^2) dx dy$. This is a non-elementary integral to evaluate in x. But observe that our bounds are equivalent to $y \in [0, x]$ and $x \in [0, 3]$, so we may re-write this as $\int_0^3 \int_0^x \sin(x^2) dy dx$.

We pick up an x, not, after integrating WRT y, so this is easy to evaluate!

DEF 2.8 A set $S \subseteq \mathbb{R}^n$ is called *path-connected* if, for every $a, b \in S$, there exists a continuous mapping containing a and b (i.e., there exists a path between them).

DEF 2.9 In $\mathcal{D} \subseteq \mathbb{R}^n$, we call $\mathcal{D}$ *elementary* if it is closed, bounded, and both its interior and boundary are path-connected.

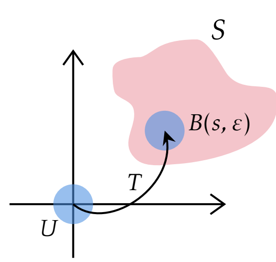
This is distinct from elementary-ness of $\mathcal{D} \subseteq \mathbb{R}^2$, which we characterized by y and x simple-ness.

DEF 2.10 Let $\mathcal{D}, \mathcal{D}^*$ be elementary subsets of $\mathbb{R}^n$. Let $T : \mathcal{D}^* \to \mathcal{D}$. We call T *onto* , or *surjective*, if the whole of $\mathcal{D}$ is mapped to, i.e. $\forall d^* \in \mathcal{D} \exists d \in \mathcal{D} : T(d) = d'$.

DEF 2.11 Using the same notation, we call T *one-to-one*, or *injective*, if no two points share a mapping, i.e. $\forall d_1^*, d_2^* \in \mathcal{D}^*$, we have $T(d_1^*) = T(d_2^*) \implies d_1^* = d_2^*$.

DEF 2.12 $S \subseteq \mathbb{R}^n$ is a *hypersurface* if, $\forall s \in S$, $\exists \varepsilon > 0$, an open set $\vec{0} \in U$, and a function $T : U \to B(s, \varepsilon)$ such that

- T is injective on $\text{Int}(\mathcal{D}^*)$ and also surjective
- $T(U \cap \{s = \langle x_1, ..., x_n \rangle : x_n = 0\}) = S \cap B(s, \varepsilon)$



PROP 2.6 For $\mathcal{D} \subseteq \mathbb{R}^n$ and F integrable, $\int_{\mathcal{D}} F dV^n = \int_{\text{Int}(\mathcal{D})} F dV^n$.

2.6 Change of Variables

Let $T : \mathcal{D}^* \to \mathcal{D}$ be C^1 and injective on $\text{Int}(\mathcal{D}^*)$. Let $F : \mathcal{D} \to \mathbb{R}$ be integrable over $\mathcal{D}$. Let $[T]$ be the Jacobian induced by T. Let $F^* : \mathcal{D}^* \to \mathbb{R} = F \circ T$. Then F^* is integrable over $\mathcal{D}^*$ and

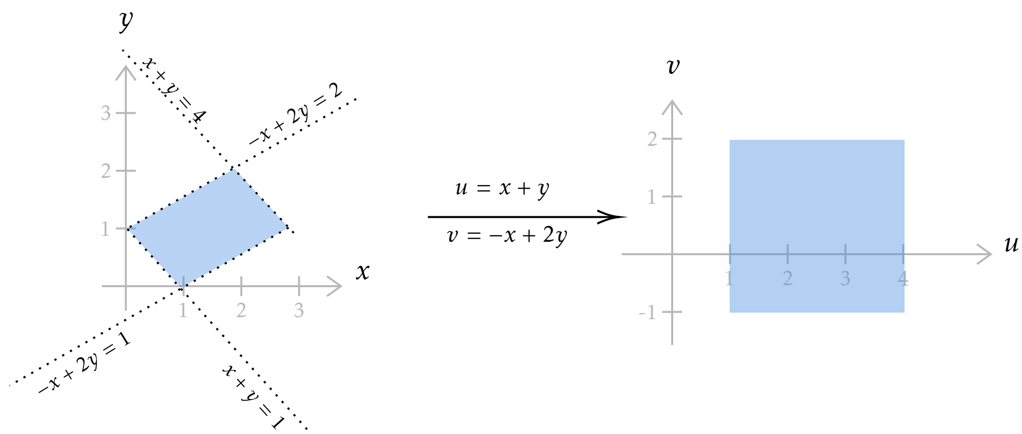
$$\int_{\mathcal{D}} F dV = \int_{\mathcal{D}^*} F^* |\det(T)| dV$$

E.G. 2.2 ♠ *Examples* ♣

In polar coordinates, $\int\limits_{\mathcal{D}} F dA = \int\limits_{\mathcal{D}^*} F^* r dA$. For this, see that the relevant Jacobian is

$$T' = \begin{bmatrix} \partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} \implies |\det(T')| = |r| = r$$

Consider the area of the following parallelogram:



Then, $x = \frac{2u-v}{3}$ and $y = \frac{u+v}{3}$. Hence, we compute our Jacobian and conclude that $\det(T') = \frac{1}{3}$. However, we may also compute the determinate of the *inverse*'s Jacobian, i.e. $u = x + y$ and $v = -x + 2y$, which will yield 3, and invert the result. Hence, since the area of the left rectangle is 9, we get an area of 3 for the parallelogram.

2.7 Mean Value Theorem in $\mathbb{R}^n$

Let $F : \mathcal{D} \to \mathbb{R}$ be integrable over an elementary region $\mathcal{D} \subseteq \mathbb{R}^n$. Let $\overline{F} := \int\limits_{\mathcal{D}} F dV \frac{1}{\text{vol}(\mathcal{D})}$ be the mean value of F. Then

$$\exists c \in \mathcal{D} : F(c) = \overline{F}$$

III Vector Fields

REGULAR PATHS

$\vec{r} : [a, b] \to \mathbb{R}^n$ is called a *regular path* if it is C^1 and $\|r'(t)\| > 0$ over $[a, b]$. DEF 3.1

$\mathcal{C} \subseteq \mathbb{R}^n$ is called a *regular curve* if it is the image of a regular path. DEF 3.2

If $\mathcal{C}$ is a regular curve, then there exists and unique arc length parameterization $\rho : [0, l] \to \mathbb{R}^n$ of $\mathcal{C}$. PROP 3.1

A regular path $\vec{r} : [a, b] \to \mathbb{R}^n$ is *simple* if it is injective (except possibly at its endpoints). DEF 3.3

A regular path $\vec{r} : [a, b] \to \mathbb{R}^n$ is called *closed* if $r(a) = r(b)$. DEF 3.4

A regular curve $\mathcal{C} \subseteq \mathbb{R}^n$ is called *simple* or *closed* if it is the image of a simple or closed path, respectively. DEF 3.5

Center of Mass

Regular curves have zero measure, and hence zero n-dimensional volume, but we *can* measure 1-dimensional volume, i.e. length. Hence, $\mathrm{vol}_1(C) := \int\limits_C 1\,ds = l$.

Let $\delta : \mathcal{D} \to \mathbb{R}_+$ be an integrable density function. Then $\mathrm{mass}(\mathcal{D}) = \int_{\mathcal{D}} \delta\, dV$. The *center of mass* $\vec{x} \in \mathcal{D}$ is given by DEF 3.6

$$x_i = \frac{1}{\mathrm{mass}(\mathcal{D})} \int\limits_{\mathcal{D}} x_i \delta\, dV$$

The mean value theorem gives the fact that $\exists c : \delta(c) = \overline{\delta}$, where $\overline{\delta} = \frac{\mathrm{mass}(\mathcal{D})}{\mathrm{vol}(\vec{D})}$.

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a curve parameterized by $r : [a, b] \to \mathbb{R}^n$. Let $\delta : \mathcal{C} \to \mathbb{R}_+$ be a density function. Then PROP 3.2

$$\mathrm{mass}(\mathcal{C}) = \int\limits_a^b \delta(r(t))\|r'(t)\|\,dt$$

PROOF.

$$\mathrm{mass}(\mathcal{C}) = \int\limits_{\mathcal{C}} \delta\, ds = \int\limits_0^l \delta(\rho(s))\, ds \underset{\text{ch. of var's}}{=} \int\limits_a^b \delta(r(t))\|r'(t)\|\,dt$$

where $\rho : [0, l] \to \mathbb{R}^n$ is the arc length parameterization of $\mathcal{C}$. □

PROP 3.3 If $\mathcal{D} = \mathcal{C}$, a curve in $\mathbb{R}^n$, then the center of mass $\vec{x}$ of $\mathcal{C}$ with respect to $\delta : \mathcal{C} \to \mathbb{R}_+$ is given by

$$x_i = \frac{1}{\text{mass}(\mathcal{C})} \int_a^b r_i(t) \circ \delta(r(t)) \| r'(t) \| dt$$

where $r(t) = \langle r_1(t), ..., r_n(t) \rangle : t \in [a, b]$ parameterizes $\mathcal{C} \subseteq \mathbb{R}^n$.

PROOF.

$$x_i = \left(\int_C x_i \delta ds \right) \frac{1}{\text{mass}(C)} = \frac{1}{\text{mass}(\mathcal{C})} \int_a^b r_i(t) \circ \delta(r(t)) \| r'(t) \| dt \quad \square$$

VECTOR FIELDS

All curves $\mathcal{C} \subseteq \mathbb{R}^n$ henceforth are regular and simple.

DEF 3.7 An *orientation* on a regular, simple curve $\mathcal{C}$ is a continuous function $T : \mathcal{C} \to \mathbb{R}^n$ which gives the unit tangent vector to $\mathcal{C}$.

PROP 3.4 There exist exactly two orientations on $\mathcal{C} \subseteq \mathbb{R}^n$, $T : \mathcal{C} \to \mathbb{R}^n$ and $-T$.

DEF 3.8 A *vector field* is a function $F : \mathbb{R}^n \to \mathbb{R}^n$.

DEF 3.9 Fix an orientation T on a curve $\mathcal{C} \subseteq \mathbb{R}^n$. The *integral* of F over $\mathcal{C}$ is given by

$$\int_{\mathcal{C}} F \cdot T ds := \int_0^l (F \circ \rho) \cdot \rho'$$

where ρ is the arc length parameterization of $\mathcal{C}$.

PROP 3.5 Under the conditions of Def 3.9, we have

$$\int_{\mathcal{C}} F \cdot T ds = \int_a^b (F \circ r(t)) \cdot r' dt$$

where $r : [a, b] \to \mathbb{R}^n$ is a parameterization of $\mathcal{C}$.

E.G. 3.1 ♠ *Examples* ♣

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = \langle 2x, 2y, 2z \rangle = 2 \langle x, y, z \rangle$. Hence, at any point, the vector generated by F will go through the line between the origin and that point (away).

We want to integrate over the triangle $C \subseteq \mathbb{R}^3$ bounded by $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. We orient this path as $(1, 0, 0) \to (0, 1, 0) \to (0, 0, 1)$.

Then, we split C up into 3 parts (the lines traversing each point)

$$
\begin{aligned}
C_1 &= r_1(t)\langle 1,0,0\rangle + t\langle -1,1,0\rangle \\
C_2 &= r_2(t) = \langle 0,1,0\rangle + t\langle 0,-1,1\rangle \\
C_3 &= r_3(t) = \langle 0,0,1\rangle + t\langle 1,0,-1\rangle
\end{aligned}
$$

$$
\implies \int_{C_1} F \cdot T\,ds = \int_0^1 \langle 2(1-t), 2t, 2(0)\rangle \cdot \langle -1,1,0\rangle\,dt = \int_0^1 4t - 2dt
$$
$$
= [2t^2 - 2t]_0^1 = 0
$$

By symmetry, the integral across C_2, C_3 will be the same, i.e. $3 \cdot 0 = 0$.

3.1 Line Integrals on Gradient Fields

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $\varphi : \mathcal{U} \to \mathbb{R}^n$ be C^1 continuous. Let $\mathcal{C} \subseteq \mathcal{U}$ be a curve with a parameterization $r : [a,b] \to \mathcal{U}$ and orientation T. Let $A = r(a)$ and $B = r(b)$. Then

$$
\int_C \nabla\varphi \cdot T\,ds = \varphi(B) - \varphi(A)
$$

PROOF.

$$
\begin{aligned}
\int_C \nabla\varphi \cdot T\,ds &= \int_a^b \nabla\varphi(r(t)) \cdot r'(t)dt \\
&\overset{\text{CR}}{=} \int_a^b (\varphi \circ r)'(t)dt \overset{\text{FTC}}{=} [\varphi \circ r]_a^b \\
&= \varphi(r(b)) - \varphi(r(a)) = \varphi(B) - \varphi(A) \quad \square
\end{aligned}
$$

DEF 3.10 A vector field T is called *unit tangent* for a curve $\mathcal{C} \subseteq \mathbb{R}^n$ if $T = \langle T_1, T_2\rangle$ is exactly the unit tangent vector to $\mathcal{C}$ (AKA its orientation). Similarly, a vector field n is called *unit normal* for $\mathcal{C}$ if $n = \langle T_2, -T_1\rangle$.

3.2 Jordan Curve Theorem

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a curve. Then there exists an elementary region $\mathcal{D} \subseteq \mathbb{R}^2$ such that $\mathcal{C}$ is the boundary of $\mathcal{D}$.

PROOF.

The proof of this is beyond the scope of this course.

3.3 Green's Theorem

Let $\mathcal{D} \subseteq \mathcal{U}$ be an elementary region. Fix an orientation $T = \langle T_1, T_2 \rangle$ on $\partial\mathcal{D}$. Let $F : \mathcal{U} \to \mathbb{R}^2$ be a C^1 vector field. Then

$$\int\limits_{\partial\mathcal{D}} F \cdot T ds = \iint\limits_{\mathcal{D}} \partial_1 F_2 - \partial_1 F_1 dA = \iint\limits_{\mathcal{D}} \mathrm{curl}_2(F) dA$$

where $\mathrm{curl}_2 = \det\begin{pmatrix} \partial_1 & \partial_2 \\ F_1 & F_2 \end{pmatrix}$. Let $n = \langle T_2, -T_1 \rangle$. Then

$$\int\limits_{\partial D} F \cdot n ds = \iint\limits_{D} \partial_1 F_1 + \partial_2 F_2 = \iint\limits_{D} \mathrm{div}_2(F) dA$$

Conceptually, the curl of F at a point $\vec{a}$ gives how much "spinning" is occurring about $\vec{a}$, and the divergence of F measures the tendency of nearby vectors to "move away" from $\vec{a}$. (Or, toward, if negative).

DEF 3.11 Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $\varphi : \mathcal{U} \to \mathbb{R}^n$ be C^2 continuous. Then, if F is a vector field and $F = \nabla\varphi$, then F is called a *gradient field*.

DEF 3.12 A vector field $F : \mathbb{R}^m \to \mathbb{R}^n$ is *conservative* if $\partial_i F_j = \partial_j F_i \ \forall i \neq j$ and $F = \langle F_1, ..., F_n \rangle$.

DEF 3.13 An open set $\mathcal{U} \subseteq \mathbb{R}^n$ is called *convex* if all line segments between points in $\mathcal{U}$ are contained in $\mathcal{U}$.

3.4 Conservative $\iff$ Gradient: $2D$

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be convex. Let $F : \mathcal{U} \to \mathbb{R}^2$ be a C^1 vector field. Then

$$F \text{ is conservative} \iff F \text{ is a gradient field}$$

PROOF. We show this for $m = 2$. Fix $a \in U$. For any $x \in U$, let $[a, x]$ denote the line segment from a to x (oriented). Define $\varphi : U \to \mathbb{R} : x \mapsto \int\limits_{[a,x]} F \cdot T ds$.

We claim that $\partial_1 \varphi(x) = F_1(x)$. An identical proof for F_2 will establish $F = \nabla\varphi$.

Expanding

$$
\begin{aligned}
x = \langle x_1, x_2 \rangle \implies \partial_1 \varphi(x) &= \lim_{h\to 0} \frac{\varphi(x_1 + h, x_2) - \varphi(x_1, x_2)}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int\limits_{[a, x+he_1]} F \cdot T\,ds - \int\limits_{[a,x]} F \cdot T\,ds \right) \quad \text{by def.} \\
&= \lim_{h\to 0} \frac{1}{h} \int\limits_{[x, x+he_1]} F \cdot T\,ds \quad \text{by Green}
\end{aligned}
$$

At this point, observe that $\operatorname{curl}(F) = \partial_1 F_2 - \partial_2 F_1 = 0$, since F is conservative, so consider C the curve bounded by $a \to x + he_1 \to x \to a$. Then

$$
\int\limits_{[x+he_1, x]} + \int\limits_{[x,a]} + \int\limits_{[a, x+he_1]} = \int\limits_{C} F \cdot T\,ds \iint\limits_{D} \operatorname{curl}(F) = 0
$$

Then, continuing from above:

$$
\partial_1 \varphi(x) = \lim_{h\to 0} \int_{x_1}^{x_1+h} F_1(t, x_2)\,dt \overset{\text{FTC}}{=} F_1(x_1, x_2) = F_1(x) \quad \square
$$

SURFACES

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be an elementary region. Then $\rho : \mathcal{D} \to \mathbb{R}^3$ be called a *regular, 2D parameterization* if it is injective and $\|\partial_1 \rho \times \partial_2 \rho\| > 0$. DEF 3.14

$S \subseteq \mathbb{R}^3$ is called a *regular surface* if it is closed, bounded, and $\forall x \in S, \exists \varepsilon > 0$ such that $B(x, \varepsilon) \cap S$ is the image of a 2D parameterization. DEF 3.15

If $S \subseteq \mathbb{R}^3$ is the image of a regular 2D parameterization, it is a regular surface. PROP 3.6

Let S be a regular surface with a parameterization $\rho : \mathcal{D} \to \mathbb{R}^3$ for some $\mathcal{D} \subseteq \mathbb{R}^2$. Then, for a scalar function $\varphi : S \to \mathbb{R}$, the integral of φ over S is given by

$$
\iint\limits_{S} \varphi\,d\sigma = \iint\limits_{\mathcal{D}} (\varphi \circ p) \|\partial_1 p \times \partial_2 p\|\,dA
$$

Given a surface $S \subseteq \mathbb{R}^3$ which is path-connected, $\mu \to \mathbb{R}^3$ is called an *orientation representative* if it is continuous and $\mu(\vec{a})$ is nontrivial and normal to S at $\vec{a}$ DEF 3.16

S is *orientable* if an orientation representative exists. DEF 3.17

Two orientation representatives μ, ν for S are *equivalent* if $\mu(\vec{a}) \cdot \nu(\vec{a}) > 0\ \forall \vec{a} \in S$. DEF 3.18

PROP 3.7 If S is orientable, then it has exactly 2 distinct orientations O and $\overline{O}$, and hence two unit normal vector fields $\vec{n}$ and $-\vec{n}$, and 2 area elements $d\sigma$ and $-d\sigma$.

DEF 3.19 Fix an orientation $\vec{n}$ on a regular surface $S \subseteq \mathbb{R}^3$, consisting of the unit normal vector field. Let $\rho : \mathcal{D} \to \mathbb{R}^3$ be its 2D parameterization. Then

$$\iint_S F \cdot n d\sigma = \iint_{\mathcal{D}} (F \circ \rho) \cdot (\partial_1 \rho \times \partial_2 \rho) dA$$

where, in particular $n = \partial_1 \rho \times \partial_2 \rho$. Otherwise, dot instead with $\partial_2 p \times \partial_1 p$.

3.5 Stoke's Theorem

Let $\mathcal{U} \subseteq \mathbb{R}^3$ be open and $S \subseteq \mathcal{U}$ be a C^2-regular surface. Let $F : \mathcal{U} \to \mathbb{R}^3$ be a C^1 vector field. Fix an orientation T for ∂S. Then

$$\int_{\partial S} \vec{F} \cdot \vec{T} ds = \iint_S \mathrm{curl}_3(\vec{F}) \cdot n dS$$

where $\mathrm{curl}_3(\vec{F})$ denotes $\nabla \times \vec{F}$, i.e.

$$\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{pmatrix} \quad \text{with } \vec{F} = \langle F_1, F_2, F_3 \rangle$$

3.6 Conservative $\iff$ Gradient, 3D

Let $\mathcal{U} \subseteq \mathbb{R}^3$ be open and convex. Let $F : \mathcal{U} \setminus X \to \mathbb{R}^3$ be a C^1 vector field, where X is finite. Then

$$\mathrm{curl}_3(F) = 0 \iff F = \nabla \varphi$$

for some C^2 function $\varphi : \mathcal{U} \setminus X \to \mathbb{R}$.

We call a vector field G in $\mathbb{R}^3$ *solenoidal* if $\mathrm{div}(G) = 0$.

3.7 Solenoidal $\iff$ curl_3

Let $\mathcal{U} \subseteq \mathbb{R}^3$ be open and convex. Let $G : \mathcal{U} \to \mathbb{R}^3$ be a C^2 vector field. Then

$$\mathrm{div}(G) = 0 \iff G = \mathrm{curl}_3(H)$$

for some other C^2 vector field $H : \mathcal{U} \to \mathbb{R}^3$.

3.8 Gauss's Theorem

Let $\mathcal{U} \subseteq \mathbb{R}^3$ be open, $R \subseteq \mathcal{U}$ be elementary, and $G : \mathcal{U} \to \mathbb{R}^3$ be a C^1 vector field. Then

$$\iint\limits_{\partial R} G \cdot n d\sigma = \iiint\limits_{R} \operatorname{div}(G) dV$$

3.9 Stoke's Theorem For Manifolds

Let $U \subseteq \mathbb{R}^n$ be open, $S \subseteq U$ be a regular, C^2 surface. Let ω be a C^1 1-form on U. Then

$$\int\limits_{\partial S} \omega = \iint\limits_{S} d\omega$$

We also have the even more general form: $\int\limits_{\partial M} \omega = \int\limits_{M} d\omega$.