ALGEBRA 3 NOTES

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I Groups

In Algebra 3 we will study abstract algebraic structures. Chiefly among them, we have (1) *groups*, which are useful in representing symmetries, (2) *rings & fields*, which help us think about number systems, and (3) *vector spaces & modules*, which encode physical space.sdfg

A *group* is a set G endowed with a binary composition $G \times G \to G$ such that the following axioms hold:

1. $\exists e \in G$, an identity element, such that $e * a = a * e = a \ \forall a \in G$.
2. $\forall a \in G, \exists a' \in G$ such that $a * a' = a' * a = e$.
3. $a * (b * c) = (a * b) * c \ \forall a, b, c \in G$.

If $a * b = b * a \ \forall a, b \in G$, we call G *commutative*.

Why do we care about groups? If X is an object, we call a *symmetry* of X a function $X \to X$ which preserves the structure of the object.

e.g. a polygon, graphs, tilings, "crystal," "molecules," rings, vector spaces, metric spaces, manifolds

The collection of symmetries, $\mathrm{Aut}(X) = \{f : X \to X\}$, we can structure as a group: $* = \circ$, composition of functions; $e = \mathrm{Id}$; $f \in \mathrm{Aut}(X)$ must be bijective.

A note on notation: for non-commutative groups, we write $a * b = ab$, $e = 1$ or $\mathbb{1}$, $a' = a^{-1}$, and $a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}}$. This is called *multiplicative notation*. For commutative rings, we write $a * b = a + b$, $e = 0$ or $\mathbb{0}$, $a' = -a$, and $na = \underbrace{a + \ldots + a}_{n \text{ times}}$.

Examples

1.