

Geometry & Topology 1

MATH576

Nicholas Hayek

Taught by Prof. Przytycki at McGill University

CONTENTS

I	Topologies	1
----------	-------------------	----------

I Topologies

We will study continuous mappings between objects in \mathbb{R}^n . These, in contrast with isogenies (rotations, reflections, translations), will allow us to identify circles with squares, doughnuts with mugs, etc. A guiding question: how do we describe deformation mathematically?

A *metric* on a space X is a function $d : X \times X \rightarrow \mathbb{R}$ satisfying

DEF 1.1

- (1) $d(x, y) \geq 0$, and $d(x, y) = 0 \iff x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \leq d(x, z) + d(y, z)$

A *metric space* (X, d) is a space with a specific metric.

The *ball centered at x of radius ε* , denoted $B(x, \varepsilon)$, is the set

DEF 1.2
or $B_d(x, \varepsilon)$
w.r.t. a metric
 d

$$\{y \in X : d(x, y) < \varepsilon\} \quad \text{with} \quad \varepsilon > 0$$

$U \subseteq X$ is called *open* if, for each $x \in X$, there exists $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq U$.

DEF 1.3

Denote by \mathcal{T} the family of all open sets in X .

DEF 1.4

Given (X, d) :

PROP 1.1

- (1) $\emptyset, X \in \mathcal{T}$
- (2) For any (possibly infinite) collection of sets in \mathcal{T} , their union is in \mathcal{T}
- (3) For any finite collection of sets in \mathcal{T} , their intersection is in \mathcal{T}

(1) follows by definition. For (2), let U be some union of sets in \mathcal{T} , and $x \in U$. Then x belongs to some open set U_i comprising the union. Then $B(x, \varepsilon_i) \subseteq U_i \subseteq U$ for some ε_i .

PROOF.

For (3), let $\{U_1, \dots, U_n\} \subseteq \mathcal{T}$. Pick $x \in \bigcup_{i \in [n]} U_i =: U$. It belongs to all sets $U_i : i \in [n]$. Since these sets are open, there exists some $\varepsilon_i : B(x, \varepsilon_i) \subseteq U_i$. Let $\varepsilon := \min\{\varepsilon_i\}$. Then $B(x, \varepsilon) \subseteq U_i$ for each i , and hence $B(x, \varepsilon) \subseteq U$. \square

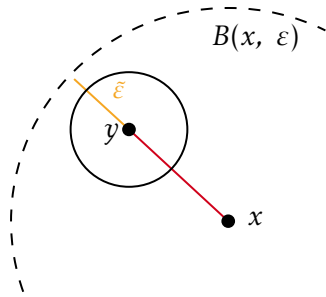
A *topology* on (X, d) is a collection \mathcal{T} of subsets of X satisfying [Prop 1.1](#). Elements of \mathcal{T} are called *open sets*.

DEF 1.5

If $\mathcal{T} \subseteq \mathcal{T}'$, we say that \mathcal{T} is *coarser* than \mathcal{T}' . Conversely, \mathcal{T}' is *finer* than \mathcal{T} .

Eg. 1.1.1 $B(x, \varepsilon)$ is open. Let y be some internal point. Take $\tilde{\varepsilon} := \varepsilon - d(x, y)$. Then if $z \in B(y, \tilde{\varepsilon})$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon \implies z \in B(x, \varepsilon)$$



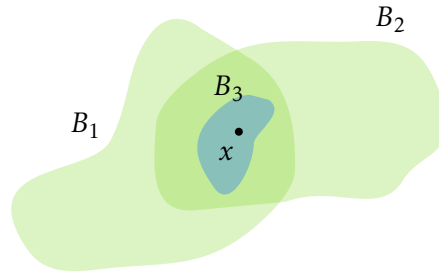
Eg. 1.1.2 If $X = \{a, b, c\}$, we can consider $\mathcal{T} = \mathcal{P}(X)$, i.e. the powerset of X . This is a topology. We can also arrive at this by considering the metric $d(x, y) = 1, x \neq y$. The topology $\mathcal{T}' = \{\emptyset, \{a, b\}, \{b, c\}, \{b\}, X\}$ has a less obvious metric-theoretical explanation.

Eg. 1.1.3 Condition 3 of [Prop 1.1](#) fails for infinite collections (e.g. balls of radius approaching 0).

DEF 1.6 The pair (\mathcal{T}, X) is called a *topological space*.

DEF 1.7 A *basis* \mathcal{B} on a topology (X, \mathcal{T}) is a collection of subsets of X satisfying

- (1) For each $x \in X$, there exists $B \in \mathcal{B}$ with $x \in B$
- (2) If $x \in B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq (B_1 \cap B_2)$.



DEF 1.8 The *topology generated by* \mathcal{B} is the collection of sets

$$\{U : \forall x \in U, \exists B \in \mathcal{B} : x \in B \subseteq U\}$$

Eg. 1.2.1 $\mathcal{B} = \{B(x, \varepsilon), x \in X, \varepsilon > 0\}$ is a basis. Clearly condition (1) of [Def 1.7](#) is satisfied. For condition (2), suppose $x \in B(x_i, \varepsilon_i)$ for $i = 1, 2$. Take $\varepsilon_3 := \min\{\varepsilon_i - d(x, x_i)\}$. Then $B(x, \varepsilon_3)$ is contained in both balls $B(x_i, \varepsilon_i)$.

PROP 1.2 The topology generated by a basis \mathcal{B} is a topology.

Let \mathcal{T} be generated by \mathcal{B} . We will go through each condition in [Prop 1.1](#):

PROOF.

- (1) Condition (1) of [Def 1.7](#) provides for $\emptyset, X \in \mathcal{T}$
- (2) Let $U_\alpha : \alpha \in J$ be a collection in \mathcal{T} , for some index J . Let $U := \cup_{\alpha \in J} U_\alpha$. Given $x \in U$, we know $x \in U_\alpha$ for some $\alpha \in J$. Hence, since $U_\alpha \in \mathcal{T}$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq U_\alpha \subseteq U$.
- (3) It suffices to show $U_1 \cap U_2 \in \mathcal{T}$, since we only consider finite collections. Fix $x \in U_1 \cap U_2$. There exists $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. We know there exists, then, $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq (B_1 \cap B_2) \subseteq (U_1 \cap U_2)$. \square

Let \mathcal{B} be a basis for X generating \mathcal{T} . Then \mathcal{T} is the collection of all possible unions of elements in \mathcal{B} . PROP 1.3

Let \mathcal{T}' denote all possible unions of elements in \mathcal{B} . Then clearly $\mathcal{T}' \subseteq \mathcal{T}$, since $\mathcal{B} \subseteq \mathcal{T}$.
Conversely, let $U \in \mathcal{T}$. Then each $x \in U$ belongs to some $B_x \subseteq U \in \mathcal{B}$. Hence, $U \subseteq \cup_{x \in U} B_x$. But each $B_x \subseteq U$, so U is exactly this union, and hence $U \in \mathcal{T}'$. \square

PROOF.

Let X, Y be topological spaces. Then the *product topology on $X \times Y$* is the topology generated by the basis DEF 1.9

$$\{U \times V : U \text{ open in } X, V \text{ open in } Y\}$$

Indeed, this is a proper basis. For (1) of [Def 1.7](#), if $(x, y) \in X \times Y$, we take $U = X$ and $V = Y$. For (2), if $(x, y) \in U_1 \times V_1, U_2 \times V_2$, then $U_1 \cap U_2 \in \mathcal{T}_X$ and $V_1 \cap V_2 \in \mathcal{T}_Y$ are open. Hence, $x \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$.

Let $Y \subseteq (X, \mathcal{T})$. Then the collection $\{Y \cap U : U \in \mathcal{T}\}$ is called the *subspace topology* with respect to Y . DEF 1.10

Let \mathcal{S} be a collection of subsets of X . Let \mathcal{B} be the collection of all finite intersections of elements in \mathcal{S} . Then \mathcal{B} is a basis in X , as in [Def 1.7](#). PROP 1.4

\mathcal{S} , as above, is called a *sub-basis* of the topology in X generated by \mathcal{B} . DEF 1.11

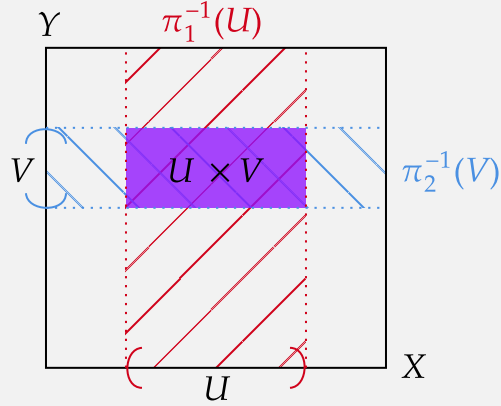
Eg. 1.3.1 Let $\pi_1 : X \times Y \rightarrow X, \pi_2 : X \times Y \rightarrow Y$ be the coordinate projections onto the X, Y subspaces of $X \times Y$. Then

$$\{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ open in } Y\} =: \mathcal{S}$$

is a sub-basis of the product topology.

PROOF.

The intersection of any two elements in \mathcal{S} may be written as $U \times V$, $U \times Y$, or $X \times V$ for open subsets U, V of X, Y , respectively.



Eg. 1.3.2 Let $X = \prod_{\alpha \in J} X_\alpha$, with $\pi_\alpha : X \rightarrow X_\alpha$ the coordinate projection. Then the product topology on X will have a sub-basis

$$\mathcal{S} = \{\pi_\alpha^{-1}(U) : U \text{ open in } X_\alpha : \alpha \in J\}$$

with corresponding basis

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \text{ open in } X_\alpha \right\}$$

where $U_\alpha = X_\alpha$ except at finitely many indices.

DEF 1.12 A subset $A \subseteq X$ is called *closed* if $X \setminus A$ is open.

PROP 1.5 Let X be a topological space. Then

1. X, \emptyset are closed
2. Finite unions of closed sets are closed
3. Infinite intersections of closed sets are closed

PROOF.

We'll take a look at (2). Let A_1, A_2 be closed. Then $(A_1 \cup A_2)^C = A_1^C \cap A_2^C$ by De Morgan's. Then, This is open, since A_i^C is open, and hence $A_1 \cup A_2$ is closed. \square

PROP 1.6 If X is a metric space, then $A \subseteq X$ is closed \iff for each $x_n \rightarrow x$ (i.e. $d(x_n, x) \rightarrow 0$), with $x_i \in A$, then $x \in A$.

(\implies) Suppose $x \notin A$, and hence $x \in X \setminus A$, which is open. Hence, $B(x, \varepsilon) \subseteq X \setminus A$ for some $\varepsilon > 0$. Choose $x_n : d(x_n, x) < \varepsilon$. Then $x_n \in B(x, \varepsilon) \subseteq X \setminus A$, which is a contradiction.

PROOF.

(\impliedby) Let $x \in X \setminus A$. Fix some n . We wish to show that there exists $B(x, \frac{1}{n}) \subseteq X \setminus A$. If not, then $\exists x_n \in B(x, \frac{1}{n})$. Repeating this, we have a sequence $x_n \rightarrow x$ with $x_n \in A$ and $x \notin A$, a contradiction. \square

Let A be a subset of a topological space X . The *interior* of A , denoted $\text{Int}(A)$, is the union of all open sets contained in A . Hence, it is open. An alternative characterization of $\text{Int}(A)$ is: the largest open subset of A .

DEF 1.13

Conversely, for $A \subseteq X$, the *closure* of A , denoted \overline{A} , is the intersection of all closed sets containing A . Hence, it is closed. An alternative characterization of \overline{A} is: the smallest closed set containing A .

DEF 1.14

We say A is *dense* in X if $\overline{A} = X$.

DEF 1.15

For $x \in X$, U is called a *neighborhood* of x if $x \in \text{Int}(U)$.

DEF 1.16

Eg. 1.4.1 $[a, b] \subseteq \mathbb{R}$ is closed, as the complement, $(-\infty, a) \cup (b, \infty)$, is open.

Eg. 1.4.2 "Closed balls," i.e. $A = \{y : d(x, y) \leq \varepsilon\}$ for $x \in X, \varepsilon > 0$, are closed. For $y \in X \setminus A$, choose $\tilde{\varepsilon} = d(x, y) - \varepsilon$ to show openness.

Eg. 1.4.3 $\text{Int}(\mathbb{Q}) = \emptyset$ and $\overline{\mathbb{Q}} = \mathbb{R}$, i.e. \mathbb{Q} is dense in \mathbb{R} .

Eg. 1.4.4 $X \setminus \text{Int}(A) = \overline{X \setminus A}$

Eg. 1.4.5 $U = \{y : d(x, y) \leq \varepsilon\}$ is a neighborhood of x , since $B(x, \varepsilon) \subseteq U$. In general, any open set containing x is a neighborhood of x .

Let $A \subseteq X$ be a topological space. Then $x \in \overline{A} \iff$ each neighborhood U of x intersects A (i.e. $U \cap A \neq \emptyset$).

PROP 1.7

We'll prove the contrapositive, i.e. $x \notin \overline{A} \iff$ there exists a neighborhood U with $U \cap A = \emptyset$.

PROOF.

(\implies) Take $U = X \setminus \overline{A}$, since $x \in X \setminus \overline{A}$. U is open, and hence a neighborhood. Since $A \subseteq \overline{A}$, $U \cap A = \emptyset$.

(\impliedby) Let U be a neighborhood of x with $U \cap A = \emptyset$. Then $X \setminus \text{Int}(U)$ is closed and contains A . Hence, $\overline{A} \subseteq X \setminus \text{Int}(U)$. But $x \in \text{Int}(U)$, so $x \notin \overline{A}$. \square

Let X, Y be topological spaces. A function $f : X \rightarrow Y$ is *continuous at* $x \in X$ if, for each neighborhood A of $f(x)$, the preimage $f^{-1}(A)$ is a neighborhood of x .

DEF 1.17

If $f : X \rightarrow Y$ is continuous for all $x \in X$, we simply call f *continuous*.

DEF 1.18

PROP 1.8 f is continuous \iff for each open set $U \subseteq Y$, $f^{-1}(U)$ is open in X .

1.1 boop

test!