Higher Algebra 2

MATH 571

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CONTENTS

Review

Much of the content in this section should be familiar from MATH 457. We will require a cursory understanding of tensor products, categories, and functors. The official prerequisite for this course is MATH 570 (which includes category theory, commutative algebra, Noetherian rings), but these notes will be written from the point of view of someone (me) who has not studied these topics.

COMMUTATOR SUBGROUPS

DEF 1.1 Let G be a group. The *commutator* of $a, b \in G$, denoted by $[a, b]$, is the element $aba^{-1}b^{-1}$. Clearly, $[a, b] = 1 \iff a$ and b commute. Let $G' \subseteq G$ be generated by all finite multiplications of commutators, i.e.

$$G' = \langle [a, b] : a, b \in G \rangle$$

DEF 1.2 G' is called the *commutator subgroup* of G.

PROP 1.1 The commutator subgroup of G is normal.

PROOF. Note that $g[a, b]g^{-1} = gaba^{-1}b^{-1}g^{-1} = [gag^{-1}, gbg^{-1}]$. Then,

$$g[a_1, b_1] \cdots [a_N, b_N]g^{-1} = g[a_1, b_1]g^{-1} \cdot g[a_2, b_2]g^{-1} \cdot \ldots \cdot g[a_N, b_N]g^{-1} \in G' \quad \square$$

PROP 1.2 If $H \triangleleft G$, then G/H is abelian $\iff G' \subseteq H$.

PROOF. Suppose G/H is abelian. Consider $aba^{-1}b^{-1} = [a, b] \in G'$. Then

$$aba^{-1}b^{-1}H = aH \cdot bH \cdot a^{-1}H \cdot b^{-1}H = aa^{-1}H \cdot bb^{-1}H = H$$

Hence, $[a, b] \in H$, so $G' \subseteq H$. Conversely, suppose $G' \subseteq H$. Then

$$a^{-1}b^{-1}abH = H \implies abH = baH$$

so G/H is abelian. $\square$

PROP 1.3 G/G' is the largest abelian subgroup of the form G/H for $H \triangleleft G$. In other words, G' is the smallest normal subgroup of G such that G/G' is abelian.

PROOF. Suppose G/H is abelian. Then $G' \subseteq H$ by Prop 1.2. Thus, $|G/G'| \geq |G/H|$. $\square$

DEF 1.3 $G^{ab} := G/G'$ is called the *abelianization* of G.

Theorem 1.1 Unique Factoring Over Abelianizations

Let $\varphi : G \to A$ be a homomorphism into an abelian group. Then φ factors uniquely into $\varphi = \psi \circ \pi$, where $\pi : G \twoheadrightarrow G^{ab}$ is the natural quotient and $\psi : G^{ab} \to A$.

PROOF. Recall the homomorphism theorem, of which the isomorphism theorem is a special case. Let $\varphi : G \to H$. Let $N \subseteq \ker(\varphi)$ be a normal subgroup of G. Then $\varphi = \psi \circ \pi$, where $\pi : G \twoheadrightarrow G/N$ is the natural quotient and $\psi : G/N \to H$ is a homomorphism (surjective into $\mathrm{Im}(\varphi)$). Moreover, this decomposition is unique.

We apply this directly to the theorem above. Since A is abelian, so is $\mathrm{Im}(\varphi)$. But $\mathrm{Im}(\varphi) \cong G/\ker(\varphi)$. By Prop 1.2, it follows that $G' \subseteq \ker(\varphi)$. Since G' is normal, the homomorphism theorem applies. □

TENSOR PRODUCTS

Let $\mathbf{Mod_R}$ and $\mathbf{{}_RMod}$ denote the categories of left and right modules over a ring R, respectively. Recall that, for an R-module M, $r \in R$, and $m \in M$, left modules act by $(r, m) \mapsto rm$ and right modules act by $(r, m) \mapsto mr$.

If a module is both a left and right module, and obeys all respective module axioms, we call it a *bimodule*, and write $\mathbf{{}_SMod_R}$ for the category of bimodules.

If $A \in \mathbf{Mod_R}$ and $B \in \mathbf{{}_RMod}$, an *R-biadditive* map is a function

$$f : A \times B \to G$$

where H is a abelian. Additionally, we require that

- $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$
- $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$
- $f(ar, b) = f(a, rb)$

As H is a group, we do not impose any scaling qualities for f with respect to R.

We would like to construct an abelian group G and associated R-biadditive function φ such that, for any R-biadditive function f, there is a unique group homomorphism g with

$$\begin{tikzcd} A \times B \arrow[r, "\varphi"] \arrow[dr, "f"'] & G =: A \otimes_R B \arrow[d, "g"] \\ & H \end{tikzcd}$$

commuting. If such a pair (G, φ) exists, we say it satisfies the *universal property*.

Construction

We will construct a group G which satisfies the universal property, as above

Consider $H = \mathbb{Z} \cdot (A \times B)$, the $\mathbb{Z}$-module, and hence free abelian group. In other words,

$$H \ni h = \oplus_{(a,b) \in A \times B} k_{(a,b)} \cdot (a, b) \quad \text{where} \quad k_{(a,b)} \in \mathbb{Z}$$

Furthermore, consider the subgroup $N < H$ by

$$N = \{(a_1 + a_2, b) - (a_1, b) - (a_2, b)\} \cup \{(a, b_1 + b_2) - (a, b_1) - (a, b_2)\} \cup \{(ar, b) - (a, rb)\}$$

under $a, a_i \in A$, $b, b_i \in B$, and $r \in R$. One shows manually that this is a group

Define $A \otimes_R B := H/N$, and call this the *tensor product* of A and B over R. DEF 1.7

Let $\varphi : A \times B \to A \otimes_R B$ be the natural map formed by viewing (a, b) as an element of the $\mathbb{Z}$-module H, and modding out by N as above.

Immediately, we see that the subgroup N ensures that φ is biadditive.

We denote the image of (a, b) under φ by $a \otimes b$, and call the result a *tensor*. DEF 1.8

$(\varphi, A \otimes_R B)$ has the universal property. PROP 1.4

$V^* = \mathrm{Hom}_k(V, k)$ is called the *dual vector space*. Recall that $\dim_k(V^*) = \dim_k(V)$. DEF 1.9

Theorem 1.2 Properties of the Tensor Product

1. $\mathrm{Hom}_k(V, W) \cong V^* \otimes_k W$, V, W are finite dimensional vector spaces over k.
2. $\dim(V \otimes_k W) = \dim_k(V) \cdot \dim_k(W)$
3. If $f \in \mathrm{Hom}_R(A, A')$, $g \in \mathrm{Hom}_R(B, B')$, then
$$f \otimes g : A \otimes_R B \to A' \otimes_R B' \text{ given by } (a \otimes b) \mapsto f(a) \otimes f(b)$$
is a homomorphism.
4. If $A \cong A'$ and $B \cong B'$, then $A \otimes_R B \cong A' \otimes_R B'$
5. $A \otimes_R R = A$ and $R \otimes_R B = B$
6. $(\oplus_{i \in I} A_i) \otimes_R B \cong \oplus_{i \in I}(A_i \otimes_R B)$ and $A \otimes (\oplus_{i \in I} B_i) = \oplus_{i \in I} A \oplus_R B_i$
7. If R is commutative, then $A \otimes_R B = B \otimes_R A$.
8. If R is commutative, then $A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$.

REPRESENTATIONS OF FINITE GROUPS

A *linear representation* of a finite group G is a vector space V over a field $\mathbb{F}$ equipped with a group action DEF 1.10

$$G \times V \to V$$

that respects the vector space, i.e. $m_g : V \to V$ with $m_g(v) = gv$ is a linear transformation. We make the following assumptions unless otherwise stated:

1. G is finite.
2. V is finite dimensional.

3. $\mathbb{F}$ is algebraically closed and of characteristic 0. We write $\mathbb{F} = \mathbb{C}$.

Since V is a G-set, $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ which sends $g \mapsto m_g$ is a homomorphism.

Relatedly, if $\dim(V) < \infty$, then $\rho : G \mapsto \mathrm{Aut}_{\mathbb{C}}(V) = \mathrm{GL}_n(\mathbb{C})$.

The *group ring* $\mathbb{C}[G]$ is a (typically) non-commutative ring consisting of all finite linear combinations $\{\sum_{g\in G} \lambda_g g : \lambda_g \in \mathbb{C}\}$, with $1 \cdot \mathbb{1}_G = \mathbb{1}_{\mathbb{C}[G]}$. It's endowed with the multiplication rule

$$\left(\sum_{g\in G} \alpha_g g\right)\left(\sum_{h\in G} \beta_h h\right) = \sum_{(g,h)\in G\times G} \alpha_g \beta_h (gh)$$

We can view representations as a module over the group ring $\mathbb{C}[G]$.

Let V be a $\mathbb{C}[G]$-module. Consider $g \in G \subseteq \mathbb{C}[G]$, $\lambda \mathbb{1}_G \in \mathbb{C}[G]$, and $v_1, v_2 \in V$. Since V is a $\mathbb{C}[G]$-module,

$$g(v_1 + v_2) = gv_1 + gv_2 \qquad (gh)v_1 = g(hv_1)$$

Then: $(g\lambda\mathbb{1}_G)v_1 = (\lambda(g\mathbb{1}_G))v_1 = (\lambda g)v_1$. But also, $(g\lambda\mathbb{1}_G)v_1 = g(\lambda\mathbb{1}_G v_1) = g(\lambda v_1)$. Hence, the map $v \mapsto gv$ is a linear transformation on V over $\mathbb{C}$. $\square$

We will frequently return to this view when module theory is more convenient.

Eg. 1.1 Consider $\rho : G \to \{\mathbb{1}\}$, the *trivial representation*, which maps $\rho(g)(v) = v$. We will denote the trivial representation simply by $\mathbb{1}$, subject to context.

Eg. 1.2 We call $\rho^{\mathrm{reg}} : h \mapsto \left[\sum_{g\in G} \alpha_g g \mapsto \sum_{g\in G} \alpha_g hg\right]$ the *regular representation*, with $G \circlearrowright \mathbb{C}[G]$ by left multiplication.

Over $\mathbb{C}$, $\mathbb{C}[G]$ has basis $\{g_1, \ldots, g_n\}$, where $n = |G|$. Then $\chi(h) = \{g_i \in G : hg_i = g_i\}$. If $h = 1$, then $\chi(h) = |G|$. Otherwise, it is impossible for $hg_i = g_i$.

Generally, recall that the trace counts the number of basis vectors which are fixed by a transformation

We conclude that

$$\chi(g) = \begin{cases} |G| & g = 1 \\ 0 & \text{o.w.} \end{cases}$$

Special Representations

RESTRICTED AND INDUCED REPRESENTATIONS

Let $H < G$ be a subgroup. Then we consider a functor between the categories of representations of G and H,

$$\mathrm{Res}^G_H : \mathbf{Rep}(G) \to \mathbf{Rep}(H) : \rho \mapsto \rho|_H = \mathrm{Res}^G_H(\rho)$$

called the *restricted representation* of G to H. Analogously, this sends a $\mathbb{C}[G]$-module V to the submodule W defined over $\mathbb{C}[H]$.

Similarly, we consider a functor

$$\mathrm{Ind}_H^G : \mathbf{Rep}(H) \to \mathbf{Rep}(G) : V \mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

called the *induced representation* of H to G, where we view V as a $\mathbb{C}[G]$-module. Observe that $\dim_{\mathbb{C}[H]}(\mathbb{C}[G]) = [G:H]$, so $\dim(\mathrm{Ind}_H^G) = [G:H]\dim(V)$. DEF 1.15

Eg. 1.3 Consider $H = \{\mathbb{1}\}$ with the representation $V = \mathbb{C}$. Then $\mathrm{Ind}_H^G(\mathbb{C}) = \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[G]$.

DUAL REPRESENTATIONS

Let ρ, V be a representation of G. Recall the dual, $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, the set of linear transformations from $V \to C$. Given an endomorphism $T : V \to V$, we call

$$T^t : V^* \to V^* : (T^t\varphi)(v) := \varphi(Tv)$$

the *transpose*. DEF 1.16 If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V, then we construct the *dual basis* $\beta^* = \{\varphi_1, \ldots, \varphi_n\}$ for V^*, where $\varphi_i(v_j) = \delta_{ij}$. In the dual basis, we have DEF 1.17

$$[T^t]_{\beta^*} = [T]_\beta^t \implies \mathrm{tr}(T) = \mathrm{tr}(T^t)$$ PROP 1.6

PROOF. See MATH 251 notes. □

When $T = \rho(g) : V \to V$, we also observe

$$(\rho(gh)^t\varphi)(v) = (\rho(h)^t\rho(g)^t\varphi)(v) \implies \rho(gh)^t = \rho(h)^t\rho(g)^t$$

Given a representation ρ, $\rho^* : G \to \mathrm{GL}(V^*)$ by $g \mapsto \rho(g^{-1})^t$ is called the *dual representation*. DEF 1.18

PROP 1.7 $\chi_{\rho^*} = \overline{\chi_\rho}$

PROOF. If $g \in G$ has order n, then $\rho(g)$ has order $m|n$, since $\rho(g)^n = \rho(g^n) = \rho(\mathbb{1}) = I$. Hence, in a certain basis,

$$\rho(g) = \begin{pmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{pmatrix} \quad \text{where} \quad \xi_i^m = 1$$

It follows that

$$\rho(g^{-1}) = \begin{pmatrix} \xi_1^{-1} & & \\ & \ddots & \\ & & \xi_n^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\xi_1} & & \\ & \ddots & \\ & & \overline{\xi_n} \end{pmatrix}$$

Thus, $\mathrm{tr}(\rho^*(g)) = \mathrm{tr}(\rho(g^{-1})^t) = \mathrm{tr}(\rho(g^{-1})) = \overline{\mathrm{tr}(\rho(g))}$, using Prop 1.2. □

1-DIM REPRESENTATIONS

A *1-dim representation* (ρ, V) is a representation with $\dim(V) = 1$. In this case, as V is a $\mathbb{C}$-vector space and $\rho(g) \in GL(V)$, we write $V = \mathbb{C}^\times$. Also observe that $\chi_\rho = \rho$.

$G^* = \mathrm{Hom}(G, \mathbb{C}^\times)$, as groups, is called the *group of multiplicative characters*.

If G is a finite, abelian group, then every irreducible representation has dimension 1.

See MATH 457. □

$(G^{ab})^* \cong G^*$

If $f \in (G^{ab})^*$ is a homomorphism $f : G^{ab} \to \mathbb{C}^\times$, then $f \circ \pi : G \twoheadrightarrow G/N \to \mathbb{C}^\times$ is also a homomorphism. Conversely, any $F : G \to \mathbb{C}^\times$ must factor uniquely into $f \circ \pi$ by Thm 1.1, where $f : G^{ab} \to C^\times$. See the following diagram:

$$\begin{tikzcd} G \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathbb{C}^\times \\ & G^{ab} \arrow[ur, "f"'] & \end{tikzcd}$$

TENSOR REPRESENTATIONS

If ρ is a finite representation of G and τ is a 1-dim representation, we can generate a new representation

$$\rho \otimes \tau : G \to GL(V \otimes_{\mathbb{C}} \mathbb{C}) \cong GL(V) : g \mapsto \tau(g) \otimes \rho(g)$$

Note that $\tau(g) \in \mathbb{C}^\times$, so $\chi_{\rho\otimes\tau} = \tau\chi_\rho$.

In generality, given two representations ρ_1, ρ_2, we generate the tensor product representation $\rho_1 \otimes \rho_2$ over $V_1 \otimes_{\mathbb{C}} V_2$, with dimension $\dim(V_1)\dim(V_2)$ and trace $\chi_{\rho_1}\chi_{\rho_2}$.

Irreducible Representations

Let (ρ, V) be a representation. It is called an *irreducible representation* if there are no G-stable, nontrivial subspaces of V (i.e. no nontrivial subrepresentations). In the language of modules, irreducible representations are simple $\mathbb{C}[G]$-modules.

Theorem 1.3 Semi-Simplicity of Representations

Every finite dimensional, non-zero representation of G is a direct sum of irreducible representations.

This is "Maschke's Theorem" without uniqueness

PRO

Pick any Hermitian inner product $\langle \cdot, \cdot \rangle$ on V. Define

$$\langle u, v \rangle^* = \frac{1}{|G|} \sum_{g \in G} \langle gu, gv \rangle$$

It can be easily verified that $\langle \cdot, \cdot \rangle^*$ is an inner product which is G-equivariant. If $W \subseteq V$ is a subrepresentation, then set $W^\perp = \{u : \langle u, v \rangle = 0 \ \forall v \in W\}$, i.e. the orthogonal complement of W with respect to $\langle \cdot, \cdot \rangle^*$. It follows that $V = W \otimes W^\perp$, with $W^\perp$ being G-stable by the G-equivariance of the inner product.

We then argue by induction to yield a direct sum of irreducible representations. See MATH 457 for more details on semi-simplicity. $\square$

SCHUR'S LEMMA

In this section, we will build up the intuition necessary for proving Schur's Lemma.

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Def 1.22

Let (ρ, V) be a representation. Let $V^G = \{v : \rho(g)(v) = v : \ \forall g \in G\}$ be the space of *invariant vectors*. Notice that V^G is a subrepresentation of V equivalent to $\underbrace{\mathbb{1} \oplus \cdots \oplus \mathbb{1}}_{\dim(V^G) \text{ times}}$.

Prop 1.10

$\dim(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$

PROOF. Let $\pi : v \mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)(v)$. Writing $\rho(h)\pi = \frac{1}{|G|} \sum_{g \in G} \rho(hg) = \frac{1}{|G|} \sum_{g \in G} \rho(g)$ verifies that $\mathrm{Im}(\pi) \subseteq V^G$. It is also easy to verify that $\pi|_{V^G} = \mathrm{Id}_{V^G}$. Hence, we may write $V = \ker(\pi) \oplus V^G$. It follows that, in some basis,

$$\pi = \begin{pmatrix} 0 & 0 \\ 0 & I_{\dim(V^G)} \end{pmatrix}$$

and thus $\mathrm{tr}(\pi) = \dim(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$. $\square$

Let $(\rho_1, V_1), (\rho_2, V_2)$ be two representations. Consider

$$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) = \{T : V_1 \to V_2 \text{ with } T \ \mathbb{C}\text{-linear}\}$$

This is a $\mathbb{C}$-vector space of dimension $\dim(V_1)\dim(V_2)$. Similarly, we consider

Def 1.23

$$\mathit{Hom}_G(V_1, V_2) = \{T : V_1 \to V_2 \text{ with } T\rho_1(g) = \rho_2(g)T\}$$

It is often more natural to think of $\mathrm{Hom}_G(V_1, V_2)$ as transformations which satisfy $T(gv) = gT(v) \ \forall v \in V_1$, noting the distinct actions of g on V_1 and V_2, respectively.

Prop 1.11

Over the vector space $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, $\rho : g \mapsto [T \mapsto \rho_2(g^{-1})T\rho_1(g)]$ is a G-representation.

PROOF.

Clearly $\rho_2(g^{-1})T\rho_1(g) \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$. Also note

$$\rho_2((gh)^{-1})T\rho_1(gh) = \rho_2(h^{-1})\rho_2(g^{-1})T\rho_1(g)\rho_2(h)$$

so $\rho(gh) = \rho(g)\rho(h)$. $\square$

$\mathrm{Hom}_G(V_1, V_2) = (\mathrm{Hom}_{\mathbb{C}}(V_1, V_2))^G$

Let $T \in (\mathrm{Hom}_{\mathbb{C}}(V_1, V_2))^G$. Let $g \in G$. Then

$$gT = T \implies \rho_2(g^{-1})T\rho_1(g) = T \implies T\rho_1(g) = \rho_2(g)T \qquad \square$$

$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \cong V_1^* \otimes V_2$ as vector spaces and as G-representations.

$\dim(\mathrm{Hom}_G(V_1, V_2)) = \frac{1}{|G|}\sum_{g\in G} \chi_1(g)\overline{\chi_2(g)}$

$$\begin{aligned}\dim(\mathrm{Hom}_G(V_1, V_2)) &= \dim(\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)^G) = \dim((V_1^* \otimes V_2)^G) \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_1^*\otimes\rho_2} = \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_1^*}\chi_{\rho_2} = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_{\rho_1}}\chi_{\rho_2} \\ &= \frac{1}{|G|}\sum_{g\in G} \chi_{\rho_1}\overline{\chi_{\rho_2}}\end{aligned}$$

In the last step, we use the fact that the dimension is always real, so $\overline{\dim} = \dim$. $\square$

Eg. 1.4 Let $G = S_n$, $V = \mathbb{C}^n$, and let ρ be the standard representation (i.e. permuting indices). Then $V^G = \{\langle x, ..., x\rangle : x \in \mathbb{C}\}$. This implies that

$$1 = \dim(V^G) = \frac{1}{|G|}\sum_{\sigma\in S_n} \chi_\rho(\sigma) \qquad \text{Prop 1.10}$$

But the trace of $\rho(\sigma)$ is exactly the number of fixed points of σ. To see this, note that σ permutes $i \to j$ if the i^{th} row is equal to e_j. Hence

$$1 = \frac{1}{n!}\sum_{\sigma\in S_n} \#\text{FP of } \sigma$$

On average, then, a random permutation has 1 fixed point.

Theorem 1.4 Schur's Lemma

Let $(\rho, V), (\tau, W)$ be irreducible representations of G. Then

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathbb{C} & \rho \cong \tau \\ 0 & \rho \ncong \tau \end{cases}$$

PRO

We claim that any nonzero $T \in \text{Hom}_G(V, W)$ is an isomorphism.

$\ker(T)$ and $\text{Im}(T)$ are subrepresentations of ρ and τ, respectively. Since both ρ and τ are irreducible, it follows that $\ker(T) = V \vee 0$, and $\text{Im}(T) = W \vee 0$.

Equivalently, the kernel and image of a homomorphism of modules are submodules

$\ker(T) = 0$, since T is nonzero. $\text{Im}(T) \neq 0$ for the same reason, so $\text{Im}(T) = W$. It follows that T is an isomorphism. Immediately, $\text{Hom}_G(V, W) = 0$ when $\rho \neq \tau$.

Suppose $\rho \cong \tau$. We can write $\text{Hom}_G(V, W) = \text{End}_G(V)$. Let $T \in \text{End}_G(V)$ be nonzero. Let λ be some eigenvalue of T with corresponding eigenspace $U_\lambda \subseteq V$. Then

$$T(gu) = g(Tu) = g(\lambda u) = \lambda gu \ \forall u \in U, g \in G$$

It follows that $gu \in U_\lambda$. Hence, U_λ is a G-stable subspace, and hence a subrepresentation. By irreducibly, $U_\lambda = V$, so $T = \lambda$.

By this argument, we can map $T \mapsto \lambda_T = \text{tr}(T)/\dim(V) \in \mathbb{C}$. The converse map $\lambda \mapsto \lambda I$ completes the proof. The well-structuredness of this map derives from the fact that

$$\frac{\text{tr}(T+G)}{\dim(V)} = \frac{\text{tr}(T)}{\dim(V)} + \frac{\text{tr}(G)}{\dim(V)}$$

□

Class Functions

A function $f : G \to \mathbb{C}$ is called a *class function* if $f(hgh^{-1}) = f(g)$. In other words, f is constant on each conjugacy class of G. DEF 1.24

We will denote by $h(G)$ the number of conjugacy classes of G. Similarly,

$$\textit{Class(G)} = \{f : f \text{ is a class function on } G\}$$ DEF 1.25

Note that $\text{Class}(G)$ is a $\mathbb{C}$-vector space with dimension $h(G)$.

Eg. 1.5 If G is abelian, then $h(G) = |G|$.

Eg. 1.6 $h(S_n)$ is the number of permutations of n.

We can endow $\text{Class}(G)$ with the inner product

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\psi(g)}$$

Note also that χ_ρ is a class function. For the following theorems and propositions, we will denote by $(p_1, V_1), \dots, (p_n, V_n)$ the irreducible representations of G, along with their dimensions d_i and characters λ_i for $i \in [n]$.

Theorem 1.5 Irreducible Characters Form Orthonormal Basis of Class(G)

Up to isomorphism, the irreducible characters of G form an orthonormal basis for $\text{Class}(G)$. We conclude that $\#h(G) = \#\text{irreducible representations of } G$.

PROOF. Let σ, τ be irreducible representations. We do not distinguish between σ, τ and their associated vector spaces. Then, by Schur's Lemma (Thm 1.4) and Prop 1.14,

$$\delta_{\sigma,\tau} = \dim(\mathrm{Hom}_{\mathbb{C}}(\sigma, \tau)^G) = \frac{1}{|G|}\sum_{g\in G} \chi_\sigma \overline{\chi_\tau} = \langle \chi_\sigma, \chi_\tau \rangle$$

From this, we conclude that the irreducible characters are orthonormal in Class(G). It remains to show that they are a basis. From linear algebra (see MATH 251), we recall the criterion

$$\langle \chi_i, \beta \rangle = 0 \ \forall i \in [n] \implies \beta \in \mathrm{Class}(G) \equiv 0$$

This ensures that Fourier coefficients always exist using the irreducible characters provided, which establishes spanning-ness. Let $\alpha \in \mathrm{Class}(G) : G \to \mathbb{C}$. Consider

$$A_\rho = \sum_{g\in G} \alpha(g)\rho(g) \in \mathrm{End}_{\mathbb{C}}(V)$$

We claim that $A_p \in \mathrm{End}_G(V)$. Write

$$\begin{aligned}\rho(h)A_\rho\rho(h^{-1}) &= \sum_{g\in G} \alpha(g)\rho(hgh^{-1}) = \sum_{g\in G} \alpha(hgh^{-1})\rho(hgh^{-1})\\ &= \sum_{g\in G} \alpha(g)\rho(g) = A_\rho\end{aligned}$$

We claim that, if $\alpha = \overline{\beta}$, with ρ irreducible, then $A_\rho = 0$. Schur's Lemma gives the map

$$\mathrm{End}_G(V) \to \mathbb{C} : T \mapsto \frac{\mathrm{tr}(T)}{\dim(\rho)}$$

Which we apply to A_ρ

$$A_\rho \mapsto \frac{\mathrm{tr}(\sum_{g\in G} \alpha(g)\rho(g))}{\dim(\rho)} = \frac{|G|}{\dim(\rho)}\frac{1}{|G|}\sum_{g\in G} \chi_\rho(g)\beta(g) = \frac{|G|}{\dim(\rho)}\left\langle \chi_\rho, \beta \right\rangle = 0$$

This holds for any irreducible representation, so, in particular, A_{ρ_i} $\forall i \in [n]$. It must also hold for the regular representation. Hence, $A_{\rho^{\mathrm{reg}}} = 0$ on $\mathrm{End}_G(\mathbb{C}[G])$. Consider $\mathbb{1}_G$. Then we must have

$$\sum_{g\in G} \alpha(g)\rho^{\mathrm{reg}}(g)(\mathbb{1}_G) = \sum_{g\in G} \alpha(g)[g] = 0$$

Since $[g] : g \in G$ is a basis for $\mathbb{C}[G]$, it must be that $\alpha(g) = 0$ $\forall g \in G$. As $\alpha = \overline{\beta}$, the result follows. $\square$

Theorem 1.6 Mascke's Theorem

If (ρ, V) is a representation of G, then it has a unique decomposition

$$p \cong p_1^{a_1} \oplus \cdots \oplus p_n^{a_n}$$

where $a_i = \langle \lambda_\rho, \lambda_i \rangle$.

PROOF. Letting a_i be as in Thm 1.3, we know $\chi_\rho = \sum_{i=1}^n a_i \lambda_i$. It remains to show that a_i are unique. But we can compute

$$\langle \chi_\rho, \chi_i \rangle = \sum_{j=1}^n a_i \underbrace{\langle \chi_j, \chi_i \rangle}_{\delta_{ij} \text{ by Thm 1.5}} = a_i$$

PROP 1.15 $\rho \cong \tau \iff \chi_\rho = \chi_\tau$

PROOF. We only need to consider the ($\Longleftarrow$) direction. In this case, we write

$$\langle \chi_\rho, \chi_i \rangle = \langle \chi_\tau, \chi_i \rangle \ \forall i$$

But these are the multiplicities of the irreducible characters in ρ and τ, so $\rho \cong \tau$. □

PROP 1.16 Let ρ^{reg} be the regular representation of G on $\mathbb{C}[G]$. We have

$$\rho^{\text{reg}} \cong \rho_1^{d_1} \oplus \cdots \oplus \rho_n^{d_n}$$

Consequently, $|G| = \sum_{i=1}^n d_i^2$.

PROOF.

$$\langle \chi^{\text{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi^{\text{reg}}(g) \overline{\chi_i(g)} = \frac{1}{|G|} |G| \chi_i(1) = d_i$$

□

PROP 1.17 ρ is irreducible $\iff \|\chi_\rho\|^2 = 1$. Similarly, ρ is the direct sum of two irreducible representations $\iff \|\chi_\rho\|^2 = 2$.

PROOF. We know $\|\chi_\rho\|^2 = \sum_{i=1}^h a_i^2$, where a_i is the multiplicity of the i-th irreducible representation in ρ's decomposition. Recall

$$\begin{aligned}\|\chi_\rho\|^2 = \left\langle \chi_\rho, \chi_\rho \right\rangle &= \langle a_1\chi_1 + ... + a_h\chi_h, a_1\chi_1 + ... + a_h\chi_h\rangle \\ &= \sum_{i=1}^{h} \langle a_i\chi_i, a_1\chi_1 + ... + a_h\chi_h\rangle \\ &= \sum_{i=1}^{h} a_i^2 \langle \chi_i, \chi_i\rangle = \sum_{i=1}^{h} a_i^2\end{aligned}$$

It follows that $\|\chi_\rho\|^2 = 1$ if and only if exactly one of $a_i^2 = 1$, i.e. χ_ρ is irreducible. If $\|\chi_\rho\|^2 = 2$, we must have some $i \neq j$ with $a_i^2 = a_j^2 = 1$, and so $\rho = \rho_i \otimes \rho_j$, where ρ_i, ρ_j are irreducible. □

Eg. 1.7 Consider $S_n : n \geq 2$. We consider ρ^{std}, the natural action of S_n on a set of n elements (e.g. permuting the indices of $v \in \mathbb{C}^n$). Recall Example 1.4, where we derived

$$1 = \frac{1}{n!}\sum_{\sigma\in S_n} \chi^{\text{std}}(\sigma) = \frac{1}{n!}\sum_{\sigma\in S_n} \#\text{FP of } \sigma$$

But also

$$\|\chi^{\text{std}}\|^2 = \frac{1}{n!}\sum_{\sigma\in S_n} (\chi^{\text{std}}(\sigma))^2 = \frac{1}{n!}\sum_{\sigma\in S_n} (\#\text{FP of } \sigma)^2$$

To analyze this equation, we define an action of S_n on $[n]^2$, which sends $\sigma(i,j) = (\sigma(i), \sigma(j))$. We observe exactly 2 orbits under this action: $\{(i,i) : i \in [n]\}$ and $\{(i,j) : i \neq j \in [n]\}$. By Burnside's Lemma,

$$2 = \frac{1}{n!}\sum_{\sigma\in S_n} (\#\text{FP of } \sigma \text{ on } [n]^2)$$

Observe that, σ has a fixed point $(k,\ell) \in F$ on $[n]^2$ if and only if it is fixed on each coordinate of each fixed point. In this way, we have a $n - to - n^2$ mapping, and conclude that $\|\chi^{\text{std}}\|^2 = 2$.

Note that the trivial representation is a G-stable subrepresentation of ρ^{std}. By "subtracting" $\mathbb{1}$ from ρ^{std} we can recover the other irreducible representation implied by the computation above, which we denote by $\rho^{\text{std},0}$. In particular

$$\rho^{\text{std}} = \mathbb{1} \oplus \rho^{\text{std},0}$$

Every irreducible representation of an abelian group is one dimensional.

As G is abelian, $h = |G|$. Then, $|G| = \sum_{i=1}^{|G|} d_i^2$, from which we conclude $d_i = 1\ \forall i$. □

Character Tables