Pseudorandom Generators in Proof Complexity

Based on the 2001 article of the same name by Alekhnovich, Ben-Sasson, Razborov, & Wigderson

N. Hayek &
B. Kyle

April 1st 2025

Pseudorandom Generators

Definition (Generator)

A mapping $G_n : \{0,1\}^n \to \{0,1\}^m$ is called a *generator*.

Pseudorandom Generators

Definition (Generator)

A mapping $G_n : \{0,1\}^n \to \{0,1\}^m$ is called a *generator*.

Definition (Pseudorandomness)

A deterministic generator is *pseudorandom* if no efficient algorithm can differentiate between the probability distributions of $G_n(\vec{x})$ and $\vec{y}$, where $\vec{x}$ and $\vec{y}$ are truly random.

Pseudorandom Generators

Definition (Generator)

A mapping $G_n : \{0,1\}^n \to \{0,1\}^m$ is called a *generator*.

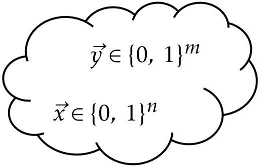
Definition (Pseudorandomness)

A deterministic generator is *pseudorandom* if no efficient algorithm can differentiate between the probability distributions of $G_n(\vec{x})$ and $\vec{y}$, where $\vec{x}$ and $\vec{y}$ are truly random.

Definition (Hardness)

Let P be a proof system. Let $b \in \{0,1\}^m$ be arbitrary. A generator is *hard for P* if P cannot prove in polynomial size that $b \notin \mathrm{Im}(G_n)$.

Pseudorandom G_n



Uniformally Random (God)

$P(G_n(\vec{x})) = 1$

$P(\vec{y}) = 1$

(I'm a polynomial algorithm)

Hardness of G_n

$\exists\, \vec{x} \,:\, P(G_n(\vec{x})) = b$

(I'm a polynomial P – proof)

Motivation 1

▹ If no proof system can deduce the **most basic property** of G_n efficiently (notably, what is in its image), then it certainly can't **distinguish** between the image of G_n and a proper random distribution efficiently.

$$\text{Hard in } \underline{\text{Especially Strong}} \text{ System } P \overset{?}{\Longrightarrow} \text{Pseudorandom}$$

Motivation 1

▹ If no proof system can deduce the **most basic property** of G_n efficiently (notably, what is in its image), then it certainly can't **distinguish** between the image of G_n and a proper random distribution efficiently.

$$\text{Hard for } \underline{\text{Especially Strong}} \text{ System } P \overset{?}{\Longrightarrow} \text{Pseudorandom}$$

▹ Conversely, if G_n is pseudorandom, it is hard for most proof systems (if there existed an algorithm that could efficiently prove $b \notin \mathrm{Im}(G_n)$, we could use this to distinguish G_n from random, and break the generator).

$$\text{Pseudorandom} \overset{?}{\Longrightarrow} \text{Hard for } \underline{\text{all}}\ P$$

Motivation 2

- The existence of hard generators for P (in particular, when $m > n$) provides lower bounds for a class of tautologies in P.

Tseitin Generators

Example

Tseitin tautologies provide a good context for constructing hard generators. Let G be a connected graph, with $|E| = n$ and $|V| = m$. Let $\vec{x}$ be a vector of variables on E. Enumerate V arbitrarily $v_1, \dots, v_m$. Then we have the generator

$$T_G : \{0,1\}^n \to \{0,1\}^m \quad \vec{x} \mapsto \begin{bmatrix} \bigoplus_{e \ni v_1} x_e \\ \vdots \\ \bigoplus_{e \ni v_m} x_e \end{bmatrix}$$

where $\oplus$ is the typical XOR (i.e. $\equiv_2 1$). One can show that $\vec{\sigma} \in \{0,1\}^m$ is *not* in the image of $T_G \iff \oplus_{i=1}^m \sigma_i = 1$ (i.e. "odd"). When is it hard, then, for a proof system to show $\vec{\sigma} \notin \mathrm{Im}(T_G)$?

T_G hard $\equiv$ Tseitin hard to refute

Example (cont.)

Fix $\vec{\sigma}$ which is *odd*. Then

$$T_G(\vec{x}) = \vec{\sigma}$$

cannot happen, since $\vec{\sigma} \notin \mathrm{Im}(T_G)$. In other words, there is no satisfying assignment to $\vec{x}$: $T_G(\vec{x}) = \vec{\sigma}$, and so the tautologies

$$\oplus_{e \ni v_1} x_e = \sigma_1$$
$$\vdots$$
$$\oplus_{e \ni v_m} x_e = \sigma_m$$

are unsatisfiable.

We know of some good (exponential) lower bounds on refuting the Tseitin tautologies, e.g. in resolution. We conclude that, in these systems, T_G is a hard generator, since the choice of mapping $\vec{\sigma}$ is arbitrary when proving such lower bounds.

Main Example: Nisan-Wigderson Generators

Let $g_i : \vec{x} \rightarrow \{0,1\}$ be a function on n-dimensional vector of variables $\langle x_1,\ldots,x_n\rangle$. We call each g_i a **base function**.

Main Example: Nisan-Wigderson Generators

Let $g_i : \vec{x} \rightarrow \{0,1\}$ be a function on n-dimensional vector of variables $\langle x_1, \ldots, x_n \rangle$. We call each g_i a **base function**.

(Caveat: Fix a binary matrix A of dimensions $m \times n$. We impose that g_i depend only on variables x_j for which the j-th entry in the i-th row is 1.)

Main Example: Nisan-Wigderson Generators

Let $g_i : \vec{x} \to \{0,1\}$ be a function on n-dimensional vector of variables $\langle x_1, \ldots, x_n \rangle$. We call each g_i a **base function**.

(Caveat: Fix a binary matrix A of dimensions $m \times n$. We impose that g_i depend only on variables x_j for which the j-th entry in the i-th row is 1.)

Then define

$$G_n(\vec{x}) = \langle g_1(\vec{x}), \ldots, g_m(\vec{x}) \rangle$$

These are believed to be pseudorandom in certain contexts.

Subject to conditions on A and g_i, we will show that these are hard for some standard proof systems.

Propositionalizing NW Generators

Definition (Matrix Restriction)

$$J_i(A) = \{j \in [n] : a_{ij} = 1\} \qquad X_i(A) = \{x_j : j \in J_i(A)\}$$

Above, we related the hardness of T_n to the Tseitin tautologies. We are interested now in the hardness of NW generators, i.e. refuting the tautologies

$$\begin{cases} g_1(\vec{x}) = 1 \\ \quad\vdots \\ g_m(\vec{x}) = 1 \end{cases} \qquad \mathrm{Vars}(g_i) \subseteq X_i(A) \tag{1}$$

in some common proof systems.

Propositionalizing NW Generators

Definition (Matrix Restriction)

$$J_i(A) = \{j \in [n] : a_{ij} = 1\} \qquad X_i(A) = \{x_j : j \in J_i(A)\}$$

Above, we related the hardness of T_n to the Tseitin tautologies. We are interested now in the hardness of NW generators, i.e. refuting the tautologies

$$\begin{cases} g_1(\vec{x}) = 1 \\ \quad \vdots \\ g_m(\vec{x}) = 1 \end{cases} \qquad \mathrm{Vars}(g_i) \subseteq X_i(A) \tag{1}$$

in some common proof systems.

(Caveat: we later impose hardness conditions on g_i. By allowing these conditions to be satsified by $g_i \iff$ they are satisfied by $\overline{g_i}$, it is sufficient to consider (1), i.e. $b = \langle 1, \ldots, 1 \rangle$, for the sake of refuting $\vec{b} \notin \mathrm{Im}(G_n)$ for any $\vec{b}$.)

Background Definitions

We're interested in propositionalizing (1). In our paper, circuit-based, linear, and functional encodings are provided. We will focus on the latter.

Definition (Extension Variable)

Fix $i \in [m]$. Let f be a boolean function for which $\mathrm{Vars}(f) \subseteq X_i(A)$. Then y_f is an *extension variable* for f.

Denote by $\mathrm{Vars}(A) = \{y_f : \exists i \in [m] : \mathrm{Vars}(f) \subseteq X_i(A)\}$ all possible extension variables (and hence functions) on the variables $X_i(A) : i \in [m]$.

Example

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then $\mathrm{Vars}(A)$ is in correspondence with all boolean functions on a subset of $\{x_1, x_3\}$ or $\{x_2\}$.

More definitions...

Definition (Vars(A) $\to \vec{x}$)

Let $C = y_{f_1}^{\varepsilon_1} \vee \cdots \vee y_{f_k}^{\varepsilon_k}$ be a clause on Vars(A). Then

$$\|C\| := f_1^{\varepsilon_1} \vee \cdots \vee f_k^{\varepsilon_k}$$

is a boolean function on the variables $\vec{x}$.

Example

With A as above, let $f_1 = x_1 \wedge \overline{x_3}$ and $f_2 = x_2$. Let $C = \overline{y_{f_1}} \vee y_{f_2}$. Then

$$\|C\| = \overline{x_1 \wedge \overline{x_3}} \vee x_2 = \overline{x_1} \vee x_3 \vee x_2$$

The Functional Encoding

Let $g_1, \ldots, g_m$ be functions on $\vec{x}$ which constitute a generator G_n. Note that $\mathrm{Vars}(g_i) \subseteq X_i(A)$. We encode (1) as follows:

Definition (Functional Encoding of NW Generator Hardness)

Fix A. Let $\tau(A, G_n)$ denote the collection of clauses of the form $C = y_{f_1}^{\varepsilon_1} \vee \cdots \vee y_{f_k}^{\varepsilon_k}$ for which

$$\mathrm{Vars}(f_i) \subseteq X_i(A) \quad i = 1, \ldots, k \qquad \text{and} \qquad g_i \models \|C\|$$

$\tau(A, G_n)$ is the *functional encoding of (1)*.

The Functional Encoding is Correct

Theorem ($\tau(A, G_n)$ Corresponds with (1))

$\tau(A, G_n)$ is satisfiable $\iff$ (1) has a mapping that satisfies it.

Proof.

$(\Longleftarrow)$ Let $\vec{x_0}$ be a solution to the system

$$\begin{cases} g_1(\vec{x_0}) = 1 \\ \quad\vdots \\ g_m(\vec{x_0}) = 1 \end{cases}$$

Consider $f_y \in \mathrm{Vars}(A)$. Let $\rho : \mathrm{Vars}(A) \to \{0,1\}$ be the truth assignment $y_f \mapsto f(\vec{x_0})$. Let $C \in \tau(A, G_n)$, i.e.

$$C = y_{f_1}^{\varepsilon_1} \vee \cdots \vee y_{f_k}^{\varepsilon_k} \quad \text{with} \quad \mathrm{Vars}(f_j) \subseteq X_i(A)\ \forall j, \text{ some } i$$

Since $g_i \vDash \|C\|$, and $g_i(\vec{x_0}) = 1$, we have $\|C\| = f_1^{\varepsilon_1} \vee \cdots \vee f_k^{\varepsilon_k} = 1$, so $\exists i : f_i^{\varepsilon_i}(\vec{x_0}) = 1$. Therefore, ρ will satisfy $y_{f_i}^{\varepsilon_i}$, and hence C. □

The Functional Encoding is Correct

Proof.

$(\implies)$ Let ρ be an assignment on $\mathrm{Vars}(A)$ satisfying $\tau(A, G_n)$. Define

$$\vec{x_0} = \begin{bmatrix} y_{x_1} \\ \vdots \\ y_{x_n} \end{bmatrix}$$

Note that the formula x_i belong to $\mathrm{Vars}(A)$ so long as we have no zero columns in A (we will impose this later). One can show by induction that $\rho(y_f) = f(\vec{x_0})$ as above. Since $\mathrm{Vars}(g_i) \subseteq X_i(A)$ and clearly $g_i \vDash g_i$, we have $g_i \in \tau(A, G_n)$ as a clause. ρ is satisfying for $\tau(A, G_n)$, so $\rho(g_i) = 1$ (as a bit assignment). But then $\rho(g_i) = g_i(\vec{x_0}) = 1$, as desired. □

Main Result: $\tau(A, G_n)$ Width Bounds in Resolution

Without yet defining $r, s, c \in \mathbb{R}$, (r, s, c)-expanders, or ℓ-robustness, we state the following theorem:

Theorem (Width of $\tau(A, G_n)$ in Resolution)

Let $A \in M_{m\times n}(\{0,1\})$ be an (r,s,c)-expander, and let g_i be ℓ-robust for $i = 1, \ldots, m$. Let $c + \ell \geq s + 1$. Then

$$w_{Res}(\tau(A, G_n)) > \frac{r(c+\ell-s)}{2\ell} = \Omega(r)$$

The foreign terms constitute the "hardness conditions" on A and g_i. We will define the following

1. (r, s, c)-expanders: these are sparse matrices which generalize well-connectedness for graphs. In such a way, tight groupings of variables between base functions are discouraged, preventing localized contradictions.
2. ℓ-robust functions g_i resist partial assignments.

Hardness of $A \leftrightarrow (r,s,c)$-expanders

Definition ((r,s,c)-expanders)

Let $A \in M_{m\times n}(\{0,1\})$. For a set of rows $I \subseteq [m]$, let $\partial_A(I)$ (called the *boundary* of I) denote all columns which, when restricted to I, contain one "1."

Then, A is called an (r,s,c)-*expander* if $|J_i(A)| \leq s$ and, for all choices I as above, $|I| \leq r \implies |\partial_A(I)| \geq c|I|$.

What does this say: first, the number of 1s in any given row is bounded (by s). This allows sparseness.

Hardness of $A \leftrightarrow (r,s,c)$-expanders

Definition ((r,s,c)-expanders)

Let $A \in M_{m\times n}(\{0,1\})$. For a set of rows $I \subseteq [m]$, let $\partial_A(I)$ (called the *boundary* of I) denote all columns which, when restricted to I, contain one "1."

Then, A is called an (r,s,c)-*expander* if $|J_i(A)| \leq s$ and, for all choices I as above, $|I| \leq r \implies |\partial_A(I)| \geq c|I|$.

What does this say: first, the number of 1s in any given row is bounded (by s). This allows sparseness. Secondly, up to a selectivity threshold (r), we may lower bound the density (c) of boundary columns in I by a linear factor.

Hardness of $A \leftrightarrow (r,s,c)$-expanders

Definition ((r,s,c)-expanders)

Let $A \in M_{m\times n}(\{0,1\})$. For a set of rows $I \subseteq [m]$, let $\partial_A(I)$ (called the *boundary* of I) denote all columns which, when restricted to I, contain one "1."

Then, A is called an (r,s,c)-*expander* if $|J_i(A)| \leq s$ and, for all choices I as above, $|I| \leq r \implies |\partial_A(I)| \geq c|I|$.

What does this say: first, the number of 1s in any given row is bounded (by s). This allows sparseness. Secondly, up to a selectivity threshold (r), we may lower bound the density (c) of boundary columns in I by a linear factor. For instance, a $(1,2,.5)$-expander could look like:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Hardness of $g_i \leftrightarrow \ell$-robustness

Our hardness condition on g_i (and its motivation) is more straight-forward:

Definition (ℓ-robustness)

A function g_i is called *ℓ-robust* if every assignment ρ such that $g_i(\rho) \in \{0, 1\}$ (i.e. not $\star$) satisfies $|\rho| \geq \ell$.

In other words, no short assignments satisfy or *dis*satisfy ℓ. For instance...

Hardness of $g_i \leftrightarrow \ell$-robustness

Our hardness condition on g_i (and its motivation) is more straight-forward:

Definition (ℓ-robustness)

A function g_i is called *ℓ-robust* if every assignment ρ such that $g_i(\rho) \in \{0, 1\}$ (i.e. not $\star$) satisfies $|\rho| \geq \ell$.

In other words, no short assignments satisfy or *dis*satisfy ℓ. For instance...

Example

$x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is n-robust, since we need to map *all* variables to determine if the sum is $\equiv_2 1$.

Conversely, $\ell_1 \vee \cdots \vee \ell_n$ is only 1-robust (take the assignment $\ell_1 = 1$).

By selecting robust (i.e. large enough ℓ) functions, we increase the number of variable assignments a prover would need to check implicitly (no shortcuts).

Proof of Theorem: Measure

Theorem (Width of $\tau(A, G_n)$ in Resolution)

Let $A \in M_{m\times n}(\{0,1\})$ be an (r,s,c)-expander, and let g_i be ℓ-robust for $i = 1,\dots,m$. Let $c+\ell \geq s+1$. Then

$$w_{Res}(\tau(A, G_n)) > \frac{r(c+\ell-s)}{2\ell}$$

Proof. We will first define a measure μ on clauses.

Definition (μ)

For a clause C in $\mathrm{Vars}(A)$, $\mu(C)$ is the size of a minimal $I \subseteq [m]$ such that:

(a) $\forall y_f^{\varepsilon} \in C\ \exists i \in I : \mathrm{Vars}(f) \subseteq X_i(A)$

(b) $\{g_i \mid i \in I\} \vDash \|C\|$

Remark. μ is sub-additive: if C_0 and C_1 resolve to C then $\mu(C) \leq \mu(C_0) + \mu(C_1)$. Furthermore, $\mu(C) = 1$ for $C \in \tau(A, G_n)$.

Proof of Theorem: Roadmap

Theorem (Width of $\tau(A, G_n)$ in Resolution)

Let $A \in M_{m\times n}(\{0,1\})$ be an (r,s,c)-expander, and let g_i be ℓ-robust for $i = 1, \ldots, m$. Let $c + \ell \geq s + 1$. Then

$$w_{Res}(\tau(A,G_n)) > \frac{r(c+\ell-s)}{2\ell}$$

1. We'll first establish a connection between $\mu(C)$ and $w(C)$: medium-μ clauses are wide.
2. We'll then show that $\mu(\perp)$ is large, and therefore, by our remark, that any resolution refutation of τ must contain a medium-μ clause which is wide.

Proof of Theorem: Claim #1

For a clause C in $\mathrm{Vars}(A)$, $\mu(C)$ is the size of a minimal $I \subseteq [m]$ such that:

(a) $\forall y_f^{\varepsilon} \in C\ \exists i \in I : \mathrm{Vars}(f) \subseteq X_i(A)$ (b) $\{ g_i \mid i \in I \} \vDash \|C\|$

Claim (#1)

For a clause C with $\frac{r}{2} < \mu(C) \leq r$, $w(C) > \frac{r(c+\ell-s)}{2\ell}$.

Proof.

Let I satisfy $\mu(C)$. Let $I_0 \subseteq I$ be minimal such that (a) still holds. Then, for, $I_1 := I \setminus I_0$, $\{g_i : i \in I \setminus k\} \nvDash \|C\|$ for any $k \in I_1$.

Fix $k \in I_1$. We make the sub-claim that $|J_k(A) \cap \partial_A(I)| \leq s - \ell$. Let α be an assignment such that $g_i(\alpha) = 1\ \forall i \in I \setminus k$, but $\|C\|(\alpha) = 0$. (This exists by the above). Then

$$\rho(x_i) := \begin{cases} \alpha(x_i) & i \notin \partial_A(I) \cap J_k(A) \\ \star & \text{otherwise} \end{cases}$$

Proof of Theorem: Claim #1

cont.

$$\rho(x_i) := \begin{cases} \alpha(x_i) & i \notin \partial_A(I) \cap J_k(A) \\ \star & \text{otherwise} \end{cases}$$

Let $i \neq k$ be arbitrary. We claim that, if $x_s \in \mathrm{Vars}(g_i)$, then $s \notin \partial_A(I) \cap J_k(A)$, and hence ρ is defined totally on each g_i. Let $x_s \in \mathrm{Vars}(g_i)$. Suppose $s \in \partial_A(I)$. Then s is a column in which only one "1" exists. But $s \in J_i(A)$, so $s \notin J_k(A)$ for any $k \neq i$ (since this would constitute a second "1"). A similar argument shows that variables $x_s \in \|C\|$ are such that $s \notin \partial_A(I) \cap J_k(A)$, with the additional rationale that $k \notin I_0$.

$\implies g_i|\rho = 1$ and $C|\rho = 0$. By (b), $g_k|\rho = 0$. But g_k is ℓ-robust:

$$\implies \#J_k(A) \setminus [\partial_A(I) \cap J_k(A)] \geq \ell$$
$$\implies s - |\partial_A(I) \cap J_k(A)| \geq \ell \implies |\partial_A(I) \cap J_k(A)| \leq s - \ell$$

Proof of Theorem: Claim #1

cont.

To restate: so far, we have, for any $k \in I_1$, the inequality

$$|\partial_A(I) \cap J_k(A)| \leq s - \ell$$

Hence, we sum up

$$\begin{aligned}
c|I| &\leq |\partial_A(I)| && \text{by } (r,s,c)\text{-properties of } A \\
&\leq \sum_{i \in I_0} |J_i(A) \cap \partial_A(I)| + \sum_{i \in I_1} |J_i(A) \cap \partial_A(I)| \\
&\leq \sum_{i \in I_0} |J_i(A)| + (s-\ell)|I_1| && \text{by sub-claim above} \\
&\leq s|I_0| + (s-\ell)|I_1| && \text{by prop of } A \text{ (sparseness)} \\
&= (s-\ell)|I| + \ell|I_0| \\
&\leq (s-\ell)|I| + \ell \cdot w(C)
\end{aligned}$$

Proof of Theorem: Claim #1

cont.

In this last step, we use $|I_0| \leq w(C)$. Recall that $I_0 \subseteq I$ is minimal such that

$$\text{(a)} \quad \forall y_f^\varepsilon \in C \; \exists i \in I : \mathrm{Vars}(f) \subseteq X_i(A)$$

But then $I_0 \subseteq \{i : \mathrm{Vars}(f) \subseteq X_i$ for some $f \leftrightarrow y_f^\varepsilon \in C\}$, and the magnitude of this set is bounded by $\{f \leftrightarrow y_f^\varepsilon\}$, which is bounded by $w(C)$.

To restate: $c|I| \leq (s-\ell)|I| + \ell \cdot w(C)$. But $|I| = \mu(C) > \frac{r}{2}$ by assumption, so $w(C) \geq \frac{(c+\ell-s)|I|}{\ell} > \frac{r(c+\ell-s)}{2\ell}$. □

This is the bulk of our theorem! Now that we have shown that medium-μ clauses attain our width bound, we just need to show they exist in a resolution proof.

Proof of Theorem: Claim #2

Claim (#2)

Any resolution refutation Π of τ contains a clause C with $\frac{r}{2} < \mu(C) \leq r$.

Proof.

We first show $\mu(\perp) > r$. Suppose not: $\mu(\perp) \leq r$.

Then we arrive at the same inequalities, i.e. $c \cdot |I| \leq (s - \ell)|I| + \ell|I_0|$

This time, I_0 is empty, and we get $c \leq s - l$. But the expansion property was that $c \geq s - \ell + 1$ ※

Now, since Π derives $\perp$ from clauses C_i in τ with $\mu(C_i) = 1$, and μ is sub-additive, we are done. □

Putting It All Together

Adding Claim 1 with Claim 2 completes our lower bound on resolution width:

$$w_{\text{Res}}(\tau(A_, G_n)) > \frac{r(c+\ell-s)}{2\ell}$$

□

Later in the paper, it is shown that nearly all matrices satisfy $c > 0.9s$, and that most functions satisfy $\ell > 0.9s$. Observe that when this is true, our bound is linear in r.

$$w_{\text{Res}}(\tau(A_, G_n)) = \Omega(r)$$

Note that r is bounded above my m, but can be taken to be roughly $\frac{n}{s}$. This width lower-bound to a strong size-lower bound by the known relation from class. In Section 4 of the paper, the method of random restriction in Polynomial Calculus is used to extend a width lower bound to a size one. The results are *stronger* bounds on the *weaker* linear encoding.

Corollary: Size Lower Bound

Corollary

Let $\varepsilon > 0$ be an arbitrary fixed constant, A be an $(r, s, \varepsilon s)$-expander of size $(m \times n)$, and $g_1, \ldots, g_m$ be $(1 - \varepsilon/2)s$-robust functions. Then every resolution refutation of $\tau(A, \vec{g})$ must have size $exp(\Omega(\frac{r^2}{m \cdot 2^{2^s}}))/2^s$

Concluding Remarks

- We have strong lower bounds for both resolution and algebraic systems in refuting (1), and the authors believe the same for stronger systems.
- These strong lower bounds on proving these tautologies imply even stronger lower bounds on breaking the generator itself, further affirming the strength of Nisan's and Wigderson's construction.
- For proof systems P with the Efficient Interpolation Property, there is an easy way of converting any computationally secure generator to another generator which is hard for P. But in general no such method exists.