Stochastic Processes

MATH 447

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CONTENTS

We assume working knowledge of probability and no knowledge of measure theory (though a grasp of analysis is essential). See these MATH 356 notes, also taught by Louigi!

I Markov Chains

Before providing definitions, we give some examples of stochastic processes:

Eg. 1.1 A simple random walk: $S_{i+1} = S_i + X_i$, where $X_i \sim \text{Ber}(p)$ and $S_0 = 0$. We might ask: does S_i ever return to 0, i.e.

$$\mathbb{P}(\exists i > 0 : S_i = 0)$$

Eg. 1.2 A branching process: as in asexual reproduction, we have an initial node. Each node n has a number of children X_n, where $\frac{X_n}{2} \sim \text{Ber}(p)$. We denote Z_i to be the number of inididuals in the i-th generation. We might ask: does Z_i ever have no children, i.e.

$$\mathbb{P}(\exists i > 0 : Z_i = 0)$$

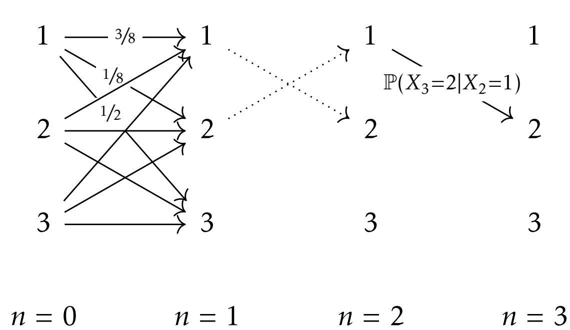
Eg. 1.3 Choose k independent random points in the square $[0, \sqrt{k}]^2$. On average, then, there is 1 point within any unit square $U \subseteq [0, \sqrt{k}]^2$.

DEF 1.1 DEF 1.2 Given a finite or countable set V, a *Markov chain* with *state space* V is a sequence $X_n : n \geq 0$ of random variables, with $X_n \in V$, such that:

$$\mathbb{P}(\underbrace{X_{n+1} = v_{n+1}}_{\text{future}} \,|\, \underbrace{X_0 = v_0, \ldots, X_{n-1} = v_{n-1}}_{\text{past}}, \underbrace{X_n = v_n}_{\text{present}}) = \mathbb{P}(X_{n+1} = v_{n+1} | X_n = v_n)$$

In other words, the future only depends on the past via the present. This is called the
DEF 1.3 *Markov property*.

Sometimes we allow Markov chains to be only finitely large (i.e. $0 \leq n \leq m$). For instance, we limit ourselves to one weekend of gambling in Las Vegas. A graphical example would look something like:



By repeated Bayes' Law, we observe **PROP 1.1**

$$\begin{aligned}&\mathbb{P}(X_1 = v_1, ..., X_n = v_n | X_0 = v_0)\\ =&\mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_0 = v_0, X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_0 = v_0, ..., X_{n-1} = v_{n-1})\\ =&\mathbb{P}(X_1 = v_1 | X_0 = v_0) \cdot \mathbb{P}(X_2 = v_2 | X_1 = v_1) \cdots \mathbb{P}(X_n = v_n | X_{n-1} = v_{n-1}) \quad \text{by Markov property}\end{aligned}$$

Time-Homogeneous Markov Chains

We often write THMC

We say that a Markov chain is *time-homogeneous* if, for all $u, v \in V$ and $n \geq 0$ DEF 1.4

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u)$$

In other words, the chain's behavior is described entirely by $\mathbb{P}(X_1 = v | X_0 = u)$ for each $(v, u) \in V \times V$. In this case, we can describe such probabilities in a *transition matrix* P: DEF 1.5

$$P = \left(p_{u,v}\right)_{(u,v) \in V^2} = \left(\mathbb{P}(X_1 = v | X_0 = u)\right)_{(u,v) \in V^2}$$

Eg. 1.4 Recall the game Snakes and Ladders. A 6×6 grid is indexed $1, ..., 36$. Players start at the 1 cell. The game ends when a player reaches the 36 cell. A die roll dictates how many spots one advances. The are some directed edges between cells (increasing: "ladders", decreasing: "snakes"). One must follow these edges when one lands at its tail. Suppose a ladder exists from 11 to 27. Then

$$\mathbb{P}(X_{11} = 27 | X_{10} = 6, X_9 = 3) = \frac{1}{6} = \mathbb{P}(X_{11} = 27 | X_{10} = 6) = \mathbb{P}(X_2 = 27 | X_1 = 6)$$

We see that Snakes and Ladders is naturally modeled as a time-homogeneous Markov chain.

Eg. 1.5 Sampling without replacement is *not* a Markov chain. If we sample from $|X| = 10$, we have

$$\begin{aligned}&\mathbb{P}(X_3 = a\,| X_2 = b) = 1/9\\ &\mathbb{P}(X_3 = a\,| X_2 = b, X_1 = c) = 1/8\\ &\mathbb{P}(X_3 = a\,| X_2 = b, X_1 = a) = 0\end{aligned}$$

so we do not satisfy the Markov property.

Eg. 1.6 Returning to the Snakes and Ladders example, consider $S \subseteq V$. Let $T_S = \inf\{n \geq 0 : X_n \in S\}$. We may ask...

- What is the average number of rounds to finite? We can write this as $\mathbb{E}[T_{\{36\}} | X_0 = 1]$.
- What is the probability of landing on 18 or 19 before the game ends? We can write this as $\mathbb{P}(T_{\{18,19\}} < T_{\{36\}} | X_0 = 1)$.
- What is the average number of visits to 6 before the game ends? We can write this as
$$\mathbb{E}[\#\{n \in [T_{\{36\}}] : X_n = 6\} | X_0 = 1]$$

- What is the expected proportion of time spent on state 5 before the game ends?
- If we allow two players, what is the probability that player 1 wins? Is this still a Markov chain?

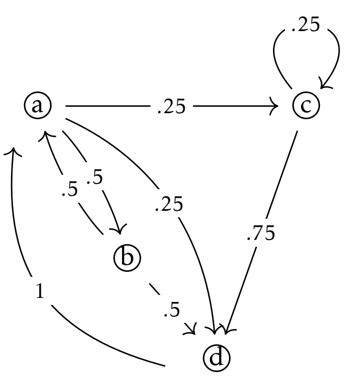
DEF 1.6 A matrix $P = \left(p_{u,v}\right)_{(u,v)\in V^2}$ is called a *stochastic matrix* if every row sums to 1, i.e.

$$\forall\ u \in V, \sum_{v\in V} p_{u,v} = 1$$

Note that any stochastic matrix is the transition matrix of some time-homogeneous Markov chain with state space V and transition probabilities

$$\mathbb{P}(X_{n+1} = v | X_n = u) = \mathbb{P}(X_1 = v | X_0 = u) = p_{u,v}$$

A directed graph, together with its stochastic matrix, can visualize any THMC:



$$\longleftrightarrow \begin{bmatrix} 0 & .5 & .25 & .25 \\ .5 & 0 & 0 & .5 \\ 0 & 0 & .25 & .75 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Eg. 1.7 Random walks on an undirected weighted graph, where edge weights dictate the proportional probability of transitioning between two states, are a special class of THMCs. In particular, given a graph $G = (V, E)$ with weights $w(e) > 0 : e \in E$, we set

$$p_{u,v} = \frac{w(\{u,v\})}{\sum_{z\in N(u)} w(\{u,z\})}$$

If there are no edges $u \leftrightarrow v$, we write $p_{u,v} = 0$.

Not every THMC can be represented by a random walk on an undirected weighted graph. In particular, see the directed graph listed above, or any transition matrix which is not symmetric.

As a concrete example, we can consider a random walk on the number line $\mathbb{Z}$, where, if $w(k, k+1) = \alpha$, $w(k-1, k) = \frac{\alpha}{2}$.

$$\cdots \overset{\frac{1}{16}}{\text{——}} -3 \overset{\frac{1}{8}}{\text{——}} -2 \overset{\frac{1}{4}}{\text{——}} -1 \overset{\frac{1}{2}}{\text{——}} 0 \overset{1}{\text{——}} 1 \overset{2}{\text{——}} 2 \overset{4}{\text{——}} 3 \overset{8}{\text{——}} \cdots$$

Multi-Step Transition Probabilities

Given a THMC $X = X_n : n \geq 0$ with a transition matrix P, we write

$$\begin{aligned}\mathbb{P}(X_2 = v|X_0 = u) &= \sum_{w \in V} \mathbb{P}(X_2 = v, X_1 = w|X_0 = u) \\ &= \sum_{w \in V} \mathbb{P}(X_1 = w|X_0 = u)\mathbb{P}(X_2 = v|X_1 = w, \cancel{X_0 = u}) \quad \text{by Markov property} \\ &= \sum_{w \in V} p_{u,w}p_{w,v} = (P^2)_{u,v} \quad \text{or write } P^2_{u,v}\end{aligned}$$

Hence, to determine a two-step transition probability, and by extension an n-step transition probability from u to w, we consider $P^n_{u,v}$. **PROP 1.2**

PROOF. See Prop 1.1 to expand probabilities, using Bayes', as needed. We get that

$$\begin{aligned}\mathbb{P}(X_n = v|X_0 = u) &= \sum_{v_1,\dots,v_{n-1} \in V} \mathbb{P}(X_1 = v_1, \dots, X_{n-1} = v_{n-1}, X_n = v|X_0 = u) \\ &= \sum_{v_1,\dots,v_{n-1} \in V} p_{u,v_1}p(v_1, v_2)\cdots p(v_{n-1}, v) = (P^n)_{u,v} \qquad \square\end{aligned}$$

Thus, if P is a stochastic matrix, then so is P^n. **PROP 1.3**

PROOF. $\sum_{v \in V} P^n_{u,v} = \sum_{v \in V} \mathbb{P}(X_n = v|X_0 = u) = 1$. $\square$

Theorem 1.1 Markov Property

If $X_n : n \geq 0$ is a THMC with state space V, then for all $u_0, \dots, u_{n-1}, u, v \in V$,

$$\mathbb{P}(X_{n+m} = v|X_0 = u_0, \dots, X_{n-1} = u_{n-1}, X_n = u) = \mathbb{P}(X_{n+m} = v|X_n = u) = P^m_{u,v}$$

PROOF. One shows this by combining the Markov property with Prop 1.2 via induction. $\square$

Somewhat nonsensically, we *also* call this the Markov property. When talking about THMCs, this will be the default notion.

We say that a Markov chain has an *initial distribution* $\alpha = (\alpha_v : v \in V)$ if $\mathbb{P}(X_0 = v) = \alpha_v$ for each $v \in V$. If this is the case, we often write α as a subscript of our state probabilities. For instance, DEF 1.7

$$\mathbb{P}_\alpha(X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u, X_n = v) = \sum_{u \in V} \mathbb{P}_\alpha(X_0 = u)\mathbb{P}_\alpha(X_n = v|X_0 = u) = \sum_{u \in V} \alpha_u P^n_{u,v}$$

For any event E depending only on $X_0, \dots, X_n$, with $\mathbb{P}(X_n = u, E) > 0$, we have **PROP 1.4**

$$\mathbb{P}(X_{n+m} = v|X_n = u, E) = P^m_{u,v}$$

PROOF. For any such event E, we can determine whether E occurs exactly when we know the realized values u_i of X_i for $i = 1, \ldots n-1$. Hence, we may write $\mathcal{S}$ to be the set of tuples $(u_0, \ldots, u_{n-1})$ that guarantee E. It follows that

$$\mathbb{P}(X_n = u, E) = \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(\mathbf{x} = \mathbf{s}, X_n = u)$$

Similarly, we have

$$\begin{aligned}\mathbb{P}(X_{n+m} = v, X_n = u, E) &= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v, X_n = u, \mathbf{x} = \mathbf{s}) \\ &= \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_{n+m} = v | X_n = u, \mathbf{x} = \mathbf{s}) \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) \\ &= P^m_{u,v} \sum_{\mathbf{s} \in \mathcal{S}} \mathbb{P}(X_n = u, \mathbf{x} = \mathbf{s}) = P^m_{u,v} \mathbb{P}(X_n = u, E)\end{aligned}$$

Divide and use Bayes, and the result follows. □

PROP 1.5 If X is a THMC with transition matrix P, then, for all $k \geq 1$, $X_{kn} : n \geq 0$ is a THMC with transition matrix P^k.

PROOF. For any $n \neq 0$, any sequence $u_0, \ldots, u_{n+1} \in V$ satisfies

$$\mathbb{P}(X_{(n+1)k} = u_{n+1} | X_0 = u_0, X_k = u_1, \ldots, X_{nk} = u_n) = P^k_{u_n, u_{n+1}} \qquad \square$$

Theorem 1.2 Chapman-Kolmogorov

For any Markov chain X with state space V, any $m, n \geq 0$, and $u, v \in V$,

$$\mathbb{P}(X_{m+n} = v | X_0 = u) = \sum_{w \in V} \mathbb{P}(X_n = w | X_0 = u) \mathbb{P}(X_{m+n} = v | X_n = w)$$

If the X is time homogeneous, then this is $P^{n+m}_{u,v}$, which agrees with Prop 1.2.

Long Term Behavior

DEF 1.8 Recall from probability the *law of large numbers*: if $Y_n : n \geq 1$ are IID with common mean μ, then $\frac{S_n}{n} \to \mu$ in probability, where $S_n = \sum_{i=1}^n Y_i$, i.e. $\forall \varepsilon > 0$,

$$\limsup_{n \to \infty} \mathbb{P}\left(\left| \frac{S_n}{n} - \mu \right| > \varepsilon \right) = 0$$

If $Y_i \in \mathbb{Z}$ then, for $k, \ell, u_i \in \mathbb{Z}$ and $i = 1, \ldots, n-1$,

$$\begin{aligned}\mathbb{P}(S_{n+1} = \ell | S_n = k, S_i = u_i \ \forall i) &= \mathbb{P}(Y_{n+1} = \ell - k | S_n = k, S_i = u_i \ \forall i) \\ &= \mathbb{P}(Y_{n+1} = \ell - k | Y_1 = u_1 - u_0, Y_2 = u_2 - u_1, \ldots, Y_n = k - u_{n-1}) \\ &= \mathbb{P}(Y_{n+1} | \ell - k) = \mathbb{P}(Y_1 = \ell - k) = P_{k,\ell}\end{aligned}$$

where $S_n : n \geq 0$ has transition matrix P, noting that it may be viewed as a THMC.

From now on, we denote by $\mathbb{P}_v(E)$ the probability $\mathbb{P}(E|v)$.

Eg. 1.8 A general two-state chain, with states A and B, can be described by

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Let $q_n = \mathbb{P}_A(X_n = A) = \mathbb{P}(X_n = A|X_0 = A)$. Then

$$\begin{aligned} q_{n+1} &= \mathbb{P}_A(X_{n+1} = A, X_n = A) + \mathbb{P}_A(X_{n+1} = A, X_n = B) \\ &= \mathbb{P}_A(X_{n+1} = A|X_n = A)\mathbb{P}_A(X_n = A) + \mathbb{P}_A(X_{n+1} = A|X_n = B)\mathbb{P}_A(X_n = B) \\ &= (1-\alpha)q_n + \beta(1-q_n) = \beta + (1-\alpha-\beta)q_n \end{aligned}$$

This recurrence has a unique solution. In particular, one can find

$$q_n = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^n \frac{\alpha}{\alpha+\beta}$$

It follows that $q_n \to \frac{\beta}{\alpha+\beta}$, and hence $\mathbb{P}_A(X_n = B) = 1 - q_n \to \frac{\alpha}{\alpha+\beta}$. Likewise:

$$\mathbb{P}_B(X_n = B) = \frac{\alpha}{\alpha+\beta} + (1_\alpha - \beta)^n \frac{\beta}{\alpha+\beta}$$

So $\mathbb{P}_B(X_n = B) \to \frac{\alpha}{\alpha+\beta}$.

Let $\pi := (\pi_A, \pi_B) := \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$ be the distribution of our initial state X_0, Then

$$\mathbb{P}_\pi(X_1 = A) = \pi_A \mathbb{P}_A(X_1 = A) + \pi_B \mathbb{P}_B(X_1 = A) = \pi_A$$

and, similarly, $\mathbb{P}_\pi(X_1 = B) = \pi_B$. Hence, if X_0 has initial distribution π, then X_1 also has distribution π. By induction, X_n has distribution π $\forall n \geq 0$.

When we say $X = \text{Markov}(P)$, we mean that X is a THMC with transition matrix P.

A probability distribution π is called *stationary* if $\pi P = \pi$. Similarly, a probability DEF 1.9
distribution λ is called a *limiting distribution* if, for each $u, v \in V$ DEF 1.10

$$(P^n)_{u,v} \to \lambda_v \text{ as } n \to \infty$$

In other words, $\mathbb{P}_u(X_n = v) \to \lambda_v$. Note that, for any initial distribution α, we have $\alpha P^n \to \lambda$, i.e. $(\alpha P^n)_v \to \lambda_v$, where λ is limiting.

If λ is a limiting distribution for P, then λ is stationary for P. **PROP 1.6**

PROOF. Fix any initial distribution α, we have

$$\lambda = \lim_{n\to\infty}(\alpha P^n) = \lim_{n\to\infty}(\alpha P^{n-1}P) = (\lim_{n\to\infty}\alpha P^{n-1})P = \lambda P \qquad \square$$

Stationary distributions need not be unique, but limiting distributions are (as the limit $\lim_{n\to\infty}\alpha P^n$ is well-defined). In general, then, stationary distributions need not be limiting distributions.

DEF 1.11 A stochastic matrix P is called *regular* if $\exists n \geq 1$ such that all entries of P^n are positive.

When $n = 0$, $P^n = I$, which encapsulates the idea that, at timestep 0, we will be at our initial positions.

Theorem 1.3 Fundamental Theorem of Markov Chains

Every regular stochastic matrix P has a limiting distribution π.

Incorporating some of the formulations above, this is equivalent to saying: *For a regular stochastic matrix, there exists a unique distribution $\pi = (\pi_v : v \in V)$ such that $\pi P = \pi$ and $\mathbb{P}_u(X_n = v) \to \pi_v\ \forall u, v \in V$.*

In this case, there is a unique stationary distribution, and it is the unique limiting distribution.

Let $\rho = \langle 1, ..., 1\rangle$. Then note that $P\rho = \rho$, since the sum of any row in P must be 1. Hence, P has eigenvalue 1. It follows that it has a left eigenvector, i.e. $\pi : \pi P = \pi$. This is exactly a stationary distribution (as long as we scale suitably such that π is a distribution).

Classification of States

DEF 1.12 For $u, v \in V$, we say that v is *accessible* from u if $\exists n \geq 0$ such that $(P^n)_{u,v} > 0$. Equivalently, in the directed graph generated by P, there is a directed path from u to v. When v is accessible from u, we write $u \to v$.

DEF 1.13 States u and v *communicate* if $u \to v$ and $v \to u$. When u and v communicate, we write $u \leftrightarrow v$. Observe that communication is a equivalence relation. Hence, the state space V can be written as a disjoint union of mutually-communicating states, called a

DEF 1.14 *communication class*. Note that, in the directed graph generated by P, these correspond to the strongly connected components.

DEF 1.15 We say that P is *irreducible* if there is only one communication class.

Clearly, if P is regular, then it is irreducible

PROP 1.7 $u \to v \iff \mathbb{P}_u(T_{\{v\}} < \infty) > 0$.

DEF 1.16 The *period* of a state $u \in V$ is

$$d(u) := \gcd(n > 0 : P^n_{u,u} > 0)$$

DEF 1.17 If $d(u) = 1$, we call u *aperiodic*. By extension, P is aperiodic if $d(u) = 1\ \forall u \in V$, and X is

PROP 1.8 aperiodic if $X = \text{Markov}(P)$ for P aperiodic. If $u \leftrightarrow v$, then $d(u) = d(v)$.

PROOF. Let $I = \{n > 0 : P^n_{u,u} > 0\}$, and similarly J for v. Hence, $d(u) = \gcd(I)$ and $d(v) = \gcd(J)$.

Let $a, b > 0$ such that $P^a_{u,v} > 0$ and $P^b_{v,u} > 0$. Then

$$P^{a+b}_{u,u} \geq P^a_{u,v} P^b_{v,u} > 0$$

$\implies a + b \in I$, so $d(u)|a + b$. Now, if $n \in J$, then

$$P^{a+b+n}_{u,u} \geq P^a_{u,v} P^n_{v,v} P^b_{v,u} > 0$$

$\implies a + b + n \in I$, so $d(u)|n + a + b$. But, by the previous line, $d(u)|n$. Since $n \in J$ is arbitrary, we can write $d(u)| \gcd(J) = d(v)$.

Symmetrically, we could conclude that $d(v)|d(u)$, so indeed $d(v) = d(u)$. □

We now have the tools to prove Thm 1.3, the fundamental theorem of Markov chains.

PF. OF 1.3

Recall the statement: every regular stochastic matrix P has a limiting distribution. □

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