ASSIGNMENT 4

MATH 251

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QUESTION 1

Consider a matrix $A \in M_{m\times n}(\mathbb{F})$. Let $L_A : \mathbb{F}^n \to \mathbb{F}^m$ be such that $L_A(v) = A \cdot v$ for any $v \in \mathbb{F}^n$. In particular, A is of the form

$$A := \begin{bmatrix} | & | & & | \\ L_A(e_1) & L_A(e_2) & \cdots & L_A(e_n) \\ | & | & & | \end{bmatrix}$$

where $\{e_i\}$ form the standard basis for $\mathbb{F}^n$. Then, $\text{rank}(A) = \text{rank}(L_A) = \dim(\text{Im}(L_A)) = \dim(\text{Span}(L_A(e_1), ..., L_A(e_n)))$. Thus, $\text{rank}(A)$ is the size of a maximally independent subset of $\{L_A(e_1), ..., L_A(e_n)\} = \{A^{(1)}, ..., A^{(n)}\}$ from above, and this is c-rank(A).

QUESTION 2

Part (a): Let $I : V \to V$ be the identity transformation, α, β be two ordered bases for V with size n, and $v \in V$. We have $v = a_1\alpha_1 + ... + a_n\alpha_n$, where $\alpha_i \in \alpha$. Let $\alpha_i = b_1^i\beta_1 + ... + b_n^i\beta_n$ be the unique representation of α_i in β, where $\beta_j \in \beta$. Then:

$$\begin{aligned} v &= a_1(b_1^1\beta_1 + ... + b_n^1\beta_n) + ... + a_n(b_1^n\beta_1 + ... + b_n^n\beta_n) \\ &= (a_1b_1^1 + ... + a_nb_1^n)\beta_1 + ... + (a_1b_n^1 + ... + a_nb_n^n)\beta_n \end{aligned}$$

Hence, $[v]_\beta$ is

$$\begin{bmatrix} a_1b_1^1 + ... + a_nb_1^n \\ \vdots \\ a_1b_n^1 + ... + a_nb_n^n \end{bmatrix} = \begin{bmatrix} | \\ a_1[\alpha_1]_\beta + ... + a_n[\alpha_n]_\beta \\ | \end{bmatrix}$$

Now, we just observe that

$$\begin{aligned} [I]_\alpha^\beta \cdot [v]_\alpha &= \begin{bmatrix} | & | & & | \\ [I(\alpha_1)]_\beta & [I(\alpha_2)]_\beta & \cdots & [I(\alpha_n)]_\beta \\ | & | & & | \end{bmatrix} \cdot [v]_\alpha \\ &= \begin{bmatrix} | \\ a_1[\alpha_1]_\beta + ... + a_n[\alpha_n]_\beta \\ | \end{bmatrix} \qquad \text{since } I(\alpha_i) = \alpha_i \\ &= [v]_\beta \quad \text{from above} \end{aligned}$$

Part (b): We already know that, for V, W, U finite-dimensional, with bases α, β, γ, respectively, and $R : V \to W$, $S : W \to U$, we have $[S \circ R]_\alpha^\gamma = [S]_\beta^\gamma[R]_\alpha^\beta$. Taking $V = W = U$, and $R = S = I$, it follows that $[I \circ I]_\alpha^\gamma = [I]_\alpha^\gamma = [I]_\beta^\gamma[I]_\alpha^\beta$.

c.f. Lecture 13

Part (c): Using (b), we have

$$[I]_\beta^\alpha[I]_\alpha^\beta = [I]_\alpha^\alpha = \begin{bmatrix} | & & | \\ [I(\alpha_1)]_\alpha & \cdots & [I(\alpha_n)]_\alpha \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ [\alpha_1]_\alpha & \cdots & [\alpha_n]_\alpha \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} = I$$

(Found two-sided inverse)

$[I]_\alpha^\beta[I]_\beta^\alpha = [I]_\beta^\beta = I$ holds identically, and so $[I]_\alpha^\beta$ is invertible, with $([I]_\alpha^\beta)^{-1} = [I]_\beta^\alpha$.

Part (d): As above, we consider $[S \circ R]_\alpha^\gamma = [S]_\beta^\gamma [R]_\alpha^\beta$, where $\gamma = \beta$, $S = T$, $R = I$, and α, β are bases of V. From this, we immediately observe that $[T]_\beta^\beta [I]_\alpha^\beta = [T \circ I]_\alpha^\beta = [T]_\alpha^\beta$. Similarly, we have $[I]_\beta^\alpha [T]_\alpha^\beta = [T]_\alpha^\alpha$. Since $([I]_\alpha^\beta)^{-1} = [I]_\beta^\alpha$ from part (c), we conclude

$$([I]_\alpha^\beta)^{-1}[T]_\beta^\beta [I]_\alpha^\beta = [T]_\alpha^\alpha \implies [T]_\beta^\beta \sim [T]_\alpha^\alpha \text{ i.e. } [T]_\beta \sim [T]_\alpha$$

Part (e): We'll first show that $Q = [I]_\alpha^\beta$ for bases $\alpha, \beta \in \mathbb{F}^n$, and generalize this to V via an isomorphism $T : \mathbb{F}^n \to V$. Let $[L_Q] = Q$, and observe

$$Q = \begin{bmatrix} | & & | \\ L_Q(e_1) & \cdots & L_Q(e_n) \\ | & & | \end{bmatrix}$$

where $e_i \in \mathrm{St_n}$. Since Q is invertible, c-rank$(Q) = \text{rank}(Q) = n = \dim(\mathbb{F}^n)$, which necessitates that $\alpha := \{L_Q(e_1), ..., L_Q(e_n)\}$ form a basis for $\mathbb{F}^n$. Let $\{e_1, ..., e_n\} =: \beta$.

$$[I]_\alpha^\beta = \begin{bmatrix} | & & | \\ [I(L_Q(e_1))]_\beta & \cdots & [I(L_Q(e_n))]_\beta \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ [L_Q(e_1)]_\beta & \cdots & [L_Q(e_n)]_\beta \\ | & & | \end{bmatrix}$$

Recall that $[v]_\beta$, where β is the standard basis, is simply v, so we conclude that

$$[I]_\alpha^\beta = \begin{bmatrix} | & & | \\ L_Q(e_1) & \cdots & L_Q(e_n) \\ | & & | \end{bmatrix} = Q$$

Since V and $\mathbb{F}^n$ have dimension n, they are isomorphic, so consider some isomorphism $T : \mathbb{F}^n \to V$. Since T is bijective, $T(\alpha)$ and $T(\beta)$ are bases for V.

$$[I]_{T(\alpha)}^{T(\beta)} = \begin{bmatrix} | & & | \\ [T(\alpha_1)]_{T(\beta)} & \cdots & [T(\alpha_n)]_{T(\beta)} \\ | & & | \end{bmatrix} \qquad \alpha_i = L_Q(e_i)$$

Then $T(\alpha_i) = a_1 T(e_1) + ... + a_n T(e_n) \implies \alpha_i = a_1 e_1 + ... + a_n e_n$ by linearity and injectivity of T. We conclude that $[T(\alpha_i)]_{T(\beta)} = [\alpha_i]_\beta$, and in particular $[T(L_Q(e_i))]_{T(\beta)} = [L_Q(e_i)]_\beta = L_Q(e_i)$, i.e. $Q = [I]_\alpha^\beta = [I]_{T(\alpha)}^{T(\beta)}$ as desired.

QUESTION 3

Part (a): We have $\operatorname{tr}(cA+B) = \sum_{i=1}^{n}(ca_{ii}+b_{ii}) = c\sum_{i=1}^{n}a_{ii} + \sum_{i=1}^{n}b_{ii} = c\operatorname{tr}(A)+\operatorname{tr}(B)$, $c \in \mathbb{F}$, where we note that the diagonal elements of $cA+B$ are $ca_{ii}+b_{ii}$.

Part (b): Observe the following picture

$$AB = \begin{bmatrix} \text{——} A_{(1)} \text{——} \\ \vdots \\ \text{——} A_{(n)} \text{——} \end{bmatrix} \begin{bmatrix} | & & | \\ B^{(1)} & \cdots & B^{(n)} \\ | & & | \end{bmatrix}$$

to conclude that the diagonal elements of AB are $A_{(i)}B^{(i)}$ (i.e. dot product). Then:

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} A_{(i)}B^{(i)} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n} B_{(j)}A^{(j)} = \operatorname{tr}(BA)$$

as desired.

Part (c): Since $A \sim B$, $Q^{-1}AQ = B$ for some $Q \in GL_n(\mathbb{F})$. Then $\operatorname{tr}(Q^{-1}AQ) = \operatorname{tr}(B)$, but from (b) we have $\operatorname{tr}(\underbrace{Q^{-1}A}\,\underbrace{Q}) = \operatorname{tr}(\underbrace{Q}\,\underbrace{Q^{-1}A}) = \operatorname{tr}(A)$, so we find $\operatorname{tr}(A) = \operatorname{tr}(B)$.

QUESTION 4

Let V be a vector space, $U \subseteq V$, and V/U be the quotient ring, i.e. $\{v+U : v \in V\} = \{\overline{v} : v \in V\}$. We've seen previously that $\overline{v_1 + v_2} = \overline{v_1} + \overline{v_2}$ is a well-defined notion. Consider $T : (V/U)^* \to U^\perp$, where $T(f)(v) = f(v+U) = f(\overline{v})$, and $f : V/U \to \mathbb{F}$ linear. We observe that, indeed, $T(f) \in U^\perp$: for $u \in U$, $T(f)(u) = f(\overline{u}) = f(\overline{0}) = 0$.

T is linear: $T(af+g)(v) = (af+g)(\overline{v})$. Since f, g are linear, this is $af(\overline{v}) + g(\overline{v}) = aT(f)(v) + T(g)(v)$ for all $v \in V$, $a \in \mathbb{F}$, and $f, g \in (V/U)^*$.

T is injective: Suppose $T(f)(v) = T(g)(v)\ \forall v \in V$. Then $f(\overline{v}) = g(\overline{v})$ for all cosets $\overline{v} \in V/U$, so in particular $f = g$.

T is surjective: Consider $f \in U^\perp$ and define $g \in (V/U)^*$ such that g sends $\overline{v} \to f(v)$.

> g is well defined: let $\overline{v_1} = \overline{v_2}$. Then $g(\overline{v_1}) = f(v_1)$ and $g(\overline{v_2}) = f(v_2)$. Since $\{v_1 + u : u \in U\} = \{v_2 + u : u \in U\}$, $\{f(v_1) + f(u) : u \in U\} = \{f(v_2) + f(u) : u \in U\}$ by linearity of f. But $f(u) = 0\ \forall u \in U$, so we conclude that $f(v_1) = f(v_2)$.
>
> g is linear: $g(c\overline{v_1} + \overline{v_2}) = g(\overline{cv_1 + v_2}) = f(cv_1 + v_2) = cf(v_1) + f(v_2) = cg(\overline{v_1}) + g(\overline{v_2})$ by linearity of f.

Then $T(g)(v) = g(\overline{v}) = f(v)\ \forall v \in V$, so $T(g) = f$, as desired.

T is an isomorphism between $(V/U)^*$ and $U^\perp$, so $(V/U)^* \cong U^\perp$.

Question 5

Proof follows from tutorial

We've seen previously that $\mathrm{Im}(T^t) \subseteq (\ker(T))^\perp$. Let $f \in (\ker(T))^\perp$. We define $g : W \to \mathbb{F}$ that takes $w \to f(x)$, where $w = T(x) + y$ uniquely. We see immediately that $T^t(g)(v) = g(T(v)) = f(v)$, as desired, but we still need to show $W = \mathrm{Im}(T) \oplus \mathrm{Span}(\gamma)$ for some set γ, and that g is linear.

Let β be a basis for $\mathrm{Im}(T)$. This is linearly independent, so we can extend it to a basis $\beta \cup \gamma$ for W. Clearly $\mathrm{Span}(\beta) \cap \mathrm{Span}(\gamma) = \{\mathbb{0}\}$, else $\beta \cup \gamma$ would lose independence. Since $\beta \cup \gamma$ form a basis, we can write $w = a_1\beta_1 + ... + a_n\beta_n + b_1\gamma_1 + ... + b_m\gamma_m$ $\forall w \in W$ with $a_i, b_i \in \mathbb{F}$ i.e. $x + y$ with $x \in \mathrm{Span}(\beta)$ and $y \in \mathrm{Span}(\gamma)$. Since $\mathrm{Span}(\beta) = \mathrm{Im}(T)$, we can conclude $W = \mathrm{Im}(T) \oplus \mathrm{Span}(\gamma)$.

Note that g is well defined: let $w_1 = w_2$. Then $w_1 = w_2 = T(x) + y$ for some $y \in \mathrm{Span}(\gamma)$, $x \in V$, and thus $g(w_1) = g(w_2) = f(x)$.

Let g be defined as above. Let w_1 and w_2 be expressed as $T(x_1) + y_1$ and $T(x_2) + y_2$, respectively, where $y_1, y_2 \in \mathrm{Span}(\gamma)$. Then $aw_1 + w_2 = aT(x_1) + T(x_2) + ay_1 + y_2 = T(ax_1 + x_2) + y_1 + y_2$, so we conclude $g(aw_1 + w_2) = f(ax_1 + x_2)$, and since $f \in V^*$, this breaks out linearly to $af(x_1) + f(x_2) = ag(w_1) + g(w_2)$.

$\implies g \in W^*$ and $T^t(g)(v) = f(v)$ as shown $\implies f \in \mathrm{Im}(T^t)$

Then $(\ker(T))^\perp \subseteq \mathrm{Im}(T^t) \implies (\ker(T))^\perp = \mathrm{Im}(T^t)$

QUESTION 6

Part (a): We've shown that all $A \in GL_n(\mathbb{F})$ can be written as a product of elementary matrices, so write

$$A = E_1 \cdot ... \cdot E_k \implies E_k^{-1} \cdot ... \cdot E_1^{-1} A = I_n \quad \text{AND} \quad E_1 \cdot ... \cdot E_k A^{-1} = I_n$$

Combining, we find $E_k^{-1} \cdot .... \cdot E_1^{-1} A = E_1 \cdot ... \cdot E_k A^{-1} \implies E_k^{-1} \cdot ... \cdot E_1^{-1} E_k^{-1} \cdot ... \cdot E_1^{-1} A = A^{-1}$, i.e. $EA = A^{-1}$, where E is a product of elementary matrices.

Part (b): We know $A = E_1 \cdot ... \cdot E_k$, i.e. $A^{-1} = E_k^{-1} \cdot ... \cdot E_1^{-1}$. Then $E_k^{-1} \cdot ... \cdot E_1^{-1}(A|I_n) = (E_k^{-1} \cdot ... \cdot E_1^{-1} A | E_k^{-1} \cdot ... \cdot E_1^{-1}) = (A^{-1}A|E_k^{-1} \cdot ... \cdot E_1^{-1}) = (I_n|A^{-1})$, as desired. Note that, since we multiplied on the left, we are effectively performing row operations.

Part (c): Suppose E is a product of elementary operations s.t. $E(A|I) = (I|B)$. Then $(EA|E) = (I|B)$, so $EA = I$ and $E = B \implies E = IA^{-1} \implies B = A^{-1}$.

QUESTION 7

Part (a): Observe that $c_1 A^{(1)} + ... + c_n A^{(n)} = \vec{0} \iff A_{(i)}\vec{c} = 0 \ \forall 1 \le i \le n$, where $\vec{c}$ is the column vector of our coefficients $c_i \in \mathbb{F}$. If we (i) swap any two $A_{(i)} \leftrightarrow A_{(j)}$, both rows will still satisfy $A_{(i)}\vec{c} = 0$, as before. If we (ii) multiply row i by some scalar α, then $\alpha A_{(i)}\vec{c} = \alpha \cdot 0 = 0$ as before. Lastly (iii), $[A_{(i)} - \alpha A_{(j)}]\vec{c} = A_{(i)}\vec{c} - \alpha A_{(j)}\vec{c} = 0 - 0 = 0$, so all elementary row operations E will satisfy $(EA)_{(i)}\vec{c} = 0$, and thus

$$c_1 A^{(1)} + ... + c_n A^{(n)} = 0 \implies c_1 (EA)^{(1)} + ... + c_1 (EA)^{(n)} = 0 \implies c_1 B^{(1)} + ... + c_1 B^{(n)} = 0$$

If we instead assume $c_1 B^{(1)} + ... + c_n B^{(n)} = 0$, notice that $EA = B \implies E^{-1}B = A$, i.e. A is obtained from B via some *other* row operation, and the proof is identical.

Part (b): Let $EA = B$. We can apply (a) as follows

$$c_1 A^{(j_1)} + ... + c_k A^{(j_k)} = 0 \iff c_1 B^{(j_1)} + ... + c_k B^{(j_k)} = 0 \qquad j_i \in J$$

as the matrix $\{A^{(j_i)} : j_i \in J\}$ is obtained from $\{B^{(j_i)} : j_i \in J\}$ via the same row operation E. Thus, if the columns of A are linearly independent over an index J, all $c_i = 0$, and thus B is linearly independent over J as well, and vice versa.

Now suppose $A^{(m)} \in \mathrm{Span}(A^{(j_i)} : j_i \in J)$, where $m \notin J$. Then

$$c_1 A^{(j_1)} + ... + c_m A^{(j_m)} - A^{(m)} = 0$$

where $j_i \in J$. From (a), this happens IFF

$$c_1 B^{(j_1)} + ... + c_m B^{(j_m)} - B^{(m)} = 0 \implies B^{(m)} \in \mathrm{Span}(B^{(j_i)} : j_i \in J)$$

Once again, we can apply (a) because we can obtain $\{B^{(i)} : i \in J \cup \{m\}\}$ from $\{A^{(i)} : i \in J \cup \{m\}\}$ via E.

If we start by taking $B^{(m)} \in \mathrm{Span}(B^{(j_i)} : j_i \in J)$, the ($\Longleftarrow$) direction follows identically. Also note that, had we considered $m \in J$, then we'd have nothing to show, as $A^{(m)} = 1 \cdot A^{(m)}$ and $B^{(m)} = 1 \cdot B^{(m)}$ always.

QUESTION 8

Let $A \in M_{m\times n}(\mathbb{F})$, and B, C be two matrices in RREF, obtained from A. We know that, if $\{B^{(j)} : j \in J\}$ is the set of all pivot columns in B, it is linearly independent. We also know that all *non-pivot* columns are contained in the span of those previous, so in particular $\{B^{(j)} : j \in J\}$ is maximally independent, and $|J| = \text{c-rank}(B) = \text{rank}(B) = \text{rank}(A)$, since row operations are rank-preserving.

From (7a), $\{C^{(j)} : j \in J\}$ is linearly independent. We know $\text{rank}(C) = |J|$ as well, so in fact $\{C^{(j)} : j \in J\}$ describes all pivots of C, i.e. B and C contain the same pivot columns.

The arguments in (7) extend easily to n row operations, so if $E_B A = B$ and $E_C A = C$ for a series of row opeartions, the columns $A^{(j)} : j \in J$ are linearly independent $\iff$ $B^{(j)}$ are $\iff$ $C^{(j)}$ are.

Thus, we only need to show that the non-pivot columns are equal. We can express a non-pivot column in B, $B^{(k)}$, as a combination of columns in $\{B^{(j)} : j \in J\} = \{C^{(j)} : j \in J\} = \{e_j : j \in J\}$. From (7b), we know $B^{(k)} = c_1 B^{(j_1)} + ... + c_l B^{(j_l)} \iff C^{(k)} = c_1 C^{(j_1)} + ... + c_l C^{(j_l)}$, but $C^{j_i} = B^{j_i} = e_{j_i}$, so $B^{(k)} = C^{(k)}$.

$\implies$ B and C contain pivots in the same columns (which are of the form e_j), and have equal non-pivot columns, so we conclude $B = C$.

Question 9

Part (a): Consider $B = EA$, where E swaps row i and $i+1$. Then

$$A' := \begin{pmatrix} \text{——} & A_{(1)} & \text{——} \\ & \vdots & \\ \text{——} & A_{(i)} + A_{(i+1)} & \text{——} \\ \text{——} & A_{(i+1)} + A_{(i)} & \text{——} \\ & \vdots & \\ \text{——} & A_{(n)} & \text{——} \end{pmatrix} \implies \delta(A') = 0$$

since rows $A'_{(i)}$ and $A'_{(i+1)}$ are equivalent. But also

$$\delta(A') = \delta\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(i)} + A_{(i+1)} \\ A_{(i+1)} + A_{(i)} \\ \vdots \\ A_{(n)} \end{pmatrix} = \delta\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(i)} \\ A_{(i+1)} + A_{(i)} \\ \vdots \\ A_{(n)} \end{pmatrix} + \delta\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(i+1)} \\ A_{(i+1)} + A_{(i)} \\ \vdots \\ A_{(n)} \end{pmatrix}$$

$$= \delta\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(i)} \\ A_{(i+1)} \\ \vdots \\ A_{(n)} \end{pmatrix} + \delta\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(i+1)} \\ A_{(i)} \\ \vdots \\ A_{(n)} \end{pmatrix} + \delta\cancelto{0}{\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(i+1)} \\ A_{(i+1)} \\ \vdots \\ A_{(n)} \end{pmatrix}} + \delta\cancelto{0}{\begin{pmatrix} A_{(1)} \\ \vdots \\ A_{(i)} \\ A_{(i)} \\ \vdots \\ A_{(n)} \end{pmatrix}}$$

$$= \delta(A) + \delta(B)$$

Thus, $\delta(A) + \delta(B) = 0 \implies \delta(A) = -\delta(B)$

Part (b): Let A be such that $A_{(i)} = A_{(j)}$ for $j > i$. Denote by E_k the elementary operation which swaps the rows k and $k+1$. When we let $E_{j-1}A =: A'$, we get that $A'_{(i)} = A'_{(j-1)}$, and subsequently $E_k \cdot \ldots \cdot E_{j-2}E_{j-1}A =: A' \implies A'_{(i)} = A'_{(k)}$.

Thus, we set $E_{i+1} \cdot \ldots \cdot E_{j-2}E_{j-1}A =: A'$ to yield a matrix A' with $A'_{(i)} = A'_{(i+1)}$. We conclude that $\delta(A') = 0$. But also notice that

$$\delta(A) = -\delta(E_{j-1}A) = \delta(E_{j-2}E_{j-1}A) = \ldots = (-1)^{j-i-1}\delta(E_{i+1} \cdot \ldots \cdot E_{j-2}E_{j-1}A)$$

by the result of (a). Thus, $\delta(A) = (-1)^{j-i-1}\delta(E_{i+1} \cdot \ldots \cdot E_{j-1}A) = \delta(A') = 0$.

QUESTION 10

Note the following:

$$\sum_{i,j,k}^{3} \delta \begin{pmatrix} e_i \\ e_j \\ e_k \end{pmatrix} = \sum_{\substack{i,j,k: \\ i \neq j \neq k \neq i}}^{3} \delta \begin{pmatrix} e_i \\ e_j \\ e_k \end{pmatrix}$$

as δ is alternating, so matrices with $i = j$, $j = k$, or $i = k$ may be removed. Note also that the ordered set $\{i, j, k \in [1, 3] : i \neq j \neq k \neq i\}$ is precisely the ordered set $\{\pi(1), \pi(2), \pi(3) : \pi \in S_3\}$, so in particular we can write

$$\sum_{\substack{i,j,k: \\ i \neq j \neq k \neq i}}^{3} \delta \begin{pmatrix} e_i \\ e_j \\ e_k \end{pmatrix} = \sum_{\pi \in S_3} \delta \begin{pmatrix} e_{\pi(1)} \\ e_{\pi(2)} \\ e_{\pi(3)} \end{pmatrix} = \sum_{\pi \in S_3} \delta(\pi I_3)$$

Now, we have

$$\delta(A) = \delta \begin{pmatrix} \sum_{i=1}^{3} a_{1j} e_i \\ \text{---}A_{(2)}\text{---} \\ \text{---}A_{(3)}\text{---} \end{pmatrix} = \sum_{i=1}^{3} a_{1i} \delta \begin{pmatrix} e_i \\ \text{---}A_{(2)}\text{---} \\ \text{---}A_{(3)}\text{---} \end{pmatrix} \quad \text{by multi-linearity}$$

$$= \sum_{i=1}^{3} a_{1i} \delta \begin{pmatrix} e_i \\ \sum_{j=1}^{3} a_{2j} e_j \\ \text{---}A_{(3)}\text{---} \end{pmatrix} = \sum_{i=1}^{3} a_{1i} \sum_{j=1}^{3} a_{2j} \delta \begin{pmatrix} e_i \\ e_j \\ \text{---}A_{(3)}\text{---} \end{pmatrix}$$

$$= \sum_{i=1}^{3} a_{1i} \sum_{j=1}^{3} a_{2j} \delta \begin{pmatrix} e_i \\ e_j \\ \sum_{k=1}^{3} a_{3k} e_k \end{pmatrix} = \sum_{i=1}^{3} a_{1i} \sum_{j=1}^{3} a_{2j} \sum_{k=1}^{3} a_{3k} \delta \begin{pmatrix} e_i \\ e_j \\ e_k \end{pmatrix}$$

$$= \sum_{i,j,k}^{3} a_{1i} a_{2j} a_{3k} \delta \begin{pmatrix} e_i \\ e_j \\ e_k \end{pmatrix} = \sum_{\pi \in S_3} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \delta(\pi I_3) \quad \text{by the observations made above}$$

This generalizes easily to the n case, with slightly more notation and trust.