

---

# VECTOR CALCULUS NOTES

NICHOLAS HAYEK

*Lectures by Prof. Jean Pierre Mutanguha*

---

## CONTENTS

<b>I</b>	<b>Curves and Surfaces</b>	<b>1</b>
	Products on Vector Spaces	1
	Lines	1
	Planes	2
	Transformations and Parameterizations	2
	Differentiation and Continuity	4
	Surfaces	5
	Midterm Review	12
<b>II</b>	<b>Integration</b>	<b>14</b>
	Riemann Integration	14
	On Hypercubes . . . . .	14
	Point-Set Topology . . . . .	14
	On Arbitrary $\mathbb{R}^n$ Subsets . . . . .	15
	Orthonormal Curvilinear System	20
	Path Integrals	20
	Vector Fields	20
	Differential Forms . . . . .	22

# I Curves and Surfaces

## PRODUCTS ON VECTOR SPACES

Recall the definition of the *inner product* over a vector space  $V$ :

DEF 1.1

1.  $\langle u, v \rangle = \overline{\langle v, u \rangle} = \langle v, u \rangle$  in  $\mathbb{R}$  (where we'll be in this class)
2.  $\langle au + bw, v \rangle = a \langle u, v \rangle + b \langle w, v \rangle$
3.  $\langle u, u \rangle \geq 0$ , and  $= 0 \iff u = \mathbf{0}$

From this, we define the *norm* of  $u \in V$  to be  $\|u\| := \sqrt{\langle u, u \rangle}$ . This is well-defined, since  $\langle u, u \rangle \geq 0$ .

DEF 1.2

$$\forall u, v \in V, |\langle u, v \rangle| \leq \|u\| \|v\|$$

PROP 1.1

Cauchy-Schwartz Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

PROP 1.2

Triangle Inequality

The *cross product* of  $u, v \in \mathbb{R}^3$ , with respect to  $\mathbb{R}^3$ , is the determinate of the following “matrix”:

DEF 1.3

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where  $u = \langle u_1, u_2, u_3 \rangle$  and  $v = \langle v_1, v_2, v_3 \rangle$ . We observe the following two properties of the cross product in  $\mathbb{R}^3$ :

PROP 1.3

1.  $(u \times v) \cdot u = 0$
2.  $\|u \times v\| = \|u\| \|v\| \sin(\theta)$ , where  $\theta$  is the angle found between  $u$  and  $v$ . A conceptualization of this property is that “ $u$ -cross- $v$  is equal to the area created by the parallelogram bounded by  $u$  and  $v$ .”

## LINES

Define a *line*  $l(t) \in \mathbb{R}^n$  to be a function from  $\mathbb{R} \rightarrow \mathbb{R}^n$ , with the primary form  $l(t) = P + td$ , with  $P, d \in \mathbb{R}^n$ ,  $t \in \mathbb{R}$ . We call  $P$  the “point vector” and  $d$  the “direction vector”. An alternate form, with two points  $P, Q \in \mathbb{R}^n$ , would be  $l(t) = (1-t)P + tQ$ , where  $l(t)$  lies along the path between  $P$  and  $Q$  for  $t \in [0, 1]$ .

DEF 1.4

**Distance between a point and line** Using this definition, how can we find the shortest path between a point  $R$  and a line  $l(t)$ , which lies between  $P$  and  $Q$ ?

*Idea 1* We know the desired vector  $w = PR \sin(\theta)$ , the angle between  $PR$  and  $PQ$ . To find this value, note that  $\|PR \times PQ\| = \|PR\| \|PQ\| \sin(\theta)$ .

*Idea 2* We can project  $R$  onto  $PQ$ , and then subtract this projection from  $PR$ .

*Idea 3* We can minimize a distance function between  $R$  and a point on  $l$ , i.e.  $l(t)$ . Thus, we take  $\min_{t \in \mathbb{R}} \|R - l(t)\| = \alpha$ , and then take  $RI(\alpha)$  to be the shortest path.

*Idea 4* We can find when  $(R - l(t)) \cdot d = 0$ .

Sometimes called “skew lines”

**Distance between 2 lines** Consider two lines,  $l_1$  and  $l_2$ , which do not intersect but are not necessarily parallel. What is the minimal distance between  $l_1$  and  $l_2$ ?

*Idea 0* Conceptualize this problem as finding the distance between the parallel planes defined by  $\{l_1, l_2\}$ .

*Idea 1* We can minimize  $\|l_1(t) - l_2(s)\|$  (really, one should minimize the square to make one’s life easier).

*Idea 2* Pick any two points, say  $l_1(T)$  and  $l_2(S)$ , and project  $l_1(T)l_2(S)$  onto  $l_1 \times l_2$ .

*Idea 3* Minimize  $\text{dist}(l_1(t), l_2)$  for fixed  $t$ .

*Idea 4* Find  $t$  and  $s$  such that  $[l_1(t) - l_2(s)] \cdot \vec{d}_1 = 0$  and  $[l_1(t) - l_2(s)] \cdot \vec{d}_2 = 0$

PROP 1.4

$\|u \times v\| = \|u\| \|v\| \sin(\theta) = \text{Area of parallelogram defined by } u \text{ and } v.$

## PLANES

DEF 1.5

A plane  $r(s, t)$  is a function  $[0, 1]^2 \rightarrow \mathbb{R}^3$  defined by  $d_1, d_2 \in \mathbb{R}^3$ , two vectors, and  $P \in \mathbb{R}^3$ , a point. In particular,  $r(s, t) = P + s\vec{d}_1 + t\vec{d}_2$ . This is called the *parametric form*.

DEF 1.6

The *point-normal* form is a function  $\mathbb{R}^2 \rightarrow \mathbb{R}^3$  is given by  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ , where  $\vec{n} = \langle a, b, c \rangle$  is a vector normal to the plane, and  $P = \langle x_0, y_0, z_0 \rangle$  is a point lying on the plane.

**Distance between a point  $R$  and a plane  $r$**

*Idea 1* Minimize  $\|R - r(s, t)\|$  (or the square)

*Idea 2*  $\|\text{proj}_{\vec{n}}(P - R)\|$ , where  $\vec{n}$  and  $P$  are as given in the point-normal form.

## TRANSFORMATIONS AND PARAMETERIZATIONS

The following table give general examples of linear transformations  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

Dimension	Linear	Affine
$n = 0$	$\lambda(0) = 0$	$\lambda(0) = P$
$n = 1$	$\lambda(t) = t\vec{d}$	$\lambda(t) = P + t\vec{d}$
$n = 2$	$\lambda(t, s) = t\vec{d}_1 + s\vec{d}_2$	$\lambda(t, s) = P + t\vec{d}_1 + s\vec{d}_2$
$n = 3$	$\lambda(t, s, r) = t\vec{d}_1 + s\vec{d}_2 + r\vec{d}_3$	$\lambda(t, s, r) = P + t\vec{d}_1 + s\vec{d}_2 + r\vec{d}_3$

We also define the following important curves in  $\mathbb{R}^2$ :

Type	Explicit Form	Parametric Form
Ellipse	$x^2 + y^2 = 1$	$r(t) = \langle t, \sqrt{1-t^2} \rangle_{t \in [-1,1]} = \langle \cos(t), \sin(t) \rangle_{t \in [-\pi, \pi]}$
Hyperbola	$x^2 - y^2 = 1$	$r(t) = \langle \sqrt{1+t^2}, t \rangle_{t \in \mathbb{R}} = \langle \cosh(t), \sinh(t) \rangle_{t \in \mathbb{R}}$
Parabola	$x = y^2$	
Double Cone	$x^2 = y^2$	
Any Function	$y = F(x)$	$r(t) = \langle t, F(t) \rangle$

Define a *path* in  $\mathbb{R}^m$  to be a continuous function  $r : \mathbb{R} \rightarrow \mathbb{R}^m$ , e.g.  $[a, b] \rightarrow \mathbb{R}^m$ . DEF 1.7

Define a *curve* in  $\mathbb{R}^m$  to be the image of a path (i.e. a set of points in  $\mathbb{R}^m$ ). Recall the statement “paths parameterize curves.” DEF 1.8

For example, the unit circle  $x^2 + y^2 = 1$  is parameterized by the path  $r : \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $r(t) = \langle \cos(t), \sin(t) \rangle$ .

Define the *tangent* line of  $\vec{r}$  at  $a \in \mathbb{R}$  to be an affine transformation  $l : \mathbb{R} \rightarrow \mathbb{R}^m$  satisfying the following:

1.  $l(t) = r(a) + (t - a)\vec{d} : \vec{d} \neq 0$
2.  $\lim_{t \rightarrow a} \frac{\|r(t) - l(t)\|}{|t - a|} = 0$

♠ Examples ♣

E.G. 1.1

We'll now find the derivative of the unit circle at a point  $a \in \mathbb{R}$ : we have  $r(a) = \langle \cos(a), \sin(a) \rangle$ . Thus:

$$l(t) = \langle \cos(t), \sin(t) \rangle + (t - a) \langle d_1, d_2 \rangle$$

Where  $\langle d_1, d_2 \rangle \neq 0$ . Consider now the limit in question 2:

$$\begin{aligned}
 \lim_{t \rightarrow a} \frac{\|r(t) - l(t)\|}{|t - a|} &= \lim_{t \rightarrow a} \frac{1}{|t - a|} \sqrt{(\cos(t) - \cos(a) - (t - a)d_1)^2 + (\sin(t) - \sin(a) - (t - a)d_2)^2} \\
 &= \lim_{t \rightarrow a} \sqrt{\left(\frac{\cos(t) - \cos(a)}{t - a} - d_1\right)^2 + \left(\frac{\sin(t) - \sin(a)}{t - a} - d_2\right)^2} \\
 &\stackrel{=}{=} \sqrt{(-\sin(a) - d_1)^2 + (\cos(a) - d_2)^2} = 0 \\
 &\iff d_1 = -\sin(a) \wedge d_2 = \cos(a) \\
 &\implies l(t) = \langle -\sin(a), \cos(a) \rangle \quad \square
 \end{aligned}$$

## DIFFERENTIATION AND CONTINUITY

Frequently,  $l(t)$  is referred to as the “velocity vector” of  $r(t)$ , and is notated as  $r'(t)$ . Notice that  $r'(t)$  is equivalent to the component-wise derivative of the coordinates of  $r(t)$  w.r.t.  $t$ . Formally:

DEF 1.9

Given  $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^n$ , the *derivative* of  $\vec{r}$  at  $a \in \mathbb{R}$  is a linear transformation  $\vec{\lambda} : \mathbb{R} \rightarrow \mathbb{R}^n$  satisfying

$$\lim_{t \rightarrow a} \frac{\|r(t) - r(a) - \lambda(t - a)\|}{|t - a|} = 0 \quad \text{or equivalently} \quad \lim_{h \rightarrow 0} \frac{\|r(a + h) - r(a) - \lambda(h)\|}{|h|} = 0$$

It is denoted  $D\vec{r}_a$ , and represented by the  $n \times 1$  matrix  $r'(a)$ . One may now rewrite the tangent line in the form  $l(t) = r(a) + \lambda(t - a)$ .

DEF 1.10

The *arc length* of a curve  $r(t)$  is given by

$$s = \int_a^b \|r'(t)\| dt$$

DEF 1.11

An *arc length parameterization* of  $r(t)$  is some  $t = \alpha(s)$  such that  $r(\alpha(s))$  has a unit velocity vector, i.e.  $\|r'(\alpha(s))\| = 1$ . Alternatively, one could find an expression for arc length, and then parameterize  $r(t)$  in terms of its arc length. The resultant will be equivalent.

DEF 1.12

$\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is *continuous* at  $\vec{a}$  if, for any  $\varepsilon > 0$ , we can find  $\delta > 0$  such that

$$\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon \quad \forall \vec{x} \in \mathbb{R}^n$$

E.G. 1.2

---

♠ Examples ♣

---

We'll do an arc length parameterization of a semicircle of radius 1 with its center at the origin, i.e.  $y = \sqrt{1 - x^2}$ . We get the natural parameterization  $r(t) = \langle t, \sqrt{1 - t^2} \rangle$ , where  $t \in [-1, 1]$ . We'd like to find a change of parameters  $t = \alpha(s)$  such that  $\|r(\alpha(s))\| = 1$  and  $\alpha' \geq 0$ .

$$\begin{aligned} r(\alpha(s)) &= \langle \alpha(s), \sqrt{1 - \alpha(s)^2} \rangle \\ r'(\alpha(s)) &= \left\langle \alpha'(s), \frac{1}{2}(1 - \alpha(s)^2)^{-\frac{1}{2}} \cdot (-2\alpha(s)\alpha'(s)) \right\rangle \\ &= \alpha'(s) \left\langle 1, \frac{-\alpha(s)}{\sqrt{1 - \alpha(s)^2}} \right\rangle \end{aligned}$$

$$\begin{aligned} \text{Then } 1 = \|r'(\alpha(s))\| &= \alpha'(s) \sqrt{1 + \frac{\alpha(s)^2}{1 - \alpha(s)^2}} \\ &= \frac{\alpha'(s)}{\sqrt{1 - \alpha(s)^2}} \end{aligned}$$

Integrating with respect to  $s$ , we get  $s = \arcsin(\alpha(s)) = \arcsin(t)$ . Thus,  $t = \sin(s)$ , and  $s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , and we yield the parameterization  $\langle \sin(s), \cos(s) \rangle : s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ .

## SURFACES

We note the following quadric surfaces:

Type	Explicit Form
Ellipsoid	$x^2 + y^2 + z^2 = 1$
Elliptic Hyperboloid	$x^2 + y^2 - z^2 = 1$
Elliptic Paraboloids	$x^2 + y^2 - z^2 = -1$
Hyperbolic Paraboloids	$x = y^2 - z^2$
Double Cones	$x^2 = y^2 + z^2$

A surface  $F(x, y)$  is called *differentiable* at  $(a, b)$  if there exists some linear transformation  $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that DEF 1.13

$$\lim_{(h,k) \rightarrow (0,0)} \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k \rangle\|}$$

One may represent  $\lambda(h, k) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = uh + vk$

### ♠ Examples ♣

E.G. 1.3

Let  $F(x, y) = xy$ . We consider  $F$  at  $(a, b)$ . Then

$$\begin{aligned} 0 \leq \frac{|F(a+h, b+k) - F(a, b) - \lambda(h, k)|}{\|\langle h, k \rangle\|} &= \frac{|(a+h)(b+k) - ab - (uk + vk)|}{\|\langle h, k \rangle\|} \\ &= \frac{|bh + ak + hk - uh - vk|}{\|\langle h, k \rangle\|} = \frac{|(b-u)h + (a-v)k + hk|}{\|\langle h, k \rangle\|} \\ &\leq \frac{|b-u||h|}{|h|} + \frac{|a-v||k|}{|k|} + \frac{|h||k|}{|h|} \quad \text{since } |h|, |k| \leq \|\langle h, k \rangle\| \\ &= |b-u| + |a-v| + |k| \rightarrow |b-u| + |a-v| \\ &= 0 \quad \text{when } b = u, a = v \end{aligned}$$

Thus, the desired limit is always  $\geq$  and  $\leq 0$ , so especially it is 0. Our derivative at  $(a, b)$  is then  $\lambda(x, y) = bx + ay$ .

One may also find these coefficients as the partial derivative of  $F$ , i.e.

$$\nabla F(a, b) = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle \bigg|_{(a,b)}$$

DEF 1.14

This is called the *gradient*. Similarly,  $\alpha(x, y) = F(a, b) + \lambda(x - a, y - b)$  is called the *affine approximation* at  $(a, b)$ .

PROP 1.5

Note that the converse is *false* (as a counterexample, see  $F = \sqrt{|xy|}$ )

If  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable at  $\vec{a}$ , then all partial derivatives of  $F$  at  $\vec{a}$  exist. Furthermore,  $\lambda(\vec{a}) = F'(\vec{a}) = \left[ \partial_1 F \cdots \partial_n F \right] \Big|_{\vec{a}}$ .

### 1.1 Partial Converse

If all partial derivatives of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  exist near  $\vec{a}$  and are continuous at  $\vec{a}$ , then  $F$  is differentiable at  $\vec{a}$ .

 PROOF FOR  $n = 2$ .

Let  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$  be a linear transformation defined by  $\left[ \partial_1 F \cdots \partial_n F \right] \Big|_{\vec{a}}$ . Then

$$\lambda(\vec{h}) = \sum_{i=1}^n \partial_i F(\vec{a}) h_i$$

Let  $n = 2$ . Then

$$\begin{aligned} |F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})| &= |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) + F(a_1 + h_1, a_2) \\ &\quad - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1 - \partial_2 F(\vec{a}) h_2| \\ &\leq |F(a_1 + h_1, a_2 + h_2) - F(a_1 + h_1, a_2) - \partial_2 F(\vec{a}) h_2| \\ &\quad + |F(a_1 + h_1, a_2) - F(a_1, a_2) - \partial_1 F(\vec{a}) h_1| \\ &= |\partial_2 F(\vec{c}) h_2 - \partial_2 F(\vec{a}) h_2| + |\partial_1 F(\vec{d}) h_1 - \partial_1 F(\vec{a}) h_1| \\ &\quad \text{by mean value thm.} \\ &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| |h_2| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| |h_1| \\ \frac{|F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})|}{\|\vec{h}\|} &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{\|\vec{h}\|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{\|\vec{h}\|} \\ &\leq |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| \frac{|h_2|}{|h_2|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \frac{|h_1|}{|h_1|} \\ &\quad \text{since } |h_i| < \|\vec{h}\| \\ &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \end{aligned}$$

Then, as  $\vec{h} \rightarrow 0$ ,  $\vec{c}, \vec{d} \rightarrow \vec{a}$ . Since  $F$ , is continuous, we know  $F(\vec{c}) \rightarrow F(\vec{a})$  and similarly for  $F(\vec{d})$ . Thus,

$$|\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \rightarrow |\partial_2 F(\vec{a}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{a}) - \partial_1 F(\vec{a})| = 0$$

And we conclude that the limit, as  $\leq$  and  $\geq 0$ , is 0.  $\square$

DEF 1.15

$F : \mathbb{R}^n \rightarrow \mathbb{R}$  is called  $C^1$  continuous (or *continuously differentiable*) at  $\vec{a}$  if all partial



exists near  $\vec{a}$  and are continuous at  $\vec{a}$ .

Note that the converse to our partial converse is *not* true: i.e. if  $F$  is differentiable at  $\vec{a}$ , it is not necessarily continuously differentiable at  $\vec{a}$ . Some counter examples include  $F(x, y) = |y|$  and  $F(x) = x^2 \sin(\frac{1}{x})$  s.t.  $x \neq 0$  and 0 otherwise.

We have an alternative and equivalent definition of differentiability. Let  $E$  be continuous and  $= 0$  at 0. Let  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then PROP 1.6

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + \|\vec{h}\|E(\vec{h}) \quad \forall h$$

implies differentiability.

---

♠ Examples ♣

---

E.G. 1.4

In our previous example, we prove (laboriously) that  $F(x, y) = xy$  is differentiable for all  $(a, b)$ . We can now use Thm 1.1 to show this result: the partial derivatives  $F_x = y$  and  $F_y = x$  exist and are continuous  $\forall x, y \in \mathbb{R}$ , so  $F$  is differentiable  $\forall x, y \in \mathbb{R}$ .

## 1.2 Characterization of the Derivative

Let  $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . The derivative at  $\vec{a}$  exists if:

1.  $\exists$  a linear transformation  $\vec{\lambda} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  satisfying

$$\lim_{\vec{h} \rightarrow \vec{0}} \frac{\|F(\vec{a} + \vec{h}) - F(\vec{a}) - \vec{\lambda}(\vec{h})\|}{\|\vec{h}\|} = 0$$

2.  $\exists$  a linear transformation  $\vec{\lambda} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and a function  $E$  such that

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \vec{\lambda}(\vec{h}) + \|\vec{h}\|E(\vec{h})$$

and  $E(0) = 0$  is continuous at 0.

Such a  $\lambda$  is unique when found, and is called the derivative. We denote it by  $D\vec{F}_{\vec{a}}$ .

This follows from Def 1.12 and Thm 1.1. □

PROOF.

We may represent the partial derivatives of  $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m = \langle F_1, \dots, F_m \rangle$  using a *Jacobian* matrix, denoted  $F'(\vec{a})$ , and defined as follows: DEF 1.16

$$\begin{bmatrix} TBD \end{bmatrix}$$

PROP 1.7  
Chain Rule

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $\vec{a} \in \mathbb{R}^n$ . Let  $g : \mathbb{R}^m \rightarrow \mathbb{R}^l$  be differentiable at  $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$ . Then

$$h = g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^l \text{ is differentiable at } \vec{a}$$

and  $D\vec{h}_{\vec{a}} = D\vec{g}_{\vec{b}} \circ D\vec{f}_{\vec{a}}$ . Furthermore, their Jacobians obey

$$h'(a) = g'(b)f'(a)$$

(matrix multiplication)  
E.G. 1.5

---

♠ Examples ♣

---

1. Consider  $f(x, y) = \langle x + y, x - y \rangle$  and  $g(x, y) = \frac{1}{4}x^2 - \frac{1}{4}y^2$ . Then  $h = g \circ f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is given by

$$\frac{1}{4}(x + y)^2 - \frac{1}{4}(x - y)^2$$

Let  $\vec{a} = \langle a_1, a_2 \rangle$ . Then  $f(a) = b = \langle a_1 + a_2, a_1 - a_2 \rangle$ . What about the Jacobian of  $f$ ?

$$f'(a) = \left[ \begin{array}{cc} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{array} \right] \Big|_{(a_1, a_2)} = \left[ \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right]$$

Similarly, for  $g$  we have

$$g'(b) = \left[ \partial_1 g \quad \partial_2 g \right] \Big|_{(a_1 + a_2, a_1 - a_2)} = \left[ \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1 - a_2) \right]$$

Then, by the chain rule, we multiple these two matrices to yield

$$\left[ \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1 - a_2) \right] \cdot \left[ \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right] = \left[ a_2 \quad a_1 \right]$$

One can (less) manually find that  $h = g \circ f$  is  $xy$ , and conclude the same.

2. Let  $S$  be a surface in  $\mathbb{R}^3$  given by  $F(x, y, z) = 0$  (this is called a “level surface,” e.g.  $xy - z = 0$ ). Let  $P = (a, b, c)$  be a point on  $F$ , and let  $C$  be a curve in  $S$  containing  $P$ , parameterized by  $r(t)$ .

Denote  $r(t) = \langle x(t), y(t), z(t) \rangle$ . Then  $g = F \circ r = F(x(t), y(t), z(t)) = 0$ . By chain rule, we have  $0 = g'(t_0) = F'(P) \cdot r'(t_0)$ , where we choose  $t_0$  such that  $r(t_0) = \langle a, b, c \rangle$ . Then, we observe that

$$0 = \nabla F(P) \cdot \vec{v}(t_0) \implies \nabla F(P) \perp \vec{v}(t_0)$$

Where  $\vec{v} = r'$  is the velocity vector of  $r$ . By considering all curves that satisfy our construction  $C \subset S$ , we yield the tangent plane of  $S$  at  $P$  with normal vector  $\vec{n} = \nabla F(P)$ . In particular, the point-normal form of the tangent plane of a surface  $F$  at  $P = (a, b, c)$  is given by

$$\partial_x F(P)(x - a) + \partial_y F(P)(y - b) + \partial_z F(P)(z - c) = 0$$

3. Generally, we can consider  $S^{n-1} \subset \mathbb{R}^n$  of  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ . (This is called a *hypersurface*). Suppose this is differentiable at  $P \in S$ . Let  $C \subset S$  be a curve in  $S$  through  $P$ , parameterized by  $r : \mathbb{R} \rightarrow \mathbb{R}^n$  and differentiable at  $t_0$  with  $r(t_0) = P$ .

Then, by the chain rule,  $v(t_0) \perp \nabla F(P)$ . If  $v(t_0) \neq 0$ , then the tangent line to  $C$  at  $P$  has derivative  $r(t_0)$ . If  $\nabla F(P) \neq 0$ , then the tangent hyperplane to  $S$  at  $P$  has a normal vector  $n = \nabla F(P)$ .

Let  $\mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\vec{a}, \vec{h} \in \mathbb{R}^n$ . Let  $l(t) = a + th$ . Then the *directional derivative* of  $F$  along  $h$  at  $a$ , denoted  $\partial_{\vec{h}} F(\vec{a})$ , is given by

DEF 1.17

$$\lim_{t \rightarrow 0} \frac{F(a + th) - F(a)}{t}$$

Then, if  $F$  is differentiable at  $a$ , we have the more useful form

Thus, if  $h = e_1$ , then  $\partial_{e_1} F(\vec{a}) = \partial_1 F(\vec{a})$ .

$$\partial_{\vec{h}} F(\vec{a}) = \vec{h} \cdot \nabla F(\vec{a}) = \sum_{i=1}^n h_i \partial_i F(\vec{a})$$

Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable, and let  $a, h \in \mathbb{R}^n$ , with  $h \neq 0$ . Then

PROP 1.8  
Mean Value Thm.

$$F(a + h) - F(a) = \partial_{\vec{h}} F(c_h) = h \nabla F(c_h) \quad c_h \in [a, a + h]$$

Note that, since  $a, h$  are vectors, by  $c_h \in [a, a + h]$  we mean that  $c_h$  lies along the line segment connecting  $a$  and  $a + h$ .

We now restate the chain rule:

### 1.3 Chain Rule

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $\vec{a}$ . Let  $g : \mathbb{R}^m \rightarrow \mathbb{R}^l$  be differentiable at  $\vec{b} = f(\vec{a})$ . Then

$$h = g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^l$$

is differentiable at  $\vec{a}$  and  $h'(\vec{a}) = g'(\vec{b}) \circ f'(\vec{a})$ .

Let  $\lambda$  be the derivative of  $f$ . Let  $\vec{t}, \vec{s}$  be arbitrary. Then we have

PROOF.

$$f(\vec{a} + \vec{t}) - f(\vec{a}) = \lambda(\vec{t}) + \|\vec{t}\| \varepsilon_1(\vec{t})$$

where  $\varepsilon_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous and  $\vec{0} @ \vec{0}$ . Similarly, for  $g$ :

$$g(\vec{b} + \vec{s}) - g(\vec{b}) = \mu(\vec{s}) + \|\vec{s}\| \varepsilon_2(\vec{s})$$

where  $\mu$  is the derivative of  $g$ , and  $\varepsilon_2$  is as above. Our goal is to write  $h = g \circ f$

in the same manner. Let  $\nu = \mu \circ \lambda$ . Then

$$\begin{aligned}
 h(\vec{a} + \vec{t}) - h(\vec{a}) &= g(f(\vec{a} + \vec{t})) - g(f(\vec{a})) \\
 &= g(f(\vec{a}) + \underbrace{\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})}_{:=\vec{s}}) - g(f(\vec{a})) \\
 &= \mu(\vec{s}) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
 &= \mu(\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
 &= \mu(\lambda(\vec{t})) + \|\vec{t}\|\mu(\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
 &= \nu(\vec{t}) + \|\vec{t}\|\underbrace{\left(\mu(\varepsilon_1(\vec{t})) + \frac{\|\vec{s}\|}{\|\vec{t}\|}\varepsilon_2(\vec{s})\right)}_{=\varepsilon_3(\vec{t})} \quad \text{if } \vec{t} \neq 0 \\
 \vec{t} \neq 0 \implies 0 \leq \|\varepsilon_3(\vec{t})\| &\leq \|\mu(\varepsilon_1(\vec{t}))\| + \frac{\|\lambda(\vec{t})\| + \|\vec{t}\|\|\varepsilon_1(\vec{t})\|}{\|\vec{t}\|}\|\varepsilon_2(\vec{s})\| \\
 &\leq M\|\varepsilon_1(\vec{t})\| + (L + \|\varepsilon_1(\vec{t})\|)\|\varepsilon_2(\vec{s})\| \\
 &\quad (\text{where } \lambda(\vec{t}) \leq L\|\vec{x}\| \text{ and } \mu(\vec{x}) \leq M\|\vec{x}\|) \\
 \implies \lim_{\vec{t} \rightarrow 0} \varepsilon_3(\vec{t}) &= 0 \quad \square
 \end{aligned}$$

DEF 1.18  
Iterated Partial Derivatives

Suppose  $g = \partial_i f$  is defined near  $\vec{a} \in \mathbb{R}^n$ , where  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then if  $\partial_j g$  exists at  $\vec{a}$ , we call it a  $2^{nd}$  order partial derivative of  $f$  at  $\vec{a}$ . We denote this  $\partial_j \partial_i f(\vec{a})$ , where  $i, j \in [1, n]$ .

#### 1.4 Mixed Partial Derivatives are Equal

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $\vec{a} = \langle a_1, a_2 \rangle$ . Let  $\partial_1 f, \partial_2 \partial_1 f$  exist near  $\vec{a}$ , with  $\partial_2 \partial_1 f$  continuous at  $\vec{a}$ . Suppose further that  $\partial f(x, a_2)$  is defined near  $x = a_1$ .

$\implies \partial_1 \partial_2 f$  is defined at  $\vec{a}$  and  $\partial_1 \partial_2 f(\vec{a}) = \partial_2 \partial_1 f(\vec{a})$ .

PROOF.

$$\begin{aligned}
 \partial_1 \partial_2 f(\vec{a}) &= \lim_{h_1 \rightarrow 0} \underbrace{\frac{\partial_2 f(a_1 + h_1, a_2) - \partial_2 f(a_1, a_2)}{h_1}}_{\beta(h_1): \mathbb{R}_{\neq 0} \rightarrow \mathbb{R}} \\
 \implies \beta(h_1) &= \frac{\lim_{h_2 \rightarrow 0} \frac{f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2)}{h_2} - \lim_{h_2 \rightarrow 0} \frac{f(a_1, a_2 + h_2) - f(a_1, a_2)}{h_2}}{h_1} \\
 &= \lim_{h_2 \rightarrow 0} \underbrace{\frac{1}{h_2} \frac{(f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2)) - (f(a_1, a_2 + h_2) - f(a_1, a_2))}{h_1}}_{\alpha(h_1, h_2): \mathbb{R}_{\neq 0}^2 \rightarrow \mathbb{R}}
 \end{aligned}$$

Now, for a break...

If  $\lim_{\vec{h} \rightarrow \vec{0}} \alpha(\vec{h})$  exists, then  $\lim_{h_1 \rightarrow 0} \beta(h_1)$  exists, where  $\beta(h_1) = \lim_{\vec{h} \setminus h_1 \rightarrow 0} \alpha(h_1, (\vec{h} \setminus h_1))$ . Furthermore, we conclude PROP 1.9

$$\lim_{h_1 \rightarrow 0} \beta(h_1) = \lim_{\vec{h} \rightarrow \vec{0}} \alpha(\vec{h})$$

Now, it's enough to show that  $\lim_{\vec{h} \rightarrow \vec{0}} \alpha(\vec{h}) = \partial_2 \partial_1 f(\vec{a})$ . By the Mean Value Thm, we have

PROOF (CONTINUED).

$$\begin{aligned} \alpha(\vec{h}) &= \frac{1}{h_2} (\partial_1 f(c_1, a_2 + h_2) - \partial_1 f(c_1, a_2)) \\ &= \partial_2 \partial_1 f(c_1, c_2) : c_2 \in [a_2, a_2 + h] \end{aligned}$$

Let  $\vec{c} = \langle c_1, c_2 \rangle$ . Then as  $\vec{h} \rightarrow \vec{0}$ , we have  $\vec{c} \rightarrow \vec{a}$ . Thus

$$\lim_{\vec{h} \rightarrow \vec{0}} \alpha(\vec{h}) = \lim_{\vec{c} \rightarrow \vec{a}} \partial_2 \partial_1 f(\vec{c}) = \partial_2 \partial_1 f(\vec{a}) \quad \square$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $k$ -times continuously differentiable at  $\vec{a}$  if all  $k^{th}$ -order partial derivatives exist near  $\vec{a}$  and are continuous at  $\vec{a}$ . DEF 1.19

We say that  $f$  is  $k$ -times continuously differentiable *near*  $\vec{a}$  if it is continuously differentiable at  $\vec{a}$  and all  $k$ -th order partial derivatives are continuous near  $\vec{a}$ .

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is twice continuously differentiable at  $\vec{a}$ , then all mixed partial derivatives are equal at  $\vec{a}$ . PROP 1.10

If  $f$  is  $k$ -times continuously differentiable at  $\vec{a}$ , then the  $(k-1)$ -order partial derivatives are continuously differentiable (hence differentiable and continuous) at  $\vec{a}$ . PROP 1.11

is the following a proof? proposition?

Let  $\vec{h} \in \mathbb{R}^n, \vec{l} : \mathbb{R} \rightarrow \mathbb{R}^n$  given by  $\vec{l}(t) = \vec{a} + t\vec{h}$ . Set  $g := f \circ \vec{l} : \mathbb{R} \rightarrow \mathbb{R}$ , i.e.  $g(t) = f(\vec{a} + t\vec{h})$ . PROOF.

Then let  $f$  be  $k$ -times continuously differentiable at  $\vec{a}$ . Then  $g$  is  $k$ -times differentiable at 0, and we have

$$\partial_{\vec{h}}^i f(\vec{a}) = g^{(i)}(0) \underset{\text{CR}}{=} (\vec{h} \cdot \nabla)^i f \Big|_{\vec{a}}$$

For example, with  $n = 2$ , we have

$$\partial_{\vec{h}}^2 = (\vec{h} \cdot \nabla)(\vec{h} \cdot \nabla) = (h_1 \partial_1 + h_2 \partial_2)(h_1 \partial_1 + h_2 \partial_2)$$



### 1.5 Multivariable Taylor's Theorem

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be  $k$ -times continuously differentiable near  $\vec{a}$  with  $\vec{a} \in \mathbb{R}^n$ . Let  $\alpha_j : \mathbb{R}^n \rightarrow \mathbb{R}$  be a degree  $j$  homogeneous polynomial, i.e. all non-zero terms have the same degree.

Let  $E : \mathbb{R}^n \rightarrow \mathbb{R}$  be such that

$$\begin{cases} \bullet f(\vec{a} + \vec{h}) - f(\vec{a}) = \alpha_1(\vec{h}) + \dots + \alpha_k(\vec{h}) + \underbrace{\|h\|^k E(\vec{h})}_{R_k(\vec{h})} \quad \forall \vec{h} \\ \bullet E(\vec{0}) = 0 \end{cases}$$

To find such an  $E$ , we can take

$$E(\vec{h}) = \begin{cases} \frac{1}{\|h\|^k} (f(\vec{a} + \vec{h}) - f(\vec{a}) - \alpha_1(\vec{h}) - \dots - \alpha_k(\vec{h})) & \vec{h} \neq 0 \\ 0 & \vec{h} = 0 \end{cases}$$

Then Taylor's Theorem states:

$$E \text{ continuous at } \vec{0} \iff \alpha_j(\vec{h}) = \frac{1}{j!} \partial_{\vec{h}}^j f(\vec{a}) = \frac{1}{j!} (\vec{h} \cdot \nabla)^j f(\vec{a}) \quad \forall j \in [1, k]$$

If  $E$  is continuous at  $\vec{a}$  and  $\vec{h} \neq \vec{0}$  is near  $\vec{0}$ , then:

$$R_{k-1}(\vec{h}) = \frac{1}{k!} \partial_{\vec{h}}^k f(\vec{c}_h)$$

where  $\vec{c} \in [\vec{a}, \vec{a} + \vec{h}]$ .

### MIDTERM REVIEW

Recall that the directional derivative is defined as follows

$$\partial_{\vec{h}} f(\vec{a}) := \lim_{t \rightarrow 0} \frac{f(\vec{a} + t\vec{h}) - f(\vec{a})}{t} = \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t} = g'(0) \quad g(t) := f(\vec{a} + t\vec{h})$$

An iterated directional derivative, denoted  $\partial_{\vec{h}}^i f(\vec{a})$ , is then

$$g^{(i)}(0)$$

If  $f$  is  $i$ -times continuously differentiable at  $\vec{a}$ , then we can write

$$\partial_{\vec{h}}^i(\vec{a}) = (\vec{h} \cdot \nabla)^i f(\vec{a})$$

## II Integration

### RIEMANN INTEGRATION

#### *On Hypercubes*

DEF 2.1 Let  $\mathcal{B}$  be a box in  $\mathbb{R}^n$ . Choose  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  which is bounded on the box. Then, informally,  $F$  is *integrable* if the limit of its Riemann summation is equivalent across all orderings of tagged partitions.

PROP 2.1 By the extreme value theorem, if  $F$  is continuous on  $\mathcal{B}$ , then  $F$  is bounded on  $\mathcal{B}$ .

#### 2.1 Integrability Criterion

If  $F$  is continuous on  $\mathcal{B}$ , then  $F$  is integrable over  $\mathcal{B}$ .

#### 2.2 Fubini

Let  $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$ . Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous on  $\mathcal{B}$ . Then

$$\int_{\mathcal{B}} F dV^n = \int_{x_n=a_n}^{x_n=b_n} \cdots \left( \int_{x_1=a_1}^{x_1=b_1} F(x_1, \dots, x_n) dx_1 \right) \cdots dx_n$$

Furthermore, the order of integration doesn't matter.

PROP 2.2  $\int_a^b g(x) dx = g(c)(b-a)$  where  $a < c < b$ .

PROOF.  $\frac{G(b)-G(a)}{b-a} = G'(c) = g(c)$  by the mean value theorem and the FTC.  $\square$

#### 2.3

The set of discontinuities of  $F$  in  $\mathcal{B}$  has zero measure  $\iff F$  is integrable over  $\mathcal{B}$ .

Note that this theorem is not useful in MATH 248, and its proof is out of the scope of this course.

#### *Point-Set Topology*

DEF 2.2 A set  $S \subseteq \mathbb{R}^n$  has *zero measure* if  $\forall \varepsilon > 0$  we can choose a set of open balls such that



$S \subseteq \bigcup B(x_i, \varepsilon_i)$  where  $\sum \text{vol}(B(x_i, \varepsilon_i)) < \varepsilon$ .

In general, hypersurfaces in  $\mathbb{R}^n$  have zero measure. Thus, if  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous except on a hypersurface,  $F$  is still integrable.

$\vec{p} \in \text{Int}(S)$  is called an *interior point* of  $S$  if  $\exists \varepsilon > 0$  such that  $B(\vec{p}, \varepsilon) \subseteq S$ .

DEF 2.3

1. If  $S \subseteq \mathbb{R}^n$  has zero measure and  $S' \subseteq S$ , then  $S'$  has zero measure.

PROP 2.3

2. If  $S \subseteq \mathbb{R}^n$  has zero measure, then  $S$  has no interior points.

Let  $S \subseteq \mathbb{R}^n$ . Then

DEF 2.4

1.  $\text{Int}(S)$ , the *interior* of  $S$ , is the set of all interior points of  $S$

2.  $S$  is called *open* if  $S = \text{Int}(S)$ .

3.  $S^c$ , the *complement* of  $S$ , is  $\mathbb{R}^n \setminus S$ .

4.  $p \in S^c$  is called an *exterior point* of  $S$  if  $\exists \varepsilon > 0$  with  $B(p, \varepsilon) \subseteq S^c$ .

5.  $\text{Ext}(S)$ , the *exterior* of  $S$ , is the set of all exterior points of  $S$ .

6.  $S$  is *closed* if  $S^c = \text{Ext}(S)$ .

7.  $p \in \mathbb{R}^n$  is called a *boundary point* of  $S$  if  $p \notin \text{Int}(S) \wedge p \notin \text{Ext}(S)$ .

8. The *boundary* of  $S$ , denoted  $\partial S$ , is the set of all boundary points of  $S$ .

9.  $S$  is *bounded* if  $\exists \mathcal{B}$  with  $S \subseteq \mathcal{B} \subsetneq \mathbb{R}^n$ .

$S$  is closed  $\iff S^c$  is open  $\iff S$  contains its boundary.

PROP 2.4

### On Arbitrary $\mathbb{R}^n$ Subsets

Let  $\mathcal{D} \subseteq \mathbb{R}^n$  be closed and bounded. Let  $f : \mathcal{D} \rightarrow \mathbb{R}^n$  be some function.  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\hat{f}(x) = \begin{cases} f(x) & x \in \mathcal{D} \\ 0 & \text{o.w.} \end{cases}$$

is called the *trivial extension* of  $f$ .

$f$  is integrable over  $\mathcal{D}$  if its trivial extension is integrable over a box  $\mathcal{B} \supseteq \mathcal{D}$ .

PROP 2.5

## 2.4

Let  $\mathcal{D} \subseteq \mathbb{R}^n$  be closed and bounded, with a boundary that has zero measure. Then, if  $f : \mathcal{D} \rightarrow \mathbb{R}$  is continuous on  $\mathcal{D}$ , then  $f$  is integrable.

PROOF.

If  $f$  is continuous on  $\mathcal{D}$ , then  $\hat{f}$  is continuous on both  $\text{Int}(\mathcal{D})$  and  $\text{Ext}(\mathcal{D})$  (for any point in either of these sets, we can find epsilon balls centered at the point and contained in the set—within these intervals  $\hat{f} = f$ ). Thus, since  $\mathcal{D} = \text{Int}(\mathcal{D}) \cup \text{Ext}(\mathcal{D}) \cup \partial D$ , the set of discontinuities of  $\hat{f}$  has at most measure 0. Hence,  $\hat{f}$  is integrable over any box containing  $\mathcal{D}$ , and hence  $f$  is integrable over  $\mathcal{D}$  by Prop 2.5.  $\square$

DEF 2.5

$\mathcal{D} \subseteq \mathbb{R}^2$  is called *y-simple* if, for  $a, b \in \mathbb{R}$  and  $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$  continuous, we may write

$$\mathcal{D} = \left\{ \begin{array}{l} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{array} \right.$$

Similarly,  $\mathcal{D}$  is *x-simple* if

$$\mathcal{D} = \left\{ \begin{array}{l} a \leq y \leq b \\ g_1(y) \leq x \leq g_2(y) \end{array} \right.$$

Note that, since  $x \in [a, b]$  is closed (hence compact),  $g_1(x)$  and  $g_2(x)$  are bounded. We reason similarly for *x-simple* domains.

DEF 2.6

$\mathcal{D} \subseteq \mathbb{R}^2$  is *elementary* if it is *y-* or *x-simple*. It is *simple* if it is both.

## 2.5 Fubini

If  $\mathcal{D} \subseteq \mathbb{R}^n$  is elementary and  $f : \mathcal{D} \rightarrow \mathbb{R}$  is continuous, then

- $\mathcal{D}$  is *y-simple*  $\implies \iint_{\mathcal{D}} f dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy dx$
- $\mathcal{D}$  is *x-simple*  $\implies \iint_{\mathcal{D}} f dA = \int_{y=a}^{y=b} \int_{x=g_1(y)}^{x=g_2(y)} f(x, y) dx dy$

E.G. 2.1

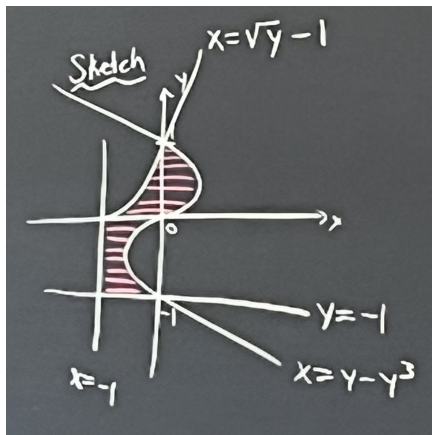
### ♠ Examples ♣

1. Consider  $\iint_{\mathcal{D}} (1 + 2y) dA$ , where  $\mathcal{D}$  is bounded by  $y = 2x^2$  and  $y = 1 + x^2$ . We first find the intersection between these two curves:  $2x^2 = 1 + x^2 \implies x = \pm 1$ .

Then, by Thm 2.5 ( $\mathcal{D}$  is  $y$ -simple), we write

$$\begin{aligned}
 \iint_{\mathcal{D}} (1+2y) dA &= \int_{x=-1}^{x=1} \int_{2x^2}^{1+x^2} (1+2y) dy dx = \int_{-1}^1 y + y^2 \Big|_{2x^2}^{1+x^2} \\
 &= \int_{-1}^1 (1+x^2) + (1+x^2)^2 - 2x^2 - 4x^4 \\
 &= \int_{-1}^1 1 + x^2 + 1 + x^4 + 2x^2 - 2x^2 - 4x^4 \\
 &= \int_{-1}^1 -3x^4 + x^2 + 2 = \left. -\frac{3}{5}x^5 + \frac{1}{3}x^3 + 2x \right|_{-1}^1 = 2\frac{-3}{5} + 2\frac{1}{3} + 4 \\
 &= 2\left(\frac{-9}{15} + \frac{5}{15} + \frac{30}{15}\right) = \frac{52}{15}
 \end{aligned}$$

2. Consider  $\iint_{\mathcal{D}} y dA$ , where  $\mathcal{D}$  is bounded by  $x = y - y^3$ ,  $x = \sqrt{y} - 1$ ,  $x = -1$ , and  $y = -1$  (OOF). By Thm 2.5 ( $y$ -simple):



We split this up into two  $x$ -simple graphs, one in  $y \in [-1, 0]$ , and one in  $y \in [0, 1]$ . Then we have  $\iint_{\mathcal{D}} = I_1 + I_2$ , with

$$I_1 = \int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y dx dy \quad I_2 = \int_{-1}^0 \int_{-1}^{y-y^3} y dx dy$$

Computing this integral a hassle. Try it yourself.

3. We may also flip the bounds of integration using Thm 2.5. For example, consider  $\int_0^3 \int_y^3 \sin(x^2) dx dy$ . This is a non-elementary integral to evaluate in  $x$ . But observe that our bounds are equivalent to  $y \in [0, x]$  and  $x \in [0, 3]$ , so we may re-write this as  $\int_0^3 \int_0^x \sin(x^2) dy dx$ .

We pick up an  $x$ , not, after integrating WRT  $y$ , so this is easy to evaluate!

DEF 2.7

A set  $S \subseteq \mathbb{R}^n$  is called *path-connected* if, for every  $a, b \in S$ , there exists a continuous mapping containing  $a$  and  $b$  (i.e., there exists a path between them).

DEF 2.8

This is distinct from elementary-ness of  $\mathcal{D} \subseteq \mathbb{R}^2$ , which we characterized by  $y$  and  $x$  simple-ness.

DEF 2.10

In  $\mathcal{D} \subseteq \mathbb{R}^n$ , we call  $\mathcal{D}$  *elementary* if it is closed, bounded, and both its interior and boundary are path-connected.

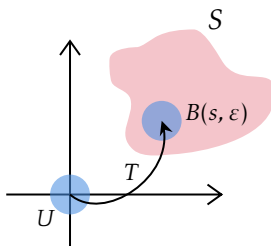
Let  $\mathcal{D}, \mathcal{D}^*$  be elementary subsets of  $\mathbb{R}^n$ . Let  $T : \mathcal{D}^* \rightarrow \mathcal{D}$ . We call  $T$  *onto*, or *surjective*, if the whole of  $\mathcal{D}$  is mapped to, i.e.  $\forall d^* \in \mathcal{D}^* \exists d \in \mathcal{D} : T(d) = d^*$ .

Using the same notation, we call  $T$  *one-to-one*, or *injective*, if no two points share a mapping, i.e.  $\forall d_1^*, d_2^* \in \mathcal{D}^*$ , we have  $T(d_1^*) = T(d_2^*) \implies d_1^* = d_2^*$ .

DEF 2.11

$S \subseteq \mathbb{R}^n$  is a *hypersurface* if,  $\forall s \in S$ ,  $\exists \varepsilon > 0$ , an open set  $U \subseteq \mathbb{R}^n$ , and a function  $T : U \rightarrow B(s, \varepsilon)$  such that

- $T$  is injective on  $\text{Int}(\mathcal{D}^*)$  and also surjective
- $T(U \cap \{s = \langle x_1, \dots, x_n \rangle : x_n = 0\}) = S \cap B(s, \varepsilon)$



## 2.6 Change of Variables

Let  $T : \mathcal{D}^* \rightarrow \mathcal{D}$  be continuously differentiable on  $\text{Int}(\mathcal{D}^*)$  (i.e. all partial derivatives exist and are continuous on  $\text{Int}(\mathcal{D}^*)$ ). Let  $T'$  be the Jacobian induced by  $T$ . Let  $F^* = F \circ T$ .

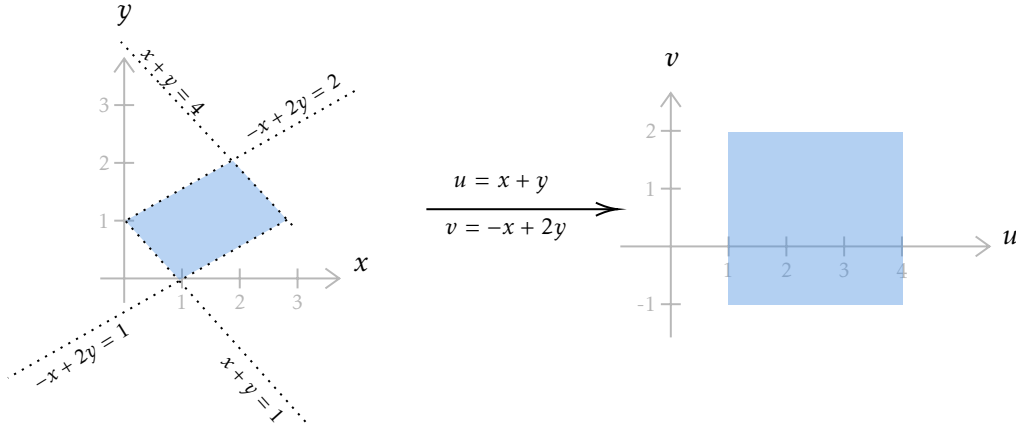
If  $F : \mathcal{D} \rightarrow \mathbb{R}$  is integrable over  $\mathcal{D}$ , then  $F^*$  is integrable over  $\mathcal{D}^*$  and

$$\int_{\mathcal{D}} F dV = \int_{\mathcal{D}^*} F^* |\det(T)| dV$$

For example, in  $n = 2$  polar coordinates,  $\int_{\mathcal{D}} F dA = \int_{\mathcal{D}^*} F^* r dA$ . For this, see that the relevant Jacobian is

$$T' = \begin{bmatrix} \partial_r x & \partial_\theta x \\ \partial_r y & \partial_\theta y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{bmatrix} \Rightarrow |\det(T')| = |r| = r$$

Consider the area of the following parallelogram:



Then,  $x = \frac{2u-v}{3}$  and  $y = \frac{u+v}{3}$ . Hence, we compute our Jacobian and conclude that  $\det(T') = \frac{1}{3}$ . However, we may also compute the determinate of the *inverse's* Jacobian, i.e.  $u = x + y$  and  $v = -x + 2y$ , which will yield 3, and invert the result.

Hence, since the area of the left rectangle is 9, we get an area of 3 for the parallelogram.

## 2.7 Mean Value Theorem in $\mathbb{R}^n$

Let  $F : \mathcal{D} \rightarrow \mathbb{R}$  be integrable over an elementary region  $\mathcal{D} \subseteq \mathbb{R}^n$ . Let  $\bar{F} := \int_{\mathcal{D}} F dV \frac{1}{\text{vol}(\mathcal{D})}$  be the mean value of  $F$ . Then

$$\exists c \in \mathcal{D} : F(c) = \bar{F}$$

Let  $\delta : \mathcal{D} \rightarrow \mathbb{R}_+$  be a density function (which is integrable). Then define  $\text{mass}(\mathcal{D}) = \int_{\mathcal{D}} \delta dV$ . Then the center of mass  $x \in \mathcal{D}$  is given by

$$x_i = \frac{\int_{\mathcal{D}} x_i \delta dV}{\text{mass}(\mathcal{D})}$$

The mean value theorem gives the fact that  $\exists c : \delta(c) = \bar{\delta}$ , where  $\bar{\delta} = \frac{\text{mass}(\mathcal{D})}{\text{vol}(\mathcal{D})}$

## ORTHONORMAL CURVILINEAR SYSTEM

DEF 2.12 Let  $T : U^* \rightarrow U$  be given by  $T(\vec{u}) = \langle x_1(\vec{u}), \dots, x_n(\vec{u}) \rangle$ . Then we call  $T$  *orthogonal* if  $\partial_j T \perp \partial_i T \ \forall i \neq j$ . We then call the Jacobian of  $T$  an *orthonormal system*. Intuitively, we have the grid-lines in  $U^*$  being mapped to (not necessarily straight) lines in  $U$ , such that all their intersections are perpendicular.

## PATH INTEGRALS

DEF 2.13  $\vec{r} : [a, b] \rightarrow \mathbb{R}^n$  is called a *regular path* if it is  $C^1$  continuous and  $\|r'(t)\| > 0 \ \forall a < t < b$

DEF 2.14  $C \subset \mathbb{R}^n$  is called a *regular curve* if it is the image of a regular path. Thus, since  $\|r'(t)\| > 0$  on  $(a, b)$ , there exists some arc length parameterization  $p : [0, l] \rightarrow \mathbb{R}^n$ .

Regular curves have zero measure, and hence zero  $n$ -dimensional volume, but we *can* measure 1-dimensional volume, i.e. length. Hence,  $\text{vol}_1(C) := \int_C 1 ds = l$ .

In practice, we have

$$\text{mass}(C) = \int_C \delta ds = \int_0^l \delta(p(s)) ds \stackrel{\text{ch. of var's}}{=} \int_a^b \delta(r(t)) \|r'(t)\| dt$$

where  $p(s)$  is an arc length parameterization.

Hence, the center of mass across a curve is going to be given by

$$x_i = \left( \int_C x_i \delta ds \right) \frac{1}{\text{mass}(C)}$$

DEF 2.15 A regular path  $r$  is *simple* if it is injective (except possibly  $r(a) = r(b)$ ).  $r$  is called *closed* if, in particular,  $r(a) = r(b)$ .

DEF 2.16 A regular curve  $C$  is simple or closed if its pre-image path  $r$  is simple or closed, respectively.

PROP 2.6 If a curve is regular, there exists a unique arc length parameterization  $p : [0, l] \rightarrow \mathbb{R}^n$  of it.

## VECTOR FIELDS

DEF 2.17 An *orientation* on a simple, regular curve  $C$  is a function  $\mu : C \rightarrow \mathbb{R}^n$  which gives the tangent vector to  $C$  at  $a$ . Two orientations are equivalent if they differ by a positive scaling function.  $C$  then has exactly two orientations ( $\mathcal{O}$  for “forwards”

and  $\bar{O}$  for “backwards”). Given an orientation, there exists a unique equivalent unit orientation  $T$ . (The other unique unit will be  $-T$ ).

The boundary of the oriented curve will be a pair of “oriented points,” i.e.  $\{A^+, B^-\}$ .

A *vector field* is a function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

DEF 2.18

Fix an orientation on a simple curve  $C$ . The *integral* of  $F$  over  $C$ ,  $\int_C F \cdot T ds := \int_{[0,l]} (F \circ \rho) \cdot \rho'$ , where  $\rho$  is some arc length parameterization of  $C$ .

DEF 2.19

$\int_C F \cdot T ds = \int_a^b (F \circ r) \cdot r' dt$ , where  $r$  is *any* parameterization of  $C$ .

PROP 2.7

---

♠ *Examples* ♣

---

E.G. 2.2

Let  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined by  $F(x, y, z) = \langle 2x, 2y, 2z \rangle = 2\langle x, y, z \rangle$ . Hence, at any point, the vector generated by  $F$  will go through the line between the origin and that point (away).

We want to integrate over the triangle  $C \subseteq \mathbb{R}^3$  bounded by  $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ . We orient this path as  $(1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 0, 1)$ .

Then, we split  $C$  up into 3 parts (the lines traversing each point)

$$\begin{aligned} C_1 &= r_1(t) \langle 1, 0, 0 \rangle + t \langle -1, 1, 0 \rangle \\ C_2 &= r_2(t) = \langle 0, 1, 0 \rangle + t \langle 0, -1, 1 \rangle \\ C_3 &= r_3(t) = \langle 0, 0, 1 \rangle + t \langle 1, 0, -1 \rangle \end{aligned}$$

Then

$$\begin{aligned} \int_{C_1} F \cdot T ds &= \int_0^1 \langle 2(1-t), 2t, 2(0) \rangle \cdot \langle -1, 1, 0 \rangle dt = \int_0^1 4t - 2 dt \\ &= [2t^2 - 2t]_0^1 = 0 \end{aligned}$$

By symmetry, the integral across  $C_2, C_3$  will be the same, i.e.  $3 \cdot 0 = 0$ .

## 2.8 Line Integrals on Gradient Fields

Let  $U \subseteq \mathbb{R}^n$  be open and  $\varphi : U \rightarrow \mathbb{R}$  by  $C^1$  continuous. Then  $\nabla \varphi : U \rightarrow \mathbb{R}^n$  is a continuous gradient field. Let  $C \subseteq U$  be a regular oriented curve with a parameterization  $r : [a, b] \rightarrow U$  and an orientation  $T$ . Let  $A = r(a)$  and  $B = r(b)$ .

$$\implies \int_C \nabla \varphi \cdot T ds = \varphi(B) - \varphi(A)$$

PROOF.

$$\begin{aligned}
\int_C \nabla \varphi \cdot T ds &= \int_a^b \nabla \varphi(r(t)) \cdot r'(t) dt \\
&\stackrel{\text{CR}}{=} \int_a^b (\varphi \circ r)'(t) dt \stackrel{\text{FTC}}{=} [\varphi \circ r]_a^b \\
&= \varphi(r(b)) - \varphi(r(a)) = \varphi(B) - \varphi(A)
\end{aligned}$$

□

### Differential Forms

DEF 2.20 A differential 0-form on  $C$  (or any open set  $U$ ) is a scalar function  $F : C \rightarrow \mathbb{R}$  (or  $F : U \rightarrow \mathbb{R}$ ).

DEF 2.21 A differential 1-form on  $C$  (or  $U$ ) is a set of functions such that, for fixed  $a \in C$  (or  $U$ ),  $\omega_a : T_a C \rightarrow \mathbb{R}$  is linear.

E.G. 2.3

#### ♠ Examples ♣

1. Consider  $dx$  on  $U \subseteq \mathbb{R}^2$ . Then  $\forall a \in U$ , we define  $dx_a : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $dx_a(u, v) = u$ . We also have  $dy$  on  $U$ , with  $dy_a : \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $dy_a(u, v) = v$ .
2. Any differential 1-form  $\omega$  can be written as  $\omega = g ds$  for some scalar function  $g : C \rightarrow \mathbb{R}$ , where  $ds$  is the length element of the oriented curve.
3. Let  $\omega$  be a 1-form on  $U \subseteq \mathbb{R}^2$ . Then  $\omega = \alpha dx + \beta dy$  for some scalar functions  $\alpha, \beta$ .

DEF 2.22 A differential 1-form on  $U$  is a set of functions such that, for fixed  $a \in U$ ,  $d\varphi_a : \mathbb{R}^n \rightarrow \mathbb{R}$  is such that  $d\varphi_a = D\varphi_a$ , i.e. the derivative of  $\varphi$  at  $a$ .

The chain rule dictates that  $d\varphi_a(\vec{v}) = \nabla \varphi(a) \cdot \vec{v}$ . Hence

$$d\varphi_a(\vec{a}) = \partial_1 \varphi(a) \underbrace{v_1}_{dx_1(\vec{v})} + \dots + \partial_n \varphi(a) v_n$$

Hence,  $d\varphi = \partial_1 \varphi dx_1 + \dots + \partial_n \varphi dx_n$ . On a point on an oriented curve  $C, T$ , we have  $d\varphi = \nabla \varphi \cdot T ds$ .

DEF 2.23 A gradient field  $F$  is a  $C^1$  vector field such that  $F = \nabla \varphi$ , and  $\partial_i \partial_j \varphi = \partial_j \partial_i \varphi$ .



### 2.9 Jordan Curve Theorem

Let  $C \subseteq \mathbb{R}^2$  be a simple closed curve. Then there exists an elementary region  $D \subseteq \mathbb{R}^2$  such that  $C$  is the boundary of  $D$ .

### 2.10 Green

Given an open  $U \subseteq \mathbb{R}^2$ , an elementary region  $D \subseteq U$ , and a  $C^1$  vector field  $F : U \rightarrow \mathbb{R}^2$ , we have

$$\int_{\partial D} F \cdot T ds = \iint_D \partial_1 F_2 - \partial_2 F_1 dA$$

Note that  $\partial_1 F_2 - \partial_2 F_1 = \det \begin{pmatrix} \partial_1 & \partial_2 \\ F_1 & F_2 \end{pmatrix} =: \text{curl}(F)$ , so we can restate Green's as

$$\int_{\partial D} F \cdot T ds = \iint_D \text{curl}(F) dA$$