VECTOR CALCULUS NOTES

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CONTENTS

I Curves and Surfaces

PRODUCTS ON VECTOR SPACES

Recall the definition of the *inner product* over a vector space V: DEF 1.1

1. $\langle u, v\rangle = \overline{\langle v, u\rangle} = \langle v, u\rangle$ in $\mathbb{R}$ (where we'll be in this class)
2. $\langle au + bw, v\rangle = a\langle u, v\rangle + v\langle w, v\rangle$
3. $\langle u, u\rangle \geq 0$, and $= 0 \iff u = \mathbb{0}$

From this, we define the *norm* of $u \in V$ to be $\|u\| := \sqrt{\langle u, u\rangle}$. This is well-defined, since $\langle u, u\rangle \geq 0$. DEF 1.2

$$\forall u, v \in V, |\langle u, v\rangle| \leq \|u\|\|v\|$$

PROP 1.1 Cauchy-Schwartz Inequality

$$\forall u, v \in V, \|u + v\| \leq \|u\| + \|v\|$$

PROP 1.2 Triangle Inequality

The *cross product* of $u, v \in \mathbb{R}$, with respect to $\mathbb{R}^3$, is the determinate of the following "matrix": DEF 1.3

$$u \times v := \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where $u = \langle u_1, u_2, u_3\rangle$ and $v = \langle v_1, v_2, v_3\rangle$. We observe the following two properties of the cross product in $\mathbb{R}^3$: PROP 1.3

1. $(u \times v) \cdot u = 0$
2. $\|u \times v\| = \|u\|\|v\|\sin(\theta)$, where θ is the angle found between u and v. A conceptualization of this property is that "u-cross-v is equal to the area created by the parallelogram bounded by u and v."

LINES

Define a *line* $l(t) \in \mathbb{R}^n$ to be a function from $\mathbb{R} \to \mathbb{R}^n$, with the primary form $l(t) = P + td$, with $P, d \in \mathbb{R}^n$, $t \in \mathbb{R}$. We call P the "point vector" and d the "direction vector" An alternate form, with two points $P, Q \in \mathbb{R}^n$, would be $l(t) = (1-t)P + tQ$, where $l(t)$ lies along the path between P and Q for $t \in [0, 1]$. DEF 1.4

Distance between a point and line Using this definition, how an we find the shortest path between a point R and a line $l(t)$, which lies between P and Q?

Idea 1 We know the desired vector $w = PR\sin(\theta)$, the angle between PR and PQ. To find this value, note that $\|PR \times PQ\| = \|PR\|\|PQ\|\sin(\theta)$.

Idea 2 We can project R onto PQ, and then subtract this projection from PR.

Idea 3 We can minimize a distance function between R and a point on l, i.e. $l(t)$. Thus, we take $\min_{t\in\mathbb{R}} \|R - l(t)\| = \alpha$, and then take $Rl(\alpha)$ to be the shortest path.

Idea 4 We can find when $(R - l(t)) \cdot d = 0$.

Distance between 2 lines Consider two lines, l_1 and l_2, which do not intersect but are not necessarily parallel. What is the minimal distance between l_1 and l_2?

Sometimes called "skew lines"

Idea 0 Conceptualize this problem as finding the distance between the parallel planes defined by $\{l_1, l_2\}$.

Idea 1 We can minimize $\|l_1(t) - l_2(s)\|$ (really, one should minimize the square to make one's life easier).

Idea 2 Pick any two points, say $l_1(T)$ and $l_2(S)$, and project $l_1(T)l_2(S)$ onto $l_1 \times l_2$.

Idea 3 Minimize $\text{dist}(l_1(t), l_2)$ for fixed t.

Idea 4 Find t and s such that $[l_1(t) - l_2(s)] \cdot \vec{d_1} = 0$ and $[l_1(t) - l_2(s)] \cdot \vec{d_2} = 0$

PROP 1.4 $\|u \times v\| = \|u\|\|v\| \sin(\theta) =$ Area of parallelogram defined by u and v.

Planes

DEF 1.5 A plane $r(s,t)$ is a function $[0,1]^2 \to \mathbb{R}^3$ defined by $d_1, d_2 \in \mathbb{R}^3$, two vectors, and $P \in \mathbb{R}^3$, a point. In particular, $r(s,t) = P + s\vec{d_1} + t\vec{d_2}$. This is called the *parametric form*.

DEF 1.6 The *point-normal* form is a function $\mathbb{R}^2 \to \mathbb{R}^3$ is given by $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, where $\vec{n} = \langle a, b, c\rangle$ is a vector normal to the plane, and $P = \langle x_0, y_0, z_0\rangle$ is a point lying on the plane.

Distance between a point R and a plane r

Idea 1 Minimize $\|R - r(s,t)\|$ (or the square)

Idea 2 $\|\text{proj}_{\vec{n}}(P - R)\|$, where $\vec{n}$ and P are as given in the point-normal form.

Transformations and Parameterizations

The following table give general examples of linear transformations $\lambda : \mathbb{R}^n \to \mathbb{R}^m$.

Dimension	Linear	Affine
$n = 0$	$\lambda(0) = 0$	$\lambda(0) = P$
$n = 1$	$\lambda(t) = t\vec{d}$	$\lambda(t) = P + t\vec{d}$
$n = 2$	$\lambda(t,s) = t\vec{d_1} + s\vec{d_2}$	$\lambda(t,s) = P + t\vec{d_1} + s\vec{d_2}$
$n = 3$	$\lambda(t,s,r) = t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$	$\lambda(t,s,r) = P + t\vec{d_1} + s\vec{d_2} + r\vec{d_3}$

We also define the following important curves in $\mathbb{R}^2$:

Type	Explicit Form	Parametric Form
Ellipse	$x^2 + y^2 = 1$	$r(t) = \left\langle t, \sqrt{1-t^2} \right\rangle_{t\in[-1,1]} = \langle \cos(t), \sin(t) \rangle_{t\in[-\pi,\pi]}$
Hyperbola	$x^2 - y^2 = 1$	$r(t) = \left\langle \sqrt{1+t^2}, t \right\rangle_{t\in\mathbb{R}} = \langle \cosh(t), \sinh(t) \rangle_{t\in\mathbb{R}}$
Parabola	$x = y^2$	
Double Cone	$x^2 = y^2$	
Any Function	$y = F(x)$	$r(t) = \langle t, F(t) \rangle$

Define a *path* in $\mathbb{R}^m$ to be a continuous function $r : \mathbb{R} \to \mathbb{R}^m$, e.g. $[a, b] \to \mathbb{R}^m$. DEF 1.7

Define a *curve* in $\mathbb{R}^m$ to be the image of a path (i.e. a set of points in $\mathbb{R}^m$). Recall the statement "paths parameterize curves." DEF 1.8

For example, the unit circle $x^2 + y^2 = 1$ is parameterized by the path $r : \mathbb{R} \to \mathbb{R}^2$ given by $r(t) = \langle \cos(t), \sin(t) \rangle$.

Define the *tangent* line of $\vec{r}$ at $a \in \mathbb{R}$ to be an affine transformation $l : \mathbb{R} \to \mathbb{R}^m$ satisfying the following:

1. $l(t) = r(a) + (t-a)\vec{d} : \vec{d} \neq 0$
2. $\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} = 0$

♠ *Examples* ♣ E.G. 1.1

We'll now find the derivative of the unit circle at a point $a \in \mathbb{R}$: we have $r(a) = \langle \cos(a), \sin(a) \rangle$. Thus:

$$l(t) = \langle \cos(t), \sin(t) \rangle + (t-a) \langle d_1, d_2 \rangle$$

Where $\langle d_1, d_2 \rangle \neq 0$. Consider now the limit in question 2:

$$\begin{aligned}
\lim_{t\to a} \frac{\|r(t)-l(t)\|}{|t-a|} &= \lim \frac{1}{|t-a|} \sqrt{(\cos(t) - \cos(a) - (t-a)d_1)^2 + (\sin(t) - \sin(a) - (t-a)d_2)^2} \\
&= \lim \sqrt{\left(\frac{\cos(t)-\cos(a)}{t-a} - d_1\right)^2 + \left(\frac{\sin(t)-\sin(a)}{t-a} - d_2\right)^2} \\
&\underset{t\to a}{=} \sqrt{(-\sin(a) - d_1)^2 + (\cos(a) - d_2)^2} = 0 \\
&\iff d_1 = -\sin(a) \wedge d_2 = \cos(a) \\
&\implies l(t) = \langle -\sin(a), \cos(a) \rangle \quad \square
\end{aligned}$$

DIFFERENTIATION AND CONTINUITY

Frequently, $l(t)$ is referred to as the "velocity vector" of $r(t)$, and is notated as $r'(t)$. Notice that $r'(t)$ is equivalent to the component-wise derivative of the coordinates of $r(t)$ w.r.t. t. Formally:

DEF 1.9 Given $\vec{r}: \mathbb{R} \to \mathbb{R}^n$, the *derivative* of $\vec{r}$ at $a \in \mathbb{R}$ is a linear transformation $\vec{\lambda}: \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\lim_{t \to a} \frac{\|r(t) - r(a) - \lambda(t-a)\|}{|t-a|} = 0 \quad \text{or equivalently} \quad \lim_{h \to 0} \frac{\|r(a+h) - r(a) - \lambda(a)\|}{|h|} = 0$$

It is denoted $D\vec{r}_a$, and represented by the $n \times 1$ matrix $r'(a)$. One may now rewrite the tangent line in the form $l(t) = r(a) + \lambda(t-a)$.

DEF 1.10 The *arc length* of a curve $r(t)$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

DEF 1.11 An *arc length parameterization* of $r(t)$ is some $t = \alpha(s)$ such that $r(\alpha(s))$ has a unit velocity vector, i.e. $\|r'(\alpha(s))\| = 1$. Alternatively, one could find an expression for arc length, and then parameterize $r(t)$ in terms of its arc length. The resultant will be equivalent.

DEF 1.12 $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $\vec{a}$ if, for any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon \ \forall \vec{x} \in \mathbb{R}^n$$

E.G. 1.2 ♠ *Examples* ♣

We'll do an arc length parameterization of a semicircle of radius 1 with its center at the origin, i.e. $y = \sqrt{1-x^2}$. We get the natural parameterization $r(t) = \left\langle t, \sqrt{1-t^2} \right\rangle$, where $t \in [-1, 1]$. We'd like to find a change of parameters $t = \alpha(s)$ such that $\|r(\alpha(s))\| = 1$ and $\alpha' \geq 0$.

$$\begin{aligned}
r(\alpha(s)) &= \left\langle \alpha(s), \sqrt{1-\alpha(s)^2} \right\rangle \\
r'(\alpha(s)) &= \left\langle \alpha'(s), \frac{1}{2}(1-\alpha(s)^2)^{-\frac{1}{2}} \cdot (-2\alpha(s)\alpha'(s)) \right\rangle \\
&= \alpha'(s) \left\langle 1, \frac{-\alpha(s)}{\sqrt{1-\alpha(s)^2}} \right\rangle \\
\text{Then } 1 = \|r'(\alpha(s))\| &= \alpha'(s) \sqrt{1 + \frac{\alpha(s)^2}{1-\alpha(s)^2}} \\
&= \frac{\alpha'(s)}{\sqrt{1-\alpha(s)^2}}
\end{aligned}$$

Integrating with respect to s, we get $s = \arcsin(\alpha(s)) = \arcsin(t)$. Thus, $t = \sin(s)$, and $s \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, and we yield the parameterization $\langle \sin(s), \cos(s) \rangle : s \in [\frac{-\pi}{2}, \frac{\pi}{2}]$.

SURFACES

We note the following quadric surfaces:

Type	Explicit Form
Ellipsoid	$x^2 + y^2 + z^2 = 1$
Elliptic Hyperboloid	$x^2 + y^2 - z^2 = 1$
Elliptic Paraboloids	$x^2 + y^2 - z^2 = -1$
Hyperbolic Paraboloids	$x = y^2 - z^2$
Double Cones	$x^2 = y^2 + z^2$

A surface $F(x, y)$ is called *differentiable* at (a, b) if there exists some linear transformation $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that DEF 1.13

$$\lim_{(h,k)\to(0,0)} \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\| \langle h, k \rangle \|}$$

One may represent $\lambda(h, k) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = uh + vk$

♠ *Examples* ♣ E.G. 1.3

Let $F(x, y) = xy$. We consider F at (a, b). Then

$$\begin{aligned} 0 \leq \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\| \langle h, k \rangle \|} &= \frac{|(a+h)(b+k) - ab - (uk + vk)|}{\| \langle h, k \rangle \|} \\ &= \frac{|bh + ak + hk - uh - vk|}{\| \langle h, k \rangle \|} = \frac{|(b-u)h + (a-v)k + hk|}{\| \langle h, k \rangle \|} \\ &\leq \frac{|b-u||h|}{|h|} + \frac{|a-v||k|}{|k|} + \frac{|h||k|}{|h|} \quad \text{since } |h|, |k| \leq \| \langle h, k \rangle \| \\ &= |b-u| + |a-v| + |k| \to |b-u| + |a-v| \\ &= 0 \quad \text{when } b = u, a = v \end{aligned}$$

Thus, the desired limit is always $\geq$ and ≤ 0, so especially it is 0. Our derivative at (a, b) is then $\lambda(x, y) = bx + ay$.

One may also find these coefficients as the partial derivative of F, i.e.

$$\nabla F(a, b) = \left. \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle \right|_{(a,b)}$$

DEF 1.14

This is called the *gradient*. Similarly, $\alpha(x,y) = F(a,b) + \lambda(x-a, y-b)$ is called the *affine approximation* at (a,b).

PROP 1.5

Note that the converse is *false* (as a counterexample, see $F = \sqrt{|xy|}$)

If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then all partial derivatives of F at $\vec{a}$ exist. Furthermore, $\lambda(\vec{a}) = F'(\vec{a}) = \left[\partial_1 F \cdots \partial_n F\right]\Big|_{\vec{a}}$.

1.1 Partial Converse

If all partial derivatives of $F : \mathbb{R}^n \to \mathbb{R}$ exist near $\vec{a}$ and are continuous at $\vec{a}$, then F is differentiable at $\vec{a}$.

PROOF FOR $n = 2$.

Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a linear transformation defined by $\left[\partial_1 F \cdots \partial_n F\right]\Big|_{\vec{a}}$. Then

$$\lambda(\vec{h}) = \sum_{i=1}^{n} \partial_i F(\vec{a}) h_i$$

Let $n = 2$. Then

$$\begin{aligned}
|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})| &= |F(a_1+h_1, a_2+h_2) - F(a_1+h_1, a_2) + F(a_1+h_1, a_2) \\
&\quad - F(a_1, a_2) - \partial_1 F(\vec{a})h_1 - \partial_2 F(\vec{a})h_2| \\
&\le |F(a_1+h_1, a_2+h_2) - F(a_1+h_1, a_2) - \partial_2 F(\vec{a})h_2| \\
&\quad + |F(a_1+h_1, a_2) - F(a_1, a_2) - \partial_1 F(\vec{a})h_1| \\
&= |\partial_2 F(\vec{c})h_2 - \partial_2 F(\vec{a})h_2| + |\partial_1 F(\vec{d})h_1 - \partial_1 F(\vec{a})h_1| \\
&\qquad \text{by mean value thm.} \\
&= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})||h_2| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})||h_1| \\
\frac{|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})|}{||\vec{h}||} &= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})|\frac{|h_2|}{||\vec{h}||} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})|\frac{|h_1|}{||\vec{h}||} \\
&\le |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})|\frac{|h_2|}{|h_2|} + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})|\frac{|h_1|}{|h_1|} \\
&\qquad \text{since } |h_i| < ||\vec{h}|| \\
&= |\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})|
\end{aligned}$$

Then, as $\vec{h} \to 0$, $\vec{c}, \vec{d} \to \vec{a}$. Since F, is continuous, we know $F(\vec{c}) \to F(\vec{a})$ and similarly for $F(\vec{d})$. Thus,

$$|\partial_2 F(\vec{c}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{d}) - \partial_1 F(\vec{a})| \to |\partial_2 F(\vec{a}) - \partial_2 F(\vec{a})| + |\partial_1 F(\vec{a}) - \partial_1 F(\vec{a})| = 0$$

And we conclude that the limit, as $\le$ and ≥ 0, is 0. □

DEF 1.15

$F : \mathbb{R}^n \to \mathbb{R}$ is called C^1 continuous (or *continuously differentiable*) at $\vec{a}$ if all partial exists near $\vec{a}$ and are continuous at $\vec{a}$.

Note that the converse to our partial converse is *not* true: i.e. if F is differentiable at $\vec{a}$, it is not necessarily continuously differentiable at $\vec{a}$. Some counter examples include $F(x,y) = |y|$ and $F(x) = x^2 \sin(\frac{1}{x})$ s.t. $x \neq 0$ and 0 otherwise.

We have an alternative and equivalent definition of differentiability. Let E be continuous and $= 0$ at 0. Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a linear transformation. Then PROP 1.6

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + \|\vec{h}\| E(\vec{h}) \qquad \forall h$$

implies differentiability.

♠ *Examples* ♣ E.G. 1.4

In our previous example, we prove (laboriously) that $F(x,y) = xy$ is differentiable for all (a,b). We can now use Thm 1.1 to show this result: the partial derivatives $F_x = y$ and $F_y = x$ exist and are continuous $\forall x, y \in \mathbb{R}$, so F is differentiable $\forall x, y \in \mathbb{R}$.

1.2 Characterization of the Derivative

Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$. The derivative at $\vec{a}$ exists if:

1. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$\lim_{\vec{h} \to \vec{0}} \frac{\|F(\vec{a} + \vec{h}) - F(\vec{a}) - \lambda(\vec{h})\|}{\|\vec{h}\|} = 0$$

2. $\exists$ a linear transformation $\vec{\lambda} : \mathbb{R}^n \to \mathbb{R}^m$ and a function E such that

$$F(\vec{a} + \vec{h}) - F(\vec{a}) = \lambda(\vec{h}) + \|\vec{h}\| E(\vec{h})$$

and $E(0) = 0$ is continuous at 0.

Such a λ is unique when found, and is called the derivative. We denote it by $D\vec{F}_{\vec{a}}$.

This follows from Def 1.12 and Thm 1.1. □ PROOF.

We may represent the partial derivatives of $\vec{F} : \mathbb{R}^n \to \mathbb{R}^m = \langle F_1, ..., F_m \rangle$ using a *Jacobian* matrix, denoted $F'(\vec{a})$, and defined as follows: DEF 1.16

$$\left[TBD \right]$$

PROP 1.7
Chain Rule

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{a} \in \mathbb{R}^n$. Let $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{b} = f(\vec{a}) \in \mathbb{R}^m$. Then

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l \quad \text{is differentiable at } \vec{a}$$

and $D\vec{h}_{\vec{a}} = D\vec{g}_{\vec{b}} \circ D\vec{f}_{\vec{a}}$. Furthermore, their Jacobians obey

$$h'(a) = g'(b)f'(a)$$

(matrix multiplication)

E.G. 1.5

♠ *Examples* ♣

1. Consider $f(x, y) = \langle x + y, x - y \rangle$ and $g(x, y) = \frac{1}{4}x^2 - \frac{1}{4}y^2$. Then $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}$ is given by
$$\frac{1}{4}(x + y)^2 - \frac{1}{4}(x - y)^2$$
Let $\vec{a} = \langle a_1, a_2 \rangle$. Then $f(a) = b = \langle a_1 + a_2, a_1 - a_2 \rangle$. What about the Jacobian of f?
$$f'(a) = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{bmatrix}\Bigg|_{(a_1, a_2)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
Similarly, for g we have
$$g'(b) = \begin{bmatrix} \partial_1 g & \partial_2 g \end{bmatrix}\Big|_{(a_1+a_2, a_1-a_2)} = \begin{bmatrix} \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1 - a_2) \end{bmatrix}$$
Then, by the chain rule, we multiple these two matrices to yield
$$\begin{bmatrix} \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1 - a_2) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$
One can (less) manually find that $h = g \circ f$ is xy, and conclude the same.

2. Let S be a surface in R^3 given by $F(x, y, z) = 0$ (this is called a "level surface," e.g. $xy - z = 0$). Let $P = (a, b, c)$ be a point on F, and let C be a curve in S containing P, parameterized by $r(t)$.

   Denote $r(t) = \langle x(t), y(t), z(t) \rangle$. Then $g = F \circ r = F(x(t), y(t), z(t)) = 0$. By chain rule, we have $0 = g'(t_0) = F'(P) \cdot r'(t_0)$, where we choose t_0 such that $r(t_0) = \langle a, b, c \rangle$. Then, we observe that
$$0 = \nabla F(P) \bullet \vec{v}(t_0) \implies \nabla F(P) \perp \vec{v}(t_0)$$
   Where $\vec{v} = r'$ is the velocity vector of r. By considering all curves that satisfy our construction $C \subset S$, we yield the tangent plane of S at P with normal vector $\vec{n} = \nabla F(P)$. In particular, the point-normal form of the tangent plane of a surface F at $P = (a, b, c)$ is given by
$$\boxed{\partial_x F(P)(x - a) + \partial_y F(P)(y - b) + \partial_z F(P)(z - c) = 0}$$

3. Generally, we can consider $S^{n-1} \subset \mathbb{R}^n$ of $F : \mathbb{R}^n \to \mathbb{R}$. (This is called a *hypersurface*). Suppose this is differentiable at $P \in S$. Let $C \subset S$ be a curve in S through P, parameterized by $r : \mathbb{R} \to \mathbb{R}^n$ and differentiable at t_0 with $r(t_0) = P$.

   Then, by the chain rule, $v(t_0) \perp \nabla F(P)$. If $v(t_0) \neq 0$, then the tangent line to C at P has derivative $r(t_0)$. If $\nabla F(P) \neq 0$, then the tangent hyperplane to S at P has a normal vector $n = \nabla F(P)$.

DEF 1.17

Let $\mathbb{R}^n \to \mathbb{R}, \vec{a}, \vec{h} \in \mathbb{R}^n$. Let $l(t) = a + th$. Then the *directional derivative* of F along h at a, denoted $\partial_{\vec{h}} F(\vec{a})$, is given by

$$\lim_{t \to 0} \frac{F(a+th) - F(a)}{t}$$

Then, if F is differentiable at a, we have the more useful form

Thus, if $h = e_1$, then $\partial_{e_1} F(\vec{a}) = \partial_1 F(\vec{a})$.

$$\partial_{\vec{h}} F(\vec{a}) = \vec{h} \cdot \nabla F(\vec{a}) = \sum_{i=1}^{n} h_i \partial_i F(\vec{a})$$

PROP 1.8
Mean Value Thm.

Let $F : \mathbb{R}^n \to R$ be differentiable, and let $a, h \in \mathbb{R}^n$, with $h \neq 0$. Then

$$F(a+h) - F(a) = \partial_{\vec{h}} F(c_h) = h \nabla F(c_h) \quad c_h \in [a, a+h]$$

Note that, since a, h are vectors, by $c_h \in [a, a+h]$ we mean that c_h lies along the line segment connecting a and $a+h$.

We now restate the chain rule:

1.3 Chain Rule

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{a}$. Let $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{b} = F(\vec{a})$. Then

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l$$

is differentiable at $\vec{a}$ and $h'(\vec{a}) = g'(\vec{b}) \circ f'(\vec{a})$.

PROOF.

Let λ be the derivative of f. Let $\vec{t}, \vec{s}$ be arbitrary. Then we have

$$f(\vec{a} + \vec{t}) - f(\vec{a}) = \lambda(\vec{t}) + \|\vec{t}\| \varepsilon_1(\vec{t})$$

where $\varepsilon_1 : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $\vec{0}@\vec{0}$. Similarly, for g:

$$g(\vec{b} + \vec{s}) - g(\vec{b}) = \mu(\vec{s}) + \|\vec{s}\| \varepsilon_2(\vec{s})$$

where μ is the derivative of g, and ε_2 is as above. Our goal is to write $h = g \circ f$

in the same manner. Let $\nu = \mu \circ \lambda$. Then

$$\begin{aligned}
h(\vec{a}+\vec{t}) - h(\vec{a}) &= g(f(\vec{a}+\vec{t})) - g(f(\vec{a})) \\
&= g(f(\vec{a}) + \underbrace{\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})}_{:=\vec{s}}) - g(f(\vec{a})) \\
&= \mu(\vec{s}) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \mu(\lambda(\vec{t}) + \|\vec{t}\|\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \mu(\lambda(\vec{t})) + \|\vec{t}\|\mu(\varepsilon_1(\vec{t})) + \|\vec{s}\|\varepsilon_2(\vec{s}) \\
&= \nu(\vec{t}) + \|\vec{t}\|\underbrace{\left(\mu(\varepsilon_1(\vec{t})) + \frac{\|\vec{s}\|}{\|\vec{t}\|}\varepsilon_2(\vec{s})\right)}_{=\varepsilon_3(\vec{t})} \quad \text{if } \vec{t} \neq 0
\end{aligned}$$

$$\begin{aligned}
\vec{t} \neq 0 \implies 0 \leq \|\varepsilon_3(\vec{t})\| &\leq \|\mu(\varepsilon_1(\vec{t}))\| + \frac{\|\lambda(\vec{t})\| + \|\vec{t}\|\|\varepsilon_1(\vec{t})\|}{\|\vec{t}\|}\|\varepsilon_2(\vec{s})\| \\
&\leq M\|\varepsilon_1(\vec{t})\| + (L + \|\varepsilon_1(\vec{t})\|)\|\varepsilon_2(\vec{s})\| \\
&(\text{where} \lambda(\vec{t}) \leq L\|\vec{x}\| \text{ and } \mu(\vec{x})) \leq M\|\vec{x}\|) \\
&\implies \lim_{\vec{t}\to 0} \varepsilon_3(\vec{t}) = 0 \quad \square
\end{aligned}$$

DEF 1.18
Iterated Partial Derivatives

Suppose $g = \partial_i f$ is defined near $\vec{a} \in \mathbb{R}^n$, where $F : \mathbb{R}^n \to \mathbb{R}$. Then if $\partial_j g$ exists at $\vec{a}$, we call it a 2^{nd} order partial derivative of f at $\vec{a}$. We denote this $\partial_j \partial_i f(\vec{a})$, where $i, j \in [1, n]$.

1.4 Mixed Partials are Equal

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $\vec{a} = \langle a_1, a_2 \rangle$. Let $\partial_1 f, \partial_2 \partial_1 f$ exist near $\vec{a}$, with $\partial_2 \partial_1 f$ continuous at $\vec{a}$. Suppose further that $\partial f(x, a_2)$ is defined near $x = a_1$.

$\implies \partial_1 \partial_2 f$ is defined at $\vec{a}$ and $\partial_1 \partial_2 f(\vec{a}) = \partial_2 \partial_1 f(\vec{a})$.

PROOF.

$$\partial_1 \partial_2 f(\vec{a}) = \lim_{h_1 \to 0} \underbrace{\frac{\partial_2 f(a_1 + h_2) - \partial_2 f(a_1, a_2)}{h_1}}_{\beta(h_1):\mathbb{R}_{\neq 0} \to \mathbb{R}}$$

$$\begin{aligned}
\implies \beta(h_1) &= \frac{\lim_{h_2 \to 0} \frac{f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2)}{h_2} - \lim_{h_2 \to 0} \frac{f(a_1, a_2+h_2) - f(a_1, a_2)}{h_2}}{h_1} \\
&= \lim_{h_2 \to 0} \underbrace{\frac{1}{h_2} \frac{(f(a_1+h_1, a_2+h_2) - f(a_1+h_1, a_2)) - (f(a_1, a_2+h_2) - f(a_1, a_2))}{h_1}}_{\alpha(h_1, h_2):\mathbb{R}^2_{\neq 0} \to \mathbb{R}}
\end{aligned}$$

Now, for a break...

PROP 1.9

If $\lim_{\vec{h}\to\vec{0}} \alpha(\vec{h})$ exists, then $\lim_{h_1\to 0} \beta(h_1)$ exists, where $\beta(h_1) = \lim_{\vec{h}\setminus h_1\to 0} \alpha(h_1, (\vec{h}\setminus h_1))$. Furthermore, we conclude

$$\lim_{h_1\to 0} \beta(h_1) = \lim_{\vec{h}\to\vec{0}} \alpha(\vec{h})$$

PROOF (CONTINUED).

Now, it's enough to show that $\lim_{\vec{h}\to\vec{0}} \alpha(\vec{h}) = \partial_2\partial_1 f(\vec{a})$. By the Mean Value Thm, we have

$$\begin{aligned}\alpha(\vec{h}) &= \frac{1}{h_2}\left(\partial_1 f(c_1, a_2 + h_2) - \partial_1 f(c_1, a_2)\right)\\ &= \partial_2\partial_1 f(c_1, c_2) : c_2 \in [a_2, a_2 + h]\end{aligned}$$

Let $\vec{c} = \langle c_1, c_2\rangle$. Then as $\vec{h} \to \vec{0}$, we have$\vec{c} \to \vec{a}$. Thus

$$\lim_{\vec{h}\to\vec{0}} = \lim_{\vec{c}\to\vec{a}} \partial_2\partial_1 f(\vec{c}) = \partial_2\partial_1\vec{a} \qquad \square$$

DEF 1.19

$f : \mathbb{R}^n \to \mathbb{R}$ is *k-times continuously differentiable* at $\vec{a}$ if all k^{th}-order partial derivatives exist near $\vec{a}$ and are continuous at $\vec{a}$.

We say that f is k-times continuously differentiable *near* $\vec{a}$ if it is continuously differentiable at $\vec{a}$ and all k-th order partial derivatives are continuous near $\vec{a}$.

PROP 1.10

If $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable at $\vec{a}$, then all mixed partial derivatives are equal at $\vec{a}$.

PROP 1.11

If f is k-time continuously differentiable at $\vec{a}$, then the $(k-1)$-order partial derivatives are continuously differentiable (hence differentiable and continuous) at $\vec{a}$.

is the following a proof? proposition?

PROOF.

Let $\vec{h} \in \mathbb{R}^n, \vec{l} : \mathbb{R} \to \mathbb{R}^n$ given by $\vec{l}(t) = \vec{a} + t\vec{h}$. Set $g := f \circ \vec{l} : \mathbb{R} \to \mathbb{R}$, i.e. $g(t) = f(\vec{a} + t\vec{h})$.

Then let f be k-times continuously differentiable at $\vec{a}$. Then g is k-times differentiable at 0, and we have

$$\partial^i_{\vec{h}} f(\vec{a}) = g^{(i)}(0) \underset{\text{CR}}{=} (\vec{h}\cdot\nabla)^i f\Big|_{\vec{a}}$$

For example, with $n = 2$, we have

$$\partial^2_{\vec{h}} = (\vec{h}\cdot\nabla)(\vec{h}\cdot\nabla) = (h_1\partial_1 + h_2\partial_2)(h_1\partial_1 + h_2\partial_2)$$

$\square$

1.5 Multivariable Taylor's Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be k-times continuously differentiable near $\vec{a}$ with $\vec{a} \in \mathbb{R}^n$. Let $\alpha_j : \mathbb{R}^n \to \mathbb{R}$ be a degree j homogeneous polynomial, i.e. all non-zero terms have the same degree.

Let $E : \mathbb{R}^n \to \mathbb{R}$ be such that

$$\begin{cases} \bullet\ f(\vec{a}+\vec{h}) - f(\vec{a}) = \alpha_1(\vec{h}) + \ldots + \overbrace{\alpha_k(\vec{h}) + \underbrace{\|h\|^k E(\vec{h})}_{R_k(\vec{h})}}^{R_{k-1}(\vec{h})} \ \forall \vec{h} \\ \bullet\ E(\vec{0}) = 0 \end{cases}$$

To find such an E, we can take

$$E(\vec{h}) = \begin{cases} \frac{1}{\|h\|^k}(f(\vec{a}+\vec{h}) - f(\vec{a}) - \alpha_1(\vec{h}) - \ldots - \alpha_k(\vec{h})) & \vec{h} \neq 0 \\ \vec{0} \quad \vec{h} = 0 \end{cases}$$

Then Taylor's Theorem states:

$$E \text{ continuous at } \vec{0} \iff \alpha_j(\vec{h}) = \frac{1}{j!}\partial^j_{\vec{h}} f(\vec{a}) = \frac{1}{j!}(\vec{h}\cdot\nabla)^j f(\vec{a}) \quad \forall j \in [1,k]$$

If E is continuous at $\vec{a}$ and $\vec{h} \neq \vec{0}$ is near $\vec{0}$, then:

$$R_{k-1}(\vec{h}) = \frac{1}{k!}\partial^k_{\vec{h}} f(\vec{c}_h)$$

where $\vec{c} \in [\vec{a}, \vec{a}+\vec{h}]$.

HW Note: Let $F(x,y)$ be defined like

$$F(x,y) = \begin{cases} \frac{\cdots}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

then to find $\partial_1 F$, use quotient rule to find the top piecewise, and direct computation (i.e. using definitions) to find the bottom. Then show the limits of the partials are equal.

For HW, convert limit to polar coordinates! Much easier!

INTEGRATION

Let $\mathcal{B}$ be a box in $\mathbb{R}^n$. Choose $F : \mathbb{R}^n \to \mathbb{R}$ which is bounded on the box. Then, informally, F is *integrable* if the limit of its Riemann summation is equivalent across all orderings of tagged partitions. DEF 1.20

By the extreme value theorem, if F is continuous on $\mathcal{B}$, then F is bounded on $\mathcal{B}$. PROP 1.12

1.6 Integrability Criterion

If F is continuous on $\mathcal{B}$, then F is integrable over $\mathcal{B}$.

1.7 Fubini

Let $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathcal{B}$. Then

$$\int_{\mathcal{B}} F dV^n = \int_{x_n=a_n}^{x_n=b_n} \cdots \left(\int_{x_1=a_1}^{x_1=b_1} F(x_1, \ldots, x_n) dx_1 \right) \cdots dx_n$$

Furthermore, the order of integration doesn't matter.

$\int_a^b g(x)dx = g(c)(b-a)$ where $a < c < b$. PROP 1.13

PROOF. $\frac{G(b)-G(a)}{b-a} = G'(c) = g(c)$ by the mean value theorem and the FTC. □

1.8

The set of discontinuities of F in $\mathcal{B}$ has zero measure $\iff$ F is integrable over $\mathcal{B}$.

Note that this theorem is not useful in MATH 248, and its proof is out of the scope of this course.

A set $S \subseteq \mathbb{R}^n$ has *zero measure* if $\forall \varepsilon > 0$ we can choose a set of open balls such that $S \subseteq \bigcup B(x_i, \varepsilon_i)$ where $\sum \text{vol}(B(x_i, \varepsilon_i)) < \varepsilon$. DEF 1.21

In general, hypersurfaces in $\mathbb{R}^n$ have zero measure. Thus, if $F : \mathbb{R}^n \to \mathbb{R}$ is continuous except on a hypersurface, F is still integrable.

$\vec{p} \in \text{Int}(S)$ is called an *interior point* of S if $\exists \varepsilon > 0$ such that $B(\vec{p}, \varepsilon) \subseteq S$. DEF 1.22

1. If $S \subseteq \mathbb{R}^n$ has zero measure and $S' \subseteq S$, then S' has zero measure. PROP 1.14
2. If $S \subseteq \mathbb{R}^n$ has zero measure, then S has no interior points.