ASSIGNMENT 1

MATH 356

QUESTION 1

Let X be a random variable with countably many possible values. Let S be a countable set such that $\mathbb{P}(X \in S) = 1$, and let $Q(A) = \mathbb{P}(X \in A)$.

1. For any strict subset of the reals, X is either *in* or *out* of $A \subset \mathbb{R}$. Fix X. Then, $\mathbb{P}(X \in A) = 1 \vee 0$.

   $\implies 0 \leq Q(A) \leq 1.$

2. We let $\mathbb{R} = \Omega$. Since X takes on a real number, we have $\mathbb{P}(X \in \mathbb{R}) = 1 = Q(\Omega)$. Also, since X is well defined with $\emptyset \nsubseteq S$, $\mathbb{P}(X \in \emptyset) = 0 = Q(\emptyset)$

3. Let A_n, $n \geq 1$ be disjoint events.

   Case 1: $X \notin A_n \forall n \geq 1$. Then trivially

   $$Q\left(\cup_{n\geq 1} A_n\right) = \mathbb{P}(X \in \cup_{n\geq 1} A_n) = 0 = \sum_{n\geq 1} \mathbb{P}(X \in A_n) = \sum_{n\geq 1} Q(A_n)$$

   Case 2: without loss of generality, let $X \in A_1$

   $\implies X \notin A_2 \vee A_3 \vee \ldots$, since A_n are all disjoint

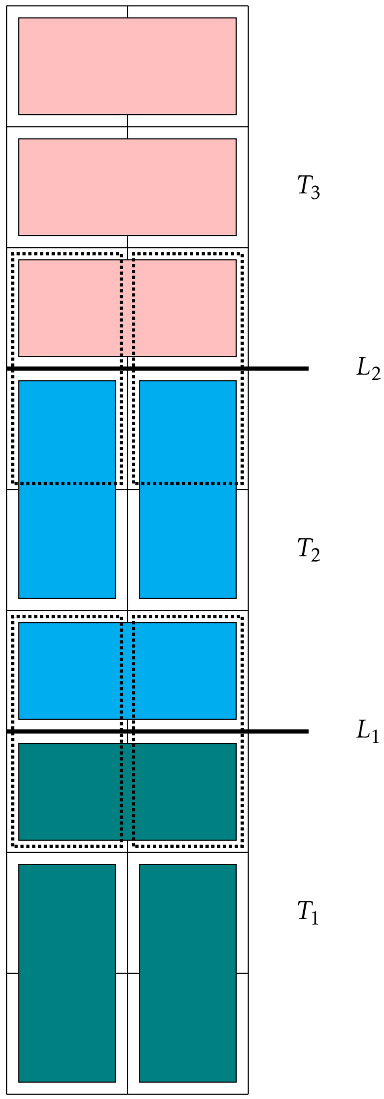
   $\implies \mathbb{P}(X \in A_2 \vee A_3 \vee \ldots) = 0 = \mathbb{P}(X \in A_2) + \mathbb{P}(X \in A_3) + \ldots$

   $\implies \mathbb{P}(X \in A_1 \vee A_2 \vee A_3 \vee \ldots) = 1 = \underbrace{\mathbb{P}(X \in A_1)}_{=1} + \underbrace{\mathbb{P}(X \in A_2) + \mathbb{P}(X \in A_3) + \ldots}_{=0}$

   $\implies \mathbb{P}(X \in \cup_{n\geq 1} A_n) = \sum_{n\geq 1} \mathbb{P}(X \in A_n)$

   Finally, we have $Q\left(\cup_{n\geq 1} A_n\right) = \sum_{n\geq 1} Q(A_n)$

QUESTION 2 AND 3



Quick note: I made this figure before realizing that a 2x9 grid should be oriented hamburger-style. Oh well.

Let's split the 2x9 grid into equal 2x3 sections, separated by lines L_1 and L_2. To count the total number of possible tilings, we'll consider 2 cases.

For the first case, have that all dominoes stay within their respective chunk. There are three tilings possible in each chunk (you can see them pictured in T_1, T_2, and T_3), and thus 27 possible combinations across the whole grid.

Now consider the case where dominoes cross L_1, L_2, or both. These are drawn with dotted lines on the figure. Note that, if one domino crosses on a line, an adjacent one must as well (or else a corner of 3 blocks, un-tileable, will result).

Thus, we have the following:

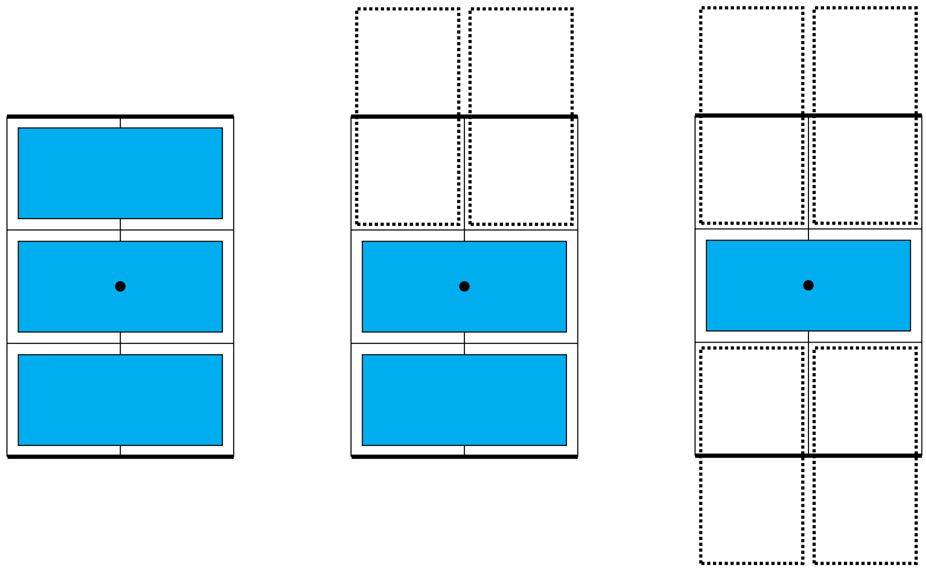
1. **Dominoes crossing L_1 only**: remaining slots in T_1 and T_2 may be tiled either HH or VV, producing 4 combinations. T_3 is left as it was, introducing a factor of 3 to make $4(3) = 12$.
2. **Dominoes crossing L_2 only**: same as above, with 12 arrangements.
3. **Dominoes crossing L_1 and L_2**: the remaining slot in T_2 is determinedly vertical. In both T_1 and T_3, remaining arrangements are HH and VV, producing 4 combinations.

$\implies 27 + 12 + 12 + 4 = 55$ total combinations

There is one unique arrangement in which all dominoes are placed vertically on the board. Thus,

$$\mathbb{P}(\{\text{all tiles vertical}\}) = \frac{1}{\#\Omega} = \frac{1}{55}$$

Fix a vertical domino in column 5, as pictured below.



We have three cases, here:

1. **Midsection filled**: the 2x3 chunks above and below retain their 3 arrangements, and so we have 9 tilings.
2. **Crossing at the top/bottom only**: for the chunk into which the dominoes cross, there are 2 remaining arrangements (HH or VV). The opposite chunk retains 3 arrangements, so we have $2(3) = 6$ tilings. Repeating this for both the top and bottom crossings makes $6(2) = 12$ total tilings.
3. **Crossing at both top and bottom**: here, the chunks above and below may be arranged either HH or VV, making $2(2) = 4$ arrangements.

$\implies 9 + 12 + 4 = 25$ tilings where column 5 is occupied by a vertical domino.

Since $\#\Omega = 55$ as shown above, $P(\{\text{Vertical in middle}\}) = \frac{25}{55} = .4545...$

Question 4

Consider the case where, out of $A_1, A_2, ...A_n$ people, $A_2, A_2, ...A_l$ get their hats back and people $A_{l+1}, A_{l+2}, ..., A_n$ do not.

We can express this probability as $P = \mathbb{P}(A_1 A_2 ... A_l A_{l+1}^C A_{l+2}^C ... A_n^C)$

If there are $G_i\,|_{1 \leq i \leq n!}$ ways of "lining up" people to pick up their hats, the probability that no one receives a hat in that *fixed* arrangement G can be represented as $\mathbb{P}$(no one receives hat, with G fixed) $\cdot\, \mathbb{P}(G)$. Suppose that we fix the arrangement G such that the first l people are guaranteed to receive their hats. Then we have that:

1. $\mathbb{P}(G) = \frac{1}{l!}$
2. $\mathbb{P}$(no one receives hat, with G fixed) = $\mathbb{P}$(no one receives hat out of $n-l$ people), since the first l people have $\mathbb{P} = 0$ of not receiving a hat.

$$= \sum_{k=0}^{n-l} \frac{(-1)^k}{k!}$$

Thus, $\mathbb{P} = \sum\limits_{k=0}^{n-l} \frac{(-1)^k}{k! \cdot l!}$

As $n \to \infty$, we have $\frac{e^{-1}}{l!}$, per the example from class.

QUESTION 5

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability set with points $\omega : \mathbb{P}(\omega) > 0$. Let the set of ω for which $\mathbb{P}(\omega) > 0$ be notated Ω^*. Also note that, since $\omega \in \Omega^*$ are distinct points, they can be thought of as disjoint events.

Assume that Ω^* is uncountable. We have:

$$\forall \omega \in \Omega^* \quad \exists n : \mathbb{P}(\omega) > \frac{1}{n}$$

Fix $\omega^* \in \Omega^*$ such that $B(\mathbb{P}(\omega^*), \varepsilon)$ corrosponds to an infinite subset $A \in \Omega^*$ for which $\mathbb{P}(\omega) \in B(\mathbb{P}(\omega^*), \varepsilon)\ \forall \omega \in A$

or

such that an infinite subset $A \in \Omega^*$ exists with $\mathbb{P}(\omega) = \mathbb{P}(\omega^*)$

Note that we can always find such an ω^*, since, if not, there would exist, at most, a finite number of finite subsets B which make up Ω^*

With ω^* successfully fixed, let N be such that $\mathbb{P}(\omega^*) > \frac{1}{N}$

It is also true that, $\forall \omega \in A \quad \mathbb{P}(\omega) \geq \frac{1}{N}$, by our construction of A

$\mathbb{P}(A) \geq \sum_{i=1}^{\infty} \frac{1}{N}$, which (since N is fixed) will diverge.

$\therefore$ we have found $A \subset \Omega^* \subset \Omega$ such that $\mathbb{P}(A) > 1 \implies \mathbb{P}(\Omega^*) > 1 \implies \mathbb{P}(\Omega) > 1$

This contradicts one of our axioms, and we are done $\lightning$

Question 6

Let I be the event that a set of twins is identical, G be the event that both twins are girls. For this expirement, we assume that only pairs of girls or boys may be identical. From Bayes' formula, we have:

$$\mathbb{P}(G) = \mathbb{P}(G|I)\mathbb{P}(I) + \mathbb{P}(G|I^C)\mathbb{P}(I^C)$$

Given that twins are identical, they may either be a pair of girls or boys, and so $\mathbb{P}(G|I) = 1/2$. If the twins are *not* identical, we lose the condition that they may only be GG or BB, and so $\mathbb{P}(G|I^C) = 1/4$, as there are 4 equally-likely arrangements (GG, GB, BG, BB).

$$\implies \mathbb{P}(G) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{3}$$

Assuming that the set of twins are girls, we can use conditionality to figure the odds of them being identical:

$$\mathbb{P}(I|G) = \frac{\mathbb{P}(IG)}{\mathbb{P}(G)} = \frac{1/3 \cdot 1/2}{1/4} = \frac{2}{3}$$

Note that $\mathbb{P}(IG)$ may be derived, to be pedantic, by further conditional logic: $\mathbb{P}(GI) = \mathbb{P}(G|I)\mathbb{P}(I)$