Geometry & Topology 1

MATH576

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CONTENTS

I Topologies

We will study continuous mappings between objects in $\mathbb{R}^n$. These, in contrast with isogenies (rotations, reflections, translations), will allow us to identify circles with squares, doughnuts with mugs, etc. A guiding question: how do we describe deformation mathematically?

A *metric* on a space X is a function $d : X \times X \to \mathbb{R}$ satisfying DEF 1.1

(1) $d(x,y) \geq 0$, and $d(x,y) = 0 \iff x = y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) \leq d(x,z) + d(y,z)$

A *metric space* (X, d) is a space with a specific metric.

The *ball centered at x of radius ε*, denoted $B(x,\varepsilon)$, is the set DEF 1.2

or $B_d(x,\varepsilon)$ w.r.t. a metric d

$$\{y \in X : d(x,y) < \varepsilon\} \quad \text{with} \quad \varepsilon > 0$$

$U \subseteq X$ is called *open* if, for each $x \in X$, there exists $\varepsilon > 0$ with $B(x,\varepsilon) \subseteq X$. DEF 1.3

Denote by $\mathcal{T}$ the family of all open sets in X. DEF 1.4

Given (X, d): **PROP 1.1**

(1) $\varnothing, X \in \mathcal{T}$

(2) For any (possibly infinite) collection of sets in $\mathcal{T}$, their union is in $\mathcal{T}$

(3) For any finite collection of sets in $\mathcal{T}$, their intersection is in $\mathcal{T}$

PROOF. (1) follows by definition. For (2), let U be some union of sets in $\mathcal{T}$, and $x \in U$. Then x belongs to some open set U_i comprising the union. Then $B(x,\varepsilon_i) \subseteq U_i \subseteq U$ for some ε_i.

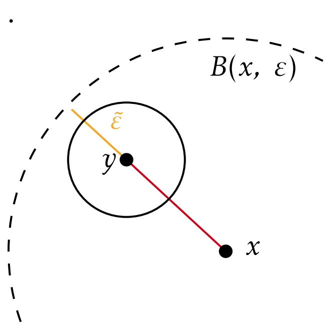
For (3), let $\{U_1, ..., U_n\} \subseteq \mathcal{T}$. Pick $x \in \cup_{i\in[n]} U_i =: U$. It belongs to all sets $U_i : i \in [n]$. Since these sets are open, there exists some $\varepsilon_i : B(x,\varepsilon_i) \subseteq U_i$. Let $\varepsilon := \min\{\varepsilon_i\}$. Then $B(x,\varepsilon) \subseteq U_i$ for each i, and hence $B(x,\varepsilon) \subseteq \mathcal{T}$. □

A *topology* on (X, d) is a collection $\mathcal{T}$ of subsets of X satisfying Prop 1.1. Elements of $\mathcal{T}$ are called *open sets*. DEF 1.5

If $\mathcal{T} \subseteq \mathcal{T}'$, we say that $\mathcal{T}$ is *coarser* than $\mathcal{T}'$. Conversely, $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

Eg. 1.1.1 $B(x,\varepsilon)$ is open. Let y be some internal point. Take $\tilde{\varepsilon} := \varepsilon - d(x,y)$. Then if $z \in B(y,\tilde{\varepsilon})$, we have

$$d(x,z) \leq d(x,y) + d(z,y) < d(x,y) + \varepsilon - d(x,y) = \varepsilon \implies z \in B(x,\varepsilon)$$



Eg. 1.1.2 If $X = \{a, b, c\}$, we can consider $\mathcal{T} = \mathcal{P}(X)$, i.e. the powerset of X. This is a topology. We can also arrive at this by considering the metric $d(x, y) = 1, x \neq y$. The topology $\mathcal{T}' = \{\varnothing, \{a, b\}, \{b, c\}, \{b\}, X\}$ has a less obvious metric-theoretical explanation.

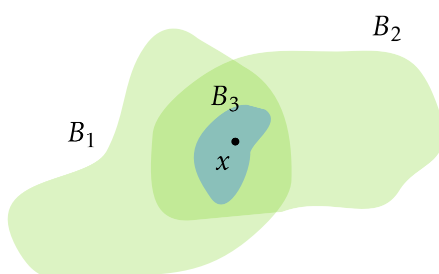
Eg. 1.1.3 Condition 3 of Prop 1.1 fails for infinite collections (e.g. balls of radius approaching 0).

DEF 1.6 The pair $(\mathcal{T}, X)$ is called a *topological space*.

DEF 1.7 A *basis* $\mathcal{B}$ on a topology $(X, \mathcal{T})$ is a collection of subsets of X satisfying

(1) For each $x \in X$, there exists $B \in \mathcal{B}$ with $x \in B$

(2) If $x \in B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq (B_1 \cap B_2)$.



DEF 1.8 The *topology generated by* $\mathcal{B}$ is the collection of sets

$$\{U : \ \forall x \in U, \ \exists B \in \mathcal{B} : x \in B \subseteq U\}$$

Eg. 1.2.1 $\mathcal{B} = \{B(x, \varepsilon), x \in X, \varepsilon > 0\}$ is a basis. Clearly condition (1) of Def 1.7 is satisfied. For condition (2), suppose $x \in B(x_i, \varepsilon_i)$ for $i = 1, 2$. Take $\varepsilon_3 := \min\{\varepsilon_i - d(x, x_i)\}$. Then $B(x, \varepsilon_3)$ is contained in both balls $B(x_i, \varepsilon_i)$.

1.1 $\mathcal{B}$ Induces A Topology The topology generated by a basis $\mathcal{B}$ is a topology.

PROOF. Let $\mathcal{T}$ be generated by $\mathcal{B}$. We will go through each condition in Prop 1.1:

(1) Condition (1) of Def 1.7 provides for $\varnothing, X \in \mathcal{T}$

(2) Let $U_\alpha : \alpha \in J$ be a collection in $\mathcal{T}$, for some index J. Let $U := \cup_{\alpha \in J} U_\alpha$. Given $x \in U$, we know $x \in U_\alpha$ for some $\alpha \in J$. Hence, since $U_\alpha \in \mathcal{T}$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq U_\alpha \subseteq U$.

(3) It suffices to show $U_1 \cap U_2 \in \mathcal{T}$, since we only consider finite collections. Fix $x \in U_1 \cap U_2$. There exists $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. We know there exists, then, $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq (B_1 \cap B_2) \subseteq (U_1 \cap U_2)$. □

PROP 1.2 Let $\mathcal{B}$ be a basis for X generating $\mathcal{T}$. Then $\mathcal{T}$ is the collection of all possible unions of elements in $\mathcal{B}$.

PROOF. Let $\mathcal{T}'$ denote all possible unions of elements in $\mathcal{B}$. Then clearly $\mathcal{T}' \subseteq \mathcal{T}$, since $\mathcal{B} \subseteq \mathcal{T}$.

Conversely, let $U \in \mathcal{T}$. Then each $x \in U$ belongs to some $B_x \subseteq U \in \mathcal{B}$. Hence, $U \subseteq \cup_{x \in U} B_x$. But each $B_x \subseteq U$, so U is exactly this union, and hence $U \in \mathcal{T}'$. □

DEF 1.9 Let X, Y be topological spaces. Then the *product topology on* $X \times Y$ is the topology generated by the basis

$$\{U \times V : U \text{ open in } X, V \text{ open in } Y\}$$

Indeed, this is a proper basis. For (1) of Def 1.7, if $(x, y) \in X \times Y$, we take $U = X$ and $V = Y$. For (2), if $(x, y) \in U_1 \times V_1, U_2 \times V_2$, then $U_1 \cap U_2 \in \mathcal{T}_X$ and $V_1 \cap V_2 \in \mathcal{T}_Y$ are open. Hence, $x \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$.

DEF 1.10 Let $Y \subseteq (X, \mathcal{T})$. Then the collection $\{Y \cap U : U \in \mathcal{T}\}$ is called the *subspace topology* with respect to Y.

PROP 1.3 Let $\mathcal{S}$ be a collection of subsets of X. Let $\mathcal{B}$ be the collection of all finite intersections of elements in $\mathcal{S}$. Then $\mathcal{B}$ is a basis in X, as in Def 1.7.

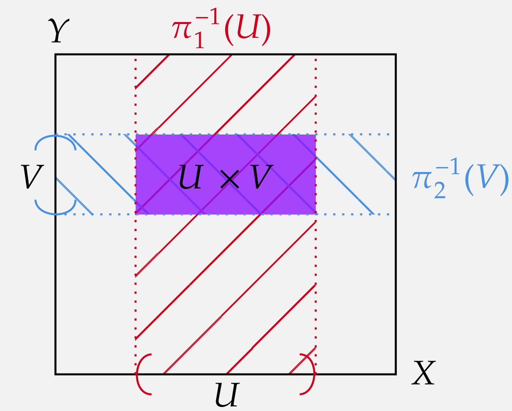
DEF 1.11 $\mathcal{S}$, as above, is called a *sub-basis* of the topology in X generated by $\mathcal{B}$.

Eg. 1.3.1 Let $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ be the coordinate projections onto the X, Y subspaces of $X \times Y$. Then

$$\{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ open in } Y\} =: \mathcal{S}$$

is a sub-basis of the product topology.

PROOF. The intersection of any two elements in $\mathcal{S}$ may be written as $U \times V$, $U \times Y$, or $X \times V$ for open subsets U, V of X, Y, respectively.



Eg. 1.3.2 Let $X = \prod_{\alpha \in J}$, with $\pi_\alpha : X \to X_\alpha$ the coordinate projection. Then the product topology on X will have a sub-basis

$$\mathcal{S} = \{\pi_\alpha^{-1}(U) : U \text{ open in } X_\alpha : \alpha \in J\}$$

with corresponding basis

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha : U_\alpha \text{ open in } X_\alpha \right\}$$

where $U_\alpha = X_\alpha$ except at finitely many indices.

DEF 1.12 A subset $A \subseteq X$ is called *closed* if $X \setminus A$ is open.

PROP 1.4 Let X be a topological space. Then

1. $X, \varnothing$ are closed
2. Finite unions of closed sets are closed
3. Infinite intersections of closed sets are closed

PROOF. We'll take a look at (2). Let A_1, A_2 be closed. Then $(A_1 \cup A_2)^C = A_1^C \cap A_2^C$ by De Morgan's. Then, This is open, since A_i^C is open, and hence $A_1 \cap A_2$ is closed. □

PROP 1.5 If X is a metric space, then $A \subseteq X$ is closed $\iff$ for each $x_n \to x$ (i.e. $d(x_n, x) \to 0$), with $x_i \in A$, then $x \in A$.

PROOF.

($\Longrightarrow$) Suppose $x \notin A$, and hence $x \in X \setminus A$, which is open. Hence, $B(x, \varepsilon) \subseteq X \setminus A$ for some $\varepsilon > 0$. Choose $x_n : d(x_n, x) < \varepsilon$. Then $x_n \in B(x, \varepsilon) \subseteq X \setminus A$, which is a contradiction.

($\Longleftarrow$) Let $x \in X \setminus A$. Fix some n. We wish to show that there exists $B(x, \frac{1}{n}) \subseteq X \setminus A$. If not, then $\exists x_n \in B(x, \frac{1}{n})$. Repeating this, we have a sequence $x_n \to x$ with $x_n \in A$ and $x \notin A$, a contradiction. □

DEF 1.13 Let A be a subset of a topological space X. The *interior of* A, denoted $\text{Int}(A)$, is the union of all open sets contained in A. Hence, it is open. An alternative characterization of $\text{Int}(A)$ is: the largest open subset of A.

DEF 1.14 Conversely, for $A \subseteq X$, the *closure of* A, denoted $\overline{A}$, is the intersection of all closed sets containing A. Hence, it is closed. An alternative characterization of $\overline{A}$ is: the smallest closed set containing A.

DEF 1.15 We say A is *dense* in X if $\overline{A} = X$.

DEF 1.16 For $x \in X$, U is called a *neighborhood of* x if $x \in \text{Int}(U)$.

Eg. 1.4.1 $[a, b] \subseteq \mathbb{R}$ is closed, as the compliment, $(-\infty, a) \cup (b, \infty)$, is open.

Eg. 1.4.2 "Closed balls," i.e. $A = \{y : d(x, y) \leq \varepsilon\}$ for $x \in X, \varepsilon > 0$, are closed. For $y \in X \setminus A$, choose $\tilde{\varepsilon} = d(x, y) - \varepsilon$ to show openness.

Eg. 1.4.3 $\text{Int}(\mathbb{Q}) = \varnothing$ and $\overline{\mathbb{Q}} = \mathbb{R}$, i.e. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Eg. 1.4.4 $X \setminus \text{Int}(A) = \overline{X \setminus A}$

Eg. 1.4.5 $U = \{y : d(x, y) \leq \varepsilon\}$ is a neighborhood of x, since $B(x, \varepsilon) \subseteq U$. In general, any open set containing x is a neighborhood of x.

PROP 1.6 Let $A \subseteq X$ be a topological space. Then $x \in \overline{A} \iff$ each neighborhood U of x intersects A (i.e. $U \cap A \neq \varnothing$).

PROOF.

We'll prove the contrapositive, i.e. $x \notin \overline{A} \iff$ there exists a neighborhood U with $U \cap A = \varnothing$.

($\Longrightarrow$) Take $U = X \setminus \overline{A}$, since $x \in X \setminus \overline{A}$. U is open, and hence a neighborhood. Since $A \subseteq \overline{A}$, $U \cap A = \varnothing$.

($\Longleftarrow$) Let U be a neighborhood of x with $U \cap A = \varnothing$. Then $X \setminus \text{Int}(U)$ is closed and contains A. Hence, $\overline{A} \subseteq X \setminus \text{Int}(U)$. But $x \in \text{Int}(U)$, so $x \notin \overline{A}$. □

DEF 1.17 Let X, Y be topological spaces. A function $f : X \to Y$ is *continuous at* $x \in X$ if, for each neighborhood A of $f(x)$, the preimage $f^{-1}(A)$ is a neighborhood of x.

DEF 1.18 If $f : X \to Y$ is continuous for all $x \in X$, we simply call f *continuous.*

PROP 1.7 f is continuous $\iff$ for each open set $U \subseteq Y$, $f^{-1}(U)$ is open in X.