Vectors

Definition 1

An *inner product* on a vector space is such that

1. $\langle u, v\rangle = \langle v, u\rangle$
2. $\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$
3. $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$
4. $\langle u, u\rangle \geq 0$, and $= 0 \iff u = 0$

Definition 2

A *norm* is $\|v\| = \sqrt{\langle v, v\rangle}$.

Definition 3

A *line* $l(t) : \mathbb{R} \to \mathbb{R}^n$ is $l(t) = P + td$, where $P, d \in \mathbb{R}^n$. It may also be given by $l(t) = (1-t)Q + tP$, where $P, Q \in \mathbb{R}^n$, and $t \in [0, 1]$.

Definition 4

A *plane* $p(s, t) : \mathbb{R}^2 \to \mathbb{R}^3$ is $p(s, t) = P + sd_1 + td_2$, where $d_1, d_2 \in \mathbb{R}^3$.

It may also be given in point-normal form, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $\langle a, b, c\rangle \in \mathbb{R}^3$ is normal to the plane, and $(x_0, y_0, z_0) \in \mathbb{R}^3$ lies on the plane.

Definition 5

A *linear transformation* $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is such that $\lambda(\vec{0}) = \vec{0}$, and $\lambda(\vec{a} + \vec{b}) = \lambda(\vec{a}) + \lambda(\vec{b})$.

Alternatively, write $\lambda(x_1, ..., x_n) = x_1\vec{d_1} + ... + x_n\vec{d_n}$, where $\vec{d_i} \in \mathbb{R}^m$.

Definition 6

An *affine* transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation plus a point $P \in \mathbb{R}^m$.

Definition 7

The *projection* of v onto u, denoted $\text{proj}_u(v)$, is given by $(u \cdot v)\frac{u}{\|u\|^2}$.

Definition 8

The *tangent line* of $r : \mathbb{R} \to \mathbb{R}^n$ at $a \in \mathbb{R}$ is an affine transformation λ satisfying

$$\lim_{t\to a} \frac{\|r(t) - \lambda(t)\|}{|t - a|} = 0$$

$$\lambda(t) = r(a) + (t - a)\vec{d} \quad \vec{d} \neq 0$$

Prop 1 (Inequalities)

$\|v + u\| \leq \|v\| + \|u\|$ ($\triangle$) and

$|\langle u, v\rangle| \leq \|u\|\,\|v\|$ (Cauchy-Schwartz).

Prop 2 (Cross/Dot Products)

$(u \times v) \cdot u = 0$

$\|u \times v\|$ is the area of the parallelogram bounded by u, v.

$\|u \times v\| = \|u\|\,\|v\| \sin(\theta)$

$u \cdot v = \|u\|\,\|v\| \cos(\theta)$

Prop 3 (Distances)

(a) The distance between a point R and a plane may be given by $\|\text{proj}_{\vec{n}}(P - R)\|$, where $P, \vec{n}$ are as in point-normal form.

(b) The distance between skew lines may be given by projecting [a third line which contains points of both] onto [the normal vector of the skew lines].

(c) The distance between R and PQ may be given by $\|PR - \text{proj}_{PQ}(R)\|$.

Differentiation

Definition 1

(a) $r : \mathbb{R} \to \mathbb{R}^n$ is *differentiable* at $\vec{a}$ if $\exists$ a linear transformation $\lambda : \mathbb{R} \to \mathbb{R}^n$ such that

$$\lim_{h\to 0} \frac{\|r(\vec{a}+h) - r(\vec{a}) - \lambda(\vec{h})\|}{|h|} = 0$$

(b) Similarly, $F : \mathbb{R}^2 \to \mathbb{R}$ is *differentiable* at $\vec{a}$ if $\exists$ a linear transformation $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(h,k)\to(0,0)} \frac{|F(a+h, b+k) - F(a,b) - \lambda(h,k)|}{\|\langle h,k\rangle\|}$$

(c) Generally, $F : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $\vec{a}$ if $\exists$ a linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{\|F(\vec{a}+\vec{h}) - F(\vec{a}) - \lambda(\vec{h})\|}{\|\vec{h}\|}$$

We denote the derivative $\lambda(\vec{a})$ as $DF_{\vec{a}}$

Definition 2

The *arc length* of a curve $r(t) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$s = \int_a^b \|r'(t)\| dt$$

The arc length parameterization is some $t = \alpha(s)$ such that $\|r'(\alpha(s))\| = 1$.

Definition 3

The *Jacobian* matrix of $F : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$F' = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \cdots & \frac{\partial F}{\partial x_n} \end{bmatrix}$$

Note that, where $m > 1$, each element is a column vector.

Definition 4

A *level surface* is $F : \mathbb{R}^n \to 0$.

Definition 5

The *gradient* of $F : \mathbb{R}^n \to \mathbb{R}$ is $\left\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle$ (i.e. its Jacobian) and denoted by ∇F.

Definition 6

$\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* at $\vec{a}$ if, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|\vec{x} - \vec{a}\| < \delta \implies \|\lambda(\vec{x}) - \lambda(\vec{a})\| < \varepsilon$.

Definition 7

$F : \mathbb{R}^n \to \mathbb{R}$ is called C^k-*continuous* (or k-times continuously differentiable) at $\vec{a}$ if all k^{th}-order partial derivatives exist near $\vec{a}$ and are continuous at $\vec{a}$.

Definition 8

(a) The k^{th} *partial derivative* of $F : \mathbb{R}^n \to \mathbb{R}$ at $\vec{a}$, denoted by $\frac{\partial F}{\partial x_k}(\vec{a})$ is given by

$$\lim_{t\to 0} \frac{F(\vec{a}+te_k) - F(\vec{a})}{t}$$

where e_k is the standard basis vector, e.g. $\vec{i}$ for $k = 1$.

(b) The *directional derivative* of $F : \mathbb{R}^n \to \mathbb{R}$ along $\vec{h}$, denoted by $\partial_{\vec{h}} F(\vec{a})$, is given by

$$\lim_{t\to 0} \frac{F(\vec{a}+t\vec{h}) - F(\vec{a})}{t}$$

(c) The j^{th} *iterated directional derivative* of $F : \mathbb{R}^n \to \mathbb{R}$ at $\vec{a}$ along $\vec{h}$, denoted by $\partial^j_{\vec{h}} F(\vec{a})$, is given by $g^{(j)}(0)$, where $g(t) = F(\vec{a}+t\vec{h})$. Note that $g'(0) = \partial_{\vec{h}} F(\vec{a})$.

Prop 1 (Differentiability and Partials)

(a) If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then all partial derivatives of F exist at $\vec{a}$.

(b) If all partial derivatives of $F : \mathbb{R}^n \to \mathbb{R}$ exist near $\vec{a}$ and are continuous at $\vec{a}$, then F is differentiable.

(c) If $F : \mathbb{R}^n \to \mathbb{R}$ is k-times continuously differentiable, then all $(k-1)^{th}$-order partial derivative are continuously differentiable.

Prop 2 (Chain Rule)

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ be differentiable at $\vec{a} \in \mathbb{R}^n$ and $f(\vec{a}) \in \mathbb{R}^m$, respectively. Then

$$h = g \circ f : \mathbb{R}^n \to \mathbb{R}^l$$

is differentiable. Furthermore, $Dh_{\vec{a}} = Dg_{f(\vec{a})} \circ Df_{\vec{a}}$, and $h'(\vec{a}) = g'(f(\vec{a}))f'(\vec{a})$.

Prop 3 (Iterated Partials)

(a) If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a}$, then

$$\partial_{\vec{h}}(\vec{a}) = \vec{h} \cdot \nabla F = \sum_{i=1}^{n} h_i \frac{\partial F}{\partial x_i}(\vec{a})$$

(b) If $F : \mathbb{R}^n \to \mathbb{R}$ if k-times continuously differentiable, then

$$\partial^j_{\vec{h}}(\vec{a}) = (\vec{h} \cdot \nabla)^j F(\vec{a})$$

Prop 4 (Mixed Partials)

If $F : \mathbb{R}^2 \to \mathbb{R}$ is C^2-continuous, then $\partial_x \partial_y F = \partial_y \partial_x F$.

Prop 5 (Taylor)

If $F : \mathbb{R}^n \to \mathbb{R}$ is C^k continuous, let

$$\alpha_j(\vec{h}) = \frac{1}{j!}\partial^j_{\vec{h}}(\vec{a}) = \frac{1}{j!}(\vec{h} \cdot \nabla)^j F(\vec{a})$$

Then $F(\vec{a}) + \sum_{i=1}^{k} \alpha_i(\vec{x} - \vec{a})$ is the best degree k approximation of F near $\vec{a}$.

Integration

Definition 1

Let $\mathcal{B} = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Then $F : \mathbb{R}^n \to \mathbb{R}$ is *integrable* over $\mathcal{B}$ if

$$\lim_{m \to \infty} \sum_{i=1}^{m} F(c_j^m) \frac{\mathrm{vol}(\mathcal{B})}{m^n} \text{ exists}$$

and is equivalent up to choices of c_j^m, where $c_j^m \in [a_j^k, b_j^k]$, $k \in [1, m]$ (slice of size $\frac{1}{m}$).

Definition 2

A set $S \subseteq \mathbb{R}^n$ has *zero measure* if $\forall \varepsilon > 0$ we can choose a countable set of open balls such that $S \subseteq \cup B(x_i, \varepsilon_i)$, where $\sum \mathrm{vol} B(x_i, \varepsilon_i) < \varepsilon$.

Definition 3

$S \subseteq \mathbb{R}^n$ is *path connected* if there exists a mapping contained in S between any two points $a, b \in S$. Then $\mathcal{D} \subseteq \mathbb{R}^n$ is then *elementary* if it is closed, bounded, path connected, and its boundary is path connected.

Prop 1 (Integrability)

(a) If $F : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathcal{B}$, then F is integrable on $\mathcal{B}$.

(b) The set of discontinuities of F in $\mathcal{B}$ has zero measure if and only if F is integrable.

(c) If f is continuous on a closed and bounded set $\mathcal{D}$ whose boundary has zero measure, then f is integrable over $\mathcal{D}$.

Prop 2 (Fubini)

Let $\mathcal{B} = [a_1, b_1] \times \cdots \times [a_n, b_n]$. If $F : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathcal{B}$, then

$$\int_{\mathcal{B}} F = \int_{x_n = a_n}^{x_n = b_n} \cdots \left(\int_{x_1 = a_1}^{x_1 = b_1} F(x_1, \ldots, x_n) dx_1 \right) \cdots dx_n$$

The order of integration doesn't matter.

Prop 3 (MVT)

(a) $\int_a^b g(x) = g(c)(b-a)$ for some $c \in [a,b]$

(b) Let F be integrable over an elementary region $\mathcal{D} \subseteq \mathbb{R}^n$. Let $\overline{F} = \frac{1}{\text{vol}(\mathcal{D})}\int_{\mathcal{D}} F dV^n$. Then

$$\exists c \in \mathcal{D} : F(c) = \overline{F}$$

Prop 3 (Change of Variables)

Let $T : \mathcal{D}^* \to \mathcal{D}$ be C^1 and injective on $\text{Int}(\mathcal{D}^*)$. Let $F : \mathcal{D} \to \mathbb{R}$ be integrable over $\mathcal{D}$. Let $[T]$ be the Jacobian induced by T. Let $F^* : \mathcal{D}^* \to \mathbb{R} = F \circ T$. Then F^* is integrable over $\mathcal{D}^*$ and

$$\int_{\mathcal{D}} F dV = \int_{\mathcal{D}^*} F^* |\det(T)| dV$$

Vector Fields

Definition 1

$\vec{r} : [a,b] \to \mathbb{R}^n$ is a *regular path* if it is C^1 and $\|r'(t)\| > 0$ on $[a,b]$. A regular path $\vec{r}$ is *simple* if it is injective, except possibly at its endpoints. A regular path $\vec{r}$ is *closed* if $r(a) = r(b)$.

Definition 2

A curve $\mathcal{C} \subseteq \mathbb{R}^n$ is *regular/simple/closed* if it is the image of a regular/simple/closed path.

Definition 3

Respective to an integrable density function $\delta : \mathcal{D} \to \mathbb{R}_+$, $\text{mass}(\mathcal{D}) = \int_{\mathcal{D}} \delta dV^n$. If $\mathcal{D} = \mathcal{C}$, a curve, then this is $\text{mass}(\mathcal{C}) = \int_a^b \delta \circ r(t) \|r'(t)\| dt$.

Definition 4 (Line Integrals)

$$\int_{\mathcal{C}} F \cdot T ds = \int_0^l (F \circ \rho) \cdot \rho' dt = \int_a^b (F \circ r) \cdot r' dt$$

$$\int_{\mathcal{C}} F \cdot n ds = \int_a^b (F \circ r) \cdot \langle r_2', -r_1' \rangle dt$$

Definition 5 (Surface Integrals)

$$\iint_S F \cdot n d\sigma = \iint_{\mathcal{D}} (F \circ \rho) \cdot (\partial_1 \rho \times \partial_2 \rho) dA$$

Proposition 1

The coordinates of the center of mass $\mathcal{D}$ with respect to an integrable density function $\delta : \mathcal{D} \to \mathbb{R}_+$, is given by

$$x_i = \frac{1}{\text{mass}(\mathcal{D})} \int_{\mathcal{D}} x_i \delta dV$$

In the case where $\mathcal{D} = \mathcal{C}$, a curve, then

$$x_i = \frac{1}{\text{mass}(\mathcal{C})} \int_a^b [r_i(t) \circ \delta \circ r(t)] \|r'(t)\| dt$$

Proposition 2

Let $\varphi : \mathcal{U} \to \mathbb{R}^n$ be C^1. Let $r : [a,b] \to \mathcal{U}$ parameterize a curve $\mathcal{C}$. Then

$$\int_{\mathcal{C}} \nabla \varphi \cdot T ds = \varphi(B) - \varphi(A)$$

Proposition 3 (Green)

Let $F : \mathcal{U} \to \mathbb{R}^n$ be a C^1 vector field and $\mathcal{D} \subseteq \mathcal{U}$ be elementary. Then

$$\int_{\partial \mathcal{D}} F \cdot T ds = \iint_{\mathcal{D}} \partial_1 F_2 - \partial_1 F_1 dA = \iint_{\mathcal{D}} \text{curl}_2(F) dA$$

$$\int_{\partial\mathcal{D}} F \bullet n ds = \iint_{\mathcal{D}} \partial_1 F_1 + \partial_2 F_2 dA = \iint_{\mathcal{D}} \operatorname{div}_2(F) dA$$

Proposition 4 (Cons $\iff$ ∇)

(a) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and convex. Let $F : \mathcal{U} \to \mathbb{R}^2$ be a C^1 vector field. Then F is conservative $\iff$ it is gradient.

(b) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and convex. Let $F : \mathcal{U} \setminus X \to \mathbb{R}^3$ be a C^1 vector field, where $|X| < \infty$. Then $\operatorname{curl}_3(F) = 0 \iff F$ is gradient.

Proposition 5 (Integrating Scalars)

(a) $\int_{\mathcal{C}} \varphi ds = \int_a^b (\varphi \circ r) \|r'(t)\| dt$

(b) $\iint_{\S} \varphi d\sigma = \iint_{\mathcal{D}} (\varphi \circ \rho) \|\partial_1 \rho \times \partial_2 \rho\| dA$