

---

# ASSIGNMENT 1

MATH 251

NICHOLAS HAYEK

---



## QUESTION 1

Define  $V := (0, \infty) \subset \mathbb{R}$  with the operations  $r \oplus s = rs$  and  $\lambda \otimes r = r^\lambda$ , where  $rs$  and  $r^\lambda$  are defined as usual, and  $\mathbb{0}_V = 1$ . This is a vector space over  $\mathbb{R}$ .

We are given that  $V$  is abelian. Let  $\alpha, \beta \in \mathbb{R}$  and  $r, s \in V$ . Then

1. Where  $\mathbb{R}$  is a field,  $\mathbb{1}_{\mathbb{R}} = 1$ . Then for any  $r \in V$ ,  $\mathbb{1}_{\mathbb{R}} \otimes r = r^1 = r$
2.  $\alpha \otimes (\beta \otimes r) = \alpha \otimes (r^\beta) = (r^\beta)^\alpha = r^{\beta\alpha} = (\alpha\beta) \otimes r$ .
3.  $(\alpha + \beta) \otimes r = r^{\alpha+\beta}$ . Similarly,  $\alpha \otimes r \oplus \beta \otimes r = r^\alpha r^\beta = r^{\alpha+\beta}$ .
4.  $\alpha \otimes (r \oplus s) = \alpha \otimes (rs) = (rs)^\alpha$ . Similarly,  $\alpha \otimes r \oplus \alpha \otimes s = r^\alpha s^\alpha = (rs)^\alpha$

QUESTION 2

**Part (a):**  $M_{n \times m}(\mathbb{F})$  is an abelian group, where  $(a_{ij}) \oplus (b_{ij}) = (a_{ij} + b_{ij})$  for all  $(a_{ij}), (b_{ij}) \in M_{n \times m}(\mathbb{F})$ , and  $+$  is defined as usual over  $\mathbb{F}$ .

*Group axioms:* Define the neutral element  $\mathbb{0}_M$  to be the matrix of  $\mathbb{0}_{\mathbb{F}}$  entries. Choose  $(a_{ij}) \in M_{n \times m}(\mathbb{F})$ . Then  $(a_{ij}) \oplus \mathbb{0}_M = (a_{ij} + \mathbb{0}_{\mathbb{F}}) = (a_{ij})$ .

Furthermore, we have the inverse element  $(-a_{ij})$  for  $(a_{ij})$ , the matrix of additive inverses in  $\mathbb{F}$  for each coordinate  $a_{ij}$ .

$M_{n \times m}(\mathbb{F})$  is associative:  $[(a_{ij}) \oplus (b_{ij})] \oplus (c_{ij}) = (a_{ij} + b_{ij} + c_{ij}) = (a_{ij}) \oplus [(b_{ij}) \oplus (c_{ij})]$ ; and commutative:  $(a_{ij} \oplus b_{ij}) = (a_{ij} + b_{ij}) = (b_{ij}) \oplus (a_{ij})$ .

$M_{n \times m}(\mathbb{F})$  is a vector space over  $\mathbb{F}$ , where scalar multiplication is defined  $\lambda \otimes (a_{ij}) = (\lambda a_{ij})$ , and  $\lambda a_{ij}$  is defined as usual over  $\mathbb{F}$ . Let  $\alpha, \beta \in \mathbb{F}$ . We check the axioms:

1.  $\mathbb{1}_{\mathbb{F}} \otimes (a_{ij}) = (\mathbb{1}_{\mathbb{F}} a_{ij}) = (a_{ij})$ , since  $a_{ij} \in \mathbb{F}$ .
2.  $\alpha \otimes [\beta \otimes (a_{ij})] = \alpha \otimes (\beta a_{ij}) = (\alpha \beta a_{ij}) = [\alpha \beta] \otimes (a_{ij})$
3.  $[\alpha + \beta] \otimes (a_{ij}) = ([\alpha + \beta] a_{ij})$ . Similarly,  $\alpha \otimes (a_{ij}) \oplus \beta \otimes (a_{ij}) = (\alpha a_{ij}) \oplus (\beta a_{ij}) = (\alpha a_{ij} + \beta a_{ij}) = ([\alpha + \beta] a_{ij})$
4.  $\alpha \otimes [(a_{ij}) \oplus (b_{ij})] = \alpha \otimes (a_{ij} + b_{ij}) = (\alpha [a_{ij} + b_{ij}]) = (\alpha a_{ij} + \alpha b_{ij}) = (\alpha a_{ij}) \oplus (\alpha b_{ij}) = \alpha \otimes (a_{ij}) \oplus \alpha \otimes (b_{ij})$

For notational reasons,  $[ ]$  will be used for associativity.  $( )$  are all sequences.

Dropping the  $\oplus, \otimes$  notation now.

**Part (b):** Consider  $M_n(\mathbb{F})$  and  $\text{Sym}_n(\mathbb{F})$ , the set of matrices for which  $(a_{ij}) = (a_{ji})$ . This is a subspace of  $M_n(\mathbb{F})$ . To check:

1. As above,  $\mathbb{0}_M$  is the  $n \times n$  matrix of  $\mathbb{0}_{\mathbb{F}}$  entries. This is clearly symmetric, so  $\mathbb{0}_M \in \text{Sym}_n(\mathbb{F})$
  2. Suppose  $(a_{ij})$  and  $(b_{ij})$  are in  $\text{Sym}_n(\mathbb{F})$ . Then  $(a_{ij} + b_{ij}) = (a_{ij}) + (b_{ij}) = (a_{ji}) + (b_{ji}) = (a_{ji} + b_{ji})$ , so  $(a_{ij}) + (b_{ij})$  is symmetric, i.e.  $x, y \in \text{Sym}_n(\mathbb{F}) \implies x + y \in \text{Sym}_n(\mathbb{F})$
  3. Suppose  $(a_{ij}) \in \text{Sym}_n(\mathbb{F})$ , and let  $\alpha \in \mathbb{F}$ . Then  $\alpha(a_{ij}) = (\alpha a_{ij})$ . Similarly,  $\alpha(a_{ij}) = \alpha(a_{ji}) = (\alpha a_{ji})$ , so especially  $(\alpha a_{ij}) = (\alpha a_{ji})$ , and  $\alpha(a_{ij}) \in \text{Sym}_n(\mathbb{F})$ .
- $\implies \text{Sym}_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$  is a subspace.

## QUESTION 3

- (a) Consider  $U := \{(x, y, z) : x, y, z > 0\}$ . For any  $a, b > 0$ , we know that  $a + b > 0$ , and thus if  $(x, y, z), (x', y', z') \in U$ , their sum  $(x + x', y + y', z + z') \in U$ . Thus,  $U$  is closed under addition.

This is not closed under scalar multiplication: take  $\lambda = 0$ , for example.  $0 \cdot (x, y, z) = (0, 0, 0)$ , and clearly 0 is not positive, so  $0(x, y, z) \notin U$

- (b) Consider  $V := \{(x, y, z) : x, y, z \in \mathbb{N}\}$ . This is closed under addition, since the naturals are closed under addition. However,  $\lambda(x, y, z) \notin V$ , where  $\lambda$  is irrational, for example, so it is not closed under scalar multiplication.

- (c) Consider  $W := \{(0, 0, z) : z \in \mathbb{R}\}$ . This is closed under addition:  $(0, 0, z) + (0, 0, z') = (0, 0, z + z') \in W$ . This is also closed under scalar multiplication:  $\lambda \cdot (0, 0, z) = (0, 0, \lambda z)$ . Lastly,  $0_{\mathbb{R}^3} \in W$ , so  $W \subseteq \mathbb{R}^3$  is a subspace. It is proper and nontrivial.

Proper:  $(1, 0, 0)$  cannot be represented  
Non-trivial:  $(0, 0, 1)$  exists

## QUESTION 4

It has been shown in class that  $X + Y$  is a subspace. Also,  $X + Y \supseteq X \cup Y$ , as any element in  $X \cup Y$  can be represented as  $x + \mathbf{0}_V$  or  $\mathbf{0}_V + y$ .

Suppose that  $U \supseteq X \cup Y$  is a subspace. Let  $v \in X + Y$ . Then  $v = x + y$  for  $x \in X$  and  $y \in Y$ . We know that  $x, y \in U$ , since  $x, y \in X \cup Y \subseteq U$ . Since  $U$  is a subspace, it is closed under addition, and so  $x + y \in U$ , or  $v \in U$ , and we conclude that  $U \supseteq X + Y$ .

## QUESTION 5

Let  $V$  be a vector space over  $\mathbb{F}$  and  $S \subseteq V$  be a subspace.

( $\implies$ ) Let  $S$  be linearly dependent. Then there exists at least one  $v_j \in S$  such that  $v_j \in \text{Span}(S \setminus v_j)$ , so  $v_j = b_1 v_1 + \dots + b_{j-1} v_{j-1} + b_{j+1} v_{j+1} + \dots + b_n v_n$ , where  $b_i$  not all zero and  $\{v_i\}_{i \in [n]} = S$

Let  $w \in \text{Span}(S)$ . Then we can write  $w = a_1 v_1 + \dots + a_n v_n$ , where  $a_i$  not all zero. Substituting our expression for  $v_j$  in the linear combination for  $w$  yields

$$w = a_1 v_1 + \dots + a_{j-1} v_{j-1} + (b_1 v_1 + \dots + b_{j-1} v_{j-1} + b_{j+1} v_{j+1} + \dots + b_n v_n) + a_{j+1} v_{j+1} + \dots + a_n v_n$$

This is a linear combination that does not contain  $v_j$ , so we conclude that  $w \in \text{Span}(S \setminus v_j) \implies \text{Span}(S) \subseteq \text{Span}(S \setminus v_j)$ , where clearly  $S \setminus v_j \subsetneq S$ . Furthermore,  $\text{Span}(S \setminus v_j) \subseteq \text{Span}(S)$  automatically, so  $\text{Span}(S) = \text{Span}(S \setminus v_j)$

( $\impliedby$ ) Suppose there exists a proper subset  $S' \subsetneq S$  such that  $\text{Span}(S) = \text{Span}(S')$ . Let  $w \in \text{Span}(S) = \text{Span}(S')$  be defined as  $v_1 + \dots + v_n$ . Then  $w = \sum_{i \in I} a_i v_i$ , where  $I \subsetneq [n]$ , and WLOG, all  $a_i$  are non-zero.

$$\implies v_1 + \dots + v_n = \sum_{i \in I} a_i v_i \implies v_1 + \dots + v_n - \sum_{i \in I} a_i v_i = 0$$

Since  $S' \subsetneq S$ , there exists some vectors in  $\{v_1, \dots, v_n\}$  which are *not* in  $\{v_i : i \in I\}$ . In particular, these vectors have the coefficient 1 in front. Thus, we've found a non-trivial linear combination = 0, and  $S$  is linearly dependent.

To clarify: one "throws out" any zero coefficients in  $w$ 's representation. Indices  $i : a_i \neq 0$  then make up the set  $I$ , which we know to be non-empty.

# QUESTION 6

Let  $V$  be a v.s. over a field  $\mathbb{F}$  for which  $2 \neq 0$ .

(  $\implies$  ) Suppose  $x, y \in V$  are independent. Consider  $a(x + y) + b(x - y) = 0$ , and assume that at least one of  $a, b \neq 0$ . Rearranging, we get  $(a + b)x + (a - b)y = 0$ . Since  $x, y$  are independent, we know  $a + b = 0$  and  $a - b = 0$ . This implies  $b = -b \implies 2 = 0$ , which is a contradiction. Thus,  $a = b = 0$ .

$\implies x + y, x - y$  are independent.

(  $\impliedby$  ) Suppose  $x + y, x - y$  are independent. As we did above, we can rearrange  $ax + by = 0$ , which we assume to be non-trivial.

$$ax + by = (x + y)\frac{a + b}{2} + (x - y)\frac{a - b}{2} = 0$$

This is non dividing by 0, since our assumption is that  $2 \neq 0$ !

Thus, since  $x + y, x - y$  are independent, we need  $\frac{a+b}{2} = 0$  and  $\frac{a-b}{2} = 0$ . This implies  $a + b = 0$  and  $a - b = 0 \implies b = -b \implies 2 = 0 \nmid$ .

$\implies x, y$  are independent.



## QUESTION 7

Let  $S \subseteq \mathbb{F}[t]$  be a possibly infinite set containing non-zero polynomials with pairwise different degrees. This is linearly independent. Consider an arbitrary, finite subset  $S_0 \subset S$ . If  $S_0$  is linearly independent, so is  $S$ . Take  $f \in S_0$  to have degree  $n$ , and assume  $f \in \text{Span}(S_0 \setminus f)$ . Since  $S_0 \subset S$  contains functions of distinct degrees, one writes  $f = \sum_{i \in I} a_i t^i$ , where  $n \notin I$ .

However, this is impossible: the summation would have degree  $\max(i \in I)$ , which is especially  $\neq n$ , while  $f$  has degree  $n$ .

$\implies f \notin \text{Span}(S_0 \setminus v)$ , so  $S_0$ , and thus  $S$ , is linearly independent.

## QUESTION 8

Consider the set  $S \subseteq \text{Sym}_n(\mathbb{F})$  of matrices with all zero entries except at a two particular locations,  $ij$  and  $ji$ , where a 1 is placed. If  $i = j$ , i.e. lie on the diagonal, only one 1 will be placed. This set is  $S = \{(z_{ij} + z_{ji}) : ij \in I\} \cup \{z_{ij} : ij \in D\}$ , where  $I$  is the set of coordinates in the upper triangle ( $i \neq j$ ), and  $D$  is the set of coordinates on the diagonal ( $i = j$ ).

Notice also that all elements of  $S$  are symmetric:  $(z_{ij} + z_{ji}) = (z_{ji} + z_{ij})$ , where  $i \neq j$ , and  $z_{ij} = z_{ji} = z_{ii}$  where  $i = j$ . The following is a rough enumeration of these elements:

$$S = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right\}$$

This is independent. Write an arbitrary linear combination in  $S$  as follows:

$$\sum_{ij \in I} \lambda_{ij}(z_{ij} + z_{ji}) + \sum_{ij \in D} \lambda_{ij}(z_{ij}) = \mathbf{0}_{\text{Sym}_n(\mathbb{F})}$$

This is exactly the following matrix:

$$\begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} & \cdots & \lambda_{n1} \\ \lambda_{21} & \lambda_{22} & \lambda_{32} & \cdots & \lambda_{n2} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \cdots & \lambda_{n3} \\ \vdots & \vdots & \vdots & \ddots & \\ \lambda_{n1} & \lambda_{n2} & \lambda_{n3} & \cdots & \lambda_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\Rightarrow \lambda_{ij} = 0$  for all  $i, j$ , so  $S$  is linearly independent. Now consider a nonzero symmetric matrix  $(a_{ij}) \in \text{Sym}_n(\mathbb{F}) \setminus S$ , and let  $\varphi(ij)$  be a function returning the value of  $(a_{ij})$  at each coordinate. Then we have:

$$(a_{ij}) - \sum_{ij \in I} \varphi(ij)(z_{ij} + z_{ji}) - \sum_{ij \in D} \varphi(ij)(z_{ij}) = \mathbf{0}_{\text{Sym}_n(\mathbb{F})}$$

This is a nontrivial linear combination, so  $S \cup \{(a_{ij})\}$  is linearly dependent.

$\Rightarrow S$  is maximally independent, and thus a basis for  $\text{Sym}_n(\mathbb{F})$ .

## QUESTION 9

Consider  $t$  and  $e^t$  in the vector space of continuous functions,  $C[\mathbb{R}]$ , over the field  $\mathbb{R}$ . Let  $at + be^t = 0$ , and assume WLOG that one of  $a, b \neq 0$ . Choose  $t = 0$ . Then  $b = 0$ . We update our assumption to find  $at = 0 \implies a = 0$ , since  $t$  is arbitrary.  $a = b = 0$ , so  $t, e^t$  are linearly independent.

$t$  and  $e^t$  do not form a basis. Consider  $t^2 = at + be^t$ . At  $t = 0$  we find that  $b = 0$ . We update our assumption to  $t^2 = at$ . Choosing  $t = 1$  yields  $a = 1$ , and we are left with  $t^2 = t$ , which is clearly not true in generality. Thus, we've found  $f \in C[\mathbb{R}]$   $t^2$  is continuous such that  $f \notin \text{Span}(t, e^t)$ , so  $\{t, e^t\}$  does not span  $C[\mathbb{R}]$ , and is not a basis.