ASSIGNMENT 2

MATH 251

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QUESTION 1

Let $V := \mathbb{R}^3$, $S = \{(1,1,1), (-1,1,0), (1,0,1), (0,1,1)\}$, and $I = \{(1,0,0), (1,1,0)\}$. First, to verify that S is spanning and I is linearly independent:

S spanning Consider $(a,b,c) \in \mathbb{R}^3$. This is $a[(1,1,1) - (0,1,1)] + b[(1,1,1) - (1,0,1)] + c[(1,0,1) - (1,1,1) + (0,1,1)]$. Observe that each combination of vectors associated with a, b, c is equal to $(1,0,0), (0,1,0), (0,0,1)$, respectively.

I linearly indep. Suppose $x(1,0,0) + y(1,1,0) = (0,0,0)$. Then $y(1) = 0$. But then $x(1) = 0$.

Let $S' \subseteq S := \{(-1,1,0), (1,0,1)\}$, and consider $S' \cup I = \{(-1,1,0), (1,0,1), (1,0,0), (1,1,0)\}$. Vectors from $S' \cup I$ produce the standard basis of $\mathbb{R}^3$ through linear combinations, and so $S' \cup I$ is spanning. As proof:

For any $(a,b,c) \in \mathbb{R}^3$, $(a,b,c) = a(1,0,0) + b[(-1,1,0) + (1,0,0)] + c[(1,0,1) - (1,0,0)]$.

Question 2

Let U and V be finite dimensional v.s. over $\mathbb{F}$, with bases $B_U := \{u_1, ..., u_m\}$ and $B_V := \{v_1, ..., v_n\}$, respectively. Consider $U \times V$ and the set

$$B_{U\times V} \subseteq U \times V := \{(u_i, 0), (0, v_i) : u_i \in B_U, v_i \in B_V\}$$

We'll show that this is a basis for $U \times V$, which means that $\dim(U \times V) = \dim(U) + \dim(V)$, since $B_{U\times V}$ has exactly $|B_U| + |B_V|$ elements.

$B_{U\times V}$ is linearly independent. Let $a_1(u_1, 0) + ... + a_m(u_m, 0) + b_1(0, v_1) + ... + b_n(0, v_n) = (0, 0)$, where $a_i, b_i \in \mathbb{F}$. We then have

$$a_1 u_1 + ... + a_m u_m = 0 \quad \text{and} \quad b_1 v_1 + ... + b_n v_n = 0$$

But B_U and B_V are bases, so they are linearly independent, and we conclude that $a_i = 0$ and $b_i = 0 \ \forall i$.

$B_{U\times V}$ is spanning for $U \times V$. Take an arbitrary (u, v), and let $u = a_1 u_1 + ... + a_m u_m$ and $v = b_1 v_1 + ... + b_n v_n$ be the unique representation of u and v in their bases. Then we can write the following

If $u \in B_U$ or $v \in B_V$, this combination is $u = u_i$ or $v = v_i$ for some i, and the argument remains the same

$$(u, v) = a_1(u_1, 0) + ... + a_m(u_m, 0) + b_1(0, v_1) + ... + b_n(0, v_n)$$

Thus, $B_{U\times V}$ is linearly independent and spanning for $U \times V$, so it is a basis, and we are done.

QUESTION 3

Let $p_i(t)$ be Lagrange polynomials, and note $p_i(c_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$

Consider the set $B = \{p_i(t) : i \in [0, n]\}$, which has $n + 1$ elements. Since the standard basis for $\mathbb{F}[t]_n$, $\{1, t, ..., t_n\}$, also has $n + 1$ elements, if one shows that B is either spanning or independent for $\mathbb{F}[t]_n$, it will be a basis. We'll show that it's independent:

Also note that $B \subseteq \mathbb{F}[t]_n$.

c.f. corollary from substitution lemma, Lec. 7

Suppose B is linearly dependent, and write $0 = a_0 p_0(t) + ... + a_n p_n(t)$, where $a_i \neq 0$ at least once. This must hold for all $t \in \mathbb{F}$, so it especially holds for $t = c_0$. Then we conclude that $0 = a_0$, as $p_i(c_0) = 0$ where $i \neq 0$, and $p_0(c_0) = 1$.

We continue: let $0 = a_1 p_1(t) + ... + a_n p_n(t)$, set $t = c_1$, and find $0 = a_1$.

One may repeat this process, evaluating for $t \in \{c_i\}_{i \leq n}$, each time concluding that $a_i = 0$. B is then linearly independent by contradiction, and is a basis from above.

Question 4

Let V be finite dimensional, $W_1, W_2 \subseteq V$ be subspaces. Then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Let B be a basis for $W_1 \cap W_2$. Then $B \subseteq W_1$ is linearly independent, so $\exists B_1$ such that $B \cup B_1$ is a basis for W_1. Similarly, $\exists B_2$ with $B \cup B_2$ being a basis for W_2.

B, B_1, and B_2 are mutually disjoint sets. Suppose $B_1 \cap W_1 \cap W_2 \neq \varnothing$, and let b be in this set. Let $w \in W_1$ have a unique representation containing b. Then one expresses b as a further combination of vectors in B, since B is a basis for $W_1 \cap W_2$. This violates unique representation. ↯ Similarly, we conclude $B_2 \cap W_1 \cap W_2 = \varnothing$.

We can guarentee such a w exists, since $B \cup B_1$ is mimimally spanning.

Why does this violate unique representation? If $B \cup B_1$ is a basis for W_1, then our chosen $w \in W_1$ can be expressed as a combination containing b, and a combination not containing b.

Thus, $B_1 \cap B_2 = \varnothing$, since $B_1 \subseteq W_1$, $B_2 \subseteq W_2$, and neither is contained in the intersection. By definition, $B_1 \cap B = \varnothing$ and $B_2 \cap B = \varnothing$ also. We conclude that $|B \cup B_1 \cup B_2| = |B \cup B_1| + |B \cup B_2| - |B| = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

To show that $B \cup B_1 \cup B_2$ is a basis for $W_1 + W_2$, we show it is spanning and linearly independent:

Spanning Consider $w_1 + w_2 \in W_1 + W_2$. Note the notational trick $\{B \cup B_1 \cup B_2\} = \{B \cup B_1\} \cup \{B \cup B_2\}$. Since $w_1 \in W_1$, it has a unique representation in $B \cup B_1$, and similarly w_2 has a unique representation in $B \cup B_2$. Thus, $w_1 + w_2$ can be written as a combination of vectors in $B \cup B_1 \cup B_2$, i.e. $W_1 + W_2 \subseteq \text{Span}(B \cup B_1 \cup B_2)$. The converse holds trivially, so $B \cup B_1 \cup B_2$ is spanning.

"converse holds trivially"... if $w \in \text{Span}(B \cup B_1 \cup B_2)$, then $w = w_1 + w_2$, where $w_1 \in \text{Span}(B \cup B_1) \subseteq W_1, w_2 \in \text{Span}(B_2) \subseteq W_2$.

Independence We claim $\text{Span}(B_1) \cap \text{Span}(B \cup B_2) = \varnothing$. We have $\text{Span}(B \cup B_2) = W_2$, since $B \cup B_2$ is a basis. Furthermore, $\text{Span}(B_1) \subseteq W_1 \setminus (W_1 \cap W_2)$ by the same arguments expressed above: if $\exists b \in \text{Span}(B_1)$ which is also contained in $W_1 \cap W_2$, then it could instead be written as a combination of vectors in B, violating unique representation in the basis. Thus, our claim holds.

With this in mind, let $x_i \in B_1$, $y_i \in B_2$, $z_i \in B$. By the disjointedness shown above, we can let these elements be distinct. Suppose $B \cup B_1 \cup B_2$ is linearly dependent, and write the following for not-all-zero $\alpha_i \in \mathbb{F}$:

$$\alpha_1 x_1 + \dots + \alpha_l x_l + \alpha_{l+1} y_{l+1} + \dots + \alpha_m y_m + \alpha_{m+1} z_{m+1} + \dots + \alpha_n z_n = 0$$

There is some subtlety here: what if $\alpha_1, \dots, \alpha_l$ were zero? Then this would show linear dependence of $B \cup B_2$, which contradicts this set being a basis.

Re-ordering, we get:

$$\alpha_{l+1} y_{l+1} + \dots + \alpha_m y_m + \alpha_{m+1} z_{m+1} + \dots + \alpha_n z_n = -\alpha_1 x_1 - \dots - \alpha_l x_l$$

This implies that there exists an element in $\text{Span}(B_1)$ that is also in $\text{Span}(B \cup B_2)$, which is not the case ↯.

QUESTION 5

Part (a): Suppose that $v_1 + W = v_2 + W$. Then $\exists w \in W$ such that $v_1 = v_2 w$.

We wish to show that $(\alpha v_1) + W$ and $(\alpha v_2) + W$ are equivalent. This holds iff one can write $\alpha v_1 = \alpha v_2 w$ for some $w \in W$. Multiplying by α^{-1}, this condition is reduced to $v_1 = v_2 w$, and this is true from above.

Part (b): We know that V/W is abelian, and have a well-defined notion of scalar multiplication over $\mathbb{F}$. Thus, we just check the axioms:

1. $\mathbb{1}_{\mathbb{F}}\overline{v} = \overline{\mathbb{1}_{\mathbb{F}}v} = \overline{v}$
2. $\alpha(\beta\overline{v}) = \alpha(\overline{\beta v}) = \overline{\alpha\beta v} = (\alpha\beta)\overline{v}$
3. $(\alpha+\beta)\overline{v} = \overline{(\alpha+\beta)v} = \overline{\alpha v + \beta v}$ (v.s. properties of V) $= \overline{\alpha v} + \overline{\beta v} = \alpha\overline{v} + \beta\overline{v}$
4. $\alpha\overline{u+v} = \overline{\alpha(u+v)} = \overline{\alpha u + \alpha v} = \overline{\alpha u} + \overline{\alpha v} = \alpha\overline{u} + \alpha\overline{v}$

Part (c): Consider $\overline{v} \in V/W$. Since v has a unique representation in B:

$$\begin{aligned} \overline{v} &= \overline{\alpha_1 w_1 + ... + \alpha_k w_k + \beta_1 u_1 + ... + \beta_m u_m} \\ &= \overline{\alpha_1 w_1} + ... + \overline{\alpha_k w_k} + \overline{\beta_1 u_1} + ... + \overline{\beta_m u_m} \\ &= \alpha_1\overline{w_1} + ... + \alpha_k\overline{w_k} + \beta_1\overline{u_1} + ... + \beta_m\overline{u_m} \end{aligned}$$

where $\alpha_i, \beta_i \in \mathbb{F}$. However, $\{w_1, ..., w_k\} \subseteq W$, so $\overline{w_i} = \overline{0}$ in the quotient space $\forall i \in [1,k]$. Thus, one can write $\overline{v} = \beta_1\overline{u_1} + ... + \beta_m\overline{u_m}$ only, so $\overline{v} \in \text{Span}\{\overline{u_1}, ..., \overline{u_m}\}$. Clearly, $\text{Span}\{\overline{u_1}, ..., \overline{u_m}\} \subseteq V/W$, so this set is spanning.

Moreover, this set is independent. Suppose otherwise, and write

$$\overline{0} = \beta_1\overline{u_1} + ... + \beta_m\overline{u_m} = \overline{\beta_1 u_1 + ... + \beta_m u_m}$$

where not all $\beta_i = 0$. This implies that $\beta_1 u_1 + ... + \beta_m u_m \in W$. However, we know this to be false: let $w \in W$ be this element. B is minimally spanning, so there exists a $v \in V$ whose unique representation contains w. But we claim one can write w as a combination of vectors in $\{u_1, ..., u_m\}$, so this representation loses uniqueness.

$\implies \beta_i = 0\ \forall i$, and $\{u_1, ..., u_m\}$ is a basis for V/W

Part (d) From above $\{u_1, ..., u_m\}$ is a basis for V/W, so $\dim(V/W) = m$. We know that B_W is a basis for W with k elements, so $\dim(W) = k$. Lastly, the basis B for V has $k+m$ elements, so $\dim(V) = k+m$.

$\implies \dim(V/W) = \dim(V) - \dim(W)$ $\square$

Question 6

Let V be a vector space over $\mathbb{F}$, and let $S \subseteq V$ be a finite spanning set. Then $\exists$ a basis $B \subseteq S$ for V. We'll show by induction. Denote $|S| = n$:

$n = 1$: If S is a spanning singleton, it is linearly independent, and thus a basis. One handles the case $S = \{0\}$ separately: if this set is spanning, then $V = \{0\}$, and is thus minimally spanning, so S is a basis.

$n \to n+1$: Let $|S| = n+1$, where S is spanning for V. Suppose S is *not* minimally spanning (otherwise, S would be a basis, and we are done). Then $\exists s \in S$ such that $S \setminus \{s\}$ is still spanning.

Notably, $|S \setminus \{s\}| = n \overset{\text{ind. hyp.}}{\Longrightarrow} \exists$ a basis $B \subseteq S$ for V. $\square$

QUESTION 7

$T(-3,0) = -3T(1,0) = -3[T(1,-1)+T(0,1)] = -3[(2,3)+(0,0)] = -3(2,3) = (-6,-9)$

QUESTION 8

Let B and C be finite, countable, or uncountable. Let $T_0 : B \to C$ be a bijection between these bases. We guarantee the existence of a linear transformation $T : V \to W$ which extends T_0. It is left to show that T is bijective (i.e. is an isomorphism).

Injective Let $T(x) = T(y)$. Both x, y have a unique representation in the basis B. We'll write these as follows, where $\alpha_i, \beta_i \in \mathbb{F}$, and $b_i \in B$.

$$x = \alpha_1 b_1 + ... + \alpha_t b_t + \alpha_l b_l + ... + \alpha_k b_k$$

$$y = \beta_1 b_1 + ... + \beta_n b_t + \beta_m b_m + ... + \beta_n b_n$$

Notationally, what we have is that $b_1 ... b_t$ are (possibly) shared between the representations of x and y, while $b_l, ..., b_k$ belong exclusively to x and $b_m, ..., b_n$ belong exclusively to y. This covers all possible cases, and we wish to show that $b_l = ... = b_k = b_m = ... = b_n = \mathbb{0}_V$ and $\alpha_i = \beta_i$ for $i \in [1, t]$.

Why do we have to be so careful? If B, C were finite, then we could enumerate all b_i no problem, but this is not the case.

For the vectors expressed above, let $i \to \sigma(i)$ be the bijective mapping such that $T(b_i) = c_{\sigma(i)}$, where $c_{\sigma(i)} \in C$. Breaking apart $T(x)$ and $T(y)$, we get:

We use T here, instead of T_0, as $T_0 = T$ for $b_i \in B$. This is where we also use the fact that T_0, and thus $\sigma(i)$, is bijective.

$$\begin{aligned} T(x) &= T(\alpha_1 b_1 + ... + \alpha_t b_t + \alpha_l b_l + ... + \alpha_k b_k) \\ &= \alpha_1 T(b_1) + ... + \alpha_t T(b_t) + \alpha_l T(b_l) + ... + \alpha_k T(b_k) \\ &= \alpha_1 c_{\sigma(1)} + ... + \alpha_t c_{\sigma(t)} + \alpha_l c_{\sigma(l)} + ... + \alpha_k c_{\sigma(k)} \\ T(y) &= \beta_1 c_{\sigma(1)} + ... + \beta_t c_{\sigma(t)} + \beta_m c_{\sigma(m)} + ... + \beta_n c_{\sigma(n)} \end{aligned}$$

Similar derviation for $T(y)$

Given that $T(x) = T(y) \implies T(x) - T(y) = \mathbb{0}_V$, we write

$$(\alpha_1 - \beta_1)c_{\sigma(1)} + ... + (\alpha_t - \beta_t)c_{\sigma(t)} + \alpha_l c_{\sigma(l)} + ... + \alpha_k c_{\sigma(k)} - \beta_m c_{\sigma(m)} - ... - \beta_n c_{\sigma(n)} = \mathbb{0}_V$$

C is a basis, and thus independent, so all these coefficients must be 0, i.e. $\alpha_i = \beta_i$ for $i \in [1, t]$, $\alpha_i = 0$ for $i \in [l, k]$, and $\beta_i = 0$ for $i \in [m, n]$. We conclude that $x = y = \alpha_1 b_1 + ... + \alpha_t b_t$.

Surjective Consider $w \in W$. This has a unique representation in the basis C, say $w = \alpha_1 c_1 + ... + \alpha_n c_n$. One then has

$$T(\alpha_1 b_{\sigma^{-1}(1)} + ... + \alpha_n b_{\sigma^{-1}(n)}) = \alpha_1 T(b_{\sigma^{-1}(1)}) + ... + \alpha_n T(b_{\sigma^{-1}(n)}) = \alpha_1 c_1 + ... + \alpha_n c_n = w$$

Thus, T is bijective, and we are done.

QUESTION 9

We'll construct a basis for $\ker(T)$, where $T : M_n(\mathbb{F}) \to \mathbb{F}$ sends $A \to \operatorname{tr}(A)$, and $\operatorname{tr}(A)$ is defined to be $a_{11} + \ldots + a_{nn}$, the sum of diagonal entries. Note that the $\ker(\operatorname{tr}(A))$ is just the set of matrices whose diagonal entries sum to zero.

Let (a_{ij}) denote the matrix with a 1 entry at coordinate ij and 0s elsewhere. Let (z_i) be the matrix with a 1 entry at position ii, and a -1 at position $(i-1)(i-1)$. Consider the following set of $n^2 - 1$ elements:

$$B = \underbrace{\left\{(a_{ij}) : i \neq j\right\}}_{n^2-n \text{ elements}} \cup \underbrace{\{(z_i) : i \in [1, n-1]\}}_{n-1 \text{ elements}}$$

Here is a rough enumeration of B:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix}$$

$$\bigcup$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

Note that $B \subseteq \ker(\operatorname{tr}(A))$, since $\{(a_{ij}) : i \neq j\} \subseteq B$ can only place 1 in positions $i \neq j$, i.e. not on the diagonal. $\{(z_i) : i \in [1, n-1]\} \subseteq B$ is defined such that the trace of these matrices is always $1 - 1 + 0 + \ldots + 0 = 0$.

B **is spanning**: suppose we have a matrix A whose diagonal elements add up to 0. Consider only non-diagonal elements, and suppose they have a $\beta_{ij} \in \mathbb{F}$ entry at position ij. Then $\sum_{i \neq j} \beta_{ij}(a_{ij})$ will fill all non-diagonal entries.

$\beta_{ij}(a_{ij})$ is just the matrix with β_{ij} at position ij, and 0s elsewhere. Summing over $i \neq j$ yields a matrix with all non-diagonal elements of A, and 0s on the diagonal.

As for the diagonal entries, denoted $\beta_i \in \mathbb{F}$, we have

$$\sum_{i=1}^{n-1} \sum_{k=1}^{i} \beta_k(z_i) = \begin{bmatrix} \beta_1 & & & & & \\ & \beta_2 & & & & \\ & & \beta_3 & & \text{0s elsewhere!} & \\ & & & \ddots & & \\ & & & & \beta_{n-1} & \\ & & & & & -\beta_1 - \beta_2 - \ldots - \beta_{n-1} \end{bmatrix}$$

Written explicitly, this is: $\beta_1(z_1) + (\beta_1 + \beta_2)(z_2) + \ldots$
$\ldots + (\beta_1 + \ldots + \beta_{n-1})(z_{n-1})$

One passes along $-\beta_i$ to the $i+1^{\text{th}}$ diagonal, so one must add back β_i each iteration. This leads to a large negative overflow in the last entry.

However, diagonal entries must add to 0, so $\beta_n = -\beta_1 - \beta_2 - \ldots - \beta_{n-1}$, and so this bottom entry is really β_n, as desired. Combining both sums, we have that any matrix with $\text{tr}(A) = 0$ can be represented as

$$\sum_{i \neq j} \beta_{ij}(a_{ij}) + \sum_{i=1}^{n-1} \sum_{k=1}^{i} \beta_k(z_i)$$

B **is linearly independent**: consider an arbitrary linear combination

$$\star \quad \sum_{i=1}^{n-1} \beta_i(z_i) + \sum_{i \neq j} \beta_{ij}(a_{ij}) \qquad \beta_{ij}, \beta_i \in \mathbb{F}$$

The first part of this summation yields the matrix

$$\sum_{i=1}^{n-1} \beta_i(z_i) = \begin{bmatrix} \beta_1 & & & & & \\ -\beta_1 + \beta_2 & & & & & \\ & -\beta_2 + \beta_3 & & & \text{0s elsewhere!} & \\ & & \ddots & & & \\ & & & -\beta_{n-2} + \beta_{n-1} & & \\ & & & & -\beta_{n-1} & \end{bmatrix}$$

The second part is easier to visualize: all non-diagonal entries will have a single β_{ij} entry corresponding to the ij-th coordinate. Clearly, setting $\star$ to the zero matrix will force all non-diagonal entries to be 0 as well, so $\beta_{ij} = 0$ for $i \neq j$.

For the non-diagonal entries, we immediately have that $\beta_1 = 0$. But then $-\beta_1 + \beta_2 = 0$, so $\beta_2 = 0$. Then $-\beta_2 + \beta_3 = 0 \implies \beta_3 = 0$, and so on. One deduces that all $\beta_i = 0$ for $i \in [1, n-1]$.

All coefficients are 0, so B is linearly independent, and is thus a basis. In particular, $\dim(\ker(T)) = n^2 - 1$, as $|B| = n^2 - 1$ from above.

Extraneous: $\text{Im}(\text{tr}(A))$ is precisely $\mathbb{F}$, which has dimension 1 ($\{\mathbb{1}\}$ is a basis.) $\dim(M_n(\mathbb{F})) = n^2$, so by rank-nullity, $\dim(\ker(T)) = n^2 - 1$.