Higher Algebra 2

MATH 571

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CONTENTS

I Review

Much of the content in this section should be familiar from MATH 457. We will require a cursory understanding of tensor products, categories, and functors. The official prerequisite for this course is MATH 570 (which includes category theory, commutative algebra, Noetherian rings), but these notes will be written from the point of view of someone (me) who has not studied these topics.

TENSOR PRODUCTS

Let $\mathbf{Mod_R}$ and ${}_\mathbf{R}\mathbf{Mod}$ denote the categories of left and right modules over a ring R, respectively. Recall that, for an R-module M, $r \in R$, and $m \in M$, left modules act by $(r, m) \mapsto rm$ and right modules act by $(r, m) \mapsto mr$.

If a module is both a left and right module, and obeys all respective module axioms, we
DEF 1.1 call it a *bimodule*, and write ${}_\mathbf{S}\mathbf{Mod_R}$ for the category of bimodules.

DEF 1.2 If $A \in \mathbf{Mod_R}$ and $B \in {}_\mathbf{R}\mathbf{Mod}$, an *R-biadditive* map is a function

$$f : A \times B \to G$$

where H is a abelian. Additionally, we require that

- $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$
- $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$
- $f(ar, b) = f(a, rb)$

As H is a group, we do not impose any scaling qualities for f with respect to R.

We would like to construct an abelian group G and associated R-biadditive function φ such that, for any R-biadditive function f, there is a unique group homomorphism g with

$$\begin{tikzcd} A \times B \arrow[r, "\varphi"] \arrow[dr, "f"'] & G =: A \otimes_R B \arrow[d, "g"] \\ & H \end{tikzcd}$$

DEF 1.3 commuting. If such a pair (G, φ) exists, we say it satisfies the *universal property*.

Construction

We will construct a group G which satisfies the universal property, as above

Consider $H = \mathbb{Z} \cdot (A \times B)$, the $\mathbb{Z}$-module, and hence free abelian group. In other words,

$$H \ni h = \oplus_{(a,b) \in A \times B} k_{(a,b)} \cdot (a, b) \quad \text{where} \quad k_{(a,b)} \in \mathbb{Z}$$

Furthermore, consider the subgroup $N < H$ by

$$N = \{(a_1 + a_2, b) - (a_1, b) - (a_2, b)\} \cup \{(a, b_1 + b_2) - (a, b_1) - (a, b_2)\} \cup \{(ar, b) - (a, rb)\}$$

One shows manually that this is a group

under $a, a_i \in A$, $b, b_i \in B$, and $r \in R$.

Define $A \otimes_R B := H/N$, and call this the *tensor product* of A and B over R. DEF 1.4

Let $\varphi : A \times B \to A \otimes_R B$ be the natural map formed by viewing (a, b) as an element of the $\mathbb{Z}$-module H, and modding out by N as above.

Immediately, we see that the subgroup N ensures that φ is biadditive.

We denote the image of (a, b) under φ by $a \otimes b$, and call the result a *tensor*. DEF 1.5

$(\varphi, A \otimes_R B)$ has the universal property. **PROP 1.1**

$V^* = \mathrm{Hom}_k(V, k)$ is called the *dual vector space*. DEF 1.6

Theorem 1.1 Properties of the Tensor Product

1. $\mathrm{Hom}_k(V, W) = V^* \otimes_k W$, where k is a field.
2. $\dim(V \otimes_k W) = \dim_k(V) \cdot \dim_k(W)$
3. If $f \in \mathrm{Hom}_R(A, A')$, $g \in \mathrm{Hom}_R(B, B')$, then

   $$f \otimes g : A \otimes_R B \to A' \otimes_R B' \text{ given by } (a \otimes b) \mapsto f(a) \otimes f(b)$$

   is a homomorphism.
4. If $A \cong A'$ and $B \cong B'$, then $A \otimes_R B \cong A' \otimes_R B'$
5. $A \otimes_R R = A$ and $R \otimes_R B = B$
6. $(\oplus_{i \in I} A_i) \otimes B \cong \otimes_{i \in I} A_i \otimes B$

REPRESENTATIONS OF FINITE GROUPS

A *linear representation* of a finite group G is a vector space V over a field $\mathbb{F}$ equipped with a group action DEF 1.7

$$G \times V \to V$$

that respects the vector space, i.e. $m_g : V \to V$ with $m_g(v) = gv$ is a linear transformation. We make the following assumptions unless otherwise stated:

1. G is finite.
2. V is finite dimensional.
3. $\mathbb{F}$ is algebraically closed and of characteristic 0. We write $\mathbb{F} = \mathbb{C}$.

Since V is a G-set, $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ which sends $g \mapsto m_g$ is a homomorphism.

Relatedly, if $\dim(V) < \infty$, then $\rho : G \mapsto \mathrm{Aut}_{\mathbb{C}}(V) = \mathrm{GL}_n(\mathbb{C})$.

DEF 1.8 The *group ring* $\mathbb{C}[G]$ is a (typically) non-commutative ring consisting of all finite linear combinations $\{\sum_{g\in G}\lambda_g g : \lambda_g \in \mathbb{C}\}$, with $1 \cdot \mathbb{1}_G = \mathbb{1}_{\mathbb{C}[G]}$. It's endowed with the multiplication rule

$$\left(\sum_{g\in G}\alpha_g g\right)\left(\sum_{h\in G}\beta_h h\right) = \sum_{g,h\in G\times G}\alpha_g\beta_h(gh)$$

PROP 1.2 We can view representations as a module over the group ring $\mathbb{C}[G]$.

PROOF. Let V be a $\mathbb{C}[G]$-module. Consider $g \in G \subseteq \mathbb{C}[G]$, $\lambda\mathbb{1}_G \in \mathbb{C}[G]$, and $v_1, v_2 \in V$. Since V is a $\mathbb{C}[G]$-module,

$$g(v_1 + v_2) = gv_1 + gv_2 \qquad (gh)v_1 = g(hv_1)$$

Then: $(g\lambda\mathbb{1}_G)v_1 = (\lambda(g\mathbb{1}_G))v_1 = (\lambda g)v_1$. But also, $(g\lambda\mathbb{1}_G)v_1 = g(\lambda\mathbb{1}_G v_1) = g(\lambda v_1)$. Hence, the map $v \mapsto gv$ is a linear transformation on V over $\mathbb{C}$. □

We will frequently return to this view when module theory is more convenient.

DEF 1.9 As a preliminary example, consider $\rho : G \to \{\mathbb{1}\}$, the *trivial representation*, which maps $\rho(g)(v) = v$. We will denote the trivial representation simply by $\mathbb{1}$, sometimes.

Restricted and Induced Representations

Let $H < G$ be a subgroup. Then we consider a functor between the categories of representations of G and H,

$$\mathrm{Res}_H^G : \mathbf{Rep}(G) \to \mathbf{Rep}(H) : \rho \mapsto \rho|_H = \mathrm{Res}_H^G(\rho)$$

DEF 1.10 called the *restricted representation* of G to H. Analogously, this sends a $\mathbb{C}[G]$-module V to the submodule W defined over $\mathbb{C}[H]$.

Similarly, we consider a functor

$$\mathrm{Ind}_H^G : \mathbf{Rep}(H) \to \mathbf{Rep}(G) : V \mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

DEF 1.11 called the *induced representation* of H to G, where we view V as a $\mathbb{C}[G]$-module. Observe that $\dim_{\mathbb{C}[H]}(\mathbb{C}[G]) = [G : H]$, so $\dim(\mathrm{Ind}_H^G) = [G : H]\dim(V)$.

As an example, consider $H = \{\mathbb{1}\}$ with the representation $V = \mathbb{C}$. Then $\mathrm{Ind}_H^G(\mathbb{C}) = \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[G]$.

Dual Representations

Let ρ, V be a representation of G. Recall the dual, $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, the set of linear transformations from $V \to C$. Given an endomorphism $T : V \to V$, we call

$$T^t : V^* \to V^* : (T^t\varphi)(v) := \varphi(Tv)$$

the *transpose*. If $\beta = \{v_1, ..., v_n\}$ is a basis for V, then we construct the *dual basis* $\beta^* = \{\varphi_1, ..., \varphi_n\}$ for V^*, where $\varphi_i(v_j) = \delta_{ij}$. In the dual basis, we have DEF 1.12 DEF 1.13

$$[T^t]_{\beta^*} = [T]_\beta^t \implies \mathrm{tr}(T) = \mathrm{tr}(T^t)$$

PROP 1.3

When $T = \rho(g) : V \to V$, we also observe

$$(\rho(gh)^t \varphi)(v) = (\rho(h)^t \rho(g)^t \varphi)(v) \implies \rho(gh)^t = \rho(h)^t \rho(g)^t$$

Given a representation ρ, $\rho^* : G \to \mathrm{GL}(V^*)$ by $g \mapsto \rho(g^{-1})^t$ is called the *dual representation*. DEF 1.14

$\chi_{\rho^*} = \overline{\chi_\rho}$ **PROP 1.4**

PROOF. If $g \in G$ has order n, then $\rho(g)$ has order $m|n$, since $\rho(g)^n = \rho(g^n) = \rho(\mathbb{1}) = I$. Hence, in a certain basis,

$$\rho(g) = \begin{pmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{pmatrix} \quad \text{where} \quad \xi_i^m = 1$$

If ξ is a root of unity, $\xi\overline{\xi} = 1$ (try viewing this geometrically)

It follows that

$$\rho(g^{-1}) = \begin{pmatrix} \xi_1^{-1} & & \\ & \ddots & \\ & & \xi_n^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\xi_1} & & \\ & \ddots & \\ & & \overline{\xi_n} \end{pmatrix}$$

Thus, $\mathrm{tr}(\rho^*(g)) = \mathrm{tr}(\rho(g^{-1})^t) = \mathrm{tr}(\rho(g^{-1})) = \overline{\mathrm{tr}(\rho(g))}$, using Prop 1.2. $\square$

1-Dim Representations

A *1-dim representation* (ρ, V) is a representation with $\dim(V) = 1$. In this case, as V is a $\mathbb{C}$-vector space and $\rho(g) \in \mathrm{GL}(V)$, we write $V = \mathbb{C}^\times$. Also observe that $\chi_\rho = \rho$. DEF 1.15

$G^* = \mathrm{Hom}(G, \mathbb{C}^\times)$, as groups, is called the *group of multiplicative characters*. DEF 1.16

$(G^{ab})^* \cong G^*$ **PROP 1.5**

PROOF. Recall that $G^{ab} = G/N$, where N is the smallest normal subgroup such that G/N is abelian. Hence, if $f \in (G^{ab})^*$ is a homomorphism $f : G^{ab} \to \mathbb{C}^\times$, then $f \circ \pi : G \twoheadrightarrow G/N \to \mathbb{C}^\times$ is also a homomorphism. Furthermore, any $F : G \to \mathbb{C}^\times$ must factor uniquely into $f \circ \pi$. See the following diagram:

$$\begin{tikzcd} G \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathbb{C}^\times \\ & G^{ab} \arrow[ur, "f"'] & \end{tikzcd}$$

$\square$

Tensor Products

If ρ is a finite representation of G and τ is a 1-dim representation, we can generate a new representation

$$\rho \otimes \tau : G \to \mathrm{GL}(V \otimes_{\mathbb{C}} \mathbb{C}) \cong \mathrm{GL}(V) : g \mapsto \tau(g)\rho(g)$$

Note that $\tau(g) \in \mathbb{C}^{\times}$. Note that, then, $\chi_{\rho\otimes\tau} = \tau\chi_{\rho}$.

In generality, given two representations ρ_1, ρ_2, we generate the tensor product representation $\rho_1 \otimes \rho_2$ over $V_1 \otimes_{\mathbb{C}} V_2$, with dimension $\dim(V_1)\dim(V_2)$ and trace $\chi_{\rho_1}\chi_{\rho_2}$.

Irreducible Representations

DEF 1.17 Let (ρ, V) be a representation. It is called an *irreducible representation* if there are no G-stable, nontrivial subspaces of V (i.e. no nontrivial subrepresentations). In the language of modules, irreducible representations are simple $\mathbb{C}[G]$-modules.

Theorem 1.2 Semi-Simplicity of Representations

Every finite dimensional, non-zero representation of G is a direct sum of irreducible representations.

PROOF. Pick any Hermitian inner product $\langle\cdot,\cdot\rangle$ on V. Define

$$\langle u, v\rangle^* = \frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle$$

It can be easily verified that $\langle\cdot,\cdot\rangle^*$ is an inner product which is G-equivariant. If $W \subseteq V$ is a subrepresentation, then set $W^{\perp} = \{u : \langle u, v\rangle = 0 \ \forall v \in W\}$, i.e. the orthogonal complement of W with respect to $\langle\cdot,\cdot\rangle^*$. It follows that $V = W \otimes W^{\perp}$, with $W^{\perp}$ being G-stable by the G-equivariance of the inner product.

We then argue by induction to yield a direct sum of irreducible representations. See MATH 457 for more details on semi-simplicity. □

SCHUR'S LEMMA

DEF 1.18 Let (ρ, V) be a representation. Let $V^G = \{v : \rho(g)(v) = v : \ \forall g \in G\}$ be the space of *invariant vectors*. Note that V^G is a subrepresentation of V equivalent to $\underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{\dim(V)\text{ times}}$.

$\dim(V^G) = \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)$

PROOF. Let $\pi : v \mapsto \frac{1}{|G|}\sum_{g\in G}\rho(g)(v)$. Writing $\rho(h)\pi = \frac{1}{|G|}\sum_{g\in G}\rho(hg) = \frac{1}{|G|}\sum_{g\in G}\rho(g)$ verifies that $\mathrm{Im}(\pi) \subseteq V^G$. It is also easy to verify that $\pi|_{V^G} = \mathrm{Id}_{V^G}$. Hence, we may write

$V = \ker(\pi) \oplus V^G$. It follows that, in some basis,

$$\pi = \begin{pmatrix} 0 & 0 \\ 0 & I_{\dim(V^G)} \end{pmatrix}$$

and thus $\mathrm{tr}(\pi) = \dim(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$. □

Let $(\rho_1, V_1), (\rho_2, V_2)$ be two representations. Consider

$$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) = \{T : V_1 \to V_2 \text{ with } T\ \mathbb{C}\text{-linear}\}$$

This is a $\mathbb{C}$-vector space of dimension $\dim(V_1)\dim(V_2)$. Similarly, we consider

$$\mathrm{Hom}_G(V_1, V_2) = \{T : V_1 \to V_2 \text{ with } T\rho_1(g) = \rho_2(g)T\}$$

We may also view $\mathrm{Hom}_G(V_1, V_2)$ as the transformations which satisfy $\rho_2(g^{-1})T\rho_1(g) = T$.

Over the vector space $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, $T \mapsto \rho_2(g^{-1})T\rho_1(g)$ is a G-representation. **PROP 1.6**

We can view $\mathrm{Hom}_G(V_1, V_2) = (\mathrm{Hom}_{\mathbb{C}}(V_1, V_2))^G$. **PROP 1.7**

$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \cong V_1^* \otimes V_2$ as vector spaces and as G-representations. **PROP 1.8**

$\dim(\mathrm{Hom}_G(V_1, V_2)) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}$ **PROP 1.9**

PROOF.

$$\begin{aligned} \dim(\mathrm{Hom}_G(V_1, V_2)) &= \dim(\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)^G) = \dim((V_1^* \otimes V_2)^G) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1^* \otimes \rho_2} = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1^*}\chi_{\rho_2} = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_1}}\chi_{\rho_2} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}\overline{\chi_{\rho_2}} \end{aligned}$$

In the last step, we use the fact that the dimension is always real, so $\overline{\dim} = \dim$. □