Equivalent kernels and their moments

Michal Kolesár

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This package also computes equivalent kernels for local linear regression and their moments. For orders $r = 1, 2$ (local linear and local quadratic), selected moments and equivalent kernels are computed analytically for the the uniform, triangular and Epanechnikov kernels. Higher-order moments, and moments for other kernels with bounded support normalized to $[-1, 1]$ are computed numerically.

The uniform, triangular and Epanechnikov kernels are defined as

$$k_{\mathrm{U}}(u) = \frac{1}{2} I(|u| < 1), \qquad k_{\mathrm{T}}(u) = (1 - |u|)I(|u| < 1), \qquad k_{\mathrm{E}}(u) = \frac{3}{4}(1 - u^2)I(|u| < 1).$$

We compute the following moments

$$\mu_j(k) = \int_{\mathcal{X}} u^j k(u)\, du, \qquad \nu_j(k) = \int_{\mathcal{X}} u^j k^2(u)\, du, \qquad \pi_j(k) = \int_{\mathcal{X}} |u^j k^2(u)|\, du,$$

where $\mathcal{X}$ is $[0, 1]$ for boundary regression, and $[-1, 1]$ for interior regression, and k is a kernel of equivalent kernel.

Equivalent kernels

Define $M_r(k) = \int_{\mathcal{X}} p_r(u)p_r(u)'k(u)\, du$, where $p_r(u) = (1, u, \dots, u^r)$, so that $(M_r(k))_{i,j} = \mu_{i+j-2}(k)$. Then for local polynomial regression of order r using a kernel k, the equivalent kernel is given by

$$k^* \colon u \mapsto e_1' M_r(k)^{-1} p(u)k(u), \qquad u \in \mathcal{X}$$

where e_1 is a vector of zeros with 1 in the first position. This assumes that the parameter of interest is the value of the regression function f at a point, which we assume throughout. For $r = 0$, the equivalent kernel is the same as the original kernel, except it is normalized so that it integrates to one over $\mathcal{X}$, $\int_{\mathcal{X}} k = 1$.

For the three kernels and boundary regression of order $r = 0, 1, 2$, the equivalent kernels are the following functions with domain $[0, 1]$

Kernel	Order 0	Order 1	Order 2
U	1	$(4 - 6u)$	$3(3 - 12u + 10u^2)$
T	$2(1 - u)$	$6(1 - 2u)(1 - u)$	$12(1 - 5u + 5u^2)(1 - u)$
E	$\frac{3}{2}(1 - u^2)$	$\frac{12}{19}(8 - 15u)(1 - u^2)$	$\frac{5}{8}(17 - 80u + 77u^2)(1 - u^2)$

For interior regression, the equivalent kernels are the following functions with domain $[-1, 1]$ (orders 0 and 1 are the same)

Kernel	Order 0 or 1	Order 2
U	$\frac{1}{2}$	$\frac{1}{8}(9 - 15u^2)$
T	$(1 - \|u\|)$	$\frac{6}{7}(2 - 5u^2)(1 - \|u\|)$
E	$\frac{3}{4}(1 - u^2)$	$\frac{15}{32}(3 - 7u^2)(1 - u^2)$

The equivalent kernels can be computed with the function `EqKern`, which returns the equivalent kernel function

```
library("RDHonest")
EqKern("uniform", boundary = TRUE, order = 2)(0.5)
# Equivalent call
EqKern(function(u) u <= 1, boundary = TRUE, order = 2 )(0.5)
```

Kernel Moments

The package stores analytically-computed low-order moments for the uniform, triangular, and Epanechnikov kernels for fast access in the dataframe `kernC`. The moments for boundary kernels are as follows:

Kernel	Order	μ_0	μ_1	μ_2	μ_3	μ_4	ν_0	ν_1	ν_2	ν_3	ν_4
U	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
T	0	1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{4}{105}$
E	0	1	$\frac{3}{8}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{3}{35}$	$\frac{6}{5}$	$\frac{3}{8}$	$\frac{6}{35}$	$\frac{3}{32}$	$\frac{2}{35}$
U	1	1	0	$-\frac{1}{6}$	$-\frac{1}{5}$	$-\frac{1}{5}$	4	1	$\frac{8}{15}$	$\frac{2}{5}$	$\frac{12}{15}$
T	1	1	0	$-\frac{1}{10}$	$-\frac{1}{10}$	$-\frac{3}{35}$	$\frac{24}{5}$	$\frac{3}{5}$	$\frac{6}{35}$	$\frac{3}{35}$	$\frac{2}{35}$
E	1	1	0	$-\frac{11}{95}$	$-\frac{16}{133}$	$-\frac{141}{1330}$	$\frac{56832}{12635}$	$\frac{1770}{2527}$	$\frac{2976}{12635}$	$\frac{324}{2527}$	$\frac{12352}{138985}$
U	2	1	0	0	$\frac{1}{20}$	$\frac{3}{35}$	9	$\frac{3}{2}$	$\frac{27}{35}$	$\frac{81}{140}$	$\frac{17}{35}$
T	2	1	0	0	$\frac{1}{35}$	$\frac{3}{70}$	$\frac{72}{7}$	$\frac{6}{7}$	$\frac{8}{35}$	$\frac{4}{35}$	$\frac{4}{35}$
E	2	1	0	0	$\frac{15}{448}$	$\frac{13}{252}$	$\frac{9895}{1008}$	$\frac{4105}{4032}$	$\frac{325}{1008}$	$\frac{2825}{16128}$	$\frac{16795}{144144}$

For interior regression (recall order 0 and 1 give the same kernel)

Kernel	Order	μ_0	μ_2	μ_4	ν_0	ν_2	ν_4
U	0	1	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{10}$
T	0	1	$\frac{1}{6}$	$\frac{1}{15}$	$\frac{2}{3}$	$\frac{1}{15}$	$\frac{2}{105}$
E	0	1	$\frac{1}{5}$	$\frac{3}{35}$	$\frac{3}{5}$	$\frac{3}{35}$	$\frac{1}{35}$
U	2	1	0	$-\frac{3}{35}$	$\frac{9}{8}$	$-\frac{9}{56}$	$-\frac{23}{280}$
T	2	1	0	$-\frac{19}{490}$	$\frac{456}{343}$	$-\frac{106}{1715}$	$-\frac{256}{18865}$
E	2	1	0	$-\frac{1}{21}$	$\frac{5}{4}$	$-\frac{25}{308}$	$-\frac{85}{4004}$

We also store absolute moments. At the boundary:

Kernel	Order	π_0	π_1	π_2	π_3	π_4
U	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
T	0	1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{10}$	$\frac{1}{15}$
E	0	1	$\frac{3}{8}$	$\frac{3}{5}$	$\frac{1}{8}$	$\frac{3}{35}$
U	1	$\frac{5}{3}$	$\frac{16}{27}$	$\frac{59}{162}$	$\frac{113}{405}$	$\frac{857}{3545}$
T	1	$\frac{3}{2}$	$\frac{3}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{3}{32}$
E	1	$\frac{100483}{64125}$	$\frac{702464}{1603125}$	$\frac{16520549}{72140625}$	$\frac{235792912}{1514953125}$	$\frac{1792724653}{15149531250}$
U	2	$1+12\frac{\sqrt{6}}{5^2}$	$\frac{36\sqrt{6}}{5^3}$	$\frac{558\sqrt{6}}{5^5}$	$\frac{1782\sqrt{6}}{5^6}+1/20$	0.268931
T	2	$1+\frac{2}{\sqrt{5}}$	$\frac{24\sqrt{5}}{5^3}$	$\frac{12\sqrt{5}}{5^3}$	$\frac{218\sqrt{5}}{4375}+1/35$	0.102656
E	2	2.0051585	0.50792888	0.26617935	0.17770885	0.1321148

In the interior:

Kernel	Order	π_0	π_1	π_2	π_3	π_4
U	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
T	0	1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$
E	0	1	$\frac{3}{8}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{3}{35}$
U	2	1	0.4875	0.27885480	0.1975	0.157419807
T	2	1	0.31155912	0.13986945	0.08443054	0.060140887
E	2	1	0.36033164	0.17177502	0.10671010	0.077066194

Finally, in the column `kernC$pMSE`, the package stores the optimal constant for pointwise MSE optimal bandwidth (see Fan and Gijbels, page 67):

$$\left(\frac{(p+1)!^2\nu_0}{2(p+1)\mu_{p+1}^2}\right)^{\frac{1}{2p+3}},$$

where p is the order of local polynomial.

For other kernels, the moments can be computed using the `KernMoment` function:

```
## mu_1, should be 0
KernMoment(function(u) 4-6*u, moment = 1, boundary = TRUE, type = "raw")
```

```
## [1] -5.551115e-17
```

```
## nu_1, should be 0
KernMoment(function(u) 4-6*u, moment = 1, boundary = TRUE, type = "raw2")
```

```
## [1] 1
```

```
## pi_1, should be 16/27
KernMoment(function(u) 4-6*u, moment = 1, boundary = TRUE, type = "absolute")
```

```
## [1] 0.5925926
```