PHAS2443: Practical Mathematics II 2018

# Homework 3

1. Jupiter is the largest planet in the solar system. It is surrounded by a large, rotating disc of *plasma* (ionized gas) which comes originally from the volcanoes on the surface of the moon, Io. Disc material situated at different distances $r$ from the centre of Jupiter rotates at different values of angular velocity $\omega$. The variation of angular velocity of the disc material as a function of distance is approximately described by the following differential equation:

$$\frac{1}{2} r \frac{d\omega}{dr} + \omega = \frac{(1-\omega)\,\kappa_0}{\dot{M}\, r^4}$$

The constant $\kappa_0 = 10^9$ represents the efficiency of the interaction between the rotating planet and the plasma disc.

The constant $\dot{M}$ is the rate at which material is added to the disc, in units of kg / s. The units for $\omega$ are chosen so that a value $\omega$=1 indicates perfect corotation with the planet. The units for $r$ are chosen so that a value $r$=1 represents one planet radius.

Use `NDSolve` to obtain and plot the five solutions of the above differential equation for the following five values of $\dot{M}$: 500 kg/s, 1000 kg/s, 2000 kg/s, 4000 kg/s and the hypothetical case $\dot{M}=\infty$ (infinite mass loading). Use the boundary condition: $\omega$=1 at $r$=6 (Io orbit), and obtain the solutions over the range $r$=6 to 40.

Plot your five solutions on the *same* graphics panel, label the axes appropriately, and clearly indicate which plot corresponds to which value of the mass loading rate, $\dot{M}$. For each of the five solutions, determine the value of $r$ for which $\omega$=0.7 - if such a point exists.

For the $\dot{M}=\infty$ solution only, make a separate plot of the quantity $l = \omega\, r^2$ (specific angular momentum) as a function of $r$. Label the axes appropriately, and comment on the appearance of the plot.

**[Q1: 19 marks]**

**2.** The differential equation in part **1** can be expressed as a *finite-difference* equation. If we replace the exact derivative with a centred-difference 'derivative', and rearrange, this finite-difference version of the equation is equivalent to the set of equations:

$$\frac{1}{2} r_j \left(\omega_{j+1} - \omega_{j-1}\right) + 2\,\Delta r \left(\frac{\kappa_0}{\dot{M}\, r_j{}^4} + 1\right)\omega_j = \frac{2\,\kappa_0\,\Delta r}{\dot{M}\, r_j{}^4}$$

Where these equations are defined for the interior points ($j$=2, 3, ..., ($n$-1)) of a uniformly spaced grid in radial distance, given by

$r_k = (6 + (k - 1)\,\Delta r)$ ($k$=1, 2, ..., $n$), with grid spacing $\Delta r$ = (40-6)/($n$-1), and $\omega_j$ is the value of the solution at radial distance $r_j$.

This set of ($n$-2) equations is thus equivalent to the matrix equation:

$$\mathbf{M} \begin{pmatrix} \omega_2 \\ \cdots \\ \omega_{n-1} \end{pmatrix} = \mathbf{b},$$

where **M** is a ($n$-2) × ($n$-2) tridiagonal matrix, whose elements are related to the constant coefficients of the angular velocity values. **b** is a vector of constant values, some of which involve the exterior values $\omega = \omega_1 = 1$ at $r = r_1 = 6$ (corotation with the planet at the closest distance) and $\omega = \omega_n$ at $r = r_n = 40$.

Using a grid of $n$=23 points, $\dot{M}$=500 kg/s, and $\omega_n$ equal to the corresponding value from the `NDSolve` solution (obtained in question **1**), compute all the elements of the matrix **M** and the vector **b**. Then solve the matrix equation to obtain the vector of angular velocity values at the interior points of the grid.

Plot the set of $n$ points given by $\{r_k, \omega_k\}$ ($k$=1,2,...,$n$) which represent the matrix solution - label the axes appropriately. On the same graphics panel, plot a continuous curve illustrating the `NDSolve` solution you obtained for $\dot{M}$=500 kg/s in question **1**. Calculate the maximum value of the absolute relative difference $|(\omega_k - \omega(r_k))/\omega(r_k)|$ between the matrix solution $\omega_k$ and the `NDSolve` solution evaluated at the same points, denoted here by $\omega(r_k)$. Is the matrix solution in agreement with `NDSolve` to better than 0.5 percent?

**[Q2: 19 marks]**

**Solutions are due by 5pm on Monday March 12th. Make a copy of your final solutions with the output deleted (`Cell|Delete All Output`), name that file `yourname_hw2` with an appropriate substitution for yourname, upload that file to Moodle and submit it. The file will be evaluated using `Evaluation|Evaluate Notebook`, so make sure the final file you submit will produce the results you expect when that is done. If you submit a file without deleting the output, the output will be deleted and regenerated for purposes of assessment.**