Time Series Forecasting

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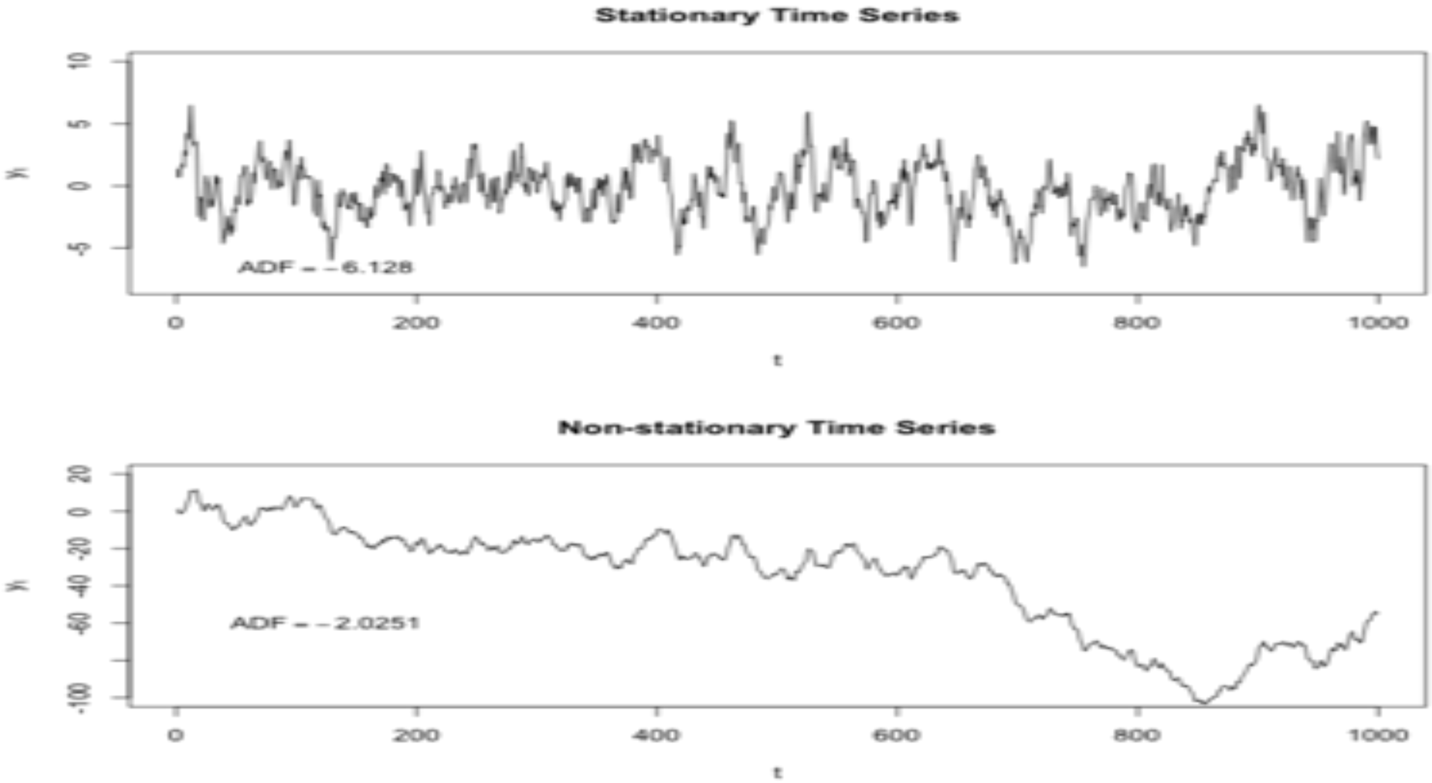
Why is stationarity important?

In the most intuitive sense, stationarity means that the statistical properties of a process generating a time series do not change over time.

It does not mean that the series does not change over time, just that the ***way*** it changes does not itself change over time.

The algebraic equivalent is thus a linear function, perhaps, and not a constant one; the value of a linear function changes as x grows, but the way it changes remains constant — it has a constant slope; one value that captures that rate of change.

Stationary vs non-stationary graphs



Why is this important?

1. Stationary processes are easier to analyze
2. In many cases simple models can be surprisingly useful, either as building blocks in constructing more elaborate ones, or as helpful approximations to complex phenomena

Time series and Lag definitions

- **Time series**: Commonly, a time series $(x_1, \ldots, x_e)$ is assumed to be a sequence of real values taken at successive equally spaced points in time, from time t=1 to time t=e.
- **Lag**: For some specific time point r, the observation x_{r-i} (i periods back) is called the i-th lag of x_r. A time series Y generated by back-shifting another time series X by i time steps is also sometime called the i-th lag of X, or an i-lag of X. This transformation is called both the backshifting operator, commonly denoted as B(·),and the lag operator, commonly denoted as L(·); thus, $L(X_r)=X_{r-1}$. Powers of the operators are defined as $L^i(X_r)=X_{r-i}$.

Stochastic Processes

A common approach in the analysis of time series data is to consider the observed time series as part of a ***realization*** of a ***stochastic process***.

A real *stochastic process* is a family of real random variables $\boldsymbol{X}=\{x_i(\omega);\ i \in T\}$, all defined on the same probability space (Ω, F, P). The set T is called the index set of the process. If $T \subset \mathbb{Z}$, then the process is called a discrete stochastic process. If T is an interval of $\mathbb{R}$, then the process is called a continuous stochastic process.

Finite Dimensional Distribution

For a finite set of integers T={t_1, …,tn}, the joint distribution function of $\boldsymbol{X}$={X_i(ω); i∈T} is defined by

Which for a stochastic process $\boldsymbol{X}$ is also commonly denoted as:

Finite dimensional distribution

The finite dimensional distribution of a stochastic process is then defined to be the set of all such joint distribution functions for all such finite integer sets T of any size n. For a discrete process it is thus the set:

Intuitively, this represents a projection of the process onto a finite-dimensional vector space (in this case, a finite set of time points).

Definitions of stationarity

Intuitively, ***stationarity*** means that the statistical properties of the process do not change over time.

Strong stationarity concerns the shift-invariance (in time) of its finite-dimensional distributions.

Weak stationarity only concerns the shift-invariance (in time) of first and second moments of a process.

Strongly stationary stochastic processes

Definition. The process $\{x_t; t \in Z\}$ is strongly stationary if $F_{t1+k, t2+k, \cdots, ts+k}(b1, b2, \cdot\cdot\cdot, bs) = F_{t1,t2,\cdots,ts}(b1, b2, \cdot\cdot\cdot, bs)$ for any finite set of indices $\{t_1, t_2, \cdot\cdot\cdot, t_s\} \subset Z$ with $s \in Z^+$, and any $k \in Z$. Thus the process $\{x_t; t \in Z\}$ is strongly stationary if the joint distribution function of the vector $(x_{t1+k}, x_{t2+k}, ..., x_{ts+k})$ is equal with the one of $(x_{t1}, x_{t2}, ..., x_{ts})$ for any finite set of indices $\{t_1, t_2, \cdot\cdot\cdot, t_s\} \subset Z$ with $s \in Z^+$, and any $k \in Z$.

Strongly stationary stochastic processes

The meaning of the strongly stationarity is that the distribution of a number of random variables of the stochastic process is the same as we shift them along the time index axis.

IID process

An iid process is a strongly stationary process. This follows almost immediate from the definition.

Since the random variables x_{t1+k}, x_{t2+k}, ..., x_{ts+k} are **iid**, we have that

$$F_{t1+k, t2+k, \cdots, ts+k}(b1, b2, \cdots, bs) = F(b1)F(b2) \cdots F(bs)$$

On the other hand, also the random variables x_{t1}, x_{t2}, ..., x_{ts} are iid and hence

$$F_{t1, t2, \cdots, ts}(b1, b2, \cdots, bs) = F(b1)F(b2) \cdots F(bs).$$

We can conclude that

$$F_{t1+k, t2+k, \cdots, ts+k}(b1, b2, \cdots, bs) = F_{t1, t2, \cdots, ts}(b1, b2, \cdots, bs)$$

Weakly stationary stochastic processes

Definition: The process $\{x_t; t \in Z\}$ is weakly stationary, or covariance-stationary if

1. the second moment of x_t is finite for all t, that is $\mathbf{E|x_t|^2 < \infty}$ for all t
2. the first moment of x_t is independent of t, that is $\mathbf{E(x_t) = \mu \ \forall t}$
3. the cross moment $E(x_{t1}\ x_{t2})$ depends only on $t_1 - t_2$, that is $\mathbf{cov(x_{t1}, x_{t2}) = cov(x_{t1+h}, x_{t2+h})\ \forall t_1, t_2, h}$

Weakly stationary stochastic processes

Thus a stochastic process is covariance-stationary if

- it has the same mean value, μ, at all time points;
- it has the same variance, γ_0, at all time points; and
- the covariance between the values at any two time points, $t, t - k$, depend only on k, the difference between the two times, and not on the location of the points along the time axis.

Weakly stationary stochastic processes

An important example of covariance-stochastic process is the so-called white noise process.

Definition: A stochastic process $\{u_t ;t \in Z\}$ in which the random variables ut , t = 0 ± 1, ±2... are such that

- $E(u_t) = 0 \ \forall t$
- $Var(u_t) = \sigma^2_u < \infty \ \forall t$
- $Cov(u_t, u_{t-k}) = 0 \ \forall t, \ \forall k$

is called white noise with mean 0 and variance σ^2_u , written $u_t \sim WN(0, \sigma^2_u)$.

Weakly stationary stochastic processes

First condition establishes that the expectation is always constant and equal to zero. Second condition establishes that variance is constant. Third condition establishes that the variables of the process are uncorrelated for all lags.

If the random variables ut are independently and identically distributed with mean 0 and variance σ^2_u then we will write

$$\mathbf{u_t \sim IID(0, \sigma^2_u)}$$

White Noise process

Figure shows a possible realization of an IID(0,1) process.

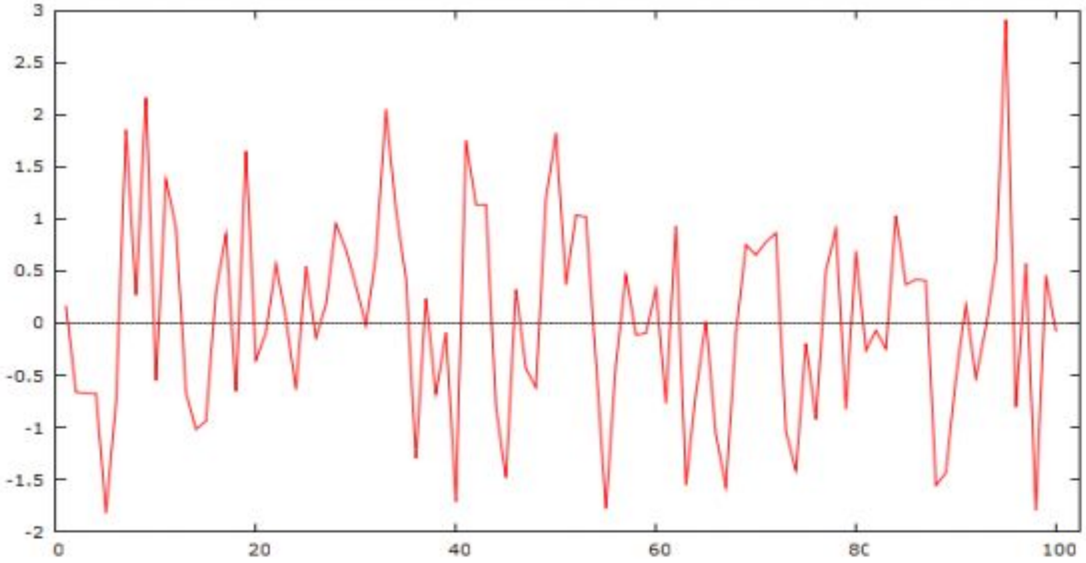


Figure : A realization of an $IID(0,1)$.

Random Walk process

An important example of weakly non-stationary stochastic processes is the following.

Let $\{y_t ; t = 0, 1, 2, ...\}$ be a stochastic process where $y_0 = \delta < \infty$ and $y_t = y_{t-1} + u_t$ for $t = 1,2,...$, with $u_t \sim WN(0, \sigma^2_u)$.

This process is called **random walk**.

Random Walk process

The mean of y_t is given by $E(y_t) = \delta$ and its variance is $Var(y_t) = t\sigma^2_u$

Thus a random walk is not weakly stationary process.

Random Walk process

Figure shows a possible realization of a random walk.

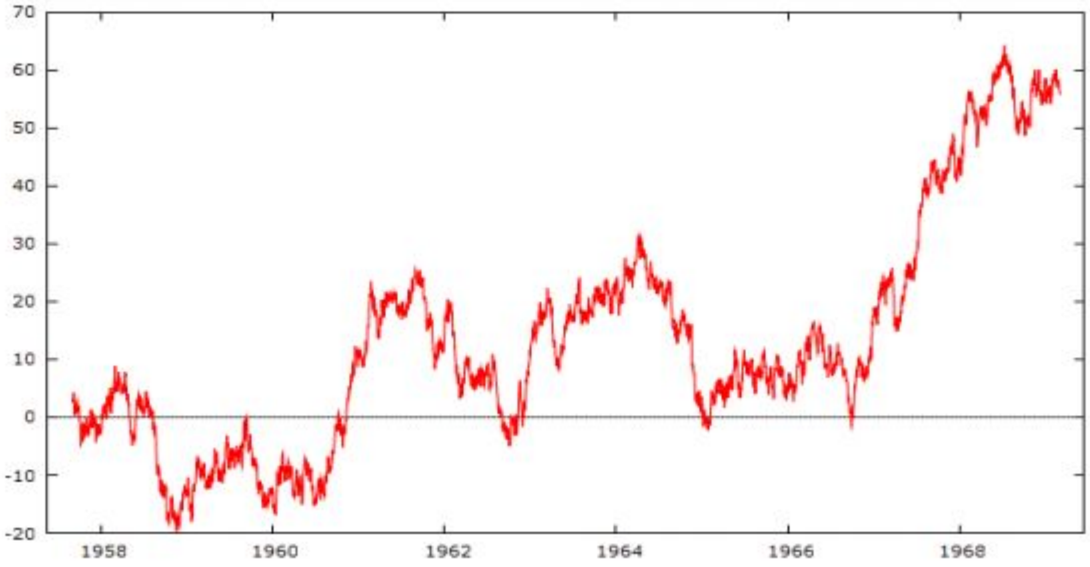


Figure : A realization of a random walk.

Relation between strong and weak Stationarity

First note that finite second moments are not assumed in the definition of strong stationarity, therefore, strong stationarity does not necessarily imply weak stationarity.

For example, an iid process with standard Cauchy distribution is strictly stationary but not weak stationary because the second moment of the process is not finite

Relation between strong and weak Stationarity

If the process $\{x_t ;t \in Z\}$ is strongly stationary and has finite second moment, then $\{x_t ;t \in Z\}$ is weakly stationary.

References

- Stationarity
- http://www.phdeconomics.sssup.it/documents/Lesson4.pdf