considered under decomposition methods. Under iterative methods, the initial approximate solution is assumed to be known and is improved towards the exact solution in an iterative way. We consider *Jacobi, Gauss–Seidel* and *relaxation methods* under iterative methods. All these methods are easily adoptable to computers and can be used to solve even hundred or more simultaneous linear equations.

3.2 GAUSSIAN ELIMINATION METHOD

In the Gaussian elimination method, the solution to the system of Eqs. (3.2) is obtained in two stages. In the first stage, the given system of equations is reduced to an equivalent upper triangular form using elementary transformations. In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order x_n, $x_{n-1}, x_{n-2}, \ldots, x_2, x_1$.

This method is explained by considering a system of n equations in n unknowns in the form as follows

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \tag{3.3}$$

Stage I: We divide the first equation by a_{11} and then subtract this equation multiplied by $a_{21}, a_{31}, \ldots, a_{n1}$ from the 2nd, 3rd, ..., n th equation. Then the system (3.3) reduces to the following form:

$$\left.\begin{array}{l} x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n = b'_1 \\ \qquad a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2 \\ \qquad \vdots \qquad \qquad \vdots \qquad \vdots \\ \qquad a'_{n2}x_2 + \cdots + a'_{nn}x_n = b'_n \end{array}\right\} \tag{3.4}$$

Here, we can observe that the last $(n - 1)$ equations are independent of x_1, that is, x_1 is eliminated from the last $(n - 1)$ equations.

This procedure is repeated with the second equation of (3.4), that is, we divide the second equation by a'_{22} and then x_2 is eliminated from 3rd, 4th, ..., nth equations of (3.4). The same procedure is repeated again and again till the given system assumes the following upper triangular form:

$$\left.\begin{array}{l} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n = d_1 \\ \qquad c_{22}x_2 + \cdots + c_{2n}x_n = d_2 \\ \qquad \qquad \qquad \vdots \qquad \vdots \\ \qquad \qquad \qquad c_{nn}x_n = d_n \end{array}\right\} \tag{3.5}$$

Stage II: Now, the values of the unknowns are determined by back substitution procedure, in which we obtain x_n from the last equation of (3.5) and then substituting this value of x_n in the preceding equation, we get the value of x_{n-1}. Continuing this way, we can find the values of all other unknowns in the order $x_n, x_{n-1}, \ldots, x_2, x_1$.

In this method, we observe that the determinant of the coefficient matrix is obtained as a by-product, that is,

$$|A| = c_{11}c_{22}\cdots c_{nn} \tag{3.6}$$

To familiarize with the method, we consider the following example:

Example 3.1 Solve the following system of equations using Gaussian elimination method

$$2x + 3y - z = 5$$
$$4x + 4y - 3z = 3$$
$$-2x + 3y - z = 1$$

Solution The given system of equations is solved in two stages.

Stage I (Reduction to upper-triangular form): We divide the first equation by 2 and then subtract the resulting equation (multiplied by 4 and –2) from the second and third equations respectively. Thus, we eliminate x from the 2nd and 3rd equations. The resulting new system is given by

$$\left.\begin{aligned} x + \frac{3}{2}y - \frac{z}{2} &= \frac{5}{2} \\ -2y - z &= -7 \\ 6y - 2z &= 6 \end{aligned}\right\} \tag{1}$$

Now, we divide the second equation of (1) by –2 and eliminate y from the last equation and the modified system is given by

$$\left.\begin{aligned} x + \frac{3}{2}y - \frac{z}{2} &= \frac{5}{2} \\ y + \frac{z}{2} &= \frac{7}{2} \\ -5z &= -15 \end{aligned}\right\} \tag{2}$$

Stage II (Back substitution): From the last equation of (2), we immediately get

$$z = 3 \tag{3}$$

using this value of z, the second equation of (2) gives

$$y = \frac{7}{2} - \frac{3}{2} = 2 \tag{4}$$

Using these values of y and z in the first equation of (2), we get

$$x = \frac{5}{2} + \frac{3}{2} - 3 = 1 \tag{5}$$

Thus, the solution of the given system is given by Eqs. (3) – (5).

Partial and full pivoting

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order of equations is called *pivoting*.

We now introduce the concept of partial pivoting. In this technique, if the pivot a_{ii} happens to be zero, then the ith column elements are searched for the numerically largest element. Let the jth row ($j > i$) contains this element, then we interchange the ith equation with the jth equation and proceed for elimination. This process is continued whenever pivots become zero during elimination. For example, let us examine the solution of the following simple system

$$10^{-5}x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

Using Gaussian elimination method with and without partial pivoting, assuming that we require the solution accurate to only four decimal places. The solution by Gaussian elimination gives $x_1 = 0$, $x_2 = 1$. If we use partial pivoting, the system takes the form

$$x_1 + x_2 = 2$$
$$10^{-5}x_1 + x_2 = 1$$

Using Gaussian elimination method, the solution is found to be $x_1 = 1$, $x_2 = 1$, which is a meaningful and perfect result.

In full pivoting which is also known as *complete pivoting*, we interchange rows as well as columns, such that the largest element in the matrix of the system becomes the pivot element. In this process, the position of the unknown variables also get changed. Full pivoting, in fact, is more complicated than the partial pivoting. Partial pivoting is preferred for hand computation.

Example 3.2 Solve the system of equations

$$x + y + z = 7$$
$$3x + 3y + 4z = 24$$
$$2x + y + 3z = 16$$

by Gaussian elimination method with partial pivoting.

Solution In matrix notation, the given system can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \\ 16 \end{pmatrix} \tag{1}$$

To start with, we observe that the pivot element $a_{11} = 1\ (\neq 0)$. However, a glance at the first column reveals that the numerically largest element is 3 which is in the second row. Hence, we interchange the first row with the second row and then proceed for elimination. Thus, Eq. (1) takes the form

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 \\ 7 \\ 16 \end{pmatrix} \tag{2}$$

after partial pivoting.

Stage I (Reduction to upper triangular form): By dividing the first row of the system (2) by 3 and then subtracting the resulting row, multiplied by 1 and 2 from the second and third rows of the system (2), we get

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} \tag{3}$$

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - 2R_1$

The second row in Eq. (3) cannot be used as the pivot row, as $a_{22} = 0$. Interchanging the second and third rows, we obtain

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ & -1 & \frac{1}{3} \\ & & -\frac{1}{3} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} \tag{4}$$

which is in the upper triangular form.

Stage II (Back substitution): From the last row of Eq. (4), we at once get

$$z = 3 \tag{5}$$

The second row of Eq. (4) with this value of z gives

$$-y + 1 = 0 \quad \text{or} \quad y = 1 \tag{6}$$

Using these values of y and z, the first row of Eq. (4) gives

$$x + 1 + 4 = 8 \quad \text{or} \quad x = 3 \tag{7}$$

Thus, Eqs. (5)–(7) constitute the solution to the given system of equations.

Example 3.3 Solve by Gaussian elimination method with partial pivoting, the following system of equations:

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$
$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$
$$4x_1 + 5x_2 + 6.5x_3 + 2x_4 = 26$$
$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

Solution In matrix notation, the given system can be written as

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 24 \\ 32 \\ 26 \\ 21 \end{pmatrix} \tag{1}$$

To start with, we observe that the pivot row, that is, the first row has a zero pivot element ($a_{11} = 0$). This row should be interchanged with any row following it, which on becoming a pivot row should not have a zero pivot element. While interchanging rows it is better to interchange with a row having largest pivotal element. Thus, we interchange the first and fourth rows, which is called partial pivoting and get,

$$\begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 21 \\ 32 \\ 26 \\ 24 \end{pmatrix} \tag{2}$$

We observe that, in partial pivoting, the unknown vector remains unaltered, while the right-hand side vector gets changed.

Now, we shall carry out Gaussian elimination process in two stages.

Stage I (Reduction to upper-triangular form): In this stage, by dividing the first row of the system (2) by 9 and then subtracting this resulting row, multiplied by 4 and 4 from the second and third rows of Eq. (2), we get

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 8.2222 & 3.2222 & 4 \\ 0 & 3.2222 & 4.7222 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.3333 \\ 22.6666 \\ 16.6666 \\ 24 \end{pmatrix} \tag{3}$$

Now, we divide the second pivot row by 8.2222 and subtract the resultant row multiplied by 3.2222 and 4 from the third and fourth rows of Eq. (3) to get

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 3.4594 & 0.4324 \\ 0 & 0 & 0.4324 & 6.0540 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.3333 \\ 2.7568 \\ 7.7836 \\ 12.9728 \end{pmatrix} \tag{4}$$

Finally, we divide the third pivot row by 3.4594 and subtract the resultant row multiplied by 0.4324 from fourth row of Eq. (4), thus getting the upper triangular form

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 1 & 0.1250 \\ 0 & 0 & 0 & 5.9999 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.3333 \\ 2.7568 \\ 2.2500 \\ 11.9999 \end{pmatrix} \tag{5}$$

Stage II (Back substitution): From the last row of Eq. (5), we immediately get $x_4 = 2.0000$. Using this value of x_4 into the third row of Eq. (5), we obtain

$$x_3 + 0.25 = 2.25 \quad \text{or} \quad x_3 = 2.0000 \tag{6}$$

Similarly, we get

$$x_2 = 1.0000, \quad x_1 = 1.0000$$

Thus, the solution of the given system is given by

$$x_1 = 1.0, \quad x_2 = 1.0, \quad x_3 = 2.0, \quad x_4 = 2.0$$

3.3 GAUSS–JORDAN ELIMINATION METHOD

This method is a variation of Gaussian elimination method. In this method, the elements above and below the diagonal are simultaneously made zero and thereby the given system is reduced to an equivalent diagonal form using elementary transformations. Then the solution of the resulting diagonal system can be readily obtained.

Sometimes, we normalize the pivot row with respect to the pivot element, before elimination. Partial pivoting is also used whenever the pivot element becomes zero.

This method is illustrated through the following examples.

Example 3.4 Solve the system of equations

$$\left.\begin{aligned} x + 2y + z &= 8 \\ 2x + 3y + 4z &= 20 \\ 4x + 3y + 2z &= 16 \end{aligned}\right\} \tag{1}$$

using Gauss–Jordan elimination method.

Solution In matrix notation, the given system (1) can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 16 \end{pmatrix} \tag{2}$$

We subtract the first row multiplied by 2 and 4 from the second and third rows respectively of Eq. (2), and eliminate x

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -16 \end{pmatrix} \tag{3}$$

Now, we eliminate y from the first and third rows using the second row. Thus, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ -36 \end{pmatrix} \tag{4}$$

Before, eliminating z from the first and second row, normalizing the third row with respect to the pivot element, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ 3 \end{pmatrix} \tag{5}$$

Using the third row of Eq. (5), eliminating z from the first and second rows of Eq. (5), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \tag{6}$$

From Eq. (16), we get the solution directly as $x = 1$, $y = 2$, $z = 3$.

3.4 CROUT'S REDUCTION METHOD

This method is based on the fact that the coefficient matrix $[A]$ of the system of equations (3.3) can be decomposed into the product of two matrices $[L]$ and $[U]$, where $[L]$ is a lower-triangular matrix and $[U]$ is an upper-triangular matrix with 1's on its main diagonal. The rules for getting $[L]$ and $[U]$ can be obtained from the fact

$$[L]\,[U] = [A] \tag{3.7}$$

For the purpose of illustration, let us consider a (3×3) general matrix in the form

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{3.8}$$

The sequence of steps for getting $[L]$ and $[U]$ are given below:

Step I: Multiplying all the rows of $[L]$ by the first column of $[U]$, we get

$$l_{11} = a_{11}, \quad l_{21} = a_{21}, \quad l_{31} = a_{31} \tag{3.9}$$

Thus, we observe that the first column of $[L]$ is same as the first column of $[A]$.

Step II: Now, multiplying the first row of $[L]$ by the second and third columns of $[U]$, we obtain

$$l_{11}u_{12} = a_{12}, \quad l_{11}u_{13} = a_{13}$$

or

$$u_{12} = \frac{a_{12}}{l_{11}}, \quad u_{13} = \frac{a_{13}}{l_{11}} \tag{3.10}$$

Thus, the first row of $[U]$ is obtained. Now, we continue this process, thus getting alternately the column of $[L]$ and a row of $[U]$.

Step III: Multiply the second and third rows of $[L]$ by the second column of $[U]$ to get

$$l_{21}u_{12} + l_{22} = a_{22}, \quad l_{31}u_{12} + l_{32} = a_{32}$$

which gives

$$l_{22} = a_{22} - l_{21}u_{12}, \quad l_{32} = a_{32} - l_{31}u_{12} \tag{3.11}$$

Thus, the second column of $[L]$ is obtained.

Step IV: Now, multiply the second row of $[L]$ by the third column of $[U]$ which yields

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \quad \text{or} \quad u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} \tag{3.12}$$

Step V: Lastly, we multiply the third row of $[L]$ by the third column of $[U]$ and get

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}$$

which gives

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \tag{3.13}$$

Thus, the above five steps determine $[L]$ and $[U]$. This algorithm can be generalized to any linear system of order n.

Now, to obtain the solution of the linear system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \tag{3.14}$$

in matrix notation as $[A]\,(X) = (B)$. Let $[A] = [L]\,[U]$, then we get,

$$[L]\,[U]\,(X) = (B) \tag{3.15}$$

Substituting $[U]\,(X) = (Z)$ in Eq. (3.15), we obtain

$$[L]\,(Z) = (B) \tag{3.16}$$

Now, Eq. (3.16) is equivalent to

$$\left.\begin{aligned} l_{11}z_1 &= b_1 \\ l_{21}z_1 + l_{22}z_2 &= b_2 \\ l_{31}z_1 + l_{32}z_2 + l_{33}z_3 &= b_3 \end{aligned}\right\} \tag{3.17}$$

The first of these equations gives z_1. Knowing z_1, the second equation of (3.17) gives z_2; then the third equation of (3.17) can be solved and z_3 is obtained. Having computed z_1, z_2 and z_3, we can compute x_1, x_2 and x_3 from equation $[U]\,(X) = (Z)$ or from

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

This method is also known as *Cholesky reduction method*. This technique is widely used in the numerical solutions of partial differential equations.

This method is very popular from computer programming point of view, since the storage space reserved for matrix $[A]$ can be used to store the elements of $[L]$ and $[U]$ at the end of computation.

It may be noted that this method fails if any $a_{ii} = 0$. In that case, the system is singular. In order to familiarize with the Crout's reduction method, we consider the following examples.

Example 3.5 Solve the following system of equations

$$5x_1 - 2x_2 + x_3 = 4$$
$$7x_1 + x_2 - 5x_3 = 8$$
$$3x_1 + 7x_2 + 4x_3 = 10$$

by Crout's reduction method using hand computation. Also, check the solution using computer simulation Package MATLAB.

Solution Let the coefficient matrix $[A]$ be written as $[L]\,[U]$. Thus,

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} \tag{1}$$

Step I: Multiply all the rows of $[L]$ by the first column of $[U]$, we get

$$l_{11} = 5, \qquad l_{21} = 7, \qquad l_{31} = 3 \tag{2}$$

Step II: Multiply the first row of $[L]$ by the second and third columns of $[U]$, we have

$$l_{11}u_{12} = -2, \qquad l_{11}u_{13} = 1$$

Using Eq. (2), we get

$$u_{12} = -\frac{2}{5}, \qquad u_{13} = \frac{1}{5} \tag{3}$$

Step III: Multiply the second and third rows of $[L]$ by the second column of $[U]$. Using Eqs. (2) and (3), we get

$$\left.\begin{aligned} l_{21}u_{12} + l_{22} = 1 \quad &\text{or} \quad l_{22} = 1 + \frac{14}{5} = \frac{19}{5} \\ l_{31}u_{12} + l_{32} = 7 \quad &\text{or} \quad l_{32} = 7 + \frac{6}{5} = \frac{41}{5} \end{aligned}\right\} \tag{4}$$

Step IV: Multiply the second row of $[L]$ by the third column of $[U]$ which yields $l_{21}u_{13} + l_{22}u_{23} = -5$. Using Eqs. (2)–(4), we get

$$\frac{19}{5}u_{23} = -5 - \frac{7}{5}$$

Therefore,

$$u_{23} = -\frac{32}{19} \tag{5}$$

Step V: Finally, multiply the third row of $[L]$ with the third column of $[U]$, we obtain

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4$$

Using Eqs. (2)–(5), we get

$$l_{33} = \frac{327}{19} \tag{6}$$

Thus, the given system of equations takes the form $[L]$ $[U]$ $(X) = (B)$. That is,

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{32}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix} \tag{7}$$

Let $[U](X) = (Z)$, then

$$[L](Z) = (4 \ 8 \ 10)^T$$

or

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 10 \end{pmatrix} \tag{8}$$

which gives

$$z_1 = \frac{4}{5}, \quad z_2 = \frac{12}{19}, \quad z_3 = \frac{46}{327} \tag{9}$$

utilizing these values of z, Eq. (7) becomes

$$\begin{bmatrix} 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{32}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{12}{19} \\ \frac{46}{327} \end{pmatrix} \tag{10}$$

By back substitution method, we obtain

$$x_3 = \frac{46}{327}, \qquad x_2 = \frac{284}{327}, \qquad x_1 = \frac{366}{327}$$

This is the required solution. MATLAB simulation gives $x_1 = 1.1193$, $x_2 = 0.8685, x_3 = 0.1407$.

3.5 JACOBI'S METHOD

Jacobi's method is an iterative method, where initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way. In general, when the coefficient matrix of the system of equations is a sparse matrix (many elements are zero), iterative methods have definite advantage over direct methods in respect of economy in computer memory. Such sparse matrices arise in computing the numerical solution of partial differential equations.

To illustrate Jacobi's method, let us consider a linear system given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \tag{3.18}$$

In this method, we assume that the coefficient matrix $[A]$ is strictly diagonally dominant, that is, in each row of $[A]$ the modulus of the diagonal element exceeds the sum of the off-diagonal elements. We also assume that the diagonal element a_{ii} do not vanish. If any diagonal element vanishes, the equations can always be rearranged to satisfy this condition. Now the system (3.18) can be written as

$$\left.\begin{array}{l} x_1 = \dfrac{b_1}{a_{11}} - \dfrac{a_{12}}{a_{11}}x_2 - \cdots - \dfrac{a_{1n}}{a_{11}}x_n \\ x_2 = \dfrac{b_2}{a_{22}} - \dfrac{a_{21}}{a_{22}}x_1 - \cdots - \dfrac{a_{2n}}{a_{22}}x_n \\ \;\vdots \quad\; \vdots \qquad \vdots \qquad\qquad\quad \vdots \\ x_n = \dfrac{b_n}{a_{nn}} - \dfrac{a_{n1}}{a_{nn}}x_1 - \cdots - \dfrac{a_{n(n-1)}}{a_{nn}}x_{n-1} \end{array}\right\} \tag{3.19}$$

Solution At first, we rewrite the given system in the form

$$\left.\begin{aligned} x &= \frac{95}{83} - \frac{11}{83}y + \frac{4}{83}z \\ y &= \frac{104}{52} - \frac{7}{52}x - \frac{13}{52}z \\ z &= \frac{71}{29} - \frac{3}{29}x - \frac{8}{29}y \end{aligned}\right\} \tag{1}$$

Taking the initial starting of solution vector as $(0, 0, 0)^T$, from Eq. (1), we have the first approximation as

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{pmatrix} = \begin{pmatrix} 1.1446 \\ 2.0000 \\ 2.4483 \end{pmatrix} \tag{2}$$

Now, using Eq. (1), the second approximation is computed from the equations

$$\left.\begin{aligned} x^{(2)} &= 1.1446 - 0.1325y^{(1)} + 0.0482z^{(1)} \\ y^{(2)} &= 2.0 - 0.1346x^{(1)} - 0.25z^{(1)} \\ z^{(2)} &= 2.4483 - 0.1035x^{(1)} - 0.2759y^{(1)} \end{aligned}\right\} \tag{3}$$

Substituting Eq. (2) into Eq. (3), we get the second approximation as

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{pmatrix} = \begin{pmatrix} 0.9976 \\ 1.2339 \\ 1.7424 \end{pmatrix} \tag{4}$$

Similar procedure yields the third, fourth and fifth approximations to the required solution and they are tabulated as below:

Iteration number, r	Variables		
	x	y	z
1	1.1446	2.0000	2.4483
2	0.9976	1.2339	1.7424
3	1.0651	1.4301	2.0046
4	1.0517	1.3555	1.9435
5	1.0587	1.3726	1.9655

3.6 GAUSS–SEIDEL ITERATION METHOD

It is another well-known iterative method for solving a system of linear equations of the form of system (3.18). In Jacobi method, the $(r + 1)$th approximation to the system (3.18) is given by Eqs. (3.21), from which we can observe that no element of $x_i^{(r+1)}$ replaces $x_i^{(r)}$ entirely for the next cycle of computation.

However, in Gauss–Seidel method, the corresponding elements of $x_i^{(r+1)}$ replaces those of $x_i^{(r)}$ as soon as they become available. Hence, it is called the method of *successive displacements*. For illustration, consider the system (3.1). In Gauss–Seidel iteration, the $(r + 1)$th approximation or iteration is computed from

$$\left.\begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(r)} - \dots - \frac{a_{1n}}{a_{11}}x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(r+1)} - \dots - \frac{a_{2n}}{a_{22}}x_n^{(r)} \\ &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(r+1)} - \dots - \frac{a_{n,(n-1)}}{a_{nn}}x_{n-1}^{(r+1)} \end{aligned}\right\} \quad (3.24)$$

Thus, the general procedure can be written in the following compact form

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1}\frac{a_{ij}}{a_{ii}}x_j^{(r+1)} - \sum_{j=i+1}^{n}\frac{a_{ij}}{a_{ii}}x_j^{(r)} \quad (3.25)$$

for all $i = 1, 2, \dots, n$ and $r = 1, 2, \dots$

To describe system (3.24); in the first equation, we substitute the rth approximation into the right-hand side and denote the result by $x_1^{(r+1)}$. In the second equation, we substitute $(x_1^{(r+1)}, x_3^{(r)}, \dots, x_n^{(r)})$ and denote the result by $x_2^{(r+1)}$. In the third equation, we substitute $(x_1^{(r+1)}, x_2^{(r+1)}, x_4^{(r)}, \dots, x_n^{(r)})$ and denote the result by $x_3^{(r+1)}$, and so on. This process is continued till we arrive at the desired result. For illustration, we consider the following example.

Example 3.7 Find the solution of the following system of equations

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$

using Gauss–Seidel method and perform the first-five iterations.

Solution The given system of equations can be rewritten as

$$\left.\begin{aligned} x_1 &= 0.5 + 0.25x_2 + 0.25x_3 \\ x_2 &= 0.5 + 0.25x_1 + 0.25x_4 \\ x_3 &= 0.25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 0.25 + 0.25x_2 + 0.25x_3 \end{aligned}\right\} \quad (1)$$

Taking $x_2 = x_3 = x_4 = 0$ on the right-hand side of the first equation of system (1), we get $x_1^{(1)} = 0.5$. Taking $x_3 = x_4 = 0$ and the current value of x_1, we get

$$x_2^{(1)} = 0.5 + (0.25)\,(0.5) + 0 = 0.625$$

from the second equation of system (1). Further, we take $x_4 = 0$ and the current value of x_1, we obtain

$$x_3^{(1)} = 0.25 + (0.25)\,(0.5) + 0 = 0.375$$

from the third equation of system (1). Now, using the current values of x_2 and x_3, the fourth equation of system (1) gives

$$x_4^{(1)} = 0.25 + (0.25)\,(0.625) + (0.25)\,(0.375) = 0.5$$

The Gauss–Seidel iterations for the given set of equations can be written as

$$\begin{aligned} x_1^{(r+1)} &= 0.5 + 0.25x_2^{(r)} + 0.25x_3^{(r)} \\ x_2^{(r+1)} &= 0.5 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)} \\ x_3^{(r+1)} &= 0.25 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)} \\ x_4^{(r+1)} &= 0.25 + 0.25x_2^{(r+1)} + 0.25x_3^{(r+1)} \end{aligned}$$

Now, by Gauss–Seidel procedure, the second and subsequent approximations can be obtained and the sequence of the first-five approximations are tabulated as below:

Iteration number r	Variables			
	x_1	x_2	x_3	x_4
1	0.5	0.625	0.375	0.5
2	0.75	0.8125	0.5625	0.59375
3	0.84375	0.85938	0.60938	0.61719
4	0.86719	0.87110	0.62110	0.62305
5	0.87305	0.87402	0.62402	0.62451

3.7 THE RELAXATION METHOD

This method is an iterative method and is due to Southwell. To explain the details, consider again the system of equations (3.18). Let

$$X^{(p)} = (x_1^{(p)}, x_2^{(p)}, \ldots, x_n^{(p)})^T$$

be the solution vector obtained iteratively after pth iteration. If $R_i^{(p)}$ denotes the residual of the ith equation of system (3.18), that is of