Chapter 6

Interpolation

6.1 INTRODUCTION

Finite differences play an important role in numerical techniques, where tabulated values of the functions are available. For instance, consider a function $y = f(x)$. As x takes values $x_0, x_1, x_2, \ldots, x_n$, let the corresponding values of y be $y_0, y_1, y_2, \ldots, y_n$. That is, for a given table of values, (x_k, y_k), $k = 0, 1, 2, \ldots, n$; the process of estimating the value of y, for any intermediate value of x, is called *interpolation*. However, the method of computing the value of y, for a given value of x, lying outside the table of values of x is known as *extrapolation*. It may be noted that if the function $f(x)$ is known, the value of y corresponding to any x can be readily computed to the desired accuracy. But, in practice, it may be difficult or sometimes impossible to know the function $y = f(x)$ in its exact form.

To look at a practical example, let us consider the computation of trajectory of a rocket flight, where we solve the Euler's dynamical equations of motion to compute its position and velocity vectors at specified times during the flight. Under the same conditions, suppose, we require the position and velocity vector, at some other intermediate times; we need not compute the trajectory again by solving the dynamical equations. Instead, we can use the best known interpolation technique to get the desired values.

In general, for interpolation of a tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

6.2 FINITE DIFFERENCE OPERATORS

6.2.1 Forward Differences

For a given table of values (x_k, y_k), $k = 0, 1, 2, \ldots, n$ with equally-spaced abscissas of a function $y = f(x)$, we define the *forward difference* operator Δ as follows: The first forward difference is usually expressed as

$$\Delta y_i = y_{i+1} - y_i, \quad i = 0, 1, \ldots, (n-1) \tag{6.1}$$

To be explicit, we write

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

These differences are called *first differences of the function y* and are denoted by the symbol Δy_i. Here, Δ is called *forward difference operator.*

Similarly, the differences of the first differences are called *second differences*, defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

Thus, in general

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i \tag{6.2}$$

Here Δ^2 is called the *second difference operator*. Thus, continuing, we can define, *r*th difference of *y*, as

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i \tag{6.3}$$

By defining a difference table as a convenient device for displaying various differences, the above defined differences can be written down systematically by constructing a difference table for values (x_k, y_k), $k = 0, 1, \ldots, 6$ as shown below:

Table 6.1 Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

This difference table is called *forward difference table or diagonal difference table.* Here, each difference is located in its appropriate column, mid-way between the elements of the previous column. It can be noted that the subscript remains constant along each diagonal of the table. The first term in the table, that is y_0 is called the *leading term*, while the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ are called *leading differences*.

Example 6.1 Construct a forward difference table for the following values of *x* and *y*:

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

Example 6.2 Express $\Delta^2 y_0$ and $\Delta^3 y_0$ in terms of the values of the function y.

Solution Noting that each higher order difference is defined in terms of the lower order difference, we have

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

and

$$\begin{aligned}\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)\\ &= (y_3 - y_2) - (y_2 - y_1) - (y_2 - y_1) + (y_1 - y_0)\\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

Hence, we observe that the coefficients of the values of y, in the expansion of $\Delta^2 y_0$, $\Delta^3 y_0$ are binomial coefficients. Thus, in general, we arrive at the following result.

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0 \qquad (6.4)$$

Example 6.3 Show that the value of y_n can be expressed in terms of the leading value y_0 and the leading differences Δy_0, $\Delta^2 y_0$, $\ldots$, $\Delta^n y_0$.

Solution We have from the forward difference table

$$\left.\begin{aligned} y_1 - y_0 &= \Delta y_0 \quad \text{or} \quad y_1 = y_0 + \Delta y_0\\ y_2 - y_1 &= \Delta y_1 \quad \text{or} \quad y_2 = y_1 + \Delta y_1\\ y_3 - y_2 &= \Delta y_2 \quad \text{or} \quad y_3 = y_2 + \Delta y_2\end{aligned}\right\} \qquad (6.5)$$

and so on. Similarly

$$\left.\begin{aligned} \Delta y_1 - \Delta y_0 &= \Delta^2 y_0 \quad \text{or} \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0\\ \Delta y_2 - \Delta y_1 &= \Delta^2 y_1 \quad \text{or} \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1\end{aligned}\right\} \qquad (6.6)$$

and so on. Similarly, we can write

$$\left.\begin{aligned} \Delta^2 y_1 - \Delta^2 y_0 &= \Delta^3 y_0 \quad \text{or} \quad \Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0\\ \Delta^2 y_2 - \Delta^2 y_1 &= \Delta^3 y_1 \quad \text{or} \quad \Delta^2 y_2 = \Delta^2 y_1 + \Delta^3 y_1\end{aligned}\right\} \qquad (6.7)$$

and so on. Also, from Eqs. (6.6) and (6.7), we can rewrite Δy_2 as

$$\Delta y_2 = (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$
$$= \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0 \tag{6.8}$$

From Eqs. (6.5)–(6.8), y_3 can be rewritten

$$\begin{aligned} y_3 &= y_2 + \Delta y_2 \\ &= (y_1 + \Delta y_1) + (\Delta y_1 + \Delta^2 y_1) \\ &= (y_0 + \Delta y_0) + 2(\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0) \\ &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 \\ &= (1 + \Delta)^3 y_0 \end{aligned} \tag{6.9}$$

Similarly, we can symbolically write

$$y_1 = (1 + \Delta)y_0, \qquad y_2 = (1 + \Delta)^2 y_0, \qquad y_3 = (1 + \Delta)^3 y_0$$

Continuing this procedure, we can show, in general

$$y_n = (1 + \Delta)^n y_0$$

Hence, we obtain

$$y_n = y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + \cdots + \Delta^n y_0 \tag{6.10}$$

Equivalently, we can also write the result as

$$y_n = \sum_{i=0}^{n} {}^nC_i \, \Delta^i y_0 \tag{6.11}$$

6.2.2 Backward Differences

For a given table of values (x_k, y_k), $k = 0, 1, 2, \ldots, n$ of a function $y = f(x)$ with equally spaced abscissas, the first backward differences are usually expressed in terms of the backward difference operator ∇ as

$$\nabla y_i = y_i - y_{i-1}, \qquad i = n, (n - 1), \ldots, 1 \tag{6.12}$$

Thus,

$$\begin{aligned} \nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ \vdots \quad & \quad \vdots \quad \vdots \\ \nabla y_n &= y_n - y_{n-1} \end{aligned}$$

The differences of these differences are called *second differences* and they are denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$. That is,

$$\begin{aligned} \nabla^2 y_2 &= \nabla y_2 - \nabla y_1 \\ \nabla^2 y_3 &= \nabla y_3 - \nabla y_2 \\ \vdots \quad & \quad \vdots \quad \vdots \\ \nabla^2 y_n &= \nabla y_n - \nabla y_{n-1} \end{aligned}$$

Thus, in general, the second backward differences are

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}, \qquad i = n, (n-1), \ldots, 2 \tag{6.13}$$

while the kth backward differences are given as

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \qquad i = n, (n-1), \ldots, k \tag{6.14}$$

These backward differences can be systematically arranged for a table of values (x_k, y_k), $k = 0, 1, \ldots, 6$ as indicated in Table 6.2.

Table 6.2 Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0	∇y_1	$\nabla^2 y_2$				
x_1	y_1	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_3$	$\nabla^4 y_4$		
x_2	y_2	∇y_3	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_5$	$\nabla^5 y_5$	$\nabla^6 y_6$
x_3	y_3	∇y_4	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_6$	$\nabla^5 y_6$	
x_4	y_4	∇y_5	$\nabla^2 y_6$	$\nabla^3 y_6$			
x_5	y_5	∇y_6					
x_6	y_6						

From this table, it can be observed that the subscript remains constant along every backward diagonal.

Example 6.4 Show that any value of y can be expressed in terms of y_n and its backward differences.

Solution From Eq. (6.12) we have

$$y_{n-1} = y_n - \nabla y_n \quad \text{and} \quad y_{n-2} = y_{n-1} - \nabla y_{n-1}.$$

Also from the definition as given in Eq. (6.13), we get

$$\nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$$

From these equations, we obtain

$$y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$$

Similarly, we can show that

$$y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$$

Symbolically, these results can be rewritten as follows:

$$y_{n-1} = (1 - \nabla) y_n, \qquad y_{n-2} = (1 - \nabla)^2 y_n, \qquad y_{n-3} = (1 - \nabla)^3 y_n$$

Thus is general, we can write

$$y_{n-r} = (1 - \nabla)^r y_n \tag{6.15}$$

That is,

$$y_{n-r} = y_n - {}^rC_1 \nabla y_n + {}^rC_2 \nabla^2 y_n - \cdots + (-1)^r \nabla^r y_n$$

6.2.3 Central Differences

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol δ to represent central difference operator and the subscript of δy for any difference as the average of the subscripts of the two members of the difference. Thus, we write

$$\delta y_{1/2} = y_1 - y_0, \qquad \delta y_{3/2} = y_2 - y_1, \text{ etc.}$$

In general

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)} \tag{6.16}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)} \tag{6.17}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)} \tag{6.18}$$

These central differences can be systematically arranged as indicated in Table 6.3:

Table 6.3 Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

Thus, we observe that all the odd differences have a fractional suffix and all the even differences with the same subscript lie horizontally.

The following alternative notation may also be adopted to introduce finite difference operators. Let $y = f(x)$ be a functional relation between x and y, which is also denoted by y_x. Suppose, we are given consecutive values of x differing by h say $x, x + h, x + 2h, x + 3h$, etc. The corresponding values of y are $y_x, y_{x+h}, y_{x+2h}, y_{x+3h}$, etc. As before, we can form the differences of these values. Thus

$$\Delta y_x = y_{x+h} - y_x = f(x + h) - f(x) \tag{6.19}$$

Similarly

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

and

$$\nabla y_x = y_x - y_{x-h} = f(x) - f(x - h) \tag{6.20}$$

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \tag{6.21}$$

Shift operator, E

Let $y = f(x)$ be a function of x, and let x takes the consecutive values $x, x + h, x + 2h$, etc. We then define an operator E having the property

$$E\,f(x) = f(x + h) \tag{6.22}$$

Thus, when E operates on $f(x)$, the result is the next value of the function. Here, E is called the *shift operator*. If we apply the operator E twice on $f(x)$, we get

$$E^2 f(x) = E\,[E\,f(x)] = E\,[f(x + h)] = f(x + 2h)$$

Thus, in general, if we apply the operator E n times on $f(x)$, we arrive at

$$E^n f(x) = f(x + nh)$$

In terms of new notation, we can write

$$E^n y_x = y_{x+nh}$$

or

$$E^n f(x) = f(x + nh) \tag{6.23}$$

for all real values of n. Also, if $y_0, y_1, y_2, y_3, \ldots$ are the consecutive values of the function y_x, then we can also write

$$E y_0 = y_1, \qquad E^2 y_0 = y_2, \qquad E^4 y_0 = y_4, \qquad \ldots, \qquad E^2 y_2 = y_4$$

and so on. The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x - h)$$

and similarly

$$E^{-n} f(x) = f(x - nh) \tag{6.24}$$

Average operator, μ

The average operator μ is defined as

$$\mu\,f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right] = \frac{1}{2}[y_{x+(h/2)} + y_{x-(h/2)}] \tag{6.25}$$

Differential operator, D

It is known that D represents a differential operator having a property

$$\left.\begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned}\right\} \tag{6.26}$$

Having defined various difference operators Δ, ∇, δ, E, μ and D, we can obtain the following relations easily:

From the definition of operators Δ and E, we have

$$\Delta y_x = y_{x+h} - y_x = E y_x - y_x = (E - 1) y_x$$

Therefore,

$$\Delta = E - 1 \tag{6.27}$$

Following the definition of operators ∇ and E^{-1}, we have

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x = (1 - E^{-1})\, y_x$$

Therefore,

$$\nabla = 1 - E^{-1} = \frac{E-1}{E} \tag{6.28}$$

The definition of operators δ and E gives

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = E^{1/2}y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})\, y_x$$

Hence,

$$\delta = E^{1/2} - E^{-1/2} \tag{6.29}$$

The definition of μ and E similarly yields

$$\mu y_x = \frac{1}{2}[y_{x+(h/2)} + y_{x-(h/2)}] = \frac{1}{2}(E^{1/2} + E^{-1/2})y_x$$

Therefore,

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) \tag{6.30}$$

It is known that

$$Ey_x = y_{x+h} = f(x+h)$$

using Taylor series expansion, we have

$$Ey_x = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!}D^2f(x) + \cdots$$

$$= \left(1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \cdots\right) f(x) = e^{hD}\, y_x$$

Thus,

$$hD = \log E \tag{6.31}$$

Hence, all the operators are expressed in terms of E.

Example 6.5 Prove that

$$hD = \log(1 + \Delta) = -\log(1 - \nabla) = \sinh^{-1}(\mu\delta)$$

Solution Using the standard relations (6.27)–(6.31), we have

$$hD = \log E = \log(1 + \Delta) = -\log E^{-1} = -\log(1 - \nabla) \tag{1}$$

Also,

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$$

Solution At the outset, we shall construct Newton's backward difference table for the given data as

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
1976	43	3			
1978	48	5	2		
1980	52	4	–1	–3	
1982	57	5	1	2	5

In this example,

$$p = \frac{1979 - 1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1, \quad \nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

Newton's backward interpolation formula gives

$$y_{1979} = 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1)$$

$$+ \frac{(-1.5)(-0.5)(0.5)}{6}(2) + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5)$$

$$= 57 - 7.5 + 0.375 + 0.125 + 0.1172$$

Therefore,

$$y_{1979} = 50.1172$$

6.5 LAGRANGE'S INTERPOLATION FORMULA

Newton's interpolation formulae developed in the earlier sections can be used only when the values of the independent variable x are equally spaced. Also the differences of y must ultimately become small. If the values of the independent variable are not given at equidistant intervals, then we have the basic formula associated with the name of *Lagrange* which is derived as follows:

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$. Since there are $(n + 1)$ values of y corresponding to $(n + 1)$ values of x, we can represent the function $f(x)$ by a polynomial of degree n. Suppose we write this polynomial in the form

$$f(x) = A_0x^n + A_1x^{n-1} + \cdots + A_n$$

or, more conveniently, in the form

$$y = f(x) = a_0 (x - x_1)(x - x_2) \cdots (x - x_n) + a_1 (x - x_0)(x - x_2) \cdots (x - x_n)$$
$$+ a_2 (x - x_0)(x - x_1) \cdots (x - x_n) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \quad (6.36)$$

Here, the coefficients a_k are so chosen as to satisfy Eq. (6.36) by the $(n + 1)$ pairs

(x_i, y_i). Thus, Eq. (6.36) yields

$$y_0 = f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)$$

Therefore,

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)}$$

Similarly, we obtain

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}$$

$$a_i = \frac{y_i}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}$$

and

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})}$$

Now, substituting the values of $a_0, a_1, \dots, a_n$ into Eq. (6.36), we get

$$y = f(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} y_0$$

$$+ \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)} y_1 + \dots$$

$$+ \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} y_i + \dots$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})} y_n \tag{6.37}$$

Equation (6.37) is Lagrange's formula for interpolation. This formula can be used whether the values $x_0, x_1, x_2, \dots, x_n$ are equally spaced or not. Alternatively, Eq. (6.37) can also be written in compact form as

$$y = f(x) = L_0(x)\, y_0 + L_1(x)\, y_1 + \dots + L_i(x)\, y_i + \dots + L_n(x)\, y_n$$

$$= \sum_{k=0}^{n} L_k(x) y_k$$

$$= \sum_{k=0}^{n} L_k(x) f(x_k) \tag{6.38}$$

where,

$$L_i(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} \tag{6.39}$$

we can easily observe that, $L_i(x_i) = 1$ and $L_i(x_j) = 0$, $i \neq j$. Thus introducing

Kronecker delta notation

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Further, if we introduce the notation

$$\Pi(x) = \prod_{i=0}^{n} (x - x_i) = (x - x_0)(x - x_1) \cdots (x - x_n) \tag{6.40}$$

that is, $\Pi(x)$ is a product of $(n + 1)$ factors. Clearly, its derivative $\Pi'(x)$ contains a sum of $(n + 1)$ terms in each of which one of the factors of $\Pi(x)$ will be absent. We also define,

$$P_k(x) = \prod_{i \neq k} (x - x_i) \tag{6.41}$$

which is same as $\Pi(x)$ except that the factor $(x - x_k)$ is absent. Then

$$\Pi'(x) = P_0(x) + P_1(x) + \cdots + P_n(x) \tag{6.42}$$

But, when $x = x_k$, all terms in the above sum vanishes except $P_k(x_k)$. Hence,

$$\Pi'(x_k) = P_k(x_k) = (x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n) \tag{6.43}$$

Therefore, using Eqs. (6.40)–(6.43), Eq. (6.39) can be rewritten as

$$L_k(x) = \frac{P_k(x)}{P_k(x_k)} = \frac{P_k(x)}{\Pi'(x_k)} = \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} \tag{6.44}$$

Finally, the Lagrange's interpolation polynomial of degree n can be written as

$$y(x) = f(x) = \sum_{k=0}^{n} \frac{\Pi(x)}{(x - x_k)\Pi'(x_k)} f(x_k) = \sum_{k=0}^{n} L_k(x) f(x_k) = \sum_{k=0}^{n} L_k(x) y_k \tag{6.45}$$

Lagrange's interpolation is illustrated through the following examples.

Example 6.14 Find Lagrange's interpolation polynomial fitting the points $y(1) = -3$, $y(3) = 0$, $y(4) = 30$, $y(6) = 132$. Hence find $y(5)$.

Solution The given data can be arranged as follows:

x	1	3	4	6
$y = f(x)$	−3	0	30	132

using Lagrange's interpolation formula (6.37), we have

$$y(x) = f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(0)$$
$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

$$= \frac{x^3 - 13x^2 + 54x - 72}{-30}(-3) + \frac{x^3 - 11x^2 + 34x - 24}{6}(0)$$
$$+ \frac{x^3 - 10x^2 + 27x - 18}{-6}(30) + \frac{x^3 - 8x^2 + 19x - 12}{30}(132)$$

On simplification, we get

$$y(x) = \frac{1}{10}\left(-5x^3 + 135x^2 - 460x + 300\right) = \frac{1}{2}(-x^3 + 27x^2 - 92x + 60)$$

which is the required Lagrange's interpolation polynomial. Now, $y(5) = 75$.

Example 6.15 Given the following data, evaluate $f(3)$ using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

Solution Using Lagrange's interpolation formula given by Eq. (6.37), we have

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$
$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Therefore,

$$f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1) + \frac{(3-1)(3-5)}{(2-1)(2-5)}(4) + \frac{(3-1)(3-2)}{(5-1)(5-2)}(10) = 6.4999$$

6.6 DIVIDED DIFFERENCES

When the function values are given at non-equispaced points, we have already developed the Lagrange's interpolation formula for interpolation in Section 6.5. Now, we shall introduce the concept of divided differences and then develop Newton's divided difference interpolation formula, whose accuracy is same as that of Lagrange's formula, but has the advantage of being computationally economical in the sense that it involves less number of arithmetic operations.

Let us assume that the function $y = f(x)$ is known for several values of x, (x_i, y_i), for $i = 0\,(1)\,n$. The divided differences of orders 0, 1, 2, …, n are defined recursively as follows:

$$y\,[x_0] = y\,(x_0) = y_0$$

is the 0th order divided difference. The first order divided difference is defined as

Similarly, the higher order divided differences are defined in terms of lower order divided differences by the relations (Hildebrand, 1982) of the form

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0}$$

while

$$y[x_0, x_1, \cdots, x_n] = \frac{y[x_1, x_2, \cdots, x_n] - y[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \tag{6.46}$$

The standard format of the divided differences are displayed in Table 6.4.

Table 6.4 Divided Differences

x	$y(x)$	1st order	2nd order	3rd order	4th order
x_0	y_0	$y[x_1, x_0]$	$y[x_0, x_1, x_2]$	$y[x_0, x_1, x_2, x_3]$	$y[x_0, x_1, x_2, x_3, x_4]$
x_1	y_1	$y[x_2, x_1]$	$y[x_1, x_2, x_3]$	$y[x_1, x_2, x_3, x_4]$	
x_2	y_2	$y[x_3, x_2]$	$y[x_2, x_3, x_4]$		
x_3	y_3	$y[x_4, x_3]$			
x_4	y_4				

We can easily verify that the divided difference is a symmetric function of its arguments. That is,

$$y[x_1, x_0] = y[x_0, x_1] = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0}$$

Now,

$$y[x_0, x_1, x_2] = \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} = \frac{1}{x_2 - x_0}\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

Therefore,

$$y[x_0, x_1, x_2] = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

which is a symmetric form, hence suggests the general result as

$$y[x_0, \cdots, x_k] = \frac{y_0}{(x_0 - x_1)\cdots(x_0 - x_k)} + \frac{y_1}{(x_1 - x_0)\cdots(x_1 - x_k)} + \cdots$$

$$+ \frac{y_k}{(x_k - x_0)\cdots(x_k - x_{k-1})} \tag{6.47}$$

$$= \sum_{i=0}^{k} \frac{y_i}{\prod\limits_{\substack{i=0 \\ i\neq j}}^{k} (x_i - x_j)}$$

In Eq. (6.47), it can be noted that zero factor $(x_i - x_i)$ is omitted in the denominator of each term of the sum.

...mple 6.16 Find the interpolating polynomial by (i) Lagrange's formula, and ...ewton's divided difference formula for the following data, and hence show they represent the same interpolating polynomial.

x	0	1	2	4
y	1	1	2	5

Solution The divided difference table for the given data is constructed as ...ws:

x	y	1st divided difference	2nd divided difference	3rd divided difference
0	1	0	1/2	−1/12
1	1	1	1/6	
2	2	3/2		
4	5			

(i) Lagrange's interpolation formula (6.37) gives

$$y = f(x) = \frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(1)$$

$$+ \frac{(x-0)(x-1)(x-4)}{(2)(2-1)(2-4)}(2) + \frac{(x-0)(x-1)(x-2)}{4(4-1)(4-2)}(5)$$

$$= \frac{-(x^3-7x^2+14x-8)}{8} + \frac{x^3-6x^2+8x}{3} - \frac{x^3-5x^2+4x}{2}$$

$$+ \frac{5(x^3-3x^2+2x)}{24}$$

$$= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1 \qquad (1)$$

(ii) Newton's divided difference formula gives

$$y = f(x) = 1 + (x-0)(0) + (x-0)(x-1)\left(\frac{1}{2}\right) + (x-0)(x-1)(x-2)\left(-\frac{1}{12}\right)$$

$$= -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1 \qquad (2)$$

From Eqs. (1) and (2) we observe that the interpolating polynomial by both Lagrange's and Newton's divided difference formulae is one and the same. Also Newton's formula involves less number of arithmetic operations than that of Lagrange's.

Example 6.17 Using Newton's divided difference formula, find the quadratic equation for the following data. Hence find $y(2)$.

x	0	1	4
y	2	1	4

Solution The divided difference table for the given data is constructed as follows:

x	y	1st divided difference	2nd divided difference
0	2	−1	1/2
1	1	1	
4	4		

$-\frac{1}{1}$

$-\frac{1}{1}, \frac{3}{3}$

$+2/4$

Now, using Newton's divided difference formula, we have

$$y = 2 + (x - 0)(-1) + (x - 0)(x - 1)\left(\frac{1}{2}\right) = \frac{1}{2}(x^2 - 3x + 4)$$

Hence, $y(2) = 1$.

Example 6.18 A function $y = f(x)$ is given at the sample points $x = x_0, x_1$ and x_2. Show that the Newton's divided difference interpolation formula and the corresponding Lagrange's interpolation formula are identical.

Solution For the function $y = f(x)$, we have the data (x_i, y_i), $i = 0, 1, 2$. The interpolation polynomial using Newton's divided difference formula is given as

$$y = f(x) = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] \qquad (1)$$

Using the definition of divided differences and Eq. (6.47), we can rewrite Eq. (1) in the form

$$y = y_0 + (x - x_0)\frac{(y_1 - y_0)}{(x_1 - x_0)} + (x - x_0)(x - x_1)\left[\frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}\right]$$

$$= \left[1 - \frac{(x_0 - x)}{(x_0 - x_1)} + \frac{(x - x_0)(x - x_1)}{(x_0 - x_1)(x_0 - x_2)}\right] y_0 + \left[\frac{(x - x_0)}{(x_1 - x_0)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)}\right] y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\, y_2$$