2.3 REGULA–FALSI METHOD

This method is also known as the *method of false position.* In this method, we choose two points x_n and x_{n-1} such that $f(x_n)$ and $f(x_{n-1})$ are of opposite signs. Intermediate value property suggests that the graph of $y = f(x)$ crosses the x-axis between these two points, and therefore, a root say $x = \xi$ lies between these two points. Thus, to find a real root of $f(x) = 0$ using Regula-Falsi method, we replace the part of the curve between the points $A[x_n, f(x_n)]$ and $B[x_{n-1}, f(x_{n-1})]$ by a chord in that interval and we take the point of intersection of this chord with the x-axis as a first approximation to the root (see Fig. 2.3).

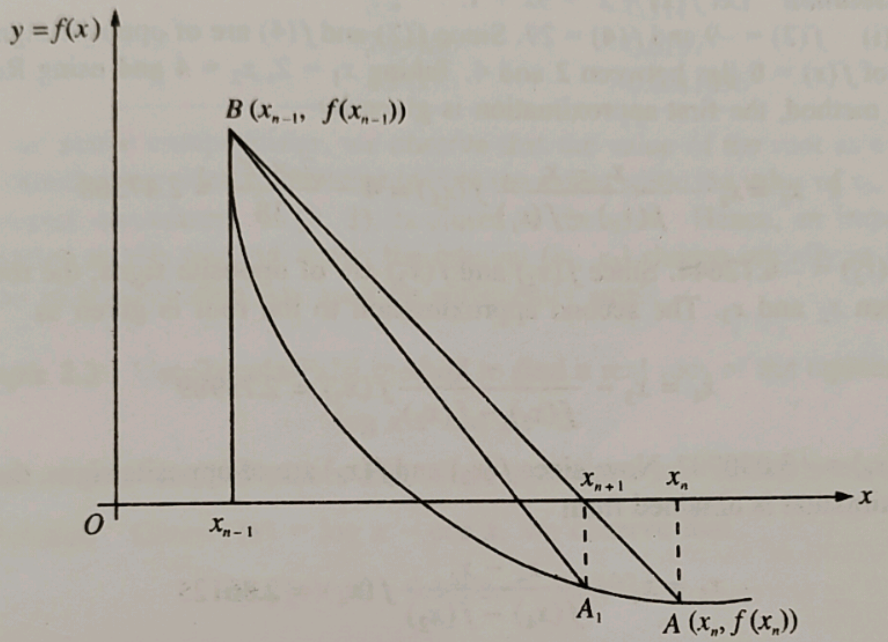

Fig. 2.3 Geometrical illustration of Regula-Falsi method.

Now, the equation of the chord joining the points A and B is

$$\frac{y - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n} \tag{2.1}$$

Setting $y = 0$ in Eq. (2.1), we get

$$x = x_n - \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} f(x_n)$$

Hence, the first approximation to the root of $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \tag{2.2}$$

From Fig. 2.3, we observe that $f(x_{n-1})$ and $f(x_{n+1})$ are of opposite sign. Thus, it is possible to apply the above procedure, to determine the line through B and A_1 and so on. Hence, the successive approximations to the root of $f(x) = 0$ is given by Eq. (2.2). This method can best be understood through the following examples.

Example 2.2 Use the Regula-Falsi method to compute a real root of the equation $x^3 - 9x + 1 = 0$,

(i) if the root lies between 2 and 4
(ii) if the root lies between 2 and 3.

Comment on the results.

Solution Let $f(x) = x^3 - 9x + 1$.

(i) $f(2) = -9$ and $f(4) = 29$. Since $f(2)$ and $f(4)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 4. Taking $x_1 = 2$, $x_2 = 4$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 4 - \frac{2 \times 29}{38} = 2.47368$$

and $f(x_3) = -6.12644$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given as

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.73989$$

and $f(x_4) = -3.090707$. Now, since $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.86125$$

and $f(x_5) = -1.32686$. This procedure can be continued till we get the desired result. The first three iterations are shown as in the table.

n	x_{n+1}	$f(x_{n+1})$
2	2.47368	–6.12644
3	2.73989	–3.090707
4	2.86125	–1.32686

(ii) $f(2) = -9$ and $f(3) = 1$. Since $f(2)$ and $f(3)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 3. Taking $x_1 = 2$, $x_2 = 3$ and using Regula-Falsi method, the first approximation is given by

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{1}{10} = 2.9$$

and $f(x_3) = -0.711$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3. The second approximation to the root is given as

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.94156$$

and $f(x_4) = -0.0207$. Now, we observe that $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$x_5 = x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.94275$$

and $f(x_5) = -0.0011896$. This procedure can be continued till we get the desired result. The first three iterations are shown as in the table.

n	x_{n+1}	$f(x_{n+1})$
2	2.9	−0.711
3	2.94156	−0.0207
4	2.94275	−0.0011896

From the above computations, we observe that the value of the root as a third approximation is evidently different in both the cases, while the value of x_5, when the interval considered is (2, 3), is closer to the root. Hence, an important observation in this method is that the interval (x_1, x_2) chosen initially in which the root of the equation lies must be sufficiently small.

Example 2.3 Use Regula-Falsi method to find a real root of the equation

$$\log x - \cos x = 0$$

accurate to four decimal places after three successive approximations.

Solution Given $f(x) = \log x - \cos x$. We observe that

$$f(1) = 0 - 0.5403 = -0.5403$$

and

$$f(2) = 0.69315 + 0.41615 = 1.1093$$

Since $f(1)$ and $f(2)$ are of opposite signs, the root lies between $x_1 = 1, x_2 = 2$. The first approximation is obtained from

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2 - \frac{1.1093}{1.6496} = 1.3275$$

and

$$f(x_3) = 0.2833 - 0.2409 = 0.0424$$

Now, since $f(x_1)$ and $f(x_3)$ are of opposite signs, the second approximation is obtained as

$$x_4 = 1.3275 - \frac{(.3275)\,(.0424)}{0.0424 + 0.5403} = 1.3037$$

and

$$f(x_4) = 1.24816 \times 10^{-3}$$

Similarly, we observe that $f(x_1)$ and $f(x_4)$ are of opposite signs, so, the third approximation is given by

$$x_5 = 1.3037 - \frac{(0.3037)\,(0.001248)}{0.001248 + 0.5403} = 1.3030$$

and
$$f(x_5) = 0.62045 \times 10^{-4}$$

Hence, the required real root is 1.3030.

Example 2.4 Using Regula-Falsi method, find the real root of the following equation correct to three decimal places:

$$x \log_{10} x = 1.2$$

Solution Let $f(x) = x \log_{10} x - 1.2$. We observe that $f(2) = -0.5979$, $f(3) = 0.2314$. Since $f(2)$ and $f(3)$ are of opposite signs, the real root lies between $x_1 = 2$, $x_2 = 3$. The first approximation is obtained from

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{0.2314}{0.8293} = 2.72097$$

and $f(x_3) = -0.01713$. Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root of $f(x) = 0$ lies between x_2 and x_3. Now, the second approximation is given by

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.7402$$

and $f(x_4) = -3.8905 \times 10^{-4}$. Thus, the root of the given equation correct to three decimal places is 2.740.

2.4 METHOD OF ITERATION

The method of iteration can be applied to find a real root of the equation $f(x) = 0$ by rewriting the same in the form,

$$x = \phi(x) \tag{2.3}$$

For example, $f(x) = \cos x - 2x + 3 = 0$. It can be rewritten as

$$x = \frac{1}{2}(\cos x + 3) = \phi(x)$$

Let $x = \xi$ is the desired root of Eq. (2.3). Suppose x_0 is its initial approximation. The first and successive approximations to the root can be obtained as

$$\left.\begin{aligned} x_1 &= \phi(x_0) \\ x_2 &= \phi(x_1) \\ &\vdots \\ x_{n+1} &= \phi(x_n) \end{aligned}\right\} \tag{2.4}$$

Definition 2.1 Let $\{x_i\}$ be the sequence obtained by a given method and let $x = \xi$ denotes the root of the equation $f(x) = 0$. Then, the method is said to be *convergent*, if and only if

Lt $|x$ ...

The conver...

Therefore,

$$\phi'(x) = -\frac{1}{2(x+1)^{3/2}}$$

We note that $|\phi'(x)| < 1$, for all x in (0, 1). Hence, the method of iteration is applicable here.

Taking the initial value $x_0 = 1$, we successively obtain the following values:

$$\begin{aligned}
x_1 &= \phi(x_0) = 1/\sqrt{2} = 0.70711, & f(x_1) &= -0.14644\\
x_2 &= \phi(x_1) = 0.76537, & f(x_2) &= 0.03414\\
x_3 &= \phi(x_2) = 0.75263, & f(x_3) &= 7.2213 \times 10^{-3}\\
x_4 &= \phi(x_3) = 0.75536, & f(x_4) &= 1.55658 \times 10^{-3}\\
x_5 &= \phi(x_4) = 0.75477, & f(x_5) &= -3.44323 \times 10^{-4}\\
x_6 &= \phi(x_5) = 0.7549, & f(x_6) &= 7.38295 \times 10^{-5}
\end{aligned}$$

Hence, the required root is 0.7549.

Note: The given equation can be rewritten in many ways. Suppose, we rewrite

$$x^2 = 1 - x^3 \quad \text{or} \quad x = (1 - x^3)^{1/2} = \phi(x)$$

Then

$$|\phi'(x)| = \frac{3x^2}{2(1 - x^3)^{1/2}}$$

if we take $x = 1$, in the interval (0, 1), $|\phi'(x)| = \infty$, then the condition $|\phi'(x)| < 1$ is violated.

2.5 NEWTON–RAPHSON METHOD

This is a very powerful method for finding the real root of an equation in the form, $f(x) = 0$. Suppose, x_0 is an approximate root of $f(x) = 0$. Let $x_1 = x_0 + h$, where h is small, be the exact root of $f(x) = 0$, then $f(x_1) = 0$. Now, expanding $f(x_0 + h)$ by Taylor's theorem, we get

$$f(x_0 + h) = f(x_0) + h\,f'(x_0) + \frac{h^2}{2} f''(x_0) + \cdots = 0 \tag{2.5}$$

Since h is small, we neglect terms containing h^2 and its higher powers, then

$$f(x_0) + h\,f'(x_0) = 0 \quad \text{or} \quad h = \frac{-f(x_0)}{f'(x_0)}$$

Therefore, a better approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Still better and successive approximations $x_2, x_3, \ldots, x_n$ to the root can obviously be obtained from the iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.6}$$

This is known as Newton–Raphson iteration formula, which has the following geometrical interpretation:

Suppose, the graph of the function $y = f(x)$ crosses the x-axis at α (see Fig. 2.4), then $x = \alpha$ is the root of the equation $f(x) = 0$.

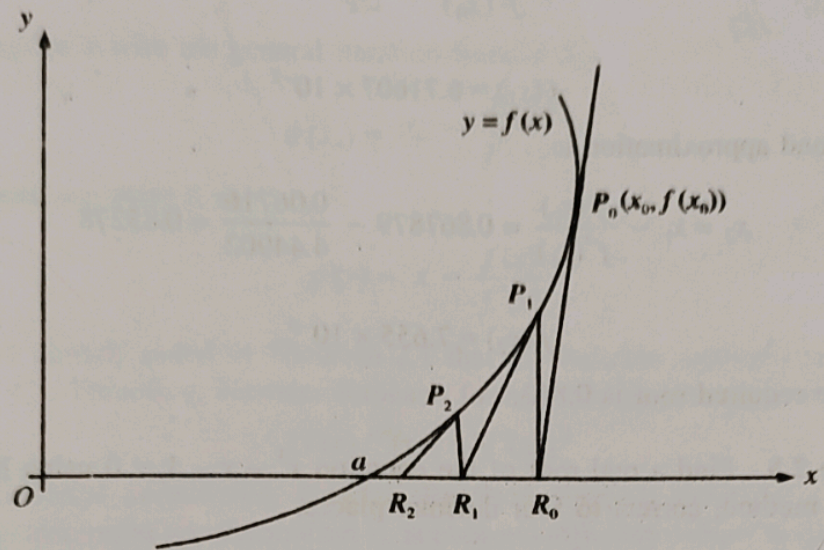

Fig. 2.4 Geometrical interpretation of Newton–Raphson method.

Let x_0 be a point closer to the root α, then the equation of the tangent at $P_0(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)\,(x - x_0) \tag{2.7}$$

This tangent cuts the x-axis at $R_0\ (x_1, 0)$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.8}$$

which is a first approximation to the root α. If P_1 is a point on the curve corresponding to x_1, then the tangent at P_1 cuts the x-axis at $R_1(x_2, 0)$, which is still closer to α, than x_1. Therefore, x_2 is a second approximation to the root. Continuing this process, we arrive at the root α, very rapidly, which is evident from Fig. 2.4. Thus, in this method, we have replaced the part of the curve between the point P_0 and x-axis by a tangent to the curve at P_0 and so on. In order to illustrate this method, we shall consider the following examples.

Example 2.7 Find the real root of the equation $xe^x - 2 = 0$ correct to two decimal places, using Newton–Raphson method.

Solution Given $f(x) = xe^x - 2$, we have

$$f'(x) = xe^x + e^x \text{ and } f''(x) = xe^x + 2e^x$$

clearly, we have

$$f(0) = -2 \quad \text{and} \quad f(1) = e - 2 = 0.71828$$

Hence, the required root lies in the interval (0, 1) and is nearer to 1.

Also, $f(x)$ and $f'(x)$ do not vanish in (0, 1) and $f(x)$ and $f''(x)$ will have the same sign at $x = 1$. Therefore, we take the first approximation $x_0 = 1$, and using Newton–Raphson method, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{e+2}{2e} = 0.867879$$

and

$$f(x_1) = 6.71607 \times 10^{-2}$$

The second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.867879 - \frac{0.06716}{4.44902} = 0.85278$$

and

$$f(x_2) = 7.655 \times 10^{-4}$$

Thus, the required root is 0.853.

Example 2.8 Find a real root of the equation $x^3 - x - 1 = 0$ using Newton–Raphson method, correct to four decimal places.

Solution Let $f(x) = x^3 - x - 1$, then we observe that $f(1) = -1, f(2) = 5$. Therefore, the root lies in the interval (1, 2). We also observe

$$f'(x) = 3x^2 - 1, \qquad f''(x) = 6x$$

and

$$f(1) = -1, \quad f''(1) = 6, \quad f(2) = 5, \quad f''(2) = 12$$

Since $f(2)$ and $f''(2)$ are of the same sign, we choose $x_0 = 2$ as the first approximation to the root. The second approximation is computed using Newton–Raphson method as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{5}{11} = 1.54545 \quad \text{and} \quad f(x_1) = 1.14573$$

The successive approximations are

$$x_2 = 1.54545 - \frac{1.14573}{6.16525} = 1.35961, \qquad f(x_2) = 0.15369$$

$$x_3 = 1.35961 - \frac{0.15369}{4.54562} = 1.32579, \qquad f(x_3) = 4.60959 \times 10^{-3}$$

$$x_4 = 1.32579 - \frac{4.60959 \times 10^{-3}}{4.27316} = 1.32471, \qquad f(x_4) = -3.39345 \times 10^{-5}$$

$$x_5 = 1.32471 + \frac{3.39345 \times 10^{-5}}{4.26457} = 1.324718, \qquad f(x_5) = 1.823 \times 10^{-7}$$

Hence, the required root is 1.3247.