

Hazel Phi: 11-type-constructors

July 26, 2021

SYNTAX

Kind	κ	$::=$	$\mathbf{Type} \mid \mathbf{KHole} \mid \mathbf{S}_\kappa(\tau) \mid \Pi_{t::\kappa_1.\kappa_2}$
User Types	$\hat{\tau}$	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \hat{\tau} \rangle^u \mid \lambda t::\mathbf{Type}.\hat{\tau} \mid \tau_1 \tau_2$
Internal Types	τ	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \tau \rangle^u \mid \langle t \rangle^u \mid \lambda t::\kappa.\tau \mid \tau_1 \tau_2$
Base Types	\mathbf{bse}	$::=$	$\mathbf{Int} \mid \mathbf{Float} \mid \mathbf{Bool}$
BinOp	\oplus	$::=$	$\times \mid + \mid \rightarrow$
Type Pattern			
User Expression			
Internal Expression			

DECLARATIVES

$\boxed{\Delta; \Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\begin{array}{c}
 \frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathbf{bse} ::> \mathbf{S}_{\mathbf{Type}}(\mathbf{bse})} \quad (1) \qquad \frac{\Delta; \Phi_1, t::\kappa, \Phi_2 \vdash \text{OK}}{\Delta; \Phi \vdash t ::> \mathbf{S}_\kappa(t)} \quad (2) \qquad \frac{\Delta; \Phi \vdash \tau_1::\mathbf{Type} \quad \Delta; \Phi \vdash \tau_2::\mathbf{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathbf{S}_{\mathbf{Type}}(\tau_1 \oplus \tau_2)} \quad (3) \\
 \\
 \frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \langle \rangle^u ::> \mathbf{S}_\kappa(\langle \rangle^u)} \quad (4) \qquad \frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad \Delta; \Phi \vdash \tau::\kappa_1}{\Delta; \Phi \vdash \langle \tau \rangle^u ::> \mathbf{S}_\kappa(\langle \tau \rangle^u)} \quad (5) \\
 \\
 \frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad t \notin \text{dom}(\Phi)}{\Delta; \Phi \vdash \langle t \rangle^u ::> \mathbf{S}_\kappa(\langle t \rangle^u)} \quad (6) \qquad \frac{\Delta; \Phi, t::\kappa_1 \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau ::> \mathbf{S}_{\Pi_{t::\kappa_1.\kappa_2}}(\lambda t::\kappa_1.\tau)} \quad (7) \\
 \\
 \frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \quad \Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1.\kappa_2} \quad \Delta; \Phi \vdash \tau_2::\kappa_1}{\Delta; \Phi \vdash \tau_1 \tau_2 ::> [\tau_2/t]\kappa_2} \quad (8)
 \end{array}$$

$\boxed{\Delta; \Phi \vdash \tau::\kappa}$ τ is well formed at kind κ

$$\begin{array}{c}
 \frac{\Delta; \Phi \vdash \tau ::> \kappa}{\Delta; \Phi \vdash \tau::\kappa} \quad (9) \qquad \frac{\Delta; \Phi \vdash \tau ::> \mathbf{S}_\kappa(\tau)}{\Delta; \Phi \vdash \tau::\kappa} \quad (10) \qquad \frac{\Delta; \Phi \vdash \tau::\mathbf{S}_\kappa(\tau_1) \quad \Delta; \Phi \vdash \tau_1::\kappa}{\Delta; \Phi \vdash \tau::\kappa} \quad (11) \\
 \\
 \frac{\Delta; \Phi \vdash \tau ::> \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau::\kappa} \quad (12) \\
 \\
 \frac{\Delta; \Phi \vdash \tau::\kappa}{\Delta; \Phi \vdash \tau::\mathbf{S}_\kappa(\tau)} \quad (13)
 \end{array}$$

$\Delta; \Phi \vdash \kappa \blacktriangleright_{\Pi} \Pi_{t::\kappa_1} \cdot \kappa_2$ κ has matched Π -kind $\Pi_{t::\kappa_1} \cdot \kappa_2$

$$\frac{}{\Delta; \Phi \vdash \mathbf{KHole} \blacktriangleright_{\Pi} \Pi_{t::\mathbf{KHole}} \cdot \mathbf{KHole}} \quad (14)$$

$$\frac{\Delta; \Phi \vdash \kappa \equiv \mathbf{S}_{\mathbf{KHole}}(\tau_1)}{\Delta; \Phi \vdash \kappa \blacktriangleright_{\Pi} \Pi_{t::\kappa} \cdot \kappa} \quad (15)$$

$$\frac{\Delta; \Phi \vdash \kappa \equiv \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \kappa \blacktriangleright_{\Pi} \Pi_{t::\kappa_1} \cdot \kappa_2} \quad (16)$$

$\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$ κ_1 is equivalent to κ_2

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \equiv \kappa} \quad (17)$$

$$\frac{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \quad (18)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \quad (19)$$

$$\frac{\Delta; \Phi \vdash \tau::\mathbf{S}_{\kappa}(\tau_1)}{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{S}_{\kappa}(\tau_1)}(\tau) \equiv \mathbf{S}_{\kappa}(\tau_1)} \quad (20)$$

$$\frac{\Delta; \Phi \vdash \tau::\Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \mathbf{S}_{\Pi_{t::\kappa_1} \cdot \kappa_2}(\tau) \equiv \Pi_{t_1::\kappa_1} \cdot \mathbf{S}_{[t_1/t]_{\kappa_2}}(\tau \ t_1)} \quad (21)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1} \cdot \kappa_2 \equiv \Pi_{t::\kappa_3} \cdot \kappa_4} \quad (22)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau_1) \equiv \mathbf{S}_{\kappa}(\tau_2)} \quad (23)$$

$\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{KHole} \lesssim \kappa} \quad (24)$$

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{KHole}} \quad (25)$$

$$\frac{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \text{ OK} \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \lesssim \kappa} \quad (26)$$

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK} \quad \Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{S}_{\mathbf{KHole}}(\tau)} \quad (27)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \quad (28)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \lesssim \kappa_4 \quad \Delta; \Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \quad (29)$$

$$\frac{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \lesssim \kappa} \quad (30)$$

$$\frac{\Delta; \Phi \vdash \kappa_3 \lesssim \kappa_1 \quad \Delta; \Phi, t::\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1} \cdot \kappa_2 \lesssim \Pi_{t::\kappa_3} \cdot \kappa_4} \quad (31)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2 \quad \Delta; \Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa_1}(\tau_1) \lesssim \mathbf{S}_{\kappa_2}(\tau_2)} \quad (32)$$

$\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \equiv S_{\kappa}(\tau_2)}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \quad (33)$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \Pi_{t::\kappa_1} \cdot \kappa_3 \quad \Delta; \Phi \vdash \tau_2 :: \Pi_{t::\kappa_1} \cdot \kappa_4 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \stackrel{\kappa_2}{=} \tau_2 \quad t}{\Delta; \Phi \vdash \tau_1 \stackrel{\Pi_{t::\kappa_1} \cdot \kappa_2}{=} \tau_2} \quad (34)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{S_{\kappa}(\tau)}{=} \tau_2}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \quad (35)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{\Pi_{t::\kappa_1} \cdot \kappa_2}{=} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \equiv^{\kappa_1} \tau_4}{\Delta; \Phi \vdash \tau_1 \tau_2 \stackrel{[\tau_2/t] \kappa_2}{=} \tau_3 \tau_4} \quad (36)$$

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau \equiv^{\kappa} \tau} \quad (37)$$

$$\frac{\Delta; \Phi \vdash \tau_2 \equiv^{\kappa} \tau_1}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \quad (38)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_3 \quad \Delta; \Phi \vdash \tau_3 \equiv^{\kappa} \tau_1}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \quad (39)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{\text{Type}}{=} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \stackrel{\text{Type}}{=} \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \stackrel{\text{Type}}{=} \tau_3 \oplus \tau_4} \quad (40)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \equiv^{\kappa} \tau_2}{\Delta; \Phi \vdash \lambda t::\kappa_1. \tau_1 \stackrel{\Pi_{t::\kappa_1} \cdot \kappa}{=} \lambda t::\kappa_2. \tau_2} \quad (41)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2 \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \quad (42)$$

$\Delta; \Phi \vdash \kappa \text{ OK}$ κ is well formed

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{Type OK}} \quad (43)$$

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{KHole OK}} \quad (44)$$

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash S_{\kappa}(\tau) \text{ OK}} \quad (45)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \text{ OK} \quad \Delta; \Phi, t::\kappa_1 \vdash \kappa_2 \text{ OK}}{\Delta; \Phi \vdash \Pi_{t::\kappa_1} \cdot \kappa_2 \text{ OK}} \quad (46)$$

$\Delta; \Phi \vdash \text{OK}$ Context is well formed

$$\frac{}{; \vdash \text{OK}} \quad (47)$$

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi, t::\kappa \vdash \text{OK}} \quad (48)$$

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta, u::\kappa; \Phi \vdash \text{OK}} \quad (49)$$

Variables implicitly assumed to be fresh as necessary

METATHEORY

Lemma 1 (Weakening). *If $\Delta; \Phi \vdash \tau :: \kappa$, then $\Delta; \Phi, t::\kappa_1 \vdash \tau :: \kappa$ when $\Delta; \Phi, t::\kappa_1 \vdash \text{OK}$*

Proof. By rule induction/length of proof.

L1. (9)

□

Proof. By rule induction/length of proof.

L2. (9)

□

Lemma 2 (OK-PK). *If $\Delta; \Phi \vdash \tau ::> \kappa$, then $\Delta; \Phi \vdash \text{OK}$ and $\Delta; \Phi \vdash \kappa \text{ OK}$*

Lemma 3 (OK-WFaK). *If $\Delta; \Phi \vdash \tau :: \kappa$, then $\Delta; \Phi \vdash OK$ and $\Delta; \Phi \vdash \kappa OK$*

Lemma 4 (OK-MatchPi). *If $\Delta; \Phi \vdash \kappa \Pi_{t::\kappa_1.\kappa_2}$, then $\Delta; \Phi \vdash OK$ and $\Delta; \Phi \vdash \kappa OK$ and $\Delta; \Phi \vdash \Pi_{t::\kappa_1.\kappa_2} OK$*

Lemma 5 (OK-KEquiv). *If $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, then $\Delta; \Phi \vdash OK$ and $\Delta; \Phi \vdash \kappa_1 OK$ and $\Delta; \Phi \vdash \kappa_2 OK$*

Lemma 6 (OK-CSK). *If $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, then $\Delta; \Phi \vdash OK$ and $\Delta; \Phi \vdash \kappa_1 OK$ and $\Delta; \Phi \vdash \kappa_2 OK$*

Lemma 7 (OK-KWF). *If $\Delta; \Phi \vdash \kappa OK$, then $\Delta; \Phi \vdash OK$*

Lemma 8 (OK-Substitution).

*If $\Delta; \Phi \vdash \tau_L :: \kappa_{L1}$ and $\Delta; \Phi, t_L :: \kappa_{L1} \vdash \kappa_{L2} OK$, then $\Delta; \Phi \vdash OK$ and $\Delta; \Phi \vdash [\tau_L/t_L] \kappa_{L2} OK$
(induction on $\Delta; \Phi, t_L :: \kappa_{L1} \vdash \kappa_{L2} OK$)*

Lemma 9 (K-Substitution).

*If $\Delta; \Phi \vdash \tau_{L1} :: \kappa_{L1}$ and $\Delta; \Phi, t_L :: \kappa_{L1} \vdash \tau_{L2} :: \kappa_{L2}$, then $\Delta; \Phi \vdash [\tau_{L1}/t_L] \tau_{L2} :: [\tau_{L1}/t_L] \kappa_{L2}$
(induction on $\Delta; \Phi, t_L :: \kappa_{L1} \vdash \tau_{L2} :: \kappa_{L2}$)*

Proof. By simultaneous rule induction/length of proof.

The interesting cases per lemma:

OK-PK.	(1)	$\Delta; \Phi \vdash \mathbf{bse} :: \mathbf{S_{Type}}(\mathbf{bse})$	by (9)
		$\Delta; \Phi \vdash \mathbf{bse} :: \mathbf{Type}$	by (10)
	*	$\Delta; \Phi \vdash \mathbf{S_{Type}}(\mathbf{bse}) OK$	by (43)
	*	$\Delta; \Phi \vdash OK$	by premiss
	(8)		bad
OK-WFaK.	(12)	$\Delta; \Phi \vdash \tau_2 :: \kappa$	by (10)
	*	$\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau_2) OK$	by (43)
OK-KEquiv.	(22)	$\Delta; \Phi \vdash \tau \ t :: > \kappa$	
OK-Substitution.	(41)	$\Delta; \Phi, t_L :: \kappa_{L1} \vdash OK$	premiss (41)
		$\Delta; \Phi \vdash \kappa_{L1} OK$	by subderivation premiss (46)
	*	$\Delta; \Phi \vdash OK$	by OK-KWF
	*	$\Delta; \Phi \vdash [\tau_L/t_L] \mathbf{Type} OK$	by (41) and degenerate subst
	(43)	$\Delta; \Phi, t_L :: \kappa_{L1} \vdash \tau :: \kappa$	premiss (43)
		$\Delta; \Phi, t_L :: \kappa_{L1} \vdash OK$	by OK-WFaK
		$\Delta; \Phi \vdash \kappa_{L1} OK$	by subderivation premiss (46)
	*	$\Delta; \Phi \vdash OK$	by OK-KWF
		$\Delta; \Phi \vdash [\tau_{L1}/t_L] \tau :: [\tau_{L1}/t_L] \kappa$	by K-Substitution on premiss
	*	$\Delta; \Phi \vdash [\tau_L/t_L] \mathbf{S}_{\kappa}(\tau) OK$	by (43)

□

Lemma 10 (PK-Unicity). *If $\Delta; \Phi \vdash \tau_L :: > \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L :: > \kappa_{L2}$ then κ_{L1} is κ_{L2}*

Lemma 11. *If $\Delta; \Phi \vdash \tau :: > \kappa_1$ and $\Delta; \Phi \vdash \tau :: \kappa_2$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*