

# Algebraic Data Types for Hazel

Eric Griffis  
egriffis@umich.edu

## 1 Syntax

$$\begin{aligned}
 \text{HType} \quad \tau &::= \emptyset \mid \tau \rightarrow \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid \langle \rangle \mid \langle \alpha \rangle \\
 \text{HTypePat} \quad \pi &::= \alpha \mid \langle \rangle \\
 \text{HExp} \quad e &::= x \mid \lambda x:\tau.e \mid e(e) \mid e:\tau \mid \text{inj}_C(e) \mid \text{roll}(e) \mid \text{unroll}(e) \\
 &\quad \mid \langle \rangle^u \mid \langle e \rangle^u \mid \langle e \rangle^{u\blacktriangleright} \\
 \text{HExp} \quad d &::= \emptyset \mid x \mid \lambda x:\tau.d \mid d(d) \mid \text{inj}_C^\tau(d) \mid \text{roll}^{\mu\alpha.\tau}(d) \mid \text{unroll}(d) \\
 &\quad \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow \langle \rangle \not\Rightarrow \tau \rangle \mid \langle \rangle_\sigma^u \mid \langle d \rangle_\sigma^u \mid \langle d \rangle_\sigma^{u\blacktriangleright} \\
 \text{HTag} \quad C &::= \mathbf{C} \mid \langle \rangle^u \mid \langle c \rangle^u
 \end{aligned}$$

### 1.1 Context Extension

We write  $\Theta, \pi$  to denote the extension of type variable context  $\Theta$  with optional type variable name  $\pi$ .

$$\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \langle \rangle \end{cases}$$

## 2 Static Semantics

$$\boxed{[\tau/\pi]\tau' = \tau''} \quad \tau'' \text{ is obtained by substituting } \tau \text{ for } \pi \text{ in } \tau'$$

$$\begin{aligned}
 [\tau/\langle \rangle]\tau' &= \tau' \\
 [\tau/\alpha]\emptyset &= \emptyset \\
 [\tau/\alpha](\tau_1 \rightarrow \tau_2) &= [\tau/\alpha]\tau_1 \rightarrow [\tau/\alpha]\tau_2 \\
 [\tau/\alpha]\alpha &= \tau \\
 [\tau/\alpha]\alpha_1 &= \tau' && \text{when } \alpha \neq \alpha_1 \\
 [\tau/\alpha]\mu\alpha_1.\tau_2 &= \mu\alpha_1.[\tau/\alpha]\tau_2 && \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \text{FV}(\tau) \\
 [\tau/\alpha]\mu\langle \rangle.\tau_2 &= \mu\langle \rangle.[\tau/\alpha]\tau_2 \\
 [\tau/\alpha]+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} &= +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} \\
 [\tau/\alpha]\langle \rangle &= \langle \rangle \\
 [\alpha'/\alpha]\langle \alpha \rangle &= \langle \alpha' \rangle^u \\
 [\alpha'/\alpha]\langle \alpha' \rangle^u &= \langle \alpha' \rangle^u && \text{when } \alpha \neq \alpha'
 \end{aligned}$$

$$\boxed{\Theta \vdash \tau \text{ valid}} \quad \tau \text{ is a valid type}$$

$$\begin{array}{c}
 \text{TVUNIT} \quad \frac{}{\Theta \vdash \emptyset \text{ valid}} \quad \text{TVARR} \quad \frac{\Theta \vdash \tau_1 \text{ valid} \quad \Theta \vdash \tau_2 \text{ valid}}{\Theta \vdash \tau_1 \rightarrow \tau_2 \text{ valid}} \quad \text{TVVAR} \quad \frac{\alpha \in \Theta}{\Theta \vdash \alpha \text{ valid}} \quad \text{TVREC} \quad \frac{\Theta, \pi \vdash \tau \text{ valid}}{\Theta \vdash \mu\pi.\tau \text{ valid}} \quad \text{TVSUM} \quad \frac{\{\Theta \vdash \tau_i \text{ valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \text{ valid}} \\
 \text{TVEHOLE} \quad \frac{}{\Theta \vdash \langle \rangle \text{ valid}}
 \end{array}$$

$\boxed{\tau \sim \tau'}$   $\tau$  and  $\tau'$  are consistent

|   |  |  |   |   |  |  |
|---|--|--|---|---|--|--|
| $\text{TCREFL}$<br>$\frac{}{\tau \sim \tau}$                              | $\text{TCEHOLE1}$<br>$\frac{}{\langle\!\rangle \sim \tau}$   | $\text{TCEHOLE2}$<br>$\frac{}{\tau \sim \langle\!\rangle}$   | $\text{TCNEHOLE1}$<br>$\frac{}{\langle\!\alpha\!\rangle \sim \tau}$   | $\text{TCNEHOLE2}$<br>$\frac{}{\tau \sim \langle\!\alpha\!\rangle}$ | $\text{TCUNIT}$<br>$\frac{}{\emptyset \sim \emptyset}$ | $\text{TCARR}$<br>$\frac{\tau_1 \sim \tau'_1 \quad \tau_2 \sim \tau'_2}{\tau_1 \rightarrow \tau_2 \sim \tau'_1 \rightarrow \tau'_2}$ |
| $\text{TCREC}$<br>$\frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'}$ | $\text{TCRECHOLE1}$<br>$\frac{\alpha \notin \text{FV}(\tau) \quad \tau \sim \tau'}{\mu\langle\!\rangle.\tau \sim \mu\alpha.\tau'}$ | $\text{TCRECHOLE2}$<br>$\frac{\alpha \notin \text{FV}(\tau) \quad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu\langle\!\rangle.\tau'}$ | $\text{TCSUM}$<br>$\frac{\{\tau_i \sim \tau'_i\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}}}$ |   |  |  |

$\boxed{C \text{ valid}}$   $C$  is a valid tag

|  |   |
|--|---|
| $\text{CVTAG}$<br>$\frac{}{C \text{ valid}}$ | $\text{CVEHOLE}$<br>$\frac{}{\langle\!\rangle^u \text{ valid}}$ |
|--|---|

## 2.1 Bidirectional Typing

We call  $[\mu\pi.\tau/\pi]\tau$  the *unrolling* of recursive type  $\mu\pi.\tau$ .

**Theorem 1** (Synthetic Type Validity). *If  $\Gamma \vdash e \Rightarrow \tau$  then  $\emptyset \vdash \tau \text{ valid}$ .*

**Theorem 2** (Consistency Preserves Validity). *If  $\Theta \vdash \tau \text{ valid}$  and  $\tau \sim \tau'$  then  $\Theta \vdash \tau' \text{ valid}$ .*

$\boxed{\tau \blacktriangleright \rightarrow \tau_1 \rightarrow \tau_2}$   $\tau$  has matched arrow type  $\tau_1 \rightarrow \tau_2$

|  |  |
|--|--|
| $\text{MAHOLE}$<br>$\frac{}{\langle\!\rangle \blacktriangleright \rightarrow \langle\!\rangle \rightarrow \langle\!\rangle}$ | $\text{MAARR}$<br>$\frac{}{\tau_1 \rightarrow \tau_2 \blacktriangleright \rightarrow \tau_1 \rightarrow \tau_2}$ |
|--|--|

$\boxed{\tau \blacktriangleright_\mu \mu\pi.\tau'}$   $\tau$  has matched recursive type  $\mu\pi.\tau'$

|  |   |
|--|---|
| $\text{MRREC}$<br>$\frac{}{\mu\pi.\tau \blacktriangleright_\mu \mu\pi.\tau}$ | $\text{MRHOLE}$<br>$\frac{}{\langle\!\rangle \blacktriangleright_\mu \mu\langle\!\rangle.\langle\!\rangle}$ |
|--|---|

$\boxed{\Gamma \vdash e \Rightarrow \tau}$   $e$  synthesizes type  $\tau$

|   |   |   |   |
|---|---|---|---|
| $\text{SUNIT}$<br>$\frac{}{\Gamma \vdash \emptyset \Rightarrow \emptyset}$  | $\text{SVAR}$<br>$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau}$   | $\text{SVARFREE}$<br>$\frac{x \notin \text{dom}(\Gamma)}{\Gamma \vdash \langle\!x\!\rangle^u \Rightarrow \langle\!\rangle}$   | $\text{SLAM}$<br>$\frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \tau \rightarrow \tau'}$ |
| $\text{SLAMINVALID}$<br>$\frac{\Gamma, x : \langle\!\rangle \vdash e \Rightarrow \tau}{\Gamma \vdash \lambda x : \langle\!\rangle. e \Rightarrow \langle\!\rangle \rightarrow \tau}$  |   | $\text{SAPP}$<br>$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright \rightarrow \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau}$                              |   |
| $\text{SAPPNOTARR}$<br>$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \approx \langle\!\rangle \rightarrow \langle\!\rangle \quad \Gamma \vdash e_2 \Leftarrow \langle\!\rangle}{\Gamma \vdash \langle\!e_1\!\rangle^u \blacktriangleright (e_2) \Rightarrow \langle\!\rangle}$ |   | $\text{SASC}$<br>$\frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau}$  | $\text{SASCINVALID}$<br>$\frac{}{\Gamma \vdash e : \langle\!\alpha\!\rangle \Rightarrow \langle\!\rangle}$  |
| $\text{SROLLERR}$<br>$\frac{\Gamma \vdash e \Leftarrow \langle\!\rangle}{\Gamma \vdash \langle\!\text{roll}(e)\!\rangle^u \Rightarrow \mu\langle\!\rangle.\langle\!\rangle}$  | $\text{SUNROLL}$<br>$\frac{\Gamma \vdash e \Rightarrow \tau \quad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \langle\!\text{unroll}(e)\!\rangle \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$ | $\text{SUNROLLNOTREC}$<br>$\frac{\Gamma \vdash e \Rightarrow \tau \quad \tau \approx \mu\langle\!\rangle.\langle\!\rangle}{\Gamma \vdash \langle\!\text{unroll}(\langle\!e\!\rangle^u \blacktriangleright) \!\rangle \Rightarrow \langle\!\rangle}$ |   |
| $\text{SINJERR}$<br>$\frac{C \text{ valid} \quad \Gamma \vdash e \Leftarrow \langle\!\rangle}{\Gamma \vdash \langle\!\text{inj}_C(e)\!\rangle^u \Rightarrow \langle\!\rangle}$  | $\text{SINJTAGERR}$<br>$\frac{}{\Gamma \vdash \langle\!\text{inj}_{\langle\!c\!\rangle^u}(e)\!\rangle \Rightarrow \langle\!\rangle}$  | $\text{SEHOLE}$<br>$\frac{}{\Gamma \vdash \langle\!\rangle^u \Rightarrow \langle\!\rangle}$   | $\text{SNEHOLE}$<br>$\frac{}{\Gamma \vdash \langle\!e\!\rangle^u \Rightarrow \langle\!\rangle}$   |

$\boxed{\Gamma \vdash e \Leftarrow \tau}$      $e$  analyzes against type  $\tau$

$$\begin{array}{c}
\text{AROLL} \\
\frac{\tau \blacktriangleright_{\mu} \mu\pi.\tau' \quad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \text{roll}(e) \Leftarrow \tau}
\end{array}
\quad
\begin{array}{c}
\text{AROLLNOTREC} \\
\frac{\tau \approx \mu(\emptyset).\emptyset \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash (\text{roll}(e))^u \Leftarrow \tau}
\end{array}
\quad
\begin{array}{c}
\text{AINJHOLE} \\
\frac{\Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash \text{inj}_C(e) \Leftarrow \emptyset}
\end{array}$$

$$\begin{array}{c}
\text{AINJ} \\
\frac{C_j \in \mathcal{C} \quad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \text{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}
\quad
\begin{array}{c}
\text{AINJUNEXPECTEDBODY} \\
\frac{C_j \in \mathcal{C} \quad \tau_j = \emptyset \quad e \neq \emptyset \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash (\text{inj}_{C_j}(e))^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}$$

$$\begin{array}{c}
\text{AINJEXPECTEDBODY} \\
\frac{C_j \in \mathcal{C} \quad \tau_j \neq \emptyset}{\Gamma \vdash (\text{inj}_{C_j}(\emptyset))^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}
\quad
\begin{array}{c}
\text{AINJBADTAG} \\
\frac{C \text{ valid} \quad C \notin \mathcal{C} \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash (\text{inj}_C(e))^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}$$

$$\begin{array}{c}
\text{AINJTAGErr} \\
\frac{(\llbracket c \rrbracket)^u \notin \mathcal{C} \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash \text{inj}_{(\llbracket c \rrbracket)^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}
\quad
\begin{array}{c}
\text{ASUBSUME} \\
\frac{\Gamma \vdash e \Rightarrow \tau' \quad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}
\end{array}$$

## 2.2 Typed Elaboration

**Theorem 3** (Synthetic Typed Elaboration Validity). *If  $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$  then  $\emptyset \vdash \tau$  valid.*

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$      $e$  synthesizes type  $\tau$  and elaborates to  $d$

$$\begin{array}{c}
\text{ESUNIT} \\
\frac{}{\Gamma \vdash \emptyset \Rightarrow \emptyset \rightsquigarrow \emptyset \dashv \emptyset}
\end{array}
\quad
\begin{array}{c}
\text{ESVAR} \\
\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}
\end{array}
\quad
\begin{array}{c}
\text{ESVARFREE} \\
\frac{x \notin \text{dom}(\Gamma)}{\Gamma \vdash (\llbracket x \rrbracket)^u \Rightarrow \emptyset \rightsquigarrow (\llbracket x \rrbracket_{\text{id}(\Gamma)}^u \dashv u :: \emptyset)[\Gamma]}
\end{array}$$

$$\begin{array}{c}
\text{ESLAM} \\
\frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \tau \rightarrow \tau' \rightsquigarrow \lambda x : \tau. d \dashv \Delta}
\end{array}
\quad
\begin{array}{c}
\text{ESLAMINVALID} \\
\frac{\Gamma, x : \emptyset \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x : (\emptyset). e \Rightarrow \emptyset \rightarrow \tau \rightsquigarrow \lambda x : (\emptyset). d \dashv \Delta}
\end{array}$$

$$\begin{array}{c}
\text{ESAPP} \\
\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright_{\mu} \tau_2 \rightarrow \tau \quad \Gamma \vdash e_1 \Leftarrow \tau_2 \rightarrow \tau \rightsquigarrow d_1 : \tau'_1 \dashv \Delta_1 \quad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau'_2 \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1 \langle \tau'_1 \Rightarrow \tau_1 \rangle) (d_2 \langle \tau'_2 \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2}
\end{array}$$

$$\begin{array}{c}
\text{ESAPPNOTARR} \\
\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \quad \tau_1 \approx \emptyset \rightarrow \emptyset \quad \Gamma \vdash e_2 \Leftarrow \emptyset \rightsquigarrow d_2 : \tau'_2 \dashv \Delta_2}{\Gamma \vdash (e_1)^{u \blacktriangleright} (e_2) \Rightarrow \emptyset \rightsquigarrow (\llbracket d_1 \rrbracket_{\text{id}(\Gamma)}^{u \blacktriangleright} (d_2 \langle \tau'_2 \Rightarrow \emptyset \rangle)) \dashv \Delta_1 \cup \Delta_2, u :: \emptyset \rightarrow \emptyset[\Gamma]}
\end{array}$$

$$\begin{array}{c}
\text{ESASC} \\
\frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d \langle \tau' \Rightarrow \tau \rangle \dashv \Delta}
\end{array}
\quad
\begin{array}{c}
\text{ESASCINVALID} \\
\frac{\Gamma \vdash e \Leftarrow \emptyset \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash e : (\emptyset) \Rightarrow \emptyset \rightsquigarrow d \langle \tau \Rightarrow \emptyset \rangle \dashv \Delta}
\end{array}$$

$$\begin{array}{c}
\text{ESROLLErr} \\
\frac{\Gamma \vdash e \Leftarrow \emptyset \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash (\text{roll}(e))^u \Rightarrow \mu(\emptyset).\emptyset \rightsquigarrow (\text{roll}^{\mu(\emptyset).\emptyset} (d \langle \tau \Rightarrow \emptyset \rangle))^u_{\text{id}(\Gamma)} \dashv \Delta, u :: \mu(\emptyset).\emptyset[\Gamma]}
\end{array}$$

$$\begin{array}{c}
\text{ESUNROLL} \\
\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \quad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \text{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \text{unroll}(d \langle \tau \Rightarrow \mu\pi.\tau' \rangle) \dashv \Delta}
\end{array}$$

$$\begin{array}{c}
\text{ESUNROLLNOTREC} \\
\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \quad \tau \approx \mu(\mathbb{O}).\mathbb{O}}{\Gamma \vdash \text{unroll}(\langle e \rangle^u \blacktriangleright) \Rightarrow \mathbb{O} \rightsquigarrow \text{unroll}(\langle d \rangle_{\text{id}(\Gamma)}^u \blacktriangleright) \dashv \Delta, u :: \mu(\mathbb{O}).\mathbb{O}[\Gamma]} \\
\\
\text{ESINJERR} \\
\frac{C \text{ valid} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\{C(\tau)\}}{\Gamma \vdash \langle \text{inj}_C(e) \rangle^u \Rightarrow \mathbb{O} \rightsquigarrow \langle \text{inj}_C^{\tau'}(d \langle \tau \Rightarrow \mathbb{O} \rangle) \rangle_{\text{id}(\Gamma)}^u \dashv \Delta, u :: \mathbb{O}[\Gamma]} \\
\\
\text{ESINJTAGER} \quad \text{ESEHOLE} \\
\frac{\Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\{\langle c \rangle^u(\tau)\}}{\Gamma \vdash \text{inj}_{\langle c \rangle^u}(e) \Rightarrow \mathbb{O} \rightsquigarrow \text{inj}_{\langle c \rangle^u}^{\tau'}(d \langle \tau \Rightarrow \mathbb{O} \rangle) \dashv \Delta} \quad \frac{}{\Gamma \vdash \mathbb{O}^u \Rightarrow \mathbb{O} \rightsquigarrow \mathbb{O}_{\text{id}(\Gamma)}^u \dashv u :: \mathbb{O}[\Gamma]} \\
\\
\text{ESNEHOLE} \\
\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle e \rangle^u \Rightarrow \mathbb{O} \rightsquigarrow \langle d \rangle_{\text{id}(\Gamma)}^u \dashv \Delta, u :: \mathbb{O}[\Gamma]} \\
\\
\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta} \quad e \text{ analyzes against type } \tau_1 \text{ and elaborates to } d \text{ of consistent type } \tau_2 \\
\\
\text{EAROLL} \\
\frac{\tau \blacktriangleright_{\mu} \mu\pi.\tau' \quad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \text{roll}(e) \Leftarrow \tau \rightsquigarrow \text{roll}^{\mu\pi.\tau'}(d \langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rangle) : \mu\pi.\tau' \dashv \Delta} \\
\\
\text{EAROLLNOTREC} \\
\frac{\tau \approx \mu(\mathbb{O}).\mathbb{O} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash \langle \text{roll}(e) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{roll}^{\mu(\mathbb{O}).\mathbb{O}}(d) \rangle_{\text{id}(\Gamma)}^u : \mu(\mathbb{O}).\mathbb{O} \dashv \Delta, u :: \mu(\mathbb{O}).\mathbb{O}[\Gamma]} \\
\\
\text{EAINJHOLE} \quad \text{EAINJ} \\
\frac{\Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\{C(\tau)\}}{\Gamma \vdash \text{inj}_C(e) \Leftarrow \mathbb{O} \rightsquigarrow \text{inj}_C^{\tau'}(d) : \tau' \dashv \Delta} \quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \Gamma \vdash e \Leftarrow \tau_j \rightsquigarrow d : \tau'_j \dashv \Delta}{\Gamma \vdash \text{inj}_{C_j}(e) \Leftarrow \tau \rightsquigarrow \text{inj}_{C_j}^{\tau}(d \langle \tau'_j \Rightarrow \tau_j \rangle) : \tau \dashv \Delta} \\
\\
\text{EAINJUNEXPECTEDBODY} \\
\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \tau_j = \emptyset \quad e \neq \emptyset \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau_j \dashv \Delta \quad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\right\}}{\Gamma \vdash \langle \text{inj}_{C_j}(e) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{inj}_{C_j}^{\tau'}(d) \rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]} \\
\\
\text{EAINJEXPECTEDBODY} \\
\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \tau_j \neq \emptyset \quad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\emptyset)\}\right\}}{\Gamma \vdash \langle \text{inj}_{C_j}(\emptyset) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{inj}_{C_j}^{\tau'}(\emptyset) \rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]} \\
\\
\text{EAINJBADTAG} \\
\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C \notin \mathcal{C} \quad C \text{ valid} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau' \dashv \Delta \quad \tau'' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{C(\tau')\}\right\}}{\Gamma \vdash \langle \text{inj}_C(e) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{inj}_C^{\tau''}(d) \rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]} \\
\\
\text{EAINJTAGER} \\
\frac{\langle c \rangle^u \notin \mathcal{C} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{\langle c \rangle^u(\tau)\}\right\}}{\Gamma \vdash \text{inj}_{\langle c \rangle^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \rightsquigarrow \text{inj}_{\langle c \rangle^u}^{\tau'}(d \langle \tau \Rightarrow \mathbb{O} \rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta} \\
\\
\text{EASUBSUME} \quad \text{EAEHOLE} \\
\frac{e \neq \mathbb{O}^u \quad e \neq \langle e' \rangle^u \quad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \quad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta} \quad \frac{}{\Gamma \vdash \mathbb{O}^u \Leftarrow \tau \rightsquigarrow \mathbb{O}_{\text{id}(\Gamma)}^u : \tau \dashv u :: \tau[\Gamma]}
\end{array}$$

$$\frac{\text{EANEHOLE} \quad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle e \rangle^u \Leftarrow \tau \rightsquigarrow \langle d \rangle_{\text{id}(\Gamma)}^u : \tau \dashv u :: \tau[\Gamma]}$$

## 2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$   $d$  is assigned type  $\tau$

$$\begin{array}{c} \text{TANIT} \\ \hline \Delta; \Gamma \vdash \emptyset : \emptyset \end{array} \quad \begin{array}{c} \text{TAVAR} \\ x : \tau \in \Gamma \\ \hline \Delta; \Gamma \vdash x : \tau \end{array} \quad \begin{array}{c} \text{TALAM} \\ \tau \neq \langle \alpha \rangle \quad \Delta; \Gamma, x : \tau \vdash d : \tau' \\ \hline \Delta; \Gamma \vdash \lambda x : \tau. d : \tau \rightarrow \tau' \end{array} \quad \begin{array}{c} \text{TALAMFREE} \\ \Delta; \Gamma, x : \langle \rangle \vdash d : \tau \\ \hline \Delta; \Gamma \vdash \lambda x : \langle \alpha \rangle. d : \langle \rangle \rightarrow \tau \end{array}$$

$$\begin{array}{c} \text{TAAAPP} \\ \Delta; \Gamma \vdash d_1 : \tau_2 \rightarrow \tau \quad \Delta; \Gamma \vdash d_2 : \tau_2 \\ \hline \Delta; \Gamma \vdash d_1(d_2) : \tau \end{array} \quad \begin{array}{c} \text{TAROLL} \\ \emptyset \vdash \mu\pi.\tau \text{ valid} \quad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau \\ \hline \Delta; \Gamma \vdash \text{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau \end{array}$$

$$\begin{array}{c} \text{TAUNROLL} \\ \Delta; \Gamma \vdash d : \mu\pi.\tau \\ \hline \Delta; \Gamma \vdash \text{unroll}(d) : [\mu\pi.\tau/\pi]\tau \end{array} \quad \begin{array}{c} \text{TAInj} \\ \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \Delta; \Gamma \vdash d : \tau_j \\ \hline \Delta; \Gamma \vdash \text{inj}_{C_j}^\tau(d) : \tau \end{array}$$

$$\begin{array}{c} \text{TAEHOLE} \\ u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma' \\ \hline \Delta; \Gamma \vdash \langle \rangle_\sigma^u : \tau \end{array} \quad \begin{array}{c} \text{TANEHOLE} \\ \Delta; \Gamma \vdash d : \tau' \quad u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma' \\ \hline \Delta; \Gamma \vdash \langle d \rangle_\sigma^u : \tau \end{array}$$

$$\begin{array}{c} \text{TAMHOLE} \\ \Delta; \Gamma \vdash d : \tau' \quad u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma' \\ \hline \Delta; \Gamma \vdash \langle d \rangle_\sigma^{u \blacktriangleright} : \tau \end{array} \quad \begin{array}{c} \text{TACAST} \\ \Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \sim \tau_2 \\ \hline \Delta; \Gamma \vdash d \langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2 \end{array}$$

$$\begin{array}{c} \text{TAFaILEDCAST} \\ \Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2 \\ \hline \Delta; \Gamma \vdash d \langle \tau_1 \Rightarrow \langle \rangle \nRightarrow \tau_2 \rangle : \tau_2 \end{array}$$

## 3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$   $\tau$  is a ground type

$$\begin{array}{c} \text{GARR} \\ \hline \langle \rangle \rightarrow \langle \rangle \text{ ground} \end{array} \quad \begin{array}{c} \text{GREC} \\ \hline \mu \langle \rangle . \langle \rangle \text{ ground} \end{array} \quad \begin{array}{c} \text{GSUM} \\ \{\tau_i = \emptyset \vee \tau_i = \langle \rangle\}_{C_i \in \mathcal{C}} \\ \hline +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \text{ ground} \end{array}$$

$\boxed{\tau \blacktriangleright_{\text{ground}} \tau'}$   $\tau$  has matched ground type  $\tau'$

$$\begin{array}{c} \text{MGARR} \\ \tau_1 \rightarrow \tau_2 \neq \langle \rangle \rightarrow \langle \rangle \\ \hline \tau_1 \rightarrow \tau_2 \blacktriangleright_{\text{ground}} \langle \rangle \rightarrow \langle \rangle \end{array} \quad \begin{array}{c} \text{MGREC} \\ \tau \neq \langle \rangle \\ \hline \mu\pi.\tau \blacktriangleright_{\text{ground}} \mu \langle \rangle . \langle \rangle \end{array}$$

$$\begin{array}{c} \text{MGSUM} \\ +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \quad \{(\tau_i = \emptyset \Rightarrow \tau'_i = \emptyset) \wedge (\tau_i \neq \emptyset \Rightarrow \tau'_i = \langle \rangle)\}_{C_i \in \mathcal{C}} \\ \hline +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\text{ground}} +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \end{array}$$

$\boxed{d \text{ final}}$   $d$  is final

$$\frac{\text{FBOXEDVAL} \quad d \text{ boxedval}}{d \text{ final}}$$

$$\frac{\text{FINDET} \quad d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$   $d$  is a value

$$\frac{\text{VUNIT}}{\emptyset \text{ val}}$$

$$\frac{\text{VLAM}}{\lambda x:\tau. d \text{ val}}$$

$$\frac{\text{VROLL} \quad d \text{ val}}{\text{roll}^{\mu\pi.\tau}(d) \text{ val}}$$

$$\frac{\text{VINJ} \quad d \text{ val}}{\text{inj}_C^\tau(d) \text{ val}}$$

$\boxed{d \text{ boxedval}}$   $d$  is a boxed value

$$\frac{\text{BVVAL} \quad d \text{ val}}{d \text{ boxedval}}$$

$$\frac{\text{BVROLL} \quad d \text{ boxedval}}{\text{roll}^{\mu\pi.\tau}(d) \text{ boxedval}}$$

$$\frac{\text{BVINJ} \quad d \text{ boxedval}}{\text{inj}_C^\tau(d) \text{ boxedval}}$$

$$\frac{\text{BVARRCast} \quad \tau_1 \rightarrow \tau_2 \neq \tau_3 \rightarrow \tau_4 \quad d \text{ boxedval}}{d\langle \tau_1 \rightarrow \tau_2 \Rightarrow \tau_3 \rightarrow \tau_4 \rangle \text{ boxedval}}$$

$$\frac{\text{BVRECCast} \quad \mu\pi.\tau \neq \mu\pi'.\tau' \quad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}}$$

$$\frac{\text{BVSUMCast} \quad \begin{array}{l} \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \end{array} \quad d \text{ boxedval}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}$$

$$\frac{\text{BVHOLECast} \quad d \text{ boxedval} \quad \tau \text{ ground}}{d\langle \tau \Rightarrow \emptyset \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$   $d$  is indeterminate

$$\frac{\text{IEHOLE}}{\langle \emptyset \rangle_\sigma^u \text{ indet}}$$

$$\frac{\text{INEHOLE} \quad d \text{ final}}{\langle d \rangle_\sigma^u \text{ indet}}$$

$$\frac{\text{IAPP} \quad d_1 \neq d'_1 \langle \tau_1 \rightarrow \tau_2 \Rightarrow \tau_3 \rightarrow \tau_4 \rangle \quad d_1 \text{ indet} \quad d_2 \text{ final}}{d_1(d_2) \text{ indet}}$$

$$\frac{\text{IROLL} \quad d \text{ indet}}{\text{roll}^{\mu\pi.\tau}(d) \text{ indet}}$$

$$\frac{\text{IUNROLL} \quad d \text{ indet}}{\text{unroll}(d) \text{ indet}}$$

$$\frac{\text{IINJ} \quad d \text{ indet}}{\text{inj}_C^\tau(d) \text{ indet}}$$

$$\frac{\text{IINJHOLE} \quad C \neq \mathbf{C} \quad d \text{ final}}{\text{inj}_C^\tau(d) \text{ indet}}$$

$$\frac{\text{ICASTGROUNDHOLE} \quad d \text{ indet} \quad \tau \text{ ground}}{d\langle \tau \Rightarrow \emptyset \rangle \text{ indet}}$$

$$\frac{\text{ICASTHOLEGROUND} \quad d \neq d' \langle \tau' \Rightarrow \emptyset \rangle \quad d \text{ indet} \quad \tau \text{ ground}}{d\langle \emptyset \Rightarrow \tau \rangle \text{ indet}}$$

$$\frac{\text{ICASTARR} \quad \tau_1 \rightarrow \tau_2 \neq \tau_3 \rightarrow \tau_4 \quad d \text{ indet}}{d\langle \tau_1 \rightarrow \tau_2 \Rightarrow \tau_3 \rightarrow \tau_4 \rangle \text{ indet}}$$

$$\frac{\text{ICASTREC} \quad \mu\pi.\tau \neq \mu\pi'.\tau' \quad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}}$$

$$\frac{\text{ICASTSUM} \quad \begin{array}{l} \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \end{array} \quad d \text{ indet}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}$$

$$\frac{\text{IFAILEDCAST} \quad d \text{ final} \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow \emptyset \rangle \neq \tau_2 \rangle \text{ indet}}$$

$\boxed{d \longrightarrow d'}$   $d$  takes an instruction transition to  $d'$

$$\frac{\text{ITAPP} \quad [d_2 \text{ final}]}{(\lambda x:\tau. d_1)(d_2) \longrightarrow [d_2/x]d_1}$$

$$\frac{\text{ITUNROLL} \quad [d \text{ final}]}{\text{unroll}(\text{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\frac{\text{ITAPPCast} \quad [d_1 \text{ final}] \quad [d_2 \text{ final}] \quad \tau_1 \rightarrow \tau_2 \neq \tau'_1 \rightarrow \tau'_2}{d_1\langle \tau_1 \rightarrow \tau_2 \Rightarrow \tau'_1 \rightarrow \tau'_2 \rangle(d_2) \longrightarrow (d_1(d_2\langle \tau'_1 \Rightarrow \tau_1 \rangle))\langle \tau_2 \Rightarrow \tau'_2 \rangle}$$

$$\begin{array}{c}
\text{ITUnrollCast} \\
\frac{[d \text{ final}] \quad \mu\pi.\tau \neq \mu\pi'.\tau'}{\text{unroll}(d\langle\mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \text{unroll}(d)\langle[\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle}
\end{array}
\qquad
\begin{array}{c}
\text{ITCastId} \\
\frac{[d \text{ final}]}{d\langle\tau \Rightarrow \tau\rangle \longrightarrow d}
\end{array}$$

$$\begin{array}{c}
\text{ITCastSucceed} \\
\frac{[d \text{ final}] \quad \tau \text{ ground}}{d\langle\tau \Rightarrow \emptyset \Rightarrow \tau\rangle \longrightarrow d}
\end{array}
\qquad
\begin{array}{c}
\text{ITCastFail} \\
\frac{[d \text{ final}] \quad \tau_1 \neq \tau_2 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground}}{d\langle\tau_1 \Rightarrow \emptyset \Rightarrow \tau_2\rangle \longrightarrow d\langle\tau_1 \Rightarrow \emptyset \nRightarrow \tau_2\rangle}
\end{array}$$

$$\begin{array}{c}
\text{ITGround} \\
\frac{[d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle\tau \Rightarrow \emptyset\rangle \longrightarrow d\langle\tau \Rightarrow \tau' \Rightarrow \emptyset\rangle}
\end{array}
\qquad
\begin{array}{c}
\text{ITExpand} \\
\frac{[d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle\emptyset \Rightarrow \tau\rangle \longrightarrow d\langle\emptyset \Rightarrow \tau' \Rightarrow \tau\rangle}
\end{array}$$

$$\text{EvalCtx } \mathcal{E} ::= \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \text{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \text{unroll}(\mathcal{E}) \mid \text{inj}_C^\tau(\mathcal{E}) \mid \langle\mathcal{E}\rangle_\sigma^u \mid \langle\mathcal{E}\rangle_\sigma^{u\blacktriangleright} \mid \mathcal{E}\langle\tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle\tau \Rightarrow \emptyset \nRightarrow \tau\rangle$$

$$\boxed{d = \mathcal{E}\{d'\}} \quad d \text{ is obtained by placing } d' \text{ at the mark in } \mathcal{E}$$

$$\begin{array}{c}
\text{FHOuter} \\
\frac{}{d = \circ\{d\}}
\end{array}
\qquad
\begin{array}{c}
\text{FHApp1} \\
\frac{d_1 = \mathcal{E}\{d'_1\}}{d_1(d_2) = \mathcal{E}(d_2)\{d'_1\}}
\end{array}
\qquad
\begin{array}{c}
\text{FHApp2} \\
\frac{[d_1 \text{ final}] \quad d_2 = \mathcal{E}\{d'_2\}}{d_1(d_2) = d_1(\mathcal{E})\{d'_2\}}
\end{array}
\qquad
\begin{array}{c}
\text{FHRoll} \\
\frac{d = \mathcal{E}\{d'\}}{\text{roll}^{\mu\pi.\tau}(d) = \text{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}
\end{array}$$

$$\begin{array}{c}
\text{FHUnroll} \\
\frac{d = \mathcal{E}\{d'\}}{\text{unroll}(d) = \text{unroll}(\mathcal{E})\{d'\}}
\end{array}
\qquad
\begin{array}{c}
\text{FHIJ} \\
\frac{d = \mathcal{E}\{d'\}}{\text{inj}_C^\tau(d) = \text{inj}_C^\tau(\mathcal{E})\{d'\}}
\end{array}
\qquad
\begin{array}{c}
\text{FNEHoleInside} \\
\frac{d = \mathcal{E}\{d'\}}{\langle d \rangle_\sigma^u = \langle \mathcal{E} \rangle_\sigma^u\{d'\}}
\end{array}
\qquad
\begin{array}{c}
\text{FMHoleInside} \\
\frac{d = \mathcal{E}\{d'\}}{\langle d \rangle_\sigma^{u\blacktriangleright} = \langle \mathcal{E} \rangle_\sigma^{u\blacktriangleright}\{d'\}}
\end{array}$$

$$\begin{array}{c}
\text{FHCastInside} \\
\frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow \tau_2\rangle\{d'\}}
\end{array}
\qquad
\begin{array}{c}
\text{FHFailedCast} \\
\frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow \emptyset \nRightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow \emptyset \nRightarrow \tau_2\rangle\{d'\}}
\end{array}$$

$$\boxed{d \mapsto d'} \quad d \text{ steps to } d'$$

$$\begin{array}{c}
\text{STEP} \\
\frac{d = \mathcal{E}\{d_0\} \quad d_0 \longrightarrow d'_0 \quad d' = \mathcal{E}\{d'_0\}}{d \mapsto d'}
\end{array}$$