

# Algebraic Data Types for Hazel

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## 1 Syntax

$$\begin{array}{ll}
 \text{HTyp} & \tau ::= \emptyset \mid \tau \rightarrow \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid \langle \rangle \mid \langle \alpha \rangle \\
 \text{HTypPat} & \pi ::= \alpha \mid \langle \rangle \\
 \text{HExp} & e ::= \emptyset \mid x \mid \lambda x:\tau.e \mid e(e) \mid e:\tau \mid \text{inj}_C(e) \mid \text{roll}(e) \mid \text{unroll}(e) \\
 & \quad \mid \langle \rangle^u \mid \langle e \rangle^u \mid \langle e \rangle^{u\blacktriangleright} \\
 \text{IHExp} & d ::= \emptyset \mid x \mid \lambda x:\tau.d \mid d(d) \mid \text{inj}_C^\tau(d) \mid \text{roll}^{\mu\alpha.\tau}(d) \mid \text{unroll}(d) \\
 & \quad \mid d\langle\tau \Rightarrow \tau\rangle \mid d\langle\tau \Rightarrow \langle \rangle \not\Rightarrow \tau\rangle \mid \langle d \rangle_\sigma^u \mid \langle d \rangle_\sigma^u \mid \langle d \rangle_\sigma^{u\blacktriangleright} \\
 \text{HTag} & C ::= \mathbf{C} \mid \langle \rangle^u \mid \langle c \rangle^u
 \end{array}$$

### 1.1 Context Extension

We write  $\Theta, \pi$  to denote the extension of type variable context  $\Theta$  with optional type variable name  $\pi$ .

$$\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \langle \rangle \end{cases}$$

## 2 Static Semantics

$$\boxed{[\tau/\pi]\tau' = \tau''} \quad \tau'' \text{ is obtained by substituting } \tau \text{ for } \pi \text{ in } \tau'$$

$$\begin{array}{ll}
 [\tau/\langle \rangle]\tau' & = \tau' \\
 [\tau/\alpha]\emptyset & = \emptyset \\
 [\tau/\alpha](\tau_1 \rightarrow \tau_2) & = [\tau/\alpha]\tau_1 \rightarrow [\tau/\alpha]\tau_2 \\
 [\tau/\alpha]\alpha & = \tau \\
 [\tau/\alpha]\alpha_1 & = \tau' & \text{when } \alpha \neq \alpha_1 \\
 [\tau/\alpha]\mu\alpha_1.\tau_2 & = \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \text{FV}(\tau) \\
 [\tau/\alpha]\mu\langle \rangle.\tau_2 & = \mu\langle \rangle.[\tau/\alpha]\tau_2 \\
 [\tau/\alpha]+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} \\
 [\tau/\alpha]\langle \rangle & = \langle \rangle \\
 [\alpha'/\alpha]\langle \alpha \rangle & = \langle \alpha' \rangle^u \\
 [\alpha'/\alpha]\langle \alpha' \rangle^u & = \langle \alpha' \rangle^u & \text{when } \alpha \neq \alpha'
 \end{array}$$

$$\boxed{\Theta \vdash \tau \text{ valid}} \quad \tau \text{ is a valid type}$$

$$\begin{array}{c}
 \text{TVUNIT} \quad \text{TVARR} \quad \text{TVVAR} \quad \text{TVREC} \quad \text{TVSUM} \\
 \frac{}{\Theta \vdash \emptyset \text{ valid}} \quad \frac{\Theta \vdash \tau_1 \text{ valid} \quad \Theta \vdash \tau_2 \text{ valid}}{\Theta \vdash \tau_1 \rightarrow \tau_2 \text{ valid}} \quad \frac{\alpha \in \Theta}{\Theta \vdash \alpha \text{ valid}} \quad \frac{\Theta, \pi \vdash \tau \text{ valid}}{\Theta \vdash \mu\pi.\tau \text{ valid}} \quad \frac{\{\Theta \vdash \tau_i \text{ valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \text{ valid}} \\
 \text{TVEHOLE} \\
 \frac{}{\Theta \vdash \langle \rangle \text{ valid}}
 \end{array}$$

$\boxed{\tau \sim \tau'}$   $\tau$  and  $\tau'$  are consistent

$$\begin{array}{c}
\text{TCREFL} \quad \frac{}{\tau \sim \tau} \quad \text{TCEHOLE1} \quad \frac{}{\langle \rangle \sim \tau} \quad \text{TCEHOLE2} \quad \frac{}{\tau \sim \langle \rangle} \quad \text{TCNEHOLE1} \quad \frac{}{\langle \alpha \rangle \sim \tau} \quad \text{TCNEHOLE2} \quad \frac{}{\tau \sim \langle \alpha \rangle} \quad \text{TCARR} \quad \frac{\tau_1 \sim \tau'_1 \quad \tau_2 \sim \tau'_2}{\tau_1 \rightarrow \tau_2 \sim \tau'_1 \rightarrow \tau'_2} \\
\\
\text{TCREC} \quad \frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \quad \text{TCRECHOLE1} \quad \frac{\alpha \notin \text{FV}(\tau) \quad \tau \sim \tau'}{\mu\langle \rangle.\tau \sim \mu\alpha.\tau'} \quad \text{TCRECHOLE2} \quad \frac{\alpha \notin \text{FV}(\tau) \quad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu\langle \rangle.\tau'} \quad \text{TCSUM} \quad \frac{\{\tau_i \sim \tau'_i\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}}}
\end{array}$$

$\boxed{C \text{ valid}}$   $C$  is a valid tag

$$\begin{array}{c}
\text{CVTAG} \quad \frac{}{\mathbf{C} \text{ valid}} \quad \text{CVEHOLE} \quad \frac{}{\langle \rangle^u \text{ valid}}
\end{array}$$

## 2.1 Bidirectional Typing

We call  $[\mu\pi.\tau/\pi]\tau$  the *unrolling* of recursive type  $\mu\pi.\tau$ .

**Theorem 1** (Synthetic Type Validity). *If  $\Gamma \vdash e \Rightarrow \tau$  then  $\emptyset \vdash \tau \text{ valid}$ .*

**Theorem 2** (Consistency Preserves Validity). *If  $\Theta \vdash \tau \text{ valid}$  and  $\tau \sim \tau'$  then  $\Theta \vdash \tau' \text{ valid}$ .*

$\boxed{\tau \blacktriangleright_{\rightarrow} \tau_1 \rightarrow \tau_2}$   $\tau$  has matched arrow type  $\tau_1 \rightarrow \tau_2$

$$\begin{array}{c}
\text{MAHOLE} \quad \frac{}{\langle \rangle \blacktriangleright_{\rightarrow} \langle \rangle \rightarrow \langle \rangle} \quad \text{MAARR} \quad \frac{}{\tau_1 \rightarrow \tau_2 \blacktriangleright_{\rightarrow} \tau_1 \rightarrow \tau_2}
\end{array}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$   $\tau$  has matched recursive type  $\mu\pi.\tau'$

$$\begin{array}{c}
\text{MRREC} \quad \frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau} \quad \text{MRHOLE} \quad \frac{}{\langle \rangle \blacktriangleright_{\mu} \mu\langle \rangle.\langle \rangle}
\end{array}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$   $e$  synthesizes type  $\tau$

$$\begin{array}{c}
\text{SUNIT} \quad \frac{}{\Gamma \vdash \emptyset \Rightarrow \emptyset} \quad \text{SVAR} \quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \quad \text{SVARFREE} \quad \frac{x \notin \text{dom}(\Gamma)}{\Gamma \vdash \langle x \rangle^u \Rightarrow \langle \rangle} \quad \text{SLAM} \quad \frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \tau \rightarrow \tau'} \\
\\
\text{SLAMINVALID} \quad \frac{\neg(\emptyset \vdash \tau \text{ valid}) \quad \Gamma, x : \langle \rangle \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \langle \rangle \rightarrow \tau'} \quad \text{SAPP} \quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright_{\rightarrow} \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \\
\\
\text{SAPPNOTARR} \quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \approx \langle \rangle \rightarrow \langle \rangle \quad \Gamma \vdash e_2 \Leftarrow \langle \rangle}{\Gamma \vdash \langle e_1 \rangle^u \blacktriangleright (e_2) \Rightarrow \langle \rangle} \quad \text{SASC} \quad \frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \quad \text{SASCINVALID} \quad \frac{\neg(\emptyset \vdash \tau \text{ valid}) \quad \Gamma \vdash e \Leftarrow \langle \rangle}{\Gamma \vdash e : \tau \Rightarrow \langle \rangle} \\
\\
\text{SROLLERR} \quad \frac{\Gamma \vdash e \Leftarrow \langle \rangle}{\Gamma \vdash \langle \text{roll}(e) \rangle^u \Rightarrow \mu\langle \rangle.\langle \rangle} \quad \text{SUNROLL} \quad \frac{\Gamma \vdash e \Rightarrow \tau \quad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \langle \text{unroll}(e) \rangle \Rightarrow [\mu\pi.\tau'/\pi]\tau'} \quad \text{SUNROLLNOTREC} \quad \frac{\Gamma \vdash e \Rightarrow \tau \quad \tau \approx \mu\langle \rangle.\langle \rangle}{\Gamma \vdash \langle \text{unroll}(\langle e \rangle^u \blacktriangleright) \rangle \Rightarrow \langle \rangle} \\
\\
\text{SINJERR} \quad \frac{\mathbf{C} \text{ valid} \quad \Gamma \vdash e \Leftarrow \langle \rangle}{\Gamma \vdash \langle \text{inj}_C(e) \rangle^u \Rightarrow \langle \rangle} \quad \text{SINJTAGER} \quad \frac{}{\Gamma \vdash \langle \text{inj}_{\langle c \rangle^u}(e) \rangle \Rightarrow \langle \rangle} \quad \text{SEHOLE} \quad \frac{}{\Gamma \vdash \langle \rangle^u \Rightarrow \langle \rangle} \quad \text{SNEHOLE} \quad \frac{}{\Gamma \vdash \langle e \rangle^u \Rightarrow \langle \rangle}
\end{array}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$   $e$  analyzes against type  $\tau$

$$\begin{array}{c}
\text{AROLL} \\
\frac{\tau \blacktriangleright_{\mu} \mu\pi.\tau' \quad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \text{roll}(e) \Leftarrow \tau}
\end{array}
\quad
\begin{array}{c}
\text{AROLLNOTREC} \\
\frac{\tau \approx \mu(\emptyset).\emptyset \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash (\text{roll}(e))^u \Leftarrow \tau}
\end{array}
\quad
\begin{array}{c}
\text{AINJHOLE} \\
\frac{\Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash \text{inj}_C(e) \Leftarrow \emptyset}
\end{array}$$

$$\begin{array}{c}
\text{AINJ} \\
\frac{C_j \in \mathcal{C} \quad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \text{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}
\quad
\begin{array}{c}
\text{AINJTAGER} \\
\frac{\langle d \rangle^u \notin \mathcal{C} \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash \text{inj}_{\langle d \rangle^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}
\quad
\begin{array}{c}
\text{AINJBADTAG} \\
\frac{C' \text{ valid} \quad C' \notin \mathcal{C} \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash (\text{inj}_{C'}(e))^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}$$

$$\begin{array}{c}
\text{AINJUNEXPECTEDBODY} \\
\frac{C_j \in \mathcal{C} \quad \tau_j = \emptyset \quad e \neq \emptyset \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash (\text{inj}_{C_j}(e))^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}
\quad
\begin{array}{c}
\text{AINJEXPECTEDBODY} \\
\frac{C_j \in \mathcal{C} \quad \tau_j \neq \emptyset}{\Gamma \vdash (\text{inj}_{C_j}(\emptyset))^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}
\end{array}$$

$$\begin{array}{c}
\text{ASUBSUME} \\
\frac{\Gamma \vdash e \Rightarrow \tau' \quad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}
\end{array}$$

## 2.2 Typed Elaboration

**Theorem 3** (Synthetic Typed Elaboration Validity). *If  $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$  then  $\emptyset \vdash \tau$  valid.*

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$   $e$  synthesizes type  $\tau$  and elaborates to  $d$

$$\begin{array}{c}
\text{ESUNIT} \\
\frac{}{\Gamma \vdash \emptyset \Rightarrow \emptyset \rightsquigarrow \emptyset \dashv \emptyset}
\end{array}
\quad
\begin{array}{c}
\text{ESVAR} \\
\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}
\end{array}
\quad
\begin{array}{c}
\text{ESVARFREE} \\
\frac{x \notin \text{dom}(\Gamma)}{\Gamma \vdash (x)^u \Rightarrow \emptyset \rightsquigarrow (x)_{\text{id}(\Gamma)}^u \dashv u :: \emptyset[\Gamma]}
\end{array}$$

$$\begin{array}{c}
\text{ESLAM} \\
\frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \tau \rightarrow \tau' \rightsquigarrow \lambda x : \tau. d \dashv \Delta}
\end{array}
\quad
\begin{array}{c}
\text{ESLAMINVALID} \\
\frac{\neg(\emptyset \vdash \tau \text{ valid}) \quad \Gamma, x : \emptyset \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \emptyset \rightarrow \tau' \rightsquigarrow \lambda x : \tau. d \dashv \Delta}
\end{array}$$

$$\begin{array}{c}
\text{ESAPP} \\
\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright_{\mu} \tau_2 \rightarrow \tau \quad \Gamma \vdash e_1 \Leftarrow \tau_2 \rightarrow \tau \rightsquigarrow d_1 : \tau'_1 \dashv \Delta_1 \quad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau'_2 \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1 \langle \tau'_1 \Rightarrow \tau_1 \rangle)(d_2 \langle \tau'_2 \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2}
\end{array}$$

$$\begin{array}{c}
\text{ESAPPNOTARR} \\
\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \quad \tau_1 \approx \emptyset \rightarrow \emptyset \quad \Gamma \vdash e_2 \Leftarrow \emptyset \rightsquigarrow d_2 : \tau'_2 \dashv \Delta_2}{\Gamma \vdash (e_1)^u \blacktriangleright_{\mu} (e_2) \Rightarrow \emptyset \rightsquigarrow (d_1)_{\text{id}(\Gamma)}^u \blacktriangleright_{\mu} (d_2 \langle \tau'_2 \Rightarrow \emptyset \rangle) \dashv \Delta_1 \cup \Delta_2, u :: \emptyset \rightarrow \emptyset[\Gamma]}
\end{array}$$

$$\begin{array}{c}
\text{ESASC} \\
\frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d \langle \tau' \Rightarrow \tau \rangle \dashv \Delta}
\end{array}
\quad
\begin{array}{c}
\text{ESASCINVALID} \\
\frac{\neg(\emptyset \vdash \tau \text{ valid}) \quad \Gamma \vdash e \Leftarrow \emptyset \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \emptyset \rightsquigarrow d \langle \tau' \Rightarrow \emptyset \rangle \dashv \Delta}
\end{array}$$

$$\begin{array}{c}
\text{ESROLLE} \\
\frac{\Gamma \vdash e \Leftarrow \emptyset \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash (\text{roll}(e))^u \Rightarrow \mu(\emptyset).\emptyset \rightsquigarrow (\text{roll}^{\mu(\emptyset).\emptyset}(d \langle \tau \Rightarrow \emptyset \rangle))^u_{\text{id}(\Gamma)} \dashv \Delta, u :: \mu(\emptyset).\emptyset[\Gamma]}
\end{array}$$

$$\begin{array}{c}
\text{ESUNROLL} \\
\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \quad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \text{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \text{unroll}(d \langle \tau \Rightarrow \mu\pi.\tau' \rangle) \dashv \Delta}
\end{array}$$

$$\frac{\text{ESUNROLLNOTREC} \quad \Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \quad \tau \approx \mu(\mathbb{O}).\mathbb{O}}{\Gamma \vdash \text{unroll}(\langle e \rangle^u) \Rightarrow \mathbb{O} \rightsquigarrow \text{unroll}(\langle d \rangle_{\text{id}(\Gamma)}^u) \dashv \Delta, u :: \mu(\mathbb{O}).\mathbb{O}[\Gamma]}$$

$$\frac{\text{ESINJERR} \quad C \text{ valid} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\{C(\tau)\}}{\Gamma \vdash \langle \text{inj}_C(e) \rangle^u \Rightarrow \mathbb{O} \rightsquigarrow \langle \text{inj}_C^{\tau'}(d \langle \tau \Rightarrow \mathbb{O} \rangle) \rangle_{\text{id}(\Gamma)}^u \dashv \Delta, u :: \mathbb{O}[\Gamma]}$$

$$\frac{\text{ESINJTAGER} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\{\langle c \rangle^u(\tau)\}}{\Gamma \vdash \text{inj}_{\langle c \rangle^u}(e) \Rightarrow \mathbb{O} \rightsquigarrow \text{inj}_{\langle c \rangle^u}^{\tau'}(d \langle \tau \Rightarrow \mathbb{O} \rangle) \dashv \Delta} \quad \text{ESEHOLE} \quad \frac{}{\Gamma \vdash \mathbb{O}^u \Rightarrow \mathbb{O} \rightsquigarrow \mathbb{O}_{\text{id}(\Gamma)}^u \dashv u :: \mathbb{O}[\Gamma]}$$

$$\frac{\text{ESNEHOLE} \quad \Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle e \rangle^u \Rightarrow \mathbb{O} \rightsquigarrow \langle d \rangle_{\text{id}(\Gamma)}^u \dashv \Delta, u :: \mathbb{O}[\Gamma]}$$

$$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta} \quad e \text{ analyzes against type } \tau_1 \text{ and elaborates to } d \text{ of consistent type } \tau_2$$

$$\frac{\text{EAROLL} \quad \tau \blacktriangleright_{\mu} \mu\pi.\tau' \quad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \text{roll}(e) \Leftarrow \tau \rightsquigarrow \text{roll}^{\mu\pi.\tau'}(d \langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rangle) : \mu\pi.\tau' \dashv \Delta}$$

$$\frac{\text{EAROLLNOTREC} \quad \tau \approx \mu(\mathbb{O}).\mathbb{O} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash \langle \text{roll}(e) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{roll}^{\mu(\mathbb{O}).\mathbb{O}}(d) \rangle_{\text{id}(\Gamma)}^u : \mu(\mathbb{O}).\mathbb{O} \dashv \Delta, u :: \mu(\mathbb{O}).\mathbb{O}[\Gamma]}$$

$$\frac{\text{EAINJHOLE} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\{C(\tau)\}}{\Gamma \vdash \text{inj}_C(e) \Leftarrow \mathbb{O} \rightsquigarrow \text{inj}_C^{\tau'}(d) : \tau' \dashv \Delta} \quad \text{EAINJ} \quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \Gamma \vdash e \Leftarrow \tau_j \rightsquigarrow d : \tau'_j \dashv \Delta}{\Gamma \vdash \text{inj}_{C_j}(e) \Leftarrow \tau \rightsquigarrow \text{inj}_{C_j}^{\tau}(d \langle \tau'_j \Rightarrow \tau_j \rangle) : \tau \dashv \Delta}$$

$$\frac{\text{EAINJTAGER} \quad \langle c \rangle^u \notin \mathcal{C} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau \dashv \Delta \quad \tau' = +\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{\langle c \rangle^u(\tau)\}\}}{\Gamma \vdash \text{inj}_{\langle c \rangle^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \rightsquigarrow \text{inj}_{\langle c \rangle^u}^{\tau'}(d \langle \tau \Rightarrow \mathbb{O} \rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta}$$

$$\frac{\text{EAINJBADTAG} \quad \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C \notin \mathcal{C} \quad C \text{ valid} \quad \Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau' \dashv \Delta \quad \tau'' = +\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{C(\tau')\}\}}{\Gamma \vdash \langle \text{inj}_C(e) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{inj}_C^{\tau''}(d) \rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\frac{\text{EAINJUNEXPECTEDBODY} \quad \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \tau_j = \emptyset \quad e \neq \emptyset}{\Gamma \vdash e \Leftarrow \mathbb{O} \rightsquigarrow d : \tau_j \dashv \Delta \quad \tau' = +\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\}} \quad \frac{}{\Gamma \vdash \langle \text{inj}_{C_j}(e) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{inj}_{C_j}^{\tau'}(d) \rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\frac{\text{EAINJEXPECTEDBODY} \quad \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \tau_j \neq \emptyset \quad \tau' = +\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\emptyset)\}\}}{\Gamma \vdash \langle \text{inj}_{C_j}(\emptyset) \rangle^u \Leftarrow \tau \rightsquigarrow \langle \text{inj}_{C_j}^{\tau'}(\emptyset) \rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\frac{\text{EASUBSUME} \quad e \neq \mathbb{O}^u \quad e \neq \langle e' \rangle^u \quad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \quad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta} \quad \text{EAEHOLE} \quad \frac{}{\Gamma \vdash \mathbb{O}^u \Leftarrow \tau \rightsquigarrow \mathbb{O}_{\text{id}(\Gamma)}^u : \tau \dashv u :: \tau[\Gamma]}$$

$$\frac{\text{EANEHOLE} \quad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle e \rangle^u \Leftarrow \tau \rightsquigarrow \langle d \rangle_{\text{id}(\Gamma)}^u : \tau \dashv u :: \tau[\Gamma]}$$

## 2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$   $d$  is assigned type  $\tau$

$$\begin{array}{c} \text{TAAUNIT} \quad \frac{}{\Delta; \Gamma \vdash \emptyset : \emptyset} \quad \text{TAVAR} \quad \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau} \quad \text{TALAM} \quad \frac{\emptyset \vdash \tau \text{ valid} \quad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x : \tau. d : \tau \rightarrow \tau'} \quad \text{TALAMINVALID} \quad \frac{\neg(\emptyset \vdash \tau \text{ valid}) \quad \Delta; \Gamma, x : \emptyset \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x : \tau. d : \emptyset \rightarrow \tau'} \\[10pt] \text{TAAAPP} \quad \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \rightarrow \tau \quad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau} \quad \text{TAROLL} \quad \frac{\emptyset \vdash \mu\pi.\tau \text{ valid} \quad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \text{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau} \\[10pt] \text{TAUNROLL} \quad \frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \text{unroll}(d) : [\mu\pi.\tau/\pi]\tau} \quad \text{TAInj} \quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \text{inj}_{C_j}^\tau(d) : \tau} \\[10pt] \text{TAEHOLE} \quad \frac{u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \langle \rangle_\sigma^u : \tau} \quad \text{TANEHOLE} \quad \frac{\Delta; \Gamma \vdash d : \tau' \quad u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \langle d \rangle_\sigma^u : \tau} \\[10pt] \text{TAMHOLE} \quad \frac{\Delta; \Gamma \vdash d : \tau' \quad u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \langle d \rangle_\sigma^{u\blacktriangleright} : \tau} \quad \text{TACAST} \quad \frac{\Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d \langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2} \\[10pt] \text{TAFaILEDCAST} \quad \frac{\Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d \langle \tau_1 \Rightarrow \emptyset \nRightarrow \tau_2 \rangle : \tau_2} \end{array}$$

## 3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$   $\tau$  is a ground type

$$\begin{array}{c} \text{GARR} \quad \frac{}{\langle \rangle \rightarrow \langle \rangle \text{ ground}} \quad \text{GREC} \quad \frac{}{\mu \langle \rangle . \langle \rangle \text{ ground}} \quad \text{GSUM} \quad \frac{\{\tau_i = \emptyset \vee \tau_i = \langle \rangle\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \text{ ground}} \\[10pt] \boxed{\tau \blacktriangleright_{\text{ground}} \tau'} \quad \tau \text{ has matched ground type } \tau' \end{array}$$

$$\begin{array}{c} \text{MGARR} \quad \frac{\tau_1 \rightarrow \tau_2 \neq \langle \rangle \rightarrow \langle \rangle}{\tau_1 \rightarrow \tau_2 \blacktriangleright_{\text{ground}} \langle \rangle \rightarrow \langle \rangle} \quad \text{MGREC} \quad \frac{\tau \neq \langle \rangle}{\mu\pi.\tau \blacktriangleright_{\text{ground}} \mu \langle \rangle . \langle \rangle} \\[10pt] \text{MGSUM} \quad \frac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \quad \{(\tau_i = \emptyset \Rightarrow \tau'_i = \emptyset) \wedge (\tau_i \neq \emptyset \Rightarrow \tau'_i = \langle \rangle)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\text{ground}} +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}}} \end{array}$$

$\boxed{d \text{ final}}$   $d$  is final

$$\frac{\text{FBOXEDVAL} \quad d \text{ boxedval}}{d \text{ final}}$$

$$\frac{\text{FINDET} \quad d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$   $d$  is a value

$$\frac{\text{VUNIT}}{\emptyset \text{ val}}$$

$$\frac{\text{VLAM}}{\lambda x:\tau.d \text{ val}}$$

$$\frac{\text{VROLL} \quad d \text{ val}}{\text{roll}^{\mu\pi.\tau}(d) \text{ val}}$$

$$\frac{\text{VINJ} \quad d \text{ val}}{\text{inj}_C^\tau(d) \text{ val}}$$

$\boxed{d \text{ boxedval}}$   $d$  is a boxed value

$$\frac{\text{BVVAL} \quad d \text{ val}}{d \text{ boxedval}}$$

$$\frac{\text{BVROLL} \quad d \text{ boxedval}}{\text{roll}^{\mu\pi.\tau}(d) \text{ boxedval}}$$

$$\frac{\text{BVINJ} \quad d \text{ boxedval}}{\text{inj}_C^\tau(d) \text{ boxedval}}$$

$$\frac{\text{BVARRCast} \quad \tau_1 \rightarrow \tau_2 \neq \tau_3 \rightarrow \tau_4 \quad d \text{ boxedval}}{d\langle\tau_1 \rightarrow \tau_2 \Rightarrow \tau_3 \rightarrow \tau_4\rangle \text{ boxedval}}$$

$$\frac{\text{BVRECCast} \quad \mu\pi.\tau \neq \mu\pi'.\tau' \quad d \text{ boxedval}}{d\langle\mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle \text{ boxedval}}$$

$$\frac{\text{BVSUMCast} \quad \begin{array}{l} \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \end{array} \quad d \text{ boxedval}}{d\langle\tau \Rightarrow \tau'\rangle \text{ boxedval}}$$

$$\frac{\text{BVHOLECast} \quad d \text{ boxedval} \quad \tau \text{ ground}}{d\langle\tau \Rightarrow \emptyset\rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$   $d$  is indeterminate

$$\frac{\text{IEHOLE}}{\langle\emptyset\rangle_\sigma^u \text{ indet}}$$

$$\frac{\text{INEHOLE} \quad d \text{ final}}{\langle d \rangle_\sigma^u \text{ indet}}$$

$$\frac{\text{IAPP} \quad d_1 \neq d'_1 \langle\tau_1 \rightarrow \tau_2 \Rightarrow \tau_3 \rightarrow \tau_4\rangle \quad d_1 \text{ indet} \quad d_2 \text{ final}}{d_1(d_2) \text{ indet}}$$

$$\frac{\text{IROLL} \quad d \text{ indet}}{\text{roll}^{\mu\pi.\tau}(d) \text{ indet}}$$

$$\frac{\text{IUNROLL} \quad d \text{ indet}}{\text{unroll}(d) \text{ indet}}$$

$$\frac{\text{IINJ} \quad d \text{ indet}}{\text{inj}_C^\tau(d) \text{ indet}}$$

$$\frac{\text{IINJHOLE} \quad C \neq \mathbf{C} \quad d \text{ final}}{\text{inj}_C^\tau(d) \text{ indet}}$$

$$\frac{\text{ICASTGROUNDHOLE} \quad d \text{ indet} \quad \tau \text{ ground}}{d\langle\tau \Rightarrow \emptyset\rangle \text{ indet}}$$

$$\frac{\text{ICASTHOLEGROUND} \quad d \neq d' \langle\tau' \Rightarrow \emptyset\rangle \quad d \text{ indet} \quad \tau \text{ ground}}{d\langle\emptyset \Rightarrow \tau\rangle \text{ indet}}$$

$$\frac{\text{ICASTARR} \quad \tau_1 \rightarrow \tau_2 \neq \tau_3 \rightarrow \tau_4 \quad d \text{ indet}}{d\langle\tau_1 \rightarrow \tau_2 \Rightarrow \tau_3 \rightarrow \tau_4\rangle \text{ indet}}$$

$$\frac{\text{ICASTREC} \quad \mu\pi.\tau \neq \mu\pi'.\tau' \quad d \text{ indet}}{d\langle\mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle \text{ indet}}$$

$$\frac{\text{ICASTSUM} \quad \begin{array}{l} \tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \end{array} \quad d \text{ indet}}{d\langle\tau \Rightarrow \tau'\rangle \text{ indet}}$$

$$\frac{\text{IFAILEDCAST} \quad d \text{ final} \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2}{d\langle\tau_1 \Rightarrow \emptyset \neq \tau_2\rangle \text{ indet}}$$

$\boxed{d \longrightarrow d'}$   $d$  takes an instruction transition to  $d'$

$$\frac{\text{ITAPP} \quad [d_2 \text{ final}]}{(\lambda x:\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1}$$

$$\frac{\text{ITUNROLL} \quad [d \text{ final}]}{\text{unroll}(\text{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\frac{\text{ITAPPCast} \quad [d_1 \text{ final}] \quad [d_2 \text{ final}] \quad \tau_1 \rightarrow \tau_2 \neq \tau'_1 \rightarrow \tau'_2}{d_1\langle\tau_1 \rightarrow \tau_2 \Rightarrow \tau'_1 \rightarrow \tau'_2\rangle(d_2) \longrightarrow (d_1(d_2\langle\tau'_1 \Rightarrow \tau_1\rangle))\langle\tau_2 \Rightarrow \tau'_2\rangle}$$

$$\frac{\text{ITUnrollCast} \quad [d \text{ final}] \quad \mu\pi.\tau \neq \mu\pi'.\tau'}{\text{unroll}(d\langle\mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \text{unroll}(d)\langle[\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle} \quad \frac{\text{ITCastId} \quad [d \text{ final}]}{d\langle\tau \Rightarrow \tau\rangle \longrightarrow d}$$

$$\frac{\text{ITCastSucceed} \quad [d \text{ final}] \quad \tau \text{ ground}}{d\langle\tau \Rightarrow \emptyset \Rightarrow \tau\rangle \longrightarrow d} \quad \frac{\text{ITCastFail} \quad [d \text{ final}] \quad \tau_1 \neq \tau_2 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground}}{d\langle\tau_1 \Rightarrow \emptyset \Rightarrow \tau_2\rangle \longrightarrow d\langle\tau_1 \Rightarrow \emptyset \nRightarrow \tau_2\rangle}$$

$$\frac{\text{ITGround} \quad [d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle\tau \Rightarrow \emptyset\rangle \longrightarrow d\langle\tau \Rightarrow \tau' \Rightarrow \emptyset\rangle} \quad \frac{\text{ITExpand} \quad [d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle\emptyset \Rightarrow \tau\rangle \longrightarrow d\langle\emptyset \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\text{EvalCtx } \mathcal{E} ::= \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \text{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \text{unroll}(\mathcal{E}) \mid \text{inj}_C^\tau(\mathcal{E}) \mid \langle\mathcal{E}\rangle_\sigma^u \mid \langle\mathcal{E}\rangle_\sigma^{u\blacktriangleright} \mid \mathcal{E}\langle\tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle\tau \Rightarrow \emptyset \nRightarrow \tau\rangle$$

$$\boxed{d = \mathcal{E}\{d'\}} \quad d \text{ is obtained by placing } d' \text{ at the mark in } \mathcal{E}$$

$$\begin{array}{c} \text{FHOuter} \quad \frac{}{d = \circ\{d\}} \quad \text{FHApp1} \quad \frac{d_1 = \mathcal{E}\{d'_1\}}{d_1(d_2) = \mathcal{E}(d_2)\{d'_1\}} \quad \text{FHApp2} \quad \frac{[d_1 \text{ final}] \quad d_2 = \mathcal{E}\{d'_2\}}{d_1(d_2) = d_1(\mathcal{E})\{d'_2\}} \quad \text{FHRoll} \quad \frac{d = \mathcal{E}\{d'\}}{\text{roll}^{\mu\pi.\tau}(d) = \text{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}} \\ \text{FHUnroll} \quad \frac{d = \mathcal{E}\{d'\}}{\text{unroll}(d) = \text{unroll}(\mathcal{E})\{d'\}} \quad \text{FHIInj} \quad \frac{d = \mathcal{E}\{d'\}}{\text{inj}_C^\tau(d) = \text{inj}_C^\tau(\mathcal{E})\{d'\}} \quad \text{FNEHoleInside} \quad \frac{d = \mathcal{E}\{d'\}}{\langle d \rangle_\sigma^u = \langle \mathcal{E} \rangle_\sigma^u\{d'\}} \quad \text{FMHoleInside} \quad \frac{d = \mathcal{E}\{d'\}}{\langle d \rangle_\sigma^{u\blacktriangleright} = \langle \mathcal{E} \rangle_\sigma^{u\blacktriangleright}\{d'\}} \\ \text{FHCastInside} \quad \frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow \tau_2\rangle\{d'\}} \quad \text{FHFailedCast} \quad \frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow \emptyset \nRightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow \emptyset \nRightarrow \tau_2\rangle\{d'\}} \end{array}$$

$$\boxed{d \mapsto d'} \quad d \text{ steps to } d'$$

$$\frac{\text{STEP} \quad d = \mathcal{E}\{d_0\} \quad d_0 \longrightarrow d'_0 \quad d' = \mathcal{E}\{d'_0\}}{d \mapsto d'}$$