# Algebraic Data Types for Hazel


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## 1 Syntax

$$
\begin{array}{llcl}
\mathsf{HTyp} & \tau & ::= & \varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid \llparenthesis\rrparenthesis \mid \llparenthesis\alpha\rrparenthesis \\
\mathsf{HTypPat} & \pi & ::= & \alpha \mid \llparenthesis\rrparenthesis \\
\mathsf{HExp} & e & ::= & \varnothing \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \mathtt{inj}_C(e) \mid \mathtt{roll}(e) \mid \mathtt{unroll}(e) \\
 & & & \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^{u\blacktriangleright} \\
\mathsf{IHExp} & d & ::= & \varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \mathtt{inj}^{\tau}_C(d) \mid \mathtt{roll}^{\mu\alpha.\tau}(d) \mid \mathtt{unroll}(d) \\
 & & & \mid d\langle\tau \Rightarrow \tau\rangle \mid d\langle\tau \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau\rangle \mid \llparenthesis\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^{u\blacktriangleright}_\sigma \\
\mathsf{HTag} & C & ::= & \mathbf{C} \mid \llparenthesis\rrparenthesis \mid \llparenthesis\mathbf{C}\rrparenthesis
\end{array}
$$

### 1.1 Context Extension

We write $\Theta, \pi$ to denote the extension of type variable context $\Theta$ with optional type variable name $\pi$.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \llparenthesis\rrparenthesis \end{cases}
$$

## 2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ $\tau''$ is obtained by substituting $\tau$ for $\pi$ in $\tau'$

$$
\begin{array}{lcll}
[\tau/\llparenthesis\rrparenthesis]\tau' & = & \tau' & \\
[\tau/\alpha]\varnothing & = & \varnothing & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu\llparenthesis\rrparenthesis.\tau_2 & = & \mu\llparenthesis\rrparenthesis.[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha]\llparenthesis\rrparenthesis & = & \llparenthesis\rrparenthesis & \\
[\alpha'/\alpha]\llparenthesis\alpha\rrparenthesis & = & \llparenthesis\alpha'\rrparenthesis & \\
[\alpha'/\alpha]\llparenthesis\alpha'\rrparenthesis & = & \llparenthesis\alpha'\rrparenthesis & \text{when } \alpha \neq \alpha'
\end{array}
$$

$\boxed{\Theta \vdash \tau \; \mathsf{valid}}$ $\tau$ is a valid type

$$
\textsc{TVUnit}\;\frac{}{\Theta \vdash \varnothing \; \mathsf{valid}}
\qquad
\textsc{TVArr}\;\frac{\Theta \vdash \tau_1 \; \mathsf{valid} \qquad \Theta \vdash \tau_2 \; \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2 \; \mathsf{valid}}
\qquad
\textsc{TVVar}\;\frac{\alpha \in \Theta}{\Theta \vdash \alpha \; \mathsf{valid}}
\qquad
\textsc{TVRec}\;\frac{\Theta, \pi \vdash \tau \; \mathsf{valid}}{\Theta \vdash \mu\pi.\tau \; \mathsf{valid}}
\qquad
\textsc{TVSum}\;\frac{\{\Theta \vdash \tau_i \; \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \; \mathsf{valid}}
$$

$$
\textsc{TVEHole}\;\frac{}{\Theta \vdash \llparenthesis\rrparenthesis \; \mathsf{valid}}
\qquad
\textsc{TVNEHole}\;\frac{\alpha \notin \Theta}{\Theta \vdash \llparenthesis\alpha\rrparenthesis \; \mathsf{valid}}
$$

$\boxed{\tau \sim \tau'}$ $\tau$ and $\tau'$ are consistent

$$\text{TCRefl}\ \frac{}{\tau \sim \tau} \qquad \text{TCEHole1}\ \frac{}{\llparenthesis\rrparenthesis \sim \tau} \qquad \text{TCEHole2}\ \frac{}{\tau \sim \llparenthesis\rrparenthesis} \qquad \text{TCNEHole1}\ \frac{}{\llparenthesis\alpha\rrparenthesis \sim \tau} \qquad \text{TCNEHole2}\ \frac{}{\tau \sim \llparenthesis\alpha\rrparenthesis} \qquad \text{TCArr}\ \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

$$\text{TCRec}\ \frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \qquad \text{TCRecHole1}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\llparenthesis\rrparenthesis.\tau \sim \mu\alpha.\tau'} \qquad \text{TCRecHole2}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu\llparenthesis\rrparenthesis.\tau'} \qquad \text{TCSum}\ \frac{\{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

## 2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

**Theorem 1** (Synthetic Type Validity)**.** *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

**Theorem 2** (Consistency Preserves Validity)**.** *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ $\tau$ has matched arrow type $\tau_1 \to \tau_2$

$$\text{MAHole}\ \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\to} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis} \qquad \text{MAArr}\ \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ $\tau$ has matched recursive type $\mu\pi.\tau'$

$$\text{MRRec}\ \frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau} \qquad \text{MRHole}\ \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\mu} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ $e$ synthesizes type $\tau$

$$\text{SUnit}\ \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing} \qquad \text{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \text{SVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis} \qquad \text{SLam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}$$

$$\text{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \qquad \text{SAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis}$$

$$\text{SAsc}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \qquad \text{SRollErr}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{roll}(e)\rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis} \qquad \text{SUnroll}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$$

$$\text{SUnrollNotRec}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}\big(\llparenthesis e\rrparenthesis^{u\blacktriangleright}\big) \Rightarrow \llparenthesis\rrparenthesis} \qquad \text{SInjErr}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{inj}_C(e)\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis} \qquad \text{SInjTagErr}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{inj}_{\llparenthesis\mathbf{C}\rrparenthesis}(e) \Rightarrow \llparenthesis\rrparenthesis} \qquad \text{SEHole}\ \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}$$

$$\text{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash \llparenthesis e\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ $e$ analyzes against type $\tau$

$$\textsc{ARoll}\quad \frac{\tau \blacktriangleright_{\mu} \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau}$$

$$\textsc{ARollNotRec}\quad \frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\texttt{roll}(e)\rrparenthesis^{u} \Leftarrow \tau}$$

$$\textsc{AInjHole}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{inj}_{C}(e) \Leftarrow \llparenthesis\rrparenthesis}$$

$$\textsc{AInj}\quad \frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjTagErr}\quad \frac{\mathbf{C} \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{inj}_{\llparenthesis\mathbf{C}\rrparenthesis}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjUnexpectedBody}\quad \frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\texttt{inj}_{C_j}(e)\rrparenthesis^{u} \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjExpectedBody}\quad \frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash \llparenthesis\texttt{inj}_{C_j}(\varnothing)\rrparenthesis^{u} \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{ASubsume}\quad \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

## 2.2 Typed Elaboration

**Theorem 3** (Synthetic Typed Elaboration Validity). *If* $\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta$ *then* $\emptyset \vdash \tau\,\mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta}$ $e$ synthesizes type $\tau$ and elaborates to $d$

$$\textsc{ESUnit}\quad \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \leadsto \varnothing \dashv \emptyset}$$

$$\textsc{ESVar}\quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \leadsto x \dashv \emptyset}$$

$$\textsc{ESVarFree}\quad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis x\rrparenthesis^{u}_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESLam}\quad \frac{\emptyset \vdash \tau\,\mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \leadsto \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESApp}\quad \frac{\begin{array}{c}\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \\ \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \leadsto d_1 : \tau_1' \dashv \Delta_1 \\ \Gamma \vdash e_2 \Leftarrow \tau_2 \leadsto d_2 : \tau_2' \dashv \Delta_2\end{array}}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \leadsto (d_1\langle \tau_1' \Rightarrow \tau_1\rangle)(d_2\langle \tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\textsc{ESAppNotArr}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \leadsto d_1 \dashv \Delta_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis \leadsto d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis d_1\rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow \llparenthesis\rrparenthesis\rangle) \dashv \Delta_1 \cup \Delta_2, u :: \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESAsc}\quad \frac{\begin{array}{c}\emptyset \vdash \tau\,\mathsf{valid} \\ \Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta\end{array}}{\Gamma \vdash e : \tau \Rightarrow \tau \leadsto d\langle \tau' \Rightarrow \tau\rangle \dashv \Delta}$$

$$\textsc{ESRollErr}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta}{\Gamma \vdash \llparenthesis\texttt{roll}(e)\rrparenthesis^{u} \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \leadsto \llparenthesis\texttt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle)\rrparenthesis^{u}_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESUnroll}\quad \frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \texttt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \leadsto \texttt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}$$

ESUnrollNotRec

$$\frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{unroll}\left(\llparenthesis e \rrparenthesis^{u\blacktriangleright}\right) \Rightarrow \llparenthesis\rrparenthesis \leadsto \texttt{unroll}\left(\llparenthesis d \rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\right) \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

ESInjErr

$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \llparenthesis \texttt{inj}_C(e) \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis \texttt{inj}^{\tau'}_C(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

ESInjTagErr

$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{\llparenthesis \mathbf{C} \rrparenthesis(\tau)\}}{\Gamma \vdash \texttt{inj}_{\llparenthesis \mathbf{C} \rrparenthesis}(e) \Rightarrow \llparenthesis\rrparenthesis \leadsto \texttt{inj}^{\tau'}_{\llparenthesis \mathbf{C} \rrparenthesis}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle) \dashv \Delta}$$

ESEHole

$$\frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

ESNEHole

$$\frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \leadsto d : \tau_2 \dashv \Delta}$ $e$ analyzes against type $\tau_1$ and elaborates to $d$ of consistent type $\tau_2$

EARoll

$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \leadsto d : \tau'' \dashv \Delta}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau \leadsto \texttt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}$$

EARollNotRec

$$\frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau' \dashv \Delta}{\Gamma \vdash \llparenthesis \texttt{roll}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

EAInjHole

$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow \llparenthesis\rrparenthesis \leadsto \texttt{inj}^{\tau'}_C(d) : \tau' \dashv \Delta}$$

EAInj

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \leadsto d : \tau_j' \dashv \Delta}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow \tau \leadsto \texttt{inj}^{\tau}_{C_j}\left(d\langle \tau_j' \Rightarrow \tau_j\rangle\right) : \tau \dashv \Delta}$$

EAInjTagErr

$$\frac{\llparenthesis \mathbf{C} \rrparenthesis \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{\llparenthesis \mathbf{C} \rrparenthesis(\tau)\}\right\}}{\Gamma \vdash \texttt{inj}_{\llparenthesis \mathbf{C} \rrparenthesis}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \leadsto \texttt{inj}^{\tau'}_{\llparenthesis \mathbf{C} \rrparenthesis}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta}$$

EAInjUnexpectedBody

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau_j \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\right\}\end{array}}{\Gamma \vdash \llparenthesis \texttt{inj}_{C_j}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}^{\tau'}_{C_j}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjExpectedBody

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\right\}}{\Gamma \vdash \llparenthesis \texttt{inj}_{C_j}(\varnothing) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}^{\tau'}_{C_j}(\varnothing) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EASubsume

$$\frac{e \neq \llparenthesis\rrparenthesis^u \qquad e \neq \llparenthesis e' \rrparenthesis^u \qquad \Gamma \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta}$$

EAEHole

$$\frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

EANEHole

$$\frac{\Gamma \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

### 2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ $d$ is assigned type $\tau$

$$\text{TAUnit}\quad \frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

$$\text{TAVar}\quad \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

$$\text{TALam}\quad \frac{\emptyset \vdash \tau \text{ valid} \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}$$

$$\text{TAApp}\quad \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

$$\text{TARoll}\quad \frac{\emptyset \vdash \mu\pi.\tau \text{ valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \texttt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}$$

$$\text{TAUnroll}\quad \frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \texttt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

$$\text{TAInj}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \texttt{inj}^{\tau}_{C_j}(d) : \tau}$$

$$\text{TAEHole}\quad \frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|\!)^{u}_{\sigma} : \tau}$$

$$\text{TANEHole}\quad \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^{u}_{\sigma} : \tau}$$

$$\text{TAMHole}\quad \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^{u\blacktriangleright}_{\sigma} : \tau}$$

$$\text{TACast}\quad \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

$$\text{TAFailedCast}\quad \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2 \rangle : \tau_2}$$

## 3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$ $\tau$ is a ground type

$$\text{GArr}\quad \frac{}{(\!|\!) \to (\!|\!) \text{ ground}}$$

$$\text{GRec}\quad \frac{}{\mu(\!|\!).(\!|\!) \text{ ground}}$$

$$\text{GSum}\quad \frac{\{\tau_i = \varnothing \lor \tau_i = (\!|\!)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \text{ ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ $\tau$ has matched ground type $\tau'$

$$\text{MGArr}\quad \frac{\tau_1 \to \tau_2 \neq (\!|\!) \to (\!|\!)}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} (\!|\!) \to (\!|\!)}$$

$$\text{MGRec}\quad \frac{\tau \neq (\!|\!)}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu(\!|\!).(\!|\!)}$$

$$\text{MGSum}\quad \frac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \qquad \{(\tau_i = \varnothing \implies \tau_i' = \varnothing) \land (\tau_i \neq \varnothing \implies \tau_i' = (\!|\!))\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \text{ final}}$ $d$ is final

$$\text{FBoxedVal}\quad \frac{d \text{ boxedval}}{d \text{ final}}$$

$$\text{FIndet}\quad \frac{d \text{ indet}}{d \text{ final}}$$

$\boxed{d\ \mathsf{val}}$ $d$ is a value

$$\textsc{VUnit}\ \frac{}{\varnothing\ \mathsf{val}} \qquad \textsc{VLam}\ \frac{}{\lambda x{:}\tau.d\ \mathsf{val}} \qquad \textsc{VRoll}\ \frac{d\ \mathsf{val}}{\mathtt{roll}^{\mu\pi.\tau}(d)\ \mathsf{val}} \qquad \textsc{VInj}\ \frac{d\ \mathsf{val}}{\mathtt{inj}^{\tau}_{\mathbf{C}}(d)\ \mathsf{val}}$$

$\boxed{d\ \mathsf{boxedval}}$ $d$ is a boxed value

$$\textsc{BVVal}\ \frac{d\ \mathsf{val}}{d\ \mathsf{boxedval}} \qquad \textsc{BVRoll}\ \frac{d\ \mathsf{boxedval}}{\mathtt{roll}^{\mu\pi.\tau}(d)\ \mathsf{boxedval}} \qquad \textsc{BVInj}\ \frac{d\ \mathsf{boxedval}}{\mathtt{inj}^{\tau}_{\mathbf{C}}(d)\ \mathsf{boxedval}}$$

$$\textsc{BVArrCast}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d\ \mathsf{boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle\ \mathsf{boxedval}}$$

$$\textsc{BVRecCast}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d\ \mathsf{boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle\ \mathsf{boxedval}} \qquad \textsc{BVSumCast}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i\in\mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i\in\mathcal{C}} \\ \tau \neq \tau' \qquad d\ \mathsf{boxedval}\end{array}}{d\langle \tau \Rightarrow \tau'\rangle\ \mathsf{boxedval}} \qquad \textsc{BVHoleCast}\ \frac{d\ \mathsf{boxedval} \qquad \tau\ \mathsf{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle\ \mathsf{boxedval}}$$

$\boxed{d\ \mathsf{indet}}$ $d$ is indeterminate

$$\textsc{IEHole}\ \frac{}{\llparenthesis\rrparenthesis^u_\sigma\ \mathsf{indet}} \qquad \textsc{INEHole}\ \frac{d\ \mathsf{final}}{\llparenthesis d\rrparenthesis^u_\sigma\ \mathsf{indet}} \qquad \textsc{IApp}\ \frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle \qquad d_1\ \mathsf{indet} \qquad d_2\ \mathsf{final}}{d_1(d_2)\ \mathsf{indet}} \qquad \textsc{IRoll}\ \frac{d\ \mathsf{indet}}{\mathtt{roll}^{\mu\pi.\tau}(d)\ \mathsf{indet}}$$

$$\textsc{IUnroll}\ \frac{d\ \mathsf{indet}}{\mathtt{unroll}(d)\ \mathsf{indet}} \qquad \textsc{IInj}\ \frac{d\ \mathsf{indet}}{\mathtt{inj}^{\tau}_{\mathbf{C}}(d)\ \mathsf{indet}} \qquad \textsc{IInjHole}\ \frac{C \neq \mathbf{C} \qquad d\ \mathsf{final}}{\mathtt{inj}^{\tau}_{C}(d)\ \mathsf{indet}} \qquad \textsc{ICastGroundHole}\ \frac{d\ \mathsf{indet} \qquad \tau\ \mathsf{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle\ \mathsf{indet}}$$

$$\textsc{ICastHoleGround}\ \frac{d \neq d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis\rangle \qquad d\ \mathsf{indet} \qquad \tau\ \mathsf{ground}}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau\rangle\ \mathsf{indet}} \qquad \textsc{ICastArr}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d\ \mathsf{indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle\ \mathsf{indet}} \qquad \textsc{ICastRec}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d\ \mathsf{indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle\ \mathsf{indet}}$$

$$\textsc{ICastSum}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i\in\mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i\in\mathcal{C}} \\ \tau \neq \tau' \qquad d\ \mathsf{indet}\end{array}}{d\langle \tau \Rightarrow \tau'\rangle\ \mathsf{indet}} \qquad \textsc{IFailedCast}\ \frac{d\ \mathsf{final} \qquad \tau_1\ \mathsf{ground} \qquad \tau_2\ \mathsf{ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle\ \mathsf{indet}}$$

$\boxed{d \longrightarrow d'}$ $d$ takes an instruction transition to $d'$

$$\textsc{ITApp}\ \frac{[d_2\ \mathsf{final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1} \qquad \textsc{ITUnroll}\ \frac{[d\ \mathsf{final}]}{\mathtt{unroll}(\mathtt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\textsc{ITAppCast}\ \frac{[d_1\ \mathsf{final}] \qquad [d_2\ \mathsf{final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1\rangle))\langle \tau_2 \Rightarrow \tau_2'\rangle}$$

$$\textsc{ITUnrollCast}\ \frac{[d\ \mathsf{final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\mathtt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \mathtt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle} \qquad \textsc{ITCastId}\ \frac{[d\ \mathsf{final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastSucceed}\quad \frac{[d\ \text{final}] \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow (\!|\!) \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastFail}\quad \frac{[d\ \text{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground}}{d\langle \tau_1 \Rightarrow (\!|\!) \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2\rangle}$$

$$\textsc{ITGround}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow (\!|\!)\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow (\!|\!)\rangle}$$

$$\textsc{ITExpand}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle (\!|\!) \Rightarrow \tau\rangle \longrightarrow d\langle (\!|\!) \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\begin{array}{llll} \mathsf{EvalCtx} & \mathcal{E} & ::= & \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \mathtt{unroll}(\mathcal{E}) \mid \mathtt{inj}_C^{\tau}(\mathcal{E}) \mid (\!|\mathcal{E}|\!)^u_\sigma \mid (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma \\ & & & \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow (\!|\!) \not\Rightarrow \tau\rangle \end{array}$$

$\boxed{d = \mathcal{E}\{d'\}}$ $d$ is obtained by placing $d'$ at the mark in $\mathcal{E}$

$$\textsc{FHOuter}\quad \frac{}{d = \circ\{d\}}$$

$$\textsc{FHApp1}\quad \frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}$$

$$\textsc{FHApp2}\quad \frac{[d_1\ \text{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

$$\textsc{FHRoll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{roll}^{\mu\pi.\tau}(d) = \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FHUnroll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{unroll}(d) = \mathtt{unroll}(\mathcal{E})\{d'\}}$$

$$\textsc{FHInj}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{inj}_C^{\tau}(d) = \mathtt{inj}_C^{\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FNEHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^u_\sigma = (\!|\mathcal{E}|\!)^u_\sigma\{d'\}}$$

$$\textsc{FMHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u\blacktriangleright}_\sigma = (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma\{d'\}}$$

$$\textsc{FHCastInside}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}}$$

$$\textsc{FHFailedCast}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ $d$ steps to $d'$

$$\textsc{Step}\quad \frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$