Algebraic Data Types for Hazel

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1 Syntax

$$
\begin{array}{llcl}
\mathsf{HTyp} & \tau & ::= & \varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid (\!|\,|\!) \mid (\!|\alpha|\!) \\
\mathsf{HTypPat} & \pi & ::= & \alpha \mid (\!|\,|\!) \\
\mathsf{HExp} & e & ::= & \varnothing \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \mathtt{inj}_C(e) \mid \mathtt{roll}(e) \mid \mathtt{unroll}(e) \\
 & & & \mid (\!|\,|\!)^u \mid (\!|e|\!)^u \mid (\!|e|\!)^{u\blacktriangleright} \\
\mathsf{IHExp} & d & ::= & \varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \mathtt{inj}^{\tau}_C(d) \mid \mathtt{roll}^{\mu\alpha.\tau}(d) \mid \mathtt{unroll}(d) \\
 & & & \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow (\!|\,|\!) \nRightarrow \tau \rangle \mid (\!|\,|\!)^u_\sigma \mid (\!|d|\!)^u_\sigma \mid (\!|d|\!)^{u\blacktriangleright}_\sigma \\
\mathsf{HTag} & C & ::= & \mathbf{C} \mid (\!|\,|\!)^u \mid (\!|c|\!)^u
\end{array}
$$

1.1 Context Extension

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = (\!|\,|\!) \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{lcll}
[\tau/(\!|\,|\!)]\tau' & = & \tau' & \\
[\tau/\alpha]\varnothing & = & \varnothing & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu(\!|\,|\!).\tau_2 & = & \mu(\!|\,|\!).[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]{+}\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha](\!|\,|\!) & = & (\!|\,|\!) & \\
[\alpha'/\alpha](\!|\alpha|\!) & = & (\!|\alpha'|\!)^u & \\
[\alpha'/\alpha](\!|\alpha'|\!)^u & = & (\!|\alpha'|\!)^u & \text{when } \alpha \neq \alpha'
\end{array}
$$

$\boxed{\Theta \vdash \tau\ \mathsf{valid}}$ τ is a valid type

$$
\textsc{TVUnit}\ \frac{}{\Theta \vdash \varnothing\ \mathsf{valid}}
\qquad
\textsc{TVArr}\ \frac{\Theta \vdash \tau_1\ \mathsf{valid} \qquad \Theta \vdash \tau_2\ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2\ \mathsf{valid}}
\qquad
\textsc{TVVar}\ \frac{\alpha \in \Theta}{\Theta \vdash \alpha\ \mathsf{valid}}
\qquad
\textsc{TVRec}\ \frac{\Theta, \pi \vdash \tau\ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau\ \mathsf{valid}}
\qquad
\textsc{TVSum}\ \frac{\{\Theta \vdash \tau_i\ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \mathsf{valid}}
$$

$$
\textsc{TVEHole}\ \frac{}{\Theta \vdash (\!|\,|\!)\ \mathsf{valid}}
\qquad
\textsc{TVNEHole}\ \frac{\alpha \notin \Theta}{\Theta \vdash (\!|\alpha|\!)\ \mathsf{valid}}
$$

$\boxed{\tau \sim \tau'}$ τ and τ' are consistent

$$\textsc{TCRefl}\ \frac{}{\tau \sim \tau} \qquad \textsc{TCEHole1}\ \frac{}{(\!|\!) \sim \tau} \qquad \textsc{TCEHole2}\ \frac{}{\tau \sim (\!|\!)} \qquad \textsc{TCNEHole1}\ \frac{}{(\!|\alpha|\!) \sim \tau} \qquad \textsc{TCNEHole2}\ \frac{}{\tau \sim (\!|\alpha|\!)}$$

$$\textsc{TCArr}\ \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

$$\textsc{TCRec}\ \frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \qquad \textsc{TCRecHole1}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu(\!|\!).\tau \sim \mu\alpha.\tau'} \qquad \textsc{TCRecHole2}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu(\!|\!).\tau'}$$

$$\textsc{TCSum}\ \frac{\{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{C\ \mathsf{valid}}$ C is a valid tag

$$\textsc{CVTag}\ \frac{}{\mathbf{C}\ \mathsf{valid}} \qquad \textsc{CVEHole}\ \frac{}{(\!|\!)^u\ \mathsf{valid}}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

Theorem 2 (Consistency Preserves Validity). *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ τ has matched arrow type $\tau_1 \to \tau_2$

$$\textsc{MAHole}\ \frac{}{(\!|\!) \blacktriangleright_{\to} (\!|\!) \to (\!|\!)} \qquad \textsc{MAArr}\ \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ τ has matched recursive type $\mu\pi.\tau'$

$$\textsc{MRRec}\ \frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau} \qquad \textsc{MRHole}\ \frac{}{(\!|\!) \blacktriangleright_{\mu} \mu(\!|\!).(\!|\!)}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ e synthesizes type τ

$$\textsc{SUnit}\ \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing} \qquad \textsc{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \textsc{SVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\!)}$$

$$\textsc{SLam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}$$

$$\textsc{SLamInvalid}\ \frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma, x : (\!|\!) \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow (\!|\!) \to \tau'}$$

$$\textsc{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau}$$

$$\textsc{SAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim (\!|\!) \to (\!|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\!)}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\!)}$$

$$\textsc{SAsc}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \qquad \textsc{SAscInvalid}\ \frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash e : \tau \Rightarrow (\!|\!)}$$

$$\textsc{SRollErr}\ \frac{\Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Rightarrow \mu(\!|\!).(\!|\!)} \qquad \textsc{SUnroll}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$$

$$\textsc{SUnrollNotRec}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu(\!|\!).(\!|\!)}{\Gamma \vdash \mathtt{unroll}\big((\!|e|\!)^{u\blacktriangleright}\big) \Rightarrow (\!|\!)}$$

$$\textsc{SInjErr}\ \frac{C\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash (\!|\mathtt{inj}_C(e)|\!)^u \Rightarrow (\!|\!)} \qquad \textsc{SInjTagErr}\ \frac{\Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash \mathtt{inj}_{(\!|c|\!)^u}(e) \Rightarrow (\!|\!)}$$

$$\textsc{SEHole}\ \frac{}{\Gamma \vdash (\!|\!)^u \Rightarrow (\!|\!)} \qquad \textsc{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\!)}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

$$\textsc{ARoll}\quad \frac{\tau \blacktriangleright_{\mu} \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau}$$

$$\textsc{ARollNotRec}\quad \frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\texttt{roll}(e)\rrparenthesis^{u} \Leftarrow \tau}$$

$$\textsc{AInjHole}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{inj}_{C}(e) \Leftarrow \llparenthesis\rrparenthesis}$$

$$\textsc{AInj}\quad \frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjTagErr}\quad \frac{\llparenthesis c\rrparenthesis^{u} \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{inj}_{\llparenthesis c\rrparenthesis^{u}}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjBadTag}\quad \frac{C\ \mathsf{valid} \qquad C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\texttt{inj}_{C}(e)\rrparenthesis^{u} \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjUnexpectedBody}\quad \frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\texttt{inj}_{C_j}(e)\rrparenthesis^{u} \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjExpectedBody}\quad \frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash \llparenthesis\texttt{inj}_{C_j}(\varnothing)\rrparenthesis^{u} \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{ASubsume}\quad \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity). *If* $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

$$\textsc{ESUnit}\quad \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \rightsquigarrow \varnothing \dashv \emptyset}$$

$$\textsc{ESVar}\quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

$$\textsc{ESVarFree}\quad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis x\rrparenthesis^{u}_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESLam}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESLamInvalid}\quad \frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma, x : \llparenthesis\rrparenthesis \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \llparenthesis\rrparenthesis \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESApp}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_1\rangle)(d_2\langle \tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\textsc{ESAppNotArr}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d_1\rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow \llparenthesis\rrparenthesis\rangle) \dashv \Delta_1 \cup \Delta_2, u :: \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESAsc}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau\rangle \dashv \Delta}$$

$$\textsc{ESAscInvalid}\quad \frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow d\langle \tau' \Rightarrow \llparenthesis\rrparenthesis\rangle \dashv \Delta}$$

$$\textsc{ESRollErr}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash \llparenthesis\texttt{roll}(e)\rrparenthesis^{u} \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis\texttt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle)\rrparenthesis^{u}_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESUnroll}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \texttt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \texttt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}$$

ESUnrollNotRec

$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \nsim \mu (\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash \mathtt{unroll}\left((\!|e|\!)^{u\blacktriangleright}\right) \Rightarrow (\!|\,|\!) \rightsquigarrow \mathtt{unroll}\left((\!|d|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\right) \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}$$

ESInjErr

$$\frac{C\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash (\!|\mathtt{inj}_C(e)|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|\mathtt{inj}^{\tau'}_C(d\langle\tau \Rightarrow (\!|\,|\!)\rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: (\!|\,|\!)[\Gamma]}$$

ESInjTagErr

$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{(\!|c|\!)^u(\tau)\}}{\Gamma \vdash \mathtt{inj}_{(\!|c|\!)^u}(e) \Rightarrow (\!|\,|\!) \rightsquigarrow \mathtt{inj}^{\tau'}_{(\!|c|\!)^u}(d\langle\tau \Rightarrow (\!|\,|\!)\rangle) \dashv \Delta}$$

ESEHole

$$\frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|\,|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\,|\!)[\Gamma]}$$

ESNEHole

$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: (\!|\,|\!)[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

EARoll

$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau \rightsquigarrow \mathtt{roll}^{\mu\pi.\tau'}(d\langle\tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}$$

EARollNotRec

$$\frac{\tau \nsim \mu(\!|\,|\!).(\!|\,|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\mathtt{roll}^{\mu(\!|\,|\!).(\!|\,|\!)}(d)|\!)^u_{\mathsf{id}(\Gamma)} : \mu(\!|\,|\!).(\!|\,|\!) \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}$$

EAInjHole

$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow (\!|\,|\!) \rightsquigarrow \mathtt{inj}^{\tau'}_C(d) : \tau' \dashv \Delta}$$

EAInj

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \rightsquigarrow d : \tau'_j \dashv \Delta}{\Gamma \vdash \mathtt{inj}_{C_j}(e) \Leftarrow \tau \rightsquigarrow \mathtt{inj}^{\tau}_{C_j}\left(d\langle\tau'_j \Rightarrow \tau_j\rangle\right) : \tau \dashv \Delta}$$

EAInjTagErr

$$\frac{(\!|c|\!)^u \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{(\!|c|\!)^u(\tau)\}\right\}}{\Gamma \vdash \mathtt{inj}_{(\!|c|\!)^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \rightsquigarrow \mathtt{inj}^{\tau'}_{(\!|c|\!)^u}(d\langle\tau \Rightarrow (\!|\,|\!)\rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta}$$

EAInjBadTag

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C \notin \mathcal{C} \qquad C\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau' \dashv \Delta \qquad \tau'' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{C(\tau')\}\right\}}{\Gamma \vdash (\!|\mathtt{inj}_C(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\mathtt{inj}^{\tau''}_C(d)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjUnexpectedBody

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau_j \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\right\}\end{array}}{\Gamma \vdash (\!|\mathtt{inj}_{C_j}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\mathtt{inj}^{\tau'}_{C_j}(d)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjExpectedBody

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\right\}}{\Gamma \vdash (\!|\mathtt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\mathtt{inj}^{\tau'}_{C_j}(\varnothing)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EASubsume

$$\frac{e \neq (\!|\,|\!)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

EAEHole

$$\frac{}{\Gamma \vdash (\!|\,|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\,|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$$\textsc{EANEHole}\quad \dfrac{\Gamma \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

$$\textsc{TAUnit}\quad \dfrac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

$$\textsc{TAVar}\quad \dfrac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

$$\textsc{TALam}\quad \dfrac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}$$

$$\textsc{TALamInvalid}\quad \dfrac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Delta; \Gamma, x : \llparenthesis\rrparenthesis \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \llparenthesis\rrparenthesis \to \tau'}$$

$$\textsc{TAApp}\quad \dfrac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

$$\textsc{TARoll}\quad \dfrac{\emptyset \vdash \mu\pi.\tau\ \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}$$

$$\textsc{TAUnroll}\quad \dfrac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

$$\textsc{TAInj}\quad \dfrac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(d) : \tau}$$

$$\textsc{TAEHole}\quad \dfrac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis\rrparenthesis^u_\sigma : \tau}$$

$$\textsc{TANEHole}\quad \dfrac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^u_\sigma : \tau}$$

$$\textsc{TAMHole}\quad \dfrac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^{u\blacktriangleright}_\sigma : \tau}$$

$$\textsc{TACast}\quad \dfrac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

$$\textsc{TAFailedCast}\quad \dfrac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle : \tau_2}$$

3 Dynamic Semantics

$\boxed{\tau\ \text{ground}}$ τ is a ground type

$$\textsc{GArr}\quad \dfrac{}{\llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis\ \text{ground}}$$

$$\textsc{GRec}\quad \dfrac{}{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis\ \text{ground}}$$

$$\textsc{GSum}\quad \dfrac{\{\tau_i = \varnothing \vee \tau_i = \llparenthesis\rrparenthesis\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \text{ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

$$\textsc{MGArr}\quad \dfrac{\tau_1 \to \tau_2 \neq \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}$$

$$\textsc{MGRec}\quad \dfrac{\tau \neq \llparenthesis\rrparenthesis}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$$\textsc{MGSum}\quad \dfrac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \qquad \{(\tau_i = \varnothing \implies \tau_i' = \varnothing) \wedge (\tau_i \neq \varnothing \implies \tau_i' = \llparenthesis\rrparenthesis)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \text{ final}}$ d is final

$$\textsc{FBoxedVal}\ \frac{d \text{ boxedval}}{d \text{ final}} \qquad \textsc{FIndet}\ \frac{d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$ d is a value

$$\textsc{VUnit}\ \frac{}{\varnothing \text{ val}} \qquad \textsc{VLam}\ \frac{}{\lambda x{:}\tau.d \text{ val}} \qquad \textsc{VRoll}\ \frac{d \text{ val}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ val}} \qquad \textsc{VInj}\ \frac{d \text{ val}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ val}}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

$$\textsc{BVVal}\ \frac{d \text{ val}}{d \text{ boxedval}} \qquad \textsc{BVRoll}\ \frac{d \text{ boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}} \qquad \textsc{BVInj}\ \frac{d \text{ boxedval}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ boxedval}}$$

$$\textsc{BVArrCast}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}$$

$$\textsc{BVRecCast}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}} \qquad \textsc{BVSumCast}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \qquad d \text{ boxedval}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}} \qquad \textsc{BVHoleCast}\ \frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\,|\!) \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$ d is indeterminate

$$\textsc{IEHole}\ \frac{}{(\!|\,|\!)^u_\sigma \text{ indet}} \qquad \textsc{INEHole}\ \frac{d \text{ final}}{(\!|d|\!)^u_\sigma \text{ indet}} \qquad \textsc{IApp}\ \frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \qquad d_1 \text{ indet} \qquad d_2 \text{ final}}{d_1(d_2) \text{ indet}} \qquad \textsc{IRoll}\ \frac{d \text{ indet}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ indet}}$$

$$\textsc{IUnroll}\ \frac{d \text{ indet}}{\texttt{unroll}(d) \text{ indet}} \qquad \textsc{IInj}\ \frac{d \text{ indet}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ indet}} \qquad \textsc{IInjHole}\ \frac{C \neq \mathbf{C} \qquad d \text{ final}}{\texttt{inj}^{\tau}_{C}(d) \text{ indet}} \qquad \textsc{ICastGroundHole}\ \frac{d \text{ indet} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\,|\!) \rangle \text{ indet}}$$

$$\textsc{ICastHoleGround}\ \frac{d \neq d'\langle \tau' \Rightarrow (\!|\,|\!) \rangle \qquad d \text{ indet} \qquad \tau \text{ ground}}{d\langle (\!|\,|\!) \Rightarrow \tau \rangle \text{ indet}} \qquad \textsc{ICastArr}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ indet}} \qquad \textsc{ICastRec}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}}$$

$$\textsc{ICastSum}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \qquad d \text{ indet}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}} \qquad \textsc{IFailedCast}\ \frac{d \text{ final} \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2 \rangle \text{ indet}}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\textsc{ITApp}\ \frac{[d_2 \text{ final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1} \qquad \textsc{ITUnroll}\ \frac{[d \text{ final}]}{\texttt{unroll}(\texttt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\textsc{ITAppCast}\ \frac{[d_1 \text{ final}] \qquad [d_2 \text{ final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2' \rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1 \rangle))\langle \tau_2 \Rightarrow \tau_2' \rangle}$$

$$\textsc{ITUnrollCast}\quad \frac{[d\ \text{final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\texttt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \texttt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle}$$

$$\textsc{ITCastId}\quad \frac{[d\ \text{final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastSucceed}\quad \frac{[d\ \text{final}] \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow (\!|\!) \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastFail}\quad \frac{[d\ \text{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground}}{d\langle \tau_1 \Rightarrow (\!|\!) \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2\rangle}$$

$$\textsc{ITGround}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow (\!|\!)\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow (\!|\!)\rangle}$$

$$\textsc{ITExpand}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle (\!|\!) \Rightarrow \tau\rangle \longrightarrow d\langle (\!|\!) \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\begin{aligned}\mathsf{EvalCtx}\ \ \mathcal{E} \ ::=\ & \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \texttt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \texttt{unroll}(\mathcal{E}) \mid \texttt{inj}_C^{\tau}(\mathcal{E}) \mid (\!|\mathcal{E}|\!)^u_\sigma \mid (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma \\ & \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow (\!|\!) \not\Rightarrow \tau\rangle\end{aligned}$$

$\boxed{d = \mathcal{E}\{d'\}}$ d is obtained by placing d' at the mark in $\mathcal{E}$

$$\textsc{FHOuter}\quad \frac{}{d = \circ\{d\}}$$

$$\textsc{FHApp1}\quad \frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}$$

$$\textsc{FHApp2}\quad \frac{[d_1\ \text{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

$$\textsc{FHRoll}\quad \frac{d = \mathcal{E}\{d'\}}{\texttt{roll}^{\mu\pi.\tau}(d) = \texttt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FHUnroll}\quad \frac{d = \mathcal{E}\{d'\}}{\texttt{unroll}(d) = \texttt{unroll}(\mathcal{E})\{d'\}}$$

$$\textsc{FHInj}\quad \frac{d = \mathcal{E}\{d'\}}{\texttt{inj}_C^{\tau}(d) = \texttt{inj}_C^{\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FNEHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^u_\sigma = (\!|\mathcal{E}|\!)^u_\sigma\{d'\}}$$

$$\textsc{FMHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u\blacktriangleright}_\sigma = (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma\{d'\}}$$

$$\textsc{FHCastInside}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}}$$

$$\textsc{FHFailedCast}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ d steps to d'

$$\textsc{Step}\quad \frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$