

# Algebraic Data Types for Hazel

Eric Griffis  
egriffis@umich.edu

## 1 Syntax

HTyp	$\tau$	$::= \tau \rightarrow \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \mid \langle \rangle \mid \langle \alpha \rangle^u$
HTypPat	$\pi$	$::= \alpha \mid \langle \rangle$
HExp	$e$	$::= x \mid \lambda x:\tau.e \mid e(e) \mid e : \tau \mid \text{inj}_C(E) \mid \text{roll}(e) \mid \text{unroll}(e) \mid \langle \rangle^u \mid \langle e \rangle^u \mid \langle e \rangle^u \blacktriangleright$
HTag	$C$	$::= \mathbf{C} \mid ?^u$
HTagTyp	$T$	$::= \tau \mid \emptyset$
HTagArg	$E$	$::= e \mid \emptyset$
IHExp	$d$	$::= x \mid \lambda x:\tau.d \mid d(d) \mid \text{inj}_C^\tau(D) \mid \text{roll}^{\mu\alpha.\tau}(d) \mid \text{unroll}(d) \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow \langle \rangle \not\Rightarrow \tau \rangle \mid \langle \rangle_\sigma^u \mid \langle d \rangle_\sigma^u \mid \langle d \rangle_\sigma^u \blacktriangleright$
IHTagArg	$D$	$::= d \mid \emptyset$

### 1.1 Context Extension

We write  $\Gamma, X : T$  to denote the extension of typing context  $\Gamma$  with optional variable  $X$  of optional type  $T$ .

$$\Gamma, X : T = \begin{cases} \Gamma, x : \tau & X = x \wedge T = \tau \\ \Gamma, x : \langle \rangle & X = x \wedge T = \emptyset \\ \Gamma & X = \emptyset \end{cases}$$

We write  $\Theta, \pi$  to denote the extension of type variable context  $\Theta$  with optional type variable name  $\pi$ .

$$\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \langle \rangle \end{cases}$$

## 2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$   $\tau''$  is obtained by substituting  $\tau$  for  $\pi$  in  $\tau'$

$[\tau/\langle \rangle]\tau'$	$=$	$\tau'$	
$[\tau/\alpha](\tau_1 \rightarrow \tau_2)$	$=$	$[\tau/\alpha]\tau_1 \rightarrow [\tau/\alpha]\tau_2$	
$[\tau/\alpha]\alpha$	$=$	$\tau$	
$[\tau/\alpha]\alpha_1$	$=$	$\tau'$	when $\alpha \neq \alpha_1$
$[\tau/\alpha]\mu\alpha_1.\tau_2$	$=$	$\mu\alpha_1.[\tau/\alpha]\tau_2$	when $\alpha \neq \alpha_1$ and $\alpha_1 \notin \text{FV}(\tau)$
$[\tau/\alpha]\mu\langle \rangle.\tau_2$	$=$	$\mu\langle \rangle.[\tau/\alpha]\tau_2$	
$[\tau/\alpha]+\{C_i(T_i)\}_{C_i \in \mathcal{C}}$	$=$	$+\{C_i([\tau/\alpha]T_i)\}_{C_i \in \mathcal{C}}$	
$[\tau/\alpha]\langle \rangle$	$=$	$\langle \rangle$	
$[\alpha'/\alpha]\langle \alpha \rangle^u$	$=$	$\langle \alpha' \rangle^u$	
$[\alpha'/\alpha]\langle \alpha_1 \rangle^u$	$=$	$\langle \alpha_1 \rangle^u$	when $\alpha \neq \alpha_1$

$\boxed{[\tau/\pi]T = \tau'}$   $\tau'$  is obtained by substituting  $\tau$  for  $\pi$  in  $T$

$$[\tau/\pi]T = \begin{cases} [\tau/\pi]\tau' & \text{when } T = \tau' \\ \emptyset & \text{when } T = \emptyset \end{cases}$$

$\boxed{\Theta \vdash \tau \text{ valid}}$   $\tau$  is a valid type

$$\begin{array}{c} \text{TVARR} \\ \frac{\Theta \vdash \tau_1 \text{ valid} \quad \Theta \vdash \tau_2 \text{ valid}}{\Theta \vdash \tau_1 \rightarrow \tau_2 \text{ valid}} \end{array} \quad \begin{array}{c} \text{TVVAR} \\ \frac{\alpha \in \Theta}{\Theta \vdash \alpha \text{ valid}} \end{array} \quad \begin{array}{c} \text{TVREC} \\ \frac{\Theta, \pi \vdash \tau \text{ valid}}{\Theta \vdash \mu\pi.\tau \text{ valid}} \end{array} \quad \begin{array}{c} \text{TVSUM} \\ \frac{\{\Theta \vdash T_i \text{ valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \text{ valid}} \end{array} \quad \begin{array}{c} \text{TVEHOLE} \\ \frac{}{\Theta \vdash \emptyset \text{ valid}} \end{array}$$

$$\begin{array}{c} \text{TVNEHOLE} \\ \frac{\alpha \notin \Theta}{\Theta \vdash \langle \alpha \rangle^u \text{ valid}} \end{array}$$

$\boxed{\Theta \vdash T \text{ valid}}$   $T$  is a valid optional type

$$\begin{array}{c} \text{TVSOME} \\ \frac{T = \tau \quad \Theta \vdash \tau \text{ valid}}{\Theta \vdash T \text{ valid}} \end{array} \quad \begin{array}{c} \text{TVNONE} \\ \frac{}{\Theta \vdash \emptyset \text{ valid}} \end{array}$$

$\boxed{\tau \sim \tau'}$   $\tau$  and  $\tau'$  are consistent

$$\begin{array}{c} \text{TCREFL} \\ \frac{}{\tau \sim \tau} \end{array} \quad \begin{array}{c} \text{TCEHOLE1} \\ \frac{}{\emptyset \sim \tau} \end{array} \quad \begin{array}{c} \text{TCEHOLE2} \\ \frac{}{\tau \sim \emptyset} \end{array} \quad \begin{array}{c} \text{TCNEHOLE1} \\ \frac{}{\langle \alpha \rangle^u \sim \tau} \end{array} \quad \begin{array}{c} \text{TCNEHOLE2} \\ \frac{}{\tau \sim \langle \alpha \rangle^u} \end{array} \quad \begin{array}{c} \text{TCARR} \\ \frac{\tau_1 \sim \tau'_1 \quad \tau_2 \sim \tau'_2}{\tau_1 \rightarrow \tau_2 \sim \tau'_1 \rightarrow \tau'_2} \end{array}$$

$$\begin{array}{c} \text{TCREC} \\ \frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \end{array} \quad \begin{array}{c} \text{TCRECHOLE1} \\ \frac{\alpha \notin \text{FV}(\tau) \quad \tau \sim \tau'}{\mu\emptyset.\tau \sim \mu\alpha.\tau'} \end{array} \quad \begin{array}{c} \text{TCRECHOLE2} \\ \frac{\alpha \notin \text{FV}(\tau) \quad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu\emptyset.\tau'} \end{array} \quad \begin{array}{c} \text{TCSUM} \\ \frac{\{T_i \sim T'_i\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(T'_i)\}_{C_i \in \mathcal{C}}} \end{array}$$

$\boxed{T \sim T'}$   $T$  and  $T'$  are consistent

$$\begin{array}{c} \text{TCSOME} \\ \frac{\tau \sim \tau'}{\tau \sim \tau'} \end{array} \quad \begin{array}{c} \text{TCNONE} \\ \frac{}{\emptyset \sim \emptyset} \end{array}$$

## 2.1 Bidirectional Typing

We call  $[\mu\pi.\tau/\pi]\tau$  the *unrolling* of recursive type  $\mu\pi.\tau$ .

**Theorem 1** (Synthetic Type Validity). *If  $\Gamma \vdash e \Rightarrow \tau$  then  $\emptyset \vdash \tau \text{ valid}$ .*

**Theorem 2** (Consistency Preserves Validity). *If  $\Theta \vdash \tau \text{ valid}$  and  $\tau \sim \tau'$  then  $\Theta \vdash \tau' \text{ valid}$ .*

$\boxed{\tau \blacktriangleright \rightarrow \tau_1 \rightarrow \tau_2}$   $\tau$  has matched arrow type  $\tau_1 \rightarrow \tau_2$

$$\begin{array}{c} \text{MAHOLE} \\ \frac{}{\emptyset \blacktriangleright \rightarrow \emptyset \rightarrow \emptyset} \end{array} \quad \begin{array}{c} \text{MAARR} \\ \frac{}{\tau_1 \rightarrow \tau_2 \blacktriangleright \rightarrow \tau_1 \rightarrow \tau_2} \end{array}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$   $\tau$  has matched recursive type  $\mu\pi.\tau'$

$$\frac{\text{MRREC}}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau}$$

$$\frac{\text{MRHOLE}}{\emptyset \blacktriangleright_{\mu} \mu(\emptyset).\emptyset}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$   $e$  synthesizes type  $\tau$

$$\frac{\text{SVAR} \quad x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau}$$

$$\frac{\text{SVARFREE} \quad x \notin \text{dom}(\Gamma)}{\Gamma \vdash \langle x \rangle^u \Rightarrow \emptyset}$$

$$\frac{\text{SLAM} \quad \emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \tau \rightarrow \tau'}$$

$$\frac{\text{SAPP} \quad \Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright_{\rightarrow} \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau}$$

$$\frac{\text{SAPPNOTARR} \quad \Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \approx \emptyset \rightarrow \emptyset \quad \Gamma \vdash e_2 \Leftarrow \emptyset}{\Gamma \vdash \langle e_1(e_2) \rangle^u \Rightarrow \emptyset}$$

$$\frac{\text{SASC} \quad \emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau}$$

$$\frac{\text{SROLLError} \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash \langle \text{roll}(e) \rangle^u \Rightarrow \mu(\emptyset).\emptyset}$$

$$\frac{\text{SUNROLL} \quad \Gamma \vdash e \Rightarrow \tau \quad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \text{unroll}(e) \Rightarrow [\mu\pi.\tau' / \pi]\tau'}$$

$$\frac{\text{SUNROLLNOTREC} \quad \Gamma \vdash e \Rightarrow \tau \quad \tau \approx \mu(\emptyset).\emptyset}{\Gamma \vdash \text{unroll}(\langle e \rangle^{u\blacktriangleright}) \Rightarrow \emptyset}$$

$$\frac{\text{SINJERROR} \quad \Gamma \vdash E \text{ valid}}{\Gamma \vdash \langle \text{inj}_C(E) \rangle^u \Rightarrow \emptyset}$$

$$\frac{\text{SEHOLE}}{\Gamma \vdash \emptyset^u \Rightarrow \emptyset}$$

$$\frac{\text{SNEHOLE} \quad \Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash \langle e \rangle^u \Rightarrow \emptyset}$$

$$\frac{\text{SMHOLE} \quad \Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash \langle e \rangle^{u\blacktriangleright} \Rightarrow \emptyset}$$

$\boxed{\Gamma \vdash E \text{ valid}}$   $E$  is a valid optional expression

$$\frac{\text{EVSOME} \quad \Gamma \vdash e \Leftarrow \emptyset}{\Gamma \vdash e \text{ valid}}$$

$$\frac{\text{EVNONE}}{\Gamma \vdash \emptyset \text{ valid}}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$      $e$  analyzes against type  $\tau$

$$\frac{\text{AROLL} \quad \tau \blacktriangleright_{\mu} \mu\pi.\tau' \quad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \text{roll}(e) \Leftarrow \tau}$$

$$\frac{\text{AROLLNOTREC} \quad \tau \approx \mu\langle\emptyset\rangle.\langle\emptyset\rangle \quad \Gamma \vdash e \Leftarrow \langle\emptyset\rangle}{\Gamma \vdash \text{roll}(\langle e \rangle^{u\blacktriangleright}) \Leftarrow \tau}$$

$$\frac{\text{AINJHOLE} \quad \Gamma \vdash E \text{ valid}}{\Gamma \vdash \text{inj}_C(E) \Leftarrow \langle\emptyset\rangle}$$

$$\frac{\text{AINJ} \quad C_j \in \mathcal{C} \quad \Gamma \vdash E \Leftarrow T_j}{\Gamma \vdash \text{inj}_{C_j}(E) \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\frac{\text{AINJUNEXPECTEDBODY} \quad C_j \in \mathcal{C} \quad T_j = \emptyset \quad \Gamma \vdash e \Leftarrow \langle\emptyset\rangle}{\Gamma \vdash \langle \text{inj}_{C_j}(e) \rangle^u \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\frac{\text{AINJEXPECTEDBODY} \quad C_j \in \mathcal{C} \quad T_j = \tau}{\Gamma \vdash \langle \text{inj}_{C_j}(\emptyset) \rangle^u \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\frac{\text{AINJBADTAG} \quad C \notin \mathcal{C} \quad \Gamma \vdash E \text{ valid}}{\Gamma \vdash \langle \text{inj}_C(E) \rangle^u \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\frac{\text{ASUBSUME} \quad \Gamma \vdash e \Rightarrow \tau' \quad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

$\boxed{\Gamma \vdash E \Leftarrow T}$      $E$  analyzes against optional type  $T$

$$\frac{\text{ASOME} \quad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e \Leftarrow \tau}$$

$$\frac{\text{ANONE}}{\Gamma \vdash \emptyset \Leftarrow \emptyset}$$

## 2.2 Typed Elaboration

**Theorem 3** (Synthetic Typed Elaboration Validity). *If  $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$  then  $\emptyset \vdash \tau$  valid.*

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$      $e$  synthesizes type  $\tau$  and elaborates to  $d$

$$\begin{array}{c}
\text{ESVAR} \quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset} \quad \text{ESVARFREE} \quad \frac{x \notin \text{dom}(\Gamma)}{\Gamma \vdash \langle x \rangle^u \Rightarrow \langle \rangle \rightsquigarrow \langle x \rangle_{\text{id}(\Gamma)}^u \dashv u :: \langle \rangle [\Gamma]} \quad \text{ESLAM} \quad \frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x : \tau. e \Rightarrow \tau \rightsquigarrow \tau' \rightsquigarrow \lambda x : \tau. d \dashv \Delta} \\
\\
\text{ESAPP} \quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright \rightarrow \tau_2 \rightarrow \tau \quad \Gamma \vdash e_1 \Leftarrow \tau_2 \rightarrow \tau \rightsquigarrow d_1 : \tau'_1 \dashv \Delta_1 \quad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau'_2 \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1 \langle \tau'_1 \Rightarrow \tau_2 \rightarrow \tau \rangle)(d_2 \langle \tau'_2 \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2} \\
\\
\text{ESAPPNOTARR} \quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \quad \tau_1 \approx \langle \rangle \rightarrow \langle \rangle \quad \Gamma \vdash e_2 \Leftarrow \langle \rangle \rightsquigarrow d_2 : \tau'_2 \dashv \Delta_2}{\Gamma \vdash \langle e_1 \rangle^{u \blacktriangleright}(e_2) \Rightarrow \langle \rangle \rightsquigarrow \langle d_1 \rangle_{\text{id}(\Gamma)}^{u \blacktriangleright}(d_2 \langle \tau'_2 \Rightarrow \langle \rangle \rangle) \dashv \Delta_1 \cup \Delta_2, u :: \langle \rangle \rightarrow \langle \rangle [\Gamma]} \\
\\
\text{ESASC} \quad \frac{\emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d \langle \tau' \Rightarrow \tau \rangle \dashv \Delta} \\
\\
\text{ESROLLERROR} \quad \frac{\Gamma \vdash e \Leftarrow \langle \rangle \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash \langle \text{roll}(e) \rangle^u \Rightarrow \mu \langle \rangle . \langle \rangle \rightsquigarrow \langle \text{roll}^{\mu \langle \rangle . \langle \rangle}(d \langle \tau \Rightarrow \langle \rangle \rangle) \rangle_{\text{id}(\Gamma)}^u \dashv \Delta, u :: \mu \langle \rangle . \langle \rangle [\Gamma]} \\
\\
\text{ESUNROLL} \quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \quad \tau \blacktriangleright_{\mu} \mu \pi . \tau'}{\Gamma \vdash \text{unroll}(e) \Rightarrow [\mu \pi . \tau' / \pi] \tau' \rightsquigarrow \text{unroll}(d \langle \tau \Rightarrow \mu \pi . \tau' \rangle) \dashv \Delta} \\
\\
\text{ESUNROLLNOTREC} \quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \quad \tau \approx \mu \langle \rangle . \langle \rangle}{\Gamma \vdash \text{unroll}(\langle e \rangle^{u \blacktriangleright}) \Rightarrow \langle \rangle \rightsquigarrow \text{unroll}(\langle d \rangle_{\text{id}(\Gamma)}^{u \blacktriangleright}) \dashv \Delta, u :: \mu \langle \rangle . \langle \rangle [\Gamma]} \\
\\
\text{ESINJERROR} \quad \frac{\Gamma \vdash E \Leftarrow \langle \rangle \rightsquigarrow D : T \dashv \Delta \quad T = \text{frob}(E)}{\Gamma \vdash \langle \text{inj}_C(E) \rangle^u \Rightarrow +\{C(T)\} \rightsquigarrow \langle \text{inj}_C^{+\{C(T)\}}(D \langle T \Rightarrow \langle \rangle \rangle) \rangle_{\text{id}(\Gamma)}^u \dashv \Delta, u :: +\{C(T)\}[\Gamma]} \\
\\
\text{ESEHOLE} \quad \frac{}{\Gamma \vdash \langle \rangle^u \Rightarrow \langle \rangle \rightsquigarrow \langle \rangle_{\text{id}(\Gamma)}^u \dashv u :: \langle \rangle [\Gamma]} \quad \text{ESNEHOLE} \quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle e \rangle^u \Rightarrow \langle \rangle \rightsquigarrow \langle d \rangle_{\text{id}(\Gamma)}^u \dashv \Delta, u :: \langle \rangle [\Gamma]}
\end{array}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$      $e$  analyzes against type  $\tau_1$  and elaborates to  $d$  of consistent type  $\tau_2$

$$\frac{\text{EAROLL} \quad \tau \blacktriangleright_{\mu} \mu\pi.\tau' \quad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \text{roll}(e) \Leftarrow \tau \rightsquigarrow \text{roll}^{\mu\pi.\tau'}(d\langle\tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}$$

$$\frac{\text{EAROLLNOTREC} \quad \tau \approx \mu(\emptyset).\emptyset \quad \Gamma \vdash e \Leftarrow \emptyset \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash \langle\text{roll}(e)\rangle^u \Leftarrow \tau \rightsquigarrow \text{roll}^{\mu\emptyset.\emptyset}(\langle d \rangle_{\text{id}(\Gamma)}^u \blacktriangleright_{\mu}) : \mu\emptyset.\emptyset \dashv \Delta, u :: \mu\emptyset.\emptyset[\Gamma]}$$

$$\frac{\text{EAINJHOLE} \quad \Gamma \vdash E \Leftarrow \emptyset \rightsquigarrow D : T \dashv \Delta \quad \tau = +\{C(T)\}}{\Gamma \vdash \text{inj}_C(E) \Leftarrow \emptyset \rightsquigarrow \text{inj}_C^{\tau}(D) : \tau \dashv \Delta}$$

$$\frac{\text{EAINJ} \quad \tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \Gamma \vdash E \Leftarrow T_j \rightsquigarrow D : T'_j \dashv \Delta}{\Gamma \vdash \text{inj}_{C_j}(E) \Leftarrow \tau \rightsquigarrow \text{inj}_{C_j}^{T'_j}(D\langle T'_j \Rightarrow T_j \rangle) : \tau \dashv \Delta}$$

$$\frac{\text{EAINJUNEXPECTEDBODY} \quad \tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad T_j = \emptyset \quad \Gamma \vdash e \Leftarrow \emptyset \rightsquigarrow d : \tau_j \dashv \Delta \quad \tau' = +\left\{\{C_i(T_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\right\}}{\Gamma \vdash \langle\text{inj}_{C_j}(e)\rangle^u \Leftarrow \tau \rightsquigarrow \langle\text{inj}_{C_j}^{T'_j}(d)\rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\frac{\text{EAINJEXPECTEDBODY} \quad \tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad T_j = \tau_j \quad \tau' = +\left\{\{C_i(T_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\emptyset)\}\right\}}{\Gamma \vdash \langle\text{inj}_{C_j}(\emptyset)\rangle^u \Leftarrow \tau \rightsquigarrow \langle\text{inj}_{C_j}^{T'_j}(\emptyset)\rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\frac{\text{EAINJBADTAG} \quad \tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \quad C \notin \mathcal{C} \quad \Gamma \vdash E \Leftarrow \emptyset \rightsquigarrow D : T \dashv \Delta \quad \tau' = +\left\{\{C_i(T_i)\}_{C_i \in \mathcal{C}} \cup \{C(T)\}\right\}}{\Gamma \vdash \langle\text{inj}_C(E)\rangle^u \Leftarrow \tau \rightsquigarrow \langle\text{inj}_C^{T'_j}(D)\rangle_{\text{id}(\Gamma)}^u : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\frac{\text{EASUBSUME} \quad e \neq \emptyset^u \quad e \neq \langle e' \rangle^u \quad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \quad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

$$\frac{\text{EAEHOLE}}{\Gamma \vdash \emptyset^u \Leftarrow \tau \rightsquigarrow \emptyset_{\text{id}(\Gamma)}^u : \tau \dashv u :: \tau[\Gamma]}$$

$$\frac{\text{EANEHOLE} \quad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle e \rangle^u \Leftarrow \tau \rightsquigarrow \langle d \rangle_{\text{id}(\Gamma)}^u : \tau \dashv u :: \tau[\Gamma]}$$

$$\frac{\text{EAMHOLE} \quad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle e \rangle^{u\blacktriangleright} \Leftarrow \tau \rightsquigarrow \langle d \rangle_{\text{id}(\Gamma)}^{u\blacktriangleright} : \tau \dashv u :: \tau[\Gamma]}$$

$\boxed{\Gamma \vdash E \Leftarrow T_1 \rightsquigarrow D : T_2 \dashv \Delta}$      $E$  analyzes against optional type  $T_1$  and elaborates to  $D$  of consistent optional type  $T_2$

$$\frac{\text{EASOME} \quad \Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$$

$$\frac{\text{EANONE}}{\Gamma \vdash \emptyset \Leftarrow \emptyset \rightsquigarrow \emptyset : \emptyset \dashv \emptyset}$$

### 2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$   $d$  is assigned type  $\tau$

$$\begin{array}{c}
\text{TAVAR} \quad \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau} \quad \text{TALAM} \quad \frac{\Delta; \Gamma, x : \tau_1 \vdash d : \tau_2}{\Delta; \Gamma \vdash \lambda x : \tau_1. d : \tau_1 \rightarrow \tau_2} \quad \text{TAAAPP} \quad \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \rightarrow \tau \quad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau} \\
\\
\text{TAROLL} \quad \frac{\emptyset \vdash \mu\pi.\tau \text{ valid} \quad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \text{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau} \quad \text{TAUNROLL} \quad \frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \text{unroll}(d) : [\mu\pi.\tau/\pi]\tau} \\
\\
\text{TAINJ} \quad \frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \quad C_j \in \mathcal{C} \quad \Delta; \Gamma \vdash D : T_j}{\Delta; \Gamma \vdash \text{inj}_{C_j}^\tau(D) : \tau} \quad \text{TAEHOLE} \quad \frac{u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llbracket d \rrbracket_\sigma^u : \tau} \\
\\
\text{TANEHOLE} \quad \frac{\Delta; \Gamma \vdash d : \tau' \quad u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llbracket d \rrbracket_\sigma^u : \tau} \quad \text{TAMHOLE} \quad \frac{\Delta; \Gamma \vdash d : \tau' \quad u :: \tau[\Gamma'] \in \Delta \quad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llbracket d \rrbracket_\sigma^{u\blacktriangleright} : \tau} \\
\\
\text{TACAST} \quad \frac{\Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d \langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2} \quad \text{TAFALLEDCAST} \quad \frac{\Delta; \Gamma \vdash d : \tau_1 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d \langle \tau_1 \Rightarrow \llbracket \rrbracket \nRightarrow \tau_2 \rangle : \tau_2}
\end{array}$$

$\boxed{\Delta; \Gamma \vdash D : T}$   $D$  is assigned optional type  $T$

$$\begin{array}{c}
\text{TASOME} \quad \frac{\Delta; \Gamma \vdash d : \tau}{\Delta; \Gamma \vdash d : \tau} \quad \text{TANONE} \quad \frac{}{\Delta; \Gamma \vdash \emptyset : \emptyset}
\end{array}$$

### 3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$   $\tau$  is a ground type

$$\begin{array}{c}
\text{GARR} \quad \frac{}{\llbracket \rrbracket \rightarrow \llbracket \rrbracket \text{ ground}} \quad \text{GREC} \quad \frac{}{\mu\llbracket \rrbracket.\llbracket \rrbracket \text{ ground}} \quad \text{GSUM} \quad \frac{\{T_i = \llbracket \rrbracket \vee T_i = \emptyset\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \text{ ground}}
\end{array}$$

$\boxed{\tau \blacktriangleright_{\text{ground}} \tau'}$   $\tau$  has matched ground type  $\tau'$

$$\begin{array}{c}
\text{MGARR} \quad \frac{\tau_1 \rightarrow \tau_2 \neq \llbracket \rrbracket \rightarrow \llbracket \rrbracket}{\tau_1 \rightarrow \tau_2 \blacktriangleright_{\text{ground}} \llbracket \rrbracket \rightarrow \llbracket \rrbracket} \quad \text{MGREC} \quad \frac{\tau \neq \llbracket \rrbracket}{\mu\pi.\tau \blacktriangleright_{\text{ground}} \mu\llbracket \rrbracket.\llbracket \rrbracket} \\
\\
\text{MGSUM} \quad \frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T'_i)\}_{C_i \in \mathcal{C}} \quad \{T_i = \tau_i \implies T'_i = \llbracket \rrbracket \wedge T_i = \emptyset \implies T'_i = \emptyset\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\text{ground}} +\{C_i(T'_i)\}_{C_i \in \mathcal{C}}}
\end{array}$$

$\boxed{d \text{ final}}$   $d$  is final

$$\frac{\text{FBoxedVal}}{d \text{ boxedval}} \quad d \text{ final}$$

$$\frac{\text{FIndet}}{d \text{ indet}} \quad d \text{ final}$$

$\boxed{d \text{ val}}$   $d$  is a value

$$\frac{\text{VLam}}{\lambda x:\tau. d \text{ val}}$$

$$\frac{\text{VRoll}}{d \text{ val}} \quad \text{roll}^{\mu\pi.\tau}(d) \text{ val}$$

$$\frac{\text{VINJSOME}}{d \text{ val}} \quad \text{inj}_{\mathcal{C}}^{\tau}(d) \text{ val}$$

$$\frac{\text{VINJNONE}}{\text{inj}_{\mathcal{C}}^{\tau}(\emptyset) \text{ val}}$$

$\boxed{d \text{ boxedval}}$   $d$  is a boxed value

$$\frac{\text{BVVal}}{d \text{ val}} \quad d \text{ boxedval}$$

$$\frac{\text{BVRoll}}{d \text{ boxedval}} \quad \text{roll}^{\mu\pi.\tau}(d) \text{ boxedval}$$

$$\frac{\text{BVInj}}{d \text{ boxedval}} \quad \text{inj}_{\mathcal{C}}^{\tau}(d) \text{ boxedval}$$

$$\frac{\text{BVArrCast}}{\tau_1 \rightarrow \tau_2 \neq \tau_3 \rightarrow \tau_4 \quad d \text{ boxedval}} \quad d\langle \tau_1 \rightarrow \tau_2 \Rightarrow \tau_3 \rightarrow \tau_4 \rangle \text{ boxedval}$$

$$\frac{\text{BVRecCast}}{\mu\pi.\tau \neq \mu\pi'.\tau' \quad d \text{ boxedval}} \quad d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}$$

$$\frac{\text{BVSumCast}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T'_i)\}_{C_i \in \mathcal{C}} \quad d \text{ boxedval}} \quad d\langle +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \Rightarrow +\{C_i(T'_i)\}_{C_i \in \mathcal{C}} \rangle \text{ boxedval}$$

$$\frac{\text{BVHoleCast}}{d \text{ boxedval} \quad \tau \text{ ground}} \quad d\langle \tau \Rightarrow \emptyset \rangle \text{ boxedval}$$

$\boxed{d \text{ indet}}$   $d$  is indeterminate

$$\frac{\text{IRoll}}{d \text{ indet}} \quad \text{roll}^{\mu\pi.\tau}(d) \text{ indet}$$

$$\frac{\text{IUnroll}}{d \text{ indet}} \quad \text{unroll}(d) \text{ indet}$$

$$\frac{\text{IInj}}{d \text{ indet}} \quad \text{inj}_{\mathcal{C}}^{\tau}(d) \text{ indet}$$

$$\frac{\text{IInJSOME}}{d \text{ final}} \quad \text{inj}_{\tau_u}^{\tau}(d) \text{ indet}$$

$$\frac{\text{IInjNONE}}{\text{inj}_{\tau_u}^{\tau}(\emptyset) \text{ indet}}$$

$$\frac{\text{ICastRec}}{\mu\pi.\tau \neq \mu\pi'.\tau' \quad d \text{ indet}} \quad d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}$$

$$\frac{\text{ICastSum}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T'_i)\}_{C_i \in \mathcal{C}} \quad d \text{ indet}} \quad d\langle +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \Rightarrow +\{C_i(T'_i)\}_{C_i \in \mathcal{C}} \rangle \text{ indet}$$

$\boxed{d \longrightarrow d'}$   $d$  takes an instruction transition to  $d'$



$$\begin{array}{c}
\text{ITAPP} \quad \frac{[d_2 \text{ final}]}{(\lambda x:\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1} \\
\text{ITUNROLL} \quad \frac{[d \text{ final}]}{\text{unroll}(\text{roll}^{\mu\pi.\tau}(d)) \longrightarrow d} \\
\text{ITAPPCAST} \quad \frac{[d_1 \text{ final}] \quad [d_2 \text{ final}] \quad \tau_1 \rightarrow \tau_2 \neq \tau'_1 \rightarrow \tau'_2}{d_1 \langle \tau_1 \rightarrow \tau_2 \Rightarrow \tau'_1 \rightarrow \tau'_2 \rangle (d_2) \longrightarrow (d_1 (d_2 \langle \tau'_1 \Rightarrow \tau_1 \rangle)) \langle \tau_2 \Rightarrow \tau'_2 \rangle} \\
\text{ITUNROLLCAST} \quad \frac{[d \text{ final}] \quad \mu\pi.\tau \neq \mu\pi'.\tau'}{\text{unroll}(d \langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle) \longrightarrow \text{unroll}(d) \langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau' \rangle} \quad \text{ITCASTID} \quad \frac{[d \text{ final}]}{d \langle \tau \Rightarrow \tau \rangle \longrightarrow d} \\
\text{ITCASTSUCCEED} \quad \frac{[d \text{ final}] \quad \tau \text{ ground}}{d \langle \tau \Rightarrow \emptyset \Rightarrow \tau \rangle \longrightarrow d} \quad \text{ITCASTFAIL} \quad \frac{[d \text{ final}] \quad \tau_1 \neq \tau_2 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground}}{d \langle \tau_1 \Rightarrow \emptyset \Rightarrow \tau_2 \rangle \longrightarrow d \langle \tau_1 \Rightarrow \emptyset \nRightarrow \tau_2 \rangle} \\
\text{ITGROUND} \quad \frac{[d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d \langle \tau \Rightarrow \emptyset \rangle \longrightarrow d \langle \tau \Rightarrow \tau' \Rightarrow \emptyset \rangle} \quad \text{ITEXPAND} \quad \frac{[d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d \langle \emptyset \Rightarrow \tau \rangle \longrightarrow d \langle \emptyset \Rightarrow \tau' \Rightarrow \tau \rangle}
\end{array}$$

$$\text{EvalCtx } \mathcal{E} ::= \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \text{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \text{unroll}(\mathcal{E}) \mid \text{inj}_{\mathcal{C}}^-(\mathcal{E}) \mid \langle \mathcal{E} \rangle_{\sigma}^u \mid \langle \mathcal{E} \rangle_{\sigma}^{u\blacktriangleright} \mid \mathcal{E} \langle \tau \Rightarrow \tau \rangle \mid \mathcal{E} \langle \tau \Rightarrow \emptyset \nRightarrow \tau \rangle$$

$$\boxed{d = \mathcal{E}\{d'\}} \quad d \text{ is obtained by placing } d' \text{ at the mark in } \mathcal{E}$$

$$\begin{array}{c}
\text{FHOUTER} \quad \frac{}{d = \circ\{d\}} \quad \text{FHAPP1} \quad \frac{d_1 = \mathcal{E}\{d'_1\}}{d_1(d_2) = \mathcal{E}(d_2)\{d'_1\}} \quad \text{FHAPP2} \quad \frac{[d_1 \text{ final}] \quad d_2 = \mathcal{E}\{d'_2\}}{d_1(d_2) = d_1(\mathcal{E})\{d'_2\}} \quad \text{FHROLL} \quad \frac{d = \mathcal{E}\{d'\}}{\text{roll}^{\mu\pi.\tau}(d) = \text{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}} \\
\text{FHUNROLL} \quad \frac{d = \mathcal{E}\{d'\}}{\text{unroll}(d) = \text{unroll}(\mathcal{E})\{d'\}} \quad \text{FHINJ} \quad \frac{d = \mathcal{E}\{d'\}}{\text{inj}_{\mathcal{C}}^-(d) = \text{inj}_{\mathcal{C}}^-(\mathcal{E})\{d'\}} \quad \text{FNEHOLEINSIDE} \quad \frac{d = \mathcal{E}\{d'\}}{\langle d \rangle_{\sigma}^u = \langle \mathcal{E} \rangle_{\sigma}^u\{d'\}} \quad \text{FMHOLEINSIDE} \quad \frac{d = \mathcal{E}\{d'\}}{\langle d \rangle_{\sigma}^{u\blacktriangleright} = \langle \mathcal{E} \rangle_{\sigma}^{u\blacktriangleright}\{d'\}} \\
\text{FHCASINSIDE} \quad \frac{d = \mathcal{E}\{d'\}}{d \langle \tau_1 \Rightarrow \tau_2 \rangle = \mathcal{E} \langle \tau_1 \Rightarrow \tau_2 \rangle \{d'\}} \quad \text{FHFAILEDCAST} \quad \frac{d = \mathcal{E}\{d'\}}{d \langle \tau_1 \Rightarrow \emptyset \nRightarrow \tau_2 \rangle = \mathcal{E} \langle \tau_1 \Rightarrow \emptyset \nRightarrow \tau_2 \rangle \{d'\}}
\end{array}$$

$$\boxed{d \mapsto d'} \quad d \text{ steps to } d'$$

$$\begin{array}{c}
\text{STEP} \\
\frac{d = \mathcal{E}\{d_0\} \quad d_0 \longrightarrow d'_0 \quad d' = \mathcal{E}\{d'_0\}}{d \mapsto d'}
\end{array}$$