Hazel Phi: 11-type-constructors

July 30, 2021

syntax

$$
\begin{array}{rlll}
\text{Kind} & \kappa & ::= & \mathtt{Type} \mid \mathtt{KHole} \mid \mathsf{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2 \\
\text{User Types} & \hat{\tau} & ::= & t \mid \mathtt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\hat{\tau}\rrparenthesis^{\mathtt{u}} \mid \lambda t::\mathtt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2} \\
\text{Internal Types} & \tau & ::= & t \mid \mathtt{bse} \mid \tau_1 \oplus \tau_2 \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\tau\rrparenthesis^{\mathtt{u}} \mid \llparenthesis t\rrparenthesis^{\mathtt{u}} \mid \lambda t::\kappa.\tau \mid \tau_1\ \tau_2 \\
\text{Base Types} & \mathtt{bse} & ::= & \mathtt{Int} \mid \mathtt{Float} \mid \mathtt{Bool} \\
\text{BinOp} & \oplus & ::= & \times \mid + \mid \rightarrow \\
\text{Type Pattern} & & & \\
\text{User Expression} & & & \\
\text{Internal Expression} & & &
\end{array}
$$

Declaratives

$\boxed{\Delta;\Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{bse} ::> \mathsf{S}_{\mathtt{Type}}(\mathtt{bse})}\ \texttt{PK-Base} \qquad \frac{\Delta;\Phi_1, t::\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta;\Phi \vdash t ::> \mathsf{S}_{\kappa}(t)}\ \texttt{PK-Var} \qquad \frac{\Delta;\Phi \vdash \tau_1::\mathtt{Type} \qquad \Delta;\Phi \vdash \tau_2::\mathtt{Type}}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 ::> \mathsf{S}_{\mathtt{Type}}(\tau_1 \oplus \tau_2)}\ \texttt{PK-}\oplus$$

$$\frac{\Delta_1, \mathtt{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \llparenthesis\rrparenthesis^{\mathtt{u}} ::> \mathsf{S}_{\kappa}(\llparenthesis\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-EHole} \qquad \frac{\Delta_1, \mathtt{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK} \qquad \Delta;\Phi \vdash \tau::\kappa_1}{\Delta;\Phi \vdash \llparenthesis\tau\rrparenthesis^{\mathtt{u}} ::> \mathsf{S}_{\kappa}(\llparenthesis\tau\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-NEHole}$$

$$\frac{\Delta_1, \mathtt{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK} \qquad t \notin \Phi}{\Delta;\Phi \vdash \llparenthesis t\rrparenthesis^{\mathtt{u}} ::> \mathsf{S}_{\kappa}(\llparenthesis t\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-Unbound} \qquad \frac{\Delta;\underline{\Phi, t::\kappa_1} \vdash \tau ::> \kappa_2}{\Delta;\Phi \vdash \lambda t::\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)}\ \texttt{PK-}\lambda$$

$$\frac{\Delta;\Phi \vdash \tau_1 ::> \kappa \qquad \Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \vdash \tau_2::\kappa_1}{\Delta;\Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ \texttt{PK-Ap}$$

$\boxed{\Delta;\Phi \vdash \tau::\kappa}$ τ is well formed at kind κ

$$\frac{\Delta;\Phi \vdash \tau ::> \mathsf{S}_{\kappa}(\tau)}{\Delta;\Phi \vdash \tau::\kappa}\ \texttt{WFaK-1} \qquad \frac{\Delta;\Phi \vdash \tau ::> \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \lesssim \kappa}{\Delta;\Phi \vdash \tau::\kappa}\ \texttt{WFaK-Subsump}$$

$$\frac{\Delta;\Phi \vdash \tau ::> \kappa}{\Delta;\Phi \vdash \tau::\kappa}\ \texttt{WFaK-Reit} \qquad \frac{\Delta;\Phi \vdash \tau::\kappa}{\Delta;\Phi \vdash \tau::\mathsf{S}_{\kappa}(\tau)}\ \texttt{WFaK-Self}$$

$$\frac{\Delta;\Phi \vdash \tau::\Pi_{t::\kappa_3}.\kappa_4 \qquad \Delta;\Phi \vdash \Pi_{t::\kappa_3}.\kappa_4 \lesssim \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \tau::\Pi_{t::\kappa_1}.\kappa_2}\ \texttt{WFaK-}\Pi\texttt{CSKTrans} \qquad \frac{\Delta;\Phi \vdash \tau::\mathsf{S}_{\kappa}(\tau_1) \qquad \Delta;\Phi \vdash \tau_1::\kappa}{\Delta;\Phi \vdash \tau::\kappa}\ \texttt{WFaK-Flatten}$$

$\boxed{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{KHole} \overset{\blacktriangleright}{\Pi} \Pi_{t::\mathtt{KHole}}.\mathtt{KHole}}\ \overset{\blacktriangleright}{\Pi}\texttt{-KHole} \qquad \frac{\Delta;\Phi \vdash \kappa \equiv \mathsf{S}_{\mathtt{KHole}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\mathsf{S}_{\mathtt{KHole}}(\tau)}.\mathsf{S}_{\mathtt{KHole}}(\tau\ t)}\ \overset{\blacktriangleright}{\Pi}\texttt{-SKHole} \qquad \frac{\Delta;\Phi \vdash \kappa \equiv \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}\ \overset{\blacktriangleright}{\Pi}\texttt{-}\Pi$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \equiv \kappa}\ \texttt{KEquiv-Refl} \qquad \frac{\Delta;\Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Symm} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Trans}$$

$$\frac{\Delta;\Phi \vdash \tau::\mathsf{S}_{\kappa}(\tau_1)}{\Delta;\Phi \vdash \mathsf{S}_{\mathsf{S}_{\kappa}(\tau_1)}(\tau) \equiv \mathsf{S}_{\kappa}(\tau_1)}\ \texttt{KEquiv-SKind}_{\texttt{SKind}} \qquad \frac{\Delta;\Phi \vdash \tau::\Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau) \equiv \Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau\ t_1)}\ \texttt{KEquiv-SKind}_{\Pi}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\underline{\Phi, t::\kappa_1} \vdash \kappa_3 \equiv \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{KEquiv-}\Pi \qquad \frac{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \equiv \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{KEquiv-SKind}$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{KHole} \lesssim \kappa}\ \texttt{CSK-KHoleL} \qquad \frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \mathtt{KHole}}\ \texttt{CSK-KHoleR}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\mathtt{KHole}}(\tau)\ \mathsf{OK} \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\mathtt{KHole}}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{L} \qquad \frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK} \qquad \Delta;\Phi \vdash \mathsf{S}_{\mathtt{KHole}}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \mathsf{S}_{\mathtt{KHole}}(\tau)}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{R}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-KEquiv} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta;\Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-Normal}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind} \qquad \frac{\Delta;\Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta;\underline{\Phi, t::\kappa_3} \vdash \kappa_2 \lesssim \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{CSK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta;\Phi \vdash \tau_1 \stackrel{\kappa_1}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{CSK-?}$$

$\boxed{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta;\Phi \vdash \tau::\kappa}{\Delta;\Phi \vdash \tau \stackrel{\kappa}{\equiv} \tau}\ \texttt{EquivAK-Refl} \qquad \frac{\Delta;\Phi \vdash \tau_2 \stackrel{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Symm} \qquad \frac{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_3 \stackrel{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Trans}$$

$$\frac{\Delta;\Phi \vdash \tau_1 ::> \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \mathsf{S}_{\kappa}(\tau_2)}{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-SKind} \qquad \frac{\Delta;\Phi \vdash \tau_1::\Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta;\Phi \vdash \tau_2::\Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta;\underline{\Phi, t::\kappa_1} \vdash \tau_1\ t \stackrel{\kappa_2}{\equiv} \tau_2\ t}{\Delta;\Phi \vdash \tau_1 \stackrel{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2}\ \texttt{EquivAK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \tau_1 \stackrel{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \stackrel{\kappa_1}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1\ \tau_2 \stackrel{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4}\ \texttt{EquivAK-Ap} \qquad \frac{\Delta;\Phi \vdash \tau_1 \stackrel{\mathsf{S}_{\kappa}(\tau)}{\equiv} \tau_2}{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}\ (1)$$

$$\frac{\Delta;\Phi \vdash \tau_1 \stackrel{\mathtt{Type}}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \stackrel{\mathtt{Type}}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 \stackrel{\mathtt{Type}}{\equiv} \tau_3 \oplus \tau_4}\ (2) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\underline{\Phi, t::\kappa_1} \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}{\Delta;\Phi \vdash \lambda t::\kappa_1.\tau_1 \stackrel{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t::\kappa_2.\tau_2}\ (3) \qquad \frac{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa}{\Delta;\Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}\ (4)$$

$\boxed{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}$ κ is well formed

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{Type}\ \mathsf{OK}}\ \texttt{KWF-Type} \qquad \frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{KHole}\ \mathsf{OK}}\ \texttt{KWF-KHole} \qquad \frac{\Delta;\Phi \vdash \tau::\kappa}{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau)\ \mathsf{OK}}\ \texttt{KWF-SKind} \qquad \frac{\Delta;\underline{\Phi, t::\kappa_1} \vdash \kappa_2\ \mathsf{OK}}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathsf{OK}}\ \texttt{KWF-}\Pi$$

$\boxed{\Delta;\Phi \vdash \mathsf{OK}}$ Context is well formed

$$\frac{}{\cdot;\cdot \vdash \mathsf{OK}}\ \texttt{CWF-Nil} \qquad \frac{t \notin \Phi \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\underline{\Phi, t::\kappa} \vdash \mathsf{OK}}\ \texttt{CWF-TypVar} \qquad \frac{\mathtt{u} \notin \Delta \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\underline{\Delta, \mathtt{u}::\kappa};\Phi \vdash \mathsf{OK}}\ \texttt{CWF-Hole}$$

Metatheory

Lemma 1 (COK)**.** *If* $\Delta;\Phi \vdash \mathcal{J}$*, then* $\Delta;\Phi \vdash \mathit{OK}$

Proof. By simultaneous induction on derivations.
No interesting cases. □

Lemma 2 (Exchange)**.**
If $\Delta;\Phi_1, t_{L1}::\kappa_{L1}, t_{L2}::\kappa_{L2}, \Phi_2 \vdash \mathcal{J}$ *and* $\Delta;\Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash \mathit{OK}$*, then* $\Delta;\Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

Proof. By induction on derivations.
No interesting cases.
(Only rules with Φ extended in the consequent are interesting, which is only CWF-TypVar, but when $\mathcal{J}$ is CWF, Exchange is identity) □

Corollary 3 (Marked-Exchange)**.**
If $\Delta;\underline{\underline{\Phi, t_{L1}::\kappa_{L1}}, t_{L2}::\kappa_{L2}} \vdash \mathcal{J}$ *and* $\Delta;\underline{\underline{\Phi, t_{L2}::\kappa_{L2}}, t_{L1}::\kappa_{L1}} \vdash \mathit{OK}$*, then* $\Delta;\underline{\underline{\Phi, t_{L2}::\kappa_{L2}}, t_{L1}::\kappa_{L1}} \vdash \mathcal{J}$

Lemma 4 (Weakening)**.**
If $\Delta;\Phi \vdash \mathcal{J}$ *and* $\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \mathit{OK}$ *and* $t_L \notin \mathcal{J}$ *and* $\forall t \in \kappa_L, t \notin \mathcal{J}$*, then* $\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \mathcal{J}$

Proof. By induction on derivations.
When applying Weakening in the induction, check that the left premiss is always a subderivation.
(PoS = premiss of subderivation)

Weakening

$$
\dfrac{
\dfrac{
\dfrac{}{\Delta;\underline{\Phi, t::\kappa_1} \vdash \tau ::> \kappa_2}\,\texttt{premiss}
\quad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \mathsf{OK}}\,\texttt{IH}}{\Delta;\Phi \vdash \kappa_L\ \mathsf{OK}}\,\texttt{PoS}
\quad
\dfrac{\dfrac{}{\Delta;\underline{\Phi, t::\kappa_1} \vdash \tau ::> \kappa_2}\,\texttt{premiss}}{\Delta;\underline{\Phi, t::\kappa_1} \vdash \mathsf{OK}}\,\texttt{COK}
}{\Delta;\underline{\Phi, t::\kappa_1} \vdash \kappa_L\ \mathsf{OK}}\,\texttt{Weakening}
\quad
\dfrac{
\dfrac{\dfrac{}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \mathsf{OK}}\,\texttt{IH}}{t_L \notin \Phi}\,\texttt{PoS}
\quad
\dfrac{\dfrac{}{t_L \notin \mathcal{J}}\,\texttt{IH} \quad \dfrac{}{t \in \mathcal{J}}}{t_L \neq t}
\quad
\dfrac{\dfrac{}{t_L \notin \mathcal{J}}\,\texttt{IH}}{t_L \notin \kappa_1}
}{t_L \notin \underline{\Phi, t::\kappa_1}}
}{\Delta;\underline{\underline{\Phi, t::\kappa_1}, t_L::\kappa_L} \vdash \mathsf{OK}}\,\texttt{CWF-TypVar}
}{\Delta;\underline{\underline{\Phi, t::\kappa_1}, t_L::\kappa_L} \vdash \tau ::> \kappa_2}\,\texttt{Weakening}
\quad
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\Delta;\underline{\Phi, t::\kappa_1} \vdash \tau ::> \kappa_2}\,\texttt{premiss}}{\Delta;\underline{\Phi, t::\kappa_1} \vdash \mathsf{OK}}\,\texttt{COK}}{t \notin \Phi}\,\texttt{PoS}
\quad
\dfrac{\dfrac{}{t_L \notin \mathcal{J}}\,\texttt{IH} \quad \dfrac{}{t \in \mathcal{J}}}{t \neq t_L}
\quad
\dfrac{\dfrac{}{\forall \dot{t} \in \kappa_L, \dot{t} \notin \mathcal{J}}\,\texttt{IH} \quad \dfrac{}{t \in \mathcal{J}}}{t \notin \kappa_L}
}{t \notin \underline{\Phi, t_L::\kappa_L}}
\quad
\dfrac{
\dfrac{\dfrac{}{\Delta;\underline{\Phi, t::\kappa_1} \vdash \mathsf{OK}}\,\texttt{premiss}}{\Delta;\Phi \vdash \kappa_1\ \mathsf{OK}}\,\texttt{PoS}
\quad
\dfrac{}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \mathsf{OK}}\,\texttt{premiss}
}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \kappa_1\ \mathsf{OK}}\,\texttt{Weakening}
}{\Delta;\underline{\underline{\Phi, t_L::\kappa_L}, t::\kappa_1} \vdash \mathsf{OK}}\,\texttt{CWF-TypVar}
}{
\dfrac{\Delta;\underline{\underline{\Phi, t_L::\kappa_L}, t::\kappa_1} \vdash \tau ::> \kappa_2}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \lambda t::\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)}\,\texttt{PK-}\lambda
}\,\texttt{Marked-Exchange}
$$

$$
\dfrac{
\dfrac{
\dfrac{}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\,\texttt{premiss}
\quad
\dfrac{}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \mathsf{OK}}\,\texttt{IH}
}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \kappa_1 \equiv \kappa_2}\,\texttt{Weakening}
\quad
\dfrac{
\dfrac{}{\Delta;\underline{\underline{\Phi, t::\kappa_1}, t_L::\kappa_L} \vdash \kappa_3 \equiv \kappa_4}
\quad
\dfrac{}{\Delta;\underline{\underline{\Phi, t_L::\kappa_L}, t::\kappa_1} \vdash \mathsf{OK}}
}{\Delta;\underline{\underline{\Phi, t_L::\kappa_L}, t::\kappa_1} \vdash \kappa_3 \equiv \kappa_4}\,\texttt{Marked-Exchange}
}{\Delta;\underline{\Phi, t_L::\kappa_L} \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4}\,\texttt{KEquiv-}\Pi
$$

□

Lemma 5 (OK-PK). *If* $\Delta;\Phi \vdash \tau ::> \kappa$, *then* $\Delta;\Phi \vdash \kappa$ *OK*

Lemma 6 (OK-WFaK). *If* $\Delta;\Phi \vdash \tau :: \kappa$, *then* $\Delta;\Phi \vdash \kappa$ *OK*

Lemma 7 (OK-MatchPi). *If* $\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2$, *then* $\Delta;\Phi \vdash \kappa$ *OK and* $\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2$ *OK*

Lemma 8 (OK-KEquiv). *If* $\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2$, *then* $\Delta;\Phi \vdash \kappa_1$ *OK and* $\Delta;\Phi \vdash \kappa_2$ *OK*

Lemma 9 (OK-CSK). *If* $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$, *then* $\Delta;\Phi \vdash \kappa_1$ *OK and* $\Delta;\Phi \vdash \kappa_2$ *OK*

Lemma 10 (OK-EquivAK). *If* $\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2$, *then* $\Delta;\Phi \vdash \tau_1 :: \kappa$ *and* $\Delta;\Phi \vdash \tau_2 :: \kappa$ *and* $\Delta;\Phi \vdash \kappa$ *OK*

Lemma 11 (OK-Substitution).
If $\Delta;\Phi \vdash \tau_L :: \kappa_{L1}$ *and* $\Delta;\Phi, t_L :: \kappa_{L1} \vdash \kappa_{L2}$ *OK, then* $\Delta;\Phi \vdash [\tau_L/t_L]\kappa_{L2}$ *OK*
(induction on $\Delta;\Phi, t_L :: \kappa_{L1} \vdash \kappa_{L2}$ *OK)*

Lemma 12 (K-Substitution).
If $\Delta;\Phi \vdash \tau_{L1} :: \kappa_{L1}$ *and* $\Delta;\Phi, t_L :: \kappa_{L1} \vdash \tau_{L2} :: \kappa_{L2}$, *then* $\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau_{L2} :: [\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta;\Phi, t_L :: \kappa_{L1} \vdash \tau_{L2} :: \kappa_{L2}$*)*

Proof. By simultaneous induction on derivations.
The interesting cases per lemma:

	OK-PK.	PK-Base	$\Delta;\Phi \vdash \mathtt{bse} :: \mathsf{S}_{\mathtt{Type}}(\mathtt{bse})$	by (9)
			$\Delta;\Phi \vdash \mathtt{bse} :: \mathtt{Type}$	by (10)
		*	$\Delta;\Phi \vdash \mathsf{S}_{\mathtt{Type}}(\mathtt{bse})$ OK	by (43)
		*	$\Delta;\Phi \vdash$ OK	by premiss
		PK-Ap		bad
	OK-WFaK.	(12)	$\Delta;\Phi \vdash \tau_2 :: \kappa$	by (10)
		*	$\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau_2)$ OK	by (43)
	OK-KEquiv.	(22)	$\Delta;\Phi \vdash \tau\ t ::> \kappa$	
(PoS = premiss of subderivation)	OK-Substitution.	(41)	$\Delta;\Phi, t_L :: \kappa_{L1} \vdash$ OK	premiss (41)
			$\Delta;\Phi \vdash \kappa_{L1}$ OK	by subderivation premiss (46)
		*	$\Delta;\Phi \vdash$ OK	by OK-KWF
		*	$\Delta;\Phi \vdash [\tau_L/t_L]\mathtt{Type}$ OK	by (41) and degenerate subst
		(43)	$\Delta;\Phi, t_L :: \kappa_{L1} \vdash \tau :: \kappa$	premiss (43)
			$\Delta;\Phi, t_L :: \kappa_{L1} \vdash$ OK	by OK-WFaK
			$\Delta;\Phi \vdash \kappa_{L1}$ OK	by subderivation premiss (46)
		*	$\Delta;\Phi \vdash$ OK	by OK-KWF
			$\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau :: [\tau_{L1}/t_L]\kappa$	by K-Substitution on premiss
		*	$\Delta;\Phi \vdash [\tau_L/t_L]\mathsf{S}_{\kappa}(\tau)$ OK	by (43)

□

Lemma 13 (PK-Unicity). *If* $\Delta;\Phi \vdash \tau_L ::> \kappa_{L1}$ *and* $\Delta;\Phi \vdash \tau_L ::> \kappa_{L2}$ *then* κ_{L1} *is* κ_{L2}

Lemma 14. *If* $\Delta;\Phi \vdash \tau ::> \kappa_1$ *and* $\Delta;\Phi \vdash \tau :: \kappa_2$, *then* $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$

Lemma 15. *If* $\Delta;\Phi \vdash \kappa_1 \lesssim \mathsf{S}_{\kappa_2}(\tau)$, *then* $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$