Algebraic Data Types for Hazel

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1 Syntax

$$
\begin{array}{llcl}
\textsf{HTyp} & \tau & ::= & \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \mid \llparenthesis\rrparenthesis \mid \llparenthesis\alpha\rrparenthesis^u \\
\textsf{HTypPat} & \pi & ::= & \alpha \mid \llparenthesis\rrparenthesis \\
\textsf{HExp} & e & ::= & x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \texttt{inj}_C(E) \mid \texttt{roll}(e) \mid \texttt{unroll}(e) \\
 & & & \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^u \\
\textsf{HTag} & C & ::= & \mathbf{C} \mid ?^u \\
\textsf{HTagTyp} & T & ::= & \tau \mid \varnothing \\
\textsf{HTagArg} & E & ::= & e \mid \varnothing \\
\textsf{IHExp} & d & ::= & x \mid \lambda x{:}\tau.d \mid d(d) \mid \texttt{inj}^{\tau}_C(D) \mid \texttt{roll}^{\mu\alpha.\tau}(d) \mid \texttt{unroll}(d) \\
 & & & \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau \rangle \mid \llparenthesis\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^u_\sigma \\
\textsf{IHTagArg} & D & ::= & d \mid \varnothing
\end{array}
$$

1.1 Context Extension

We write $\Gamma, X : T$ to denote the extension of typing context Γ with optional variable X of optional type T.

$$
\Gamma, X : T = \begin{cases} \Gamma, x : \tau & X = x \wedge T = \tau \\ \Gamma, x : \llparenthesis\rrparenthesis & X = x \wedge T = \varnothing \\ \Gamma & X = \varnothing \end{cases}
$$

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \llparenthesis\rrparenthesis \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{lcll}
[\tau/\llparenthesis\rrparenthesis]\tau' & = & \tau' & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu\llparenthesis\rrparenthesis.\tau_2 & = & \mu\llparenthesis\rrparenthesis.[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]+\{C_i(T_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]T_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha]\llparenthesis\rrparenthesis & = & \llparenthesis\rrparenthesis & \\
[\alpha'/\alpha]\llparenthesis\alpha\rrparenthesis^u & = & \llparenthesis\alpha'\rrparenthesis^u & \\
[\alpha'/\alpha]\llparenthesis\alpha_1\rrparenthesis^u & = & \llparenthesis\alpha_1\rrparenthesis^u & \text{when } \alpha \neq \alpha_1
\end{array}
$$

$\boxed{[\tau/\pi]T = \tau'}$ τ' is obtained by substituting τ for π in T

$$[\tau/\pi]T = \begin{cases} [\tau/\pi]\tau' & \text{when } T = \tau' \\ \varnothing & \text{when } T = \varnothing \end{cases}$$

$\boxed{\Theta \vdash \tau \; \mathsf{valid}}$ τ is a valid type

$$\textsc{TVArr}\ \frac{\Theta \vdash \tau_1 \; \mathsf{valid} \qquad \Theta \vdash \tau_2 \; \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2 \; \mathsf{valid}} \qquad \textsc{TVVar}\ \frac{\alpha \in \Theta}{\Theta \vdash \alpha \; \mathsf{valid}} \qquad \textsc{TVRec}\ \frac{\Theta, \pi \vdash \tau \; \mathsf{valid}}{\Theta \vdash \mu\pi.\tau \; \mathsf{valid}}$$

$$\textsc{TVSum}\ \frac{\{\Theta \vdash T_i \; \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \; \mathsf{valid}} \qquad \textsc{TVEHole}\ \frac{}{\Theta \vdash (\!|\!) \; \mathsf{valid}}$$

$$\textsc{TVNEHole}\ \frac{\alpha \notin \Theta}{\Theta \vdash (\!|\alpha|\!)^u \; \mathsf{valid}}$$

$\boxed{\Theta \vdash T \; \mathsf{valid}}$ T is a valid optional type

$$\textsc{TVSome}\ \frac{T = \tau \qquad \Theta \vdash \tau \; \mathsf{valid}}{\Theta \vdash T \; \mathsf{valid}} \qquad \textsc{TVNone}\ \frac{}{\Theta \vdash \varnothing \; \mathsf{valid}}$$

$\boxed{\tau \sim \tau'}$ τ and τ' are consistent

$$\textsc{TCHole1}\ \frac{}{(\!|\!) \sim \tau} \qquad \textsc{TCHole2}\ \frac{}{\tau \sim (\!|\!)} \qquad \textsc{TCRefl}\ \frac{}{\tau \sim \tau} \qquad \textsc{TCArr}\ \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

$$\textsc{TCRec}\ \frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \qquad \textsc{TCRecHole1}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu(\!|\!).\tau \sim \mu\alpha.\tau'}$$

$$\textsc{TCRecHole2}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu(\!|\!).\tau'} \qquad \textsc{TCSum}\ \frac{\{T_i \sim T_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(T_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{T \sim T'}$ T and T' are consistent

$$\textsc{TCSome}\ \frac{\tau \sim \tau'}{\tau \sim \tau'} \qquad \textsc{TCNone}\ \frac{}{\varnothing \sim \varnothing}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau \; \mathsf{valid}$.

Theorem 2 (Consistency Preserves Validity). *If* $\Theta \vdash \tau \; \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau' \; \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ τ has matched arrow type $\tau_1 \to \tau_2$

$$\textsc{MAHole}\ \frac{}{(\!|\!) \blacktriangleright_{\to} (\!|\!) \to (\!|\!)} \qquad \textsc{MAArr}\ \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}$$

$\boxed{\tau \blacktriangleright_\mu \mu\pi.\tau'}$ τ has matched recursive type $\mu\pi.\tau'$

$$\frac{}{\mu\pi.\tau \blacktriangleright_\mu \mu\pi.\tau}\ \textsc{MRRec} \qquad \frac{}{(\!|\,|\!) \blacktriangleright_\mu \mu(\!|\,|\!).(\!|\,|\!)}\ \textsc{MRHole}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ e synthesizes type τ

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau}\ \textsc{SVar} \qquad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\,|\!)}\ \textsc{SVarFree} \qquad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}\ \textsc{SLam}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau}\ \textsc{SApp} \qquad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim (\!|\,|\!) \to (\!|\,|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|e_1(e_2)|\!)^u \Rightarrow (\!|\,|\!)}\ \textsc{SAppNotArr}$$

$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau}\ \textsc{SAsc} \qquad \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Rightarrow \mu(\!|\,|\!).(\!|\,|\!)}\ \textsc{SRollError} \qquad \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}\ \textsc{SUnroll}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu(\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash (\!|\mathtt{unroll}(e)|\!)^u \Rightarrow (\!|\,|\!)}\ \textsc{SUnrollNotRec} \qquad \frac{\Gamma \vdash E\ \mathsf{valid}}{\Gamma \vdash (\!|\mathtt{inj}_C(E)|\!)^u \Rightarrow (\!|\,|\!)}\ \textsc{SInjError} \qquad \frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow (\!|\,|\!)}\ \textsc{SEHole} \qquad \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\,|\!)}\ \textsc{SNEHole}$$

$\boxed{\Gamma \vdash E\ \mathsf{valid}}$ E is a valid optional expression

$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash e\ \mathsf{valid}}\ \textsc{EVSome} \qquad \frac{}{\Gamma \vdash \varnothing\ \mathsf{valid}}\ \textsc{EVNone}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

$$\text{ARoll}\ \frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau}$$

$$\text{ARollNotRec}\ \frac{\tau \nsim \mu(\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Leftarrow \tau}$$

$$\text{AInjHole}\ \frac{\Gamma \vdash E\ \mathsf{valid}}{\Gamma \vdash \texttt{inj}_C(E) \Leftarrow (\!|\,|\!)}$$

$$\text{AInj}\ \frac{C_j \in \mathcal{C} \qquad \Gamma \vdash E \Leftarrow T_j}{\Gamma \vdash \texttt{inj}_{C_j}(E) \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\text{AInjUnexpectedBody}\ \frac{C_j \in \mathcal{C} \qquad T_j = \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^u \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\text{AInjExpectedBody}\ \frac{C_j \in \mathcal{C} \qquad T_j = \tau}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\text{AInjBadTag}\ \frac{C \notin \mathcal{C} \qquad \Gamma \vdash E\ \mathsf{valid}}{\Gamma \vdash (\!|\texttt{inj}_C(E)|\!)^u \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\text{ASubsume}\ \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

$\boxed{\Gamma \vdash E \Leftarrow T}$ E analyzes against optional type T

$$\text{ASome}\ \frac{\Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e \Leftarrow \tau}$$

$$\text{ANone}\ \frac{}{\Gamma \vdash \varnothing \Leftarrow \varnothing}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity). *If* $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ $\quad e$ synthesizes type τ and elaborates to d

$$\textsc{ESVar}\quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

$$\textsc{ESVarFree}\quad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|x|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\,|\!)[\Gamma]}$$

$$\textsc{ESLam}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESApp}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_2 \to \tau\rangle)(d_2\langle \tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\textsc{ESAppNotArr}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim (\!|\,|\!) \to (\!|\,|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\,|\!) \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|d_1|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow (\!|\,|\!)\rangle) \dashv \Delta_1 \cup \Delta_2, u :: (\!|\,|\!) \to (\!|\,|\!)[\Gamma]}$$

$$\textsc{ESAsc}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau\rangle \dashv \Delta}$$

$$\textsc{ESRollError}\quad \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Rightarrow \mu(\!|\,|\!).(\!|\,|\!) \rightsquigarrow (\!|\mathtt{roll}^{\mu(\!|\,|\!).(\!|\,|\!)}(d\langle \tau \Rightarrow (\!|\,|\!)\rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}$$

$$\textsc{ESUnroll}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \mathtt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}$$

$$\textsc{ESUnrollNotRec}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \nsim \mu(\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash \mathtt{unroll}\big((\!|e|\!)^{u\blacktriangleright}\big) \Rightarrow (\!|\,|\!) \rightsquigarrow \mathtt{unroll}\Big((\!|d|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\Big) \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}$$

$$\textsc{ESInjError}\quad \frac{\Gamma \vdash E \Leftarrow (\!|\,|\!) \rightsquigarrow D : T \dashv \Delta \qquad T = \mathit{frob}(E)}{\Gamma \vdash (\!|\mathtt{inj}_C(E)|\!)^u \Rightarrow +\{C(T)\} \rightsquigarrow (\!|\mathtt{inj}_C^{+\{C(T)\}}(D\langle T \Rightarrow (\!|\,|\!)\rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: +\{C(T)\}[\Gamma]}$$

$$\textsc{ESEHole}\quad \frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|\,|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\,|\!)[\Gamma]}$$

$$\textsc{ESNEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: (\!|\,|\!)[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

$$\textsc{EARoll}\quad \frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau \rightsquigarrow \texttt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}$$

$$\textsc{EARollNotRec}\quad \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau' \dashv \Delta \qquad \tau \nsim \mu(\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{roll}^{\mu(\!|\,|\!).(\!|\,|\!)}(d\langle \tau' \Rightarrow (\!|\,|\!)\rangle)|\!)^u_{\mathsf{id}(\Gamma)} : \mu(\!|\,|\!).(\!|\,|\!) \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}$$

$$\textsc{EAInjHole}\quad \frac{\Gamma \vdash E \Leftarrow (\!|\,|\!) \rightsquigarrow D : T \dashv \Delta \qquad \tau = +\{C(T)\}}{\Gamma \vdash \texttt{inj}_C(E) \Leftarrow (\!|\,|\!) \rightsquigarrow \texttt{inj}^\tau_C(D) : \tau \dashv \Delta}$$

$$\textsc{EAInj}\quad \frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash E \Leftarrow T_j \rightsquigarrow D : T'_j \dashv \Delta}{\Gamma \vdash \texttt{inj}_{C_j}(E) \Leftarrow \tau \rightsquigarrow \texttt{inj}^\tau_{C_j}\big(D\langle T'_j \Rightarrow T_j\rangle\big) : \tau \dashv \Delta}$$

$$\textsc{EAInjUnexpectedBody}\quad \frac{\begin{array}{c}\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \\ C_j \in \mathcal{C} \qquad T_j = \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau_j \dashv \Delta \qquad \tau' = +\Big\{\{C_i(T_i)\}_{C_i \in \mathcal{C}\setminus C_j} \cup \{C_j(\tau_j)\}\Big\}\end{array}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(d)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjExpectedBody}\quad \frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad T_j = \tau_j \qquad \tau' = +\Big\{\{C_i(T_i)\}_{C_i \in \mathcal{C}\setminus C_j} \cup \{C_j(\varnothing)\}\Big\}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(\varnothing)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjBadTag}\quad \frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C \notin \mathcal{C} \qquad \Gamma \vdash E \Leftarrow (\!|\,|\!) \rightsquigarrow D : T \dashv \Delta \qquad \tau' = +\big\{\{C_i(T_i)\}_{C_i \in \mathcal{C}} \cup \{C(T)\}\big\}}{\Gamma \vdash (\!|\texttt{inj}_C(E)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_C(D)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EASubsume}\quad \frac{e \neq (\!|\,|\!)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta} \qquad \textsc{EAEHole}\quad \frac{}{\Gamma \vdash (\!|\,|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\,|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$$\textsc{EANEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$\boxed{\Gamma \vdash E \Leftarrow T_1 \rightsquigarrow D : T_2 \dashv \Delta}$ E analyzes against optional type T_1 and elaborates to D of consistent optional type T_2

$$\textsc{EASome}\quad \frac{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta} \qquad \textsc{EANone}\quad \frac{}{\Gamma \vdash \varnothing \Leftarrow \varnothing \rightsquigarrow \varnothing : \varnothing \dashv \emptyset}$$

2.3 Type Assignment

$\boxed{\Delta;\Gamma \vdash d : \tau}$ d is assigned type τ

$$\textsc{TAVar}\quad \frac{x : \tau \in \Gamma}{\Delta;\Gamma \vdash x : \tau} \qquad \textsc{TALam}\quad \frac{\Delta;\Gamma, x : \tau_1 \vdash d : \tau_2}{\Delta;\Gamma \vdash \lambda x{:}\tau_1.d : \tau_1 \to \tau_2} \qquad \textsc{TAApp}\quad \frac{\Delta;\Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta;\Gamma \vdash d_2 : \tau_2}{\Delta;\Gamma \vdash d_1(d_2) : \tau}$$

$$\textsc{TARoll}\quad \frac{\emptyset \vdash \mu\pi.\tau\ \mathsf{valid} \qquad \Delta;\Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta;\Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau} \qquad \textsc{TAUnroll}\quad \frac{\Delta;\Gamma \vdash d : \mu\pi.\tau}{\Delta;\Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

$$\textsc{TAInj}\quad \frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta;\Gamma \vdash D : T_j}{\Delta;\Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(D) : \tau} \qquad \textsc{TAEHole}\quad \frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta;\Gamma \vdash \sigma : \Gamma'}{\Delta;\Gamma \vdash \llparenthesis\rrparenthesis^u_\sigma : \tau}$$

$$\textsc{TANEHole}\quad \frac{\Delta;\Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta;\Gamma \vdash \sigma : \Gamma'}{\Delta;\Gamma \vdash \llparenthesis d \rrparenthesis^u_\sigma : \tau} \qquad \textsc{TAMHole}\quad \frac{\Delta;\Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta;\Gamma \vdash \sigma : \Gamma'}{\Delta;\Gamma \vdash \llparenthesis d \rrparenthesis^{u\blacktriangleright}_\sigma : \tau}$$

$$\textsc{TACast}\quad \frac{\Delta;\Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta;\Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2} \qquad \textsc{TAFailedCast}\quad \frac{\Delta;\Gamma \vdash d : \tau_1 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground} \qquad \tau_1 \neq \tau_2}{\Delta;\Gamma \vdash d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle : \tau_2}$$

$\boxed{\Delta;\Gamma \vdash D : T}$ D is assigned optional type T

$$\textsc{TASome}\quad \frac{\Delta;\Gamma \vdash d : \tau}{\Delta;\Gamma \vdash d : \tau} \qquad \textsc{TANone}\quad \frac{}{\Delta;\Gamma \vdash \varnothing : \varnothing}$$

3 Dynamic Semantics

$\boxed{\tau\ \text{ground}}$ τ is a ground type

$$\textsc{GArr}\quad \frac{}{\llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis\ \text{ground}} \qquad \textsc{GRec}\quad \frac{}{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis\ \text{ground}} \qquad \textsc{GSum}\quad \frac{\{T_i = \llparenthesis\rrparenthesis \vee T_i = \varnothing\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}}\ \text{ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

$$\textsc{MGArr}\quad \frac{\tau_1 \to \tau_2 \neq \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis} \qquad \textsc{MGRec}\quad \frac{\tau \neq \llparenthesis\rrparenthesis}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$$\textsc{MGSum}\quad \frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \qquad \{T_i = \tau_i \implies T_i' = \llparenthesis\rrparenthesis \wedge T_i = \varnothing \implies T_i' = \varnothing\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(T_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \text{ final}}$ d is final

$$\frac{d \text{ boxedval}}{d \text{ final}}\ \textsc{FBoxedVal} \qquad \frac{d \text{ indet}}{d \text{ final}}\ \textsc{FIndet}$$

$\boxed{d \text{ val}}$ d is a value

$$\frac{}{\lambda x{:}\tau.d \text{ val}}\ \textsc{VLam} \qquad \frac{d \text{ val}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ val}}\ \textsc{VRoll} \qquad \frac{d \text{ val}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ val}}\ \textsc{VInjSome} \qquad \frac{}{\texttt{inj}^{\tau}_{\mathbf{C}}(\varnothing) \text{ val}}\ \textsc{VInjNone}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

$$\frac{d \text{ val}}{d \text{ boxedval}}\ \textsc{BVVal} \qquad \frac{d \text{ boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}}\ \textsc{BVRoll} \qquad \frac{d \text{ boxedval}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ boxedval}}\ \textsc{BVInj}$$

$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}\ \textsc{BVArrCast}$$

$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}}\ \textsc{BVRecCast} \qquad \frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \qquad d \text{ boxedval}}{d\langle +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \Rightarrow +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \rangle \text{ boxedval}}\ \textsc{BVSumCast}$$

$$\frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\!) \rangle \text{ boxedval}}\ \textsc{BVHoleCast}$$

$\boxed{d \text{ indet}}$ d is indeterminate

$$\frac{d \text{ indet}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ indet}}\ \textsc{IRoll} \qquad \frac{d \text{ indet}}{\texttt{unroll}(d) \text{ indet}}\ \textsc{IUnroll} \qquad \frac{d \text{ indet}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ indet}}\ \textsc{IInj} \qquad \frac{d \text{ final}}{\texttt{inj}^{\tau}_{?^u}(d) \text{ indet}}\ \textsc{IInjSome} \qquad \frac{}{\texttt{inj}^{\tau}_{?^u}(\varnothing) \text{ indet}}\ \textsc{IInjNone}$$

$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}}\ \textsc{ICastRec} \qquad \frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \qquad d \text{ indet}}{d\langle +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \Rightarrow +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \rangle \text{ indet}}\ \textsc{ICastSum}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\frac{[d_2 \text{ final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1}\ \text{ITApp} \qquad \frac{[d \text{ final}]}{\texttt{unroll}(\texttt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}\ \text{ITUnroll}$$

$$\frac{[d_1 \text{ final}] \quad [d_2 \text{ final}] \quad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1\rangle))\langle \tau_2 \Rightarrow \tau_2'\rangle}\ \text{ITAppCast}$$

$$\frac{[d \text{ final}] \quad \mu\pi.\tau \neq \mu\pi'.\tau'}{\texttt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \texttt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle}\ \text{ITUnrollCast} \qquad \frac{[d \text{ final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}\ \text{ITCastId}$$

$$\frac{[d \text{ final}] \quad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\,|\!) \Rightarrow \tau\rangle \longrightarrow d}\ \text{ITCastSucceed} \qquad \frac{[d \text{ final}] \quad \tau_1 \neq \tau_2 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground}}{d\langle \tau_1 \Rightarrow (\!|\,|\!) \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow (\!|\,|\!) \nRightarrow \tau_2\rangle}\ \text{ITCastFail}$$

$$\frac{[d \text{ final}] \quad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow (\!|\,|\!)\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow (\!|\,|\!)\rangle}\ \text{ITGround} \qquad \frac{[d \text{ final}] \quad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle (\!|\,|\!) \Rightarrow \tau\rangle \longrightarrow d\langle (\!|\,|\!) \Rightarrow \tau' \Rightarrow \tau\rangle}\ \text{ITExpand}$$

$$\mathsf{EvalCtx}\ \mathcal{E} \ ::= \ \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \texttt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \texttt{unroll}(\mathcal{E}) \mid \texttt{inj}^{\tau}_{C}(\mathcal{E}) \mid (\!|\mathcal{E}|\!)^{u}_{\sigma} \mid (\!|\mathcal{E}|\!)^{u\blacktriangleright}_{\sigma} \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow (\!|\,|\!) \nRightarrow \tau\rangle$$

$\boxed{d = \mathcal{E}\{d'\}}$ d is obtained by placing d' at the mark in $\mathcal{E}$

$$\frac{}{d = \circ\{d\}}\ \text{FHOuter} \qquad \frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}\ \text{FHApp1} \qquad \frac{[d_1 \text{ final}] \quad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}\ \text{FHApp2} \qquad \frac{d = \mathcal{E}\{d'\}}{\texttt{roll}^{\mu\pi.\tau}(d) = \texttt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}\ \text{FHRoll}$$

$$\frac{d = \mathcal{E}\{d'\}}{\texttt{unroll}(d) = \texttt{unroll}(\mathcal{E})\{d'\}}\ \text{FHUnroll} \qquad \frac{d = \mathcal{E}\{d'\}}{\texttt{inj}^{\tau}_{C}(d) = \texttt{inj}^{\tau}_{C}(\mathcal{E})\{d'\}}\ \text{FHInj} \qquad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u}_{\sigma} = (\!|\mathcal{E}|\!)^{u}_{\sigma}\{d'\}}\ \text{FNEHoleInside} \qquad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u\blacktriangleright}_{\sigma} = (\!|\mathcal{E}|\!)^{u\blacktriangleright}_{\sigma}\{d'\}}\ \text{FMHoleInside}$$

$$\frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}}\ \text{FHCastInside} \qquad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow (\!|\,|\!) \nRightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow (\!|\,|\!) \nRightarrow \tau_2\rangle\{d'\}}\ \text{FHFailedCast}$$

$\boxed{d \mapsto d'}$ d steps to d'

$$\frac{d = \mathcal{E}\{d_0\} \quad d_0 \longrightarrow d_0' \quad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}\ \text{Step}$$