Algebraic Data Types for Hazel

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1 Syntax

$$
\begin{array}{llll}
\mathsf{HTyp} & \tau & ::= & \varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid (\!|\,|\!) \mid (\!|\alpha|\!) \\
\mathsf{HTypPat} & \pi & ::= & \alpha \mid (\!|\,|\!) \\
\mathsf{HExp} & e & ::= & \varnothing \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \mathtt{inj}_C(e) \mid \mathtt{roll}(e) \mid \mathtt{unroll}(e) \\
 & & & \mid (\!|\,|\!)^u \mid (\!|e|\!)^u \mid (\!|e|\!)^{u\blacktriangleright} \\
\mathsf{IHExp} & d & ::= & \varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \mathtt{inj}^{\tau}_C(d) \mid \mathtt{roll}^{\mu\alpha.\tau}(d) \mid \mathtt{unroll}(d) \\
 & & & \mid d\langle \tau \Rightarrow \tau\rangle \mid d\langle \tau \Rightarrow (\!|\,|\!) \not\Rightarrow \tau\rangle \mid (\!|\,|\!)^u_\sigma \mid (\!|d|\!)^u_\sigma \mid (\!|d|\!)^{u\blacktriangleright}_\sigma \\
\mathsf{HTag} & C & ::= & \mathbf{C} \mid (\!|\,|\!)^u \mid (\!|c|\!)^u
\end{array}
$$

1.1 Context Extension

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = (\!|\,|\!) \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{lcll}
[\tau/(\!|\,|\!)]\tau' & = & \tau' & \\
[\tau/\alpha]\varnothing & = & \varnothing & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu(\!|\,|\!).\tau_2 & = & \mu(\!|\,|\!).[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]{+}\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha](\!|\,|\!) & = & (\!|\,|\!) & \\
[\alpha'/\alpha](\!|\alpha|\!) & = & (\!|\alpha'|\!) & \\
[\alpha'/\alpha](\!|\alpha'|\!) & = & (\!|\alpha'|\!) & \text{when } \alpha \neq \alpha'
\end{array}
$$

$\boxed{\Theta \vdash \tau\ \mathsf{valid}}$ τ is a valid type

$$
\frac{}{\Theta \vdash \varnothing\ \mathsf{valid}}\ \textsc{TVUnit}
\qquad
\frac{\Theta \vdash \tau_1\ \mathsf{valid} \qquad \Theta \vdash \tau_2\ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2\ \mathsf{valid}}\ \textsc{TVArr}
\qquad
\frac{\alpha \in \Theta}{\Theta \vdash \alpha\ \mathsf{valid}}\ \textsc{TVVar}
\qquad
\frac{\Theta, \pi \vdash \tau\ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau\ \mathsf{valid}}\ \textsc{TVRec}
\qquad
\frac{\{\Theta \vdash \tau_i\ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \mathsf{valid}}\ \textsc{TVSum}
$$

$$
\frac{}{\Theta \vdash (\!|\,|\!)\ \mathsf{valid}}\ \textsc{TVEHole}
\qquad
\frac{\alpha \notin \Theta}{\Theta \vdash (\!|\alpha|\!)\ \mathsf{valid}}\ \textsc{TVNEHole}
$$

$\boxed{\tau \sim \tau'}$ $\quad\tau$ and τ' are consistent

$$\textsc{TCRefl}\ \frac{}{\tau \sim \tau} \qquad \textsc{TCEHole1}\ \frac{}{\llparenthesis\rrparenthesis \sim \tau} \qquad \textsc{TCEHole2}\ \frac{}{\tau \sim \llparenthesis\rrparenthesis} \qquad \textsc{TCNEHole1}\ \frac{}{\llparenthesis\alpha\rrparenthesis \sim \tau} \qquad \textsc{TCNEHole2}\ \frac{}{\tau \sim \llparenthesis\alpha\rrparenthesis}$$

$$\textsc{TCArr}\ \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

$$\textsc{TCRec}\ \frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \qquad \textsc{TCRecHole1}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\llparenthesis\rrparenthesis.\tau \sim \mu\alpha.\tau'} \qquad \textsc{TCRecHole2}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu\llparenthesis\rrparenthesis.\tau'}$$

$$\textsc{TCSum}\ \frac{\{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

Theorem 2 (Consistency Preserves Validity). *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ $\quad\tau$ has matched arrow type $\tau_1 \to \tau_2$

$$\textsc{MAHole}\ \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\to} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis} \qquad \textsc{MAArr}\ \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ $\quad\tau$ has matched recursive type $\mu\pi.\tau'$

$$\textsc{MRRec}\ \frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau} \qquad \textsc{MRHole}\ \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\mu} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ $\quad e$ synthesizes type τ

$$\textsc{SUnit}\ \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing} \qquad \textsc{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \textsc{SVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SLam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}$$

$$\textsc{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \qquad \textsc{SAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis}$$

$$\textsc{SAsc}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \qquad \textsc{SRollErr}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{roll}(e)\rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis} \qquad \textsc{SUnroll}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$$

$$\textsc{SUnrollNotRec}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}(\llparenthesis e\rrparenthesis^{u\blacktriangleright}) \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SInjErr}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{inj}_C(e)\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SInjTagErr}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{inj}_{\llparenthesis c\rrparenthesis^u}(e) \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SEHole}\ \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}$$

$$\textsc{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash \llparenthesis e\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ $\quad e$ analyzes against type τ

$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau}\textsc{ARoll}$$

$$\frac{\tau \nsim \mu(\!|\!).(\!|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Leftarrow \tau}\textsc{ARollNotRec}$$

$$\frac{\Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow (\!|\!)}\textsc{AInjHole}$$

$$\frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}\textsc{AInj}$$

$$\frac{(\!|c|\!)^u \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash \texttt{inj}_{(\!|c|\!)^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}\textsc{AInjTagErr}$$

$$\frac{C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash (\!|\texttt{inj}_C(e)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}\textsc{AInjBadTag}$$

$$\frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\!)}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}\textsc{AInjUnexpectedBody}$$

$$\frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}\textsc{AInjExpectedBody}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}\textsc{ASubsume}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity). *If $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ then $\emptyset \vdash \tau\ \mathsf{valid}$.*

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

$$\frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \rightsquigarrow \varnothing \dashv \emptyset}\textsc{ESUnit}$$

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}\textsc{ESVar}$$

$$\frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\!) \rightsquigarrow (\!|x|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\!)[\Gamma]}\textsc{ESVarFree}$$

$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}\textsc{ESLam}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \\ \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \\ \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2\end{array}}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_1 \rangle)(d_2\langle \tau_2' \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2}\textsc{ESApp}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim (\!|\!) \to (\!|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\!) \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\!) \rightsquigarrow (\!|d_1|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow (\!|\!) \rangle) \dashv \Delta_1 \cup \Delta_2, u :: (\!|\!) \to (\!|\!)[\Gamma]}\textsc{ESAppNotArr}$$

$$\frac{\begin{array}{c}\emptyset \vdash \tau\ \mathsf{valid} \\ \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta\end{array}}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau \rangle \dashv \Delta}\textsc{ESAsc}$$

$$\frac{\Gamma \vdash e \Leftarrow (\!|\!) \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Rightarrow \mu(\!|\!).(\!|\!) \rightsquigarrow (\!|\texttt{roll}^{\mu(\!|\!).(\!|\!)}(d\langle \tau \Rightarrow (\!|\!) \rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu(\!|\!).(\!|\!)[\Gamma]}\textsc{ESRollErr}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \texttt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \texttt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau' \rangle) \dashv \Delta}\textsc{ESUnroll}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \nsim \mu(\!|\!).(\!|\!)}{\Gamma \vdash \texttt{unroll}\Big((\!|e|\!)^{u\blacktriangleright}\Big) \Rightarrow (\!|\!) \rightsquigarrow \texttt{unroll}\Big((\!|d|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\Big) \dashv \Delta, u :: \mu(\!|\!).(\!|\!)[\Gamma]}\textsc{ESUnrollNotRec}$$

$$\textsc{ESInjErr}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \llparenthesis \texttt{inj}_C(e) \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis \texttt{inj}_C^{\tau'}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESInjTagErr}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{\llparenthesis c \rrparenthesis^u(\tau)\}}{\Gamma \vdash \texttt{inj}_{\llparenthesis c \rrparenthesis^u}(e) \Rightarrow \llparenthesis\rrparenthesis \leadsto \texttt{inj}^{\tau'}_{\llparenthesis c \rrparenthesis^u}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) \dashv \Delta}$$

$$\textsc{ESEHole}\quad \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESNEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \leadsto d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

$$\textsc{EARoll}\quad \frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \leadsto d : \tau'' \dashv \Delta}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau \leadsto \texttt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rangle) : \mu\pi.\tau' \dashv \Delta}$$

$$\textsc{EARollNotRec}\quad \frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau' \dashv \Delta}{\Gamma \vdash \llparenthesis \texttt{roll}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{EAInjHole}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow \llparenthesis\rrparenthesis \leadsto \texttt{inj}_C^{\tau'}(d) : \tau' \dashv \Delta}$$

$$\textsc{EAInj}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \leadsto d : \tau_j' \dashv \Delta}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow \tau \leadsto \texttt{inj}^{\tau}_{C_j}\big(d\langle \tau_j' \Rightarrow \tau_j \rangle\big) : \tau \dashv \Delta}$$

$$\textsc{EAInjTagErr}\quad \frac{\llparenthesis c \rrparenthesis^u \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{\llparenthesis c \rrparenthesis^u(\tau)\}\big\}}{\Gamma \vdash \texttt{inj}_{\llparenthesis c \rrparenthesis^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \leadsto \texttt{inj}^{\tau'}_{\llparenthesis c \rrparenthesis^u}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta}$$

$$\textsc{EAInjBadTag}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau' \dashv \Delta \qquad \tau'' = +\big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{C(\tau')\}\big\}}{\Gamma \vdash \llparenthesis \texttt{inj}_C(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}_C^{\tau''}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjUnexpectedBody}\quad \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau_j \dashv \Delta \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\Big\}\end{array}}{\Gamma \vdash \llparenthesis \texttt{inj}_{C_j}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}^{\tau'}_{C_j}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjExpectedBody}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\Big\}}{\Gamma \vdash \llparenthesis \texttt{inj}_{C_j}(\varnothing) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}^{\tau'}_{C_j}(\varnothing) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EASubsume}\quad \frac{e \neq \llparenthesis\rrparenthesis^u \qquad e \neq \llparenthesis e' \rrparenthesis^u \qquad \Gamma \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta}$$

$$\textsc{EAEHole}\quad \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$$\textsc{EANEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

$$\textsc{TAUnit}\ \frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing} \qquad \textsc{TAVar}\ \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau} \qquad \textsc{TALam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'} \qquad \textsc{TAApp}\ \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

$$\textsc{TARoll}\ \frac{\emptyset \vdash \mu\pi.\tau\ \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau} \qquad \textsc{TAUnroll}\ \frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

$$\textsc{TAInj}\ \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(d) : \tau} \qquad \textsc{TAEHole}\ \frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis\rrparenthesis^u_\sigma : \tau}$$

$$\textsc{TANEHole}\ \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^u_\sigma : \tau} \qquad \textsc{TAMHole}\ \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^{u\blacktriangleright}_\sigma : \tau}$$

$$\textsc{TACast}\ \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2} \qquad \textsc{TAFailedCast}\ \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle : \tau_2}$$

3 Dynamic Semantics

$\boxed{\tau\ \text{ground}}$ τ is a ground type

$$\textsc{GArr}\ \frac{}{\llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis\ \text{ground}} \qquad \textsc{GRec}\ \frac{}{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis\ \text{ground}} \qquad \textsc{GSum}\ \frac{\{\tau_i = \varnothing \vee \tau_i = \llparenthesis\rrparenthesis\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \text{ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

$$\textsc{MGArr}\ \frac{\tau_1 \to \tau_2 \neq \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis} \qquad \textsc{MGRec}\ \frac{\tau \neq \llparenthesis\rrparenthesis}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$$\textsc{MGSum}\ \frac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \qquad \{(\tau_i = \varnothing \implies \tau_i' = \varnothing) \wedge (\tau_i \neq \varnothing \implies \tau_i' = \llparenthesis\rrparenthesis)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{d\ \text{final}}$ d is final

$$\textsc{FBoxedVal}\ \frac{d\ \text{boxedval}}{d\ \text{final}} \qquad \textsc{FIndet}\ \frac{d\ \text{indet}}{d\ \text{final}}$$

$\boxed{d\ \text{val}}$ d is a value

$$\frac{}{\varnothing\ \mathsf{val}}\ \textsc{VUnit} \qquad \frac{}{\lambda x{:}\tau.d\ \mathsf{val}}\ \textsc{VLam} \qquad \frac{d\ \mathsf{val}}{\mathtt{roll}^{\mu\pi.\tau}(d)\ \mathsf{val}}\ \textsc{VRoll} \qquad \frac{d\ \mathsf{val}}{\mathtt{inj}^{\tau}_{\mathbf{C}}(d)\ \mathsf{val}}\ \textsc{VInj}$$

$\boxed{d\ \mathsf{boxedval}}$ d is a boxed value

$$\frac{d\ \mathsf{val}}{d\ \mathsf{boxedval}}\ \textsc{BVVal} \qquad \frac{d\ \mathsf{boxedval}}{\mathtt{roll}^{\mu\pi.\tau}(d)\ \mathsf{boxedval}}\ \textsc{BVRoll} \qquad \frac{d\ \mathsf{boxedval}}{\mathtt{inj}^{\tau}_{\mathbf{C}}(d)\ \mathsf{boxedval}}\ \textsc{BVInj}$$

$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d\ \mathsf{boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle\ \mathsf{boxedval}}\ \textsc{BVArrCast}$$

$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d\ \mathsf{boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle\ \mathsf{boxedval}}\ \textsc{BVRecCast}$$

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i\in\mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i\in\mathcal{C}} \\ \tau \neq \tau' \qquad d\ \mathsf{boxedval}\end{array}}{d\langle \tau \Rightarrow \tau'\rangle\ \mathsf{boxedval}}\ \textsc{BVSumCast}$$

$$\frac{d\ \mathsf{boxedval} \qquad \tau\ \mathsf{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle\ \mathsf{boxedval}}\ \textsc{BVHoleCast}$$

$\boxed{d\ \mathsf{indet}}$ d is indeterminate

$$\frac{}{\llparenthesis\rrparenthesis^u_\sigma\ \mathsf{indet}}\ \textsc{IEHole} \qquad \frac{d\ \mathsf{final}}{\llparenthesis d\rrparenthesis^u_\sigma\ \mathsf{indet}}\ \textsc{INEHole}$$

$$\frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle \qquad d_1\ \mathsf{indet} \qquad d_2\ \mathsf{final}}{d_1(d_2)\ \mathsf{indet}}\ \textsc{IApp} \qquad \frac{d\ \mathsf{indet}}{\mathtt{roll}^{\mu\pi.\tau}(d)\ \mathsf{indet}}\ \textsc{IRoll}$$

$$\frac{d\ \mathsf{indet}}{\mathtt{unroll}(d)\ \mathsf{indet}}\ \textsc{IUnroll} \qquad \frac{d\ \mathsf{indet}}{\mathtt{inj}^{\tau}_{\mathbf{C}}(d)\ \mathsf{indet}}\ \textsc{IInj} \qquad \frac{C \neq \mathbf{C} \qquad d\ \mathsf{final}}{\mathtt{inj}^{\tau}_{C}(d)\ \mathsf{indet}}\ \textsc{IInjHole} \qquad \frac{d\ \mathsf{indet} \qquad \tau\ \mathsf{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle\ \mathsf{indet}}\ \textsc{ICastGroundHole}$$

$$\frac{d \neq d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis\rangle \qquad d\ \mathsf{indet} \qquad \tau\ \mathsf{ground}}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau\rangle\ \mathsf{indet}}\ \textsc{ICastHoleGround}$$

$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d\ \mathsf{indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle\ \mathsf{indet}}\ \textsc{ICastArr} \qquad \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d\ \mathsf{indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle\ \mathsf{indet}}\ \textsc{ICastRec}$$

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i\in\mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i\in\mathcal{C}} \\ \tau \neq \tau' \qquad d\ \mathsf{indet}\end{array}}{d\langle \tau \Rightarrow \tau'\rangle\ \mathsf{indet}}\ \textsc{ICastSum} \qquad \frac{d\ \mathsf{final} \qquad \tau_1\ \mathsf{ground} \qquad \tau_2\ \mathsf{ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle\ \mathsf{indet}}\ \textsc{IFailedCast}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\frac{[d_2\ \mathsf{final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1}\ \textsc{ITApp} \qquad \frac{[d\ \mathsf{final}]}{\mathtt{unroll}(\mathtt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}\ \textsc{ITUnroll}$$

$$\frac{[d_1\ \mathsf{final}] \qquad [d_2\ \mathsf{final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1\rangle))\langle \tau_2 \Rightarrow \tau_2'\rangle}\ \textsc{ITAppCast}$$

$$\frac{[d\ \mathsf{final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\mathtt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \mathtt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle}\ \textsc{ITUnrollCast} \qquad \frac{[d\ \mathsf{final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}\ \textsc{ITCastId}$$

$$\textsc{ITCastSucceed}\quad \frac{[d\ \text{final}] \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow (\!|\!|\!) \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastFail}\quad \frac{[d\ \text{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground}}{d\langle \tau_1 \Rightarrow (\!|\!|\!) \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow (\!|\!|\!) \nRightarrow \tau_2\rangle}$$

$$\textsc{ITGround}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow (\!|\!|\!)\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow (\!|\!|\!)\rangle}$$

$$\textsc{ITExpand}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle (\!|\!|\!) \Rightarrow \tau\rangle \longrightarrow d\langle (\!|\!|\!) \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\begin{array}{lcl} \mathsf{EvalCtx}\ \ \mathcal{E} & ::= & \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \mathtt{unroll}(\mathcal{E}) \mid \mathtt{inj}_C^{\tau}(\mathcal{E}) \mid (\!|\mathcal{E}|\!)^u_\sigma \mid (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma \\ & & \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow (\!|\!|\!) \nRightarrow \tau\rangle \end{array}$$

$\boxed{d = \mathcal{E}\{d'\}}$ d is obtained by placing d' at the mark in $\mathcal{E}$

$$\textsc{FHOuter}\quad \frac{}{d = \circ\{d\}} \qquad \textsc{FHApp1}\quad \frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}} \qquad \textsc{FHApp2}\quad \frac{[d_1\ \text{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

$$\textsc{FHRoll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{roll}^{\mu\pi.\tau}(d) = \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FHUnroll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{unroll}(d) = \mathtt{unroll}(\mathcal{E})\{d'\}} \qquad \textsc{FHInj}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{inj}_C^{\tau}(d) = \mathtt{inj}_C^{\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FNEHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^u_\sigma = (\!|\mathcal{E}|\!)^u_\sigma\{d'\}} \qquad \textsc{FMHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u\blacktriangleright}_\sigma = (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma\{d'\}}$$

$$\textsc{FHCastInside}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}} \qquad \textsc{FHFailedCast}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow (\!|\!|\!) \nRightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow (\!|\!|\!) \nRightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ d steps to d'

$$\textsc{Step}\quad \frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$