

Hazel Phi: 11-type-constructors

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NOTES

need to finish up OK* proofs now that unicity is done

SYNTAX

Kind	κ	$::=$	$\mathbf{Type} \mid \mathbf{KHole} \mid \mathbf{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2$
User Types	$\hat{\tau}$	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \hat{\tau} \rangle^u \mid \lambda t::\mathbf{Type}.\hat{\tau} \mid \tau_1 \tau_2$
Internal Types	τ	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \tau \rangle^u \mid \langle t \rangle^u \mid \lambda t::\kappa.\tau \mid \tau_1 \tau_2$
Base Types	\mathbf{bse}	$::=$	$\mathbf{Int} \mid \mathbf{Float} \mid \mathbf{Bool}$
BinOp	\oplus	$::=$	$\times \mid + \mid \rightarrow$
Type Pattern			
User Expression			
Internal Expression			

DECLARATIVES

$\Delta; \Phi \vdash \tau ::> \kappa$ τ has principal (well formed) kind κ

$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathbf{bse} ::> \mathbf{S}_{\mathbf{Type}}(\mathbf{bse})} \text{PK-Base}$	$\frac{\Delta; \Phi_1, t::\kappa, \Phi_2 \vdash \text{OK}}{\Delta; \Phi \vdash t ::> \mathbf{S}_{\kappa}(t)} \text{PK-Var}$
$\frac{\Delta; \Phi \vdash \tau_1 :: \mathbf{Type} \quad \Delta; \Phi \vdash \tau_2 :: \mathbf{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathbf{S}_{\mathbf{Type}}(\tau_1 \oplus \tau_2)} \text{PK-}\oplus$	$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \langle \rangle^u ::> \mathbf{S}_{\kappa}(\langle \rangle^u)} \text{PK-EHole}$
$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad \Delta; \Phi \vdash \tau :: \kappa_1}{\Delta; \Phi \vdash \langle \tau \rangle^u ::> \mathbf{S}_{\kappa}(\langle \tau \rangle^u)} \text{PK-NEHole}$	$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad t \notin \Phi}{\Delta; \Phi \vdash \langle t \rangle^u ::> \mathbf{S}_{\kappa}(\langle t \rangle^u)} \text{PK-Unbound}$
$\frac{\Delta; \Phi, t::\kappa_1 \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau ::> \mathbf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)} \text{PK-}\lambda$	
$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \quad \Delta; \Phi \vdash \kappa \blacktriangleright \Pi_{t::\kappa_1}.\kappa_2 \quad \Delta; \Phi \vdash \tau_2 :: \kappa_1}{\Delta; \Phi \vdash \tau_1 \tau_2 ::> [\tau_2/t]\kappa_2} \text{PK-Ap}$	

$\Delta; \Phi \vdash \tau :: \kappa$ τ is well formed at kind κ

$$\frac{\Delta; \Phi \vdash \tau :: > \mathbf{S}_{\kappa}(\tau)}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-1} \quad \frac{\Delta; \Phi \vdash \tau :: > \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Subsump}$$

$$\frac{\Delta; \Phi \vdash \tau :: > \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Reit} \quad \frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau :: \mathbf{S}_{\kappa}(\tau)} \text{WFaK-Self}$$

$$\frac{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_3} \cdot \kappa_4 \quad \Delta; \Phi \vdash \Pi_{t::\kappa_3} \cdot \kappa_4 \lesssim \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_1} \cdot \kappa_2} \text{WFaK-PCSKTrans}$$

$$\frac{\Delta; \Phi \vdash \tau :: \mathbf{S}_{\kappa}(\tau_1) \quad \Delta; \Phi \vdash \tau_1 :: \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Flatten}$$

$\Delta; \Phi \vdash \kappa \dashv \vdash \Pi_{t::\kappa_1} \cdot \kappa_2$ κ has matched Π -kind $\Pi_{t::\kappa_1} \cdot \kappa_2$

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathbf{KHole} \dashv \vdash \Pi_{t::\mathbf{KHole}} \cdot \mathbf{KHole}} \dashv \vdash \text{-KHole} \quad \frac{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} > \mathbf{S}_{\mathbf{KHole}}(\tau)}{\Delta; \Phi \vdash \kappa \dashv \vdash \Pi_{t::\mathbf{SKHole}(\tau)} \cdot \mathbf{SKHole}(\tau \ t)} \dashv \vdash \text{-SKHole}$$

$$\frac{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} > \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \kappa \dashv \vdash \Pi_{t::\kappa_1} \cdot \kappa_2} \dashv \vdash \text{-}\Pi$$

$\Delta; \Phi \vdash \kappa_1 \equiv^* > \kappa_2$ κ_1 singleton reduces to κ_2

$$\frac{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{S}_{\kappa}(\tau_1)}(\tau) \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{S}_{\kappa}(\tau_1)}(\tau) \equiv^* > \mathbf{S}_{\kappa}(\tau_1)} \equiv^* \text{-1} \quad \frac{\Delta; \Phi \vdash \kappa_1 \equiv^* > \kappa_2 \quad \Delta; \Phi \vdash \kappa_2 \equiv^* > \kappa_3}{\Delta; \Phi \vdash \kappa_1 \equiv^* > \kappa_3} \equiv^* \text{-Trans}$$

$\Delta; \Phi \vdash \kappa_1 \equiv^{\text{norm}} > \kappa_2$ κ_1 has singleton normal form κ_2

$$\frac{\Delta; \Phi \vdash \kappa \equiv^* > \mathbf{S}_{\text{Type}}(\tau)}{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} > \mathbf{S}_{\text{Type}}(\tau)} \equiv^{\text{norm}} \text{-Type} \quad \frac{\Delta; \Phi \vdash \kappa \equiv^* > \mathbf{S}_{\mathbf{KHole}}(\tau)}{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} > \mathbf{S}_{\mathbf{KHole}}(\tau)} \equiv^{\text{norm}} \text{-KHole}$$

$$\frac{\Delta; \Phi \vdash \kappa \equiv^* > \mathbf{S}_{\Pi_{t::\kappa_1} \cdot \kappa_2}(\tau)}{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} > \Pi_{t::\kappa_1} \cdot \mathbf{S}_{[t_1/t]\kappa_2}(\tau \ t_1)} \equiv^{\text{norm}} \text{-}\Pi$$

$\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$ κ_1 is equivalent to κ_2

$$\begin{array}{c}
\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \equiv \kappa} \text{KEquiv-Ref1} \qquad \frac{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-Symm} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-Trans} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv^* \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-SReduc} \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv^{\text{norm}} \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-SNorm} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4} \text{KEquiv-}\Pi \\
\\
\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2 \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa_1}(\tau_1) \equiv \mathbf{S}_{\kappa_2}(\tau_2)} \text{KEquiv-SKind}
\end{array}$$

$\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$ κ_1 is a consistent subkind of κ_2

$$\begin{array}{c}
\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{KHole} \lesssim \kappa} \text{CSK-KHoleL} \qquad \frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{KHole}} \text{CSK-KHoleR} \\
\\
\frac{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \text{ OK} \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \lesssim \kappa} \text{CSK-SKind}_{\mathbf{KHoleL}} \\
\\
\frac{\Delta; \Phi \vdash \kappa \text{ OK} \quad \Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{S}_{\mathbf{KHole}}(\tau)} \text{CSK-SKind}_{\mathbf{KHoleR}} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \text{CSK-KEquiv} \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \lesssim \kappa_4 \quad \Delta; \Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \text{CSK-Normal} \\
\\
\frac{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \lesssim \kappa} \text{CSK-SKind} \qquad \frac{\Delta; \Phi \vdash \kappa_3 \lesssim \kappa_1 \quad \Delta; \Phi, t::\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4} \text{CSK-}\Pi \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2 \quad \Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa_1}(\tau_1) \lesssim \mathbf{S}_{\kappa_2}(\tau_2)} \text{CSK-?}
\end{array}$$

$\boxed{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau \equiv^{\kappa} \tau} \text{EquivAK-Ref1} \qquad \frac{\Delta; \Phi \vdash \tau_2 \equiv^{\kappa} \tau_1}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \text{EquivAK-Symm}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_3 \quad \Delta; \Phi \vdash \tau_3 \equiv^{\kappa} \tau_1}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \text{EquivAK-Trans}$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \equiv \mathbf{S}_{\kappa}(\tau_2)}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \text{EquivAK-SKind}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \Pi_{t::\kappa_1} \kappa_3 \quad \Delta; \Phi \vdash \tau_2 :: \Pi_{t::\kappa_1} \kappa_4 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \equiv^{\kappa_2} \tau_2 \ t}{\Delta; \Phi \vdash \tau_1 \equiv^{\Pi_{t::\kappa_1} \kappa_2} \tau_2} \text{EquivAK-}\Pi$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\Pi_{t::\kappa_1} \kappa_2} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \equiv^{\kappa_1} \tau_4}{\Delta; \Phi \vdash \tau_1 \tau_2 \equiv^{[\tau_2/t]\kappa_2} \tau_3 \tau_4} \text{EquivAK-Ap}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\text{Type}} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \equiv^{\text{Type}} \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \equiv^{\text{Type}} \tau_3 \oplus \tau_4} \text{EquivAK-}\oplus$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\mathbf{S}_{\kappa}(\tau)} \tau_2}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} (1)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \equiv^{\kappa} \tau_2}{\Delta; \Phi \vdash \lambda t::\kappa_1. \tau_1 \equiv^{\Pi_{t::\kappa_1} \kappa} \lambda t::\kappa_2. \tau_2} (2)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2 \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} (3)$$

$\boxed{\Delta; \Phi \vdash \kappa \text{ OK}}$ κ is well formed

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{Type OK}} \text{KWF-Type}$$

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{KHole OK}} \text{KWF-KHole}$$

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \text{ OK}} \text{KWF-SKind}$$

$$\frac{\Delta; \Phi, t::\kappa_1 \vdash \kappa_2 \text{ OK}}{\Delta; \Phi \vdash \Pi_{t::\kappa_1} \kappa_2 \text{ OK}} \text{KWF-}\Pi$$

$\boxed{\Delta; \Phi \vdash \text{OK}}$ Context is well formed

$$\frac{}{\cdot; \cdot \vdash \text{OK}} \text{CWF-Nil}$$

$$\frac{t \notin \Phi \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi, t::\kappa \vdash \text{OK}} \text{CWF-TypVar}$$

$$\frac{u \notin \Delta \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta, u::\kappa; \Phi \vdash \text{OK}} \text{CWF-Hole}$$

ALGORITHM

(syntactically distinguished up to α -equivalence... when needed)

(TODO: remove the ‘... when needed’. The bound variable renamings should get adjusted)

(NOTE: current implementation has explicit \equiv_{α} checks which are not written in these rules since we eventually want to use De Bruijn indices, hence the above)

$\Delta; \Phi \triangleright \tau_1 \stackrel{\kappa}{=} \tau_2$ τ_1 is equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \triangleright \tau_1 \stackrel{\kappa}{\Rightarrow} \tau_\omega \quad \Delta; \Phi \triangleright \tau_2 \stackrel{\kappa}{\Rightarrow} \tau_\omega}{\Delta; \Phi \triangleright \tau_1 \stackrel{\kappa}{=} \tau_2}$$

$\Delta; \Phi \triangleright \tau \uparrow \kappa$ path τ has natural kind κ

$$\frac{}{\Delta; \Phi \triangleright \mathbf{bse} \uparrow \mathbf{Type}} \quad \frac{\Phi_1, t::\kappa, \Phi_2}{\Delta; \Phi \triangleright t \uparrow \kappa} \quad \frac{}{\Delta; \Phi \triangleright \tau_1 \oplus \tau_2 \uparrow \mathbf{Type}} \quad \frac{\Delta_1, u::\kappa, \Delta_2}{\Delta; \Phi \triangleright \langle \rangle^u \uparrow \kappa}$$

$$\frac{\Delta_1, u::\kappa, \Delta_2}{\Delta; \Phi \triangleright \langle \tau \rangle^u \uparrow \kappa} \quad \frac{\Delta; \Phi \triangleright \tau_1 \uparrow \kappa \quad \Delta; \Phi \triangleright \kappa \Rightarrow \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \triangleright \tau_1 \tau_2 \uparrow [\tau_2/t] \kappa_2}$$

$\Delta; \Phi \triangleright \mathcal{E}[\tau_1] \rightsquigarrow \mathcal{E}[\tau_2]$ $\mathcal{E}[\tau_1]$ single step weak head reduces to $\mathcal{E}[\tau_2]$

$$\frac{TODO : \text{ check } \tau_1 \text{ against } \kappa}{\Delta; \Phi \triangleright \mathcal{E}[(\lambda t::\kappa. \tau) \tau_1] \rightsquigarrow \mathcal{E}[[\tau_1/t] \tau]} \quad \frac{\Delta; \Phi \triangleright t \uparrow \mathbf{S}_\kappa(\tau)}{\Delta; \Phi \triangleright \mathcal{E}[t] \rightsquigarrow \mathcal{E}[\tau]} \quad \frac{}{\Delta; \Phi \triangleright \mathbf{bse} \rightsquigarrow \mathbf{bse}}$$

$$\frac{}{\Delta; \Phi \triangleright \tau_1 \oplus \tau_2 \rightsquigarrow \tau_1 \oplus \tau_2} \quad \frac{}{\Delta; \Phi \triangleright \mathcal{E}[\langle \rangle^u] \rightsquigarrow \mathcal{E}[\langle \rangle^u]} \quad \frac{}{\Delta; \Phi \triangleright \mathcal{E}[\langle \tau \rangle^u] \rightsquigarrow \mathcal{E}[\langle \tau \rangle^u]}$$

$$\frac{}{\Delta; \Phi \triangleright \mathcal{E}[\langle t \rangle^u] \rightsquigarrow \mathcal{E}[\langle t \rangle^u]} \quad \frac{}{\Delta; \Phi \triangleright \lambda t::\kappa. \tau \rightsquigarrow \lambda t::\kappa. \tau}$$

$\Delta; \Phi \triangleright \tau \Downarrow \tau_\psi$ τ weak head normalizes to τ_ψ

$$\frac{\Delta; \Phi \triangleright \tau \rightsquigarrow \tau_\chi \quad \Delta; \Phi \triangleright \tau_\chi \Downarrow \tau_\psi}{\Delta; \Phi \triangleright \tau \Downarrow \tau_\psi} \quad \frac{\Delta; \Phi \triangleright \tau \rightsquigarrow \tau}{\Delta; \Phi \triangleright \tau \Downarrow \tau}$$

$\Delta; \Phi \triangleright \tau \stackrel{\kappa}{\Rightarrow} \tau_\omega$ τ normalizes to τ_ω at kind κ

$$\frac{\Delta; \Phi \triangleright \kappa \Rightarrow \mathbf{Type} \quad \Delta; \Phi \triangleright \tau \Downarrow \tau_\psi \quad \Delta; \Phi \triangleright \tau_\psi \xrightarrow{\mathbf{Type}} \tau_\omega}{\Delta; \Phi \triangleright \tau \stackrel{\kappa}{\Rightarrow} \tau_\omega} \quad \frac{\Delta; \Phi \triangleright \kappa \Rightarrow \mathbf{KHole} \quad TODO :}{\Delta; \Phi \triangleright \tau \stackrel{\kappa}{\Rightarrow} \tau_\omega}$$

$$\frac{\Delta; \Phi \triangleright \kappa \Rightarrow \mathbf{S}_{\mathbf{Type}}(\tau_s) \quad \Delta; \Phi \triangleright \tau \Downarrow \tau_\psi \quad \Delta; \Phi \triangleright \tau_\psi \xrightarrow{\mathbf{Type}} \tau_\omega \quad TODO : \text{ prop : } \tau_\omega = \tau_s}{\Delta; \Phi \triangleright \tau \stackrel{\kappa}{\Rightarrow} \tau_\omega}$$

$$\frac{\Delta; \Phi \triangleright \kappa \Rightarrow \mathbf{SKHole}(\tau_s) \quad TODO :}{\Delta; \Phi \triangleright \tau \stackrel{\kappa}{\Rightarrow} \tau_\omega} \quad \frac{\Delta; \Phi \triangleright \kappa \Rightarrow \Pi_{t::\kappa_{\omega_1}} \cdot \kappa_{\omega_2} \quad \Delta; \Phi, t_1::\kappa_{\omega_1} \triangleright \tau \ t_1 \stackrel{[t_1/t] \kappa_2}{\Rightarrow} \tau_\omega}{\Delta; \Phi \triangleright \tau \stackrel{\kappa}{\Rightarrow} \lambda t_1::\kappa_{\omega_1} \cdot \tau_\omega}$$

$\boxed{\Delta; \Phi \triangleright \tau_\psi \longrightarrow^\kappa \tau_\omega}$ path τ_ψ normalizes to τ_ω with kind κ

$$\begin{array}{c}
\frac{}{\Delta; \Phi \triangleright \mathbf{bse} \longrightarrow^{\mathbf{Type}} \mathbf{bse}} \quad \frac{\Phi_1, t::\kappa, \Phi_2}{\Delta; \Phi \triangleright t \longrightarrow^\kappa t} \quad \frac{\Delta; \Phi \triangleright \tau_1 \xRightarrow{\mathbf{Type}} \tau_{\omega_1} \quad \Delta; \Phi \triangleright \tau_2 \xRightarrow{\mathbf{Type}} \tau_{\omega_2}}{\Delta; \Phi \triangleright \tau_1 \oplus \tau_2 \xrightarrow{\mathbf{Type}} \tau_{\omega_1} \oplus \tau_{\omega_2}} \\
\\
\frac{\Delta_1, u::\kappa, \Delta_2}{\Delta; \Phi \triangleright \langle \rangle^u \longrightarrow^\kappa \langle \rangle^u} \quad \frac{\Delta_1, u::\kappa, \Delta_2}{\Delta; \Phi \triangleright \langle \tau \rangle^u \longrightarrow^\kappa \langle \tau \rangle^u} \\
\\
\frac{\Delta; \Phi \triangleright \tau_1 \longrightarrow^\kappa \tau_{\omega_1} \quad \Delta; \Phi \triangleright \kappa \Longrightarrow \Pi_{t::\kappa_1}. \kappa_2 \quad \Delta; \Phi \triangleright \tau_2 \xRightarrow{\kappa_1} \tau_{\omega_2}}{\Delta; \Phi \triangleright \tau_1 \tau_2 \longrightarrow^{[\tau_{\omega_2}/t]\kappa_2} \tau_{\omega_1} \tau_{\omega_2}}
\end{array}$$

$\boxed{\Delta; \Phi \triangleright \kappa \Longrightarrow \kappa_\omega}$ κ normalizes to κ_ω

$$\begin{array}{c}
\frac{}{\Delta; \Phi \triangleright \mathbf{Type} \Longrightarrow \mathbf{Type}} \quad \frac{}{\Delta; \Phi \triangleright \mathbf{KHole} \Longrightarrow \mathbf{KHole}} \quad \frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \mathbf{Type} \quad \Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega}{\Delta; \Phi \triangleright \mathbf{S}_\kappa(\tau) \Longrightarrow \mathbf{S}_{\mathbf{Type}}(\tau_\omega)} \\
\\
\frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \mathbf{KHole} \quad \Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega}{\Delta; \Phi \triangleright \mathbf{S}_\kappa(\tau) \Longrightarrow \mathbf{S}_{\mathbf{KHole}}(\tau_\omega)} \quad \frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \mathbf{S}_{\kappa_1}(\tau_1) \quad \Delta; \Phi \triangleright \mathbf{S}_{\kappa_1}(\tau_1) \Longrightarrow \kappa_\omega}{\Delta; \Phi \triangleright \mathbf{S}_\kappa(\tau) \Longrightarrow \kappa_\omega} \\
\\
\frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \Pi_{t::\kappa_1}. \kappa_2 \quad \Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega \quad \Delta; \Phi, t_1::\kappa_1 \triangleright \tau_\omega \quad t_1 \xRightarrow{[t_1/t]\kappa_2} \tau_{\omega_1}}{\Delta; \Phi \triangleright \mathbf{S}_\kappa(\tau) \Longrightarrow \Pi_{t_1::\kappa_1}. \mathbf{S}_{[t_1/t]\kappa_2}(\tau_{\omega_1})} \\
\\
\frac{\Delta; \Phi \triangleright \kappa_1 \Longrightarrow \kappa_{\omega_1} \quad \Delta; \Phi, t::\kappa_{\omega_1} \triangleright \kappa_2 \Longrightarrow \kappa_{\omega_2}}{\Delta; \Phi \triangleright \Pi_{t::\kappa_1}. \kappa_2 \Longrightarrow \Pi_{t::\kappa_{\omega_1}}. \omega_2}
\end{array}$$

METATHEORY

subderivation preserving inferences:

- premiss
- COK (Context OK)
- PoS (premiss of subderivation)

Lemma 1 (COK). *If $\Delta; \Phi \vdash \mathcal{J}$, then $\Delta; \Phi \vdash OK$ in a subderivation (where $\Delta; \Phi \vdash \mathcal{J} \neq \Delta; \Phi \vdash OK$)*

Proof. By induction on derivations.

No interesting cases. □

Lemma 2 (Exchange).

If $\Delta; \Phi_1, t_{L1}::\kappa_{L1}, t_{L2}::\kappa_{L2}, \Phi_2 \vdash \mathcal{J}$ and $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash OK$, then $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

Proof. By induction on derivations.

No interesting cases.

(Only rules with Φ extended in the consequent are interesting, which is only CWF-TypVar, but when \mathcal{J} is CWF, Exchange is identity) □

Corollary 3 (Marked-Exchange).

If $\Delta; \Phi, \underline{t_{L1}::\kappa_{L1}}, \underline{t_{L2}::\kappa_{L2}} \vdash \mathcal{J}$ and $\Delta; \Phi, \underline{t_{L2}::\kappa_{L2}}, \underline{t_{L1}::\kappa_{L1}} \vdash OK$, then $\Delta; \Phi, \underline{t_{L2}::\kappa_{L2}}, \underline{t_{L1}::\kappa_{L1}} \vdash \mathcal{J}$

Proof. Exchange when $\Phi_2 = \cdot$ □

Lemma 4 (Weakening).

If $\Delta; \Phi \vdash \mathcal{J}$ and $\Delta; \Phi, \underline{t_L :: \kappa_L} \vdash OK$ and $t_L \notin \mathcal{J}$ and $\forall t \in \kappa_L, t \notin \mathcal{J}$, then $\Delta; \Phi, \underline{t_L :: \kappa_L} \vdash \mathcal{J}$

Proof. see addendum □

Lemma 5 (K-Substitution).

If $\Delta; \Phi \vdash \tau_{L1} :: \kappa_{L1}$ and $\Delta; \Phi, \underline{t_L :: \kappa_{L1}} \vdash \tau_{L2} :: \kappa_{L2}$, then $\Delta; \Phi \vdash [\tau_{L1}/t_L] \tau_{L2} :: [\tau_{L1}/t_L] \kappa_{L2}$
(induction on $\Delta; \Phi, \underline{t_L :: \kappa_{L1}} \vdash \tau_{L2} :: \kappa_{L2}$)

Lemma 6 (PK-Substitution). If $\Delta; \Phi \vdash \tau_{L1} :: \kappa_{L1}$ and $\Delta; \Phi, \underline{t_L :: \kappa_{L1}} \vdash \tau_{L2} :: \kappa_{L2}$ and $\Delta; \Phi \vdash [\tau_{L1}/t_L] \tau_{L2} :: \kappa_{L3}$, then $\Delta; \Phi \vdash [\tau_{L2}/t_L] \kappa_{L2} \equiv \kappa_{L3}$

Lemma 7 (OK-Substitution).

If $\Delta; \Phi \vdash \tau_L :: \kappa_{L1}$ and $\Delta; \Phi, \underline{t_L :: \kappa_{L1}} \vdash \kappa_{L2} OK$, then $\Delta; \Phi \vdash [\tau_L/t_L] \kappa_{L2} OK$
(induction on $\Delta; \Phi, \underline{t_L :: \kappa_{L1}} \vdash \kappa_{L2} OK$)

Theorem 8 (OK-PK). If $\Delta; \Phi \vdash \tau :: \kappa$, then $\Delta; \Phi \vdash \kappa OK$

Theorem 9 (OK-WFaK). If $\Delta; \Phi \vdash \tau :: \kappa$, then $\Delta; \Phi \vdash \kappa OK$

Theorem 10 (OK-MatchPi). If $\Delta; \Phi \vdash \kappa \blacktriangleright \Pi_{t :: \kappa_1} \cdot \kappa_2$, then $\Delta; \Phi \vdash \kappa OK$ and $\Delta; \Phi \vdash \Pi_{t :: \kappa_1} \cdot \kappa_2 OK$

Theorem 11 (OK-KEquiv). If $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, then $\Delta; \Phi \vdash \kappa_1 OK$ and $\Delta; \Phi \vdash \kappa_2 OK$

Theorem 12 (OK-CSK). If $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, then $\Delta; \Phi \vdash \kappa_1 OK$ and $\Delta; \Phi \vdash \kappa_2 OK$

Theorem 13 (OK-EquivAK). If $\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2$, then $\Delta; \Phi \vdash \tau_1 :: \kappa$ and $\Delta; \Phi \vdash \tau_2 :: \kappa$ and $\Delta; \Phi \vdash \kappa OK$

Proof. see addendum □

Proof.

Weakening

By induction on derivations.

Note: When applying Weakening in the induction, check that the left premis is always a subderivation, and check variable exclusion conditions are satisfied (usually checked elsewhere in the derivation).

$$\frac{\frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l_L \notin \Phi} \text{IH}}{\text{PoS}} \quad \frac{\frac{l_L \notin \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l \in \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l_L \notin \mathcal{J}}{l_L \notin \kappa_L} \text{IH}}{l_L \notin \Phi, l_L \sqsupseteq \kappa_L} \text{IH} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi \vdash \kappa_L \text{ OK}} \text{IH}}{\text{PoS}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l \notin \Phi} \text{PoS} \quad \frac{l_L \notin \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{\forall l \in \kappa_L, l \notin \mathcal{J}}{l \notin \kappa_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \notin \mathcal{J}} \text{IH}}{l \notin \Phi, l_L \sqsupseteq \kappa_L} \text{IH} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Marked-Exchange} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{PK-}\lambda$$
$$\frac{\frac{\frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l_L \notin \Phi} \text{IH}}{\text{PoS}} \quad \frac{l_L \notin \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l \in \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l_L \notin \mathcal{J}}{l_L \notin \kappa_L} \text{IH} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi \vdash \kappa_L \text{ OK}} \text{IH}}{\text{PoS}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l \notin \Phi} \text{PoS} \quad \frac{l_L \notin \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{\forall l \in \kappa_L, l \notin \mathcal{J}}{l \notin \kappa_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \notin \mathcal{J}} \text{IH}}{l \notin \Phi, l_L \sqsupseteq \kappa_L} \text{IH} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Marked-Exchange} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{KEquiv-II} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CSK-II} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{EquivAK-II} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{KWF-II}$$

O?K-*

By simultaneous induction on derivations.

The interesting cases per theorem:

K-Substitution by type size??

OK-Substitution

OK-PK

$$\frac{\frac{\Delta; \Phi \vdash \text{base} \sqsupseteq \Phi_{\text{Type}}(\text{base})}{\Delta; \Phi \vdash \text{base}::\text{Type}} \text{premiss} \quad \text{VFaK-I} \quad \frac{\Delta; \Phi \vdash \text{base}::\text{Type}}{\Delta; \Phi \vdash \text{S}_{\text{Type}}(\text{base}) \text{ OK}} \text{KWF-SKInd}$$

$$\Delta; \Phi \vdash [\tau_2/\theta] \kappa_R \text{ OK-Substitution}$$

OK-WFaK

□

Definition 1 (Singleton Depth).

$$SSize : \{\kappa\}'' \rightarrow \mathbb{N}$$

$$SSize(\kappa_x) = \begin{cases} SSize(\kappa) + 1 & \text{if } \kappa_x = \mathbf{S}_\kappa(\tau) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 14 ($\equiv^*>$ -diminution). *If $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L1}$, then $SSize(\kappa_L) > SSize(\kappa_{L1})$*

Proof. By induction on derivations (and transitivity of $>$ on \mathbb{N}) □

Lemma 15 ($\equiv^*>-n+1$ -nicity). *If $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L2}$ where $SSize(\kappa_L) = n+1$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. By $\equiv^*>$ -diminution, $\equiv^*>$ -Trans cannot be the last inference of a derivation of $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L1}$ since $SSize(\kappa_1) \geq SSize(\kappa_2) + 2$ (in $\equiv^*>$ -Trans). Thus, $\equiv^*>-1$ must have been the last inference. Similarly for $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L2}$, thus $\kappa_{L1} = \kappa_{L2}$ □

Lemma 16 ($\equiv^*>$ -stepwise). *If $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L1}$ where $SSize(\kappa_L) = m$ and $SSize(\kappa_{L1}) = n$ and $m > n + 1$, then the derivation must contain subderivations of each singleton depth inbetween*

Proof. More precisely this says, where $m > n$ by $\equiv^*>$ -diminution, the derivation must contain subderivations of each $\Delta; \Phi \vdash \kappa_i \equiv^*> \kappa_j$ where $m \geq i > j \geq n$, $SSize(\kappa_k) = k$ when $m \geq k \geq n$, $\kappa_m = \kappa_L$, $\kappa_n = \kappa_{L1}$.

By induction on derivations (base case is where $m = n + 2$, which necessitates a last inference of $\equiv^*>$ -Trans. Each premiss must have SSize difference of 1, fulfilling hypothesis) □

Lemma 17 ($\equiv^*>-m + n$ -nicity). *If $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \equiv^*> \kappa_{L2}$ where $SSize(\kappa_L) = m + n$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. By $\equiv^*>$ -stepwise and $\equiv^*>-n + 1$ -nicity when $m > n + 1$.

By $\equiv^*>-n + 1$ -nicity when $m = n + 1$.

No other cases by $\equiv^*>$ -diminution. □

Theorem 18 ($\equiv^{\text{norm}}>$ -Unicity). *If $\Delta; \Phi \vdash \kappa_L \equiv^{\text{norm}}> \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \equiv^{\text{norm}}> \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)

All $\equiv^{\text{norm}}>$ rules have $\equiv^*>$ premiss with rhs singleton depth 1. By $\equiv^*>-m + n$ -nicity, where $n = 1$. □

Theorem 19 ($\blacktriangleright_{\Pi}$ -Unicity). *If $\Delta; \Phi \vdash \tau_L \blacktriangleright_{\Pi} \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L \blacktriangleright_{\Pi} \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)

By unicity of $\equiv^{\text{norm}}>$. □

Theorem 20 (PK-Unicity). *If $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)

As PK is syntax directed, proof is by inspection for all rules except PK- λ (variables in contexts are unique— see context rules), which is by induction on derivations, and PK-Ap, which requires of unicity of $\blacktriangleright_{\Pi}$ (above theorem). □

Theorem 21 (PK-Principality). *If $\Delta; \Phi \vdash \tau ::> \kappa_1$ and $\Delta; \Phi \vdash \tau ::> \kappa_2$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

Proof. From definition of $\Delta; \Phi \vdash \tau ::> \kappa$ and CSK-SKind □

Theorem 22 (why is this here?). *If $\Delta; \Phi \vdash \kappa_1 \lesssim \mathbf{S}_{\kappa_2}(\tau)$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

ELABORATION

TODO