Algebraic Data Types for Hazel

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1 Syntax

$$
\begin{array}{llcl}
\mathsf{HTyp} & \tau & ::= & \varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid +\{C(\tau); \ldots\} \mid \llparenthesis\rrparenthesis \mid \llparenthesis\alpha\rrparenthesis \\
\mathsf{HTypPat} & \pi & ::= & \alpha \mid \llparenthesis\rrparenthesis \\
\mathsf{HExp} & e & ::= & \varnothing \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \mathtt{inj}_C(e) \mid \mathtt{roll}(e) \mid \mathtt{unroll}(e) \\
 & & & \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^{u\blacktriangleright} \\
\mathsf{IHExp} & d & ::= & \varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \mathtt{inj}_C^{\tau}(d) \mid \mathtt{roll}^{\mu\alpha.\tau}(d) \mid \mathtt{unroll}(d) \\
 & & & \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \nRightarrow \tau \rangle \mid \llparenthesis\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^{u\blacktriangleright}_\sigma \\
\mathsf{HTag} & C & ::= & \mathbf{C} \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis\mathbf{C}\rrparenthesis^u
\end{array}
$$

1.1 Context Extension

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \llparenthesis\rrparenthesis \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{lcll}
[\tau/\llparenthesis\rrparenthesis]\tau' & = & \tau' & \\
[\tau/\alpha]\varnothing & = & \varnothing & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu\llparenthesis\rrparenthesis.\tau_2 & = & \mu\llparenthesis\rrparenthesis.[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha]+\{C(\tau'); \ldots\} & = & +\{C([\tau/\alpha]\tau'); \ldots\} & \\
[\tau/\alpha]\llparenthesis\rrparenthesis & = & \llparenthesis\rrparenthesis & \\
[\alpha'/\alpha]\llparenthesis\alpha\rrparenthesis & = & \llparenthesis\alpha'\rrparenthesis & \\
[\alpha'/\alpha]\llparenthesis\alpha'\rrparenthesis & = & \llparenthesis\alpha'\rrparenthesis & \text{when } \alpha \neq \alpha'
\end{array}
$$

$\boxed{\Theta \vdash \tau\ \mathsf{valid}}$ τ is a valid type

$$\textsc{TVUnit}\ \frac{}{\Theta \vdash \varnothing\ \mathsf{valid}} \qquad \textsc{TVArr}\ \frac{\Theta \vdash \tau_1\ \mathsf{valid} \qquad \Theta \vdash \tau_2\ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2\ \mathsf{valid}} \qquad \textsc{TVVar}\ \frac{\alpha \in \Theta}{\Theta \vdash \alpha\ \mathsf{valid}} \qquad \textsc{TVRec}\ \frac{\Theta, \pi \vdash \tau\ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau\ \mathsf{valid}} \qquad \textsc{TVSum1}\ \frac{\{\Theta \vdash \tau_i\ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \mathsf{valid}}$$

$$\textsc{TVSum2}\ \frac{\Theta \vdash \tau\ \mathsf{valid}}{\Theta \vdash +\{C(\tau); ...\}\ \mathsf{valid}} \qquad \textsc{TVEHole}\ \frac{}{\Theta \vdash \llparenthesis\rrparenthesis\ \mathsf{valid}} \qquad \textsc{TVNEHole}\ \frac{\alpha \notin \Theta}{\Theta \vdash \llparenthesis\alpha\rrparenthesis\ \mathsf{valid}}$$

$\boxed{\tau \sim \tau'}$ τ and τ' are consistent

$$\textsc{TCRefl}\ \frac{}{\tau \sim \tau} \qquad \textsc{TCEHole1}\ \frac{}{\llparenthesis\rrparenthesis \sim \tau} \qquad \textsc{TCEHole2}\ \frac{}{\tau \sim \llparenthesis\rrparenthesis} \qquad \textsc{TCNEHole1}\ \frac{}{\llparenthesis\alpha\rrparenthesis \sim \tau} \qquad \textsc{TCNEHole2}\ \frac{}{\tau \sim \llparenthesis\alpha\rrparenthesis} \qquad \textsc{TCArr}\ \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

$$\textsc{TCRec}\ \frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \qquad \textsc{TCRecHole1}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\llparenthesis\rrparenthesis.\tau \sim \mu\alpha.\tau'} \qquad \textsc{TCRecHole2}\ \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu\llparenthesis\rrparenthesis.\tau'} \qquad \textsc{TCSum1}\ \frac{\{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$$\textsc{TCSum2}\ \frac{C \neq C' \vee \tau \sim \tau'}{+\{C(\tau); ...\} \sim +\{C'(\tau'); ...\}} \qquad \textsc{TCSum12}\ \frac{C_j \in \mathcal{C} \implies \tau \sim \tau_j}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_j(\tau); ...\}} \qquad \textsc{TCSum21}\ \frac{C_j \in \mathcal{C} \implies \tau \sim \tau_j}{+\{C_j(\tau); ...\} \sim +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

Theorem 2 (Consistency Preserves Validity). *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ τ has matched arrow type $\tau_1 \to \tau_2$

$$\textsc{MAHole}\ \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\to} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis} \qquad \textsc{MAArr}\ \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ τ has matched recursive type $\mu\pi.\tau'$

$$\textsc{MRRec}\ \frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau} \qquad \textsc{MRHole}\ \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\mu} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ e synthesizes type τ

$$\textsc{SUnit}\ \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing} \qquad \textsc{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \textsc{SVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SLam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}$$

$$\textsc{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \qquad \textsc{SAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis}$$

SAsc
$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau}$$

SRollErr
$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{roll}(e)\rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

SUnroll
$$\frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$$

SUnrollNotRec
$$\frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}\big(\llparenthesis e\rrparenthesis^{u\blacktriangleright}\big) \Rightarrow \llparenthesis\rrparenthesis}$$

SInjErr
$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{inj}_C(e)\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}$$

SEHole
$$\frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}$$

SNEHole
$$\frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash \llparenthesis e\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

ARoll
$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau}$$

ARollNotRec
$$\frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{roll}(e)\rrparenthesis^u \Leftarrow \tau}$$

AInjHole
$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow \llparenthesis\rrparenthesis}$$

AInj
$$\frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \mathtt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AInjTagErr
$$\frac{C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AInjUnexpectedArg
$$\frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{inj}_{C_j}(e)\rrparenthesis^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AInjExpectedArg
$$\frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash \llparenthesis\mathtt{inj}_{C_j}(\varnothing)\rrparenthesis^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

ASubsume
$$\frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity). *If* $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

ESUnit
$$\frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \rightsquigarrow \varnothing \dashv \emptyset}$$

ESVar
$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

ESVarFree
$$\frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis x\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

ESLam
$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

ESApp
$$\frac{\begin{array}{c}\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \\ \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \\ \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2\end{array}}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle\tau_1' \Rightarrow \tau_1\rangle)(d_2\langle\tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

ESAppNotArr
$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d_1\rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle\tau_2' \Rightarrow \llparenthesis\rrparenthesis\rangle) \dashv \Delta_1 \cup \Delta_2, u :: \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis[\Gamma]}$$

ESAsc
$$\frac{\begin{array}{c}\emptyset \vdash \tau\ \mathsf{valid} \\ \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta\end{array}}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle\tau' \Rightarrow \tau\rangle \dashv \Delta}$$

ESRollErr
$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash \llparenthesis\mathtt{roll}(e)\rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis\mathtt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d\langle\tau \Rightarrow \llparenthesis\rrparenthesis\rangle)\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \texttt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \texttt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}\ \textsc{ESUnroll}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \nsim \mu(\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash \texttt{unroll}\Big((\!|e|\!)^{u\blacktriangleright}\Big) \Rightarrow (\!|\,|\!) \rightsquigarrow \texttt{unroll}\Big((\!|d|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\Big) \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}\ \textsc{ESUnrollNotRec}$$

$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau); ...\}}{\Gamma \vdash (\!|\texttt{inj}_C(e)|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|\texttt{inj}^{\tau'}_C(d\langle \tau \Rightarrow (\!|\,|\!)\rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: (\!|\,|\!)[\Gamma]}\ \textsc{ESInjErr}$$

$$\frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|\,|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\,|\!)[\Gamma]}\ \textsc{ESEHole}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: (\!|\,|\!)[\Gamma]}\ \textsc{ESNEHole}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau \rightsquigarrow \texttt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}\ \textsc{EARoll}$$

$$\frac{\tau \nsim \mu(\!|\,|\!).(\!|\,|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{roll}^{\mu(\!|\,|\!).(\!|\,|\!)}(d)|\!)^u_{\mathsf{id}(\Gamma)} : \mu(\!|\,|\!).(\!|\,|\!) \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}\ \textsc{EARollNotRec}$$

$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow (\!|\,|\!) \rightsquigarrow \texttt{inj}^{\tau'}_C(d) : \tau' \dashv \Delta}\ \textsc{EAInjHole}$$

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \rightsquigarrow d : \tau'_j \dashv \Delta}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow \tau \rightsquigarrow \texttt{inj}^{\tau}_{C_j}\big(d\langle \tau'_j \Rightarrow \tau_j\rangle\big) : \tau \dashv \Delta}\ \textsc{EAInj}$$

$$\frac{C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \rightsquigarrow \texttt{inj}^{\tau'}_C(d)\langle \tau' \Rightarrow (\!|\,|\!)\rangle : (\!|\,|\!) \dashv \Delta}\ \textsc{EAInjTagErr}$$

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau_j \dashv \Delta \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}\setminus C_j} \cup \{C_j(\tau_j)\}\Big\}\end{array}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(d)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}\ \textsc{EAInjUnexpectedArg}$$

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}\setminus C_j} \cup \{C_j(\varnothing)\}\Big\}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(\varnothing)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}\ \textsc{EAInjExpectedArg}$$

$$\frac{e \neq (\!|\,|\!)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}\ \textsc{EASubsume}$$

$$\frac{}{\Gamma \vdash (\!|\,|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\,|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}\ \textsc{EAEHole}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}\ \textsc{EANEHole}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

$$\textsc{TAUnit}\quad \frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

$$\textsc{TAVar}\quad \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

$$\textsc{TALam}\quad \frac{\emptyset \vdash \tau \ \mathsf{valid} \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}$$

$$\textsc{TAApp}\quad \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

$$\textsc{TARoll}\quad \frac{\emptyset \vdash \mu\pi.\tau \ \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}$$

$$\textsc{TAUnroll}\quad \frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

$$\textsc{TAInj1}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(d) : \tau}$$

$$\textsc{TAInj2}\quad \frac{\tau = +\{C(\tau'); \ldots\} \qquad \Delta; \Gamma \vdash d : \tau'}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C}(d) : \tau}$$

$$\textsc{TAEHole}\quad \frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|\!)^u_\sigma : \tau}$$

$$\textsc{TANEHole}\quad \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^u_\sigma : \tau}$$

$$\textsc{TAMHole}\quad \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^{u\blacktriangleright}_\sigma : \tau}$$

$$\textsc{TACast}\quad \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

$$\textsc{TAFailedCast}\quad \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2 \rangle : \tau_2}$$

3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$ τ is a ground type

$$\textsc{GArr}\quad \frac{}{(\!|\!) \to (\!|\!) \text{ ground}}$$

$$\textsc{GRec}\quad \frac{}{\mu(\!|\!).(\!|\!) \text{ ground}}$$

$$\textsc{GSum1}\quad \frac{\{\tau_i = \varnothing \vee \tau_i = (\!|\!)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \text{ ground}}$$

$$\textsc{GSum2}\quad \frac{\tau = \varnothing \vee \tau = (\!|\!)}{+\{C(\tau); \ldots\} \text{ ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

$$\textsc{MGArr}\quad \frac{\tau_1 \to \tau_2 \neq (\!|\!) \to (\!|\!)}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} (\!|\!) \to (\!|\!)}$$

$$\textsc{MGRec}\quad \frac{\tau \neq (\!|\!)}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu(\!|\!).(\!|\!)}$$

$$\textsc{MGSum1}\quad \frac{\{(\tau_i = \varnothing \implies \tau'_i = \varnothing) \wedge (\tau_i \neq \varnothing \implies \tau'_i = (\!|\!))\}_{C_i \in \mathcal{C}} \qquad +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{MGSum2}\quad \frac{\tau \neq \varnothing \qquad \tau \neq (\!|\!)}{+\{C(\tau); \ldots\} \blacktriangleright_{\mathsf{ground}} +\{C((\!|\!)); \ldots\}}$$

$\boxed{d \text{ final}}$ d is final

$$\textsc{FBoxedVal}\quad \frac{d \text{ boxedval}}{d \text{ final}}$$

$$\textsc{FIndet}\quad \frac{d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$ d is a value

$$\textsc{VUnit}\ \frac{}{\varnothing \text{ val}} \qquad \textsc{VLam}\ \frac{}{\lambda x{:}\tau.d \text{ val}} \qquad \textsc{VRoll}\ \frac{d \text{ val}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ val}} \qquad \textsc{VInj}\ \frac{d \text{ val}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ val}}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

$$\textsc{BVVal}\ \frac{d \text{ val}}{d \text{ boxedval}} \qquad \textsc{BVRoll}\ \frac{d \text{ boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}} \qquad \textsc{BVInj}\ \frac{d \text{ boxedval}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ boxedval}}$$

$$\textsc{BVArrCast}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}$$

$$\textsc{BVRecCast}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}}$$

$$\textsc{BVSum1Cast}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \qquad d \text{ boxedval}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}$$

$$\textsc{BVSum2Cast}\ \frac{\begin{array}{c}\tau = +\{C_1(\tau_1); ...\} \\ \tau' = +\{C_1'(\tau_1'); ...\} \\ C_1' = C_1 \implies \tau_1' \sim \tau_1 \qquad d \text{ boxedval}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}$$

$$\textsc{BVSum12Cast}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad \tau' = +\{C(\tau''); ...\} \\ C = C_i \implies \tau'' \sim \tau_i \qquad d \text{ boxedval}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}$$

$$\textsc{BVSum21Cast}\ \frac{\begin{array}{c}\tau = +\{C(\tau''); ...\} \qquad \tau' = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ C_i = C \implies \tau_i \sim \tau'' \qquad d \text{ boxedval}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}$$

$$\textsc{BVHoleCast}\ \frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$ d is indeterminate

$$\textsc{IEHole}\ \frac{}{\llparenthesis\rrparenthesis^u_\sigma \text{ indet}} \qquad \textsc{INEHole}\ \frac{d \text{ final}}{\llparenthesis d \rrparenthesis^u_\sigma \text{ indet}}$$

$$\textsc{IApp}\ \frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \qquad d_1 \text{ indet} \qquad d_2 \text{ final}}{d_1(d_2) \text{ indet}} \qquad \textsc{IRoll}\ \frac{d \text{ indet}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ indet}}$$

$$\textsc{IUnroll}\ \frac{d \text{ indet}}{\texttt{unroll}(d) \text{ indet}} \qquad \textsc{IInj}\ \frac{d \text{ indet}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ indet}} \qquad \textsc{IInjHole}\ \frac{C \neq \mathbf{C} \qquad d \text{ final}}{\texttt{inj}^{\tau}_{C}(d) \text{ indet}}$$

$$\textsc{ICastGroundHole}\ \frac{d \text{ indet} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle \text{ indet}}$$

$$\textsc{ICastHoleGround}\ \frac{d \neq d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \rangle \qquad d \text{ indet} \qquad \tau \text{ ground}}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau \rangle \text{ indet}}$$

$$\textsc{ICastArr}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ indet}} \qquad \textsc{ICastRec}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}}$$

$$\textsc{ICastSum1}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \qquad d \text{ indet}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}$$

$$\textsc{ICastSum2}\ \frac{\begin{array}{c}\tau = +\{C_1(\tau_1); ...\} \\ \tau' = +\{C_1'(\tau_1'); ...\} \\ C_1 = C_1' \implies \tau_1 \sim \tau_1' \qquad d \text{ indet}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}$$

$$\textsc{ICastSum12}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad \tau' = +\{C(\tau''); ...\} \\ C = C_i \implies \tau'' \sim \tau_i \qquad d \text{ indet}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}$$

$$\textsc{ICastSum21}\ \frac{\begin{array}{c}\tau = +\{C(\tau''); ...\} \qquad \tau' = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ C_i = C \implies \tau_i \sim \tau'' \qquad d \text{ indet}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}$$

$$\textsc{IFailedCast}\ \frac{d \text{ final} \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \nsim \tau_2}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle \text{ indet}}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\textsc{ITApp}\quad \frac{[d_2\ \mathsf{final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1} \qquad \textsc{ITUnroll}\quad \frac{[d\ \mathsf{final}]}{\mathtt{unroll}(\mathtt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\textsc{ITAppCast}\quad \frac{[d_1\ \mathsf{final}] \qquad [d_2\ \mathsf{final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1\rangle))\langle \tau_2 \Rightarrow \tau_2'\rangle}$$

$$\textsc{ITUnrollCast}\quad \frac{[d\ \mathsf{final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\mathtt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \mathtt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle} \qquad \textsc{ITCastId}\quad \frac{[d\ \mathsf{final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastSucceed}\quad \frac{[d\ \mathsf{final}] \qquad \tau\ \mathsf{ground}}{d\langle \tau \Rightarrow (\!|\!|\!) \Rightarrow \tau\rangle \longrightarrow d} \qquad \textsc{ITCastFail}\quad \frac{[d\ \mathsf{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \mathsf{ground} \qquad \tau_2\ \mathsf{ground}}{d\langle \tau_1 \Rightarrow (\!|\!|\!) \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow (\!|\!|\!) \not\Rightarrow \tau_2\rangle}$$

$$\textsc{ITGround}\quad \frac{[d\ \mathsf{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow (\!|\!|\!)\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow (\!|\!|\!)\rangle} \qquad \textsc{ITExpand}\quad \frac{[d\ \mathsf{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle (\!|\!|\!) \Rightarrow \tau\rangle \longrightarrow d\langle (\!|\!|\!) \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\begin{array}{lll} \mathsf{EvalCtx} & \mathcal{E} ::= & \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \mathtt{unroll}(\mathcal{E}) \mid \mathtt{inj}_C^{\tau}(\mathcal{E}) \mid (\!|\mathcal{E}|\!)^u_\sigma \mid (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma \\ & & \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow (\!|\!|\!) \not\Rightarrow \tau\rangle \end{array}$$

$\boxed{d = \mathcal{E}\{d'\}}$ d is obtained by placing d' at the mark in $\mathcal{E}$

$$\textsc{FHOuter}\quad \frac{}{d = \circ\{d\}} \qquad \textsc{FHApp1}\quad \frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}} \qquad \textsc{FHApp2}\quad \frac{[d_1\ \mathsf{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}} \qquad \textsc{FHRoll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{roll}^{\mu\pi.\tau}(d) = \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FHUnroll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{unroll}(d) = \mathtt{unroll}(\mathcal{E})\{d'\}} \qquad \textsc{FHInj}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{inj}_C^{\tau}(d) = \mathtt{inj}_C^{\tau}(\mathcal{E})\{d'\}} \qquad \textsc{FNEHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^u_\sigma = (\!|\mathcal{E}|\!)^u_\sigma\{d'\}} \qquad \textsc{FMHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u\blacktriangleright}_\sigma = (\!|\mathcal{E}|\!)^{u\blacktriangleright}_\sigma\{d'\}}$$

$$\textsc{FHCastInside}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}} \qquad \textsc{FHFailedCast}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow (\!|\!|\!) \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow (\!|\!|\!) \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ d steps to d'

$$\textsc{Step}\quad \frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$