Hazel Phi: 11-type-constructors

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NOTES

Writing up the proof for unicity

SYNTAX

$$
\begin{array}{rcll}
\text{Kind} & \kappa & ::= & \mathtt{Type} \mid \mathtt{KHole} \mid \mathsf{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2 \\
\text{User Types} & \hat{\tau} & ::= & t \mid \mathtt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\hat{\tau}\rrparenthesis^{\mathtt{u}} \mid \lambda t{::}\mathtt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2} \\
\text{Internal Types} & \tau & ::= & t \mid \mathtt{bse} \mid \tau_1 \oplus \tau_2 \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\tau\rrparenthesis^{\mathtt{u}} \mid \llparenthesis t\rrparenthesis^{\mathtt{u}} \mid \lambda t{::}\kappa.\tau \mid \tau_1\ \tau_2 \\
\text{Base Types} & \mathtt{bse} & ::= & \mathtt{Int} \mid \mathtt{Float} \mid \mathtt{Bool} \\
\text{BinOp} & \oplus & ::= & \times \mid + \mid \to \\
\text{Type Pattern} & & & \\
\text{User Expression} & & & \\
\text{Internal Expression} & & &
\end{array}
$$

DECLARATIVES

$\boxed{\Delta; \Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash \mathtt{bse} ::> \mathsf{S}_{\mathtt{Type}}(\mathtt{bse})}\ \texttt{PK-Base} \qquad \frac{\Delta; \Phi_1, t{::}\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta; \Phi \vdash t ::> \mathsf{S}_{\kappa}(t)}\ \texttt{PK-Var}$$

$$\frac{\Delta; \Phi \vdash \tau_1{::}\mathtt{Type} \qquad \Delta; \Phi \vdash \tau_2{::}\mathtt{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathsf{S}_{\mathtt{Type}}(\tau_1 \oplus \tau_2)}\ \texttt{PK-}\oplus \qquad \frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash \llparenthesis\rrparenthesis^{\mathtt{u}} ::> \mathsf{S}_{\kappa}(\llparenthesis\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-EHole}$$

$$\frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad \Delta; \Phi \vdash \tau{::}\kappa_1}{\Delta; \Phi \vdash \llparenthesis\tau\rrparenthesis^{\mathtt{u}} ::> \mathsf{S}_{\kappa}(\llparenthesis\tau\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-NEHole} \qquad \frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad t \notin \Phi}{\Delta; \Phi \vdash \llparenthesis t\rrparenthesis^{\mathtt{u}} ::> \mathsf{S}_{\kappa}(\llparenthesis t\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-Unbound}$$

$$\frac{\Delta; \underline{\Phi, t{::}\kappa_1} \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t{::}\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t{::}\kappa_1.\tau)}\ \texttt{PK-}\lambda$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \qquad \Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta; \Phi \vdash \tau_2{::}\kappa_1}{\Delta; \Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ \texttt{PK-Ap}$$

$\boxed{\Delta;\Phi \vdash \tau{::}\kappa}$ τ is well formed at kind κ

$$\frac{\Delta;\Phi \vdash \tau ::> \mathsf{S}_{\kappa}(\tau)}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-1} \qquad \frac{\Delta;\Phi \vdash \tau ::> \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \lesssim \kappa}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-Subsump}$$

$$\frac{\Delta;\Phi \vdash \tau ::> \kappa}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-Reit} \qquad \frac{\Delta;\Phi \vdash \tau{::}\kappa}{\Delta;\Phi \vdash \tau{::}\mathsf{S}_{\kappa}(\tau)}\ \texttt{WFaK-Self}$$

$$\frac{\Delta;\Phi \vdash \tau{::}\Pi_{t::\kappa_3}.\kappa_4 \qquad \Delta;\Phi \vdash \Pi_{t::\kappa_3}.\kappa_4 \lesssim \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \tau{::}\Pi_{t::\kappa_1}.\kappa_2}\ \texttt{WFaK-}\Pi\texttt{CSKTrans}$$

$$\frac{\Delta;\Phi \vdash \tau{::}\mathsf{S}_{\kappa}(\tau_1) \qquad \Delta;\Phi \vdash \tau_1{::}\kappa}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-Flatten}$$

$\boxed{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \texttt{KHole} \overset{\blacktriangleright}{\Pi} \Pi_{t::\texttt{KHole}}.\texttt{KHole}}\ \overset{\blacktriangleright}{\Pi}\texttt{-KHole} \qquad \frac{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\mathsf{S}_{\texttt{KHole}}(\tau)}.\mathsf{S}_{\texttt{KHole}}(\tau\ t)}\ \overset{\blacktriangleright}{\Pi}\texttt{-SKHole}$$

$$\frac{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}\ \overset{\blacktriangleright}{\Pi}\texttt{-}\Pi$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv>} \kappa_2}$ κ_1 singleton reduces to κ_2

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\mathsf{S}_{\kappa}(\tau_1)}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\mathsf{S}_{\kappa}(\tau_1)}(\tau) \overset{*}{\equiv>} \mathsf{S}_{\kappa}(\tau_1)}\ \overset{*}{\equiv>}\texttt{-1} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv>} \kappa_2 \qquad \Delta;\Phi \vdash \kappa_2 \overset{*}{\equiv>} \kappa_3}{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv>} \kappa_3}\ \overset{*}{\equiv>}\texttt{-Trans}$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \overset{\text{norm}}{\equiv>} \kappa_2}$ κ_1 has singleton normal form κ_2

$$\frac{\Delta;\Phi \vdash \kappa \overset{*}{\equiv>} \mathsf{S}_{\texttt{Type}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \mathsf{S}_{\texttt{Type}}(\tau)}\ \overset{\text{norm}}{\equiv>}\texttt{-Type} \qquad \frac{\Delta;\Phi \vdash \kappa \overset{*}{\equiv>} \mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \mathsf{S}_{\texttt{KHole}}(\tau)}\ \overset{\text{norm}}{\equiv>}\texttt{-KHole}$$

$$\frac{\Delta;\Phi \vdash \kappa \overset{*}{\equiv>} \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau\ t_1)}\ \overset{\text{norm}}{\equiv>}\texttt{-}\Pi$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \equiv \kappa}\ \texttt{KEquiv-Refl} \qquad \frac{\Delta;\Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Symm}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Trans}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv\!>} \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SReduc} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \overset{\text{norm}}{\equiv\!>} \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SNorm}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\Phi, \underline{t{::}\kappa_1} \vdash \kappa_3 \equiv \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{KEquiv-}\Pi$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \equiv \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{KEquiv-SKind}$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \texttt{KHole} \lesssim \kappa}\ \texttt{CSK-KHoleL} \qquad \frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \texttt{KHole}}\ \texttt{CSK-KHoleR}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau)\ \mathsf{OK} \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{L}$$

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK} \qquad \Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \mathsf{S}_{\texttt{KHole}}(\tau)}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{R}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-KEquiv} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta;\Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-Normal}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind} \qquad \frac{\Delta;\Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta;\Phi, \underline{t{::}\kappa_3} \vdash \kappa_2 \lesssim \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{CSK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{CSK}-?$$

$\boxed{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta;\Phi \vdash \tau{::}\kappa}{\Delta;\Phi \vdash \tau \overset{\kappa}{\equiv} \tau}\ \texttt{EquivAK-Refl} \qquad \frac{\Delta;\Phi \vdash \tau_2 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Symm}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_3 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Trans}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \ {::>}\ \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \mathsf{S}_{\kappa}(\tau_2)}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-SKind}$$

$$\frac{\Delta;\Phi \vdash \tau_1{::}\Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta;\Phi \vdash \tau_2{::}\Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta;\underline{\Phi, t{::}\kappa_1} \vdash \tau_1\ t \overset{\kappa_2}{\equiv} \tau_2\ t}{\Delta;\Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2}\ \texttt{EquivAK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \overset{\kappa_1}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1\ \tau_2 \overset{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4}\ \texttt{EquivAK-Ap}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\mathsf{S}_{\kappa}(\tau)}{\equiv} \tau_2}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (1)$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\mathsf{Type}}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \overset{\mathsf{Type}}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 \overset{\mathsf{Type}}{\equiv} \tau_3 \oplus \tau_4}\ (2) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\underline{\Phi, t{::}\kappa_1} \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta;\Phi \vdash \lambda t{::}\kappa_1.\tau_1 \overset{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t{::}\kappa_2.\tau_2}\ (3)$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (4)$$

$\boxed{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}$ κ is well formed

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{Type}\ \mathsf{OK}}\ \texttt{KWF-Type} \qquad \frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{KHole}\ \mathsf{OK}}\ \texttt{KWF-KHole} \qquad \frac{\Delta;\Phi \vdash \tau{::}\kappa}{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau)\ \mathsf{OK}}\ \texttt{KWF-SKind}$$

$$\frac{\Delta;\underline{\Phi, t{::}\kappa_1} \vdash \kappa_2\ \mathsf{OK}}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathsf{OK}}\ \texttt{KWF-}\Pi$$

$\boxed{\Delta;\Phi \vdash \mathsf{OK}}$ Context is well formed

$$\frac{}{\cdot;\cdot \vdash \mathsf{OK}}\ \texttt{CWF-Nil} \qquad \frac{t \notin \Phi \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\underline{\Phi, t{::}\kappa} \vdash \mathsf{OK}}\ \texttt{CWF-TypVar} \qquad \frac{\mathrm{u} \notin \Delta \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\underline{\Delta, \mathrm{u}{::}\kappa};\Phi \vdash \mathsf{OK}}\ \texttt{CWF-Hole}$$

Metatheory

subderivation preserving inferences:

- premiss
- COK (Context OK)
- PoS (premiss of subderivation)

Lemma 1 (COK). *If $\Delta;\Phi \vdash \mathcal{J}$, then $\Delta;\Phi \vdash \mathit{OK}$ in a subderivation (where $\Delta;\Phi \vdash \mathcal{J} \neq \Delta;\Phi \vdash \mathit{OK}$)*

Proof. By induction on derivations.
No interesting cases. □

Lemma 2 (Exchange).
If $\Delta;\Phi_1, t_{L1}{::}\kappa_{L1}, t_{L2}{::}\kappa_{L2}, \Phi_2 \vdash \mathcal{J}$ and $\Delta;\Phi_1, t_{L2}{::}\kappa_{L2}, t_{L1}{::}\kappa_{L1}, \Phi_2 \vdash \mathit{OK}$, then $\Delta;\Phi_1, t_{L2}{::}\kappa_{L2}, t_{L1}{::}\kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

Proof. By induction on derivations.
No interesting cases.
(Only rules with Φ extended in the consequent are interesting, which is only CWF-TypVar, but when $\mathcal{J}$ is CWF, Exchange is identity) □

Corollary 3 (Marked-Exchange).
If $\Delta;\underline{\underline{\Phi, t_{L1}{::}\kappa_{L1}}, t_{L2}{::}\kappa_{L2}} \vdash \mathcal{J}$ and $\Delta;\underline{\underline{\Phi, t_{L2}{::}\kappa_{L2}}, t_{L1}{::}\kappa_{L1}} \vdash \mathit{OK}$, then $\Delta;\underline{\underline{\Phi, t_{L2}{::}\kappa_{L2}}, t_{L1}{::}\kappa_{L1}} \vdash \mathcal{J}$

Proof. Exchange when $\Phi_2 = \cdot$ □

Lemma 4 (Weakening).
If $\Delta;\Phi \vdash \mathcal{J}$ and $\Delta;\underline{\Phi, t_L{::}\kappa_L} \vdash \mathit{OK}$ and $t_L \notin \mathcal{J}$ and $\forall t \in \kappa_L, t \notin \mathcal{J}$, then $\Delta;\underline{\Phi, t_L{::}\kappa_L} \vdash \mathcal{J}$

Proof. see addendum □

Lemma 5 (K-Substitution).
If $\Delta;\Phi \vdash \tau_{L1}{::}\kappa_{L1}$ and $\Delta;\underline{\Phi, t_L{::}\kappa_{L1}} \vdash \tau_{L2}{::}\kappa_{L2}$, then $\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau_{L2}{::}[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta;\underline{\Phi, t_L{::}\kappa_{L1}} \vdash \tau_{L2}{::}\kappa_{L2}$)

Lemma 6 (PK-Substitution). *If $\Delta;\Phi \vdash \tau_{L1}{::}\kappa_{L1}$ and $\Delta;\underline{\Phi, t_L{::}\kappa_{L1}} \vdash \tau_{L2} ::> \kappa_{L2}$ and $\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau_{L2} ::> \kappa_{L3}$, then $\Delta;\Phi \vdash [\tau_{L2}/t_L]\kappa_{L2} \equiv \kappa_{L3}$*

Lemma 7 (OK-Substitution).
If $\Delta;\Phi \vdash \tau_L{::}\kappa_{L1}$ and $\Delta;\underline{\Phi, t_L{::}\kappa_{L1}} \vdash \kappa_{L2}$ OK, then $\Delta;\Phi \vdash [\tau_L/t_L]\kappa_{L2}$ OK
(induction on $\Delta;\underline{\Phi, t_L{::}\kappa_{L1}} \vdash \kappa_{L2}$ OK)

Theorem 8 (OK-PK). *If $\Delta;\Phi \vdash \tau ::> \kappa$, then $\Delta;\Phi \vdash \kappa$ OK*

Theorem 9 (OK-WFaK). *If $\Delta;\Phi \vdash \tau{::}\kappa$, then $\Delta;\Phi \vdash \kappa$ OK*

Theorem 10 (OK-MatchPi). *If $\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\scriptstyle\Pi} \Pi_{t::\kappa_1}.\kappa_2$, then $\Delta;\Phi \vdash \kappa$ OK and $\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2$ OK*

Theorem 11 (OK-KEquiv). *If $\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2$, then $\Delta;\Phi \vdash \kappa_1$ OK and $\Delta;\Phi \vdash \kappa_2$ OK*

Theorem 12 (OK-CSK). *If $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$, then $\Delta;\Phi \vdash \kappa_1$ OK and $\Delta;\Phi \vdash \kappa_2$ OK*

Theorem 13 (OK-EquivAK). *If $\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2$, then $\Delta;\Phi \vdash \tau_1{::}\kappa$ and $\Delta;\Phi \vdash \tau_2{::}\kappa$ and $\Delta;\Phi \vdash \kappa$ OK*

Proof. see addendum □

Proof.

Weakening

By induction on derivations.

Note: When applying Weakening in the induction, check that the left premiss is always a subderivation, and check variable exclusion conditions are satisfied (usually checked elsewhere in the derivation).

$$\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\tau ::> \kappa_2}\,\texttt{premiss}\quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{IH}}{t_L\notin\Phi}\,\texttt{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\texttt{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t_L\neq t}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\texttt{IH}}{t_L\notin\kappa_1}}{t_L\notin\Phi,t{::}\kappa_1}\quad\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{IH}}{\Delta;\Phi\vdash\kappa_L\ \mathsf{OK}}\,\texttt{PoS}\quad\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\tau ::> \kappa_2}\,\texttt{premiss}}{\Delta;\Phi,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{COK}}{\Delta;\Phi,t{::}\kappa_1\vdash\kappa_L\ \mathsf{OK}}\,\texttt{Weakening}}{\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{CWF-TypVar}}{\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L\vdash\tau ::> \kappa_2}\,\texttt{Weakening}\quad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\tau ::> \kappa_2}\,\texttt{premiss}}{\Delta;\Phi,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{COK}}{t\notin\Phi}\,\texttt{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\texttt{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\neq t_L}\quad\dfrac{\dfrac{}{\forall\dot{t}\in\kappa_L,\dot{t}\notin\mathcal{J}}\,\texttt{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\notin\kappa_L}}{t\notin\Phi,t_L{::}\kappa_L}\quad\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\tau ::> \kappa_2}\,\texttt{premiss}}{\Delta;\Phi,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{COK}}{\Delta;\Phi\vdash\kappa_1\ \mathsf{OK}}\,\texttt{PoS}\quad\dfrac{}{\Delta;\Phi,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{IH}}{\Delta;\Phi,t_L{::}\kappa_L\vdash\kappa_1\ \mathsf{OK}}\,\texttt{Weakening}}{\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{CWF-TypVar}}{\dfrac{\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1\vdash\tau ::> \kappa_2}{\Delta;\Phi,t_L{::}\kappa_L\vdash\lambda t{::}\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t{::}\kappa_1.\tau)}\,\texttt{PK-}\lambda}\,\texttt{Marked-Exchange}$$

$$\dfrac{\dfrac{\dfrac{}{\Delta;\Phi\vdash\kappa_1\equiv\kappa_2}\,\texttt{premiss}\quad\dfrac{}{\Delta;\Phi,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{IH}}{\Delta;\Phi,t_L{::}\kappa_L\vdash\kappa_1\equiv\kappa_2}\,\texttt{Weakening}\quad\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\tau ::> \kappa_2}\,\texttt{premiss}\quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{IH}}{t_L\notin\Phi}\,\texttt{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\texttt{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t_L\neq t}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\texttt{IH}}{t_L\notin\kappa_1}}{t_L\notin\Phi,t{::}\kappa_1}\quad\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{IH}}{\Delta;\Phi\vdash\kappa_L\ \mathsf{OK}}\,\texttt{PoS}\quad\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\texttt{premiss}}{\Delta;\Phi,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{COK}}{\Delta;\Phi,t{::}\kappa_1\vdash\kappa_L\ \mathsf{OK}}\,\texttt{Weakening}}{\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{CWF-TypVar}}{\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L\vdash\tau ::> \kappa_2}\,\texttt{Weakening}\quad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\texttt{premiss}}{\Delta;\Phi,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{COK}}{t\notin\Phi}\,\texttt{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\texttt{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\neq t_L}\quad\dfrac{\dfrac{}{\forall\dot{t}\in\kappa_L,\dot{t}\notin\mathcal{J}}\,\texttt{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\notin\kappa_L}}{t\notin\Phi,t_L{::}\kappa_L}\quad\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t{::}\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\texttt{premiss}}{\Delta;\Phi,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{COK}}{\Delta;\Phi\vdash\kappa_1\ \mathsf{OK}}\,\texttt{PoS}\quad\dfrac{}{\Delta;\Phi,t_L{::}\kappa_L\vdash\mathsf{OK}}\,\texttt{IH}}{\Delta;\Phi,t_L{::}\kappa_L\vdash\kappa_1\ \mathsf{OK}}\,\texttt{Weakening}}{\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1\vdash\mathsf{OK}}\,\texttt{CWF-TypVar}}{\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\texttt{Marked-Exchange}}{\Delta;\Phi,t_L{::}\kappa_L\vdash\Pi_{t::\kappa_1}.\kappa_2\equiv\Pi_{t::\kappa_3}.\kappa_4}\,\texttt{KEquiv-}\Pi$$

$$\dfrac{}{sim}\,\texttt{CSK-}\Pi\qquad\dfrac{}{sim}\,\texttt{EquivAK-}\Pi\qquad\dfrac{}{sim}\,\texttt{KWF-}\Pi$$

O?K-.*

By simultaneous induction on derivations.

The interesting cases per lemma:

K-Substitution by type size??

OK-Substitution

OK-PK

$$\dfrac{\dfrac{\dfrac{}{\Delta;\Phi\vdash\texttt{bse} ::> \mathsf{S}_{\texttt{Type}}(\texttt{bse})}\,\texttt{premiss}}{\Delta;\Phi\vdash\texttt{bse}{::}\texttt{Type}}\,\texttt{WFaK-1}}{\Delta;\Phi\vdash\mathsf{S}_{\texttt{Type}}(\texttt{bse})\ \mathsf{OK}}\,\texttt{KWF-SKind}\qquad\qquad\dfrac{}{\Delta;\Phi\vdash[\tau_2/t]\kappa_2\ \mathsf{OK}}\,\texttt{OK-Substitution}$$

OK-WFaK

□

Definition 1 (Singleton Depth).

$$SSize : \text{“}\{\kappa\}\text{”} \to \mathbb{N}$$

$$SSize(\kappa_x) = \begin{cases} SSize(\kappa) + 1 & \text{if } \kappa_x = \mathtt{S}_\kappa(\tau) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 14 ($\overset{*}{\equiv>}$-diminution). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$, then $SSize(\kappa_L) > SSize(\kappa_{L1})$*

Proof. By induction on derivations (and transitivity of $>$ on $\mathbb{N}$) □

Lemma 15 ($\overset{*}{\equiv>}$-n+1-nicity). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L2}$ where $SSize(\kappa_L) = n{+}1$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. By $\overset{*}{\equiv>}$-diminution, $\overset{*}{\equiv>}$-Trans cannot be the last inference of a derivation of $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ since $SSize(\kappa_1) \geq SSize(\kappa_3) + 2$ (in $\overset{*}{\equiv>}$-Trans). Thus, $\overset{*}{\equiv>}$-1 must have been the last inference. Similarly for $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L2}$, thus $\kappa_{L1} = \kappa_{L2}$ □

Lemma 16 ($\overset{*}{\equiv>}$-stepwise). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ where $SSize(\kappa_L) = m$ and $SSize(\kappa_{L1}) = n$ and $m > n + 1$, then the derivation must contain subderivations of each singleton depth inbetween*

Proof. More precisely this says, where $m > n$ by $\overset{*}{\equiv>}$-diminution, the derivation must contain subderivations of each $\Delta; \Phi \vdash \kappa_i \overset{*}{\equiv>} \kappa_j$ where $m \geq i > j \geq n$, $SSize(\kappa_k) = k$ when $m \geq k \geq n$, $\kappa_m = \kappa_L$, $\kappa_n = \kappa_{L1}$.
By induction on derivations (base case is where $m = n + 2$, which necessitates a last inference of $\overset{*}{\equiv>}$-Trans. Each premiss must have SSize difference of 1, fulfilling hypothesis) □

Lemma 17 ($\overset{*}{\equiv>}$-$m+n$-nicity). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L2}$ where $SSize(\kappa_L) = m + n$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. By $\overset{*}{\equiv>}$-stepwise and $\overset{*}{\equiv>}$-$n+1$-nicity when $m > n + 1$.
By $\overset{*}{\equiv>}$-$n + 1$-nicity when $m = n + 1$.
No other cases by $\overset{*}{\equiv>}$-diminution. □

Theorem 18 ($\overset{\text{norm}}{\equiv>}$-Unicity). *If $\Delta; \Phi \vdash \kappa_L \overset{norm}{\equiv>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{norm}{\equiv>} \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)
All $\overset{\text{norm}}{\equiv>}$ rules have $\overset{*}{\equiv>}$ premiss with rhs singleton depth 1. By $\overset{*}{\equiv>}$-$m + n$-nicity, where $n = 1$. □

Theorem 19 ($\overset{\blacktriangleright}{\Pi}$-Unicity). *If $\Delta; \Phi \vdash \tau_L \overset{\blacktriangleright}{\Pi} \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L \overset{\blacktriangleright}{\Pi} \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)
By unicity of $\overset{\text{norm}}{\equiv>}$. □

Theorem 20 (PK-Unicity). *If $\Delta; \Phi \vdash \tau_L :: > \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L :: > \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)
As PK is syntax directed, proof is by inspection for all rules except PK-λ (variables in contexts are unique– see context rules), which is by induction on derivations, and PK-Ap, which requires of unicity of $\overset{\blacktriangleright}{\Pi}$ (above theorem). □

Theorem 21 (PK-Principality). *If $\Delta; \Phi \vdash \tau :: > \kappa_1$ and $\Delta; \Phi \vdash \tau :: \kappa_2$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

Proof. From definition of $\Delta; \Phi \vdash \tau :: \kappa$ and CSK-SKind □

Theorem 22 (why is this here?). *If $\Delta; \Phi \vdash \kappa_1 \lesssim \mathtt{S}_{\kappa_2}(\tau)$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

Elaboration

TODO