Algebraic Data Types for Hazel

Eric Griffis
egriffis@umich.edu

1 Syntax

HTyp	τ	::=	$\varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid (\!\|\!) \mid (\!|\alpha|\!)$
HTypPat	π	::=	$\alpha \mid (\!\|\!)$
HExp	e	::=	$x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \mathtt{inj}_C(e) \mid \mathtt{roll}(e) \mid \mathtt{unroll}(e)$
			$\mid (\!\|\!)^u \mid (\!|e|\!)^u \mid (\!|e|\!)^{u\blacktriangleright}$
IHExp	d	::=	$\varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \mathtt{inj}_C^\tau(d) \mid \mathtt{roll}^{\mu\alpha.\tau}(d) \mid \mathtt{unroll}(d)$
			$\mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow (\!\|\!) \not\Rightarrow \tau \rangle \mid (\!\|\!)_\sigma^u \mid (\!|d|\!)_\sigma^u \mid (\!|d|\!)_\sigma^{u\blacktriangleright}$
HTag	C	::=	$\mathbf{C} \mid (\!\|\!)^u \mid (\!|c|\!)^u$

1.1 Context Extension

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = (\!\|\!) \end{cases}$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{aligned}
[\tau/(\!\|\!)]\tau' &= \tau' \\
[\tau/\alpha]\varnothing &= \varnothing \\
[\tau/\alpha](\tau_1 \to \tau_2) &= [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 \\
[\tau/\alpha]\alpha &= \tau \\
[\tau/\alpha]\alpha_1 &= \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 &= \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu(\!\|\!).\tau_2 &= \mu(\!\|\!).[\tau/\alpha]\tau_2 \\
[\tau/\alpha]+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} &= +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} \\
[\tau/\alpha](\!\|\!) &= (\!\|\!) \\
[\alpha'/\alpha](\!|\alpha|\!) &= (\!|\alpha'|\!)^u \\
[\alpha'/\alpha](\!|\alpha_1|\!)^u &= (\!|\alpha_1|\!)^u & \text{when } \alpha \neq \alpha_1
\end{aligned}
$$

$\boxed{\Theta \vdash \tau \,\mathsf{valid}}$ τ is a valid type

$$
\frac{}{\Theta \vdash \varnothing \,\mathsf{valid}} \; \text{TVUNIT}
\qquad
\frac{\Theta \vdash \tau_1 \,\mathsf{valid} \qquad \Theta \vdash \tau_2 \,\mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2 \,\mathsf{valid}} \; \text{TVARR}
\qquad
\frac{\alpha \in \Theta}{\Theta \vdash \alpha \,\mathsf{valid}} \; \text{TVVAR}
$$

$$
\frac{\Theta, \pi \vdash \tau \,\mathsf{valid}}{\Theta \vdash \mu\pi.\tau \,\mathsf{valid}} \; \text{TVREC}
\qquad
\frac{\{\Theta \vdash \tau_i \,\mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \,\mathsf{valid}} \; \text{TVSUM}
$$

$$
\frac{}{\Theta \vdash (\!\|\!) \,\mathsf{valid}} \; \text{TVEHOLE}
$$

1

$\boxed{\tau \sim \tau'}$ τ and τ' are consistent

$$\frac{\text{TCRefl}}{\tau \sim \tau} \qquad \frac{\text{TCEHole1}}{(\!|\!|) \sim \tau} \qquad \frac{\text{TCEHole2}}{\tau \sim (\!|\!|)} \qquad \frac{\text{TCNEHole1}}{(\!|\alpha|\!) \sim \tau} \qquad \frac{\text{TCNEHole2}}{\tau \sim (\!|\alpha|\!)} \qquad \frac{\text{TCUnit}}{\varnothing \sim \varnothing} \qquad \frac{\text{TCArr} \quad \tau_1 \sim \tau_1' \quad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

$$\frac{\text{TCRec} \quad \tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'} \qquad \frac{\text{TCRecHole1} \quad \alpha \notin \mathsf{FV}(\tau) \quad \tau \sim \tau'}{\mu(\!|\!|).\tau \sim \mu\alpha.\tau'} \qquad \frac{\text{TCRecHole2} \quad \alpha \notin \mathsf{FV}(\tau) \quad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu(\!|\!|).\tau'} \qquad \frac{\text{TCSum} \quad \{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{C \text{ valid}}$ C is a valid tag

$$\frac{\text{CVTag}}{\mathbf{C} \text{ valid}} \qquad\qquad \frac{\text{CVEHole}}{(\!|\!|)^u \text{ valid}}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If $\Gamma \vdash e \Rightarrow \tau$ then $\emptyset \vdash \tau$ valid.*

Theorem 2 (Consistency Preserves Validity). *If $\Theta \vdash \tau$ valid and $\tau \sim \tau'$ then $\Theta \vdash \tau'$ valid.*

$\boxed{\tau \blacktriangleright_\to \tau_1 \to \tau_2}$ τ has matched arrow type $\tau_1 \to \tau_2$

$$\frac{\text{MAHole}}{(\!|\!|) \blacktriangleright_\to (\!|\!|) \to (\!|\!|)} \qquad\qquad \frac{\text{MAArr}}{\tau_1 \to \tau_2 \blacktriangleright_\to \tau_1 \to \tau_2}$$

$\boxed{\tau \blacktriangleright_\mu \mu\pi.\tau'}$ τ has matched recursive type $\mu\pi.\tau'$

$$\frac{\text{MRRec}}{\mu\pi.\tau \blacktriangleright_\mu \mu\pi.\tau} \qquad\qquad \frac{\text{MRHole}}{(\!|\!|) \blacktriangleright_\mu \mu(\!|\!|).(\!|\!|)}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ e synthesizes type τ

$$\frac{\text{SUnit}}{\Gamma \vdash \varnothing \Rightarrow \varnothing} \qquad \frac{\text{SVar} \quad x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \frac{\text{SVarFree} \quad x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\!|)} \qquad \frac{\text{SLam} \quad \emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}$$

$$\frac{\text{SLamInvalid} \quad \Gamma, x : (\!|\!|) \vdash e \Rightarrow \tau}{\Gamma \vdash \lambda x{:}(\!|\alpha|\!).e \Rightarrow (\!|\!|) \to \tau} \qquad\qquad \frac{\text{SApp} \quad \Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \quad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau}$$

$$\frac{\text{SAppNotArr} \quad \Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \not\sim (\!|\!|) \to (\!|\!|) \quad \Gamma \vdash e_2 \Leftarrow (\!|\!|)}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\!|)} \qquad \frac{\text{SAsc} \quad \emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \qquad \frac{\text{SAscInvalid} \quad \Gamma \vdash e \Leftarrow (\!|\!|)}{\Gamma \vdash e : (\!|\alpha|\!) \Rightarrow (\!|\!|)}$$

$$\frac{\text{SRollErr} \quad \Gamma \vdash e \Leftarrow (\!|\!|)}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Rightarrow \mu(\!|\!|).(\!|\!|)} \qquad \frac{\text{SUnroll} \quad \Gamma \vdash e \Rightarrow \tau \quad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'} \qquad \frac{\text{SUnrollNotRec} \quad \Gamma \vdash e \Rightarrow \tau \quad \tau \not\sim \mu(\!|\!|).(\!|\!|)}{\Gamma \vdash \mathtt{unroll}\big((\!|e|\!)^{u\blacktriangleright}\big) \Rightarrow (\!|\!|)}$$

$$\frac{\text{SInjErr} \quad C \text{ valid} \quad \Gamma \vdash e \Leftarrow (\!|\!|)}{\Gamma \vdash (\!|\mathtt{inj}_C(e)|\!)^u \Rightarrow (\!|\!|)} \qquad \frac{\text{SInjTagErr} \quad \Gamma \vdash e \Leftarrow (\!|\!|)}{\Gamma \vdash \mathtt{inj}_{(\!|c|\!)^u}(e) \Rightarrow (\!|\!|)} \qquad \frac{\text{SEHole}}{\Gamma \vdash (\!|\!|)^u \Rightarrow (\!|\!|)} \qquad \frac{\text{SNEHole} \quad \Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\!|)}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

AROLL
$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau}$$

AROLLNOTREC
$$\frac{\tau \nsim \mu(\!|\,|\!).(\!|\,|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Leftarrow \tau}$$

AINJHOLE
$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow (\!|\,|\!)}$$

AINJ
$$\frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \mathtt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AINJUNEXPECTEDBODY
$$\frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\mathtt{inj}_{C_j}(e)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AINJEXPECTEDBODY
$$\frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash (\!|\mathtt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AINJBADTAG
$$\frac{C\ \mathsf{valid} \qquad C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\mathtt{inj}_{C}(e)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AINJTAGERR
$$\frac{(\!|c|\!)^u \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash \mathtt{inj}_{(\!|c|\!)^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

ASUBSUME
$$\frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity). *If $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ then $\emptyset \vdash \tau\ \mathsf{valid}$.*

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

ESUNIT
$$\frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \rightsquigarrow \varnothing \dashv \emptyset}$$

ESVAR
$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

ESVARFREE
$$\frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|x|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\,|\!)\,[\Gamma]}$$

ESLAM
$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

ESLAMINVALID
$$\frac{\Gamma, x : (\!|\,|\!) \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}(\!|\alpha|\!).e \Rightarrow (\!|\,|\!) \to \tau \rightsquigarrow \lambda x{:}(\!|\alpha|\!).d \dashv \Delta}$$

ESAPP
$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_1 \rangle)(d_2\langle \tau_2' \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2}$$

ESAPPNOTARR
$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim (\!|\,|\!) \to (\!|\,|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\,|\!) \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|d_1|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow (\!|\,|\!) \rangle) \dashv \Delta_1 \cup \Delta_2, u :: (\!|\,|\!) \to (\!|\,|\!)\,[\Gamma]}$$

ESASC
$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau \rangle \dashv \Delta}$$

ESASCINVALID
$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash e : (\!|\alpha|\!) \Rightarrow (\!|\,|\!) \rightsquigarrow d\langle \tau \Rightarrow (\!|\,|\!) \rangle \dashv \Delta}$$

ESROLLERR
$$\frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Rightarrow \mu(\!|\,|\!).(\!|\,|\!) \rightsquigarrow (\!|\mathtt{roll}^{\mu(\!|\,|\!).(\!|\,|\!)}(d\langle \tau \Rightarrow (\!|\,|\!) \rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)\,[\Gamma]}$$

ESUNROLL
$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \mathtt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau' \rangle) \dashv \Delta}$$

ESUnrollNotRec

$$\frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta \qquad \tau \nsim \mu \llparenthesis \rrparenthesis . \llparenthesis \rrparenthesis}{\Gamma \vdash \mathtt{unroll}\!\left(\llparenthesis e \rrparenthesis^{u \blacktriangleright}\right) \Rightarrow \llparenthesis \rrparenthesis \leadsto \mathtt{unroll}\!\left(\llparenthesis d \rrparenthesis^{u \blacktriangleright}_{\mathsf{id}(\Gamma)}\right) \dashv \Delta, u :: \mu \llparenthesis \rrparenthesis . \llparenthesis \rrparenthesis [\Gamma]}$$

ESInjErr

$$\frac{C \ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \llparenthesis \rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \llparenthesis \mathtt{inj}_C(e) \rrparenthesis^u \Rightarrow \llparenthesis \rrparenthesis \leadsto \llparenthesis \mathtt{inj}_C^{\tau'}(d \langle \tau \Rightarrow \llparenthesis \rrparenthesis \rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis \rrparenthesis [\Gamma]}$$

ESInjTagErr

$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis \rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{\llparenthesis c \rrparenthesis^u (\tau)\}}{\Gamma \vdash \mathtt{inj}_{\llparenthesis c \rrparenthesis^u}(e) \Rightarrow \llparenthesis \rrparenthesis \leadsto \mathtt{inj}_{\llparenthesis c \rrparenthesis^u}^{\tau'}(d \langle \tau \Rightarrow \llparenthesis \rrparenthesis \rangle) \dashv \Delta}$$

ESEHole

$$\frac{}{\Gamma \vdash \llparenthesis \rrparenthesis^u \Rightarrow \llparenthesis \rrparenthesis \leadsto \llparenthesis \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis \rrparenthesis [\Gamma]}$$

ESNEHole

$$\frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis \rrparenthesis \leadsto \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis \rrparenthesis [\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \leadsto d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

EARoll

$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \leadsto d : \tau'' \dashv \Delta}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau \leadsto \mathtt{roll}^{\mu\pi.\tau'}(d \langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rangle) : \mu\pi.\tau' \dashv \Delta}$$

EARollNotRec

$$\frac{\tau \nsim \mu \llparenthesis \rrparenthesis . \llparenthesis \rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis \rrparenthesis \leadsto d : \tau' \dashv \Delta}{\Gamma \vdash \llparenthesis \mathtt{roll}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \mathtt{roll}^{\mu \llparenthesis \rrparenthesis . \llparenthesis \rrparenthesis}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \mu \llparenthesis \rrparenthesis . \llparenthesis \rrparenthesis \dashv \Delta, u :: \mu \llparenthesis \rrparenthesis . \llparenthesis \rrparenthesis [\Gamma]}$$

EAInjHole

$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis \rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow \llparenthesis \rrparenthesis \leadsto \mathtt{inj}_C^{\tau'}(d) : \tau' \dashv \Delta}$$

EAInj

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \leadsto d : \tau'_j \dashv \Delta}{\Gamma \vdash \mathtt{inj}_{C_j}(e) \Leftarrow \tau \leadsto \mathtt{inj}_{C_j}^{\tau}(d \langle \tau'_j \Rightarrow \tau_j \rangle) : \tau \dashv \Delta}$$

EAInjUnexpectedBody

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow \llparenthesis \rrparenthesis \leadsto d : \tau_j \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\right\}\end{array}}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C_j}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \mathtt{inj}_{C_j}^{\tau'}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjExpectedBody

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\right\}}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C_j}(\varnothing) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \mathtt{inj}_{C_j}^{\tau'}(\varnothing) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjBadTag

$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C \notin \mathcal{C} \qquad C \ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \llparenthesis \rrparenthesis \leadsto d : \tau' \dashv \Delta \qquad \tau'' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{C(\tau')\}\right\}}{\Gamma \vdash \llparenthesis \mathtt{inj}_C(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \mathtt{inj}_C^{\tau''}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjTagErr

$$\frac{\llparenthesis c \rrparenthesis^u \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis \rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{\llparenthesis c \rrparenthesis^u (\tau)\}\right\}}{\Gamma \vdash \mathtt{inj}_{\llparenthesis c \rrparenthesis^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \leadsto \mathtt{inj}_{\llparenthesis c \rrparenthesis^u}^{\tau'}(d \langle \tau \Rightarrow \llparenthesis \rrparenthesis \rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta}$$

EASubsume

$$\frac{e \neq \llparenthesis \rrparenthesis^u \qquad e \neq \llparenthesis e' \rrparenthesis^u \qquad \Gamma \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta}$$

EAEHole

$$\frac{}{\Gamma \vdash \llparenthesis \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

4

EANEHOLE
$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

TAUNIT
$$\frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

TAVAR
$$\frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

TALAM
$$\frac{\tau \neq (\!|\alpha|\!) \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}$$

TALAMFREE
$$\frac{\Delta; \Gamma, x : (\!|\!|\!) \vdash d : \tau}{\Delta; \Gamma \vdash \lambda x{:}(\!|\alpha|\!).d : (\!|\!|\!) \to \tau}$$

TAAPP
$$\frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

TAROLL
$$\frac{\emptyset \vdash \mu\pi.\tau \, \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}$$

TAUNROLL
$$\frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

TAINJ
$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \mathtt{inj}^\tau_{C_j}(d) : \tau}$$

TAEHOLE
$$\frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|\!|\!)^u_\sigma : \tau}$$

TANEHOLE
$$\frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^u_\sigma : \tau}$$

TAMHOLE
$$\frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^{u \blacktriangleright}_\sigma : \tau}$$

TACAST
$$\frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

TAFAILEDCAST
$$\frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \, \mathrm{ground} \qquad \tau_2 \, \mathrm{ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow (\!|\!|\!) \not\Rightarrow \tau_2 \rangle : \tau_2}$$

3 Dynamic Semantics

$\boxed{\tau \, \mathrm{ground}}$ τ is a ground type

GARR
$$\frac{}{(\!|\!|\!) \to (\!|\!|\!) \, \mathrm{ground}}$$

GREC
$$\frac{}{\mu(\!|\!|\!).(\!|\!|\!) \, \mathrm{ground}}$$

GSUM
$$\frac{\{\tau_i = \varnothing \vee \tau_i = (\!|\!|\!)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \, \mathrm{ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

MGARR
$$\frac{\tau_1 \to \tau_2 \neq (\!|\!|\!) \to (\!|\!|\!)}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} (\!|\!|\!) \to (\!|\!|\!)}$$

MGREC
$$\frac{\tau \neq (\!|\!|\!)}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu(\!|\!|\!).(\!|\!|\!)}$$

MGSUM
$$\frac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \qquad \{(\tau_i = \varnothing \implies \tau_i' = \varnothing) \wedge (\tau_i \neq \varnothing \implies \tau_i' = (\!|\!|\!))\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \text{ final}}$ d is final

FBoxedVal
$$\frac{d \text{ boxedval}}{d \text{ final}}$$

FIndet
$$\frac{d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$ d is a value

VUnit
$$\frac{}{\varnothing \text{ val}}$$

VLam
$$\frac{}{\lambda x{:}\tau.d \text{ val}}$$

VRoll
$$\frac{d \text{ val}}{\mathtt{roll}^{\mu\pi.\tau}(d) \text{ val}}$$

VInj
$$\frac{d \text{ val}}{\mathtt{inj}_{\mathbf{C}}^{\tau}(d) \text{ val}}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

BVVal
$$\frac{d \text{ val}}{d \text{ boxedval}}$$

BVRoll
$$\frac{d \text{ boxedval}}{\mathtt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}}$$

BVInj
$$\frac{d \text{ boxedval}}{\mathtt{inj}_{\mathbf{C}}^{\tau}(d) \text{ boxedval}}$$

BVArrCast
$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}$$

BVRecCast
$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}}$$

BVSumCast
$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \quad \tau \neq \tau' \qquad d \text{ boxedval}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}$$

BVHoleCast
$$\frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$ d is indeterminate

IEHole
$$\frac{}{\llparenthesis\rrparenthesis_\sigma^u \text{ indet}}$$

INEHole
$$\frac{d \text{ final}}{\llparenthesis d \rrparenthesis_\sigma^u \text{ indet}}$$

IApp
$$\frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \qquad d_1 \text{ indet} \qquad d_2 \text{ final}}{d_1(d_2) \text{ indet}}$$

IRoll
$$\frac{d \text{ indet}}{\mathtt{roll}^{\mu\pi.\tau}(d) \text{ indet}}$$

IUnroll
$$\frac{d \text{ indet}}{\mathtt{unroll}(d) \text{ indet}}$$

IInj
$$\frac{d \text{ indet}}{\mathtt{inj}_{\mathbf{C}}^{\tau}(d) \text{ indet}}$$

IInjHole
$$\frac{C \neq \mathbf{C} \qquad d \text{ final}}{\mathtt{inj}_{C}^{\tau}(d) \text{ indet}}$$

ICastGroundHole
$$\frac{d \text{ indet} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle \text{ indet}}$$

ICastHoleGround
$$\frac{d \neq d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \rangle \qquad d \text{ indet} \qquad \tau \text{ ground}}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau \rangle \text{ indet}}$$

ICastArr
$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ indet}}$$

ICastRec
$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}}$$

ICastSum
$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \quad \tau \neq \tau' \qquad d \text{ indet}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}$$

IFailedCast
$$\frac{d \text{ final} \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle \text{ indet}}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

ITApp
$$\frac{[d_2 \text{ final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1}$$

ITUnroll
$$\frac{[d \text{ final}]}{\mathtt{unroll}(\mathtt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

ITAppCast
$$\frac{[d_1 \text{ final}] \qquad [d_2 \text{ final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2' \rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1 \rangle))\langle \tau_2 \Rightarrow \tau_2' \rangle}$$

ITUNROLLCAST

$$\frac{[d \text{ final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\texttt{unroll}(d\langle\mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \texttt{unroll}(d)\langle[\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle}$$

ITCASTID

$$\frac{[d \text{ final}]}{d\langle\tau \Rightarrow \tau\rangle \longrightarrow d}$$

ITCASTSUCCEED

$$\frac{[d \text{ final}] \qquad \tau \text{ ground}}{d\langle\tau \Rightarrow \text{\textcircled{$\Vert$}} \Rightarrow \tau\rangle \longrightarrow d}$$

ITCASTFAIL

$$\frac{[d \text{ final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground}}{d\langle\tau_1 \Rightarrow \text{\textcircled{$\Vert$}} \Rightarrow \tau_2\rangle \longrightarrow d\langle\tau_1 \Rightarrow \text{\textcircled{$\Vert$}} \not\Rightarrow \tau_2\rangle}$$

ITGROUND

$$\frac{[d \text{ final}] \qquad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle\tau \Rightarrow \text{\textcircled{$\Vert$}}\rangle \longrightarrow d\langle\tau \Rightarrow \tau' \Rightarrow \text{\textcircled{$\Vert$}}\rangle}$$

ITEXPAND

$$\frac{[d \text{ final}] \qquad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle\text{\textcircled{$\Vert$}} \Rightarrow \tau\rangle \longrightarrow d\langle\text{\textcircled{$\Vert$}} \Rightarrow \tau' \Rightarrow \tau\rangle}$$

EvalCtx $\mathcal{E}$ ::= $\circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \texttt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \texttt{unroll}(\mathcal{E}) \mid \texttt{inj}_C^\tau(\mathcal{E}) \mid (\!|\mathcal{E}|\!)_\sigma^u \mid (\!|\mathcal{E}|\!)_\sigma^{u\blacktriangleright}$
$\mid \mathcal{E}\langle\tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle\tau \Rightarrow \text{\textcircled{$\Vert$}} \not\Rightarrow \tau\rangle$

$\boxed{d = \mathcal{E}\{d'\}}$ $\quad d$ is obtained by placing d' at the mark in $\mathcal{E}$

FHOUTER

$$\frac{}{d = \circ\{d\}}$$

FHAPP1

$$\frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}$$

FHAPP2

$$\frac{[d_1 \text{ final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

FHROLL

$$\frac{d = \mathcal{E}\{d'\}}{\texttt{roll}^{\mu\pi.\tau}(d) = \texttt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

FHUNROLL

$$\frac{d = \mathcal{E}\{d'\}}{\texttt{unroll}(d) = \texttt{unroll}(\mathcal{E})\{d'\}}$$

FHINJ

$$\frac{d = \mathcal{E}\{d'\}}{\texttt{inj}_C^\tau(d) = \texttt{inj}_C^\tau(\mathcal{E})\{d'\}}$$

FNEHOLEINSIDE

$$\frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)_\sigma^u = (\!|\mathcal{E}|\!)_\sigma^u\{d'\}}$$

FMHOLEINSIDE

$$\frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)_\sigma^{u\blacktriangleright} = (\!|\mathcal{E}|\!)_\sigma^{u\blacktriangleright}\{d'\}}$$

FHCASTINSIDE

$$\frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow \tau_2\rangle\{d'\}}$$

FHFAILEDCAST

$$\frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow \text{\textcircled{$\Vert$}} \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow \text{\textcircled{$\Vert$}} \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ $\quad d$ steps to d'

STEP

$$\frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$