Hazel Phi: 11-type-constructors

September 3, 2021

NOTES

This is (sadly) the last version for summer 2021.
See markdown for a brief TODO list.

SYNTAX

$$\begin{array}{rrcl}
\text{Kind} & \kappa & ::= & \mathtt{Type} \mid \mathtt{KHole} \mid \mathsf{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2 \\
\text{User Types} & \hat{\tau} & ::= & t \mid \mathtt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid (\!|\,|\!)^{\mathtt{u}} \mid (\!|\hat{\tau}|\!)^{\mathtt{u}} \mid \lambda t::\mathtt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2} \\
\text{Internal Types} & \tau & ::= & t \mid \mathtt{bse} \mid \tau_1 \oplus \tau_2 \mid (\!|\,|\!)^{\mathtt{u}} \mid (\!|\tau|\!)^{\mathtt{u}} \mid (\!|t|\!)^{\mathtt{u}} \mid \lambda t::\kappa.\tau \mid \tau_1\ \tau_2 \\
\text{Base Types} & \mathtt{bse} & ::= & \mathtt{Int} \mid \mathtt{Float} \mid \mathtt{Bool} \\
\text{BinOp} & \oplus & ::= & \times \mid + \mid \to \\
\text{Type Pattern} & & & \\
\text{User Expression} & & & \\
\text{Internal Expression} & & &
\end{array}$$

Declaratives

$\boxed{\Delta;\Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{bse} ::> \mathsf{S}_{\mathtt{Type}}(\mathtt{bse})}\ \texttt{PK-Base} \qquad \frac{\Delta;\Phi_1, t::\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta;\Phi \vdash t ::> \mathsf{S}_{\kappa}(t)}\ \texttt{PK-Var}$$

$$\frac{\Delta;\Phi \vdash \tau_1::\mathtt{Type} \qquad \Delta;\Phi \vdash \tau_2::\mathtt{Type}}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 ::> \mathsf{S}_{\mathtt{Type}}(\tau_1 \oplus \tau_2)}\ \texttt{PK-}\oplus \qquad \frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash (\!|\,|\!)^{\mathtt{u}} ::> \mathsf{S}_{\kappa}((\!|\,|\!)^{\mathtt{u}})}\ \texttt{PK-EHole}$$

$$\frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK} \qquad \Delta;\Phi \vdash \tau::\kappa_1}{\Delta;\Phi \vdash (\!|\tau|\!)^{\mathtt{u}} ::> \mathsf{S}_{\kappa}((\!|\tau|\!)^{\mathtt{u}})}\ \texttt{PK-NEHole} \qquad \frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK} \qquad t \notin \Phi}{\Delta;\Phi \vdash (\!|t|\!)^{\mathtt{u}} ::> \mathsf{S}_{\kappa}((\!|t|\!)^{\mathtt{u}})}\ \texttt{PK-Unbound}$$

$$\frac{\Delta;\underline{\Phi, t::\kappa_1} \vdash \tau ::> \kappa_2}{\Delta;\Phi \vdash \lambda t::\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)}\ \texttt{PK-}\lambda$$

$$\frac{\Delta;\Phi \vdash \tau_1 ::> \kappa \qquad \Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\scriptstyle\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \vdash \tau_2::\kappa_1}{\Delta;\Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ \texttt{PK-Ap}$$

$\boxed{\Delta;\Phi\vdash\tau{::}\kappa}$ τ is well formed at kind κ

$$\frac{\Delta;\Phi\vdash\tau ::> \mathsf{S}_{\kappa}(\tau)}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-1}\qquad\frac{\Delta;\Phi\vdash\tau ::> \kappa_1\qquad\Delta;\Phi\vdash\kappa_1\lesssim\kappa}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-Subsump}$$

$$\frac{\Delta;\Phi\vdash\tau ::> \kappa}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-Reit}\qquad\frac{\Delta;\Phi\vdash\tau{::}\kappa}{\Delta;\Phi\vdash\tau{::}\mathsf{S}_{\kappa}(\tau)}\ \texttt{WFaK-Self}$$

$$\frac{\Delta;\Phi\vdash\tau{::}\Pi_{t::\kappa_3}.\kappa_4\qquad\Delta;\Phi\vdash\Pi_{t::\kappa_3}.\kappa_4\lesssim\Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi\vdash\tau{::}\Pi_{t::\kappa_1}.\kappa_2}\ \texttt{WFaK-}\Pi\texttt{CSKTrans}$$

$$\frac{\Delta;\Phi\vdash\tau{::}\mathsf{S}_{\kappa}(\tau_1)\qquad\Delta;\Phi\vdash\tau_1{::}\kappa}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-Flatten}$$

$\boxed{\Delta;\Phi\vdash\kappa\overset{\blacktriangleright}{_{\Pi}}\Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{\Delta;\Phi\vdash\mathsf{OK}}{\Delta;\Phi\vdash\texttt{KHole}\overset{\blacktriangleright}{_{\Pi}}\Pi_{t::\texttt{KHole}}.\texttt{KHole}}\ \overset{\blacktriangleright}{_{\Pi}}\texttt{-KHole}\qquad\frac{\Delta;\Phi\vdash\kappa\equiv\mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta;\Phi\vdash\kappa\overset{\blacktriangleright}{_{\Pi}}\Pi_{t::\mathsf{S}_{\texttt{KHole}}(\tau)}.\mathsf{S}_{\texttt{KHole}}(\tau\ t)}\ \overset{\blacktriangleright}{_{\Pi}}\texttt{-SKHole}$$

$$\frac{\Delta;\Phi\vdash\kappa\overset{*}{\equiv>}\Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi\vdash\kappa\overset{\blacktriangleright}{_{\Pi}}\Pi_{t::\kappa_1}.\kappa_2}\ \overset{\blacktriangleright}{_{\Pi}}\texttt{-}\Pi$$

$\boxed{\Delta;\Phi\vdash\kappa_1\equiv>\kappa_2}$ κ_1 singleton reduces to κ_2

$$\frac{\Delta;\Phi\vdash\tau{::}\mathsf{S}_{\kappa}(\tau_1)}{\Delta;\Phi\vdash\mathsf{S}_{\mathsf{S}_{\kappa}(\tau_1)}(\tau)\equiv>\mathsf{S}_{\kappa}(\tau_1)}\ \texttt{KEquiv-SKind}_{\texttt{SKind}}\qquad\frac{\Delta;\Phi\vdash\kappa_1\equiv>\kappa_2\qquad\Delta;\Phi\vdash\kappa_2\equiv>\kappa_3}{\Delta;\Phi\vdash\kappa_1\equiv>\kappa_3}$$

$\boxed{\Delta;\Phi\vdash\kappa_1\overset{*}{\equiv>}\kappa_2}$ κ_1 has singleton normal form κ_2

$$\frac{\Delta;\Phi\vdash\kappa\equiv>\mathsf{S}_{\texttt{Type}}(\tau)}{\Delta;\Phi\vdash\kappa\overset{*}{\equiv>}\mathsf{S}_{\texttt{Type}}(\tau)}\qquad\frac{\Delta;\Phi\vdash\kappa\equiv>\mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta;\Phi\vdash\kappa\overset{*}{\equiv>}\mathsf{S}_{\texttt{KHole}}(\tau)}\qquad\frac{\Delta;\Phi\vdash\kappa\equiv>\mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau)}{\Delta;\Phi\vdash\kappa\overset{*}{\equiv>}\Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau\ t_1)}$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta;\Phi \vdash \kappa \ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \equiv \kappa}\ \texttt{KEquiv-Refl} \qquad \frac{\Delta;\Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Symm}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Trans}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv> \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SReduc} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv>} \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SNorm}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\underline{\Phi, t{::}\kappa_1} \vdash \kappa_3 \equiv \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{KEquiv-}\Pi$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \equiv \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{KEquiv-SKind}$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta;\Phi \vdash \kappa \ \mathsf{OK}}{\Delta;\Phi \vdash \texttt{KHole} \lesssim \kappa}\ \texttt{CSK-KHoleL} \qquad \frac{\Delta;\Phi \vdash \kappa \ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \texttt{KHole}}\ \texttt{CSK-KHoleR}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau)\ \mathsf{OK} \qquad \Delta;\Phi \vdash \kappa \ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{L}$$

$$\frac{\Delta;\Phi \vdash \kappa \ \mathsf{OK} \qquad \Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \mathsf{S}_{\texttt{KHole}}(\tau)}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{R}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-KEquiv} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta;\Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-Normal}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind} \qquad \frac{\Delta;\Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta;\underline{\Phi, t{::}\kappa_3} \vdash \kappa_2 \lesssim \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{CSK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{CSK-?}$$

$\boxed{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta;\Phi \vdash \tau :: \kappa}{\Delta;\Phi \vdash \tau \overset{\kappa}{\equiv} \tau}\ \texttt{EquivAK-Refl} \qquad \frac{\Delta;\Phi \vdash \tau_2 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Symm}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_3 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Trans}$$

$$\frac{\Delta;\Phi \vdash \tau_1 ::> \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \mathsf{S}_\kappa(\tau_2)}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-SKind}$$

$$\frac{\Delta;\Phi \vdash \tau_1 :: \Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta;\Phi \vdash \tau_2 :: \Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta;\underline{\Phi, t::\kappa_1} \vdash \tau_1\ t \overset{\kappa_2}{\equiv} \tau_2\ t}{\Delta;\Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2}\ \texttt{EquivAK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \overset{\kappa_1}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1\ \tau_2 \overset{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4}\ \texttt{EquivAK-Ap}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\mathsf{S}_\kappa(\tau)}{\equiv} \tau_2}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (1)$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\mathtt{Type}}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \overset{\mathtt{Type}}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 \overset{\mathtt{Type}}{\equiv} \tau_3 \oplus \tau_4}\ (2) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\underline{\Phi, t::\kappa_1} \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta;\Phi \vdash \lambda t::\kappa_1.\tau_1 \overset{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t::\kappa_2.\tau_2}\ (3)$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (4)$$

$\boxed{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}$ κ is well formed

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{Type}\ \mathsf{OK}}\ \texttt{KWF-Type} \qquad \frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{KHole}\ \mathsf{OK}}\ \texttt{KWF-KHole} \qquad \frac{\Delta;\Phi \vdash \tau :: \kappa}{\Delta;\Phi \vdash \mathsf{S}_\kappa(\tau)\ \mathsf{OK}}\ \texttt{KWF-SKind}$$

$$\frac{\Delta;\underline{\Phi, t::\kappa_1} \vdash \kappa_2\ \mathsf{OK}}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathsf{OK}}\ \texttt{KWF-}\Pi$$

$\boxed{\Delta;\Phi \vdash \mathsf{OK}}$ Context is well formed

$$\frac{}{\cdot;\cdot \vdash \mathsf{OK}}\ \texttt{CWF-Nil} \qquad \frac{t \notin \Phi \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\underline{\Phi, t::\kappa} \vdash \mathsf{OK}}\ \texttt{CWF-TypVar} \qquad \frac{u \notin \Delta \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\underline{\Delta, u::\kappa};\Phi \vdash \mathsf{OK}}\ \texttt{CWF-Hole}$$

Metatheory

subderivation preserving inferences:

- premiss
- COK (Context OK)
- PoS (premiss of subderivation)

Lemma 1 (COK). *If* $\Delta; \Phi \vdash \mathcal{J}$, *then* $\Delta; \Phi \vdash \mathit{OK}$ *in a subderivation (where* $\Delta; \Phi \vdash \mathcal{J} \neq \Delta; \Phi \vdash \mathit{OK}$*)*

Proof. By induction on derivations.
No interesting cases. □

Lemma 2 (Exchange).
If $\Delta; \Phi_1, t_{L1}{::}\kappa_{L1}, t_{L2}{::}\kappa_{L2}, \Phi_2 \vdash \mathcal{J}$ *and* $\Delta; \Phi_1, t_{L2}{::}\kappa_{L2}, t_{L1}{::}\kappa_{L1}, \Phi_2 \vdash \mathit{OK}$, *then* $\Delta; \Phi_1, t_{L2}{::}\kappa_{L2}, t_{L1}{::}\kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

Proof. By induction on derivations.
No interesting cases.
(Only rules with Φ extended in the consequent are interesting, which is only CWF-TypVar, but when $\mathcal{J}$ is CWF, Exchange is identity) □

Corollary 3 (Marked-Exchange).
If $\Delta; \underline{\underline{\Phi, t_{L1}{::}\kappa_{L1}}, t_{L2}{::}\kappa_{L2}} \vdash \mathcal{J}$ *and* $\Delta; \underline{\underline{\Phi, t_{L2}{::}\kappa_{L2}}, t_{L1}{::}\kappa_{L1}} \vdash \mathit{OK}$, *then* $\Delta; \underline{\underline{\Phi, t_{L2}{::}\kappa_{L2}}, t_{L1}{::}\kappa_{L1}} \vdash \mathcal{J}$

Proof. Exchange when $\Phi_2 = \cdot$ □

Lemma 4 (Weakening).
If $\Delta; \Phi \vdash \mathcal{J}$ *and* $\Delta; \underline{\Phi, t_L{::}\kappa_L} \vdash \mathit{OK}$ *and* $t_L \notin \mathcal{J}$ *and* $\forall t \in \kappa_L, t \notin \mathcal{J}$, *then* $\Delta; \underline{\Phi, t_L{::}\kappa_L} \vdash \mathcal{J}$

Proof. see addendum □

Lemma 5 (K-Substitution).
If $\Delta; \Phi \vdash \tau_{L1}{::}\kappa_{L1}$ *and* $\Delta; \underline{\Phi, t_L{::}\kappa_{L1}} \vdash \tau_{L2}{::}\kappa_{L2}$, *then* $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2}{::}[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta; \underline{\Phi, t_L{::}\kappa_{L1}} \vdash \tau_{L2}{::}\kappa_{L2}$*)*

Lemma 6 (PK-Substitution). *If* $\Delta; \Phi \vdash \tau_{L1}{::}\kappa_{L1}$ *and* $\Delta; \underline{\Phi, t_L{::}\kappa_{L1}} \vdash \tau_{L2} ::> \kappa_{L2}$ *and* $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2} ::> \kappa_{L3}$, *then* $\Delta; \Phi \vdash [\tau_{L2}/t_L]\kappa_{L2} \equiv \kappa_{L3}$

Lemma 7 (OK-Substitution).
If $\Delta; \Phi \vdash \tau_L{::}\kappa_{L1}$ *and* $\Delta; \underline{\Phi, t_L{::}\kappa_{L1}} \vdash \kappa_{L2}\ \mathit{OK}$, *then* $\Delta; \Phi \vdash [\tau_L/t_L]\kappa_{L2}\ \mathit{OK}$
(induction on $\Delta; \underline{\Phi, t_L{::}\kappa_{L1}} \vdash \kappa_{L2}\ \mathit{OK}$*)*

Theorem 8 (OK-PK). *If* $\Delta; \Phi \vdash \tau ::> \kappa$, *then* $\Delta; \Phi \vdash \kappa\ \mathit{OK}$

Theorem 9 (OK-WFaK). *If* $\Delta; \Phi \vdash \tau{::}\kappa$, *then* $\Delta; \Phi \vdash \kappa\ \mathit{OK}$

Theorem 10 (OK-MatchPi). *If* $\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{{}_{\Pi}} \Pi_{t::\kappa_1}.\kappa_2$, *then* $\Delta; \Phi \vdash \kappa\ \mathit{OK}$ *and* $\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathit{OK}$

Theorem 11 (OK-KEquiv). *If* $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, *then* $\Delta; \Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta; \Phi \vdash \kappa_2\ \mathit{OK}$

Theorem 12 (OK-CSK). *If* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, *then* $\Delta; \Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta; \Phi \vdash \kappa_2\ \mathit{OK}$

Theorem 13 (OK-EquivAK). *If* $\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2$, *then* $\Delta; \Phi \vdash \tau_1{::}\kappa$ *and* $\Delta; \Phi \vdash \tau_2{::}\kappa$ *and* $\Delta; \Phi \vdash \kappa\ \mathit{OK}$

Proof. see addendum □

Proof.

Weakening
By induction on derivations.

Note: When applying Weakening in the induction, check that the left premiss is always a subderivation, and check variable exclusion conditions are satisfied (usually checked elsewhere in the derivation).

$$
\begin{prooftree}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \tau ::> \kappa_2$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$t_L \notin \Phi$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$t_L \notin \mathcal{J}$}\AxiomC{}\UnaryInfC{$t \in \mathcal{J}$}\BinaryInfC{$t_L \neq t$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$t_L \notin \mathcal{J}$}\UnaryInfC{$t_L \notin \kappa_1$}
\TrinaryInfC{$t_L \notin \Phi, t{::}\kappa_1$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$\Delta;\Phi \vdash \kappa_L\ \mathsf{OK}$}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \tau ::> \kappa_2$}\RightLabel{COK}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{Weakening}\BinaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \kappa_L\ \mathsf{OK}$}
\RightLabel{CWF-TypVar}\BinaryInfC{$\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L \vdash \mathsf{OK}$}
\RightLabel{Weakening}\BinaryInfC{$\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L \vdash \tau ::> \kappa_2$}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \tau ::> \kappa_2$}\RightLabel{COK}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$t \notin \Phi$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$t_L \notin \mathcal{J}$}\AxiomC{}\UnaryInfC{$t \in \mathcal{J}$}\BinaryInfC{$t \neq t_L$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\forall \dot{t} \in \kappa_L, \dot{t} \notin \mathcal{J}$}\AxiomC{}\UnaryInfC{$t \in \mathcal{J}$}\BinaryInfC{$t \notin \kappa_L$}
\TrinaryInfC{$t \notin \Phi, t_L{::}\kappa_L$}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \tau ::> \kappa_2$}\RightLabel{COK}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$\Delta;\Phi \vdash \kappa_1\ \mathsf{OK}$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \mathsf{OK}$}
\RightLabel{Weakening}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \kappa_1\ \mathsf{OK}$}
\RightLabel{CWF-TypVar}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{Marked-Exchange}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1 \vdash \tau ::> \kappa_2$}
\RightLabel{PK-λ}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \lambda t{::}\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t{::}\kappa_1.\tau)$}
\end{prooftree}
$$

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\begin{prooftree}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \mathsf{OK}$}
\RightLabel{Weakening}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \kappa_1 \equiv \kappa_3$}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \tau ::> \kappa_2$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$t_L \notin \Phi$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$t_L \notin \mathcal{J}$}\AxiomC{}\UnaryInfC{$t \in \mathcal{J}$}\BinaryInfC{$t_L \neq t$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$t_L \notin \mathcal{J}$}\UnaryInfC{$t_L \notin \kappa_1$}
\TrinaryInfC{$t_L \notin \Phi, t{::}\kappa_1$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$\Delta;\Phi \vdash \kappa_L\ \mathsf{OK}$}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \kappa_3 \equiv \kappa_4$}\RightLabel{COK}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{Weakening}\BinaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \kappa_L\ \mathsf{OK}$}
\RightLabel{CWF-TypVar}\BinaryInfC{$\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L \vdash \mathsf{OK}$}
\RightLabel{Weakening}\BinaryInfC{$\Delta;\Phi,t{::}\kappa_1,t_L{::}\kappa_L \vdash \tau ::> \kappa_2$}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \kappa_3 \equiv \kappa_4$}\RightLabel{COK}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$t \notin \Phi$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$t_L \notin \mathcal{J}$}\AxiomC{}\UnaryInfC{$t \in \mathcal{J}$}\BinaryInfC{$t \neq t_L$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\forall \dot{t} \in \kappa_L, \dot{t} \notin \mathcal{J}$}\AxiomC{}\UnaryInfC{$t \in \mathcal{J}$}\BinaryInfC{$t \notin \kappa_L$}
\TrinaryInfC{$t \notin \Phi, t_L{::}\kappa_L$}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \kappa_3 \equiv \kappa_4$}\RightLabel{COK}\UnaryInfC{$\Delta;\Phi,t{::}\kappa_1 \vdash \mathsf{OK}$}\RightLabel{PoS}\UnaryInfC{$\Delta;\Phi \vdash \kappa_1\ \mathsf{OK}$}
\AxiomC{}\RightLabel{IH}\UnaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \mathsf{OK}$}
\RightLabel{Weakening}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \kappa_1\ \mathsf{OK}$}
\RightLabel{CWF-TypVar}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{Marked-Exchange}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L,t{::}\kappa_1 \vdash \kappa_3 \equiv \kappa_4$}
\RightLabel{KEquiv-Π}\BinaryInfC{$\Delta;\Phi,t_L{::}\kappa_L \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4$}
\end{prooftree}
\qquad
\begin{prooftree}\AxiomC{}\RightLabel{COK-Π}\UnaryInfC{sim}\end{prooftree}
\qquad
\begin{prooftree}\AxiomC{}\RightLabel{EquivAK-Π}\UnaryInfC{sim}\end{prooftree}
\qquad
\begin{prooftree}\AxiomC{}\RightLabel{KWF-Π}\UnaryInfC{sim}\end{prooftree}
$$

O?K-.*
By simultaneous induction on derivations.

The interesting cases per lemma:

K-Substitution by type size??

OK-Substitution

OK-PK

$$
\begin{prooftree}
\AxiomC{}\RightLabel{premiss}\UnaryInfC{$\Delta;\Phi \vdash \mathtt{bse} ::> \mathsf{S}_{\mathsf{Type}}(\mathtt{bse})$}
\RightLabel{WFaK-1}\UnaryInfC{$\Delta;\Phi \vdash \mathtt{bse}{::}\mathsf{Type}$}
\RightLabel{KWF-SKind}\UnaryInfC{$\Delta;\Phi \vdash \mathsf{S}_{\mathsf{Type}}(\mathtt{bse})\ \mathsf{OK}$}
\end{prooftree}
\qquad\qquad
\begin{prooftree}
\AxiomC{}\RightLabel{OK-Substitution}\UnaryInfC{$\Delta;\Phi \vdash [\tau_2/t]\kappa_2\ \mathsf{OK}$}
\end{prooftree}
$$

OK-WFaK $\square$

Theorem 14 (PK-Unicity)**.** *If* $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$ *and* $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$ *then* $\kappa_{L1} = \kappa_{L2}$

Proof. There's actually a slight problem here. In PK-Ap, $\overset{\blacktriangleright}{\Pi}$ doesn't have unicity since $\overset{\blacktriangleright}{\Pi}$-$\Pi$ uses $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, where "the same left side can have mutliple right sides" (transitivity), causing unicity to fail for PK.
Only thought about this quickly but we might need a seperate HO singleton normalization scheme that is essentially the KEquiv-SKind$_{\text{SKind}}$, KEquiv-SKind$_{\Pi}$ fragment of kind equality, and have the HO singleton normalization judgment have unicity.
Kind of gross but...
(and add other HO kinds, namely Σ, when they get here)
And as cleanup, we can remove the normalization fragment from kind equality and add a single rule (yay more sim judgments!)
Also we need some unicity lemmas for contexts like for PK-EHole and co.

$\square$

Theorem 15 (PK-Principality)**.** *If* $\Delta; \Phi \vdash \tau ::> \kappa_1$ *and* $\Delta; \Phi \vdash \tau :: \kappa_2$*, then* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$

Proof. From definition of $\Delta; \Phi \vdash \tau :: \kappa$ and CSK-SKind $\square$

Theorem 16 (why is this here?)**.** *If* $\Delta; \Phi \vdash \kappa_1 \lesssim \mathtt{S}_{\kappa_2}(\tau)$*, then* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$

Elaboration

TODO