

Hazel Phi: 11-type-constructors

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NOTES

This is (sadly) the last version for summer 2021.

See markdown for a brief TODO list.

SYNTAX

Kind	κ	$::=$	$\mathbf{Type} \mid \mathbf{KHole} \mid \mathbf{S}_\kappa(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2$
User Types	$\hat{\tau}$	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \hat{\tau} \rangle^u \mid \lambda t::\mathbf{Type}.\hat{\tau} \mid \tau_1 \tau_2$
Internal Types	τ	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \tau \rangle^u \mid \langle t \rangle^u \mid \lambda t::\kappa.\tau \mid \tau_1 \tau_2$
Base Types	\mathbf{bse}	$::=$	$\mathbf{Int} \mid \mathbf{Float} \mid \mathbf{Bool}$
BinOp	\oplus	$::=$	$\times \mid + \mid \rightarrow$
Type Pattern			
User Expression			
Internal Expression			

DECLARATIVES

$\boxed{\Delta; \Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathbf{bse} ::> \mathbf{S}_{\mathbf{Type}}(\mathbf{bse})} \text{PK-Base}$	$\frac{\Delta; \Phi_1, t::\kappa, \Phi_2 \vdash \text{OK}}{\Delta; \Phi \vdash t ::> \mathbf{S}_\kappa(t)} \text{PK-Var}$
$\frac{\Delta; \Phi \vdash \tau_1::\mathbf{Type} \quad \Delta; \Phi \vdash \tau_2::\mathbf{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathbf{S}_{\mathbf{Type}}(\tau_1 \oplus \tau_2)} \text{PK-}\oplus$	$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \langle \rangle^u ::> \mathbf{S}_\kappa(\langle \rangle^u)} \text{PK-EHole}$
$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad \Delta; \Phi \vdash \tau::\kappa_1}{\Delta; \Phi \vdash \langle \tau \rangle^u ::> \mathbf{S}_\kappa(\langle \tau \rangle^u)} \text{PK-NEHole}$	$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad t \notin \Phi}{\Delta; \Phi \vdash \langle t \rangle^u ::> \mathbf{S}_\kappa(\langle t \rangle^u)} \text{PK-Unbound}$
$\frac{\Delta; \Phi, t::\kappa_1 \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau ::> \mathbf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)} \text{PK-}\lambda$	
$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \quad \Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2 \quad \Delta; \Phi \vdash \tau_2::\kappa_1}{\Delta; \Phi \vdash \tau_1 \tau_2 ::> [\tau_2/t]\kappa_2} \text{PK-Ap}$	

$\Delta; \Phi \vdash \tau :: \kappa$ τ is well formed at kind κ

$$\frac{\Delta; \Phi \vdash \tau :: > \mathbf{S}_{\kappa}(\tau)}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-1} \quad \frac{\Delta; \Phi \vdash \tau :: > \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Subsump}$$

$$\frac{\Delta; \Phi \vdash \tau :: > \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Reit} \quad \frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau :: \mathbf{S}_{\kappa}(\tau)} \text{WFaK-Self}$$

$$\frac{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_3} \cdot \kappa_4 \quad \Delta; \Phi \vdash \Pi_{t::\kappa_3} \cdot \kappa_4 \lesssim \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_1} \cdot \kappa_2} \text{WFaK-PCSKTrans}$$

$$\frac{\Delta; \Phi \vdash \tau :: \mathbf{S}_{\kappa}(\tau_1) \quad \Delta; \Phi \vdash \tau_1 :: \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Flatten}$$

$\Delta; \Phi \vdash \kappa \Pi \Pi_{t::\kappa_1} \cdot \kappa_2$ κ has matched Π -kind $\Pi_{t::\kappa_1} \cdot \kappa_2$

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{KHole} \Pi \Pi_{t::\text{KHole}} \cdot \text{KHole}} \Pi\text{-KHole} \quad \frac{\Delta; \Phi \vdash \kappa \equiv \mathbf{S}_{\text{KHole}}(\tau)}{\Delta; \Phi \vdash \kappa \Pi \Pi_{t::\mathbf{S}_{\text{KHole}}(\tau)} \cdot \mathbf{S}_{\text{KHole}}(\tau \ t)} \Pi\text{-SKHole}$$

$$\frac{\Delta; \Phi \vdash \kappa \equiv \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \kappa \Pi \Pi_{t::\kappa_1} \cdot \kappa_2} \Pi\text{-}\Pi$$

$\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$ κ_1 is equivalent to κ_2

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \equiv \kappa} \text{KEquiv-Ref1} \quad \frac{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-Symm}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-Trans}$$

$$\frac{\Delta; \Phi \vdash \tau :: \mathbf{S}_{\kappa}(\tau_1)}{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{S}_{\kappa}(\tau_1)}(\tau) \equiv \mathbf{S}_{\kappa}(\tau_1)} \text{KEquiv-SKind}_{\text{SKind}}$$

$$\frac{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \mathbf{S}_{\Pi_{t::\kappa_1} \cdot \kappa_2}(\tau) \equiv \Pi_{t_1::\kappa_1} \cdot \mathbf{S}_{[t_1/t]\kappa_2}(\tau \ t_1)} \text{KEquiv-SKind}_{\Pi}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1} \cdot \kappa_2 \equiv \Pi_{t::\kappa_3} \cdot \kappa_4} \text{KEquiv-}\Pi$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2 \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa_1}(\tau_1) \equiv \mathbf{S}_{\kappa_2}(\tau_2)} \text{KEquiv-SKind}$$

$\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{KHole} \lesssim \kappa} \text{CSK-KHoleL} \quad \frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{KHole}} \text{CSK-KHoleR}$$

$$\frac{\Delta; \Phi \vdash \mathbf{SKHole}(\tau) \text{ OK} \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{SKHole}(\tau) \lesssim \kappa} \text{CSK-SKind}_{\text{KHoleL}}$$

$$\frac{\Delta; \Phi \vdash \kappa \text{ OK} \quad \Delta; \Phi \vdash \mathbf{SKHole}(\tau) \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{SKHole}(\tau)} \text{CSK-SKind}_{\text{KHoleR}}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \text{CSK-KEquiv} \quad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \lesssim \kappa_4 \quad \Delta; \Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \text{CSK-Normal}$$

$$\frac{\Delta; \Phi \vdash \mathbf{S}_\kappa(\tau) \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_\kappa(\tau) \lesssim \kappa} \text{CSK-SKind} \quad \frac{\Delta; \Phi \vdash \kappa_3 \lesssim \kappa_1 \quad \Delta; \Phi, t::\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1} \cdot \kappa_2 \lesssim \Pi_{t::\kappa_3} \cdot \kappa_4} \text{CSK-}\Pi$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2 \quad \Delta; \Phi \vdash \tau_1 \stackrel{\kappa_1}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa_1}(\tau_1) \lesssim \mathbf{S}_{\kappa_2}(\tau_2)} \text{CSK-?}$$

$\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \vdash \tau::\kappa}{\Delta; \Phi \vdash \tau \equiv \tau} \text{EquivAK-Ref1} \quad \frac{\Delta; \Phi \vdash \tau_2 \stackrel{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \equiv \tau_2} \text{EquivAK-Symm}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_3 \quad \Delta; \Phi \vdash \tau_3 \stackrel{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2} \text{EquivAK-Trans}$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \equiv \mathbf{S}_\kappa(\tau_2)}{\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2} \text{EquivAK-SKind}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \Pi_{t::\kappa_1} \cdot \kappa_3 \quad \Delta; \Phi \vdash \tau_2 :: \Pi_{t::\kappa_1} \cdot \kappa_4 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \stackrel{\kappa_2}{t} \equiv \tau_2 \ t}{\Delta; \Phi \vdash \tau_1 \stackrel{\Pi_{t::\kappa_1} \cdot \kappa_2}{\equiv} \tau_2} \text{EquivAK-}\Pi$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{\Pi_{t::\kappa_1} \cdot \kappa_2}{\equiv} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \stackrel{\kappa_1}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1 \tau_2 \stackrel{[\tau_2/t]\kappa_2}{\equiv} \tau_3 \tau_4} \text{EquivAK-Ap}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{\mathbf{S}_\kappa(\tau)}{\equiv} \tau_2}{\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2} (1)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{\text{Type}}{\equiv} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \stackrel{\text{Type}}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \stackrel{\text{Type}}{\equiv} \tau_3 \oplus \tau_4} (2)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}{\Delta; \Phi \vdash \lambda t::\kappa_1. \tau_1 \stackrel{\Pi_{t::\kappa_1} \cdot \kappa}{\equiv} \lambda t::\kappa_2. \tau_2} (3)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \stackrel{\kappa_1}{\equiv} \tau_2 \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa}{\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2} (4)$$

$\boxed{\Delta; \Phi \vdash \kappa \text{ OK}}$ κ is well formed

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{Type OK}} \text{KWF-Type}$$

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{KHole OK}} \text{KWF-KHole}$$

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \text{S}_\kappa(\tau) \text{ OK}} \text{KWF-SKind}$$

$$\frac{\Delta; \Phi, t :: \kappa_1 \vdash \kappa_2 \text{ OK}}{\Delta; \Phi \vdash \Pi_{t :: \kappa_1} \kappa_2 \text{ OK}} \text{KWF-}\Pi$$

$\boxed{\Delta; \Phi \vdash \text{OK}}$ Context is well formed

$$\frac{}{\cdot; \cdot \vdash \text{OK}} \text{CWF-Nil}$$

$$\frac{t \notin \Phi \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi, t :: \kappa \vdash \text{OK}} \text{CWF-TypVar}$$

$$\frac{u \notin \Delta \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta, u :: \kappa; \Phi \vdash \text{OK}} \text{CWF-Hole}$$

METATHEORY

subderivation preserving inferences:

- premiss
- COK (Context OK)
- PoS (premiss of subderivation)

Lemma 1 (COK). *If $\Delta; \Phi \vdash \mathcal{J}$, then $\Delta; \Phi \vdash \text{OK}$ in a subderivation (where $\Delta; \Phi \vdash \mathcal{J} \neq \Delta; \Phi \vdash \text{OK}$)*

Proof. By induction on derivations.

No interesting cases. □

Lemma 2 (Exchange).

If $\Delta; \Phi_1, t_{L1} :: \kappa_{L1}, t_{L2} :: \kappa_{L2}, \Phi_2 \vdash \mathcal{J}$ and $\Delta; \Phi_1, t_{L2} :: \kappa_{L2}, t_{L1} :: \kappa_{L1}, \Phi_2 \vdash \text{OK}$, then $\Delta; \Phi_1, t_{L2} :: \kappa_{L2}, t_{L1} :: \kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

Proof. By induction on derivations.

No interesting cases.

(Only rules with Φ extended in the consequent are interesting, which is only CWF-TypVar, but when \mathcal{J} is CWF, Exchange is identity) □

Corollary 3 (Marked-Exchange).

If $\Delta; \Phi, \underline{t_{L1} :: \kappa_{L1}}, \underline{t_{L2} :: \kappa_{L2}} \vdash \mathcal{J}$ and $\Delta; \Phi, \underline{t_{L2} :: \kappa_{L2}}, \underline{t_{L1} :: \kappa_{L1}} \vdash \text{OK}$, then $\Delta; \Phi, \underline{t_{L2} :: \kappa_{L2}}, \underline{t_{L1} :: \kappa_{L1}} \vdash \mathcal{J}$

Proof. Exchange when $\Phi_2 = \cdot$ □

Lemma 4 (Weakening).

If $\Delta; \Phi \vdash \mathcal{J}$ and $\Delta; \Phi, \underline{t_L :: \kappa_L} \vdash \text{OK}$ and $t_L \notin \mathcal{J}$ and $\forall t \in \kappa_L, t \notin \mathcal{J}$, then $\Delta; \Phi, \underline{t_L :: \kappa_L} \vdash \mathcal{J}$

Proof. see addendum □

Lemma 5 (K-Substitution).

*If $\Delta; \Phi \vdash \tau_{L1} :: \kappa_{L1}$ and $\Delta; \Phi, \underline{t_L :: \kappa_{L1}} \vdash \tau_{L2} :: \kappa_{L2}$, then $\Delta; \Phi \vdash [\tau_{L1}/t_L] \tau_{L2} :: [\tau_{L1}/t_L] \kappa_{L2}$
(induction on $\Delta; \Phi, \underline{t_L :: \kappa_{L1}} \vdash \tau_{L2} :: \kappa_{L2}$)*

Lemma 6 (PK-Substitution). *If $\Delta; \Phi \vdash \tau_{L1} :: \kappa_{L1}$ and $\Delta; \Phi, \underline{t_L} :: \kappa_{L1} \vdash \tau_{L2} :: > \kappa_{L2}$ and $\Delta; \Phi \vdash [\tau_{L1}/t_L] \tau_{L2} :: > \kappa_{L3}$, then $\Delta; \Phi \vdash [\tau_{L2}/t_L] \kappa_{L2} \equiv \kappa_{L3}$*

Lemma 7 (OK-Substitution).

If $\Delta; \Phi \vdash \tau_L :: \kappa_{L1}$ and $\Delta; \Phi, \underline{t_L} :: \kappa_{L1} \vdash \kappa_{L2}$ OK, then $\Delta; \Phi \vdash [\tau_L/t_L] \kappa_{L2}$ OK (induction on $\Delta; \Phi, \underline{t_L} :: \kappa_{L1} \vdash \kappa_{L2}$ OK)

Theorem 8 (OK-PK). *If $\Delta; \Phi \vdash \tau :: > \kappa$, then $\Delta; \Phi \vdash \kappa$ OK*

Theorem 9 (OK-WFaK). *If $\Delta; \Phi \vdash \tau :: \kappa$, then $\Delta; \Phi \vdash \kappa$ OK*

Theorem 10 (OK-MatchPi). *If $\Delta; \Phi \vdash \kappa \blacktriangleright \Pi_{t :: \kappa_1} . \kappa_2$, then $\Delta; \Phi \vdash \kappa$ OK and $\Delta; \Phi \vdash \Pi_{t :: \kappa_1} . \kappa_2$ OK*

Theorem 11 (OK-KEquiv). *If $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, then $\Delta; \Phi \vdash \kappa_1$ OK and $\Delta; \Phi \vdash \kappa_2$ OK*

Theorem 12 (OK-CSK). *If $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, then $\Delta; \Phi \vdash \kappa_1$ OK and $\Delta; \Phi \vdash \kappa_2$ OK*

Theorem 13 (OK-EquivAK). *If $\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2$, then $\Delta; \Phi \vdash \tau_1 :: \kappa$ and $\Delta; \Phi \vdash \tau_2 :: \kappa$ and $\Delta; \Phi \vdash \kappa$ OK*

Proof. see addendum □

Note: When applying Weakening in the induction, check that the left premiss is always a subderivation, and check variable exclusion conditions are satisfied (usually checked elsewhere in the derivation).

□

Theorem 14 (PK-Unicity). *If $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$ then $\kappa_{L1} = \kappa_{L2}$*

Proof. There's actually a slight problem here. In PK-Ap, $\blacktriangleright_{\Pi}$ doesn't have unicity since $\blacktriangleright_{\Pi}$ - Π uses $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, where "the same left side can have mutiple right sides" (transitivity), causing unicity to fail for PK.

Only thought about this quickly but we might need a separate HO singleton normalization scheme that is essentially the $\text{KEquiv-SKind}_{\text{SKind}}$, $\text{KEquiv-SKind}_{\Pi}$ fragment of kind equality, and have the HO singleton normalization judgment have unicity.

Kind of gross but...

(and add other HO kinds, namely Σ , when they get here)

And as cleanup, we can remove the normalization fragment from kind equality and add a single rule (yay more sim judgments!)

Also we need some unicity lemmas for contexts like for PK-EHole and co.

□

Theorem 15 (PK-Principality). *If $\Delta; \Phi \vdash \tau ::> \kappa_1$ and $\Delta; \Phi \vdash \tau ::\kappa_2$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

Proof. From definition of $\Delta; \Phi \vdash \tau ::\kappa$ and CSK-SKind

□

Theorem 16 (why is this here?). *If $\Delta; \Phi \vdash \kappa_1 \lesssim \mathbf{S}_{\kappa_2}(\tau)$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

ELABORATION

TODO