Algebraic Data Types for Hazel

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1 Syntax

$$
\begin{array}{llcl}
\mathsf{HTyp} & \tau & ::= & \varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid \llparenthesis\rrparenthesis \mid \llparenthesis\alpha\rrparenthesis \\
\mathsf{HTypPat} & \pi & ::= & \alpha \mid \llparenthesis\rrparenthesis \\
\mathsf{HExp} & e & ::= & \varnothing \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \mathtt{inj}_C(e) \mid \mathtt{roll}(e) \mid \mathtt{unroll}(e) \\
 & & & \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^{u\blacktriangleright} \\
\mathsf{IHExp} & d & ::= & \varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \mathtt{inj}^{\tau}_{C}(d) \mid \mathtt{roll}^{\mu\alpha.\tau}(d) \mid \mathtt{unroll}(d) \\
 & & & \mid d\langle\tau \Rightarrow \tau\rangle \mid d\langle\tau \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau\rangle \mid \llparenthesis\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^{u\blacktriangleright}_\sigma \\
\mathsf{HTag} & C & ::= & \mathbf{C} \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis c\rrparenthesis^u
\end{array}
$$

1.1 Context Extension

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \llparenthesis\rrparenthesis \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{lcll}
[\tau/\llparenthesis\rrparenthesis]\tau' & = & \tau' & \\
[\tau/\alpha]\varnothing & = & \varnothing & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu\llparenthesis\rrparenthesis.\tau_2 & = & \mu\llparenthesis\rrparenthesis.[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha]\llparenthesis\rrparenthesis & = & \llparenthesis\rrparenthesis & \\
[\alpha'/\alpha]\llparenthesis\alpha\rrparenthesis & = & \llparenthesis\alpha'\rrparenthesis^u & \\
[\alpha'/\alpha]\llparenthesis\alpha'\rrparenthesis^u & = & \llparenthesis\alpha'\rrparenthesis^u & \text{when } \alpha \neq \alpha'
\end{array}
$$

$\boxed{\Theta \vdash \tau\ \mathsf{valid}}$ τ is a valid type

$$
\textsc{TVUnit}\ \frac{}{\Theta \vdash \varnothing\ \mathsf{valid}}
\qquad
\textsc{TVArr}\ \frac{\Theta \vdash \tau_1\ \mathsf{valid} \qquad \Theta \vdash \tau_2\ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2\ \mathsf{valid}}
\qquad
\textsc{TVVar}\ \frac{\alpha \in \Theta}{\Theta \vdash \alpha\ \mathsf{valid}}
\qquad
\textsc{TVRec}\ \frac{\Theta, \pi \vdash \tau\ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau\ \mathsf{valid}}
\qquad
\textsc{TVSum}\ \frac{\{\Theta \vdash \tau_i\ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \mathsf{valid}}
$$

$$
\textsc{TVEHole}\ \frac{}{\Theta \vdash \llparenthesis\rrparenthesis\ \mathsf{valid}}
$$

$\boxed{\tau \sim \tau'}$ $\quad \tau$ and τ' are consistent

$$\frac{}{\tau \sim \tau}\ \textsc{TCRefl} \qquad \frac{}{\llparenthesis\rrparenthesis \sim \tau}\ \textsc{TCEHole1} \qquad \frac{}{\tau \sim \llparenthesis\rrparenthesis}\ \textsc{TCEHole2} \qquad \frac{}{\llparenthesis\alpha\rrparenthesis \sim \tau}\ \textsc{TCNEHole1} \qquad \frac{}{\tau \sim \llparenthesis\alpha\rrparenthesis}\ \textsc{TCNEHole2}$$

$$\frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}\ \textsc{TCArr}$$

$$\frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'}\ \textsc{TCRec} \qquad \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\llparenthesis\rrparenthesis.\tau \sim \mu\alpha.\tau'}\ \textsc{TCRecHole1} \qquad \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu\llparenthesis\rrparenthesis.\tau'}\ \textsc{TCRecHole2}$$

$$\frac{\{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}\ \textsc{TCSum}$$

$\boxed{C\ \mathsf{valid}}$ $\quad C$ is a valid tag

$$\frac{}{\mathbf{C}\ \mathsf{valid}}\ \textsc{CVTag} \qquad \frac{}{\llparenthesis\rrparenthesis^u\ \mathsf{valid}}\ \textsc{CVEHole}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity)**.** *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

Theorem 2 (Consistency Preserves Validity)**.** *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ $\quad \tau$ has matched arrow type $\tau_1 \to \tau_2$

$$\frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\to} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}\ \textsc{MAHole} \qquad \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}\ \textsc{MAArr}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ $\quad \tau$ has matched recursive type $\mu\pi.\tau'$

$$\frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau}\ \textsc{MRRec} \qquad \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\mu} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}\ \textsc{MRHole}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ $\quad e$ synthesizes type τ

$$\frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing}\ \textsc{SUnit} \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau}\ \textsc{SVar} \qquad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SVarFree}$$

$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}\ \textsc{SLam}$$

$$\frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma, x : \llparenthesis\rrparenthesis \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \llparenthesis\rrparenthesis \to \tau'}\ \textsc{SLamInvalid} \qquad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau}\ \textsc{SApp}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SAppNotArr}$$

$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau}\ \textsc{SAsc} \qquad \frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash e : \tau \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SAscInvalid}$$

$$\frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{roll}(e)\rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}\ \textsc{SRollErr} \qquad \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}\ \textsc{SUnroll}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}\big(\llparenthesis e\rrparenthesis^{u\blacktriangleright}\big) \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SUnrollNotRec}$$

$$\frac{C\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\mathtt{inj}_C(e)\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SInjErr} \qquad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{inj}_{\llparenthesis C\rrparenthesis^u}(e) \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SInjTagErr}$$

$$\frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SEHole} \qquad \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash \llparenthesis e\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis}\ \textsc{SNEHole}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

$$\textsc{ARoll}\quad \frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau}$$

$$\textsc{ARollNotRec}\quad \frac{\tau \not\sim \mu(\!|\,|\!).(\!|\,|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Leftarrow \tau}$$

$$\textsc{AInjHole}\quad \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow (\!|\,|\!)}$$

$$\textsc{AInj}\quad \frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjUnexpectedBody}\quad \frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjExpectedBody}\quad \frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjBadTag}\quad \frac{C\ \mathsf{valid} \qquad C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\texttt{inj}_C(e)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjTagErr}\quad \frac{(\!|c|\!)^u \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash \texttt{inj}_{(\!|c|\!)^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{ASubsume}\quad \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity). *If* $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

$$\textsc{ESUnit}\quad \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \rightsquigarrow \varnothing \dashv \emptyset}$$

$$\textsc{ESVar}\quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

$$\textsc{ESVarFree}\quad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|x|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\,|\!)[\Gamma]}$$

$$\textsc{ESLam}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESLamInvalid}\quad \frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma, x : (\!|\,|\!) \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow (\!|\,|\!) \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESApp}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_1 \rangle)(d_2\langle \tau_2' \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\textsc{ESAppNotArr}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \not\sim (\!|\,|\!) \to (\!|\,|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\,|\!) \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\,|\!) \rightsquigarrow (\!|d_1|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow (\!|\,|\!) \rangle) \dashv \Delta_1 \cup \Delta_2, u :: (\!|\,|\!) \to (\!|\,|\!)[\Gamma]}$$

$$\textsc{ESAsc}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau \rangle \dashv \Delta}$$

$$\textsc{ESAscInvalid}\quad \frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow (\!|\,|\!) \rightsquigarrow d\langle \tau' \Rightarrow (\!|\,|\!) \rangle \dashv \Delta}$$

$$\textsc{ESRollErr}\quad \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Rightarrow \mu(\!|\,|\!).(\!|\,|\!) \rightsquigarrow (\!|\texttt{roll}^{\mu(\!|\,|\!).(\!|\,|\!)}(d\langle \tau \Rightarrow (\!|\,|\!) \rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}$$

$$\textsc{ESUnroll}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \texttt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \texttt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau' \rangle) \dashv \Delta}$$

$$\textsc{ESUnrollNotRec}\quad \frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta \qquad \tau \not\sim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{unroll}\left(\llparenthesis e \rrparenthesis^{u\blacktriangleright}\right) \Rightarrow \llparenthesis\rrparenthesis \leadsto \texttt{unroll}\left(\llparenthesis d \rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\right) \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESInjErr}\quad \frac{C\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \llparenthesis \texttt{inj}_C(e) \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis \texttt{inj}^{\tau'}_C(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESInjTagErr}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{\llparenthesis c \rrparenthesis^u(\tau)\}}{\Gamma \vdash \texttt{inj}_{\llparenthesis c \rrparenthesis^u}(e) \Rightarrow \llparenthesis\rrparenthesis \leadsto \texttt{inj}^{\tau'}_{\llparenthesis c \rrparenthesis^u}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) \dashv \Delta}$$

$$\textsc{ESEHole}\quad \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESNEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \leadsto d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

$$\textsc{EARoll}\quad \frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \leadsto d : \tau'' \dashv \Delta}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau \leadsto \texttt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rangle) : \mu\pi.\tau' \dashv \Delta}$$

$$\textsc{EARollNotRec}\quad \frac{\tau \not\sim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau' \dashv \Delta}{\Gamma \vdash \llparenthesis \texttt{roll}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{EAInjHole}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow \llparenthesis\rrparenthesis \leadsto \texttt{inj}^{\tau'}_C(d) : \tau' \dashv \Delta}$$

$$\textsc{EAInj}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \leadsto d : \tau_j' \dashv \Delta}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow \tau \leadsto \texttt{inj}^{\tau}_{C_j}\left(d\langle \tau_j' \Rightarrow \tau_j \rangle\right) : \tau \dashv \Delta}$$

$$\textsc{EAInjUnexpectedBody}\quad \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau_j \dashv \Delta \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\Big\}\end{array}}{\Gamma \vdash \llparenthesis \texttt{inj}_{C_j}(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}^{\tau'}_{C_j}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjExpectedBody}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\Big\}}{\Gamma \vdash \llparenthesis \texttt{inj}_{C_j}(\varnothing) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}^{\tau'}_{C_j}(\varnothing) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjBadTag}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C \notin \mathcal{C} \qquad C\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau' \dashv \Delta \qquad \tau'' = +\big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{C(\tau')\}\big\}}{\Gamma \vdash \llparenthesis \texttt{inj}_C(e) \rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis \texttt{inj}^{\tau''}_C(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjTagErr}\quad \frac{\llparenthesis c \rrparenthesis^u \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \leadsto d : \tau \dashv \Delta \qquad \tau' = +\big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{\llparenthesis c \rrparenthesis^u(\tau)\}\big\}}{\Gamma \vdash \texttt{inj}_{\llparenthesis c \rrparenthesis^u}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \leadsto \texttt{inj}^{\tau'}_{\llparenthesis c \rrparenthesis^u}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta}$$

$$\textsc{EASubsume}\quad \frac{e \neq \llparenthesis\rrparenthesis^u \qquad e \neq \llparenthesis e' \rrparenthesis^u \qquad \Gamma \vdash e \Rightarrow \tau' \leadsto d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta}$$

$$\textsc{EAEHole}\quad \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Leftarrow \tau \leadsto \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}\;\textsc{EANEHole}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

$$\frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}\;\textsc{TAUnit} \qquad \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}\;\textsc{TAVar} \qquad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}\;\textsc{TALam}$$

$$\frac{\neg(\emptyset \vdash \tau\ \mathsf{valid}) \qquad \Delta; \Gamma, x : (\!|\,|\!) \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : (\!|\,|\!) \to \tau'}\;\textsc{TALamInvalid}$$

$$\frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}\;\textsc{TAApp} \qquad \frac{\emptyset \vdash \mu\pi.\tau\ \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}\;\textsc{TARoll}$$

$$\frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}\;\textsc{TAUnroll} \qquad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(d) : \tau}\;\textsc{TAInj}$$

$$\frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|\,|\!)^u_\sigma : \tau}\;\textsc{TAEHole} \qquad \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^u_\sigma : \tau}\;\textsc{TANEHole}$$

$$\frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^{u\blacktriangleright}_\sigma : \tau}\;\textsc{TAMHole} \qquad \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}\;\textsc{TACast}$$

$$\frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2 \rangle : \tau_2}\;\textsc{TAFailedCast}$$

3 Dynamic Semantics

$\boxed{\tau\ \text{ground}}$ τ is a ground type

$$\frac{}{(\!|\,|\!) \to (\!|\,|\!)\ \text{ground}}\;\textsc{GArr} \qquad \frac{}{\mu(\!|\,|\!).(\!|\,|\!)\ \text{ground}}\;\textsc{GRec} \qquad \frac{\{\tau_i = \varnothing \lor \tau_i = (\!|\,|\!)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \text{ground}}\;\textsc{GSum}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

$$\frac{\tau_1 \to \tau_2 \neq (\!|\,|\!) \to (\!|\,|\!)}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} (\!|\,|\!) \to (\!|\,|\!)}\;\textsc{MGArr} \qquad \frac{\tau \neq (\!|\,|\!)}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu(\!|\,|\!).(\!|\,|\!)}\;\textsc{MGRec}$$

$$\frac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \qquad \{(\tau_i = \varnothing \implies \tau_i' = \varnothing) \land (\tau_i \neq \varnothing \implies \tau_i' = (\!|\,|\!))\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}\;\textsc{MGSum}$$

$\boxed{d\ \text{final}}$ d is final

$$\textsc{FBoxedVal}\ \frac{d\ \text{boxedval}}{d\ \text{final}} \qquad \textsc{FIndet}\ \frac{d\ \text{indet}}{d\ \text{final}}$$

$\boxed{d\ \text{val}}$ d is a value

$$\textsc{VUnit}\ \frac{}{\varnothing\ \text{val}} \qquad \textsc{VLam}\ \frac{}{\lambda x{:}\tau.d\ \text{val}} \qquad \textsc{VRoll}\ \frac{d\ \text{val}}{\texttt{roll}^{\mu\pi.\tau}(d)\ \text{val}} \qquad \textsc{VInj}\ \frac{d\ \text{val}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d)\ \text{val}}$$

$\boxed{d\ \text{boxedval}}$ d is a boxed value

$$\textsc{BVVal}\ \frac{d\ \text{val}}{d\ \text{boxedval}} \qquad \textsc{BVRoll}\ \frac{d\ \text{boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d)\ \text{boxedval}} \qquad \textsc{BVInj}\ \frac{d\ \text{boxedval}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d)\ \text{boxedval}}$$

$$\textsc{BVArrCast}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d\ \text{boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle\ \text{boxedval}}$$

$$\textsc{BVRecCast}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d\ \text{boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle\ \text{boxedval}}$$

$$\textsc{BVSumCast}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i\in\mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i\in\mathcal{C}} \\ \tau \neq \tau' \qquad d\ \text{boxedval}\end{array}}{d\langle \tau \Rightarrow \tau'\rangle\ \text{boxedval}}$$

$$\textsc{BVHoleCast}\ \frac{d\ \text{boxedval} \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle\ \text{boxedval}}$$

$\boxed{d\ \text{indet}}$ d is indeterminate

$$\textsc{IEHole}\ \frac{}{\llparenthesis\rrparenthesis^u_\sigma\ \text{indet}} \qquad \textsc{INEHole}\ \frac{d\ \text{final}}{\llparenthesis d\rrparenthesis^u_\sigma\ \text{indet}}$$

$$\textsc{IApp}\ \frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle \qquad d_1\ \text{indet} \qquad d_2\ \text{final}}{d_1(d_2)\ \text{indet}} \qquad \textsc{IRoll}\ \frac{d\ \text{indet}}{\texttt{roll}^{\mu\pi.\tau}(d)\ \text{indet}}$$

$$\textsc{IUnroll}\ \frac{d\ \text{indet}}{\texttt{unroll}(d)\ \text{indet}} \qquad \textsc{IInj}\ \frac{d\ \text{indet}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d)\ \text{indet}} \qquad \textsc{IInjHole}\ \frac{C \neq \mathbf{C} \qquad d\ \text{final}}{\texttt{inj}^{\tau}_{C}(d)\ \text{indet}}$$

$$\textsc{ICastGroundHole}\ \frac{d\ \text{indet} \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle\ \text{indet}}$$

$$\textsc{ICastHoleGround}\ \frac{d \neq d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis\rangle \qquad d\ \text{indet} \qquad \tau\ \text{ground}}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau\rangle\ \text{indet}}$$

$$\textsc{ICastArr}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d\ \text{indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle\ \text{indet}} \qquad \textsc{ICastRec}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d\ \text{indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle\ \text{indet}}$$

$$\textsc{ICastSum}\ \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i\in\mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i\in\mathcal{C}} \\ \tau \neq \tau' \qquad d\ \text{indet}\end{array}}{d\langle \tau \Rightarrow \tau'\rangle\ \text{indet}}$$

$$\textsc{IFailedCast}\ \frac{d\ \text{final} \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle\ \text{indet}}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\textsc{ITApp}\ \frac{[d_2\ \text{final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1} \qquad \textsc{ITUnroll}\ \frac{[d\ \text{final}]}{\texttt{unroll}(\texttt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\textsc{ITAppCast}\ \frac{[d_1\ \text{final}] \qquad [d_2\ \text{final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1\rangle))\langle \tau_2 \Rightarrow \tau_2'\rangle}$$

ITUnrollCast
$$\frac{[d\ \text{final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\texttt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \texttt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle}$$

ITCastId
$$\frac{[d\ \text{final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}$$

ITCastSucceed
$$\frac{[d\ \text{final}] \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow (\!|\,|\!) \Rightarrow \tau\rangle \longrightarrow d}$$

ITCastFail
$$\frac{[d\ \text{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground}}{d\langle \tau_1 \Rightarrow (\!|\,|\!) \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle}$$

ITGround
$$\frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow (\!|\,|\!)\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow (\!|\,|\!)\rangle}$$

ITExpand
$$\frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle (\!|\,|\!) \Rightarrow \tau\rangle \longrightarrow d\langle (\!|\,|\!) \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\begin{array}{llcl} \mathsf{EvalCtx} & \mathcal{E} & ::= & \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \texttt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \texttt{unroll}(\mathcal{E}) \mid \texttt{inj}^{\tau}_{C}(\mathcal{E}) \mid (\!|\mathcal{E}|\!)^{u}_{\sigma} \mid (\!|\mathcal{E}|\!)^{u\blacktriangleright}_{\sigma} \\ & & & \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow (\!|\,|\!) \not\Rightarrow \tau\rangle \end{array}$$

$\boxed{d = \mathcal{E}\{d'\}}$ d is obtained by placing d' at the mark in $\mathcal{E}$

FHOuter
$$\frac{}{d = \circ\{d\}}$$

FHApp1
$$\frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}$$

FHApp2
$$\frac{[d_1\ \text{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

FHRoll
$$\frac{d = \mathcal{E}\{d'\}}{\texttt{roll}^{\mu\pi.\tau}(d) = \texttt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

FHUnroll
$$\frac{d = \mathcal{E}\{d'\}}{\texttt{unroll}(d) = \texttt{unroll}(\mathcal{E})\{d'\}}$$

FHInj
$$\frac{d = \mathcal{E}\{d'\}}{\texttt{inj}^{\tau}_{C}(d) = \texttt{inj}^{\tau}_{C}(\mathcal{E})\{d'\}}$$

FNEHoleInside
$$\frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u}_{\sigma} = (\!|\mathcal{E}|\!)^{u}_{\sigma}\{d'\}}$$

FMHoleInside
$$\frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^{u\blacktriangleright}_{\sigma} = (\!|\mathcal{E}|\!)^{u\blacktriangleright}_{\sigma}\{d'\}}$$

FHCastInside
$$\frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}}$$

FHFailedCast
$$\frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ d steps to d'

Step
$$\frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$