Hazel Phi: 11-type-constructors

July 27, 2021

SYNTAX

$$
\begin{array}{rccl}
\text{Kind} & \kappa & ::= & \mathtt{Type} \mid \mathtt{KHole} \mid \mathtt{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2 \\
\text{User Types} & \hat{\tau} & ::= & t \mid \mathtt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\hat{\tau}\rrparenthesis^{\mathtt{u}} \mid \lambda t::\mathtt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2} \\
\text{Internal Types} & \tau & ::= & t \mid \mathtt{bse} \mid \tau_1 \oplus \tau_2 \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\tau\rrparenthesis^{\mathtt{u}} \mid \llparenthesis t\rrparenthesis^{\mathtt{u}} \mid \lambda t::\kappa.\tau \mid \tau_1\ \tau_2 \\
\text{Base Types} & \mathtt{bse} & ::= & \mathtt{Int} \mid \mathtt{Float} \mid \mathtt{Bool} \\
\text{BinOp} & \oplus & ::= & \times \mid + \mid \to \\
\text{Type Pattern} & & & \\
\text{User Expression} & & & \\
\text{Internal Expression} & & &
\end{array}
$$

DECLARATIVES

$\boxed{\Delta; \Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash \mathtt{bse} ::> \mathtt{S}_{\mathtt{Type}}(\mathtt{bse})}\ (1) \qquad \frac{\Delta; \Phi_1, t::\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta; \Phi \vdash t ::> \mathtt{S}_{\kappa}(t)}\ (2) \qquad \frac{\Delta; \Phi \vdash \tau_1::\mathtt{Type} \qquad \Delta; \Phi \vdash \tau_2::\mathtt{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathtt{S}_{\mathtt{Type}}(\tau_1 \oplus \tau_2)}\ (3)$$

$$\frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash \llparenthesis\rrparenthesis^{\mathtt{u}} ::> \mathtt{S}_{\kappa}(\llparenthesis\rrparenthesis^{\mathtt{u}})}\ (4) \qquad \frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad \Delta; \Phi \vdash \tau::\kappa_1}{\Delta; \Phi \vdash \llparenthesis\tau\rrparenthesis^{\mathtt{u}} ::> \mathtt{S}_{\kappa}(\llparenthesis\tau\rrparenthesis^{\mathtt{u}})}\ (5)$$

$$\frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad t \notin \mathsf{dom}(\Phi)}{\Delta; \Phi \vdash \llparenthesis t\rrparenthesis^{\mathtt{u}} ::> \mathtt{S}_{\kappa}(\llparenthesis t\rrparenthesis^{\mathtt{u}})}\ (6) \qquad \frac{\Delta; \Phi, t::\kappa_1 \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau ::> \mathtt{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)}\ (7)$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \qquad \Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta; \Phi \vdash \tau_2::\kappa_1}{\Delta; \Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ (8)$$

$\boxed{\Delta; \Phi \vdash \tau::\kappa}$ τ is well formed at kind κ

$$\frac{\Delta; \Phi \vdash \tau ::> \kappa}{\Delta; \Phi \vdash \tau::\kappa}\ (9) \qquad \frac{\Delta; \Phi \vdash \tau ::> \mathtt{S}_{\kappa}(\tau)}{\Delta; \Phi \vdash \tau::\kappa}\ (10)$$

$$\frac{\Delta; \Phi \vdash \tau ::> \kappa_1 \qquad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau::\kappa}\ (11)$$

$$\frac{\Delta; \Phi \vdash \tau::\kappa}{\Delta; \Phi \vdash \tau::\mathtt{S}_{\kappa}(\tau)}\ (12) \qquad \frac{\Delta; \Phi \vdash \tau::\Pi_{t::\kappa_3}.\kappa_4 \qquad \Delta; \Phi \vdash \Pi_{t::\kappa_3}.\kappa_4 \lesssim \Pi_{t::\kappa_1}.\kappa_2}{\Delta; \Phi \vdash \tau::\Pi_{t::\kappa_1}.\kappa_2}\ (13)$$

$$\frac{\Delta; \Phi \vdash \tau::\mathtt{S}_{\kappa}(\tau_1) \qquad \Delta; \Phi \vdash \tau_1::\kappa}{\Delta; \Phi \vdash \tau::\kappa}\ (14)$$

$\boxed{\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{_\Pi} \Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{}{\Delta; \Phi \vdash \texttt{KHole} \overset{\blacktriangleright}{_\Pi} \Pi_{t::\texttt{KHole}}.\texttt{KHole}} \;(15) \qquad \frac{\Delta; \Phi \vdash \kappa \equiv \mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{_\Pi} \Pi_{t::\mathsf{S}_{\texttt{KHole}}(\tau)}.\mathsf{S}_{\texttt{KHole}}(\tau\ t)} \;(16) \qquad \frac{\Delta; \Phi \vdash \kappa \equiv \Pi_{t::\kappa_1}.\kappa_2}{\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{_\Pi} \Pi_{t::\kappa_1}.\kappa_2} \;(17)$$

$\boxed{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \equiv \kappa} \;(18) \qquad \frac{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \;(19) \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta; \Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \;(20)$$

$$\frac{\Delta; \Phi \vdash \tau :: \mathsf{S}_\kappa(\tau_1)}{\Delta; \Phi \vdash \mathsf{S}_{\mathsf{S}_\kappa(\tau_1)}(\tau) \equiv \mathsf{S}_\kappa(\tau_1)} \;(21) \qquad \frac{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_1}.\kappa_2}{\Delta; \Phi \vdash \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau) \equiv \Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau\ t_1)} \;(22)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta; \Phi, t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4} \;(23) \qquad \frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathsf{S}_\kappa(\tau_1) \equiv \mathsf{S}_\kappa(\tau_2)} \;(24)$$

$\boxed{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \texttt{KHole} \lesssim \kappa} \;(25) \qquad \frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \lesssim \texttt{KHole}} \;(26) \qquad \frac{\Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \ \mathsf{OK} \qquad \Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \lesssim \kappa} \;(27)$$

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK} \qquad \Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathsf{S}_{\texttt{KHole}}(\tau)} \;(28)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \;(29) \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta; \Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta; \Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \;(30)$$

$$\frac{\Delta; \Phi \vdash \mathsf{S}_\kappa(\tau) \ \mathsf{OK}}{\Delta; \Phi \vdash \mathsf{S}_\kappa(\tau) \lesssim \kappa} \;(31) \qquad \frac{\Delta; \Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta; \Phi, t::\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4} \;(32)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta; \Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)} \;(33)$$

$\boxed{\Delta;\Phi\vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta;\Phi\vdash \tau_1 ::> \kappa_1 \qquad \Delta;\Phi\vdash \kappa_1 \equiv \mathtt{S}_\kappa(\tau_2)}{\Delta;\Phi\vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (34)$$

$$\frac{\Delta;\Phi\vdash \tau_1::\Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta;\Phi\vdash \tau_2::\Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta;\Phi, t::\kappa_1\vdash \tau_1\ t \overset{\kappa_2}{\equiv} \tau_2\ t}{\Delta;\Phi\vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2}\ (35) \qquad \frac{\Delta;\Phi\vdash \tau_1 \overset{\mathtt{S}_\kappa(\tau)}{\equiv} \tau_2}{\Delta;\Phi\vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (36)$$

$$\frac{\Delta;\Phi\vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta;\Phi\vdash \tau_2 \overset{\kappa_1}{\equiv} \tau_4}{\Delta;\Phi\vdash \tau_1\ \tau_2 \overset{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4}\ (37)$$

$$\frac{\Delta;\Phi\vdash \tau::\kappa}{\Delta;\Phi\vdash \tau \overset{\kappa}{\equiv} \tau}\ (38) \qquad \frac{\Delta;\Phi\vdash \tau_2 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi\vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (39) \qquad \frac{\Delta;\Phi\vdash \tau_1 \overset{\kappa}{\equiv} \tau_3 \qquad \Delta;\Phi\vdash \tau_3 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi\vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (40)$$

$$\frac{\Delta;\Phi\vdash \tau_1 \overset{\mathtt{Type}}{\equiv} \tau_3 \qquad \Delta;\Phi\vdash \tau_2 \overset{\mathtt{Type}}{\equiv} \tau_4}{\Delta;\Phi\vdash \tau_1\oplus\tau_2 \overset{\mathtt{Type}}{\equiv} \tau_3\oplus\tau_4}\ (41) \qquad \frac{\Delta;\Phi\vdash \kappa_1\equiv\kappa_2 \qquad \Delta;\Phi, t::\kappa_1\vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta;\Phi\vdash \lambda t::\kappa_1.\tau_1 \overset{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t::\kappa_2.\tau_2}\ (42)$$

$$\frac{\Delta;\Phi\vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi\vdash \kappa_1\equiv\kappa}{\Delta;\Phi\vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (43)$$

$\boxed{\Delta;\Phi\vdash \kappa\ \mathsf{OK}}$ κ is well formed

$$\frac{\Delta;\Phi\vdash \mathsf{OK}}{\Delta;\Phi\vdash \mathtt{Type}\ \mathsf{OK}}\ (44) \qquad \frac{\Delta;\Phi\vdash \mathsf{OK}}{\Delta;\Phi\vdash \mathtt{KHole}\ \mathsf{OK}}\ (45) \qquad \frac{\Delta;\Phi\vdash \tau::\kappa}{\Delta;\Phi\vdash \mathtt{S}_\kappa(\tau)\ \mathsf{OK}}\ (46)$$

$$\frac{\Delta;\Phi\vdash \kappa_1\ \mathsf{OK} \qquad \Delta;\Phi, t::\kappa_1\vdash \kappa_2\ \mathsf{OK}}{\Delta;\Phi\vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathsf{OK}}\ (47)$$

$\boxed{\Delta;\Phi\vdash \mathsf{OK}}$ Context is well formed

$$\frac{}{;\vdash \mathsf{OK}}\ (48) \qquad \frac{\Delta;\Phi\vdash \kappa\ \mathsf{OK}}{\Delta;\Phi, t::\kappa\vdash \mathsf{OK}}\ (49) \qquad \frac{\Delta;\Phi\vdash \kappa\ \mathsf{OK}}{\Delta, \mathrm{u}::\kappa;\Phi\vdash \mathsf{OK}}\ (50)$$

Variables implicitly assumed to be fresh as necessary

Metatheory

Lemma 1 (Weakening). *If $\Delta;\Phi\vdash\tau::\kappa$, then $\Delta;\Phi, t::\kappa_1\vdash\tau::\kappa$ when $\Delta;\Phi, t::\kappa_1\vdash \mathsf{OK}$*

Proof. By rule induction/length of proof.
L1. (9)

□

Proof. By rule induction/length of proof.
L2. (9)

□

Lemma 2 (OK-PK). *If $\Delta;\Phi\vdash\tau ::> \kappa$, then $\Delta;\Phi\vdash \mathsf{OK}$ and $\Delta;\Phi\vdash\kappa\ \mathsf{OK}$*

Lemma 3 (OK-WFaK). *If* $\Delta;\Phi \vdash \tau{::}\kappa$, *then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$

Lemma 4 (OK-MatchPi). *If* $\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2$, *then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$ *and* $\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathit{OK}$

Lemma 5 (OK-KEquiv). *If* $\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2$, *then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_2\ \mathit{OK}$

Lemma 6 (OK-CSK). *If* $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$, *then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_2\ \mathit{OK}$

Lemma 7 (OK-TEquivAK). *If* $\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2$, *then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \tau_1{::}\kappa$ *and* $\Delta;\Phi \vdash \tau_2{::}\kappa$ *and* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$

Lemma 8 (OK-KWF). *If* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$, *then* $\Delta;\Phi \vdash \mathit{OK}$

Lemma 9 (OK-Substitution).
If $\Delta;\Phi \vdash \tau_L{::}\kappa_{L1}$ *and* $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \kappa_{L2}\ \mathit{OK}$, *then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash [\tau_L/t_L]\kappa_{L2}\ \mathit{OK}$
(induction on $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \kappa_{L2}\ \mathit{OK}$*)*

Lemma 10 (K-Substitution).
If $\Delta;\Phi \vdash \tau_{L1}{::}\kappa_{L1}$ *and* $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \tau_{L2}{::}\kappa_{L2}$, *then* $\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau_{L2}{::}[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \tau_{L2}{::}\kappa_{L2}$*)*

Proof. By simultaneous rule induction/length of proof.
The interesting cases per lemma:

OK-PK.	(1)	$\Delta;\Phi \vdash \mathtt{bse}{::}\mathtt{S}_{\mathtt{Type}}(\mathtt{bse})$	by (9)
		$\Delta;\Phi \vdash \mathtt{bse}{::}\mathtt{Type}$	by (10)
	*	$\Delta;\Phi \vdash \mathtt{S}_{\mathtt{Type}}(\mathtt{bse})\ \mathsf{OK}$	by (43)
	*	$\Delta;\Phi \vdash \mathsf{OK}$	by premiss
	(8)		bad
OK-WFaK.	(12)	$\Delta;\Phi \vdash \tau_2{::}\kappa$	by (10)
	*	$\Delta;\Phi \vdash \mathtt{S}_\kappa(\tau_2)\ \mathsf{OK}$	by (43)
OK-KEquiv.	(22)	$\Delta;\Phi \vdash \tau\ t ::> \kappa$	
OK-Substitution.	(41)	$\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \mathsf{OK}$	premiss (41)
		$\Delta;\Phi \vdash \kappa_{L1}\ \mathsf{OK}$	by subderivation premiss (46)
	*	$\Delta;\Phi \vdash \mathsf{OK}$	by OK-KWF
	*	$\Delta;\Phi \vdash [\tau_L/t_L]\mathtt{Type}\ \mathsf{OK}$	by (41) and degenerate subst
	(43)	$\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \tau{::}\kappa$	premiss (43)
		$\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \mathsf{OK}$	by OK-WFaK
		$\Delta;\Phi \vdash \kappa_{L1}\ \mathsf{OK}$	by subderivation premiss (46)
	*	$\Delta;\Phi \vdash \mathsf{OK}$	by OK-KWF
		$\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau{::}[\tau_{L1}/t_L]\kappa$	by K-Substitution on premiss
	*	$\Delta;\Phi \vdash [\tau_L/t_L]\mathtt{S}_\kappa(\tau)\ \mathsf{OK}$	by (43)

□

Lemma 11 (PK-Unicity). *If* $\Delta;\Phi \vdash \tau_L ::> \kappa_{L1}$ *and* $\Delta;\Phi \vdash \tau_L ::> \kappa_{L2}$ *then* κ_{L1} *is* κ_{L2}

Lemma 12. *If* $\Delta;\Phi \vdash \tau ::> \kappa_1$ *and* $\Delta;\Phi \vdash \tau{::}\kappa_2$, *then* $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$