Algebraic Data Types for Hazel

Eric Griffis
egriffis@umich.edu

1 Syntax

$$
\begin{array}{llcl}
\mathsf{HTyp} & \tau & ::= & \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \mid \llparenthesis\rrparenthesis \mid \llparenthesis\alpha\rrparenthesis \\
\mathsf{HTypPat} & \pi & ::= & \alpha \mid \llparenthesis\rrparenthesis \\
\mathsf{HExp} & e & ::= & x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \texttt{inj}_C(E) \mid \texttt{roll}(e) \mid \texttt{unroll}(e) \\
 & & & \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^u \mid \llparenthesis e\rrparenthesis^{u\blacktriangleright} \\
\mathsf{HTag} & C & ::= & \mathbf{C} \mid ?^u \\
\mathsf{HTagTyp} & T & ::= & \tau \mid \varnothing \\
\mathsf{HTagArg} & E & ::= & e \mid \varnothing \\
\mathsf{IHExp} & d & ::= & x \mid \lambda x{:}\tau.d \mid d(d) \mid \texttt{inj}^{\tau}_C(D) \mid \texttt{roll}^{\mu\alpha.\tau}(d) \mid \texttt{unroll}(d) \\
 & & & \mid d\langle\tau \Rightarrow \tau\rangle \mid d\langle\tau \Rightarrow \llparenthesis\rrparenthesis \nRightarrow \tau\rangle \mid \llparenthesis\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^u_\sigma \mid \llparenthesis d\rrparenthesis^{u\blacktriangleright}_\sigma \\
\mathsf{IHTagArg} & D & ::= & d \mid \varnothing
\end{array}
$$

1.1 Context Extension

We write $\Gamma, X : T$ to denote the extension of typing context Γ with optional variable X of optional type T.

$$
\Gamma, X : T = \begin{cases} \Gamma, x : \tau & X = x \wedge T = \tau \\ \Gamma, x : \llparenthesis\rrparenthesis & X = x \wedge T = \varnothing \\ \Gamma & X = \varnothing \end{cases}
$$

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = \llparenthesis\rrparenthesis \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{lcll}
[\tau/\llparenthesis\rrparenthesis]\tau' & = & \tau' & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu\llparenthesis\rrparenthesis.\tau_2 & = & \mu\llparenthesis\rrparenthesis.[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]+\{C_i(T_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]T_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha]\llparenthesis\rrparenthesis & = & \llparenthesis\rrparenthesis & \\
[\alpha'/\alpha]\llparenthesis\alpha\rrparenthesis & = & \llparenthesis\alpha'\rrparenthesis^u & \\
[\alpha'/\alpha]\llparenthesis\alpha_1\rrparenthesis^u & = & \llparenthesis\alpha_1\rrparenthesis^u & \text{when } \alpha \neq \alpha_1
\end{array}
$$

$\boxed{[\tau/\pi]T = \tau'}$ τ' is obtained by substituting τ for π in T

$$[\tau/\pi]T = \begin{cases} [\tau/\pi]\tau' & \text{when } T = \tau' \\ \varnothing & \text{when } T = \varnothing \end{cases}$$

$\boxed{\Theta \vdash \tau\ \mathsf{valid}}$ τ is a valid type

$$\frac{\Theta \vdash \tau_1\ \mathsf{valid} \qquad \Theta \vdash \tau_2\ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2\ \mathsf{valid}}\ \text{TVArr} \qquad \frac{\alpha \in \Theta}{\Theta \vdash \alpha\ \mathsf{valid}}\ \text{TVVar} \qquad \frac{\Theta, \pi \vdash \tau\ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau\ \mathsf{valid}}\ \text{TVRec} \qquad \frac{\{\Theta \vdash T_i\ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(T_i)\}_{C_i \in \mathcal{C}}\ \mathsf{valid}}\ \text{TVSum} \qquad \frac{}{\Theta \vdash (\!|\!|\!)\ \mathsf{valid}}\ \text{TVEHole}$$

$\boxed{\Theta \vdash T\ \mathsf{valid}}$ T is a valid optional type

$$\frac{T = \tau \qquad \Theta \vdash \tau\ \mathsf{valid}}{\Theta \vdash T\ \mathsf{valid}}\ \text{TVSome} \qquad \frac{}{\Theta \vdash \varnothing\ \mathsf{valid}}\ \text{TVNone}$$

$\boxed{\tau \sim \tau'}$ τ and τ' are consistent

$$\frac{}{\tau \sim \tau}\ \text{TCRefl} \qquad \frac{}{(\!|\!|\!) \sim \tau}\ \text{TCEHole1} \qquad \frac{}{\tau \sim (\!|\!|\!)}\ \text{TCEHole2} \qquad \frac{}{(\!|\alpha|\!) \sim \tau}\ \text{TCNEHole1} \qquad \frac{}{\tau \sim (\!|\alpha|\!)}\ \text{TCNEHole2} \qquad \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}\ \text{TCArr}$$

$$\frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'}\ \text{TCRec} \qquad \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu(\!|\!|\!).\tau \sim \mu\alpha.\tau'}\ \text{TCRecHole1} \qquad \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu(\!|\!|\!).\tau'}\ \text{TCRecHole2} \qquad \frac{\{T_i \sim T_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(T_i')\}_{C_i \in \mathcal{C}}}\ \text{TCSum}$$

$\boxed{T \sim T'}$ T and T' are consistent

$$\frac{\tau \sim \tau'}{\tau \sim \tau'}\ \text{TCSome} \qquad \frac{}{\varnothing \sim \varnothing}\ \text{TCNone}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

Theorem 2 (Consistency Preserves Validity). *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ τ has matched arrow type $\tau_1 \to \tau_2$

$$\frac{}{(\!|\!|\!) \blacktriangleright_{\to} (\!|\!|\!) \to (\!|\!|\!)}\ \text{MAHole} \qquad \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}\ \text{MAArr}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ τ has matched recursive type $\mu\pi.\tau'$

$$\textsc{MRRec}\ \frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau} \qquad \textsc{MRHole}\ \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\mu} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ e synthesizes type τ

$$\textsc{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \textsc{SVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x \rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SLam}\ \frac{\emptyset \vdash \tau\,\mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'} \qquad \textsc{SLamInvalid}\ \frac{\Gamma, x : \llparenthesis\rrparenthesis \vdash e \Rightarrow \tau}{\Gamma \vdash \lambda x{:}\llparenthesis\alpha\rrparenthesis.e \Rightarrow \llparenthesis\rrparenthesis \to \tau}$$

$$\textsc{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \qquad \textsc{SAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis e_1(e_2) \rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis}$$

$$\textsc{SAsc}\ \frac{\emptyset \vdash \tau\,\mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \qquad \textsc{SAscInvalid}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash e : \llparenthesis\alpha\rrparenthesis \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SRollError}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis \mathtt{roll}(e) \rrparenthesis^{u} \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis} \qquad \textsc{SUnroll}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$$

$$\textsc{SUnrollNotRec}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}\big(\llparenthesis e \rrparenthesis^{u\blacktriangleright}\big) \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SInjError}\ \frac{\Gamma \vdash E\,\mathsf{valid}}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C}(E) \rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SEHole}\ \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis} \qquad \textsc{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash \llparenthesis e \rrparenthesis^{u} \Rightarrow \llparenthesis\rrparenthesis}$$

$\boxed{\Gamma \vdash E\,\mathsf{valid}}$ E is a valid optional expression

$$\textsc{EVSome}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash e\,\mathsf{valid}} \qquad \textsc{EVNone}\ \frac{}{\Gamma \vdash \varnothing\,\mathsf{valid}}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

$$\textsc{ARoll}\ \frac{\tau \blacktriangleright_{\mu} \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau} \qquad \textsc{ARollNotRec}\ \frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{roll}\big(\llparenthesis e \rrparenthesis^{u\blacktriangleright}\big) \Leftarrow \tau} \qquad \textsc{AInjHole}\ \frac{\Gamma \vdash E\,\mathsf{valid}}{\Gamma \vdash \mathtt{inj}_{C}(E) \Leftarrow \llparenthesis\rrparenthesis}$$

$$\textsc{AInj}\ \frac{C_j \in \mathcal{C} \qquad \Gamma \vdash E \Leftarrow T_j}{\Gamma \vdash \mathtt{inj}_{C_j}(E) \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}} \qquad \textsc{AInjUnexpectedBody}\ \frac{C_j \in \mathcal{C} \qquad T_j = \varnothing \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C_j}(e) \rrparenthesis^{u} \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjExpectedBody}\ \frac{C_j \in \mathcal{C} \qquad T_j = \tau}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C_j}(\varnothing) \rrparenthesis^{u} \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}} \qquad \textsc{AInjBadTag}\ \frac{C \notin \mathcal{C} \qquad \Gamma \vdash E\,\mathsf{valid}}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C}(E) \rrparenthesis^{u} \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}} \qquad \textsc{ASubsume}\ \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

$\boxed{\Gamma \vdash E \Leftarrow T}$ E analyzes against optional type T

$$\textsc{ASome}\ \frac{\Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e \Leftarrow \tau} \qquad \textsc{ANone}\ \frac{}{\Gamma \vdash \varnothing \Leftarrow \varnothing}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity)**.** *If* $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

$$\textsc{ESVar}\quad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

$$\textsc{ESVarFree}\quad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis x \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESLam}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESLamInvalid}\quad \frac{\Gamma, x : \llparenthesis\rrparenthesis \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\llparenthesis \alpha \rrparenthesis.e \Rightarrow \llparenthesis\rrparenthesis \to \tau \rightsquigarrow \lambda x{:}\llparenthesis \alpha \rrparenthesis.d \dashv \Delta}$$

$$\textsc{ESApp}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_2 \to \tau\rangle)(d_2\langle \tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\textsc{ESAppNotArr}\quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash \llparenthesis e_1 \rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d_1 \rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow \llparenthesis\rrparenthesis\rangle) \dashv \Delta_1 \cup \Delta_2, u :: \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESAsc}\quad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau\rangle \dashv \Delta}$$

$$\textsc{ESAscInvalid}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash e : \llparenthesis \alpha \rrparenthesis \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle \dashv \Delta}$$

$$\textsc{ESRollError}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash \llparenthesis \mathtt{roll}(e) \rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis \mathtt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESUnroll}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \mathtt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}$$

$$\textsc{ESUnrollNotRec}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}\big(\llparenthesis e \rrparenthesis^{u\blacktriangleright}\big) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \mathtt{unroll}\Big(\llparenthesis d \rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\Big) \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESInjErrSome}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \llparenthesis \mathtt{inj}_C(e) \rrparenthesis^u \Rightarrow \tau' \rightsquigarrow \llparenthesis \mathtt{inj}^{\tau'}_C(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \tau'[\Gamma]}$$

$$\textsc{ESInjErrNone}\quad \frac{\tau = +\{C(\varnothing)\}}{\Gamma \vdash \llparenthesis \mathtt{inj}_c(\varnothing) \rrparenthesis^u \Rightarrow \tau \rightsquigarrow \llparenthesis \mathtt{inj}^{\tau}_C(\varnothing) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{ESEHole}\quad \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESNEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

EARoll
$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau \rightsquigarrow \texttt{roll}^{\mu\pi.\tau'}(d\langle\tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}$$

EARollNotRec
$$\frac{\tau \nsim \mu(\!|\,|\!).(\!|\,|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Leftarrow \tau \rightsquigarrow \texttt{roll}^{\mu(\!|\,|\!).(\!|\,|\!)}\left((\!|d|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\right) : \mu(\!|\,|\!).(\!|\,|\!) \dashv \Delta, u :: \mu(\!|\,|\!).(\!|\,|\!)[\Gamma]}$$

EAInjHole
$$\frac{\Gamma \vdash E \Leftarrow (\!|\,|\!) \rightsquigarrow D : T \dashv \Delta \qquad \tau = +\{C(T)\}}{\Gamma \vdash \texttt{inj}_C(E) \Leftarrow (\!|\,|\!) \rightsquigarrow \texttt{inj}^\tau_C(D) : \tau \dashv \Delta}$$

EAInj
$$\frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash E \Leftarrow T_j \rightsquigarrow D : T'_j \dashv \Delta}{\Gamma \vdash \texttt{inj}_{C_j}(E) \Leftarrow \tau \rightsquigarrow \texttt{inj}^\tau_{C_j}\big(D\langle T'_j \Rightarrow T_j\rangle\big) : \tau \dashv \Delta}$$

EAInjUnexpectedBody
$$\frac{\begin{array}{c}\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \\ C_j \in \mathcal{C} \qquad T_j = \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!) \rightsquigarrow d : \tau_j \dashv \Delta \qquad \tau' = +\Big\{\{C_i(T_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\Big\}\end{array}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(d)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjExpectedBody
$$\frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad T_j = \tau_j \qquad \tau' = +\Big\{\{C_i(T_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\Big\}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(\varnothing)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjBadTag
$$\frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C \notin \mathcal{C} \qquad \Gamma \vdash E \Leftarrow (\!|\,|\!) \rightsquigarrow D : T \dashv \Delta \qquad \tau' = +\big\{\{C_i(T_i)\}_{C_i \in \mathcal{C}} \cup \{C(T)\}\big\}}{\Gamma \vdash (\!|\texttt{inj}_C(E)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_C(D)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EASubsume
$$\frac{e \neq (\!|\,|\!)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

EAEHole
$$\frac{}{\Gamma \vdash (\!|\,|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\,|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

EANEHole
$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$\boxed{\Gamma \vdash E \Leftarrow T_1 \rightsquigarrow D : T_2 \dashv \Delta}$ E analyzes against optional type T_1 and elaborates to D of consistent optional type T_2

EASome
$$\frac{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$$

EANone
$$\frac{}{\Gamma \vdash \varnothing \Leftarrow \varnothing \rightsquigarrow \varnothing : \varnothing \dashv \emptyset}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

TAVar
$$\frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

TALam
$$\frac{\tau \neq \llparenthesis \alpha \rrparenthesis \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}$$

TALamFree
$$\frac{\Delta; \Gamma, x : \llparenthesis\rrparenthesis \vdash d : \tau}{\Delta; \Gamma \vdash \lambda x{:}\llparenthesis \alpha \rrparenthesis.d : \llparenthesis\rrparenthesis \to \tau}$$

TAApp
$$\frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

TARoll
$$\frac{\emptyset \vdash \mu\pi.\tau \ \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}$$

TAUnroll
$$\frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

TAInj
$$\frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash D : T_j}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(D) : \tau}$$

TAEHole
$$\frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis\rrparenthesis^{u}_{\sigma} : \tau}$$

TANEHole
$$\frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^{u}_{\sigma} : \tau}$$

TAMHole
$$\frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^{u\blacktriangleright}_{\sigma} : \tau}$$

TACast
$$\frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

TAFailedCast
$$\frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle : \tau_2}$$

$\boxed{\Delta; \Gamma \vdash D : T}$ D is assigned optional type T

TASome
$$\frac{\Delta; \Gamma \vdash d : \tau}{\Delta; \Gamma \vdash d : \tau}$$

TANone
$$\frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$ τ is a ground type

GArr
$$\frac{}{\llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \text{ ground}}$$

GRec
$$\frac{}{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \text{ ground}}$$

GSum
$$\frac{\{T_i = \llparenthesis\rrparenthesis \vee T_i = \varnothing\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \text{ ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

MGArr
$$\frac{\tau_1 \to \tau_2 \neq \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}$$

MGRec
$$\frac{\tau \neq \llparenthesis\rrparenthesis}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

MGSum
$$\frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \qquad \{T_i = \tau_i \implies T_i' = \llparenthesis\rrparenthesis \wedge T_i = \varnothing \implies T_i' = \varnothing\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(T_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \text{ final}}$ d is final

$$\textsc{FBoxedVal}\ \frac{d \text{ boxedval}}{d \text{ final}} \qquad \textsc{FIndet}\ \frac{d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$ d is a value

$$\textsc{VLam}\ \frac{}{\lambda x{:}\tau.d \text{ val}} \qquad \textsc{VRoll}\ \frac{d \text{ val}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ val}} \qquad \textsc{VInjSome}\ \frac{d \text{ val}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ val}} \qquad \textsc{VInjNone}\ \frac{}{\texttt{inj}^{\tau}_{\mathbf{C}}(\varnothing) \text{ val}}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

$$\textsc{BVVal}\ \frac{d \text{ val}}{d \text{ boxedval}} \qquad \textsc{BVRoll}\ \frac{d \text{ boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}} \qquad \textsc{BVInj}\ \frac{d \text{ boxedval}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ boxedval}}$$

$$\textsc{BVArrCast}\ \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}$$

$$\textsc{BVRecCast}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}} \qquad \textsc{BVSumCast}\ \frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \qquad d \text{ boxedval}}{d\langle +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \Rightarrow +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \rangle \text{ boxedval}}$$

$$\textsc{BVHoleCast}\ \frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$ d is indeterminate

$$\textsc{IRoll}\ \frac{d \text{ indet}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ indet}} \qquad \textsc{IUnroll}\ \frac{d \text{ indet}}{\texttt{unroll}(d) \text{ indet}} \qquad \textsc{IInj}\ \frac{d \text{ indet}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ indet}} \qquad \textsc{IInjSome}\ \frac{d \text{ final}}{\texttt{inj}^{\tau}_{?^u}(d) \text{ indet}} \qquad \textsc{IInjNone}\ \frac{}{\texttt{inj}^{\tau}_{?^u}(\varnothing) \text{ indet}}$$

$$\textsc{ICastRec}\ \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}} \qquad \textsc{ICastSum}\ \frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \qquad d \text{ indet}}{d\langle +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \Rightarrow +\{C_i(T_i')\}_{C_i \in \mathcal{C}} \rangle \text{ indet}}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\textsc{ITApp}\quad \frac{[d_2\ \text{final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1} \qquad \textsc{ITUnroll}\quad \frac{[d\ \text{final}]}{\texttt{unroll}(\texttt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\textsc{ITAppCast}\quad \frac{[d_1\ \text{final}] \qquad [d_2\ \text{final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1\rangle))\langle \tau_2 \Rightarrow \tau_2'\rangle}$$

$$\textsc{ITUnrollCast}\quad \frac{[d\ \text{final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\texttt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \texttt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle} \qquad \textsc{ITCastId}\quad \frac{[d\ \text{final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastSucceed}\quad \frac{[d\ \text{final}] \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \Rightarrow \tau\rangle \longrightarrow d} \qquad \textsc{ITCastFail}\quad \frac{[d\ \text{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground}}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle}$$

$$\textsc{ITGround}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow \llparenthesis\rrparenthesis\rangle} \qquad \textsc{ITExpand}\quad \frac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau\rangle \longrightarrow d\langle \llparenthesis\rrparenthesis \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\mathsf{EvalCtx}\ \ \mathcal{E} \ ::= \ \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \texttt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \texttt{unroll}(\mathcal{E}) \mid \texttt{inj}^{\tau}_{C}(\mathcal{E}) \mid \llparenthesis\mathcal{E}\rrparenthesis^{u}_{\sigma} \mid \llparenthesis\mathcal{E}\rrparenthesis^{u\blacktriangleright}_{\sigma} \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau\rangle$$

$\boxed{d = \mathcal{E}\{d'\}}$ d is obtained by placing d' at the mark in $\mathcal{E}$

$$\textsc{FHOuter}\quad \frac{}{d = \circ\{d\}} \qquad \textsc{FHApp1}\quad \frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}} \qquad \textsc{FHApp2}\quad \frac{[d_1\ \text{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}} \qquad \textsc{FHRoll}\quad \frac{d = \mathcal{E}\{d'\}}{\texttt{roll}^{\mu\pi.\tau}(d) = \texttt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FHUnroll}\quad \frac{d = \mathcal{E}\{d'\}}{\texttt{unroll}(d) = \texttt{unroll}(\mathcal{E})\{d'\}} \qquad \textsc{FHInj}\quad \frac{d = \mathcal{E}\{d'\}}{\texttt{inj}^{\tau}_{C}(d) = \texttt{inj}^{\tau}_{C}(\mathcal{E})\{d'\}} \qquad \textsc{FNEHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{\llparenthesis d\rrparenthesis^{u}_{\sigma} = \llparenthesis\mathcal{E}\rrparenthesis^{u}_{\sigma}\{d'\}} \qquad \textsc{FMHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{\llparenthesis d\rrparenthesis^{u\blacktriangleright}_{\sigma} = \llparenthesis\mathcal{E}\rrparenthesis^{u\blacktriangleright}_{\sigma}\{d'\}}$$

$$\textsc{FHCastInside}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}} \qquad \textsc{FHFailedCast}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ d steps to d'

$$\textsc{Step}\quad \frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$