Hazel Phi: 9-type-aliases

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SYNTAX

Kind	κ	::=	$\texttt{Type} \mid \texttt{KHole} \mid \mathsf{S}_\kappa(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2$
User Types	$\hat{\tau}$	::=	$t \mid \texttt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid (\!|\,|\!)^{\mathtt{u}} \mid (\!|\hat{\tau}|\!)^{\mathtt{u}} \mid \lambda t{::}\texttt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2}$
Internal Types	τ	::=	$t \mid \texttt{bse} \mid \tau_1 \oplus \tau_2 \mid (\!|\,|\!)^{\mathtt{u}} \mid (\!|\tau|\!)^{\mathtt{u}} \mid (\!|t|\!)^{\mathtt{u}} \mid \lambda t{::}\kappa.\tau \mid \tau_1\ \tau_2$
Base Types	$\texttt{bse}$	::=	$\texttt{Int} \mid \texttt{Float} \mid \texttt{Bool}$
BinOp	$\oplus$	::=	$\times \mid + \mid \rightarrow$
Type Pattern			
User Expression			
Internal Expression			

DECLARATIVES

$\boxed{\Delta; \Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash \texttt{bse} ::> \mathsf{S}_{\texttt{Type}}(\texttt{bse})}\ (1) \qquad \frac{\Delta; \Phi_1, t{::}\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta; \Phi \vdash t ::> \mathsf{S}_\kappa(t)}\ (2) \qquad \frac{\Delta; \Phi \vdash \tau_1 {::} \texttt{Type} \qquad \Delta; \Phi \vdash \tau_2 {::} \texttt{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathsf{S}_{\texttt{Type}}(\tau_1 \oplus \tau_2)}\ (3)$$

$$\frac{\Delta_1, \mathtt{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash (\!|\,|\!)^{\mathtt{u}} ::> \mathsf{S}_\kappa((\!|\,|\!)^{\mathtt{u}})}\ (4) \qquad \frac{\Delta_1, \mathtt{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad \Delta; \Phi \vdash \tau {::} \kappa_1}{\Delta; \Phi \vdash (\!|\tau|\!)^{\mathtt{u}} ::> \mathsf{S}_\kappa((\!|\tau|\!)^{\mathtt{u}})}\ (5)$$

$$\frac{\Delta_1, \mathtt{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad t \notin \mathsf{dom}(\Phi)}{\Delta; \Phi \vdash (\!|t|\!)^{\mathtt{u}} ::> \mathsf{S}_\kappa((\!|t|\!)^{\mathtt{u}})}\ (6) \qquad \frac{\Delta; \Phi, t{::}\kappa_1 \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t{::}\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t{::}\kappa_1.\tau)}\ (7)$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \qquad \Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta; \Phi \vdash \tau_2 {::} \kappa_1}{\Delta; \Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ (8)$$

$\boxed{\Delta; \Phi \vdash \tau {::} \kappa}$ τ is well formed at kind κ

$$\frac{\Delta; \Phi \vdash \tau ::> \kappa}{\Delta; \Phi \vdash \tau {::} \kappa}\ (9) \qquad \frac{\Delta; \Phi \vdash \tau ::> \mathsf{S}_\kappa(\tau)}{\Delta; \Phi \vdash \tau {::} \kappa}\ (10) \qquad \frac{\Delta; \Phi \vdash \tau {::} \mathsf{S}_\kappa(\tau_1) \qquad \Delta; \Phi \vdash \tau_1 {::} \kappa}{\Delta; \Phi \vdash \tau {::} \kappa}\ (11)$$

$$\frac{\Delta; \Phi \vdash \tau ::> \kappa_1 \qquad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau {::} \kappa}\ (12)$$

$$\frac{\Delta; \Phi \vdash \tau {::} \kappa}{\Delta; \Phi \vdash \tau {::} \mathsf{S}_\kappa(\tau)}\ (13)$$

$$\boxed{\Delta; \Phi \vdash \kappa \stackrel{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2} \quad \kappa \text{ has matched } \Pi\text{-kind } \Pi_{t::\kappa_1}.\kappa_2$$

$$\frac{}{\Delta; \Phi \vdash \texttt{KHole} \stackrel{\blacktriangleright}{\Pi} \Pi_{t::\texttt{KHole}}.\texttt{KHole}} \ (14) \qquad \frac{\Delta; \Phi \vdash \kappa \equiv \mathsf{S}_{\texttt{KHole}}(\tau_1)}{\Delta; \Phi \vdash \kappa \stackrel{\blacktriangleright}{\Pi} \Pi_{t::\kappa}.\kappa} \ (15) \qquad \frac{\Delta; \Phi \vdash \kappa \equiv \Pi_{t::\kappa_1}.\kappa_2}{\Delta; \Phi \vdash \kappa \stackrel{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2} \ (16)$$

$$\boxed{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \quad \kappa_1 \text{ is equivalent to } \kappa_2$$

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \equiv \kappa} \ (17) \qquad \frac{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \ (18) \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta; \Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \ (19)$$

$$\frac{\Delta; \Phi \vdash \tau::\mathsf{S}_\kappa(\tau_1)}{\Delta; \Phi \vdash \mathsf{S}_{\mathsf{S}_\kappa(\tau_1)}(\tau) \equiv \mathsf{S}_\kappa(\tau_1)} \ (20) \qquad \frac{\Delta; \Phi \vdash \tau::\Pi_{t::\kappa_1}.\kappa_2}{\Delta; \Phi \vdash \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau) \equiv \Pi_{t::\kappa_1}.\mathsf{S}_{\kappa_2}(\tau\ t)} \ (21)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta; \Phi, t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4} \ (22) \qquad \frac{\Delta; \Phi \vdash \tau_1 \stackrel{\kappa}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathsf{S}_\kappa(\tau_1) \equiv \mathsf{S}_\kappa(\tau_2)} \ (23)$$

$$\boxed{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \quad \kappa_1 \text{ is a consistent subkind of } \kappa_2$$

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \texttt{KHole} \lesssim \kappa} \ (24) \qquad \frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \lesssim \texttt{KHole}} \ (25) \qquad \frac{\Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \ \mathsf{OK} \qquad \Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \lesssim \kappa} \ (26)$$

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK} \qquad \Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathsf{S}_{\texttt{KHole}}(\tau)} \ (27)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \ (28) \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta; \Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta; \Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \ (29)$$

$$\frac{\Delta; \Phi \vdash \mathsf{S}_\kappa(\tau) \ \mathsf{OK}}{\Delta; \Phi \vdash \mathsf{S}_\kappa(\tau) \lesssim \kappa} \ (30) \qquad \frac{\Delta; \Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta; \Phi, t::\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4} \ (31)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta; \Phi \vdash \tau_1 \stackrel{\kappa_1}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)} \ (32)$$

$\boxed{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa_1 \qquad \Delta; \Phi \vdash \kappa_1 \equiv S_\kappa(\tau_2)}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \quad (33)$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta; \Phi \vdash \tau_2 :: \Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta; \Phi, t::\kappa_1 \vdash \tau_1\ t \overset{\kappa_2}{\equiv} \tau_2\ t}{\Delta; \Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2} \quad (34) \qquad \frac{\Delta; \Phi \vdash \tau_1 \overset{S_\kappa(\tau)}{\equiv} \tau_2}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \quad (35)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_2 \overset{\kappa_1}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1\ \tau_2 \overset{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4} \quad (36)$$

$$\frac{\Delta; \Phi \vdash \tau::\kappa}{\Delta; \Phi \vdash \tau \overset{\kappa}{\equiv} \tau} \quad (37) \qquad \frac{\Delta; \Phi \vdash \tau_2 \overset{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \quad (38) \qquad \frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_3 \overset{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \quad (39)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\text{Type}}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_2 \overset{\text{Type}}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \overset{\text{Type}}{\equiv} \tau_3 \oplus \tau_4} \quad (40) \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau_1 \overset{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t::\kappa_2.\tau_2} \quad (41)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta; \Phi \vdash \kappa_1 \equiv \kappa}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \quad (42)$$

$\boxed{\Delta; \Phi \vdash \kappa\ \text{OK}}$ κ is well formed

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{Type OK}} \quad (43) \qquad \frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{KHole OK}} \quad (44) \qquad \frac{\Delta; \Phi \vdash \tau::\kappa}{\Delta; \Phi \vdash S_\kappa(\tau)\ \text{OK}} \quad (45)$$

$$\frac{\Delta; \Phi \vdash \kappa_1\ \text{OK} \qquad \Delta; \Phi, t::\kappa_1 \vdash \kappa_2\ \text{OK}}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \text{OK}} \quad (46)$$

$\boxed{\Delta; \Phi \vdash \text{OK}}$ Context is well formed

$$\frac{}{;\vdash \text{OK}} \quad (47) \qquad \frac{\Delta; \Phi \vdash \kappa\ \text{OK}}{\Delta; \Phi, t::\kappa \vdash \text{OK}} \quad (48) \qquad \frac{\Delta; \Phi \vdash \kappa\ \text{OK}}{\Delta, u::\kappa; \Phi \vdash \text{OK}} \quad (49)$$

Variables implicitly assumed to be fresh as necessary

Metatheory

Lemma 1 (Weakening). *If $\Delta; \Phi \vdash \tau::\kappa$, then $\Delta; \Phi, t::\kappa_1 \vdash \tau::\kappa$ when $\Delta; \Phi, t::\kappa_1 \vdash OK$*

Proof. By rule induction/length of proof.
 L1.   (9)

$\square$

Proof. By rule induction/length of proof.
 L2.   (9)

$\square$

Lemma 2 (OK-PK). *If $\Delta; \Phi \vdash \tau ::> \kappa$, then $\Delta; \Phi \vdash OK$ and $\Delta; \Phi \vdash \kappa\ OK$*

Lemma 3 (OK-WFaK)**.** *If* $\Delta; \Phi \vdash \tau::\kappa$, *then* $\Delta; \Phi \vdash$ *OK and* $\Delta; \Phi \vdash \kappa$ *OK*

Lemma 4 (OK-MatchPi)**.** *If* $\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2$, *then* $\Delta; \Phi \vdash$ *OK and* $\Delta; \Phi \vdash \kappa$ *OK and* $\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2$ *OK*

Lemma 5 (OK-KEquiv)**.** *If* $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, *then* $\Delta; \Phi \vdash$ *OK and* $\Delta; \Phi \vdash \kappa_1$ *OK and* $\Delta; \Phi \vdash \kappa_2$ *OK*

Lemma 6 (OK-CSK)**.** *If* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, *then* $\Delta; \Phi \vdash$ *OK and* $\Delta; \Phi \vdash \kappa_1$ *OK and* $\Delta; \Phi \vdash \kappa_2$ *OK*

Lemma 7 (OK-KWF)**.** *If* $\Delta; \Phi \vdash \kappa$ *OK, then* $\Delta; \Phi \vdash$ *OK*

Lemma 8 (OK-Substitution)**.**
If $\Delta; \Phi \vdash \tau_L::\kappa_{L1}$ *and* $\Delta; \Phi, t_L::\kappa_{L1} \vdash \kappa_{L2}$ *OK, then* $\Delta; \Phi \vdash$ *OK and* $\Delta; \Phi \vdash [\tau_L/t_L]\kappa_{L2}$ *OK*
(induction on $\Delta; \Phi, t_L::\kappa_{L1} \vdash \kappa_{L2}$ *OK)*

Lemma 9 (K-Substitution)**.**
If $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$ *and* $\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau_{L2}::\kappa_{L2}$, *then* $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2}::[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau_{L2}::\kappa_{L2}$*)*

Proof. By simultaneous rule induction/length of proof.
The interesting cases per lemma:

OK-PK.	(1)	$\Delta; \Phi \vdash \text{bse}::S_{\text{Type}}(\text{bse})$	by (9)
		$\Delta; \Phi \vdash \text{bse}::\text{Type}$	by (10)
	*	$\Delta; \Phi \vdash S_{\text{Type}}(\text{bse})$ OK	by (43)
	*	$\Delta; \Phi \vdash$ OK	by premiss
	(8)		bad
OK-WFaK.	(12)	$\Delta; \Phi \vdash \tau_2::\kappa$	by (10)
	*	$\Delta; \Phi \vdash S_\kappa(\tau_2)$ OK	by (43)
OK-KEquiv.	(22)	$\Delta; \Phi \vdash \tau\ t ::> \kappa$	
OK-Substitution.	(41)	$\Delta; \Phi, t_L::\kappa_{L1} \vdash$ OK	premiss (41)
		$\Delta; \Phi \vdash \kappa_{L1}$ OK	by subderivation premiss (46)
	*	$\Delta; \Phi \vdash$ OK	by OK-KWF
	*	$\Delta; \Phi \vdash [\tau_L/t_L]\text{Type}$ OK	by (41) and degenerate subst
	(43)	$\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau::\kappa$	premiss (43)
		$\Delta; \Phi, t_L::\kappa_{L1} \vdash$ OK	by OK-WFaK
		$\Delta; \Phi \vdash \kappa_{L1}$ OK	by subderivation premiss (46)
	*	$\Delta; \Phi \vdash$ OK	by OK-KWF
		$\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau::[\tau_{L1}/t_L]\kappa$	by K-Substitution on premiss
	*	$\Delta; \Phi \vdash [\tau_L/t_L]S_\kappa(\tau)$ OK	by (43)

$\square$

Lemma 10 (PK-Unicity)**.** *If* $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$ *and* $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$ *then* κ_{L1} *is* κ_{L2}

Lemma 11. *If* $\Delta; \Phi \vdash \tau ::> \kappa_1$ *and* $\Delta; \Phi \vdash \tau::\kappa_2$, *then* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$