Hazel Phi: 11-type-constructors

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SYNTAX

Kind	κ	$::=$	$\mathtt{Type} \mid \mathtt{KHole} \mid \mathtt{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2$
User Types	$\hat{\tau}$	$::=$	$t \mid \mathtt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid (\!\mid\,\mid\!)^{\mathtt{u}} \mid (\!\mid\hat{\tau}\mid\!)^{\mathtt{u}} \mid \lambda t{::}\mathtt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2}$
Internal Types	τ	$::=$	$t \mid \mathtt{bse} \mid \tau_1 \oplus \tau_2 \mid (\!\mid\,\mid\!)^{\mathtt{u}} \mid (\!\mid\tau\mid\!)^{\mathtt{u}} \mid (\!\mid t \mid\!)^{\mathtt{u}} \mid \lambda t{::}\kappa.\tau \mid \tau_1\ \tau_2$
Base Types	$\mathtt{bse}$	$::=$	$\mathtt{Int} \mid \mathtt{Float} \mid \mathtt{Bool}$
BinOp	$\oplus$	$::=$	$\times \mid + \mid \rightarrow$
Type Pattern			
User Expression			
Internal Expression			

DECLARATIVES

$\boxed{\Delta; \Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash \mathtt{bse} ::> \mathtt{S}_{\mathtt{Type}}(\mathtt{bse})}\ (1) \qquad \frac{\Delta; \Phi_1, t{::}\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta; \Phi \vdash t ::> \mathtt{S}_{\kappa}(t)}\ (2) \qquad \frac{\Delta; \Phi \vdash \tau_1{::}\mathtt{Type} \qquad \Delta; \Phi \vdash \tau_2{::}\mathtt{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathtt{S}_{\mathtt{Type}}(\tau_1 \oplus \tau_2)}\ (3)$$

$$\frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash (\!\mid\,\mid\!)^{\mathtt{u}} ::> \mathtt{S}_{\kappa}((\!\mid\,\mid\!)^{\mathtt{u}})}\ (4) \qquad \frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad \Delta; \Phi \vdash \tau{::}\kappa_1}{\Delta; \Phi \vdash (\!\mid\tau\mid\!)^{\mathtt{u}} ::> \mathtt{S}_{\kappa}((\!\mid\tau\mid\!)^{\mathtt{u}})}\ (5)$$

$$\frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad t \notin \mathsf{dom}(\Phi)}{\Delta; \Phi \vdash (\!\mid t\mid\!)^{\mathtt{u}} ::> \mathtt{S}_{\kappa}((\!\mid t\mid\!)^{\mathtt{u}})}\ (6) \qquad \frac{\Delta; \Phi, t{::}\kappa_1 \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t{::}\kappa_1.\tau ::> \mathtt{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t{::}\kappa_1.\tau)}\ (7)$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \qquad \Delta; \Phi \vdash \kappa \blacktriangleright_{\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta; \Phi \vdash \tau_2{::}\kappa_1}{\Delta; \Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ (8)$$

$\boxed{\Delta; \Phi \vdash \tau{::}\kappa}$ τ is well formed at kind κ

$$\frac{\Delta; \Phi \vdash \tau ::> \kappa}{\Delta; \Phi \vdash \tau{::}\kappa}\ (9) \qquad \frac{\Delta; \Phi \vdash \tau ::> \mathtt{S}_{\kappa}(\tau)}{\Delta; \Phi \vdash \tau{::}\kappa}\ (10)$$

$$\frac{\Delta; \Phi \vdash \tau ::> \kappa_1 \qquad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau{::}\kappa}\ (11)$$

$$\frac{\Delta; \Phi \vdash \tau{::}\kappa}{\Delta; \Phi \vdash \tau{::}\mathtt{S}_{\kappa}(\tau)}\ (12) \qquad \frac{\Delta; \Phi \vdash \tau{::}\Pi_{t::\kappa_3}.\kappa_4 \qquad \Delta; \Phi \vdash \Pi_{t::\kappa_3}.\kappa_4 \lesssim \Pi_{t::\kappa_1}.\kappa_2}{\Delta; \Phi \vdash \tau{::}\Pi_{t::\kappa_1}.\kappa_2}\ (13)$$

$$\frac{\Delta; \Phi \vdash \tau{::}\mathtt{S}_{\kappa}(\tau_1) \qquad \Delta; \Phi \vdash \tau_1{::}\kappa}{\Delta; \Phi \vdash \tau{::}\kappa}\ (14)$$

$\boxed{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{}{\Delta;\Phi \vdash \texttt{KHole} \overset{\blacktriangleright}{\Pi} \Pi_{t::\texttt{KHole}}.\texttt{KHole}}\ (15) \qquad \frac{\Delta;\Phi \vdash \kappa \equiv \mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\mathsf{S}_{\texttt{KHole}}(\tau)}.\mathsf{S}_{\texttt{KHole}}(\tau\ t)}\ (16) \qquad \frac{\Delta;\Phi \vdash \kappa \equiv \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}\ (17)$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \equiv \kappa}\ (18) \qquad \frac{\Delta;\Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ (19) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ (20)$$

$$\frac{\Delta;\Phi \vdash \tau{::}\mathsf{S}_\kappa(\tau_1)}{\Delta;\Phi \vdash \mathsf{S}_{\mathsf{S}_\kappa(\tau_1)}(\tau) \equiv \mathsf{S}_\kappa(\tau_1)}\ (21) \qquad \frac{\Delta;\Phi \vdash \tau{::}\Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau) \equiv \Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau\ t_1)}\ (22)$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\Phi, t{::}\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4}\ (23) \qquad \frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathsf{S}_\kappa(\tau_1) \equiv \mathsf{S}_\kappa(\tau_2)}\ (24)$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \texttt{KHole} \lesssim \kappa}\ (25) \qquad \frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \texttt{KHole}}\ (26) \qquad \frac{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau)\ \mathsf{OK} \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \lesssim \kappa}\ (27)$$

$$\frac{\Delta;\Phi \vdash \kappa\ \mathsf{OK} \qquad \Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \mathsf{S}_{\texttt{KHole}}(\tau)}\ (28)$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ (29) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta;\Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ (30)$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_\kappa(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_\kappa(\tau) \lesssim \kappa}\ (31) \qquad \frac{\Delta;\Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta;\Phi, t{::}\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4}\ (32)$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)}\ (33)$$

$\boxed{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa_1 \qquad \Delta; \Phi \vdash \kappa_1 \equiv \mathtt{S}_\kappa(\tau_2)}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \text{ (34)}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta; \Phi \vdash \tau_2 :: \Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta; \Phi, t::\kappa_1 \vdash \tau_1\ t \overset{\kappa_2}{\equiv} \tau_2\ t}{\Delta; \Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2} \text{ (35)}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_2 \overset{\kappa_1}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1\ \tau_2 \overset{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4} \text{ (36)} \qquad \frac{\Delta; \Phi \vdash \tau_1 \overset{\mathtt{S}_\kappa(\tau)}{\equiv} \tau_2}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \text{ (37)}$$

$$\frac{\Delta; \Phi \vdash \tau::\kappa}{\Delta; \Phi \vdash \tau \overset{\kappa}{\equiv} \tau} \text{ (38)} \qquad \frac{\Delta; \Phi \vdash \tau_2 \overset{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \text{ (39)} \qquad \frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_3 \overset{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \text{ (40)}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\mathtt{Type}}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_2 \overset{\mathtt{Type}}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \overset{\mathtt{Type}}{\equiv} \tau_3 \oplus \tau_4} \text{ (41)} \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau_1 \overset{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t::\kappa_2.\tau_2} \text{ (42)}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta; \Phi \vdash \kappa_1 \equiv \kappa}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2} \text{ (43)}$$

$\boxed{\Delta; \Phi \vdash \kappa \text{ OK}}$ κ is well formed

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathtt{Type} \text{ OK}} \text{ (44)} \qquad \frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathtt{KHole} \text{ OK}} \text{ (45)} \qquad \frac{\Delta; \Phi \vdash \tau::\kappa}{\Delta; \Phi \vdash \mathtt{S}_\kappa(\tau) \text{ OK}} \text{ (46)}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \text{ OK} \qquad \Delta; \Phi, t::\kappa_1 \vdash \kappa_2 \text{ OK}}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \text{ OK}} \text{ (47)}$$

$\boxed{\Delta; \Phi \vdash \text{OK}}$ Context is well formed

$$\frac{}{; \vdash \text{OK}} \text{ (48)} \qquad \frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi, t::\kappa \vdash \text{OK}} \text{ (49)} \qquad \frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta, \mathrm{u}::\kappa; \Phi \vdash \text{OK}} \text{ (50)}$$

Variables implicitly assumed to be fresh as necessary

Metatheory

Lemma 1 (Weakening)**.** *If* $\Delta; \Phi \vdash \tau::\kappa$*, then* $\Delta; \Phi, t::\kappa_1 \vdash \tau::\kappa$ *when* $\Delta; \Phi, t::\kappa_1 \vdash OK$

Proof. By rule induction/length of proof.
L1. (9)

□

Proof. By rule induction/length of proof.
L2. (9)

□

Lemma 2 (OK-PK)**.** *If* $\Delta;\Phi \vdash \tau ::> \kappa$*, then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$

Lemma 3 (OK-WFaK)**.** *If* $\Delta;\Phi \vdash \tau{::}\kappa$*, then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$

Lemma 4 (OK-MatchPi)**.** *If* $\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2$*, then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$ *and* $\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathit{OK}$

Lemma 5 (OK-KEquiv)**.** *If* $\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2$*, then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_2\ \mathit{OK}$

Lemma 6 (OK-CSK)**.** *If* $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$*, then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta;\Phi \vdash \kappa_2\ \mathit{OK}$

Lemma 7 (OK-TEquivAK)**.** *If* $\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2$*, then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash \tau_1{::}\kappa$ *and* $\Delta;\Phi \vdash \tau_2{::}\kappa$ *and* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$

Lemma 8 (OK-KWF)**.** *If* $\Delta;\Phi \vdash \kappa\ \mathit{OK}$*, then* $\Delta;\Phi \vdash \mathit{OK}$

Lemma 9 (OK-Substitution)**.**
If $\Delta;\Phi \vdash \tau_L{::}\kappa_{L1}$ *and* $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \kappa_{L2}\ \mathit{OK}$*, then* $\Delta;\Phi \vdash \mathit{OK}$ *and* $\Delta;\Phi \vdash [\tau_L/t_L]\kappa_{L2}\ \mathit{OK}$
(induction on $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \kappa_{L2}\ \mathit{OK}$*)*

Lemma 10 (K-Substitution)**.**
If $\Delta;\Phi \vdash \tau_{L1}{::}\kappa_{L1}$ *and* $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \tau_{L2}{::}\kappa_{L2}$*, then* $\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau_{L2}{::}[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \tau_{L2}{::}\kappa_{L2}$*)*

Proof. By simultaneous rule induction/length of proof.
The interesting cases per lemma:

OK-PK.	(1)	$\Delta;\Phi \vdash \mathtt{bse}{::}\mathtt{S}_{\mathtt{Type}}(\mathtt{bse})$	by (9)
		$\Delta;\Phi \vdash \mathtt{bse}{::}\mathtt{Type}$	by (10)
	*	$\Delta;\Phi \vdash \mathtt{S}_{\mathtt{Type}}(\mathtt{bse})\ \mathsf{OK}$	by (43)
	*	$\Delta;\Phi \vdash \mathsf{OK}$	by premiss
	(8)		bad
OK-WFaK.	(12)	$\Delta;\Phi \vdash \tau_2{::}\kappa$	by (10)
	*	$\Delta;\Phi \vdash \mathtt{S}_\kappa(\tau_2)\ \mathsf{OK}$	by (43)
OK-KEquiv.	(22)	$\Delta;\Phi \vdash \tau\ t ::> \kappa$	
OK-Substitution.	(41)	$\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \mathsf{OK}$	premiss (41)
		$\Delta;\Phi \vdash \kappa_{L1}\ \mathsf{OK}$	by subderivation premiss (46)
	*	$\Delta;\Phi \vdash \mathsf{OK}$	by OK-KWF
	*	$\Delta;\Phi \vdash [\tau_L/t_L]\mathtt{Type}\ \mathsf{OK}$	by (41) and degenerate subst
	(43)	$\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \tau{::}\kappa$	premiss (43)
		$\Delta;\Phi, t_L{::}\kappa_{L1} \vdash \mathsf{OK}$	by OK-WFaK
		$\Delta;\Phi \vdash \kappa_{L1}\ \mathsf{OK}$	by subderivation premiss (46)
	*	$\Delta;\Phi \vdash \mathsf{OK}$	by OK-KWF
		$\Delta;\Phi \vdash [\tau_{L1}/t_L]\tau{::}[\tau_{L1}/t_L]\kappa$	by K-Substitution on premiss
	*	$\Delta;\Phi \vdash [\tau_L/t_L]\mathtt{S}_\kappa(\tau)\ \mathsf{OK}$	by (43)

$\square$

Lemma 11 (PK-Unicity)**.** *If* $\Delta;\Phi \vdash \tau_L ::> \kappa_{L1}$ *and* $\Delta;\Phi \vdash \tau_L ::> \kappa_{L2}$ *then* κ_{L1} *is* κ_{L2}

Lemma 12. *If* $\Delta;\Phi \vdash \tau ::> \kappa_1$ *and* $\Delta;\Phi \vdash \tau{::}\kappa_2$*, then* $\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2$