# Algebraic Data Types for Hazel

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## 1 Syntax

$$
\begin{array}{llcl}
\textsf{HTyp} & \tau & ::= & \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \mid (\!|\,|\!) \mid (\!|\alpha|\!)^u \\
\textsf{HTypPat} & \pi & ::= & \alpha \mid (\!|\,|\!) \\
\textsf{HExp} & e & ::= & x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \texttt{inj}_C(E) \mid \texttt{roll}(e) \mid \texttt{unroll}(e) \\
 & & & \mid (\!|\,|\!)^u \mid (\!|e|\!)^u \mid (\!|e|\!)^{u\blacktriangleright} \\
\textsf{HTag} & C & ::= & \mathbf{C} \mid ?^u \\
\textsf{HTagTyp} & T & ::= & \tau \mid \varnothing \\
\textsf{HTagArg} & E & ::= & e \mid \varnothing \\
\textsf{IHExp} & d & ::= & x \mid \lambda x{:}\tau.d \mid d(d) \mid \texttt{inj}^{\tau}_C(D) \mid \texttt{roll}^{\mu\alpha.\tau}(d) \mid \texttt{unroll}(d) \\
 & & & \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow (\!|\,|\!) \not\Rightarrow \tau \rangle \mid (\!|\,|\!)^u_\sigma \mid (\!|d|\!)^u_\sigma \mid (\!|d|\!)^{u\blacktriangleright}_\sigma \\
\textsf{IHTagArg} & D & ::= & d \mid \varnothing
\end{array}
$$

### 1.1 Context Extension

We write $\Gamma, X : T$ to denote the extension of typing context $\Gamma$ with optional variable $X$ of optional type $T$.

$$
\Gamma, X : T = \begin{cases} \Gamma, x : \tau & X = x \wedge T = \tau \\ \Gamma, x : (\!|\,|\!) & X = x \wedge T = \varnothing \\ \Gamma & X = \varnothing \end{cases}
$$

We write $\Theta, \pi$ to denote the extension of type variable context $\Theta$ with optional type variable name $\pi$.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = (\!|\,|\!) \end{cases}
$$

## 2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ $\tau''$ is obtained by substituting $\tau$ for $\pi$ in $\tau'$

$$
\begin{array}{lcll}
[\tau/(\!|\,|\!)]\tau' & = & \tau' & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu(\!|\,|\!).\tau_2 & = & \mu(\!|\,|\!).[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]{+}\{C_i(T_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]T_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha](\!|\,|\!) & = & (\!|\,|\!) & \\
[\alpha'/\alpha](\!|\alpha|\!)^u & = & (\!|\alpha'|\!)^u & \\
[\alpha'/\alpha](\!|\alpha_1|\!)^u & = & (\!|\alpha_1|\!)^u & \text{when } \alpha \neq \alpha_1
\end{array}
$$

$\boxed{[\tau/\pi]T = \tau'}$ $\tau'$ is obtained by substituting $\tau$ for $\pi$ in $T$

$$[\tau/\pi]T = \begin{cases} [\tau/\pi]\tau' & \text{when } T = \tau' \\ \varnothing & \text{when } T = \varnothing \end{cases}$$

$\boxed{\Theta \vdash \tau\ \mathsf{valid}}$ $\tau$ is a valid type

$$\frac{\Theta \vdash \tau_1\ \mathsf{valid} \qquad \Theta \vdash \tau_2\ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2\ \mathsf{valid}}\ \textsc{TVArr} \qquad \frac{\alpha \in \Theta}{\Theta \vdash \alpha\ \mathsf{valid}}\ \textsc{TVVar} \qquad \frac{\Theta, \pi \vdash \tau\ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau\ \mathsf{valid}}\ \textsc{TVRec}$$

$$\frac{\{\Theta \vdash T_i\ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(T_i)\}_{C_i \in \mathcal{C}}\ \mathsf{valid}}\ \textsc{TVSum} \qquad \frac{}{\Theta \vdash (\!|\!|\!)\ \mathsf{valid}}\ \textsc{TVEHole}$$

$$\frac{\alpha \notin \Theta}{\Theta \vdash (\!|\alpha|\!)^u\ \mathsf{valid}}\ \textsc{TVNEHole}$$

$\boxed{\Theta \vdash T\ \mathsf{valid}}$ $T$ is a valid optional type

$$\frac{T = \tau \qquad \Theta \vdash \tau\ \mathsf{valid}}{\Theta \vdash T\ \mathsf{valid}}\ \textsc{TVSome} \qquad \frac{}{\Theta \vdash \varnothing\ \mathsf{valid}}\ \textsc{TVNone}$$

$\boxed{\tau \sim \tau'}$ $\tau$ and $\tau'$ are consistent

$$\frac{}{(\!|\!|\!) \sim \tau}\ \textsc{TCHole1} \qquad \frac{}{\tau \sim (\!|\!|\!)}\ \textsc{TCHole2} \qquad \frac{}{\tau \sim \tau}\ \textsc{TCRefl} \qquad \frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}\ \textsc{TCArr}$$

$$\frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'}\ \textsc{TCRec} \qquad \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu(\!|\!|\!).\tau \sim \mu\alpha.\tau'}\ \textsc{TCRecHole1}$$

$$\frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu(\!|\!|\!).\tau'}\ \textsc{TCRecHole2} \qquad \frac{\{T_i \sim T_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(T_i')\}_{C_i \in \mathcal{C}}}\ \textsc{TCSum}$$

$\boxed{T \sim T'}$ $T$ and $T'$ are consistent

$$\frac{\tau \sim \tau'}{\tau \sim \tau'}\ \textsc{TCSome} \qquad \frac{}{\varnothing \sim \varnothing}\ \textsc{TCNone}$$

## 2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

**Theorem 1** (Synthetic Type Validity). *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

**Theorem 2** (Consistency Preserves Validity). *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ $\tau$ has matched arrow type $\tau_1 \to \tau_2$

$$\frac{}{(\!|\!|\!) \blacktriangleright_{\to} (\!|\!|\!) \to (\!|\!|\!)}\ \textsc{MAHole} \qquad \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}\ \textsc{MAArr}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ $\tau$ has matched recursive type $\mu\pi.\tau'$

$$\textsc{MRRec}\ \frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau} \qquad \textsc{MRHole}\ \frac{}{(\!|\,|\!) \blacktriangleright_{\mu} \mu(\!|\,|\!).(\!|\,|\!)}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ $e$ synthesizes type $\tau$

$$\textsc{SVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \textsc{SVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\,|\!)} \qquad \textsc{SLam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}$$

$$\textsc{SApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \qquad \textsc{SAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim (\!|\,|\!) \to (\!|\,|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|e_1(e_2)|\!)^u \Rightarrow (\!|\,|\!)}$$

$$\textsc{SAsc}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \qquad \textsc{SRollError}\ \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Rightarrow \mu(\!|\,|\!).(\!|\,|\!)} \qquad \textsc{SUnroll}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$$

$$\textsc{SUnrollNotRec}\ \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu(\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash \mathtt{unroll}\big((\!|e|\!)^{u\blacktriangleright}\big) \Rightarrow (\!|\,|\!)} \qquad \textsc{SInjError}\ \frac{\Gamma \vdash E\ \mathsf{valid}}{\Gamma \vdash (\!|\mathtt{inj}_C(E)|\!)^u \Rightarrow (\!|\,|\!)} \qquad \textsc{SEHole}\ \frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow (\!|\,|\!)} \qquad \textsc{SNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\,|\!)} \qquad \textsc{SMHole}\ \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^{u\blacktriangleright} \Rightarrow (\!|\,|\!)}$$

$\boxed{\Gamma \vdash E\ \mathsf{valid}}$ $E$ is a valid optional expression

$$\textsc{EVSome}\ \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash e\ \mathsf{valid}} \qquad \textsc{EVNone}\ \frac{}{\Gamma \vdash \varnothing\ \mathsf{valid}}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ $e$ analyzes against type $\tau$

$$\textsc{ARoll}\ \frac{\tau \blacktriangleright_{\mu} \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau}$$

$$\textsc{ARollNotRec}\ \frac{\tau \nsim \mu(\!|\,|\!).(\!|\,|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash \texttt{roll}\big((\!|e|\!)^{u\blacktriangleright}\big) \Leftarrow \tau}$$

$$\textsc{AInjHole}\ \frac{\Gamma \vdash E\ \mathsf{valid}}{\Gamma \vdash \texttt{inj}_{C}(E) \Leftarrow (\!|\,|\!)}$$

$$\textsc{AInj}\ \frac{C_j \in \mathcal{C} \qquad \Gamma \vdash E \Leftarrow T_j}{\Gamma \vdash \texttt{inj}_{C_j}(E) \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjUnexpectedBody}\ \frac{C_j \in \mathcal{C} \qquad T_j = \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^{u} \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjExpectedBody}\ \frac{C_j \in \mathcal{C} \qquad T_j = \tau}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^{u} \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjBadTag}\ \frac{C \notin \mathcal{C} \qquad \Gamma \vdash E\ \mathsf{valid}}{\Gamma \vdash (\!|\texttt{inj}_{C}(E)|\!)^{u} \Leftarrow +\{C_i(T_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{ASubsume}\ \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

$\boxed{\Gamma \vdash E \Leftarrow T}$ $E$ analyzes against optional type $T$

$$\textsc{ASome}\ \frac{\Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e \Leftarrow \tau}$$

$$\textsc{ANone}\ \frac{}{\Gamma \vdash \varnothing \Leftarrow \varnothing}$$

## 2.2 Typed Elaboration

**Theorem 3** (Synthetic Typed Elaboration Validity). *If* $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ $e$ synthesizes type $\tau$ and elaborates to $d$

$$\textsc{ESVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

$$\textsc{ESVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis x \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESLam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESApp}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle \tau_1' \Rightarrow \tau_2 \to \tau\rangle)(d_2\langle \tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\textsc{ESAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash \llparenthesis e_1 \rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d_1 \rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle \tau_2' \Rightarrow \llparenthesis\rrparenthesis\rangle) \dashv \Delta_1 \cup \Delta_2, u :: \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESAsc}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle \tau' \Rightarrow \tau\rangle \dashv \Delta}$$

$$\textsc{ESRollError}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash \llparenthesis \mathtt{roll}(e) \rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis \mathtt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESUnroll}\ \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \mathtt{unroll}(d\langle \tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}$$

$$\textsc{ESUnrollNotRec}\ \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}\left(\llparenthesis e \rrparenthesis^{u\blacktriangleright}\right) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \mathtt{unroll}\left(\llparenthesis d \rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\right) \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESInjError}\ \frac{\Gamma \vdash E \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow D : T \dashv \Delta \qquad T = \mathit{frob}(E)}{\Gamma \vdash \llparenthesis \mathtt{inj}_C(E) \rrparenthesis^u \Rightarrow +\{C(T)\} \rightsquigarrow \llparenthesis \mathtt{inj}_C^{+\{C(T)\}}(D\langle T \Rightarrow \llparenthesis\rrparenthesis\rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: +\{C(T)\}[\Gamma]}$$

$$\textsc{ESEHole}\ \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESNEHole}\ \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$ $e$ analyzes against type $\tau_1$ and elaborates to $d$ of consistent type $\tau_2$

EARoll
$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau \rightsquigarrow \texttt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}$$

EARollNotRec
$$\frac{\tau \nsim \mu (\!|\!).(\!|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\!) \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Leftarrow \tau \rightsquigarrow \texttt{roll}^{\mu(\!|\!).(\!|\!)}\left((\!|d|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\right) : \mu(\!|\!).(\!|\!) \dashv \Delta, u :: \mu(\!|\!).(\!|\!)[\Gamma]}$$

EAInjHole
$$\frac{\Gamma \vdash E \Leftarrow (\!|\!) \rightsquigarrow D : T \dashv \Delta \qquad \tau = +\{C(T)\}}{\Gamma \vdash \texttt{inj}_C(E) \Leftarrow (\!|\!) \rightsquigarrow \texttt{inj}^\tau_C(D) : \tau \dashv \Delta}$$

EAInj
$$\frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash E \Leftarrow T_j \rightsquigarrow D : T'_j \dashv \Delta}{\Gamma \vdash \texttt{inj}_{C_j}(E) \Leftarrow \tau \rightsquigarrow \texttt{inj}^\tau_{C_j}\left(D\langle T'_j \Rightarrow T_j\rangle\right) : \tau \dashv \Delta}$$

EAInjUnexpectedBody
$$\frac{\begin{array}{c}\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \\ C_j \in \mathcal{C} \qquad T_j = \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\!) \rightsquigarrow d : \tau_j \dashv \Delta \qquad \tau' = +\left\{\{C_i(T_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\right\}\end{array}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(e)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(d)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjExpectedBody
$$\frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad T_j = \tau_j \qquad \tau' = +\left\{\{C_i(T_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\right\}}{\Gamma \vdash (\!|\texttt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_{C_j}(\varnothing)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EAInjBadTag
$$\frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C \notin \mathcal{C} \qquad \Gamma \vdash E \Leftarrow (\!|\!) \rightsquigarrow D : T \dashv \Delta \qquad \tau' = +\left\{\{C_i(T_i)\}_{C_i \in \mathcal{C}} \cup \{C(T)\}\right\}}{\Gamma \vdash (\!|\texttt{inj}_C(E)|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\texttt{inj}^{\tau'}_C(D)|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

EASubsume
$$\frac{e \neq (\!|\!)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

EAEHole
$$\frac{}{\Gamma \vdash (\!|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

EANEHole
$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

EAMHole
$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^{u\blacktriangleright} \Leftarrow \tau \rightsquigarrow (\!|d|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$\boxed{\Gamma \vdash E \Leftarrow T_1 \rightsquigarrow D : T_2 \dashv \Delta}$ $E$ analyzes against optional type $T_1$ and elaborates to $D$ of consistent optional type $T_2$

EASome
$$\frac{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$$

EANone
$$\frac{}{\Gamma \vdash \varnothing \Leftarrow \varnothing \rightsquigarrow \varnothing : \varnothing \dashv \emptyset}$$

## 2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ $d$ is assigned type $\tau$

$$\textsc{TAVar}\ \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau} \qquad \textsc{TALam}\ \frac{\Delta; \Gamma, x : \tau_1 \vdash d : \tau_2}{\Delta; \Gamma \vdash \lambda x{:}\tau_1.d : \tau_1 \to \tau_2} \qquad \textsc{TAApp}\ \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

$$\textsc{TARoll}\ \frac{\emptyset \vdash \mu\pi.\tau \ \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau} \qquad \textsc{TAUnroll}\ \frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

$$\textsc{TAInj}\ \frac{\tau = +\{C_i(T_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash D : T_j}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(D) : \tau} \qquad \textsc{TAEHole}\ \frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|\!)^u_\sigma : \tau}$$

$$\textsc{TANEHole}\ \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^u_\sigma : \tau} \qquad \textsc{TAMHole}\ \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|d|\!)^{u\blacktriangleright}_\sigma : \tau}$$

$$\textsc{TACast}\ \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2} \qquad \textsc{TAFailedCast}\ \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow (\!|\!) \nRightarrow \tau_2 \rangle : \tau_2}$$

$\boxed{\Delta; \Gamma \vdash D : T}$ $D$ is assigned optional type $T$

$$\textsc{TASome}\ \frac{\Delta; \Gamma \vdash d : \tau}{\Delta; \Gamma \vdash d : \tau} \qquad \textsc{TANone}\ \frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

# 3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$ $\tau$ is a ground type

$$\textsc{GArr}\ \frac{}{(\!|\!) \to (\!|\!) \text{ ground}} \qquad \textsc{GRec}\ \frac{}{\mu(\!|\!).(\!|\!) \text{ ground}} \qquad \textsc{GSum}\ \frac{\{T_i = (\!|\!) \lor T_i = \varnothing\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \text{ ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ $\tau$ has matched ground type $\tau'$

$$\textsc{MGArr}\ \frac{\tau_1 \to \tau_2 \neq (\!|\!) \to (\!|\!)}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} (\!|\!) \to (\!|\!)} \qquad \textsc{MGRec}\ \frac{\tau \neq (\!|\!)}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu(\!|\!).(\!|\!)}$$

$$\textsc{MGSum}\ \frac{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(T'_i)\}_{C_i \in \mathcal{C}} \qquad \{T_i = \tau_i \implies T'_i = (\!|\!) \land T_i = \varnothing \implies T'_i = \varnothing\}_{C_i \in \mathcal{C}}}{+\{C_i(T_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(T'_i)\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \text{ final}}$ $d$ is final

$$\frac{d \text{ boxedval}}{d \text{ final}}\ \textsc{FBoxedVal} \qquad \frac{d \text{ indet}}{d \text{ final}}\ \textsc{FIndet}$$

$\boxed{d \text{ val}}$ $d$ is a value

$$\frac{}{\lambda x{:}\tau.d \text{ val}}\ \textsc{VLam} \qquad \frac{d \text{ val}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ val}}\ \textsc{VRoll} \qquad \frac{d \text{ val}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ val}}\ \textsc{VInjSome} \qquad \frac{}{\texttt{inj}^{\tau}_{\mathbf{C}}(\varnothing) \text{ val}}\ \textsc{VInjNone}$$

$\boxed{d \text{ boxedval}}$ $d$ is a boxed value

$$\frac{d \text{ val}}{d \text{ boxedval}}\ \textsc{BVVal} \qquad \frac{d \text{ boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}}\ \textsc{BVRoll} \qquad \frac{d \text{ boxedval}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ boxedval}}\ \textsc{BVInj} \qquad \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4\rangle \text{ boxedval}}\ \textsc{BVArrCast}$$

$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle \text{ boxedval}}\ \textsc{BVRecCast} \qquad \frac{+\{C_i(T_i)\}_{C_i\in\mathcal{C}} \neq +\{C_i(T_i')\}_{C_i\in\mathcal{C}} \qquad d \text{ boxedval}}{d\langle +\{C_i(T_i)\}_{C_i\in\mathcal{C}} \Rightarrow +\{C_i(T_i')\}_{C_i\in\mathcal{C}}\rangle \text{ boxedval}}\ \textsc{BVSumCast}$$

$$\frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\!)\rangle \text{ boxedval}}\ \textsc{BVHoleCast}$$

$\boxed{d \text{ indet}}$ $d$ is indeterminate

$$\frac{d \text{ indet}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ indet}}\ \textsc{IRoll} \qquad \frac{d \text{ indet}}{\texttt{unroll}(d) \text{ indet}}\ \textsc{IUnroll} \qquad \frac{d \text{ indet}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ indet}}\ \textsc{IInj} \qquad \frac{d \text{ final}}{\texttt{inj}^{\tau}_{?^u}(d) \text{ indet}}\ \textsc{IInjSome} \qquad \frac{}{\texttt{inj}^{\tau}_{?^u}(\varnothing) \text{ indet}}\ \textsc{IInjNone}$$

$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle \text{ indet}}\ \textsc{ICastRec} \qquad \frac{+\{C_i(T_i)\}_{C_i\in\mathcal{C}} \neq +\{C_i(T_i')\}_{C_i\in\mathcal{C}} \qquad d \text{ indet}}{d\langle +\{C_i(T_i)\}_{C_i\in\mathcal{C}} \Rightarrow +\{C_i(T_i')\}_{C_i\in\mathcal{C}}\rangle \text{ indet}}\ \textsc{ICastSum}$$

$\boxed{d \longrightarrow d'}$ $d$ takes an instruction transition to $d'$

$$\textsc{ITApp}\quad \frac{[d_2\ \mathsf{final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1}
\qquad
\textsc{ITUnroll}\quad \frac{[d\ \mathsf{final}]}{\mathtt{unroll}(\mathtt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\textsc{ITAppCast}\quad \frac{[d_1\ \mathsf{final}] \qquad [d_2\ \mathsf{final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2'\rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1\rangle))\langle \tau_2 \Rightarrow \tau_2'\rangle}$$

$$\textsc{ITUnrollCast}\quad \frac{[d\ \mathsf{final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\mathtt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau'\rangle) \longrightarrow \mathtt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau'\rangle}
\qquad
\textsc{ITCastId}\quad \frac{[d\ \mathsf{final}]}{d\langle \tau \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastSucceed}\quad \frac{[d\ \mathsf{final}] \qquad \tau\ \mathsf{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \Rightarrow \tau\rangle \longrightarrow d}
\qquad
\textsc{ITCastFail}\quad \frac{[d\ \mathsf{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \mathsf{ground} \qquad \tau_2\ \mathsf{ground}}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle}$$

$$\textsc{ITGround}\quad \frac{[d\ \mathsf{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow \llparenthesis\rrparenthesis\rangle}
\qquad
\textsc{ITExpand}\quad \frac{[d\ \mathsf{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau\rangle \longrightarrow d\langle \llparenthesis\rrparenthesis \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\mathsf{EvalCtx}\ \ \mathcal{E} \ ::= \ \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \mathtt{unroll}(\mathcal{E}) \mid \mathtt{inj}_C^{\tau}(\mathcal{E}) \mid \llparenthesis\mathcal{E}\rrparenthesis_\sigma^u \mid \llparenthesis\mathcal{E}\rrparenthesis_\sigma^{u\blacktriangleright} \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau\rangle$$

$\boxed{d = \mathcal{E}\{d'\}}$ $d$ is obtained by placing $d'$ at the mark in $\mathcal{E}$

$$\textsc{FHOuter}\quad \frac{}{d = \circ\{d\}}
\qquad
\textsc{FHApp1}\quad \frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}
\qquad
\textsc{FHApp2}\quad \frac{[d_1\ \mathsf{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}
\qquad
\textsc{FHRoll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{roll}^{\mu\pi.\tau}(d) = \mathtt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FHUnroll}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{unroll}(d) = \mathtt{unroll}(\mathcal{E})\{d'\}}
\qquad
\textsc{FHInj}\quad \frac{d = \mathcal{E}\{d'\}}{\mathtt{inj}_C^{\tau}(d) = \mathtt{inj}_C^{\tau}(\mathcal{E})\{d'\}}
\qquad
\textsc{FNEHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{\llparenthesis d\rrparenthesis_\sigma^u = \llparenthesis\mathcal{E}\rrparenthesis_\sigma^u\{d'\}}
\qquad
\textsc{FMHoleInside}\quad \frac{d = \mathcal{E}\{d'\}}{\llparenthesis d\rrparenthesis_\sigma^{u\blacktriangleright} = \llparenthesis\mathcal{E}\rrparenthesis_\sigma^{u\blacktriangleright}\{d'\}}$$

$$\textsc{FHCastInside}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}}
\qquad
\textsc{FHFailedCast}\quad \frac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ $d$ steps to $d'$

$$\textsc{Step}\quad \frac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$