

$\text{BinOp } \oplus ::= \text{Product} \mid \text{Sum} \mid \text{Arrow}$
 $\text{Kind } \kappa ::= \text{Ty} \mid \text{KHole} \mid \text{S}_\kappa(\tau)$
 $\text{ConstantTypes } c ::= \text{Int} \mid \text{Float} \mid \text{Bool}$
 $\text{UserHTyp } \hat{\tau} ::= c \mid \hat{\tau}_1 \oplus \hat{\tau}_2 \mid \text{list}(\hat{\tau}) \mid \langle \rangle^u \mid \langle \hat{\tau} \rangle^u$
 $\text{InternalHTyp } \tau ::= c \mid \tau_1 \oplus \tau_2 \mid \text{list}(\tau) \mid \langle \rangle^u \mid \langle \tau \rangle^u$
 $\text{TypeVars } t$
 $\text{TypePattern } \rho ::= t \mid \langle \rangle \mid \langle t \rangle$
 $\text{UserExpression } e ::= \text{type } \rho = \hat{\tau} \text{ in } e \mid \text{elided}$
 $\text{InternalExpression } \tau ::= \text{type } \rho = \tau : \kappa \text{ in } d \mid \text{elided}$

$\boxed{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$\frac{\text{KHoleL}}{\Delta; \Phi \vdash \text{KHole} \lesssim \kappa}$	$\frac{\text{KHoleR}}{\Delta; \Phi \vdash \kappa \lesssim \text{KHole}}$	$\frac{\text{KRespectEquiv} \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}$
$\frac{\text{KSubsumption} \quad \Delta; \Phi \vdash \tau \Leftarrow \text{Ty}}{\Delta; \Phi \vdash \text{S}_\kappa(\tau) \lesssim \text{Ty}}$		

$\boxed{t \text{ valid}}$ t is a valid type variable

t is valid if it is not a builtin-type or keyword, begins with an alpha char or underscore, and only contains alphanumeric characters, underscores, and primes.

$\boxed{\Delta; \Phi \vdash \kappa \text{ kind}}$ κ forms a kind

$\frac{\text{KFTy}}{\Delta; \Phi \vdash \text{Ty} \text{ kind}}$	$\frac{\text{KFHole}}{\Delta; \Phi \vdash \text{KHole} \text{ kind}}$	$\frac{\text{KFSing} \quad \Delta; \Phi \vdash \tau \Leftarrow \text{Ty}}{\Delta; \Phi \vdash \text{S}_\kappa(\tau) \text{ kind}}$
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$\boxed{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\begin{array}{c}
\text{KESym} \\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1} \\
\\
\text{KESingEquiv} \\
\frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash S_{\kappa_1}(\tau_1) \equiv S_{\kappa_2}(\tau_2)} \\
\\
\text{KESingSing} \\
\frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash S_{S_{\kappa_1}(\tau_1)}(\tau_3) \equiv S_{\kappa_2}(\tau_2)} \\
\\
\text{KESym} \\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi \vdash \kappa_2 \equiv \kappa_3}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3} \\
\\
\text{KESym} \\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1} \\
\\
\text{KESym} \\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi \vdash \kappa_2 \equiv \kappa_3}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3}
\end{array}$$

$\boxed{\Delta; \Phi \vdash \tau \Rightarrow \kappa}$ τ synthesizes kind κ

$$\begin{array}{c}
\text{KSConst} \\
\frac{}{\Delta; \Phi \vdash c \Rightarrow S_{\kappa}(c)} \\
\\
\text{KSVar} \\
\frac{t : \kappa \in \Phi}{\Delta; \Phi \vdash t \Rightarrow S_{\kappa}(t)} \\
\\
\text{KSUVar} \\
\frac{t \notin \text{dom}(\Phi)}{\Delta; \Phi \vdash t \Rightarrow \text{KHole}} \\
\\
\text{KSBinOp} \\
\frac{\Delta; \Phi \vdash \tau_1 \Leftarrow S_{\kappa}(\tau_1) \quad \Delta; \Phi \vdash \tau_2 \Leftarrow S_{\kappa}(\tau_2)}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \Rightarrow S_{\kappa}(\tau_1 \oplus \tau_2)} \\
\\
\text{KSList} \\
\frac{\Delta; \Phi \vdash \tau \Leftarrow S_{\kappa}(\tau)}{\Delta; \Phi \vdash \text{list}(\tau) \Rightarrow S_{\kappa}(\text{list}(\tau))} \\
\\
\text{KSEHole} \\
\frac{u :: \kappa \in \Delta}{\Delta; \Phi \vdash \langle \rangle^u \Rightarrow \kappa} \\
\\
\text{KSNEHole} \\
\frac{u :: \kappa \in \Delta \quad \Delta; \Phi \vdash \tau \Rightarrow \kappa'}{\Delta; \Phi \vdash \langle \tau \rangle^u \Rightarrow \kappa}
\end{array}$$

$\boxed{\Delta; \Phi \vdash \tau \Leftarrow \kappa}$ τ analyzes against kind κ

$$\begin{array}{c}
\text{KAASubsume} \\
\frac{\Phi \vdash \tau \Rightarrow \kappa' \quad \Delta; \Phi \vdash \kappa' \lesssim \kappa}{\Delta; \Phi \vdash \tau \Leftarrow \kappa}
\end{array}$$

$\boxed{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}$ τ_1 is equivalent to τ_2

$$\begin{array}{c}
\text{KCESymm} \\
\frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash \tau_2 \equiv \tau_1}
\end{array}
\quad
\begin{array}{c}
\text{KCETrans} \\
\frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2 \quad \Delta; \Phi \vdash \tau_2 \equiv \tau_3}{\Delta; \Phi \vdash \tau_1 \equiv \tau_3}
\end{array}$$

$$\begin{array}{c}
\text{KCESingEquiv} \\
\frac{\Delta; \Phi \vdash \tau_1 \Leftarrow S_\kappa(\tau_2)}{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}
\end{array}
\quad
\begin{array}{c}
\text{KCEConst} \\
\frac{}{\Delta; \Phi \vdash c \equiv c}
\end{array}
\quad
\begin{array}{c}
\text{KCEVar} \\
\frac{t : \kappa \in \Phi}{\Delta; \Phi \vdash t \equiv t}
\end{array}$$

$$\begin{array}{c}
\text{KCEBinOp} \\
\frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2 \quad \Delta; \Phi \vdash \tau_3 \equiv \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_3 \equiv \tau_2 \oplus \tau_4}
\end{array}
\quad
\begin{array}{c}
\text{KCEList} \\
\frac{\Delta; \Phi \vdash \tau_1 \equiv \tau_2}{\Delta; \Phi \vdash \text{list}(\tau_1) \equiv \text{list}(\tau_2)}
\end{array}$$

$$\begin{array}{c}
\text{KCEEHole} \\
\frac{u :: \kappa \in \Delta}{\Delta; \Phi \vdash (\text{hole})^u \equiv (\text{hole})^u}
\end{array}
\quad
\begin{array}{c}
\text{KCENEHole} \\
\frac{u :: \kappa \in \Delta \quad \Delta; \Phi \vdash \tau \Leftarrow \kappa'}{\Delta; \Phi \vdash (\text{hole})^u \equiv (\text{hole})^u}
\end{array}$$

$\boxed{\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta}$ $\hat{\tau}$ synthesizes kind κ and elaborates to τ

TElabSConst

$$\frac{}{\Phi \vdash c \Rightarrow S_\kappa(c) \rightsquigarrow c \dashv \cdot}$$

TElabSBinOp

$$\frac{\Phi \vdash \hat{\tau}_1 \Leftarrow \text{Ty} \rightsquigarrow \tau_1 \dashv \Delta_1 \quad \Phi \vdash \hat{\tau}_2 \Leftarrow \text{Ty} \rightsquigarrow \tau_2 \dashv \Delta_2}{\Phi \vdash \hat{\tau}_1 \oplus \hat{\tau}_2 \Rightarrow S_\kappa(\tau_1 \oplus \tau_2) \rightsquigarrow \tau_1 \oplus \tau_2 \dashv \Delta_1 \cup \Delta_2}$$

TElabSList

$$\frac{\Phi \vdash \hat{\tau} \Leftarrow \text{Ty} \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash \text{list}(\hat{\tau}) \Rightarrow S_\kappa(\text{list}(\tau)) \rightsquigarrow \text{list}(\tau) \dashv \Delta}$$

TElabSVar

$$\frac{t : \kappa \in \Phi}{\Phi \vdash t \Rightarrow S_\kappa(t) \rightsquigarrow t \dashv \cdot}$$

TElabSUNVar

$$\frac{t \notin \text{dom}(\Phi)}{\Phi \vdash t \Rightarrow \text{KHole} \rightsquigarrow \langle t \rangle^u \dashv u :: \text{KHole}}$$

TElabSHole

$$\frac{}{\Phi \vdash \langle \rangle^u \Rightarrow \text{KHole} \rightsquigarrow \langle \rangle^u \dashv u :: \text{KHole}}$$

TElabSNEHole

$$\frac{\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash \langle \hat{\tau} \rangle^u \Rightarrow \text{KHole} \rightsquigarrow \langle \tau \rangle^u \dashv \Delta, u :: \text{KHole}}$$

$\boxed{\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta}$ $\hat{\tau}$ analyzes against kind κ_1 and elaborates to τ

TElabASubsume

$$\frac{\hat{\tau} \neq \langle \rangle^u \quad \hat{\tau} \neq \langle \hat{\tau}' \rangle^u \quad \Phi \vdash \hat{\tau} \Rightarrow \kappa' \rightsquigarrow \tau \dashv \Delta \quad \Delta; \Phi \vdash \kappa' \lesssim \kappa}{\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta}$$

TElabAEHole

$$\frac{}{\Phi \vdash \langle \rangle^u \Leftarrow \kappa \rightsquigarrow \langle \rangle^u \dashv u :: \kappa}$$

TElabANEHole

$$\frac{\Phi \vdash \hat{\tau} \Rightarrow \kappa' \rightsquigarrow \tau \dashv \Delta}{\Phi \vdash \langle \hat{\tau} \rangle^u \Leftarrow \kappa \rightsquigarrow \langle \tau \rangle^u \dashv \Delta, u :: \kappa}$$

$\boxed{\Phi_1 \vdash \tau : \kappa \triangleright \rho \dashv \Phi_2}$ ρ matches against $\tau : \kappa$ extending Φ if necessary

RESVar

$$\frac{t \text{ valid}}{\Phi \vdash \tau : \kappa \triangleright t \dashv \Phi, t :: \kappa}$$

RESEHole

$$\frac{}{\Phi \vdash \tau : \kappa \triangleright \langle \rangle \dashv \Phi}$$

RESVarHole

$$\frac{\neg(t \text{ valid})}{\Phi \vdash \tau : \kappa \triangleright \langle t \rangle \dashv \Phi}$$

$\boxed{\Gamma; \Phi \vdash e \Rightarrow \hat{\tau} \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

ESDefine

$$\frac{\Phi_1 \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta_1 \quad \Phi_1 \vdash \tau : \kappa \triangleright \rho \dashv \Phi_2 \quad \Gamma; \Phi_2 \vdash e \Rightarrow \tau_1 \rightsquigarrow d \dashv \Delta_2}{\Gamma; \Phi_1 \vdash \mathbf{type} \ \rho = \hat{\tau} \ \mathbf{in} \ e \Rightarrow \tau_1 \rightsquigarrow \mathbf{type} \ \rho = \tau : \kappa \ \mathbf{in} \ d \dashv \Delta_1 \cup \Delta_2}$$

$\boxed{\Delta; \Gamma; \Phi \vdash d : \tau}$ d is assigned type τ

DEDefine

$$\frac{\Phi_1 \vdash \tau_1 : \kappa \triangleright \rho \dashv \Phi_2 \quad \Delta; \Gamma; \Phi_2 \vdash d : \tau_2}{\Delta; \Gamma; \Phi_1 \vdash \mathbf{type} \ \rho = \tau_1 : \kappa \ \mathbf{in} \ d : \tau_2}$$

Theorem 1 (Well-Kinded Elaboration)

- (1) If $\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta$ then $\Delta; \Phi \vdash \tau \Rightarrow \kappa$
- (2) If $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta$ then $\Delta; \Phi \vdash \tau \Leftarrow \kappa$

This is like the Typed Elaboration theorem in the POPL19 paper.

Theorem 2 (Elaborability)

- (1) $\exists \Delta$ s.t. if $\Delta; \Phi \vdash \tau \Rightarrow \kappa$ then $\exists \hat{\tau}$ such that $\Phi \vdash \hat{\tau} \Rightarrow \kappa \rightsquigarrow \tau \dashv \Delta$
- (2) $\exists \Delta$ s.t. if $\Delta; \Phi \vdash \tau \Leftarrow \kappa$ then $\exists \hat{\tau}$ such that $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau \dashv \Delta$

This is similar but a little different from Elaborability theorem in the POPL19 paper. Choose the Δ that is emitted from elaboration and then there's an $\hat{\tau}$ that elaborates to any of the τ forms. Elaborability and Well-Kinded Elaboration implies we can just rely on the elaboration forms for the premises of any rules that demand kind synthesis/analysis.

Theorem 3 (Type Elaboration Unicity)

- (1) If $\Phi \vdash \hat{\tau} \Rightarrow \kappa_1 \rightsquigarrow \tau_1 \dashv \Delta_1$ and $\Phi \vdash \hat{\tau} \Rightarrow \kappa_2 \rightsquigarrow \tau_2 \dashv \Delta_2$ then $\kappa_1 = \kappa_2$, $\tau_1 = \tau_2$, $\Delta_1 = \Delta_2$
- (2) If $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau_1 \dashv \Delta_1$ and $\Phi \vdash \hat{\tau} \Leftarrow \kappa \rightsquigarrow \tau_2 \dashv \Delta_2$ then $\tau_1 = \tau_2$, $\Delta_1 = \Delta_2$

This is like the Elaboration Unicity theorem in the POPL19 paper.

Theorem 4 (Kind Synthesis Precision)

If $\Delta; \Phi \vdash \tau \Rightarrow \kappa_1$ and $\Delta; \Phi \vdash \tau \Leftarrow \kappa_2$ then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$

Kind Synthesis Precision says that synthesis finds the most precise kappa possible for a given input type. This is somewhat trivial, but interesting to note because it means we can expect singletons wherever possible.