Hazel Phi: 11-type-constructors

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NOTES

need to finish up OK* proofs now that unicity is done

SYNTAX

Kind	κ	$::=$	$\texttt{Type} \mid \texttt{KHole} \mid \mathsf{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2$
User Types	$\hat{\tau}$	$::=$	$t \mid \texttt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid (\!\|\!)^{\mathtt{u}} \mid (\!\|\hat{\tau}\|\!)^{\mathtt{u}} \mid \lambda t::\texttt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2}$
Internal Types	τ	$::=$	$t \mid \texttt{bse} \mid \tau_1 \oplus \tau_2 \mid (\!\|\!)^{\mathtt{u}} \mid (\!\|\tau\|\!)^{\mathtt{u}} \mid (\!\|t\|\!)^{\mathtt{u}} \mid \lambda t::\kappa.\tau \mid \tau_1\ \tau_2$
Base Types	$\texttt{bse}$	$::=$	$\texttt{Int} \mid \texttt{Float} \mid \texttt{Bool}$
BinOp	$\oplus$	$::=$	$\times \mid + \mid \rightarrow$
Type Pattern			
User Expression			
Internal Expression			

DECLARATIVES

$\boxed{\Delta; \Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash \texttt{bse} ::> \mathsf{S}_{\texttt{Type}}(\texttt{bse})}\ \texttt{PK-Base} \qquad \frac{\Delta; \Phi_1, t::\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta; \Phi \vdash t ::> \mathsf{S}_{\kappa}(t)}\ \texttt{PK-Var}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \texttt{Type} \qquad \Delta; \Phi \vdash \tau_2 :: \texttt{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathsf{S}_{\texttt{Type}}(\tau_1 \oplus \tau_2)}\ \texttt{PK-}\oplus \qquad \frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2; \Phi \vdash \mathsf{OK}}{\Delta; \Phi \vdash (\!\|\!)^{\mathtt{u}} ::> \mathsf{S}_{\kappa}((\!\|\!)^{\mathtt{u}})}\ \texttt{PK-EHole}$$

$$\frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad \Delta; \Phi \vdash \tau::\kappa_1}{\Delta; \Phi \vdash (\!\|\tau\|\!)^{\mathtt{u}} ::> \mathsf{S}_{\kappa}((\!\|\tau\|\!)^{\mathtt{u}})}\ \texttt{PK-NEHole} \qquad \frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2; \Phi \vdash \mathsf{OK} \qquad t \notin \Phi}{\Delta; \Phi \vdash (\!\|t\|\!)^{\mathtt{u}} ::> \mathsf{S}_{\kappa}((\!\|t\|\!)^{\mathtt{u}})}\ \texttt{PK-Unbound}$$

$$\frac{\Delta; \underline{\Phi, t::\kappa_1} \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)}\ \texttt{PK-}\lambda$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \qquad \Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\scriptstyle\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta; \Phi \vdash \tau_2::\kappa_1}{\Delta; \Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ \texttt{PK-Ap}$$

$\boxed{\Delta;\Phi \vdash \tau{::}\kappa}$ τ is well formed at kind κ

$$\frac{\Delta;\Phi \vdash \tau \mathrel{::>} \mathsf{S}_{\kappa}(\tau)}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-1} \qquad \frac{\Delta;\Phi \vdash \tau \mathrel{::>} \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \lesssim \kappa}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-Subsump}$$

$$\frac{\Delta;\Phi \vdash \tau \mathrel{::>} \kappa}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-Reit} \qquad \frac{\Delta;\Phi \vdash \tau{::}\kappa}{\Delta;\Phi \vdash \tau{::}\mathsf{S}_{\kappa}(\tau)}\ \texttt{WFaK-Self}$$

$$\frac{\Delta;\Phi \vdash \tau{::}\Pi_{t::\kappa_3}.\kappa_4 \qquad \Delta;\Phi \vdash \Pi_{t::\kappa_3}.\kappa_4 \lesssim \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \tau{::}\Pi_{t::\kappa_1}.\kappa_2}\ \texttt{WFaK-}\Pi\texttt{CSKTrans}$$

$$\frac{\Delta;\Phi \vdash \tau{::}\mathsf{S}_{\kappa}(\tau_1) \qquad \Delta;\Phi \vdash \tau_1{::}\kappa}{\Delta;\Phi \vdash \tau{::}\kappa}\ \texttt{WFaK-Flatten}$$

$\boxed{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \texttt{KHole} \overset{\blacktriangleright}{\Pi} \Pi_{t::\texttt{KHole}}.\texttt{KHole}}\ \overset{\blacktriangleright}{\Pi}\texttt{-KHole} \qquad \frac{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\mathsf{S}_{\texttt{KHole}}(\tau)}.\mathsf{S}_{\texttt{KHole}}(\tau\ t)}\ \overset{\blacktriangleright}{\Pi}\texttt{-SKHole}$$

$$\frac{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}\ \overset{\blacktriangleright}{\Pi}\texttt{-}\Pi$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv>} \kappa_2}$ κ_1 singleton reduces to κ_2

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\mathsf{S}_{\kappa}(\tau_1)}(\tau)\ \mathsf{OK}}{\Delta;\Phi \vdash \mathsf{S}_{\mathsf{S}_{\kappa}(\tau_1)}(\tau) \overset{*}{\equiv>} \mathsf{S}_{\kappa}(\tau_1)}\ \overset{*}{\equiv>}\texttt{-1} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv>} \kappa_2 \qquad \Delta;\Phi \vdash \kappa_2 \overset{*}{\equiv>} \kappa_3}{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv>} \kappa_3}\ \overset{*}{\equiv>}\texttt{-Trans}$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \overset{\text{norm}}{\equiv>} \kappa_2}$ κ_1 has singleton normal form κ_2

$$\frac{\Delta;\Phi \vdash \kappa \overset{*}{\equiv>} \mathsf{S}_{\texttt{Type}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \mathsf{S}_{\texttt{Type}}(\tau)}\ \overset{\text{norm}}{\equiv>}\texttt{-Type} \qquad \frac{\Delta;\Phi \vdash \kappa \overset{*}{\equiv>} \mathsf{S}_{\texttt{KHole}}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \mathsf{S}_{\texttt{KHole}}(\tau)}\ \overset{\text{norm}}{\equiv>}\texttt{-KHole}$$

$$\frac{\Delta;\Phi \vdash \kappa \overset{*}{\equiv>} \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau)}{\Delta;\Phi \vdash \kappa \overset{\text{norm}}{\equiv>} \Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau\ t_1)}\ \overset{\text{norm}}{\equiv>}\texttt{-}\Pi$$

$\boxed{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \equiv \kappa}\ \texttt{KEquiv-Refl} \qquad \frac{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Symm}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta; \Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Trans}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \overset{*}{\equiv\!>} \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SReduc} \qquad \frac{\Delta; \Phi \vdash \kappa_1 \overset{\text{norm}}{\equiv\!>} \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SNorm}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta; \underline{\Phi, t{::}\kappa_1} \vdash \kappa_3 \equiv \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{KEquiv-}\Pi$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \equiv \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{KEquiv-SKind}$$

$\boxed{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \texttt{KHole} \lesssim \kappa}\ \texttt{CSK-KHoleL} \qquad \frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \lesssim \texttt{KHole}}\ \texttt{CSK-KHoleR}$$

$$\frac{\Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \ \mathsf{OK} \qquad \Delta; \Phi \vdash \kappa \ \mathsf{OK}}{\Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{L}$$

$$\frac{\Delta; \Phi \vdash \kappa \ \mathsf{OK} \qquad \Delta; \Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \ \mathsf{OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathsf{S}_{\texttt{KHole}}(\tau)}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{R}$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-KEquiv} \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta; \Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta; \Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-Normal}$$

$$\frac{\Delta; \Phi \vdash \mathsf{S}_{\kappa}(\tau) \ \mathsf{OK}}{\Delta; \Phi \vdash \mathsf{S}_{\kappa}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind} \qquad \frac{\Delta; \Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta; \underline{\Phi, t{::}\kappa_3} \vdash \kappa_2 \lesssim \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{CSK-}\Pi$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta; \Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta; \Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{CSK-?}$$

$\boxed{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta;\Phi \vdash \tau :: \kappa}{\Delta;\Phi \vdash \tau \overset{\kappa}{\equiv} \tau}\ \texttt{EquivAK-Refl} \qquad \frac{\Delta;\Phi \vdash \tau_2 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Symm}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_3 \overset{\kappa}{\equiv} \tau_1}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Trans}$$

$$\frac{\Delta;\Phi \vdash \tau_1 ::> \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \mathsf{S}_\kappa(\tau_2)}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-SKind}$$

$$\frac{\Delta;\Phi \vdash \tau_1 :: \Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta;\Phi \vdash \tau_2 :: \Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta;\underline{\Phi, t::\kappa_1} \vdash \tau_1\ t \overset{\kappa_2}{\equiv} \tau_2\ t}{\Delta;\Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2}\ \texttt{EquivAK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \overset{\kappa_1}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1\ \tau_2 \overset{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4}\ \texttt{EquivAK-Ap}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\texttt{Type}}{\equiv} \tau_3 \qquad \Delta;\Phi \vdash \tau_2 \overset{\texttt{Type}}{\equiv} \tau_4}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 \overset{\texttt{Type}}{\equiv} \tau_3 \oplus \tau_4}\ \texttt{EquivAK} - \oplus$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\mathsf{S}_\kappa(\tau)}{\equiv} \tau_2}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (1)$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\underline{\Phi, t::\kappa_1} \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta;\Phi \vdash \lambda t::\kappa_1.\tau_1 \overset{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t::\kappa_2.\tau_2}\ (2) \qquad \frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa}{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (3)$$

$\boxed{\Delta;\Phi \vdash \kappa\ \mathsf{OK}}$ κ is well formed

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \texttt{Type}\ \mathsf{OK}}\ \texttt{KWF-Type} \qquad \frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \texttt{KHole}\ \mathsf{OK}}\ \texttt{KWF-KHole} \qquad \frac{\Delta;\Phi \vdash \tau :: \kappa}{\Delta;\Phi \vdash \mathsf{S}_\kappa(\tau)\ \mathsf{OK}}\ \texttt{KWF-SKind}$$

$$\frac{\Delta;\underline{\Phi, t::\kappa_1} \vdash \kappa_2\ \mathsf{OK}}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathsf{OK}}\ \texttt{KWF-}\Pi$$

$\boxed{\Delta;\Phi \vdash \mathsf{OK}}$ Context is well formed

$$\frac{}{\cdot;\cdot \vdash \mathsf{OK}}\ \texttt{CWF-Nil} \qquad \frac{t \notin \Phi \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\Delta;\underline{\Phi, t::\kappa} \vdash \mathsf{OK}}\ \texttt{CWF-TypVar} \qquad \frac{u \notin \Delta \qquad \Delta;\Phi \vdash \kappa\ \mathsf{OK}}{\underline{\Delta, u::\kappa};\Phi \vdash \mathsf{OK}}\ \texttt{CWF-Hole}$$

Algorithm

(syntactically distinguished up to α-equivalence... when needed)
(TODO: remove the '... when needed'. The bound variable renamings should get adjusted)
(NOTE: current implementation has explicit $\equiv_\alpha$ checks which are not written in these rules since we eventually want to use De Bruijn indices, hence the above)

Elimination contexts

$$
\begin{array}{rcl}
\mathcal{E} & ::= & \diamond \\
 & | & \mathcal{E}\ \tau
\end{array}
$$

$\boxed{\Delta; \Phi \triangleright \tau_1 \stackrel{\kappa}{\equiv} \tau_2}$ τ_1 is equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \triangleright \tau_1 \stackrel{\kappa}{\Longrightarrow} \tau_\omega \qquad \Delta; \Phi \triangleright \tau_2 \stackrel{\kappa}{\Longrightarrow} \tau_\omega}{\Delta; \Phi \triangleright \tau_1 \stackrel{\kappa}{\equiv} \tau_2} \ (4)$$

$\boxed{\Delta; \Phi \triangleright \tau \uparrow \kappa}$ path τ has natural kind κ

$$\frac{}{\Delta; \Phi \triangleright \mathtt{bse} \uparrow \mathtt{Type}} \ (5) \qquad \frac{\Phi_1, t{::}\kappa, \Phi_2}{\Delta; \Phi \triangleright t \uparrow \kappa} \ (6) \qquad \frac{}{\Delta; \Phi \triangleright \tau_1 \oplus \tau_2 \uparrow \mathtt{Type}} \ (7) \qquad \frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2}{\Delta; \Phi \triangleright \llparenthesis\rrparenthesis^{\mathtt{u}} \uparrow \kappa} \ (8)$$

$$\frac{\Delta_1, \mathrm{u}{::}\kappa, \Delta_2}{\Delta; \Phi \triangleright \llparenthesis\tau\rrparenthesis^{\mathtt{u}} \uparrow \kappa} \ (9) \qquad \frac{\Delta; \Phi \triangleright \tau_1 \uparrow \kappa \qquad \Delta; \Phi \triangleright \kappa \Longrightarrow \Pi_{t::\kappa_1}.\kappa_2}{\Delta; \Phi \triangleright \tau_1\ \tau_2 \uparrow [\tau_2/t]\kappa_2} \ (10)$$

$\boxed{\Delta; \Phi \triangleright \mathcal{E}[\tau_1] \rightsquigarrow \mathcal{E}[\tau_2]}$ $\mathcal{E}[\tau_1]$ single step weak head reduces to $\mathcal{E}[\tau_2]$

$$\frac{TODO:\ \ check\ \tau_1\ against\ \kappa}{\Delta; \Phi \triangleright \mathcal{E}[(\lambda t{::}\kappa.\tau)\ \tau_1] \rightsquigarrow \mathcal{E}[[\tau_1/t]\tau]} \ (11) \qquad \frac{\Delta; \Phi \triangleright t \uparrow \mathtt{S}_\kappa(\tau)}{\Delta; \Phi \triangleright \mathcal{E}[t] \rightsquigarrow \mathcal{E}[\tau]} \ (12) \qquad \frac{}{\Delta; \Phi \triangleright \mathtt{bse} \not\rightsquigarrow \mathtt{bse}} \ (13)$$

$$\frac{}{\Delta; \Phi \triangleright \tau_1 \oplus \tau_2 \not\rightsquigarrow \tau_1 \oplus \tau_2} \ (14) \qquad \frac{}{\Delta; \Phi \triangleright \mathcal{E}[\llparenthesis\rrparenthesis^{\mathtt{u}}] \not\rightsquigarrow \mathcal{E}[\llparenthesis\rrparenthesis^{\mathtt{u}}]} \ (15) \qquad \frac{}{\Delta; \Phi \triangleright \mathcal{E}[\llparenthesis\tau\rrparenthesis^{\mathtt{u}}] \not\rightsquigarrow \mathcal{E}[\llparenthesis\tau\rrparenthesis^{\mathtt{u}}]} \ (16)$$

$$\frac{}{\Delta; \Phi \triangleright \mathcal{E}[\llparenthesis t\rrparenthesis^{\mathtt{u}}] \not\rightsquigarrow \mathcal{E}[\llparenthesis t\rrparenthesis^{\mathtt{u}}]} \ (17) \qquad \frac{}{\Delta; \Phi \triangleright \lambda t{::}\kappa.\tau \not\rightsquigarrow \lambda t{::}\kappa.\tau} \ (18)$$

$\boxed{\Delta; \Phi \triangleright \tau \Downarrow \tau_\psi}$ τ weak head normalizes to τ_ψ

$$\frac{\Delta; \Phi \triangleright \tau \rightsquigarrow \tau_\chi \qquad \Delta; \Phi \triangleright \tau_\chi \Downarrow \tau_\psi}{\Delta; \Phi \triangleright \tau \Downarrow \tau_\psi} \ (19) \quad \frac{\Delta; \Phi \triangleright \tau \not\rightsquigarrow \tau}{\Delta; \Phi \triangleright \tau \Downarrow \tau} \ (20)$$

$\boxed{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega}$ τ normalizes to τ_ω at kind κ

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \texttt{Type} \qquad \Delta;\Phi \rhd \tau \Downarrow \tau_\psi \qquad \Delta;\Phi \rhd \tau_\psi \longrightarrow^{\texttt{Type}} \tau_\omega}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (21)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \texttt{KHole} \qquad TODO:}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (22)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathsf{S}_{\texttt{Type}}(\tau_s) \qquad \Delta;\Phi \rhd \tau \Downarrow \tau_\psi \qquad \Delta;\Phi \rhd \tau_\psi \longrightarrow^{\texttt{Type}} \tau_\omega \qquad TODO:\ prop: \tau_\omega = \tau_s}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (23)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathsf{S}_{\texttt{KHole}}(\tau_s) \qquad TODO:}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (24)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \Pi_{t::\kappa_{\omega_1}}.\kappa_{\omega_2} \qquad \Delta;\Phi, t_1::\kappa_{\omega_1} \rhd \tau\ t_1 \overset{[t_1/t]\kappa_2}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \lambda t_1::\kappa_{\omega_1}.\tau_\omega} \quad (25)$$

$\boxed{\Delta;\Phi \rhd \tau_\psi \longrightarrow^{\kappa} \tau_\omega}$ path τ_ψ normalizes to τ_ω with kind κ

$$\frac{}{\Delta;\Phi \rhd \texttt{bse} \longrightarrow^{\texttt{Type}} \texttt{bse}} \quad (26) \qquad \frac{\Phi_1, t::\kappa, \Phi_2}{\Delta;\Phi \rhd t \longrightarrow^{\kappa} t} \quad (27) \qquad \frac{\Delta;\Phi \rhd \tau_1 \overset{\texttt{Type}}{\Longrightarrow} \tau_{\omega_1} \qquad \Delta;\Phi \rhd \tau_2 \overset{\texttt{Type}}{\Longrightarrow} \tau_{\omega_2}}{\Delta;\Phi \rhd \tau_1 \oplus \tau_2 \longrightarrow^{\texttt{Type}} \tau_{\omega_1} \oplus \tau_{\omega_2}} \quad (28)$$

$$\frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2}{\Delta;\Phi \rhd (\!|\,|\!)^{\mathrm{u}} \longrightarrow^{\kappa} (\!|\,|\!)^{\mathrm{u}}} \quad (29) \qquad \frac{\Delta_1, \mathrm{u}::\kappa, \Delta_2}{\Delta;\Phi \rhd (\!|\tau|\!)^{\mathrm{u}} \longrightarrow^{\kappa} (\!|\tau|\!)^{\mathrm{u}}} \quad (30)$$

$$\frac{\Delta;\Phi \rhd \tau_1 \longrightarrow^{\kappa} \tau_{\omega_1} \qquad \Delta;\Phi \rhd \kappa \Longrightarrow \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \rhd \tau_2 \overset{\kappa_1}{\Longrightarrow} \tau_{\omega_2}}{\Delta;\Phi \rhd \tau_1\ \tau_2 \longrightarrow^{[\tau_{\omega_2}/t]\kappa_2} \tau_{\omega_1}\ \tau_{\omega_2}} \quad (31)$$

$\boxed{\Delta;\Phi \rhd \kappa \Longrightarrow \kappa_\omega}$ κ normalizes to κ_ω

$$\frac{}{\Delta;\Phi \rhd \texttt{Type} \Longrightarrow \texttt{Type}} \quad (32) \qquad \frac{}{\Delta;\Phi \rhd \texttt{KHole} \Longrightarrow \texttt{KHole}} \quad (33)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \texttt{Type} \qquad \Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \mathsf{S}_{\texttt{Type}}(\tau_\omega)} \quad (34) \qquad \frac{\Delta;\Phi \rhd \kappa \Longrightarrow \texttt{KHole} \qquad \Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \mathsf{S}_{\texttt{KHole}}(\tau_\omega)} \quad (35)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathsf{S}_{\kappa_1}(\tau_1) \qquad \Delta;\Phi \rhd \mathsf{S}_{\kappa_1}(\tau_1) \Longrightarrow \kappa_\omega}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \kappa_\omega} \quad (36)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega \qquad \Delta;\Phi, t_1::\kappa_1 \rhd \tau_\omega\ t_1 \overset{[t_1/t]\kappa_2}{\Longrightarrow} \tau_{\omega_1}}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau_{\omega_1})} \quad (37)$$

$$\frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \kappa_{\omega_1} \qquad \Delta;\Phi, t::\kappa_{\omega_1} \rhd \kappa_2 \Longrightarrow \kappa_{\omega_2}}{\Delta;\Phi \rhd \Pi_{t::\kappa_1}.\kappa_2 \Longrightarrow \Pi_{t::\kappa_{\omega_1}}.\omega_2} \quad (38)$$

Metatheory

subderivation preserving inferences:

- premiss
- COK (Context OK)
- PoS (premiss of subderivation)

Lemma 1 (COK). *If $\Delta; \Phi \vdash \mathcal{J}$, then $\Delta; \Phi \vdash OK$ in a subderivation (where $\Delta; \Phi \vdash \mathcal{J} \neq \Delta; \Phi \vdash OK$)*

Proof. By induction on derivations.
No interesting cases. ☐

Lemma 2 (Exchange).
If $\Delta; \Phi_1, t_{L1}::\kappa_{L1}, t_{L2}::\kappa_{L2}, \Phi_2 \vdash \mathcal{J}$ and $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash OK$, then $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

Proof. By induction on derivations.
No interesting cases.
(Only rules with Φ extended in the consequent are interesting, which is only CWF-TypVar, but when $\mathcal{J}$ is CWF, Exchange is identity) ☐

Corollary 3 (Marked-Exchange).
If $\Delta; \underline{\underline{\Phi, t_{L1}::\kappa_{L1}}, t_{L2}::\kappa_{L2}} \vdash \mathcal{J}$ and $\Delta; \underline{\underline{\Phi, t_{L2}::\kappa_{L2}}, t_{L1}::\kappa_{L1}} \vdash OK$, then $\Delta; \underline{\underline{\Phi, t_{L2}::\kappa_{L2}}, t_{L1}::\kappa_{L1}} \vdash \mathcal{J}$

Proof. Exchange when $\Phi_2 = \cdot$ ☐

Lemma 4 (Weakening).
If $\Delta; \Phi \vdash \mathcal{J}$ and $\Delta; \underline{\Phi, t_L::\kappa_L} \vdash OK$ and $t_L \notin \mathcal{J}$ and $\forall t \in \kappa_L, t \notin \mathcal{J}$, then $\Delta; \underline{\Phi, t_L::\kappa_L} \vdash \mathcal{J}$

Proof. see addendum ☐

Lemma 5 (K-Substitution).
If $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$ and $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \tau_{L2}::\kappa_{L2}$, then $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2}::[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \tau_{L2}::\kappa_{L2}$)

Lemma 6 (PK-Substitution). *If $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$ and $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \tau_{L2} ::> \kappa_{L2}$ and $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2} ::> \kappa_{L3}$, then $\Delta; \Phi \vdash [\tau_{L2}/t_L]\kappa_{L2} \equiv \kappa_{L3}$*

Lemma 7 (OK-Substitution).
If $\Delta; \Phi \vdash \tau_L::\kappa_{L1}$ and $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \kappa_{L2}$ OK, then $\Delta; \Phi \vdash [\tau_L/t_L]\kappa_{L2}$ OK
(induction on $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \kappa_{L2}$ OK)

Theorem 8 (OK-PK). *If $\Delta; \Phi \vdash \tau ::> \kappa$, then $\Delta; \Phi \vdash \kappa$ OK*

Theorem 9 (OK-WFaK). *If $\Delta; \Phi \vdash \tau::\kappa$, then $\Delta; \Phi \vdash \kappa$ OK*

Theorem 10 (OK-MatchPi). *If $\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2$, then $\Delta; \Phi \vdash \kappa$ OK and $\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2$ OK*

Theorem 11 (OK-KEquiv). *If $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, then $\Delta; \Phi \vdash \kappa_1$ OK and $\Delta; \Phi \vdash \kappa_2$ OK*

Theorem 12 (OK-CSK). *If $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, then $\Delta; \Phi \vdash \kappa_1$ OK and $\Delta; \Phi \vdash \kappa_2$ OK*

Theorem 13 (OK-EquivAK). *If $\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2$, then $\Delta; \Phi \vdash \tau_1::\kappa$ and $\Delta; \Phi \vdash \tau_2::\kappa$ and $\Delta; \Phi \vdash \kappa$ OK*

Proof. see addendum ☐

Proof.

Weakening

By induction on derivations.

Note: When applying Weakening in the induction, check that the left premiss is always a subderivation, and check variable exclusion conditions are satisfied (usually checked elsewhere in the derivation).

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{\texttt{premiss}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \tau ::> \kappa_2$}
\AxiomC{}
\RightLabel{\texttt{IH}}
\UnaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \mathsf{OK}$}
\RightLabel{\texttt{PoS}}
\UnaryInfC{$t_L \notin \Phi$}
\AxiomC{}
\RightLabel{\texttt{IH}}
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\AxiomC{}
\UnaryInfC{$t \in \mathcal{J}$}
\BinaryInfC{$t_L \neq t$}
\AxiomC{}
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\TrinaryInfC{$t_L \notin \Phi,t::\kappa_1$}
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\UnaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \mathsf{OK}$}
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\UnaryInfC{$\Delta;\Phi \vdash \kappa_L\ \mathsf{OK}$}
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\RightLabel{\texttt{premiss}}
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\RightLabel{\texttt{COK}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{\texttt{Weakening}}
\BinaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \kappa_L\ \mathsf{OK}$}
\RightLabel{\texttt{CWF-TypVar}}
\BinaryInfC{$\Delta;\Phi,t::\kappa_1,t_L::\kappa_L \vdash \mathsf{OK}$}
\RightLabel{\texttt{Weakening}}
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\RightLabel{\texttt{COK}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \mathsf{OK}$}
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\UnaryInfC{$t \in \mathcal{J}$}
\BinaryInfC{$t \notin \kappa_L$}
\TrinaryInfC{$t \notin \Phi,t_L::\kappa_L$}
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\RightLabel{\texttt{premiss}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \tau ::> \kappa_2$}
\RightLabel{\texttt{COK}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{\texttt{PoS}}
\UnaryInfC{$\Delta;\Phi \vdash \kappa_1\ \mathsf{OK}$}
\AxiomC{}
\RightLabel{\texttt{IH}}
\UnaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \mathsf{OK}$}
\RightLabel{\texttt{Weakening}}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \kappa_1\ \mathsf{OK}$}
\RightLabel{\texttt{CWF-TypVar}}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L,t::\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{\texttt{Marked-Exchange}}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L,t::\kappa_1 \vdash \tau ::> \kappa_2$}
\RightLabel{\texttt{PK-}λ}
\UnaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \lambda t::\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)$}
\end{prooftree}
$$

$$
\begin{prooftree}
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\RightLabel{\texttt{premiss}}
\UnaryInfC{$\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2$}
\AxiomC{}
\RightLabel{\texttt{IH}}
\UnaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \mathsf{OK}$}
\RightLabel{\texttt{Weakening}}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \kappa_1 \equiv \kappa_2$}
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\RightLabel{\texttt{premiss}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \tau ::> \kappa_2$}
\AxiomC{}
\RightLabel{\texttt{IH}}
\UnaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \mathsf{OK}$}
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\UnaryInfC{$t \in \mathcal{J}$}
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\TrinaryInfC{$t_L \notin \Phi,t::\kappa_1$}
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\RightLabel{\texttt{PoS}}
\UnaryInfC{$\Delta;\Phi \vdash \kappa_L\ \mathsf{OK}$}
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\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{\texttt{Weakening}}
\BinaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \kappa_L\ \mathsf{OK}$}
\RightLabel{\texttt{CWF-TypVar}}
\BinaryInfC{$\Delta;\Phi,t::\kappa_1,t_L::\kappa_L \vdash \mathsf{OK}$}
\RightLabel{\texttt{Weakening}}
\BinaryInfC{$\Delta;\Phi,t::\kappa_1,t_L::\kappa_L \vdash \tau ::> \kappa_2$}
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\RightLabel{\texttt{premiss}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4$}
\RightLabel{\texttt{COK}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{\texttt{PoS}}
\UnaryInfC{$t \notin \Phi$}
\AxiomC{}
\RightLabel{\texttt{IH}}
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\UnaryInfC{$t \in \mathcal{J}$}
\BinaryInfC{$t \neq t_L$}
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\UnaryInfC{$t \in \mathcal{J}$}
\BinaryInfC{$t \notin \kappa_L$}
\TrinaryInfC{$t \notin \Phi,t_L::\kappa_L$}
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\RightLabel{\texttt{premiss}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4$}
\RightLabel{\texttt{COK}}
\UnaryInfC{$\Delta;\Phi,t::\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{\texttt{PoS}}
\UnaryInfC{$\Delta;\Phi \vdash \kappa_1\ \mathsf{OK}$}
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\RightLabel{\texttt{IH}}
\UnaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \mathsf{OK}$}
\RightLabel{\texttt{Weakening}}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \kappa_1\ \mathsf{OK}$}
\RightLabel{\texttt{CWF-TypVar}}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L,t::\kappa_1 \vdash \mathsf{OK}$}
\RightLabel{\texttt{Marked-Exchange}}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L,t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4$}
\RightLabel{\texttt{KEquiv-}Π}
\BinaryInfC{$\Delta;\Phi,t_L::\kappa_L \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4$}
\end{prooftree}
\qquad
\frac{}{sim}\ \texttt{CSK-}\Pi
\qquad
\frac{}{sim}\ \texttt{EquivAK-}\Pi
\qquad
\frac{}{sim}\ \texttt{KWF-}\Pi
$$

O?K-.*

By simultaneous induction on derivations.

The interesting cases per theorem:

K-Substitution by type size??

OK-Substitution

OK-PK

$$
\begin{prooftree}
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\RightLabel{\texttt{WFaK-1}}
\UnaryInfC{$\Delta;\Phi \vdash \texttt{bse}::\texttt{Type}$}
\RightLabel{\texttt{KWF-SKind}}
\UnaryInfC{$\Delta;\Phi \vdash \mathsf{S}_{\texttt{Type}}(\texttt{bse})\ \mathsf{OK}$}
\end{prooftree}
\qquad\qquad
\frac{}{\Delta;\Phi \vdash [\tau_2/t]\kappa_2\ \mathsf{OK}}\ \texttt{OK-Substitution}
$$

OK-WFaK

$\square$

Definition 1 (Singleton Depth).

$$SSize : \text{“}\{\kappa\}\text{''} \to \mathbb{N}$$

$$SSize(\kappa_x) = \begin{cases} SSize(\kappa) + 1 & \text{if } \kappa_x = \mathtt{S}_\kappa(\tau) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 14 ($\overset{*}{\equiv>}$-diminution). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$, then $SSize(\kappa_L) > SSize(\kappa_{L1})$*

Proof. By induction on derivations (and transitivity of $>$ on $\mathbb{N}$) $\square$

Lemma 15 ($\overset{*}{\equiv>}$-n+1-nicity). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L2}$ where $SSize(\kappa_L) = n{+}1$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. By $\overset{*}{\equiv>}$-diminution, $\overset{*}{\equiv>}$-Trans cannot be the last inference of a derivation of $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ since $SSize(\kappa_1) \geq SSize(\kappa_3) + 2$ (in $\overset{*}{\equiv>}$-Trans). Thus, $\overset{*}{\equiv>}$-1 must have been the last inference. Similarly for $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L2}$, thus $\kappa_{L1} = \kappa_{L2}$ $\square$

Lemma 16 ($\overset{*}{\equiv>}$-stepwise). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ where $SSize(\kappa_L) = m$ and $SSize(\kappa_{L1}) = n$ and $m > n + 1$, then the derivation must contain subderivations of each singleton depth inbetween*

Proof. More precisely this says, where $m > n$ by $\overset{*}{\equiv>}$-diminution, the derivation must contain subderivations of each $\Delta; \Phi \vdash \kappa_i \overset{*}{\equiv>} \kappa_j$ where $m \geq i > j \geq n$, $SSize(\kappa_k) = k$ when $m \geq k \geq n$, $\kappa_m = \kappa_L$, $\kappa_n = \kappa_{L1}$.
By induction on derivations (base case is where $m = n + 2$, which necessitates a last inference of $\overset{*}{\equiv>}$-Trans. Each premiss must have SSize difference of 1, fulfilling hypothesis) $\square$

Lemma 17 ($\overset{*}{\equiv>}$-$m + n$-nicity). *If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv>} \kappa_{L2}$ where $SSize(\kappa_L) = m + n$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. By $\overset{*}{\equiv>}$-stepwise and $\overset{*}{\equiv>}$-$n + 1$-nicity when $m > n + 1$.
By $\overset{*}{\equiv>}$-$n + 1$-nicity when $m = n + 1$.
No other cases by $\overset{*}{\equiv>}$-diminution. $\square$

Theorem 18 ($\overset{\text{norm}}{\equiv>}$-Unicity). *If $\Delta; \Phi \vdash \kappa_L \overset{norm}{\equiv>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{norm}{\equiv>} \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)
All $\overset{\text{norm}}{\equiv>}$ rules have $\overset{*}{\equiv>}$ premiss with rhs singleton depth 1. By $\overset{*}{\equiv>}$-$m + n$-nicity, where $n = 1$. $\square$

Theorem 19 ($\overset{\blacktriangleright}{\Pi}$-Unicity). *If $\Delta; \Phi \vdash \tau_L \overset{\blacktriangleright}{\Pi} \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L \overset{\blacktriangleright}{\Pi} \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)
By unicity of $\overset{\text{norm}}{\equiv>}$. $\square$

Theorem 20 (PK-Unicity). *If $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$*

Proof. (this is a really quick sketch)
As PK is syntax directed, proof is by inspection for all rules except PK-λ (variables in contexts are unique– see context rules), which is by induction on derivations, and PK-Ap, which requires of unicity of $\overset{\blacktriangleright}{\Pi}$ (above theorem). $\square$

Theorem 21 (PK-Principality). *If $\Delta; \Phi \vdash \tau ::> \kappa_1$ and $\Delta; \Phi \vdash \tau :: \kappa_2$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

Proof. From definition of $\Delta; \Phi \vdash \tau :: \kappa$ and CSK-SKind $\square$

Theorem 22 (why is this here?). *If $\Delta; \Phi \vdash \kappa_1 \lesssim \mathtt{S}_{\kappa_2}(\tau)$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

Elaboration

TODO