Hazel Phi: 9-type-aliases

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SYNTAX

$$
\begin{array}{rrcl}
\text{Kind} & \kappa & ::= & \texttt{Type} \mid \texttt{KHole} \mid \mathsf{S}_\kappa(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2 \\
\text{User Types} & \hat{\tau} & ::= & t \mid \texttt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid (\!|\,|\!)^{\mathtt{u}} \mid (\!|\hat{\tau}|\!)^{\mathtt{u}} \mid \lambda t::\texttt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2} \\
\text{Internal Types} & \tau & ::= & t \mid \texttt{bse} \mid \tau_1 \oplus \tau_2 \mid (\!|\,|\!)^{\mathtt{u}} \mid (\!|\tau|\!)^{\mathtt{u}} \mid (\!|t|\!)^{\mathtt{u}} \mid \lambda t::\kappa.\tau \mid \tau_1\ \tau_2 \\
\text{Base Types} & \texttt{bse} & ::= & \texttt{Int} \mid \texttt{Float} \mid \texttt{Bool} \\
\text{BinOp} & \oplus & ::= & \times \mid + \mid \to \\
\text{Type Pattern} & & & \\
\text{User Expression} & & & \\
\text{Internal Expression} & & &
\end{array}
$$

DECLARATIVES

$\boxed{\Delta;\Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash \texttt{bse} ::> \mathsf{S}_{\texttt{Type}}(\texttt{bse})}\ (1) \qquad \frac{\Delta;\Phi_1, t::\kappa, \Phi_2 \vdash \mathsf{OK}}{\Delta;\Phi \vdash t ::> \mathsf{S}_\kappa(t)}\ (2) \qquad \frac{\Delta;\Phi \vdash \tau_1::\texttt{Type} \qquad \Delta;\Phi \vdash \tau_2::\texttt{Type}}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 ::> \mathsf{S}_{\texttt{Type}}(\tau_1 \oplus \tau_2)}\ (3)$$

$$\frac{\Delta_1, \mathtt{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK}}{\Delta;\Phi \vdash (\!|\,|\!)^{\mathtt{u}} ::> \mathsf{S}_\kappa((\!|\,|\!)^{\mathtt{u}})}\ (4) \qquad \frac{\Delta_1, \mathtt{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK} \qquad \Delta;\Phi \vdash \tau::\kappa_1}{\Delta;\Phi \vdash (\!|\tau|\!)^{\mathtt{u}} ::> \mathsf{S}_\kappa((\!|\tau|\!)^{\mathtt{u}})}\ (5)$$

$$\frac{\Delta_1, \mathtt{u}::\kappa, \Delta_2;\Phi \vdash \mathsf{OK} \qquad t \notin \mathsf{dom}(\Phi)}{\Delta;\Phi \vdash (\!|t|\!)^{\mathtt{u}} ::> \mathsf{S}_\kappa((\!|t|\!)^{\mathtt{u}})}\ (6) \qquad \frac{\Delta;\Phi, t::\kappa_1 \vdash \tau ::> \kappa_2}{\Delta;\Phi \vdash \lambda t::\kappa_1.\tau ::> \mathsf{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)}\ (7)$$

$$\frac{\Delta;\Phi \vdash \tau_1 ::> \kappa \qquad \Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\underset{\Pi}{}} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \vdash \tau_2::\kappa_1}{\Delta;\Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ (8)$$

$\boxed{\Delta;\Phi \vdash \tau::\kappa}$ τ is well formed at kind κ

$$\frac{\Delta;\Phi \vdash \tau ::> \kappa}{\Delta;\Phi \vdash \tau::\kappa}\ (9) \qquad \frac{\Delta;\Phi \vdash \tau::\mathsf{S}_\kappa(\tau_1) \qquad \Delta;\Phi \vdash \tau_1::\kappa}{\Delta;\Phi \vdash \tau::\kappa}\ (10)$$

$$\frac{\Delta;\Phi \vdash \tau ::> \kappa_1 \qquad \Delta;\Phi \vdash \kappa_1 \lesssim \kappa}{\Delta;\Phi \vdash \tau::\kappa}\ (11)$$

$$\frac{\Delta;\Phi \vdash \tau_2::\mathsf{S}_\kappa(\tau_1)}{\Delta;\Phi \vdash \tau_1::\mathsf{S}_\kappa(\tau_2)}\ (12) \qquad \frac{\Delta;\Phi \vdash \tau_1::\mathsf{S}_\kappa(\tau_3) \qquad \Delta;\Phi \vdash \tau_3::\mathsf{S}_\kappa(\tau_2)}{\Delta;\Phi \vdash \tau_1::\mathsf{S}_\kappa(\tau_2)}\ (13)$$

$$\frac{\Delta;\Phi \vdash \tau::\kappa}{\Delta;\Phi \vdash \tau::\mathsf{S}_\kappa(\tau)}\ (14)$$

$\boxed{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{\Delta;\Phi \vdash \kappa \equiv \mathtt{KHole}}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\mathtt{KHole}}.\mathtt{KHole}} \; (15) \qquad \frac{\Delta;\Phi \vdash \kappa \equiv \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2} \; (16)$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta;\Phi \vdash \kappa \ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \equiv \kappa} \; (17) \qquad \frac{\Delta;\Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2} \; (18) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2} \; (19)$$

$$\frac{\Delta;\Phi \vdash \tau :: \mathtt{S}_\kappa(\tau_1)}{\Delta;\Phi \vdash \mathtt{S}_{\mathtt{S}_\kappa(\tau_1)}(\tau) \equiv \mathtt{S}_\kappa(\tau_1)} \; (20) \qquad \frac{\Delta;\Phi \vdash \tau :: \mathtt{S}_{\mathtt{KHole}}(\tau_1)}{\Delta;\Phi \vdash \mathtt{S}_{\mathtt{KHole}}(\tau) \equiv \mathtt{KHole}} \; (21)$$

$$\frac{\Delta;\Phi \vdash \tau :: \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \vdash \mathtt{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau) \equiv \Pi_{t::\kappa_1}.\mathtt{S}_{\kappa_2}(\tau\ t)} \; (22) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\Phi, t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4} \; (23)$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathtt{S}_\kappa(\tau_1) \equiv \mathtt{S}_\kappa(\tau_2)} \; (24)$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta;\Phi \vdash \kappa \ \mathsf{OK}}{\Delta;\Phi \vdash \mathtt{KHole} \lesssim \kappa} \; (25) \qquad \frac{\Delta;\Phi \vdash \kappa \ \mathsf{OK}}{\Delta;\Phi \vdash \kappa \lesssim \mathtt{KHole}} \; (26)$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2} \; (27) \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta;\Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2} \; (28)$$

$$\frac{\Delta;\Phi \vdash \tau :: \kappa}{\Delta;\Phi \vdash \mathtt{S}_\kappa(\tau) \lesssim \kappa} \; (29) \qquad \frac{\Delta;\Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta;\Phi, t::\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4} \; (30)$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathtt{S}_{\kappa_1}(\tau_1) \lesssim \mathtt{S}_{\kappa_2}(\tau_2)} \; (31)$$

$\boxed{\Delta;\Phi\vdash\tau_1\overset{\kappa}{\equiv}\tau_2}$ τ_1 is equivalent to τ_2 at kind κ

$$\frac{\Delta;\Phi\vdash\tau_1{::}\mathtt{S}_\kappa(\tau_2)}{\Delta;\Phi\vdash\tau_1\overset{\kappa}{\equiv}\tau_2}\ (32)$$

$$\frac{\Delta;\Phi\vdash\tau{::}\kappa}{\Delta;\Phi\vdash\tau\overset{\kappa}{\equiv}\tau}\ (33)\qquad\frac{\Delta;\Phi\vdash\tau_2\overset{\kappa}{\equiv}\tau_1}{\Delta;\Phi\vdash\tau_1\overset{\kappa}{\equiv}\tau_2}\ (34)\qquad\frac{\Delta;\Phi\vdash\tau_1\overset{\kappa}{\equiv}\tau_3\qquad\Delta;\Phi\vdash\tau_3\overset{\kappa}{\equiv}\tau_1}{\Delta;\Phi\vdash\tau_1\overset{\kappa}{\equiv}\tau_2}\ (35)$$

$$\frac{\Delta;\Phi\vdash\tau_1\overset{\mathtt{Type}}{\equiv}\tau_3\qquad\Delta;\Phi\vdash\tau_2\overset{\mathtt{Type}}{\equiv}\tau_4}{\Delta;\Phi\vdash\tau_1\oplus\tau_2\overset{\mathtt{Type}}{\equiv}\tau_3\oplus\tau_4}\ (36)\qquad\frac{\Delta;\Phi\vdash\kappa_1\equiv\kappa_2\qquad\Delta;\Phi,t{::}\kappa_1\vdash\tau_1\overset{\kappa}{\equiv}\tau_2}{\Delta;\Phi\vdash\lambda t{::}\kappa_1.\tau_1\overset{\Pi_{t::\kappa_1}.\kappa}{\equiv}\lambda t{::}\kappa_2.\tau_2}\ (37)$$

$$\frac{\Delta;\Phi\vdash\tau_1\overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv}\tau_3\qquad\Delta;\Phi\vdash\tau_2\overset{\kappa_1}{\equiv}\tau_4}{\Delta;\Phi\vdash\tau_1\ \tau_2\overset{[\tau_1/t]\kappa_2}{\equiv}\tau_3\ \tau_4}\ (38)$$

$$\frac{\Delta;\Phi\vdash\tau_1{::}\Pi_{t::\kappa_1}.\kappa_3\qquad\Delta;\Phi\vdash\tau_2{::}\Pi_{t::\kappa_1}.\kappa_4\qquad\Delta;\Phi,t{::}\kappa_1\vdash\tau_1\ t\overset{\kappa_2}{\equiv}\tau_2\ t}{\Delta;\Phi\vdash\tau_1\overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv}\tau_2}\ (39)$$

$$\frac{\Delta;\Phi\vdash\tau_1\overset{\kappa_1}{\equiv}\tau_2\qquad\Delta;\Phi\vdash\kappa_1\equiv\kappa}{\Delta;\Phi\vdash\tau_1\overset{\kappa}{\equiv}\tau_2}\ (40)$$

$\boxed{\Delta;\Phi\vdash\kappa\ \mathsf{OK}}$ κ is well formed

$$\frac{\Delta;\Phi\vdash\mathsf{OK}}{\Delta;\Phi\vdash\mathtt{Type}\ \mathsf{OK}}\ (41)\qquad\frac{\Delta;\Phi\vdash\mathsf{OK}}{\Delta;\Phi\vdash\mathtt{KHole}\ \mathsf{OK}}\ (42)\qquad\frac{\Delta;\Phi\vdash\tau{::}\kappa}{\Delta;\Phi\vdash\mathtt{S}_\kappa(\tau)\ \mathsf{OK}}\ (43)$$

$$\frac{\Delta;\Phi\vdash\kappa_1\ \mathsf{OK}\qquad\Delta;\Phi,t{::}\kappa_1\vdash\kappa_2\ \mathsf{OK}}{\Delta;\Phi\vdash\Pi_{t::\kappa_1}.\kappa_2\ \mathsf{OK}}\ (44)$$

$\boxed{\Delta;\Phi\vdash\mathsf{OK}}$ Context is well formed

$$\frac{}{;\vdash\mathsf{OK}}\ (45)\qquad\frac{\Delta;\Phi\vdash\kappa\ \mathsf{OK}}{\Delta;\Phi,t{::}\kappa\vdash\mathsf{OK}}\ (46)\qquad\frac{\Delta;\Phi\vdash\kappa\ \mathsf{OK}}{\Delta,\mathrm{u}{::}\kappa;\Phi\vdash\mathsf{OK}}\ (47)$$

Variables implicitly assumed to be fresh as necessary

Metatheory

Lemma 1 (Weakening)**.** *If* $\Delta;\Phi\vdash\tau{::}\kappa$*, then* $\Delta;\Phi,t{::}\kappa_1\vdash\tau{::}\kappa$ *when* $\Delta;\Phi,t{::}\kappa_1\vdash OK$

Proof. By rule induction/length of proof.
L1. (9)

□

Proof. By rule induction/length of proof.
L2. (9)

□

Lemma 2 (OK-PK)**.** *If* $\Delta;\Phi\vdash\tau ::> \kappa$*, then* $\Delta;\Phi\vdash OK$ *and* $\Delta;\Phi\vdash\kappa\ OK$

Lemma 3 (OK-WFaK)**.** *If* $\Delta;\Phi\vdash\tau{::}\kappa$*, then* $\Delta;\Phi\vdash OK$ *and* $\Delta;\Phi\vdash\kappa\ OK$

Lemma 4 (OK-MatchPi)**.** *If* $\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2$, *then* $\Delta; \Phi \vdash \mathit{OK}$ *and* $\Delta; \Phi \vdash \kappa\ \mathit{OK}$ *and* $\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2\ \mathit{OK}$

Lemma 5 (OK-KEquiv)**.** *If* $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, *then* $\Delta; \Phi \vdash \mathit{OK}$ *and* $\Delta; \Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta; \Phi \vdash \kappa_2\ \mathit{OK}$

Lemma 6 (OK-CSK)**.** *If* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, *then* $\Delta; \Phi \vdash \mathit{OK}$ *and* $\Delta; \Phi \vdash \kappa_1\ \mathit{OK}$ *and* $\Delta; \Phi \vdash \kappa_2\ \mathit{OK}$

Lemma 7 (OK-KWF)**.** *If* $\Delta; \Phi \vdash \kappa\ \mathit{OK}$, *then* $\Delta; \Phi \vdash \mathit{OK}$

Lemma 8 (OK-Substitution)**.**
If $\Delta; \Phi \vdash \tau_L::\kappa_{L1}$ *and* $\Delta; \Phi, t_L::\kappa_{L1} \vdash \kappa_{L2}\ \mathit{OK}$, *then* $\Delta; \Phi \vdash \mathit{OK}$ *and* $\Delta; \Phi \vdash [\tau_L/t_L]\kappa_{L2}\ \mathit{OK}$
(induction on $\Delta; \Phi, t_L::\kappa_{L1} \vdash \kappa_{L2}\ \mathit{OK}$*)*

Lemma 9 (K-Substitution)**.**
If $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$ *and* $\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau_{L2}::\kappa_{L2}$, *then* $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2}::[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau_{L2}::\kappa_{L2}$*)*

Proof. By simultaneous rule induction/length of proof.
The interesting cases per lemma:

OK-PK.	(1)	$\Delta; \Phi \vdash \mathtt{bse}::\mathtt{S}_{\mathtt{Type}}(\mathtt{bse})$	by (9)
		$\Delta; \Phi \vdash \mathtt{bse}::\mathtt{Type}$	by (10)
	*	$\Delta; \Phi \vdash \mathtt{S}_{\mathtt{Type}}(\mathtt{bse})\ \mathsf{OK}$	by (43)
	*	$\Delta; \Phi \vdash \mathsf{OK}$	by premiss
	(8)		bad
OK-WFaK.	(12)	$\Delta; \Phi \vdash \tau_2::\kappa$	by (10)
	*	$\Delta; \Phi \vdash \mathtt{S}_{\kappa}(\tau_2)\ \mathsf{OK}$	by (43)
OK-KEquiv.	(22)	$\Delta; \Phi \vdash \tau\ t ::> \kappa$	
OK-Substitution.	(41)	$\Delta; \Phi, t_L::\kappa_{L1} \vdash \mathsf{OK}$	premiss (41)
		$\Delta; \Phi \vdash \kappa_{L1}\ \mathsf{OK}$	by subderivation premiss (46)
	*	$\Delta; \Phi \vdash \mathsf{OK}$	by OK-KWF
	*	$\Delta; \Phi \vdash [\tau_L/t_L]\mathtt{Type}\ \mathsf{OK}$	by (41) and degenerate subst
	(43)	$\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau::\kappa$	premiss (43)
		$\Delta; \Phi, t_L::\kappa_{L1} \vdash \mathsf{OK}$	by OK-WFaK
		$\Delta; \Phi \vdash \kappa_{L1}\ \mathsf{OK}$	by subderivation premiss (46)
	*	$\Delta; \Phi \vdash \mathsf{OK}$	by OK-KWF
		$\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau::[\tau_{L1}/t_L]\kappa$	by K-Substitution on premiss
	*	$\Delta; \Phi \vdash [\tau_L/t_L]\mathtt{S}_{\kappa}(\tau)\ \mathsf{OK}$	by (43)

□

Lemma 10 (PK-Unicity)**.** *If* $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$ *and* $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$ *then* κ_{L1} *is* κ_{L2}

Lemma 11. *If* $\Delta; \Phi \vdash \tau ::> \kappa_1$ *and* $\Delta; \Phi \vdash \tau::\kappa_2$, *then* $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$