Hazel Phi: 11-type-constructors

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notes

- proofs are dilapidated

syntax

Kind	κ	$::=$	$\mathtt{Type} \mid \mathtt{KHole} \mid \mathtt{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2$
User Types	$\hat{\tau}$	$::=$	$t \mid \mathtt{bse} \mid \hat{\tau_1} \oplus \hat{\tau_2} \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\hat{\tau}\rrparenthesis^{\mathtt{u}} \mid \lambda t{::}\mathtt{Type}.\hat{\tau} \mid \hat{\tau_1}\ \hat{\tau_2}$
Internal Types	τ	$::=$	$t \mid \mathtt{bse} \mid \tau_1 \oplus \tau_2 \mid \llparenthesis\rrparenthesis^{\mathtt{u}} \mid \llparenthesis\tau\rrparenthesis^{\mathtt{u}} \mid \llparenthesis t\rrparenthesis^{\mathtt{u}} \mid \lambda t{::}\kappa.\tau \mid \tau_1\ \tau_2$
Base Types	$\mathtt{bse}$	$::=$	$\mathtt{Int} \mid \mathtt{Float} \mid \mathtt{Bool}$
BinOp	$\oplus$	$::=$	$\times \mid + \mid \to$
Type Pattern			
User Expression			
Internal Expression			

Declaratives

$\boxed{\Delta;\Phi \vdash \tau ::> \kappa}$ τ has principal (well formed) kind κ

$$\frac{\Delta;\Phi \vdash \mathrm{OK}}{\Delta;\Phi \vdash \mathtt{bse} ::> \mathtt{S}_{\mathtt{Type}}(\mathtt{bse})}\ \texttt{PK-Base} \qquad \frac{\Delta;\Phi = \Phi_1, t{::}\kappa, \Phi_2 \vdash \mathrm{OK}}{\Delta;\Phi \vdash t ::> \mathtt{S}_{\kappa}(t)}\ \texttt{PK-Var}$$

$$\frac{\Delta;\Phi \vdash \tau_1{::}\mathtt{Type} \qquad \Delta;\Phi \vdash \tau_2{::}\mathtt{Type}}{\Delta;\Phi \vdash \tau_1 \oplus \tau_2 ::> \mathtt{S}_{\mathtt{Type}}(\tau_1 \oplus \tau_2)}\ \texttt{PK-}\oplus \qquad \frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathrm{OK}}{\Delta;\Phi \vdash \llparenthesis\rrparenthesis^{\mathtt{u}} ::> \mathtt{S}_{\kappa}(\llparenthesis\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-EHole}$$

$$\frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathrm{OK} \qquad \Delta;\Phi \vdash \tau{::}\kappa_1}{\Delta;\Phi \vdash \llparenthesis\tau\rrparenthesis^{\mathtt{u}} ::> \mathtt{S}_{\kappa}(\llparenthesis\tau\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-NEHole}$$

$$\frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2; \Phi \vdash \mathrm{OK} \qquad t \notin \Phi}{\Delta;\Phi \vdash \llparenthesis t\rrparenthesis^{\mathtt{u}} ::> \mathtt{S}_{\kappa}(\llparenthesis t\rrparenthesis^{\mathtt{u}})}\ \texttt{PK-Unbound} \qquad \frac{\Delta;\Phi, \underline{t{::}\kappa_1} \vdash \tau ::> \kappa_2}{\Delta;\Phi \vdash \lambda t{::}\kappa_1.\tau ::> \mathtt{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t{::}\kappa_1.\tau)}\ \texttt{PK-}\lambda$$

$$\frac{\Delta;\Phi \vdash \tau_1 ::> \kappa \qquad \Delta;\Phi \vdash \kappa \overset{\blacktriangleright}{_{\Pi}} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \vdash \tau_2{::}\kappa_1}{\Delta;\Phi \vdash \tau_1\ \tau_2 ::> [\tau_2/t]\kappa_2}\ \texttt{PK-Ap}$$

$\boxed{\Delta;\Phi\vdash\tau{::}\kappa}$ τ is well formed at kind κ

$$\frac{\Delta;\Phi\vdash\tau ::> \mathtt{S}_{\kappa}(\tau)}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-1}\qquad\frac{\Delta;\Phi\vdash\tau ::> \kappa_1\qquad\Delta;\Phi\vdash\kappa_1\lesssim\kappa}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-Subsump}$$

$$\frac{\Delta;\Phi\vdash\tau ::> \kappa}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-Reit}\qquad\frac{\Delta;\Phi\vdash\tau{::}\kappa}{\Delta;\Phi\vdash\tau{::}\mathtt{S}_{\kappa}(\tau)}\ \texttt{WFaK-Self}$$

$$\frac{\Delta;\Phi\vdash\tau{::}\Pi_{t::\kappa_3}.\kappa_4\qquad\Delta;\Phi\vdash\Pi_{t::\kappa_3}.\kappa_4\lesssim\Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi\vdash\tau{::}\Pi_{t::\kappa_1}.\kappa_2}\ \texttt{WFaK-}\Pi\texttt{CSKTrans}$$

$$\frac{\Delta;\Phi\vdash\tau{::}\mathtt{S}_{\kappa}(\tau_1)\qquad\Delta;\Phi\vdash\tau_1{::}\kappa}{\Delta;\Phi\vdash\tau{::}\kappa}\ \texttt{WFaK-Flatten}$$

$\boxed{\Delta;\Phi\vdash\kappa\overset{\blacktriangleright}{_\Pi}\Pi_{t::\kappa_1}.\kappa_2}$ κ has matched Π-kind $\Pi_{t::\kappa_1}.\kappa_2$

$$\frac{\Delta;\Phi\vdash\mathrm{OK}}{\Delta;\Phi\vdash\mathtt{KHole}\overset{\blacktriangleright}{_\Pi}\Pi_{t::\mathtt{KHole}}.\mathtt{KHole}}\ \overset{\blacktriangleright}{_\Pi}\texttt{-KHole}\qquad\frac{\Delta;\Phi\vdash\kappa\overset{\text{norm}}{\equiv>}\mathtt{S}_{\mathtt{KHole}}(\tau)}{\Delta;\Phi\vdash\kappa\overset{\blacktriangleright}{_\Pi}\Pi_{t::\mathtt{S}_{\mathtt{KHole}}(\tau)}.\mathtt{S}_{\mathtt{KHole}}(\tau\ t)}\ \overset{\blacktriangleright}{_\Pi}\texttt{-SKHole}$$

$$\frac{\Delta;\Phi\vdash\kappa\overset{\text{norm}}{\equiv>}\Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi\vdash\kappa\overset{\blacktriangleright}{_\Pi}\Pi_{t::\kappa_1}.\kappa_2}\ \overset{\blacktriangleright}{_\Pi}\text{-}\Pi$$

$\boxed{\Delta;\Phi\vdash\kappa_1\overset{*}{\equiv>}\kappa_2}$ κ_1 singleton reduces to κ_2

$$\frac{\Delta;\Phi\vdash\mathtt{S}_{\mathtt{S}_{\kappa}(\tau_1)}(\tau)\ \mathrm{OK}}{\Delta;\Phi\vdash\mathtt{S}_{\mathtt{S}_{\kappa}(\tau_1)}(\tau)\overset{*}{\equiv>}\mathtt{S}_{\kappa}(\tau_1)}\ \overset{*}{\equiv>}\texttt{-1}\qquad\frac{\Delta;\Phi\vdash\kappa_1\overset{*}{\equiv>}\kappa_2\qquad\Delta;\Phi\vdash\kappa_2\overset{*}{\equiv>}\kappa_3}{\Delta;\Phi\vdash\kappa_1\overset{*}{\equiv>}\kappa_3}\ \overset{*}{\equiv>}\texttt{-Trans}$$

$\boxed{\Delta;\Phi\vdash\kappa_1\overset{\text{norm}}{\equiv>}\kappa_2}$ κ_1 has singleton normal form κ_2

$$\frac{\Delta;\Phi\vdash\kappa\overset{*}{\equiv>}\mathtt{S}_{\mathtt{Type}}(\tau)}{\Delta;\Phi\vdash\kappa\overset{\text{norm}}{\equiv>}\mathtt{S}_{\mathtt{Type}}(\tau)}\ \overset{\text{norm}}{\equiv>}\texttt{-Type}\qquad\frac{\Delta;\Phi\vdash\kappa\overset{*}{\equiv>}\mathtt{S}_{\mathtt{KHole}}(\tau)}{\Delta;\Phi\vdash\kappa\overset{\text{norm}}{\equiv>}\mathtt{S}_{\mathtt{KHole}}(\tau)}\ \overset{\text{norm}}{\equiv>}\texttt{-KHole}$$

$$\frac{\Delta;\Phi\vdash\kappa\overset{*}{\equiv>}\mathtt{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\tau)}{\Delta;\Phi\vdash\kappa\overset{\text{norm}}{\equiv>}\Pi_{t_1::\kappa_1}.\mathtt{S}_{[t_1/t]\kappa_2}(\tau\ t_1)}\ \overset{\text{norm}}{\equiv>}\text{-}\Pi$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta;\Phi \vdash \kappa \text{ OK}}{\Delta;\Phi \vdash \kappa \equiv \kappa}\ \texttt{KEquiv-Refl} \qquad \frac{\Delta;\Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Symm}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-Trans}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \overset{*}{\equiv\!>} \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SReduc} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \overset{\text{norm}}{\equiv\!>} \kappa_2}{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}\ \texttt{KEquiv-SNorm}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta;\underline{\Phi, t{::}\kappa_1} \vdash \kappa_3 \equiv \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{KEquiv-}\Pi$$

$$\frac{\Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \equiv \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{KEquiv-SKind}$$

$\boxed{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta;\Phi \vdash \kappa \text{ OK}}{\Delta;\Phi \vdash \texttt{KHole} \lesssim \kappa}\ \texttt{CSK-KHoleL} \qquad \frac{\Delta;\Phi \vdash \kappa \text{ OK}}{\Delta;\Phi \vdash \kappa \lesssim \texttt{KHole}}\ \texttt{CSK-KHoleR}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \text{ OK} \qquad \Delta;\Phi \vdash \kappa \text{ OK}}{\Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{L}$$

$$\frac{\Delta;\Phi \vdash \kappa \text{ OK} \qquad \Delta;\Phi \vdash \mathsf{S}_{\texttt{KHole}}(\tau) \text{ OK}}{\Delta;\Phi \vdash \kappa \lesssim \mathsf{S}_{\texttt{KHole}}(\tau)}\ \texttt{CSK-SKind}_{\texttt{KHole}}\texttt{R}$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-KEquiv} \qquad \frac{\Delta;\Phi \vdash \kappa_1 \equiv \kappa_3 \qquad \Delta;\Phi \vdash \kappa_3 \lesssim \kappa_4 \qquad \Delta;\Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2}\ \texttt{CSK-Normal}$$

$$\frac{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau) \text{ OK}}{\Delta;\Phi \vdash \mathsf{S}_{\kappa}(\tau) \lesssim \kappa}\ \texttt{CSK-SKind} \qquad \frac{\Delta;\Phi \vdash \kappa_3 \lesssim \kappa_1 \qquad \Delta;\underline{\Phi, t{::}\kappa_3} \vdash \kappa_2 \lesssim \kappa_4}{\Delta;\Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4}\ \texttt{CSK-}\Pi$$

$$\frac{\Delta;\Phi \vdash \kappa_1 \lesssim \kappa_2 \qquad \Delta;\Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2}{\Delta;\Phi \vdash \mathsf{S}_{\kappa_1}(\tau_1) \lesssim \mathsf{S}_{\kappa_2}(\tau_2)}\ \texttt{CSK-?}$$

$\boxed{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is provably equivalent to τ_2 at kind κ

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau \overset{\kappa}{\equiv} \tau}\ \texttt{EquivAK-Refl} \qquad \frac{\Delta; \Phi \vdash \tau_2 \overset{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Symm}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_3 \overset{\kappa}{\equiv} \tau_1}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-Trans}$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa_1 \qquad \Delta; \Phi \vdash \kappa_1 \equiv \mathtt{S}_\kappa(\tau_2)}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ \texttt{EquivAK-SKind}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \Pi_{t::\kappa_1}.\kappa_3 \qquad \Delta; \Phi \vdash \tau_2 :: \Pi_{t::\kappa_1}.\kappa_4 \qquad \Delta; \underline{\Phi, t::\kappa_1} \vdash \tau_1\ t \overset{\kappa_2}{\equiv} \tau_2\ t}{\Delta; \Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_2}\ \texttt{EquivAK-}\Pi$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\Pi_{t::\kappa_1}.\kappa_2}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_2 \overset{\kappa_1}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1\ \tau_2 \overset{[\tau_2/t]\kappa_2}{\equiv} \tau_3\ \tau_4}\ \texttt{EquivAK-Ap}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\mathtt{Type}}{\equiv} \tau_3 \qquad \Delta; \Phi \vdash \tau_2 \overset{\mathtt{Type}}{\equiv} \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \overset{\mathtt{Type}}{\equiv} \tau_3 \oplus \tau_4}\ \texttt{EquivAK} - \oplus$$

$$\frac{\Delta; \Phi \vdash \tau_1 \overset{\mathtt{S}_\kappa(\tau)}{\equiv} \tau_2}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (1)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \qquad \Delta; \underline{\Phi, t::\kappa_1} \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau_1 \overset{\Pi_{t::\kappa_1}.\kappa}{\equiv} \lambda t::\kappa_2.\tau_2}\ (2) \qquad \frac{\Delta; \Phi \vdash \tau_1 \overset{\kappa_1}{\equiv} \tau_2 \qquad \Delta; \Phi \vdash \kappa_1 \equiv \kappa}{\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2}\ (3)$$

$\boxed{\Delta; \Phi \vdash \kappa \text{ OK}}$ κ is well formed

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathtt{Type} \text{ OK}}\ \texttt{KWF-Type} \qquad \frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathtt{KHole} \text{ OK}}\ \texttt{KWF-KHole} \qquad \frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \mathtt{S}_\kappa(\tau) \text{ OK}}\ \texttt{KWF-SKind}$$

$$\frac{\Delta; \underline{\Phi, t::\kappa_1} \vdash \kappa_2 \text{ OK}}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \text{ OK}}\ \texttt{KWF-}\Pi$$

$\boxed{\Delta; \Phi \vdash \text{OK}}$ Context is well formed

$$\frac{}{\cdot; \cdot \vdash \text{OK}}\ \texttt{CWF-Nil} \qquad \frac{t \notin \Phi \qquad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \underline{\Phi, t::\kappa} \vdash \text{OK}}\ \texttt{CWF-TypVar} \qquad \frac{\mathrm{u} \notin \Delta \qquad \Delta; \Phi \vdash \kappa \text{ OK}}{\underline{\Delta, \mathrm{u}::\kappa}; \Phi \vdash \text{OK}}\ \texttt{CWF-Hole}$$

Algorithm

Elimination contexts

$$\begin{array}{rcl} \mathcal{E} & ::= & \diamond \\ & | & \mathcal{E}\ \tau \end{array}$$

$\boxed{\Delta;\Phi \triangleright \tau_1 \overset{\kappa}{\equiv} \tau_2}$ τ_1 is equivalent to τ_2 at kind κ

$$\frac{\Delta;\Phi \triangleright \tau_1 \overset{\kappa}{\Longrightarrow} \tau_\omega \qquad \Delta;\Phi \triangleright \tau_2 \overset{\kappa}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \triangleright \tau_1 \overset{\kappa}{\equiv} \tau_2} \ (4)$$

$\boxed{\Delta;\Phi \triangleright \tau \uparrow \kappa}$ path τ has natural kind κ

$$\frac{}{\Delta;\Phi \triangleright \texttt{bse} \uparrow \texttt{Type}} \ (5) \qquad \frac{\Phi = \Phi_1, t{::}\kappa, \Phi_2}{\Delta;\Phi \triangleright t \uparrow \kappa} \ (6) \qquad \frac{}{\Delta;\Phi \triangleright \tau_1 \oplus \tau_2 \uparrow \texttt{Type}} \ (7) \qquad \frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2}{\Delta;\Phi \triangleright (\!|\,|\!)^{\mathrm{u}} \uparrow \kappa} \ (8)$$

$$\frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2}{\Delta;\Phi \triangleright (\!|\tau|\!)^{\mathrm{u}} \uparrow \kappa} \ (9) \qquad \frac{\Delta;\Phi \triangleright \tau_1 \uparrow \kappa \qquad \Delta;\Phi \triangleright \kappa \Longrightarrow \kappa_\omega \qquad \Delta;\Phi \vdash \kappa_\omega \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2}{\Delta;\Phi \triangleright \tau_1\ \tau_2 \uparrow [\tau_2/t]\kappa_2} \ (10)$$

$\boxed{\Delta;\Phi \triangleright \mathcal{E}[\tau]}$ $\mathcal{E}[\tau]$ is a path

$$\frac{}{\Delta;\Phi \triangleright \diamond[\texttt{bse}]} \ (11) \qquad \frac{\Phi = \Phi_1, t{::}\kappa, \Phi_2}{\Delta;\Phi \triangleright \mathcal{E}[t]} \ (12) \qquad \frac{}{\Delta;\Phi \triangleright \diamond[\tau_1 \oplus \tau_2]} \ (13)$$

$$\frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2;}{\Delta;\Phi \triangleright \mathcal{E}[(\!|\,|\!)^{\mathrm{u}}]} \ (14) \qquad \frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2;}{\Delta;\Phi \triangleright \mathcal{E}[(\!|\tau|\!)^{\mathrm{u}}]} \ (15)$$

$\boxed{\Delta;\Phi \triangleright \mathcal{E}[\tau_1] \rightsquigarrow \mathcal{E}[\tau_2]}$ $\mathcal{E}[\tau_1]$ single step weak head reduces to $\mathcal{E}[\tau_2]$

$\boxed{\Delta;\Phi \triangleright \mathcal{E}[\tau] \not\rightsquigarrow}$ $\mathcal{E}[\tau]$ does not weak head reduce

$$\frac{}{\Delta;\Phi \triangleright \mathcal{E}[(\lambda t{::}\kappa.\tau)\ \tau_1] \rightsquigarrow \mathcal{E}[[\tau_1/t]\tau]} \ (16) \qquad \frac{\Delta;\Phi \triangleright \mathcal{E}[\tau] \qquad \Delta;\Phi \triangleright \tau \uparrow \kappa \qquad \Delta;\Phi \triangleright \kappa \Longrightarrow \mathrm{S}_\kappa(\tau_\psi)}{\Delta;\Phi \triangleright \mathcal{E}[\tau] \rightsquigarrow \mathcal{E}[\tau_\psi]} \ (17)$$

$$\frac{\Delta;\Phi \triangleright \mathcal{E}[\tau] \qquad \Delta;\Phi \triangleright \tau \uparrow \kappa \qquad \Delta;\Phi \triangleright \kappa \Longrightarrow \kappa_\omega \qquad \kappa_\omega \neq \mathrm{S}_\kappa(\tau_\psi)}{\Delta;\Phi \triangleright \mathcal{E}[\tau] \not\rightsquigarrow} \ (18) \qquad \frac{}{\Delta;\Phi \triangleright \diamond[\lambda t{::}\kappa.\tau] \not\rightsquigarrow} \ (19)$$

$\boxed{\Delta;\Phi \triangleright \tau \Downarrow \tau_\psi}$ τ weak head normalizes to τ_ψ

$$\frac{\Delta;\Phi \triangleright \tau \rightsquigarrow \tau_\chi \qquad \Delta;\Phi \triangleright \tau_\chi \Downarrow \tau_\psi}{\Delta;\Phi \triangleright \tau \Downarrow \tau_\psi} \ (20) \qquad \frac{\Delta;\Phi \triangleright \tau \not\rightsquigarrow}{\Delta;\Phi \triangleright \tau \Downarrow \tau} \ (21)$$

$\boxed{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega}$ τ normalizes to τ_ω at kind κ

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathtt{Type} \qquad \Delta;\Phi \rhd \tau \Downarrow \tau_\psi \qquad \Delta;\Phi \rhd \tau_\psi \longrightarrow^{\kappa_\psi} \tau_\omega \qquad \Delta;\Phi \rhd \kappa_\psi \lesssim \mathtt{Type}}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (22)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathtt{KHole} \qquad \Delta;\Phi \rhd \tau \Downarrow \tau_\psi \qquad \Delta;\Phi \rhd \tau_\psi \qquad \Delta;\Phi \rhd \tau_\psi \longrightarrow^{\kappa_\psi} \tau_\omega}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (23)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathtt{KHole} \qquad \Delta;\Phi \rhd \tau \Downarrow \lambda t{::}\kappa_1.\tau_1 \qquad \Delta;\Phi \rhd \kappa_1 \Longrightarrow \kappa_\omega \qquad \Delta;\Phi, t{::}\kappa_1 \rhd \tau_1 \overset{\kappa_1}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \lambda t{::}\kappa_\omega.\tau_\omega} \quad (24)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathsf{S}_{\mathtt{Type}}(\tau_s) \qquad \Delta;\Phi \rhd \tau \overset{\mathtt{Type}}{\Longrightarrow} \tau_\omega \qquad \tau_\omega = \tau_s}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (25)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathsf{S}_{\mathtt{KHole}}(\tau_s) \qquad \Delta;\Phi \rhd \tau \overset{\mathtt{KHole}}{\Longrightarrow} \tau_\omega \qquad \tau_\omega = \tau_s}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega} \quad (26)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \Pi_{t::\kappa_{\omega_1}}.\kappa_{\omega_2} \qquad \Delta;\Phi, t_1{::}\kappa_{\omega_1} \rhd \tau\ t_1 \overset{[t_1/t]\kappa_{\omega_2}}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \lambda t_1{::}\kappa_{\omega_1}.\tau_\omega} \quad (27)$$

$\boxed{\Delta;\Phi \rhd \tau_\psi \longrightarrow^{\kappa} \tau_\omega}$ path τ_ψ normalizes to τ_ω with kind κ

$$\frac{}{\Delta;\Phi \rhd \mathtt{bse} \longrightarrow^{\mathtt{Type}} \mathtt{bse}} \quad (28) \qquad \frac{\Phi = \Phi_1, t{::}\kappa, \Phi_2}{\Delta;\Phi \rhd t \longrightarrow^{\kappa} t} \quad (29) \qquad \frac{\Delta;\Phi \rhd \tau_1 \overset{\mathtt{Type}}{\Longrightarrow} \tau_{\omega_1} \qquad \Delta;\Phi \rhd \tau_2 \overset{\mathtt{Type}}{\Longrightarrow} \tau_{\omega_2}}{\Delta;\Phi \rhd \tau_1 \oplus \tau_2 \longrightarrow^{\mathtt{Type}} \tau_{\omega_1} \oplus \tau_{\omega_2}} \quad (30)$$

$$\frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2}{\Delta;\Phi \rhd (\!|\!)^{\mathrm{u}} \longrightarrow^{\kappa} (\!|\!)^{\mathrm{u}}} \quad (31) \qquad \frac{\Delta = \Delta_1, \mathrm{u}{::}\kappa, \Delta_2}{\Delta;\Phi \rhd (\!|\tau|\!)^{\mathrm{u}} \longrightarrow^{\kappa} (\!|\tau|\!)^{\mathrm{u}}} \quad (32)$$

$$\frac{\Delta;\Phi \rhd \tau_1 \longrightarrow^{\kappa} \tau_{\omega_1} \qquad \Delta;\Phi \rhd \kappa \Longrightarrow \kappa_\omega \qquad \Delta;\Phi \vdash \kappa_\omega \overset{\blacktriangleright}{\Pi} \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \rhd \tau_2 \overset{\kappa_1}{\Longrightarrow} \tau_{\omega_2}}{\Delta;\Phi \rhd \tau_1\ \tau_2 \longrightarrow^{[\tau_{\omega_2}/t]\kappa_2} \tau_{\omega_1}\ \tau_{\omega_2}} \quad (33)$$

$\boxed{\Delta;\Phi \rhd \kappa \Longrightarrow \kappa_\omega}$ κ normalizes to κ_ω

$$\frac{}{\Delta;\Phi \rhd \mathtt{Type} \Longrightarrow \mathtt{Type}}\ (34) \qquad \frac{}{\Delta;\Phi \rhd \mathtt{KHole} \Longrightarrow \mathtt{KHole}}\ (35)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathtt{Type} \qquad \Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \mathsf{S}_{\mathtt{Type}}(\tau_\omega)}\ (36) \qquad \frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathtt{KHole} \qquad \Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \mathsf{S}_{\mathtt{KHole}}(\tau_\omega)}\ (37)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \mathsf{S}_{\kappa_1}(\tau_1) \qquad \Delta;\Phi \rhd \mathsf{S}_{\kappa_1}(\tau_1) \Longrightarrow \kappa_\omega}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \kappa_\omega}\ (38)$$

$$\frac{\Delta;\Phi \rhd \kappa \Longrightarrow \Pi_{t::\kappa_1}.\kappa_2 \qquad \Delta;\Phi \rhd \tau \overset{\kappa}{\Longrightarrow} \tau_\omega \qquad \Delta;\Phi, t_1::\kappa_1 \rhd \tau_\omega\ t_1 \overset{[t_1/t]\kappa_2}{\Longrightarrow} \tau_{\omega_1}}{\Delta;\Phi \rhd \mathsf{S}_\kappa(\tau) \Longrightarrow \Pi_{t_1::\kappa_1}.\mathsf{S}_{[t_1/t]\kappa_2}(\tau_{\omega_1})}\ (39)$$

$$\frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \kappa_{\omega_1} \qquad \Delta;\Phi, t::\kappa_{\omega_1} \rhd \kappa_2 \Longrightarrow \kappa_{\omega_2}}{\Delta;\Phi \rhd \Pi_{t::\kappa_1}.\kappa_2 \Longrightarrow \Pi_{t::\kappa_{\omega_1}}.\omega_2}\ (40)$$

$\boxed{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}$ κ_1 is a consistent subkind of κ_2

$$\frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \mathtt{KHole}}{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}\ (41) \qquad \frac{\Delta;\Phi \rhd \kappa_2 \Longrightarrow \mathtt{KHole}}{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}\ (42) \qquad \frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \mathsf{S}_{\mathtt{KHole}}(\tau)}{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}\ (43)$$

$$\frac{\Delta;\Phi \rhd \kappa_2 \Longrightarrow \mathsf{S}_{\mathtt{KHole}}(\tau)}{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}\ (44) \qquad \frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \mathsf{S}_{\mathtt{Type}}(\tau) \qquad \Delta;\Phi \rhd \kappa_2 \Longrightarrow \mathtt{Type}}{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}\ (45)$$

$$\frac{\begin{array}{c}\Delta;\Phi \rhd \kappa_1 \Longrightarrow \Pi_{t_1::\kappa_{\omega_1}}.\kappa_{\omega_2} \\ \Delta;\Phi \rhd \kappa_2 \Longrightarrow \Pi_{t_2::\kappa_{\omega_3}}.\kappa_{\omega_4} \qquad \Delta;\Phi \rhd \kappa_{\omega_3} \lesssim \kappa_{\omega_1} \qquad \Delta;\Phi, t_3::\kappa_{\omega_3} \rhd [t_3/t_1]\kappa_{\omega_2} \lesssim [t_3/t_2]\kappa_{\omega_4}\end{array}}{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}\ (46)$$

$$\frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \kappa_{\omega_1} \qquad \Delta;\Phi \rhd \kappa_2 \Longrightarrow \kappa_{\omega_2} \qquad \Delta;\Phi \rhd \kappa_{\omega_1} \equiv \kappa_{\omega_2}}{\Delta;\Phi \rhd \kappa_1 \lesssim \kappa_2}\ (47)$$

$\boxed{\Delta;\Phi \rhd \kappa_1 \equiv \kappa_2}$ κ_1 is equivalent to κ_2

$$\frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \mathsf{S}_{\kappa_{\omega_1}}(\tau_1) \qquad \Delta;\Phi \rhd \kappa_2 \Longrightarrow \mathsf{S}_{\kappa_{\omega_2}}(\tau_2) \qquad \Delta;\Phi \rhd \kappa_{\omega_1} \equiv \kappa_{\omega_2} \qquad \Delta;\Phi \rhd \tau_1 \overset{\kappa_{\omega_1}}{\equiv} \tau_2}{\Delta;\Phi \rhd \kappa_1 \equiv \kappa_2}\ (48)$$

$$\frac{\begin{array}{c}\Delta;\Phi \rhd \kappa_1 \Longrightarrow \Pi_{t_1::\kappa_{\omega_1}}.\kappa_{\omega_2} \\ \Delta;\Phi \rhd \kappa_2 \Longrightarrow \Pi_{t_2::\kappa_{\omega_3}}.\kappa_{\omega_4} \qquad \Delta;\Phi \rhd \kappa_{\omega_3} \equiv \kappa_{\omega_1} \qquad \Delta;\Phi, t_3::\kappa_{\omega_3} \rhd [t_3/t_1]\kappa_{\omega_2} \equiv [t_3/t_2]\kappa_{\omega_4}\end{array}}{\Delta;\Phi \rhd \kappa_1 \equiv \kappa_2}\ (49)$$

$$\frac{\Delta;\Phi \rhd \kappa_1 \Longrightarrow \kappa_{\omega_1} \qquad \Delta;\Phi \rhd \kappa_2 \Longrightarrow \kappa_{\omega_2} \qquad \kappa_{\omega_1} = \kappa_{\omega_2}}{\Delta;\Phi \rhd \kappa_1 \equiv \kappa_2}\ (50)$$

Metatheory

subderivation preserving inferences:

- premiss
- COK (Context OK)
- PoS (premiss of subderivation)

Lemma 1 (COK). If $\Delta; \Phi \vdash \mathcal{J}$, then $\Delta; \Phi \vdash \mathrm{OK}$ in a subderivation (where $\Delta; \Phi \vdash \mathcal{J} \neq \Delta; \Phi \vdash \mathrm{OK}$)

Proof. By induction on derivations.
No interesting cases. □

Lemma 2 (Exchange).
If $\Delta; \Phi_1, t_{L1}::\kappa_{L1}, t_{L2}::\kappa_{L2}, \Phi_2 \vdash \mathcal{J}$ and $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash \mathrm{OK}$, then $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

Proof. By induction on derivations.
No interesting cases.
(Only rules with Φ extended in the conclusion are interesting, which is only CWF-TypVar, but when $\mathcal{J}$ is CWF, Exchange is identity) □

Corollary 3 (Marked-Exchange).
If $\Delta; \underline{\underline{\Phi, t_{L1}::\kappa_{L1}}, t_{L2}::\kappa_{L2}} \vdash \mathcal{J}$ and $\Delta; \underline{\underline{\Phi, t_{L2}::\kappa_{L2}}, t_{L1}::\kappa_{L1}} \vdash \mathrm{OK}$, then $\Delta; \underline{\underline{\Phi, t_{L2}::\kappa_{L2}}, t_{L1}::\kappa_{L1}} \vdash \mathcal{J}$

Proof. Exchange when $\Phi_2 = \cdot$ □

Lemma 4 (Weakening).
If $\Delta; \Phi \vdash \mathcal{J}$ and $\Delta; \underline{\Phi, t_L::\kappa_L} \vdash \mathrm{OK}$ and $t_L \notin \mathcal{J}$ and $\forall t \in \kappa_L, t \notin \mathcal{J}$, then $\Delta; \underline{\Phi, t_L::\kappa_L} \vdash \mathcal{J}$

Proof. see addendum □

Lemma 5 (K-Substitution).
If $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$ and $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \tau_{L2}::\kappa_{L2}$, then $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2}::[\tau_{L1}/t_L]\kappa_{L2}$
(induction on $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \tau_{L2}::\kappa_{L2}$)

Lemma 6 (PK-Substitution). If $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$ and $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \tau_{L2} ::> \kappa_{L2}$ and $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2} ::> \kappa_{L3}$, then $\Delta; \Phi \vdash [\tau_{L2}/t_L]\kappa_{L2} \equiv \kappa_{L3}$

Lemma 7 (OK-Substitution).
If $\Delta; \Phi \vdash \tau_L::\kappa_{L1}$ and $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \kappa_{L2}$ OK, then $\Delta; \Phi \vdash [\tau_L/t_L]\kappa_{L2}$ OK
(induction on $\Delta; \underline{\Phi, t_L::\kappa_{L1}} \vdash \kappa_{L2}$ OK)

Theorem 8 (OK-PK). If $\Delta; \Phi \vdash \tau ::> \kappa$, then $\Delta; \Phi \vdash \kappa$ OK

Theorem 9 (OK-WFaK). If $\Delta; \Phi \vdash \tau::\kappa$, then $\Delta; \Phi \vdash \kappa$ OK

Theorem 10 (OK-MatchPi). If $\Delta; \Phi \vdash \kappa \overset{\blacktriangleright}{\underset{\Pi}{}} \Pi_{t::\kappa_1}.\kappa_2$, then $\Delta; \Phi \vdash \kappa$ OK and $\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2$ OK

Theorem 11 (OK-KEquiv). If $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$, then $\Delta; \Phi \vdash \kappa_1$ OK and $\Delta; \Phi \vdash \kappa_2$ OK

Theorem 12 (OK-CSK). If $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$, then $\Delta; \Phi \vdash \kappa_1$ OK and $\Delta; \Phi \vdash \kappa_2$ OK

Theorem 13 (OK-EquivAK). If $\Delta; \Phi \vdash \tau_1 \overset{\kappa}{\equiv} \tau_2$, then $\Delta; \Phi \vdash \tau_1::\kappa$ and $\Delta; \Phi \vdash \tau_2::\kappa$ and $\Delta; \Phi \vdash \kappa$ OK

Proof. see addendum □

Proof.

Weakening

By induction on derivations.

Note: When applying Weakening in the induction, check that the left premiss is always a subderivation, and check variable exclusion conditions are satisfied (usually checked elsewhere in the derivation).

$$\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\tau::>\kappa_2}\,\text{premiss}\quad\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{IH}}{t_L\notin\Phi}\,\text{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\text{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t_L\neq t}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\text{IH}}{t_L\notin\kappa_1}}{t_L\notin\Phi,t::\kappa_1}\quad\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{IH}}{\Delta;\Phi\vdash\kappa_L\ \mathrm{OK}}\,\text{PoS}\quad\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\tau::>\kappa_2}\,\text{premiss}}{\Delta;\Phi,t::\kappa_1\vdash\mathrm{OK}}\,\text{COK}}{\Delta;\Phi,t::\kappa_1\vdash\kappa_L\ \mathrm{OK}}\,\text{Weakening}}{\Delta;\Phi,t::\kappa_1,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{CWF-TypVar}}{\Delta;\Phi,t::\kappa_1,t_L::\kappa_L\vdash\tau::>\kappa_2}\,\text{Weakening}\quad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\tau::>\kappa_2}\,\text{premiss}}{\Delta;\Phi,t::\kappa_1\vdash\mathrm{OK}}\,\text{COK}}{t\notin\Phi}\,\text{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\text{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\neq t_L}\quad\dfrac{\dfrac{}{\forall\dot{t}\in\kappa_L,\dot{t}\notin\mathcal{J}}\,\text{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\notin\kappa_L}}{t\notin\Phi,t_L::\kappa_L}\quad\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\tau::>\kappa_2}\,\text{premiss}}{\Delta;\Phi,t::\kappa_1\vdash\mathrm{OK}}\,\text{COK}}{\Delta;\Phi\vdash\kappa_1\ \mathrm{OK}}\,\text{PoS}\quad\dfrac{}{\Delta;\Phi,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{IH}}{\Delta;\Phi,t_L::\kappa_L\vdash\kappa_1\ \mathrm{OK}}\,\text{Weakening}}{\Delta;\Phi,t_L::\kappa_L,t::\kappa_1\vdash\mathrm{OK}}\,\text{CWF-TypVar}}{\dfrac{\Delta;\Phi,t_L::\kappa_L,t::\kappa_1\vdash\tau::>\kappa_2}{\Delta;\Phi,t_L::\kappa_L\vdash\lambda t::\kappa_1.\tau::>\mathtt{S}_{\Pi_{t::\kappa_1}.\kappa_2}(\lambda t::\kappa_1.\tau)}\,\text{PK-}\lambda}\,\text{Marked-Exchange}$$

$$\dfrac{\dfrac{\dfrac{}{\Delta;\Phi\vdash\kappa_1\equiv\kappa_2}\,\text{premiss}\quad\dfrac{}{\Delta;\Phi,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{IH}}{\Delta;\Phi,t_L::\kappa_L\vdash\kappa_1\equiv\kappa_2}\,\text{Weakening}\quad\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\tau::>\kappa_2}\,\text{premiss}\quad\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{IH}}{t_L\notin\Phi}\,\text{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\text{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t_L\neq t}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\text{IH}}{t_L\notin\kappa_1}}{t_L\notin\Phi,t::\kappa_1}\quad\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{IH}}{\Delta;\Phi\vdash\kappa_L\ \mathrm{OK}}\,\text{PoS}\quad\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\text{premiss}}{\Delta;\Phi,t::\kappa_1\vdash\mathrm{OK}}\,\text{COK}}{\Delta;\Phi,t::\kappa_1\vdash\kappa_L\ \mathrm{OK}}\,\text{Weakening}}{\Delta;\Phi,t::\kappa_1,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{CWF-TypVar}}{\Delta;\Phi,t::\kappa_1,t_L::\kappa_L\vdash\tau::>\kappa_2}\,\text{Weakening}\quad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\text{premiss}}{\Delta;\Phi,t::\kappa_1\vdash\mathrm{OK}}\,\text{COK}}{t\notin\Phi}\,\text{PoS}\quad\dfrac{\dfrac{}{t_L\notin\mathcal{J}}\,\text{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\neq t_L}\quad\dfrac{\dfrac{}{\forall\dot{t}\in\kappa_L,\dot{t}\notin\mathcal{J}}\,\text{IH}\quad\dfrac{}{t\in\mathcal{J}}}{t\notin\kappa_L}}{t\notin\Phi,t_L::\kappa_L}\quad\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta;\Phi,t::\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\text{premiss}}{\Delta;\Phi,t::\kappa_1\vdash\mathrm{OK}}\,\text{COK}}{\Delta;\Phi\vdash\kappa_1\ \mathrm{OK}}\,\text{PoS}\quad\dfrac{}{\Delta;\Phi,t_L::\kappa_L\vdash\mathrm{OK}}\,\text{IH}}{\Delta;\Phi,t_L::\kappa_L\vdash\kappa_1\ \mathrm{OK}}\,\text{Weakening}}{\Delta;\Phi,t_L::\kappa_L,t::\kappa_1\vdash\mathrm{OK}}\,\text{CWF-TypVar}}{\Delta;\Phi,t_L::\kappa_L,t::\kappa_1\vdash\kappa_3\equiv\kappa_4}\,\text{Marked-Exchange}}{\Delta;\Phi,t_L::\kappa_L\vdash\Pi_{t::\kappa_1}.\kappa_2\equiv\Pi_{t::\kappa_3}.\kappa_4}\,\text{KEquiv-}\Pi$$

$$\dfrac{}{sim}\,\text{CSK-}\Pi\qquad\dfrac{}{sim}\,\text{EquivAK-}\Pi\qquad\dfrac{}{sim}\,\text{KWF-}\Pi$$

O?K-.*

By simultaneous induction on derivations.

The interesting cases per theorem:

K-Substitution   by type size??

OK-Substitution

OK-PK

$$\dfrac{\dfrac{\dfrac{}{\Delta;\Phi\vdash\mathtt{bse}::>\mathtt{S}_{\mathtt{Type}}(\mathtt{bse})}\,\text{premiss}}{\Delta;\Phi\vdash\mathtt{bse}::\mathtt{Type}}\,\text{WFaK-1}}{\Delta;\Phi\vdash\mathtt{S}_{\mathtt{Type}}(\mathtt{bse})\ \mathrm{OK}}\,\text{KWF-SKind}\qquad\dfrac{}{\Delta;\Phi\vdash[\tau_2/t]\kappa_2\ \mathrm{OK}}\,\text{OK-Substitution}$$

OK-WFaK

□

Definition 1 (Singleton Depth).

$$SSize : \text{“}\{\kappa\}\text{''} \to \mathbb{N}$$

$$SSize(\kappa_x) = \begin{cases} SSize(\kappa) + 1 & \text{if } \kappa_x = \mathtt{S}_\kappa(\tau) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 14 ($\overset{*}{\equiv\!>}$-diminution). If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L1}$, then $SSize(\kappa_L) > SSize(\kappa_{L1})$

Proof. By induction on derivations (and transitivity of $>$ on $\mathbb{N}$) □

Lemma 15 ($\overset{*}{\equiv\!>}$-n+1-nicity). If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L2}$ where $SSize(\kappa_L) = n{+}1$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$

Proof. By $\overset{*}{\equiv\!>}$-diminution, $\overset{*}{\equiv\!>}$-Trans cannot be the last inference of a derivation of $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L1}$ since $SSize(\kappa_1) \geq SSize(\kappa_3) + 2$ (in $\overset{*}{\equiv\!>}$-Trans). Thus, $\overset{*}{\equiv\!>}$-1 must have been the last inference. Similarly for $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L2}$, thus $\kappa_{L1} = \kappa_{L2}$ □

Lemma 16 ($\overset{*}{\equiv\!>}$-stepwise). If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L1}$ where $SSize(\kappa_L) = m$ and $SSize(\kappa_{L1}) = n$ and $m > n + 1$, then the derivation must contain subderivations of each singleton depth inbetween

Proof. More precisely this says, where $m > n$ by $\overset{*}{\equiv\!>}$-diminution, the derivation must contain subderivations of each $\Delta; \Phi \vdash \kappa_i \overset{*}{\equiv\!>} \kappa_j$ where $m \geq i > j \geq n$, $SSize(\kappa_k) = k$ when $m \geq k \geq n$, $\kappa_m = \kappa_L$, $\kappa_n = \kappa_{L1}$.
By induction on derivations (base case is where $m = n + 2$, which necessitates a last inference of $\overset{*}{\equiv\!>}$-Trans. Each premiss must have SSize difference of 1, fulfilling hypothesis) □

Lemma 17 ($\overset{*}{\equiv\!>}$-m+n-nicity). If $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{*}{\equiv\!>} \kappa_{L2}$ where $SSize(\kappa_L) = m{+}n$ and $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$, then $\kappa_{L1} = \kappa_{L2}$

Proof. By $\overset{*}{\equiv\!>}$-stepwise and $\overset{*}{\equiv\!>}$-$n + 1$-nicity when $m > n + 1$.
By $\overset{*}{\equiv\!>}$-$n + 1$-nicity when $m = n + 1$.
No other cases by $\overset{*}{\equiv\!>}$-diminution. □

Theorem 18 ($\overset{\text{norm}}{\equiv\!>}$-Unicity). If $\Delta; \Phi \vdash \kappa_L \overset{\text{norm}}{\equiv\!>} \kappa_{L1}$ and $\Delta; \Phi \vdash \kappa_L \overset{\text{norm}}{\equiv\!>} \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$

Proof. (this is a really quick sketch)
All $\overset{\text{norm}}{\equiv\!>}$ rules have $\overset{*}{\equiv\!>}$ premiss with rhs singleton depth 1. By $\overset{*}{\equiv\!>}$-$m + n$-nicity, where $n = 1$. □

Theorem 19 ($\overset{\blacktriangleright}{\Pi}$-Unicity). If $\Delta; \Phi \vdash \tau_L \overset{\blacktriangleright}{\Pi} \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L \overset{\blacktriangleright}{\Pi} \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$

Proof. (this is a really quick sketch)
By unicity of $\overset{\text{norm}}{\equiv\!>}$. □

Theorem 20 (PK-Unicity). If $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$ and $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$, then $\kappa_{L1} = \kappa_{L2}$

Proof. (this is a really quick sketch)
As PK is syntax directed, proof is by inspection for all rules except PK-λ (variables in contexts are unique– see context rules), which is by induction on derivations, and PK-Ap, which requires of unicity of $\overset{\blacktriangleright}{\Pi}$ (above theorem). □

Theorem 21 (PK-Principality). If $\Delta; \Phi \vdash \tau ::> \kappa_1$ and $\Delta; \Phi \vdash \tau :: \kappa_2$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$

Proof. From definition of $\Delta; \Phi \vdash \tau :: \kappa$ and CSK-SKind □

Theorem 22 (why is this here?). If $\Delta; \Phi \vdash \kappa_1 \lesssim \mathtt{S}_{\kappa_2}(\tau)$, then $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$

Prop 23 (APN-1). If $\Delta; \Phi \rhd \tau_\psi \longrightarrow^{\kappa} \tau_\omega$, then $\Delta; \Phi \rhd \tau_\psi$

Proof. By inspection. □

Prop 24 (AKN-1). If $\Delta; \Phi \triangleright \kappa \Longrightarrow \mathtt{S}_{\kappa_\omega}(\tau_\omega)$, then $\Delta; \Phi \triangleright \tau_\omega \overset{\kappa}{\Longrightarrow} \tau_\omega$

Proof. Sim ind with some other simple props probably $\square$

Prop 25 (ATN-1). If $\Delta; \Phi \triangleright \kappa \Longrightarrow \kappa_\omega$ and $\Delta; \Phi \triangleright \tau \overset{\kappa}{\Longrightarrow} \tau_\omega$, then $\Delta; \Phi \triangleright \tau \overset{\kappa_\omega}{\Longrightarrow} \tau_\omega$

Proof. Sim ind with some other simple props probably $\square$

Elaboration

todo