Algebraic Data Types for Hazel

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1 Syntax

$$
\begin{array}{llcl}
\mathsf{HTyp} & \tau & ::= & \varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid (\!|\,|\!) \mid (\!|\alpha|\!) \\
\mathsf{HTypPat} & \pi & ::= & \alpha \mid (\!|\,|\!) \\
\mathsf{HExp} & e & ::= & \varnothing \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \mathtt{inj}_C(e) \mid \mathtt{roll}(e) \mid \mathtt{unroll}(e) \\
 & & & \mid (\!|\,|\!)^u \mid (\!|e|\!)^u \mid (\!|e|\!)^{u\blacktriangleright} \\
\mathsf{IHExp} & d & ::= & \varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \mathtt{inj}^{\tau}_C(d) \mid \mathtt{roll}^{\mu\alpha.\tau}(d) \mid \mathtt{unroll}(d) \\
 & & & \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow (\!|\,|\!) \not\Rightarrow \tau \rangle \mid (\!|\,|\!)^u_\sigma \mid (\!|d|\!)^u_\sigma \mid (\!|d|\!)^{u\blacktriangleright}_\sigma \\
\mathsf{HTag} & C & ::= & \mathbf{C} \mid (\!|\,|\!)^u \mid (\!|\mathbf{C}|\!)^u
\end{array}
$$

1.1 Context Extension

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = (\!|\,|\!) \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{lcll}
[\tau/(\!|\,|\!)]\tau' & = & \tau' & \\
[\tau/\alpha]\varnothing & = & \varnothing & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu(\!|\,|\!).\tau_2 & = & \mu(\!|\,|\!).[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]{+}\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha](\!|\,|\!) & = & (\!|\,|\!) & \\
[\alpha'/\alpha](\!|\alpha|\!) & = & (\!|\alpha'|\!) & \\
[\alpha'/\alpha](\!|\alpha'|\!) & = & (\!|\alpha'|\!) & \text{when } \alpha \neq \alpha'
\end{array}
$$

$\boxed{\Theta \vdash \tau \text{ valid}}$ τ is a valid type

$$
\frac{}{\Theta \vdash \varnothing \ \mathsf{valid}}\ \textsc{TVUnit}
\qquad
\frac{\Theta \vdash \tau_1 \ \mathsf{valid} \qquad \Theta \vdash \tau_2 \ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2 \ \mathsf{valid}}\ \textsc{TVArr}
\qquad
\frac{\alpha \in \Theta}{\Theta \vdash \alpha \ \mathsf{valid}}\ \textsc{TVVar}
\qquad
\frac{\Theta, \pi \vdash \tau \ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau \ \mathsf{valid}}\ \textsc{TVRec}
\qquad
\frac{\{\Theta \vdash \tau_i \ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \ \mathsf{valid}}\ \textsc{TVSum}
$$

$$
\frac{}{\Theta \vdash (\!|\,|\!) \ \mathsf{valid}}\ \textsc{TVEHole}
\qquad
\frac{\alpha \notin \Theta}{\Theta \vdash (\!|\alpha|\!) \ \mathsf{valid}}\ \textsc{TVNEHole}
$$

$\boxed{\tau \sim \tau'}$ τ and τ' are consistent

$$\frac{}{\tau \sim \tau}\ \text{TCRefl} \qquad \frac{}{(\!|\,|\!) \sim \tau}\ \text{TCEHole1} \qquad \frac{}{\tau \sim (\!|\,|\!)}\ \text{TCEHole2} \qquad \frac{}{(\!|\alpha|\!) \sim \tau}\ \text{TCNEHole1} \qquad \frac{}{\tau \sim (\!|\alpha|\!)}\ \text{TCNEHole2}$$

$$\frac{\tau_1 \sim \tau_1' \qquad \tau_2 \sim \tau_2'}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}\ \text{TCArr}$$

$$\frac{\tau \sim \tau'}{\mu\pi.\tau \sim \mu\pi.\tau'}\ \text{TCRec} \qquad \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu(\!|\,|\!).\tau \sim \mu\alpha.\tau'}\ \text{TCRecHole1} \qquad \frac{\alpha \notin \mathsf{FV}(\tau) \qquad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu(\!|\,|\!).\tau'}\ \text{TCRecHole2}$$

$$\frac{\{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}\ \text{TCSum}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If* $\Gamma \vdash e \Rightarrow \tau$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

Theorem 2 (Consistency Preserves Validity). *If* $\Theta \vdash \tau\ \mathsf{valid}$ *and* $\tau \sim \tau'$ *then* $\Theta \vdash \tau'\ \mathsf{valid}$.

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$ τ has matched arrow type $\tau_1 \to \tau_2$

$$\frac{}{(\!|\,|\!) \blacktriangleright_{\to} (\!|\,|\!) \to (\!|\,|\!)}\ \text{MAHole} \qquad \frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2}\ \text{MAArr}$$

$\boxed{\tau \blacktriangleright_{\mu} \mu\pi.\tau'}$ τ has matched recursive type $\mu\pi.\tau'$

$$\frac{}{\mu\pi.\tau \blacktriangleright_{\mu} \mu\pi.\tau}\ \text{MRRec} \qquad \frac{}{(\!|\,|\!) \blacktriangleright_{\mu} \mu(\!|\,|\!).(\!|\,|\!)}\ \text{MRHole}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ e synthesizes type τ

$$\frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing}\ \text{SUnit} \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau}\ \text{SVar} \qquad \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\,|\!)}\ \text{SVarFree} \qquad \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}\ \text{SLam}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_{\to} \tau_2 \to \tau \qquad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau}\ \text{SApp} \qquad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \nsim (\!|\,|\!) \to (\!|\,|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\,|\!)}\ \text{SAppNotArr}$$

$$\frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau}\ \text{SAsc} \qquad \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\texttt{roll}(e)|\!)^u \Rightarrow \mu(\!|\,|\!).(\!|\,|\!)}\ \text{SRollErr} \qquad \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \blacktriangleright_{\mu} \mu\pi.\tau'}{\Gamma \vdash \texttt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}\ \text{SUnroll}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \nsim \mu(\!|\,|\!).(\!|\,|\!)}{\Gamma \vdash \texttt{unroll}\big((\!|e|\!)^{u\blacktriangleright}\big) \Rightarrow (\!|\,|\!)}\ \text{SUnrollNotRec} \qquad \frac{\Gamma \vdash e \Leftarrow (\!|\,|\!)}{\Gamma \vdash (\!|\texttt{inj}_C(e)|\!)^u \Rightarrow (\!|\,|\!)}\ \text{SInjErr} \qquad \frac{}{\Gamma \vdash (\!|\,|\!)^u \Rightarrow (\!|\,|\!)}\ \text{SEHole} \qquad \frac{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!|e|\!)^u \Rightarrow (\!|\,|\!)}\ \text{SNEHole}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

$$\textsc{ARoll}\ \frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \texttt{roll}(e) \Leftarrow \tau}$$

$$\textsc{ARollNotRec}\ \frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\texttt{roll}(e)\rrparenthesis^u \Leftarrow \tau}$$

$$\textsc{AInjHole}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow \llparenthesis\rrparenthesis}$$

$$\textsc{AInj}\ \frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \texttt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjTagErr}\ \frac{C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \texttt{inj}_C(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjUnexpectedArg}\ \frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis}{\Gamma \vdash \llparenthesis\texttt{inj}_{C_j}(e)\rrparenthesis^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{AInjExpectedArg}\ \frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash \llparenthesis\texttt{inj}_{C_j}(\varnothing)\rrparenthesis^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

$$\textsc{ASubsume}\ \frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

2.2 Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity)**.** *If* $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ *then* $\emptyset \vdash \tau\ \mathsf{valid}$.

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

$$\textsc{ESUnit}\ \frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \rightsquigarrow \varnothing \dashv \emptyset}$$

$$\textsc{ESVar}\ \frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

$$\textsc{ESVarFree}\ \frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash \llparenthesis x\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis x\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESLam}\ \frac{\emptyset \vdash \tau\ \mathsf{valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

$$\textsc{ESApp}\ \frac{\begin{array}{c}\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \\ \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \\ \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2\end{array}}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle\tau_1' \Rightarrow \tau_1\rangle)(d_2\langle\tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

$$\textsc{ESAppNotArr}\ \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \qquad \Gamma \vdash e_2 \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash \llparenthesis e_1\rrparenthesis^{u\blacktriangleright}(e_2) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d_1\rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle\tau_2' \Rightarrow \llparenthesis\rrparenthesis\rangle) \dashv \Delta_1 \cup \Delta_2, u :: \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESAsc}\ \frac{\begin{array}{c}\emptyset \vdash \tau\ \mathsf{valid} \\ \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta\end{array}}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle\tau' \Rightarrow \tau\rangle \dashv \Delta}$$

$$\textsc{ESRollErr}\ \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash \llparenthesis\texttt{roll}(e)\rrparenthesis^u \Rightarrow \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis\texttt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d\langle\tau \Rightarrow \llparenthesis\rrparenthesis\rangle)\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESUnroll}\ \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \texttt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \texttt{unroll}(d\langle\tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}$$

$$\textsc{ESUnrollNotRec}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}{\Gamma \vdash \mathtt{unroll}\left(\llparenthesis e \rrparenthesis^{u\blacktriangleright}\right) \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \mathtt{unroll}\left(\llparenthesis d \rrparenthesis^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}\right) \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESInjErr}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \llparenthesis \mathtt{inj}_C(e) \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis \mathtt{inj}^{\tau'}_C(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESEHole}\quad \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{ESNEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \rightsquigarrow \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \llparenthesis\rrparenthesis[\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

$$\textsc{EARoll}\quad \frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau \rightsquigarrow \mathtt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rangle) : \mu\pi.\tau' \dashv \Delta}$$

$$\textsc{EARollNotRec}\quad \frac{\tau \nsim \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash \llparenthesis \mathtt{roll}(e) \rrparenthesis^u \Leftarrow \tau \rightsquigarrow \llparenthesis \mathtt{roll}^{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \dashv \Delta, u :: \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis[\Gamma]}$$

$$\textsc{EAInjHole}\quad \frac{\Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow \mathtt{inj}^{\tau'}_C(d) : \tau' \dashv \Delta}$$

$$\textsc{EAInj}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \rightsquigarrow d : \tau'_j \dashv \Delta}{\Gamma \vdash \mathtt{inj}_{C_j}(e) \Leftarrow \tau \rightsquigarrow \mathtt{inj}^{\tau}_{C_j}\left(d\langle \tau'_j \Rightarrow \tau_j \rangle\right) : \tau \dashv \Delta}$$

$$\textsc{EAInjTagErr}\quad \frac{C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \cup \{C(\tau)\}\right\}}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \rightsquigarrow \mathtt{inj}^{\tau'}_C(d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle) : +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \dashv \Delta}$$

$$\textsc{EAInjUnexpectedArg}\quad \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow \llparenthesis\rrparenthesis \rightsquigarrow d : \tau_j \dashv \Delta \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\right\}\end{array}}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C_j}(e) \rrparenthesis^u \Leftarrow \tau \rightsquigarrow \llparenthesis \mathtt{inj}^{\tau'}_{C_j}(d) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EAInjExpectedArg}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\left\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\right\}}{\Gamma \vdash \llparenthesis \mathtt{inj}_{C_j}(\varnothing) \rrparenthesis^u \Leftarrow \tau \rightsquigarrow \llparenthesis \mathtt{inj}^{\tau'}_{C_j}(\varnothing) \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau[\Gamma]}$$

$$\textsc{EASubsume}\quad \frac{e \neq \llparenthesis\rrparenthesis^u \qquad e \neq \llparenthesis e' \rrparenthesis^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

$$\textsc{EAEHole}\quad \frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Leftarrow \tau \rightsquigarrow \llparenthesis\rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

$$\textsc{EANEHole}\quad \frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Leftarrow \tau \rightsquigarrow \llparenthesis d \rrparenthesis^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau[\Gamma]}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

$$\text{TAUnit}\quad \frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

$$\text{TAVar}\quad \frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

$$\text{TALam}\quad \frac{\emptyset \vdash \tau \ \mathsf{valid} \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}$$

$$\text{TAApp}\quad \frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

$$\text{TARoll}\quad \frac{\emptyset \vdash \mu\pi.\tau \ \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}$$

$$\text{TAUnroll}\quad \frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

$$\text{TAInj}\quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \mathtt{inj}^{\tau}_{C_j}(d) : \tau}$$

$$\text{TAEHole}\quad \frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis\rrparenthesis^{u}_{\sigma} : \tau}$$

$$\text{TANEHole}\quad \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^{u}_{\sigma} : \tau}$$

$$\text{TAMHole}\quad \frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash \llparenthesis d \rrparenthesis^{u\blacktriangleright}_{\sigma} : \tau}$$

$$\text{TACast}\quad \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

$$\text{TAFailedCast}\quad \frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle : \tau_2}$$

3 Dynamic Semantics

$\boxed{\tau \text{ ground}}$ τ is a ground type

$$\text{GArr}\quad \frac{}{\llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \text{ ground}}$$

$$\text{GRec}\quad \frac{}{\mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis \text{ ground}}$$

$$\text{GSum}\quad \frac{\{\tau_i = \varnothing \vee \tau_i = \llparenthesis\rrparenthesis\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \text{ ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

$$\text{MGArr}\quad \frac{\tau_1 \to \tau_2 \neq \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}$$

$$\text{MGRec}\quad \frac{\tau \neq \llparenthesis\rrparenthesis}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu\llparenthesis\rrparenthesis.\llparenthesis\rrparenthesis}$$

$$\text{MGSum}\quad \frac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \qquad \{(\tau_i = \varnothing \implies \tau_i' = \varnothing) \wedge (\tau_i \neq \varnothing \implies \tau_i' = \llparenthesis\rrparenthesis)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \text{ final}}$ d is final

$$\text{FBoxedVal}\quad \frac{d \text{ boxedval}}{d \text{ final}}$$

$$\text{FIndet}\quad \frac{d \text{ indet}}{d \text{ final}}$$

$\boxed{d \text{ val}}$ d is a value

$$\frac{}{\varnothing \text{ val}}\ \textsc{VUnit} \qquad \frac{}{\lambda x{:}\tau.d \text{ val}}\ \textsc{VLam} \qquad \frac{d \text{ val}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ val}}\ \textsc{VRoll} \qquad \frac{d \text{ val}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ val}}\ \textsc{VInj}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

$$\frac{d \text{ val}}{d \text{ boxedval}}\ \textsc{BVVal} \qquad \frac{d \text{ boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}}\ \textsc{BVRoll} \qquad \frac{d \text{ boxedval}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ boxedval}}\ \textsc{BVInj}$$

$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}\ \textsc{BVArrCast}$$

$$\frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}}\ \textsc{BVRecCast} \qquad \frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \qquad d \text{ boxedval}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}\ \textsc{BVSumCast} \qquad \frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\!) \rangle \text{ boxedval}}\ \textsc{BVHoleCast}$$

$\boxed{d \text{ indet}}$ d is indeterminate

$$\frac{}{(\!|\!)^u_\sigma \text{ indet}}\ \textsc{IEHole} \qquad \frac{d \text{ final}}{(\!|d|\!)^u_\sigma \text{ indet}}\ \textsc{INEHole} \qquad \frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \qquad d_1 \text{ indet} \qquad d_2 \text{ final}}{d_1(d_2) \text{ indet}}\ \textsc{IApp} \qquad \frac{d \text{ indet}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ indet}}\ \textsc{IRoll}$$

$$\frac{d \text{ indet}}{\texttt{unroll}(d) \text{ indet}}\ \textsc{IUnroll} \qquad \frac{d \text{ indet}}{\texttt{inj}^{\tau}_{\mathbf{C}}(d) \text{ indet}}\ \textsc{IInj} \qquad \frac{C \neq \mathbf{C} \qquad d \text{ final}}{\texttt{inj}^{\tau}_{C}(d) \text{ indet}}\ \textsc{IInjHole} \qquad \frac{d \text{ indet} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\!) \rangle \text{ indet}}\ \textsc{ICastGroundHole}$$

$$\frac{d \neq d'\langle \tau' \Rightarrow (\!|\!) \rangle \qquad d \text{ indet} \qquad \tau \text{ ground}}{d\langle (\!|\!) \Rightarrow \tau \rangle \text{ indet}}\ \textsc{ICastHoleGround} \qquad \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ indet}}\ \textsc{ICastArr} \qquad \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}}\ \textsc{ICastRec}$$

$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \\ \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \\ \tau \neq \tau' \qquad d \text{ indet}\end{array}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}\ \textsc{ICastSum} \qquad \frac{d \text{ final} \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2 \rangle \text{ indet}}\ \textsc{IFailedCast}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\frac{[d_2 \text{ final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1}\ \textsc{ITApp} \qquad \frac{[d \text{ final}]}{\texttt{unroll}(\texttt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}\ \textsc{ITUnroll}$$

$$\frac{[d_1 \text{ final}] \qquad [d_2 \text{ final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2' \rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1 \rangle))\langle \tau_2 \Rightarrow \tau_2' \rangle}\ \textsc{ITAppCast}$$

$$\frac{[d \text{ final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\texttt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle) \longrightarrow \texttt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau' \rangle}\ \textsc{ITUnrollCast} \qquad \frac{[d \text{ final}]}{d\langle \tau \Rightarrow \tau \rangle \longrightarrow d}\ \textsc{ITCastId}$$

$$\textsc{ITCastSucceed}\ \dfrac{[d\ \text{final}] \qquad \tau\ \text{ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \Rightarrow \tau\rangle \longrightarrow d}$$

$$\textsc{ITCastFail}\ \dfrac{[d\ \text{final}] \qquad \tau_1 \neq \tau_2 \qquad \tau_1\ \text{ground} \qquad \tau_2\ \text{ground}}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \Rightarrow \tau_2\rangle \longrightarrow d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle}$$

$$\textsc{ITGround}\ \dfrac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis\rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow \llparenthesis\rrparenthesis\rangle}$$

$$\textsc{ITExpand}\ \dfrac{[d\ \text{final}] \qquad \tau \blacktriangleright_{\mathsf{ground}} \tau'}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau\rangle \longrightarrow d\langle \llparenthesis\rrparenthesis \Rightarrow \tau' \Rightarrow \tau\rangle}$$

$$\begin{array}{llll}\mathsf{EvalCtx} & \mathcal{E} & ::= & \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \texttt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \texttt{unroll}(\mathcal{E}) \mid \texttt{inj}_C^{\tau}(\mathcal{E}) \mid \llparenthesis\mathcal{E}\rrparenthesis_\sigma^u \mid \llparenthesis\mathcal{E}\rrparenthesis_\sigma^{u\blacktriangleright} \\ & & & \mid \mathcal{E}\langle \tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle \tau \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau\rangle\end{array}$$

$\boxed{d = \mathcal{E}\{d'\}}$ d is obtained by placing d' at the mark in $\mathcal{E}$

$$\textsc{FHOuter}\ \dfrac{}{d = \circ\{d\}} \qquad \textsc{FHApp1}\ \dfrac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}} \qquad \textsc{FHApp2}\ \dfrac{[d_1\ \text{final}] \qquad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

$$\textsc{FHRoll}\ \dfrac{d = \mathcal{E}\{d'\}}{\texttt{roll}^{\mu\pi.\tau}(d) = \texttt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FHUnroll}\ \dfrac{d = \mathcal{E}\{d'\}}{\texttt{unroll}(d) = \texttt{unroll}(\mathcal{E})\{d'\}} \qquad \textsc{FHInj}\ \dfrac{d = \mathcal{E}\{d'\}}{\texttt{inj}_C^{\tau}(d) = \texttt{inj}_C^{\tau}(\mathcal{E})\{d'\}}$$

$$\textsc{FNEHoleInside}\ \dfrac{d = \mathcal{E}\{d'\}}{\llparenthesis d\rrparenthesis_\sigma^u = \llparenthesis\mathcal{E}\rrparenthesis_\sigma^u\{d'\}} \qquad \textsc{FMHoleInside}\ \dfrac{d = \mathcal{E}\{d'\}}{\llparenthesis d\rrparenthesis_\sigma^{u\blacktriangleright} = \llparenthesis\mathcal{E}\rrparenthesis_\sigma^{u\blacktriangleright}\{d'\}}$$

$$\textsc{FHCastInside}\ \dfrac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2\rangle\{d'\}} \qquad \textsc{FHFailedCast}\ \dfrac{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2\rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ d steps to d'

$$\textsc{Step}\ \dfrac{d = \mathcal{E}\{d_0\} \qquad d_0 \longrightarrow d_0' \qquad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$