

# Hazel Phi: 11-type-constructors

May 8, 2022

## NOTES

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need to finish up OK\* proofs now that unicity is done

## SYNTAX

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Kind	$\kappa$	$::=$	$\mathbf{Type} \mid \mathbf{KHole} \mid \mathbf{S}_{\kappa}(\tau) \mid \Pi_{t::\kappa_1}.\kappa_2$
User Types	$\hat{\tau}$	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \hat{\tau} \rangle^u \mid \lambda t::\mathbf{Type}.\hat{\tau} \mid \tau_1 \tau_2$
Internal Types	$\tau$	$::=$	$t \mid \mathbf{bse} \mid \tau_1 \oplus \tau_2 \mid \langle \rangle^u \mid \langle \tau \rangle^u \mid \langle t \rangle^u \mid \lambda t::\kappa.\tau \mid \tau_1 \tau_2$
Base Types	$\mathbf{bse}$	$::=$	$\mathbf{Int} \mid \mathbf{Float} \mid \mathbf{Bool}$
BinOp	$\oplus$	$::=$	$\times \mid + \mid \rightarrow$
Type Pattern			
User Expression			
Internal Expression			

## DECLARATIVES

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$\Delta; \Phi \vdash \tau ::> \kappa$   $\tau$  has principal (well formed) kind  $\kappa$

$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \mathbf{bse} ::> \mathbf{S}_{\mathbf{Type}}(\mathbf{bse})} \text{PK-Base}$	$\frac{\Delta; \Phi_1, t::\kappa, \Phi_2 \vdash \text{OK}}{\Delta; \Phi \vdash t ::> \mathbf{S}_{\kappa}(t)} \text{PK-Var}$
$\frac{\Delta; \Phi \vdash \tau_1 :: \mathbf{Type} \quad \Delta; \Phi \vdash \tau_2 :: \mathbf{Type}}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 ::> \mathbf{S}_{\mathbf{Type}}(\tau_1 \oplus \tau_2)} \text{PK-}\oplus$	$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \langle \rangle^u ::> \mathbf{S}_{\kappa}(\langle \rangle^u)} \text{PK-EHole}$
$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad \Delta; \Phi \vdash \tau :: \kappa_1}{\Delta; \Phi \vdash \langle \tau \rangle^u ::> \mathbf{S}_{\kappa}(\langle \tau \rangle^u)} \text{PK-NEHole}$	$\frac{\Delta_1, u::\kappa, \Delta_2; \Phi \vdash \text{OK} \quad t \notin \Phi}{\Delta; \Phi \vdash \langle t \rangle^u ::> \mathbf{S}_{\kappa}(\langle t \rangle^u)} \text{PK-Unbound}$
$\frac{\Delta; \Phi, t::\kappa_1 \vdash \tau ::> \kappa_2}{\Delta; \Phi \vdash \lambda t::\kappa_1.\tau ::> \mathbf{S}_{\Pi_{t::\kappa_1}.\kappa_2}}(\lambda t::\kappa_1.\tau)} \text{PK-}\lambda$	
$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa \quad \Delta; \Phi \vdash \kappa \blacktriangleright \Pi_{t::\kappa_1}.\kappa_2 \quad \Delta; \Phi \vdash \tau_2 :: \kappa_1}{\Delta; \Phi \vdash \tau_1 \tau_2 ::> [\tau_2/t]\kappa_2} \text{PK-Ap}$	

$\Delta; \Phi \vdash \tau :: \kappa$   $\tau$  is well formed at kind  $\kappa$

$$\frac{\Delta; \Phi \vdash \tau :: > \mathbf{S}_{\kappa}(\tau)}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-1} \quad \frac{\Delta; \Phi \vdash \tau :: > \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \lesssim \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Subsump}$$

$$\frac{\Delta; \Phi \vdash \tau :: > \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Reit} \quad \frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau :: \mathbf{S}_{\kappa}(\tau)} \text{WFaK-Self}$$

$$\frac{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_3} \cdot \kappa_4 \quad \Delta; \Phi \vdash \Pi_{t::\kappa_3} \cdot \kappa_4 \lesssim \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \tau :: \Pi_{t::\kappa_1} \cdot \kappa_2} \text{WFaK-PCSKTrans}$$

$$\frac{\Delta; \Phi \vdash \tau :: \mathbf{S}_{\kappa}(\tau_1) \quad \Delta; \Phi \vdash \tau_1 :: \kappa}{\Delta; \Phi \vdash \tau :: \kappa} \text{WFaK-Flatten}$$

$\Delta; \Phi \vdash \kappa \dashv \vdash \Pi_{t::\kappa_1} \cdot \kappa_2$   $\kappa$  has matched  $\Pi$ -kind  $\Pi_{t::\kappa_1} \cdot \kappa_2$

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{KHole} \dashv \vdash \Pi_{t::\text{KHole}} \cdot \text{KHole}} \dashv \vdash \text{-KHole} \quad \frac{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} \mathbf{S}_{\text{KHole}}(\tau)}{\Delta; \Phi \vdash \kappa \dashv \vdash \Pi_{t::\mathbf{S}_{\text{KHole}}(\tau)} \cdot \mathbf{S}_{\text{KHole}}(\tau \ t)} \dashv \vdash \text{-SKHole}$$

$$\frac{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} \Pi_{t::\kappa_1} \cdot \kappa_2}{\Delta; \Phi \vdash \kappa \dashv \vdash \Pi_{t::\kappa_1} \cdot \kappa_2} \dashv \vdash \text{-}\Pi$$

$\Delta; \Phi \vdash \kappa_1 \equiv^* \kappa_2$   $\kappa_1$  singleton reduces to  $\kappa_2$

$$\frac{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{S}_{\kappa}(\tau_1)}(\tau) \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{S}_{\kappa}(\tau_1)}(\tau) \equiv^* \mathbf{S}_{\kappa}(\tau_1)} \equiv^* \text{-1} \quad \frac{\Delta; \Phi \vdash \kappa_1 \equiv^* \kappa_2 \quad \Delta; \Phi \vdash \kappa_2 \equiv^* \kappa_3}{\Delta; \Phi \vdash \kappa_1 \equiv^* \kappa_3} \equiv^* \text{-Trans}$$

$\Delta; \Phi \vdash \kappa_1 \equiv^{\text{norm}} \kappa_2$   $\kappa_1$  has singleton normal form  $\kappa_2$

$$\frac{\Delta; \Phi \vdash \kappa \equiv^* \mathbf{S}_{\text{Type}}(\tau)}{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} \mathbf{S}_{\text{Type}}(\tau)} \equiv^{\text{norm}} \text{-Type} \quad \frac{\Delta; \Phi \vdash \kappa \equiv^* \mathbf{S}_{\text{KHole}}(\tau)}{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} \mathbf{S}_{\text{KHole}}(\tau)} \equiv^{\text{norm}} \text{-KHole}$$

$$\frac{\Delta; \Phi \vdash \kappa \equiv^* \mathbf{S}_{\Pi_{t::\kappa_1} \cdot \kappa_2}(\tau)}{\Delta; \Phi \vdash \kappa \equiv^{\text{norm}} \Pi_{t::\kappa_1} \cdot \mathbf{S}_{[t_1/t]\kappa_2}(\tau \ t_1)} \equiv^{\text{norm}} \text{-}\Pi$$

$\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$   $\kappa_1$  is equivalent to  $\kappa_2$

$$\begin{array}{c}
\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \equiv \kappa} \text{KEquiv-Ref1} \qquad \frac{\Delta; \Phi \vdash \kappa_2 \equiv \kappa_1}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-Symm} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-Trans} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv^* \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-SReduc} \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv^{\text{norm}} \kappa_2}{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2} \text{KEquiv-SNorm} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \kappa_3 \equiv \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \equiv \Pi_{t::\kappa_3}.\kappa_4} \text{KEquiv-}\Pi \\
\\
\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2 \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa_1}(\tau_1) \equiv \mathbf{S}_{\kappa_2}(\tau_2)} \text{KEquiv-SKind}
\end{array}$$

$\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$   $\kappa_1$  is a consistent subkind of  $\kappa_2$

$$\begin{array}{c}
\frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{KHole} \lesssim \kappa} \text{CSK-KHoleL} \qquad \frac{\Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{KHole}} \text{CSK-KHoleR} \\
\\
\frac{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \text{ OK} \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \lesssim \kappa} \text{CSK-SKind}_{\mathbf{KHoleL}} \\
\\
\frac{\Delta; \Phi \vdash \kappa \text{ OK} \quad \Delta; \Phi \vdash \mathbf{S}_{\mathbf{KHole}}(\tau) \text{ OK}}{\Delta; \Phi \vdash \kappa \lesssim \mathbf{S}_{\mathbf{KHole}}(\tau)} \text{CSK-SKind}_{\mathbf{KHoleR}} \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \text{CSK-KEquiv} \qquad \frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_3 \quad \Delta; \Phi \vdash \kappa_3 \lesssim \kappa_4 \quad \Delta; \Phi \vdash \kappa_4 \equiv \kappa_2}{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2} \text{CSK-Normal} \\
\\
\frac{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \text{ OK}}{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \lesssim \kappa} \text{CSK-SKind} \qquad \frac{\Delta; \Phi \vdash \kappa_3 \lesssim \kappa_1 \quad \Delta; \Phi, t::\kappa_3 \vdash \kappa_2 \lesssim \kappa_4}{\Delta; \Phi \vdash \Pi_{t::\kappa_1}.\kappa_2 \lesssim \Pi_{t::\kappa_3}.\kappa_4} \text{CSK-}\Pi \\
\\
\frac{\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2 \quad \Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2}{\Delta; \Phi \vdash \mathbf{S}_{\kappa_1}(\tau_1) \lesssim \mathbf{S}_{\kappa_2}(\tau_2)} \text{CSK-?}
\end{array}$$

$\boxed{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2}$   $\tau_1$  is provably equivalent to  $\tau_2$  at kind  $\kappa$

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \tau \equiv^{\kappa} \tau} \text{EquivAK-Ref1} \qquad \frac{\Delta; \Phi \vdash \tau_2 \equiv^{\kappa} \tau_1}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \text{EquivAK-Symm}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_3 \quad \Delta; \Phi \vdash \tau_3 \equiv^{\kappa} \tau_1}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \text{EquivAK-Trans}$$

$$\frac{\Delta; \Phi \vdash \tau_1 ::> \kappa_1 \quad \Delta; \Phi \vdash \kappa_1 \equiv \mathbf{S}_{\kappa}(\tau_2)}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} \text{EquivAK-SKind}$$

$$\frac{\Delta; \Phi \vdash \tau_1 :: \Pi_{t::\kappa_1} \kappa_3 \quad \Delta; \Phi \vdash \tau_2 :: \Pi_{t::\kappa_1} \kappa_4 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \equiv^{\kappa_2} \tau_2 \ t}{\Delta; \Phi \vdash \tau_1 \equiv^{\Pi_{t::\kappa_1} \kappa_2} \tau_2} \text{EquivAK-}\Pi$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\Pi_{t::\kappa_1} \kappa_2} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \equiv^{\kappa_1} \tau_4}{\Delta; \Phi \vdash \tau_1 \tau_2 \equiv^{[\tau_2/t]\kappa_2} \tau_3 \tau_4} \text{EquivAK-Ap}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\text{Type}} \tau_3 \quad \Delta; \Phi \vdash \tau_2 \equiv^{\text{Type}} \tau_4}{\Delta; \Phi \vdash \tau_1 \oplus \tau_2 \equiv^{\text{Type}} \tau_3 \oplus \tau_4} \text{EquivAK-}\oplus$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\mathbf{S}_{\kappa}(\tau)} \tau_2}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} (1)$$

$$\frac{\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2 \quad \Delta; \Phi, t::\kappa_1 \vdash \tau_1 \equiv^{\kappa} \tau_2}{\Delta; \Phi \vdash \lambda t::\kappa_1. \tau_1 \equiv^{\Pi_{t::\kappa_1} \kappa} \lambda t::\kappa_2. \tau_2} (2)$$

$$\frac{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa_1} \tau_2 \quad \Delta; \Phi \vdash \kappa_1 \equiv \kappa}{\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2} (3)$$

$\boxed{\Delta; \Phi \vdash \kappa \text{ OK}}$   $\kappa$  is well formed

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{Type OK}} \text{KWF-Type}$$

$$\frac{\Delta; \Phi \vdash \text{OK}}{\Delta; \Phi \vdash \text{KHole OK}} \text{KWF-KHole}$$

$$\frac{\Delta; \Phi \vdash \tau :: \kappa}{\Delta; \Phi \vdash \mathbf{S}_{\kappa}(\tau) \text{ OK}} \text{KWF-SKind}$$

$$\frac{\Delta; \Phi, t::\kappa_1 \vdash \kappa_2 \text{ OK}}{\Delta; \Phi \vdash \Pi_{t::\kappa_1} \kappa_2 \text{ OK}} \text{KWF-}\Pi$$

$\boxed{\Delta; \Phi \vdash \text{OK}}$  Context is well formed

$$\frac{}{\cdot; \cdot \vdash \text{OK}} \text{CWF-Nil}$$

$$\frac{t \notin \Phi \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta; \Phi, t::\kappa \vdash \text{OK}} \text{CWF-TypVar}$$

$$\frac{u \notin \Delta \quad \Delta; \Phi \vdash \kappa \text{ OK}}{\Delta, u::\kappa; \Phi \vdash \text{OK}} \text{CWF-Hole}$$

## ALGORITHM

(syntactically distinguished up to  $\alpha$ -equivalence... when needed)

(TODO: remove the ‘... when needed’. The bound variable renamings should get adjusted)

(NOTE: current implementation has explicit  $\equiv_{\alpha}$  checks which are not written in these rules since we eventually want to use De Bruijn indices, hence the above)

$\Delta; \Phi \triangleright \tau_1 \equiv^{\kappa} \tau_2$   $\tau_1$  is equivalent to  $\tau_2$  at kind  $\kappa$

$$\frac{\Delta; \Phi \triangleright \tau_1 \xRightarrow{\kappa} \tau_\omega \quad \Delta; \Phi \triangleright \tau_2 \xRightarrow{\kappa} \tau_\omega}{\Delta; \Phi \triangleright \tau_1 \equiv^{\kappa} \tau_2}$$

$\Delta; \Phi \triangleright \tau \Downarrow \tau_\psi$   $\tau$  weak head normalizes to  $\tau_\psi$

$$\frac{\Delta; \Phi \triangleright \tau \rightsquigarrow \tau_\chi \quad \Delta; \Phi \triangleright \tau_\chi \Downarrow \tau_\psi}{\Delta; \Phi \triangleright \tau \Downarrow \tau_\psi} \quad \frac{\Delta; \Phi \triangleright \tau \rightsquigarrow \tau}{\Delta; \Phi \triangleright \tau \Downarrow \tau}$$

$\Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega$   $\tau$  normalizes to  $\tau_\omega$  at kind  $\kappa$

$$\frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \text{Type} \quad \Delta; \Phi \triangleright \tau \Downarrow \tau_\psi \quad \Delta; \Phi \triangleright \tau_\psi \xrightarrow{\text{Type}} \tau_\omega}{\Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega} \quad \frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \text{KHole} \quad \text{TODO :}}{\Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega}$$

$$\frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \text{S}_{\text{Type}}(\tau_s) \quad \Delta; \Phi \triangleright \tau \Downarrow \tau_\psi \quad \Delta; \Phi \triangleright \tau_\psi \xrightarrow{\text{Type}} \tau_\omega \quad \text{TODO : } \text{prop} : \tau_\omega = \tau_s}{\Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega}$$

$$\frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \text{SKHole}(\tau_s) \quad \text{TODO :}}{\Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega} \quad \frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \Pi_{t::\kappa_{\omega_1}}.\kappa_{\omega_2} \quad \Delta; \Phi, t_1::\kappa_{\omega_1} \triangleright \tau \quad t_1 \xRightarrow{[t_1/t]\kappa_2} \tau_\omega}{\Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \lambda t_1::\kappa_{\omega_1}.\tau_\omega}$$

$\Delta; \Phi \triangleright \kappa \Longrightarrow \kappa_\omega$   $\kappa$  normalizes to  $\kappa_\omega$

$$\frac{}{\Delta; \Phi \triangleright \text{Type} \Longrightarrow \text{Type}} \quad \frac{}{\Delta; \Phi \triangleright \text{KHole} \Longrightarrow \text{KHole}} \quad \frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \text{Type} \quad \Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega}{\Delta; \Phi \triangleright \text{S}_\kappa(\tau) \Longrightarrow \text{S}_{\text{Type}}(\tau_\omega)}$$

$$\frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \text{KHole} \quad \Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega}{\Delta; \Phi \triangleright \text{S}_\kappa(\tau) \Longrightarrow \text{SKHole}(\tau_\omega)} \quad \frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \text{S}_{\kappa_1}(\tau_1) \quad \Delta; \Phi \triangleright \text{S}_{\kappa_1}(\tau_1) \Longrightarrow \kappa_\omega}{\Delta; \Phi \triangleright \text{S}_\kappa(\tau) \Longrightarrow \kappa_\omega}$$

$$\frac{\Delta; \Phi \triangleright \kappa \Longrightarrow \Pi_{t::\kappa_1}.\kappa_2 \quad \Delta; \Phi \triangleright \tau \xRightarrow{\kappa} \tau_\omega \quad \Delta; \Phi, t_1::\kappa_1 \triangleright \tau_\omega \quad t_1 \xRightarrow{[t_1/t]\kappa_2} \tau_{\omega_1}}{\Delta; \Phi \triangleright \text{S}_\kappa(\tau) \Longrightarrow \Pi_{t_1::\kappa_1}.\text{S}_{[t_1/t]\kappa_2}(\tau_{\omega_1})}$$

$$\frac{\Delta; \Phi \triangleright \kappa_1 \Longrightarrow \kappa_{\omega_1} \quad \Delta; \Phi, t::\kappa_{\omega_1} \triangleright \kappa_2 \Longrightarrow \kappa_{\omega_2}}{\Delta; \Phi \triangleright \Pi_{t::\kappa_1}.\kappa_2 \Longrightarrow \Pi_{t::\kappa_{\omega_1}}.\omega_2}$$

## METATHEORY

subderivation preserving inferences:

- premiss
- COK (Context OK)
- PoS (premiss of subderivation)

**Lemma 1** (COK). *If  $\Delta; \Phi \vdash \mathcal{J}$ , then  $\Delta; \Phi \vdash OK$  in a subderivation (where  $\Delta; \Phi \vdash \mathcal{J} \neq \Delta; \Phi \vdash OK$ )*

*Proof.* By induction on derivations.

No interesting cases.  $\square$

**Lemma 2** (Exchange).

If  $\Delta; \Phi_1, t_{L1}::\kappa_{L1}, t_{L2}::\kappa_{L2}, \Phi_2 \vdash \mathcal{J}$  and  $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash OK$ , then  $\Delta; \Phi_1, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1}, \Phi_2 \vdash \mathcal{J}$

*Proof.* By induction on derivations.

No interesting cases.

(Only rules with  $\Phi$  extended in the consequent are interesting, which is only CWF-TypVar, but when  $\mathcal{J}$  is CWF, Exchange is identity)  $\square$

**Corollary 3** (Marked-Exchange).

If  $\Delta; \Phi, t_{L1}::\kappa_{L1}, t_{L2}::\kappa_{L2} \vdash \mathcal{J}$  and  $\Delta; \Phi, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1} \vdash OK$ , then  $\Delta; \Phi, t_{L2}::\kappa_{L2}, t_{L1}::\kappa_{L1} \vdash \mathcal{J}$

*Proof.* Exchange when  $\Phi_2 = \cdot$   $\square$

**Lemma 4** (Weakening).

If  $\Delta; \Phi \vdash \mathcal{J}$  and  $\Delta; \Phi, t_L::\kappa_L \vdash OK$  and  $t_L \notin \mathcal{J}$  and  $\forall t \in \kappa_L, t \notin \mathcal{J}$ , then  $\Delta; \Phi, t_L::\kappa_L \vdash \mathcal{J}$

*Proof.* see addendum  $\square$

**Lemma 5** (K-Substitution).

If  $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$  and  $\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau_{L2}::\kappa_{L2}$ , then  $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2}::[\tau_{L1}/t_L]\kappa_{L2}$   
(induction on  $\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau_{L2}::\kappa_{L2}$ )

**Lemma 6** (PK-Substitution). If  $\Delta; \Phi \vdash \tau_{L1}::\kappa_{L1}$  and  $\Delta; \Phi, t_L::\kappa_{L1} \vdash \tau_{L2}::\kappa_{L2}$  and  $\Delta; \Phi \vdash [\tau_{L1}/t_L]\tau_{L2}::\kappa_{L3}$ , then  $\Delta; \Phi \vdash [\tau_{L2}/t_L]\kappa_{L2} \equiv \kappa_{L3}$

**Lemma 7** (OK-Substitution).

If  $\Delta; \Phi \vdash \tau_L::\kappa_{L1}$  and  $\Delta; \Phi, t_L::\kappa_{L1} \vdash \kappa_{L2} OK$ , then  $\Delta; \Phi \vdash [\tau_L/t_L]\kappa_{L2} OK$   
(induction on  $\Delta; \Phi, t_L::\kappa_{L1} \vdash \kappa_{L2} OK$ )

**Theorem 8** (OK-PK). If  $\Delta; \Phi \vdash \tau::\kappa$ , then  $\Delta; \Phi \vdash \kappa OK$

**Theorem 9** (OK-WFaK). If  $\Delta; \Phi \vdash \tau::\kappa$ , then  $\Delta; \Phi \vdash \kappa OK$

**Theorem 10** (OK-MatchPi). If  $\Delta; \Phi \vdash \kappa \Pi_{t::\kappa_1} \kappa_2$ , then  $\Delta; \Phi \vdash \kappa OK$  and  $\Delta; \Phi \vdash \Pi_{t::\kappa_1} \kappa_2 OK$

**Theorem 11** (OK-KEquiv). If  $\Delta; \Phi \vdash \kappa_1 \equiv \kappa_2$ , then  $\Delta; \Phi \vdash \kappa_1 OK$  and  $\Delta; \Phi \vdash \kappa_2 OK$

**Theorem 12** (OK-CSK). If  $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$ , then  $\Delta; \Phi \vdash \kappa_1 OK$  and  $\Delta; \Phi \vdash \kappa_2 OK$

**Theorem 13** (OK-EquivAK). If  $\Delta; \Phi \vdash \tau_1 \equiv^{\kappa} \tau_2$ , then  $\Delta; \Phi \vdash \tau_1::\kappa$  and  $\Delta; \Phi \vdash \tau_2::\kappa$  and  $\Delta; \Phi \vdash \kappa OK$

*Proof.* see addendum  $\square$

Proof.

**Weakening**

By induction on derivations.

Note: When applying Weakening in the induction, check that the left premis is always a subderivation, and check variable exclusion conditions are satisfied (usually checked elsewhere in the derivation).

$$\frac{\frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l_L \notin \Phi} \text{IH}}{\text{PoS}} \quad \frac{\frac{l_L \notin \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l \in \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l_L \notin \mathcal{J}}{l_L \neq \kappa_L} \text{IH}}{l_L \notin \Phi, l_L \sqsupseteq \kappa_L} \text{IH} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi \vdash \kappa_L \text{ OK}} \text{IH}}{\text{PoS}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l \notin \Phi} \text{PoS}}{\text{PoS}} \quad \frac{l_L \notin \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{\forall l \in \kappa_L, l \notin \mathcal{J}}{l \notin \kappa_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \notin \mathcal{J}} \text{IH}}{l \notin \Phi, l_L \sqsupseteq \kappa_L} \text{IH} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Marked-Exchange} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{PK-}\lambda$$
$$\frac{\frac{\frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l_L \notin \Phi} \text{IH}}{\text{PoS}} \quad \frac{l_L \notin \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l \in \mathcal{J}}{l_L \neq l} \text{IH} \quad \frac{l_L \notin \mathcal{J}}{l_L \neq \kappa_L} \text{IH} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi \vdash \kappa_L \text{ OK}} \text{IH}}{\text{PoS}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\frac{\frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{l \notin \Phi} \text{PoS}}{\text{PoS}} \quad \frac{l_L \notin \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \neq l_L} \text{IH} \quad \frac{\forall l \in \kappa_L, l \notin \mathcal{J}}{l \notin \kappa_L} \text{IH} \quad \frac{l \in \mathcal{J}}{l \notin \mathcal{J}} \text{IH}}{l \notin \Phi, l_L \sqsupseteq \kappa_L} \text{IH} \quad \frac{\frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \tau \sqsupseteq \kappa_R}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{premiss}}{\text{CoK}} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Weakening} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CWF-TypVar} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{Marked-Exchange} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{KEquiv-II} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{CSK-II} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{EquivAK-II} \quad \frac{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}}{\Delta; \Phi, l_L \sqsupseteq \kappa_L \vdash \text{OK}} \text{KWF-II}$$

**O?K-\***

By simultaneous induction on derivations.

The interesting cases per theorem:

**K-Substitution** by type size??

**OK-Substitution**

**OK-PK**

$$\frac{\frac{\Delta; \Phi \vdash \text{base} \sqsupseteq \Phi_{\text{Type}}(\text{base})}{\Delta; \Phi \vdash \text{base}::\text{Type}} \text{premiss} \quad \text{VFaK-I} \quad \frac{\Delta; \Phi \vdash \text{base}::\text{Type}}{\Delta; \Phi \vdash \text{S}_{\text{Type}}(\text{base}) \text{ OK}} \text{KWF-SKInd}$$

$$\Delta; \Phi \vdash [\tau_2/\theta] \kappa_R \text{ OK-Substitution}$$

**OK-WFaK**

□

**Definition 1** (Singleton Depth).

$$SSize : \{\kappa\}'' \rightarrow \mathbb{N}$$

$$SSize(\kappa_x) = \begin{cases} SSize(\kappa) + 1 & \text{if } \kappa_x = \mathbf{S}_\kappa(\tau) \\ 0 & \text{otherwise} \end{cases}$$

**Lemma 14** ( $\equiv^*>$ -diminution). *If  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L1}$ , then  $SSize(\kappa_L) > SSize(\kappa_{L1})$*

*Proof.* By induction on derivations (and transitivity of  $>$  on  $\mathbb{N}$ ) □

**Lemma 15** ( $\equiv^*>-n+1$ -nicity). *If  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L1}$  and  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L2}$  where  $SSize(\kappa_L) = n+1$  and  $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$ , then  $\kappa_{L1} = \kappa_{L2}$*

*Proof.* By  $\equiv^*>$ -diminution,  $\equiv^*>$ -Trans cannot be the last inference of a derivation of  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L1}$  since  $SSize(\kappa_1) \geq SSize(\kappa_2) + 2$  (in  $\equiv^*>$ -Trans). Thus,  $\equiv^*>-1$  must have been the last inference. Similarly for  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L2}$ , thus  $\kappa_{L1} = \kappa_{L2}$  □

**Lemma 16** ( $\equiv^*>$ -stepwise). *If  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L1}$  where  $SSize(\kappa_L) = m$  and  $SSize(\kappa_{L1}) = n$  and  $m > n + 1$ , then the derivation must contain subderivations of each singleton depth inbetween*

*Proof.* More precisely this says, where  $m > n$  by  $\equiv^*>$ -diminution, the derivation must contain subderivations of each  $\Delta; \Phi \vdash \kappa_i \equiv^* \kappa_j$  where  $m \geq i > j \geq n$ ,  $SSize(\kappa_k) = k$  when  $m \geq k \geq n$ ,  $\kappa_m = \kappa_L$ ,  $\kappa_n = \kappa_{L1}$ .

By induction on derivations (base case is where  $m = n + 2$ , which necessitates a last inference of  $\equiv^*>$ -Trans. Each premiss must have  $SSize$  difference of 1, fulfilling hypothesis) □

**Lemma 17** ( $\equiv^*>-m + n$ -nicity). *If  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L1}$  and  $\Delta; \Phi \vdash \kappa_L \equiv^* \kappa_{L2}$  where  $SSize(\kappa_L) = m + n$  and  $SSize(\kappa_{L1}) = SSize(\kappa_{L2}) = n$ , then  $\kappa_{L1} = \kappa_{L2}$*

*Proof.* By  $\equiv^*>$ -stepwise and  $\equiv^*>-n + 1$ -nicity when  $m > n + 1$ .

By  $\equiv^*>-n + 1$ -nicity when  $m = n + 1$ .

No other cases by  $\equiv^*>$ -diminution. □

**Theorem 18** ( $\equiv^{\text{norm}}>$ -Unicity). *If  $\Delta; \Phi \vdash \kappa_L \equiv^{\text{norm}} \kappa_{L1}$  and  $\Delta; \Phi \vdash \kappa_L \equiv^{\text{norm}} \kappa_{L2}$ , then  $\kappa_{L1} = \kappa_{L2}$*

*Proof.* (this is a really quick sketch)

All  $\equiv^{\text{norm}}>$  rules have  $\equiv^*$  premiss with rhs singleton depth 1. By  $\equiv^*>-m + n$ -nicity, where  $n = 1$ . □

**Theorem 19** ( $\P$ -Unicity). *If  $\Delta; \Phi \vdash \tau_L \P \kappa_{L1}$  and  $\Delta; \Phi \vdash \tau_L \P \kappa_{L2}$ , then  $\kappa_{L1} = \kappa_{L2}$*

*Proof.* (this is a really quick sketch)

By unicity of  $\equiv^{\text{norm}}>$ . □

**Theorem 20** (PK-Unicity). *If  $\Delta; \Phi \vdash \tau_L ::> \kappa_{L1}$  and  $\Delta; \Phi \vdash \tau_L ::> \kappa_{L2}$ , then  $\kappa_{L1} = \kappa_{L2}$*

*Proof.* (this is a really quick sketch)

As PK is syntax directed, proof is by inspection for all rules except PK- $\lambda$  (variables in contexts are unique— see context rules), which is by induction on derivations, and PK-Ap, which requires of unicity of  $\P$  (above theorem). □

**Theorem 21** (PK-Principality). *If  $\Delta; \Phi \vdash \tau ::> \kappa_1$  and  $\Delta; \Phi \vdash \tau ::> \kappa_2$ , then  $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

*Proof.* From definition of  $\Delta; \Phi \vdash \tau ::> \kappa$  and CSK-SKind □

**Theorem 22** (why is this here?). *If  $\Delta; \Phi \vdash \kappa_1 \lesssim \mathbf{S}_{\kappa_2}(\tau)$ , then  $\Delta; \Phi \vdash \kappa_1 \lesssim \kappa_2$*

ELABORATION

TODO