Algebraic Data Types for Hazel

Eric Griffis
egriffis@umich.edu

1 Syntax

$$
\begin{array}{llll}
\textsf{HTyp} & \tau & ::= & \varnothing \mid \tau \to \tau \mid \alpha \mid \mu\pi.\tau \mid +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \mid (\!|\!) \mid (\!|\alpha|\!) \\
\textsf{HTypPat} & \pi & ::= & \alpha \mid (\!|\!) \\
\textsf{HExp} & e & ::= & \varnothing \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid e : \tau \mid \texttt{inj}_C(e) \mid \texttt{roll}(e) \mid \texttt{unroll}(e) \\
& & & \mid (\!|\!)^u \mid (\!|e|\!)^u \mid (\!|e|\!)^{u\blacktriangleright} \\
\textsf{IHExp} & d & ::= & \varnothing \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \texttt{inj}_C^\tau(d) \mid \texttt{roll}^{\mu\alpha.\tau}(d) \mid \texttt{unroll}(d) \\
& & & \mid d\langle \tau \Rightarrow \tau \rangle \mid d\langle \tau \Rightarrow (\!|\!) \not\Rightarrow \tau \rangle \mid (\!|\!)_\sigma^u \mid (\!|d|\!)_\sigma^u \mid (\!|d|\!)_\sigma^{u\blacktriangleright} \\
\textsf{HTag} & C & ::= & \mathbf{C} \mid (\!|\!)^u \mid (\!|\mathbf{C}|\!)^u
\end{array}
$$

1.1 Context Extension

We write Θ, π to denote the extension of type variable context Θ with optional type variable name π.

$$
\Theta, \pi = \begin{cases} \Theta, \alpha & \pi = \alpha \\ \Theta & \pi = (\!|\!) \end{cases}
$$

2 Static Semantics

$\boxed{[\tau/\pi]\tau' = \tau''}$ τ'' is obtained by substituting τ for π in τ'

$$
\begin{array}{llll}
[\tau/(\!|\!)]\tau' & = & \tau' & \\
[\tau/\alpha]\varnothing & = & \varnothing & \\
[\tau/\alpha](\tau_1 \to \tau_2) & = & [\tau/\alpha]\tau_1 \to [\tau/\alpha]\tau_1 & \\
[\tau/\alpha]\alpha & = & \tau & \\
[\tau/\alpha]\alpha_1 & = & \tau' & \text{when } \alpha \neq \alpha_1 \\
[\tau/\alpha]\mu\alpha_1.\tau_2 & = & \mu\alpha_1.[\tau/\alpha]\tau_2 & \text{when } \alpha \neq \alpha_1 \text{ and } \alpha_1 \notin \mathsf{FV}(\tau) \\
[\tau/\alpha]\mu(\!|\!).\tau_2 & = & \mu(\!|\!).[\tau/\alpha]\tau_2 & \\
[\tau/\alpha]+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} & = & +\{C_i([\tau/\alpha]\tau_i)\}_{C_i \in \mathcal{C}} & \\
[\tau/\alpha](\!|\!) & = & (\!|\!) & \\
[\alpha'/\alpha](\!|\alpha|\!) & = & (\!|\alpha'|\!) & \\
[\alpha'/\alpha](\!|\alpha'|\!) & = & (\!|\alpha'|\!) & \text{when } \alpha \neq \alpha'
\end{array}
$$

$\boxed{\Theta \vdash \tau\ \mathsf{valid}}$ τ is a valid type

$$
\frac{}{\Theta \vdash \varnothing\ \mathsf{valid}}\ \textsc{TVUnit}
\qquad
\frac{\Theta \vdash \tau_1\ \mathsf{valid} \quad \Theta \vdash \tau_2\ \mathsf{valid}}{\Theta \vdash \tau_1 \to \tau_2\ \mathsf{valid}}\ \textsc{TVArr}
\qquad
\frac{\alpha \in \Theta}{\Theta \vdash \alpha\ \mathsf{valid}}\ \textsc{TVVar}
$$

$$
\frac{\Theta, \pi \vdash \tau\ \mathsf{valid}}{\Theta \vdash \mu\pi.\tau\ \mathsf{valid}}\ \textsc{TVRec}
\qquad
\frac{\{\Theta \vdash \tau_i\ \mathsf{valid}\}_{C_i \in \mathcal{C}}}{\Theta \vdash +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}\ \mathsf{valid}}\ \textsc{TVSum}
$$

$$
\frac{}{\Theta \vdash (\!|\!)\ \mathsf{valid}}\ \textsc{TVEHole}
\qquad
\frac{\alpha \notin \Theta}{\Theta \vdash (\!|\alpha|\!)\ \mathsf{valid}}\ \textsc{TVNEHole}
$$

$\boxed{\tau \sim \tau'}$ $\quad \tau$ and τ' are consistent

$$\frac{\text{TCREFL}}{\tau \sim \tau} \qquad \frac{\text{TCEHOLE1}}{(\!\!|\,|\!\!) \sim \tau} \qquad \frac{\text{TCEHOLE2}}{\tau \sim (\!\!|\,|\!\!)} \qquad \frac{\text{TCNEHOLE1}}{(\!\!|\alpha|\!\!) \sim \tau} \qquad \frac{\text{TCNEHOLE2}}{\tau \sim (\!\!|\alpha|\!\!)} \qquad \frac{\text{TCARR}}{\tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}$$

$$\frac{\text{TCREC}}{\mu\pi.\tau \sim \mu\pi.\tau'} \qquad \frac{\text{TCRECHOLE1}}{\alpha \notin \mathsf{FV}(\tau) \quad \tau \sim \tau'}{\mu(\!\!|\,|\!\!).\tau \sim \mu\alpha.\tau'} \qquad \frac{\text{TCRECHOLE2}}{\alpha \notin \mathsf{FV}(\tau) \quad \tau \sim \tau'}{\mu\alpha.\tau \sim \mu(\!\!|\,|\!\!).\tau'} \qquad \frac{\text{TCSUM}}{\{\tau_i \sim \tau_i'\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \sim +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}}}$$

2.1 Bidirectional Typing

We call $[\mu\pi.\tau/\pi]\tau$ the *unrolling* of recursive type $\mu\pi.\tau$.

Theorem 1 (Synthetic Type Validity). *If $\Gamma \vdash e \Rightarrow \tau$ then $\emptyset \vdash \tau$ valid.*

Theorem 2 (Consistency Preserves Validity). *If $\Theta \vdash \tau$ valid and $\tau \sim \tau'$ then $\Theta \vdash \tau'$ valid.*

$\boxed{\tau \blacktriangleright_\to \tau_1 \to \tau_2}$ $\quad \tau$ has matched arrow type $\tau_1 \to \tau_2$

$$\frac{\text{MAHOLE}}{(\!\!|\,|\!\!) \blacktriangleright_\to (\!\!|\,|\!\!) \to (\!\!|\,|\!\!)} \qquad \frac{\text{MAARR}}{\tau_1 \to \tau_2 \blacktriangleright_\to \tau_1 \to \tau_2}$$

$\boxed{\tau \blacktriangleright_\mu \mu\pi.\tau'}$ $\quad \tau$ has matched recursive type $\mu\pi.\tau'$

$$\frac{\text{MRREC}}{\mu\pi.\tau \blacktriangleright_\mu \mu\pi.\tau} \qquad \frac{\text{MRHOLE}}{(\!\!|\,|\!\!) \blacktriangleright_\mu \mu(\!\!|\,|\!\!).(\!\!|\,|\!\!)}$$

$\boxed{\Gamma \vdash e \Rightarrow \tau}$ $\quad e$ synthesizes type τ

$$\frac{\text{SUNIT}}{\Gamma \vdash \varnothing \Rightarrow \varnothing} \qquad \frac{\text{SVAR}}{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau} \qquad \frac{\text{SVARFREE}}{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!\!|x|\!\!)^u \Rightarrow (\!\!|\,|\!\!)} \qquad \frac{\text{SLAM}}{\emptyset \vdash \tau \text{ valid} \quad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau'}$$

$$\frac{\text{SAPP}}{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \quad \Gamma \vdash e_2 \Leftarrow \tau_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau} \qquad \frac{\text{SAPPNOTARR}}{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \nsim (\!\!|\,|\!\!) \to (\!\!|\,|\!\!) \quad \Gamma \vdash e_2 \Leftarrow (\!\!|\,|\!\!)}{\Gamma \vdash (\!\!|e_1|\!\!)^{u \blacktriangleright}(e_2) \Rightarrow (\!\!|\,|\!\!)}$$

$$\frac{\text{SASC}}{\emptyset \vdash \tau \text{ valid} \quad \Gamma \vdash e \Leftarrow \tau}{\Gamma \vdash e : \tau \Rightarrow \tau} \qquad \frac{\text{SROLLERR}}{\Gamma \vdash e \Leftarrow (\!\!|\,|\!\!)}{\Gamma \vdash (\!\!|\mathsf{roll}(e)|\!\!)^u \Rightarrow \mu(\!\!|\,|\!\!).(\!\!|\,|\!\!)} \qquad \frac{\text{SUNROLL}}{\Gamma \vdash e \Rightarrow \tau \quad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \mathsf{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau'}$$

$$\frac{\text{SUNROLLNOTREC}}{\Gamma \vdash e \Rightarrow \tau \quad \tau \nsim \mu(\!\!|\,|\!\!).(\!\!|\,|\!\!)}{\Gamma \vdash \mathsf{unroll}((\!\!|e|\!\!)^{u \blacktriangleright}) \Rightarrow (\!\!|\,|\!\!)} \qquad \frac{\text{SINJERR}}{\Gamma \vdash e \Leftarrow (\!\!|\,|\!\!)}{\Gamma \vdash (\!\!|\mathsf{inj}_C(e)|\!\!)^u \Rightarrow (\!\!|\,|\!\!)} \qquad \frac{\text{SEHOLE}}{\Gamma \vdash (\!\!|\,|\!\!)^u \Rightarrow (\!\!|\,|\!\!)} \qquad \frac{\text{SNEHOLE}}{\Gamma \vdash e \Rightarrow \tau}{\Gamma \vdash (\!\!|e|\!\!)^u \Rightarrow (\!\!|\,|\!\!)}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau}$ e analyzes against type τ

AROLL
$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau'}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau}$$

AROLLNOTREC
$$\frac{\tau \nsim \mu(\!|\!|\!).(\!|\!|\!) \qquad \Gamma \vdash e \Leftarrow (\!|\!|\!)}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Leftarrow \tau}$$

AINJHOLE
$$\frac{\Gamma \vdash e \Leftarrow (\!|\!|\!)}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow (\!|\!|\!)}$$

AINJ
$$\frac{C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j}{\Gamma \vdash \mathtt{inj}_{C_j}(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AINJTAGERR
$$\frac{C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow (\!|\!|\!)}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AINJUNEXPECTEDARG
$$\frac{C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \qquad \Gamma \vdash e \Leftarrow (\!|\!|\!)}{\Gamma \vdash (\!|\mathtt{inj}_{C_j}(e)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

AINJEXPECTEDARG
$$\frac{C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing}{\Gamma \vdash (\!|\mathtt{inj}_{C_j}(\varnothing)|\!)^u \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}}}$$

ASUBSUME
$$\frac{\Gamma \vdash e \Rightarrow \tau' \qquad \tau' \sim \tau}{\Gamma \vdash e \Leftarrow \tau}$$

2.2   Typed Elaboration

Theorem 3 (Synthetic Typed Elaboration Validity). *If $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ then $\emptyset \vdash \tau$ valid.*

$\boxed{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}$ e synthesizes type τ and elaborates to d

ESUNIT
$$\frac{}{\Gamma \vdash \varnothing \Rightarrow \varnothing \rightsquigarrow \varnothing \dashv \emptyset}$$

ESVAR
$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \rightsquigarrow x \dashv \emptyset}$$

ESVARFREE
$$\frac{x \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash (\!|x|\!)^u \Rightarrow (\!|\!|\!) \rightsquigarrow (\!|x|\!)^u_{\mathsf{id}(\Gamma)} \dashv u :: (\!|\!|\!)[\Gamma]}$$

ESLAM
$$\frac{\emptyset \vdash \tau \text{ valid} \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau.e \Rightarrow \tau \to \tau' \rightsquigarrow \lambda x{:}\tau.d \dashv \Delta}$$

ESAPP
$$\frac{\begin{array}{c}\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright_\to \tau_2 \to \tau \\ \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \rightsquigarrow d_1 : \tau_1' \dashv \Delta_1 \\ \Gamma \vdash e_2 \Leftarrow \tau_2 \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2\end{array}}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \rightsquigarrow (d_1\langle\tau_1' \Rightarrow \tau_1\rangle)(d_2\langle\tau_2' \Rightarrow \tau_2\rangle) \dashv \Delta_1 \cup \Delta_2}$$

ESAPPNOTARR
$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \nsim (\!|\!|\!) \to (\!|\!|\!) \qquad \Gamma \vdash e_2 \Leftarrow (\!|\!|\!) \rightsquigarrow d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash (\!|e_1|\!)^{u\blacktriangleright}(e_2) \Rightarrow (\!|\!|\!) \rightsquigarrow (\!|d_1|\!)^{u\blacktriangleright}_{\mathsf{id}(\Gamma)}(d_2\langle\tau_2' \Rightarrow (\!|\!|\!)\rangle) \dashv \Delta_1 \cup \Delta_2, u :: (\!|\!|\!) \to (\!|\!|\!)[\Gamma]}$$

ESASC
$$\frac{\emptyset \vdash \tau \text{ valid} \qquad \Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash e : \tau \Rightarrow \tau \rightsquigarrow d\langle\tau' \Rightarrow \tau\rangle \dashv \Delta}$$

ESROLLERR
$$\frac{\Gamma \vdash e \Leftarrow (\!|\!|\!) \rightsquigarrow d : \tau \dashv \Delta}{\Gamma \vdash (\!|\mathtt{roll}(e)|\!)^u \Rightarrow \mu(\!|\!|\!).(\!|\!|\!) \rightsquigarrow (\!|\mathtt{roll}^{\mu(\!|\!|\!).(\!|\!|\!)}(d\langle\tau \Rightarrow (\!|\!|\!)\rangle)|\!)^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \mu(\!|\!|\!).(\!|\!|\!)[\Gamma]}$$

ESUNROLL
$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \blacktriangleright_\mu \mu\pi.\tau'}{\Gamma \vdash \mathtt{unroll}(e) \Rightarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow \mathtt{unroll}(d\langle\tau \Rightarrow \mu\pi.\tau'\rangle) \dashv \Delta}$$

ESUnrollNotRec
$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta \qquad \tau \not\sim \mu \langle\!\langle\rangle\!\rangle.\langle\!\langle\rangle\!\rangle}{\Gamma \vdash \mathtt{unroll}\big(\langle\!| e |\!\rangle^{u \blacktriangleright}\big) \Rightarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow \mathtt{unroll}\Big(\langle\!| d |\!\rangle^{u \blacktriangleright}_{\mathsf{id}(\Gamma)}\Big) \dashv \Delta, u :: \mu\langle\!\langle\rangle\!\rangle.\langle\!\langle\rangle\!\rangle \, [\Gamma]}$$

ESInjErr
$$\frac{\Gamma \vdash e \Leftarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \langle\!| \mathtt{inj}_C(e) |\!\rangle^u \Rightarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow \langle\!| \mathtt{inj}^{\tau'}_C(d\langle \tau \Rightarrow \langle\!\langle\rangle\!\rangle\rangle) |\!\rangle^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \langle\!\langle\rangle\!\rangle \, [\Gamma]}$$

ESEHole
$$\frac{}{\Gamma \vdash \langle\!\langle\rangle\!\rangle^u \Rightarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow \langle\!\langle\rangle\!\rangle^u_{\mathsf{id}(\Gamma)} \dashv u :: \langle\!\langle\rangle\!\rangle \, [\Gamma]}$$

ESNEHole
$$\frac{\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle\!| e |\!\rangle^u \Rightarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow \langle\!| d |\!\rangle^u_{\mathsf{id}(\Gamma)} \dashv \Delta, u :: \langle\!\langle\rangle\!\rangle \, [\Gamma]}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_2 \dashv \Delta}$ e analyzes against type τ_1 and elaborates to d of consistent type τ_2

EARoll
$$\frac{\tau \blacktriangleright_\mu \mu\pi.\tau' \qquad \Gamma \vdash e \Leftarrow [\mu\pi.\tau'/\pi]\tau' \rightsquigarrow d : \tau'' \dashv \Delta}{\Gamma \vdash \mathtt{roll}(e) \Leftarrow \tau \rightsquigarrow \mathtt{roll}^{\mu\pi.\tau'}(d\langle \tau'' \Rightarrow [\mu\pi.\tau'/\pi]\tau'\rangle) : \mu\pi.\tau' \dashv \Delta}$$

EARollNotRec
$$\frac{\tau \not\sim \mu\langle\!\langle\rangle\!\rangle.\langle\!\langle\rangle\!\rangle \qquad \Gamma \vdash e \Leftarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow d : \tau' \dashv \Delta}{\Gamma \vdash \langle\!| \mathtt{roll}(e) |\!\rangle^u \Leftarrow \tau \rightsquigarrow \langle\!| \mathtt{roll}^{\mu\langle\!\langle\rangle\!\rangle.\langle\!\langle\rangle\!\rangle}(d) |\!\rangle^u_{\mathsf{id}(\Gamma)} : \mu\langle\!\langle\rangle\!\rangle.\langle\!\langle\rangle\!\rangle \dashv \Delta, u :: \mu\langle\!\langle\rangle\!\rangle.\langle\!\langle\rangle\!\rangle \, [\Gamma]}$$

EAInjHole
$$\frac{\Gamma \vdash e \Leftarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow \mathtt{inj}^{\tau'}_C(d) : \tau' \dashv \Delta}$$

EAInj
$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \tau_j \rightsquigarrow d : \tau'_j \dashv \Delta}{\Gamma \vdash \mathtt{inj}_{C_j}(e) \Leftarrow \tau \rightsquigarrow \mathtt{inj}^{\tau}_{C_j}(d\langle \tau'_j \Rightarrow \tau_j\rangle) : \tau \dashv \Delta}$$

EAInjTagErr
$$\frac{C \notin \mathcal{C} \qquad \Gamma \vdash e \Leftarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow d : \tau \dashv \Delta \qquad \tau' = +\{C(\tau)\}}{\Gamma \vdash \mathtt{inj}_C(e) \Leftarrow +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \rightsquigarrow \mathtt{inj}^{\tau'}_C(d)\langle \tau' \Rightarrow \langle\!\langle\rangle\!\rangle\rangle : \langle\!\langle\rangle\!\rangle \dashv \Delta}$$

EAInjUnexpectedArg
$$\frac{\begin{array}{c}\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j = \varnothing \qquad e \neq \varnothing \\ \Gamma \vdash e \Leftarrow \langle\!\langle\rangle\!\rangle \rightsquigarrow d : \tau_j \dashv \Delta \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\tau_j)\}\Big\}\end{array}}{\Gamma \vdash \langle\!| \mathtt{inj}_{C_j}(e) |\!\rangle^u \Leftarrow \tau \rightsquigarrow \langle\!| \mathtt{inj}^{\tau'}_{C_j}(d) |\!\rangle^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau \, [\Gamma]}$$

EAInjExpectedArg
$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \tau_j \neq \varnothing \qquad \tau' = +\Big\{\{C_i(\tau_i)\}_{C_i \in \mathcal{C} \setminus C_j} \cup \{C_j(\varnothing)\}\Big\}}{\Gamma \vdash \langle\!| \mathtt{inj}_{C_j}(\varnothing) |\!\rangle^u \Leftarrow \tau \rightsquigarrow \langle\!| \mathtt{inj}^{\tau'}_{C_j}(\varnothing) |\!\rangle^u_{\mathsf{id}(\Gamma)} : \tau \dashv \Delta, u :: \tau \, [\Gamma]}$$

EASubsume
$$\frac{e \neq \langle\!\langle\rangle\!\rangle^u \qquad e \neq \langle\!| e' |\!\rangle^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

EAEHole
$$\frac{}{\Gamma \vdash \langle\!\langle\rangle\!\rangle^u \Leftarrow \tau \rightsquigarrow \langle\!\langle\rangle\!\rangle^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau \, [\Gamma]}$$

EANEHole
$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash \langle\!| e |\!\rangle^u \Leftarrow \tau \rightsquigarrow \langle\!| d |\!\rangle^u_{\mathsf{id}(\Gamma)} : \tau \dashv u :: \tau \, [\Gamma]}$$

2.3 Type Assignment

$\boxed{\Delta; \Gamma \vdash d : \tau}$ d is assigned type τ

TAUNIT
$$\frac{}{\Delta; \Gamma \vdash \varnothing : \varnothing}$$

TAVAR
$$\frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

TALAM
$$\frac{\emptyset \vdash \tau \, \mathsf{valid} \qquad \Delta; \Gamma, x : \tau \vdash d : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.d : \tau \to \tau'}$$

TAAPP
$$\frac{\Delta; \Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash d_2 : \tau_2}{\Delta; \Gamma \vdash d_1(d_2) : \tau}$$

TAROLL
$$\frac{\emptyset \vdash \mu\pi.\tau \, \mathsf{valid} \qquad \Delta; \Gamma \vdash d : [\mu\pi.\tau/\pi]\tau}{\Delta; \Gamma \vdash \mathtt{roll}^{\mu\pi.\tau}(d) : \mu\pi.\tau}$$

TAUNROLL
$$\frac{\Delta; \Gamma \vdash d : \mu\pi.\tau}{\Delta; \Gamma \vdash \mathtt{unroll}(d) : [\mu\pi.\tau/\pi]\tau}$$

TAINJ
$$\frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \qquad C_j \in \mathcal{C} \qquad \Delta; \Gamma \vdash d : \tau_j}{\Delta; \Gamma \vdash \mathtt{inj}^\tau_{C_j}(d) : \tau}$$

TAEHOLE
$$\frac{u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!|\;|\!)^u_\sigma : \tau}$$

TANEHOLE
$$\frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!| d |\!)^u_\sigma : \tau}$$

TAMHOLE
$$\frac{\Delta; \Gamma \vdash d : \tau' \qquad u :: \tau[\Gamma'] \in \Delta \qquad \Delta; \Gamma \vdash \sigma : \Gamma'}{\Delta; \Gamma \vdash (\!| d |\!)^{u \blacktriangleright}_\sigma : \tau}$$

TACAST
$$\frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow \tau_2 \rangle : \tau_2}$$

TAFAILEDCAST
$$\frac{\Delta; \Gamma \vdash d : \tau_1 \qquad \tau_1 \, \mathrm{ground} \qquad \tau_2 \, \mathrm{ground} \qquad \tau_1 \neq \tau_2}{\Delta; \Gamma \vdash d\langle \tau_1 \Rightarrow (\!|\;|\!) \not\Rightarrow \tau_2 \rangle : \tau_2}$$

3 Dynamic Semantics

$\boxed{\tau \, \mathrm{ground}}$ τ is a ground type

GARR
$$\frac{}{(\!|\;|\!) \to (\!|\;|\!) \, \mathrm{ground}}$$

GREC
$$\frac{}{\mu(\!|\;|\!).(\!|\;|\!) \, \mathrm{ground}}$$

GSUM
$$\frac{\{\tau_i = \varnothing \lor \tau_i = (\!|\;|\!)\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \, \mathrm{ground}}$$

$\boxed{\tau \blacktriangleright_{\mathsf{ground}} \tau'}$ τ has matched ground type τ'

MGARR
$$\frac{\tau_1 \to \tau_2 \neq (\!|\;|\!) \to (\!|\;|\!)}{\tau_1 \to \tau_2 \blacktriangleright_{\mathsf{ground}} (\!|\;|\!) \to (\!|\;|\!)}$$

MGREC
$$\frac{\tau \neq (\!|\;|\!)}{\mu\pi.\tau \blacktriangleright_{\mathsf{ground}} \mu(\!|\;|\!).(\!|\;|\!)}$$

MGSUM
$$\frac{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \neq +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}} \qquad \{(\tau_i = \varnothing \implies \tau'_i = \varnothing) \land (\tau_i \neq \varnothing \implies \tau'_i = (\!|\;|\!))\}_{C_i \in \mathcal{C}}}{+\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \blacktriangleright_{\mathsf{ground}} +\{C_i(\tau'_i)\}_{C_i \in \mathcal{C}}}$$

$\boxed{d \, \mathrm{final}}$ d is final

FBOXEDVAL
$$\frac{d \, \mathrm{boxedval}}{d \, \mathrm{final}}$$

FINDET
$$\frac{d \, \mathrm{indet}}{d \, \mathrm{final}}$$

$\boxed{d \text{ val}}$ d is a value

$$\frac{}{\varnothing \text{ val}} \text{ VUNIT}$$

$$\frac{}{\lambda x{:}\tau.d \text{ val}} \text{ VLAM}$$

$$\text{VROLL} \quad \frac{d \text{ val}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ val}}$$

$$\text{VINJ} \quad \frac{d \text{ val}}{\texttt{inj}_{\mathbf{C}}^{\tau}(d) \text{ val}}$$

$\boxed{d \text{ boxedval}}$ d is a boxed value

$$\text{BVVAL} \quad \frac{d \text{ val}}{d \text{ boxedval}}$$

$$\text{BVROLL} \quad \frac{d \text{ boxedval}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ boxedval}}$$

$$\text{BVINJ} \quad \frac{d \text{ boxedval}}{\texttt{inj}_{\mathbf{C}}^{\tau}(d) \text{ boxedval}}$$

$$\text{BVARRCAST} \quad \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}$$

$$\text{BVRECCAST} \quad \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ boxedval}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ boxedval}}$$

$$\text{BVSUMCAST} \quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \quad \tau \neq \tau' \qquad d \text{ boxedval}}{d\langle \tau \Rightarrow \tau' \rangle \text{ boxedval}}$$

$$\text{BVHOLECAST} \quad \frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \textcircled{\parallel} \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$ d is indeterminate

$$\text{IEHOLE} \quad \frac{}{\textcircled{\parallel}_{\sigma}^{u} \text{ indet}}$$

$$\text{INEHOLE} \quad \frac{d \text{ final}}{\langle\!\langle d \rangle\!\rangle_{\sigma}^{u} \text{ indet}}$$

$$\text{IAPP} \quad \frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \qquad d_1 \text{ indet} \qquad d_2 \text{ final}}{d_1(d_2) \text{ indet}}$$

$$\text{IROLL} \quad \frac{d \text{ indet}}{\texttt{roll}^{\mu\pi.\tau}(d) \text{ indet}}$$

$$\text{IUNROLL} \quad \frac{d \text{ indet}}{\texttt{unroll}(d) \text{ indet}}$$

$$\text{IINJ} \quad \frac{d \text{ indet}}{\texttt{inj}_{\mathbf{C}}^{\tau}(d) \text{ indet}}$$

$$\text{IINJHOLE} \quad \frac{C \neq \mathbf{C} \qquad d \text{ final}}{\texttt{inj}_{C}^{\tau}(d) \text{ indet}}$$

$$\text{ICASTGROUNDHOLE} \quad \frac{d \text{ indet} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \textcircled{\parallel} \rangle \text{ indet}}$$

$$\text{ICASTHOLEGROUND} \quad \frac{d \neq d'\langle \tau' \Rightarrow \textcircled{\parallel} \rangle \qquad d \text{ indet} \qquad \tau \text{ ground}}{d\langle \textcircled{\parallel} \Rightarrow \tau \rangle \text{ indet}}$$

$$\text{ICASTARR} \quad \frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ indet}}$$

$$\text{ICASTREC} \quad \frac{\mu\pi.\tau \neq \mu\pi'.\tau' \qquad d \text{ indet}}{d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle \text{ indet}}$$

$$\text{ICASTSUM} \quad \frac{\tau = +\{C_i(\tau_i)\}_{C_i \in \mathcal{C}} \quad \tau' = +\{C_i(\tau_i')\}_{C_i \in \mathcal{C}} \quad \tau \neq \tau' \qquad d \text{ indet}}{d\langle \tau \Rightarrow \tau' \rangle \text{ indet}}$$

$$\text{IFAILEDCAST} \quad \frac{d \text{ final} \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow \textcircled{\parallel} \not\Rightarrow \tau_2 \rangle \text{ indet}}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

$$\text{ITAPP} \quad \frac{[d_2 \text{ final}]}{(\lambda x{:}\tau.d_1)(d_2) \longrightarrow [d_2/x]d_1}$$

$$\text{ITUNROLL} \quad \frac{[d \text{ final}]}{\texttt{unroll}(\texttt{roll}^{\mu\pi.\tau}(d)) \longrightarrow d}$$

$$\text{ITAPPCAST} \quad \frac{[d_1 \text{ final}] \qquad [d_2 \text{ final}] \qquad \tau_1 \to \tau_2 \neq \tau_1' \to \tau_2'}{d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2' \rangle(d_2) \longrightarrow (d_1(d_2\langle \tau_1' \Rightarrow \tau_1 \rangle))\langle \tau_2 \Rightarrow \tau_2' \rangle}$$

$$\text{ITUNROLLCAST} \quad \frac{[d \text{ final}] \qquad \mu\pi.\tau \neq \mu\pi'.\tau'}{\texttt{unroll}(d\langle \mu\pi.\tau \Rightarrow \mu\pi'.\tau' \rangle) \longrightarrow \texttt{unroll}(d)\langle [\mu\pi.\tau/\pi]\tau \Rightarrow [\mu\pi'.\tau'/\pi']\tau' \rangle}$$

$$\text{ITCASTID} \quad \frac{[d \text{ final}]}{d\langle \tau \Rightarrow \tau \rangle \longrightarrow d}$$

$$\frac{\text{ITCastSucceed}}{[d \text{ final}] \quad \tau \text{ ground}}{d\langle \tau \Rightarrow \textcircled{\Downarrow} \Rightarrow \tau \rangle \longrightarrow d}$$

$$\frac{\text{ITCastFail}}{[d \text{ final}] \quad \tau_1 \neq \tau_2 \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground}}{d\langle \tau_1 \Rightarrow \textcircled{\Downarrow} \Rightarrow \tau_2 \rangle \longrightarrow d\langle \tau_1 \Rightarrow \textcircled{\Downarrow} \not\Rightarrow \tau_2 \rangle}$$

$$\frac{\text{ITGround}}{[d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle \tau \Rightarrow \textcircled{\Downarrow} \rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow \textcircled{\Downarrow} \rangle}$$

$$\frac{\text{ITExpand}}{[d \text{ final}] \quad \tau \blacktriangleright_{\text{ground}} \tau'}{d\langle \textcircled{\Downarrow} \Rightarrow \tau \rangle \longrightarrow d\langle \textcircled{\Downarrow} \Rightarrow \tau' \Rightarrow \tau \rangle}$$

$$\text{EvalCtx} \quad \mathcal{E} \quad ::= \quad \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid \texttt{roll}^{\mu\pi.\tau}(\mathcal{E}) \mid \texttt{unroll}(\mathcal{E}) \mid \texttt{inj}_C^\tau(\mathcal{E}) \mid (\!(\mathcal{E})\!)_\sigma^u \mid (\!(\mathcal{E})\!)_\sigma^{u\blacktriangleright}$$
$$\mid \mathcal{E}\langle \tau \Rightarrow \tau \rangle \mid \mathcal{E}\langle \tau \Rightarrow \textcircled{\Downarrow} \not\Rightarrow \tau \rangle$$

$\boxed{d = \mathcal{E}\{d'\}}$ $\quad d$ is obtained by placing d' at the mark in $\mathcal{E}$

$$\frac{\text{FHOuter}}{d = \circ\{d\}}$$

$$\frac{\text{FHApp1}}{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}$$

$$\frac{\text{FHApp2}}{[d_1 \text{ final}] \quad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

$$\frac{\text{FHRoll}}{d = \mathcal{E}\{d'\}}{\texttt{roll}^{\mu\pi.\tau}(d) = \texttt{roll}^{\mu\pi.\tau}(\mathcal{E})\{d'\}}$$

$$\frac{\text{FHUnroll}}{d = \mathcal{E}\{d'\}}{\texttt{unroll}(d) = \texttt{unroll}(\mathcal{E})\{d'\}}$$

$$\frac{\text{FHInj}}{d = \mathcal{E}\{d'\}}{\texttt{inj}_C^\tau(d) = \texttt{inj}_C^\tau(\mathcal{E})\{d'\}}$$

$$\frac{\text{FNEHoleInside}}{d = \mathcal{E}\{d'\}}{(\!(d)\!)_\sigma^u = (\!(\mathcal{E})\!)_\sigma^u\{d'\}}$$

$$\frac{\text{FMHoleInside}}{d = \mathcal{E}\{d'\}}{(\!(d)\!)_\sigma^{u\blacktriangleright} = (\!(\mathcal{E})\!)_\sigma^{u\blacktriangleright}\{d'\}}$$

$$\frac{\text{FHCastInside}}{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \tau_2 \rangle = \mathcal{E}\langle \tau_1 \Rightarrow \tau_2 \rangle\{d'\}}$$

$$\frac{\text{FHFailedCast}}{d = \mathcal{E}\{d'\}}{d\langle \tau_1 \Rightarrow \textcircled{\Downarrow} \not\Rightarrow \tau_2 \rangle = \mathcal{E}\langle \tau_1 \Rightarrow \textcircled{\Downarrow} \not\Rightarrow \tau_2 \rangle\{d'\}}$$

$\boxed{d \mapsto d'}$ $\quad d$ steps to d'

$$\frac{\text{Step}}{d = \mathcal{E}\{d_0\} \quad d_0 \longrightarrow d_0' \quad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$