Invariant Annuli and Singular Fibers

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Spring 2023

1 Introduction

This writeup is the first step toward showing that one dimensional orbits of $\mathrm{Homeo}_0(S^1)$ actions on compact three-manifolds are regular fibers of the associated Seifert fibration. In particular, the goal here is to prove that when there is an invariant annulus accumulating onto a given 1-diemnsional orbit, it must be regular. The proof of this fact proceeds by making use of a simple observation about sequences of points in invariant surfaces of $\mathrm{Homeo}_0(S^1)$ actions on spaces.

2 Main trick

We have the following fact which more or less amounts to a restatement of continuity.

Remark 1. Suppose that $\mathrm{Homeo}_0(S^1)$ acts on a compact three-manifold M, and A is an invariant annulus with standard model coordinates $\varphi : S^1 \times (0,1) \to A$ and σ a 1-dimensional orbit in $\overline{A} - A$. Then, if two sequences of points in A with constant S^1 coordinate under φ converge in M to $p \in \sigma$, then their S^1 coordinate must be the same.

Look for ways to relax conditions here, espec. on S^1 coord

Note that the proof of this fact depends on orbit structure of a point stabilizer subgroup of $\mathrm{Homeo}_0(S^1)$ on a model annulus. See Mann-Chen.

Track cites sooner rather than later

Pf of remark.

This proof is tricky to extract from the details of the following section, need to work more

□

Then, when seeking a contradiciton, the goal is to search for point sequences with distinct S^1 coordinates converging to the same $p \in \sigma$. The main motivation here is that a singular orbit σ very naturally affords us such point sequences by "wrapping the annulus around" multiple times.

3 First Result

Proposition 1. Suppose that $\mathrm{Homeo}_0(S^1)$ acts on a compact three-manifold M, A is an annulus orbit, and σ is a 1-dimensional orbit in $\overline{A} - A$. Then, σ must be a regular fiber of the associated Seifert fibration.

Because of my difficulties in parsing out and proving this trick independently, the following is a direct proof of this proposition which does not appeal to the trick. Hopefully it will become clearer as I go forward how to parse this proof into two pieces.

Proof. Let $p \in \sigma$ be the fixed point on σ of $G_0 \subset \mathrm{Homeo}_0(S^1)$. Now, let $\gamma := \{(x, 0) \in [0,1] \times S^1\}$. In particular, since γ is an orbit of G_0 acting on the model annulus, $\varphi(\gamma)$ is an orbit of G_0 acting on M. So, $\overline{\varphi(\gamma)}$ is also an invariant set, that is $\overline{\varphi(\gamma)} - \varphi(\gamma)$ is a union of orbits of G_0. Furthermore, note that since γ is a continuous curve whose projection onto the first coordinate of the model annulus is all of $(0,1)$, $\overline{\varphi(\gamma)}$ intersects every S^1 fiber in $\overline{A}$, so $\overline{\varphi(\gamma)} \cap \sigma \neq \emptyset$. Since the action of G_0 divides σ into an invariant open interval, and a fixed point, we must have that $p \in \overline{\varphi(\gamma)}$. Thus, there is a sequence of points $\{x_k\}$ in $\varphi(\gamma)$, which converges to p, and by definition, the S^1 coordinates of the x_k's under φ are all 0.

Now, to construct the second sequence of points, suppose toward a contradiction that σ is singular, then there is another θ such that G_θ fixes p. Then, apply the same construction with G_θ rather than G_0 to produce a sequence of points x'_k. Note that under any element f of G_0, we must have by continuity that $f(x'_k)$ converges to $f(p) = p$, but by sufficiently disturbing the S^1 coordinate away from 0, $f(x'_k)$ cannot converge to p. $\square$

I am suddenly uneasy about this part. In fact, it's become clear that the initial point sequence may not even be necessary. All I need to violate continuity is a point sequence whose image does not converge to what it should.

This proof can be extended to a general invariant annulus by noting that if A is such an invariant annulus and there is no annulus orbit meeting the conditions of the previous proposition, there must be a sequence of 1-dimensional orbits accumulating onto σ. Then the argument will proceed by taking the point sequences to be the respective sequences of the fixed points of G_0 and G_θ on the orbits of the sequence of 1-dimensional orbits. Then apply the same kind of map.

Again, I am concerned that the construction of the desired map is trickier than I previously thought. I'm not sure how to "sufficiently disturb" the points of the second sequence so that they no longer converge to what they're meant to.

Notes