

Orbit Bundle Theorem and Applications

Hazel Brenner

Spring 2023

1 Introduction

This writeup is an extension of what I've written in No Badly Compactified Annuli. This will prove a general version of the *Orbit Bundle Structure* which concerns the structure of connected regions consisting of orbits of a fixed dimension. As of writing, this result takes as an input that the region is a manifold (would be ideal to have settings that this can be relaxed).

The setting of the generalized bundle structure result is the following. Suppose $\text{Homeo}_0(M)$ acts on N , and X is a connected submanifold of N with orbits of fixed dimension. Fix a base point $b \in \text{Conf}_n(M)$. We frequently need to refer to both a point in $\text{Conf}_n(M)$ and the associated set of points in M . To avoid confusion, if x is a point in $\text{Conf}_n(M)$, we will denote by $x_{\{\cdot\}}$ the corresponding set of points in M . Also, note that the deck group of the maximal admissible covering $C_{b_{\{\cdot\}}} \rightarrow \text{Conf}_n(M)$ can be identified with $\text{stab}(b_{\{\cdot\}})/\text{stab}_0(b_{\{\cdot\}}) \subset \text{Homeo}_0(M)$.

Proposition 1.1 (Orbit Bundle). Suppose A a connected submanifold of N with orbits of fixed dimension, then there is a continuous map $p : A \rightarrow \text{Conf}_n(M)$ such that there's a map:

$$I : C_{b_{\{\cdot\}}} \times p^{-1}(b)/\Gamma \rightarrow A.$$

where Γ is the deck group of the covering $C_{b_{\{\cdot\}}} \rightarrow \text{Conf}_n(M)$ acting on the first factor by deck transformations and the second factor by the $\text{Homeo}_0(M)$ action restricted to the subgroup Γ acting on the fiber. The map I is an equivariant homeomorphism with the $\text{Homeo}_0(M)$ action on $C_{b_{\{\cdot\}}} \times p^{-1}(b)/\Gamma$ being the quotient of the product of the standard action on $C_{b_{\{\cdot\}}}$ with the trivial action on the fiber.

Proof. First, suppose O is an orbit in A . Recall that O is a continuous injective image of an admissible cover of $\text{Conf}_n(M)$ with a lift of the standard configuration action. So, in particular, for every point $x \in O$, there is a unique $X \in \text{Conf}_n(M)$ such that $\text{stab}_0(X) \subseteq G_x \subseteq \text{stab}(X)$. Then, define $p(x)_{\{\cdot\}} = X$. Since $G_{\rho(f)(x)} = fG_x f^{-1}$ it follows that $\text{stab}_0(f(X)) \subseteq G_{\rho(f)(x)} \subseteq \text{stab}(f(X))$. This directly implies p is equivariant.

Next we wish to show that p is continuous. We will do this by checking continuity on a nice basis. In particular, note that if \mathcal{B} is a basis for the topology on M , then the set of all products of n disjoint elements of \mathcal{B} form a basis for the topology on $\text{PConf}_n(M)$ i.e., the n -fold product of M with itself minus the fat diagonal. Then, the image of this basis under the quotient forms a basis for the topology on $\text{Conf}_n(M)$. Let U be image of one such $B_1 \times \cdots \times B_n$ under the quotient map to $\text{Conf}_n(M)$.

Claim. Let $H(B_i)$ denote the $\text{Homeo}_0(M)$ subgroup of homeomorphisms supported on B_i .

$$p^{-1}(U) = A \setminus \bigcup \text{fix}(H(B_i))$$

Proof. By the equivariance of p and the *locally continuously transitive* property of homeomorphism group actions

$$\text{fix}(H(B_i)) = \{x \in A \mid p(x)_{\{\cdot\}} \cap B_i = \emptyset\}$$

Interpreted semantically, this says that the fixed point set of $H(B_i)$ acting on A is the set of all points whose image under p has no coordinate in B_i . Then, by making purely set-theoretic manipulations, we have the

following chain of equivalences:

$$\begin{aligned}
A - \bigcup_{i=1}^n \text{fix}(H(B_i)) &= A - \bigcup_{i=1}^n \{x \in A \mid p(x)_{\{\cdot\}} \cap B_i = \emptyset\} \\
&= \bigcap_{i=1}^n A - \{x \in A \mid p(x)_{\{\cdot\}} \cap B_i = \emptyset\} \\
&= \bigcap_{i=1}^n \{x \in A \mid p(x)_{\{\cdot\}} \cap B_i \neq \emptyset\} \\
&= p^{-1}(U)
\end{aligned}$$

□

So, $p^{-1}(U)$ is open since $\bigcup_{i=1}^n \text{fix}(H(B_i))$ is a finite union of closed sets and thus closed. In particular, this shows that p is continuous.

Now, fixing some base-point $b \in \text{Conf}_n(M)$, let $F := p^{-1}(b)$. There is an action of $\text{stab}(b_{\{\cdot\}})/\text{stab}_0(b_{\{\cdot\}})$ on F given by noting that F is invariant under the restriction of the $\text{Homeo}_0(M)$ action to the subgroup $\text{stab}(b_{\{\cdot\}})$. Also, every element of $\text{stab}_0(b_{\{\cdot\}})$ fixes F pointwise, so there is an action of $\text{stab}(b_{\{\cdot\}})/\text{stab}_0(b_{\{\cdot\}})$ on F by homeomorphisms. Then, there is an action of the deck group of the covering $C_{b_{\{\cdot\}}} \rightarrow \text{Conf}_n(M)$ on $C_{b_{\{\cdot\}}} \times F$ given by deck transformations on the first factor and the previously described action on the F factor. The next step is to show that there is an equivariant homeomorphism

$$I : C_{b_{\{\cdot\}}} \times F/\Gamma \rightarrow A.$$

This map is constructed as follows. In order, we must construct a map \hat{I} from $C_{b_{\{\cdot\}}} \times F$, show that \hat{I} is well-defined, descends to a map I on the quotient, I is continuous, I is proper (thus a hmoemorphism), and I is equivariant. Let $(x, y) \in C_{b_{\{\cdot\}}} \times F$, then suppose $f \in \text{Homeo}_0(M)$ such that $f^{-1}(x) = b$, then define $\hat{I}((x, y)) = \rho(f^{-1}(x))(y)$.

□

2 Codimension 1 Orbit Bundles

For components consisting of codimension 1 orbits, the fiber must be 1-dimensional and all 1-dimensional homology manifolds are manifolds. So, codimension 1 orbit bundles are classified by actions of the deck group of Γ of the covering $C_X \rightarrow \text{Conf}_n(M)$ on 1-manifolds up to equivariant homeomorphism. Note that, in general, $|\pi_0(F)| \leq |\Gamma|$ and the components of F are homeomorphic.

It's hard to say something overly general here, but fixing $M = S^1$ and $\dim(N) = 3$ a classification is possible. Note, in this setting $C_X = \text{PConf}_2(S^1)$ and $\Gamma = \mathbb{Z}_2$.

Proposition 2.1. The classification of $\text{Homeo}_0(S^1)$ codimension 1 orbit bundles X in 3-manifolds is as follows.

F disconnected $F \cong C \sqcup C$ for a 1-manifold C , $X \cong \text{PConf}_2(S^1) \times C$ and $\pi_{SO(2)}(X) \cong I \times C$

F connected

\mathbb{Z}_2 **acts on F f.p. free** $F \cong S^1$, action is by a π rotation. $X \cong \text{PConf}_2(S^1) \times S^1$ and $\pi_{SO(2)} \cong I \times S^1$

\mathbb{Z}_2 **acts on F w/ fixed points** This depends on F , on each \mathbb{Z}_2 acts by reflection. Note in each case, the bundle is described in terms of the Seifert structure given by the $SO(2)$ action.

$F \cong \mathbb{R}$ X is $(\mathbb{D}^2, (2, 1))$

$F \cong S^1$ X is $(\mathbb{D}^2, (2, 1), (2, 1))$

$F \cong [0, 1]$ X is $(\mathbb{D}^2 \cup I, (2, 1))$ where I is a interval in the boundary of the closed disk \mathbb{D}^2

\mathbb{Z}_2 **acts trivially** $X \cong \text{Conf}_2(S^1) \times C$ *N.B. resulting bundle is nonorientable (and top dimensional), so N is as well*

A generic component C of the fiber is one of S^1 , \mathbb{R} , $[0, 1]$ or $[0, 1)$

Proof. Note, the quotient gives a 2-fold covering from $\text{PConf}_2(S^1) \times F$ to the orbit bundle X , which is connected, so $\text{PConf}_2(S^1) \times F$ and thus F has at most two components. Assume F has two components. Since the quotient is connected, the action must permute the components (homeomorphically), so in particular we can express the fiber as $F \cong C \sqcup C$. Then the action is just a homeomorphism from one component of $\text{PConf}_2(S^1) \times F$ to the other, so the quotient is $\text{PConf}_2(S^1) \times C$ where the resulting $\text{Homeo}_0(S^1)$ action is the product of the standard configuration space action on $\text{PConf}_2(S^1)$ and the trivial action on the fiber C .

If F has a single component, then the \mathbb{Z}_2 action is simply a choice of involution on F . For any possible F , we can allow \mathbb{Z}_2 to act trivially. Then, the quotient loads entirely onto the $\text{PConf}_2(S^1)$ factor, where \mathbb{Z}_2 acts as the deck group of $\text{PConf}_2(S^1) \rightarrow \text{Conf}_2(S^1)$, so the resulting quotient is $\text{Conf}_2(S^1) \times F$. This is nonorientable (regardless of F) and top-dimensional, so in general, such orbit bundles only arise when N is nonorientable. The remainder of the proof is done by exhaustion. $[0, 1)$ has no nontrivial involutions, $[0, 1]$ and \mathbb{R} each have one and S^1 has two. In each case, the quotient preserves the product structure away from fixed points and sends every $\text{PConf}_2(S^1) \times \{x\}$ for fixed point x to $\text{Conf}_2(S^1) \times \{x\}$. The result in each case is trivially the Seifert structure described in the proposition. \square