# Orbit Bundle Theorem and Applications

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Spring 2023

## 1 Introduction

This writeup is an extension of what I've written in No Badly Compactified Annuli. This will prove a general version of the *Orbit Bundle Structure* which concerns the structure of connected regions consisting of orbits of a fixed dimension. As of writing, this result takes as an input that the region is a manifold (would be ideal to have settings that this can be relaxed).

The setting of the generalized bundle structure result is the following. Suppose $\mathrm{Homeo}_0(M)$ acts on $N$, and $X$ is a connected submanifold of $N$ with orbits of fixed dimension. Fix a base point $b \in \mathrm{Conf}_n(M)$. We will abuse notation by denoting both the point in $\mathrm{Conf}_n(M)$ and the corresponding subset of $M$ as $b$. Also, note that the deck group of the maximal admissible covering $C_b \to \mathrm{Conf}_n(M)$ can be identified with $\mathrm{stab}(b)/\mathrm{stab}_0(b) \subset \mathrm{Homeo}_0(M)$.

**Proposition 1.1** (Orbit Bundle)**.** Suppose $X$ a connected submanifold of $N$ with orbits of fixed dimension, then there is a continuous map $p : X \to \mathrm{Conf}_n(M)$ such that there's a map:

$$I : C_n \times p^{-1}(b)/\Gamma \to X$$

where $\Gamma$ is the deck group of the covering $C_n \to \mathrm{Conf}_n(M)$ acting on the first factor by deck transformations and the second factor by the $\mathrm{Homeo}_0(M)$ action restricted to the subgroup $\Gamma$ acting on the fiber. The map $I$ is an equivariant homeomorphism with the $\mathrm{Homeo}_0(M)$ action on $C_n \times p^{-1}(b)/\Gamma$ be the quotient of the product of the standard action on $C_n$ with the trivial action on the fiber.

*Proof.* TODO $\square$

## 2 Codimension 1 Orbit Bundles

For components of codimension 1 orbits, the fiber must be 1-dimensional and all 1-dimensional homology manifolds are manifolds. So, codimension 1 orbit bundlses are classified by actions of the deck group of $\Gamma$ of the covering $C_X \to \mathrm{Conf}_n(M)$ on 1-manifolds up to equivariant homeomorphism. Note that, in general, $|\pi_0(F)| \leq |\Gamma|$ and the components of $F$ are homeomorphic.

It's hard to say something overly general here, but fixing $M = S^1$ and $\dim(N) = 3$ a classification is possible. Note, in this setting $C_X = \mathrm{PConf}_2(S^1)$ and $\Gamma = \mathbb{Z}_2$.

**Proposition 2.1.** The classification of $\mathrm{Homeo}_0(S^1)$ codimension 1 orbit bundles $X$ in 3-manifolds is as follows.

**$F$ disconnected** $F \cong C \sqcup C$ for a 1-manifold $C$, $X \cong \mathrm{PConf}_2(S^1) \times C$ and $\pi_{SO(2)}(X) \cong I \times C$

**$F$ connected**

- **$\mathbb{Z}_2$ acts on $F$ f.p. free** $F \cong S^1$, action is by a $\pi$ rotation. $X \cong \mathrm{PConf}_2(S^1) \times S^1$ and $\pi_{SO(2)} \cong I \times S^1$
- **$\mathbb{Z}_2$ acts on $F$ w/ fixed points** This depends on $F$, on each $\mathbb{Z}_2$ acts by reflection. Note in each case, the bundle is described in terms of the Seifert structure given by the $SO(2)$ action.
  - $F \cong \mathbb{R}$ $X$ is $(\mathring{\mathbb{D}}^2, (2, 1))$
  - $F \cong S^1$ $X$ is $(\mathring{\mathbb{D}}^2, (2, 1), (2, 1))$

$F \cong [0,1]$ $X$ is $(\mathring{\mathbb{D}}^2 \cup I, (2,1))$ where $I$ is a interval in the boundary of the closed disk $\mathbb{D}^2$

$\mathbb{Z}_2$ **acts trivially** $X \cong \mathrm{Conf}_2(S^1) \times C$ *N.B. resulting bundle is nonorientable (and top dimensional), so $N$ is as well*

A generic component $C$ of the fiber is one of $S^1$, $\mathbb{R}$, $[0,1]$ or $[0,1)$

*Proof.* Note that the quotient gives a 2-fold covering from $\mathrm{PConf}_2(S^1) \times F$ to the orbit bundle $X$, which is connected, so $\mathrm{PConf}_2(S^1) \times F$ and thus $F$ has at most two components. Assume $F$ has two components. Since the quotient is connected, the action must permute the components (homeomorphically), so in particular we can express the fiber as $F \cong C \sqcup C$. Then the action is just a homeomorphism from one compenent of $\mathrm{PConf}_2(S^1) \times F$ to the other, so the quotient is $\mathrm{PConf}_2(S^1) \times C$ where the resulting $\mathrm{Homeo}_0(S^1)$ action is the standard configuration space action on the $\mathrm{PConf}_2(S^1)$ coordinate and the trivial action on the $C$ coordinate.

If $F$ has a single component, then the $\mathbb{Z}_2$ action is simply a choice of involution on $F$. For any possible $F$ we can allow $\mathbb{Z}_2$ to act trivially. Then, the quotient loads entirely onto the $\mathrm{PConf}_2(S^1)$ factor, where $\mathbb{Z}_2$ acts as the deck group of $\mathrm{PConf}_2(S^1) \to \mathrm{Conf}_2(S^1)$, so the resulting quotient is $\mathrm{Conf}_2(S^1) \times F$. This is nonorientable (regardless of $F$) and top-dimensional, so in general, such orbit bundles only arise when $N$ is nonorientable. The remainder of the proof is done by exhaustion. $[0,1)$ has no nontrivial involutions, $[0,1]$ and $\mathbb{R}$ each have one and $S^1$ has two. In each case, the quotient preserves the product structure away from fixed points and sends every $\mathrm{PConf}_2(S^1) \times \{x\}$ for fixed point $x$ to $\mathrm{Conf}_2(S^1) \times \{x\}$. The result in each case is trivially the Seifert structure described in the proposition. $\square$