Open Regions Foliated by Annulus Orbits

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Summer 2023

1 Introduction

The following is a result about the structure of compact three-manifolds admitting fixed point-free $\mathrm{Homeo}_0(S^1)$ actions having only one dimensional and annulus orbits.

2 Global structure

Such actions have orbits consisting of a closed set with all S^1 fibers invariant whose complement is a union of open regions foliated by annulus orbits. A natural question is what can be said about the structure of a component of the complement of the 1-dimensional orbits. Two results are:

This has a 2 line justification, maybe obvious?

3 Foliation by Annuli

Proposition 1. Suppose X is a component of the open subset of M^3 in which all orbits are annuli. Then the decomposition of X into orbits is a C^0 foliation by annuli.

Proof sketch. Take a short curve segment through x, build two one paramater families of homeos, giving local product structure (do this explicitly, use flow of a bump vec field on circle). $\square$

flesh this out with details

4 Annuli Compactify Nicely

As a small point of notation, recall the following definition from point set topology:

Definition 1. The *frontier* of a set X, denoted $\mathrm{fr}(X)$ is defined as $\bar{X} - X$

Proposition 2. Suppose A is an annulus orbit in a component X of the open region foliated by annulus orbits. Then $\mathrm{fr}(A) \subset \mathrm{fr}(X)$

Proof. Suppose toward a contradiction that $\mathrm{fr}(A) \cap X \neq \emptyset$. Then, since every fiber in X is contained in some annulus orbit and $\bar{A}$ must be invariant, there is some invariant annulus $A' \subset \mathrm{fr}(A)$. Now, consider the restriction of the $\mathrm{Homeo}_0(S^1)$ action to the action of a point stabilizer subgroup G_0. Let I be a G_0 invariant interval in A. Let $\pi_{SO(2)}$ denote the projection along the orbits of the rotation subgroup to the base orbifold. Then, since $\pi_{SO(2)}(I) = \pi_{SO(2)}(A)$ and $\pi_{SO(2)}$ is continuous,

$$\pi_{SO(2)}(\bar{I}) = \overline{\pi_{SO(2)}(I)} = \overline{\pi_{SO(2)}(A)} = \pi_{SO(2)}(\bar{A}).$$

This implies that $\mathrm{fr}(I)$ intersects every fiber of $\mathrm{fr}(A)$ and in particular, every fiber of A'. Since $\bar{I}$ is closed and invariant, $\mathrm{fr}(I)$ must be a union of G_0 orbits in A' containing at least one interval orbit I' (the disk orbits are not closed). Thus, there is some sequence of points $\{x_n\} \subset I$ converging to a point $x \in I'$.

From [?], there are continuous bijections $f : I \to S^1 - \{0\}$ and $g : I' \to S^1 - \{0\}$ given by mapping the unique fixed point of $G_\theta \cap G_0$ on I to $\theta \in S^1 - \{0\}$ and similarly for I'. Then, up to a subsequence, $\{f(x_n)\}$ is a monotonic sequence converging to 0. So we can choose a monotonic element $\phi \in G_0$ satisfying $\phi(f(x_k)) = f(x_{k+1})$. In particular, we must have that $\{\rho(\phi)(x_n)\}$ converges to x, but by monotonicity, $\rho(\phi)(x) \neq x$, violating continuity, which is a contradiction. Thus $\mathrm{fr}(A) \cap X = \emptyset$ $\square$

Notes