Orbit Bundle Theorem and Applications

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1 Introduction

This writeup is an extension of what I've written in No Badly Compactified Annuli. This will prove a general version of the *Orbit Bundle Structure* which concerns the structure of connected regions consisting of orbits of a fixed dimension. As of writing, this result takes as an input that the region is a manifold (would be ideal to have settings that this can be relaxed).

The setting of the generalized bundle structure result is the following. Suppose $\mathrm{Homeo}_0(M)$ acts on N, and X is a connected submanifold of N with orbits of fixed dimension. Fix a base point $b \in \mathrm{Conf}_n(M)$. We frequently need to refer to both a point in $\mathrm{Conf}_n(M)$ and the associated set of points in M. To avoid confusion, if x is a point in $\mathrm{Conf}_n(M)$, we will denote by $x_{\{\cdot\}}$ the corresponding set of points in M. Also, note that the deck group of the maximal admissible covering $C_{b_{\{\cdot\}}} \to \mathrm{Conf}_n(M)$ can be identified with $\mathrm{stab}(b_{\{\cdot\}})/\mathrm{stab}_0(b_{\{\cdot\}}) \subset \mathrm{Homeo}_0(M)$.

Proposition 1.1 (Orbit Bundle)**.** Suppose A a connected submanifold of N with orbits of fixed dimension, then there is a continuous map $p : A \to \mathrm{Conf}_n(M)$ such that there's a map:

$$I : C_{b_{\{\cdot\}}} \times p^{-1}(b)/\Gamma \to A.$$

where Γ is the deck group of the covering $C_{b_{\{\cdot\}}} \to \mathrm{Conf}_n(M)$ acting on the first factor by deck transformations and the second factor by the $\mathrm{Homeo}_0(M)$ action restricted to the subgroup Γ acting on the fiber. The map I is an equivariant homeomorphism with the $\mathrm{Homeo}_0(M)$ action on $C_{b_{\{\cdot\}}} \times p^{-1}(b)/\Gamma$ being the quotient of the product of the standard action on $C_{b_{\{\cdot\}}}$ with the trivial action on the fiber.

Proof. First, suppose O is an orbit in A. Recall that O is a continuous injective image of an admissible cover of $\mathrm{Conf}_n(M)$ with a lift of the standard configuration action. So, in particular, for every point $x \in O$, there is a unique $X \in \mathrm{Conf}_n(M)$ such that $\mathrm{stab}_0(X) \subseteq G_x \subseteq \mathrm{stab}(X)$. Then, define $p(x)_{\{\cdot\}} = X$. Since $G_{\rho(f)(x)} = fG_xf^{-1}$ it follows that $\mathrm{stab}_0(f(X)) \subseteq G_{\rho(f)(x)} \subseteq \mathrm{stab}(f(X))$. This directly implies p is equivariant.

Next we wish to show that p is continuous. We will do this by checking continuity on a nice basis. In particular, note that if $\mathscr{B}$ is a basis for the topology on M, then the set of all products of n disjoint elements of $\mathscr{B}$ form a basis for the topology on $\mathrm{PConf}_n(M)$ i.e., the n-fold product of M with itself minus the fat diagonal. Then, the image of this basis under the quotient forms a basis for the topology on $\mathrm{Conf}_n(M)$. Let U be image of one such $B_1 \times \cdots \times B_n$ under the quotient map to $\mathrm{Conf}_n(M)$.

Claim. Let $H(B_i)$ denote the $\mathrm{Homeo}_0(M)$ subgroup of homeomorphisms supported on B_i.

$$p^{-1}(U) = A \setminus \bigcup \mathrm{fix}(H(B_i))$$

Proof. By the equivariance of p and the *locally continuously transitive* property of homeomorphism group actions

$$\mathrm{fix}(H(B_i)) = \{x \in A \,|\, p(x)_{\{\cdot\}} \cap B_i = \emptyset\}$$

Interpreted semantically, this says that the fixed point set of $H(B_i)$ acting on A is the set of all points whose image under p has no coordinate in B_i. Then, by making purely set-theoretic manipulations, we have the following chain of equivalences:

$$\begin{aligned} A - \bigcup_{i=1}^{n} \mathrm{fix}(H(B_i)) &= A - \bigcup_{i=1}^{n} \{x \in A \,|\, p(x)_{\{\cdot\}} \cap B_i = \emptyset\} \\ &= \bigcap_{i=1}^{n} A - \{x \in A \,|\, p(x)_{\{\cdot\}} \cap B_i = \emptyset\} \\ &= \bigcap_{i=1}^{n} \{x \in A \,|\, p(x)_{\{\cdot\}} \cap B_i \neq \emptyset\} \\ &= p^{-1}(U) \end{aligned}$$

□

So, $p^{-1}(U)$ is open since $\cup_{i=1}^{n} \mathrm{fix}(H(B_i))$ is a finite union of closed sets and thus closed. In particular, this shows that p is continuous.

Now, fixing some base-point $b \in \mathrm{Conf}_n(M)$, let $F \coloneqq p^{-1}(b)$. There is an action of $\mathrm{stab}(b_{\{\cdot\}})/\mathrm{stab}_0(b_{\{\cdot\}})$ on $p^{-1}(b)$ given by Then, there is an action of the deck group of the covering $C_{b_{\{\cdot\}}} \to \mathrm{Conf}_n(M)$ on $C_{b_{\{\cdot\}}} \times F$ given by deck transformations on the first factor and

□

2 Codimension 1 Orbit Bundles

For components consisting of codimension 1 orbits, the fiber must be 1-dimensional and all 1-dimensional homology manifolds are manifolds. So, codimension 1 orbit bundlses are classified by actions of the deck group of Γ of the covering $C_X \to \mathrm{Conf}_n(M)$ on 1-manifolds up to equivariant homeomorphism. Note that, in general, $|\pi_0(F)| \leq |\Gamma|$ and the components of F are homeomorphic.

It's hard to say something overly general here, but fixing $M = S^1$ and $\dim(N) = 3$ a classification is possible. Note, in this setting $C_X = \mathrm{PConf}_2(S^1)$ and $\Gamma = \mathbb{Z}_2$.

Proposition 2.1. The classification of $\mathrm{Homeo}_0(S^1)$ codimension 1 orbit bundles X in 3-manifolds is as follows.

F disconnected $F \cong C \sqcup C$ for a 1-manifold C, $X \cong \mathrm{PConf}_2(S^1) \times C$ and $\pi_{SO(2)}(X) \cong I \times C$

F connected

- **$\mathbb{Z}_2$ acts on F f.p. free** $F \cong S^1$, action is by a π rotation. $X \cong \mathrm{PConf}_2(S^1) \times S^1$ and $\pi_{SO(2)} \cong I \times S^1$
- **$\mathbb{Z}_2$ acts on F w/ fixed points** This depends on F, on each $\mathbb{Z}_2$ acts by reflection. Note in each case, the bundle is described in terms of the Seifert structure given by the $SO(2)$ action.
  - $F \cong \mathbb{R}$ X is $(\mathring{\mathbb{D}}^2, (2,1))$
  - $F \cong S^1$ X is $(\mathring{\mathbb{D}}^2, (2,1), (2,1))$
  - $F \cong [0,1]$ X is $(\mathring{\mathbb{D}}^2 \cup I, (2,1))$ where I is a interval in the boundary of the closed disk $\mathbb{D}^2$
- **$\mathbb{Z}_2$ acts trivially** $X \cong \mathrm{Conf}_2(S^1) \times C$ *N.B. resulting bundle is nonorientable (and top dimensional), so N is as well*

A generic component C of the fiber is one of S^1, $\mathbb{R}$, $[0,1]$ or $[0,1)$

Proof. Note, the quotient gives a 2-fold covering from $\mathrm{PConf}_2(S^1) \times F$ to the orbit bundle X, which is connected, so $\mathrm{PConf}_2(S^1) \times F$ and thus F has at most two components. Assume F has two components. Since the quotient is connected, the action must permute the components (homeomorphically), so in particular we can express the fiber as $F \cong C \sqcup C$. Then the action is just a homeomorphism from one compenent of $\mathrm{PConf}_2(S^1) \times F$ to the other, so the quotient is $\mathrm{PConf}_2(S^1) \times C$ where the resulting $\mathrm{Homeo}_0(S^1)$ action is the product of the standard configuration space action on $\mathrm{PConf}_2(S^1)$ and the trivial action on the fiber C.

If F has a single component, then the $\mathbb{Z}_2$ action is simply a choice of involution on F. For any possible F, we can allow $\mathbb{Z}_2$ to act trivially. Then, the quotient loads entirely onto the $\mathrm{PConf}_2(S^1)$ factor, where

$\mathbb{Z}_2$ acts as the deck group of $\mathrm{PConf}_2(S^1) \to \mathrm{Conf}_2(S^1)$, so the resulting quotient is $\mathrm{Conf}_2(S^1) \times F$. This is nonorientable (regardless of F) and top-dimensional, so in general, such orbit bundles only arise when N is nonorientable. The remainder of the proof is done by exhaustion. $[0, 1)$ has no nontrivial involutions, $[0, 1]$ and $\mathbb{R}$ each have one and S^1 has two. In each case, the quotient preserves the product structure away from fixed points and sends every $\mathrm{PConf}_2(S^1) \times \{x\}$ for fixed point x to $\mathrm{Conf}_2(S^1) \times \{x\}$. The result in each case is trivially the Seifert structure described in the proposition. $\square$