Orbit Bundle Theorem and Applications

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1 Introduction

The following is a jumble of tools and subsequently interesting examples of actions of $\mathrm{Homeo}_0(S^1)$ on three-manifolds

2 Tools

2.1 Gluing

Proposition 2.1. Suppose ρ and ρ' are actions of $\mathrm{Homeo}_0(M)$ on N and N' which are manifolds of the same dimension with boundary, and further let $X \subseteq \partial N$ be invariant with $\varphi : X \to \partial N'$ an embedding such that ρ and ρ' are conjugate via φ. Then, there is an action of $\mathrm{Homeo}_0(M)$ on $N \cup_\varphi N'$ which restricts to ρ and ρ' on N and N' respectively.

Proof. Use the pasting lemma. □

2.2 Products

Proposition 2.2. Suppose ρ and ρ' are actions of $\mathrm{Homeo}_0(M)$ on A and B. Then, there is an action denoted $\rho \times \rho'$ of $\mathrm{Homeo}_0(M)$ on $A \times B$ given by

$$\rho \times \rho'(f)(x, y) = (\rho(f)(x), \rho(f)(y))$$

2.3 Quotients

Proposition 2.3. Suppose that $q : X \to Y$ is a quotient map. The set

$$\Gamma_q \coloneqq \{f \in \mathrm{Homeo}_0(X) \mid q(x) = q(y) \implies q(f^{\pm 1}(x)) = q(f^{\pm 1}(y))\}$$

is a subgroup, and there is a group homeomorphism $q_* : \Gamma_q \to \mathrm{Homeo}_0(Y)$.

In particular,

Corollary 2.1. If $\rho : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(X)$ is an action, such that $\mathrm{im}(\rho) \subseteq \Gamma_q$, then there is an action of $\mathrm{Homeo}_0(M)$ on the quotient Y given by composing ρ and q_*. We will say that such an action *descends to the quotient*

So if q just collapses a set of homeomorphic orbits to a single orbit in a "reasonable" way, then the action descends. More precisely,

Corollary 2.2. Suppose $q : X \to Y$ is a quotient map such that there is a set C consisting entirely of homeomorphic orbits such that $q|_{X-C}$ is a homeomorphism onto its image, $q(C)$ is a single orbit of the same homeomorphism type of the orbits in C, for all orbits $O \in C$, $q|_O$ is a homeomorphism onto its image, and for all orbits $O, O' \subseteq C$ the induced homeomorphism $\varphi_q : O \to O'$ commutes with $p : C \to \mathrm{Conf}_n(M)$, i.e., $p|_O = p|_{O'} \circ \varphi_q^{-1}$ and $p|_{O'} = p|_O \circ \varphi_q$

3 Examples

3.1 Diagonal action on T^3

3.2 Product Constructions

Example 3.1.

3.3 Quotient Constructions

Example 3.2. Denote by $\Delta : \mathrm{Homeo}_0(S^1) \to \mathrm{Homeo}_0(\overline{Ann})$ the action induced by splitting T^2 with the configuration space action along the invariant circle. Then, by the product construction in Proposition 2.2, there is a continuous action $\Delta \times 0$ of $\mathrm{Homeo}_0(S^1)$ on $\overline{Ann} \times S^1$ where 0 is the trivial action. Under this action, every circle of the form $(0, \theta) \times S^1$ and $(1, \theta) \times S^1$ is invariant. Then consider the relation $(1, \theta, \varphi) \sim (1, \theta, \varphi)$ for all $\theta \in S^1$. Trivially, for all $f \in \mathrm{Homeo}_0(S^1)$, then $(\Delta \times 0)(f)(1, \theta, \varphi) = (1, \theta, f(\varphi))$, so $\Delta \times 0$ descends to the quotient. Successively performing the same quotient on $0 \times S^1 \times S^1$ yields an action on $S^2 \times S^1$ with an $Ann \times S^1$ orbit bundles such the positive frontier of all of the annuli orbit are a single invariant S^1 and the same with the negative frontiers.