Open Regions Foliated by Annulus Orbits

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1 Introduction

The following is a result about the structure of compact three-manifolds admitting fixed point-free $\mathrm{Homeo}_0(S^1)$ actions having only one dimensional and annulus orbits. As a matter of convention, denote we denote configuration space actions and their lifts to covers by $*$. A fact we will make repeated use of to prove these results is the following:

Remark 1.1. Let $\pi_{SO(2)}$ denote the projection along the orbits of the rotation subgroup to the base orbifold. Then, since M^3 and the base orbifold are compact Hausdorff spaces, by a classical fact from point-set topology, $\pi_{SO(2)}$ is a closed map.

2 Global structure

Consider the subset of M consisting of 1-dimensional orbits. The set $\mathrm{fix}(G_0)$ is closed, and $\pi_{SO(2)}$ is a closed map, so $\pi_{SO(2)}(\mathrm{fix}(G_0))$ is closed in the base surface. By continuity then $\pi^{-1}_{SO(2)}(\pi_{SO(2)}(\mathrm{fix}(G_0)))$,which is the union of all of the one-dimensional orbits, is closed. The complement is then an open subset of M which decomposes into annulus orbits. It is natural to ask about the structure of this open region. Below are two ipmortant structural results:

3 Foliation by Annuli

We seek to investigate the structure of an open region consisting of orbits of the same type. For what follows, consider X a connected component of the open region consisiting entirely of annulus orbits. Of primary relevance in our investigation will be the map $p : X \to \mathrm{Conf}_2(S^1)$ given by $p(x) = (\theta, \varphi)$ where θ and φ are the unique points in S^1 such that $x \in \mathrm{fix}(G_\theta \cap G_\varphi)$. First, an elementary property of group actions.

Remark 3.1. Suppose there is an action of group G on M by homeomorphisms and H is a subgroup of G. Then, for all $g \in G$,

$$g * \mathrm{fix}(H) = \mathrm{fix}(gHg^{-1})$$

Proof. Suppose $y \in g * \mathrm{fix}(H)$, then $g^{-1} * y \in \mathrm{fix}(H)$. I.e., $(hg^{-1}) * y = g^{-1} * y$ for all $h \in H$. So, by cancellation, $(ghg^{-1}) * y = y$ for all $h \in H$; thus, $y \in \mathrm{fix}(gHg^{-1})$. Each step was reversible, so this concludes the proof of equality. $\square$

A similar (but dual) argument shows that for any $f \in \mathrm{Homeo}_0(S^1)$ and G_θ a point stabilizer subgroup, $fG_\theta f^{-1} = G_{f(\theta)}$.

Proposition 3.1. The map p is equivariant, continuous, and its fibers are homeomorphic.

Proof. First, we check equivariance. Suppose $x \in X$, $f \in \mathrm{Homeo}_0(S^1)$ and $p(x) = (\theta, \varphi)$. Under the standard configuration space action $f * p(x) = (f(\theta), f(\varphi))$. So we, need to check that $\rho(f)(x) \in \mathrm{fix}(\rho(G_{f(\theta)} \cap G_{f(\varphi)}))$. By the remark,

$$\rho(f)(\mathrm{fix}(G_\theta \cap G_\varphi)) = \mathrm{fix}(\rho(f(G_\theta \cap G_\varphi)f^{-1}))$$

Then, pulling the conjugation through the intersection and applying the second remark,

$$\rho(f)(\mathrm{fix}(G_\theta \cap G_\varphi)) = \mathrm{fix}(\rho(G_{f(\theta)} \cap G_{f(\varphi)}))$$

Thus, $f * p(x) = p(\rho(f)(x))$.

Second, we check continuity by checking that open neighborhoods of points in $\mathrm{conf}_2(S^1)$ pull back to open subsets of X. Let (a, b) be a point $\mathrm{conf}_2(S^1)$ and $U = (I, J)$ be a small open neighborhood of (a, b) consisting of small disjoint open intervals I and J around a and b in S^1. Denote by $H(I)$ and $H(J)$ the subgroups of $\mathrm{Homeo}_0(S^1)$ supported on I and J respectively. We will show that $p^{-1}(U)$ is open by proving that

$$p^{-1}(U) = X - (\mathrm{fix}(\rho(H(I))) \cup \mathrm{fix}(\rho(H(J))))$$

Suppose $x \notin p^{-1}(U)$, then in particular for $p(x) = (\theta, \varphi)$, assume without loss of generality $\theta, \varphi \notin I$. Then by definition, every element of $H(I)$ fixes (θ, φ), so by equivariance, $x \in \mathrm{fix}(\rho(H(I)))$. In particular, x is not in the right hand side of the desired equality. Now, suppose $x \in p^{-1}(U)$, then in particular $p(x) = (\theta, \varphi)$ for some $\theta \in I, \varphi \in J$. Fix an equivariant homeomorphism φ from the $\mathrm{Homeo}_0(S^1)$ orbit of x to $\mathrm{PConf}_2(S^1)$ such that $\psi(x) = (\theta, \varphi)$. Then, in particular we know $\rho(f)(x) = \psi^{-1}(f(\theta), f(\varphi))$ and thus if $f \in H(I) - G_\theta$ then $f \notin \mathrm{Stab}(x)$. The contrapositive of this is then that $x \notin \mathrm{fix}(\rho(H(I)))$. A symmetric argument will show that $x \notin \mathrm{fix}(\rho(H(J)))$, so it is not in the union. Since fixed point sets are closed, their union is as well, thus the complement here is open. So this shows p is open.

Finally, we can demonstrate that for a point $(x, y) \in \mathrm{Conf}_2(S^1)$, there is a homeomorphism from F to $p^{-1}(x, y)$. Simply choose a homeomorphism $h \in \mathrm{Homeo}_0(S^1)$ such that $h(b) = (x, y)$, then by equivariance of p, $\rho(h)$ restricts to a homeomorphism $F \to p^{-1}(x, y)$. □

Next, rather than belabor a proof that p in fact gives a bundle structure, we will build a homeomorphism between X and a particular generalized flat bundle over $\mathrm{Conf}_2(S^1)$ that pulls its projection map back to p.

Remark 3.2. Suppose for some topological spaces X, Y and B such that $p : X \to B$ is a topological fiber bundle and $h : X \to Y$ is a homeomorphism. Suppose $q : Y \to B$ is a map such that the relevant triangle over B commutes. Then $q : Y \to B$ is a fiber bundle and in particular h is a bundle isomorphism.

The relevant bundle is constructed as follows,

Observation. For convenience, assume that the S^1 coordinates of the basepoint are antipodal. There is an $\mathbb{Z}_2$ action on $\mathrm{PConf}_2(S^1) \times p^{-1}(b)$ given by $\tau \times \rho(r_\pi)|_{p^{-1}(b)}$ where r_π is the half turn rotation and τ is the unique nontrivial deck transformation of $\mathrm{PConf}_2(S^1) \to \mathrm{Conf}_2(S^1)$. The restriction $\rho(r_\pi)|_{p^{-1}(b)}$ is a homeomorphism from $p^{-1}(b) \to p^{-1}(b)$ since every orbit contains exactly two points of $p^{-1}(b)$ and p is equivariant. This is a product of (fixed point-free) homeomorphisms, so this action is continuous and free, and $\mathrm{PConf}_2(S^1) \times p^{-1}(b)/\mathbb{Z}_2 \to \mathrm{PConf}_2(S^1) \times p^{-1}(b)$ is a two-fold covering map.

We begin by building the map which we will demonstrate to be a homeomorphism. First we check that it is a continuous bijection:

Proposition 3.2. There is a continuous bijection from X to $(\mathrm{PConf}_2(S^1) \times F)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathrm{PConf}_2(S^1) \times F$ by the involution which exchanges θ and ϕ coordinates on $\mathrm{PConf}_2(S^1)$ and swaps the all of the pairs of points in F lying in the same orbit.

Proof. To start, we fix a basepoint $b \in \mathrm{Conf}_2(S^1)$ and label the two choices of lifts in $\mathrm{PConf}_2(S^1)$ as $\tilde{b}_0$ and $\tilde{b}_1$ with the property that $\tilde{b}_0 = \tau(\tilde{b}_1)$ where $\tau : \mathrm{PConf}_2(S^1) \to \mathrm{PConf}_2(S^1)$ is the involution which swaps the two points on the circle.Then, if $x \in F = p^{-1}(b)$, by the orbit classification theorem, there are two choices of equivariant homeomorphism from the orbit of x to $\mathrm{PConf}_2(S^1)$ with the standard action; namely, Φ_{0_x} and Φ_{1_x} which send x to $\tilde{b}_0$ and $\tilde{b}_1$ respectively. We begin by constructing a map

$$\hat{I} : \mathrm{PConf}_2(S^1) \times F \to X.$$

Given $x = (\alpha, \beta) \in \mathrm{PConf}_2(S^1)$ and $y \in F$, then choose $f_x \in \mathrm{Homeo}_0(S^1)$ such that $f_x * (\alpha, \beta) = \tilde{b}_0$. Then,

$$\hat{I}((\alpha, \beta) \times y) := \rho(f_x^{-1})(y)$$

$\hat{I}$ is well-defined. Suppose f_x and g_x are two such maps. Then, $f_x \circ g_x^{-1}$ fixes b, thus also fixes y. Reworded, there is some $h \in \mathrm{stab}(y)$ such that $f_x = h \circ g_x$. Then,

$$\rho(f_x^{-1})(y) = \rho(g_x^{-1}h^{-1})(y) = \rho(g_x^{-1})(y)$$

$\hat{I}$ descends. Let $(\alpha, \beta) \times y$ and $(\beta, \alpha) \times y'$ be two points which are identified by the $\mathbb{Z}_2$ action. Notice that the map Φ_0 centered at y must take y' to $\tilde{b}_1$. Now choose a map $f_x \in \mathrm{Homeo}_0(S^1)$ which moves (α, β) to $\tilde{b}_0$, and f'_x which moves (β, α) to $\tilde{b}_0$. By working in Φ_0 coordinates at y, we see that $\rho(f_x^{-1})(y) = \Phi_{0_y}^{-1}(f_x^{-1} * \tilde{b}_0)$. By construction, this is the same as $\Phi_{0_y}^{-1}(\alpha, \beta)$. By the same argument, $\rho({f'}_x{}^{-1})(y') = \Phi_{0_{y'}}^{-1}(\beta, \alpha)$. Since the change of coordinates map is τ, these are the same point.

I is bijective. We refer to the induced map on the quotient as I. We wish to demonstrate that this map is a homeomorphism. Surjectivity is clear since every point in X belongs to some orbit of a point in $p^{-1}(b)$. To see injectivity, suppose $(\theta, \varphi) \times y$ and $(\theta', \varphi') \times y'$ map to the same point under I. First note that, trivially, y and y' must be points of $p^-1(b)$ which lie in the same orbit in X. This then proceeds by an identical argument to the argument that the map descends, ultimately showing that if y and y' are the same point, we must have $(\theta, \varphi) = (\theta', \varphi')$ and if y and y' are distinct, $(\theta, \varphi) = \tau(\theta', \varphi')$.

could write something slightly slicker with change of coordinates map here

I is continuous. Finally, we will check sequential continuity. Suppose (m_i, y_i) converges to (m, y) in $(\mathrm{PConf}_2(S^1) \times F)/\mathbb{Z}_2$. Then, we are free to choose a sequence $\{f_{m_i}\} \subset \mathrm{Homeo}_0(S^1)$ converging to f_m a representative function for producing $I(m, y)$. Then, by the continuity of the $\mathrm{Homeo}_0(S^1)$ action $\{\rho(f_{m_i})\}$ is a sequence converging to $\rho(f_m)$ in $\mathrm{Homeo}_0(X)$. In particular, since X is a manifold, convergence in $\mathrm{Homeo}_0(X)$ implies pointwise convergence, so

$$I(m_i, y_i) = \rho(f_{m_i}^{-1})(y_i) \to \rho(f_m^{-1})(y) = I(m, y).$$

□

To complete the proof that this map is, in fact, a homeomorphism, we will show that it is proper. To do this, we will need the following lemma.

Lemma 3.1. Suppose M, N closed manifolds such that $\mathrm{Homeo}_0(M)$ acts continously on N. Further, suppose K is a compact subset of N and S is a compact subset of $\mathrm{Homeo}_0(M)$, then $\rho(S)(K) := \bigcup_{\sigma \in S} \rho(\sigma)(K)$ is compact.

Proof. Let $\{x_n\}$ be a sequnce of points in $\rho(S)(K)$ converging to x. Then denote by σ_n a sequence of elements of S such that $x_n \in \rho(\sigma_n)(K)$. Using the continuity of the action map, clearly $\rho(S) \subseteq \mathrm{Homeo}_0(N)$ is compact. Since $\mathrm{Homeo}_0(N)$ is metric, we can pass to a convergent subsequence of $\{\rho(\sigma_n)\}$. In particular, since N is closed, convergence of a sequence in $\mathrm{Homeo}_0(N)$ implies uniform convergence when considered as a sequence of functions from N to N. Since $\rho(\sigma_n)$ converges to some $\rho(\sigma)$ uniformly, and inversion is continuous in $\mathrm{Homeo}_0(N)$, $\rho(\sigma_n^{-1})$ converges uniformly to $\rho(\sigma^{-1})$. Then, by the general metric space remark 3.3, $\rho(\sigma_n^{-1})(x_n)$ converges to $\rho(\sigma^{-1})(x)$. But, by choice, $\rho(\sigma_n^{-1})(x_n)$ is a sequence in K, so in particular, $\rho(\sigma^{-1})(x) \in K$ since K is closed by the Hausdorffness of N. But, this means that $x \in \rho(\sigma)(K) \subseteq \rho(S)(K)$; thus, $\rho(S)(K)$ is closed, and in particular, compact. □

For completeness, the following is a brief presentation of the metric space fact used in the above proof

Remark 3.3. Suppose X, Y are metric space, $f_n : X \to Y$ a sequence of continuous functions converging uniformly to f and $\{x_n\}$ is a sequence converging to x, then $f_n(x_n)$ converges to $f(x)$

Proof. Let $\varepsilon > 0$. By uniform convergence of $\{f_n\}$, there is some N such that for all $n \geq N$ we have $\mathrm{d}(f_n(x_n), f(x_n)) \leq \varepsilon/2$ for all $n \geq N$. Since f is continuous and x_n converges to x, there is some other N' such that for all $n \geq N'$ we have $\mathrm{d}(f(x_n), f(x)) \leq \varepsilon/2$. By taking $n \geq \max\{N, N'\}$, and applying the triangle inequality, we arrive at $\mathrm{d}(f_n(x_n), f(x)) \leq \varepsilon$ as desired. □

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We will complete the proof that I is a homeomorphism, by showing that I is *proper* since a continuous, proper bijection is a homeomorphism. Recall,

Definition 3.1. Let X and Y be metric spaces. A sequence $\{x_n\}$ *diverges to infinity* if every compact set K in X contains at most finitely many points of $\{x_n\}$.

A map $f : X \to Y$ is *proper* if $\{x_n\}$ diverges to infinity implies $\{f(x_n)\}$ diverges to infinity.

Finally,

Proposition 3.3. The map $I : (\mathrm{PConf}_2(S^1) \times F)/\mathbb{Z}_2 \to X$ is proper.

Proof. Let (x_n) be a sequence in $(\mathrm{PConf}_2(S^1) \times F)/\mathbb{Z}_2$ diverging to infinity. We consider the sequence $(I(x_n)) \subset X$. Let $K \subsetneq X$ be some compact set. We wish to show that K may contain only finitely many points of $(I(x_n))$. First, consider $\{p(I(x_n))\} \cap p(K)$. If this collection were finite, we would be finished as K could then only contain finitely many of $(I(x_n))$, so suppose without loss of generality it is infinite. Denote by (y_n) the subsequence of (x_n) such that $p(I(y_n)) \in p(K)$. Note that p is continuous, so in particular $p(K)$ is compact; furthermore, since $\pi : \mathrm{PConf}_2(S^1) \to \mathrm{Conf}_2(S^1)$ is a finite-sheeted covering, it is proper, so $\pi^{-1}(p(K)) \subsetneq \mathrm{PConf}_2(S^1)$ is compact. By the definition of the map p, if $p(I(x_n))$ is in $p(K)$, then there is a lift $\tilde{x}_n$ such that $\mathrm{pr}_{\mathrm{PConf}_2(S^1)}(\tilde{x_n})$ is in $\pi^{-1}(p(K))$. Thus there exist lifts $(\tilde{y}_n)$ such that $\pi{-}1(p(K))$ contains $(\mathrm{pr}_{\mathrm{PConf}_2(S^1)}(\tilde{y}_n))$. If there were some compact K' containing infinitely many of $(\mathrm{pr}_F(\tilde{y}_n))$ for any choice of lifts, then the image of $\pi^{-1}(p(K)) \times K'$ under the $\mathbb{Z}_2$ quotient is a compact set containing infinitely many points of $(y_n) \subseteq (x_n)$, but (x_n) diverges to infinity, so no such compact set can exist. Thus, there is no such K' and $\mathrm{pr}_F(\tilde{y}_n)$ must diverge to infinity for any choices of lifts.

The goal for what follows is to construct a large compact set $\hat{K}$ containing K with the property that $I(y_n) \in \hat{K}$ if and only if $\mathrm{pr}_F(\tilde{y}_n) \in \hat{K} \cap p^{-1}(b_0)$ as such a $\hat{K}$ may contain only finitely many of $(I(y_n))$. Consider the set $\mathcal{P}_N * SO(2) \subset \mathrm{Homeo}_0(S^1)$ defined as:

$$\mathcal{P}_N * SO(2) \coloneqq \left\{ P_t R \mid P_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} t \in [-N, N], R \in SO(2) \right\}$$

Where $SO(2)$ is the rotation subgroup and P_t is considered as an element of $PSL(2;\mathbb{R})$. Note that $\mathcal{P}_N * SO(2)$ is a group product of two continuous families in $\mathrm{Homeo}_0(S^1)$ which intersect at the identity, so it is homeomorphic to their topological product, i.e. a closed annulus; thus, $\mathcal{P}_N * SO(2)$ is compact for all $N \in \mathbb{R}$. Moreover, if $B \subset \mathrm{Conf}_2(S^1)$ is compact, there is some $N \in \mathbb{R}$ with the property that for every $x \in B$, there is some element $f \in \mathcal{P}_N * SO(2)$ such that $f * x = b$. The procedure to construct this element is as follows. Since B is compact, there is a uniform ε such that for all $(\alpha, \beta) \in B$, $|\beta - \alpha| > \epsilon$. Suppose without loss of generality that $b_0 = (0, \theta)$ for some θ. Then, let $\varepsilon' = \min(\varepsilon, \theta, 1 - \theta)$. Now, for $N = 2 * \cot(\pi\varepsilon')$, the element P_N satisfies $P_N(\varepsilon') = 1 - \varepsilon'$, so in particular, for any $\beta, \beta' \in [\varepsilon', 1 - \varepsilon']$ there is some $t_{\beta\beta'} \in [-N, N]$ such that $P_{t_{\beta\beta'}}(\beta) = \beta'$. Thus, the element of $\mathcal{P}_N * SO(2)$ taking (α, β) to $(0, \theta)$ is $P_{t_{(\beta-\alpha)\theta}} R_{-\alpha}$. Moreover, since, by construction, N was uniform over B, this is satisfied by the same $\mathcal{P}_N * SO(2)$.

Now, fix an appropriate N for $p(K)$, then by applying lemma 3.1, we get that

$$\hat{K} \coloneqq \bigcup_{f \in \mathcal{P}_N * SO(2)} \rho(f)(K)$$

is a compact set containing K. Moreover, $\hat{K}$ has the property that if $I(x) \in K$, then $\mathrm{pr}_F(\tilde{x}) \in \hat{K} \cap p^{-1}(b)$ which is a subset of the fiber F, since F is defined to be $p^{-1}(b)$. This is easy to see since fixing an orbit A, by definition the I^{-1} image of each point in A has a lift whose F coordinate is each of the points in $p^{-1}(b) \cap A$. At this point, we are essentially finished though. Since $p{-}1(b)$ is closed, $\hat{K} \cap p^{-1}(b)$ is compact in X and thus compact in $p{-}1(b)$. But, as observed earlier, any sequence $(\mathrm{pr}_F(\tilde{y}_n))$ must diverge to infinity where $(\tilde{y}_n)$ is any sequence of lifts (y_n), so in particular, $\hat{K}$ contains only finitely many F projections of lifts of points in (y_n), so K contains only finitely many of the points of (y_n) and thus (x_n) $\square$

4 2D Orbits Compactify Nicely

As a small point of notation, recall the following definition from point set topology:

Definition 4.1. The *frontier* of a set X, denoted $\mathrm{fr}(X)$ is defined as $\overline{X} - X$

When X is a 2-dimensional orbit, we use the following notation for convenience.

Definition 4.2. Let A be an annulus orbit, fix a model annulus $f : A \to S^1 \times I$. Then, denote by $\mathrm{fr}_+(A)$ the set of frontier points which can be accumulated by sequences with increasing I coordinate and $\mathrm{fr}_-(A)$ analogously. Note that $\mathrm{fr}_+(A) \cup \mathrm{fr}_-(A) = \mathrm{fr}(A)$, but they are not necessarily distinct. We can compatibly label subsets of the frontier of a G_0 invariant interval in an annulus A. For notational consistency, when X is a Möbius band orbit, let $\mathrm{fr}_+(X) = \mathrm{fr}_-(X) = \mathrm{fr}(X)$.

For the following two proofs, we want to make use of a kind of canonical coordinates on interval orbits of G_0.

Remark 4.1. [?] Suppose A is an annulus orbit of $\mathrm{Homeo}_0(S^1)$ and $I \subset A$ is an interval orbit of G_0. Then, we can define two homeomorphisms $\varphi, \psi : (0,1) \to I$ by $\varphi(\theta) = \mathrm{fix}(G_0 \cap G_\theta)$ and $\psi(\theta) = \mathrm{fix}(G_0 \cap G_{-\theta})$. One of these maps will send monotonic sequences that converge to 1 to sequences which converge to something in $\mathrm{fr}_+(I)$, we will denote the inverse of this map as f_I, which we will refer to as the *canonical homeomorphism between I and $S^1 - \{0\}$*. Note, the same construction works for the one interval orbit in a Möbius band orbit.

The following results are true for 2-dimensional orbits of any $\mathrm{Homeo}_0(S^1)$ action without fixed points on a closed 3-manifold.

Proposition 4.1. Suppose X is an 2-dimensional orbit, then $\mathrm{fr}(X)$ consists of 1-dimensional orbits.

Proof. Trivially, $\mathrm{fr}(X)$ contains no 3-dimensional orbit.

Suppose toward a contradiction that $\mathrm{fr}(X)$ contains some 2-dimensional orbit X'. Now, consider the restriction of the $\mathrm{Homeo}_0(S^1)$ action to the point stabilizer subgroup G_0. There is some interval orbit I in X. Since $\pi_{SO(2)}$ is continuous and closed,

$$\pi_{SO(2)}(\bar{I}) = \overline{\pi_{SO(2)}(I)} = \overline{\pi_{SO(2)}(A)} = \pi_{SO(2)}(\bar{A}).$$

In particular, this means that $\mathrm{fr}(I)$ must nontrivially intersect X'. The closure of every G_0 orbit in X' contains at least one interval orbit I', so $I' \subseteq \mathrm{fr}(I)$. In particular, there is some sequence of points $\{x_n\} \subset I$ converging to a point $x \in I'$.

From the remark, there are continuous bijections $f : I \to S^1 - \{0\}$ and $g : I' \to S^1 - \{0\}$ given by mapping the unique fixed point of $G_\theta \cap G_0$ on I to $\theta \in S^1 - \{0\}$ and similarly for I'. Then, up to a subsequence, $\{f(x_n)\}$ is a monotonic sequence converging to 0. So we can choose a monotonic element $\phi \in G_0$ satisfying $\phi(f(x_k)) = f(x_{k+1})$. In particular, we must have that $\{\rho(\phi)(x_n)\}$ converges to x, but by monotonicity, $\rho(\phi)(x) \neq x$, violating continuity, which is a contradiction. Thus $\mathrm{fr}(A) \cap X = \emptyset$ □

Note that, in particular if X is an annulus orbit in a component foliated by annuli, this implies that $\mathrm{fr}(X)$ is a subset of the frontier of the component.

We can in fact say something stronger.

Proposition 4.2. Suppose X is a 2-dimensional orbit. Then, $\mathrm{fr}_+(X)$ and $\mathrm{fr}_-(X)$ are (not necessarily distinct) one-dimensional orbits.

Proof. Suppose that σ and σ' are invariant circles in (WLOG) $\mathrm{fr}_+(X)$ with respective G_0 fixed points x and x'. Let I be a G_0-invariant interval in X. By the same reasoning as the previous result, $x, x' \in \mathrm{fr}(I)$, so let $\{a_k\}$ and $\{b_k\}$ be sequences in I reperesenting x and x'. Since both sequences are monotonic in I, there is a homeomorphism $f \in G_0$ such that $\rho(f)(a_k) = b_k$. By continuity, this implies that $\{b_k\}$ converges to $\rho(f)(x)$, which is x, since $x \in \mathrm{fix}(G_0)$. Thus, $x = x'$; morevoer, $\sigma = \sigma'$ since every fixed point of G_0 is part of a unique 1-dimensional orbit of $\mathrm{Homeo}_0(S^1)$. □

This result can be rephrased in a slightly more concrete way.

Corollary 4.1. When X is an annulus orbit, $\bar{X}$ is either an invariant closed annulus with boundary circles $\mathrm{fr}_+(A)$ and $\mathrm{fr}_-(A)$ or an invariant T^2 when $\mathrm{fr}_-(A) = \mathrm{fr}_+(A)$.

When X is a Möbius band orbit, $\bar{X}$ is an invariant closed Möbius band, with invariant boundary circle $\mathrm{fr}(X)$.

Notes