Open Regions Foliated by Annulus Orbits

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1 Introduction

The following is a result about the structure of compact three-manifolds admitting fixed point-free $\mathrm{Homeo}_0(S^1)$ actions having only one dimensional and annulus orbits. As a matter of convention, denote we denote configuration space actions and their lifts to covers by $*$. A fact we will make repeated use of to prove these results is the following:

Remark 1. Let $\pi_{SO(2)}$ denote the projection along the orbits of the rotation subgroup to the base orbifold. Then, since M^3 and the base orbifold are compact Hausdorff spaces, by a classical fact from point-set topology, $\pi_{SO(2)}$ is a closed map.

2 Global structure

Consider the subset of M consisting of 1-dimensional orbits. The set $\mathrm{fix}(G_0)$ is closed, and $\pi_{SO(2)}$ is a closed map, so $\pi_{SO(2)}(\mathrm{fix}(G_0))$ is closed in the base surface. By continuity then $\pi^{-1}_{SO(2)}(\pi_{SO(2)}(\mathrm{fix}(G_0)))$,which is the union of all of the one-dimensional orbits, is closed. The complement is then an open subset of M which decomposes into annulus orbits. It is natural to ask about the structure of this open region. Below are two ipmortant structural results:

3 Foliation by Annuli

We seek to investigate the structure of an open region consisting of orbits of the same type. For what follows, consider X a connected component of the open region consisiting entirely of annulus orbits. Of primary relevance in our investigation will be the map $p : X \to \mathrm{Conf}_2(S^1)$ given by $p(x) = (\theta, \varphi)$ where θ and φ are the unique points in S^1 such that $x \in \mathrm{fix}(G_\theta \cap G_\varphi)$. First, an elementary property of group actions.

Remark 2. Suppose there is an action of group G on M by homeomorphisms and H is a subgroup of G. Then, for all $g \in G$,

$$g * \mathrm{fix}(H) = \mathrm{fix}(gHg^{-1})$$

Proof. Suppose $y \in g * \mathrm{fix}(H)$, then $g^{-1} * y \in \mathrm{fix}(H)$. I.e., $(hg^{-1}) * y = g^{-1} * y$ for all $h \in H$. So, by cancellation, $(ghg^{-1}) * y = y$ for all $h \in H$; thus, $y \in \mathrm{fix}(gHg^{-1})$. Each step was reversible, so this concludes the proof of equality. □

A similar (but dual) argument shows that for any $f \in \mathrm{Homeo}_0(S^1)$ and G_θ a point stabilizer subgroup, $fG_\theta f^{-1} = G_{f(\theta)}$.

Proposition 1. The map p is equivariant, continuous, and its fibers are homeomorphic.

Proof. First, we check equivariance. Suppose $x \in X$, $f \in \mathrm{Homeo}_0(S^1)$ and $p(x) = (\theta, \varphi)$. Under the standard configuration space action $f * p(x) = (f(\theta), f(\varphi))$. So we, need to check that $\rho(f)(x) \in \mathrm{fix}(\rho(G_{f(\theta)} \cap G_{f(\varphi)}))$. By the remark,

$$\rho(f)(\mathrm{fix}(G_\theta \cap G_\varphi)) = \mathrm{fix}(\rho(f(G_\theta \cap G_\varphi)f^{-1}))$$

Then, pulling the conjugation through the intersection and applying the second remark,

$$\rho(f)(\mathrm{fix}(G_\theta \cap G_\varphi)) = \mathrm{fix}(\rho(G_{f(\theta)} \cap G_{f(\varphi)}))$$

Thus, $f * p(x) = p(\rho(f)(x))$.

Second, we check continuity by checking that open neighborhoods of points in $\mathrm{conf}_2(S^1)$ pull back to open subsets of X. Let (a, b) be a point $\mathrm{conf}_2(S^1)$ and $U = (I, J)$ be a small open neighborhood of (a, b) consisting of small disjoint open intervals I and J around a and b in S^1. Denote by $H(I)$ and $H(J)$ the subgroups of $\mathrm{Homeo}_0(S^1)$ supported on I and J respectively. We will show that $p^{-1}(U)$ is open by proving that

$$p^{-1}(U) = X - (\mathrm{fix}(\rho(H(I))) \cup \mathrm{fix}(\rho(H(J))))$$

Suppose $x \notin p^{-1}(U)$, then in particular for $p(x) = (\theta, \varphi)$, assume without loss of generality $\theta, \varphi \notin I$. Then by definition, every element of $H(I)$ fixes (θ, φ), so by equivariance, $x \in \mathrm{fix}(\rho(H(I)))$. In particular, x is not in the right hand side of the desired equality. Now, suppose $x \in p^{-1}(U)$, then in particular $p(x) = (\theta, \varphi)$ for some $\theta \in I, \varphi \in J$. Fix an equivariant homeomorphism φ from the $\mathrm{Homeo}_0(S^1)$ orbit of x to $\mathrm{PConf}_2(S^1)$ such that $\psi(x) = (\theta, \varphi)$. Then, in particular we know $\rho(f)(x) = \psi^{-1}(f(\theta), f(\varphi))$ and thus if $f \in H(I) - G_\theta$ then $f \notin \mathrm{Stab}(x)$. The contrapositive of this is then that $x \notin \mathrm{fix}(\rho(H(I)))$. A symmetric argument will show that $x \notin \mathrm{fix}(\rho(H(J)))$, so it is not in the union. Since fixed point sets are closed, their union is as well, thus the complement here is open. So this shows p is open.

Finally, we can demonstrate that for a point $(x, y) \in \mathrm{Conf}_2(S^1)$, there is a homeomorphism from F to $p^{-1}(x, y)$. Simply choose a homeomorphism $h \in \mathrm{Homeo}_0(S^1)$ such that $h(b) = (x, y)$, then by equivariance of p, $\rho(h)$ restricts to a homeomorphism $F \to p^{-1}(x, y)$. □

4 2D Orbits Compactify Nicely

As a small point of notation, recall the following definition from point set topology:

Definition 1. The *frontier* of a set X, denoted $\mathrm{fr}(X)$ is defined as $\overline{X} - X$

When X is a 2-dimensional orbit, we use the following notation for convenience.

Definition 2. Let A be an annulus orbit, fix a model annulus $f : A \to S^1 \times I$. Then, denote by $\mathrm{fr}_+(A)$ the set of frontier points which can be accumulated by sequences with increasing I coordinate and $\mathrm{fr}_-(A)$ analogously. Note that $\mathrm{fr}_+(A) \cup \mathrm{fr}_-(A) = \mathrm{fr}(A)$, but they are not necessarily distinct. We can compatibly label subsets of the frontier of a G_0 invariant interval in an annulus A. For notational consistency, when X is a Möbius band orbit, let $\mathrm{fr}_+(X) = \mathrm{fr}_-(X) = \mathrm{fr}(X)$.

For the following two proofs, we want to make use of a kind of canonical coordinates on interval orbits of G_0.

Remark 3. [?] Suppose A is an annulus orbit of $\mathrm{Homeo}_0(S^1)$ and $I \subset A$ is an interval orbit of G_0. Then, we can define two homeomorphisms $\varphi, \psi : (0, 1) \to I$ by $\varphi(\theta) = \mathrm{fix}(G_0 \cap G_\theta)$ and $\psi(\theta) = \mathrm{fix}(G_0 \cap G_{-\theta})$. One of these maps will send monotonic sequences that converge to 1 to sequences which converge to something in $\mathrm{fr}_+(I)$, we will denote the inverse of this map as f_I, which we will refer to as the *canonical homeomorphism between I and $S^1 - \{0\}$*. Note, the same construction works for the one interval orbit in a Möbius band orbit.

The following results are true for 2-dimensional orbits of any $\mathrm{Homeo}_0(S^1)$ action without fixed points on a closed 3-manifold.

Proposition 2. Suppose X is an 2-dimensional orbit, then $\mathrm{fr}(X)$ consists of 1-dimensional orbits.

Proof. Trivially, $\mathrm{fr}(X)$ contains no 3-dimensional orbit.

Suppose toward a contradiction that $\mathrm{fr}(X)$ contains some 2-dimensional orbit X'. Now, consider the restriction of the $\mathrm{Homeo}_0(S^1)$ action to the point stabilizer subgroup G_0. There is some interval orbit I in X. Since $\pi_{SO(2)}$ is continuous and closed,

$$\pi_{SO(2)}(\bar{I}) = \overline{\pi_{SO(2)}(I)} = \overline{\pi_{SO(2)}(A)} = \pi_{SO(2)}(\bar{A}).$$

In particular, this means that $\mathrm{fr}(I)$ must nontrivially intersect X'. The closure of every G_0 orbit in X' contains at least one interval orbit I', so $I' \subseteq \mathrm{fr}(I)$. In particular, there is some sequence of points $\{x_n\} \subset I$ converging to a point $x \in I'$.

From the remark, there are continuous bijections $f : I \to S^1 - \{0\}$ and $g : I' \to S^1 - \{0\}$ given by mapping the unique fixed point of $G_\theta \cap G_0$ on I to $\theta \in S^1 - \{0\}$ and similarly for I'. Then, up to a subsequence, $\{f(x_n)\}$ is a monotonic sequence converging to 0. So we can choose a monotonic element $\phi \in G_0$ satisfying $\phi(f(x_k)) = f(x_{k+1})$. In particular, we must have that $\{\rho(\phi)(x_n)\}$ converges to x, but by monotonicity, $\rho(\phi)(x) \neq x$, violating continuity, which is a contradiction. Thus $\mathrm{fr}(A) \cap X = \emptyset$ $\square$

Note that, in particular if X is an annulus orbit in a component foliated by annuli, this implies that $\mathrm{fr}(X)$ is a subset of the frontier of the component.

We can in fact say something stronger.

Proposition 3. Suppose X is a 2-dimensional orbit. Then, $\mathrm{fr}_+(X)$ and $\mathrm{fr}_-(X)$ are (not necessarily distinct) one-dimensional orbits.

Proof. Suppose that σ and σ' are invariant circles in (WLOG) $\mathrm{fr}_+(X)$ with respective G_0 fixed points x and x'. Let I be a G_0-invariant interval in X. By the same reasoning as the previous result, $x, x' \in \mathrm{fr}(I)$, so let $\{a_k\}$ and $\{b_k\}$ be sequences in I reperesenting x and x'. Since both sequences are monotonic in I, there is a homeomorphism $f \in G_0$ such that $\rho(f)(a_k) = b_k$. By continuity, this implies that $\{b_k\}$ converges to $\rho(f)(x)$, which is x, since $x \in \mathrm{fix}(G_0)$. Thus, $x = x'$; morevoer, $\sigma = \sigma'$ since every fixed point of G_0 is part of a unique 1-dimensional orbit of $\mathrm{Homeo}_0(S^1)$. $\square$

This result can be rephrased in a slightly more concrete way.

Corollary 1. When X is an annulus orbit, $\bar{X}$ is either an invariant closed annulus with boundary circles $\mathrm{fr}_+(A)$ and $\mathrm{fr}_-(A)$ or an invariant T^2 when $\mathrm{fr}_-(A) = \mathrm{fr}_+(A)$.

When X is a Möbius band orbit, $\bar{X}$ is an invariant closed Möbius band, with invariant boundary circle $\mathrm{fr}(X)$.

Notes