Open Regions Foliated by Annulus Orbits

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1 Introduction

The following is a result about the structure of compact three-manifolds admitting fixed point-free $\mathrm{Homeo}_0(S^1)$ actions having only one dimensional and annulus orbits. A fact we will make repeated use of to prove these results is the following:

Remark 1. Let $\pi_{SO(2)}$ denote the projection along the orbits of the rotation subgroup to the base orbifold. Then, since M^3 and the base orbifold are compact Hausdorff spaces, by a classical fact from point-set topology, $\pi_{SO(2)}$ is a closed map.

2 Global structure

Consider the subset of M consisting of 1-dimensional orbits. The set $\mathrm{fix}(G_0)$ is closed, and $\pi_{SO(2)}$ is a closed map, so $\pi_{SO(2)}(\mathrm{fix}(G_0))$ is closed in the base surface. By continuity then $\pi_{SO(2)}^{-1}(\pi_{SO(2)}(\mathrm{fix}(G_0)))$,which is the union of all of the one-dimensional orbits, is closed. The complement is then an open subset of M which decomposes into annulus orbits. It is natural to ask about the structure of this open region. Below are two ipmortant structural results:

3 Foliation by Annuli

Proposition 1. Suppose X is a component of the open subset of M^3 in which all orbits are annuli. Then the decomposition of X into orbits is a C^0 foliation by annuli.

Proof sketch. Take a short curve segment through x, build two one paramater families of homeos, giving local product structure (do this explicitly, use flow of a bump vec field on circle). ☐ flesh this out with details

4 Annuli Compactify Nicely

As a small point of notation, recall the following definition from point set topology:

Definition 1. The *frontier* of a set X, denoted $\mathrm{fr}(X)$ is defined as $\bar{X} - X$

When X is an annulus orbit, we use the following notation for convenience.

Definition 2. Let A be an annulus orbit, fix a model annulus $f : A \to S^1 \times I$. Then, denote by $\mathrm{fr}_+(A)$ the set of frontier points which can be accumulated by sequences with increasing I coordinate and $\mathrm{fr}_-(A)$ analogously. Note that $\mathrm{fr}_+(A) \cup \mathrm{fr}_-(A) = \mathrm{fr}(A)$, but they are not necessarily distinct. We can compatibly label subsets of the frontier of a G_0 invariant interval in an annulus A

For the following two proofs, we want to make use of a kind of canonical coordinates on interval orbits of G_0.

Remark 2. Suppose A is an annulus orbit of $\mathrm{Homeo}_0(S^1)$ and $I \subset A$ is an interval orbit of G_0. Then, we can define two homeomorphisms $\varphi, \psi : (0,1) \to I$ by $\varphi(\theta) = \mathrm{fix}(G_0 \cap G_\theta)$ and $\psi(\theta) = \mathrm{fix}(G_0 \cap G_{-\theta})$. One of these maps will send monotonic sequences that converge to 1 to sequences which converge to something in $\mathrm{fr}_+(I)$, we will denote the inverse of this map as f_I, which we will refer to as the *canonical homeomorphism between I and $S^1 - \{0\}$*

Proposition 2. Suppose A is an annulus orbit in a component X of the open region foliated by annulus orbits. Then $\mathrm{fr}(A) \subset \mathrm{fr}(X)$

Proof. Suppose toward a contradiction that $\mathrm{fr}(A) \cap X \neq \emptyset$. Then, since every fiber in X is contained in some annulus orbit and $\bar{A}$ must be invariant, there is some invariant annulus $A' \subset \mathrm{fr}(A)$. Now, consider the restriction of the $\mathrm{Homeo}_0(S^1)$ action to the action of a point stabilizer subgroup G_0. Let I be a G_0 invariant interval in A. Since $\pi_{SO(2)}$ is continuous and closed,

$$\pi_{SO(2)}(\bar{I}) = \overline{\pi_{SO(2)}(I)} = \overline{\pi_{SO(2)}(A)} = \pi_{SO(2)}(\bar{A}).$$

This implies that $\mathrm{fr}(I)$ intersects every fiber of $\mathrm{fr}(A)$ and in particular, every fiber of A'. Since $\bar{I}$ is closed and invariant, $\mathrm{fr}(I)$ must be a union of G_0 orbits in A' containing at least one interval orbit I' (the disk orbits are not closed). Thus, there is some sequence of points $\{x_n\} \subset I$ converging to a point $x \in I'$.

From [**?**], there are continuous bijections $f : I \to S^1 - \{0\}$ and $g : I' \to S^1 - \{0\}$ given by mapping the unique fixed point of $G_\theta \cap G_0$ on I to $\theta \in S^1 - \{0\}$ and similarly for I'. Then, up to a subsequence, $\{f(x_n)\}$ is a monotonic sequence converging to 0. So we can choose a monotonic element $\phi \in G_0$ satisfying $\phi(f(x_k)) = f(x_{k+1})$. In particular, we must have that $\{\rho(\phi)(x_n)\}$ converges to x, but by monotonicity, $\rho(\phi)(x) \neq x$, violating continuity, which is a contradiction. Thus $\mathrm{fr}(A) \cap X = \emptyset$ □

Note that, in particular, $\mathrm{fr}(X)$ consists entirely of one-dimensional orbits, so $\mathrm{fr}(A)$ must as well. A piece of notation which will be useful for the following,

In fact, we can say something quite strong about these sets.

Proposition 3. Suppose A is an annulus orbit. Then, $\mathrm{fr}_+(A)$ and $\mathrm{fr}_-(A)$ are (not necessarily distinct) one-dimensional orbits.

Proof. Suppose that σ and σ' are invariant circles in $\mathrm{fr}_+(A)$ with respective G_0 fixed points x and x'. Let I be a G_0-invariant interval in A. By the same reasoning as the previous result, $x, x' \in \mathrm{fr}(I)$, so let $\{a_k\}$ and $\{b_k\}$ be sequences in I reperesenting x and x'. Since both sequences are monotonic in I, there is a homeomorphism $f \in G_0$ such that $\rho(f)(a_k) = b_k$. By continuity, this implies that $\{b_k\}$ converges to $\rho(f)(x)$, which is x, since $x \in \mathrm{fix}(G_0)$. Thus, $x = x'$; morevoer, $\sigma = \sigma'$ since every fixed point of G_0 is part of a unique 1-dimensional orbit of $\mathrm{Homeo}_0(S^1)$. □

This result can be rephrased in a slightly more concrete way.

Corollary 1. The invariant subset $\bar{A}$ is either an invariant closed annulus with boundary circles $\mathrm{fr}_+(A)$ and $\mathrm{fr}_-(A)$ or an invariant T^2 when $\mathrm{fr}_-(A) = \mathrm{fr}_+(A)$.

Next steps: show that G_0 invariant intervals must have frontier $= \mathrm{fix}(G_0) \cap \mathrm{fr}(A)$?

Notes