Design and implementation of a bicycle state estimator

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Abstract

This paper presents two approaches for estimating bicycle state; 1) a reduced-state estimator and 2) a full-state estimator. Experimental results obtained on a robotic bicycle are presented for the full-state estimator; discussion of the reduced-state estimator is limited to a theoretical framework. In both cases, the plant is taken to be the Whipple bicycle model linearized about the zero lean, zero steer configuration with a constant forward speed. We also briefly discuss how the speed dependent dynamics were accounted for in the implementation of the robotic bicycle.

1 Introduction

A frequent requirement in control system design is knowledge of the state of the plant to be controlled. If the plant is observable through the available measurements, the state estimates $\hat{x}$ can be used in place of the true states x when applying the feedback control law (i.e., $u = K\hat{x}$ instead of $u = Kx$). Alternatively, if some states are directly measurable, it is possible to design a reduced-state estimator [2] with fewer states than that of the plant (a full-state estimator has the same number of states as the plant). Reduced-state estimators typically have higher bandwidth but do not filter the measurements of the state so they are more susceptible to measurement noise. From a control system implementation perspective, reduced-state estimators can have lower computational cost and therefore be attractive in applications with constrained computational capabilities or with stringent bandwidth requirements.

The four states in the linearized state space equations of the Whipple bicycle model [4] are lean ϕ, steer δ, lean rate $\dot{\phi}$, steer rate $\dot{\delta}$. Of these, the most difficult to measure directly is ϕ. Techniques to calculate or estimate ϕ include optical sensors on both sides of the rear wheel axle to measure the differential distance from the axle to the ground, mechanical trailers measuring lean with a potentiometer or encoder, and IMU based solutions which employ rate gyroscopes and/or accelerometers and Kalman filtering techniques to obtain orientation estimates[1].

We constructed a robotic bicycle to conduct system identification experiments, without a human rider, for the purpose of experimentally validating the Whipple bicycle model[6]. This required the implementation of a stabilizing state feedback controller and by extension, a state estimator. The bicycle was equipped with optical encoders to measure δ and the wheel angles θ_r and θ_f; the rear wheel angle measurement was differentiated numerically to obtain the wheel rate (and hence, the forward speed). A MEMS rate gyroscope fixed to the rear bicycle frame was used to measure roll rate $\dot{\phi}$. The bicycle was also equipped with a rear hub motor to control the speed and a steer motor to turn the fork relative to the frame (and hence balance the bicycle by steering). Using state space matrices determined from measurements of bicycle physical parameters and the presumed Whipple model, we designed a gain scheduled LQR controller which assumes full state feedback. We omit the details of the LQR gain selection, and focus instead on the design of the state estimator.

This paper is organized as follows. Section 2 describes the design of the reduced-state and full-state estimators. Section 3 presents results obtained for the full-state estimator for a single run of the robotic bicycle at $2.0\,\mathrm{m\,s^{-1}}$. We discuss the estimator performance in Section 4 and summarize our findings as well as give thoughts for future work in Section 5.

2 Methods

The basis for the estimator and control system design is the Whipple bicycle model [7] [4] . The physical parameters of the robotic bicycle were measured following the approach developed in [5]. The linearized equations of motion for the bicycle at constant forward speed can be written in state space form as

$$\begin{aligned}\dot{x} &= Ax + BT_\delta \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ b_{20} \\ b_{30} \end{bmatrix} T_\delta\end{aligned}$$

where $x = \left[\phi, \delta, \dot{\phi}, \dot{\delta}\right]^T$ and T_δ is the steer torque acting between the fork and frame. It is worth noting that the a_{20} and a_{30} entries of the system dynamics matrix are independent of forward speed, a_{21} and a_{31} depend on the square of forward speed, and the remaining a_{ij} depend linearly on forward speed.

2.1 Reduced-state estimator

If steer torque T_δ is known and measurements of $\delta, \dot{\phi}$, and $\dot{\delta}$ are available, then the equation relating the measurements z to the states x is

$$z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x.$$

Following [2], we introduce a change of variables

$$\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} k_0 & k_1 & k_2 & k_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \quad \Longrightarrow \quad x = \begin{bmatrix} \frac{1}{k_0} & -\frac{k_1}{k_0} & -\frac{k_2}{k_0} & -\frac{k_3}{k_0} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \tag{1}$$

where $k_0 \neq 0$. The equation on the right is the estimator output equation. From this change of variables, an estimator for w can be synthesized as

$$\begin{aligned}\dot{\hat{w}} = &\frac{a_{20}k_2 + a_{30}k_3}{k_0}\hat{w} + \left(a_{21}k_2 + a_{31}k_3 - \frac{k_1\left(a_{20}k_2 + a_{30}k_3\right)}{k_0}\right)\delta \\ &+ \left(a_{22}k_2 + a_{32}k_3 + k_0 - \frac{k_2\left(a_{20}k_2 + a_{30}k_3\right)}{k_0}\right)\dot{\phi} + \left(a_{23}k_2 + a_{33}k_3 + k_1 - \frac{k_3\left(a_{20}k_2 + a_{30}k_3\right)}{k_0}\right)\dot{\delta} \\ &+ \left(b_{20}k_2 + b_{30}k_3\right)T_\delta\end{aligned}$$

To stabilize the estimator state equation we must choose k_0, k_2, k_3 such that

$$\frac{a_{20}k_2 + a_{30}k_3}{k_0} < 0$$

The selection of k_0, k_2, and k_3, is guided by the control systems design principle which suggests that estimator poles be placed 3-10 times faster than the fastest pole of the closed-loop controlled plant. Since a_{20} and a_{30} are independent of speed, the estimator eigenvalues can be arbitrarily assigned by selection of fixed k_0, k_2, and k_3 that are independent of speed.

The choice of k_0, k_2, k_3 which results in a specified estimator pole is not unique. The effect of sensor measurement noise will be influenced by the choice of the k_i's. Thus care must be taken when selecting the k_i's to ensure that noise is not amplified. The transfer functions from the three measurements to the estimated lean (i.e., the first row of the estimator output equation in Equation 1) should be examined to verify that sufficient noise attenuation is achieved for the chosen k_0, k_2, and k_3.

2.2 Full-state estimator

Alternatively, a full-state estimator can be used to reconstruct the entire state vector, even including measured states. If the system is observable, a state estimator can be implemented through the following equations:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + BT_\delta + L\left(y - \hat{y}\right)\\ y &= \left[\begin{smallmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{smallmatrix}\right] x = Cx\end{aligned}$$

where $\hat{x}$ is the state estimate, A and B are given explicitly in subsection 2.1, and the measurements are $y = \left[\delta, \dot{\phi}\right]^T$. Note that the measurements do not depend directly on input steer torque.

If estimator error is defined to be

$$e = \hat{x} - x \qquad \dot{e} = \dot{\hat{x}} - \dot{x}$$

which can be rewritten as

$$\dot{e} = \left(A - LC\right)\hat{x} + BT_\delta + Ly - \left(Ax + BT_\delta\right) = \left(A - LC\right)e$$

then the error in the estimator will converge to zero asymptotically if

$$Re\left(\sigma\left(A - LC\right)\right) < 0$$

After computing the gain K for full state feedback, the closed-loop estimator poles are chosen to be significantly faster than the fastest eigenvalue of $A + BK$ with repeated pole locations avoided. Specifically

$$p_0 = 3 \times \min\left(Re\left(\sigma\left(A + BK\right)\right)\right), \quad p_1 = p_0 - 0.2\mathrm{s}^{-1}, \quad p_2 = p_1 - 0.2\mathrm{s}^{-1}, \quad p_3 = p_2 - 0.2\mathrm{s}^{-1}.$$

Here we use the same design principle as before and place estimator poles 3 times faster than the slowest pole of the controlled plant, ensuring the convergence of the estimator is faster than the controller dynamics. The other 3 poles are assigned arbitrarily to be marginally more negative than p_0. By choosing all the poles to be on the real axis, any oscillation in estimator state convergence is avoided.

Calculating the gain L is done using the MATLAB `place` command which uses an algorithm described in [3]. Since the pole placement problem is under-determined for multi-variable systems, the algorithm uses the extra degrees of freedom to determine a robust solution for L that minimizes sensitivities of the desired poles to perturbations in the system and gain matrices. Thus the estimator will converge as desired despite small modelling errors in the system.

3 Results

A number of experiments were conducted with the robotic bicycle [6]. The balancing robotic bicycle is visible in Figure 1. Representative estimator results during an experimental run with a commanded speed of $2.0\,\mathrm{m\,s^{-1}}$ are shown in Figure 2. This speed is well below the stable speed range of the uncontrolled robotic bicycle and wouldn't have been possible without active stabilization.

Figure 1: Taking the robot bicycle on a jog.

4 Discussion

The full-state estimator worked sufficiently well for the state feedback law to stabilize the bicycle outside its stable speed range. However, it is clear from Figure 2 that the steer rate estimate $\hat{\dot{\delta}}$ was not accurate. For example, at $t = 25.0\,\mathrm{s}$, the slope of the δ plot is clearly negative, yet $\hat{\dot{\delta}}$ is positive at that time.

Besides the poor estimate of $\hat{\dot{\delta}}$, the other state estimates seem reasonable. The mean lean angle estimate $\mu_{\hat{\phi}} = 0.002\,\mathrm{rad}$ over the time range presented is in agreement with our observations that the bicycle turned slightly to the right over the course of the run. The lean rate estimate $\hat{\dot{\phi}}$ closely follows the lean rate measurement $\dot{\phi}$ but with slightly lower noise magnitudes.

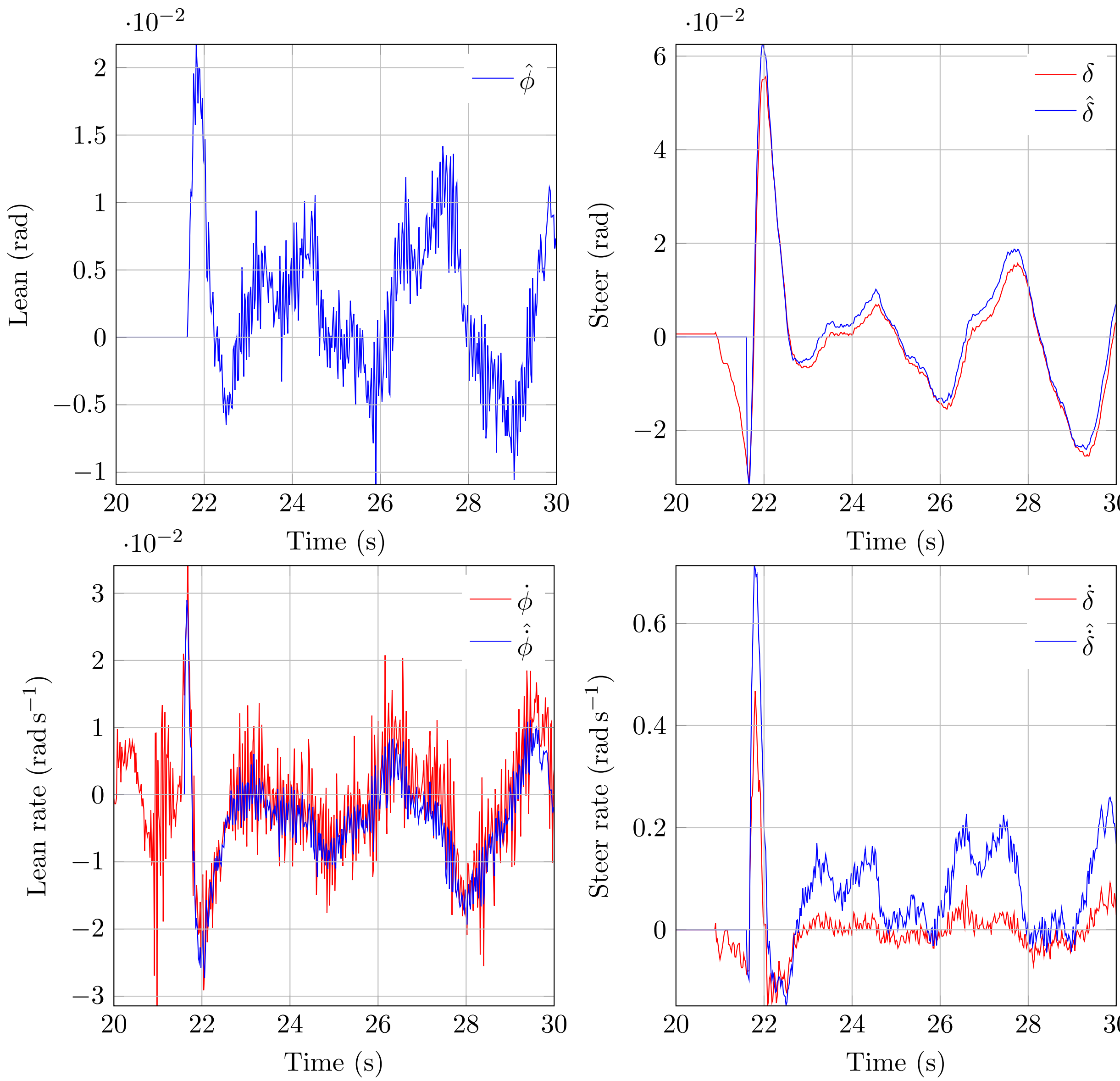

Figure 2: Estimator and measurement response during an experiment conducted at $2.0\,\mathrm{m\,s^{-1}}$. No direct measurement of lean was available to compare with lean estimate.

5 Conclusion

The state estimate, along with an LQR full-state feedback gain, was sufficient to stabilize the robotic bicycle outside the stable speed range. More tuning is needed to improve the estimate of steer rate. Additionally, a direct measurement of the lean angle to which the lean angle estimate can be compared would yield valuable information about the estimator performance. Lean angle is the most heavily weighted state in the LQR gain so reducing lean angle estimation error is especially desirable. Experimental testing with the reduced-state estimator is needed to determine whether the potential increase in bandwidth would be beneficial to balancing at speeds lower than $1.0\,\mathrm{m\,s^{-1}}$.

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