Perceptons

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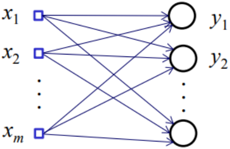
9th March 2018

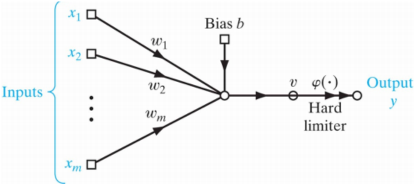
Adapted from slides provided by Prof. Michael Mandel.

Outline

Perceptrons

- Architecture: one-layer feedforward net
  - Without loss of generality, consider a single-neuron perceptron





Definition

$$y = \varphi(\nu)$$

$$\nu = \sum_{i=1}^{m} w_i x_i + b$$

$$\varphi(\nu) = \begin{cases} 1 & \text{if } \nu \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Hence a McCulloch-Pitts neuron, but with real-valued inputs

Pattern recognition

- With a bipolar output, the perceptron performs a 2-class classification problem, *i.e.*, apples vs. oranges.
- How do we learn to perform classification?
- The perceptron is given pairs of input x_p and desired output d_p
- How can we find $y_p = \varphi(x_p^T w) = d_p, \forall p$

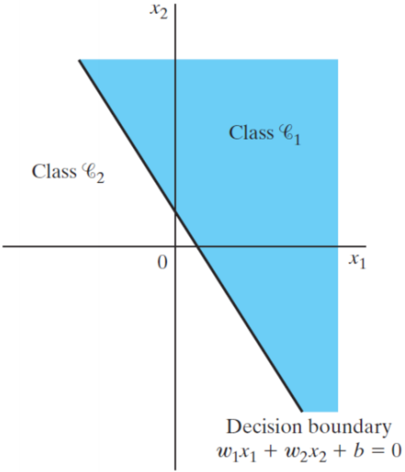
But first: decision boundary

- Can we visualize the decision the perceptron would make in classifying every potential point?
- Yes, it is called the discriminant function

$$g(x) = x^T w = \sum_{i=0}^{m} w_i x_i$$

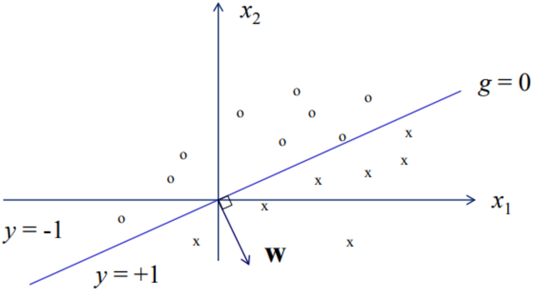
- What is the boundary between the two classes like?
- This is a linear function of x

Decision boundary example



Decision boundary

- For an m-dimensional input space, the decision boundary is an $(m-1)$-dimensional hyperplane perpendicular to w. The hyperplane separate the input space into two halves, with one half having $y = 1$, and the other half having $y = -1$
  - When $b = 0$, the hyperplane goes through the origin.



Linear separability

- For a set of input patterns x_p, if there exists at least one w that separates $d = 1$ patterns from $d = -1$ patterns, then the classification problem is linearly separable.
  - In other words, there exists a linear discriminant function that produces no classification error.
  - Examples: AND, OR, XOR

Linear separability

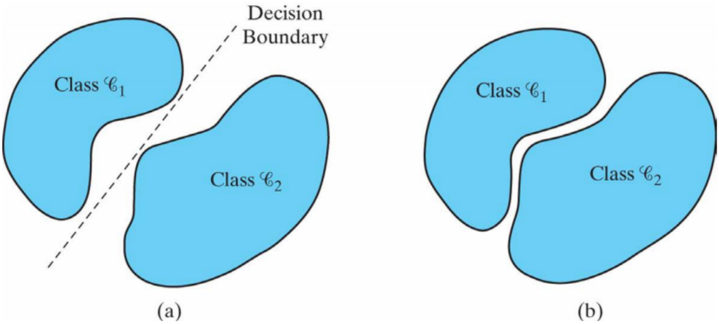


Figure: illustration: **left**: Linear separable, **right**: Not linear separable

Outline

Perceptron definition (recap.)

$$y = \varphi(\nu)$$

$$\nu = \sum_{i=1}^{m} w_i x_i + b$$

$$\varphi(\nu) = \begin{cases} 1 & \text{if } \nu \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Hence a McCulloch-Pitts neuron, but with real-valued inputs

Perceptron learning rule

- Learn parameters w from examples (x_p, d_p)
- In an online fashion, *i.e.*, one point at a time
- Adjust weights as necessary, *i.e.*, when incorrect
- Adjust weights to be more like $d = 1$ points and more like negative $d = -1$ points.

Biological analogy

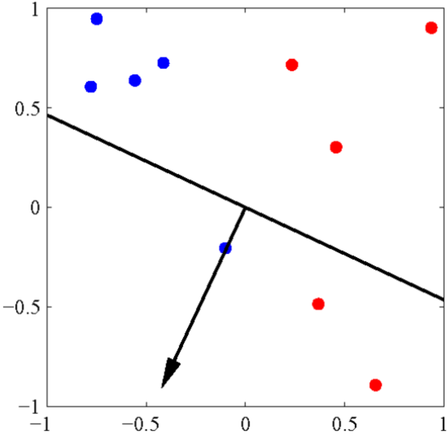
- Strengthen an active synapse if the postsynaptic neuron fails to fire when it should have fired
- Weaken an active synapse if the neuron fires when it should not have fired
- Formulated by Rosenblatt based on biological intuition

Quantitatively

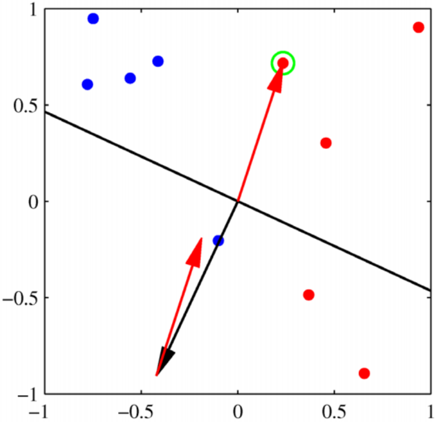
$$
\begin{aligned}
w(n+1) &= w(n) + \Delta w(n) \\
&= w(n) + \eta[d(n) - y(n)]x(n)
\end{aligned} \tag{1}
$$

- n: iteration number, iterating over points in turn
- η: step size or learning rate
- Only updates w when $y(n)$ is incorrect

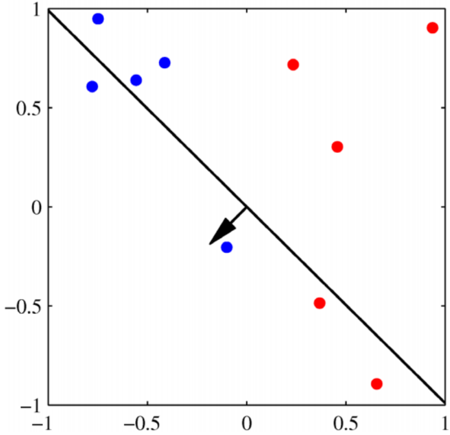
Geometric interpretation



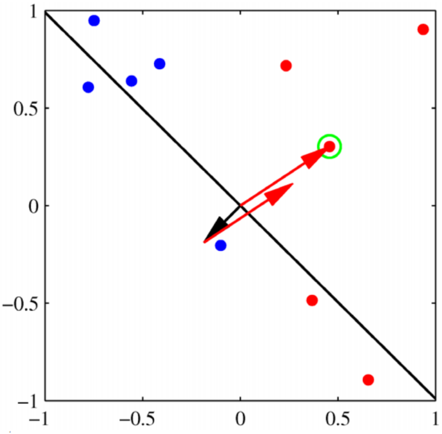
Geometric interpretation



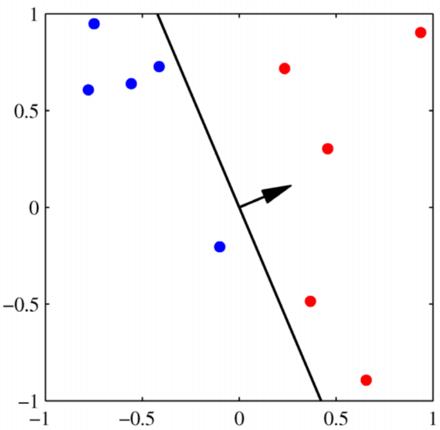
Geometric interpretation



Geometric interpretation



Geometric interpretation



Geometric interpretation

- Each weight update moves w closer to $d = 1$ patterns, or away from $d = -1$ patterns.
- Final weight vector in example solves the classification problem
- Is that true in all cases?

Summary of perceptron learning algorithm

- Definition:
  - $w(n)$: (m+1)-by-1 weight vector (including bias) at step n
- Inputs:
  - $x(n)$: n^{th} (m+1)-by-1 input vector with first element = 1
  - $d(n)$: n^{th} desired response
- Initialization: set $w(0) = 0$
- Repeat until no points are mis-classified
  - Compute response: $y(n) = \operatorname{sgn}\left[w(n)^T x(n)\right]$
  - Update: $w(n+1) = w(n) + [d(n) - y(n)]\, x(n)$

Outline

Perceptron convergence theorem

- Assume that there exists some unit vector w_0 and some α such that $d(n)w_0^T x(n) \geq \alpha$, *i.e.* the data are linearly separable.
- Assume also that there exists some R such that $\|x(n)\| = \sqrt{x(n)^T x(n)} \leq R \quad \forall n$
- Then the perceptron algorithm makes at most R^2/α^2 errors

Perceptron convergence proof outline

- Define w_k as the parameter vector when the algorithm makes its k^{th} error (note $w_1 = 0$)
- First show $k\alpha \leq \|w_{k+1}\|$ by induction
- Second show $\|w_{k+1}\|^2 \leq kR^2$ by induction
- Then it follows that $k \leq R^2/\alpha^2$, *i.e.* the perceptron makes a finite number of errors.

Fisrt show $k\alpha \leq \|w_{k+1}\|$ by induction

- Assume that the k^{th} error is made on example n.
- Because of the perceptron update rule,

$$\begin{aligned} w_{k+1}^T w_0 &= (w_k + d(n)x(n)^T) w_0 \\ &= w_k^T w_0 + d(n)x(n)^T w_0 \\ &\geq w_k^T w_0 + \alpha \end{aligned} \tag{2}$$

- Because, by assumption, $d(n)x(n)^T w_0 \geq \alpha$
- Then, by induction on k, $w_{k+1}^T w_0 \geq k\alpha$
- In addition, $\|w_{k+1}\|\|w_0\| \geq w_{k+1}^T w_0$ by Cauchy-Schwartz, with $\|w_0\| = 1$
- Thus, $\|w_{k+1}\| \geq w_{k+1}^T w_0 \geq k\alpha$

Second show $\|w_{k+1}\|^2 \leq kR^2$ by induction

- Because of the perceptron update rule

$$\begin{aligned} \|w_{k+1}\|^2 &= \|w_k + d(n)x(n)\|^2 \\ &= \|w_k\|^2 + d^2(n)\|x(n)\|^2 + 2d(n)x(n)^T w_k \end{aligned} \tag{3}$$

- By definition, $d^2(n) = 1$
- By assumption, $\|x(n)\|^2 \leq R^2$
- Because the n-th points was misclassified $2d(n)x(n)^T w_k \leq 0$
- Thus, $\|w_{k+1}\|^2 \leq \|w_k\|^2 + R^2$
- And , by induction on k, $\|w_{k+1}\|^2 \leq kR^2$

Then it follows that $k \leq R^2/\alpha^2$

- We have shown $k\alpha \leq \|w_{k+1}\|$ and $\|w_{k+1}\|^2 \leq kR^2$
- So, $k^2\alpha^2 \leq \|w_{k+1}\|^2 \leq kR^2$
- Then it follows that $k \leq R^2/\alpha^2$
- Thus the perceptron learning algorithm makes a bounded number of mistakes, *i.e.*, converges.

Perceptron learning remarks

- If data are not linearly separable
  - Algorithm will iterate forever
- Scaling w does not affect the preceptron's decision
  - So the learning rate η does not affect the perceptron's decision either, and can be set to 1
- The solution weight vector w is not unique.

Generalization

- Performance of a learning machine on test patterns not used during training
- Perceptrons generalize by deriving a decision boundary in the input space. Selection of training pattens is thus important for generalization.

Thank you!