Posts and Telecommunication Institute of Technology
Faculty of Information Technology 1

Introduction to Artificial Intelligence

Propositional logic

Ngo Xuan Bach

Outline

- Knowledge representation and reasoning
- Propositional logic
- Inference in propositional logic

The need of knowledge and reasoning

- Humans live in the environment
  - Perceive the world through the senses (ears, eyes, ...)
  - Collected information will be accumulated into knowledge
  - Use accumulated knowledge and reasoning/inference ability to make reasonable actions
- An intelligent system needs to have the ability to use knowledge and make inference
  - High flexibility
    - The combination of known knowledge and inference allows to create new knowledge
  - Can work in incomplete information cases
  - Convenience for system building
    - Just change the knowledge base; keep the same inference procedure

Knowledge representation language

Knowledge representation language
= Syntax + Semantics + reasoning mechanism

- Syntax
  - Includes symbols and rules for linking symbols (syntactic rules) to form sentences (formulas) in the language
- Semantics
  - Allows to determine the meaning of sentences in a certain domain of the real world
- Reasoning mechanism
  - A computational process
  - Input: a set of formulas (formal representation of known knowledge)
  - Output: a set of new formulas (formal representation of new knowledge)

A good knowledge representation language

- Good representation ability
  - Allows to represent all necessary knowledge of the problem
- Efficient
  - Concise knowledge representation
  - Inference procedure requires little computation time and little memory space
- Close to natural language
  - Convenient to describe knowledge

Outline

- Knowledge representation and reasoning
- Propositional logic
  - Syntax
  - Semantics
- Inference in propositional logic

Syntax of propositional logic (1/2)

- Symbols
  - Truth symbols (logical constants): True (T) and False (F)
  - Propositional symbols (Propositional variables): P, Q, …
  - Logical connectives: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$
  - Brackets (and)
- Syntactic rules
  - Truth symbols and propositional variables are formulas
  - If A and B are formulas
    - $(A \wedge B)$: "A and B" (conjunction)
    - $(A \vee B)$: "A or B" (disjunction)
    - $(\neg A)$: "not A" (negation)
    - $(A \Rightarrow B)$: "if A then B" (implication)
    - $(A \Leftrightarrow B)$: "A if and only if B" (equivalence)

    Are formulas

Syntax of propositional logic (2/2)

- Unnecessary parentheses can be removed
  - Example: $((A \vee B) \wedge C)$ can be written as $(A \vee B) \wedge C$
- The precedence of logical connectives
  - $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- Propositional symbols are atomic sentences
  - Examples: P, Q
- If P is a propositional symbol then P and $\neg P$ are literal
  - P is a positive literal, $\neg P$ is a negative literal
- Complex sentences of the form $A_1 \vee A_2 \vee \ldots \vee A_m$, where A_i are literals, are called clausal sentences

Semantics of propositional logic (1/2)

- Each propositional symbol corresponds to a propositional statement
  - P = "Paris is the capital of France"
  - Q = "Pi constant is an integer number"
- A propositional statement is either True (T) or False (F)
  - P True, Q False
- An interpretation is a way of assigning each propositional variable a truth value (True or False)

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
True	True	False	True	True	True	True
True	False	False	False	True	False	False
False	True	True	False	True	True	False
False	False	True	False	False	True	True

Semantics of propositional logic (2/2)

- A formula is satisfiable if there exists an interpretation that makes the formula true
  - $(P \wedge Q) \vee \neg R$
- A formula whose truth table contains only false in any interpretation is called unsatisfiable
  - $P \wedge \neg P$
- A formula is valid if it is true in every interpretation
  - $P \vee \neg P$
- A model of a formula is an interpretation that makes the formula true
  - $\{P \leftarrow False, Q \leftarrow True, R \leftarrow False\}$

Logical equivalences (1/2)

- Formulas A and B are said to be logical equivalent if they have the same truth value in every interpretation
  - Notation: $A \equiv B$

- Basic logical equivalences
  - $A \Rightarrow B \equiv \neg A \vee B$ (implication law)
  - $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ (equivalent law)
  - $\neg(\neg A) \equiv A$ (double negation law)

- De Morgan's laws
  - $\neg(A \vee B) \equiv \neg A \wedge \neg B$
  - $\neg(A \wedge B) \equiv \neg A \vee \neg B$

Logical equivalences (2/2)

- Commutative laws
  - $A \vee B \equiv B \vee A$
  - $A \wedge B \equiv B \wedge A$
- Associative laws
  - $(A \vee B) \vee \mathrm{C} \equiv A \vee (B \vee \mathrm{C})$
  - $(A \wedge B) \wedge \mathrm{C} \equiv A \wedge (B \wedge \mathrm{C})$
- Distributive laws
  - $A \wedge (B \vee \mathrm{C}) \equiv (A \wedge B) \vee (A \wedge C)$
  - $A \vee (B \wedge \mathrm{C}) \equiv (A \vee B) \wedge (A \vee C)$

Conjunctive normal form (1/2)

- A disjunctive clause (or simply clause) is a disjunction of literals
  - A clause has the form $P_1 \vee P_2 \vee \ldots \vee P_n$, where P_i are literals

- A conjunctive normal form (CNF) formula is a conjunction of disjunctive clauses
  - $(A \vee E \vee F \vee G) \wedge (B \vee C \vee D)$

Conjunctive normal form (2/2)

- We can convert any formula into an equivalent formula that is in CNF:
  - Eliminate equivalences: $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$
  - Eliminate implications: $A \Rightarrow B \equiv \neg A \vee B$
  - Apply De Morgan's laws to move negations close to propositional variables
  - Eliminate double negations: $\neg(\neg A) \equiv A$
  - Apply distributive laws: $A \vee (B \wedge \mathrm{C}) \equiv (A \vee B) \wedge (A \vee C)$

Exercise 1

- Prove the following basic logical equivalences using truth tables
  1. $A \Rightarrow B \equiv \neg A \vee B$ (Implication law)
  2. $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$ (Equivalent law)
  3. $\neg(\neg A) \equiv A$ (Double negation law)
  4. $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgan's law)
  5. $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan's law)
  6. $A \wedge (B \vee \mathrm{C}) \equiv (A \wedge B) \vee (A \wedge C)$ (Distributive law)
  7. $A \vee (B \wedge \mathrm{C}) \equiv (A \vee B) \wedge (A \vee C)$ (Distributive law)

Exercise 2

- Prove that the following formulas are valid (using basic logical equivalences and laws)

$a)\quad (P \wedge Q) \Rightarrow P$

$b)\quad P \Rightarrow (P \vee Q)$

$c)\quad \neg P \Rightarrow (P \Rightarrow Q)$

$d)\quad (P \wedge Q) \Rightarrow (P \Rightarrow Q)$

$e)\quad \neg(P \Rightarrow Q) \Rightarrow P$

$f)\quad \neg(P \Rightarrow Q) \Rightarrow \neg Q$

g) $\neg P \wedge (P \vee Q) \Rightarrow Q$

h) $(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$

i) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

j) $((P \vee Q) \wedge (P \Rightarrow R) \wedge (Q \Rightarrow R)) \Rightarrow R$

Exercise 3

- Prove the following logical equivalences (using basic logical equivalences and laws)

1) $(P \Leftrightarrow Q) \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$

2) $\neg P \Leftrightarrow Q \equiv P \Leftrightarrow \neg Q$

3) $\neg(P \Leftrightarrow Q) \equiv \neg P \Leftrightarrow Q$

Exercise 4

- Convert the following formula into CNF

$$(P \Rightarrow Q) \lor \neg(R \lor \neg S)$$

Outline

- Knowledge representation and reasoning
- Propositional logic
- Inference in propositional logic
  - Logical inference
  - Inference using truth tables
  - Inference rules

Logical inference

- A formula H is said to be a logical consequence of a set of formulas $G = \{G_1, \dots, G_m\}$ if in any interpretation that G is true then H is also true
- An inference procedure consists of a set of premises and a conclusion

$$\frac{set\ of\ premises}{conclusion}$$

  - Soundness: if conclusion is a logical consequence of the set of premises
  - Completeness: if can find every logical consequence of the set of premises
- Notations
  - KB : Knowledge Base, set of known formulas
  - $\mathrm{KB} \vdash \alpha$: α is a logical consequence of KB

Inference using truth tables

- Using truth tables can determine whether a formula is a logical consequence of a set of formulas in KB or not
  - Example: $KB = \{A \vee C, B \vee \neg C\}, \alpha = A \vee B$
- Properties
  - Soundness?
    - Yes
  - Completeness?
    - Yes
  - Computational complexity
    - High

Inference rules (1/2)

- Modus Ponens

$$\frac{\alpha \Rightarrow \beta,\ \alpha}{\beta}$$

- Modus Tollens

$$\frac{\alpha \Rightarrow \beta,\ \neg\beta}{\neg\alpha}$$

- And-Elimination

$$\frac{\alpha_1 \wedge \ldots \wedge \alpha_i \wedge \ldots \wedge \alpha_m}{\alpha_i}$$

- And-Introduction

$$\frac{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_m}{\alpha_1 \wedge \ldots \wedge \alpha_i \wedge \ldots \wedge \alpha_m}$$

α, β, α_i are formulas

Inference rules (2/2)

- Or-Introduction

$$\frac{\alpha_i}{\alpha_1 \vee \ldots \vee \alpha_i \vee \ldots \vee \alpha_m}$$

- Double-Negation Elimination

$$\frac{\neg(\neg\alpha)}{\alpha}$$

- Transitivity

$$\frac{\alpha \Rightarrow \beta, \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$$

$\alpha, \beta, \gamma, \alpha_i$ are formulas

- Unit Resolution

$$\frac{\alpha \vee \beta, \neg\beta}{\alpha}$$

- Resolution

$$\frac{\alpha \vee \beta, \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

Exercise 1

- Prove the following logical consequences using truth tables
  1. $\{A \Rightarrow B,\ A\} \vdash B$
  2. $\{A \Rightarrow B, \neg B\} \vdash \neg A$
  3. $\{A \Rightarrow B, B \Rightarrow C\} \vdash A \Rightarrow C$
  4. $\{A \vee B, \neg B\} \vdash A$

Exercise 2

- Given a KB:

$$Q \land S \Rightarrow G \land H \qquad (1)$$
$$P \Rightarrow Q \qquad (2)$$
$$R \Rightarrow S \qquad (3)$$
$$P \qquad (4)$$
$$R \qquad (5)$$

Prove the following logical consequence using inference rules: $\mathrm{KB} \vdash G$