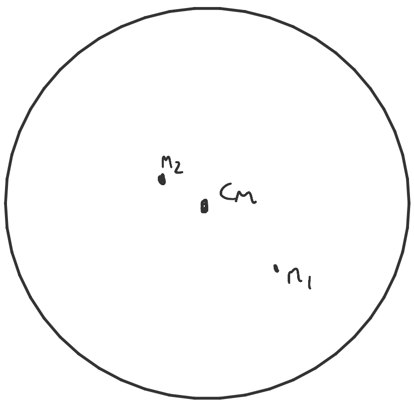
DM derivation

Thursday, October 17, 2024 10:09 PM

I would like to start off by assuming that the distribution of dark matter is distributed uniformly and spherically around the center of mass of the binary system. This is my picture.



μ - dark matter density

It actually does how the larger circle is centered. This is because, we know from the spherically symmetry that anything outside of a spherical shell will have no effect, but anything inside will merely follow Gauss's law. This means how this sphere is centered matters and does effect thing directly. For example, if the sphere was shifted slightly to the right, that would affect the forces on m1 and m2. The affect of the dark matter on m1 and m2 is purely a function of the mass's position in r in the larger sphere of dark matter. I think I would like to try to use Lagrangian mechanics for this.

I would like to start by noting that I will eventually like to express my Lagrangian in terms of r_cm and r_relative. I write

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \Rightarrow \qquad \vec{r}_1 = \vec{r}_{cm} - \frac{m_2 \vec{r}}{m_1+m_2}$$

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{r}_2 = \frac{m_1 \vec{r}}{m_1+m_2} + \vec{r}_{cm}$$

Then if I just say we are in the coordinate system where the center of mass if at the origin and not moving then my coordinates becomes

$$\vec{r}_1 = -\frac{m_2 \vec{r}}{B} \qquad B \equiv m_1 + m_2$$

$$\vec{r}_2 = \frac{m_1 \vec{r}}{B}$$

Lets go ahead and write down my Lagrangian in terms of the r1 and r2 coordinates

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} + \frac{2Gm_1m_2}{|\vec{r}_2 - \vec{r}_1|} + \frac{4\pi G m_1 \mu \dot{\vec{r}}_1^{\,2}}{3} + \frac{4\pi G m_2 \mu \dot{\vec{r}}_2^{\,2}}{3}$$

We can see the dark matter terms show up pretty simply. Lets see what this ends up looking like if we swap to our cm coordinates

$$L = \frac{1}{2} m_1 \left[\frac{-m_2 \dot{\vec{r}}}{B}\right]^2 + \frac{1}{2} m_2 \left(\frac{m_1 \dot{\vec{r}}}{B}\right)^2 + \frac{2Gm_1m_2}{r} + \frac{4\pi G m_1 \mu}{3}\left[\frac{-m_2 \dot{\vec{r}}}{B}\right]^2 + \frac{4\pi G m_2 \mu}{3}\left[\frac{m_1 \vec{r}}{B}\right]^2$$

$$= \frac{m_1 m_2^2 \dot{\vec{r}}^{\,2}}{2B^2} + \frac{m_2 m_1^2 \dot{\vec{r}}^{\,2}}{2B^2} + \frac{2Gm_1m_2}{r} + \frac{4\pi G m_1 m_2^2 \mu r^2}{3B^2} + \frac{4\pi G m_2 m_1^2 \mu r^2}{3B^2}$$

Its pretty simple to write out that r-dot squared term

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \quad \text{so} \quad \dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2\dot{\theta}^2$$

I just plug that in and get

$$L = \left[\frac{m_1m_2[m_1+m_2]}{2[m_1+m_2]^2}\right]\left[\dot{r}^2 + r^2\dot{\theta}^2\right] + \frac{2Gm_1m_2}{r} + \frac{4\pi G m_1 m_2 [m_1+m_2]\mu}{3[m_1+m_2]^2} r^2$$

This would be a good time to just go ahead and define the reduced mass. Lets use gamma for it since I already used mu for the dark matter density

$$\gamma = m_1 m_2$$

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$$\gamma \equiv \frac{m_1 m_2}{m_1 + m_2}$$

$$L = \frac{\gamma}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right] + \frac{2Gm_1m_2}{r} + \frac{4\pi G\mu\gamma}{3} r^2$$

Note all this is, is a general class of problem of the central potential. I can write it as

$$L = T - V = \frac{\gamma}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \qquad V(r) = -\left[\frac{2Gm_1m_2}{r} + \frac{4\pi G\mu\gamma}{3} r^2\right]$$

This is in Goldstein, I can look there for more

Expressed now in plane polar coordinates, the Lagrangian is

$$L = T - V = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

Perhaps I can solve this with mathematica and my life will be easy. Let me look at the theta EOM

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\gamma r^2\dot{\theta}\right) = 0$$

Tells us that

$$\gamma r^2 \dot{\theta} = L_0 = \text{Const}$$

Lets put this into our Lagrangian to obtain

$$L = \frac{\gamma}{2}\left(\dot{r}^2 + r^2\left[\frac{L_0}{\gamma r^2}\right]^2\right) - V(r) = \frac{\gamma}{2}\left[\dot{r}^2 + \frac{L_0^2}{\gamma^2 r^2}\right] - V(r)$$

Now I have something only in r and I can get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \gamma\ddot{r}$$

$$\frac{\partial L}{\partial r} = \frac{\gamma}{2}\left[-\frac{2L_0^2}{\gamma^2 r^3}\right] - \frac{2Gm_1m_2}{r^2} + \frac{8\pi G\mu\gamma r}{3}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = \gamma\ddot{r} - \left[\frac{-L_0^2}{\gamma r^3} - \frac{2Gm_1m_2}{r^2} + \frac{8\pi G\mu\gamma r}{3}\right] = \ddot{r} + \frac{L_0^2}{\gamma^2 r^3} + \frac{2Gm_1m_2}{\gamma r^2} - \frac{8\pi G\mu r}{3} = 0$$

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In[2]:= DSolve[r''[t] + a/(r[t])^3 + b/(r[t])^2 - c*r[t] == 0, r[t], t]

Out[2]= Solve[(4 (r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1]) (r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2])^2
  (EllipticPi[(Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4])/(Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]), ArcSin[Sqrt[((r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))/((r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))]],
    -((Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 3]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))/((-Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] + Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 3]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))]
   (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2]) + EllipticF[ArcSin[Sqrt[((r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))/((r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))]],
    -((Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 3]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))/((-Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] + Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 3]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4]))] Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2])^2
  (r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 3]) (r[t] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4])) / (r[t]^2 (c1 + a/r[t]^2 + 2 b/r[t] + c r[t]^2) (-Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] + Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2])^2
  (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 1] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 3]) (Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 2] - Root[a + 2 b #1 + c1 #1^2 + c #1^4 &, 4])) == (t + c2)^2, r[t]]
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Mathematica gives this when I try to solve it

I guess I can take a moment and compute the relative magnitude that the effect of the dark matter would have. I assume that both objects in the binary are 1 solar mass and they are both 1 light second apart. Also I take the dark matter density to be 2.2* 10^-27kg/m^3

$$\frac{2G\left(m^{2}\right)}{d}$$

$$= 1.7798133333 \times 10^{42}$$

$$\frac{4}{3}\pi \cdot G \cdot \left(\frac{m}{2}\right)2.2 \cdot 10^{-27} \cdot d^{2}$$

$$= 5.5355340078 \times 10^{10}$$

Oh man, that is very different