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SM-4351 Test 2

1. (a) $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

[3] $H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$

μ is estimated by the sample mean $\bar{X}$.

$\bar{X} \sim N(\mu, \sigma^2/n)$

i. if σ^2 is known, then we can use

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$ in which case $Z \sim N(0,1)$

if σ^2 is unknown, then we can use

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

Alternatively, if n is large, then the Wald test

$$Z = \frac{\bar{X} - \mu_0}{S.E(\bar{X})} = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim N(0,1)$$ or 1

can be used, since

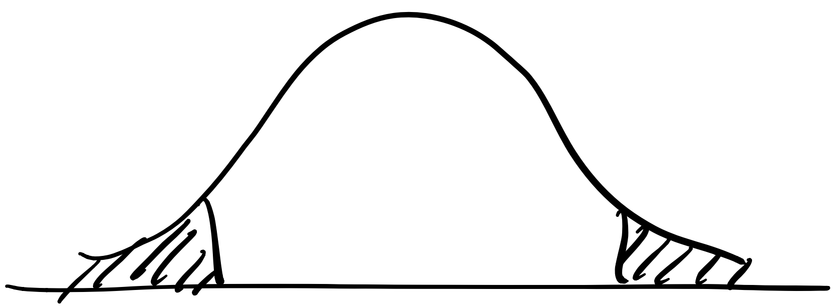
$$S.E.(\bar{X}) = \sqrt{\frac{\hat{\sigma}^2}{n}} = s/\sqrt{n}.$$

ii. p-value is the probability of observing a value equal to or more extreme than what is observed. [2]

$$p = P\left(|T| > \frac{\bar{X} - \mu_0}{s/\sqrt{n}}\right)$$

if using the t-distribution, then this probability is obtained from the tables.

1



The p-value is compared against the significant value α of the test.

if $p < \alpha$ then we have sufficient evidence to reject the null hypothesis.

If $p > \alpha$ then we do not have enough evidence to reject the null hypothesis. 1

(b) Let X_i be the outcome of the coin toss.

[5] 1/2 $X_i = \begin{cases} 1 & \text{w.p. } \pi \quad \text{(Heads)} \\ 0 & \text{w.p. } 1-\pi \quad \text{(Tails.)} \end{cases}$

we have that $\sum_{i=1}^{30} X_i = 22$

Want to test

$H_0 : \pi = 0.5$ 1/2

$H_1 : \pi \neq 0.5$

π is estimated by $\hat{\pi} = \sum X_i / 30 = \bar{X} = \frac{22}{30}$

Now, $\hat{\pi} = \bar{X} \approx N\left(\pi, \hat{\pi}(1-\hat{\pi})/n\right)$ 1

So we can use the test statistic

$$Z = \frac{\hat{\pi} - 0.5}{\sqrt{\hat{\pi}(1-\hat{\pi})/30}}$$ and compare against $N(0,1)$.

1

the value of the observed test-statistic is

$$Z = \frac{22/30 - 0.5}{\sqrt{22/3375}} = \frac{0.2333}{0.080737}$$
$$= 2.89 \quad 1$$

Since $|Z| = 2.89 > z(0.025) = 1.96$,

1 we may reject the null hypothesis at the 5% level, and conclude that the coin is biased.

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2. $Z_i \sim N(0,1)$ $i=1,\dots,n$.

(a) Then $X = Z_1^2 + \dots + Z_n^2 \sim \chi^2_n$.
[2]

(b) $E(X) = E(Z_1^2) + \dots + E(Z_n^2)$
[3]
$= 1 + \dots + 1$ 1
$= n$

since $Var(Z_i) = E(Z_i^2) - E^2(Z_i)$
$1 = E(Z_i^2) - 0$

Then, $Var(X) = E(X^2) - E^2(X)$

$$= \prod_{k=0}^{1} (n+2k) - n^2$$

$$= n(n+2) \quad - n^2$$

$$= n^2 + 2n - n^2$$

$$= 2n$$ 2

(c) Since $\frac{X}{n} = \frac{1}{n} \sum Z_i^2$,

[3]

$X/n \xrightarrow{P} E(Z_1^2) = 1$ by LLN. 1

Thus, $X/n \xrightarrow{D} 1$ 1 as well since convergence in probability implies convergence in distribution. 1

(d) $$F_{n,m} = \frac{\chi_n^2/n}{\chi_m^2/m}$$ 1

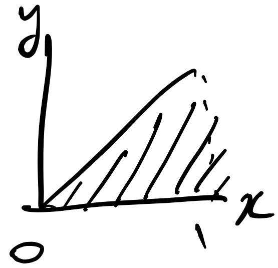
[2]

Since $\chi_n^2/n \to 1$

and $\chi_m^2/m \to 1$

$F_{n,m} \to 1/1 = 1$ as $n,m \to \infty$. 1

3. (a) $\displaystyle\int_{x=0}^{y}\int_{y=0}^{1} 4y^2\,dy\,dx$



[3]

$$= \int_0^1 \left[4y^2x\right]_0^y dy$$

$$= \int_0^1 4y^3\,dy$$

$$= \left[\frac{\cancel{4}y^4}{\cancel{4}}\right]_0^1 = 1-0 = 1$$

(b) $\displaystyle f_Y(y) = \int_{x=0}^{y} 4y^2\,dx$

[2]

$$= 4y^3$$