SM-4331 Exercise 4

1. Suppose that $X_1, ..., X_n$ and $Y_1, ..., Y_n$ are two independent random samples from two exponential distributions with mean μ_1 and μ_2 respectively.

   (a) Find the likelihood ratio test for statistic $H_0 : \mu_1 = \mu_2$ against $H_0 : \mu_1 \neq \mu_2$.

   Solution: Let $\mathcal{D} = \{X_1, ..., X_n, Y_1, ..., Y_n\}$ be the observed sample. Then the log-likelihood function is

$$
\begin{aligned}
l(\mu_1, \mu_2|\mathcal{D}) &= \log f(\mathcal{D}|\mu_1, \mu_2) \\
&= \log \{f(X_1, ..., X_n|\mu_1) f(Y_1, ..., Y_n|\mu_2)\} \\
&= \sum_{i=1}^{n} \log \left(\mu_1^{-1} e^{-X_i/\mu_1}\right) + \sum_{i=1}^{n} \log \left(\mu_2^{-1} e^{-Y_i/\mu_2}\right) \\
&= -n \log \mu_1 - \sum_{i=1}^{n} X_i/\mu_1 - n \log \mu_2 - \sum_{i=1}^{n} Y_i/\mu_2
\end{aligned}
$$

   The unconstrained MLE for μ_1 and μ_2 is

$$
\hat{\mu}_1 = \bar{X} \quad \text{and} \quad \hat{\mu}_2 = \bar{Y},
$$

   while the constrained MLE under $H_0 : \mu_1 = \mu_2 = \mu$ is the solution to

$$
\frac{\mathrm{d}}{\mathrm{d}\mu} \left(-2n \log \mu - \frac{1}{\mu} \sum_{i=1}^{n} (X_i + Y_i) \right) = -\frac{2n}{\mu} + \frac{1}{\mu^2} \sum_{i=1}^{n} (X_i + Y_i) = 0
$$

   which is $\hat{\mu} = \frac{1}{2n} \sum_{i=1}^{n} (X_i + Y_i) = \frac{1}{2}(\bar{X} + \bar{Y})$.
   The log-likelihood ratio test statistic is given by

$$
\begin{aligned}
\mathrm{LR} &= \frac{L(\hat{\mu}_1, \hat{\mu}_2)}{L(\hat{\mu})} \\
&= \frac{\hat{\mu}_1^{-n} e^{-\sum_i X_i/\hat{\mu}_1} \hat{\mu}_2^{-n} e^{-\sum_i Y_i/\hat{\mu}_2}}{\hat{\mu}^{-2n} e^{-\sum_i (X_i+Y_i)/\hat{\mu}}} \\
&= \frac{\bar{X}^{-n} e^{-n\cancel{\bar{X}}/\cancel{\bar{X}}} \bar{Y}^{-n} e^{-n\cancel{\bar{Y}}/\cancel{\bar{Y}}}}{2^{2n} (\bar{X} + \bar{Y})^{-2n} e^{-2n\cancel{(\bar{X}+\bar{Y})}/\cancel{(\bar{X}+\bar{Y})}}} \\
&= \frac{(\bar{X} + \bar{Y})^{2n}}{4^n \bar{X}^n \bar{Y}^n}
\end{aligned}
$$

   (b) Specify the asymptotic distribution of the test statistic under H_0.

   Solution: The difference in the number of estimated parameters is $2 - 1 = 1$. Hence,

$$
-2 \log \frac{L(\hat{\mu})}{L(\hat{\mu}_1, \hat{\mu}_2)} \xrightarrow{\mathrm{D}} \chi_1^2.
$$

2. A survey of the use a particular product was conducted in four areas, and a random sample of 200 potential users was interviewed in each area. In area i, for $i = 1, 2, 3, 4$, X_i of the 200 said that they used the product. Construct a likelihood ratio test to test whether the proportion of the population using the product is the same in each of the four areas. Carry out the test at 5% level for the case $X_1 = 76$, $X_2 = 53$, $X_3 = 59$ and $X_4 = 48$.

Solution: Each of the X_i is $\mathrm{Bin}(200, p_i)$. We want to test $H_0 : p_1 = p_2 = p_3 = p_4 =: \pi$ against the alternative that they are not all the same. Let $\mathbf{X} = \{X_1, X_2, X_3, X_4\}$.

The likelihood function is

$$L(p_1, p_2, p_3, p_4|\mathbf{X}) = \prod_{i=1}^{4} \frac{200!}{X_i!(200 - X_i)!} p_i^{X_i}(1 - p_i)^{200-X_i}.$$

The unconstrained MLEs are simply $\hat{p}_i = X_i/200$, $i = 1, 2, 3, 4$.

Under H_0 however, the MLE is the solution to

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}p}\left\{\text{const.} + \log p \sum_{i=1}^{4} X_i + \log(1-p) \sum_{i=1}^{4}(200 - X_i)\right\} \\
&= \frac{1}{p}\sum_{i=1}^{4} X_i - \frac{1}{1-p}\sum_{i=1}^{4}(200 - X_i) \\
&= \frac{\bar{X}}{p} - \frac{1-\bar{X}}{1-p} \\
\Rightarrow \hat{p} &= \bar{X}
\end{aligned}$$

where $\bar{X} = \frac{1}{4 \cdot 200}\sum_{i=1}^{4} X_i$. Thus, the (asymptotic) LR statistic is given by

$$\begin{aligned}
D &= -2 \log \frac{L(\hat{p})}{L(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4)} \\
&= -2\left\{\cancel{\text{const.}} + \log \hat{p} \sum_{i=1}^{4} X_i + \log(1-\hat{p}) \sum_{i=1}^{4}(200 - X_i)\right\} \\
&\quad + 2\left\{\cancel{\text{const.}} + \sum_{i=1}^{4} X_i \log \hat{p}_i + \sum_{i=1}^{4}(200 - X_i)\log(1 - \hat{p}_i)\right\}
\end{aligned}$$

Putting in the observed values for X_i, we get

$$\hat{p}_1 = 76/200, \hat{p}_2 = 53/200, \hat{p}_3 = 59/200, \hat{p}_4 = 48/200, \text{ and } \hat{p} = 236/800.$$

and $D = 153.96$. This is compared against χ^2_3 (since d.f.$= 4 - 1 = 3$). Since $D = 153.96 > \chi^2_3(0.05) = 7.815$, we reject the null at the 5% significance level.

3. Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be an independent sample from a distribution with pdf

$$\mathrm{P}(X_i = x) = f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

(a) Write down the log-likelihood function and the score function, and then obtain the ML estimate for λ.

Solution: The log-likelihood function given all the data is

$$\begin{aligned} l(\lambda|\mathbf{X}) &= \sum_{i=1}^{n} \{-\lambda + X_i \log \lambda - \log X_i!\} \\ &= \text{const.} - n\lambda + \log \lambda \sum_{i=1}^{n} X_i \\ &= \text{const.} - n\lambda + n\bar{X} \log \lambda. \end{aligned}$$

The score function is

$$\begin{aligned} S(\lambda|\mathbf{X}) &= \frac{\partial}{\partial \lambda} \left\{\text{const.} - n\lambda + n\bar{X} \log \lambda\right\} \\ &= 0 - n + n\bar{X}/\lambda. \end{aligned}$$

The MLE is the solution to $S(\lambda) = 0$, so $\hat{\lambda} = \bar{X}$.

(b) Determine the Fisher information for λ.

Solution: The Fisher information is

$$\begin{aligned} \mathcal{I}(\lambda) &= -\mathrm{E}\left[S'(\lambda|\mathbf{X})\right] \\ &= -\mathrm{E}\left[\frac{\partial}{\partial \lambda}(n\bar{X}/\lambda)\right] \\ &= \mathrm{E}\left[n\bar{X}/\lambda^2\right] \\ &= n/\lambda. \end{aligned}$$

(c) Suppose that we observe $\bar{X} = 3.5$ from $n = 10$ observations. Test at the 5% significance level that

$$H_0 : \lambda = 5 \quad \text{v.s.} \quad H_1 : \lambda \neq 5$$

i. Using the Wald test.

Solution: The asymptotic distribution of the MLE $\hat{\lambda} = \bar{X}$ is $\mathrm{N}(\lambda, \lambda/n)$.

Thus, we reject the null hypothesis in favour of the alternative if

$$Z = \left| \frac{\bar{X} - 5}{\sqrt{5/n}} \right| > z(0.025) = 1.96.$$

For the given data, our observed test statistic is $Z = 2.12$, hence we reject the null hypothesis.

ii. Using the Score test.

Solution: The score test rejects the null hypothesis if

$$Z = \left| \frac{S(\lambda_0)}{\sqrt{\mathcal{I}(\lambda_0)}} \right| > z(0.025) = 1.96.$$

where λ_0 is the value of λ under H_0. For the given data, our observed test statistic is

$$Z = \left| \frac{n\bar{X}/\lambda_0 - n}{\sqrt{n/\lambda_0}} \right| = 2.12,$$

hence we reject the null hypothesis.

4. In 1861, 10 essays appeared in the New Orleans Daily Crescent. They were signed "Quintus Curtius Snodgrass" and some people suspected they were actually written by Mark Twain. To investigate this, we will consider the proportion of three letter words found in an authors work. From 8 of Twains essays, the proportions are:

0.225 0.262 0.217 0.240 0.230 0.229 0.235 0.217

From 10 Snodgrass essays, the proportions are:

0.209 0.205 0.196 0.210 0.202 0.207 0.224 0.223 0.220 0.201

Perform a Wald test for equality of the means. Report the p-value and a 95% confidence interval for the difference of means. What do you conclude?

Solution: Let p be the proportions for Twains essays, and q the proportions for Snodgrass essays. We would like to test $H_0 : p = q$ against $H_1 : p \neq q$. Let $\delta = p - q$; so we are testing whether or not $\delta = 0$. A natural estimator for δ is $\hat{p} - \hat{q}$, where $\hat{p}$ and $\hat{q}$ are the sample means of the proportions from Twains and Snodgrass essays respectively. From the data, we have $\hat{\delta} = \hat{p} - \hat{q} = 0.2319 - 0.2097 = 0.0222$.

The variance of this estimator is $\operatorname{Var}(\hat{p} - \hat{q}) = \operatorname{Var}\hat{p} + \operatorname{Var}\hat{q}$, which is estimated by $s_p^2/n_p + s_q^2/n_q$, where s_p^2 and s_q^2 are the unbiased sample variances of the proportions from Twains and Snodgrass essays respectively. From the data, this is calculated to be $s_p^2 = 0.0002121$ and $s_q^2 = 0.00009334$, where $n_p = 8$ and $n_q = 10$. Therefore, the standard error is

$$\mathrm{SE}(\hat{\delta}) = \sqrt{s_p^2/n_p + s_q^2/n_q} = \sqrt{\frac{0.0002121}{8} + \frac{0.00009334}{10}} = 0.00599.$$

Now, asymptotically $\hat{\delta} \xrightarrow{\mathrm{D}} \mathrm{N}(0, \mathrm{SE}(\hat{\delta}))$ under H_0, so the null is rejected if its p-value given by

$$p = \mathrm{P}\left(Z > \left|\frac{\hat{\delta}}{\mathrm{SE}(\hat{\delta})}\right|\right), \text{ where } Z \sim \mathrm{N}(0,1)$$

is greater than $\alpha = 0.05$. From the data, the p-value is calculated to be $\mathrm{P}(Z > 3.7062) = 0.0002104$, so the null hypothesis is rejected at the 5% significance level.

As for the confidence interval, we can calculate it as

$$\begin{aligned}\hat{\delta} \pm z(0.025) \cdot \mathrm{SE}(\hat{\delta}) &= 0.0222 \pm 1.96 \times 0.00599 \\ &= (0.0104, 0.0339).\end{aligned}$$

We also see that 0 is not included in the 95% confidence interval.

While the Wald test rejects the hypothesis that the two works are from similar authors, we have to be wary about the fact that the sample size is small in this case. The Wald test is an asymptotic test where it relies on large sample sizes for the test to be valid.

5. A sample of 11 observations from population $\mathrm{N}(\mu, \sigma^2)$ yields the sample mean $\bar{X} = 8.68$ and the sample variance $s^2 = 1.21$. At 5% significance level, test the following hypotheses.
   (a) $H_0 : \mu = 8$ against $H_1 : \mu > 8$
   (b) $H_0 : \mu = 8$ against $H_1 : \mu < 8$
   (c) $H_0 : \mu = 8$ against $H_1 : \mu \neq 8$

   Repeat the above exercise with the additional assumption $\sigma^2 = 1.21$. Compare the results with those derived without this assumption and comment.

Solution: When σ^2 is not known, we use the t statistic $T = \sqrt{n}(\bar{X} - \mu)/s$ and compare against a t_{n-1} distribution. We reject H_0 if

(a) $T > t_{10}(0.05) = 1.81$ against $H_1 : \mu > 8$.

(b) $T < -t_{10}(0.05) = -1.81$ against $H_1 : \mu < 8$.

(c) $|T| > t_{10}(0.025) = 2.23$ against $H_1 : \mu \neq 8$.

The observed test-statistic is $T = 2.0503$ under H_0. Hence, we reject H_0 in (a) against the alternative $H_1 : \mu > 8$, but will not reject H_0 against the other two alternatives.

Now, when σ^2 is known, we can use the test statistic $Z = \sqrt{n}(\bar{X} - \mu)/\sigma$, which follows a $N(0, 1)$ distribution. So we reject H_0 if

(a) $Z > z(0.05) = 1.64$ against $H_1 : \mu > 8$.

(b) $Z < -z(0.05) = -1.64$ against $H_1 : \mu < 8$.

(c) $|Z| > z(0.025) = 1.96$ against $H_1 : \mu \neq 8$.

The observed test statistic is $Z = 2.0503$. Hence, we reject H_0 against the alternatives $H_1 : \mu > 8$ or $H_1 : \mu \neq 8$, but not the other alternative.

6. There is a theory that people can postpone their death until after an important event. To test this theory, Phillips and King (1988, Lancet, pp. 728-) collected data on deaths around the Jewish holiday Passover. Of 1919 deaths, 922 died the week before the holiday and 997 died the week after. Think of this as a binomial and test the null hypothesis that $\theta = 1/2$, where θ is the probability that a death occurs after the holiday. Also construct a confidence interval for θ.

Solution: Let $X_i = 1$ if the ith person dies after Passover for $i = 1, \ldots, 1919$, and 0 otherwise. So $X_i \sim \text{Bern}(\theta)$ and an estimator for θ is $\hat{\theta} = \bar{X} = 997/1919 = 0.5195$. The standard error of this estimator is $\text{SE}(\hat{\theta}) = \sqrt{\hat{\theta}(1 - \hat{\theta})/1919} = 0.0114$. Using the CLT, we know that $\hat{\theta} \xrightarrow{\text{D}} \text{N}(\theta, \text{SE}(\hat{\theta})^2)$. To test $H_0 : \theta = 1/2$ against $H_1 : \theta \neq 1/2$, we use the test statistic

$$Z = \left| \frac{\hat{\theta} - \theta}{\text{SE}(\hat{\theta})} \right|,$$

which is then compared against $z(0.025) = 1.96$. Under H_0, we find that the observed test statistic is $Z = (0.5195 - 0.5)/0.0114 = 1.711$. Thus, we are not able to reject the null hypothesis, and conclude that there is no significant evidence indicating that the death rates before and after the Passover are different.

An approximate 95% confidence interval for θ is

$$\hat{\theta} \pm 1.96 \cdot \text{SE}(\hat{\theta}) = (0.4972, 0.5419).$$

Remark: A more relevant setting for this problem is testing $H_0 : \theta = 1/2$ against $H_1 : \theta > 1/2$. Then the p-value is $\text{P}(Z > 1.711) = 0.0435$. We reject H_0 at the 5% level, but not at the 1% level.

7. (a) Two independent random samples, of n_1 and n_2 observations, are drawn from normal distributions with the same variance σ^2. Let s_1^2 and s_2^2 be the sample variances of the first and the second sample, respectively. Show that

$$\hat{\sigma}^2 = \frac{1}{n_1 + n_2 - 2}\left\{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right\}$$

is an unbiased estimator for σ.

> **Solution:** We use the fact that $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$. From the properties of the χ^2-distribution, $\mathrm{E}((n-1)s^2/\sigma^2) = (n-1)$. Thus,
>
> $$\begin{aligned} \mathrm{E}(\hat{\sigma}^2) &= \frac{(n_1-1)\,\mathrm{E}(s_1^2) + (n_2-1)\,\mathrm{E}(s_2^2)}{n_1+n_2-2} \\ &= \frac{(n_1-1)\sigma^2 + (n_2-1)\sigma^2}{n_1+n_2-2} \\ &= \sigma^2. \end{aligned}$$

(b) Two makes of car safety belts, A and B have breaking strengths which are normally distributed with the same variance. A sample of 140 belts of make A and a sample of 220 belts of make B were tested. The sample means and the sums of squares about the means (i.e. $\sum_i (X_i - \bar{X})^2$) of of the breaking strengths (in lbf units) were $(2685, 19000)$ for make A, and $(2680, 34000)$ for make B. Is there any significant evidence to support the hypothesis that belts of make A are stronger than belts of make B?

> **Solution:** We have $\bar{A} = 2685$ and $\bar{B} = 2680$. Also, $139s_A^2 = 19000$ and $219s_B^2 = 34000$. We wish to test $H_0 : \mu_A = \mu_B$ against $H_1 : \mu_A > \mu_B$. Note that $\bar{A} - \bar{B}$ is normal with mean $\mathrm{E}(\bar{A} - \bar{B}) = \mu_A - \mu_B$, and variance $\mathrm{Var}(\bar{A} - \bar{B}) = \mathrm{Var}(\bar{A}) + \mathrm{Var}(\bar{B}) = \sigma^2/140 + \sigma^2/220 = 0.01169\sigma^2$. Thus,
>
> $$\bar{A} - \bar{B} \sim \mathrm{N}(\mu_A - \mu_B, 0.01169\sigma^2).$$
>
> Also, $139s_A^2/\sigma^2 \sim \chi^2_{139}$, while $219s_B^2/\sigma^2 \sim \chi^2_{219}$. Therefore, $(139s_A^2 + 219s_B^2)/\sigma^2 \sim \chi^2_{358}$. The test statistic under H_0 is
>
> $$\begin{aligned} T &= \frac{\bar{A} - \bar{B}}{\sigma\sqrt{0.01169}} \Bigg/ \sqrt{\frac{139s_A^2 + 219s_B^2}{358\sigma^2}} \\ &= \frac{(\bar{A} - \bar{B})/\sqrt{0.01169}}{\sqrt{(139s_A^2 + 219s_B^2)/358}} \sim t_{358} \end{aligned}$$
>
> We observe $T = 3.801$, which is rejected against a critical value of $t_{358}(0.01) \approx z(0.01) = 2.33$. We conclude that there is evidence to suggest that Make A belts are stronger than Make B belts.

8. The table below summarized the fate of the passengers and the crew when the Titanic sank on Monday, 15 April 1912. Test the hypothesis of independence between the row variable and the column variable in the table, and interpret your findings.

	Men	Women	Boys	Girl
Survived	332	318	29	27
Died	1360	104	35	18

Solution: First, calculate the marginals

	Men	Women	Boys	Girl	Totals
Survived	332	318	29	27	706
Died	1360	104	35	18	1517
Totals	1692	422	64	45	2223

Then, the expected values are calculated as follows:

	Men	Women	Boys	Girl	Totals
Survived	$\frac{706 \cdot 1692}{2223}$	$\frac{706 \cdot 422}{2223}$	$\frac{706 \cdot 64}{2223}$	$\frac{706 \cdot 45}{2223}$	706
Died	$\frac{1517 \cdot 1692}{2223}$	$\frac{1517 \cdot 422}{2223}$	$\frac{1517 \cdot 64}{2223}$	$\frac{1517 \cdot 45}{2223}$	1517
Totals	1692	422	64	45	2223

	Men	Women	Boys	Girl	Totals
Survived	537.360	134.022	20.326	14.291	706
Died	1154.640	287.978	43.674	30.709	1517
Totals	1692	422	64	45	2223

Calculate the χ^2 test statistic as

$$T = \sum_{i,j} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = 507.084$$

which is then compared against a χ_3^2 distribution. Note that d.f.$= (4-1)(2-1) = 3$. Since $\chi_3^2(0.01) = 11.345$, we reject the independent hypothesis.

The above statistical analysis reveals that there is very significant evidence indicating that to die or not depends on the gender/age category. Looking at the difference table, one can see clearly that the number of male deaths is much higher than the expected number under the independence hypothesis. In contrast, the numbers of deaths for women, boys and girls are all smaller than the expected numbers. Of course we know why this happened!