SM-1402 Exercise 4

1. Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f(x) = 0.05, \quad 0 \leq x \leq 20$$

   (a) What is the probability that a current measurement is less than 10 milliamperes?

   (b) What is the probability that a current measurement is between 5 and 20 milliamperes?

   (c) Find the mean and variance of X.

> **Solution:**
>
> (a) $\mathrm{P}(X < 10) = 0.5$.
>
> (b) $\mathrm{P}(5 < X < 20) = 0.75$.
>
> (c) $E(X) = \int_0^{20} 0.05x \, \mathrm{d}x = 10$, $\mathrm{Var}(X) = \int_0^{20} 0.05(x - \mathrm{E}(X))^2 \, \mathrm{d}x = 33.33$.

2. Suppose that a r.v. X has pdf $f(x) = x/8$ for $3 < x < 5$. Determine the following probabilities

   (a) $\mathrm{P}(4 > X)$

   (b) $\mathrm{P}(X > 3.5)$

   (c) $\mathrm{P}(4 < X < 5)$

   (d) $\mathrm{P}(X < 4.5)$

   (e) $\mathrm{P}(X < 3.5 \text{ or } X > 4.5)$

> **Solution:**
>
> (a) $\mathrm{P}(X < 4) = 0.4375$.
>
> (b) $\mathrm{P}(X > 3.5) = 0.7969$.
>
> (c) $\mathrm{P}(4 < X < 5) = 0.5625$.
>
> (d) $\mathrm{P}(X < 4.5) = 0.7031$.
>
> (e) $\mathrm{P}(X < 3.5) + \mathrm{P}(X > 4.5) = 0.5$.

3. Suppose that a r.v. X has pdf $f(x) = 1.5x^2$ for $-1 < x < 1$.

   (a) Find the cdf of X.

   (b) $\mathrm{P}(X > 0)$

   (c) $\mathrm{P}(|X| \leq 0.5)$

(d) $\mathrm{P}(X < -2)$

(e) Determine x such that $\mathrm{P}(X > x) = 0.05$

Solution:

(a)

$$F(x) = \int_{-1}^{x} 1.5\tilde{x}^2 \, \mathrm{d}\tilde{x} = \begin{cases} 0 & \text{if } x < -1 \\ 0.5(1 + x^3) & \text{if } -1 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

.

(b) $\mathrm{P}(X > 0) = 1 - F(0) = 0.5$.

(c) $\mathrm{P}(|X| \leq 0.5) = \mathrm{P}(-0.5 \leq X \leq 0.5) = F(0.5) - F(-0.5) = 0.125$.

(d) $\mathrm{P}(X < -2) = 0$.

(e)

$$\begin{aligned} \mathrm{P}(X > x) = 1 - F(x) &= 0.05 \\ 1 - 0.5(1 + x^3) &= 0.05 \\ 0.5 - 0.5x^3 &= 0.05 \\ x^3 &= 0.45/0.5 = 0.9 \\ x &= \sqrt[3]{0.9} = 0.9655 \end{aligned}$$

4. Let $X \sim f(x)$ where $f(x) = 2x^{-3}$ for $x > 1$. Determine the mean and variance of X.

Solution: The mean of X is

$$\mathrm{E}(X) = \int_{1}^{\infty} x \cdot 2x^{-3} \, \mathrm{d}x = \left[-2x^{-1} \right]_1^{\infty} = 2.$$

However,

$$\mathrm{E}(X^2) = \int_{1}^{\infty} x^2 \cdot 2x^{-3} \, \mathrm{d}x = \int_{1}^{\infty} 2/x \, \mathrm{d}x = \left[\log x \right]_1^{\infty} \not< \infty$$

therefore the variance is undefined.

5. Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 squared milliamperes.

(a) What is the probability that a measurement will exceed 13 milliamperes?

(b) What is the probability that a current measurement is between 9 and 11 milliamperes?

Solution: Let X be the current measurements in a strip of wire. Then $X \sim \mathrm{N}(10, 4)$.

(a) $\mathrm{P}(X > 13) = 1 - \Phi(1.5) = 0.0668$.

(b) $\mathrm{P}(9 < X > 11) = \Phi(0.5) - \Phi(-0.5) = 0.3829$.

6. Assume X is normally distributed with a mean of 10 and a standard deviation of 2. Determine the following:
   (a) $\mathrm{P}(X < 13)$
   (b) $\mathrm{P}(X > 9)$
   (c) $\mathrm{P}(6 < X < 14)$
   (d) $\mathrm{P}(2 < X < 4)$
   (e) $\mathrm{P}(-2 < X < 8)$

Solution:

(a) $\mathrm{P}(X < 13) = 0.9332$.

(b) $\mathrm{P}(X > 9) = 0.6914$.

(c) $\mathrm{P}(6 < X < 14) = 0.9545$.

(d) $\mathrm{P}(2 < X < 4) = 0.001$.

(e) $\mathrm{P}(-2 < X < 8) = 0.1587$.

7. The line width for semiconductor manufacturing is assumed to be normally distributed with a mean of 0.5 μm and a standard deviation of 0.05 μm.
   (a) What is the probability that a line width is greater than 0.62 μm?
   (b) What is the probability that a line width is between 0.47 and 0.63 μm?
   (c) Determine the line width such that 90% of samples are below that value.

Solution: Let X be the line width for semiconductor manufacturing. Then $X \sim \mathrm{N}(0.5, 0.05^2)$.

(a) $\mathrm{P}(X > 0.62) = 1 - \Phi(2.4) = 0.0082$.

(b) $\mathrm{P}(0.47 < X < 0.63) = \Phi(2.6) - \Phi(-0.6) = 0.7211$.

(c) We want to know what is x such that $\mathrm{P}(X < x) = 0.9$. So,

$$\begin{aligned} \mathrm{P}(X < x) = \Phi\left(\frac{x - 0.5}{0.05}\right) &= 0.9 \\ (x - 0.5)/0.05 &= \Phi^{-1}(0.9) = 1.2816 \\ x &= 0.5641 \end{aligned}$$

8. Suppose that X is a binomial random variable with $n = 200$ and $p = 0.4$.
   (a) Approximate the probability that X is less than or equal to 70.
   (b) Approximate the probability that X is greater than 70 and less than 90.
   (c) Approximate $\mathrm{P}(X = 80)$.

Solution: Since $np > 5$ and $n(1 - p) > 5$, we can use the normal approximation. The mean and variance of the binomial r.v. is $200(0.4) = 80$ and $200(0.4)(0.6) = 48$ respectively. Therefore, $X_{cts} \sim \mathrm{N}(80, 48)$.

(a) $\mathrm{P}(X \leq 70) \approx \mathrm{P}(X_{cts} < 70.5) = 0.0853$.
(b) $\mathrm{P}(70 < X < 90) \approx \mathrm{P}(70.5 < X_{cts} < 89.5) = 0.8293$.
(c) $\mathrm{P}(X = 80) \approx \mathrm{P}(79.5 < X_{cts} < 80.5) = 0.0575$.

9. Suppose that X has a Poisson distribution with a mean of 64. Approximate the following probabilities:
   (a) $\mathrm{P}(X > 72)$
   (b) $\mathrm{P}(X < 64)$
   (c) $\mathrm{P}(60 < X < 68)$

Solution: Let $X \sim \mathrm{Pois}(\lambda = 64)$. Since $\lambda > 15$, we can use the normal approximation $X_{cts} \sim \mathrm{N}(64, 64)$.

(a) $\mathrm{P}(X > 72) \approx \mathrm{P}(X_{cts} > 71.5) = 0.1439$.
(b) $\mathrm{P}(X < 64) \approx \mathrm{P}(X_{cts} < 63.5) = 0.4749$.
(c) $\mathrm{P}(60 < X \leq 68) \approx \mathrm{P}(60.5 < X_{cts} < 68.5) = 0.3826$.

10. An electronic office product contains 5000 electronic components. Assume that the probability that each component operates without failure during the useful life of the product is 0.999, and assume that the components fail independently. Approximate the probability that 10 or more of the original 5000 components fail during the useful life of the product.

Solution: Let X be the number of components failing in an electronic office product during its life. Then $X \sim \text{Bin}(5000, 0.001)$, with $\text{E}(X) = 5000(0.001) = 5$ and $\text{Var}(X) = 5000(0.001)(0.999) = 4.995$. Note that $np = 5 \geq 5$ and $n(1-p) = 4995 > 5$ so we can use the normal approximation $X_{cts} \sim \text{N}(5, 4.995)$.

$$\begin{aligned}
\text{P}(X \geq 10) &\approx \text{P}(X_{cts} > 9.5) \\
&= 1 - \Phi\left(\frac{9.5 - 5}{\sqrt{4.995}}\right) \\
&= 1 - \Phi(2.01) \\
&= 1 - 0.9778 = 0.0222
\end{aligned}$$