SM-4331 Exercise 0

1. Let $X_1, \dots, X_n$ be a sample taken from a population with mean μ and variance σ^2.
   (a) Write down the formula for the sample mean $\bar{X}$ and unbiased sample variance s^2.
   (b) Show that
   $$\sum_{i=1}^{n}(X_i - \bar{X}) = \sum_{i=1}^{n} X_i - n\bar{X}$$
   (c) Find the expectation of $\bar{X}$ and s^2.

2. Let $X_i \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$, for $i = 1, \dots, n$.
   (a) What is $\mathrm{E}(X_i)$ and $\mathrm{Var}(X_i)$?
   (b) What is $\mathrm{Cov}(X_i, X_j)$, $i \neq j$?
   (c) Write down distributions for the following random variables:
      i. $Y = aX_i + b$, where $a, b \in \mathbb{R}$.
      ii. $Z = X_1 + \cdots + X_n$.
      iii. $W = X_1 - X_2$.

3. Let $X_1, \dots, X_n$ be a sample taken from a population with mean μ and variance σ^2. Suppose that $\bar{X}_n \sim \mathrm{N}(\mu, \sigma^2/n)$, and that $\bar{X}_n$ is the sample mean,
   (a) What is a 95% confidence interval for $\bar{X}_n$?
   (b) What is the distribution of $Z = \sum_{i=1}^{n} X_i$?
   (c) What is a 95% confidence interval for Z?

4. Let $X_1, \dots, X_n$ be independent random variables, and let $a_1, \dots, a_n$ be constants. Show that
   $$\mathrm{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \, \mathrm{E}(X_i)$$
   and
   $$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \, \mathrm{Var}(X_i)$$

5. For random variables X and Y
   (a) Write down the formula for $\mathrm{Cov}(X, Y)$.
   (b) Show that
   $$\mathrm{Cov}(X + Y, X - Y) = \mathrm{Var}(X) - \mathrm{Var}(Y)$$

6. Let $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-(x - \mu)/2\sigma^2\}$.
   (a) Show that $\log f(x) = -\frac{1}{2} \log 2\pi - \frac{1}{2} \log \sigma^2 - (x - \mu)/2\sigma^2$.

(b) Obtain $\frac{\partial}{\partial \mu} \log f(x)$.

(c) Obtain $\frac{\partial}{\partial \sigma^2} \log f(x)$.

7. A random variable X has distribution as follows:

x	-1	0	1	2
$f(x)$	3/8	2/8	2/8	1/8

(a) Is X discrete or continuous?

(b) What is $\mathrm{P}(X \leq 0)$?

(c) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

8. A continuous random variable, X, has the following cdf:

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate $P(0.5 < X < 1)$.

(b) Find x such that $P(X > x) = 0.05$.

(c) Determine $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

9. A random variable, X, has the following pdf:

$$f(x) = \begin{cases} 2x/3 & \text{for } 0 < x < 1 \\ (3-x)/3 & \text{for } 1 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Derive the cdf of X.

(b) Find the mean and standard deviation of X.