SM-1402 Exercise 3

1. A random variable is defined as follows:

x	-2	-1	0	1	2
$f(x)$	1/8	2/8	2/8	2/8	1/8

Determine the following probabilities

(a) $\mathrm{P}(X \leq 2)$

(b) $\mathrm{P}(X > -2)$

(c) $\mathrm{P}(|X| \leq 1)$

(d) $\mathrm{P}(\{X \leq -1\} \cup \{X = 2\})$

Solution: (a) $\mathrm{P}(X \leq 2) = 1$ (probability of ‘everything’).

(b) $\mathrm{P}(X > -2) = 1 - 1/8$ (all excluding $X = -2$).

(c) $\mathrm{P}(|X| \leq 1) = \mathrm{P}(-1 \leq X \leq 1) = 2/8 + 2/8 + 2/8 = 6/8$.

(d) $\mathrm{P}(\{X \leq -1\} \cup \{X = 2\}) = \mathrm{P}(X \leq -1) + \mathrm{P}(X = 2) = 1/8 + 2/8 + 1/8 = 4/8$.

2. A random variable X has the following probability mass function:

$$f(x) = \frac{2x+1}{25}, \quad x \in \{0, 1, 2, 3, 4\}.$$

Find

(a) $\mathrm{P}(X = 4)$

(b) $\mathrm{P}(X \leq 1)$

(c) $\mathrm{P}(2 \leq X < 4)$

(d) $\mathrm{P}(X > -10)$

Solution: (a) $\mathrm{P}(X = 4) = \frac{2 \cdot 4 + 1}{25} = 9/25$.

(b) $\mathrm{P}(X \leq 1) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1) == \frac{2 \cdot 0 + 1}{25} + \frac{2 \cdot 1 + 1}{25} = 4/25$.

(c) $\mathrm{P}(2 \leq X < 4) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3) == \frac{2 \cdot 2 + 1}{25} + \frac{2 \cdot 3 + 1}{25} = 12/25$.

(d) $\mathrm{P}(X > -10) = 1$.

3. In a semiconductor manufacturing process, three wafers from a lot are tested. Each wafer is classified as pass or fail. Assume that the probability that a wafer passes the test is 0.8 and that wafers are independent. Determine the mass function of the number of wafers from a lot that pass test.

Solution: Let X be the number of wafers that pass the test (out of 3). Then $X \sim \text{Bin}(3, 0.8)$, and the pmf of X is $\text{P}(X = x) = \binom{3}{x} 0.8^x 0.2^{3-x}$. We can then create the following distribution table:

x	0	1	2	3
$\text{P}(X = x)$	0.008	0.096	0.384	0.512

Alternative solutions would be to use a probability tree, but the binomial pmf is much faster.

4. Suppose that a days production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. Find the cumulative distribution function of X.

Solution: Let Y_i denote the state of the ith part, $i = 1, 2$. Let $Y_i = 1$ if it conforms, and $Y_i = 0$ otherwise.

- $\text{P}(X = 0) = \text{P}(Y_1 = 1)\,\text{P}(Y_2 = 1) = \frac{800}{850} \cdot \frac{799}{849} = 0.886.$
- $\text{P}(X = 2) = \text{P}(Y_1 = 0)\,\text{P}(Y_2 = 0) = \frac{50}{850} \cdot \frac{49}{849} = 0.003.$
- $\text{P}(X = 1) = 1 - \text{P}(X = 0) - \text{P}(X = 2) = 0.111.$

So the cdf of X is

x	0	1	2
$\text{P}(X \leq x)$	0.886	0.997	1.000

Note that the sampling without replacement means that the probability of the two parts being defective are "related".

5. Given that a discrete r.v. X has cdf

$$F(x) = \begin{cases} 0 & x < -10 \\ 0.25 & -10 \leq x < 30 \\ 0.75 & 30 \leq x < 50 \\ 1 & 50 \leq x \end{cases}$$

Find

(a) $\text{P}(X \leq 50)$

(b) $\mathrm{P}(X \leq 40)$

(c) $\mathrm{P}(40 \leq X \leq 60)$

(d) $\mathrm{P}(X < 0)$

(e) $\mathrm{P}(0 \leq X < 10)$

(f) $\mathrm{P}(-10 \leq X < 10)$

Solution: (a) $\mathrm{P}(X \leq 50) = 1$.

(b) $\mathrm{P}(X \leq 40) = F(40) = 0.75$.

(c) $\mathrm{P}(40 \leq X \leq 60) = F(60) - F(40) = 1 - 0.75 = 0.25$.

(d) $\mathrm{P}(X < 0) = F(0) = 0.25$.

(e) $\mathrm{P}(0 \leq X < 10) = F(10) - F(0) = 0.25 - 0.25 = 0$.

(f) $\mathrm{P}(-10 \leq X < 10) = F(10) - F(-10) = 0.25 - 0.25 = 0$.

6. A r.v. X has expectation $\mathrm{E}(X) = 3$ and variance $\mathrm{Var}(X) = 2.5$. Let $Y = 2X$, $Z = 4X + 2$ and $W = 3X - 1$. Find the means and variances of the three r.v. Y, Z and W.

Solution: $\mathrm{E}(Y) = 2\,\mathrm{E}(X) = 2 \cdot 3 = 6$. $\mathrm{Var}(Y) = 4\,\mathrm{Var}(X) = 4 \cdot 2.5 = 10$.

$\mathrm{E}(Z) = 4\,\mathrm{E}(X) + 2 = 4 \cdot 3 + 2 = 14$. $\mathrm{Var}(Z) = 4^2\,\mathrm{Var}(X) = 16 \cdot 2.5 = 40$.

$\mathrm{E}(W) = 3\,\mathrm{E}(X) - 1 = 3 \cdot 3 - 1 = 8$. $\mathrm{Var}(W) = 3^2\,\mathrm{Var}(X) = 9 \cdot 2.5 = 22.5$.

7. Calculate the expectation and variance of X, when X has the following distribution:

x	0	1	2	3
$\mathrm{P}(X = x)$	0.1	0.2	0.5	0.2

Solution:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{x=0}^{3} x\,\mathrm{P}(X=x) \\
&= 0(0.1) + 1(0.2) + 2(0.5) + 3(0.2) = 1.8
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{3} x^2\,\mathrm{P}(X=x) \\
&= 0^2(0.1) + 1^2(0.2) + 2^2(0.5) + 3^2(0.2) = 4
\end{aligned}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}^2(X) = 4 - 1.8^2 = 0.76$$

8. Let X be a binomial random variable with $p = 0.1$ and $n = 10$. Calculate the following probabilities.
   (a) $\mathrm{P}(X \leq 2)$
   (b) $\mathrm{P}(X > 8)$
   (c) $\mathrm{P}(X = 4)$
   (d) $\mathrm{P}(5 \leq X \leq 7)$

Solution: $X \sim \mathrm{Bin}(10, 0.1)$, $\mathrm{P}(X = x) = \binom{10}{x} 0.1^x 0.9^{10-x}$.

(a) $\mathrm{P}(X \leq 2) = 1 - \mathrm{P}(X < 2) = 1 - \mathrm{P}(X = 0, 1) = 0.2639011$.

(b) $\mathrm{P}(X > 8) = \mathrm{P}(X = 9, 10) = 9.1 \times 10^{-9}$.

(c) $\mathrm{P}(X = 4) = \binom{10}{4} 0.1^4 0.9^6 = 0.01116026$.

(d) $\mathrm{P}(5 \leq X \leq 7) = \mathrm{P}(X = 5, 6, 7) = 0.001634564$.

9. The phone lines to an airline reservation system are occupied 40% of the time. Assume that the events that the lines are occupied on successive calls are independent. Assume that 10 calls are placed to the airline.
   (a) What is the probability that for exactly three calls the lines are occupied?
   (b) What is the probability that for at least one call the lines are not occupied?
   (c) What is the expected number of calls in which the lines are all occupied?

Solution: Let X be the number of calls accepted out of 10. Each time a person calls, there is a 60% chance of the line is not being occupied. Then $X \sim \mathrm{Bin}(10, 0.6)$.

(a)$P(X=7) = \binom{10}{7}0.6^7 0.4^3 = 0.215.$

(b) $P(X \geq 1) = 1 - P(X=0) = 1 - 0.6^{10} = 0.9998951.$

(c) 40% of 10 calls are occupied, so the expected number of calls in which all lines are occupied is $40\% \times 10 = 4$.

10. Because not all airline passengers show up for their reserved seat, an airline sells 125 tickets for a flight that holds only 120 passengers. The probability that a passenger does not show up is 0.10, and the passengers behave independently.

    (a) What is the probability that every passenger who shows up can take the flight?

    Solution: This means that out of the 125 who bought the ticket, at most 120 show up. Let X be the number of passengers who show up for the flight. $X \sim \text{Bin}(125, 0.9)$. Let's use a normal approximation for this, i.e. $X \approx N(125 \times 0.9, 125 \times 0.9 \times 0.1) \equiv N(112.5, 11.25)$. The probability required is

    $$\begin{aligned} P(X \leq 120) \approx P(X < 120.5) &= P\left(Z < \frac{120.5 - 112.5}{\sqrt{11.25}}\right) \\ &= P(Z < 2.38) = 0.9913437. \end{aligned}$$

    (b) What is the probability that the flight departs with empty seats?

    Solution: This suggests not all 120 passengers show up, or in other words, at most 119 passengers show up. Let X be as above. The required probability is

    $$\begin{aligned} P(X \leq 119) \approx P(X < 119.5) &= P\left(Z < \frac{119.5 - 112.5}{\sqrt{11.25}}\right) \\ &= P(Z < 2.0869) = 0.9815514. \end{aligned}$$

11. Suppose that the number of flaws in a thin copper wire follows a Poisson distribution with a mean of 2.3 flaws per millimeters.

    (a) Determine the probability of exactly 2 flaws in 1 millimeter of wire.

    Solution: Let X be the number of flaws in 1mm of wire. Then $X \sim \text{Pois}(2.3)$. So $P(X=2) = e^{-2.3}2.3^2/2! = 0.2651846$.

    (b) Determine the probability of 10 flaws in 5 millimeters of wire.

Solution: Let Y be the number of flaws in 5mm of wire. Then $Y \sim \text{Pois}(2.3 \times 5 = 11.5)$. So $\text{P}(X = 10) = e^{-11.5} 11.5^{10}/10! = 0.1129351$.

(c) Determine the probability of at least 1 flaw in 2 millimeters of wire.

Solution: Let Z be the number of flaws in 2mm of wire. Then $Z \sim \text{Pois}(2.3 \times 2 = 4.6)$. So $\text{P}(Z \geq 1) = 1 - \text{P}(Z = 0) = e^{-11.5} 11.5^{10}/10! = 0.9899482$.

12. Astronomers treat the number of stars in a given volume of space as a Poisson random variable. The density in the Milky Way Galaxy in the vicinity of our solar system is one star per 16 cubic light years.

(a) What is the probability of two or more stars in 16 cubic light years?

Solution: Denote the number of stars in 16 cubic light years of space by X. Then $X \sim \text{Pois}(1)$. So $\text{P}(X \geq 2) = 1 - \text{P}(X = 0, 1) = 0.2642411$.

(b) How many cubic light years of space must be studied so that the probability of one or more stars exceeds 0.95?

Solution: Let $Y \sim \text{Pois}(\lambda)$. Then

$$\begin{aligned} \text{P}(Y \geq 1) = 1 - \text{P}(Y = 0) &> 0.95 \\ \Rightarrow 1 - e^{-\lambda} &> 0.95 \\ \Rightarrow e^{-\lambda} &> 0.05 \\ \Rightarrow \lambda &> -\log 0.05 \\ &= 2.995732 \end{aligned}$$

1 is to 16 as 2.99 is to x, and thus $x = 16 \times 2.99 = 47.84$ cubic lightyears.

13. Traffic flow is traditionally modelled as a Poisson distribution. A traffic engineer monitors the traffic flowing through an intersection with an average of 6 cars per minute.

(a) What is the probability of no cars through the intersection within 30 seconds?

Solution: This has follows a Pois(3) distribution, and denote this by X. Then $\text{P}(X = 0) = e^{-3} = 0.0498$.

(b) What is the probability of three or more cars through the intersection within 30 seconds?

Solution: Let X as above. Then $\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X = 0, 1, 2) = 0.5768099$.

(c) Calculate the minimum number of cars through the intersection so that the probability of this number or fewer cars in 30 seconds is at least 90%.

Solution: Let X as above. Then we are after x such that $\mathrm{P}(X \leq x) \geq 0.9$. Let's tabulate the cdf:

x	0	1	2	3	4	5	6
$\mathrm{P}(X \leq x)$	0.050	0.199	0.423	0.647	0.815	0.916	0.966

We see that the answer is 5.

(d) If the variance of the number of cars through the intersection per minute is 20, is the Poisson distribution appropriate?

Solution: The variance of a Pois(6) random variate is 6, so it seems that the data might not be suited to a Poisson distribution.