# SM-4331 Advanced Statistics
# Chapter 3 (Important Univariate and Multivariate Distributions)

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# Outline

1. $\chi^2$-distribution
2. Student's $t$-distribution
3. $F$-distribution
4. Multivariate distributions
    - Bivariate distributions
    - Multivariate distributions
5. Multinomial and categorical distribution
    - Multinomial distribution
    - Categorical distribution
6. Multivariate normal distribution

# $\chi^2$-distribution

The $\chi^2$-distribution is an important distribution in statistics. It is closely linked with the normal, Student's $t$ and $F$ distributions. Inference for the variance parameter $\sigma^2$ relies on $\chi^2$-distributions. More importantly, most goodness-of-fit tests are based on $\chi^2$-distributions.

## Definition 1 ($\chi^2$-distribution)

Let $Z_1, \dots, Z_k \overset{\text{iid}}{\sim} \text{N}(0,1)$, i.e. each $Z_i$ has pdf $f(z_i) = (2\pi)^{-1/2} e^{-z_i^2/2}$ for $i = 1, \dots, k$. Then,

$$X = Z_1^2 + \cdots + Z_k^2 = \sum_{i=1}^{k} Z_i^2$$

follows a $\chi^2$-distribution with $k$ degrees of freedom. We write $X \sim \chi^2_k$.

# $\chi^2$-distribution (cont.)

**Remark**

Out of curiosity, the pdf of a $\chi^2_k$ distribution is $f(x) = Cx^{k/2-1}e^{-x/2}$, where the normalising constant $C$ is equal to $2^{-k/2}\Gamma^{-1}(k/2)$ ($\Gamma(\cdot)$ is the gamma function). The form of the pdf is less important to know than the definition of $\chi^2_k$ distribution given in Definition 1.

Here are some important properties of the $\chi^2_k$ distribution.

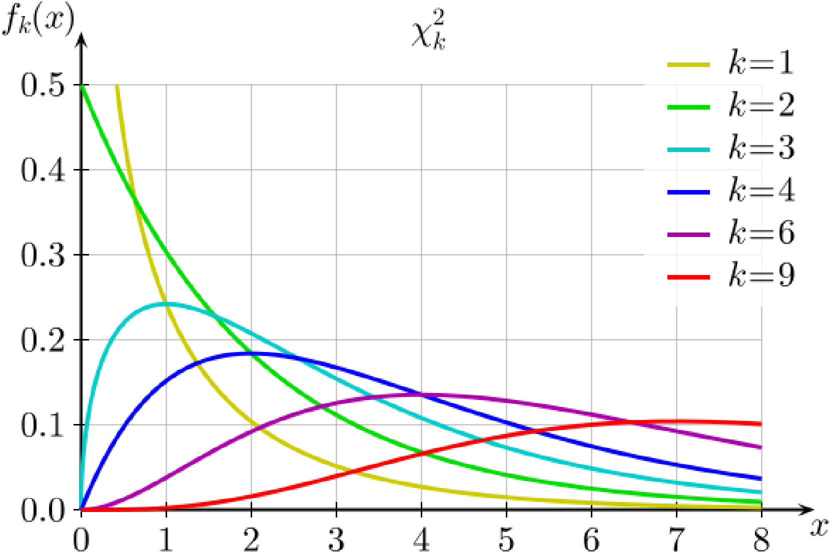
1. $X$ has support over $[0, \infty)$.
2. $\mathrm{E}(X) = k$.
3. $\mathrm{Var}(X) = 2k$.
4. If $X_1 \sim \chi^2_{k_1}$ and $X_2 \sim \chi^2_{k_2}$, and $X_1 \perp X_2$, then $X_1 + X_2 \sim \chi^2_{k_1+k_2}$.

# $\chi^2$-distribution (cont.)

Proof.

Prove properties 2–4 as an exercise.

# $\chi^2$-distribution (cont.)



# Probabilities tables for the $\chi^2$-distribution

Probabilities such as

$$\mathrm{P}(\chi^2_k \leq x) = \int_0^x f_X(\tilde{x}) \, \mathrm{d}\tilde{x}$$

where $f_X$ is the pdf of $\chi^2_k$ cannot be found in closed form. It is calculated using computer approximations for the integral above. In R, use `pchisq()`.

Alternatively, statistical tables are used. You will find tables for percentiles of the $\chi^2$ distribution. That is, you are able to find the value of $x := \chi^2_k(A)$ such that

$$\mathrm{P}(\chi^2_k \leq x) = \int_0^x f_X(\tilde{x}) \, \mathrm{d}\tilde{x} = A$$

for various values of $A$ and $k$.

# Example

## Example 2

Let $Y_1, \dots, Y_n \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2)$. Then,

$$Z_i = \frac{Y_i - \mu}{\sigma} \sim \mathrm{N}(0, 1).$$

Hence,

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \mu)^2 = \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2.$$

Note that

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \mu)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2 + \frac{n}{\sigma^2} (\bar{Y}_n - \mu)^2. \tag{1}$$

# Example (cont.)

## Example 2

Since $\bar{Y}_n \sim \mathrm{N}(\mu, \sigma^2/n)$, it must be that

$$\frac{n}{\sigma^2}(\bar{Y}_n - \mu)^2 \sim \chi^2_1.$$

It can also be proved (see Exercise Sheet 3) that

$$\frac{1}{\sigma^2}\sum_{i=1}^n (Y_i - \bar{Y}_n)^2 \sim \chi^2_{n-1}.$$

Thus, the decomposition in (1) may formally be written as

$$\chi^2_n = \chi^2_{n-1} + \chi^2_1$$

1. $\chi^2$-distribution
2. Student's $t$-distribution
3. $F$-distribution
4. Multivariate distributions
5. Multinomial and categorical distribution
6. Multivariate normal distribution

# Student's $t$-distribution

This is another important distribution in statistics, because:

- The $t$-test is perhaps the most frequently used statistical test in application.
- Confidence intervals for normal mean with unknown variance may be *accurately* constructed based on the $t$-distribution.

Historical note: The $t$-distribution was first studied by the Englishman William Sealy Gosset (1876-1937), who worked as a statistician for Guinness, writing under the pen-name "Student".

# Student's $t$-distribution (cont.)

## Definition 3

Suppose

- $Z \sim \mathrm{N}(0,1)$,
- $X \sim \chi^2_k$, and
- $X \perp Z$, i.e. $X$ and $Z$ are independent.

Then, the distribution of the random variable

$$T = \frac{Z}{\sqrt{X/k}}$$

is called the $t$-distribution with $k$ degrees of freedom. We write $T \sim t_k$.

# Student's $t$-distribution (cont.)

### Remark

The pdf for $T \sim t_k$ is given by

$$f(t) \propto \left(1 + \frac{t^2}{k}\right)^{-\frac{k+1}{2}},$$

but once again the actual form of the pdf is not as important as the definition of the $t$-distribution.

Some important properties of the $t$-distribution:

1. $T$ is continuous and symmetric over $(-\infty, \infty)$.
2. $\operatorname{E}(T) = 0$, provided $\operatorname{E}(|T|) < \infty$ $(k > 1)$.
3. $\operatorname{Var}(T) = \frac{k}{k-2}$.
4. Technically, $k \in \mathbb{R}$, but we will usually deal with $k \in \mathbb{Z} > 0$.

# Student's $t$-distribution (cont.)

5. $t_k \xrightarrow{\text{D}} \text{N}(0,1)$ as $k \to \infty$.

## Proof.

If $X \sim \chi^2_k$, then by definition $X = Z_1^2 + \cdots + Z_k^2$, where $Z_i \overset{\text{iid}}{\sim} \text{N}(0,1)$. By the LLN,

$$\frac{X}{k} = \frac{Z_1^2 + \cdots + Z_k^2}{k} \xrightarrow{\text{P}} \text{E}(Z_1^2) = 1.$$

as $k \to \infty$. Therefore, $\sqrt{X/k} \xrightarrow{\text{P}} 1$, and in particular,

$$T = \frac{Z}{\sqrt{X/k}} \xrightarrow{\text{D}} \text{N}(0,1)$$

following Slutzky's theorem. □

# Student's *t*-distribution (cont.)

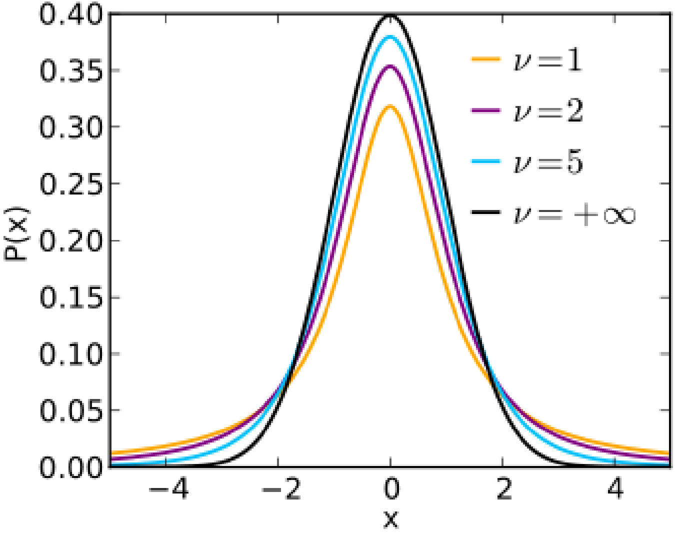
6. The *t*-distribution has **heavy tails**. That is, if $T \sim t_k$, $\text{E}(|T|^k) = \infty$. Comparing this to the normal distribution: $X \sim \text{N}(\mu, \sigma^2)$, $\text{E}(|X|^k) < \infty$ for any $k > 0$.

## Remark

This 'heavy-tails' property is a useful property in modelling abnormal phenomena or outliers (e.g. in financial or insurance data). C.f. "robust statistics".

Explore the *t*-distribution vs normal distribution here:
`https://eripoll12.shinyapps.io/t_Student/`

# Student's $t$-distribution (cont.)



# An important property of normal samples

## Theorem 4 (An important property of normal samples)

Let $\{X_1, \dots, X_n\}$ be a sample from $\text{N}(\mu, \sigma^2)$. Let

$$\bar X = \frac{1}{n}\sum_{i=1}^n X_i, \qquad s^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar X)^2, \qquad \text{and} \qquad \text{SE}(\bar X) = s/\sqrt{n}.$$

Then,

i. $\bar X \sim \text{N}(\mu, \sigma^2/n)$

ii. $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$

iii. $\bar X \perp s^2$

iv. $\frac{\sqrt{n}(\bar X - \mu)}{s} = \frac{\bar X - \mu}{\text{SE}(\bar X)} \sim t_{n-1}$

# An important property of normal samples (cont.)

## Proof.

i. follows directly from properties of normal distributions, and earlier we "proved" $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2_{n-1}$ which implies ii.

Consider any $X_j$, $j \in \{1,\dots,n\}$ and $\operatorname{Cov}(X_j - \bar{X}, \bar{X})$:

$$\begin{aligned}
\operatorname{Cov}(X_j - \bar{X}, \bar{X}) &= \operatorname{Cov}(X_j, \bar{X}) - \operatorname{Cov}(\bar{X}, \bar{X}) \\
&= \operatorname{Cov}\left(X_j, \frac{1}{n}\sum_{i=1}^n X_i\right) - \operatorname{Var}(\bar{X}) \\
&= \frac{1}{n}\sum_{i=1}^n \operatorname{Cov}(X_j, X_i) - \sigma^2/n \\
&= \sigma^2/n - \sigma^2/n = 0
\end{aligned}$$

Since the covariance is zero and they are normal, they are independent.

# An important property of normal samples (cont.)

## Proof.

Following this, if $\bar{X}$ is independent of $X_j - \bar{X}$ for any $j$, it stands to reason that $\bar{X}$ is also independent of $\tilde{\mathbf{X}} = (X_1 - \bar{X}, \dots, X_n - \bar{X})^\top$, and also of

$$\tilde{\mathbf{X}}^\top \tilde{\mathbf{X}} = \begin{pmatrix} X_1 - \bar{X} & \cdots & X_n - \bar{X} \end{pmatrix} \begin{pmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix} = \sum_{i=1}^n (X_i - \bar{X})^2 = (n-1)s^2,$$

and thus also of $s^2$.

Remark: we used the fact that if $X \perp Y_i$, then $g(X) \perp g(Y_i)$, and also $g(X) \perp \{g(Y_1) + \cdots + g(Y_n)\}$.

# An important property of normal samples (cont.)

## Proof.

Finally, putting everything together,

$$\frac{\overbrace{\sqrt{n}(\bar{X}-\mu)/\sigma}^{\text{N(0,1)}}}{\sqrt{\frac{\overbrace{(n-1)s^2/\sigma^2}^{\chi^2_{n-1}}}{n-1}}} = \frac{\bar{X}-\mu}{s/\sqrt{n}} = \frac{\bar{X}-\mu}{\text{SE}(\bar{X})} \sim t_{n-1}.$$

□

## Remark

This is why for normal distributions where $\sigma^2$ is unknown, and is estimated by the unbiased sample variance $s^2$, the standardised sample mean follows a *t*-distribution! This gives rise to the *t*-test.

1. $\chi^2$-distribution

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# $F$-distribution

The $F$-distribution is another notable distribution in statistics. It commonly arises as the null distribution of a test statistic, particularly in the analysis of variance (ANOVA).

## Definition 5

Let $X_1 \sim \chi^2_{k_1}$ and $X_2 \sim \chi^2_{k_2}$. Then, the distribution of

$$Y = \frac{X_1/k_1}{X_2/k_2}$$

is called the $F$-distribution with $(k_1, k_2)$ degrees of freedom. We write $Y \sim F_{k_1,k_2}$.

# $F$-distribution (cont.)

## Remark

Not even going to bother writing down the pdf! See for yourself: `https://en.wikipedia.org/wiki/F-distribution`. Remember the definition, though.

Some important properties of the $F$-distribution:

1. $Y$ is continuous and has support over $[0, \infty)$, provided $k_1 > 1$.
2. $\operatorname{E}(Y) = \frac{k_2}{k_2 - 2}$, provided $k_2 > 2$.
3. $\operatorname{Var}(Y) = \frac{2k_2^2(k_1 + k_2 - 2)}{k_1(k_2 - 2)^2(k_2 - 4)}$, provided $k_2 > 4$.
4. Technically, $k_1, k_2 \in \mathbb{R}_{>0}$, but we will usually deal with $k_1, k_2 \in \mathbb{Z} > 0$.

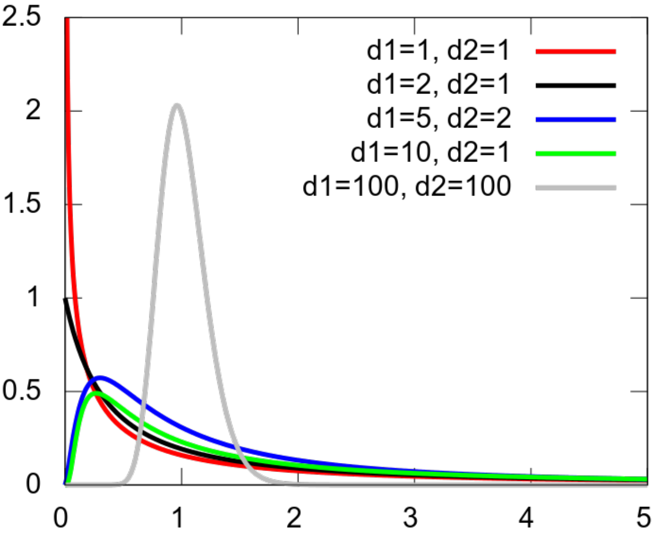
# $F$-distribution (cont.)

5. If $Y \sim F_{k_1,k_2}$, then $Y^{-1} \sim F_{k_2,k_1}$.
6. If $T \sim t_k$, then $T^2 \sim F_{1,k}$.

Proof.

**Exercise: Prove properties 5 and 6.** □

# *F*-distribution (cont.)



# The analysis of variance

The ANOVA, despite its name, is a (collection of) methods used to analyse differences among group means in a sample. The ANOVA was developed by Sir Ronald Fisher.

## Example 6

Let $X_{ij} \sim \mathrm{N}(\mu_j, \sigma^2)$, $i = 1, \dots, n_j$ and $j = 1, \dots, m$ with both $\mu_j$ and $\sigma^2$ unknown. Let $n = \sum_{j=1}^m n_j$ be the total sample size. Define the grand mean and the respective group means to be $\bar X = n^{-1} \sum_{i,j} X_{ij}$ and $\bar X_j = n_j^{-1} \sum_{i=1}^{n_j} X_{ij}$ respectively. The "total sum of squares" is

$$S = \sum_{i,j} (X_{ij} - \bar X)^2$$

# The analysis of variance

## Example 6

The total sum of squares can be decomposed into

$$S = \overbrace{\sum_{i,j}(X_{ij} - \bar{X}_j)^2}^{S_1} + \overbrace{\sum_{j} n_j(\bar{X}_j - \bar{X})^2}^{S_2}$$

where $S_1$ is called the "within sum of squares" (how much variation among individuals in each group) and $S_2$ the " between sum of squares" (how much variation in the mean among groups).

There is the concept of "degrees of freedom": $n-1$ in the TSS, $m-1$ in the BSS, and therefore $n-m$ in the WSS.

# The analysis of variance

## Example 6

This gives rise to the ANOVA table:

| Source | SS | d.f. | MSS | F-statistic |
|---|---|---|---|---|
| Between | $\sum_j n_j(\bar{X}_j - \bar{X})^2$ | $m-1$ | $\frac{\sum_j n_j(\bar{X}_j - \bar{X})^2}{m-1}$ | $\frac{\sum_j n_j(\bar{X}_j - \bar{X})^2/(m-1)}{\sum_{i,j}(X_{ij} - \bar{X}_j)^2/(n-m)}$ |
| Within | $\sum_{i,j}(X_{ij} - \bar{X}_j)^2$ | $n-m$ | $\frac{\sum_{i,j}(X_{ij} - \bar{X}_j)^2}{n-m}$ | |
| Total | $\sum_{i,j}(X_{ij} - \bar{X})^2$ | $n-1$ | | |

What is the distribution of $F$?

# The analysis of variance

## Example 6

We have seen that

$$\frac{1}{\sigma^2}\sum_{i,j}(X_{ij}-\bar{X})^2 \sim \chi^2_{n-1}.$$

In fact, we can also show similarly that

$$\frac{1}{\sigma^2}\sum_{i,j}(X_{ij}-\bar{X}_j)^2 \sim \chi^2_{n-m}.$$

Using these two facts, we deduce that

$$\frac{1}{\sigma^2}\sum_{j} n_j(\bar{X}_j-\bar{X})^2 \sim \chi^2_{m-1}$$

from the property of $\chi^2$-distributions.

# The analysis of variance

## Example 6

So now,

$$F = \frac{\cancel{1/\sigma^2}\overbrace{\sum_j n_j(\bar{X}_j - \bar{X})^2}^{\chi^2_{m-1}}/(m-1)}{\cancel{1/\sigma^2}\overbrace{\sum_{i,j}(X_{ij} - \bar{X})^2}^{\chi^2_{n-m}}/(n-m)}$$

is a ratio of two $\chi^2$-distributions, which means that $F$ follows an $F$-distribution with $(m-1, n-m)$ degrees of freedom.

# The analysis of variance

### Stop to think

What happens when there are only two groups ($m = 2$)?

- What is the distribution of $\chi^2_1$ equal to?
- What is the distribution of the test statistic $F$?
- What is the distribution of $\sqrt{F}$?

1. $\chi^2$-distribution
2. Student's $t$-distribution
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# Bivariate distributions

For a pair of r.v. $(X, Y)$, the joint pdf $f_{X,Y}(x, y)$ for $(X, Y)$ is a probability distribution that gives the probability that each of $X$ and $Y$ falls in any particular range or set of values specified for these variables.

The **marginal distributions** may be obtained by

$$f_X(x) = \begin{cases} \sum_y f_{X,Y}(x, y) & \text{if } Y \text{ is discrete} \\ \int_y f_{X,Y}(x, y) \, \mathrm{d}y & \text{if } Y \text{ is continuous} \end{cases}$$

$$f_Y(y) = \begin{cases} \sum_x f_{X,Y}(x, y) & \text{if } X \text{ is discrete} \\ \int_x f_{X,Y}(x, y) \, \mathrm{d}x & \text{if } X \text{ is continuous} \end{cases}$$

# Bivariate distributions (cont.)

The joint cdf is defined as

$$F_{X,Y}(x,y) = \mathrm{P}(X \le x, Y \le y).$$

From this, the marginal distributions (cdf) are given by

$$F_X(x) = \mathrm{P}(X \le x, Y \le \infty) = F_{X,Y}(x,\infty)$$

and

$$F_Y(y) = \mathrm{P}(Y \le \infty, Y \le y) = F_{X,Y}(\infty,y)$$

# Bivariate distributions (cont.)

To be even more specific, we define the joint pdf as follows:

## Definition 7 (Joint bivariate pdf)

In the discrete case, i.e. $X$ and $Y$ are two discrete r.v., the **joint probability mass function** (pmf) is

$$p_{X,Y}(x,y) = \mathrm{P}(X = x, Y = y)$$

In the continuous case, i.e. $X$ and $Y$ are two continuous r.v., the **joint probability density function** (pdf) is

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y).$$

# Properties of bivariate distributions

1. Since the joint pdf is a probability distribution, it must satisfy

$$\sum_x \sum_y \mathrm{P}(X = x, Y = y) = 1$$

in the discrete case, and

$$\int_x \int_y f_{X,Y}(x, y) \,\mathrm{d}x \,\mathrm{d}y = 1$$

in the continuous case.

# Properties of bivariate distributions (cont.)

2. We can write the pmf/pdf in terms of conditional distributions. For the discrete case, this is

$$
\begin{aligned}
p_{X,Y}(x,y) &= \mathrm{P}(Y=y|X=x)\,\mathrm{P}(X=x) \\
&= \mathrm{P}(X=x|Y=y)\,\mathrm{P}(Y=y).
\end{aligned}
$$

For the continuous case, this is

$$
\begin{aligned}
f_{X,Y}(x,y) &= f_{Y|X}(y|x)f_X(x) \\
&= f_{X|Y}(x|y)f_Y(y).
\end{aligned}
$$

# Properties of bivariate distributions (cont.)

3. Though we won't be looking at these, it is possible to have a "mixed" joint pdf/pmf, i.e. $X$ is continuous and $Y$ is discrete, or the other way around. The joint pdf may be written

$$\begin{aligned} f_{X,Y}(x,y) &= f_{X|Y}(x|y)\,\mathrm{P}(Y=y) \\ &= \mathrm{P}(Y=y|X=x) f_X(x) \end{aligned}$$

and its joint cdf by

$$F_{X,Y}(x,y) = \sum_{\tilde{y}\leq y} \int_{-\infty}^{x} f_{X,Y}(\tilde{x},\tilde{y})\,\mathrm{d}\tilde{x}$$

One common example is when using logistic regression in predicting probability of a binary outcome, conditional on the value of a continuously distributed outcome.

# Properties of bivariate distributions (cont.)

4. The covariance between $X$ and $Y$, denoted $\operatorname{Cov}(X, Y)$, is

$$\begin{aligned}\operatorname{Cov}(X, Y) &= \operatorname{E}[(X - \operatorname{E} X)(Y - \operatorname{E} Y)] \\ &= \operatorname{E}(XY) - \operatorname{E} X \cdot \operatorname{E} Y\end{aligned}$$

5. The correlation between $X$ and $Y$, denoted $\rho_{XY}$, is

$$\rho_{XY} = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var} X \cdot \operatorname{Var} Y}}$$

## Stop to think

- What is the covariance between a r.v. $X$ and itself?
- What possible values can $\rho$ take?

# Properties of bivariate distributions (cont.)

6. Two r.v. $X$ and $Y$ are **independent** if and only if the joint cdf satisfies

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y).$$

As for their pmf/pdf,

$$p_{X,Y}(x,y) = \mathrm{P}(X=x, Y=y) = \mathrm{P}(X=x) \cdot \mathrm{P}(Y=y)$$

in the discrete case, and

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

in the continuous case.

**Remark**

$X \perp Y \Rightarrow \operatorname{Cov}(X,Y) = 0$, but not necessarily the other way around.

# Discrete bivariate distributions

If $X$ takes discrete values $x_1, \dots, x_m$ and $Y$ takes discrete values $y_1, \dots, y_n$, their joint pmf may be presented in table:

| | $y_1$ | $y_2$ | $\cdots$ | $y_n$ | |
|---|---|---|---|---|---|
| $x_1$ | $p_{11}$ | $p_{12}$ | $\cdots$ | $p_{1n}$ | $p_{1\cdot}$ |
| $x_2$ | $p_{21}$ | $p_{22}$ | $\cdots$ | $p_{2n}$ | $p_{2\cdot}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $x_m$ | $p_{m1}$ | $p_{m2}$ | $\cdots$ | $p_{mn}$ | $p_{m\cdot}$ |
| | $p_{\cdot 1}$ | $p_{\cdot 2}$ | $\cdots$ | $p_{\cdot n}$ | |

where $p_{ij} = \mathrm{P}(X = x_i, Y = y_j)$, and $p_{i\cdot}$ and $p_{\cdot j}$ are the marginal probabilities of $X$ and $Y$ respectively.

# Discrete bivariate distributions (cont.)

So from the previous slides we know that

$$
\begin{aligned}
p_{\cdot j} &= \sum_{i=1}^{n} p_{ij} \\
p_{i\cdot} &= \sum_{j=1}^{m} p_{ij}
\end{aligned}
$$

and of course, $\sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$.

# Examples

## Example 8

Flip a fair coin two times. Let $X = 1$ if it is heads in the first flip, and $X = 0$ if it is tails. Let $Y = 1$ if the outcomes in the two flips are the same, and $Y = 0$ if the two outcomes are different. The joint probability function is

| | $Y = 1$ | $Y = 0$ |
|---|---|---|
| $X = 1$ | $1/4$ | $1/4$ |
| $X = 0$ | $1/4$ | $1/4$ |

It is easy to see that $X$ and $Y$ are independent (although we had not assumed this).

# Examples (cont.)

## Example 9

Consider a uniform distribution on the unit square $[0,1] \times [0,1]$. It has pdf given by

$$f(x,y) = \begin{cases} 1 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

This is a well-defined pdf, as $f \geq 0$ and $\int\int f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$. It is also easy to see that $X$ and $Y$ are independent.

Find $\mathrm{P}(X < 1/2, Y < 1/2)$ and $\mathrm{P}(X + Y < 1)$.

# Examples (cont.)

## Example 9

$$
\begin{aligned}
\mathrm{P}(X < 1/2, Y < 1/2) &= \int_0^{1/2} \int_0^{1/2} \mathrm{d}x \, \mathrm{d}y \\
&= \left[ [xy]_0^{1/2} \right]_0^{1/2} = 1/4.
\end{aligned}
$$

Note that the set $\{x + y < 1\}$ corresponds to $\{0 < y < 1, 0 < x < 1 - y\}$.

$$
\begin{aligned}
\mathrm{P}(X + Y < 1) &= \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}x \\
&= \int_0^1 \mathrm{d}y [x]_0^{1-y} \\
&= \int_0^1 (1 - y) \, \mathrm{d}y = \left[ y - y^2/2 \right]_0^1 = 1/2.
\end{aligned}
$$

# Conditional distributions

If $X$ and $Y$ are not independent, knowing $X$ should be helpful in determining $Y$, as $X$ may carry some information on $Y$. Therefore it makes sense to define the distribution of $Y$ given, say $X = x$. This is the concept of *conditional distributions*.

If $X$ and $Y$ are discrete, then you have probably seen that

$$\mathrm{P}(Y = y|X = x) = \frac{\mathrm{P}(Y = y, X = x)}{\mathrm{P}(X = x)} = \frac{\mathrm{P}(X = x|Y = y)\,\mathrm{P}(Y = y)}{\mathrm{P}(X = x)}$$

However, this definition does not extend to continuous random variables, because the probability of a single point has mass zero.

# Conditional distributions (cont.)

### Definition 10

For continuous r.v. $X$ and $Y$, the conditional pdf of $Y$ given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

- As a function of $y$, $f_{Y|X}(y|x)$ is a pdf, keeping the value of $X$ constant at $x$:

$$\mathrm{P}(Y \in A|X = x) = \int_A f_{Y|X}(y|x) \, \mathrm{d}y$$

- Both the conditional mean $\mathrm{E}(Y|X = x)$ and conditional variance $\mathrm{Var}(Y|X = x)$ are functions of $x$:

$$\mathrm{E}(Y|X = x) = \int y f_{Y|X}(y|x) \, \mathrm{d}y$$

$$\mathrm{Var}(Y|X = x) = \int \big(y \, \mathrm{E}(Y|X = x)\big)^2 f_{Y|X}(y|x) \, \mathrm{d}y$$

# Conditional distributions (cont.)

- If $X$ and $Y$ are independent, then $f_{Y|X}(y|x) = f_Y(y)$
- Note that

$$f_{Y|X}(y|x)f_X(x) = f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$$

which offers alternative ways to determine the joint pdf.

- For any r.v. $X$ and $Y$,

$$\mathrm{E}_X\big(\mathrm{E}(Y|X)\big) = \mathrm{E}(Y)$$

Proof.

$$\begin{aligned}\mathrm{E}\big(\mathrm{E}(Y|X)\big) &= \int \left\{ \int y\, f_{Y|X}(y|x)\,\mathrm{d}y \right\} f_X(x)\,\mathrm{d}x \\ &= \int\int y\, f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int y\, f(y)\,\mathrm{d}y = \mathrm{E}(Y)\end{aligned}$$

# Conditional distributions (cont.)

## Example 11

Let $f_{X,Y}(x,y) = e^{-y}$ for $0 < x < y < \infty$, and 0 otherwise. Find $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$.

We need to find the marginals first:

$$\begin{aligned} f_X(x) &= \int_x^\infty e^{-y}\,\mathrm{d}y = e^{-x}, \quad 0 < x < \infty \\ f_Y(y) &= \int_0^y e^{-y}\,\mathrm{d}x = ye^{-y}, \quad 0 < y < \infty \end{aligned}$$

# Conditional distributions (cont.)

## Example 11

Thus,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}, \quad x < y < \infty$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-y}}{ye^{-y}} = y^{-1}, \quad 0 < x < y$$

Note that given $Y = y$, $X|(Y = y) \sim \text{Unif}(0, y)$.

# Multivariate distributions

The bivariate results we saw earlier can be extended to more than two variables, resulting in multivariate distributions.

Let $\mathbf{X} = (X_1, \ldots, X_n)^\top$ be a random vector consisting of $n$ r.v.. The **joint cdf** is defined as

$$F(x_1, \ldots, x_n) = \mathrm{P}(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

If $X$ is continuous, the **joint pdf** satisfies

$$f_{X,Y}(x, y) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F(x_1, \ldots, x_n).$$

# Multivariate distributions (cont.)

- In general, the pdf admits the factorisation

$$\begin{aligned} f(x_1,\ldots,x_n) &= f(x_1)f(x_2,\ldots,x_n|x_1) \\ &= f(x_1)f(x_2|x_1)f(x_3,\ldots,x_n|x_1,x_2) \\ &\;\;\vdots \\ &= f(x_1)f(x_2|x_1)f(x_3|x_1,x_2)\cdots f(x_n|x_1,\ldots,x_{n-1}) \end{aligned}$$

  where $f(x_j|x_1,\ldots,x_{j-1})$ denotes the conditional pdf of $X_j$ given $X_1 = x_1,\ldots,X_{j-1} = x_{j-1}$.
- On the other hand, $X_1,\ldots,X_n$ are **independent** if and only if

$$f(x_1,\ldots,x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

# Random samples

- We will often say "$X_1, \ldots, X_n$ is a random sample from a distribution with pdf $f$". Without specifying independence, we should not assume that it is an iid sample.
- Thus, when doing inference, we should work with the **joint pdf** $f(x_1, \ldots, x_n)$. Several multivariate distributions will be discussed in the next section.
- On the other hand, if $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f(\mathbf{x})$, then and only then

$$f(\mathbf{x}) = f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f(x_i)$$

1. $\chi^2$-distribution
2. Student's $t$-distribution
3. $F$-distribution
4. Multivariate distributions
5. Multinomial and categorical distribution
6. Multivariate normal distribution

# Multinomial distribution

The multinomial distribution is an extension of the binomial distribution. Suppose we threw an $k$-sided die $n$ times, and we recorded $X_i$, the number of times the $i$th side turned up $(i = 1, \ldots, k)$. Then $\mathbf{X} = (X_1, \ldots, X_k)^\top$ follows a multinomial distribution.

### Definition 12 (Multinomial distribution)

Let $\mathbf{X} = (X_1, \ldots, X_k)^\top \sim \text{Mult}(n, p_1, \ldots, p_k)$. Then, the pmf of $\mathbf{X}$ is given by

$$f(x_1, \ldots, x_k) = \frac{n!}{x_1! x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

Here, $p_j$ is the probability of success associated with event $X_j$.

# Multinomial distribution (cont.)

- Obviously, each $p_j \geq 0$, and that $\sum_{j=1}^{k} p_j = 1$.
- We also observe that if out of the $n$ trials, $X_j$ is the number of "success" associated with the $j$th outcome, then
  - $X_1 + \cdots + X_k = n$, and therefore, the $X_j$s are **not independent**.
  - $X_j \sim \text{Bin}(n, p_j)$, and hence $\text{E}(X_j) = np_j$ and $\text{Var}(X_j) = np_j(1 - p_j)$.
- We can show that $\text{Cov}(X_j, X_{j'}) = -np_j p_{j'}$ (see Exercise 3).
- Therefore,

$$\text{E}(\mathbf{X}) = n\mathbf{p} \in \mathbb{R}^k$$
$$\text{Var}(\mathbf{X}) = n\left[\text{diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}^\top\right] \in \mathbb{R}^{k \times k}$$

where $\mathbf{p} = (p_1, \ldots, p_k)^\top$.

# Multinomial distribution (cont.)

Expanding the expectation vector and variance-covariance matrix in full:

$$\mathrm{E}(\mathbf{X}) = \begin{pmatrix} np_1 \\ \vdots \\ np_k \end{pmatrix}$$

$$\mathrm{Var}(\mathbf{X}) = \begin{pmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_k \\ -np_1p_2 & np_2(1-p_2) & \cdots & -np_2p_k \\ \vdots & \vdots & \ddots & \vdots \\ -np_1p_k & -np_2p_k & \cdots & np_k(1-p_k) \end{pmatrix}$$

# Multinomial distribution (cont.)

- Marginal distributions after collapsing categories are also multinomial. E.g. $(X_1, \dots, X_5)^\top \sim \text{Mult}(n, p_1, \dots, p_5)$, then $(X_1 + X_2, X_3 + X_4, X_5)^\top \sim \text{Mult}(n, p_1 + p_2, p_3 + p_4, p_5)$.

- Conditional distributions are also multinomial. E.g.$(X_1, \dots, X_5)^\top \sim \text{Mult}(n, p_1, \dots, p_5)$, then

$$\begin{aligned}(X_1, X_2)^\top | (X_3 = x_3, X_4 = x_4, X_5 = x_5) \\ \sim \text{Mult}\left(n - x_3 - x_4 - x_5, \frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right)\end{aligned}$$

# Multinomial distribution (cont.)

### Example 13

Suppose that in a three-way election for a large country, candidate A received 20% of the votes, candidate B received 30% of the votes, and candidate C received 50% of the votes. If six voters are selected randomly, what is the probability that there will be exactly one supporter for candidate A, two supporters for candidate B and three supporters for candidate C in the sample?

Let $(X_A, X_B, X_C)$ represent the number of voters for candidates A, B and C respectively. Then $(X_A, X_B, X_C)$ $\text{Mult}(6, 0.2, 0.3, 0.5)$. So $\text{P}(X_A = 1, X_B = 2, X_C = 3) = \frac{6!}{1!2!3!} 0.2^1 0.3^2 0.5^3 = 0.135$.

# Parameter estimation

## Question

Suppose that you observe a random sample $\mathbf{X} = (X_{i1}, \dots, X_{ik})$ from a $\text{Mult}(n, p_1, \dots, p_k)$ distribution, where each $X_{ij}$ tells you the "number of successes" for category $j$. What is an appropriate estimator for each of the $p_j$? **Work this out as an exercise.** *Hint: Each component of* $\mathbf{X}$ *is* $\text{Bern}(p_j)$*.*

# Categorical distribution

When we have only one trial ($n = 1$), we have a special case of the multinomial distribution. There is exactly one entry in $\mathbf{X} = (X_1, \dots, X_k)$ that is equal to one, while the rest is zero.

### Definition 14 (Categorical distribution)

Let $\mathbf{X} = (X_1, \dots, X_k)^\top \sim \text{Cat}(p_1, \dots, p_k)$. Then, the pmf of $\mathbf{X}$ is given by

$$f(x_1, \dots, x_k) = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

Here, $p_j$ is the probability of success associated with event $X_j$.

# Categorical distribution (cont.)

- Since each $X_j \sim \text{Bin}(1, p_j) \equiv \text{Bern}(p_j)$, the categorical distribution is effectively a generalisation of the Bernoulli distribution.

- Interestingly, we can also define

$$Y = \arg\max_j X_j$$

  so that $Y$ is a random variable taking on one distinct value $j \in \{1, \dots, k\}$ (category) with probability $p_j$. This formulation has a lot of importance in choice modelling in statistics and econometrics.

1. $\chi^2$-distribution

2. Student's $t$-distribution

3. $F$-distribution

4. Multivariate distributions

5. Multinomial and categorical distribution

6. Multivariate normal distribution

# Multivariate normal distribution

This is the multivariate extension of the regular normal distribution. It is undoubtedly the most commonly used distribution in statistics, and it has a lot of interesting properties.

### Definition 15 (Multivariate normal distribution)

Let $\mathbf{X} = (X_1, \dots, X_k)^\top$ be distributed according to a multivariate normal distribution with mean vector $\boldsymbol{\mu} \in \mathbb{R}^k$ and variance-covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{k \times k}$. It has pdf

$$f(\mathrm{x}) = (2\pi)^{-k/2} |\boldsymbol{\Sigma}|^{-1/2} \exp\left\{ -\frac{1}{2} (\mathrm{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathrm{x} - \boldsymbol{\mu}) \right\}$$

and we write $\mathbf{X} \sim \mathrm{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

# Multivariate normal distribution (cont.)

Here are some properties of the multivariate normal distribution:

1. $\boldsymbol{\mu}$ is a vector of length $k$, and $\boldsymbol{\Sigma}$ is a square $k \times k$, symmetric, positive-definite matrix.
2. The first and second moments of $\mathbf{X}$ are

$$\begin{gathered} \mathrm{E}(\mathbf{X}) = \boldsymbol{\mu} \\ \mathrm{E}(\mathbf{X}\mathbf{X}^\top) = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top \end{gathered}$$

so therefore,

$$\begin{aligned} \mathrm{Var}(\mathbf{X}) &= \mathrm{E}\left((\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^\top\right) \\ &= \mathrm{E}(\mathbf{X}\mathbf{X}^\top) - \boldsymbol{\mu}\boldsymbol{\mu}^\top = \boldsymbol{\Sigma} \end{aligned}$$

3. Let $\boldsymbol{\Sigma} = (\sigma_{ij})$. Then

$$\sigma_{ij} = \begin{cases} \mathrm{Var}(X_i) & \text{if } i = j \\ \mathrm{Cov}(X_i, X_j) & \text{if } i \neq j \end{cases}$$

# Multivariate normal distribution (cont.)

4. When $\sigma_{ij} = 0$ for all $i \neq j$, i.e. the components of $X$ are *uncorrelated*, we have $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_{11}, \dots, \sigma_{kk})$. Also, $|\boldsymbol{\Sigma}| = \prod_{i=1}^{k} \sigma_{ii}$. Hence, the pdf admits a simpler form

$$f(\mathbf{x}) = \prod_{i=1}^{k} \left\{ \frac{1}{\sqrt{2\pi\sigma_{ii}}} e^{-\frac{1}{2\sigma_{ii}}(x_i - \mu_i)^2} \right\}$$

   and therefore, $X_1, \dots, X_k$ are independent, and each $X_i \sim \mathrm{N}(\mu_i, \sigma_{ii})$.

## Remark

In general, two r.v. $X$ and $Y$ which are independent imply that $\operatorname{Cov}(X, Y) = 0$, but not necessarily the other way around. But if $X$ and $Y$ are two normal r.v., then $X$ and $Y$ are independent if and only if $\operatorname{Cov}(X, Y) = 0$.

# Multivariate normal distribution (cont.)

5. Suppose that we can partition $\mathbf{X}$ into $\mathbf{X} = (\mathbf{X}_a, \mathbf{X}_b)^\top$, and similarly,

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ab}^\top & \boldsymbol{\Sigma}_b \end{pmatrix}.$$

   Then,

   - **Marginal distributions.** $\mathbf{X}_a \sim \mathrm{N}_{\dim \mathbf{X}_a}(\boldsymbol{\mu}_a, \boldsymbol{\Sigma}_a)$ and $\mathbf{X}_b \sim \mathrm{N}_{\dim \mathbf{X}_b}(\boldsymbol{\mu}_b, \boldsymbol{\Sigma}_b)$.
   - **Conditional distributions.** $\mathbf{X}_a | \mathbf{X}_b \sim \mathrm{N}_{\dim \mathbf{X}_a}(\tilde{\boldsymbol{\mu}}_a, \tilde{\boldsymbol{\Sigma}}_a)$ and $\mathbf{X}_b \sim \mathrm{N}_{\dim \mathbf{X}_b}(\tilde{\boldsymbol{\mu}}_b, \tilde{\boldsymbol{\Sigma}}_b)$, where

$$\begin{aligned} \tilde{\boldsymbol{\mu}}_a &= \boldsymbol{\mu}_a + \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_b^{-1}(\mathbf{X}_b - \boldsymbol{\mu}_b) & \qquad \tilde{\boldsymbol{\mu}}_b &= \boldsymbol{\mu}_b + \boldsymbol{\Sigma}_{ab}^\top\boldsymbol{\Sigma}_a^{-1}(\mathbf{X}_a - \boldsymbol{\mu}_a) \\ \tilde{\boldsymbol{\Sigma}}_a &= \boldsymbol{\Sigma}_a - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_b^{-1}\boldsymbol{\Sigma}_{ab}^\top & \qquad \tilde{\boldsymbol{\Sigma}}_b &= \boldsymbol{\Sigma}_b - \boldsymbol{\Sigma}_{ab}^\top\boldsymbol{\Sigma}_a^{-1}\boldsymbol{\Sigma}_{ab} \end{aligned}$$

# Multivariate normal distribution (cont.)

6. Let $\mathbf{A}$ be a $\mathbb{R}^{q \times k}$ constant matrix, and $\mathbf{b} \in \mathbb{R}^k$ a constant vector. Then $\mathbf{Y} := \mathbf{A}\mathbf{X} + \mathbf{b} \in \mathbb{R}^q$ has distribution

   $$\mathbf{Y} \sim \mathrm{N}_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$$

   This is simply the linearity property of normal distributions for the multivariate case.

# Standard multivariate normal

A special case of the multivariate normal is when $\boldsymbol{\mu} = (0, \dots, 0)^\top$, and $\boldsymbol{\Sigma} = \mathbf{I}_k$. We would then have the standard multivariate normal distribution $\mathbf{Z} \sim \mathrm{N}_k(\mathbf{0}, \mathbf{I}_k)$.

The pdf of the standard normal is

$$\phi(\mathbf{z}) = (2\pi)^{-k/2} \exp\left\{ -\frac{1}{2}\mathbf{z}^\top \mathbf{z} \right\}$$

# Standard multivariate normal (cont.)

Suppose that $\mathbf{L}$ is a (non-singular) matrix such that $\mathbf{L}\mathbf{L}^\top = \boldsymbol{\Sigma}$. Note that, by the linearity property of the multivariate normal, $\mathbf{X} = \boldsymbol{\mu} + \mathbf{L}\mathbf{Z}$ is also normally distributed, with mean

$$\mathrm{E}(\mathbf{X}) = \boldsymbol{\mu} + \mathbf{L}\,\mathrm{E}(\mathbf{Z}) = \boldsymbol{\mu}$$

and variance

$$\mathrm{Var}(\mathbf{X}) = \mathbf{0} + \mathbf{L}\,\mathrm{Var}(\mathbf{Z})\mathbf{L}^\top = \mathbf{L}\mathbf{I}_k\mathbf{L}^\top = \boldsymbol{\Sigma}.$$

Therefore $\mathbf{X} \sim \mathrm{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

This property is often used for *simulating* from multivariate normals: 1) Generate samples from $k$ iid $\mathrm{N}(0, 1)$ distributions; then 2) apply the linearity transformation above.

# Standard multivariate normal (cont.)

Of course, the other way around works too. If $\mathbf{X} \sim \mathrm{N}_k(\boldsymbol{\mu}, \mathbf{L}\mathbf{L}^\top)$, then

$$\mathbf{Z} = \mathbf{L}^{-1}(\mathbf{X} - \boldsymbol{\mu})$$

has mean

$$\mathrm{E}(\mathbf{Z}) = \mathbf{L}^{-1}(\mathrm{E}\,\mathbf{X} - \boldsymbol{\mu}) = \mathbf{0}$$

and variance

$$\mathrm{Var}(\mathbf{Z}) = \mathbf{L}^{-1}(\mathrm{Var}\,\mathbf{X})(\mathbf{L}^{-1})^\top = \mathbf{L}^{-1}\mathbf{L}\mathbf{L}^\top(\mathbf{L}^{-1})^\top = \mathbf{I}_k.$$

So it is possible to “standardise” a multivariate normal distribution. This is useful when we want to calculate multivariate normal probabilities (although admittedly, this is a very computer-intensive problem involving numerical approximations).

# Standard multivariate normal (cont.)

Strategies for decomposing the variance-covariance matrix $\boldsymbol{\Sigma}$:

- **Eigendecomposition**. For a positive-definite matrix, we have that $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^\top$. The matrix $\boldsymbol{\Gamma}$ is a matrix of eigenvectors of $\boldsymbol{\Sigma}$, and $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_k)$ is a diagonal matrix of eigenvalues. Additionally for positive matrices, $\boldsymbol{\Gamma}\boldsymbol{\Gamma}^\top = \mathbf{I}_k$ (orthogonal).
- **Cholesky decomposition**. The Cholesky decomposition of a positive-definite matrix is a decomposition of the form $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L}$ is a *lower-triangular* matrix with real and positive diagonal entries.
- **LDL decomposition**. Closely related to the Cholesky decomposition, this is a decomposition of the form $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{D}\mathbf{L}^\top$, where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{L}$ is a *lower-unitriangular* matrix (all diagonal elements are 1).

# Example

### Example 16

Suppose that $X_1 \sim \mathrm{N}(\mu_1, \sigma_1^2)$, and $X_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$. The covariance between $X_1$ and $X_2$ is $\mathrm{Cov}(X_1, X_2) = \sigma_{12}$.

Then $\mathbf{X} = (X_1, X_2)^\top$ is a bivariate normal distribution with mean $\boldsymbol{\mu} = (\mu_1, \mu_2)^\top$, and variance

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

# Example (cont.)

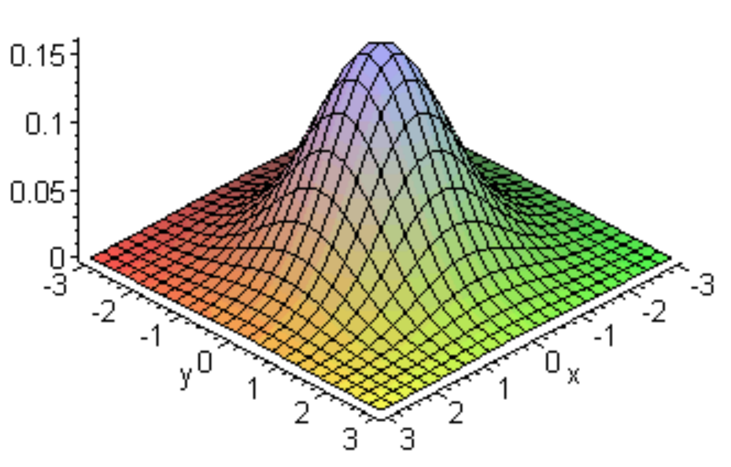
## Example 16

The pdf is

$$
\begin{aligned}
f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =& (2\pi)^{-1} \det \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}^{-1} \\
& \times \exp\left\{ -\frac{1}{2} \begin{pmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} \right\}
\end{aligned}
$$

# Example (cont.)

## Example 16



# Parameter estimation

Very briefly, suppose that you had $n$ samples from a $k$-variate normal distribution. That is, you observe $\mathbf{X}_1, \dots, \mathbf{X}_n$, where each $\mathbf{X}_i \in \mathbb{R}^k$ is a $k$-dimensional vector.

The unconstrained maximum likelihood estimators for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n} \mathsf{x}_i = (\bar{X}_1, \dots, \bar{X}_k)^\top \in \mathbb{R}^k$$

and

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n} (\mathsf{x}_i - \hat{\boldsymbol{\mu}})(\mathsf{x}_i - \hat{\boldsymbol{\mu}})^\top \in \mathbb{R}^{k \times k}.$$

It turns out also that $\hat{\boldsymbol{\mu}} \xrightarrow{\text{D}} \text{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)$, as $n \to \infty$ (multivariate CLT).

# Parameter estimation

## Question

How many total unknown parameters are there in a $k$-variate normal distribution? **Work this out as an exercise.**

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