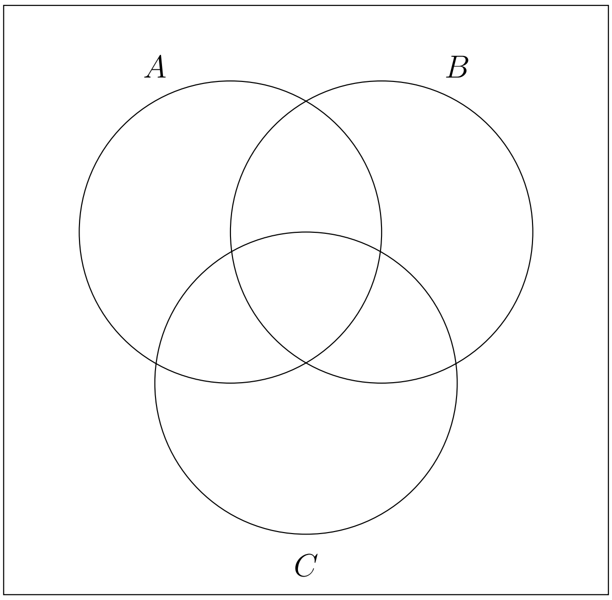
In addition to UBD's own teaching feedback, appreciate if you can fill in my feedback form, thanks! Helps me understand which aspects of my teaching I need to improve.

https://forms.gle/BG4ZWVKfBS5tMFCr6

SM-1402 Exercise 2

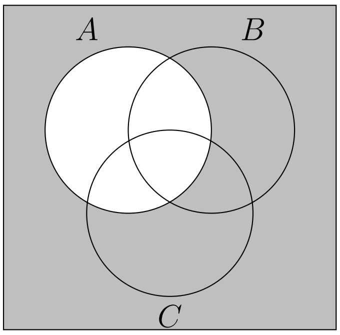
1. Three events are shown on the Venn diagram:



Reproduce the figure above and shade the region that corresponds to each of the following events:

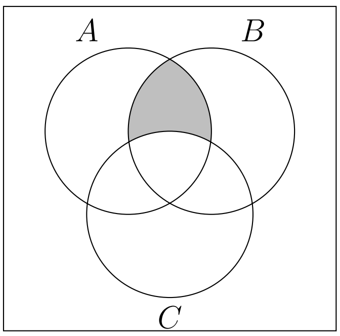
(a) A'

Solution:

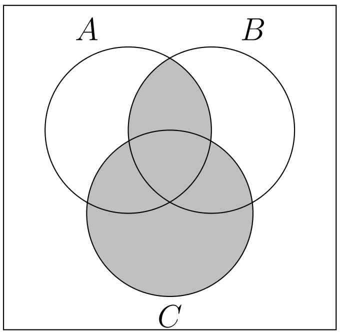


(b) $(A \cap B) \cap C'$

Solution:

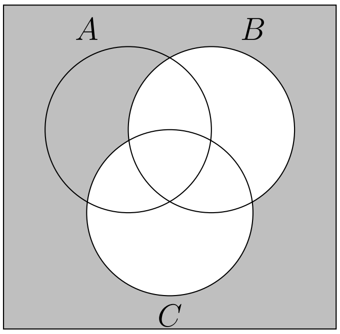


(c) $(A \cap B) \cup C$



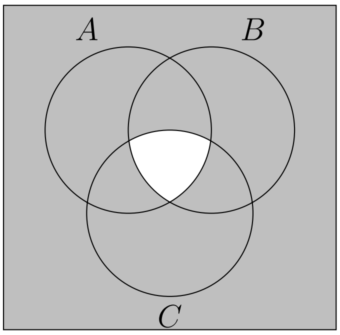
Solution:

(d) $(B \cup C)'$



Solution:

(e) $(A \cap B)' \cup C'$



Solution:

2. The sample space of a random experiment is $\mathcal{S} = \{A, B, C, D, E\}$, with probabilities

$$\mathrm{P}(\{A\}) = \mathrm{P}(\{B\}) = 0.1, \qquad \mathrm{P}(\{D\}) = 0.4$$
$$\mathrm{P}(\{C\}) = \mathrm{P}(\{E\})$$

Define the events $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A} = \{A, B, C\}$ $\mathcal{B} = \{C, D, E\}$. Calculate the following probabilities:

(a) $\mathrm{P}(\{C\})$

> **Solution:** Since $\mathrm{P}(\mathcal{S}) = \mathrm{P}(\{A\})+\mathrm{P}(\{B\})+\mathrm{P}(\{C\})+\mathrm{P}(\{D\})+\mathrm{P}(\{E\}) = 1$, and $\mathrm{P}(\{C\}) = \mathrm{P}(\{E\})$, we have that $0.1 + 0.1 + 2\,\mathrm{P}(\{C\}) + 0.4 = 1$, so $\mathrm{P}(\{C\}) = 0.2$.

(b) $\mathrm{P}(\mathcal{A})$

> **Solution:** $\mathrm{P}(\mathcal{A}) = \mathrm{P}(\{A, B, C\}) = 0.1 + 0.1 + 0.2 = 0.4$ (since they are exhaustive events).

(c) $\mathrm{P}(\mathcal{B})$

> **Solution:** $\mathrm{P}(\mathcal{B}) = \mathrm{P}(\{C, D, E\}) = 0.2 + 0.4 + 0.2 = 0.8$ (since they are exhaustive events).

(d) $\mathrm{P}(\mathcal{A}')$

> **Solution:** $\mathrm{P}(\mathcal{A}') = 1 - \mathrm{P}(\mathcal{A}) = 1 - 0.4 = 0.6$.

(e) $\mathrm{P}(\mathcal{A} \cap \mathcal{B})$

> **Solution:** $\mathrm{P}(\mathcal{A} \cap \mathcal{B}) = \mathrm{P}(\{A, B, C\} \cap \{C, D, E\}) = \mathrm{P}(\{C\}) = 0.2$.

(f) $\mathrm{P}(\mathcal{A} \cup \mathcal{B})$

> **Solution:** $\mathrm{P}(\mathcal{A} \cup \mathcal{B}) = \mathrm{P}(\{A, B, C\} \cup \{C, D, E\}) = \mathrm{P}(\mathcal{S}) = 1$.

3. If A, B, and C are mutually exclusive events with $\mathrm{P}(A) = 0.2$, $\mathrm{P}(B) = 0.3$, and $\mathrm{P}(C) = 0.4$, determine the following probabilities:

(a) $\mathrm{P}(A \cup B \cup C)$

> **Solution:** $\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) = 0.2 + 0.3 + 0.4 = 0.9$.

(b) $\mathrm{P}(A \cap B \cap C)$

> **Solution:** $\mathrm{P}(A \cap B \cap C) = 0$ since they are mutually exclusive (no overlaps).

(c) $\mathrm{P}(A \cap B)$

> **Solution:** $\mathrm{P}(A \cap B) = 0$ too.

(d) $\mathrm{P}\big((A \cup B) \cap C\big)$

Solution: $\mathrm{P}\big((A\cup B)\cap C\big) = \mathrm{P}\big((A\cap C)\cup(B\cap C)\big) = \mathrm{P}(A\cap C)+\mathrm{P}(B\cap C) = 0.$

(e) $\mathrm{P}(A' \cap B' \cap C')$

Solution: Note that from the lectures, we have the result that $\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B)$. So therefore

$$\begin{aligned}
\mathrm{P}\big(A' \cap (B' \cap C')\big) &= \mathrm{P}\big(A' \cap (B \cup C)'\big) \\
&= 1 - \mathrm{P}\big(A \cup (B \cup C)\big) \\
&= 1 - \mathrm{P}(A \cup B \cup C) \\
&= 1 - 0.9 = 0.1
\end{aligned}$$

4. Discs of polycarbonate plastic from a supplier are analysed for scratch and shock resistance. The results from 100 discs are summarised as follows:

		Shock resistance	
		High	Low
Scratch	High	70	9
resistance	Low	16	5

Let A denote the event that a disc has high shock resistance, and let B denote the event that a disc has high scratch resistance. Determine the following probabilities:

(a) $\mathrm{P}(A)$

Solution: $\mathrm{P}(A) = n(A)/100 = (70 + 16)/100 = 0.86.$

(b) $\mathrm{P}(B)$

Solution: $\mathrm{P}(B) = n(B)/100 = (70 + 9)/100 = 0.79.$

(c) $\mathrm{P}(A|B)$

Solution: $\mathrm{P}(A\cap B) = n(A\cap B)/100 = 70/100 = 0.70$. Therefore, $\mathrm{P}(A|B) = \mathrm{P}(A \cap B)/\mathrm{P}(B) = 0.70/0.79 = 0.886.$

(d) $\mathrm{P}(B|A)$

Solution: $\mathrm{P}(B|A) = \mathrm{P}(A \cap B)/\mathrm{P}(A) = 0.70/0.86 = 0.814.$

5. Samples of a cast aluminium part are classified on the basis of surface finish (in microinches) and edge finish. The results of 100 parts are summarised as follows:

		Edge finish	
		Excellent	Good
Surface finish	Excellent	80	2
	Good	10	8

What is the probability that a part selected at random

(a) will have an excellent surface finish?

(b) will have an excellent edge finish?

(c) will not have an excellent surface finish?

(d) will have both excellent surface and edge finish?

(e) will have either an excellent surface or edge finish?

(f) will have not have an excellent surface finish, but an excellent edge finish?

Solution: Let A denote the event that a randomly selected part has an excellent surface finish, and B denote the event that a randomly selected part has an excellent edge finish.
(a) $P(A) = (80 + 2)/100 = 0.82$.
(b) $P(B) = (80 + 10)/100 = 0.90$.
(c) $P(A') = 1 - 0.82 = 0.18$.
(d) $P(A \cap B) = 80/100 = 0.80$.
(e) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.82 + 0.9 - 0.8 = 0.92$.
(f) $P(A' \cap B) = 10/100 = 0.10$.

6. You are accused of a crime you did not commit. To make matters worse, during your trial, it was found that you failed a polygraph test. The judge takes this as strong evidence against you is about to convict you for your alleged wrongdoings. Luckily, you remembered what you learnt in statistics class regarding conditional probabilities, and you implore the judge not to fall into the trap of the 'Prosecutor's fallacy'.

   Your argument is as follows:

   - From data it is shown that 15 out of 100 people lie on polygraph tests.
   - The accuracy of polygraph tests detecting lying people is 0.81.
   - On the other hand, polygraph tests have a small chance of 'false detection', estimated to be 0.17.

   You therefore conclude, using Bayes' theorem, that the probability of you truly lying on the polygraph test is less than a half (worse than chance). Show your working to save yourself from lifetime in prison.

Solution: Let L denote the event that I am lying, and '+' denote the event that the polygraph tests positive. The probability of interest here is $P(L|+)$, and not $P(+|L)$. The former is the probability I am lying given a positive lie detection on the polygraph test; while the latter is the "accuracy" of the polygraph test–the probability of obtaining a positive result given that I am truly lying.

From the data, we know that

- $P(L) = 0.15$.
- Since 'lying' and 'not lying' are two mutually exclusive events, $P(L') = 1 - 0.15 = 0.85$.
- The accuracy of the polygraph is $P(+|L) = 0.81$.
- False detection rates is $P(+|L') = 0.17$.

Using Bayes' theorem,

$$\begin{aligned} P(L|+) &= \frac{P(+|L)P(L)}{P(+)} \\ &= \frac{P(+|L)P(L)}{P(L)P(+|L) + P(L')P(+|L')} \\ &= \frac{0.81 \times 0.15}{0.81 \times 0.15 + 0.17 \times 0.85} \\ &= 0.46 \end{aligned}$$

7. The probability that it is Saturday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Saturday is 0.2. What is the probability that a student is absent given that today is Saturday?

Solution: Let S denote the event that today is Saturday, and A the event that a student is absent. Then,

$$\begin{aligned} \mathrm{P}(A|S) &= \frac{P(A \cap S)}{P(S)} \\ &= \frac{0.03}{0.2} \\ &= 0.15 \end{aligned}$$

8. In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

Solution: Case 1: (4 boys selected) ${}^6C_4 = 15$ ways. Case 2: (3 boys and 1 girl selected) ${}^6C_3 \times {}^4C_1 = 80$ ways. Case 3: (2 boys and 2 girls selected) ${}^6C_2 \times {}^4C_2 = 90$ ways. Case 3: (1 boy and 3 girls selected) ${}^6C_1 \times {}^4C_3 = 24$ ways. So total ways is $15 + 80 + 90 + 24 = 209$.

Alternatively, without restrictions, there are ${}^{10}C_4 = 210$ possible ways of selecting the children, but there is only ${}^4C_4 = 1$ way of selecting children in which there are no boys. Therefore, $210 - 1 = 209$ ways have at least one boy in the selection.

9. Find the number of permutations of the letters of the word 'BARUNEI' such that the consonants always occur in odd places. EDIT: Initially it was the word 'BRUNEI'.

Solution: BRUNEI contains 6 letters, 3 vowels and 3 consonants. The restriction on the permutation of letters is that the consonants must occur at at the 1st, 3rd and 5th position of the word. There are $3! = 6$ ways of permuting the consonants in the three available spaces. Once that's fixed, the remaining 3 vowels can be permuted $3! = 6$ ways. Thus, the total number of ways is $6 \times 6 = 36$.

Solution: BARUNEI contains 7 letters, 3 of which are consonants. The restriction on the permutation of letters is that the consonants must occur at at the 1st, 3rd, 5th or 7th position of the word. There are ${}^4P_3 = 24$ ways of permuting the consonants in the four available spaces. Once that's fixed, the remaining 4 vowels can be permuted $4! = 24$ ways. Thus, the total number of ways is $24 \times 24 = 576$.