SM-4335 Advanced Probability Class Test 1
2021/22 Semester I
17 September 2022
Time allowed: 60 minutes

Instructions:

- There are three (3) questions totalling 30 marks.
- Answer **ALL** questions on a separate answer sheet.
- Ensure that you have written your name and student number on your answer sheets that you are submitting.
- The use of calculators is allowed.

Question:	1	2	3	Total
Marks:	5	5	5	15

1. (5 marks) Mark each of the following statements as either TRUE or FALSE. You do not need to provide a reason for your answer.

   (a) Two sets A and B have the same cardinality if there exists a bijection (a.k.a. one-to-one correspondence) between them.

   $\surd$ **TRUE** ☐ FALSE

   Solution: This is exactly the definition of equal cardinality.

   (b) Let A be any set. The empty set is always a subset of A.

   $\surd$ **TRUE** ☐ FALSE

   Solution: Note that $B \subseteq A$ if and only if $\forall x \in B, x \in A$. Is is true that for all $x \in \{\}$, we have that $x \in A$? Yes (vacuous truth), since we cannot actually pick any elements from $\{\}$. Consider the statement "there are no mobile phones in the room". If this is so, then the statement "all mobile phones are turned off" is also vacuously true because the antecedent cannot actually be satisfied.

   (c) Let A be any set. Then $|A| < |\mathcal{P}(A)|$.

   $\surd$ **TRUE** ☐ FALSE

   Solution: This is Cantor's Theorem.

   (d) Every algebra is also a σ-algebra.

   ☐ TRUE $\surd$ **FALSE**

   Solution: The algebra may not necessarily satisfy closure under countable unions.

(e) Let $A_1, A_2, \dots$ be sets. Suppose that the limit infimum of these sets exists. Then $x \in \liminf_{n\to\infty} A_n$ implies that x belongs to infinitely many A_k.

$\surd$ **TRUE** ☐ FALSE

> **Solution:** Since $\liminf_n A_n \subseteq \limsup A_n$, any $x \in \liminf_n A_n$ also belongs to $\limsup_n A_n$. Thus, x does belong to infinitely many A_n.

2. (a) (3 marks) Let $\mathcal{F}$ be a collection of subsets of a set Ω. Provide the three necessary conditions for $\mathcal{F}$ to be an *algebra*.

> **Solution:** Bookwork.
>
> - $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;
> - $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$; and
> - $\Omega \in \mathcal{F}$.

(b) (2 marks) Let $\Omega = \{1, 2, 3\}$. Construct the *smallest algebra* from the collection of subsets of Ω, $\mathcal{C} = \{\{1\}, \{2\}\}$.

> **Solution:**
>
> $$\mathcal{A}(\mathcal{C}) = \{\{1\}, \{2\}, \{2,3\}, \{1,3\}, \{1,2\}, \{3\}, \Omega, \varnothing\}$$

3. (5 marks) For $n \in \mathbb{N}$, let

$$A_n = \left\{1, \frac{n+1}{n}, \frac{n+2}{n}, \dots, \frac{2n-1}{n}, 2\right\}.$$

(a) Find the limit inferior and limit superior of the set A_n.

> **Solution:**
>
> - $\liminf_n A_n = \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k = \{1, 2\}$
> - $\limsup_n A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k = \mathbb{Q} \cap [1, 2]$

(b) Does the sequence A_n have a limit? Why, or why not?

> **Solution:** It does not, because the limit superior and limit inferior are not the same.