

# Limited information goodness-of-fit tests for ordinal factor models

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Limited information approaches overcome sparsity issues and computational challenges in traditional goodness-of-fit tests. This paper describes the implementation of LIGOF tests for ordinal factor models that have been fitted using the `{lavaan}` package in R. The tests are computationally efficient and reliable, and adapted to suit whichever parameter estimation procedure was used to fit the model. The implementation is available as an R package called `{lavaan.ligof}`.

## 1 Introduction

## 2 Methods

### 2.1 Ordinal data

Consider the case of analysing multivariate data  $\mathbf{y} = (y_1, \dots, y_p)^\top$ , where each item  $y_i$  is an ordinal random variable with  $m_i$  categories,  $i = 1, \dots, p$ . Let  $\mathcal{R} = \{\mathbf{c} = (c_1, \dots, c_p)^\top \mid c_i \in \{1, \dots, m_i\}\}$  be the set of all possible response patterns, and let  $R = \prod_i m_i$  be the cardinality of this set. The joint probability of observing a response pattern  $\mathbf{c}_r \in \mathcal{R}$  is given by

$$\pi_r = \Pr(\mathbf{y} = \mathbf{c}_r) = \Pr(y_1 = \mathbf{c}_{r1}, \dots, y_p = \mathbf{c}_{rp}), \quad r = 1, \dots, R, \quad (1)$$

with  $\sum_r \pi_r = 1$ . Collect all response probabilities into the vector  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_R)^\top \in [0, 1]^R$ . An example with  $p = 3$ ,  $m_1 = 2$ , and  $m_2 = m_3 = 3$  is given below. In total, there are  $R = 2 \times 3 \times 3 = 18$  response patterns as shown in Table 1.

Later on we wish to use lower-order residuals to assess the fit of a model to the data, which first requires a description of lower-order moments and its connection to the joint response probabilities. Marginally, each  $y_i$  can be viewed as a multinoulli random variable with event probabilities  $\pi_k^{(i)} = \Pr(y_i = k)$ ,  $k = 1, \dots, m_i$ , that sum to one. Therefore, this univariate distribution is characterised by its  $(m_i - 1)$  moments  $\pi_2^{(i)}, \dots, \pi_{m_i}^{(i)}$ , with the first moment

Table 1: Response patterns for  $p = 3$  with  $m_1 = 2$ , and  $m_2 = m_3 = 3$ .

$r$	$y_1$	$y_2$	$y_3$	Pattern	$r$	$y_1$	$y_2$	$y_3$	Pattern
1	1	1	1	111	10	2	1	1	211
2	1	1	2	112	11	2	1	2	212
3	1	1	3	113	12	2	1	3	213
4	1	2	1	121	13	2	2	1	221
5	1	2	2	122	14	2	2	2	222
6	1	2	3	123	15	2	2	3	223
7	1	3	1	131	16	2	3	1	231
8	1	3	2	132	17	2	3	2	232
9	1	3	3	133	18	2	3	3	233

being redundant due to the sum to unity constraint. All univariate moments can be collected into the vector  $\dot{\boldsymbol{\pi}}_1 = (\pi_k^{(i)})^\top$  whose dimension is  $S_1 = \sum_i (m_i - 1)$ . In a similar light, the bivariate distribution of  $(y_i, y_j)$  is characterised by its  $(m_i - 1)(m_j - 1)$  *joint moments*  $\pi_{k,l}^{(ij)} = \Pr(y_i = k, y_j = l)$ ,  $k = 2, \dots, m_i$ ,  $l = 2, \dots, m_j$ . Also collect all bivariate moments into the vector  $\dot{\boldsymbol{\pi}}_2 = (\pi_{k,l}^{(ij)})^\top$  whose dimension is  $S_2 = \sum_{i < j} (m_i - 1)(m_j - 1)$ . Finally, denote by  $\boldsymbol{\pi}_2 = (\dot{\boldsymbol{\pi}}_1^\top, \dot{\boldsymbol{\pi}}_2^\top)^\top$  the vector of multivariate moments up to order 2, which is a vector of length  $S = S_1 + S_2$ .

Because the lower order moments are contained in the higher order moments, the vector  $\boldsymbol{\pi}_2$  can be extracted from the joint probabilities  $\boldsymbol{\pi}$  via a linear operation  $\boldsymbol{\pi}_2 = \mathbf{T}_2 \boldsymbol{\pi}$  (Jamil et al., 2025). As an example, continuing from the  $p = 3$  instance above, the moments for the first variable  $y_1$ ,  $\Pr(y_1 = 2)$  can be obtained by *summing* over all joint probabilities whose patterns contain  $y_1 = 2$ . The positions of these joint probabilities in the vector  $\boldsymbol{\pi}$  are picked up by the first row of the matrix  $\mathbf{T}_2$ . Similarly, the two bivariate moments of  $(y_1, y_2)$ , i.e.  $\pi_{22}^{(12)}$  and  $\pi_{23}^{(12)}$  are obtained by summing over the joint probabilities whose patterns contain  $y_1 = 2$  and  $y_2 = 2$ , and  $y_1 = 2$  and  $y_2 = 3$ , respectively.

Note that this construction of lower-order moments generalises to any order  $q \leq p$ , but the total number of moments up to order  $q$  grows combinatorially in both  $p$  and the category counts  $m_i$ , yielding design matrices  $\mathbf{T}_q$  that can become computationally burdensome. Moreover, although we arbitrarily dropped the first moment in the foregoing construction, the choice of which category to omit is immaterial. This is because category probabilities sum to one, so excluding any one category produces a similar-dimensional parameterisation algebraically equivalent to excluding any other. For further details, consult Reiser (1996) and Maydeu-Olivares & Joe (2006).

	111	112	113	121	122	123	131	132	133	211	212	213	221	222	223	231	232	233
Y1=2	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
Y2=2	0	0	0	1	1	1	0	0	0	0	0	0	1	1	1	0	0	0
Y2=3	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	1	1	1
Y3=2	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
Y3=3	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
Y1=2, Y2=2	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0
Y1=2, Y2=3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
Y1=2, Y3=2	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0
Y1=2, Y3=3	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1
Y2=2, Y3=2	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
Y2=2, Y3=3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
Y2=3, Y3=2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
Y2=3, Y3=3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1

Figure 1: Matrix  $\mathbf{T}_2$  for the case of  $p = 3$  with  $m_1 = 2$ , and  $m_2 = m_3 = 3$ .

## 2.2 Confirmatory factor analysis

The confirmatory factor analysis (CFA) model imposes a structure on the joint response probabilities by assuming that the  $p$  observed variables are manifestations of a smaller set of  $q$  latent variables. In this way, the CFA may be viewed as a data-reduction technique since, effectively, the correlations among variables are modelled by a pre-specific factor structure using lower-dimensional data summaries.

CFA is typically used for continuous manifest variables, but it can also be applied to ordinal data. A common approach is the *underlying variable* (UV) approach, where the observed responses  $y_i$  are assumed to be discretised versions of continuous latent variables  $y_i^*$ . The connection is made through

$$y_i = \begin{cases} 1 & \tau_0^{(i)} < y_i^* < \tau_1^{(i)} \\ 2 & \tau_1^{(i)} < y_i^* < \tau_2^{(i)} \\ 3 & \tau_2^{(i)} < y_i^* < \tau_3^{(i)} \\ \vdots & \vdots \\ m_i & \tau_{m_i-1}^{(i)} < y_i^* < \tau_{m_i}^{(i)}, \end{cases}$$

with the *thresholds*  $\tau_k^{(i)}$  for item  $i$  satisfying the ordering

$$-\infty \equiv \tau_0^{(i)} < \tau_1^{(i)} < \tau_2^{(i)} < \cdots < \tau_{m_i-1}^{(i)} < \tau_{m_i}^{(i)} \equiv +\infty.$$

Evidently, the model is invariant to a linear transformation, since scaling and shifting the underlying variables  $y_i^*$  do not affect the outcome of the ordinal variable  $y_i$ . For this reason

it is convenient to assume, for the purposes of model identifiability, a zero mean Gaussian distribution  $\mathbf{y}^* \sim N_p(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{y}^*})$ , where  $\mathbf{\Sigma}_{\mathbf{y}^*}$  is a correlation matrix.

The underlying continuous variables  $\mathbf{y}^*$ , unlike their discrete counterparts  $\mathbf{y}$ , are now suitable to be modelled using a factor analysis model. Here, the goal is to find a set of latent factors  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_q)^\top \in \mathbb{R}^q$ , with  $q \ll p$ , that sufficiently explain the covariance structure of the  $p$ -dimensional variable space. This is achieved by the relationship

$$\mathbf{y}^* = \mathbf{\Lambda}\boldsymbol{\eta} + \boldsymbol{\epsilon},$$

where  $\mathbf{\Lambda}$  is a (often sparse)  $p \times q$  matrix of factor loadings, and  $\boldsymbol{\epsilon}$  is a vector of residuals. Certain distributional assumptions are made, namely that  $\boldsymbol{\eta} \sim N_q(\mathbf{0}, \mathbf{\Psi})$  with  $\mathbf{\Psi}$  a correlation matrix,  $\boldsymbol{\epsilon} \sim N_p(\mathbf{0}, \mathbf{\Theta}_\epsilon)$  with  $\mathbf{\Theta}_\epsilon = \mathbf{I} - \text{diag}(\mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}^\top)$ , and that  $\text{Cov}(\boldsymbol{\eta}, \boldsymbol{\epsilon}) = \mathbf{0}$ . Together, this implies that the polychoric correlation matrix of  $\mathbf{y}$  is given by

$$\mathbf{\Sigma}_{\mathbf{y}^*} = \mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}^\top + \mathbf{\Theta}_\epsilon \in \mathbb{R}^{p \times p}.$$

As a remark, the UV approach is commonly employed in the context of confirmatory factor analysis (CFA) models due to the ease of modelling, though other approaches such as item response theory (IRT) models are also available (Jöreskog & Moustaki, 2001).

For this factor analysis model, the parameters of interest are the non-zero entries  $\boldsymbol{\lambda}$  of the loading matrix  $\mathbf{\Lambda}$ , the unique non-diagonal entries  $\boldsymbol{\psi}$  in the factor correlation matrix  $\mathbf{\Psi}$ , and the thresholds  $\boldsymbol{\tau}^{(i)} = (\tau_1^{(i)}, \dots, \tau_{m_i-1}^{(i)})^\top$  for each ordinal item  $y_i$ . Collectively, these parameters are denoted by  $\theta = (\boldsymbol{\lambda}^\top, \boldsymbol{\rho}^\top, \boldsymbol{\tau}^{(1)}, \dots, \boldsymbol{\tau}^{(p)})^\top$  belonging to some parameter space  $\Theta$ . Under this CFA model, each joint response probability  $\pi_r$  from Equation 1 is now evaluated as a function of  $\theta$ :

$$\pi_r := \pi_r(\theta) = \int \cdots \int_{\mathcal{C}_r} \phi_p(\mathbf{y}^* | \mathbf{0}, \mathbf{\Sigma}_{\mathbf{y}^*}) d\mathbf{y}^*, \quad (2)$$

where the  $p$ -dimensional integral is taken over the set  $\mathcal{C}_r = \{\mathbf{y}^* \in \mathbb{R}^p \mid y_i = \mathbf{c}_{ri}, i = 1, \dots, p\}$ , the set of all continuous values that yield the response pattern  $\mathbf{c}_r$ .

## 2.3 Parameter estimation

Suppose that a sample  $\mathcal{Y} = \{\mathbf{y}^{(s)}\}_{s=1}^n$  is obtained, where  $\mathbf{y}^{(s)} = (y_1^{(s)}, \dots, y_p^{(s)})^\top$  represents the  $p$ -dimensional ordinal-data observation from subject  $s \in \{1, \dots, n\}$ . As a remark, samples may not necessarily be independent, and in such cases, corresponding sampling weights  $\omega_s$  can be used to account for the sampling design (Jamil et al., 2025), and most of what will be discussed below can be adapted to account for this.

Many methods exist to estimate the parameters  $\theta$  of the CFA model, but we are most interested in those that yield a  $\sqrt{n}$ -consistent and asymptotically normal estimator. Specifically, we assume that  $\hat{\theta}$  satisfies

$$\sqrt{n}(\hat{\theta} - \theta) = \mathbf{Q}_n \cdot \sqrt{n}(\mathbf{p} - \boldsymbol{\pi}(\theta)) + o_p(1), \quad (3)$$

where the term  $\mathbf{p} = (\pi_1, \dots, \pi_R)^\top$  is the vector of empirical joint response proportions, and  $\mathbf{Q}_n$  is some *influence matrix* that performs asymptotic linearisation from the joint response proportions  $\mathbf{p}$  to the parameters  $\theta$ . This includes a wide range of likelihood and pseudo-likelihood based methods and also generalised least squares (GLS) based methods, with the latter popularly implemented as a 3-stage procedure in software. The result also holds true whether full information methods (i.e., estimation using joint response probabilities) or limited information methods (i.e., using a lower-order subset of the response probabilities) are employed.

A neat way of viewing the parameter estimation is that most of these methods are a class of M-estimators. M-estimation provides a general and flexible framework for parameter estimation, in which estimators are obtained by minimizing an objective function  $F(\theta)$ , typically expressed as an empirical average  $\sum_{i=1}^n F(Y_i, \theta)$ , or, equivalently, by solving a system of estimating equations  $\sum_{i=1}^n \nabla_\theta F(Y_i, \theta) = \mathbf{0}$ , where  $\nabla_\theta F = \partial F / \partial \theta$ . This formulation encompasses a wide range of classical and robust procedures, including maximum likelihood, least squares, and weighted least squares methods mentioned above.

In the context of confirmatory factor analysis (CFA) with ordinal indicators, the estimating equations typically arise from a discrepancy function defined on thresholds and polychoric correlations, and M-estimation offers a principled way to derive estimators even when the full likelihood is computationally intractable. A central assumption in this framework is that there exists a parameter  $\theta_0 \in \Theta$  such that the population moment condition  $E[\nabla_\theta F(Y, \theta_0)] = 0$  holds. This condition is not a consequence of the data, but rather a theoretical premise about the underlying data-generating mechanism. It defines the parameter value to which the estimator is expected to converge. In a correctly specified model,  $\theta_0$  corresponds to the true parameter; in the presence of misspecification, it instead represents the value that best satisfies the moment condition within the assumed model class.

Under standard regularity conditions—such as continuity of  $\nabla_\theta F$  in  $\theta$ , measurability, and uniform convergence of empirical averages—the M-estimator  $\hat{\theta}$  is consistent and asymptotically normal (Huber, 1964; van der Vaart, 1998). Specifically,

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{D} N(\mathbf{0}, \mathbf{V}),$$

where the asymptotic variance is given by the sandwich formula  $\mathbf{V} = H^{-1} J H^{-T}$ , with

$$\mathbf{H} = E \left[ -\nabla_\theta^2 F(Y, \theta) \right], \quad \mathbf{J} = E \left[ \nabla_\theta F(Y, \theta) \nabla_\theta F(Y, \theta)^\top \right].$$

These properties make M-estimation particularly appealing in settings where the data are ordinal and the working model may be misspecified, as is often the case in large-scale psychometric applications. For a detailed treatment of the asymptotic theory of M-estimators in econometric and semiparametric contexts, see Newey and McFadden (1994). For the commonly used techniques to estimate CFA, the table below gives an overview for the form that  $F$  and its derivatives take.

Table 2

Table of objective functions and their derivatives.

Achieving the desired form stated in Equation 3 requires an asymptotic linearisation argument. For CFA models, a general M-estimator  $\hat{\theta}$  for  $\theta$  is obtained by solving the set of estimating equations

$$U(\theta) := n\mathbf{\Delta}(\theta)^\top \mathbf{W}_\theta(\mathbf{m} - \mu(\theta)) = 0$$

where  $\mathbf{m}$  is a vector of sample “moments”,  $\mu(\theta)$  is the vector of model-implied moments,  $\mathbf{\Delta}(\theta)$  is the Jacobian of the model-implied moments with respect to  $\theta$ , and  $\mathbf{W}_\theta$  is a weight matrix which may or may not depend on the parameters. Under correct model specification, the sensitivity matrix takes the form

$$\mathbf{H}(\theta) = \mathbf{\Delta}(\theta)^\top \mathbf{W}_\theta \mathbf{\Delta}(\theta).$$

A first-order Taylor expansion of  $U_n(\theta)$  around  $\theta$  and with a little rearranging and multiplying through by  $\sqrt{n}$  gives

$$\sqrt{n}(\hat{\theta} - \theta) = \left[ -\frac{1}{n} \frac{\partial U(\theta)}{\partial \theta} \right]^{-1} \mathbf{\Delta}(\theta)^\top \mathbf{W}_\theta \cdot \sqrt{n}(\mathbf{m} - \mu(\theta)) + o_p(1),$$

where the observed Hessian  $-\frac{1}{n} \partial U(\theta) / \partial \theta \xrightarrow{P} \mathbf{H}(\theta)$  as  $n \rightarrow \infty$ . Taking limits, we now see the influence matrix for CFA taking the form

$$\tilde{\mathbf{Q}}_n = \left[ -\frac{1}{n} \frac{\partial U(\theta)}{\partial \theta} \right]^{-1} \mathbf{\Delta}(\theta)^\top \mathbf{W}_\theta \xrightarrow{P} \mathbf{H}(\theta)^{-1} \mathbf{\Delta}(\theta)^\top \mathbf{W}_\theta \quad \text{as } n \rightarrow \infty.$$

For certain estimators like maximum likelihood, the influence matrix directly fits in with the  $R$ -vector of moment differences  $\mathbf{m} - \mu(\theta) = \mathbf{p} - \boldsymbol{\pi}(\theta)$ . Incidentally, in this case, the influence matrix is  $\mathcal{I}^{-1} \mathbf{\Delta}_\pi \mathbf{D}_\pi \in \mathbb{R}^{t \times R}$ , where  $\mathcal{I} = \mathbf{\Delta}^\top \mathbf{D}^{-1} \mathbf{\Delta}^\top$  is the unit Fisher information,  $\mathbf{\Delta}_\pi = (\partial \boldsymbol{\pi} / \partial \theta)^\top$  is the Jacobian of the joint response probabilities with respect to the parameters, and  $\mathbf{D}_\pi = \text{diag}(\boldsymbol{\pi})$  is a diagonal matrix of the joint response probabilities, agreeing with results in Maydeu-Olivares & Joe (2005). It can be further shown that  $E[\tilde{\mathbf{Q}}_n]$  simplifies to  $\mathbf{\Delta}_\pi^\top \mathcal{I}^{-1} \mathbf{\Delta}_\pi$ .

In other cases, we need to post multiply the influence matrix  $\tilde{\mathbf{Q}}_n$  by an appropriate matrix so that it fits into the matrix-vector multiplication. This depends on the vector of moment differences. When dealing with pairwise likelihood, then an appropriate design matrix must be constructed to extract the pairwise probabilities from the joint response probabilities (Jamil et al., 2025). When dealing with lower-order moments such as uni- and bi-variate moments, a similar thing is done. These represent cases where there is a linear relationship  $g$  going from the moments  $\mathbf{m}$  and  $\boldsymbol{\mu}$  to the response probabilities  $\mathbf{p}$  and  $\boldsymbol{\pi}$ . In cases where the transformation  $g : \mathbf{p} \mapsto \mathbf{m}$  is not linear, such as the commonly used WLS methods, then we appeal to the Delta method. In the DWLS case, the transformation is

$$g : \mathbf{p} \mapsto (\mathbf{s}_1, \mathbf{s}_2)^\top$$

where  $\mathbf{s}_1$  are thresholds and  $\mathbf{s}_2$  are the polychoric correlations. Such a transformation was described by Muthén (1978) in the context of dichotomous data, but extends to the case of ordinal data too. Let  $\mathbf{G} := \partial g / \partial \mathbf{p}$  be the Jacobian of the transformation. Then,

$$\begin{aligned}\sqrt{n}(\mathbf{m} - \boldsymbol{\mu}(\theta)) &= \sqrt{n}(g(\mathbf{p}) - g(\boldsymbol{\pi}(\theta))) \\ &= \mathbf{G}\sqrt{n}(\mathbf{p} - \boldsymbol{\pi}(\theta)) + o_p(1).\end{aligned}$$

Plugging this into the above equation lets us see the form of the influence matrix as  $\mathbf{Q}_n = \tilde{\mathbf{Q}}_n \mathbf{G}$ .

## 2.4 Distribution of residuals

$$\sqrt{n}(\mathbf{p} - \boldsymbol{\pi}) \xrightarrow{D} N_R(\mathbf{0}, \boldsymbol{\Sigma}) \quad (4)$$

## 2.5 Wald-type tests

## 2.6 Pearson-type tests

## 2.7 General GOF tests

## 2.8 Estimation of degrees of freedom

# 3 References

# 4 Usage

## References

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