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Chapter 2

Vector space of functions

For regression modelling with I-priors, it is assumed that the regression functions lie in some vector space of functions. The purpose of this chapter is to provide a concise review of functional analysis leading up to the theory of reproducing kernel Hilbert and Kreĭn spaces (RKHS/RKKS). The interest with these RKHSs and RKKSs is that these spaces have well established mathematical structure and offer desirable topologies. In particular, it allows the possibility of deriving the Fisher information for regression functions—this will be covered in [Chapter 3](#). As we shall see, RKHSs are also extremely convenient in that they may be specified completely via their reproducing kernels. Several of these function spaces are of interest to us, for example, spaces of linear functions, smoothing functions, and functions whose inputs are nominal values and even functions themselves. RKHSs are widely studied in the applied statistical and machine learning literature, but perhaps RKKSs are less so. To provide an early insight, RKKSs are simply a generalisation of RKHSs, whose reproducing kernels are defined as the difference between two RKHS kernels. The flexibility provided by RKKSs will prove both useful and necessary, especially when considering sums and products of scaled function spaces, as is done in I-prior modelling.

It is emphasised that a deep knowledge of functional analysis, including RKHS and RKKS theory, is not at all necessary for I-prior modelling, so perhaps the advanced reader may wish to skip [Sections 2.1 to 2.3](#). [Section 2.4](#) describes the fundamental RKHSs of interest for I-prior regression, which we refer to as the “building block” RKHSs. The reason for this is that it is possible to construct new function spaces from existing ones, and this is described in [Section 2.5](#).

Two remarks before starting, and the first is on notation. Sets and vector spaces are denoted by calligraphic letters, and as much as possible, we shall stick to the convention that \mathcal{F} denotes a function space, and \mathcal{X} denotes the set of covariates or function inputs. Occasionally, we will describe a generic Hilbert space denoted by \mathcal{H} . Elements of the

vector space of real functions over a set \mathcal{X} are denoted by $f(\cdot)$, but more commonly and simply, by f . This distinguishes them from the actual evaluation of the function at an input point $x \in \mathcal{X}$, denoted $f(x) \in \mathbb{R}$. To make for a cleaner read, we dispense with boldface notation for vectors and matrices when talking about them, without ambiguity, in the abstract sense. Secondly, on bibliography: references will be minimised throughout the presentation of this chapter, but a complete annotated bibliography is furnished at the end in [Section 2.6](#).

2.1 Some functional analysis

The core study of functional analysis revolves around the treatment of functions as objects in vector spaces over a field¹. Vector spaces, or linear spaces as they are sometimes known, may be endowed with some kind of structure so as to allow ideas such as closeness and limits to be conceived. Of particular interest to us is the structure brought about by *inner products*, which allow the rigorous mathematical study of various geometrical concepts such as lengths, directions, and orthogonality, among other things. We begin with the definition of an inner product.

Definition 2.1 (Inner products). Let \mathcal{F} be a vector space over \mathbb{R} . A function $\langle \cdot, \cdot \rangle_{\mathcal{F}} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ is said to be an inner product on \mathcal{F} if all of the following are satisfied:

- **Symmetry.** $\langle f, g \rangle_{\mathcal{F}} = \langle g, f \rangle_{\mathcal{F}}, \forall f, g \in \mathcal{F}$.
- **Linearity.** $\langle \lambda_1 f_1 + \lambda_2 f_2, g \rangle_{\mathcal{F}} = \lambda_1 \langle f_1, g \rangle_{\mathcal{F}} + \lambda_2 \langle f_2, g \rangle_{\mathcal{F}}, \forall f_1, f_2, g \in \mathcal{F}, \forall \lambda_1, \lambda_2 \in \mathbb{R}$.
- **Non-degeneracy.** $\langle f, f \rangle_{\mathcal{F}} = 0 \Leftrightarrow f = 0$.

Additionally, an inner product is said to be *positive definite* if $\langle f, f \rangle_{\mathcal{F}} \geq 0, \forall f \in \mathcal{F}$. Inner products need not necessarily be positive definite, and we shall revisit this fact later when we cover Kreĭn spaces. For the purposes of the forthcoming discussion, the inner products that are referenced are the positive-definite kind, unless otherwise stated.

We can always define a *norm* on \mathcal{F} using the inner product as

$$\|f\|_{\mathcal{F}} = \sqrt{\langle f, f \rangle_{\mathcal{F}}}. \quad (2.1)$$

Norms are another form of structure that specifically captures the notion of length. This is defined below.

Definition 2.2 (Norms). Let \mathcal{F} be a vector space over \mathbb{R} . A non-negative function $\|\cdot\|_{\mathcal{F}} : \mathcal{F} \times \mathcal{F} \rightarrow [0, \infty)$ is said to be a norm on \mathcal{F} if all of the following are satisfied:

- **Absolute homogeneity.** $\|\lambda f\|_{\mathcal{F}} = |\lambda| \|f\|_{\mathcal{F}}, \forall \lambda \in \mathbb{R}, \forall f \in \mathcal{F}$

¹In this thesis, this will be \mathbb{R} exclusively.

- **Subadditivity.** $\|f + g\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}} + \|g\|_{\mathcal{F}}, \forall f, g \in \mathcal{F}$
- **Point separating.** $\|f\|_{\mathcal{F}} = 0 \Leftrightarrow f = 0$

Note that since $\|-f\|_{\mathcal{F}} = |-1| \|f\|_{\mathcal{F}} = \|f\|_{\mathcal{F}}$, and by the subadditivity and point separating property, we have that $\|f\|_{\mathcal{F}} = \frac{1}{2}\|f\|_{\mathcal{F}} + \frac{1}{2}\|-f\|_{\mathcal{F}} \geq \frac{1}{2}\|f - f\|_{\mathcal{F}} = 0$, thus implying non-negativity of norms.

The subadditivity property is also known as the *triangle inequality*. Following this, there is also the *reverse triangle inequality*, which states $\|f - g\|_{\mathcal{F}} \geq \left| \|f\|_{\mathcal{F}} - \|g\|_{\mathcal{F}} \right|$. In fact, the general forms of these triangle inequalities (Bergsma, 2018, Lemma 10) also hold for $0 \leq a \leq 1$ and any $f, g, h \in \mathcal{F}$:

$$\|f - g\|_{\mathcal{F}}^a \leq \|f - h\|_{\mathcal{F}}^a + \|g - h\|_{\mathcal{F}}^a \quad (2.2)$$

$$\|f - g\|_{\mathcal{F}}^a \geq \left| \|f\|_{\mathcal{F}}^a - \|g\|_{\mathcal{F}}^a \right| \quad (2.3)$$

Several other important relationships involving norms and inner products hold in linear spaces, namely, the *Cauchy-Schwarz inequality*

$$|\langle f, g \rangle_{\mathcal{F}}| \leq \|f\|_{\mathcal{F}} \|g\|_{\mathcal{F}},$$

the *parallelogram law*

$$\|f + g\|_{\mathcal{F}}^2 + \|f - g\|_{\mathcal{F}}^2 = 2\|f\|_{\mathcal{F}}^2 + 2\|g\|_{\mathcal{F}}^2,$$

and the *polarisation identity* (in various forms)

$$\begin{aligned} \|f + g\|_{\mathcal{F}}^2 - \|f - g\|_{\mathcal{F}}^2 &= 4\langle f, g \rangle_{\mathcal{F}}, \\ \|f + g\|_{\mathcal{F}}^2 - \|f\|_{\mathcal{F}}^2 - \|g\|_{\mathcal{F}}^2 &= 2\langle f, g \rangle_{\mathcal{F}}, \text{ and} \\ -\|f - g\|_{\mathcal{F}}^2 + \|f\|_{\mathcal{F}}^2 + \|g\|_{\mathcal{F}}^2 &= 2\langle f, g \rangle_{\mathcal{F}}, \end{aligned}$$

for any $f, g \in \mathcal{F}$.

A vector space endowed with an inner product (c.f. norm) is called an inner product space (c.f. normed vector space). As a remark, inner product spaces can always be equipped with a norm using (2.1), but not always the other way around. A norm needs to satisfy the parallelogram law for an inner product to be properly defined.

The norm $\|\cdot\|_{\mathcal{F}}$, in turn, induces a metric (a notion of distance) on \mathcal{F} , i.e. $D(f, g) = \|f - g\|_{\mathcal{F}}$, for $f, g \in \mathcal{F}$. With these notions of distances, one may talk about sequences of functions in \mathcal{F} which are *convergent*, and sequences whose elements become arbitrarily close to one another as the sequence progresses (*Cauchy*).

Definition 2.3 (Convergent sequence). A sequence $\{f_n\}_{n=1}^{\infty}$ of elements of a normed vector space $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ is said to *converge* to some $f \in \mathcal{F}$, if for every $\epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N}$, such that $\forall n > N$, $\|f_n - f\|_{\mathcal{F}} < \epsilon$.

Definition 2.4 (Cauchy sequence). A sequence $\{f_n\}_{n=1}^{\infty}$ of elements of a normed vector space $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ is said to be a Cauchy sequence if for every $\epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N}$, such that $\forall n, m > N$, $\|f_n - f_m\|_{\mathcal{F}} < \epsilon$.

Every convergent sequence is Cauchy (from the triangle inequality), but the converse is not true. If the limit of the Cauchy sequence exists within the vector space, then the sequence converges to it. A vector space is said to be *complete* if it contains the limits of all Cauchy sequences, or in other words, if every Cauchy sequence converges. There are special names given to complete vector spaces. A complete inner product space is known as a *Hilbert space*, while a complete normed space is called a *Banach space*. Out of interest, an inner product space that is not complete is sometimes known as a *pre-Hilbert space*, since its completion with respect to the norm induced by the inner product is a Hilbert space.

A subset $\mathcal{G} \subseteq \mathcal{F}$ is a *closed subspace* of \mathcal{F} if it is closed under addition and multiplication by a scalar. That is, for any $g, g' \in \mathcal{G}$, $\lambda_1 g + \lambda_2 g'$ is also in \mathcal{G} , for $\lambda_1, \lambda_2 \in \mathbb{R}$. For Hilbert spaces, each closed subspace is also complete, and thus a Hilbert space in its own right. Although, as a remark, not every Hilbert subspace need be closed, and therefore complete.

Being vectors in a vector space, we can discuss mapping of vectors onto a another space, or in essence, having a function acted upon them. To establish terminology, we define linear and bilinear maps (operators).

Definition 2.5 (Linear map/operator). Let \mathcal{F} and \mathcal{G} be two Hilbert spaces over \mathbb{R} . An operator A is a map from \mathcal{F} to \mathcal{G} , and we denote its action on a function $f \in \mathcal{F}$ as $A(f) \in \mathcal{G}$, or simply $Af \in \mathcal{G}$. A *linear operator* satisfies $A(f + f') = A(f) + A(f')$ and $A(\lambda f) = \lambda A(f)$, for all $f, f' \in \mathcal{F}$ and $\lambda \in \mathbb{R}$. If \mathcal{G} is the base field (\mathbb{R} in our case), then the linear operator A is called a *linear functional*.

Definition 2.6 (Bilinear map/operator). Let \mathcal{F} , \mathcal{G} and \mathcal{H} be Hilbert spaces over \mathbb{R} . A *bilinear operator* $B : \mathcal{F} \times \mathcal{G} \rightarrow \mathcal{H}$ is linear in each argument separately, i.e.

- $B(\lambda_1 f + \lambda_2 f', h) = \lambda_1 B(f, h) + \lambda_2 B(f', h)$; and
- $B(f, \lambda_1 g + \lambda_2 g') = \lambda_1 B(f, g) + \lambda_2 B(f, g')$,

for all $f, f' \in \mathcal{F}$, $g, g' \in \mathcal{G}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. In other words, the mappings $B_g : f \mapsto B(f, g)$ for any $g \in \mathcal{G}$, and $B_f : g \mapsto B(f, g)$ for any $f \in \mathcal{F}$, are both linear maps. If $\mathcal{F} \equiv \mathcal{G}$, then the bilinear map is *symmetric*. If \mathcal{H} is the base field (\mathbb{R} in our case), then B is called a *bilinear form*.

An interesting property of these operators to look at, besides linearity, is whether or not they are *continuous*.

Definition 2.7 (Continuity). Let \mathcal{F} and \mathcal{G} be two Hilbert spaces. A function $A : \mathcal{F} \rightarrow \mathcal{G}$ is said to be *continuous at* $g \in \mathcal{F}$, if for every $\epsilon > 0$, $\exists \delta = \delta(\epsilon, g) > 0$ such that

$$\|f - g\|_{\mathcal{F}} < \delta \Rightarrow \|Af - Ag\|_{\mathcal{G}} < \epsilon.$$

A is *continuous on* \mathcal{F} , if it is continuous at every point $g \in \mathcal{F}$. If, in addition, δ depends on ϵ only, A is said to be *uniformly continuous*.

Continuity in the sense of linear operators here means that a convergent sequence in the domain \mathcal{F} can be mapped to a convergent sequence in the range \mathcal{G} . For a particular linear operator, the evaluation functional, this means that closeness in norm implies pointwise closeness—this relates to RKHSs, which is discussed in Section 2.2. There is an even stronger notion of continuity called *Lipschitz continuity*.

Definition 2.8 (Lipschitz continuity). Let \mathcal{F} and \mathcal{G} be two Hilbert spaces. A function $A : \mathcal{F} \rightarrow \mathcal{G}$ is *Lipschitz continuous* if $\exists M > 0$ such that $\forall f, f' \in \mathcal{F}$,

$$\|Af - Af'\|_{\mathcal{G}} \leq M\|f - f'\|_{\mathcal{F}}.$$

Clearly, Lipschitz continuity implies uniform continuity: choose $\delta = \delta(\epsilon) := \epsilon/M$ and replace this in Definition 2.7. A continuous, linear operator is also one that is bounded.

Definition 2.9 (Bounded operator). The linear operator $A : \mathcal{F} \rightarrow \mathcal{G}$ between two Hilbert spaces \mathcal{F} and \mathcal{G} is said to be *bounded* if there exists some $M > 0$ such that

$$\|Af\|_{\mathcal{G}} \leq M\|f\|_{\mathcal{F}}.$$

The smallest such M is defined to be the *operator norm*, denoted $\|A\|_{\text{op}} := \sup_{f \in \mathcal{F}} \frac{\|Af\|_{\mathcal{G}}}{\|f\|_{\mathcal{F}}}$.

Lemma 2.1 (Equivalence of boundedness and continuity). *Let \mathcal{F} and \mathcal{G} be two Hilbert spaces, and $A : \mathcal{F} \rightarrow \mathcal{G}$ a linear operator. A is bounded if and only if it is continuous.*

Proof. Suppose that A is bounded. Then, $\forall f, f' \in \mathcal{F}$, $\exists M > 0$ such that $\|A(f - f')\|_{\mathcal{G}} \leq M\|f - f'\|_{\mathcal{F}}$, so A is Lipschitz continuous. Conversely, let A be a continuous linear operator, especially at the zero vector. In other words, $\exists \delta > 0$ such that $\|A(f)\|_{\mathcal{G}} = \|A(f + 0 - 0)\|_{\mathcal{G}} = \|A(f) - A(0)\|_{\mathcal{G}} \leq 1$, $\forall f \in \mathcal{F}$ whenever $\|f\|_{\mathcal{F}} \leq \delta$. Thus, for all non-zero $f \in \mathcal{F}$,

$$\begin{aligned} \|A(f)\|_{\mathcal{G}} &= \left\| \frac{\|f\|_{\mathcal{F}}}{\delta} A\left(\frac{\delta}{\|f\|_{\mathcal{F}}} f\right) \right\|_{\mathcal{G}} \\ &= \left| \frac{\|f\|_{\mathcal{F}}}{\delta} \right| \left\| A\left(\frac{\delta}{\|f\|_{\mathcal{F}}} f\right) \right\|_{\mathcal{G}} \\ &\leq \frac{\|f\|_{\mathcal{F}}}{\delta} \cdot 1, \end{aligned}$$

and therefore A is bounded. ■

So important is the concept of linearity and continuity, that there are specially named spaces which contain linear and continuous functionals.

Definition 2.10 (Dual spaces). Let \mathcal{F} be a Hilbert space. The space \mathcal{F}^\vee of *linear functionals* is called the *algebraic dual space* of \mathcal{F} . The space \mathcal{F}^* of *continuous linear functionals* is called the *continuous dual space* or alternatively, the *topological dual space*, of \mathcal{F} .

As it turns out, the algebraic dual space and continuous dual space coincide in finite-dimensional Hilbert spaces: take any $A \in \mathcal{F}^\vee$; since A is finite dimensional, it is bounded, and therefore continuous (see [Lemma 2.1](#)), so $A \in \mathcal{F}^*$ and $\mathcal{F}^\vee \subseteq \mathcal{F}^*$; but $\mathcal{F}^* \subseteq \mathcal{F}^\vee$ trivially, so $\mathcal{F}^\vee \equiv \mathcal{F}^*$. For infinite-dimensional Hilbert spaces, this is not so, but in any case, we will only be considering the continuous dual space in this thesis. The following result is an important one, which states that continuous linear functionals of an inner product space are nothing more than just inner products.

Theorem 2.2 (Riesz-Fréchet). *Let \mathcal{F} be a Hilbert space. Every element A of the continuous dual space \mathcal{F}^* , i.e. all continuous linear functionals $A : \mathcal{F} \rightarrow \mathbb{R}$, can be uniquely written in the form $\langle \cdot, g \rangle_{\mathcal{F}} =: A_g \in \mathcal{F}^*$, for some $g \in \mathcal{F}$. Moreover, $\|g\|_{\mathcal{F}} = \|A_g\|_{\mathcal{F}^*}$.*

Proof. Omitted—see [Yamamoto \(2012, Thm. 4.2.1\)](#) for a proof. ■

Remark 2.1. The Riesz-Fréchet theorem is also commonly referred to as the *Riesz representation theorem* for Hilbert spaces.

The notion of isometry (transformation that preserves distance) is usually associated with metric spaces; two metric spaces being isometric means that they are identical as far as their metric properties are concerned. For Hilbert spaces (and more generally, for normed spaces), there is an analogous concept as well in *isometric isomorphism* (a bijective isometry), such that two Hilbert spaces being isometrically isomorphic imply that they have exactly the same geometric structure, but may very well contain fundamentally different objects.

Definition 2.11 (Isometric isomorphism). Two Hilbert spaces \mathcal{F} and \mathcal{G} are said to be *isometrically isomorphic*, symbolised $\mathcal{F} \cong \mathcal{G}$, if there is a linear bijective map $U : \mathcal{F} \rightarrow \mathcal{G}$ which preserves the inner product, i.e. for any $f, f' \in \mathcal{F}$,

$$\langle f, f' \rangle_{\mathcal{F}} = \langle Uf, Uf' \rangle_{\mathcal{G}}.$$

A consequence of the Riesz-Fréchet theorem is that it gives us a canonical isometric isomorphism $U : g \mapsto \langle \cdot, g \rangle_{\mathcal{F}} =: A_g$ between \mathcal{F} and its continuous dual \mathcal{F}^* : the map A_g is obviously linear (by the bilinear property of inner products), and using the polarisation

identity, we have that

$$\begin{aligned} 2\langle Ug, Ug' \rangle_{\mathcal{F}^*} &= \|U(g)\|_{\mathcal{F}^*}^2 + \|U(g')\|_{\mathcal{F}^*}^2 - \|U(g - g')\|_{\mathcal{F}^*}^2 \\ &= \|g\|_{\mathcal{F}}^2 + \|g'\|_{\mathcal{F}}^2 - \|g - g'\|_{\mathcal{F}}^2 \\ &= 2\langle g, g' \rangle_{\mathcal{F}}. \end{aligned}$$

Implicitly, this means that \mathcal{F}^* is a Hilbert space as well.

Another important type of mapping is the mapping P of an element in \mathcal{F} onto a closed subspace $\mathcal{G} \subset \mathcal{F}$, such that $Pf \in \mathcal{G}$ is closest to f . This mapping is called the *orthogonal projection*, due to the fact that such projections yield perpendicularity in the sense that $\langle f - Pf, g \rangle_{\mathcal{F}} = 0$ for any $g \in \mathcal{G}$. Consequently, we see that $\|f\|_{\mathcal{F}}^2 = \|Pf\|_{\mathcal{F}}^2 + \|f - Pf\|_{\mathcal{F}}^2$ from the polarisation identity. The remainder $f - Pf$ belongs to the *orthogonal complement* of \mathcal{G} .

Definition 2.12 (Orthogonal complement). Let \mathcal{F} be a Hilbert space and $\mathcal{G} \subset \mathcal{F}$ be a closed subspace. The linear subspace $\mathcal{G}^\perp = \{f \mid \langle f, g \rangle_{\mathcal{F}} = 0, \forall g \in \mathcal{G}\}$ is called the orthogonal complement of \mathcal{G} in \mathcal{F} .

Theorem 2.3 (Orthogonal decomposition). Let \mathcal{F} be a Hilbert space and $\mathcal{G} \subset \mathcal{F}$ be a closed subspace. For every $f \in \mathcal{F}$, we can write $f = g + g^c$, where $g \in \mathcal{G}$ and $g^c \in \mathcal{G}^\perp$, and this decomposition is unique.

Proof. Omitted—see Rudin (1987, Thm. 4.11) for a proof. ■

We can write $\mathcal{F} = \mathcal{G} \oplus \mathcal{G}^\perp$, where the \oplus symbol denotes the *direct sum*, and such a decomposition is called a *tensor sum decomposition*. In infinite-dimensional Hilbert spaces, some subspaces are not closed, but all orthogonal complements are closed. In such spaces, the orthogonal complement of the orthogonal complement of \mathcal{G} is the closure of \mathcal{G} , i.e. $(\mathcal{G}^\perp)^\perp =: \overline{\mathcal{G}}$, and we say that \mathcal{G} is *dense* in $\overline{\mathcal{G}}$. Another interesting fact regarding the orthogonal complement is that $\mathcal{G} \cap \mathcal{G}^\perp = \{0\}$, since any $g \in \mathcal{G} \cap \mathcal{G}^\perp$ must be orthogonal to itself, i.e. $\langle g, g \rangle_{\mathcal{G}} = 0$ implying that $g = 0$.

The following theorem states that orthogonal decompositions are unique.

Corollary 2.3.1. Let \mathcal{G} be a subspace of a Hilbert space \mathcal{F} . Then, $\mathcal{G}^\perp = \{0\}$ if and only if \mathcal{G} is dense in \mathcal{F} .

Proof. If $\mathcal{G}^\perp = \{0\}$ then $(\mathcal{G}^\perp)^\perp = \overline{\mathcal{G}} = \mathcal{F}$. Conversely, since \mathcal{G} is dense in \mathcal{F} , we have $\mathcal{G}^\perp = \overline{\mathcal{G}}^\perp = \mathcal{F}^\perp = \{0\}$. ■

Besides tensor sums, of importance is the concept of *tensor products*, which can be thought of as a generalisation of the outer product in Euclidean space.

Definition 2.13 (Tensor products). Let $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$ be two elements of two real Hilbert spaces. Then, the tensor product $x_1 \otimes x_2 : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{R}$, is a bilinear form defined as

$$(x_1 \otimes x_2)(y_1, y_2) = \langle x_1, y_1 \rangle_{\mathcal{H}_1} \langle x_2, y_2 \rangle_{\mathcal{H}_2}$$

for any $(y_1, y_2) \in \mathcal{H}_1 \times \mathcal{H}_2$.

Correspondingly, we may also define the *tensor product space*.

Definition 2.14 (Tensor product space). The tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of the space

$$\mathcal{A} = \left\{ \sum_{j=1}^J x_{1j} \otimes x_{2j} \mid x_{1j} \in \mathcal{H}_1, x_{2j} \in \mathcal{H}_2, J \in \mathbb{N} \right\}.$$

with respect to the norm induced by the inner product

$$\left\langle \sum_{j=1}^J x_{1j} \otimes x_{2j}, \sum_{k=1}^K y_{1k} \otimes y_{2k} \right\rangle_{\mathcal{A}} = \sum_{j=1}^J \sum_{k=1}^K \langle x_{1j}, y_{1k} \rangle_{\mathcal{H}_1} \langle x_{2j}, y_{2k} \rangle_{\mathcal{H}_2}.$$

Interestingly, the tensor product can be viewed as an operator between two Hilbert spaces. That is, for each pair of elements $(x_1, x_2) \in \mathcal{H}_1 \times \mathcal{H}_2$, we define the operator $A_{x_1, x_2} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ in the following way:

$$\begin{aligned} A_{x_1, x_2} : \mathcal{H}_1 &\rightarrow \mathcal{H}_2 \\ y_1 &\mapsto \langle x_1, y_1 \rangle_{\mathcal{H}_1} x_2. \end{aligned}$$

Incidentally, an operator defined in such a way is called a *rank one* operator. Indeed, for any $y_1 \in \mathcal{H}_1$ and $y_2 \in \mathcal{H}_2$, we have that

$$\begin{aligned} \langle A_{x_1, x_2}(y_1), y_2 \rangle_{\mathcal{H}_2} &= \langle \langle x_1, y_1 \rangle_{\mathcal{H}_1} x_2, y_2 \rangle_{\mathcal{H}_2} \\ &= \langle x_1, y_1 \rangle_{\mathcal{H}_1} \langle x_2, y_2 \rangle_{\mathcal{H}_2} \\ &= (x_1 \otimes x_2)(y_1, y_2). \end{aligned}$$

We now have three distinct interpretations of the tensor product. For $x_1, y_1 \in \mathcal{H}_1$ and $x_2, y_2 \in \mathcal{H}_2$, these are:

- **General form.** An element in the tensor product space,

$$x_1 \otimes x_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

- **Operator form.** An operator between two Hilbert spaces,

$$\begin{aligned} x_1 \otimes x_2 : \mathcal{H}_1 &\rightarrow \mathcal{H}_2 \\ y_1 &\mapsto \langle x_1, y_1 \rangle_{\mathcal{H}_1} x_2. \end{aligned}$$

- **Bilinear form.** As per Definition 2.13,

$$\begin{aligned} x_1 \otimes x_2 &: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{R} \\ (y_1, y_2) &\mapsto \langle x_1, y_1 \rangle_{\mathcal{H}_1} \langle x_2, y_2 \rangle_{\mathcal{H}_2}. \end{aligned}$$

Remark 2.2. As explained by Kokoszka and Reimherr (2017, Sec. 10.5, p. 227), tensors are often thought of as generalisations of matrices and outer products. For example, in Euclidean space, a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$, formed by two vectors $\mathbf{x}_1 \in \mathbb{R}^n$ and $\mathbf{x}_2 \in \mathbb{R}^m$ via $\mathbf{A} = \mathbf{x}_1 \mathbf{x}_2^\top =: \mathbf{x}_1 \otimes \mathbf{x}_2$, can be viewed in at least three ways: 1) as a traditional matrix in the space $\mathbb{R}^n \otimes \mathbb{R}^m = \mathbb{R}^{n \times m}$; 2) as a linear operator in Euclidean space $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (or the reverse) by multiplying \mathbf{A} from the left or right by a vector; or 3) as a bilinear mapping $\mathbf{A} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ in the form of $\mathbf{A}(\mathbf{y}_1, \mathbf{y}_2) = \mathbf{y}_1^\top \mathbf{A} \mathbf{y}_2 = \mathbf{y}_1^\top \mathbf{x}_1 \mathbf{x}_2^\top \mathbf{y}_2 = (\mathbf{y}_1^\top \mathbf{x}_1)(\mathbf{y}_2^\top \mathbf{x}_2)$, for some $\mathbf{y}_1 \in \mathbb{R}^n$ and $\mathbf{y}_2 \in \mathbb{R}^m$, arising often in the study of quadratic forms.

For the last part of this introductory section on functional analysis, we discuss measures on Hilbert spaces, and in particular, a probability measure. Let \mathcal{H} be a real Hilbert space. As discussed earlier, we can define a metric on \mathcal{H} using $D(x, x') = \|x - x'\|_{\mathcal{H}}$, where the norm on \mathcal{H} is the norm induced by the inner product. A collection Σ of subsets of \mathcal{H} is called a σ -algebra if $\emptyset \in \Sigma$, $S \in \Sigma$ implies its complement $S^c \in \Sigma$, and $S_j \in \Sigma$, $j \geq 1$ implies $\bigcup_{j=1}^{\infty} S_j \in \Sigma$. The smallest σ -algebra containing all open subsets of \mathcal{H} is called the *Borel σ -algebra*, and its members the Borel sets. Denote by $\mathcal{B}(\mathcal{H})$ the Borel σ -algebra of \mathcal{H} .

Recall that a function $\nu : \Sigma \rightarrow [0, \infty]$ is called a *measure* if it satisfies

- **Non-negativity:** $\nu(S) \geq 0$ for all S in Σ ;
- **Null empty set:** $\nu(\emptyset) = 0$; and
- **σ -additivity:** for all countable, mutually disjoint sets $\{S_i\}_{i=1}^{\infty}$,

$$\nu\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \nu(S_i).$$

A measure ν on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ is called a *Borel measure* on \mathcal{H} . We shall only concern ourselves with finite Borel measures. In addition, if $\nu(\mathcal{H}) = 1$ then ν is a (*Borel*) *probability measure* and the measure space $(\mathcal{H}, \mathcal{B}(\mathcal{H}), \nu)$ is a (*Borel*) *probability space*.

Let $(\Omega, \mathcal{E}, \mathbb{P})$ be a probability space. We say that a mapping $X : \Omega \rightarrow \mathcal{H}$ is a *random element* in \mathcal{H} if $X^{-1}(B) \in \mathcal{E}$ for every Borel set, i.e., X is a function such that for every $B \in \mathcal{B}(\mathcal{H})$, its preimage $X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\}$ lies in \mathcal{E} . This is simply a generalisation of the definition of random variables in regular Euclidean space. From this definition, we can also properly define random functions f in a Hilbert space of functions \mathcal{F} . In any case, every random element X induces a probability measure on \mathcal{H}

defined by

$$\nu(B) = P(X^{-1}(B)) = P(\omega \in \Omega | X(\omega) \in B) = P(X \in B).$$

The measure ν is called the *distribution* of X . The *density* p of X is a measurable function with the property that

$$P(X \in B) = \int_{X^{-1}(B)} \omega \, dP(\omega) = \int_B p(x) \, d\nu(x).$$

Definition 2.15 (Mean vector). Let ν be a Borel probability measure on a real Hilbert space \mathcal{H} . Supposing that a random element X of \mathcal{H} is *integrable*, that is to say

$$E\|X\|_{\mathcal{H}} = \int_{\mathcal{H}} \|z\|_{\mathcal{H}} \, d\nu(z) < \infty,$$

then the unique element $\mu \in \mathcal{H}$ satisfying

$$\langle \mu, x \rangle = \int_{\mathcal{H}} \langle z, x \rangle_{\mathcal{H}} \, d\nu(z) = E\langle X, x \rangle_{\mathcal{H}}$$

for all $x \in \mathcal{H}$ is called the *mean vector*.

Definition 2.16 (Covariance operator). Let ν be a Borel probability measure on a real Hilbert space \mathcal{H} . Suppose that a random element X of \mathcal{H} is *square integrable*, i.e., $E\|X\|_{\mathcal{H}}^2 < \infty$, and let μ be the mean vector of X . Then the *covariance operator* C is defined by the mapping

$$\begin{aligned} C : \mathcal{H} &\rightarrow \mathcal{H} \\ x &\mapsto E[\langle X - \mu, x \rangle_{\mathcal{H}}(X - \mu)]. \end{aligned}$$

The covariance operator C is also an element of $\mathcal{H} \otimes \mathcal{H}$ that satisfies

$$\begin{aligned} \langle C, x \otimes x' \rangle_{\mathcal{H} \otimes \mathcal{H}} &= \int_{\mathcal{H}} \langle z - \mu, x \rangle_{\mathcal{H}} \langle z - \mu, x' \rangle_{\mathcal{H}} \, d\nu(z) \\ &= E[\langle X - \mu, x \rangle_{\mathcal{H}} \langle X - \mu, x' \rangle_{\mathcal{H}}] \end{aligned}$$

for all $x, x' \in \mathcal{H}$.

From the definition of the covariance operator, we see that it induces a symmetric, bilinear form, which we shall denote by $\text{Cov} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, through

$$\begin{aligned} \langle Cx, x' \rangle_{\mathcal{H}} &= \langle E[\langle X - \mu, x \rangle_{\mathcal{H}}(X - \mu)], x' \rangle_{\mathcal{H}} \\ &= E[\langle X - \mu, x \rangle_{\mathcal{H}} \langle X - \mu, x' \rangle_{\mathcal{H}}] \\ &=: \text{Cov}(x, x'). \end{aligned}$$

Definition 2.17 (Gaussian vectors). A random element X is called *Gaussian* if $\langle X, x \rangle_{\mathcal{H}}$ has a normal distribution for all fixed $x \in \mathcal{H}$. A Gaussian vector X is characterised by its mean element $\mu \in \mathcal{H}$ and its covariance $C \in \mathcal{H} \otimes \mathcal{H}$.

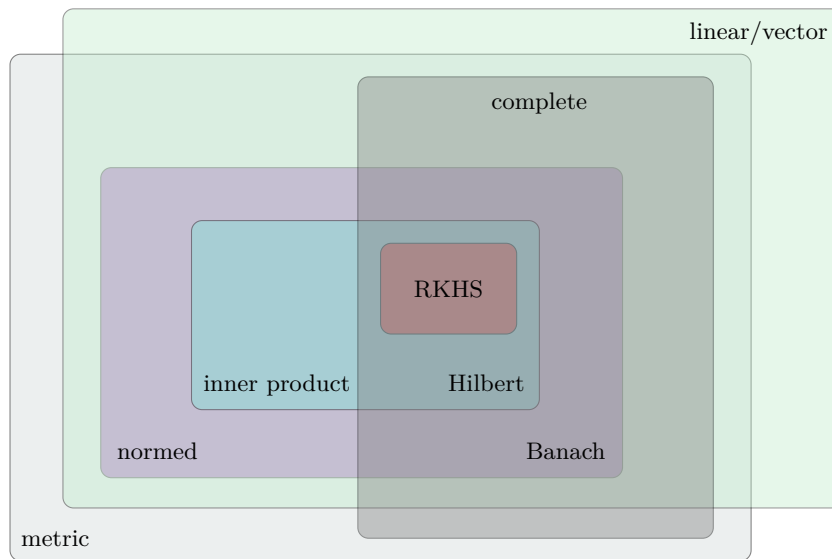


Figure 2.1: A hierarchy of vector spaces².

2.2 Reproducing kernel Hilbert space theory

The introductory section sets us up nicely to discuss the coveted reproducing kernel Hilbert space. This is a subset of Hilbert spaces for which its evaluation functionals are continuous (by definition, in fact). The majority of this section, apart from defining RKHSs, is an exercise in persuading ourselves that each and every RKHS of functions can be specified solely through its reproducing kernel. To begin, we consider a fundamental linear functional on a Hilbert space of functions \mathcal{F} , that assigns a value to $f \in \mathcal{F}$ for each $x \in \mathcal{X}$, called the *evaluation functional*.

Definition 2.18 (Evaluation functional). Let \mathcal{F} be a vector space of functions $f : \mathcal{X} \rightarrow \mathbb{R}$, defined on a non-empty set \mathcal{X} . For a fixed $x \in \mathcal{X}$, the functional $\delta_x : \mathcal{F} \rightarrow \mathbb{R}$ as defined by $\delta_x(f) = f(x)$ is called the (Dirac) evaluation functional at x .

It is easy to see that evaluation functionals are always linear: $\delta_x(\lambda f + g) = (\lambda f + g)(x) = \lambda f(x) + g(x) = \lambda \delta_x(f) + \delta_x(g)$ for $\lambda \in \mathbb{R}$, $f, g \in \mathcal{F}$ real functions over \mathcal{X} . Humble as they may seem, the entirety of the evaluation functionals over the domain \mathcal{X} determines f uniquely, and thus are of great importance in understanding the space \mathcal{F} . Core topological properties like convergence are hinged on continuity, and it is therefore important that evaluation functionals are continuous. As it turns out, RKHSs by definition provide exactly this.

²Reproduced from the lecture slides of Dino Sejdinovic and Arthur Gretton entitled “Foundations of Reproducing Kernel Hilbert Spaces: Advanced Topics in Machine Learning”, 2014. URL: http://www.stats.ox.ac.uk/~sejdinovic/teaching/atml14/Theory_slides2_2014.pdf.

Definition 2.19 (Reproducing kernel Hilbert space). A Hilbert space \mathcal{F} of real-valued functions $f : \mathcal{X} \rightarrow \mathbb{R}$ on a non-empty set \mathcal{X} is called a *reproducing kernel Hilbert space* if the evaluation functional $\delta_x : f \mapsto f(x)$ is continuous (equivalently, bounded) on \mathcal{F} , $\forall x \in \mathcal{X}$.

Continuity (boundedness) of evaluation functionals in an RKHS means that functions that are close in RKHS norm imply that they are also close pointwise, since $|\delta_x(f) - \delta_x(g)| = |\delta_x(f - g)| \leq M\|f - g\|_{\mathcal{F}}$ for some real $M > 0$. Note that the converse is not necessarily true. RKHSs are particularly well behaved in this respect, compared to other Hilbert spaces, and this property in particular has desirable consequences for a wide variety of applications, including nonparametric curve estimation, learning and decision theory, and many more.

While the continuity condition by definition is what makes an RKHS, it is neither easy to check this condition in practice, nor is it intuitive as to the meaning of its name. In fact, there isn't even any mention of what a reproducing kernel actually is. In order to benefit from the desirable continuity property of RKHS, we should look at this from another, more intuitive, perspective. By invoking the Riesz representation theorem, we see that for all $x \in \mathcal{X}$, there exists a unique element $h_x \in \mathcal{F}$ such that

$$f(x) = \delta_x(f) = \langle f, h_x \rangle_{\mathcal{F}}, \forall f \in \mathcal{F}$$

holds. Since h_x itself is a function in \mathcal{F} , it holds that for every $x' \in \mathcal{X}$ there exists a $h_{x'} \in \mathcal{F}$ such that

$$h_x(x') = \delta_{x'}(h_x) = \langle h_x, h_{x'} \rangle_{\mathcal{F}}.$$

This leads us to the definition of a *reproducing kernel* of an RKHS—the very notion that inspires its name.

Definition 2.20 (Reproducing kernels). Let \mathcal{F} be a Hilbert space of functions over a non-empty set \mathcal{X} . A symmetric, bivariate function $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is called a *kernel*, and it is a *reproducing kernel* of \mathcal{F} if h satisfies

- $\forall x \in \mathcal{X}, h(\cdot, x) \in \mathcal{F}$; and
- $\forall x \in \mathcal{X}, f \in \mathcal{F}, \langle f, h(\cdot, x) \rangle_{\mathcal{F}} = f(x)$ (the reproducing property).

In particular, for any $x, x' \in \mathcal{X}$,

$$h(x, x') = \langle h(\cdot, x), h(\cdot, x') \rangle_{\mathcal{F}}.$$

An important property for reproducing kernels of an RKHS is that they are positive definite functions. That is, $\forall \lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\forall x_1, \dots, x_n \in \mathcal{X}$,

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j h(x_i, x_j) \geq 0.$$

Lemma 2.4 (Reproducing kernels of RKHSs are positive definite). *Let $h: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be a reproducing kernel for a Hilbert space \mathcal{F} . Then h is a symmetric and positive-definite function.*

Proof.

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n a_i a_j h(x_i, x_j) &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \langle h(\cdot, x_i), h(\cdot, x_j) \rangle_{\mathcal{F}} \\ &= \left\langle \sum_{i=1}^n a_i h(\cdot, x_i), \sum_{j=1}^n a_j h(\cdot, x_j) \right\rangle_{\mathcal{F}} \\ &= \left\| \sum_{i=1}^n a_i h(\cdot, x_i) \right\|_{\mathcal{F}}^2 \\ &\geq 0 \end{aligned} \quad \blacksquare$$

Remark 2.3. In the kernel methods literature, a *kernel* $h: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is usually defined as the inner product between inputs in feature space. That is, take $\phi: \mathcal{X} \rightarrow \mathcal{V}$, $x \mapsto \phi(x)$, where \mathcal{V} is a Hilbert space. Then the kernel is defined as $h(x, x') = \langle \phi(x), \phi(x') \rangle_{\mathcal{V}}$, for any $x, x' \in \mathcal{X}$. The space \mathcal{V} is known as the *feature space* and the mapping ϕ the *feature map*. In many mathematical models involving feature space mappings, elucidation of the feature map and feature space is not necessary, and thus computation is made simpler by the use of kernels (known as the *kernel trick*—Hofmann et al., 2008). Note that kernels defined in this manner are positive definite, while in this thesis, we opt for a more general definition allowing kernels to not necessarily be positive. The relevance of this generality will be appreciated when we discuss reproducing kernel Kreĭn spaces in Section 2.3.

Introducing the following definition of the *kernel matrix* (also known as the *Gram matrix*) is useful at this point.

Definition 2.21 (Kernel matrix). Let $\{x_1, \dots, x_n\}$ be a sample of points, where each $x_i \in \mathcal{X}$, and h a kernel over \mathcal{X} . Define the *kernel matrix* \mathbf{H} for h as the $n \times n$ matrix with (i, j) entries equal to $h(x_i, x_j)$.

Obviously, \mathbf{H} is a positive definite matrix if the kernel that defines it is positive definite: $\mathbf{a}^\top \mathbf{H} \mathbf{a} = \sum_{i=1}^n \sum_{j=1}^n a_i a_j h(x_i, x_j) \geq 0$ for any choice of $a_1, \dots, a_n \in \mathbb{R}$ and $x_1, \dots, x_n \in \mathcal{X}$.

So far, we have seen that reproducing kernels of an RKHS are positive-definite functions, and that RKHSs are Hilbert spaces with continuous evaluation functionals, but one might wonder what exactly the relationship between a reproducing kernel and an RKHS is. We assert the following:

- For every RKHS \mathcal{F} of functions over a set \mathcal{X} , there corresponds a unique, positive-definite reproducing kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, and vice-versa. That is, a Hilbert space is an RKHS if it possesses a unique, reproducing kernel.
- For every positive-definite function $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, there corresponds a unique RKHS \mathcal{F} that has h as its reproducing kernel.

Pictorially, the following relationships are established:

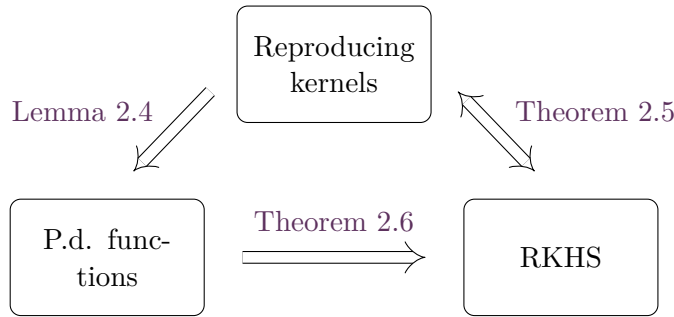


Figure 2.2: Relationships between positive definite functions, reproducing kernels, and RKHSs.

In essence, the notion of positive-definite functions and reproducing kernels of RKHSs are equivalent, and that there is a bijection between the set of positive-definite kernels and the set of RKHSs. The rest of this section is a consideration of these assertions, addressed by the two theorems that follow.

Theorem 2.5 (RKHS uniqueness). *Let \mathcal{F} be a Hilbert space of functions over \mathcal{X} . \mathcal{F} is an RKHS if and only if \mathcal{F} has a reproducing kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, and that h is unique to \mathcal{F} .*

Proof. First we tackle existence, i.e. we prove that \mathcal{F} is an RKHS if and only if \mathcal{F} has a reproducing kernel. Suppose \mathcal{F} is a Hilbert space of functions, and $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is a reproducing kernel for \mathcal{F} . Then, choosing $\delta = \epsilon / \|h(\cdot, x)\|_{\mathcal{F}}$, for any $f \in \mathcal{F}$ such that $\|f - g\|_{\mathcal{F}} < \delta$, we have

$$\begin{aligned}
 |\delta_x(f) - \delta_x(g)| &= |(f - g)(x)| \\
 &= |\langle f - g, h(\cdot, x) \rangle_{\mathcal{F}}| \quad (\text{reproducing property}) \\
 &\leq \|h(\cdot, x)\|_{\mathcal{F}} \|f - g\|_{\mathcal{F}} \quad (\text{Cauchy-Schwarz}) \\
 &= \epsilon.
 \end{aligned}$$

Thus, the evaluation functional is (uniformly) continuous on \mathcal{F} , and by definition, \mathcal{F} is an RKHS. Now suppose that \mathcal{F} is an RKHS, and h is a kernel function over $\mathcal{X} \times \mathcal{X}$. The reproducing property of h is had by following the argument preceding Definition 2.20.

As for uniqueness, assume that the RKHS \mathcal{F} has two reproducing kernels h_1 and h_2 . Then, $\forall f \in \mathcal{F}$ and $\forall x \in \mathcal{X}$,

$$\langle f, h_1(\cdot, x) - h_2(\cdot, x) \rangle_{\mathcal{F}} = f(x) - f(x) = 0.$$

In particular, if we take $f = h_1(\cdot, x) - h_2(\cdot, x)$, we obtain $\|h_1(\cdot, x) - h_2(\cdot, x)\|_{\mathcal{F}}^2 = 0$. Thus, $h_1 = h_2$. \blacksquare

Theorem 2.6 (Moore-Aronszajn). *If $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is a positive-definite function then there exists a unique RKHS whose reproducing kernel is h .*

Sketch proof. Most of the details here have been omitted, except for the parts which we feel are revealing as to the properties of an RKHS. For a complete proof, see Gu (2013, Thm. 2.3) or Berlinet and Thomas-Agnan (2004, Thm. 3). Start with the linear space

$$\mathcal{F}_0 = \left\{ f_n : \mathcal{X} \rightarrow \mathbb{R} \mid f_n = \sum_{i=1}^n w_i h(\cdot, x_i), x_i \in \mathcal{X}, w_i \in \mathbb{R}, n \in \mathbb{N} \right\}$$

and endow this linear space with the following inner product:

$$\left\langle \sum_{i=1}^n w_i h(\cdot, x_i), \sum_{j=1}^m w'_j h(\cdot, x'_j) \right\rangle_{\mathcal{F}_0} = \sum_{i=1}^n \sum_{j=1}^m w_i w'_j h(x_i, x'_j).$$

It may be shown that this is indeed a valid inner product satisfying the conditions laid in Definition 2.1. At this point, the reproducing property is already had:

$$\begin{aligned} \langle f_n, h(\cdot, x) \rangle_{\mathcal{F}_0} &= \left\langle \sum_{i=1}^n w_i h(\cdot, x_i), h(\cdot, x) \right\rangle_{\mathcal{F}_0} \\ &= \sum_{i=1}^n w_i h(x, x_i) \\ &= f_n(x), \end{aligned}$$

for any $f_n \in \mathcal{F}_0$.

Let \mathcal{F} be the completion of \mathcal{F}_0 with respect to this inner product. In other words, define \mathcal{F} to be the set of functions $f : \mathcal{X} \rightarrow \mathbb{R}$ for which there exists a Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ in \mathcal{F}_0 converging pointwise to $f \in \mathcal{F}$. The inner product for \mathcal{F} is defined to be

$$\langle f, f' \rangle_{\mathcal{F}} = \lim_{n \rightarrow \infty} \langle f_n, f'_n \rangle_{\mathcal{F}_0}.$$

The sequence $\{\langle f_n, f'_n \rangle_{\mathcal{F}_0}\}_{n=1}^\infty$ is convergent and does not depend on the sequence chosen, but only on the limits f and f' (Berlinet and Thomas-Agnan, 2004, Lemma 5). We may check that this indeeds defines a valid inner product. The reproducing property carries over to the completion:

$$\begin{aligned}\langle f, h(\cdot, x) \rangle_{\mathcal{F}} &= \lim_{n \rightarrow \infty} \langle f_n, h(\cdot, x) \rangle_{\mathcal{F}_0} \\ &= \lim_{n \rightarrow \infty} f_n(x) \\ &= f(x).\end{aligned}$$

To prove uniqueness, let \mathcal{G} be another RKHS with reproducing kernel h . \mathcal{F} has to be a closed subspace of \mathcal{G} , since $h(\cdot, x) \in \mathcal{G}$ for all $x \in \mathcal{X}$, and because \mathcal{G} is complete and contains \mathcal{F}_0 and hence its completion. Using the orthogonal decomposition theorem, we have $\mathcal{G} = \mathcal{F} \oplus \mathcal{F}^\perp$, i.e. any $g \in \mathcal{G}$ can be decomposed as $g = f + f^c$, $f \in \mathcal{F}$ and $f^c \in \mathcal{F}^\perp$. For each element $g \in \mathcal{G}$ we have that, for all $x \in \mathcal{X}$,

$$\begin{aligned}g(x) &= \langle g, h(\cdot, x) \rangle_{\mathcal{G}} \\ &= \langle f + f^c, h(\cdot, x) \rangle_{\mathcal{G}} \\ &= \langle f, h(\cdot, x) \rangle_{\mathcal{G}} + \underbrace{\langle f^c, h(\cdot, x) \rangle_{\mathcal{G}}}_{\rightarrow 0} \\ &= f(x)\end{aligned}$$

so therefore $g \in \mathcal{F}$ too. It must be that $\mathcal{F} \equiv \mathcal{G}$. ■

A consequence of the above proof is that we can show that any function f in an RKHS \mathcal{F} with kernel h can be written in the form $f(x) = \sum_{i=1}^n h(x, x_i) w_i$, with some $(w_1, \dots, w_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$. More precisely, \mathcal{F} is the completion of the space $\mathcal{G} = \text{span}\{h(\cdot, x) \mid x \in \mathcal{X}\}$ endowed with the inner product as stated in Section 2.2.

2.3 Reproducing kernel Kreĭn space theory

In this section, we review elementary Kreĭn and reproducing kernel Kreĭn space (RKKS) theory, and comment on the similarity and differences between it and RKHSs. Kreĭn spaces are linear spaces endowed with a Hilbertian topology, characterised by an inner product which is non-positive.

Definition 2.22 (Negative and indefinite inner products). Let $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ be an inner product of a vector space \mathcal{F} , as per Definition 2.1. An inner product is said to be *negative-definite* if for all $f \in \mathcal{F}$, $\langle f, f \rangle_{\mathcal{F}} \leq 0$. It is *indefinite* if it is neither positive- nor negative-definite.

Definition 2.23 (Kreĭn space). An inner product space $(\mathcal{F}, \langle \cdot, \cdot \rangle_{\mathcal{F}})$ is a *Kreĭn space* if there exists two Hilbert spaces $(\mathcal{F}_+, \langle \cdot, \cdot \rangle_{\mathcal{F}_+})$ and $(\mathcal{F}_-, \langle \cdot, \cdot \rangle_{\mathcal{F}_-})$ spanning \mathcal{F} such that

- All $f \in \mathcal{F}$ can be decomposed into $f = f_+ + f_-$, where $f_+ \in \mathcal{F}_+$ and $f_- \in \mathcal{F}_-$.
- This decomposition is orthogonal, i.e. $\mathcal{F}_+ \cap \mathcal{F}_- = \{0\}$, and $\langle f_+, f'_- \rangle_{\mathcal{F}} = 0$ for any $f, f' \in \mathcal{F}$, with the inner product on \mathcal{F} defined below.
- $\forall f, f' \in \mathcal{F}, \langle f, f' \rangle_{\mathcal{F}} = \langle f_+, f'_+ \rangle_{\mathcal{F}_+} - \langle f_-, f'_- \rangle_{\mathcal{F}_-}$.

Remark 2.4. Any Hilbert space is also a Kreĭn space, which is seen by taking $\mathcal{F}_- = \{0\}$ in the above Definition 2.23.

Let P be the projection of the Kreĭn space \mathcal{F} onto \mathcal{F}_+ , and $Q = I - P$ the projection onto \mathcal{F}_- , where I is the identity map. These are called the *fundamental projections* of \mathcal{F} . We shall refer to \mathcal{F}_+ as the *positive subspace*, and \mathcal{F}_- as the *negative subspace*. These monikers stem from the fact that for all $f, f' \in \mathcal{F}$, $\langle Pf, Pf' \rangle_{\mathcal{F}_+} \geq 0$ while $\langle Qf, Qf' \rangle_{\mathcal{F}_-} \leq 0$. We introduce the notation \ominus to refer to the Kreĭn space decomposition: $\mathcal{F} = \mathcal{F}_+ \ominus \mathcal{F}_-$. There is then a notion of an *associated Hilbert space*.

Definition 2.24 (Associated Hilbert space). Let \mathcal{F} be a Kreĭn space with decomposition into Hilbert spaces \mathcal{F}_+ and \mathcal{F}_- . Denote by $\mathcal{F}_{\mathcal{H}}$ the associated Hilbert space defined by $\mathcal{F}_{\mathcal{H}} = \mathcal{F}_+ \oplus \mathcal{F}_-$, with inner product

$$\langle f, f' \rangle_{\mathcal{F}_{\mathcal{H}}} = \langle f_+, f'_+ \rangle_{\mathcal{F}_+} + \langle f_-, f'_- \rangle_{\mathcal{F}_-},$$

for all $f, f' \in \mathcal{F}$.

The associated Hilbert space can be found via the linear operator $J = P - Q$ called the *fundamental symmetry*. That is, a Kreĭn space \mathcal{F} can be turned into its associated Hilbert space by using the positive-definite inner product of the associated Hilbert space as $\langle f, f' \rangle_{\mathcal{F}_{\mathcal{H}}} = \langle f, Jf' \rangle_{\mathcal{F}}$, for all $f, f' \in \mathcal{F}$. The converse is true too: starting from a Hilbert space $\mathcal{F}_{\mathcal{H}}$ and an operator J , the vector space endowed with the inner product $\langle f, f' \rangle_{\mathcal{F}} = \langle f, Jf' \rangle_{\mathcal{F}_{\mathcal{H}}}$, for all $f, f' \in \mathcal{F}$, is a Kreĭn space.

We realise that for a Kreĭn space \mathcal{F} , $|\langle f, f \rangle_{\mathcal{F}}| \leq \|f\|_{\mathcal{F}_{\mathcal{H}}}^2$ for all $f \in \mathcal{F}$. As such, we say that $\mathcal{F}_{\mathcal{H}}$ majorises \mathcal{F} , and in fact it is the smallest Hilbert space to do so. The strong topology on \mathcal{F} is defined to be the topology arising from the norm of $\mathcal{F}_{\mathcal{H}}$, and this does not depend on the decomposition chosen (Ong et al., 2004). Now, we define an RKKS.

Definition 2.25 (Reproducing kernel Kreĭn space). A Kreĭn space \mathcal{F} of real-valued functions $f : \mathcal{X} \rightarrow \mathbb{R}$ on a non-empty set \mathcal{X} is called a *reproducing kernel Kreĭn space* if the evaluation functional $\delta_x : f \mapsto f(x)$ is continuous on \mathcal{F} , $\forall x \in \mathcal{X}$, endowed with its strong topology (i.e. the topology of its associated Hilbert space $\mathcal{F}_{\mathcal{H}}$).

One might wonder whether the uniqueness theorem ([Theorem 2.5](#)) holds for RKKS. Indeed, for every RKKS \mathcal{F} of functions over a set \mathcal{X} , there corresponds a unique reproducing kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$.

Lemma 2.7 (Uniqueness of kernel for RKKS). *Let \mathcal{F} be an RKKS of functions over a set \mathcal{X} , with $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$. Then, \mathcal{F}_+ and \mathcal{F}_- are both RKHS with kernel h_+ and h_- , and the kernel $h = h_+ - h_-$ is a unique, symmetric, reproducing kernel for \mathcal{F} .*

Proof. Since \mathcal{F} is an RKKS, evaluation functionals are continuous on \mathcal{F} with respect to topology of the associated Hilbert space $\mathcal{F}_{\mathcal{H}} = \mathcal{F}_+ \oplus \mathcal{F}_-$. Therefore, $\mathcal{F}_{\mathcal{H}}$ is an RKHS, and so too are \mathcal{F}_+ and \mathcal{F}_- with respective kernels h_+ and h_- .

Furthermore, $h(\cdot, x) \in \mathcal{F}$ since $h_+(\cdot, x) \in \mathcal{F}_+$ and $h_-(\cdot, x) \in \mathcal{F}_-$ for some $x \in \mathcal{X}$. Then, for any $f \in \mathcal{F}$,

$$\begin{aligned}
 \langle f, h(\cdot, x) \rangle_{\mathcal{F}} &= \langle f, h_+(\cdot, x) \rangle_{\mathcal{F}_+} - \langle f, h_-(\cdot, x) \rangle_{\mathcal{F}_-} \\
 &= \langle f_+, h_+(\cdot, x) \rangle_{\mathcal{F}_+} - \langle f_-, h_+(\cdot, x) \rangle_{\mathcal{F}_-} \\
 &\quad - \langle f_+, h_-(\cdot, x) \rangle_{\mathcal{F}_+} + \langle f_-, h_-(\cdot, x) \rangle_{\mathcal{F}_-} \\
 &= f_+(x) + f_-(x) \\
 &= f(x)
 \end{aligned}$$

The last two lines are achieved by linearity of evaluation functionals ($\delta_x(f_+) + \delta_x(f_-) = \delta_x(f_+ + f_-)$), and the fact that $f = f_+ + f_-$ (by the Kreĭn space decomposition). We have that $h = h_+ - h_-$ is a reproducing kernel for \mathcal{F} . Uniqueness follows as a consequence of the non-degeneracy condition of the respective inner products for \mathcal{F}_+ and \mathcal{F}_- . ■

Remark 2.5. Unlike reproducing kernels of RKHSs, reproducing kernels of RKKSs may not be positive definite.

The analogue of the Moore-Aronszajn theorem holds partially for RKKS, up to uniqueness. That is, there is *at least* one associated RKKS with kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ if and only if h can be decomposed as the difference between two positive kernels h_+ and h_- over \mathcal{X} , i.e. $h = h_+ - h_-$. The proof of this statement is rather involved, so is omitted in the interest of maintaining coherence to the discussion at hand. This subject has been studied by various authors; one may refer to works by [Alpay \(1991, Thm. 2 & E.g. in Sec. 4\)](#), and [Mary \(2003, Thm. 2.28\)](#).

The take-away message as we close this section is that there is no bijection, but a surjection, between the set of RKKS and the set of bivariate, symmetric functions over $\mathcal{X} \times \mathcal{X}$. In any case, Hilbertian topology applies to Kreĭn spaces via the associated Hilbert space, and in particular, RKKS provide a functional space for which evaluation

functionals are continuous. The motivation for the use of Kreĭn spaces will become clear when constructing function spaces out of (scaled) building block RKHS later in Section 2.5.

2.4 RKHS building blocks

This section describes what we refer to as the “building block” RKHSs of functions. In the context of regression modelling using I-priors, we may assume that the regression function lies in any one of these single RKHSs, although it may be more appropriate to consider function spaces built upon these RKHSs for more complex models. I-priors will be presented in detail in Chapter 3, but in advance of the forthcoming discussion, the plots in this section are intended to give an impression of sample I-prior paths from the respective RKHSs. Construction of new function spaces from these building block RKHSs will be discussed in the next section.

2.4.1 The RKHS of constant functions

The vector space of constant functions \mathcal{F} over a set \mathcal{X} contains the functions $f : \mathcal{X} \rightarrow \mathbb{R}$ such that $f(x) = c_f \in \mathbb{R}, \forall x \in \mathcal{X}$. These functions would be useful to model an overall average, i.e. an “intercept effect”. The space \mathcal{F} can be equipped with a norm to form an RKHS, as shown in the following proposition.

Proposition 2.8 (RKHS of constant functions). *The space \mathcal{F} as described above endowed with the norm $\|f\|_{\mathcal{F}} = |c_f|$ forms an RKHS with the reproducing kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ as defined, rather simply, by*

$$h(x, x') = 1,$$

known as the constant kernel.

Proof. If \mathcal{F} is an RKHS with kernel h as described, then \mathcal{F} is spanned by the functions $h(\cdot, x) = 1$, so it is clear that \mathcal{F} consists of constant functions over \mathcal{X} . On the other hand, if the space \mathcal{F} is equipped with the inner product $\langle f, f' \rangle_{\mathcal{F}} = c_f c_{f'}$, then the reproducing property follows, since $\langle f, h(\cdot, x) \rangle_{\mathcal{F}} = c_f = f(x)$. Hence, $\|f\|_{\mathcal{F}} = \sqrt{\langle f, f \rangle_{\mathcal{F}}} = |c_f|$. ■

Remark 2.6. In I-prior modelling, it is simpler to consider the intercept of a regression model as a parameter to be estimated, rather than a separate function within an RKHS of constant functions for which its posterior is to be estimated. See Section 4.2.1 (p. 110) in Chapter 4 for further details.

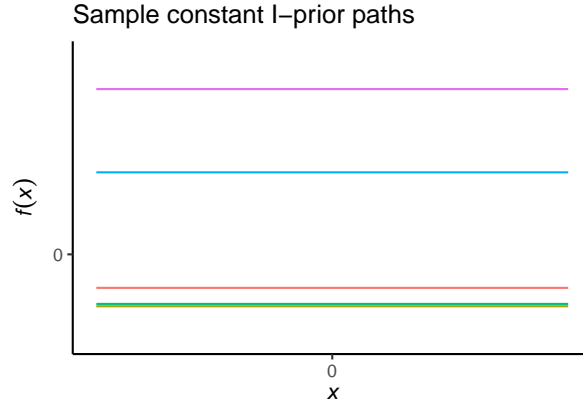


Figure 2.3: Sample I-prior paths from the RKHS of constant functions.

2.4.2 The canonical (linear) RKHS

Consider a function space \mathcal{F} over \mathcal{X} which consists of functions of the form $f_\beta : \mathcal{X} \rightarrow \mathbb{R}$, $f_\beta : x \mapsto \langle x, \beta \rangle_{\mathcal{X}}$ for some $\beta \in \mathbb{R}$. Suppose that $\mathcal{X} \equiv \mathbb{R}^p$, then \mathcal{F} consists of the linear functions $f_\beta(x) = x^\top \beta$. More generally, if \mathcal{X} is a Hilbert space, then its continuous dual consists of elements of the form $f_\beta = \langle \cdot, \beta \rangle_{\mathcal{X}}$ by the Riesz representation theorem. We can show that the continuous dual space of \mathcal{X} is an RKHS which consists of these linear functions.

Proposition 2.9 (Canonical RKHS). *The continuous dual space of a Hilbert space \mathcal{X} , denoted by \mathcal{X}^* , is an RKHS of linear functions over \mathcal{X} of the form $\langle \cdot, \beta \rangle_{\mathcal{X}}$, $\beta \in \mathcal{X}$. Its reproducing kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is defined by*

$$h(x, x') = \langle x, x' \rangle_{\mathcal{X}}.$$

Proof. Define $f_\beta := \langle \cdot, \beta \rangle_{\mathcal{X}}$ for some $\beta \in \mathcal{X}$. Clearly this is linear and continuous, so $f_\beta \in \mathcal{X}^*$, and so \mathcal{X}^* is a Hilbert space containing functions $f : \mathcal{X} \rightarrow \mathbb{R}$ of the form $f_\beta(x) = \langle x, \beta \rangle_{\mathcal{X}}$. By the Riesz representation theorem, every element of \mathcal{X}^* has the form f_β . It also gives us a natural isometric isomorphism such that the following is true:

$$\langle \beta, \beta' \rangle_{\mathcal{X}} = \langle f_\beta, f_{\beta'} \rangle_{\mathcal{X}^*}.$$

Hence, for any $f_\beta \in \mathcal{X}^*$,

$$\begin{aligned} f_\beta(x) &= \langle x, \beta \rangle_{\mathcal{X}} \\ &= \langle f_x, f_\beta \rangle_{\mathcal{X}^*} \\ &= \langle \langle \cdot, x \rangle_{\mathcal{X}}, f_\beta \rangle_{\mathcal{X}^*}. \end{aligned}$$

Thus, $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ as defined by $h(x, x') = \langle x, x' \rangle_{\mathcal{X}}$ is the reproducing kernel of \mathcal{X}^* . ■

In many other literature, the kernel $h(x, x') = \langle x, x' \rangle_{\mathcal{X}}$ is also known as the *linear kernel*. The use of the term ‘canonical’ is fitting not just due to the relation between a Hilbert space and its continuous dual space. Let $\phi : \mathcal{X} \rightarrow \mathcal{V}$ be the feature map from the space of covariates (inputs) to some feature space \mathcal{V} . Suppose both \mathcal{X} and \mathcal{V} are Hilbert spaces, then a kernel, as per [Remark 2.3](#), is defined as $h(x, x') = \langle \phi(x), \phi(x') \rangle_{\mathcal{V}}$. Taking the feature map to be $\phi(x) = \langle \cdot, x \rangle_{\mathcal{X}}$, we can prove the reproducing property to obtain $h(x, x') = \langle x, x' \rangle_{\mathcal{X}}$, which implies $\phi(x) = h(\cdot, x)$, and thus ϕ is the *canonical feature map* ([Steinwart and Christmann, 2008, Lemma 4.19](#)).

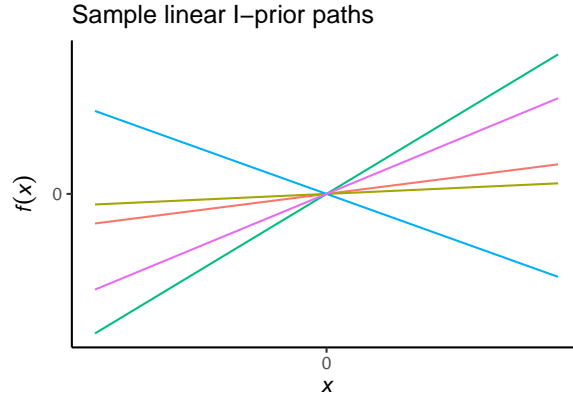


Figure 2.4: Sample I-prior paths from the canonical RKHS.

The origin of a Hilbert space may be arbitrary, in which case a centring may be appropriate. We define the centred canonical RKHS as follows.

Definition 2.26 (Centred canonical RKHS). Let \mathcal{X} be a Hilbert space, P be a probability measure over \mathcal{X} , and $\mu \in \mathcal{X}$ be the mean of a random element $X \in \mathcal{X}$. Define $(\mathcal{X} - \mu)'$, the continuous dual space of $\mathcal{X} - \mu$, to be the *centred canonical RKHS*. $(\mathcal{X} - \mu)'$ consists of the centred linear functions $f_{\beta}(x) = \langle x - \mu, \beta \rangle_{\mathcal{X}}$, for $\beta \in \mathcal{X}$, such that $E[f_{\beta}(X)] = 0$. The reproducing kernel of $(\mathcal{X} - \mu)'$ is

$$h(x, x') = \langle x - \mu, x' - \mu \rangle_{\mathcal{X}}.$$

That the centred canonical RKHS consists of zero-mean functions, $E[f_{\beta}(X)] = 0$, consider the following argument:

$$\begin{aligned} E[f_{\beta}(X)] &= E\langle X - \mu, \beta \rangle_{\mathcal{X}} \\ &= E\langle X, \beta \rangle_{\mathcal{X}} - \langle \mu, \beta \rangle_{\mathcal{X}}, \end{aligned}$$

and since $E\langle X, \beta \rangle_{\mathcal{X}} = \langle \mu, \beta \rangle_{\mathcal{X}}$ for any $\beta \in \mathcal{X}$, the results follows.

Remark 2.7. In practice, the probability measure P over \mathcal{X} is unknown, so we find it useful to use the empirical distribution over \mathcal{X} instead, so that \mathcal{X} is centred by the sample mean $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i$.

2.4.3 The fractional Brownian motion RKHS

Brownian motion, which also goes by the name Wiener process, has been an inquisitive subject in the mathematical sciences, and here, we describe a function space motivated by a generalised version of Brownian motion paths.

Suppose $B_\gamma(t)$ is a continuous-time Gaussian process on $[0, T]$, i.e. for any finite set of indices t_1, \dots, t_k , where each $t_j \in [0, T]$, $(B_\gamma(t_1), \dots, B_\gamma(t_k))$ is a multivariate normal random variable. $B_\gamma(t)$ is said to be a *fractional Brownian motion* (fBm) if $E[B_\gamma(t)] = 0$ for all $t \in [0, T]$ and

$$\text{Cov}(B_\gamma(t), B_\gamma(s)) = \frac{1}{2}(|t|^{2\gamma} + |s|^{2\gamma} - |t - s|^{2\gamma}) \quad \forall t, s \in [0, T],$$

where $\gamma \in (0, 1)$ is called the *Hurst index*, *Hurst parameter* or even *Hurst coefficient*. Introduced by Mandelbrot and Ness (1968), fBms are a generalisation of Brownian motion. The Hurst parameter plays two roles: 1) it describes the raggedness of the resultant motion, with higher values leading to smoother motion; and 2) it determines the type of process the fBm is, as past increments of $B_\gamma(t)$ are weighted by $(t - s)^{\gamma-1/2}$. When $\gamma = 1/2$ exactly, the fBm is a standard Brownian motion and its increments are independent; when $\gamma > 1/2$ (resp. $\gamma < 1/2$) its increments are positively (resp. negatively) correlated.

Now, let \mathcal{X} be a Hilbert space. Schoenberg (1937, Thm. 3) has shown that, for $0 < \gamma \leq 1$, there exists a Hilbert space \mathcal{V} and a function $\phi_\gamma : \mathcal{X} \rightarrow \mathcal{V}$ such that $\forall x, x' \in \mathcal{X}$,

$$\|\phi_\gamma(x) - \phi_\gamma(x')\|_{\mathcal{V}} = \|x - x'\|_{\mathcal{X}}^\gamma.$$

Using the polarisation identity, we find that the kernel of the RKHS with feature space \mathcal{V} and feature map ϕ_γ defines a kernel function $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ identical to the fBm covariance kernel.

Definition 2.27 (Fractional Brownian motion RKHS). The fractional Brownian motion (fBm) RKHS \mathcal{F} is the space of functions on the Hilbert space \mathcal{X} possessing the reproducing kernel $h_\gamma : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ defined by

$$h_\gamma(x, x') = \langle \phi_\gamma(x), \phi_\gamma(x') \rangle_{\mathcal{V}} = \frac{1}{2}(\|x\|_{\mathcal{X}}^{2\gamma} + \|x'\|_{\mathcal{X}}^{2\gamma} - \|x - x'\|_{\mathcal{X}}^{2\gamma}),$$

which depends on the Hurst coefficient $\gamma \in (0, 1)$. We shall reference this space as the fBm- γ RKHS.

Remark 2.8. When $\gamma = 1$, by the polarisation identity we get $h(x, x') = \langle x, x' \rangle_{\mathcal{X}}$, which is the (reproducing) kernel of the canonical RKHS.

From its construction, it is clear that the fBm kernel is positive definite, and thus defines an RKHS. That the fBm RKHS describes a space of functions is proved in [Cohen \(2002\)](#), who studied this space in depth. It is also noted in the collection of examples of [Berlinet and Thomas-Agnan \(2004, Sec 3.3, E.g. 3, p. 71 & Sec 7.4, E.g. 20, p. 319\)](#).

The Hurst coefficient γ controls the “smoothness” of the functions in the RKHS. We can talk about smoothness in the context of Hölder continuity of functions.

Definition 2.28 (Hölder condition). A function f over a set $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is said to be *Hölder continuous* of order $0 < a \leq 1$ if there exists an $M > 0$ such that $\forall x, x' \in \mathcal{X}$,

$$|f(x) - f(x')| \leq M \|x - x'\|^a.$$

Functions in the Hölder space $C^{k,a}(\mathcal{X})$, where $k \geq 0$ is an integer, consists of those functions over \mathcal{X} having continuous derivatives up to order k and such that the k 'th partial derivatives are Hölder continuous of order a . Unlike realisations of actual fBm paths with Hurst index γ , which are well-known to be almost surely Hölder continuous of order less than γ ([Embrechts and Maejima, 2002, Thm. 4.1.1](#)), functions in its namesake RKHS are strictly smoother.

Proposition 2.10 (Hölder smoothness of fBm- γ RKHS functions). *The fBm- γ RKHS \mathcal{F} of functions over $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ are Hölder continuous of order γ .*

Proof. For some $f \in \mathcal{F}$ we have $f(x) = \langle f, h(\cdot, x) \rangle_{\mathcal{F}}$ by the reproducing property of the kernel h of \mathcal{F} . It follows from the Cauchy-Schwarz inequality that for any $x, x' \in \mathcal{X}$,

$$\begin{aligned} |f(x) - f(x')| &= |\langle f, h_{\gamma}(\cdot, x) - h_{\gamma}(\cdot, x') \rangle_{\mathcal{F}}| \\ &\leq \|f\|_{\mathcal{F}} \|h_{\gamma}(\cdot, x) - h_{\gamma}(\cdot, x')\|_{\mathcal{F}} \\ &= \|f\|_{\mathcal{F}} \|x - x'\|_{\mathcal{X}}^{\gamma}, \end{aligned}$$

since

$$\begin{aligned} \|h_{\gamma}(\cdot, x) - h_{\gamma}(\cdot, x')\|_{\mathcal{F}}^2 &= \|h_{\gamma}(\cdot, x)\|_{\mathcal{F}}^2 + \|h_{\gamma}(\cdot, x')\|_{\mathcal{F}}^2 - 2\langle h_{\gamma}(\cdot, x), h_{\gamma}(\cdot, x') \rangle_{\mathcal{F}} \\ &= h_{\gamma}(x, x) + h_{\gamma}(x', x') - 2h_{\gamma}(x, x') \\ &= \|x\|_{\mathcal{X}}^{2\gamma} + \|x'\|_{\mathcal{X}}^{2\gamma} - \left(\|x\|_{\mathcal{X}}^{2\gamma} + \|x'\|_{\mathcal{X}}^{2\gamma} - \|x - x'\|_{\mathcal{X}}^{2\gamma} \right) \\ &= \|x - x'\|_{\mathcal{X}}^{2\gamma}, \end{aligned}$$

and thus proving the proposition. ■

The span of the kernels is dense in \mathcal{F} (see paragraph at the end of [Theorem 2.6](#)), and it is interesting to note that the basis functions $h(\cdot, x)$ are smoother still.

Proposition 2.11 (Hölder smoothness of fBm- γ basis functions). *For $0 < \gamma \leq 1/2$ and $z \in \mathcal{X}$, the function $h_\gamma(\cdot, z) : \mathcal{X} \rightarrow \mathbb{R}$ is Hölder continuous of order 2γ .*

Proof. Following the triangle inequalities [\(2.2\)](#) and [\(2.3\)](#), for any $x, x' \in \mathcal{X}$, we have that

$$\begin{aligned}
|h_\gamma(z, x) - h_\gamma(z, x')| &= \frac{1}{2} \left| \|z\|_{\mathcal{X}}^{2\gamma} + \|x\|_{\mathcal{X}}^{2\gamma} - \|z - x\|_{\mathcal{X}}^{2\gamma} - \|z\|_{\mathcal{X}}^{2\gamma} - \|x'\|_{\mathcal{X}}^{2\gamma} + \|z - x'\|_{\mathcal{X}}^{2\gamma} \right| \\
&\leq \frac{1}{2} \left| \|z - x\|_{\mathcal{X}}^{2\gamma} - \|z - x'\|_{\mathcal{X}}^{2\gamma} \right| + \frac{1}{2} \left| \|x\|_{\mathcal{X}}^{2\gamma} - \|x'\|_{\mathcal{X}}^{2\gamma} \right| \\
&\leq \frac{1}{2} \left| \|z\|_{\mathcal{X}}^{2\gamma} + \|x\|_{\mathcal{X}}^{2\gamma} - \|z\|_{\mathcal{X}}^{2\gamma} - \|x'\|_{\mathcal{X}}^{2\gamma} \right| + \frac{1}{2} \left| \|x\|_{\mathcal{X}}^{2\gamma} - \|x'\|_{\mathcal{X}}^{2\gamma} \right| \\
&\leq \left| \|x\|_{\mathcal{X}}^{2\gamma} - \|x'\|_{\mathcal{X}}^{2\gamma} \right| \\
&\leq \|x - x'\|_{\mathcal{X}}^{2\gamma}.
\end{aligned}$$

Remark 2.9. The above proposition can also be proven for $1/2 < \gamma < 1$, for which Hölder smoothness has to be defined for orders $1 < a \leq 2$. See ([Bergsma, 2018, Lemma 9](#)) for details.

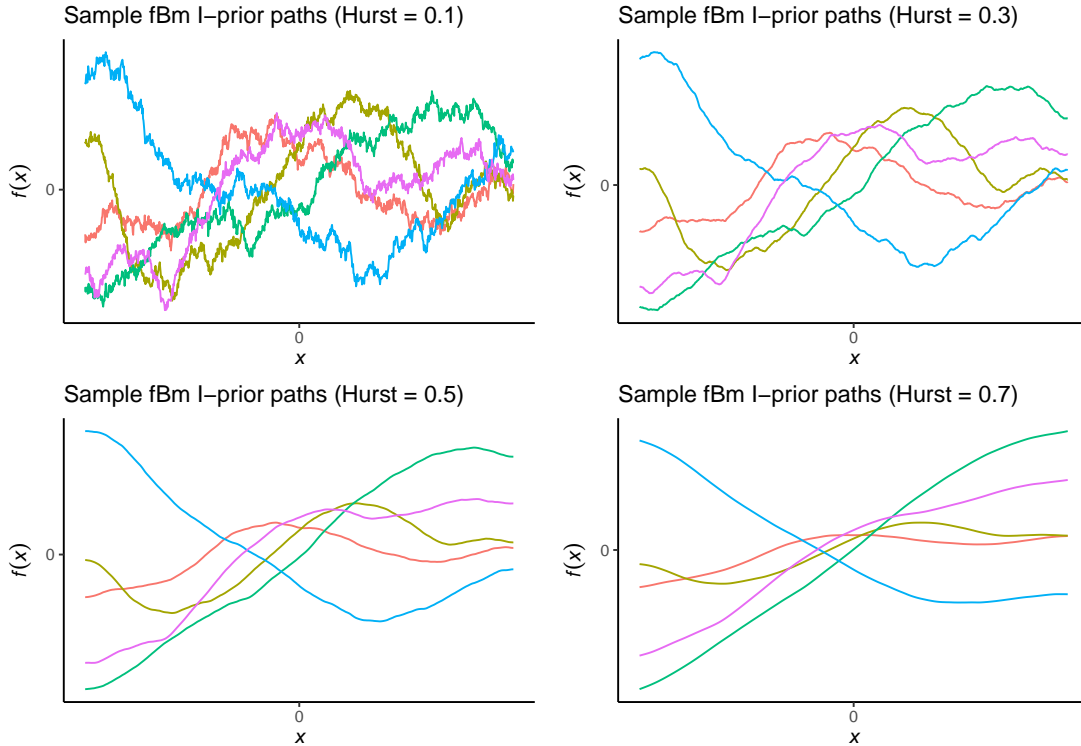


Figure 2.5: Sample I-prior paths from the fBm RKHS with varying Hurst coefficients. Note that the fBm- γ RKHS contains functions that are rougher than these I-prior paths as a consequence of [Proposition 2.11](#) and the fact that I-prior realisations are finite combinations of the basis functions $h_\eta(\cdot, x)$ (c.f. [Equation 3.7](#), p. 99).

An undesirable property of fBm- γ RKHS being spanned by the functions $h_\gamma(\cdot, x)$ is that $f(0) = 0$ for all $f \in \mathcal{F}$. We define the centred fBm RKHS as follows.

Definition 2.29 (Centred fBm RKHS). Let \mathcal{X} be a Hilbert space, P be a probability measure over \mathcal{X} , and $\mu \in \mathcal{X}$ be the mean with respect to this probability measure. The kernel $\bar{h}_\gamma : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ defined by

$$\bar{h}_\gamma(x, x') = \frac{1}{2} \mathbb{E} \left[\|x - X\|_{\mathcal{X}}^{2\gamma} + \|x' - X'\|_{\mathcal{X}}^{2\gamma} - \|x - x'\|_{\mathcal{X}}^{2\gamma} - \|X - X'\|_{\mathcal{X}}^{2\gamma} \right]$$

is the reproducing kernel of the *centred* fBm- γ RKHS, which consists of functions f in the fBm- γ RKHS such that $\mathbb{E}[f(X)] = 0$. In the above definition, $X, X' \sim P$ are two independent copies of a random vector $X \in \mathcal{X}$.

Remark 2.10. Again, when $\gamma = 1$, we get the reduction

$$\begin{aligned} \bar{h}_{\gamma=1}(x, x') &= \frac{1}{2} \mathbb{E} [\|x - X\|_{\mathcal{X}}^2 + \|x' - X'\|_{\mathcal{X}}^2 - \|x - x'\|_{\mathcal{X}}^2 - \|X - X'\|_{\mathcal{X}}^2] \\ &= \frac{1}{2} \mathbb{E} [\langle X, X \rangle_{\mathcal{X}} + \langle X', X' \rangle_{\mathcal{X}} + 2\langle x, x' \rangle_{\mathcal{X}} - 2\langle x, X \rangle_{\mathcal{X}} - 2\langle x', X' \rangle_{\mathcal{X}}] \\ &= \langle \mu, \mu \rangle_{\mathcal{X}} + \langle x, x' \rangle_{\mathcal{X}} - \langle x, \mu \rangle_{\mathcal{X}} - \langle \mu, x' \rangle_{\mathcal{X}} \\ &= \langle x - \mu, x' - \mu \rangle_{\mathcal{X}}, \end{aligned}$$

which is the (reproducing) kernel of the centred canonical RKHS.

Remark 2.11. For posterity, a general centring of any (positive-definite) kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ can be achieved via

$$\bar{h}(x, x') = h(x, x') - \mathbb{E}[h(x, X')] - \mathbb{E}[h(X, x')] + \mathbb{E}[h(X, X')],$$

where expectations are taken for the random elements $X, X' \stackrel{\text{iid}}{\sim} P$, a probability measure over \mathcal{X} . This centred kernel gives rise to the centred RKHS $\bar{\mathcal{F}}$ of centred functions $\mathbb{E}[f(X)] = 0$, $f \in \bar{\mathcal{F}}$. As per [Remark 2.7](#), the empirical distribution of P can be used to approximate the unknown, true P .

2.4.4 The squared exponential RKHS

The squared exponential (SE) kernel function is indeed known to be the default kernel used for Gaussian process regression in machine learning. It is a positive definite function, and hence defines an RKHS. The definition of the SE RKHS is as follows.

Definition 2.30 (Squared exponential RKHS). The squared exponential (SE) RKHS \mathcal{F} of functions over some set $\mathcal{X} \subseteq \mathbb{R}^p$ equipped with the 2-norm $\|\cdot\|_2$ is defined by the

positive definite kernel $h_l : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$

$$h_l(x, x') = \exp \left(-\frac{\|x - x'\|_2^2}{2l^2} \right).$$

The real-valued parameter $l > 0$ is called the *lengthscale* parameter, and is a smoothing parameter for the functions in the RKHS.

It is known by many other names, including the Gaussian kernel, due to its semblance to the kernel of the Gaussian pdf. Especially in the machine learning literature, the term Gaussian radial basis functions (RBF) is used, and commonly the simpler parameterisation $\gamma = (2l^2)^{-1}$ is utilised. [Duvenaud \(2014\)](#) remarks that “exponentiated quadratic” is a more aptly descriptive name for this kernel.

Despite being used extensively for learning algorithms using kernels, an explicit study of the RKHS defined by the SE kernel was not done until recently by [Steinwart et al. \(2006\)](#). In that work, the authors describe the nature of real-valued functions in the SE RKHS by considering a real restriction on the SE RKHS of functions over complex values. Their derivation of an orthonormal basis of such an RKHS proved the SE kernel to be the reproducing kernel for the SE RKHS.

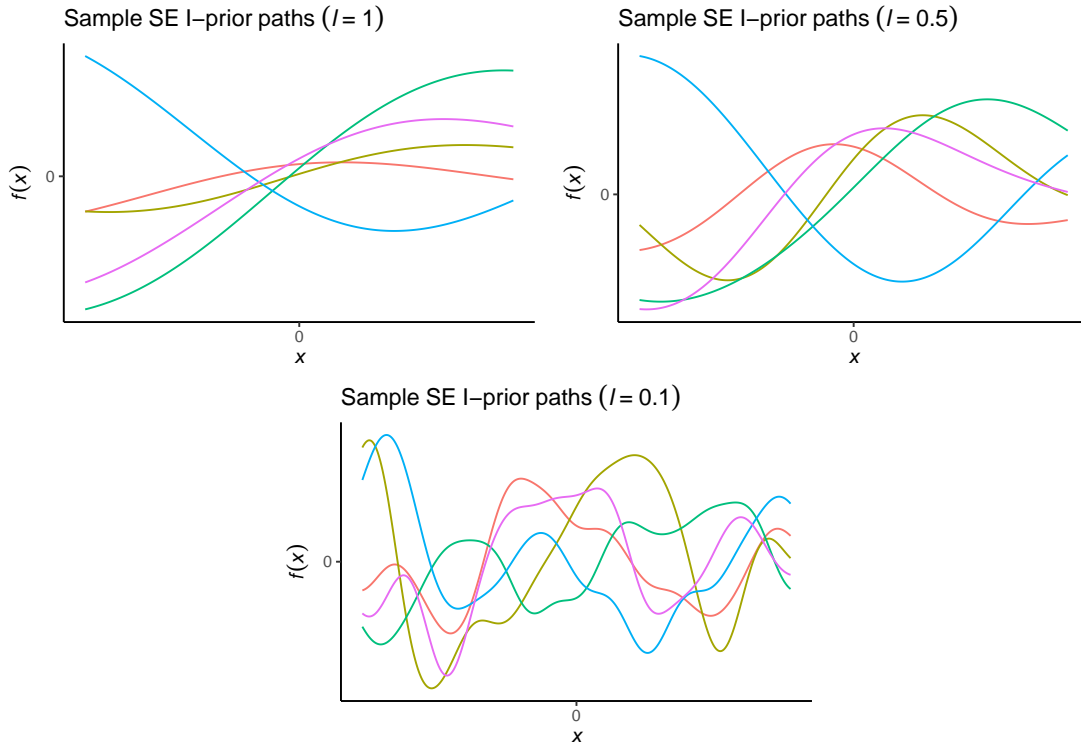


Figure 2.6: Sample paths from the SE RKHS with varying values for the lengthscale.

SE kernels are known to be “universal”. That is, it satisfies the following definition of universal kernels is due to [Micchelli et al. \(2006\)](#).

Definition 2.31 (Universal kernel). Let $C(\mathcal{X})$ be the space of all continuous, complex-valued functions $f : \mathcal{X} \rightarrow \mathbb{C}$ equipped with the maximum norm $\|\cdot\|_\infty$, and denote $\mathcal{K}(\mathcal{X})$ as the space of *kernel sections* $\overline{\text{span}}\{h(\cdot, x) | x \in \mathcal{X}\}$, where here, h is a complex-valued kernel function. A kernel h is said to be *universal* if given any compact subset $\mathcal{Z} \subset \mathcal{X}$, any positive number ϵ and any function $f \in C(\mathcal{Z})$, there is a function $g \in \mathcal{K}(\mathcal{Z})$ such that $\|f - g\|_{\mathcal{Z}} \leq \epsilon$.

The consequence of universality vis-à-vis regression modelling is that any (continuous) regression function f may be approximated very well by a function \hat{f} belonging to the SE RKHS, and these two functions can get arbitrarily close to each other in the maximum norm sense. This, together with the convenient computational advantages that the SE kernel brings (Raykar and Duraiswami, 2007), is a testament to the popularity of SE kernels, especially in machine learning methods.

In a similar manner to the two previous subsections, we may also derive the *centred* SE RKHS.

Definition 2.32 (Centred SE RKHS). Let $\mathcal{X} \subseteq \mathbb{R}^p$ be equipped with the 2-norm $\|\cdot\|_2$, and let P denote the distribution over \mathcal{X} . Assuming integrability of $h(x, X)$, for any $x \in \mathcal{X}$ and a random element $X \in \mathcal{X}$, the *centred* squared exponential (SE) RKHS (with lengthscale l) of functions over \mathcal{X} is defined by the positive definite kernel $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$

$$h(x, x') = e^{-\frac{\|x-x'\|_2^2}{2l^2}} - \mathbb{E} e^{-\frac{\|x-X'\|_2^2}{2l^2}} - \mathbb{E} e^{-\frac{\|X-x'\|_2^2}{2l^2}} + \mathbb{E} e^{-\frac{\|X-X'\|_2^2}{2l^2}},$$

where $X, X' \sim P$ are two independent random elements of \mathcal{X} . This ensures that $\mathbb{E}[f(X)] = 0$ for any f in this RKHS.

2.4.5 The Pearson RKHS

In all of the previous RKHSs of functions, the domain \mathcal{X} was taken to be some Euclidean space. The Pearson RKHS is a space of functions whose domain \mathcal{X} is a finite set. Let P be a probability measure over the finite set \mathcal{X} . The Pearson RKHS is defined as follows.

Definition 2.33 (Pearson RKHS). The *Pearson RKHS* is the RKHS of functions over a finite set \mathcal{X} defined by the reproducing kernel

$$h(x, x') = \frac{\delta_{xx'}}{P(X = x)} - 1,$$

where $X \sim P$ and δ is the Kronecker delta.

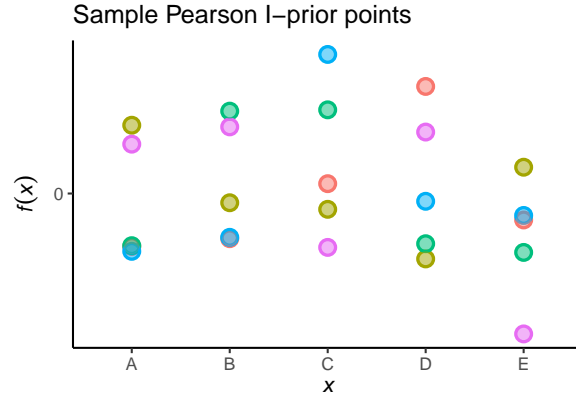


Figure 2.7: Sample I-prior “paths” from the Pearson RKHS. These are represented as points over a finite set. Similarly coloured points are from the same “path”, and since they are zero-mean functions, they sum to zero.

The Pearson RKHS contains functions which are centred, and has the desirable property that the contribution of $f(x)^2$ (the square of $f(x)$) to the squared norm of f is proportional to $P(X = x)$.

Proposition 2.12 (Mean and variance of functions in a Pearson RKHS). *Let \mathcal{F} be the Pearson RKHS of functions over a finite set \mathcal{X} . Then,*

$$\mathcal{F} = \{f : \mathcal{X} \rightarrow \mathbb{R} \mid E[f(X)] = 0\}$$

with

$$\|f\|_{\mathcal{F}}^2 = \text{Var}[f(X)] = \sum_{x \in \mathcal{X}} f(x)^2 P(X = x), \quad \forall f \in \mathcal{F}.$$

Proof. Write $p_x = P(X = x)$. The set of functions $\{h(\cdot, x) \mid x \in \mathcal{X}\}$ form a basis for \mathcal{F} , and thus each $f \in \mathcal{F}$ can be written as $f(x) = \sum_{x' \in \mathcal{X}} w_{x'} h(x, x')$ for some scalars $w_i \in \mathbb{R}$, $i \in \mathcal{X}$. But $E[h(X, x')] = E[\delta_{Xx'}]/p_{x'} - 1 = p_{x'}/p_{x'} - 1 = 0$, and thus $E[f(X)] = 0$. Conversely, suppose $f : \mathcal{X} \rightarrow \mathbb{R}$ is such that $E[f(X)] = 0$. Taking $w_x = f(x)p_x$, we see that

$$\begin{aligned} \sum_{x' \in \mathcal{X}} w_{x'} h(x, x') &= \frac{w_x}{p_x} - \sum_{x' \in \mathcal{X}} w_{x'} \\ &= \frac{f(x)p_x}{p_x} - \sum_{x' \in \mathcal{X}} f(x')p_{x'} \xrightarrow{E[f(X)] = 0} \\ &= f(x) \end{aligned}$$

and thus $h(\cdot, x)$ spans \mathcal{F} so $f \in \mathcal{F}$.

The second part is proved as follows. Noting that with the choice $w_x = p_x f(x)$ and due to the reproducing property of h for the RKHS \mathcal{F} , the squared norm is

$$\begin{aligned}
\|f\|_{\mathcal{F}}^2 &= \langle f, f \rangle_{\mathcal{F}} = \left\langle \sum_{x \in \mathcal{X}} w_x h(\cdot, x), \sum_{x' \in \mathcal{X}} w_{x'} h(\cdot, x') \right\rangle_{\mathcal{F}} \\
&= \sum_{x \in \mathcal{X}} \sum_{x' \in \mathcal{X}} w_x w_{x'} \langle h(\cdot, x), h(\cdot, x') \rangle_{\mathcal{F}} \\
&= \sum_{x \in \mathcal{X}} \sum_{x' \in \mathcal{X}} w_x w_{x'} h(x, x') \\
&= \sum_{x \in \mathcal{X}} f(x) w_x \\
&= \sum_{x \in \mathcal{X}} f(x)^2 P(X = x),
\end{aligned}$$

and this is equal to the variance of $f(X)$. ■

2.5 Constructing RKKSs from existing RKHSs

The previous section outlined all of the basic RKHSs of functions that will form the building blocks when constructing more complex function spaces. We will see, at the outset, that sums of kernels are kernels and products of kernels are also kernels. This provides us a platform for constructing new function spaces from existing ones. To be more flexible in the specification of these new function spaces, we do not restrict ourselves to positive-definite kernels only, thereby necessitating us to use the theory of RKKSs.

2.5.1 Sums, products and scaling of RKHS

Sums of positive definite kernels are also positive definite kernels, and the product of positive definite kernel is a positive definite kernel. They each, in turn, are associated with an RKHS that is defined by the sum of kernels and product of kernels, respectively. The two lemmas below formalise these two facts.

Lemma 2.13 (Sum of kernels). *If h_1 and h_2 are positive-definite kernels on \mathcal{X}_1 and \mathcal{X}_2 respectively, then $h = h_1 + h_2$ is a positive-definite kernel on $\mathcal{X}_1 \times \mathcal{X}_2$. Moreover, denote \mathcal{F}_1 and \mathcal{F}_2 the RKHS defined by h_1 and h_2 respectively. Then $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ is an RKHS defined by $h = h_1 + h_2$, where*

$$\mathcal{F}_1 \oplus \mathcal{F}_2 = \{f : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbb{R} \mid f = f_1 + f_2, f_1 \in \mathcal{F}_1 \text{ and } f_2 \in \mathcal{F}_2\}.$$

For all $f \in \mathcal{F}$,

$$\|f\|_{\mathcal{F}}^2 = \min_{f_1 + f_2 = f} \{\|f_1\|_{\mathcal{F}_1}^2 + \|f_2\|_{\mathcal{F}_2}^2\}.$$

Proof. That $h_1 + h_2$ is a positive-definite kernel should be obvious, as the sum of two positive definite functions is also positive definite. For a proof of the remaining statements, see [Berlinet and Thomas-Agnan \(2004, Thm. 5\)](#). ■

Lemma 2.14 (Products of kernels). *Let \mathcal{F}_1 and \mathcal{F}_2 be two RKHSs of functions over \mathcal{X}_1 and \mathcal{X}_2 , with respective reproducing kernels h_1 and h_2 . Then, the tensor product space $\mathcal{F}_1 \otimes \mathcal{F}_2$ is an RKHS with reproducing kernel $h = h_1 h_2$ on $\mathcal{X}_1 \times \mathcal{X}_2$.*

Proof. Fix $n \in \mathbb{N}$, and let \mathbf{H}_1 and \mathbf{H}_2 be the kernel matrices for h_1 and h_2 respectively. Since these kernel matrices are symmetric and positive definite by virtue of h_1 and h_2 being symmetric and positive-definite functions, we can write $\mathbf{H}_1 = \mathbf{A}^\top \mathbf{A}$ and $\mathbf{H}_2 = \mathbf{B}^\top \mathbf{B}$ for some matrices \mathbf{A} and \mathbf{B} : perform an (orthogonal) eigendecomposition of each of the kernel matrices, and take square roots of the eigenvalues. Let \mathbf{H} be the kernel matrix for $h = h_1 h_2$. With $x_i = (x_{i1}, x_{i2})$, its (i, j) entries are

$$\begin{aligned} h(x_i, x_j) &= h_1(x_{i1}, x_{j1}) h_2(x_{i2}, x_{j2}) \\ &= (\mathbf{A}^\top \mathbf{A})_{ij} (\mathbf{B}^\top \mathbf{B})_{ij} \\ &= \sum_{k=1}^n a_{ik} a_{jk} \sum_{l=1}^n b_{il} b_{jl}, \end{aligned}$$

where we have denoted a_{ij} and b_{ij} to be the (i, j) 'th entries of \mathbf{A} and \mathbf{B} respectively. Then,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n h(x_i, x_j) &= \sum_{k=1}^n \sum_{l=1}^n \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j a_{ik} a_{jk} b_{il} b_{jl} \\ &= \sum_{k=1}^n \sum_{l=1}^n \left(\sum_{i=1}^n \lambda_i a_{ik} b_{il} \right) \left(\sum_{j=1}^n \lambda_j a_{jk} b_{jl} \right) \\ &= \sum_{k=1}^n \sum_{l=1}^n \left(\sum_{i=1}^n \lambda_i a_{ik} b_{il} \right)^2 \\ &\geq 0 \end{aligned}$$

Again, for the remainder of the statement in the lemma, we refer to [Berlinet and Thomas-Agnan \(2004, Thm. 13\)](#). ■

A familiar fact from linear algebra is realised here from [Lemmas 2.13](#) and [2.14](#): 1) the addition of positive-(semi)definite matrices is a positive-(semi)definite matrix; and 2) the *Hadamard product*³ of two positive (semi-)definite matrices is a positive (semi-)definite matrix.

³The Hadamard product is an element-wise multiplication of two matrices \mathbf{A} and \mathbf{B} of identical dimensions, denoted $\mathbf{A} \circ \mathbf{B}$. That is, $(\mathbf{A} \circ \mathbf{B})_{ij} = \mathbf{A}_{ij} \mathbf{B}_{ij}$.

The scale of an RKHS of functions \mathcal{F} over a set \mathcal{X} with kernel h may be arbitrary. To resolve this issue, a scale parameter $\lambda \in \mathbb{R}$ for the kernel h may be introduced, which will typically need to be estimated from the data. If h is a positive definite-kernel on $\mathcal{X} \times \mathcal{X}$, and $\lambda \geq 0$ a scalar, then this yields a scaled RKHS $\mathcal{F}_\lambda = \{\lambda f \mid f \in \mathcal{F}\}$ with reproducing kernel λh , where \mathcal{F} is the RKHS defined by h .

Restricting λ to the positive reals is arbitrary and unnecessarily restrictive. Especially when considering sums and products of scaled RKHSs, having negative scale parameters also give additional flexibility. The resulting kernels from summation and/or multiplication with negative kernels may no longer be positive definite, and in such cases, they give rise to RKKSs instead.

Remark 2.12. Recall that an RKKS \mathcal{F} of functions over \mathcal{X} can be uniquely decomposed as the difference between two RKHSs \mathcal{F}_+ and \mathcal{F}_- , and its associated Hilbert space $\mathcal{F}_\mathcal{H}$ is the RKHS $\mathcal{F}_+ \oplus \mathcal{F}_-$. It is important to note that both \mathcal{F} and $\mathcal{F}_\mathcal{H}$ contain identical functions over \mathcal{X} , but different topologies. That is to say, functions that are close with respect to the norm of \mathcal{F} may not be close to each other in the norm of $\mathcal{F}_\mathcal{H}$.

2.5.2 The polynomial RKKS

A polynomial construction based on a particular RKHS building block is considered here. For example, using the canonical RKHS in the polynomial construction would allow us to easily add higher order effects of the covariates $x \in \mathcal{X}$. In particular, we only require a single scale parameter in polynomial kernel construction.

Definition 2.34 (Polynomial RKKS). Let \mathcal{X} be a Hilbert space. The kernel function $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ obtained through the d -degree polynomial construction of linear kernels is

$$h_\lambda(x, x') = (\lambda \langle x, x' \rangle_\mathcal{X} + c)^d,$$

where $\lambda \in \mathbb{R}$ is a scale parameter for the linear kernel, and $c \in \mathbb{R}$ is a real constant called the *offset*. This kernel defines the *polynomial RKKS* of degree d . Note that if $\lambda, c > 0$ then the kernel is positive definite and thus defines an RKHS.

Write

$$h_\lambda(x, x')_\mathcal{F} = \sum_{k=0}^d \frac{d!}{k!(d-k)!} c^{k-d} \lambda^k \langle x, x' \rangle_\mathcal{X}^k.$$

Evidently, as the name suggests, this is a polynomial involving the canonical kernel. In particular, each of the k -powered kernels (i.e. $\langle x, x' \rangle_\mathcal{X}^k$) defines an RKHS of their own (since these are merely products of kernels), and therefore the sum of these k -powered kernels define the polynomial RKKS.

The offset parameter influences trade-off between the higher-order versus lower-order terms in the polynomial. It is sometimes known as the bias term.

Proposition 2.15. *The polynomial RKKS \mathcal{F} of real functions over \mathcal{X} contains polynomial functions of the form $f(x) = \sum_{k=0}^d \beta_k x^k$.*

Proof. By construction, $\mathcal{F} = \mathcal{F}_\emptyset \oplus \bigoplus_{i=1}^d \bigotimes_{j=1}^i \mathcal{F}_j$, where each $\mathcal{F}_j, j \neq 0$ is the canonical RKHS, and \mathcal{F}_\emptyset is the RKHS of constant functions. Each $f \in \mathcal{F}$ can therefore be written as $f = \beta_0 + \sum_{i=1}^d \prod_{j=1}^i f_j$, and $f_j(x) = b_j x$ as they are functions from the canonical RKHS, where b_j is a constant. Therefore, $f(x) = \sum_{k=0}^d \beta_k x^k$. ■

Remark 2.13. We may opt to use other RKHSs as the building blocks of the polynomial RKKS. In particular, using the centred canonical kernel seems natural, so that each of the functions in the constituents of the direct sum of spaces is centred. However, the polynomial RKKS itself will not be centred.

2.5.3 The ANOVA RKKS

We find it useful to begin this subsection by spending some time to elaborate on the classical analysis of variance (ANOVA) decomposition, and the associated notions of main effects and interactions. This will go a long way in understanding the thinking behind constructing an ANOVA-like RKKS of functions.

The classical ANOVA decomposition

The standard one-way ANOVA is essentially a linear regression model which allows comparison of means from two or more samples. Given sets of observations $y_j = \{y_{1j}, \dots, y_{n_j j}\}$, $j = 1, \dots, m$, we consider the linear model $y_{ij} = \mu_j + \epsilon_{ij}$, where ϵ_{ij} are independent, univariate, normal random variables with a common variance. This covariate-less model is used to make inferences about the *treatment means* μ_j . Often, the model is written in the *overparameterised* form by substituting $\mu_j = \mu + \tau_j$. This gives a different, arguably better, interpretability to the model: the τ_j 's, referred to as the *treatment effects*, now represent the amount of deviation from the grand, *overall mean* μ . Estimating all τ_j 's and μ separately is not possible because there is one degree of freedom that needs to be addressed in the model: there are $p+1$ mean parameters to estimate but only information from p means. A common fix to this identification issue is to set one of the μ_j 's, say the first one μ_1 , to zero, or impose the restriction $\sum_{j=1}^m \mu_j = 0$. The former treats one of the m levels as the control, while the latter treats all treatment effects symmetrically.

Now write the ANOVA model slightly differently, as $y_i = f(x_i) + \epsilon_i$, where f is defined on the discrete domain $\mathcal{X} = \{1, \dots, m\}$, and i indexes all of the $n := \sum_{j=1}^m n_j$

observations. Here, f represents the group-level mean, returning μ_j for some $j \in \mathcal{X}$. In a similar manner, we can perform the ANOVA decomposition on f as

$$f = Af + (I - A)f = f_o + f_t,$$

where A is an averaging operator that “averages out” its argument x and returns a constant, and I is the identity operator. $f_o = Af$ is a constant function representing the *overall mean*, whereas $f_t = (I - A)f$ is a function representing the *treatment effects* τ_j . Here are two choices of A :

- $Af(x) = f(1) = \mu_1$. This implies $f(x) = f(1) + (f(x) - f(1))$. The overall mean μ is the group mean μ_1 , which corresponds to setting the restriction $\mu_1 = 0$.
- $Af(x) = \sum_{x=1}^m f(x)/m =: \bar{\alpha}$. This implies $f(x) = \bar{\alpha} + (f(x) - \bar{\alpha})$. The overall mean is $\mu = \sum_{j=1}^m \alpha_j/m$, which corresponds to the restriction $\sum_{j=1}^m \mu_j = 0$.

By definition, $AAf = A^2f = Af$, because averaging a constant returns that constant. We must have that $Af_t = A(I - A)f = Af - A^2f = 0$. The choice of A is arbitrary, as is the choice of restriction, so long as it satisfies the condition that $Af_t = 0$.

The multiway ANOVA can be motivated in a similar fashion. Let $x = (x_1, \dots, x_p) \in \prod_{k=1}^p \mathcal{X}_k$, and consider functions that map $\prod_{k=1}^p \mathcal{X}_k$ to \mathbb{R} . Let A_k be an averaging operator on \mathcal{X}_k that averages the k 'th component of x from the active argument list, i.e. $A_k f$ is constant on the \mathcal{X}_k axis but not necessarily an overall constant function. An ANOVA decomposition of f is

$$f = \left(\prod_{k=1}^p (A_k + I - A_k) \right) f = \sum_{\mathcal{K} \in \mathcal{P}_p} \left(\prod_{k \in \mathcal{K}} (I - A_k) \prod_{k \notin \mathcal{K}} A_k \right) f = \sum_{\mathcal{K} \in \mathcal{P}_p} f_{\mathcal{K}} \quad (2.4)$$

where we had denoted $\mathcal{P}_p = \mathcal{P}(\{1, \dots, p\})$ to be the power set of $\{1, \dots, p\}$ whose cardinality is 2^p . The summands $f_{\mathcal{K}}$ will compose of the overall effect, main effects, two-way interaction terms, and so on. Each of the terms will satisfy the condition $A_k f_{\mathcal{K}} = 0, \forall k \in \mathcal{K} \in \mathcal{P}_p \setminus \{\}$.

Example 2.1 (Two-way ANOVA decomposition). Let $p = 2$, $\mathcal{X}_1 = \{1, \dots, m_1\}$, and $\mathcal{X}_2 = \{1, \dots, m_2\}$. The power set \mathcal{P}_2 is $\{\{\}, \{1\}, \{2\}, \{1, 2\}\}$. The ANOVA decomposition of f (with indices derived trivially from the power set) is

$$f = f_{\emptyset} + f_1 + f_2 + f_{12}.$$

Here are two choices for the averaging operator A_k analogous to the previous illustration in the one-way ANOVA.

- Let $A_1 f(x) = f(1, x_2)$ and $A_2 f(x) = f(x_1, 1)$. Then,

$$\begin{aligned} f_{\emptyset}(x) &= A_1 A_2 f &&= f(1, 1) \\ f_1(x) &= (I - A_1) A_2 f &&= f(x_1, 1) - f(1, 1) \\ f_2(x) &= A_1 (I - A_2) f &&= f(1, x_2) - f(1, 1) \\ f_{12}(x) &= (I - A_1)(I - A_2) f &&= f(x_1, x_2) - f(x_1, 1) - f(1, x_2) + f(1, 1). \end{aligned}$$

- Let $A_k f(x) = \sum_{x_k=1}^{m_k} f(x_1, x_2) / m_k, k = 1, 2$. Then,

$$\begin{aligned} f_{\emptyset}(x) &= A_1 A_2 f &&= f.. \\ f_1(x) &= (I - A_1) A_2 f &&= f_{x_1.} - f.. \\ f_2(x) &= A_1 (I - A_2) f &&= f_{.x_2} - f.. \\ f_{12}(x) &= (I - A_1)(I - A_2) f &&= f - f_{x_1.} - f_{.x_2} + f.., \end{aligned}$$

where $f.. = \sum_{x_1, x_2} f(x_1, x_2) / m_1 m_2$, $f_{x_1.} = \sum_{x_2} f(x_1, x_2) / m_2$, and $f_{.x_1} = \sum_{x_1} f(x_1, x_2) / m_1$.

It is also easy to convince ourselves that $A_1 f_1 = A_2 f_2 = A_1 f_{12} = A_2 f_{12} = 0$ in either choice of the averaging operator A_k .

Functional ANOVA decomposition

Let us now extend the ANOVA decomposition idea to a general function $f : \mathcal{X} \rightarrow \mathbb{R}$ in some vector space \mathcal{F} . We shall jump straight into the multiway ANOVA analogue for functional decomposition, and to that end, consider $x = (x_1, \dots, x_p) \in \prod_{k=1}^p \mathcal{X}_k =: \mathcal{X}$ a measurable space, where each of the spaces \mathcal{X}_k has measure ν_k , and $\nu = \nu_1 \times \dots \times \nu_p$ is the product measure on \mathcal{X} . In the following, denote by \mathcal{F}_k the vector space of functions over the set $\mathcal{X}_k, k = 1, \dots, p$, and \mathcal{F}_{\emptyset} the vector space of constant functions.

As \mathcal{X} need not necessarily be a collection of finite sets, we need to figure out a suitable linear operator that performs an “averaging” of some sort. Consider the linear operator $A_k : \mathcal{F} \rightarrow \mathcal{F}_{-k}$, where \mathcal{F}_{-k} is a vector space of functions for which the k th component is constant over \mathcal{X} , defined by

$$A_k f(x) = \int_{\mathcal{X}_k} f(x_1, \dots, x_p) d\nu_k(x_k). \quad (2.5)$$

Thus, for the one-way ANOVA ($p = 1$), we get

$$f(x) = \overbrace{\int_{\mathcal{X}} f(x) d\nu(x)}^{f_{\emptyset}(x)} + \overbrace{\left(f - \int_{\mathcal{X}} f(x) d\nu(x) \right)}^{f_1(x)} \quad (2.6)$$

and for the two-way ANOVA ($p = 2$), we have $f = f_\emptyset + f_1 + f_2 + f_{12}$, with

$$\begin{aligned} f_\emptyset(x) &= \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} f(x_1, x_2) d\nu_1(x_1) d\nu_2(x_2) \\ f_1(x) &= \int_{\mathcal{X}_2} \left(f(x_1, x_2) - \int_{\mathcal{X}_1} f(x_1, x_2) d\nu_1(x_1) \right) d\nu_2(x_2) \\ f_2(x) &= \int_{\mathcal{X}_1} \left(f(x_1, x_2) - \int_{\mathcal{X}_2} f(x_1, x_2) d\nu_2(x_2) \right) d\nu_1(x_1) \\ f_{12}(x) &= f(x_1, x_2) - \int_{\mathcal{X}_1} f(x_1, x_2) d\nu_1(x_1) - \int_{\mathcal{X}_2} f(x_1, x_2) d\nu_2(x_2) \\ &\quad + \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} f(x_1, x_2) d\nu_1(x_1) d\nu_2(x_2). \end{aligned}$$

The averaging operator A_k defined in (2.5) generalises the concept of the previous subsection's averaging operator. We must then also have, as before, that $A_k f_K = 0, \forall k \in \mathcal{K} \in \mathcal{P}_p \setminus \{\emptyset\}$. For the one-way functional ANOVA decomposition in (2.6), it must be that f_1 is a zero-mean function. As for the two-way ANOVA, it is the case that $\int_{\mathcal{X}_k} f_K(x_1, x_2) d\nu_k(x_k) = 0, k = 1, 2$, and $\mathcal{K} \in \{\{1\}, \{2\}, \{1, 2\}\}$ (Durrande et al., 2013).

This is highly suggestive as to what the ANOVA decomposition of the space \mathcal{F} should look like in general. Starting with $p = 1$, any $f \in \mathcal{F}$ can be decomposed as a sum of a constant plus a zero-mean function, so we have that $\mathcal{F} = \mathcal{F}_\emptyset \oplus \bar{\mathcal{F}}_1$, where a bar over \mathcal{F}_k , $k = 1, \dots, p$ will be used to denote the vector space of zero-mean functions over \mathcal{X}_k . For $p \geq 2$ we can argue something similar. Take the vector space space \mathcal{F} of functions over $\prod_{k=1}^p \mathcal{X}_k$ to be the tensor product space $\mathcal{F} = \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_p$ whose elements are identified as being tensor product functions $f_1 \otimes \dots \otimes f_p$, where each $f_k : \mathcal{X}_k \rightarrow \mathbb{R}$ belongs to \mathcal{F}_k . This is constructed by repeatedly taking the completion of linear combinations of the tensor product $f_k \otimes f_j$, $k, j \in \{1, \dots, p\}$ as per Definition 2.14. Considered individually, each \mathcal{F}_k can then be decomposed as $\mathcal{F}_k = \mathcal{F}_{\emptyset k} \oplus \bar{\mathcal{F}}_k$, where $\mathcal{F}_{\emptyset k}$ is the space of functions constant along the k 'th axis. Expanding out under the distributivity rule of tensor products and rearranging slightly, we obtain

$$\begin{aligned} \mathcal{F} &= (\mathcal{F}_{\emptyset 1} \oplus \bar{\mathcal{F}}_1) \otimes \dots \otimes (\mathcal{F}_{\emptyset p} \oplus \bar{\mathcal{F}}_p) \\ &= \mathcal{F}_\emptyset \oplus \bigoplus_{j=1}^p \left(\bigotimes_{i \neq j} \mathcal{F}_{\emptyset i} \otimes \bar{\mathcal{F}}_j \right) \oplus \bigoplus_{\substack{j,k=1 \\ j < k}}^p \left(\bigotimes_{i \neq j,k} \mathcal{F}_{\emptyset i} \otimes \bar{\mathcal{F}}_j \otimes \bar{\mathcal{F}}_k \right) \\ &\quad \oplus \dots \oplus \left(\bar{\mathcal{F}}_1 \otimes \dots \otimes \bar{\mathcal{F}}_p \right), \end{aligned} \tag{2.7}$$

where ' \oplus ' and ' \otimes ' represent the summation and product operator for direct/tensor sums and products, respectively. To clarify,

- \mathcal{F}_\emptyset is the space of constant functions $f_\emptyset : \mathcal{X}_1 \times \dots \times \mathcal{X}_p \rightarrow \mathbb{R}$;

- $\left(\bigotimes_{i \neq j} \mathcal{F}_{\emptyset i} \otimes \bar{\mathcal{F}}_j\right)$ is the space of functions that are constant on all coordinates except the j 'th coordinate of x , and the functions are centred on the j 'th coordinate;
- $\left(\bigotimes_{i \neq j, k} \mathcal{F}_{\emptyset i} \otimes \bar{\mathcal{F}}_j \otimes \bar{\mathcal{F}}_k\right)$ is the space of functions that are constant on all coordinates except the j th and k th coordinate of x , and the functions are centred on these two coordinates;
- $\bar{\mathcal{F}}_1 \otimes \cdots \otimes \bar{\mathcal{F}}_p$ is the space of zero-mean functions $f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_p \rightarrow \mathbb{R}$;

and so on for the rest of the spaces in the summand, of which there are 2^p members all together. Therefore, given an arbitrary function $f \in \mathcal{F}$, the projection of f onto the above respective spaces in (2.7) leads to the *functional ANOVA representation*

$$f(x) = \alpha + \sum_{j=1}^p f_j(x_j) + \sum_{\substack{j,k=1 \\ j < k}}^p f_{jk}(x_j, x_k) + \cdots + f_{1\dots p}(x), \quad (2.8)$$

where α is the grand intercept (a constant).

Definition 2.35 (Functional ANOVA representation). Let $\mathcal{P}_p = \mathcal{P}(\{1, \dots, p\})$, the power set of $\{1, \dots, p\}$. For any function $f : \mathcal{X}_1 \times \cdots \times \mathcal{X}_p \rightarrow \mathbb{R}$, the formula for f in (2.8) is known as the *functional ANOVA representation* of f if $\forall k \in \mathcal{K} \in \mathcal{P}_p \setminus \{\}$,

$$A_k f_{\mathcal{K}}(x) = \int_{\mathcal{X}_k} f_{\mathcal{K}}(x) d\nu_k(x_k) = 0. \quad (2.9)$$

In other words, the integral of $f_{\mathcal{K}}$ with respect to any of the variables indexed by the elements in \mathcal{K} , is zero. Consequentially, each of the functional ANOVA components are centred with respect to each axis \mathcal{X}_j , $j = 1, \dots, p$.

For the constant term, main effects, and two-way interaction terms, the familiar classical expressions are obtained:

$$\begin{aligned} f_{\emptyset} &= \int f d\nu; \\ f_j &= \int f \prod_{i \neq j} d\nu_i - f_{\emptyset}; \\ f_{jk} &= \int f \prod_{i \neq j, k} d\nu_i - f_j - f_k - f_{\emptyset}. \end{aligned}$$

The ANOVA kernel

At last, we come to the section of deriving the ANOVA RKKS, and, rest assured, the preceding long build-up will prove to not be in vain. The main idea is to construct an RKKS such that the functions that lie in them will have the ANOVA representation in (2.8). The bulk of the work has been done, and in fact we know exactly how this ANOVA

RKKS should be structured—it is the space as specified in (2.7). The ANOVA RKKS will be constructed by a similar manipulation of the individual kernels representing the RKHS building blocks.

Definition 2.36 (ANOVA RKKS). For $k = 1, \dots, p$, let \mathcal{F}_k be centred RKHSs of functions over the set \mathcal{X}_k with kernel $h_k : \mathcal{X}_k \times \mathcal{X}_k \rightarrow \mathbb{R}$. Let $\lambda_k, k = 1, \dots, p$ be real-valued scale parameters. The ANOVA RKKS of functions $f : \mathcal{X}_1 \times \dots \times \mathcal{X}_p \rightarrow \mathbb{R}$ is specified by the ANOVA kernel, defined by

$$h_\lambda(x, x') = \prod_{k=1}^p (1 + \lambda_k h_k(x_k, x'_k)). \quad (2.10)$$

It is interesting to note that an ANOVA RKKS is constructed very simply through multiplication of univariate kernels. Expanding out equations (2.10), we see that it is in fact a sum of products of kernels with increasing orders of interaction:

$$h_\lambda(x, x') = 1 + \sum_{j=1}^p \lambda_j h_j(x_j, x'_j) + \sum_{\substack{j,k=1 \\ j < k}}^p \lambda_j \lambda_k h_j(x_j, x'_j) h_k(x_k, x'_k) + \dots + \prod_{j=1}^p \lambda_j h_j(x_j, x'_j).$$

It is now clear from this expansion that the ANOVA RKKS yields functions that resemble those with the ANOVA representation in (2.8): the mean value of the function stems from the ‘1’, i.e. it lies in an RKHS of constant functions; the main effects are represented by the sum of the individual kernels; the two-way interaction terms are represented by the second-order kernel interactions; and so on.

Example 2.2 (ANOVA RKKS construction). Consider two RKHSs \mathcal{F}_k with kernel h_k , $k = 1, 2$. The ANOVA kernel defining the ANOVA RKKS \mathcal{F} is

$$h_\lambda((x_1, x_2), (x'_1, x'_2)) = 1 + \lambda_1 h_1(x_1, x'_1) + \lambda_2 h_2(x_2, x'_2) + \lambda_1 \lambda_2 h_1(x_1, x'_1) h_2(x_2, x'_2).$$

Suppose that \mathcal{F}_1 and \mathcal{F}_2 are the centred canonical RKHS of functions over \mathbb{R} . Then, functions in $\mathcal{F} = \mathcal{F}_\emptyset \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus (\mathcal{F}_1 \otimes \mathcal{F}_2)$ are of the form

$$f(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2.$$

As a remark, not all of the components of the ANOVA RKKS need to be included in the construction. The selective exclusion of certain interactions characterises many interesting statistical models. Excluding certain terms of the ANOVA RKKS is equivalent to setting the scale parameter for those relevant components to be zero, i.e. they play no role in the decomposition of the function. With this in mind, the ANOVA RKKS then gives us an objective way of model-building, from linear regression, to multilevel models, longitudinal models, and so on.

2.6 Summary

The review of functional analysis allows us to describe the theory of RKHSs and RKKSs, which are of interest to us because the topology endowed on such spaces gives appreciable assurances—in particular, all evaluation functionals are continuous in these spaces. Moreover, RKHSs and RKKSs can be specified completely through kernel functions, with new and complex function spaces built simply by manipulation of these kernel functions. Of particular importance is the ANOVA functional decomposition, for which we realise provides an objective way of constructing various function spaces for regression and modelling. Such models will be described later on in detail in [Chapter 4](#).

An annotated collection of bibliographical references used for this chapter is as follows.

- **Functional analysis.** On the introductory material relating to functional analysis in [Section 2.1](#), the lecture notes by [Sejdinovic and Gretton \(2012\)](#) is recommended, and forms the basis for most of our material. Additionally, [Kokoszka and Reimherr \(2017\)](#), [Rudin \(1987\)](#), and [Yamamoto \(2012\)](#) provides a complementary reading.
- **RKHS theory.** There are certainly no shortages of introductory texts relating to the theory of RKHSs: [Berlinet and Thomas-Agnan \(2004\)](#), [Gu \(2013\)](#), and [Steinwart and Christmann \(2008\)](#), to name a few. The concise sketch proof for the Moore-Aronszajn theorem was mostly inspired by [Hein and Bousquet \(2004, Thm. 4\)](#).
- **Kreĭn space and RKKS theory.** The innovation of indefinite inner product spaces perhaps started in mathematical physics literature, for which the theory of special relativity depends. Four-dimensional space-time is an often cited example. In any case, we referred to mainly [Ong et al. \(2004\)](#), which gives an overview in the context of learning using indefinite kernels. [Alpay \(1991\)](#) and [Zafeiriou \(2012\)](#) were also useful for understanding the fundamental concepts of RKKSs.
- **RKHS building blocks.** The main building block RKHSs, i.e. the canonical RKHS, the fBm RKHS and the Pearson RKHS, are described in the manuscript of [Bergsma \(2018\)](#).
- **ANOVA and functional ANOVA.** Classical ANOVA is pretty much existent in every fundamental statistical textbook. These texts have extremely well written introductions to this very important concept: [Casella and Berger \(2002, Ch. 11\)](#), [Dean and Voss \(1999, Ch. 3\)](#). On the relation between classical ANOVA and functional ANOVA decomposition, [Gu \(2013\)](#) offers novel insights. There is diverse literature concerning functional ANOVA, namely from the fields of statistical learning (e.g. [Wahba, 1990](#)), applied mathematics (e.g. [Kuo et al., 2010](#)), and sensitivity analysis (e.g. [Durrande et al., 2013](#); [Sobol, 2001](#)).

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