Bias-reduced estimation of structural equation models

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`https://haziqj.ml/sem-bias/`

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> **Jamil, H.**, Rosseel, Y., Kemp, O., & Kosmidis, I. (2025). Bias-Reduced Estimation of Structural Equation Models. *Manuscript in Submission*. `arXiv:2509.25419`.

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SCAN ME

- Source: https://github.com/haziqj/sembias-gradsem
- Slides: https://haziqj.ml/sembias-gradsem/slides.pdf
- R Package: https://github.com/haziqj/brlavaan

Context

SEM in a nutshell

Analyse multivariate data $\mathbf{y} = (y_1, \ldots, y_p)^\top$ to measure and relate hidden variables $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_q)^\top$, $q \ll p$, and uncover complex patterns.

In the social sciences, latent variables are used to represent **constructs**—the *theoretical, unobserved* concepts of interest.

(Psychology)
Personality traits

(Healthcare)
Quality of life

(Political science)
Social trust

(Education)
Competencies

Photo credits: Unsplash @dtravisphd, @impulsq, @ev, @benmullins.

Key issue

"Using SEMs in empirical research is often challenged by small sample sizes."

- Why? Data collection is expensive, time-consuming, or difficult, or all of these!
- Rare populations:
  - **Quezada, González, and Mecott (2016):** Identifying factors of adjustment in pediatric burn patients to facilitate appropriate mental health interventions postinjury ($n = 51$).
  - **Figueroa-Jiménez et al. (2021):** Studying functional connectivity network on individuals with rare genetic disorders ($n = 22$).
  - **Fabbricatore et al. (2023)**: Assessment of psycho-social aspects and performance of elite swimmers ($n = 161$).
  - **Manuela and Sibley (2013)**: Validating self-report measures of identity on a unique cultural group ($n = 143$).
- SEM is desirable, but small $n \Rightarrow$ poor finite-sample performance (esp. bias).

Outline

1. **Brief overview of SEMs**
   - Motivating example
   - ML estimation and inference
   - Examples of SEMs
     - Two-factor SEM
     - Latent growth models
2. **Bias reducing methods**
   - What is bias?
   - A review of bias reduction methods
   - Reduced-Bias M-estimation (RBM)
     - Implicit correction
     - Explicit correction
3. **Simulation studies and results**

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Structural equation models

Motivating example

Glycemic control and kidney health

> Does poorer glycemic control lead to greater severity of kidney disease?

Observe $p = 6$ variables for each patient:

	Indicator	Description	Unit
y_1	HbA1c	3-month avg. blood glucose	%
y_2	FPG	Fasting plasma glucose	mmol/L
y_3	Insulin	Fasting insulin level	µU/mL
y_4	PCr	Plasma creatinine	µmol/L
y_5	ACR	Albumin–creatinine ratio	mg/g
y_6	BUN	Blood urea nitrogen	mmol/L

Example adapted from Song and Lee (2012).

Covariance-based approach

Sample correlation matrix looks like this[1]:

	y_1	y_2	y_3	y_4	y_5	y_6
y_1	1	0.82	0.8	0.2	0.2	0.2
y_2	0.82	1	0.81	0.19	0.21	0.21
y_3	0.8	0.81	1	0.19	0.2	0.2
y_4	0.2	0.19	0.19	1	0.82	0.82
y_5	0.2	0.21	0.2	0.82	1	0.83
y_6	0.2	0.21	0.2	0.82	0.83	1

- The data suggests clustering of variables
  - y_1, y_2, y_3 measure *glycemic control* (`GlyCon`)
  - y_4, y_5, y_6 measure *kidney health* (`KdnHlt`)
- There is an element of dimension-reduction; much needed for analysing (correlated) multivariate data.
- Easier to hypothesize relationships, e.g.

$$\texttt{KdnHlt} = \alpha + \beta\,\texttt{GlyCon} + \texttt{error}$$

- SEM is about modelling the <u>covariance structure</u> of the data,

$$\mathbf{\Sigma} = \mathbf{\Sigma}(\vartheta).$$

1. Simulated data, from a two-factor SEM ($n = 1000$).

SEM equations

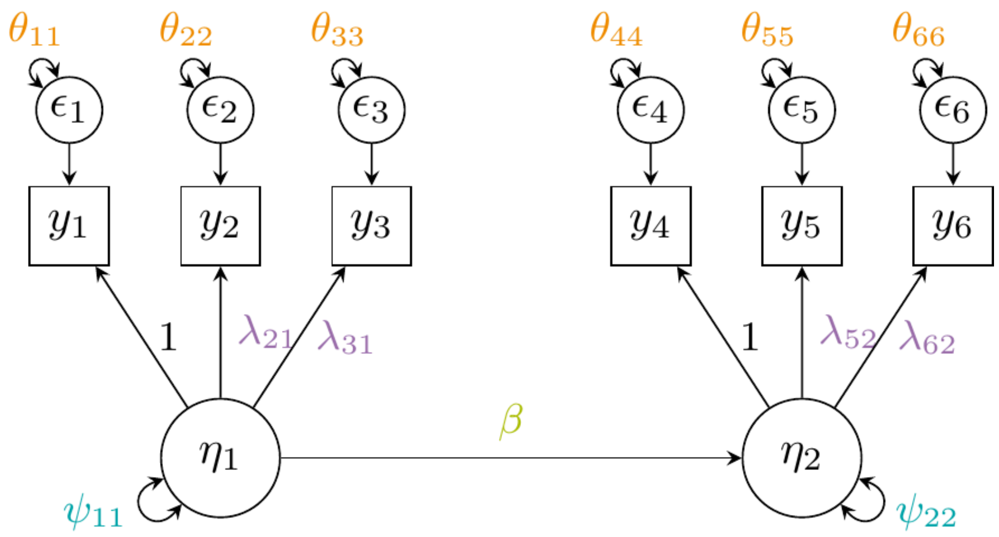
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & \lambda_{42} \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix}$$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

Or, more compactly as

$$\begin{aligned} \mathbf{y} &= \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\epsilon} \\ \boldsymbol{\eta} &= \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta} + \boldsymbol{\zeta} \end{aligned}$$

with assumptions $\boldsymbol{\epsilon} \sim \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Theta})$, $\boldsymbol{\zeta} \sim \mathrm{N}_q(\mathbf{0}, \boldsymbol{\Psi})$, and $\mathbf{cov}(\boldsymbol{\epsilon}, \boldsymbol{\zeta}) = \mathbf{0}$.



- SEM parameters include the free entries of $\boldsymbol{\nu}$, $\boldsymbol{\Lambda}$, $\boldsymbol{\Theta}$, $\boldsymbol{\alpha}$, $\mathbf{B}$, and $\boldsymbol{\Psi}$.
- Dump all in $\vartheta \in \mathbb{R}^m$, where $m < p(p+1)/2 + p$.
- Sometimes, not interested in mean structure, so $\boldsymbol{\nu}$ and $\boldsymbol{\alpha}$ are dropped.

ML estimation

- It can be shown that the normal SEM reduces to $\mathbf{y} \sim \mathrm{N}_p\big(\boldsymbol{\mu}(\vartheta), \boldsymbol{\Sigma}(\vartheta)\big)$, where

$$
\begin{aligned}
\boldsymbol{\mu}(\vartheta) &= \boldsymbol{\nu} + \boldsymbol{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\alpha} \\
\boldsymbol{\Sigma}(\vartheta) &= \boldsymbol{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\Psi}(\mathbf{I} - \mathbf{B})^{-\top}\boldsymbol{\Lambda}^\top + \boldsymbol{\Theta}
\end{aligned} \tag{1}
$$

- Suppose we observe $\mathcal{Y} = \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$. ML estimation maximises (up to a constant) the log-likelihood

$$
\ell(\vartheta) = -\frac{n}{2}\Big[\log|\boldsymbol{\Sigma}(\vartheta)| + \operatorname{tr}\big(\boldsymbol{\Sigma}(\vartheta)^{-1}\mathbf{S}\big) + \big(\bar{\mathbf{y}} - \boldsymbol{\mu}(\vartheta)\big)^\top \boldsymbol{\Sigma}(\vartheta)^{-1}\big(\bar{\mathbf{y}} - \boldsymbol{\mu}(\vartheta)\big)\Big] \tag{2}
$$

  where
  - $\mathbf{S} = \frac{1}{n}\sum_{i=1}^n (\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})^\top$ is the (biased) sample covariance matrix; and
  - $\bar{\mathbf{y}} = \frac{1}{n}\sum_{i=1}^n \mathbf{y}_i$ is the sample mean.

- Clearly, the MLE aims to minimise the discrepancy between $\mathbf{S}$ and $\boldsymbol{\Sigma}(\vartheta)$.

Properties of MLE

- Let $\bar{\vartheta}$ be the true parameter value. Subject to standard regularity conditions (Cox and Hinkley 1979), as $n \to \infty$,

$$\sqrt{n}(\hat{\vartheta} - \bar{\vartheta}) \xrightarrow{\mathrm{D}} \mathrm{N}_m\left(\mathbf{0}, \left[U(\bar{\vartheta})V(\bar{\vartheta})^{-1}U(\bar{\vartheta})\right]^{-1}\right) \tag{3}$$

  where

  - $U(\vartheta) = -\mathbb{E}\left[\nabla\nabla^\top \ell_1(\vartheta)\right]$ is the *sensitivity matrix*; and
  - $V(\vartheta) = \mathbf{var}\left[\nabla \ell_1(\vartheta)\right]$ is the *variability matrix*.

- Calculation of SEs are based off estimates of these matrices. The Godambe or "sandwich" matrix gives robust SEs (Satorra and Bentler 1994; Savalei 2014) in cases of model misspecification.

- *If* model is correctly specified, $U(\vartheta) = V(\vartheta) = I(\vartheta)$, the Fisher information.

Latent growth curve model (GCM)

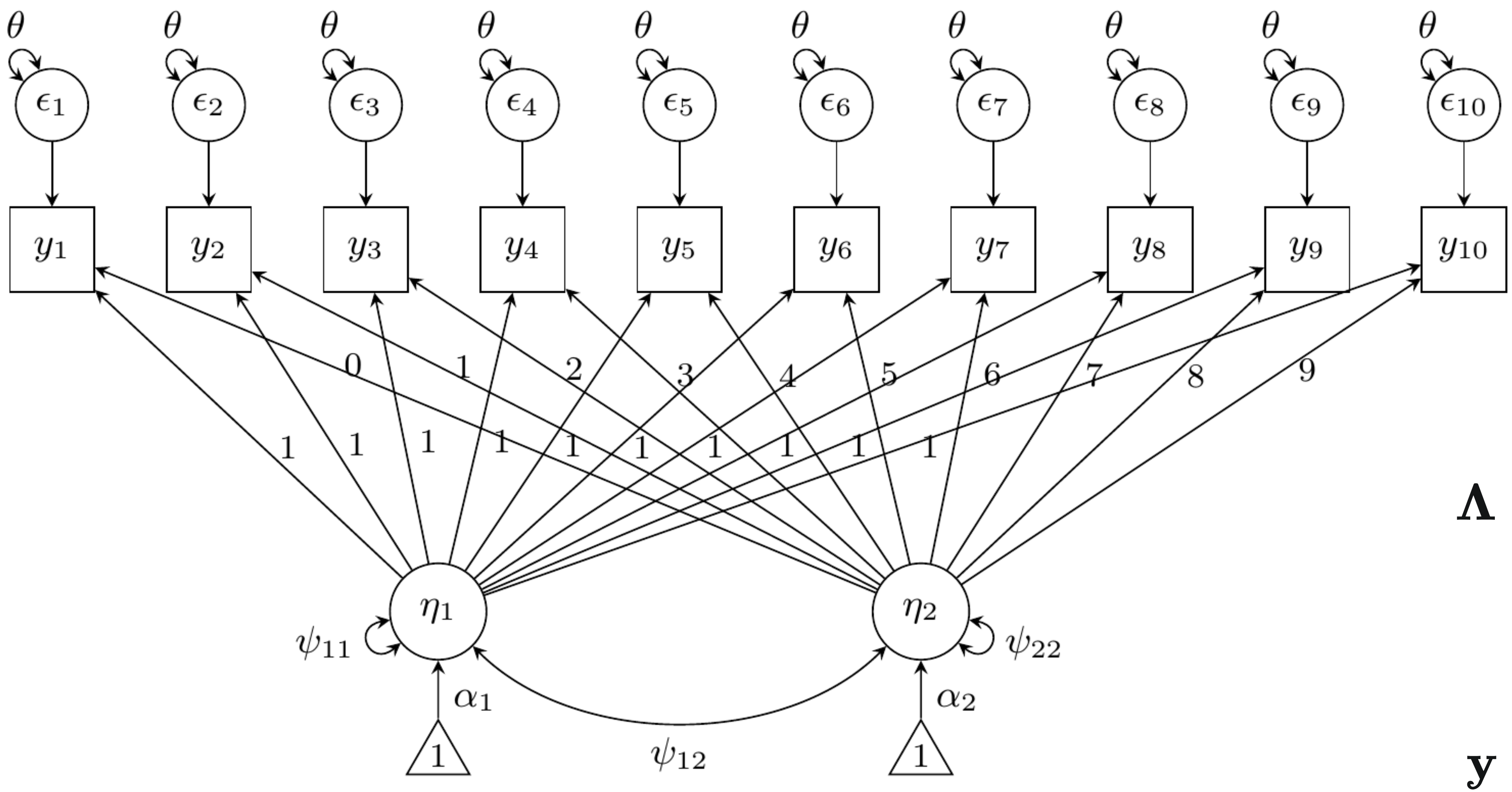
- **Longitudinal data:** repeated measurements on individuals i over time, e.g. $\mathbf{y}_i = (y_{i1}, y_{i2}, \dots, y_{i10})$, $i = 1, \dots, n$ (Rabe-Hesketh and Skrondal 2008).
- Usually, linear mixed effects models are used, where

$$y_{it} = \overbrace{(\alpha_1 + \eta_{1i})}^{\text{random int.}} + \overbrace{(\alpha_2 + \eta_{2i})}^{\text{random slope}} \cdot (t-1) + \epsilon_{it} \qquad t = 1, \dots, 10$$

$$\begin{pmatrix} \eta_{1i} \\ \eta_{2i} \end{pmatrix} \sim \mathrm{N}_2 \left(\mathbf{0}, \begin{pmatrix} \psi_{11} & \psi_{12} \\ \cdot & \psi_{22} \end{pmatrix} \right)$$

  - α_1 and α_2 are fixed effects (intercept and slope);
  - η_1 and η_2 are correlated random effects (individual deviations);
  - $\epsilon_t \sim \mathrm{N}(0, \theta)$ are measurement errors.
- Restricted ML is a popular method to estimate such models, with good parameter recovery for variance components (Corbeil and Searle 1976).

Latent GCM as SEM



$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 9 \end{bmatrix}$$

$$\mathbf{y} = \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\epsilon}$$
$$\boldsymbol{\epsilon} \sim \mathrm{N}_{10}(\mathbf{0}, \theta\mathbf{I})$$
$$\boldsymbol{\eta} \sim \mathrm{N}_2(\boldsymbol{\alpha}, \boldsymbol{\Psi})$$

Bias reduction methods

Poll

Maximum likelihood estimators are known to have which properties?

$$\hat{\vartheta} \rightarrow \vartheta$$

What is bias?

Bias of an estimator

$$\mathcal{B}_{\bar{\vartheta}}(\hat{\vartheta}) = \mathbb{E}\left[\hat{\vartheta} - \bar{\vartheta}\right] \tag{4}$$

Consider the stochastic Taylor expansion of $\boldsymbol{s}(\hat{\vartheta}) = \nabla \ell(\vartheta) = 0$ around $\bar{\vartheta}$. For many common estimators including MLE, the bias function is:

$$\mathcal{B}_{\bar{\vartheta}} = \frac{b_1(\bar{\vartheta})}{n} + \frac{b_2(\bar{\vartheta})}{n^2} + \frac{b_3(\bar{\vartheta})}{n^3} + O(n^{-4}). \tag{5}$$

Bias arises because the roots of the score equations are **not exactly centred at** $\bar{\vartheta}$, due to:

a. The curvature of the score $\boldsymbol{s}(\vartheta)$ creating asymmetry; and
b. The randomness of the score itself.

Illustration

Biased MLE estimator for σ^2

Consider $X_1, \ldots, X_n \sim \mathrm{N}(0, \sigma^2)$. The MLE for σ^2 is $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n X_i^2$.

$n = 10$ $n = 25$ $n = 1000$

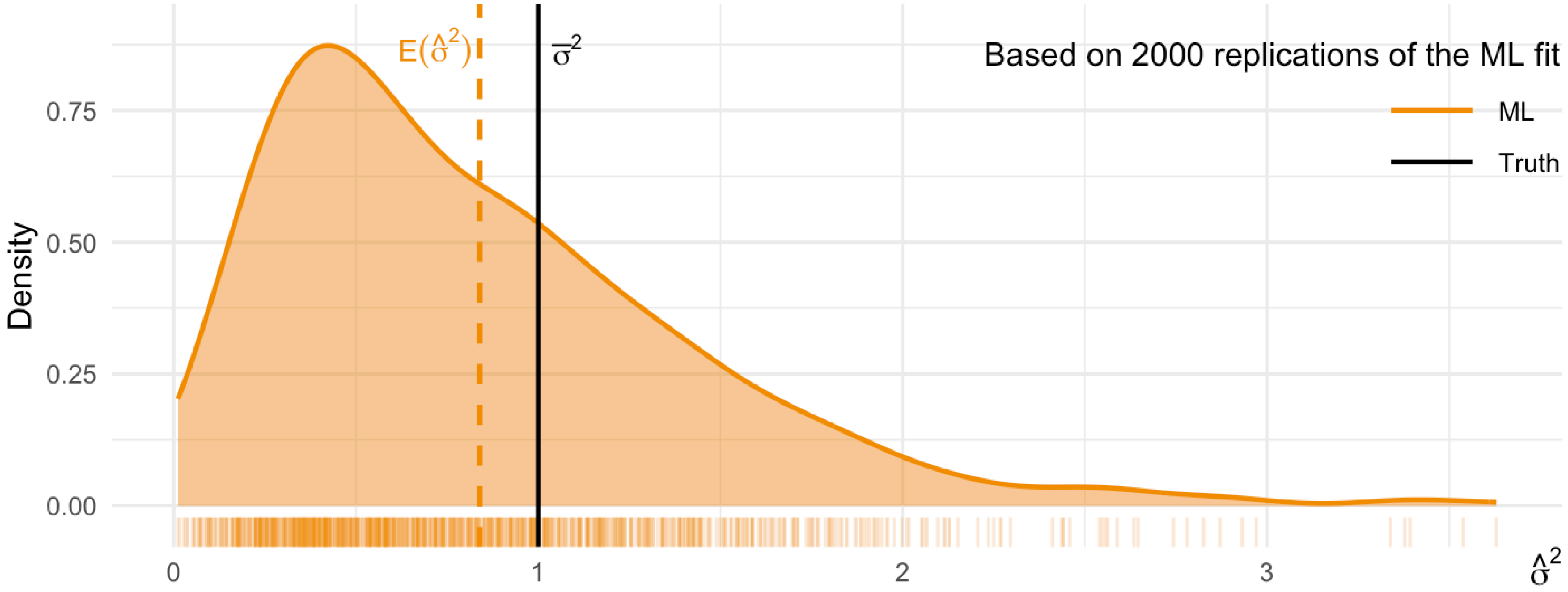
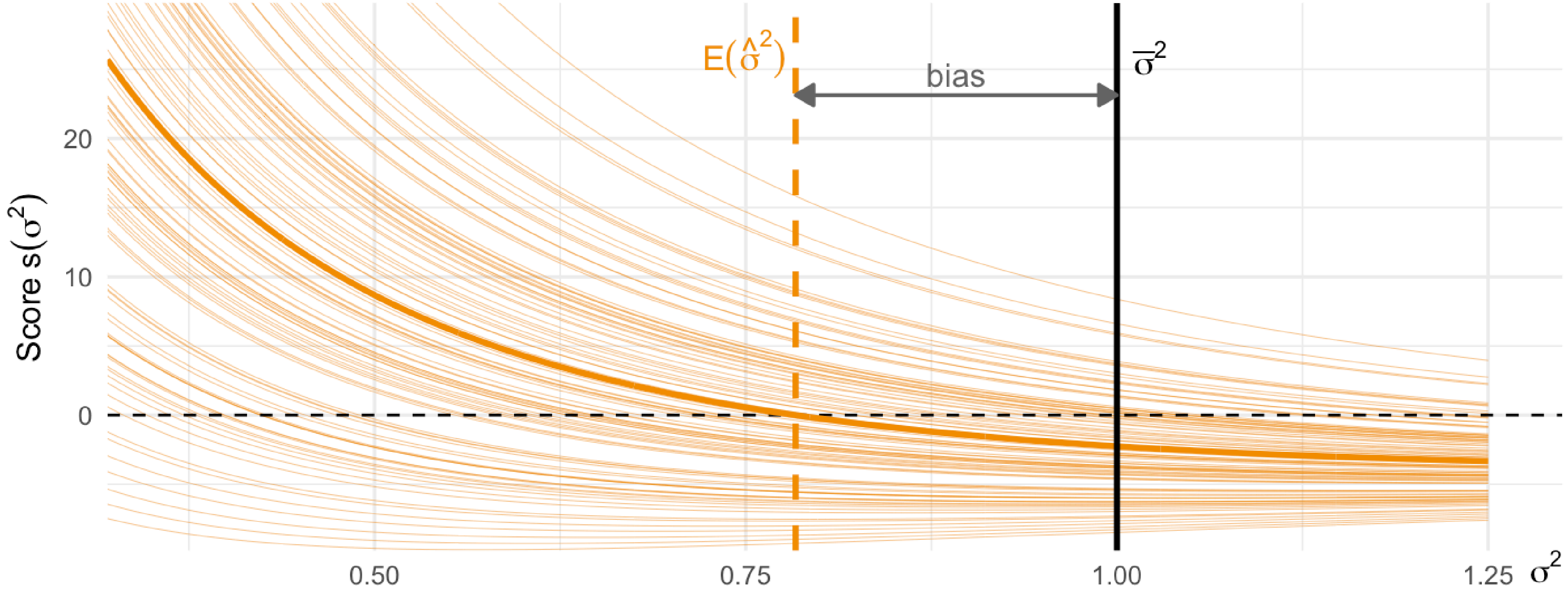


Illustration (cont.)

Score functions are random too

The score is $s(\sigma^2) = \ell'(\sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n X_i^2$.

$n = 10$ | $n = 25$ | $n = 1000$



If you're interested...

...and love differentiation ❤️🤓

For a comprehensive treatment of bias-reduction methods,

- Start here: Cox and Snell (1968)
- Follow up with: Firth (1993); Kosmidis and Firth (2009); Kosmidis (2014)

A review

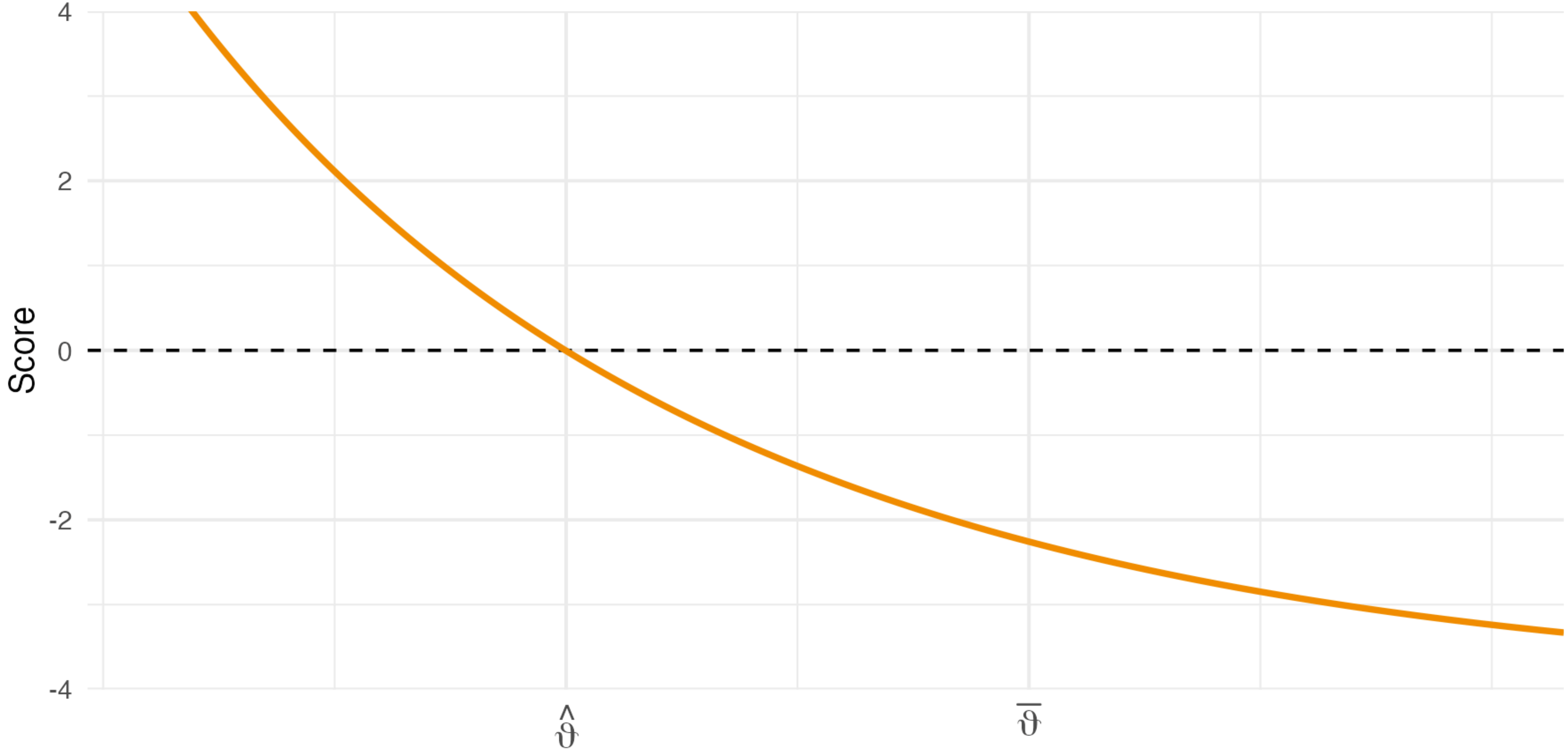
$$\overset{\text{estimator}}{\hat{\vartheta}} - \overset{\text{improved estimator}}{\tilde{\vartheta}} = \overset{\text{bias function}}{\mathcal{B}(\bar{\vartheta})} := \overset{\text{possibly intractable}}{\mathbb{E}}\left(\hat{\vartheta} - \overset{\text{unknown true value}}{\bar{\vartheta}}\right)$$

					Requirements		
	Method	**Model**	$\mathcal{B}(\bar{\vartheta})$	**Type**	$\mathbb{E}(\cdot)$	$\partial\,\cdot$	$\hat{\vartheta}$
1	Asymptotic bias correction	full	analytical	explicit	✓	✓	✓
2	Adjusted score functions	full	analytical	implicit	✓	✓	✗
3	Bootstrap	partial	simulation	explicit	✗	✗	✓
4	Jackknife	partial	simulation	explicit	✗	✗	✓
5	Indirect inference	full	simulation	implicit	✗	✗	✓
6	Explicit RBM	partial	analytical	explicit	✗	✓	✓
7	Implicit RBM	partial	analytical	implicit	✗	✓	✗

1–Efron (1975), Cordeiro and McCullagh (1991); 2–Firth (1993), Kosmidis and Firth (2009); 3–Efron and Tibshirani (1994), Hall and Martin (1988); 4–Quenouille (1956), Efron (1982); 5–Gourieroux, Monfort, and Renault (1993), MacKinnon and Smith Jr (1998)

Firth's adjusted scores methods

Instead of solving $s(\vartheta) = 0$, solve $s(\vartheta) + \overbrace{A(\vartheta)}^{\mathcal{B}(\vartheta)I(\vartheta)} = 0$.



Implicit RBM estimator

Computing $\mathcal{B}(\vartheta)$ and $I(\vartheta)$ can be difficult. Consider

$$s(\vartheta) + A(\vartheta) = 0 \;\Leftrightarrow\; \arg\max_{\vartheta} \{\ell(\vartheta) + P(\vartheta)\} \tag{6}$$

where $P(\vartheta)$ is a penalty term constructed such that $A(\vartheta) = \nabla P(\vartheta)$. Kosmidis and Lunardon (2024) show that

$$P(\vartheta) = -\frac{1}{2}\mathrm{tr}\left\{ j(\vartheta)^{-1} e(\vartheta) \right\}$$

where

- $j(\vartheta) = \sum_{i=1}^{n} \nabla\nabla^\top \ell_i(\vartheta)$ is the observed information;
- $e(\vartheta) = \sum_{i=1}^{n} \nabla\ell_i(\vartheta)\nabla^\top\ell_i(\vartheta)\nabla\ell_i(\vartheta)$ is the outer-product of scores.

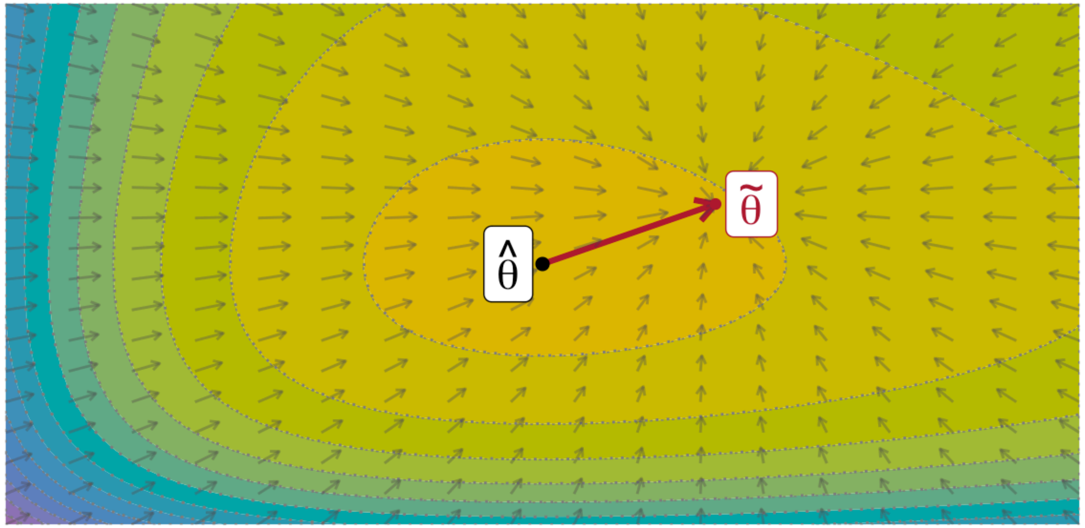
The solution $\tilde{\vartheta}$ to (6) is called the *implicit* RBM estimator (iRBM).

Explicit RBM estimator

Intuitively, by thinking in terms of a Newton-style update, an *explicit* estimator is obtained via

$$\vartheta^* = \hat{\vartheta} + j(\hat{\vartheta})^{-1} A(\hat{\vartheta}).$$

This moves $\hat{\theta}$ in the direction $A(\hat{\vartheta})$ away from the bias, with step length governed by the curvature $j(\hat{\vartheta})^{-1}$.

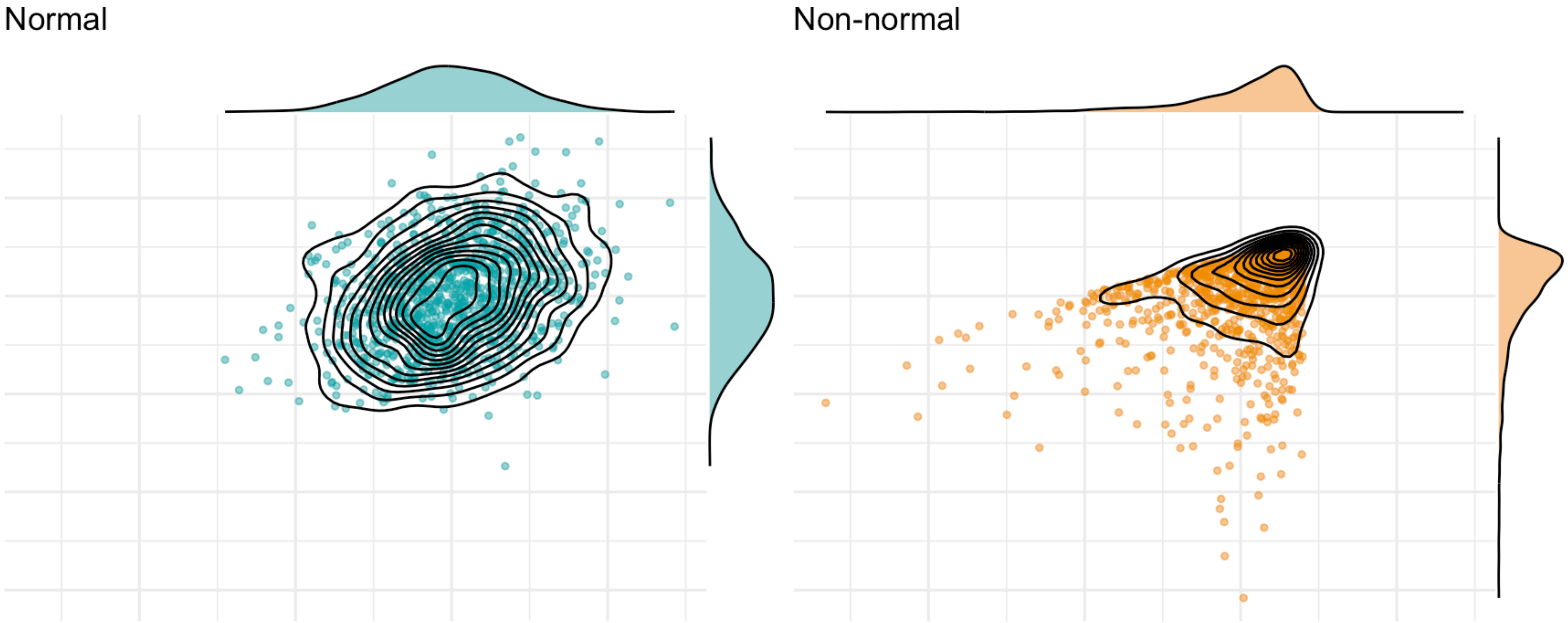


- Operationally, eRBM is simpler and quicker to compute than iRBM—no re-optimisation needed if $\hat{\theta}$ is available.
- However, unlike iRBM, no guarantees that bias correction stays inside the parameter space.

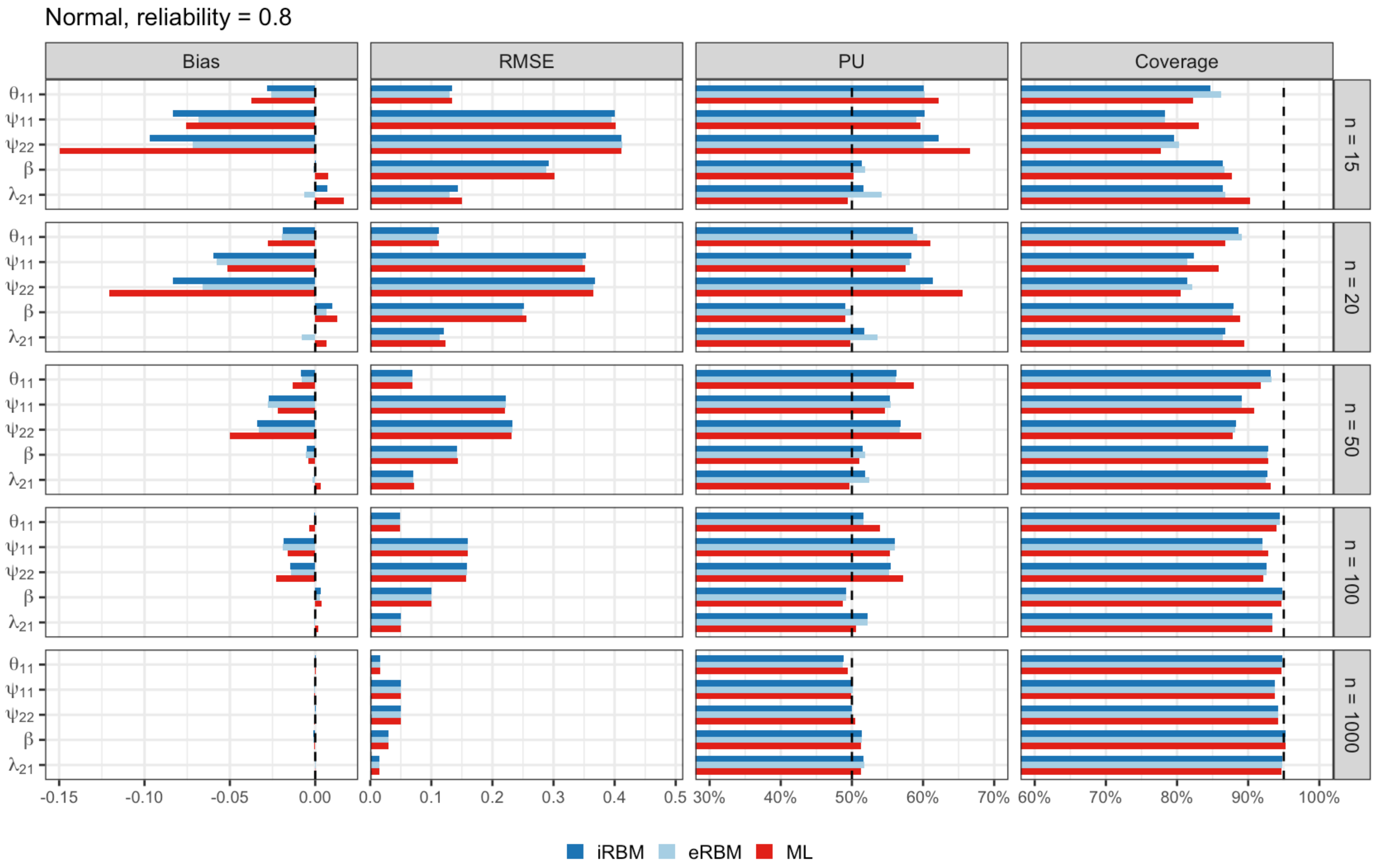
Simulation studies

Simulation design

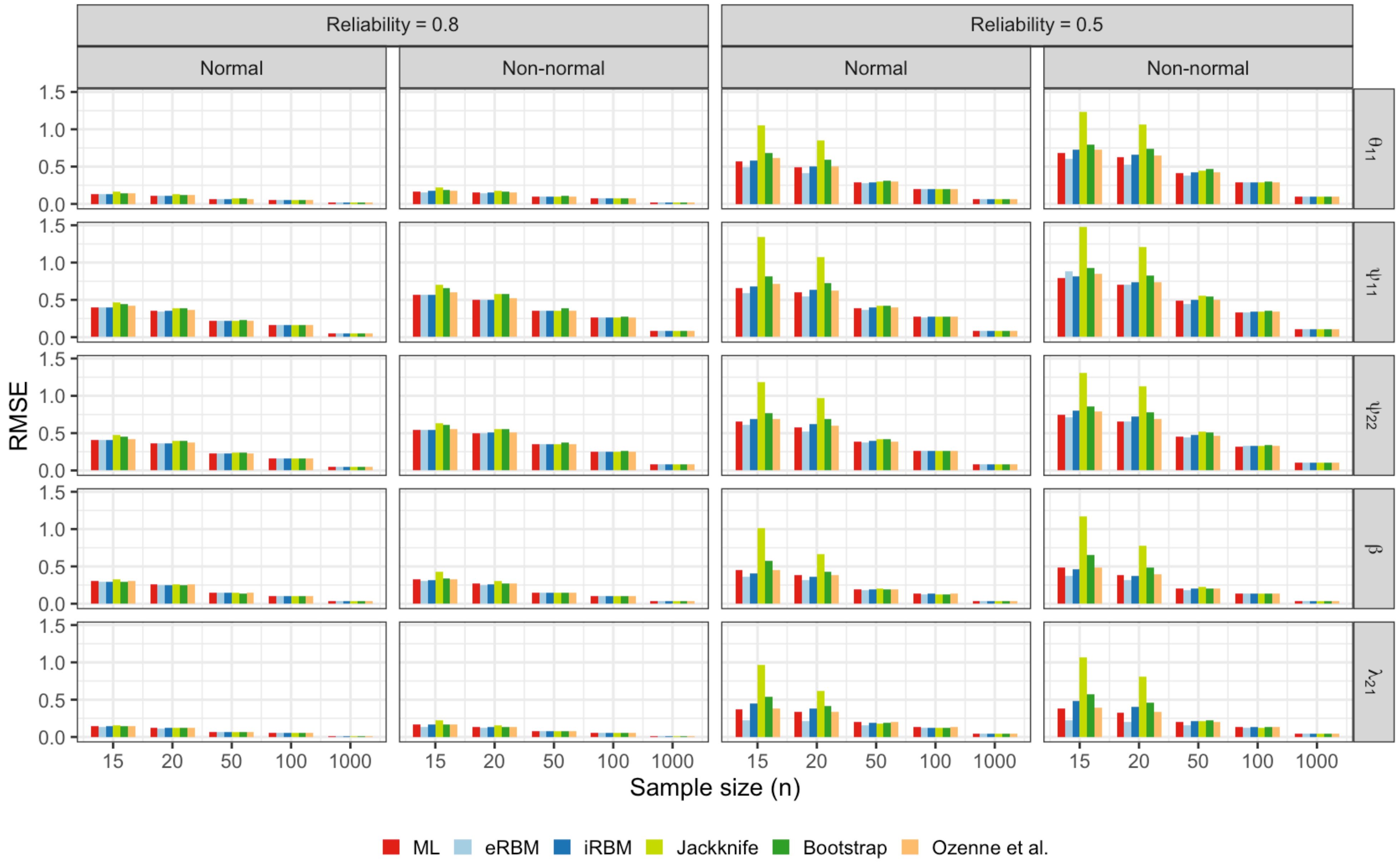
- Sample size: $n \in \{15, 20, 50, 100, 1000\}$
- Item reliability: Low or High ($\mathbf{Rel} = p^{-1}\sum_{j=1}^{p} \mathbf{\Sigma}^*_{jj}/\mathbf{\Sigma}_{jj}$)
- Distributional assumption: Normal or Non-normal



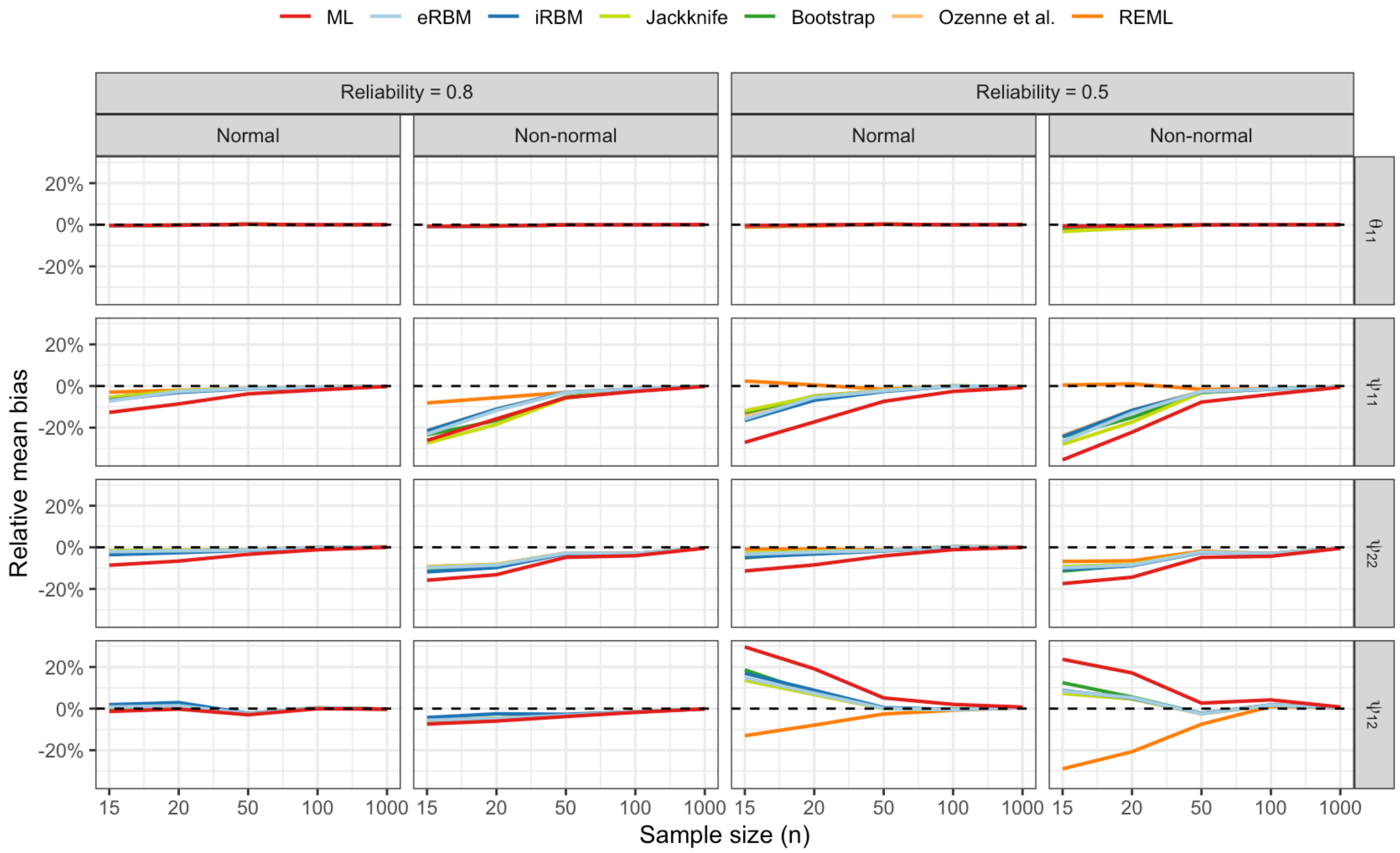
Results: Two-factor SEM



Results: Two-factor SEM (cont.)



Results: Latent GCM



Conclusion

Summary & future work

RBM applied to small sample estimation of SEM show key advantages:

- 🚀 Improved estimator performances (mean & median bias, RMSE, coverage).
- 💻 Computationally efficient (c.f. resampling methods).
- 🤖 Robust to model misspecification.

Future work include

1. Computational improvements for iRBM.
2. Plugin penalties to limit exploration of ill-conditioned regions.
3. Extension to other SEM families, e.g.
    - Path models, mediation models, latent interactions, etc.
    - Alternative to ML estimation e.g. WLS, DWLS, etc.

Software

```r
# pak::pak("haziqj/brlavaan")
library(brlavaan)

mod <- "
  eta1 =~ y1 + y2 + y3
  eta2 =~ y4 + y5 + y6
"
fit <- brsem(model = mod, data = dat, estimator.args = list(rbm = "implicit"))
summary(fit)
```

```
brlavaan 0.1.1.9008 ended normally after 88 iterations

  Estimator                                         ML
  Bias reduction method                       IMPLICIT
  Plugin penalty                                  NONE
  Optimization method                           NLMINB
  Number of model parameters                        13

  Number of observations                            50
```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https://haziqj.ml/sembias-gradsem

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