SM-1402 Basic Statistics

Chapter 4 (Continuous Distributions)

Dr. Haziq Jamil

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Continuous Distribution

- This chapter concerns **continuous random variables**.
- When a variable is continuous, it can take on infinitely many, uncountable values, for example
  - The mass (grams) of a bag of sugar packaged by a particular machine.
  - The time taken (minutes) to perform a task.
  - The height (cm) of a three-year-old girl.
  - The lifetime (hours) of a 100-watt light bulb.

Continuous Distribution

- A continuous random variable X is described by its **probability density function** (pdf).
- It is specified for the range of values for which x is valid (support).
- The pdf can be illustrated by a curve $y = f(x)$. Note that this function cannot be negative.
- Probabilities are given by the **area under the (pdf) curve**. It is sometimes possible to find an area by geometry (e.g. area of triangles, rectangles, etc.). However, often areas need to be calculated using integration.

Continuous Distribution

Example 5.1: X *is the delay in hours of a flight from Kuching, where*

$$f(x) = 0.2 - 0.02x, \qquad 0 \leq x \leq 10$$

Find

(a) the probability that the delay will be less than four hours.

(b) the probability that the delay will be between two and six hours.

Continuous Distribution

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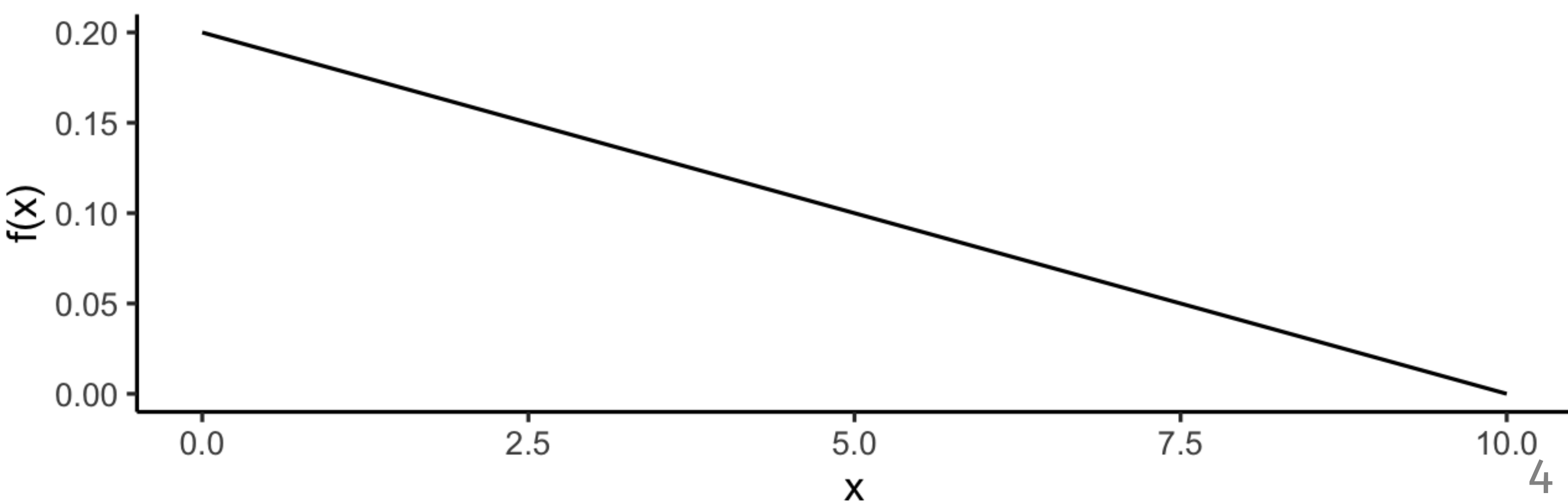
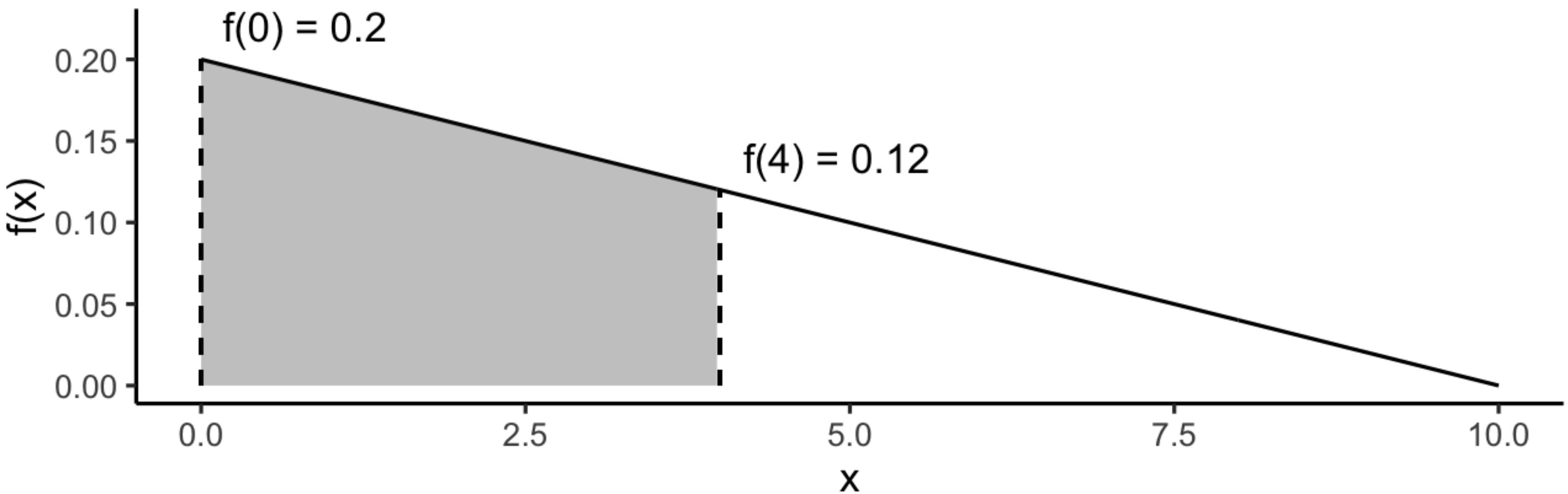


Continuous Distribution

(a) the probability that the delay will be less than four hours.



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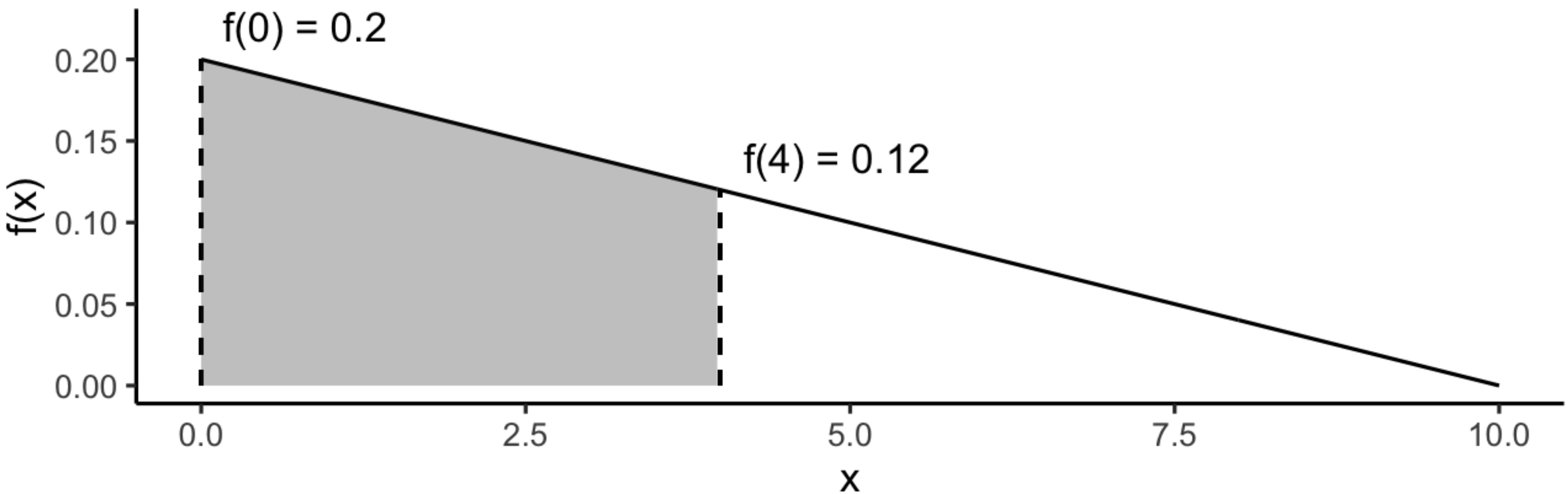

Area = $\frac{1}{2}(a+b)h = \frac{1}{2}(0.2+0.12)(4) = 0.64$

Continuous Distribution

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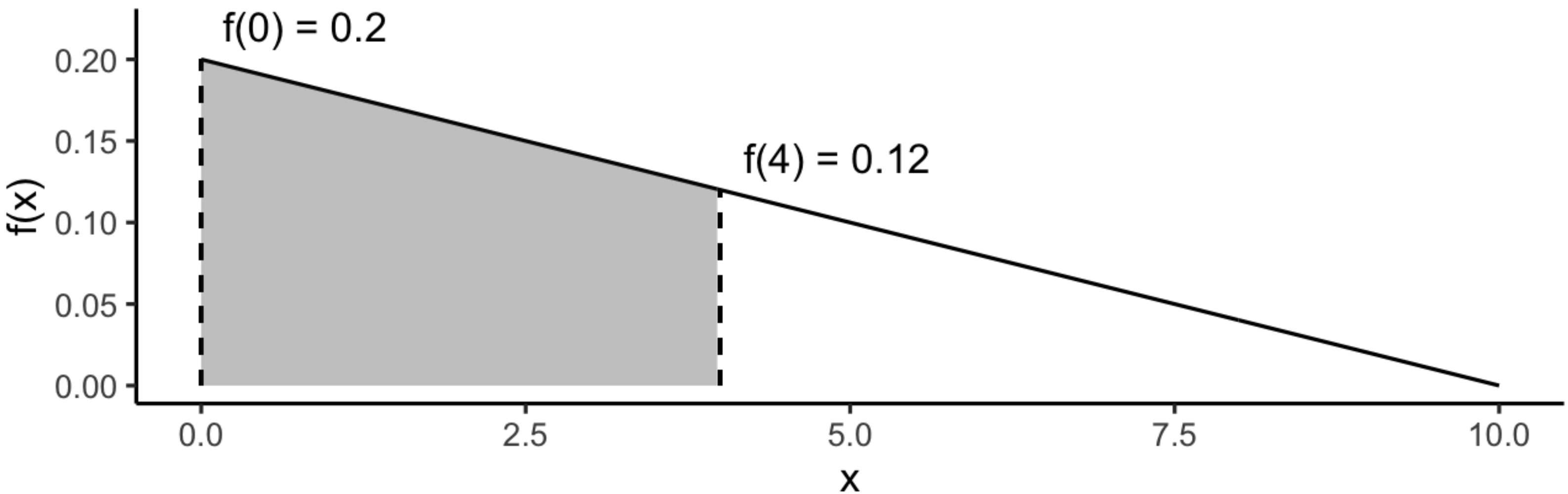

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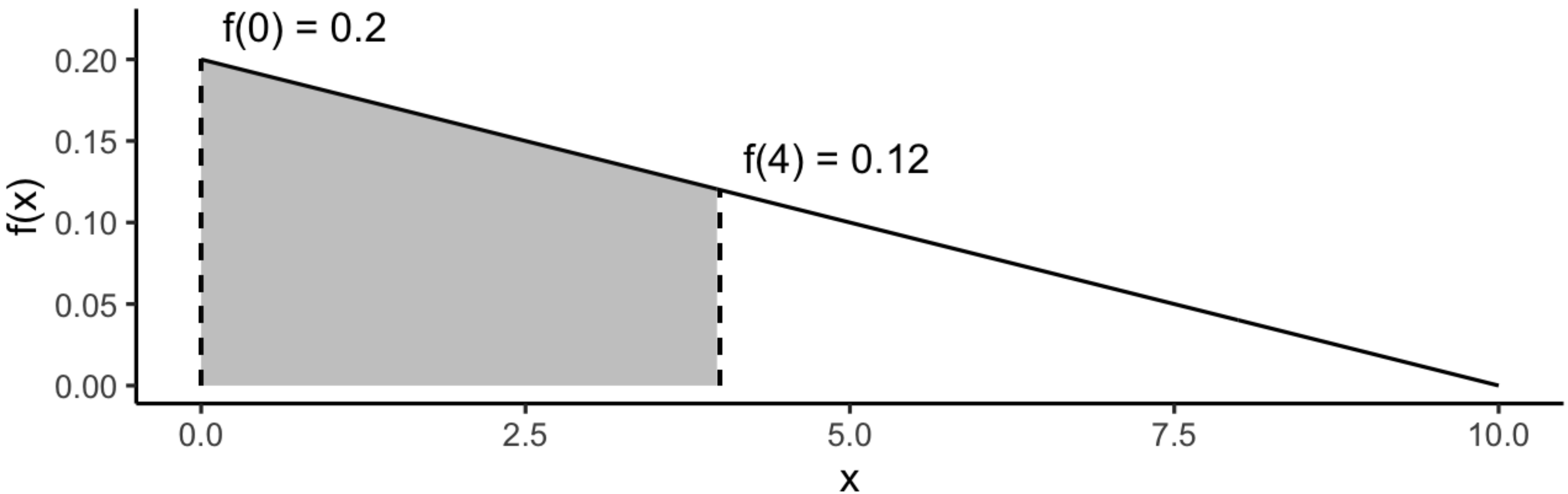

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$\therefore P(X \leq 4) = 0.64$

Continuous Distribution

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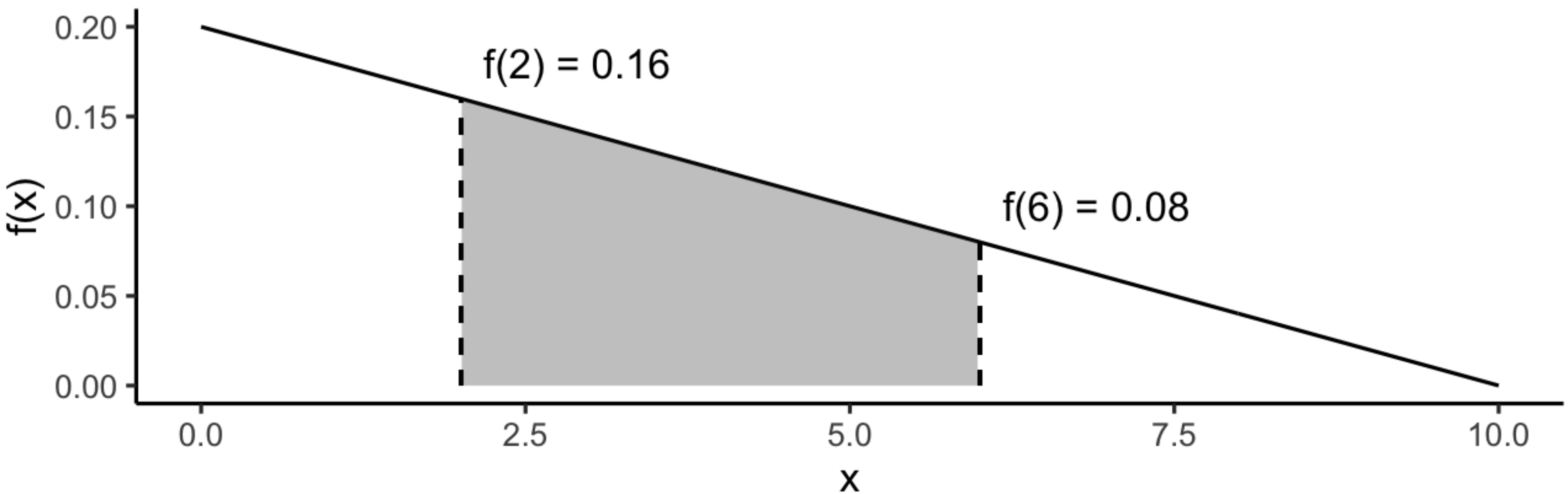

Continuous Distribution

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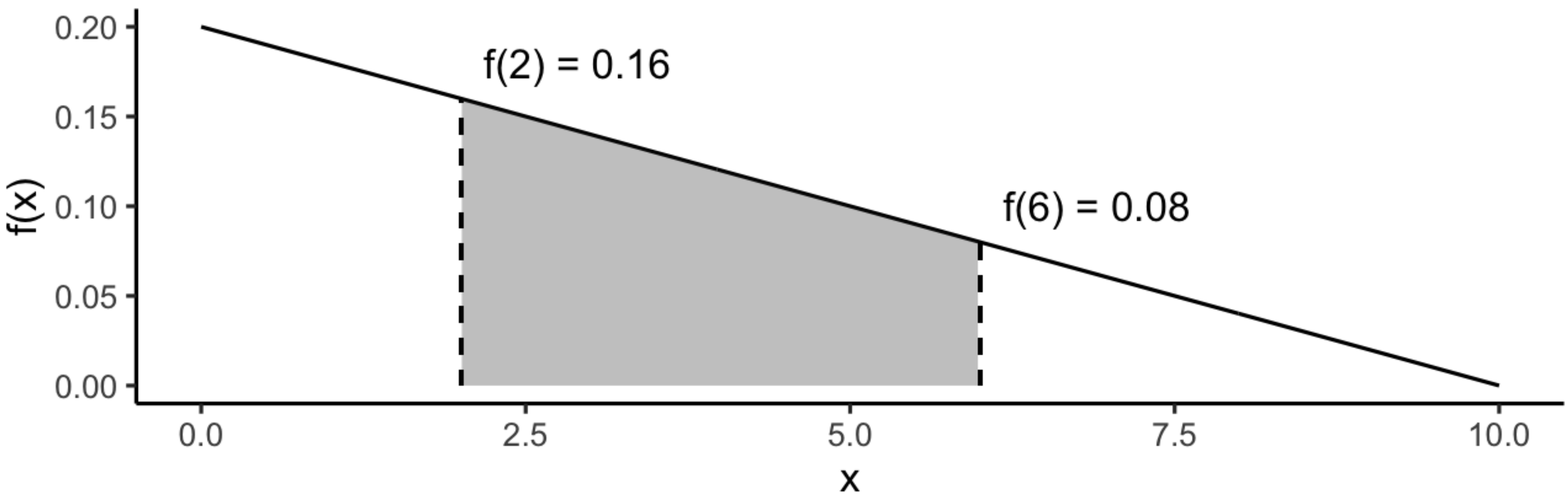

Area = $\frac{1}{2}(a+b)h = \frac{1}{2}(0.16+0.08)(4) = 0.48$

Continuous Distribution

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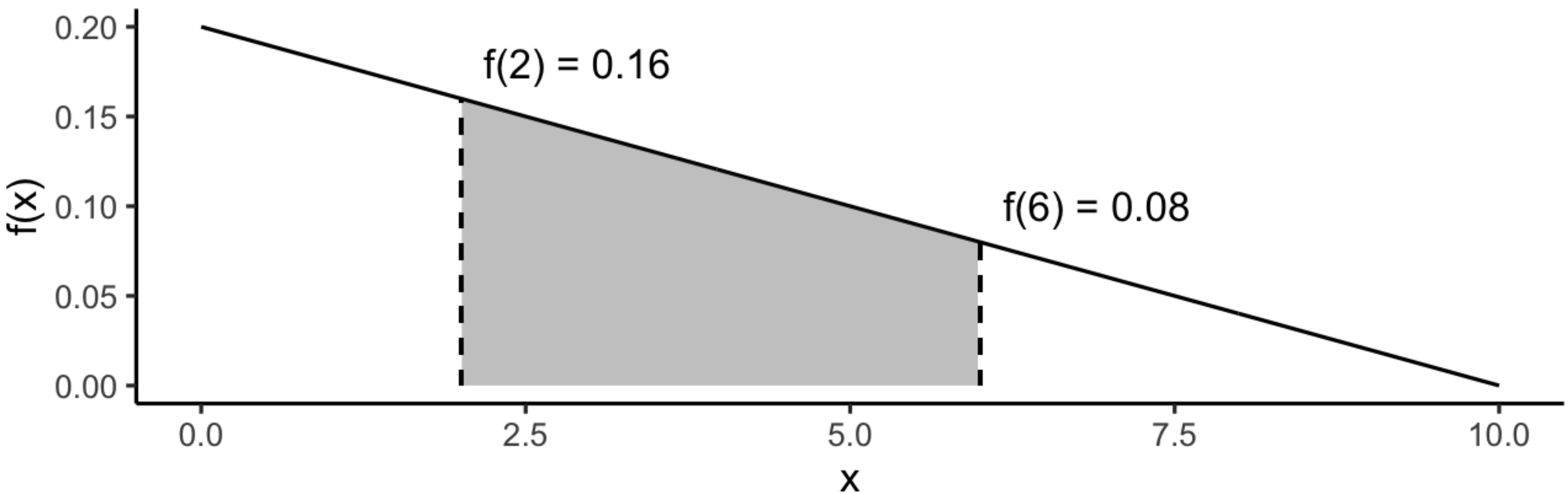

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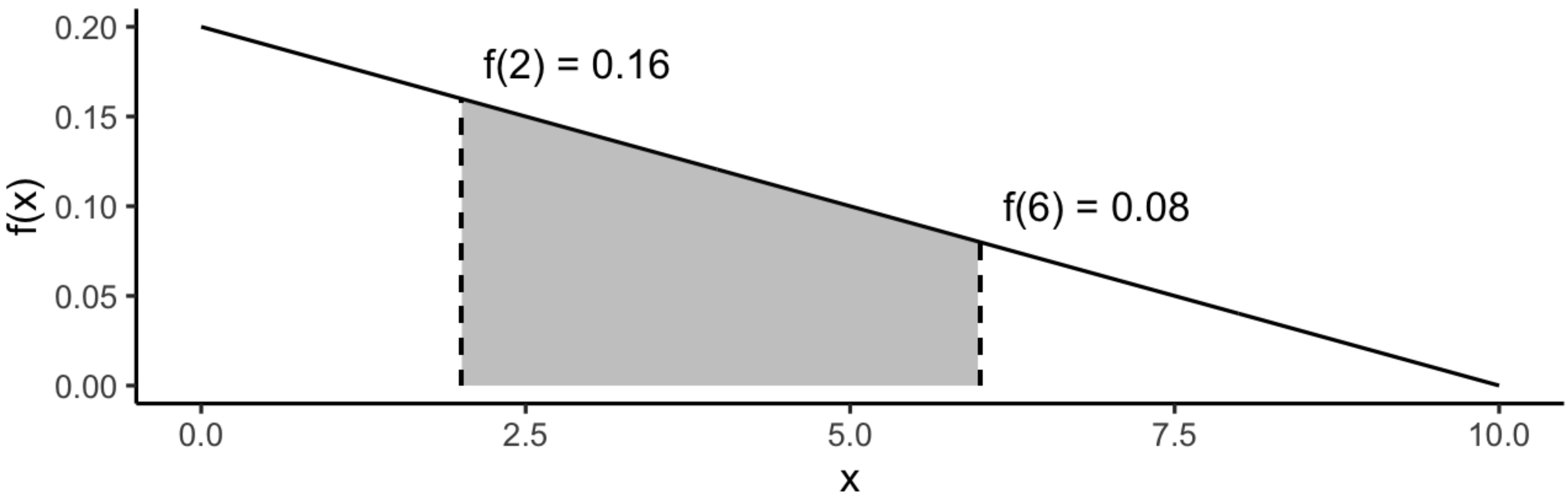

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$\therefore P(X \leq 4) = 0.48$

Continuous Distribution

- It is not possible to find the probability that the delay is, say, **exactly** three hours.
- If you try to integrate, then

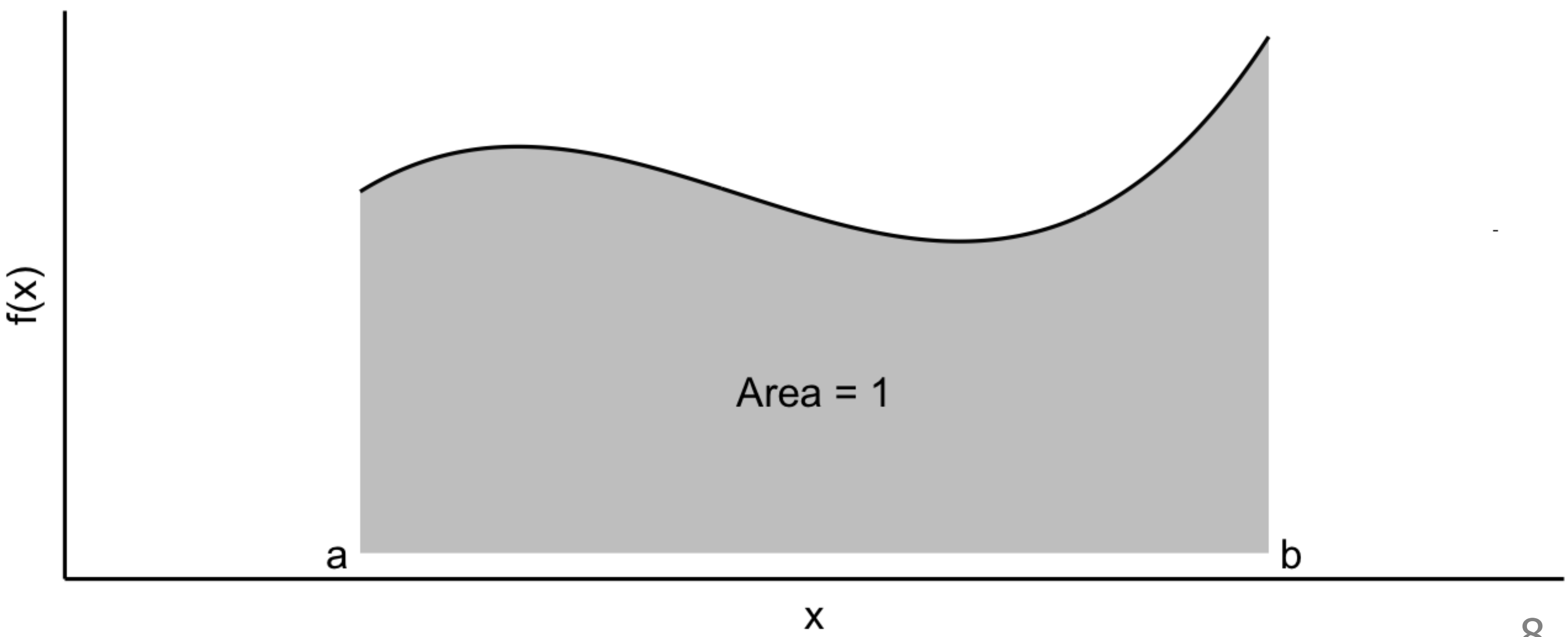
$$\int_3^3 0.2 - 0.02x\,dx = [0.2x - 0.01x^2]_3^3 = 0$$

- You can only find the probability that X lies within a range.
- Also, for the same reason, it does not matter whether $\leq$ or $<$ was used. Therefore, these are all the same
    - $P(2 \leq X \leq 6)$
    - $P(2 < X \leq 6)$
    - $P(2 \leq X < 6)$
    - $P(2 < X < 6)$

Continuous Distribution

For a continuous random variable X with pdf $f(x)$ valid over the range $[a, b]$,

$$\int_a^b f(x)\,dx = 1$$



Continuous Distribution

Example 5.2: A continuous random variable has pdf $f(x) = kx^2$ *for* $0 < x < 4$*. Find* $P(1 \leq X \leq 3)$*.*

Continuous Distribution

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First, solve for k. Since the total area must equal 1,

$$P(0 \le X \le 4) = \int_0^4 kx^2\,dx = \left[\frac{kx^3}{3}\right]_0^4 = 1$$

$$\Rightarrow k4^3/3 - 0 = 1$$

$$\Rightarrow k = 3/64$$

Continuous Distribution

Example 5.2: A continuous random variable has pdf $f(x) = kx^2$ *for* $0 < x < 4$. *Find* $P(1 \leq X \leq 3)$.

Then,

$$P(1 \leq X \leq 3) = \int_1^3 kx^2 \, dx = \left[\frac{kx^3}{3}\right]_1^3 = \frac{k}{3}(3^3 - 1) = 26k/3$$

So $P(1 \leq X \leq 3) = 3/64 \times 26/3 = 26/64 = 0.406$

Expectation

For a continuous random variable X with pdf $f(x)$, the expected value of X is given by the formula

$$E(X) = \int_x x f(x)\, dx$$

$E(X)$ is referred to as the *mean* or *expectation* of X and is often denoted by μ.

Expectation

Example 5.3: X *has pdf* $f(x) = x^2/9$ *for* $0 < x < 3$*. Find (a)* μ*; and (b)* $P(X < \mu)$.

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(a)

$$\begin{aligned} \mu = \int_0^3 x f(x)\,dx &= \int_0^3 x \cdot x^2/9\,dx \\ &= \frac{1}{9}\left[x^4/4\right]_0^3 \\ &= 81/36 = 2.25 \end{aligned}$$

Expectation

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(b)

$P(X < \mu) = P(X < 2.25) = \int_0^{2.25} x^2/9\,dx = [x^3/27]_0^{2.25} = 0.42$

Expectation

The expectation of **any function** of X can also be derived[1] as

$$E\big(g(X)\big) = \int_x g(x) f(x)\, dx$$

For example,

- $E(X^2) = \int_x x^2 f(x)\, dx$
- $E(1/X) = \int_x \frac{1}{x} f(x)\, dx$
- $E(e^{tx}) = \int_x e^{tx} f(x)\, dx$

[1] Assuming existence!

Expectation

The properties of expectations also apply to continuous random variables: In general, for constants a, b and c, and two random variables X and Y,

- $E(a) = a$
- $E(aX) = aE(X)$
- $E(aX + bY + c) = aE(X) + bE(Y) + c$

Also, for two functions of X, $g(X)$ and $h(X)$,

- $E\big(g(X) + h(X)\big) = E\big(g(X)\big) + E\big(h(X)\big)$

Variance

As in the discrete case, the variance is given by

$$\mathrm{Var}(X) = E\big((X-\mu)^2\big)$$

and for continuous r.v. X with pdf $f(x)$,

$$\begin{aligned}
\mathrm{Var}(X) &= \int_x (x-\mu)^2 f(x)\,dx \\
&= \int_x x^2 f(x)\,dx - \mu^2 \\
&= \int_x x^2 f(x)\,dx - E^2(X) \\
&= E(X^2) - E^2(X)
\end{aligned}$$

Variance

An as before, the following results also hold when X is continuous (a and b are constants)

- $\mathrm{Var}(X) \geq 0$
- $\mathrm{Var}(a) = 0$
- $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Given another r.v. Y such that X and Y are **independent**, and a constant c, then

- $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- $\mathrm{Var}(aX \pm bY + c) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

Variance

Example 5.6: The continuous random variable X *has pdf* $f(x) = x/8$, *for* $0 < x < 4$. *Find the variance of* X.

Variance

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First find $E(X)$ and $E(X^2)$

$$E(X) = \int_0^4 x \cdot \frac{x}{8}\, dx = \left[\frac{x^3}{24}\right]_0^4 = 64/24 = 8/3$$

Variance

Example 5.6: The continuous random variable X has pdf $f(x) = x/8$, for $0 < x < 4$. Find the variance of X.

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$$E(X^2) = \int_0^4 x^2 \cdot \frac{x}{8}\,dx = \left[\frac{x^4}{32}\right]_0^4 = 256/32 = 8$$

Variance

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$$E(X^2) = \int_0^4 x^2 \cdot \frac{x}{8}\,dx = \left[\frac{x^4}{32}\right]_0^4 = 256/32 = 8$$

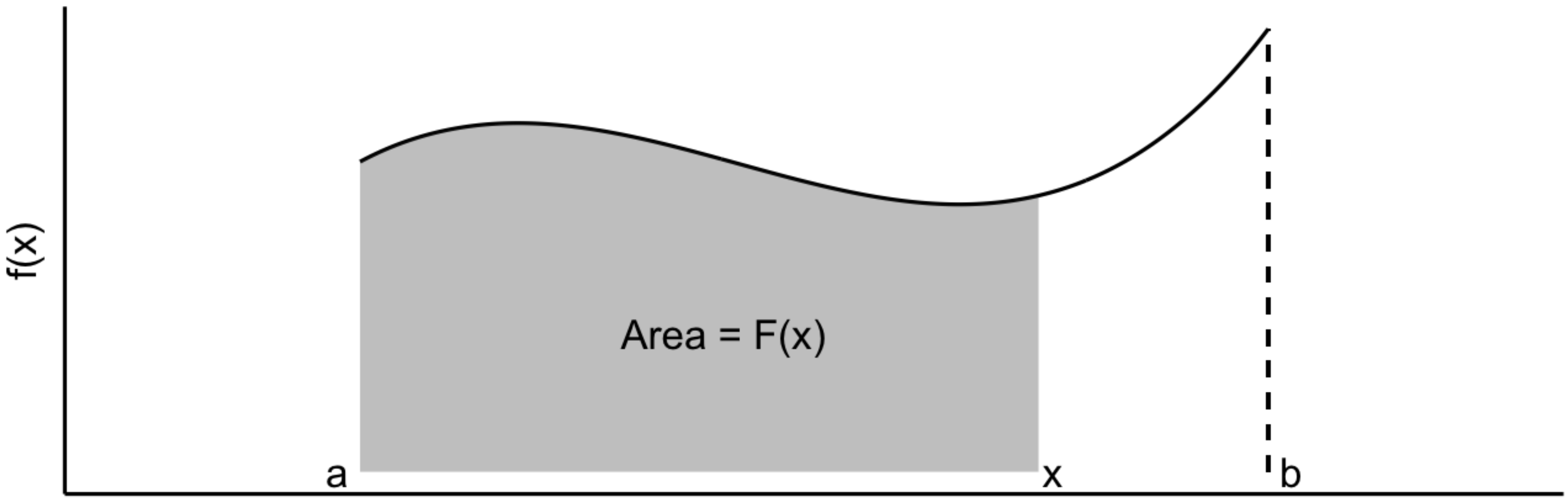
Thus, $\mathrm{Var}(X) = E(X^2) - E^2(X) = 8 - (8/3)^2 = 8/9$

Cumulative distribution function

Recall that the cdf is defined as $F(x) = P(X \leq x)$.

For a continuous r.v. X with pdf $f(x)$, the cdf is

$$F(x) = \int_{-\infty}^{x} f(\tilde{x})\, d\tilde{x}$$



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Example: X *has pdf* $f(x) = x/8$, *for* $0 < x < 4$. *Find* $P(1 < X < 3)$ *using the cdf.*

Cumulative distribution function

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First find the cdf

$$\begin{aligned} F(x) = \int_{-\infty}^{x} \tilde{x}/8 \, d\tilde{x} &= \int_{0}^{x} \tilde{x}/8 \, d\tilde{x} \\ &= [\tilde{x}^2/16]_0^x \\ &= x^2/16 \end{aligned}$$

Cumulative distribution function

Example: X has pdf $f(x) = x/8$, for $0 < x < 4$. Find $P(1 < X < 3)$ using the cdf.

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$$
\begin{aligned}
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&= [\tilde{x}^2/16]_0^x \\
&= x^2/16
\end{aligned}
$$

$$
\begin{aligned}
P(1 < X < 3) &= F(3) - F(1) \\
&= 3^2/16 - 1/16 \\
&= 8/16 = 0.5
\end{aligned}
$$

Special continuous distributions: The normal distributions

The normal distribution

The normal distribution is the most important distribution in statistics.

- Many naturally occurring phenomena can be *modelled* as following a normal distribution.
- The central limit theorem (CLT): The disribution of the mean of a sample tends to converge to a normal distribution, as more and more samples are collected.
- Often, the normal distribution is used for the error term in standard statistical models (e.g. linear regression).

The normal distribution

Let X be disributed according to a normal distribution with mean μ and variance σ^2. We write $X \sim \mathrm{N}(\mu, \sigma^2)$. Then its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

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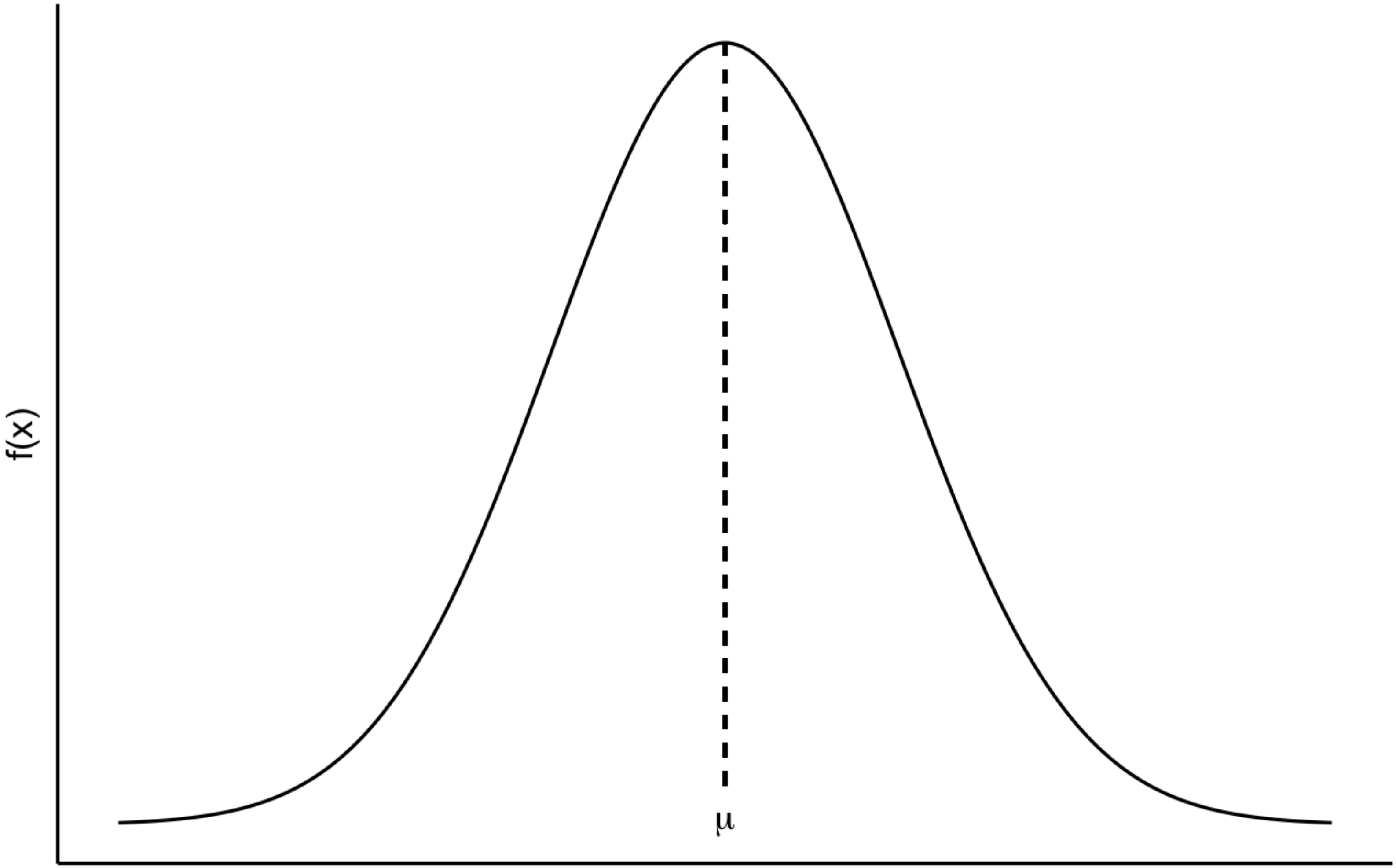
<u>Properties</u>

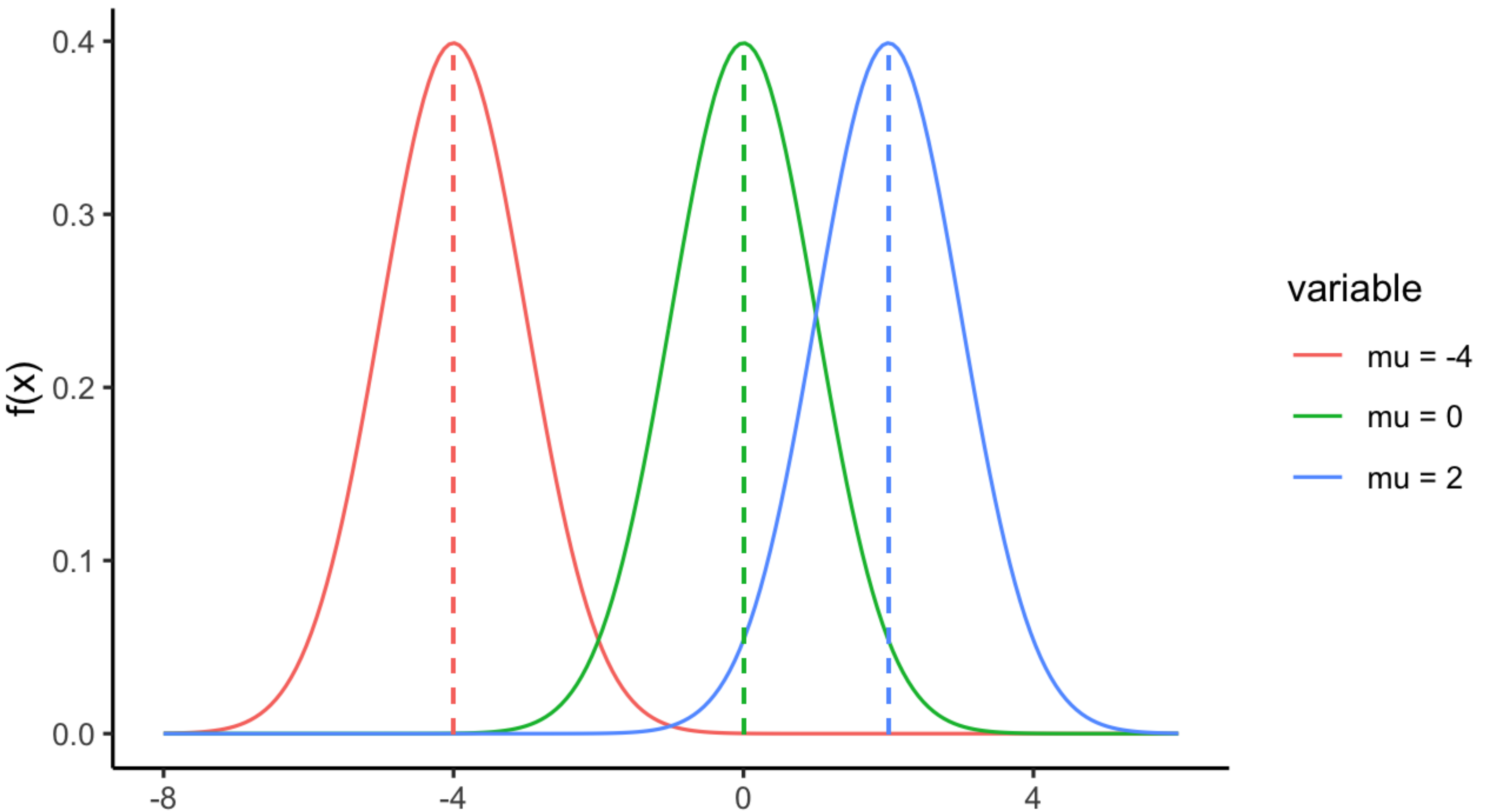
- The parameters of the normal distribution are the mean $E(X) = \mu$ and the variance $\mathrm{Var}(X) = \sigma^2$.
- Support: $x \in \mathbb{R}$.
- The normal distribution is **symmetric** about μ, and **centred** at μ.
- Due to symmetry, median = mean = μ.
- The mode is also μ (peaks at the mean).

The normal distribution



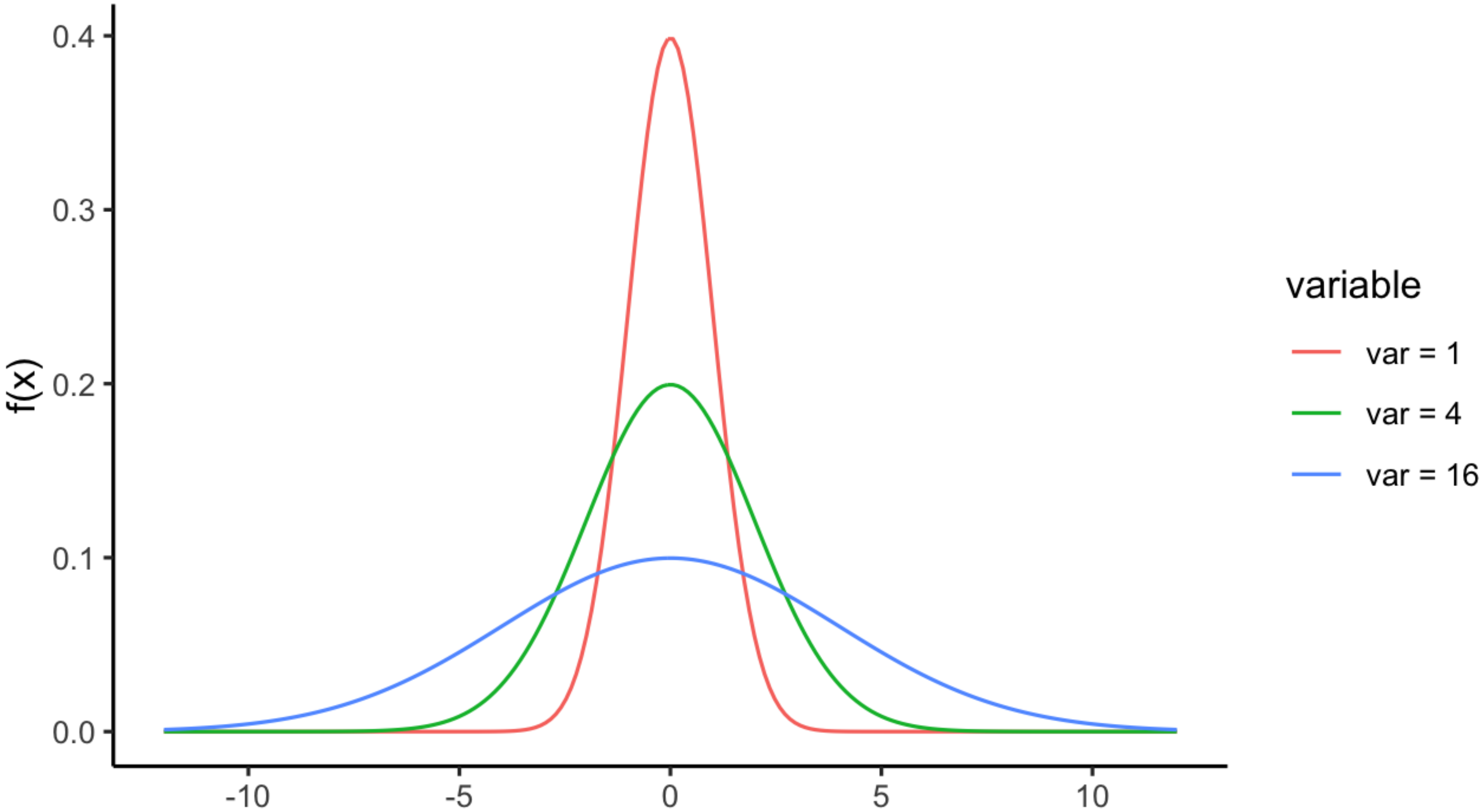
The normal distribution

Effect of changing the mean (location). Variance $\sigma^2 = 1$.



The normal distribution

Effect of changing the variance (scale/spread). Mean $\mu = 0$.



The standard normal

The most important normal distribution is the special case when $\mu = 0$ and $\sigma^2 = 1$. This is called the **standard normal** distribution, denoted

$$Z \sim \mathrm{N}(0, 1)$$

with pdf

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

The pdf of Z is denoted specially by the greek letter ϕ, pronounced 'phi'.

The standard normal

Any normal r.v. can be **standardized** by

- subtracting the mean
- dividing by the standard deviation

Let $X \sim \mathrm{N}(\mu, \sigma^2)$. Then

$$Z := \frac{X - \mu}{\sigma} \sim \mathrm{N}(0, 1)$$

The standard normal

Calculating probabilities of the (standard) normal distribution is not possible to do explicitly using integration. That is,

$$P(a \leq Z \leq b) = \int_a^b \phi(z)\, dz$$

is **not tractable** (no closed form exists).

Instead, we rely on **computer approximations**, or by manually searching for the values from a statistical table. Statistical tables will show values of the standard normal cdf

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi(\tilde{z})\, d\tilde{z}$$

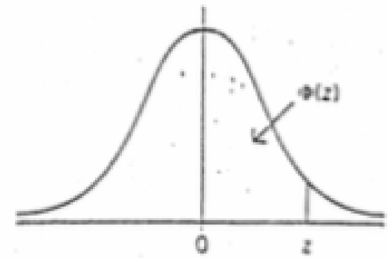
The standard normal

THE STANDARD NORMAL DISTRIBUTION FUNCTION

If Z has a normal distribution with mean 0 and variance 1 then, for each value of z, the table gives the value of $\Phi(z)$ where

$$\Phi(z) = P(Z < z).$$

For negative values of z use $\Phi(-z) = 1 - \Phi(z)$.



z	0	1	2	3	4	5	6	7	8	9	ADD								
											1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23

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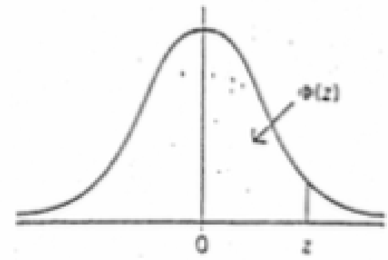

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0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23

Find (a) $P(Z < 0.85)$, (b) $P(Z > 0.85)$

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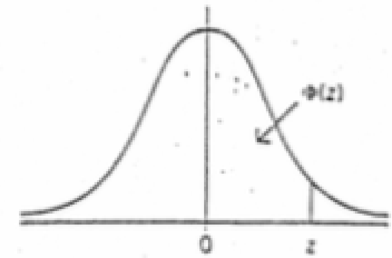

z	0	1	2	3	4	5	6	7	8	9	ADD 1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
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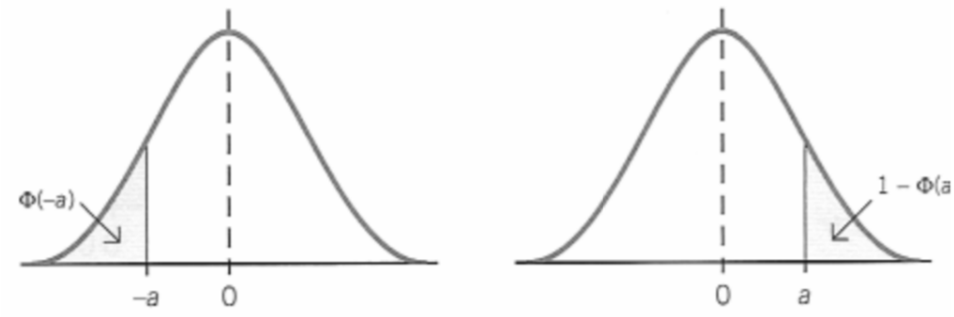
Find (a) $P(Z < 0.85)$, (b) $P(Z > 0.85)$

(a) 0.8023, (b) $1 - 0.8023 = 0.1977$

The standard normal

Finding probabilities involving negative values of z.

$$P(Z < -z) = \Phi(-z) = 1 - \Phi(z)$$



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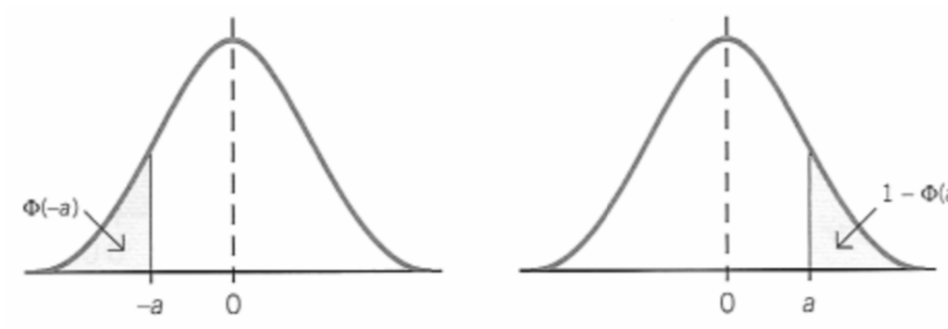
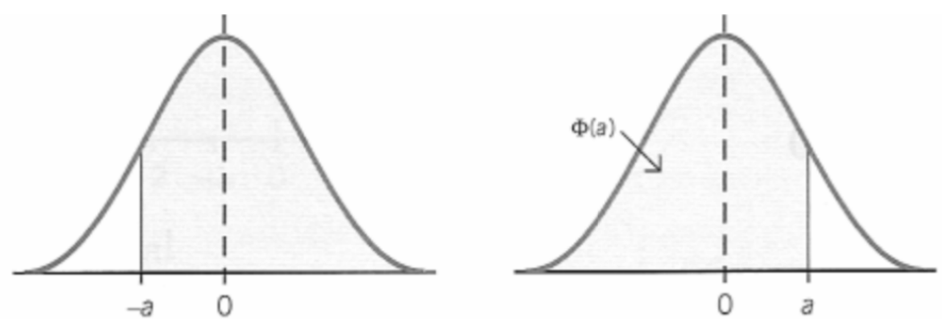


$$P(Z > -z) = 1 - P(Z < -z) = 1 - (1 - \Phi(z)) = \Phi(z)$$



The standard normal

Example: Find $P(Z < -0.913)$ *and* $P(Z > -0.913)$.

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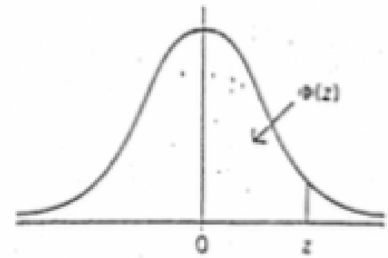

											1	2	3	4	5	6	7	8	9
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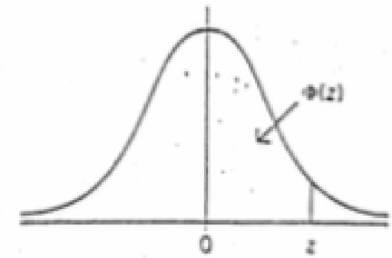

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$$P(Z < -0.913) = 1 - \Phi(0.913) = 1 - 0.8194 = 0.1806.$$

The standard normal

Example: Find $P(Z < -0.913)$ *and* $P(Z > -0.913)$.

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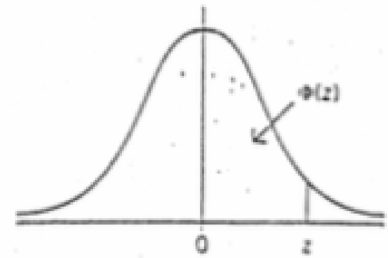

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$$P(Z > -0.913) = \Phi(0.913) = 0.8194.$$

The standard normal

Important results worth learning

- $P(Z \le -a) = 1 - \Phi(a)$
- $P(Z \le -a) = \Phi(a)$
- $P(a \le Z \le b) = \Phi(b) - \Phi(a)$
- $P(-a \le Z \le b) = \Phi(a) + \Phi(b) - 1$
- $P(-a \le Z \le -b) = \Phi(b) - \Phi(a)$
- $P(|Z| \le a) = P(-a \le Z \le a) = 2\Phi(a) - 1$
- $P(|Z| \ge a) = P(\{Z < -a\} \cup \{Z > a\}) = 2\big(1 - \Phi(a)\big)$

The standard normal

Example 6.3: Lengths of metal strips produced by a machine are normally distributed with mean length of 150cm and a standard deviation of 10cm. Find the probability that the length of a randomly selected strip is

(a) shorter than 165cm.

(b) within 5cm of the mean.

The standard normal

Example 6.3: Lengths of metal strips produced by a machine are normally distributed with mean length of 150cm and a standard deviation of 10cm. Find the probability that the length of a randomly selected strip is

(a) shorter than 165cm.

(b) within 5cm of the mean.

Let X represent the lenghts of metal strips produced. Then $X \sim \mathrm{N}(150, 10^2)$.

(a)

$$\begin{aligned} P(X < 165) = P(Z < (165 - 150)/10) = P(Z < 1.5) \\ = \Phi(1.5) = 0.9332 \end{aligned}$$

The standard normal

(b) To be within 5cm of the mean implies that the value X should not be more than 155cm and should not be less than 145cm. That is, $145 < X < 155$.

The standard normal

(b) To be within 5cm of the mean implies that the value X should not be more than 155cm and should not be less than 145cm. That is, $145 < X < 155$.

Thus,

$$\begin{aligned} P(145 < X < 155) &= P\left(\frac{145-150}{10} < Z < \frac{155-150}{10}\right) \\ &= P(-0.5 < Z < 0.5) \\ &= 2\Phi(0.5) - 1 \\ &= 2 \times 0.6915 - 1 \\ &= 0.383 \end{aligned}$$

Linear functions of normal r.v.

If $X \sim \mathrm{N}(\mu, \sigma^2)$ and a and b are constants (with $b \neq 0$), then:

$$a + bX \sim \mathrm{N}(a + b\mu, b^2\sigma^2)$$

Linear functions of normal r.v.

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If X_1 and X_2 are *independent* normal random variables, such that $X_1 \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$, then:

$$X_1 \pm X_2 \sim \mathrm{N}(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$$

Note that variances are *added* even when dealing with the difference between independent random variables.

Linear functions of normal r.v.

Example: An invester has the choice of two out of four investments: X_1*,* X_2*,* X_3 *and* X_4*. The profits (in BND 1,000 per annum) from these may be assumed to be indpendently distributed and:*

- Profit from $X_1 \sim \mathrm{N}(2, 1)$
- Profit from $X_2 \sim \mathrm{N}(3, 3)$
- Profit from $X_3 \sim \mathrm{N}(1, 0.25)$
- Profit from $X_4 \sim \mathrm{N}(2.5, 4)$

Which pair of investments should the investor choose in order to maximise the probability of making a total profit of at least BND 2,000? What is the maximum probability?

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Which pair of investments should the investor choose in order to maximise the probability of making a total profit of at least BND 2,000? What is the maximum probability?

Let's sum the ${}^4C_2 = 6$ possible pairwise combinations of normal r.v. Then calculate the Z value. We want to maximise $P(Z > 2)$ (the distribution which gives us the smallest Z value).

Linear functions of normal r.v.

- $X_1 + X_2 \sim \mathrm{N}(5, 4)$, $Z = -1.50$
- $X_1 + X_3 \sim \mathrm{N}(3, 1.25)$, $Z = -0.89$
- $X_1 + X_4 \sim \mathrm{N}(4.5, 5)$, $Z = -1.12$
- $X_2 + X_3 \sim \mathrm{N}(4, 3.25)$, $Z = -1.11$
- $X_2 + X_4 \sim \mathrm{N}(5.5, 7)$, $Z = -1.32$
- $X_3 + X_4 \sim \mathrm{N}(3.5, 4.25)$, $Z = -0.73$

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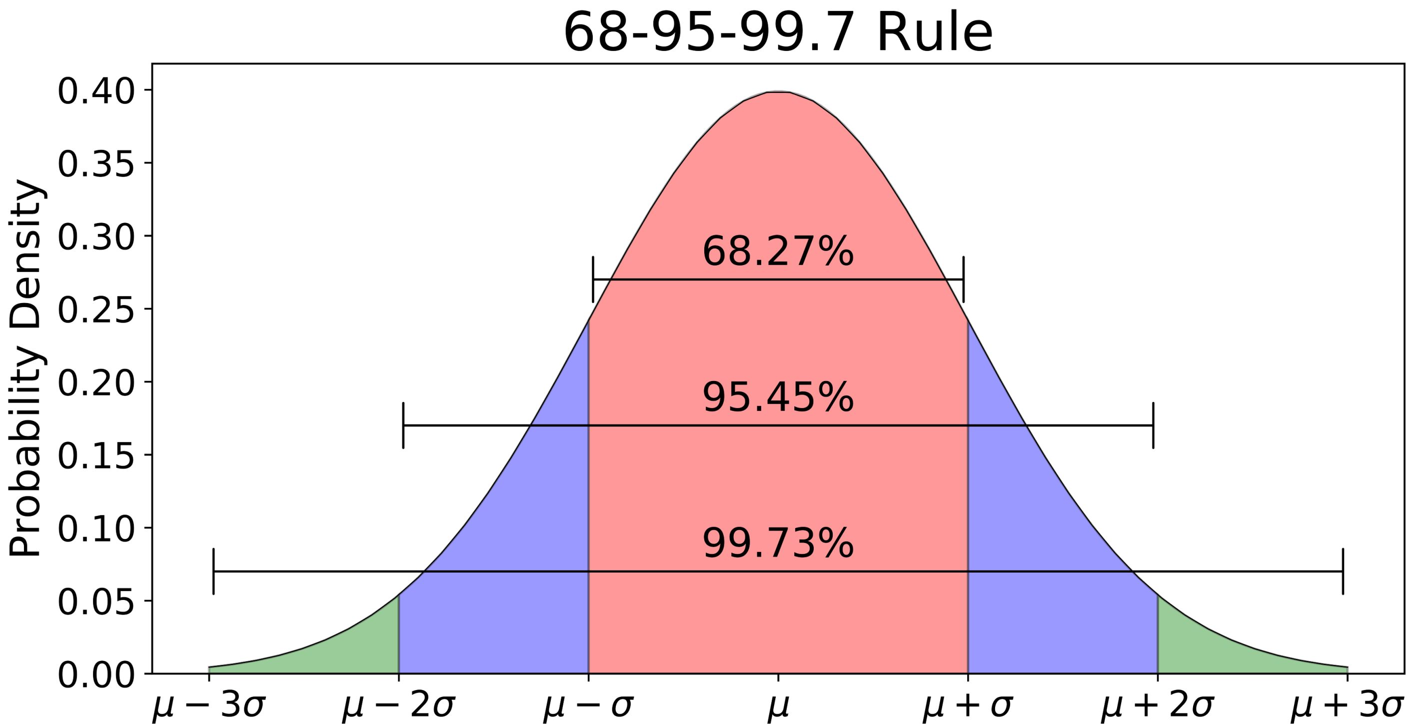
- $X_3 + X_4 \sim \mathrm{N}(3.5, 4.25)$, $Z = -0.73$

Evidently, the combination of X_1 and X_2 gives us a mean profit of BND 5,000 with a standard deviation of 2. This gives the smalles Z values and thus the highes probability of out all possible pairs of investments.

$$P(Z > -1.5) = \Phi(1.5) = 0.9332$$

68-95-99.7 rule

Also known as the empirical rule, is a shorthand to remember the percentage of values that lie within a band around the mean in a normal distribution.



Normal approximations to discrete distributions

Normal approximation to binomial

Recall that $X \sim \mathrm{Bin}(n, p)$ has these properties

- $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$
- $E(X) = np$
- $Var(X) = np(1-p)$

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- $E(X) = np$
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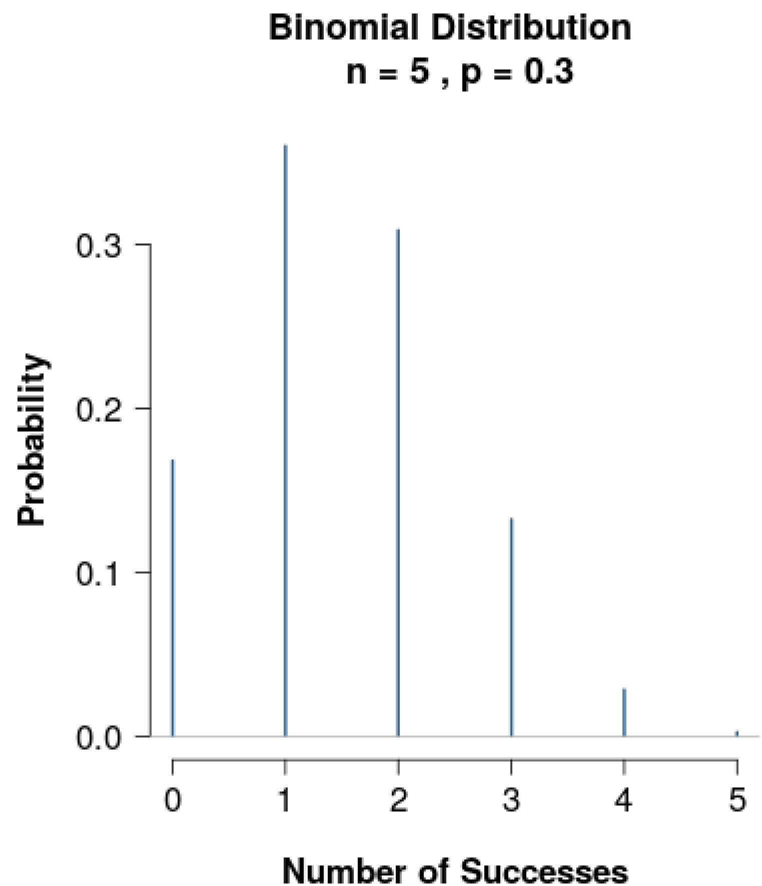
Consider computing these probabilities below, when $n = 100$:

- $P(X = 50)$ (try calculating $\binom{100}{50}$ and p^{50} for various values of p—what do you observe?)
- $P(X > 50)$
- $P(25 < X < 75)$ (how many computations must you do?)

Normal approximation to binomial

What does the distribution actually look like? Fix $p = 0.3$ and let $n \to \infty$:



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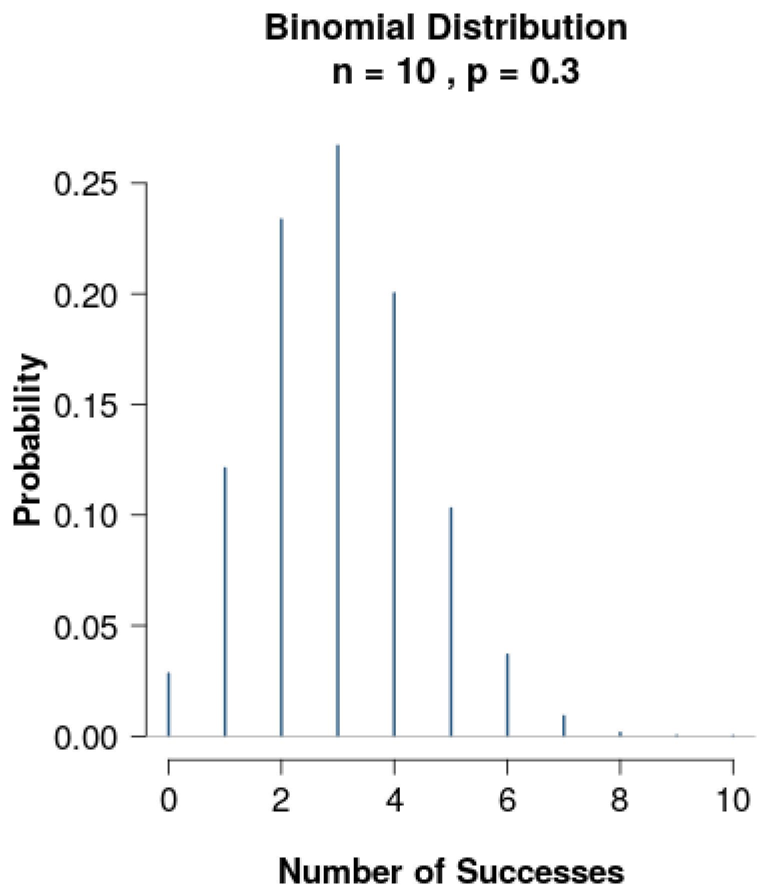

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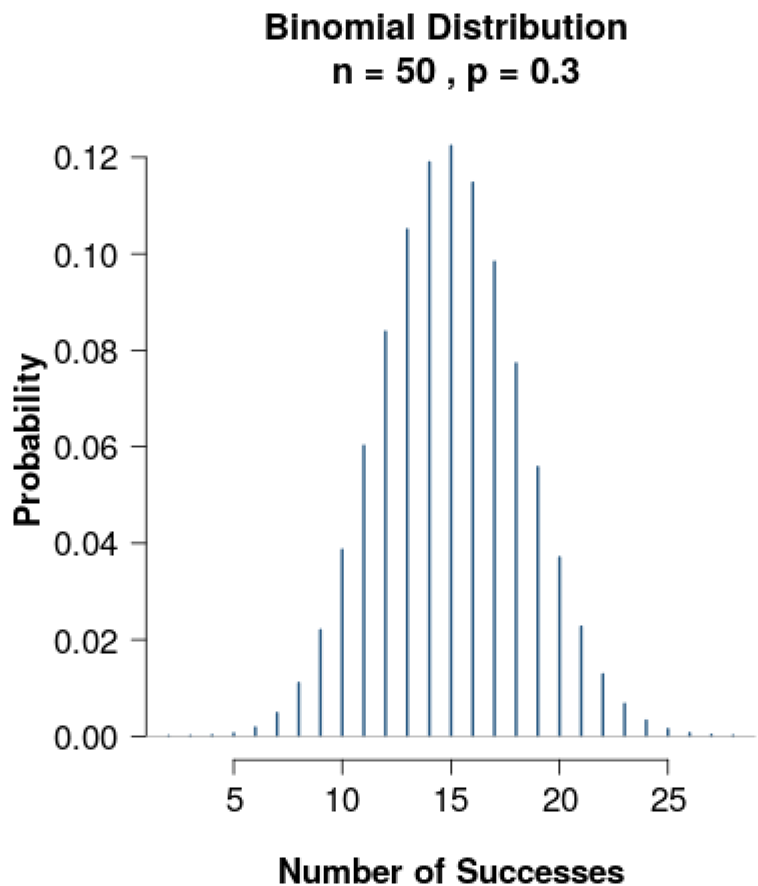

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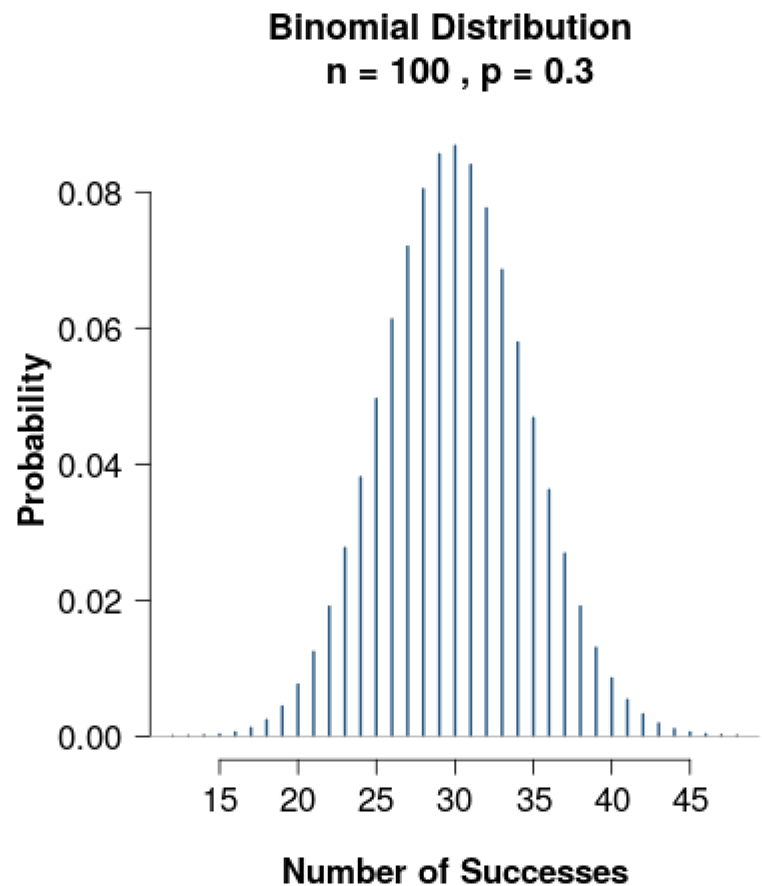

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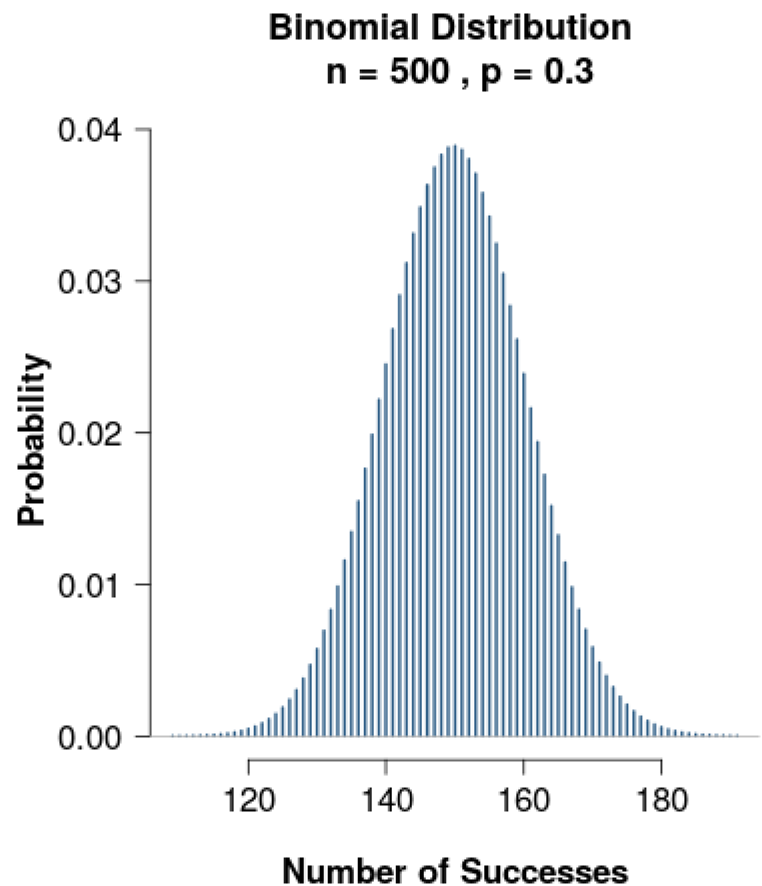

Normal approximation to binomial

When `(np >5)` and `(n(1-p)>5)`, it is appropriate to use the normal approximation to the binomial

As $n \to \infty, \mathrm{Bin}(n, p) \to N\big(np, np(1-p)\big)$.

Normal approximation to binomial

When `(np >5)` and `(n(1-p)>5)`, it is appropriate to use the normal approximation to the binomial

As $n \to \infty$, $\text{Bin}(n,p) \to N\big(np, np(1-p)\big)$. This means that for $X \sim \text{Bin}(n,p)$,

$$
\begin{aligned}
P(X < x) &\approx P\left(\frac{X - np}{\sqrt{np(1-p)}} < \frac{x - np}{\sqrt{np(1-p)}}\right) \\
&= \Phi\left(\frac{x - np}{\sqrt{np(1-p)}}\right)
\end{aligned}
$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution.

Continuity correction

For discrete distributions,

$$P(X \leq x) = P(X < x + 1)$$

But we know for continuous distributions, $P(X \leq x) = P(X < x)$.
So when a discrete X is approximated by a continuous distribution, we must apply **continuity correction**.

Continuity correction

For discrete distributions,

$$P(X \leq x) = P(X < x + 1)$$

But we know for continuous distributions, $P(X \leq x) = P(X < x)$. So when a discrete X is approximated by a continuous distribution, we must apply **continuity correction**.

Roughly speaking,

$$P(X \leq x) \approx P(X_{cts} < x + 1/2)$$

Continuity correction

Apply these continuity corrections in your calculations!

Discrete	Continous
$X = c$	$c - 0.5 < X < c + 0.5$
$X < c$	$X < c + 0.5$
$X \leq c$	$X < c + 0.5$
$X > c$	$X > c - 0.5$
$X \geq c$	$X > c - 0.5$

Continuity correction

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Questions

- What is continuity correction for $a < X < b$?
- What is $P(X > 0)$?
- What is $P(X = -1)$?

Continuity correction

Example 6.8: Find the probability of obtaining 4, 5, 6 or 7 heads when a fair coin is tossed 12 times.

Continuity correction

Example 6.8: Find the probability of obtaining 4, 5, 6 or 7 heads when a fair coin is tossed 12 times.

Let X be the number of heads when coin is tossed 12 times. $X \sim \mathrm{Bin}(12, 0.5)$. We are interested in $P(X = 4, 5, 6, 7)$. Since np and $n(1-p)$ are both greater than 5, we can use the normal approximation $X \sim N(6, 3)$.

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$$\begin{aligned}
P(4 \leq X \leq 7) &\approx P(3.5 < X < 7.5) \\
&= P\left(\frac{3.5 - 6}{\sqrt{3}} < Z < \frac{7.5 - 6}{\sqrt{3}}\right) \\
&= P(-1.44 < Z < 0.866) \\
&= \Phi(0.866) - \Phi(-1.44) \\
&= 0.8067549 - 0.0749337 = 0.732
\end{aligned}$$

Normal approximation to Poisson

Recall that $X \sim \text{Pois}(\lambda)$ has these properties

- $P(X = x) = e^{-\lambda}\lambda^x/x!$
- $E(X) = \lambda$
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This can be problematic if computations not handled carefully (just like the binomial case).

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As $n \to \infty$, $\mathrm{Pois}(\lambda) \to N\big(\lambda, \lambda\big)$. This means that for $X \sim \mathrm{Pois}(\lambda)$,

$$
\begin{aligned}
P(X < x) &\approx P\left(\frac{X-\lambda}{\sqrt{\lambda}} < \frac{x-\lambda}{\sqrt{\lambda}}\right) \\
&= \Phi\left(\frac{x-\lambda}{\sqrt{\lambda}}\right)
\end{aligned}
$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. But also don't forget *continuity correction*!

Normal approximation to Poisson

Example 6.10: A radioactive disintegration gives counts that follow a Poisson distribution with a mean count of 25 per second. Find the probability that in a one-second interval the count is between 23 and 27 inclusive.

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Denote the radioactive disintegration counts by X. Then $X \sim \mathrm{Pois}(25)$. We are interested in $P(23 \leq X \leq 27)$. Since $\lambda > 15$, we can apply a normal approximation.

Normal approximation to Poisson

Example 6.10: A radioactive disintegration gives counts that follow a Poisson distribution with a mean count of 25 per second. Find the probability that in a one-second interval the count is between 23 and 27 inclusive.

Denote the radioactive disintegration counts by X. Then $X \sim \mathrm{Pois}(25)$. We are interested in $P(23 \leq X \leq 27)$. Since $\lambda > 15$, we can apply a normal approximation.

$$\begin{aligned} P(23 \leq X \leq 27) &\approx P(22.5 < X < 27.5) \\ &= P\left(\frac{22.5 - 25}{\sqrt{25}} < Z < \frac{27.5 - 25}{\sqrt{25}}\right) \\ &= P(-0.5 < Z < 0.5) \\ &= 2\Phi(0.5) - 1 \\ &= 2(0.6914625) - 1 = 0.383 \end{aligned}$$

Special continuous distributions: The exponential distributions

Exponential distribution

Suppose you are interested in

- The amount of time until an earthquake occurs.
- The time between two lightbulbs failing.
- The length (in minutes) of faculty staff meetings at UBD.
- The average waiting time at a hospital's A&E.

Exponential distribution

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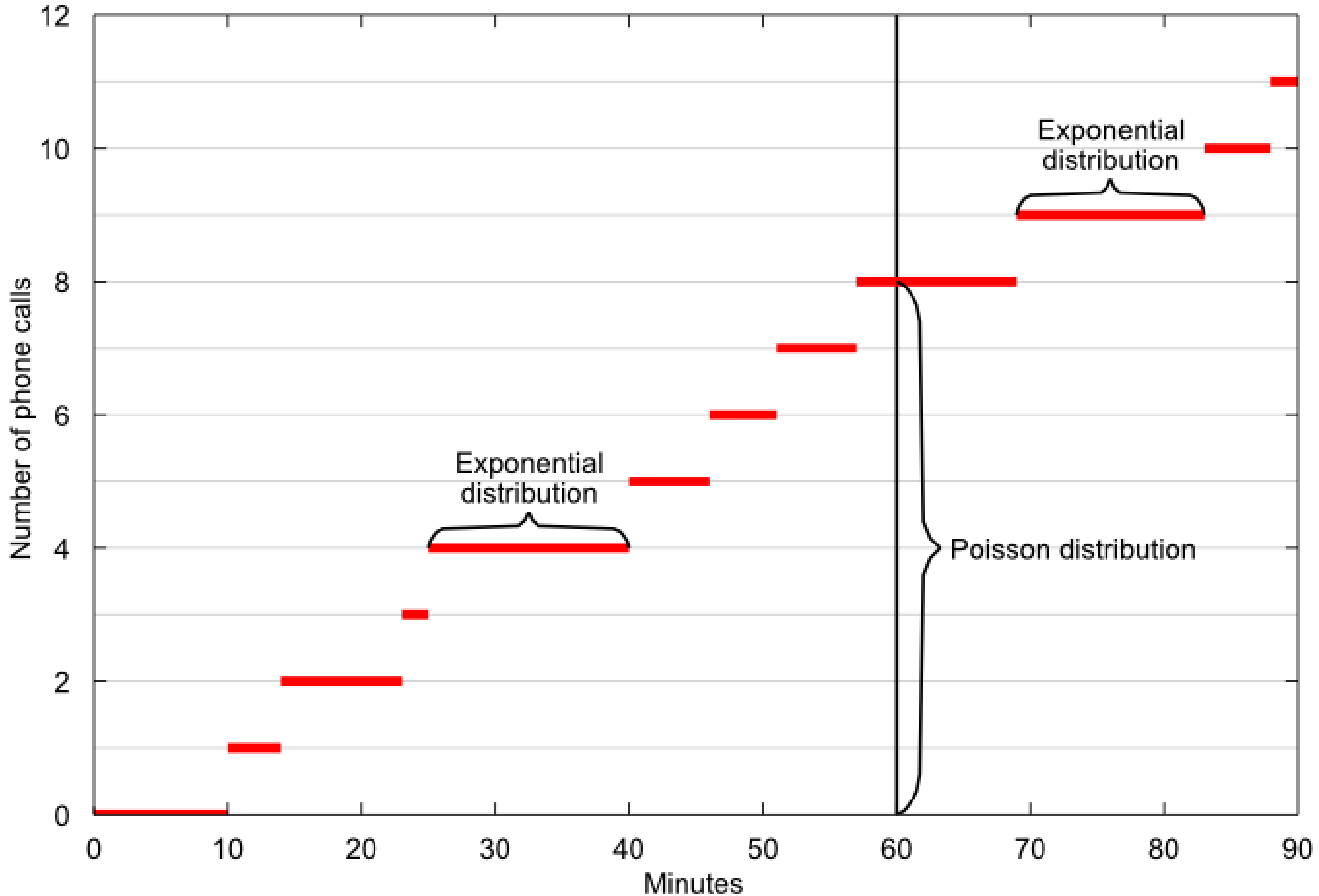
The exponential distribution is the probability distribution of the time between events in a Poisson point process, i.e., a process in which events occur continuously and independently at a constant average rate.

Exponential distribution



Exponential distribution

Let X be disributed according to an exponential distribution with rate λ. We write $X \sim \mathrm{Exp}(\lambda)$. Then its pdf is

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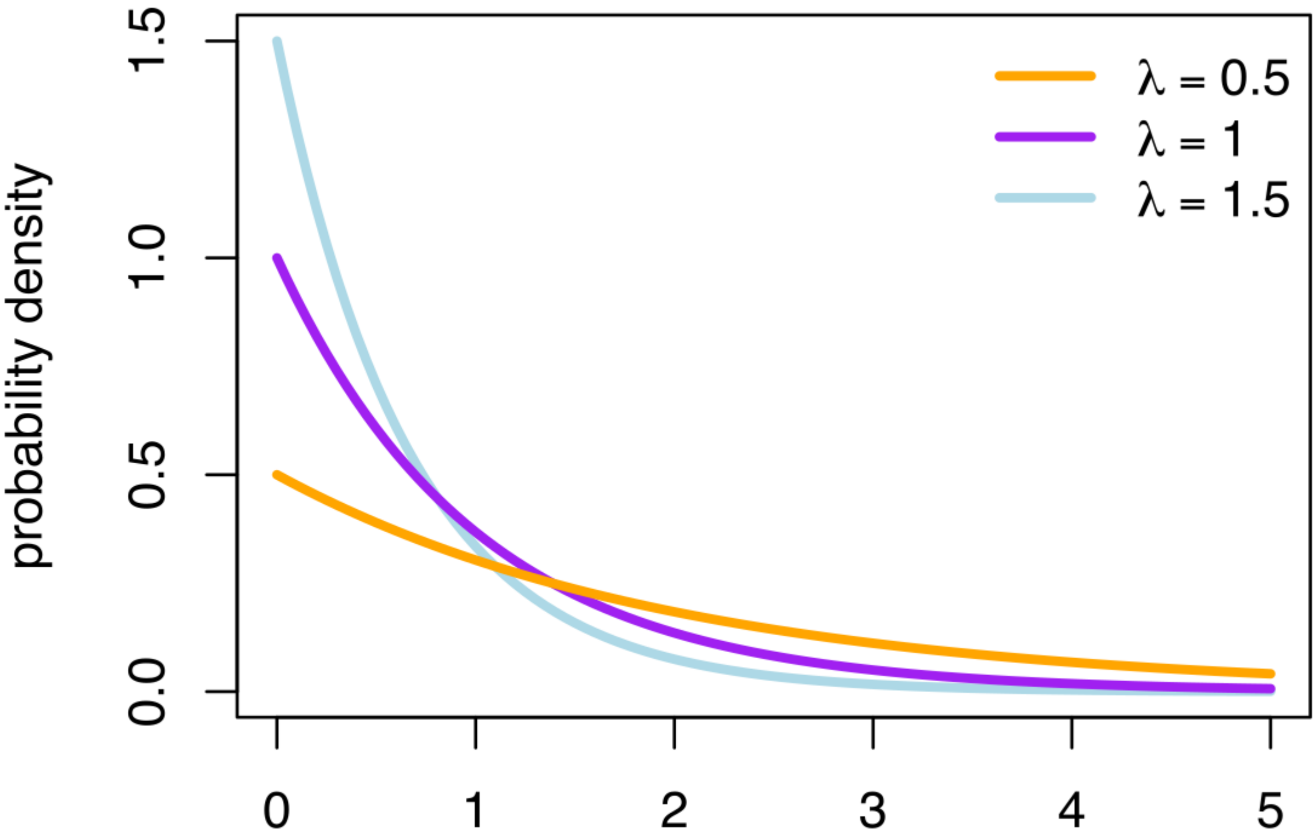
<u>Properties</u>

- The parameter of the exponential distribution is the rate $\lambda > 0$.
- Support: $x > 0$.
- $E(X) = 1/\lambda$.
- $Var(X) = 1/\lambda^2$.

Exponential distribution

The pdf experiences "exponential decay"— long wait times between two events occuring becomes more and more unlikely.



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$$\begin{aligned} P(0.5 < X < 1) &= \int_{0.5}^{1} 3e^{-3x} dx \\ &= \left[-e^{-3x}\right]_{0.5}^{1} \\ &= e^{-1.5} - e^{-3} = 0.173 \end{aligned}$$

Exponential distribution

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$$\begin{aligned} P(X < 6) &= \int_{0}^{6} 3e^{-3x} dx \\ &= [-e^{-3x}]_{0}^{6} \\ &= 1 - e^{-18} \approx 1 \end{aligned}$$

Memoryless property

The exponential distribution has what is called the "memoryless" property. Given that you know the last event occured in x amount of time, the next occurence is independent of this.

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For example, assume that bus waiting times are exponentially distributed. A memoryless wait for a bus would mean that the probability that a bus arrived in the next minute is the same whether you just got to the station or if you've been sitting there for twenty minutes already.

https://perplex.city/memorylessness-at-the-bus-stop-f2c97c59e420

END