SM-1402/CU-0304 Basic Statistics

Chapter 1 (Summarising Data)

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Representation of Data

Types of data

Numeric data can be one of two types: *discrete* or *continuous*.

Discrete data

Discrete data can take **only exact (whole) values**, for example:

- the number of lecture rooms in UBD,
- the number of students in a class,
- the number of durians on each tree in a plantations.

Example of raw discrete data: *An orange tree produces roughly a number of fruits per tree. A survey of 30 orange trees yielded the following number of fruits per tree.*

```
 [1] 31 32 34 30 32 33 31 34 32 33 35 32 32 33 32 32 33 31 32 33
[21] 32 30 31 31 32 30 33 32 33 33
```

Types of data

Numeric data can be one of two types: *discrete* or *continuous*.

Continuous data

Continuous data *cannot* take exact (whole) values, but can be given only within a specified range, or measured to a specified degree of accuracy.

- the heights of students in UBD,
- the length of classrooms in UBD,
- the travel time in minutes from home to UBD.

Example of raw continuous data: *A survey of the heights of 20 children in a sports club yielded the following measurements (rounded to the nearest centimetre)*

```
 [1] 133 136 120 138 133 131 127 141 127 143 130 131 125 144 128
[16] 134 135 137 133 129
```

Grouping data

Raw data can be grouped for a more concise presentation.

Discrete data (oranges from trees)

Count the number of times each value occurs and summarise these in a table known as a **frequency distribution**

```
 [1] 31 32 34 30 32 33 31 34 32 33 35 32 32 33 32 32 33 31 32 33
[21] 32 30 31 31 32 30 33 32 33 33
```

Number of oranges	**30**	**31**	**32**	**33**	**34**	**35**	
Frequency	3	5	11	8	2	1	Total = 30

Grouping data

Raw data can be grouped for a more concise presentation.

Continuous data (children's heights)

To form a **frequency distribution** of the heights of the 20 children, group the information into **classes or intervals**

```
 [1] 133 136 120 138 133 131 127 141 127 143 130 131 125 144 128
[16] 134 135 137 133 129
```

Height (cm)	**119.5-124.5**	**124.5-129.5**	**129.5-134.5**	**134.5-139.5**	**139.5-144.5**	
Freq.	1	5	7	4	3	Total = 20

The values 119.5, 124.5, 129.4, ... are called the **class boundaries**. The **upper class boundary** of one interval is the lower class boundary of the next interval.

Grouping data

Raw data can be grouped for a more concise presentation.

Terminology

- The **mode** (for discrete data) is the is the value which occurs most often. This can be read off directly from the frequency distribution.

  For the orange data, the mode is *32 oranges*.

- The **width of an interval** (for grouped data) is the difference between boundaries.

  For the children's height frequency distribution, the interval width used was *5cm*.

Ways of grouping data

Example 1

The length of 30 metal rods were measured to the nearest millimetres.

Length (mm)	**27-31**	**32-36**	**37-46**	**47-51**
Freq.	4	11	12	3

- The interval 27-31 means 26.5mm $\leq$ length $<$ 31.5mm, and so on.
- The class boundaries are 26.5, 31.5, 36.5, 46.5, and 51.5.
- The class widths[1] are 5, 5, 10, and 5.

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[1] Class widths do not necessarily have to be the same for all classes.

Ways of grouping data

Example 2

100 students took a test, and their marks are tabulated.

Mark	**30-39**	**40-49**	**50-59**	**60-69**	**70-79**	**80-99**
Freq.	10	14	26	20	18	12

The frequency distribution can be interpreted in two ways:

- As **discrete** data.
  - The interval 30-39 represents $30 \leq \text{mark} < 40$.
  - The class boundaries are 30, 40, 50, 60, 70, 80, 100.
  - The class widths are 10, 10, 10, 10, 10, 20.
- As **continuous** data assuming marks are to the nearest integer.
  - The interval 30-39 represents $29.5 \leq \text{mark} < 39.5$.
  - The class boundaries are 29.5, 39.5, 49.5, 59.5, 69.5, 79.5, 99.5.
  - The class widths are 10, 10, 10, 10, 10, 20.

Ways of grouping data

Example 3

Ages (in completed years) of applicants for a teaching post.

Age (years)	**21-24**	**25-28**	**29-32**	**33-40**	**41-52**
Freq.	4	2	2	1	1

- Since ages are given in *completed years* (not to the nearest year), then '21-24' means $21 \leq \text{age} < 25$[1].
- We can also write the categories as '21-', '25-', etc.
- The class bounderies are 21, 25, 29, 33, 41, 53.
- The class widths are 4, 4, 4, 8, 12.

---

[1] Someone who is 24 years and 11 months is not yet 25 years old, so would come under this category.

Histograms

Consider the following table for the distribution of ages of 118 passengers on a shuttle flight from Bandar Seri Begawan, Brunei to Kota Kinabalu, Sabah.

Age (years)	**0-19**	**20-39**	**40-49**	**50-69**	**70-100**
Frequency	4	44	36	28	6

A **histogram** provides a visual representation of *grouped data*.

Histograms

Age (years)	0-19	20-39	40-49	50-69	70-100
Frequency	4	44	36	28	6

Histograms

- In a histogram, the area of each bar is proportional to the frequency that it represents, and thus

$$\text{total area} \propto \text{total frequency}$$

- The vertical axis is not labelled frequency, but *frequency density*[1], where

$$\text{freq. density} = \frac{\text{frequency}}{\text{interval width}}$$

---

[1] You may also find histograms with the vertical axis labelled 'density' (c.f. density plots), or even 'frequency' (this type of histogram visualises where values are concentrated).

Histograms

- Using the formula given, we can calculate the complete table for the histogram as follows:

Age (years)	Interval width	Frequency	Freq. density
0-19	20	4	4 ÷ 20 = 0.2
20-39	20	44	44 ÷ 20 = 2.2
40-49	10	36	36 ÷ 10 = 3.6
50-69	20	28	28 ÷ 20 = 1.4
70-99	30	6	6 ÷ 30 = 0.2

Histograms

- Histograms may have bars of different widths, so the height of the bar must be adjusted accordingly.
- The **modal class** is the interval with the greatest frequency density, i.e. the interval represented by *the highest bar* in the histogram. For the current example, the modal class is the interval '40-49'[1].

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[1] Notice that this interval does not have the greatest frequency, but it does have the greatest frequency density.

Frequency polygons

An alternative way of displaying grouped data is using **frequency polygons**.

Frequency polygons

Age (years)	0-19	20-39	40-49	50-69	70-100
Frequency	4	44	36	28	6

Frequency polygons

To construct a frequency polygon,

- Calculate the mid-interval value, where

$$\text{mid-interval value} = \frac{1}{2}(\text{LCB} + \text{UCB})$$

- Calculate the frequency densities.
- Join points together with a straight line.

Frequency polygons

- Calculate the complete table for the frequency polygon:

Age (years)	Mid-interval value	Interval width	Frequency	Freq. density
0-19	(0 + 20) ÷ 2 = 10	20	4	4 ÷ 20 = 0.2
20-39	(20 + 40) ÷ 2 = 30	20	44	44 ÷ 20 = 2.2
40-49	(40 + 50) ÷ 2 = 45	10	36	36 ÷ 10 = 3.6
50-69	(50 + 70) ÷ 2 = 60	20	28	28 ÷ 20 = 1.4
70-99	(70 + 100) ÷ 2 = 85	30	6	6 ÷ 30 = 0.2

Pie charts

Yet another diagram to display grouped data is a pie chart.

- A circle (pie) is divided into several **sectors** (slices of the pie), with each sector corresponding to a class interval.
- The central angle θ of the sectors are *proportional* to the frequencies of the intervals they represents, according to the formula

$$\theta = \frac{\text{class freq.}}{\text{total freq.}} \times 360^\circ$$

- The area of a sector A of a circle with radius r is given by the formula

$$A = \frac{\text{class freq.}}{\text{total freq.}} \times \pi r^2 = \frac{\theta}{360} \times \pi r^2$$

Pie charts

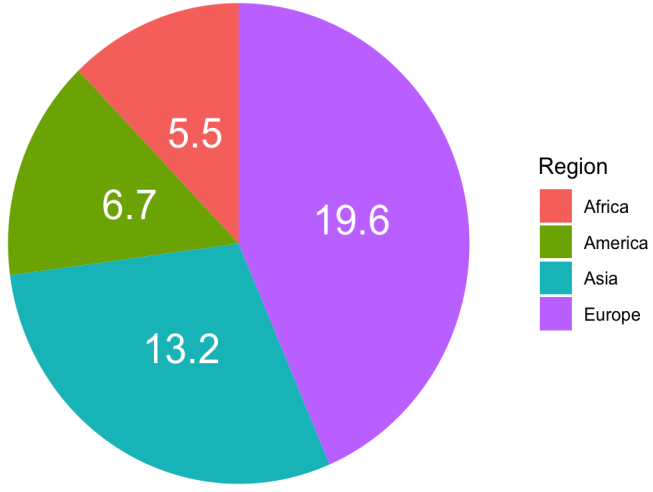
WARNING: Pie charts should only be used for grouped data with equal class intervals

Annual sales (millions of dollars) of a company by region

Region	Africa	America	Asia	Europe
Sales	5.5	6.7	13.2	19.6

Pie chart table

Region	Sales	Angle	Area (r=1)
Africa	5.5	5.5/45 x 360 = 44	5.5/45 x π = 0.38
America	6.7	6.7/45 x 360 = 54	6.7/45 x π = 0.47
Asia	13.2	13.2/45 x 360 = 106	13.2/45 x π = 0.92
Europe	19.6	19.6/45 x 360 = 156	19.6/45 x π = 1.37
Total	45.0	360	π



Data Summaries

The mean

- A 'typical' or 'average' value is useful when interpreting data. One such alue is the **mean**[1].
- The formula for the mean is

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{x_1 + \cdots + x_n}{n}$$

- Consider 5 numbers

  `0.9, 1.4, 2.8, 3.1, 5.6`

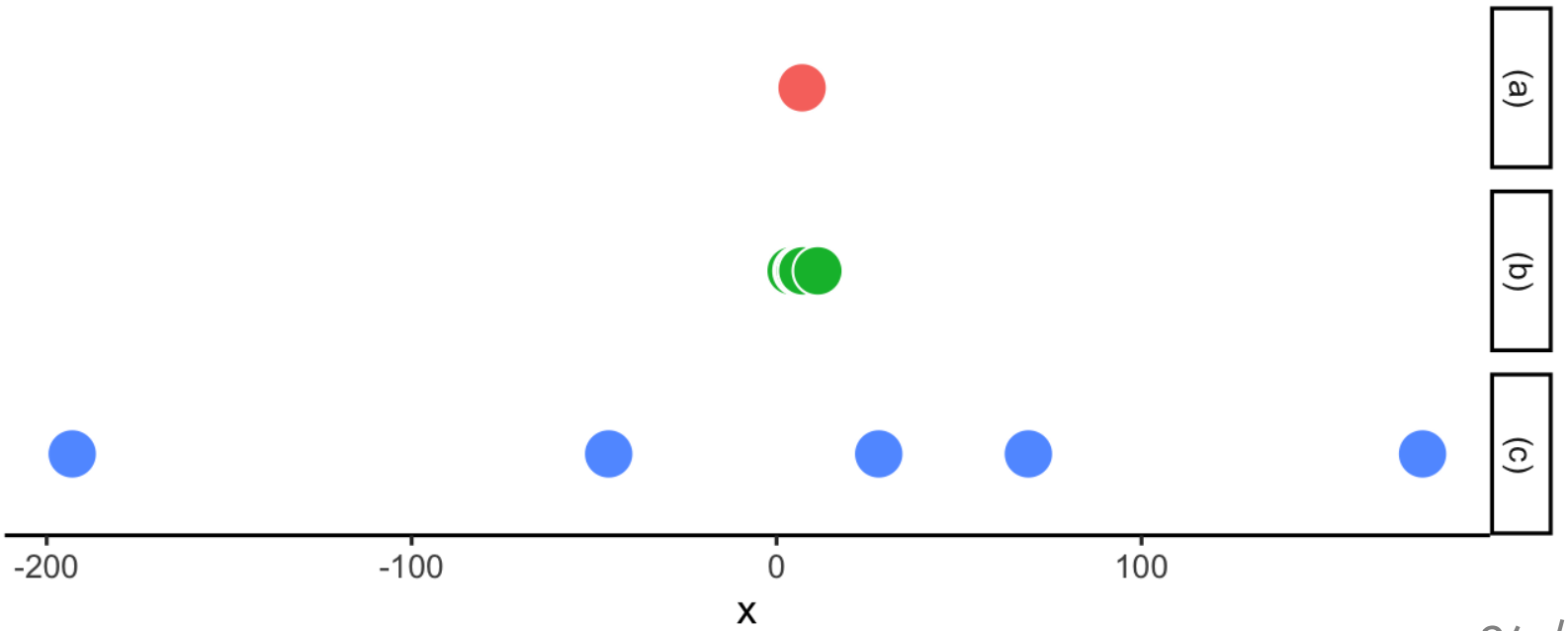
  The mean is $(0.9 + 1.4 + 2.8 + 3.1 + 5.6)/5 = 13.8/5 = 2.76$.

[1] 'Average' does not always refer to the 'mean'. There are other measures of *central tendencies*, such as the mode, median and others.

Variability

- Each of these sets of numbers has a mean of 7, but the *spread* of each set is different.

```
(a): 7, 7, 7, 7, 7
(b): 4, 6, 6.5, 7.2, 11.3
(c): -193, -46, 28, 69, 177
```



Variability

We can measure the **variability** or spread of a distribution by calculating the *range* and the *standard deviation*.

The range

- The range is based entirely on the extreme values of the distribution.

$$\text{range} = \text{highest value} - \text{lowest value}$$

```
(a): range(7, 7, 7, 7, 7)          = 7 - 7          = 0
(b): range(4, 6, 6.5, 7.2, 11.3) = 11.3 - 4      = 7.3
(c): -193, -46, 28, 69, 177        = 177 - (-193) = 370
```

Variability

We can measure the **variability** or spread of a distribution by calculating the *range* and the *standard deviation*.

The standard deviation and the variance

- The standard deviation s gives a measure of the deviations of the readings from the mean $\bar{x}$.
- Steps to calculate s (for $i = 1, \ldots, n$)
    1. For each reading x_i, calculate $x_i - \bar{x}$, its deviation from the mean
    2. Square this deviation to give $(x_i - \bar{x})^2$
    3. Sum all of these squared deviations to give $\sum_{i=1}^{n} (x_i - \bar{x})^2$
    4. Divide this sum by n to give the **variance**
    5. Finally, take the positive square root of the variance to obtain the standard deviation s

Variability

We can measure the **variability** or spread of a distribution by calculating the *range* and the *standard deviation*.

The standard deviation and the variance

- The formula for the variance is given by

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

- The formula for the standard deviation is given by

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

Calculating the s.d.

Set (a)

i	**Data** x_i	**Mean** $\bar{x}$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1	7	7	0	0
2	7	7	0	0
3	7	7	0	0
4	7	7	0	0
5	7	7	0	0
				SUM = 0

$$s^2 = 0/5 = 0$$

$$s = \sqrt{s^2} = \sqrt{0} = 0$$

Calculating the s.d.

Set (b)

i	**Data** x_i	**Mean** $\bar{x}$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1	4	7	-3	9
2	6	7	-1	1
3	6.5	7	-0.5	0.25
4	7.2	7	0.2	0.04
5	11.3	7	4.3	18.49
				SUM = 28.78

$$s^2 = 28.78/5 = 5.756$$

$$s = \sqrt{s^2} = \sqrt{5.756} = 2.4 \text{ (1 d.p.)}$$

Notes on the s.d.

- The units of the standard deviation are the *same* as the units of the data.
- Standard deviations are useful when comparing sets of data; the higher the standard deviation, the greater the variability in the data.
- Alternative formula for the s.d.

$$s = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2}$$

Cumulative frequencies

- The cumulative frequency is the **total frequency** *up to a particular item*.
- A cumulative frequency distribution can be obtained from a frequency distribution and can be illustrated
  - When the data are *discrete* and *ungrouped* (**step diagram**);
  - When the data are *continuous* or in the form of *grouped discrete distribution* (**cumulative frequency polygon/curve**).

Step diagram

Example: The table above shows the number of attempts needed to pass the driving test by 100 candidates at a particular test centre.

No. of attempts	**1**	**2**	**3**	**4**	**5**	**6**
Freq. (No. of candidates)	33	42	13	6	4	2

The cumulative frequency table is created by adding the attempts cumulatively.

No. of attempts	$\leq$ **1**	$\leq$ **2**	$\leq$ **3**	$\leq$ **4**	$\leq$ **5**	$\leq$ **6**
Cumulative freq.	33	75	88	94	98	100

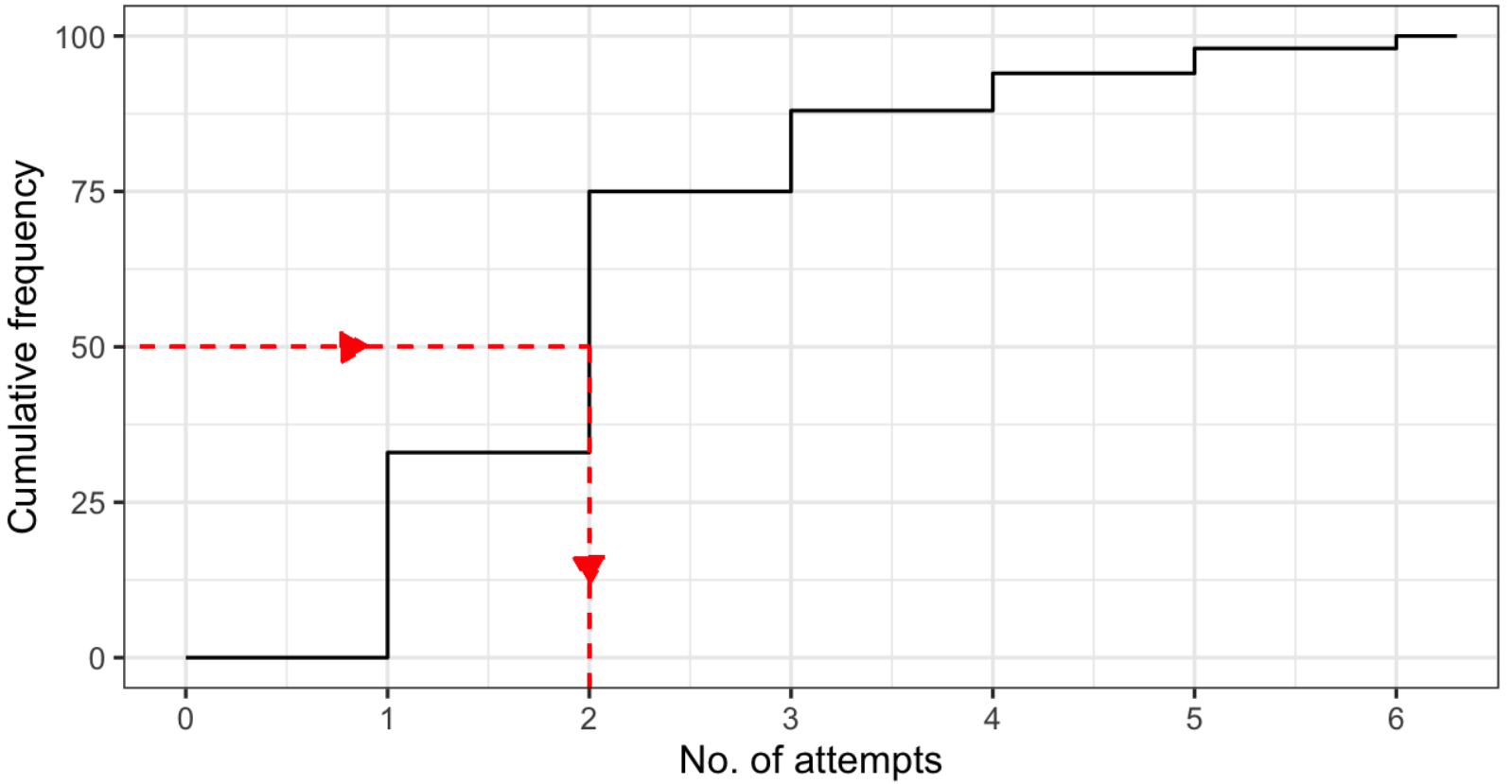
Notice that the last entry of the cumulative frequency table must equal to the number of observations in total.

Step diagram

No. of attempts	$\leq$ **1**	$\leq$ **2**	$\leq$ **3**	$\leq$ **4**	$\leq$ **5**	$\leq$ **6**
Cumulative freq.	33	75	88	94	98	100

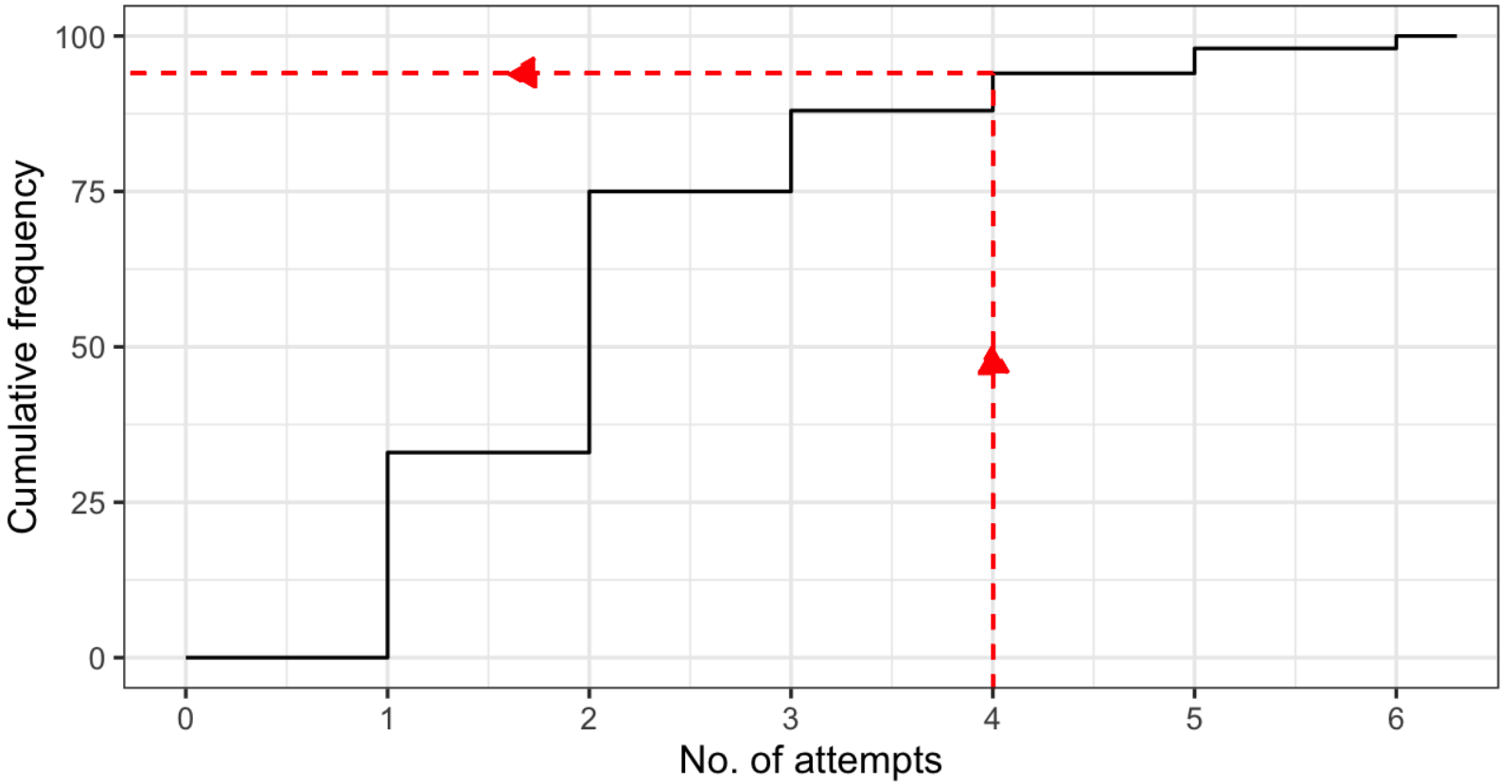
Step diagram

Arrange candidates in ascending order of number of attempts. How many tries did the 50th candidate take to pass the test?



Step diagram

How many candidates took up to four attempts?



Step diagram

- It only makes sense when you read from the discrete whole values on the horizontal axis. It would be silly to consider 3.6 attempts, for example.
- In the step diagram, the *mode* is given by the value of the variable that gives the 'steepest' step.

Cumulative frequency polygons

Example: Six weeks after planting, the heights of 30 broad bean plants were measured and the frequency distribution formed as shown.

Height x (cm)	**$3 \leq x <$ 6**	**$6 \leq x <$ 9**	**$9 \leq x <$ 12**	**$12 \leq x <$ 15**	**$15 \leq x <$ 18**	**$18 \leq x <$ 21**
Frequency	1	2	11	10	5	1

- The cumulative frequency is calculated up to each UCB (6, 9, 12, 15, 18, 21).
- The lower boundary of the first class is 3.

Height x (cm)	**< 3**	**< 6**	**< 9**	**< 12**	**< 15**	**< 18**	**< 21**
Cumulative freq.	0	1	3	14	24	29	30

Cumulative frequency polygons

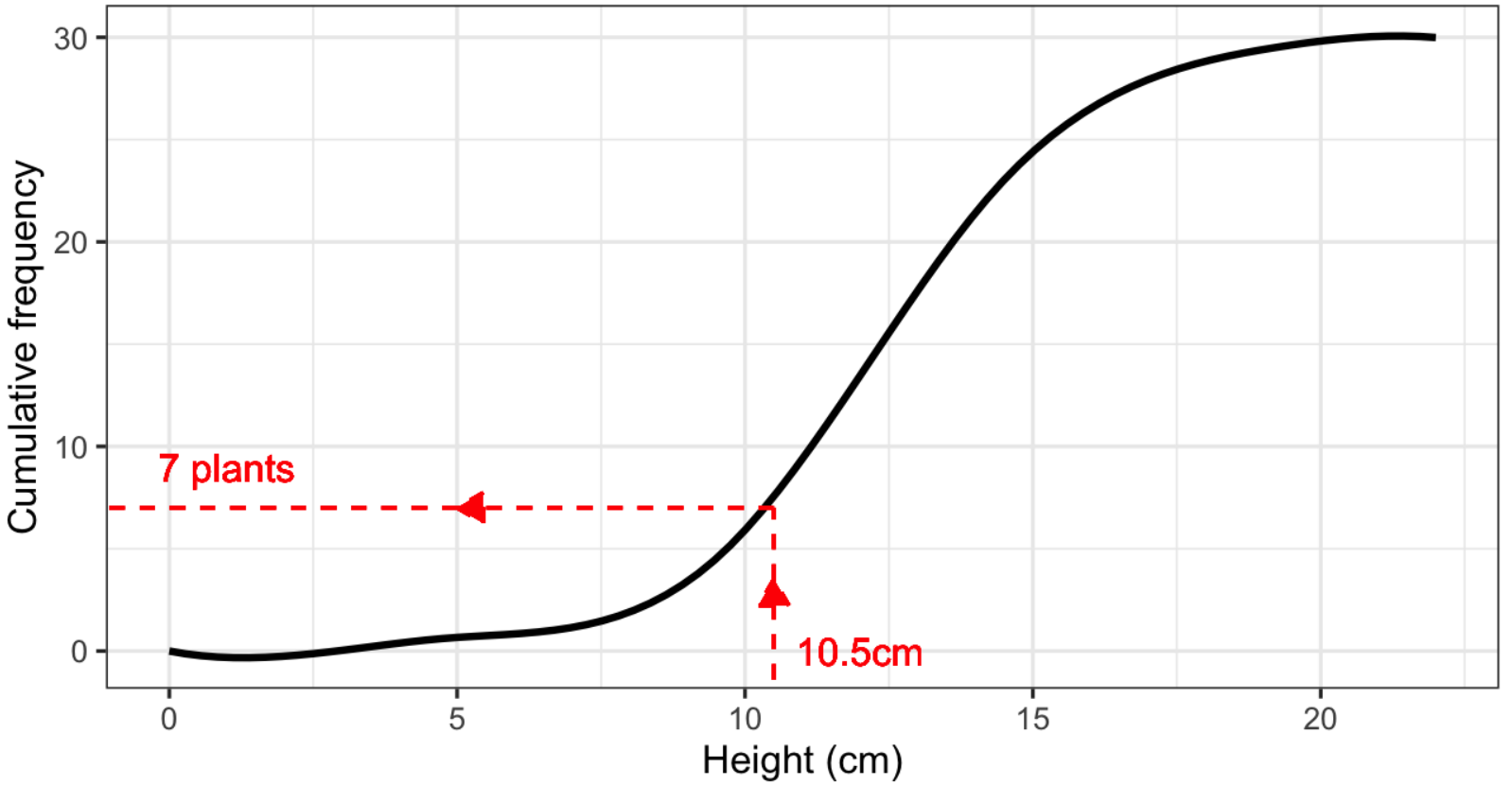
Joining all the points with a straight line gives a **cumulative frequency polygon**. This assumes that the readings are *evenly* distributed throughout the intervals.

Cumulative frequency curve

Joining all the points with a smooth line gives a **cumulative frequency curve**. This assumes that the readings are *not evenly* distributed throughout the intervals.

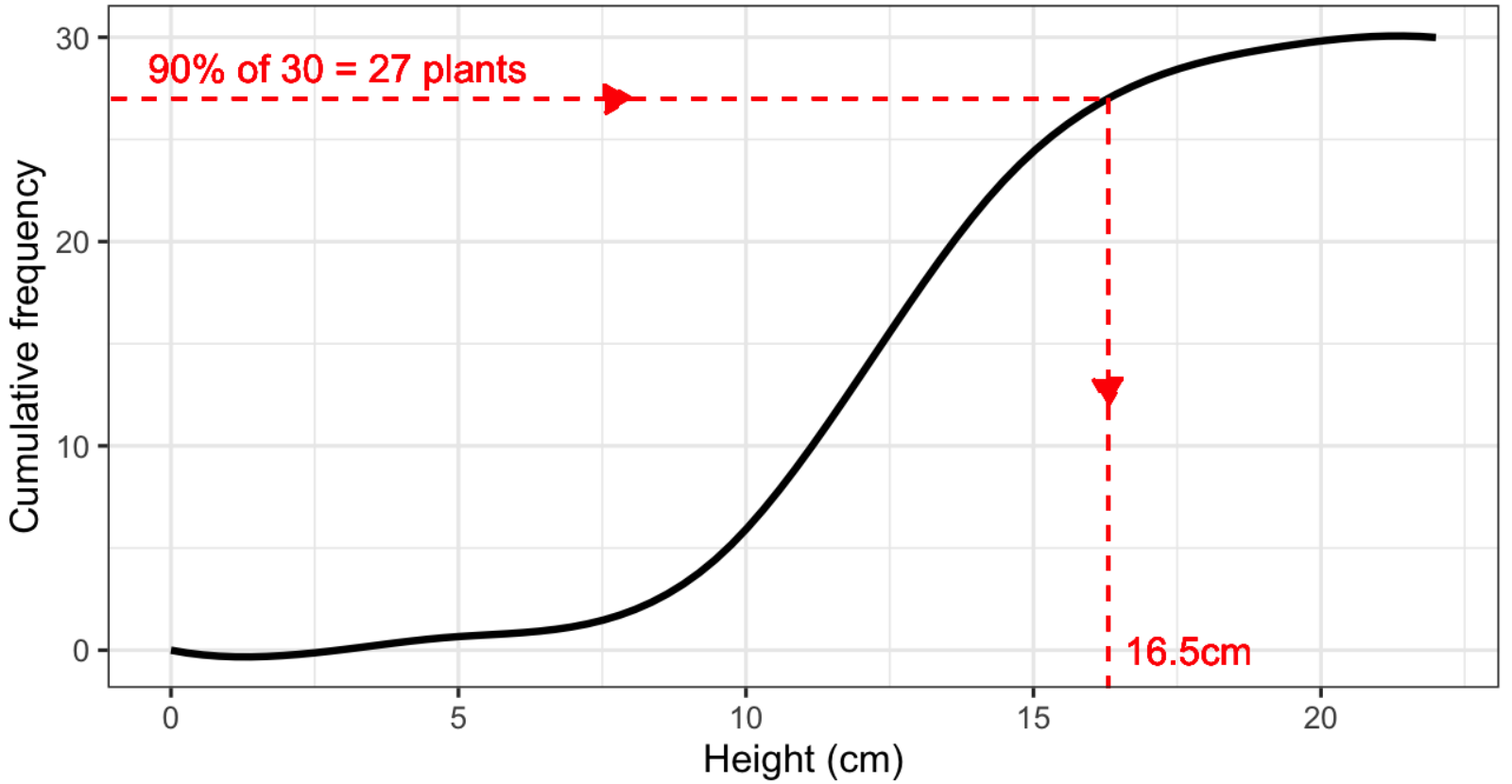
Cumulative frequency curve

How many plants were less than 10.5cm tall?



Cumulative frequency curve

Find x where 90% of the plants were less than x cm tall.



Median and quartiles

For data arranged in order of size,

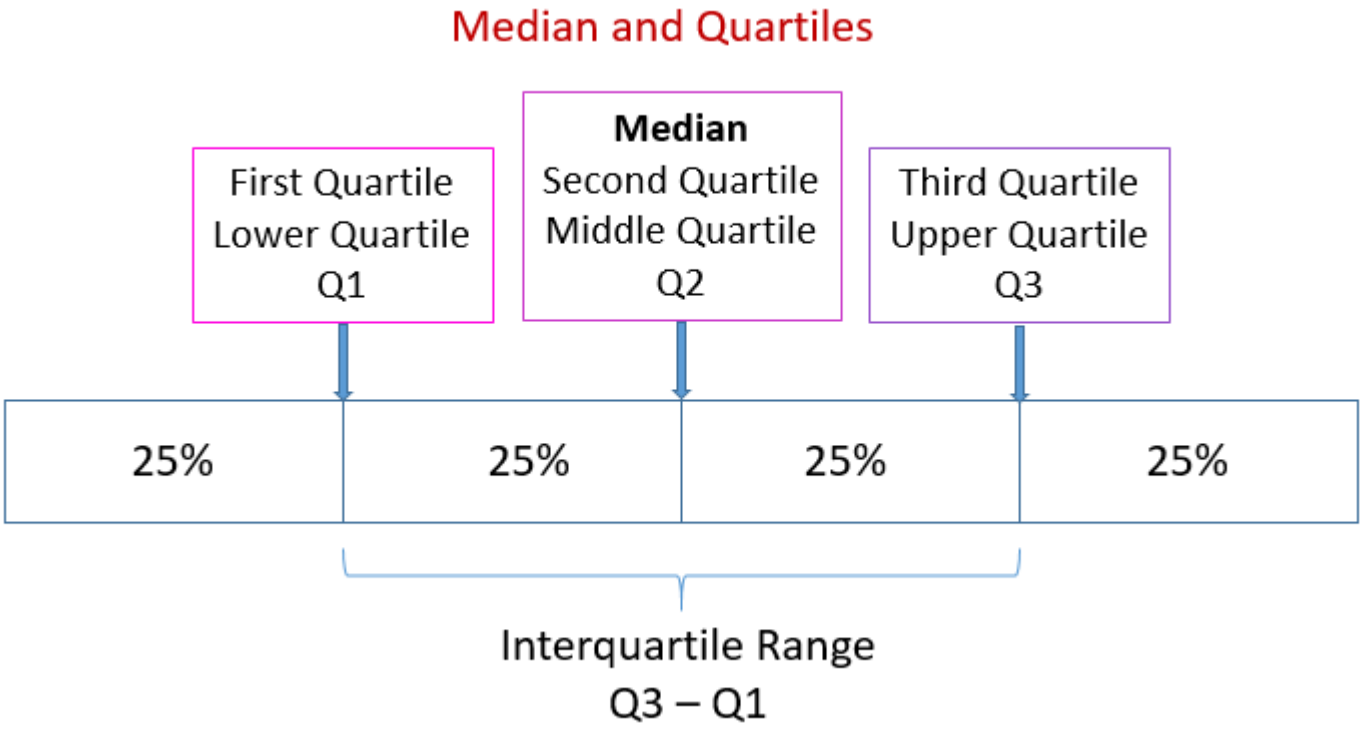
- The **lower quartile** Q_1 is the value 25% of the way through the distribution.
- The **median** Q_2 is the value 50% of the way through the distribution.
- The **upper quartile** Q_3 is the value 75% of the way through the distribution.

Therefore, the median, lower quartile and upper quartile together split the distribtion into four equal parts.

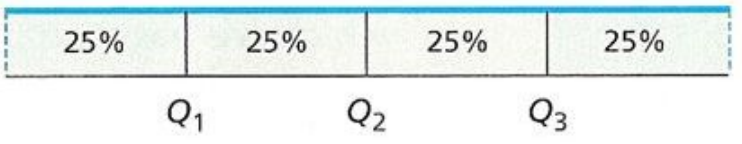
- The **inter-quartile range**, defined as the difference between the quartiles, tells you the range of the middle 50% of the distribution.

$$\mathrm{IR} = Q_3 - Q_1$$

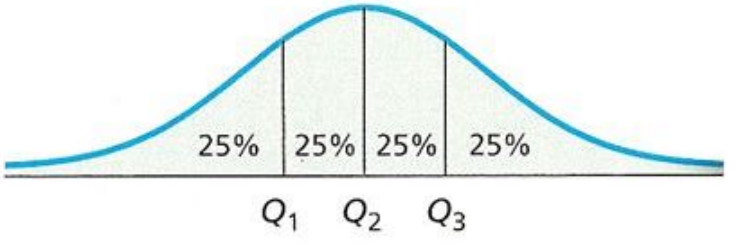
Median and quartiles



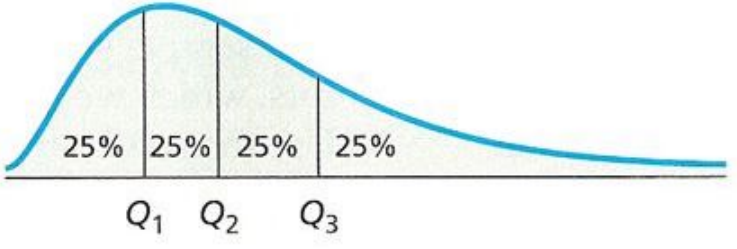
Median and quartiles



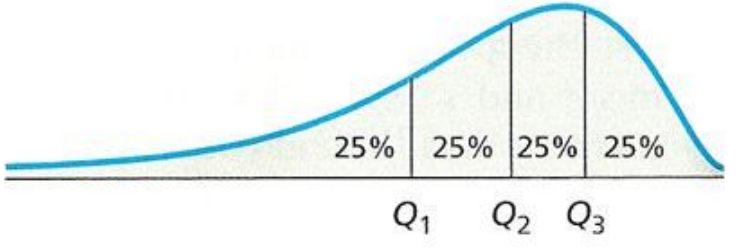
(a) Uniform



(b) Bell shaped



(c) Right skewed



(d) Left skewed

Quartiles for ungrouped data

- For ungrouped data consisting of n observations arranged in ascending order of size, the median is the

$$Q_2 = \frac{1}{2}(n+1)\text{th observation}$$

  - If there is an odd number of observations, the median is the middle value.
  - If there is an even number of observations, there are two middle values, and the median is the average of these two numbers.

- The quartiles should divide the two distributions either side of the median in half. Re-apply the formula for the median for each half to find the lower and upper quartiles.

Quartiles for ungrouped data

Example: Consider the set of numbers $\{7, 7, 2, 3, 4, 2, 7, 9, 31\}$

- Since there are 9 numbers, the median is the (9 + 1)/2 th observation, i.e. the 5th observation, which is 7.

```
2, 2, 3, 4, [7], 7, 7, 9, 31
```

- The median of the left distribution $\{2, 2, 3, 4\}$ gives the lower quartile, i.e. the middle value between 2 and 3 = (2 + 3) / 2 = 2.5.

```
2, [2, 3], 4, [7], 7, 7, 9, 31
```

- The median of the right distribution $\{7, 7, 9, 31\}$ gives the lower quartile, i.e. the middle value between 7 and 9 = (7 + 9) / 2 = 8.

```
2, 2, 3, 4, [7], 7, [7, 9], 31
```

Quartiles for ungrouped data

- Sometimes, the following rule is used to find the quartiles:

$$Q_1 = \frac{1}{4}(n+1)\text{th observation}$$

$$Q_3 = \frac{3}{4}(n+1)\text{th observation}$$

This rule agrees with the above method when n is odd, but there is a discrepancy when n is even. But it does not make a great deal of difference whichever method is used.

Quartiles for ungrouped data

To find the median and quartiles of data in the form of an ungrouped frequency distribution, it is useful to find the cumulative frequency.

Example: The table shows the cumulative number of attempts needed to pass the driving test by 100 candidates at a particular test centre.

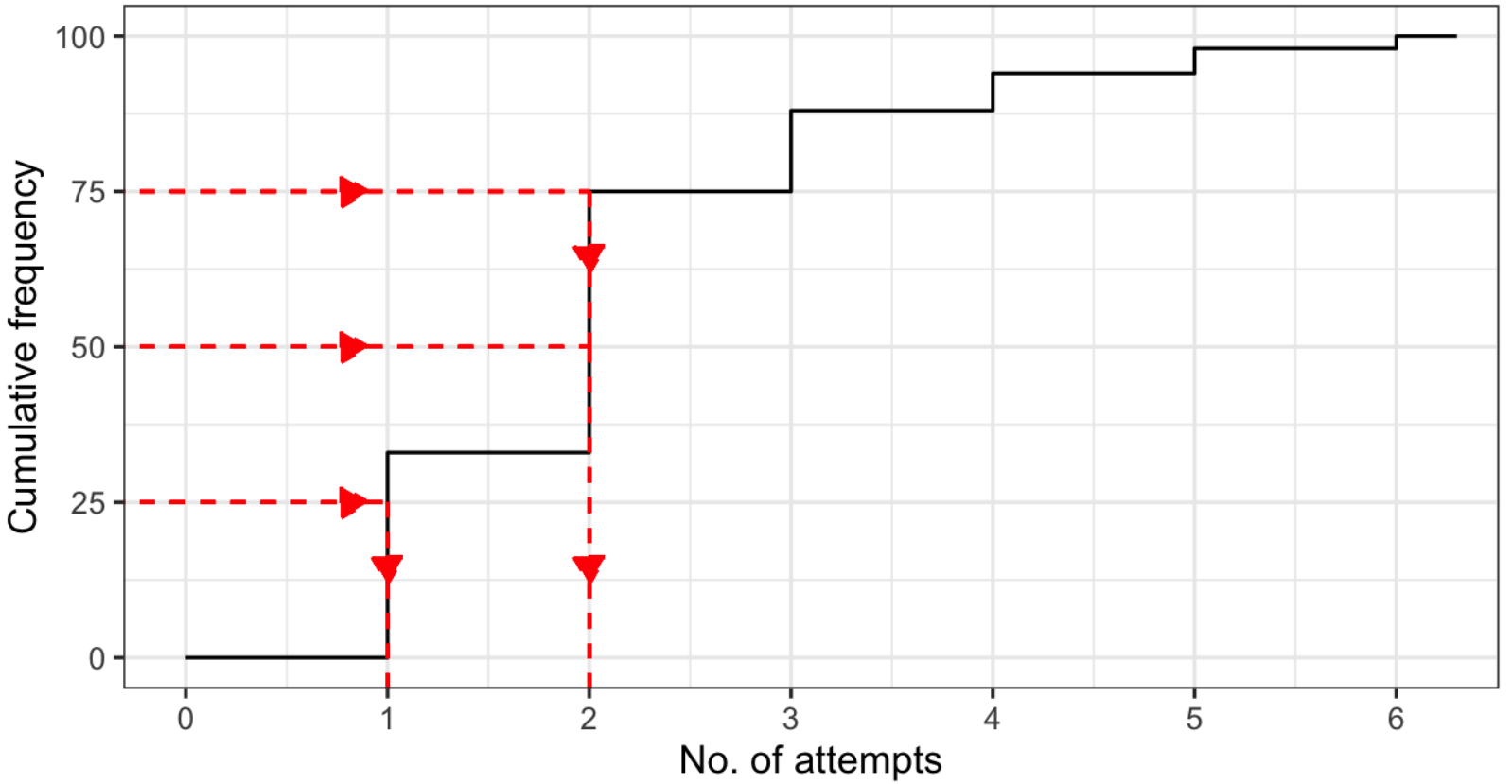
No. of attempts	$\leq$ **1**	$\leq$ **2**	$\leq$ **3**	$\leq$ **4**	$\leq$ **5**	$\leq$ **6**
Cumulative freq.	33	75	88	94	98	100

Since there are $n = 100$ observations,

- Q_2 is the middle value of 50 and 51, i.e. the 50.5th observation. This lies in the class of '≤ 2' attempts.
- Q_1 is the 25th observation, which lies in the class of '≤ 1' attempts.
- Q_3 is the 75th observation, which lies in the class of '≤ 2' attempts.

Quartiles for ungrouped data

Or, we can read from the step diagram



Quartiles for grouped data

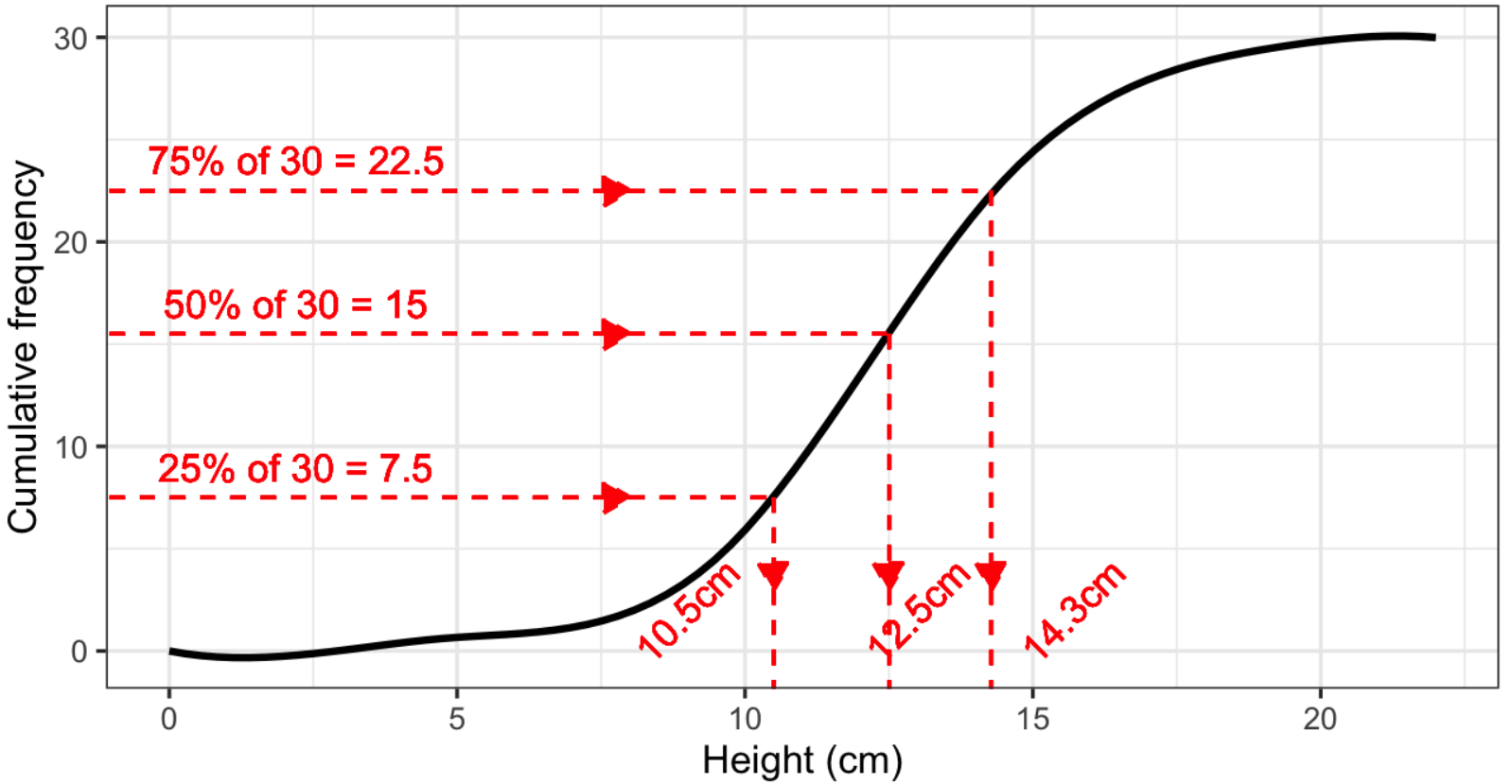
- When data have been grouped into intervals, the original information has been lost, so it is only possible to make *estimates* of the median and quartiles.

- One way of doing this is to use a cumulative frequency graph (polygon or curve).

  - Q_1 is the $\frac{1}{4}$th reading.
  - Q_2 is the $\frac{1}{2}$th reading.
  - Q_3 is the $\frac{3}{4}$th reading.

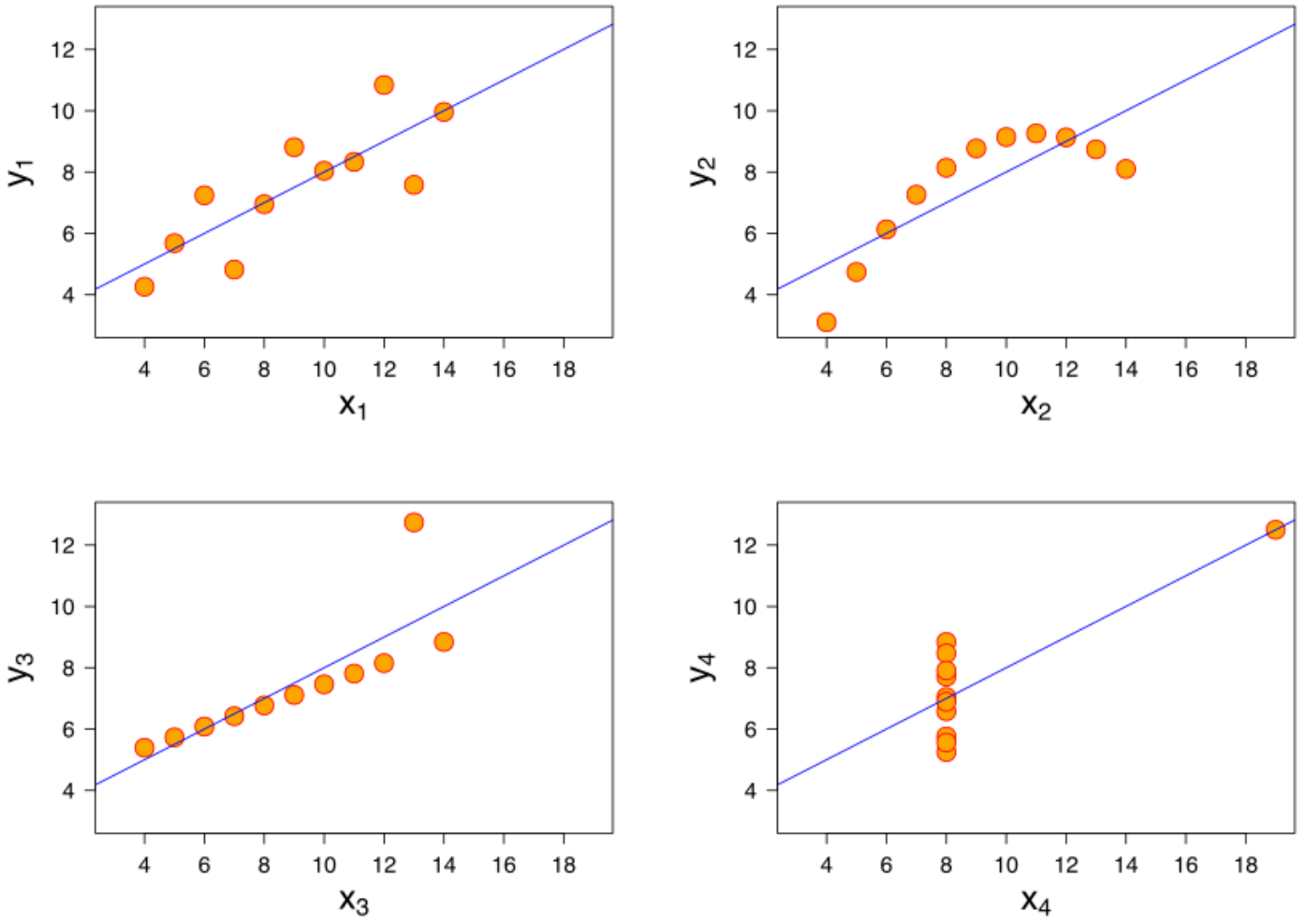
Quartiles for grouped data

Back to the heights of 30 broad beans six weeks after planting.



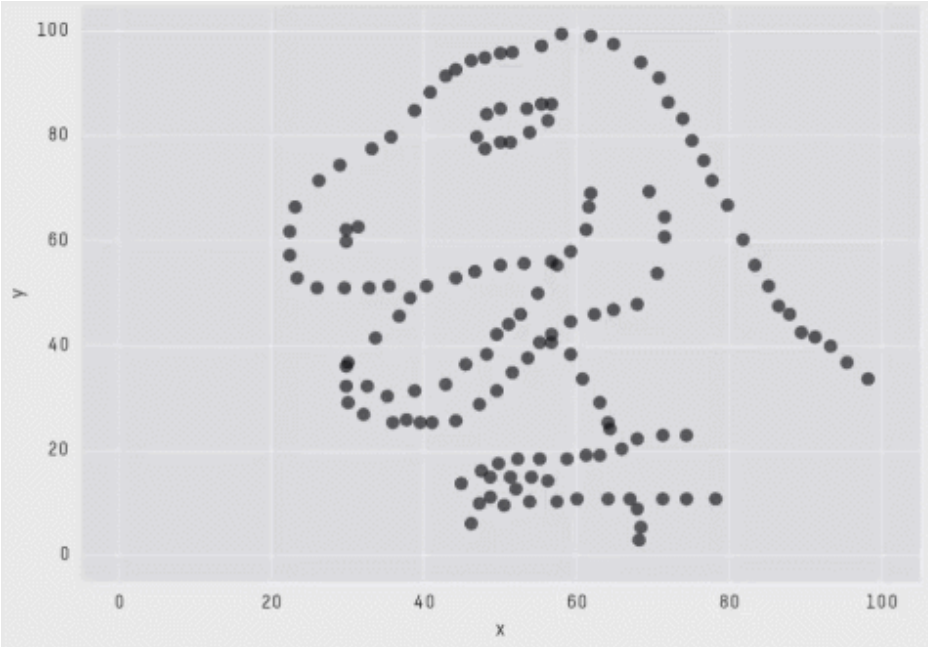
Anscombe's Quartet

A cautionary tale in summary statistics



Mean of x = 9, Variance of x = 11. Mean of y = 7.5, Variance of y = 4.125.
Correlation between x and y = 0.816.

The Datasaurus Dozen



X Mean: 54.2659224
Y Mean: 47.8313999
X SD : 16.7649829
Y SD : 26.9342120
Corr. : -0.0642526

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END