SM-1402 Basic Statistics

Chapter 3 (Probability Distributions)

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Probability Distribution

- This chapter concerns **discrete random variables**.

- When a variable is discrete, it is possible to specify or describe all possible numerical values, for example

    - *The number for females in a group of four students.* The possible values are 0, 1, 2, 3 and 4.
    - *The number of times you throw a die until a six appears.* The possible values are 1, 2, 3, ... , ∞

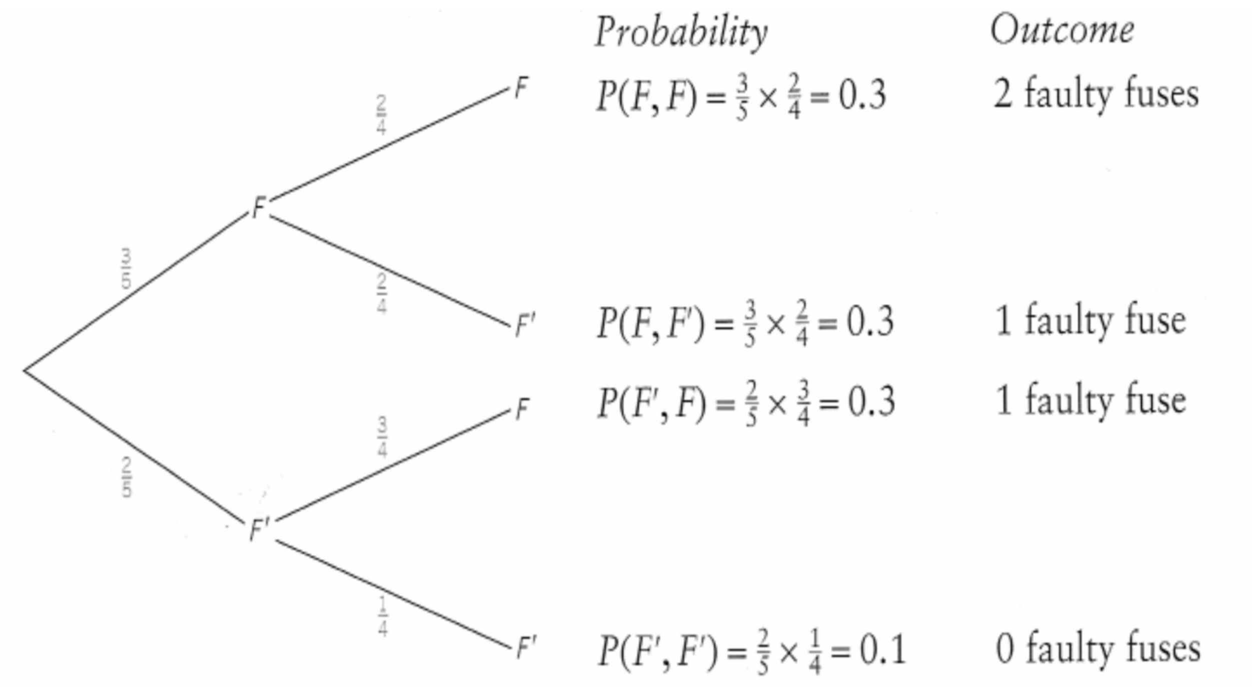
Probability Distribution

A probability distribution gives the probability of each possible value of the variable. Consider the following situation:

By mistake, three faulty fuses are put into a box containing two good fuses. The faulty and good fuses become mixed up and are indistinguishable by sight. You take two fuses from the box. What is the probability that you take

- *(a) no faulty fuses*
- *(b) one faulty fuse*
- *(c) two faulty fuses*

Probability Distribution



- (a) P(no faulty fuses) = 0.1
- (b) P(one faulty fuse) = 0.3 + 0.3 = 0.6
- (c) P(two faulty fuses) = 0.3

Probability Distribution

- A *variable* is an 'object' whose value is subject to change. Let X be the (random) variable which denotes 'the number of faulty fuses'.
- In the example, X can take on three possible values only: 0, 1, 2.
- A variable is said to be random if the value it takes cannot be predetermined from available knowledge. Its value is unknown, until it is observed.
- (Discrete) Random variables are assigned **probability** to them. For instance,
    - (a) $P(X = 0) = 0.1$
    - (b) $P(X = 1) = 0.6$
    - (c) $P(X = 2) = 0.3$

Probability Distribution

- Notation:
  - (Random) variables are denoted by *capital* letters (X, Y, Z, etc.)
  - A particular value that the variable takes is denoted by *small* letters (x, y, z, etc.)
  - Therefore, $P(X = x)$ means the 'probability that the variable X takes on the value x
- We can summarise the probability distribution for X in a table

x	**0**	**1**	**2**
$P(X = x)$	0.1	0.6	0.3

- The sum of the probabilities in a probability distribution table must add up to 1.

Probability Distribution

- Another way of summarising probability distribution is via a **probability distribution function** (pdf). For example,

$$f(x) = \begin{cases} 0.1 & \text{if } x = 0 \\ 0.6 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2 \end{cases}$$

  so $f(x)$ is called the pdf of X.

- The pdf is responsible for allocating probabilities.

- Some pdfs lists the probabilities individually (like the one above), but some pdfs summarises the probabilities in a formula.

$$f(x) = 0.1^{1(x=0)} 0.6^{1(x=1)} 0.3^{1(x=2)}$$

  where $1(\cdot)$ is the indicator function.

Probability Distribution

Example: Two tetrahedral dice, each with faces labelled 1,2,3, and 4, are thrown and the scores noted, where the score is the sum of two nombers on which the dice land. Find the pdf of X, where X is defined to be 'the score when two dice are thrown'.

The score for each possible outcome is shown in the possibility space:

Second die		First die			
	4	5	6	7	8
	3	4	5	6	7
	2	3	4	5	6
	1	2	3	4	5
		1	2	3	4

Probability Distribution

We see that possible values for X are 2, 3, 4, 5, 6, 7 and 8.

$$P(X=2) = \frac{\text{no. of times 2 occurs}}{\text{no. of possible scores}} = \frac{1}{16}$$

Similarly, $P(X=3) = 2/16$, $P(X=4) = 3/16$, etc.

Probability Distribution

The probability distribution is formed:

x	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$P(X = x)$	1/16	2/16	3/16	4/16	3/16	2/16	1/16

Probability Distribution

Notice the pattern in the probabilities

x	2	3	4	5	6	7	8
$P(X=x)$	1/16	2/16	3/16	4/16	3/16	2/16	1/16

We can summarise this in terms of a pdf as

$$P(X=x)=\begin{cases}\frac{x-1}{16} & x=2,3,4,5\\ \frac{9-x}{16} & x=6,7,8\end{cases}$$

And notice that the sum of the probabilities add up to 1.

Expectation

$E(X)$ is read as 'E of X' and it gives an average or typical value of X, known as the *expected value of* X, or the *expectation of* X.

Consider a fair, six-sided die being thrown. The pdf is $P(X = x) = 1/6$ for $x = 1, 2, 3, 4, 5, 6$, i.e. the probability is the same regardless of what value x is.

x	**1**	**2**	**3**	**4**	**5**	**6**
$P(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6

Expectation

The expected value of X is obtained by

- multiplying each score
- by its probability
- and then summing them up.

x	**1**	**2**	**3**	**4**	**5**	**6**
$P(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6

$$
\begin{aligned}
E(X) &= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} \\
&= 3.5
\end{aligned}
$$

Expectation

The formula for expectation is given as follows:

$$E(X) = \sum_x x \cdot P(X = x)$$

The symbol μ (pronounced 'miu') is often used for the expectation

$$\mu := E(X)$$

Expectation

Example: A random variable X has probability distribution as shown. Find the expectation $E(X)$.

x	**-2**	**-1**	**0**	**1**	**2**
$P(X = x)$	0.3	0.1	0.15	0.4	0.05

$$\begin{aligned} E(X) &= (-2) \times 0.3 + (-1) \times 0.1 + 0 \times 0.15 \\ &\quad + 1 \times 0.4 + 2 \times 0.05 \\ &= -0.2 \end{aligned}$$

Expectation

The definition of expectation can be extended to any function of X.

$$E\big(g(X)\big) = \sum_x g(x) \cdot P(X = x)$$

This allows us to calculate $E(10X)$, $E(X^2)$, $E(1/X)$, $E(X - 4)$, etc.

$$\begin{aligned}
E(10X) &= \sum_x 10x \cdot P(X = x) \\
E(X^2) &= \sum_x x^2 \cdot P(X = x) \\
E(1/X) &= \sum_x \frac{1}{x} \cdot P(X = x) \\
E(X - 4) &= \sum_x (x - 4) \cdot P(X = x)
\end{aligned}$$

Expectation

Example 3.6: The random variable X has pdf as shown.

x	**0**	**1**	**2**
$P(X = x)$	0.1	0.6	0.3

Calculate (a) $E(X)$, (b) $E(3)$, (c) $E(5X)$, (d) $E(5X + 3)$

(a) $E(X) = 0 \times 0.1 + 1 \times 0.6 + 2 \times 0.3 = 2.2$

(b) $E(3) = 3 \times 0.1 + 3 \times 0.6 + 3 \times 0.3 = 3$

(c) $E(5X) = (5 \times 0) \times 0.1 + (5 \times 1) \times 0.6 + (5 \times 2) \times 0.3 = 11$

$$\text{(d)} \quad \begin{aligned} E(5X + 3) &= (5 \times 0 + 3) \times 0.1 + (5 \times 1 + 3) \times 0.6 \\ &\qquad + (5 \times 2 + 3) \times 0.3 \\ &= 14 \end{aligned}$$

Expectation

- In general, for constants a, b, c, and two random variables X and Y

$$E(aX + bY + c) = aE(X) + bE(Y) + c$$

  This property is known as the *linearity of expectations*.

- For two functions of X, $g(X)$ and $h(X)$,

$$E\big(g(X) + h(X)\big) = E\big(g(X)\big) + E\big(h(X)\big)$$

  This property allows us to easily calculate the variance...

Variance

Recall that a set of observations $x_1, \ldots, x_n$, the (sample) mean and variance is given by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

For a random variable X, we saw that the mean is $\mu = E(X)$ ("the average value of X").

In the same spirit as the sample variance for observations, we define the variance of X to be

$$\mathrm{Var}(X) = E\big((X - \mu)^2\big)$$

In words, this is the "average value of the squared deviation from the mean".

Variance

Here are alternative formulae for the variance. They are all equivalent.

$$\mathrm{Var}(X) = E\big((X-\mu)^2\big)$$

$$\mathrm{Var}(X) = E(X^2) - \mu^2$$

$$\mathrm{Var}(X) = E(X^2) - \big(E(X)\big)^2$$

$$\mathrm{Var}(X) = E(X^2) - E^2(X)$$

Variance

Example 3.8: The random variable X has probability distribution as shown in the table

x	**1**	**2**	**3**	**4**	**5**
$P(X = x)$	0.1	0.3	0.2	0.3	0.1

Find (a) $E(X)$, (b) $E(X^2)$, (c) $\mathrm{Var}(X)$, (d) σ.

(a) $E(X) = 1(0.1) + 2(0.3) + 3(0.3) + 4(0.2) + 5(0.1) = 3$

(b)
$E(X^2) = 1^2(0.1) + 2^2(0.3) + 3^2(0.3) + 4^2(0.2) + 5^2(0.1) = 10.4$

(c) $\mathrm{Var}(X) = E(X^2) - E^2(X) = 10.4 - 3^2 = 1.4$

(d) $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{1.4} = 1.18$

Variance

Properties of variance: If a and b are constants, and X is a random variable,

- $\mathrm{Var}(X) \geq 0$
- $\mathrm{Var}(a) = 0$
- $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Given another r.v. Y such that X and Y are **independent**, and a constant c, then

- $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
- $\mathrm{Var}(aX \pm bY + c) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

Variance

Example 3.11: X *and* Y *are independent random variables such that* $E(X) = 10$, $\mathrm{Var}(X) = 2$, $E(Y) = 8$, *and* $\mathrm{Var}(Y) = 3$. *Find*

(a) $E(5X + 4Y)$

(b) $\mathrm{Var}(5X + 4Y)$

(c) $\mathrm{Var}(\frac{1}{2}X - Y)$

(d) $\mathrm{Var}(\frac{1}{2}X + Y)$

Variance

Example 3.11: X and Y are independent random variables such that $E(X) = 10$, $\mathrm{Var}(X) = 2$, $E(Y) = 8$, and $\mathrm{Var}(Y) = 3$. Find

(a) $E(5X + 4Y) = 5E(X) + 4E(Y) = 5(10) + 4(8) = 82$

(b) $\mathrm{Var}(5X + 4Y) = 5^2(2) + 4^2(3) = 98$

(c) $\mathrm{Var}(\frac{1}{2}X - Y) = \frac{1}{2^2}(2) + 3 = 3.5$

(d) $\mathrm{Var}(\frac{1}{2}X + Y) = 3.5$

Covariance

Given two random variables X and Y with means $E(X) = \mu$ and $E(Y) = \nu$ (ν is pronounced 'niu') respectively, the covariance between them is defined to be

$$\mathrm{Cov}(X, Y) = E\left[(X - \mu)(Y - \nu)\right]$$

Alternatively,

$$\mathrm{Cov}(X, Y) = E(XY) - \mu\nu$$

The covariance measures the joint variability of two random variables. The sign of the covariance (+ or -) shows the tendency in the linear relationship between the two variables.

Note that the covariance of X with itself is its variance! i.e. $\mathrm{Cov}(X, X) = E(X - \mu)^2 = \mathrm{Var}(X)$.

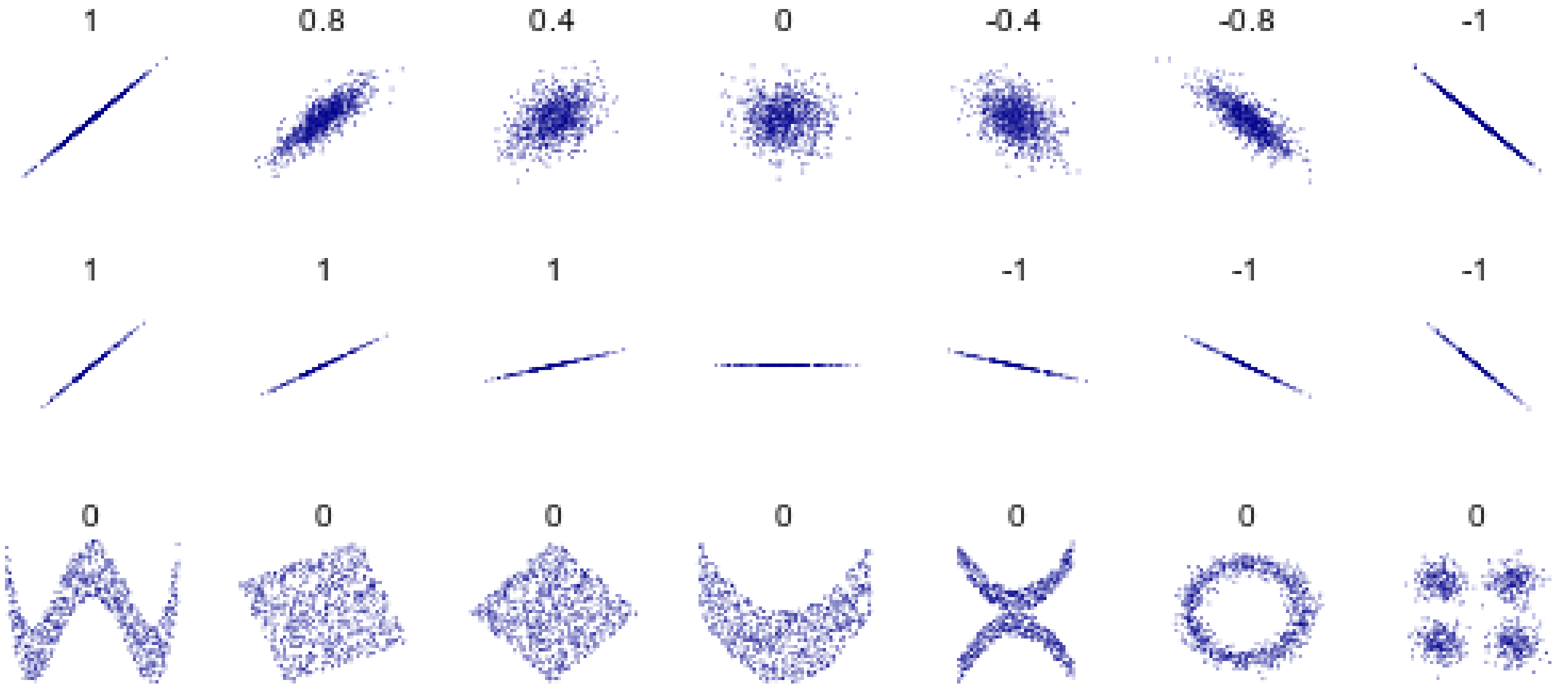
Correlation

If we scale the covariance by the product of the standard deviations of X and Y, then we get the correlation coefficient between X and Y.

$$\rho := \rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

This measures the level of association between two r.v.; it is a value between -1 and 1.

Correlation



The cumulative distribution function

Given a probability distribution, the probabilities up to a certain value are summed to give a cumulative probability. This is denoted

$$F(x) = P(X \leq x)$$

Example: Consider the following probability distribution

x	**1**	**2**	**3**	**4**	**5**
$P(X = x)$	0.05	0.4	0.3	0.15	0.1

- $F(1) = P(X \leq 1) = 0.05$
- $F(2) = P(X \leq 2) = P(X = 1, 2) = 0.05 + 0.4 = 0.45$
- $F(3) = P(X \leq 3) = 0.05 + 0.4 + 0.3 = 0.75$
- $F(4) = P(X \leq 4) = 0.9$
- $F(5) = P(X \leq 4) = 1$

The cumulative distribution function

Example: Consider the following probability distribution

x	1	2	3	4	5
$P(X = x)$	0.05	0.4	0.3	0.15	0.1

The cumulative probability distribution function is

x	1	2	3	4	5
$F(x)$	0.05	0.45	0.75	0.9	1.0

Special discrete distributions

The Bernoulli distribution

Suppose we are interested in the outcome of a (single) random trial, which can either be *success* or *failure* only. Examples include

- Flip a coin, it can land *Heads* or *Tails*.
- The colour of the suit of a randomly drawn card from a pack of playing cards, it can be either *Red* or *Black*.
- Rolling a dice, the outcome can be either an *Even* number or an *Odd* number.
- Babies being born being *Boy* or *Girl*.

The Bernoulli distribution

Let X be a random variable denoting the outcome of a success/fail trial. Then,

$$P(X = x) = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1 - p \end{cases}$$

We will write $X \sim \mathrm{Bern}(p)$.

This is read as " X is distributed according to a Bernoulli distribution with success probability p".

<u>Properties</u>

- p is the **parameter** of the Bernoulli distribution.
- Support: $x \in \{0, 1\}$
- $E(X) = p$
- $Var(X) = p(1 - p)$

The binomial distribution

Suppose we have a situation such that

- a finite number n trials are carried out.
- each trial is independent of each other.
- the outcome of each trial is either *success* or *failure*.
- the probability p of a successful outcome is the same for each trial.

Let X be the number of success outcomes in n trials. Then X has a **binomial distribution**, and this is written $X \sim \mathrm{Bin}(n, p)$.

The *probability mass function* (pmf) of X is

$$P(X = x) = {}^nC_x p^x (1-p)^{(n-x)}$$

for $x = 0, 1, 2, \ldots, n$.

The binomial distribution

Properties

- n and p are the **parameters** of the binomial distribution.
- Support: $x \in \{0, 1, 2, \ldots, n\}$
- $E(X) = np$
- $Var(X) = np(1 - p)$

- $P(X = 0) = (1 - p)^n$; $P(X = n) = p^n$

- Special case: If $X \sim \mathrm{Bin}(1, p)$ then $X \sim \mathrm{Bern}(p)$.

- Let $X_1, \ldots, X_n \sim \mathrm{Bern}(p)$ (iid), then

$$X = \sum_{i=1}^{n} X_i \sim \mathrm{Bin}(n, p)$$

The binomial distribution

Visualising the binomial distribution

https://shiny.rit.albany.edu/stat/binomial/

The binomial distribution

Pascal's Triangle

$$
\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&& \\
&&&&&&&&&1&&1&&&&&&&&& \\
&&&&&&&&1&&2&&1&&&&&&&& \\
&&&&&&&1&&3&&3&&1&&&&&&& \\
&&&&&&1&&4&&6&&4&&1&&&&&& \\
&&&&&1&&5&&10&&10&&5&&1&&&&& \\
&&&&1&&6&&15&&20&&15&&6&&1&&&& \\
&&&1&&7&&21&&35&&35&&21&&7&&1&&& \\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&& \\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1& \\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1 \\
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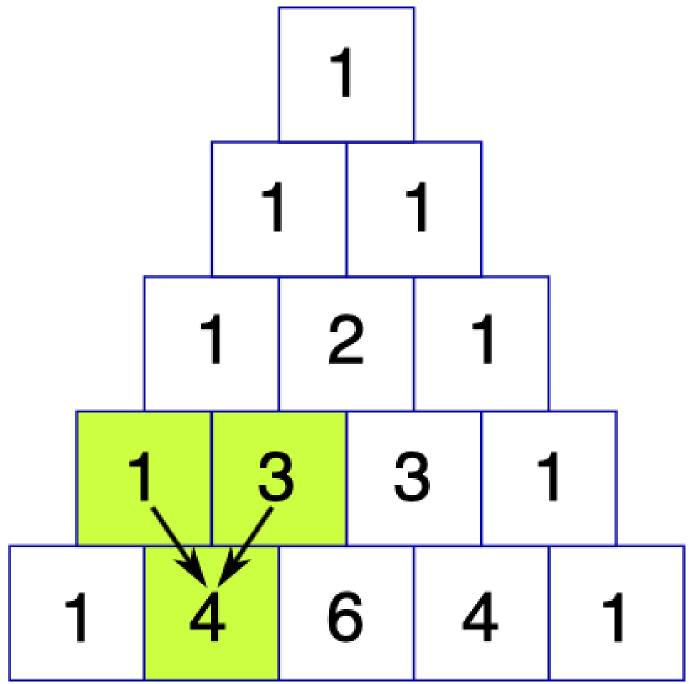
The binomial distribution

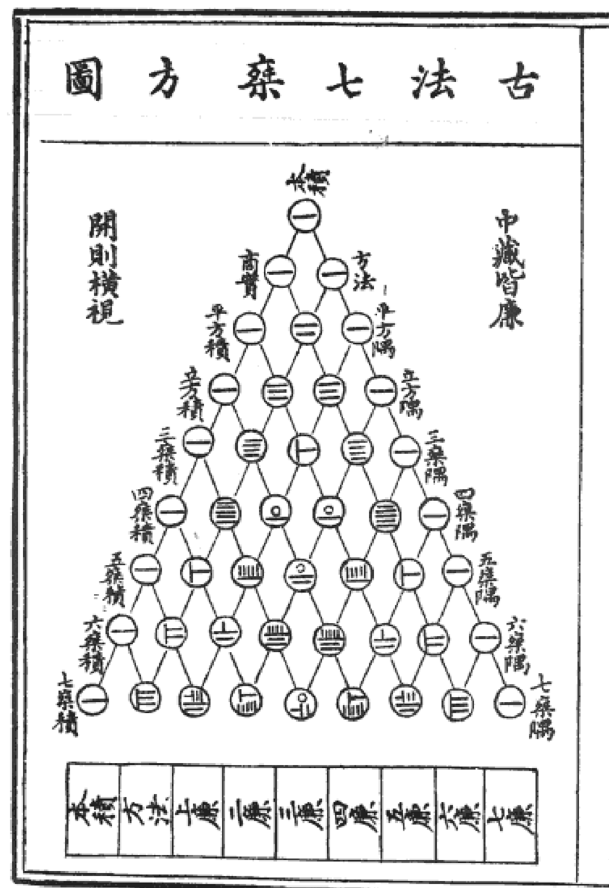
Pascal's Triangle



The binomial distribution

贾宪 (Jiǎ Xiàn) Triangle (1303 CE)

- Author: 朱世杰 (Zhū Shìjié) / Chu Shih-chieh
- Title: 四元玉鉴 (sì yuán yù jiàn) / Jade Mirror of the Four Unknowns
- He showed how to convert a problem stated verbally into a system of polynomial equations up to the 14th order.



The binomial distribution

Galton Board (The Quincunx)

The binomial distribution

Example 4.1: At a supermarket, 60% of customers pay by credit card. Find the probability that in a randomly selected sample of ten customers,

(a) exactly two pay by credit card.

(b) more than seven pay by credit card.

- Let X be the number of customers in sample of ten who pay by credit card.
- Consider 'paying by credit card' as a *success*, so $p = 0.6$.
- Assume also independence of $n = 10$ trials.

Then $X \sim \mathrm{Bin}(10, 0.6)$.

The binomial distribution

Example 4.1: At a supermarket, 60% of customers pay by credit card. Find the probability that in a randomly selected sample of ten customers,

(a) exactly two pay by credit card.

$$P(X = 2) = {}^{10}C_2(0.6^2)(0.4^8) = 0.011$$

(b) more than seven pay by credit card.

$$\begin{aligned} P(X > 7) &= P(X = 8) + P(X = 9) + P(X = 10) \\ &= {}^{10}C_8(0.6^8)(0.4^2) + {}^{10}C_9(0.6^9)(0.4^1) \\ &\quad + {}^{10}C_{10}(0.6^{10})(0.4^0) \\ &= 0.17 \end{aligned}$$

The binomial distribution

Example 4.2: $X \sim \mathrm{Bin}(7, 0.2)$. *Find* $P(1 < X \leq 4)$ *and* $P(X > 1)$. *What is* $E(X)$ *and* $\mathrm{Var}(X)$*?*

$$\begin{aligned} P(1 < X \leq 4) &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= {}^7C_2(0.2^2)(0.8^5) + {}^7C_3(0.2^3)(0.8^4) \\ &\quad + {}^7C_4(0.2^4)(0.8^3) \\ &= 0.419 \end{aligned}$$

Rather than calculate $P(X = 2)$, $P(X = 3)$, etc., we use the fact that

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - {}^7C_0(0.2^0)(0.8^7) + {}^7C_1(0.2^1)(0.8^6) \\ &= 0.423 \end{aligned}$$

The binomial distribution

Example 4.2: $X \sim \mathrm{Bin}(7, 0.2)$. *Find* $P(1 < X \leq 4)$ *and* $P(X > 1)$. *What is* $E(X)$ *and* $\mathrm{Var}(X)$*?*

$E(X) = np = 7 \times 0.2 = 1.4$

$\mathrm{Var}(X) = np(1 - p) = 7 \times 0.2 \times 0.8 = 1.12$

Poisson distribution

Suppose we have a situation such that

- Events occur singly and randomly in a given interval of time or space
- The mean number of occurrences λ in the given interval is known and is finite

Let X be the number of occurrences in the given interval. Then X has a **Poisson distribution**, and this is written $X \sim \mathrm{Poi}(\lambda)$.

Examples of Poisson outcomes

- Amount of e-mails received in 24-hour period.
- Number of calls received by a call centre per hour.
- The number of photons hitting a detector in a particular time interval.
- The number of patients arriving in an emergency room between 10pm and 11pm.

Poisson distribution

The *probability mass function* (pmf) of X is

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

for $x = 0, 1, 2, \ldots$. Here, $e = 2.71828...$ is Euler's number.

<u>Properties</u>

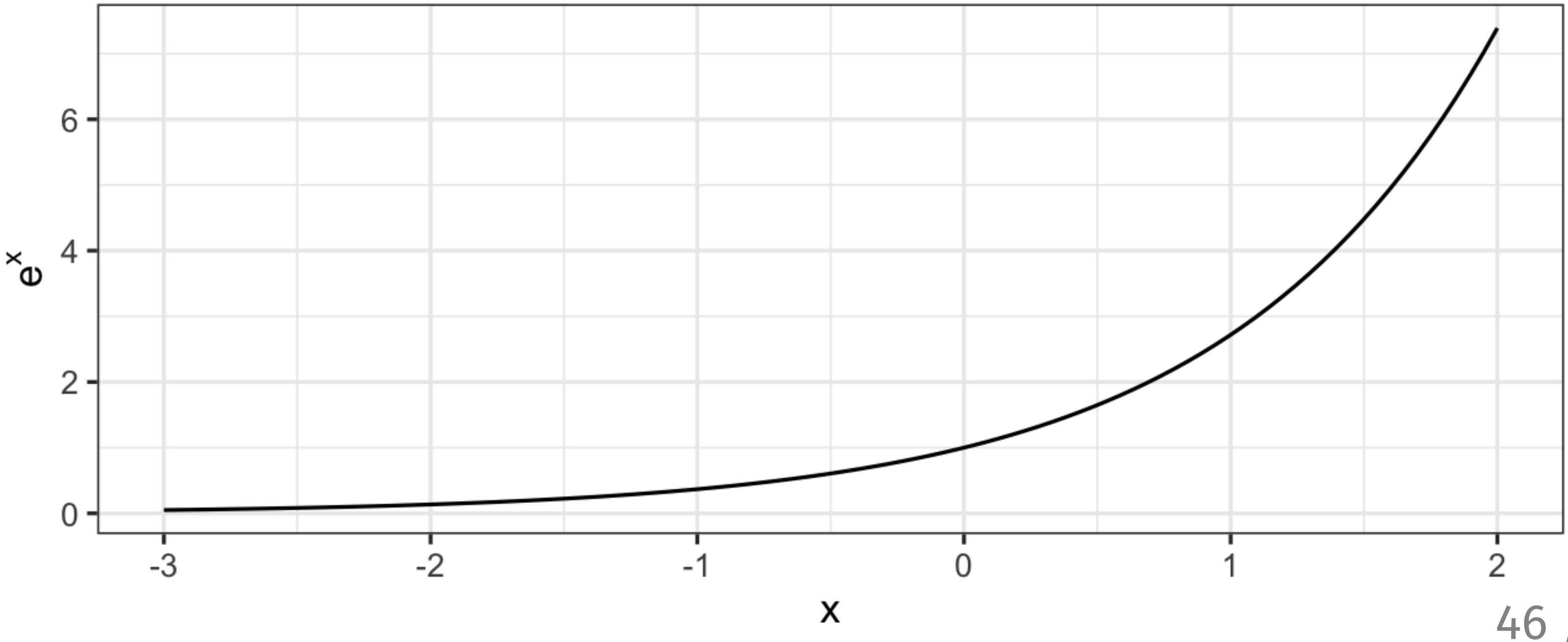
- λ (mean) is the **parameter** of the Poisson distribution.
- Support: $x \in \{0, 1, 2, \ldots\}$
- $E(X) = \lambda$
- $Var(X) = \lambda$
- $P(X = 0) = e^{-\lambda}$
- $P(X = 1) = \lambda e^{-\lambda}$

Poisson distribution

What is e?

The exponenial function is defined as

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$



Poisson distribution

What is e?

Plot of $y = (1 + 1/n)^n$. As $n \to \infty$, $y \to e^1 = 2.71828...$.

Poisson distribution

Example 4.5: A student finds that the average number of small fishes in 10ml of pond water from a particular pond is four. Assuming that the number of small fishes follows a Poisson distribution, find the probability that in a 10ml sample,

(a) there are exactly five fishes

(b) there are no small fishes

(c) there fewer than three small fishes

Poisson distribution

Let X represent this Poisson distribution. Then, $X \sim \mathrm{Poi}(4)$, and

(a) $P(X=5) = \frac{e^{-4}4^5}{5!} = 0.156$

(b) $P(X=0) = e^{-4} = 0.183$

(c)

$$\begin{aligned} P(X<3) &= P(X=0) + P(X=1) + P(X=2) \\ &= e^{-4} + 4e^{-4} + \frac{e^{-4}4^2}{2!} \\ &= e^{-4}(1+4+8) \\ &= 13e^{-4} \\ &= 0.238 \end{aligned}$$

Poisson distribution

CAUTION

- Care must be taken to specify the **unit interval** being considered.
- In the previous example, the unit interval is 10ml of pond water.
- If, for example, we want to find probabilities relating to 5ml of pond water. In this case, the **mean** number of small fish is $4 \div 2 = 2$. Let Y be the number of fish distributed in 5ml of pond water. Then $Y \sim \text{Poi}(2)$.
- In 1ml of pond water, distribution is $\text{Poi}(0.4)$.
- In 100ml of pond water, distribution is $\text{Poi}(40)$.
- etc.

END