A beginner's guide to variational inference

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Outline

Exponential families

- For the mean-field variational method, suppose that each complete conditional is in the exponential family:

$$p(\mathbf{z}^{(j)}|\mathbf{z}_{-j},\mathbf{y}) = h(\mathbf{z}^{(j)}) \exp\big(\eta_j(\mathbf{z}_{-j},\mathbf{y}) \cdot \mathbf{z}^{(j)} - A(\eta_j)\big).$$

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- Then, from (??),

$$\begin{aligned}
\tilde{q}_j(\mathbf{z}^{(j)}) &\propto \exp\big(\,\mathrm{E}_{-j}[\log p(\mathbf{z}^{(j)}|\mathbf{z}_{-j},\mathbf{y})]\big) \\
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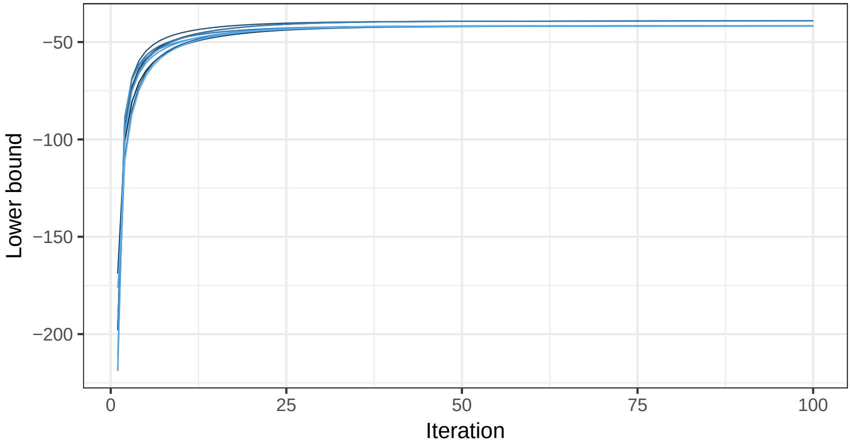
- **ISSUE**: What if not in exponential family? Importance sampling or Metropolis sampling.

Non-convexity of ELBO



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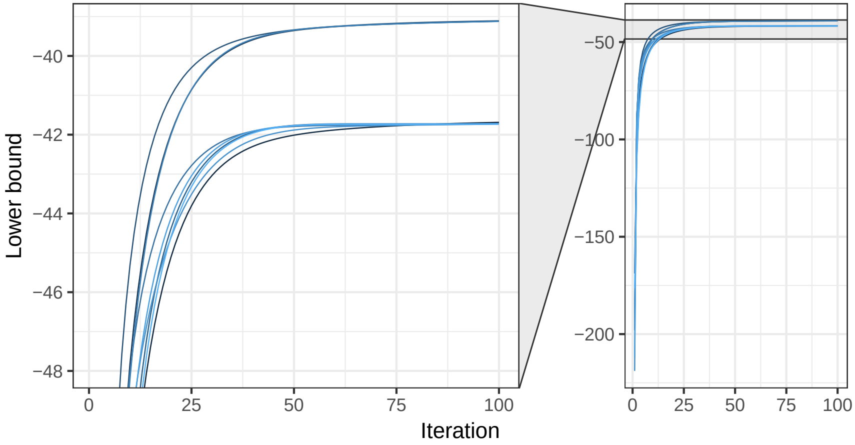

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Zero-forcing vs Zero-avoiding

- Back to the KL divergence:

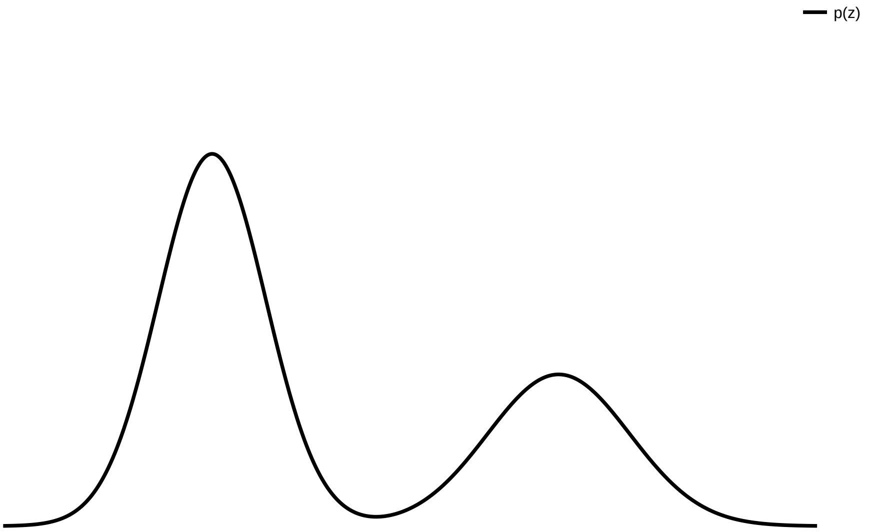
$$\mathrm{KL}(q\|p) = \int \log \frac{q(\mathbf{z})}{p(\mathbf{z}|\mathbf{y})} q(\mathbf{z})\,\mathrm{d}\mathbf{z}$$

- $\mathrm{KL}(q\|p)$ is large when $p(\mathbf{z}|\mathbf{y})$ is close to zero, unless $q(\mathbf{z})$ is also close to zero (*zero-forcing*).
- **ISSUE**: What about other measures of closeness? For instance,

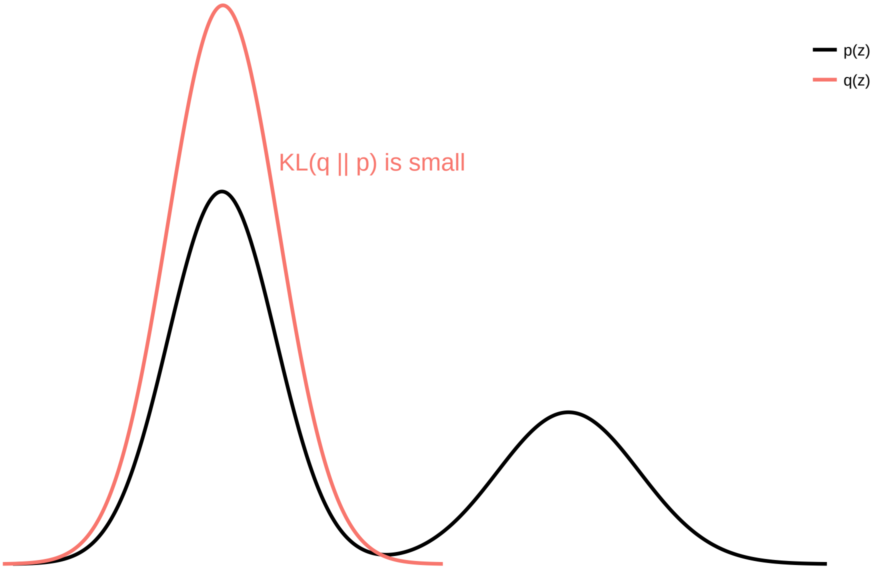
$$\mathrm{KL}(p\|q) = \int \log \frac{p(\mathbf{z}|\mathbf{y})}{q(\mathbf{z}|\mathbf{y})} p(\mathbf{z}|\mathbf{y})\,\mathrm{d}\mathbf{z}.$$

- This gives the Expectation Propagation (EP) algorithm.
- It is *zero-avoiding*, because $\mathrm{KL}(p\|q)$ is small when both $p(\mathbf{z}|\mathbf{y})$ and $q(\mathbf{z})$ are non-zero.

Zero-forcing vs Zero-avoiding (cont.)



Zero-forcing vs Zero-avoiding (cont.)



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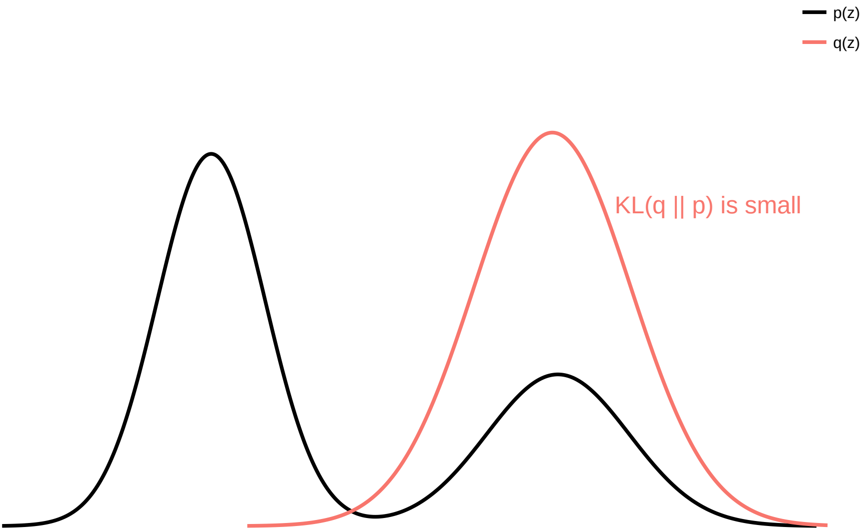

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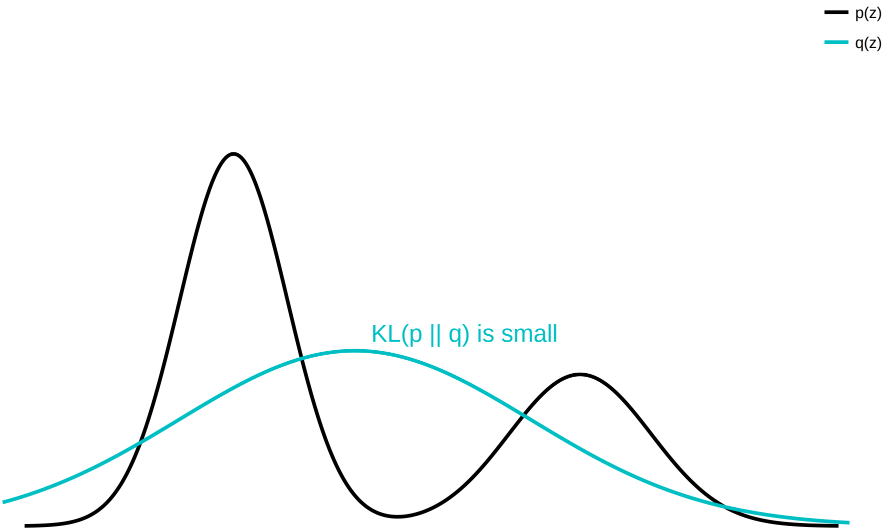
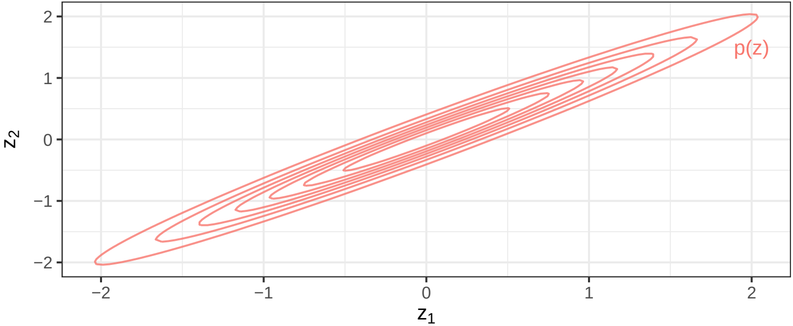


Distortion of higher order moments



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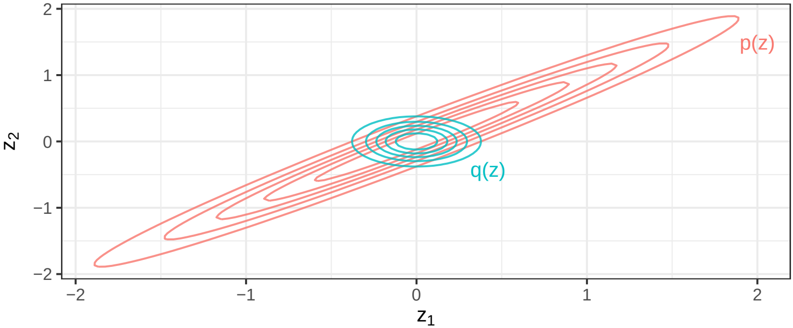

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- Approximating $p(\mathbf{z})$ by $q(\mathbf{z}) = q(z_1)q(z_2)$ yields

$$\tilde{q}(z_1) = \mathrm{N}(z_1 | \mu_1, \boldsymbol{\Psi}_{11}^{-1}) \quad \text{and} \quad \tilde{q}(z_2) = \mathrm{N}(z_2 | \mu_2, \boldsymbol{\Psi}_{22}^{-1})$$

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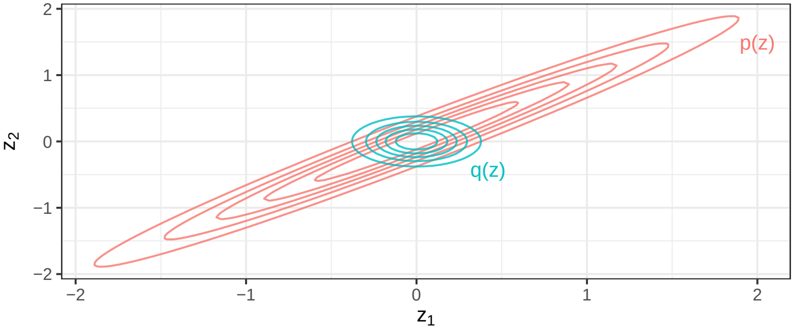

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  and by definition, $\mathrm{Cov}(z_1, z_2) = 0$ under $\tilde{q}$.
- This leads to underestimation of variances (widely reported in the literature—Zhao and Marriott 2013).

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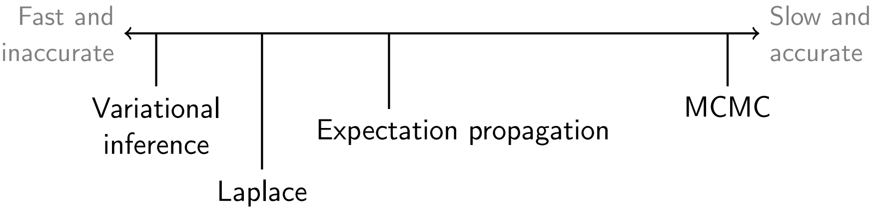
- Statistical properties not well understood—what is its statistical profile relative to the exact posterior?
- Speed trumps accuracy?



Advanced topics

- Local variational bounds
  - Not using the mean-field assumption.
  - Instead, find a bound for the marginalising integral $\mathcal{I}$.
  - Used for Bayesian logistic regression as follows:

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  - Use ideas from stochastic optimisation—gradient based improvement of ELBO from subsamples of the data.
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  - Use ideas from stochastic optimisation—gradient based improvement of ELBO from subsamples of the data.
  - Scales to massive data.
- Black box variational inference
  - Beyond exponential families and model-specific derivations.

End

Thank you!

Slides and source code are made available at: http://socialstats.haziqj.ml

References I

Gunawardana, A. and W. Byrne (2005). "Convergence theorems for generalized alternating minimization procedures". *Journal of machine learning research* 6.Dec, pp. 2049–2073.

Zhao, H. and P. Marriott (2013). "Diagnostics for Variational Bayes approximations". arXiv: 1309.5117.

4 Additional material