Calculus: Differentiation and Its Application

A statistical perspective

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`https://haziqj.ml/uitm-calculus/`

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(Almost) Everything you ought to know…

…about calculus in the first year

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Definition 1 (Differentiability) $f(x)$ is said to be *differentiable* at a point $x \in \mathcal{X}$ if the limit

$$L = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

exists. If L exists, we denote it by $f'(x)$ or $\frac{df}{dx}(x)$, and call it the *derivative* of f at x. Further, f is said to be differentiable on $\mathcal{X}$ if it is differentiable at every point in $\mathcal{X}$.

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For now, we assume $\mathcal{X} \subseteq \mathbb{R}$, and will extend to higher dimensions later.

Some examples

Function	Derivative
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = \sum_n a_n x^n$	$f'(x) = \sum_n n a_n x^{n-1}$
$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$f(x) = e^x$	$f'(x) = e^x$
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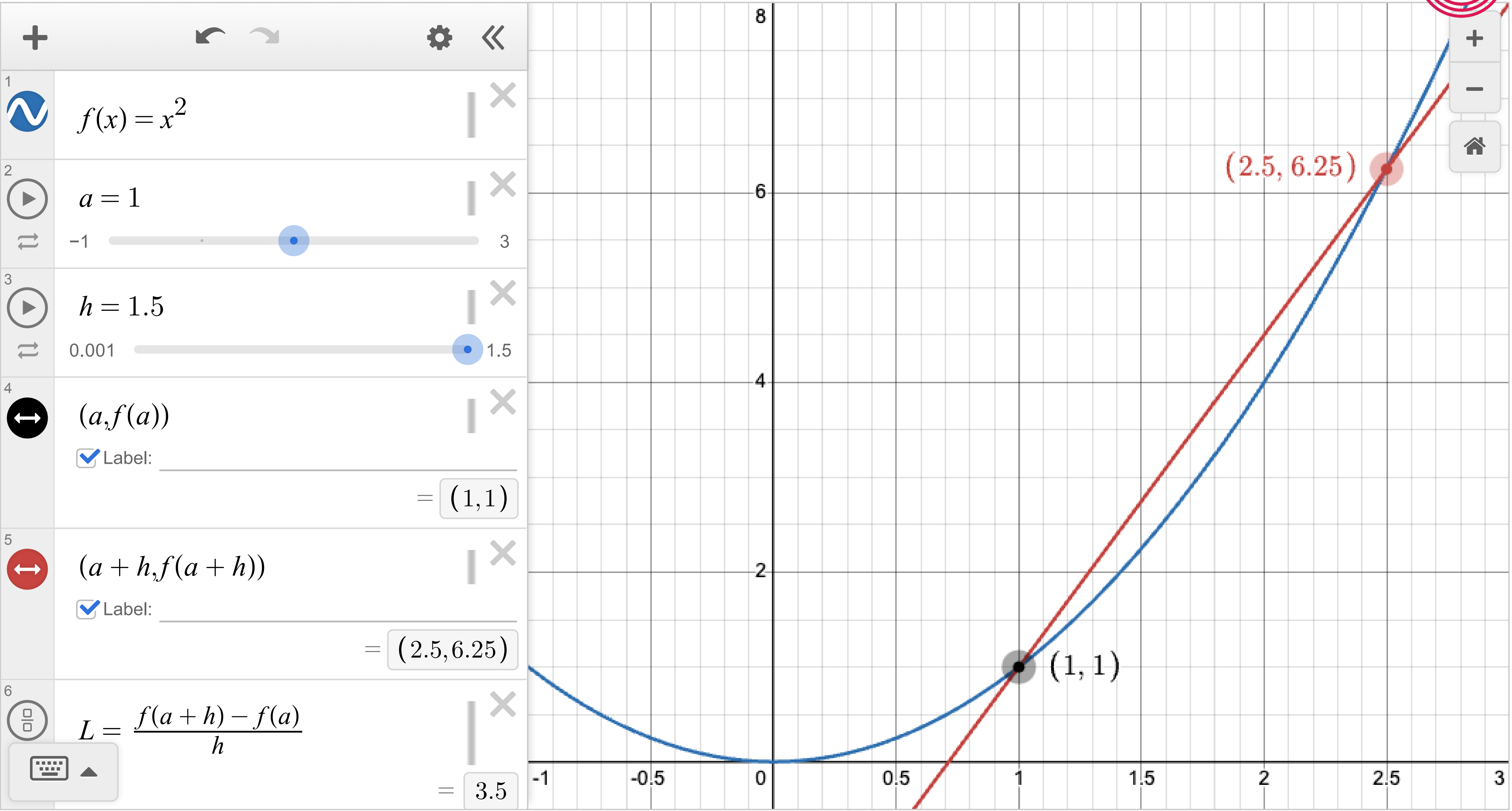
We can derive it "by hand" using the definition. Let $f(x) = x^2$. Then,

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x.
\end{aligned}$$

Taylor series is a powerful tool for approximating functions using derivatives.

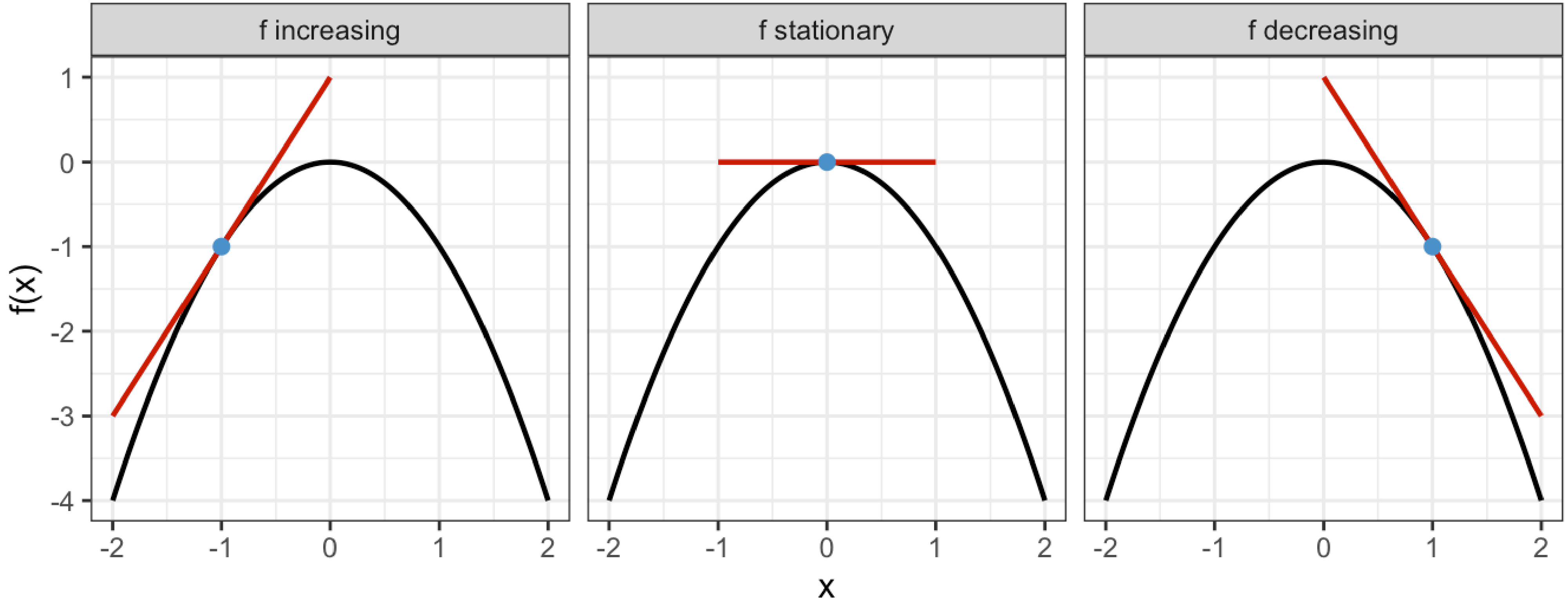
Graphically...



But what *is* a derivative?

The derivative of a function tells you:

- 🚀 How fast the function is *changing* at any point
- 📐 The **slope** of the tangent line at that point



The concept of optimisation

- When f is some kind of a "reward" function, then the value of x that maximises f is highly of interest. Some examples:
  - 💰 **Profit maximisation**: Find the price that maximises profit.
  - 🧬 **Biological processes**: Find the conditions that maximise growth or reproduction rates.
  - 👷 **Engineering**: Find the design parameters that maximise strength or efficiency.
- Derivatives help us find so-called *critical values*: Solve $f'(x) = 0$.

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$$
\begin{aligned}
f'(x) = -12x^3 + 12x^2 + 24x &= 0 \\
\Leftrightarrow 12x(2 + x - x^2) &= 0 \\
\Leftrightarrow 12x(x+1)(x-2) &= 0 \\
&\Leftrightarrow x = 0, -1, 2.
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Are all of these critical values maxima values? 🤔

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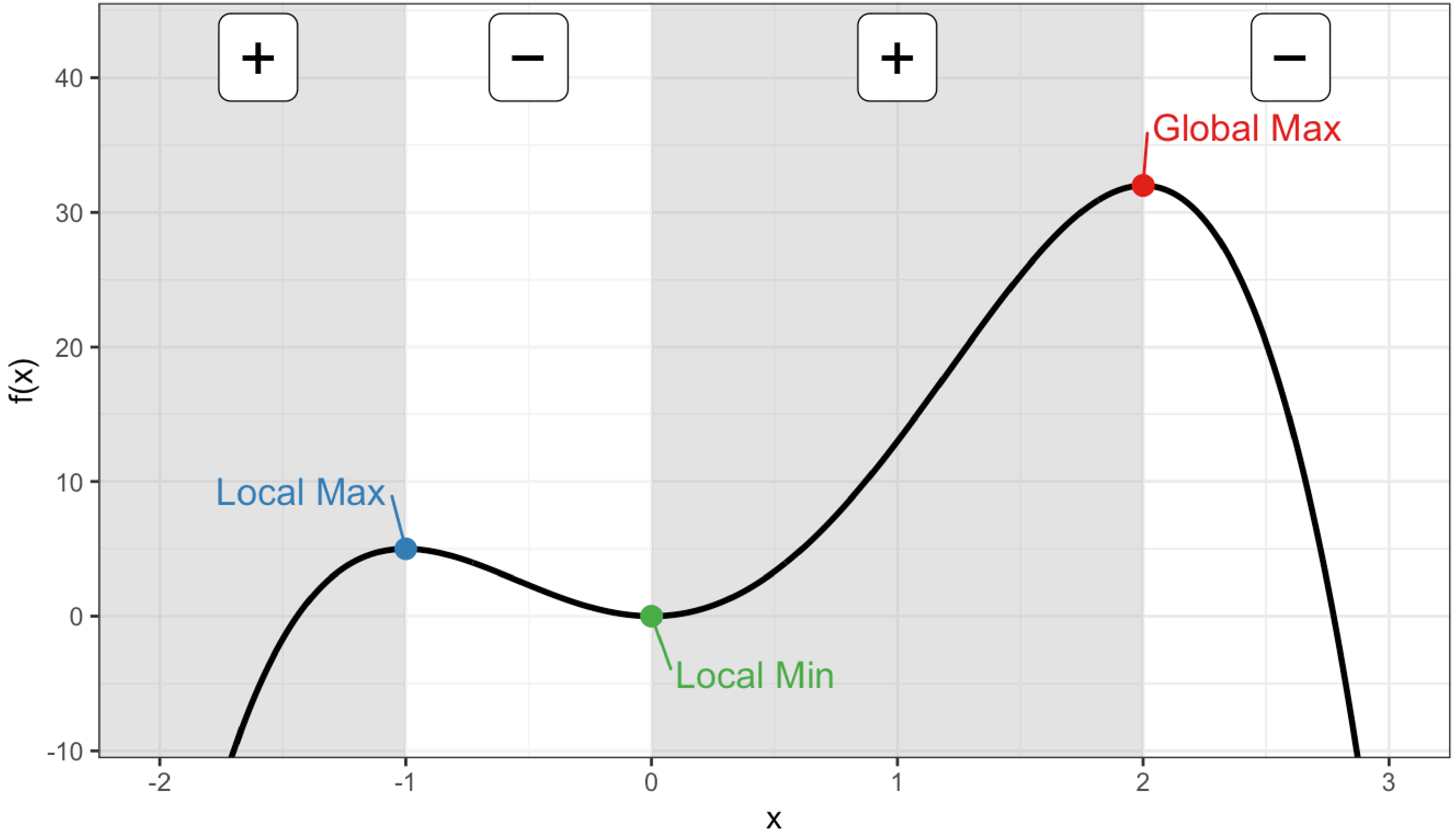

How do we know if it's a maxima or minima?

Second derivative test: Measure the **change in slope** around a point x, i.e. $f''(\hat{x}) = \frac{d}{dx}\left(\frac{df}{dx}(x)\right) = \frac{d^2 f}{dx^2}(x)$.

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Behaviour of f near $\hat{x}$	$f''(\hat{x})$	**Shape**	**Conclusion**
Increasing → Decreasing	$f''(\hat{x}) < 0$	Concave (∩)	Local maximum
Decreasing → Increasing	$f''(\hat{x}) > 0$	Convex (∪)	Local minimum
No sign change / flat region	$f''(\hat{x}) = 0$	Unknown / flat	Inconclusive

Second derivative test

From Example 1, the second derivative is given by

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-12x^3 + 12x^2 + 24x\right) \\ &= -36x^2 + 24x + 24 \end{aligned}$$

Plug in the critical points:

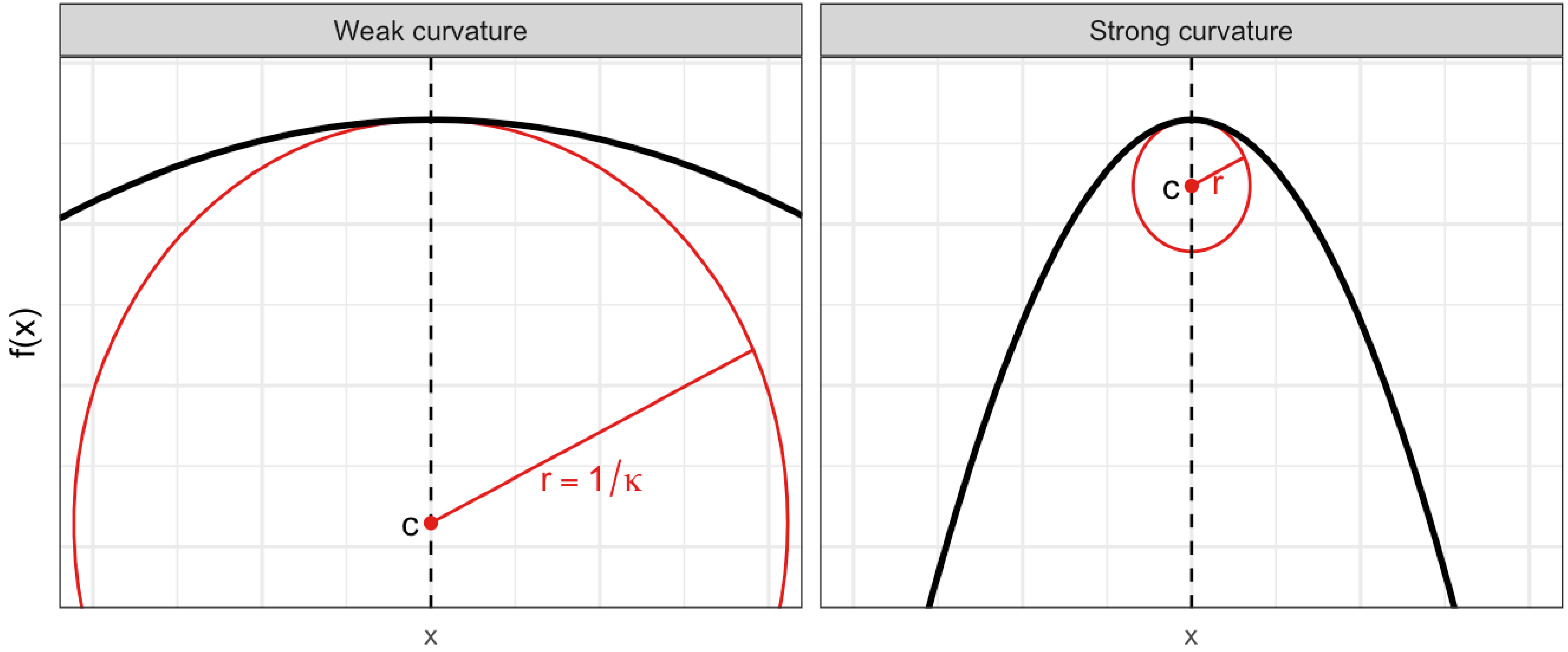
- $x = -1$: $f''(-1) = -36 - 24 + 24 = -36 < 0$, hence local maximum.
- $x = 0$: $f''(0) = 0 + 0 + 24 = 24 > 0$, hence local minimum.
- $x = 2$: $f''(2) = -144 + 48 + 24 = -72 < 0$, hence local maximum.

Tip

Often it is not enough to just differentiate once to find optima. Differentiate twice to classify critical points.

Curvature

Let $\mathcal{C}_x$ denote the *osculating circle* at x with centre c and radius r, i.e. the circle that **best approximates the graph of f at x**. Historically, the curvature κ for a graph of a function f at a point x is measured as $\kappa = \frac{1}{r}$.



Curvature and concavity

Definition 2 (Curvature) The (signed) curvature for a graph $y = f(x)$ is

$$\kappa = \frac{f''(x)}{\left(1 + [f'(x)]^2\right)^{3/2}}.$$

- The second derivative $f''(x)$ tells us how fast the slope is changing.
- The **sign** of the curvature is the same as the sign of $f''(x)$. Hence,
  - If $f''(x) > 0$, the graph is **concave up** (convex).
  - If $f''(x) < 0$, the graph is **concave down** (concave).
- The **magnitude** of the curvature is proportional to $f''(x)$. Hence,
  - If $|f''(x)|$ is large, the graph is *steep* and "curvier".
  - If $|f''(x)|$ is small, the graph is *flat* and "gentle".
- For reference, a straight line has zero curvature.

Summary so far

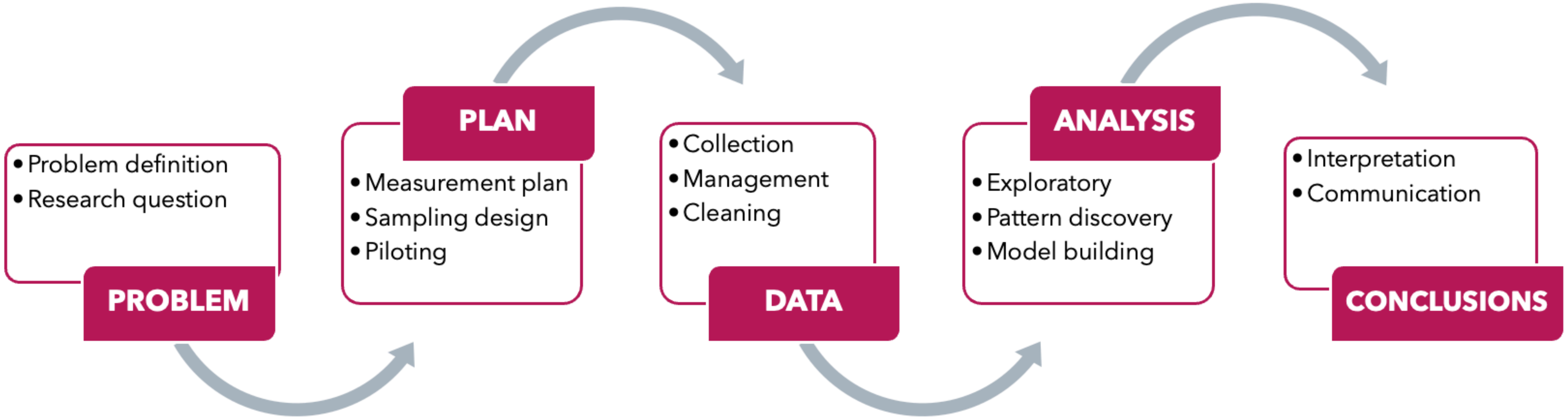
- Derivatives represent rate of change (slope) of a function $f : \mathcal{X} \to \mathbb{R}$.
- Interested in optimising an *objective* function $f(x)$ representing some kind of "reward" or "cost".
- Find critical points by solving $f'(x) = 0$.
- Use the second derivative test to classify critical points:
  - If $f''(x) < 0$, then f is concave down at x and x is a local maximum.
  - If $f''(x) > 0$, then f is concave up at x and x is a local minimum.
  - If $f''(x) = 0$, then the test is inconclusive.
- Curvature tells us how steep the curve is at its optima. In some sense, it tells us how hard or easy it is to find the optimum.

A statistical perspective

But what *is* statistics?

Statistics is a scientific subject that deals with the collection, analysis, interpretation, and presentation of data.

- **Collection** means designing experiments, questionnaires, sampling schemes, and also administration of data collection.
- **Analysis** means mathematically modelling, estimation, testing, forecasting.



See also: *The Art of Statistics: Learning from Data* by David Spiegelhalter.

Motivation

> I toss a coin n times and I wish to find p, the probability of heads. Let $X_i = 1$ if a heads turns up, and $X_i = 0$ if tails.

- I do not know the value of p, so I want to estimate it somehow.
- I have a "guess" what it might be e.g. $p = 0.5$ or $p = 0.7$.
- How do I objectively decide which value is better?

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A more high-stakes example

Think of the binary outcomes as a stock price rising or falling. You'll need to decide to invest based on what you believe (or the data suggests) the probability of the stock price rising is.

A probabilistic model

Each X_i is a *random variable* taking only two possible outcomes, i.e.

$$X_i = \begin{cases} 1 & \text{w.p.} \quad p \\ 0 & \text{w.p.} \quad 1-p \end{cases}$$

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Suppose that $X = X_1 + \cdots + X_n$. So we are counting the number of heads in n tossess. Then this becomes a **binomial** random variable. We write $X \sim \operatorname{Bin}(n,p)$, and the *probability mass function* is given by

$$f(x \mid p) = \Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n.$$

Often we might want to find quantities such as $\operatorname{E}(X) = np$ and $\operatorname{Var}(X) = np(1-p)$, but we will not go into details here.

Learning from data

p is unknown

If we do not know p, then it is not possible to calculate probabilities, expectations, variances… ☹️

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$$H, H, H, T, T, H, H, T, H, H$$

There is a total of $X = 7$ heads, and from this you surmise that (at least) the coin is *unlikely* to be fair, because:

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- If $p = 0.7$, then $\Pr(X = 7 \mid p = 0.5) = \binom{10}{7}(0.7)^7(0.3)^3 = 0.267$.

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- If $p = 0.7$, then $\Pr(X = 7 \mid p = 0.5) = \binom{10}{7}(0.7)^7(0.3)^3 = 0.267.$
- If $p = 0.9$, then $\Pr(X = 7 \mid p = 0.9) = \binom{10}{7}(0.9)^7(0.1)^3 = 0.057.$

How to formalise this idea?

The likelihood function

Definition 3 Given a probability function $x \mapsto f(x \mid \theta)$ where x is a realisation of a random variable X, the *likelihood function* is $\theta \mapsto f(x \mid \theta)$, often written $\mathcal{L}(\theta) = f(x \mid \theta)$.

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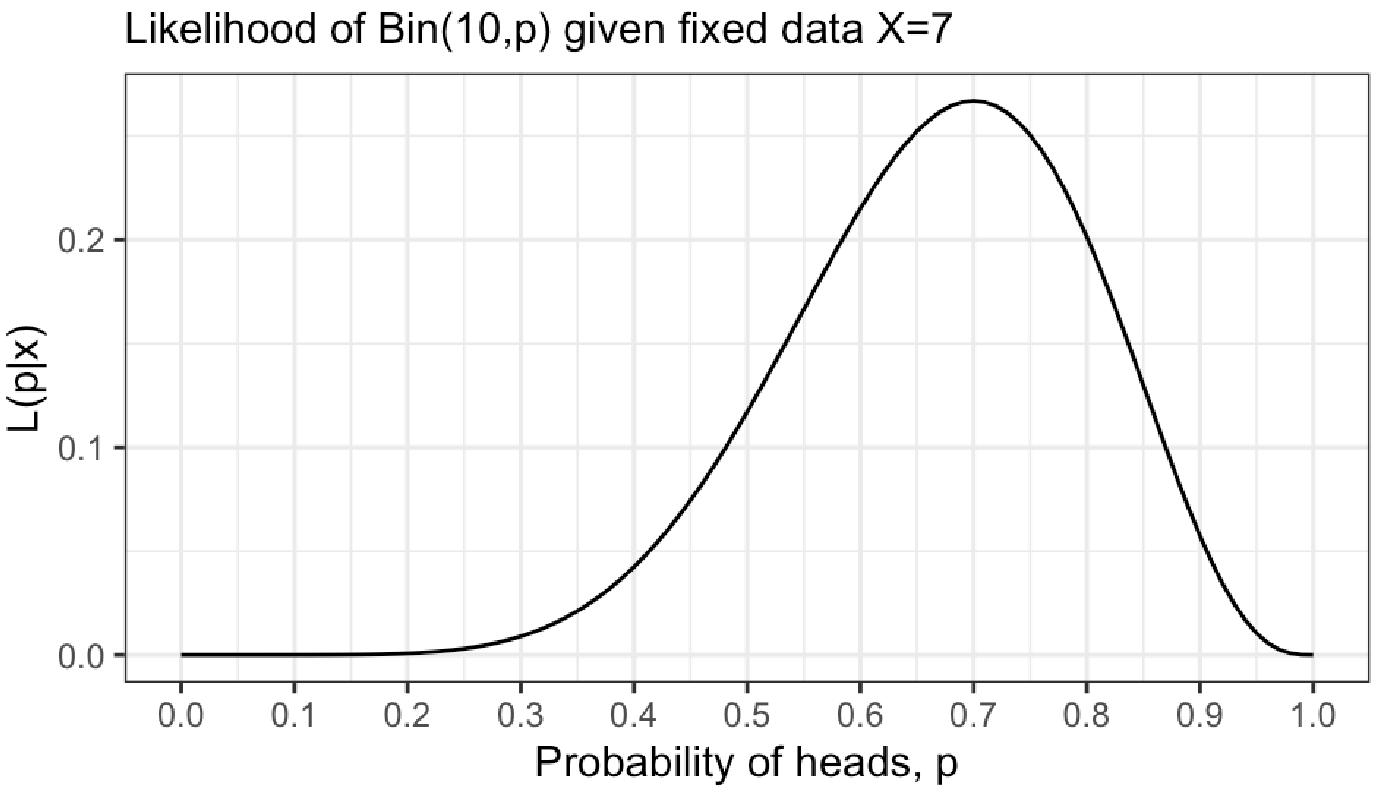

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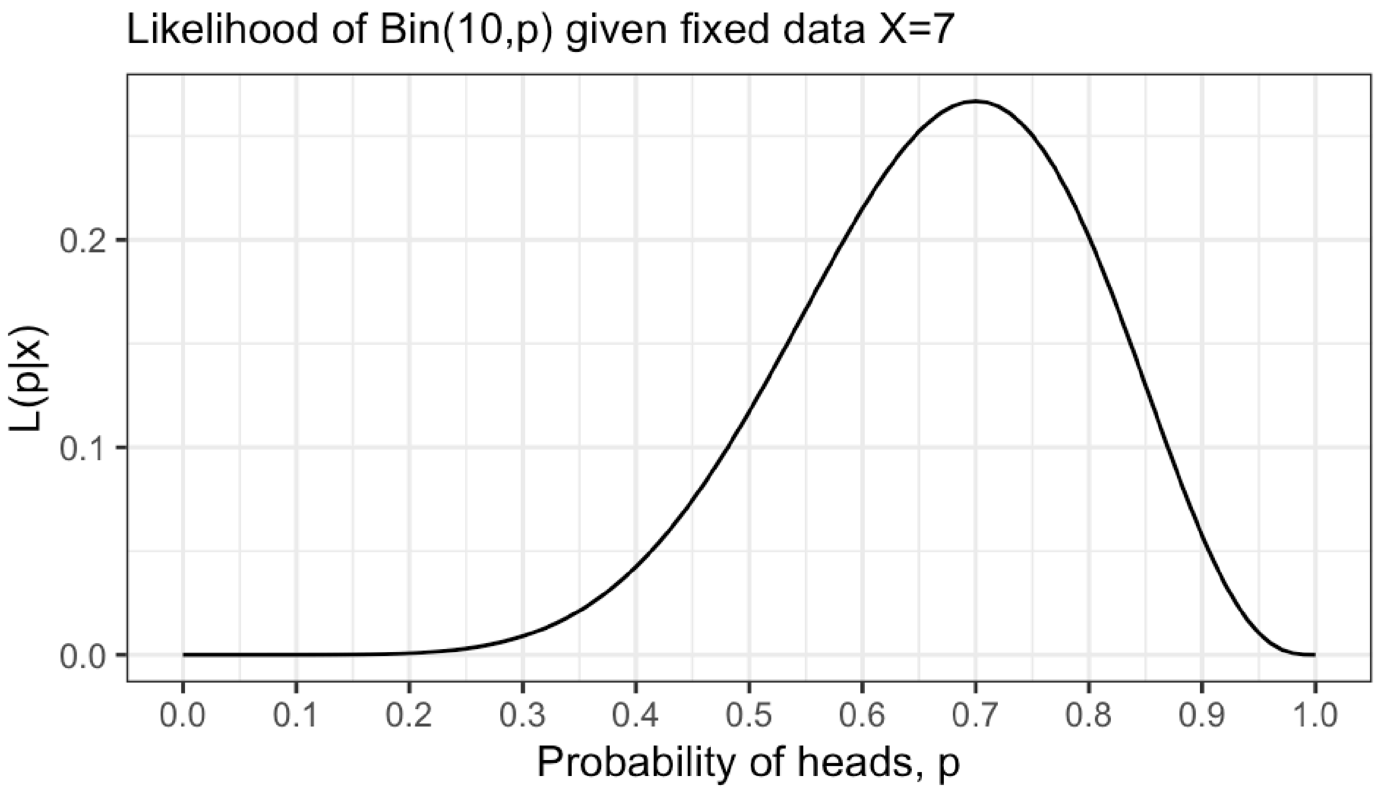

The value $\hat{\theta}$ which maximises $\mathcal{L}(\theta)$ is called the **maximum likelihood estimator** (MLE) of θ.

Parameteric statistical models

Assume that $X_i \sim f(x \mid \theta)$ independently for $i = 1, \ldots, n$. Here, functional form of f is known, but the parameter θ is unknown. Examples:

Name	$f(x \mid \theta)$	θ	**Remarks**
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$	$p \in (0,1)$	No. successes in n trials
Poisson	$\frac{\lambda^x e^{-\lambda}}{x!}$	$\lambda > 0$	Count data
Uniform	$\frac{1}{b-a}$ for $x \in [a,b]$	$a < b$	Equally likely outcomes
Exponential	$\lambda e^{-\lambda x}$ for $x \geq 0$	$\lambda > 0$	Waiting time
Normal	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$\mu \in \mathbb{R},\ \sigma^2 > 0$	Bell curve

⚠️ Everything we need to know about the distribution is captured by the parameter θ.

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Suppose we observe $X_1, \ldots, X_n$ from a normal distribution with unknown mean μ and known variance σ^2. The log-likelihood function is given by

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\ell(\mu) &= \sum_{i=1}^{n} \log f(X_i \mid \mu) \\
&= \sum_{i=1}^{n} \log \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}} \right) \\
&= \sum_{i=1}^{n} \left\{ -\frac{1}{2} \log(2\pi\sigma^2) - \frac{(X_i-\mu)^2}{2\sigma^2} \right\} \\
&= -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i-\mu)^2.
\end{aligned}
$$

Example (Normal mean, cont.)

To find the MLE of μ, we differentiate the log-likelihood function with respect to μ and set it to zero:

Example (Normal mean, cont.)

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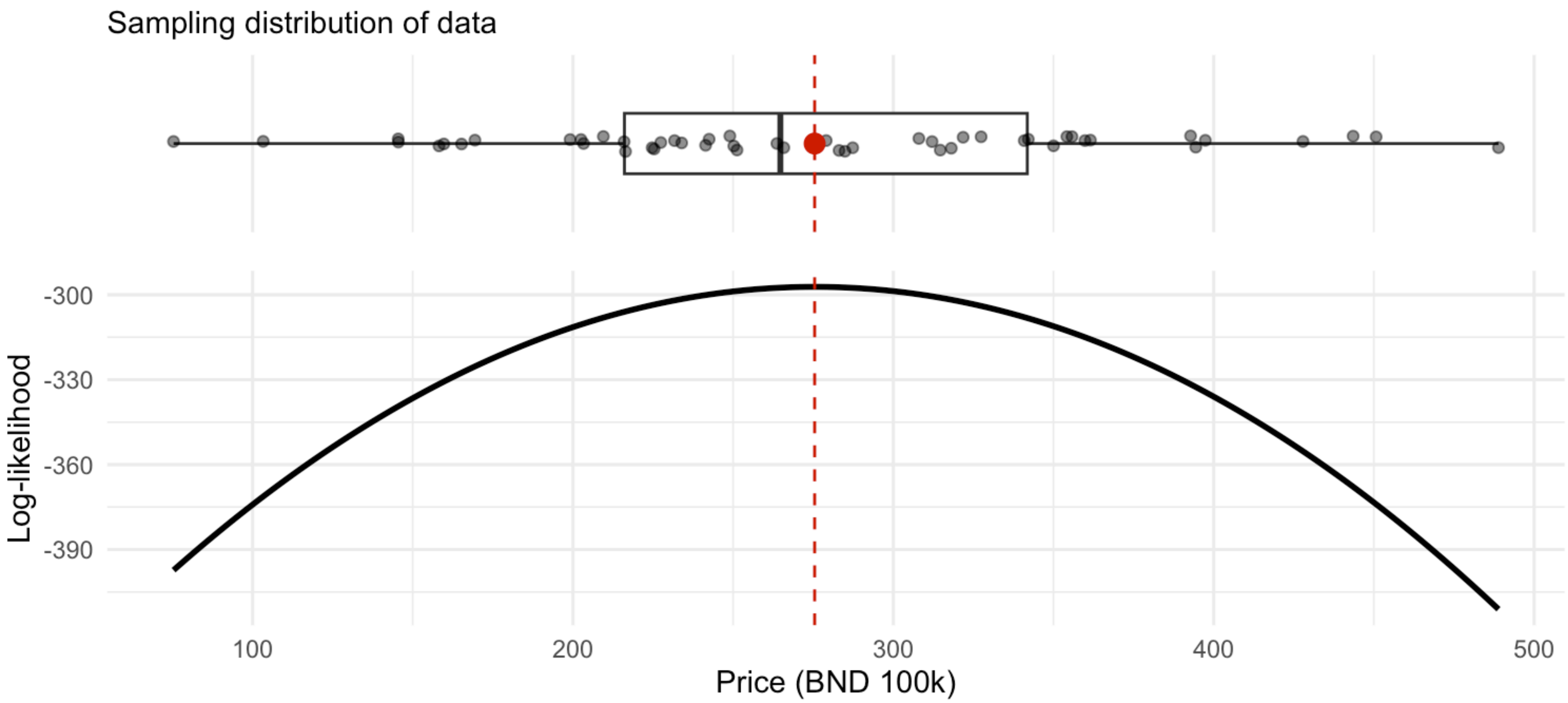
$$\begin{aligned}\frac{d}{d\mu}\ell(\mu) &= -\frac{1}{2\sigma^2}\cdot 2\sum_{i=1}^{n}(X_i-\mu)(-1)\\ &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)=0\\ &\Leftrightarrow \sum_{i=1}^{n}X_i - n\mu = 0 \Leftrightarrow \mu = \frac{1}{n}\sum_{i=1}^{n}X_i.\end{aligned}$$

Thus, the MLE for μ is $\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

A real data example

Example 2 Sample $n = 50$ house prices randomly in Brunei.

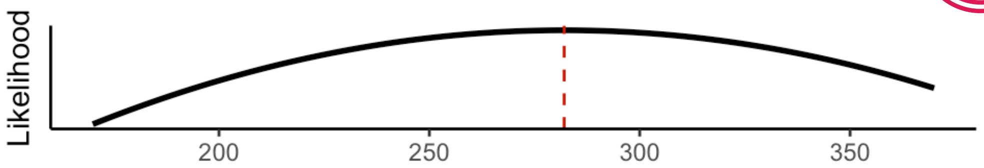


HJ (2025). A spatio-temporal analysis of house prices in Brunei Darussalam. *Qual Quant*, 1-32. DOI: 10.1007/s11135-025-02164-0.

But wait, the sample was random...

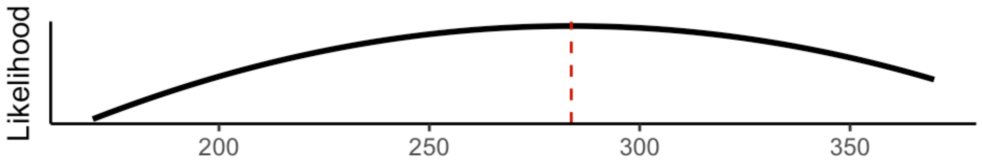
209 290 188 432 305 190
321 346 330 241 423 ...

$\bar{X} = 282.0$



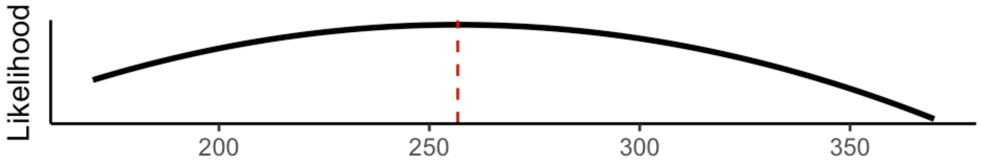
312 211 306 159 415 470
235 168 329 258 512 ...

$\bar{X} = 283.7$



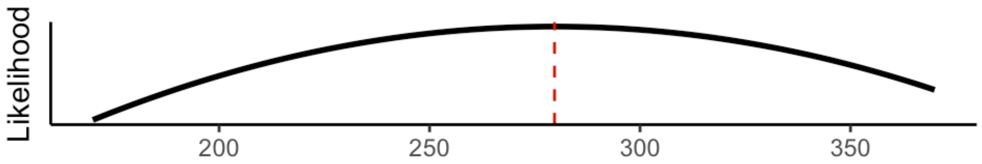
210 276 181 288 207 449
344 363 310 440 208 ...

$\bar{X} = 256.8$



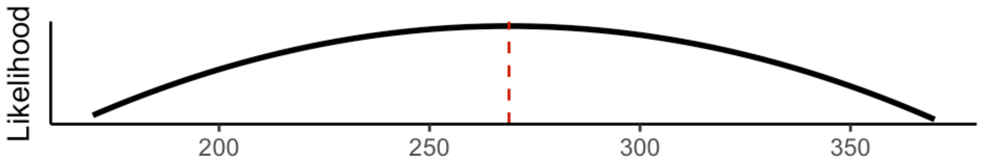
317 270 240 179 123 164
372 210 134 459 315 ...

$\bar{X} = 279.7$

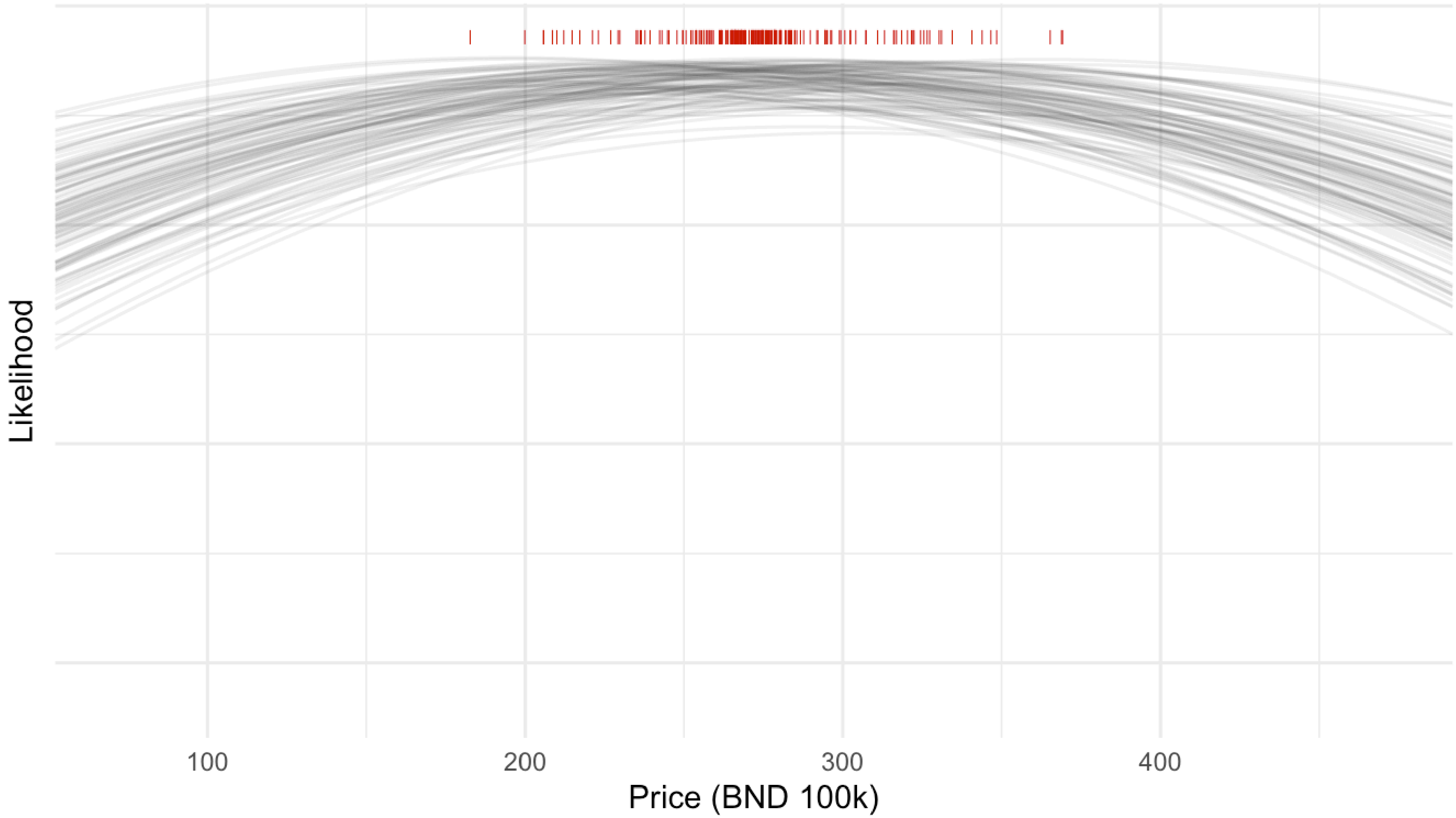


313 441 431 239 43 522
339 326 271 323 256 ...

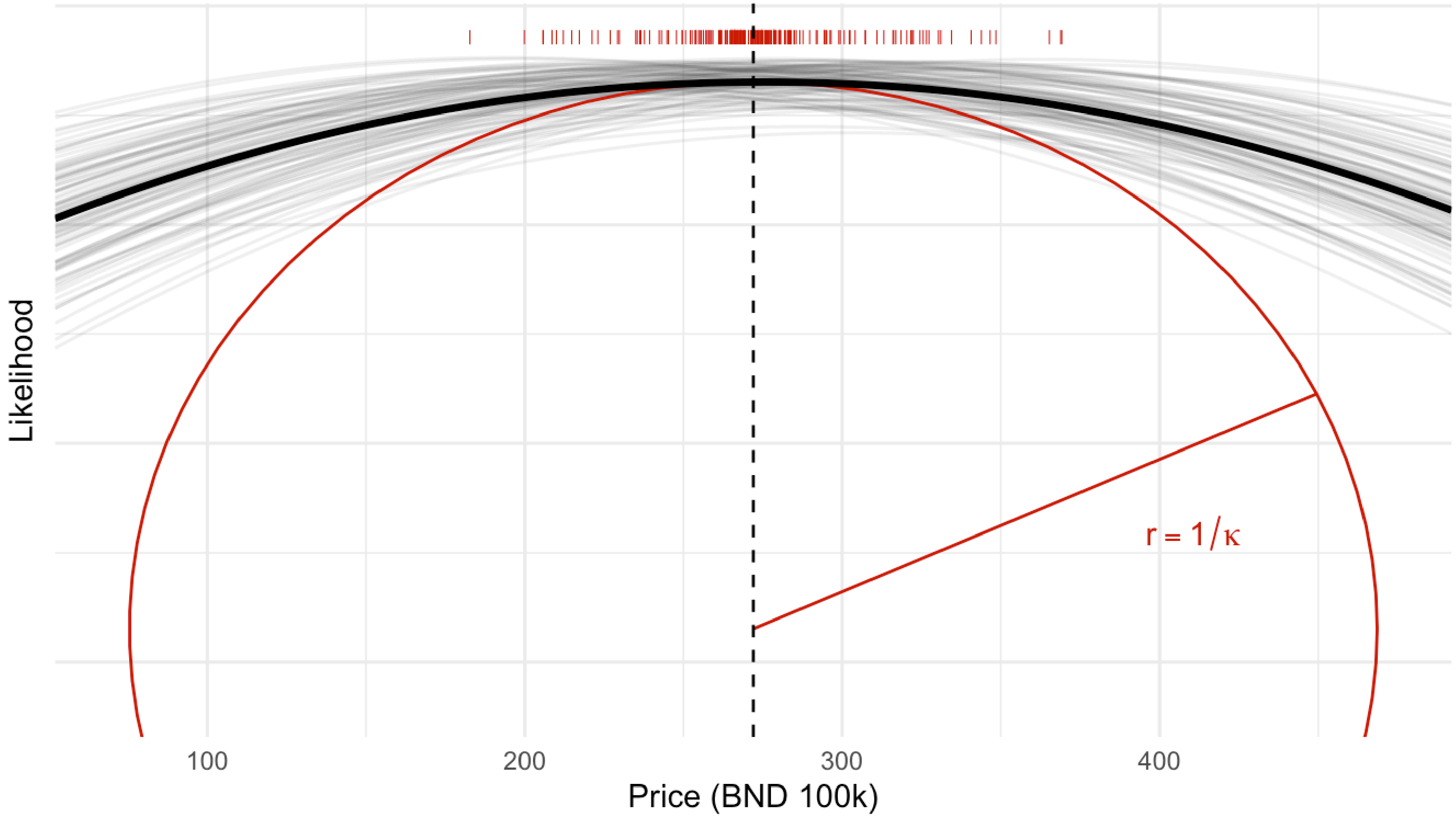
$\bar{X} = 268.9$



Averaging (hypothetical) likelihoods



Averaging (hypothetical) likelihoods, cont.



Fisher information

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Evidently the Fisher information is **proportional** to the *curvature of the (log)-likelihood function.*

> 🧠 INTUITION: Stronger curvature → Easier to find optima → More information about the parameter → Less uncertainty (and vice versa)

Extension of the concepts of "curvature" to the case of **random outcomes**!

Example (Normal mean Fisher information)

For the normal mean example, we have $\ell'(\mu) = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)$. Thus,

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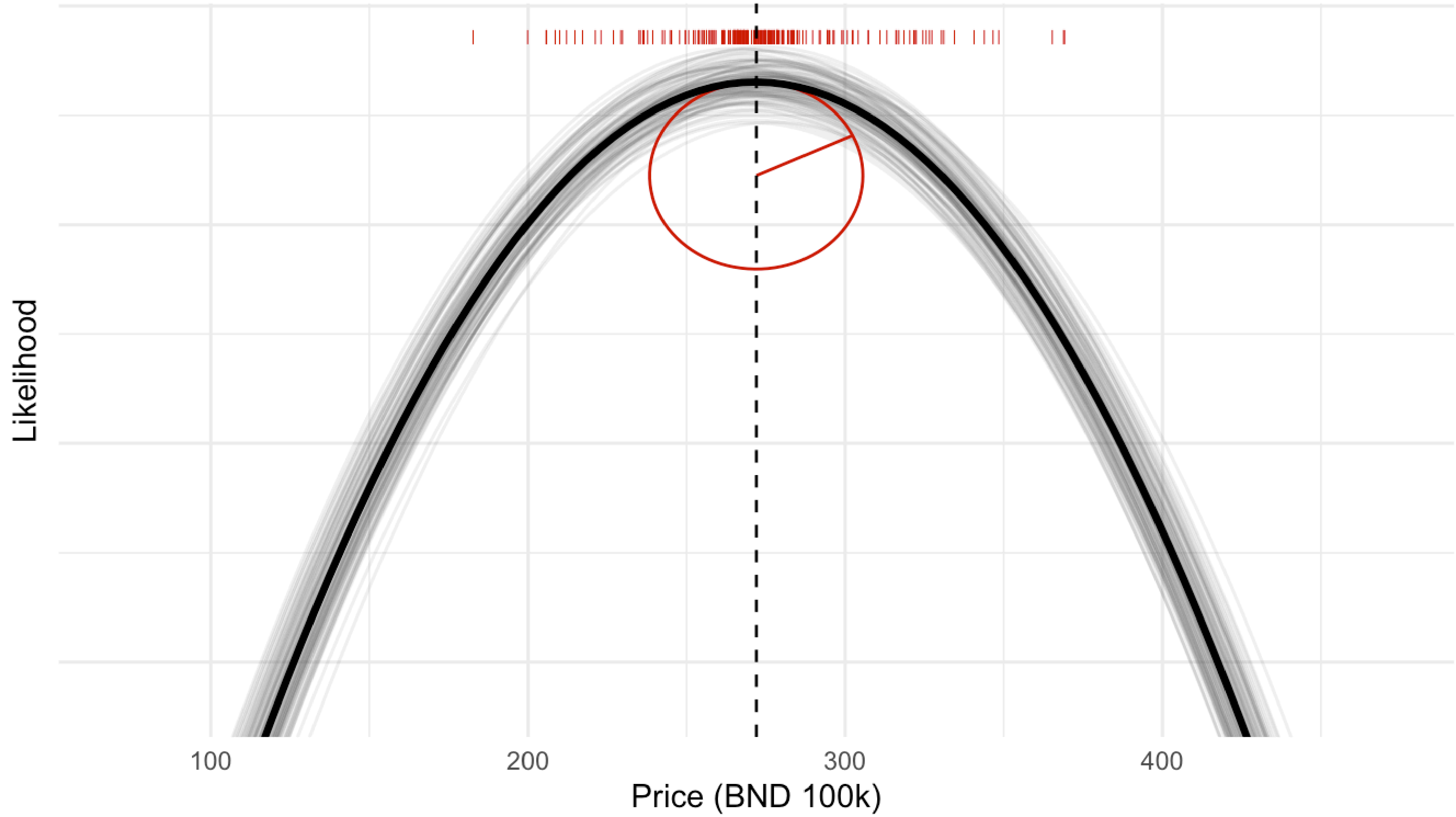
$$\ell''(\mu) = \frac{d^2\ell}{d\mu^2}(\mu) = \frac{d}{d\mu}\left[\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)\right]$$
$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(-1) = -n/\sigma^2.$$

Therefore, the Fisher information is

$$\mathcal{I}(\mu) = -\mathbf{E}\left[\ell''(\mu)\right] = \frac{n}{\sigma^2}.$$

We can improve the estimate of μ by increasing the sample size n!

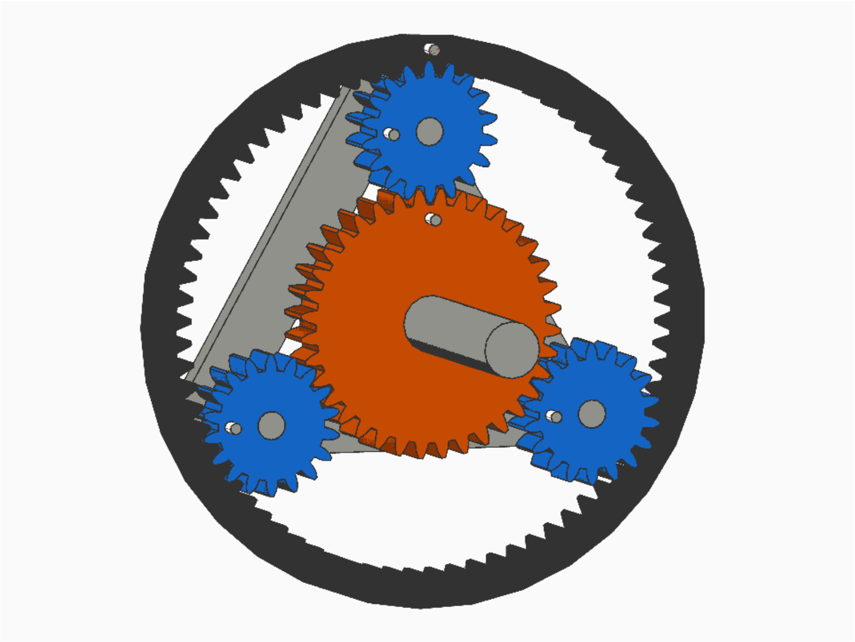
Large information ($n = 1000$)



Another example

Exponential waiting time

Example 3 (Estimating failure rate of a machine component) Suppose we collect data on how long (in hours) a machine component lasts before it fails. This could be a valve in a chemical plant, a sensor in a civil engineering structure, or a server part in a data centre.



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Assume that failure times X follow an exponential distribution:

$$f(x \mid \lambda) = \lambda e^{-\lambda x}, \quad x > 0$$

where λ is the failure **rate**. Using observed failure times from a sample of machines, we can estimate λ via **Maximum Likelihood Estimation (MLE)**.

Engineers and analysts can predict average lifetime ($1/\lambda$), schedule maintenance, and make design decisions to improve reliability.

Exponential waiting time (cont.)

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$$\begin{aligned}\ell(\lambda) = \log\left[\prod_{i=1}^{n} f(X_i \mid \lambda)\right] &= \sum_{i=1}^{n} \log f(X_i \mid \lambda) \\ &= \sum_{i=1}^{n} \log\left(\lambda e^{-\lambda X_i}\right) \\ &= n \log \lambda - \lambda \sum_{i=1}^{n} X_i.\end{aligned}$$

Exponential waiting time (cont.)

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$$\Leftrightarrow \lambda = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}}.$$

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\end{aligned}
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Tip

To obtain the Fisher information, just differentiate $\ell'(\lambda)$ once more, and take negative expectations. Verify that it is $\mathcal{I}(\lambda) = n/\lambda^2$.

Data example

Suppose $n = 50$ machines were observed, and the failure times (in hours) recorded:

293.4	339.4	392.6	84.4	36.9	792.5	88.8	844.1	182.6	103.7
364.5	73.7	578.6	101.9	143.9	459.4	200.2	206.0	461.5	301.2
199.6	218.0	76.5	89.8	324.3	240.5	2022.2	264.8	213.8	901.3
219.9	729.4	1322.9	551.3	571.2	428.1	781.1	395.7	18.2	50.9
322.6	110.6	157.4	310.5	477.7	168.4	9.2	969.1	399.5	5.4

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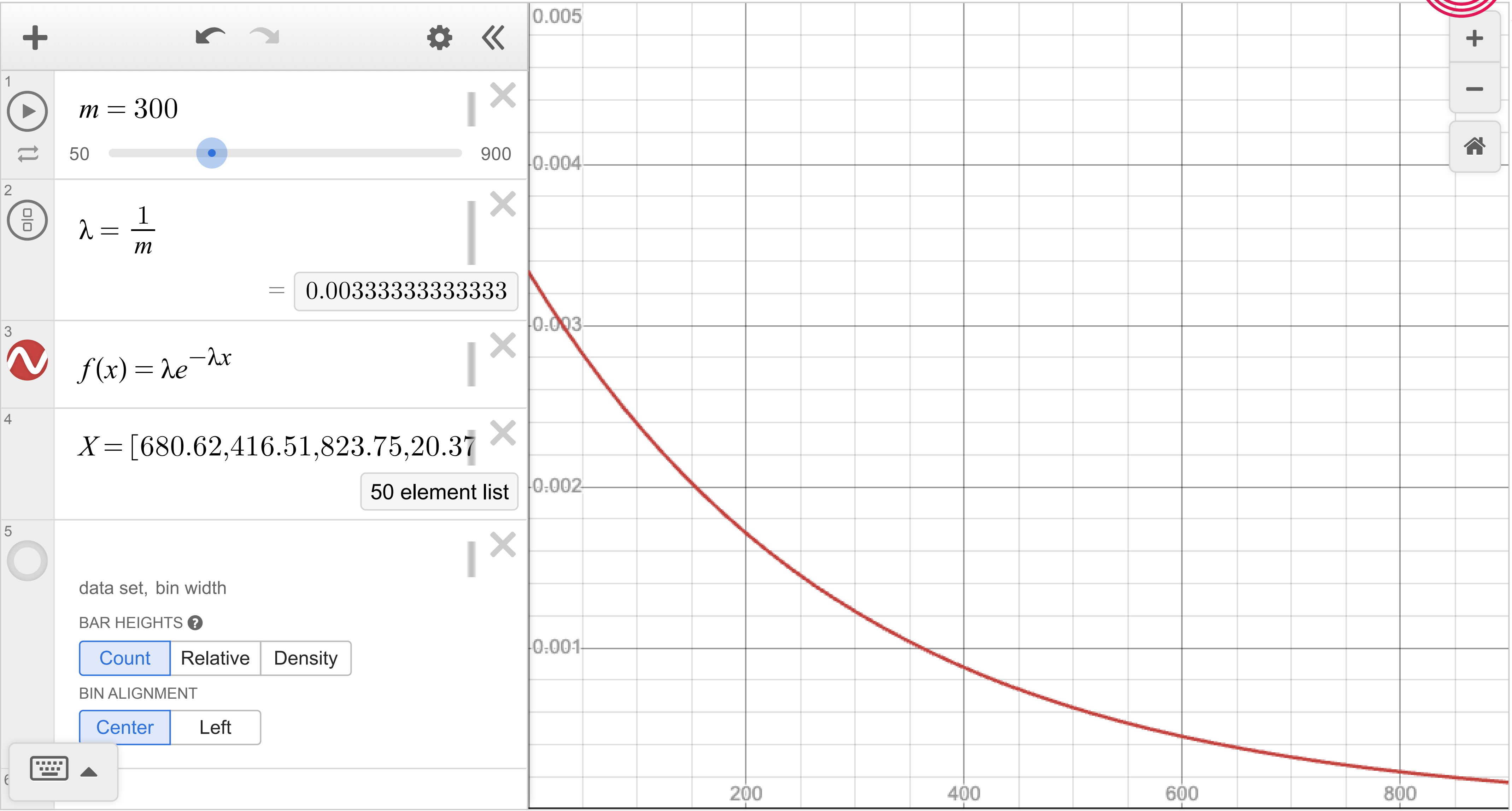
Since the MLE of λ is $\hat{\lambda} = 1/\bar{X}$, we can compute it as follows:

$$\hat{\lambda} = \frac{1}{\bar{X}} = \frac{1}{372.0} \approx 2.69e-03.$$

In other words, approximately one failure every 372 hours.

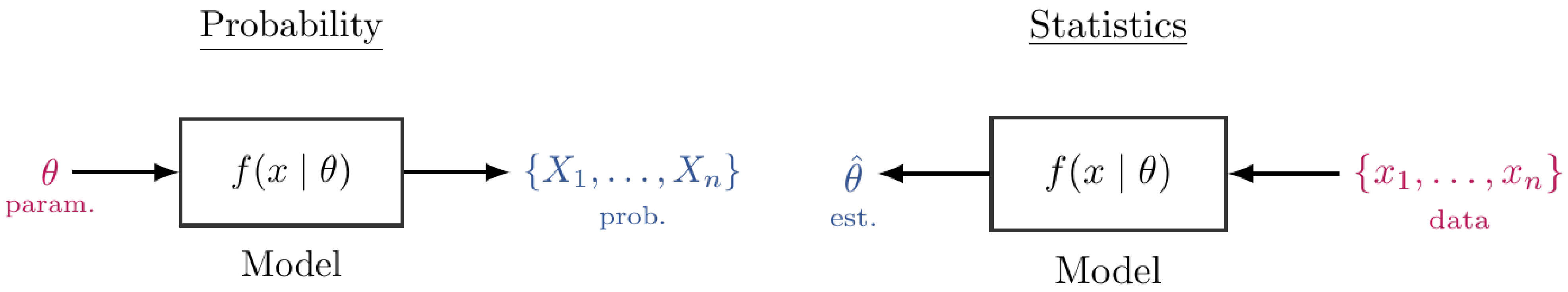
Plot of data and exponential fit



Conclusions

Summary

Given a model, probability allows us to *predict* data. Statistics on the other hand, allows us to *learn* from data.



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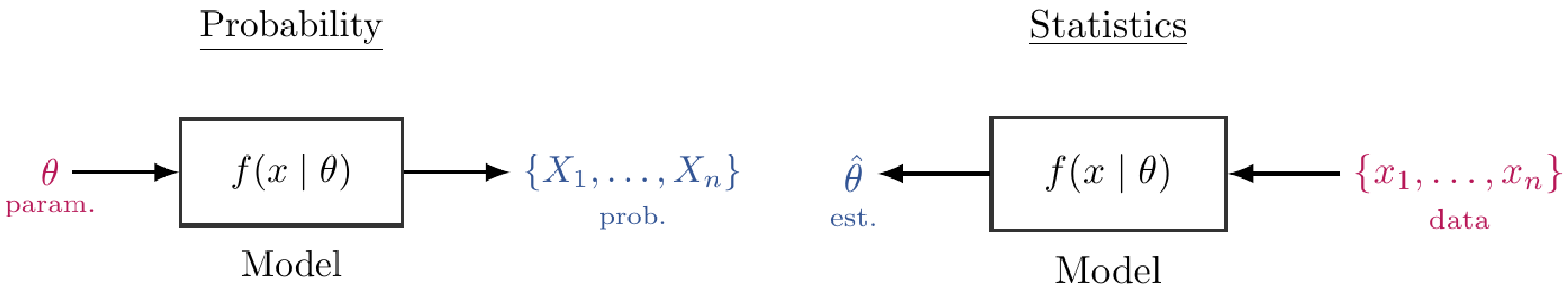

- Parameter estimation is a **central** task in statistics.
  - Finding the **Maximum Likelihood Estimator (MLE)**
  - Understanding **Fisher Information** and curvature
  - Uncovering role of sample size in estimation uncertainty
- Calculus is not just background maths–it's the **engine** driving statistical theory.

Where to go from here

🌐 **Numerical derivatives** – how computers approximate calculus

```r
fun <- function(x) x ^ 2
numDeriv::grad(fun, x = 2)
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[1] 4
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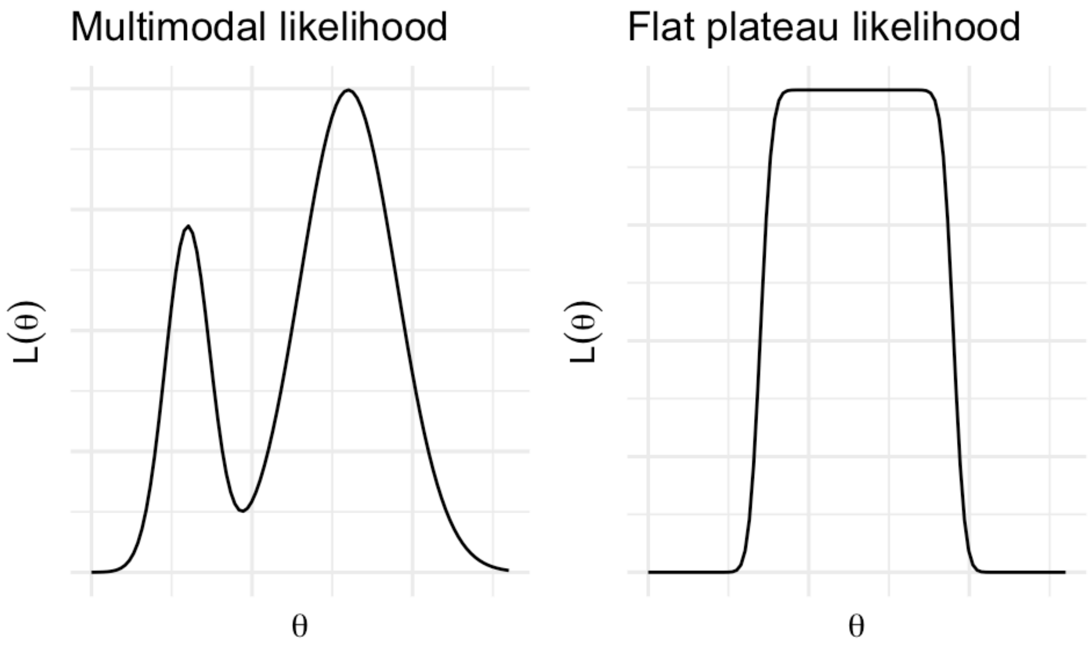
🧠 **Bimodal and non-standard distributions** when simple models break



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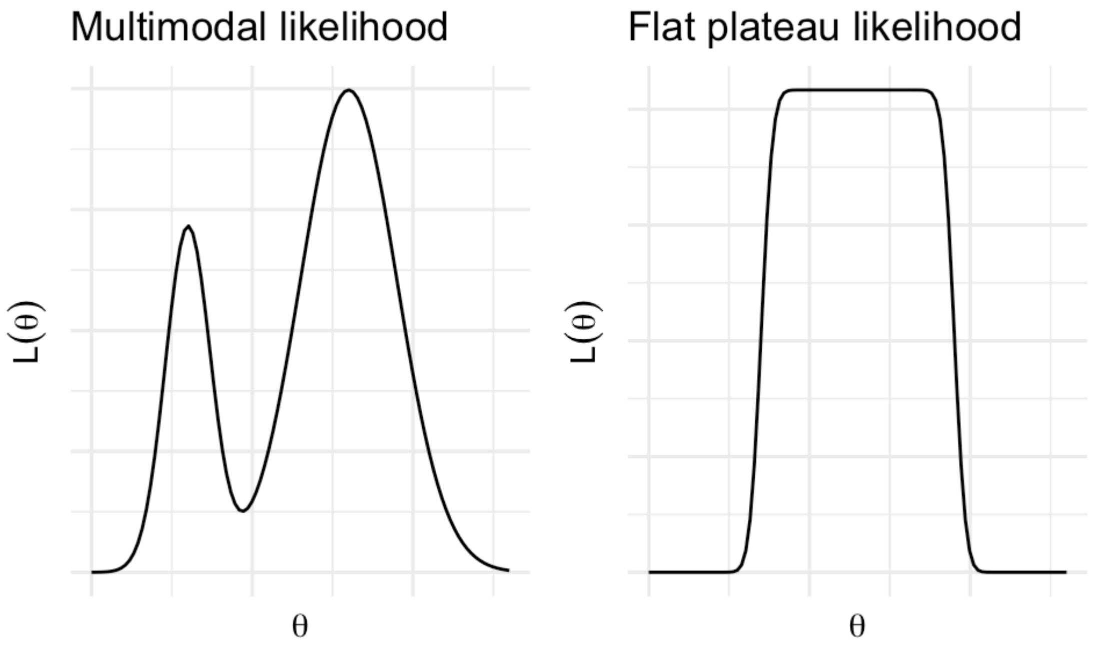
🚀 Modern statistics tackles:

- **Big data** (too much)
- **High-dimensional data** (too many variables)
- **Complex models** (real-world messiness)

🧠 **Bimodal and non-standard distributions** when simple models break



Thanks!

`https://haziqj.ml/`