APPLICATIONS OF DIFFERENTIATION

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1. RELATED RATES

● A **related rates** problem asks you to calculate the rate of change of a differentiable real function f of a continuous real variable x, if the rate of change of x is given.

● Related rates problems usually involve an application of the **Chain Rule**, which reads

$$\frac{d}{dt} f(x(t)) = \frac{df}{dx} \cdot \frac{dx}{dt}$$

Example 1: A particle is moving along a parabolic path

$$y = x^2 + 3x - 7$$

What is the rate of change dy/dt of y at the moment when $x = 1$ if $dx/dt = 3$?

Answer: According to the Chain Rule,

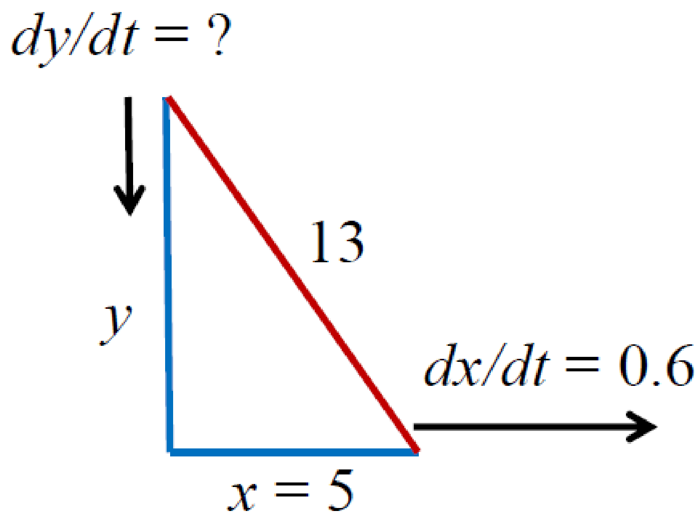
$$\frac{dy}{dt} = \frac{d}{dx}(x^2 + 3x - 7) \cdot \frac{dx}{dt} = (2x+3) \cdot \frac{dx}{dt}$$

so if $x = 1$ and $dx/dt = 3$,

$$\frac{dy}{dt} = (2\times 1+3)\times 3 = 5\times 3 = 15$$

Example 2: A ladder of length 13 metres is resting against a vertical wall, with its foot on the floor at a distance of 5 metres from the wall. If the foot of the ladder is pulled away from the wall at a speed of 0.6 metres per second, how fast is the top of the ladder sliding down the wall?

Answer: Let the distance of the ladder from the wall be x metres, and the height of the top of the ladder be y metres.
From Pythagoras' Theorem, we know that



$$x^2 + y^2 = 13^2 = 169$$

We can solve this equation for y to give:

$$y = \sqrt{169 - x^2}$$

and in particular if $x = 5$ then $y = \sqrt{169 - 5^2} = \sqrt{144} = 12$.

To calculate the rate of change dy/dt of y when $dx/dt = 0.6$, we could apply the Chain Rule in the form

$$\frac{dy}{dt} = (\frac{d}{dx}\sqrt{169 - x^2}) \cdot \frac{dx}{dt}$$

but in practice it is easier to differentiate the equation $x^2 + y^2 = 169$ with respect to t. This gives:

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(169) = 0$$

which is equivalent to

$$0 = \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dx}(x^2) \cdot \frac{dx}{dt} + \frac{d}{dy}(y^2) \cdot \frac{dy}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Solving this equation for dy/dt gives:

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

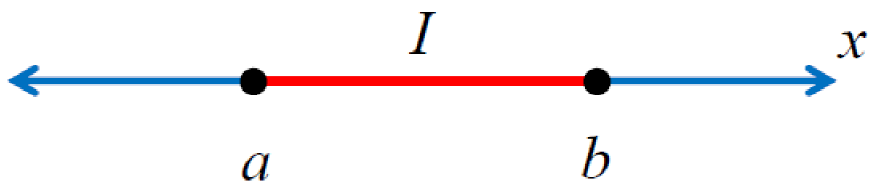
If we now let $x = 5$, $y = 12$ and $dx/dt = 0.6$, the answer is $dy/dt = -5 \times 0.6/12 = -3/12 = -0.25$.

2. ANALYSIS OF FUNCTIONS

a. Intervals of increase and decrease

As before, let f be a real function of a continuous real variable x – although we will not for the moment be assuming that f is differentiable (or even continuous).

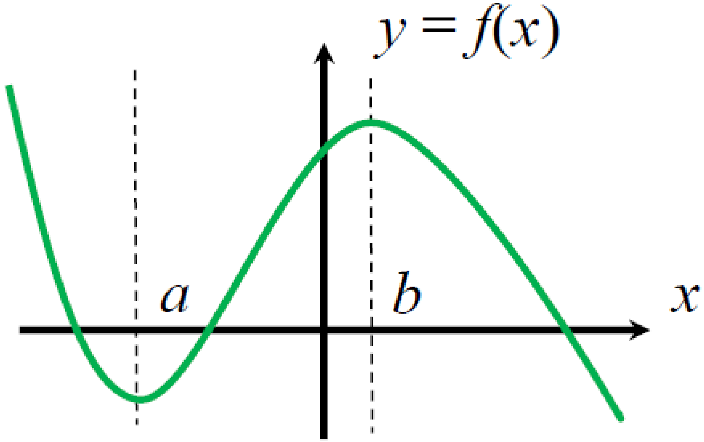
An <u>interval</u> I on the real axis is defined to be the set of all numbers x between two endpoints a and b, with $a < b$. If both endpoints are included in the interval, we write $I = [a, b]$; if only the right endpoint is included, we write $I = (a, b]$; if only the left endpoint is included, we write $I = [a, b)$; if neither endpoint is included, we write $I = (a, b)$.

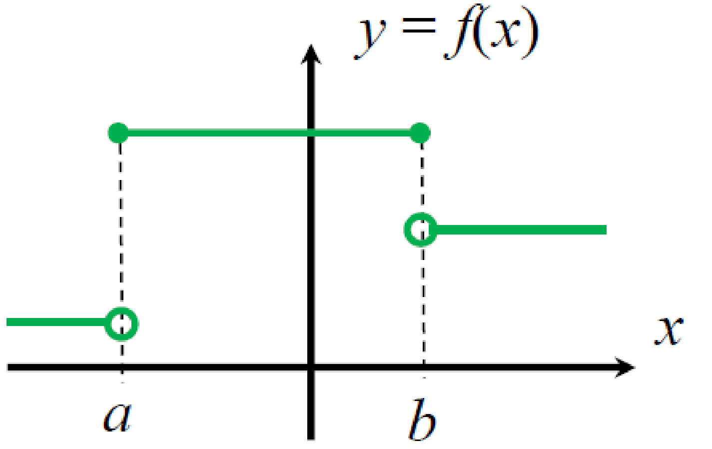


The notion of an interval can be extended to situations where one of the endpoints is at infinity, so that $I = (-\infty, b]$ or $I = (-\infty, b)$ or $I = [a, +\infty)$ or $I = (a, +\infty)$; and to the case where both endpoints are at infinity, so that $I = (-\infty, +\infty)$ is the whole real line.

Given a function f and an interval I, we say that:

- f is <u>increasing</u> on I if $f(x_2) \geq f(x_1)$ for all pairs of points $x_1, x_2 \in I$ with $x_2 > x_1$
- f is <u>decreasing</u> on I if $f(x_2) \leq f(x_1)$ for all pairs of points $x_1, x_2 \in I$ with $x_2 > x_1$
- f is <u>strictly increasing</u> on I if $f(x_2) > f(x_1)$ for all pairs of points $x_1, x_2 \in I$ with $x_2 > x_1$
- f is <u>strictly decreasing</u> on I if $f(x_2) < f(x_1)$ for all pairs of points $x_1, x_2 \in I$ with $x_2 > x_1$





In the left-hand figure, f is <u>increasing</u> and <u>strictly increasing</u> on the interval $[a, b]$, and is <u>decreasing</u> and <u>strictly decreasing</u> on the intervals $(-\infty, a]$ and $[b, +\infty)$. In the right-hand figure, f is <u>increasing</u> and <u>decreasing</u> (but not strictly) on all 3 intervals $[a, b]$, $(-\infty, a]$ and $[b, +\infty)$.

Furthermore, if f is differentiable at all points in the interval I, then:

- f is increasing on I if $f'(x) \geq 0$ for all points $x \in I$
- f is decreasing on I if $f'(x) \leq 0$ for all points $x \in I$

Example 3: Find the intervals on which the function

$$f(x) = 2x^3 + 3x^2 - 12x - 5$$

is increasing or decreasing.

Answer: f is a polynomial, so it is differentiable at all points x on the real line. Its derivative is:

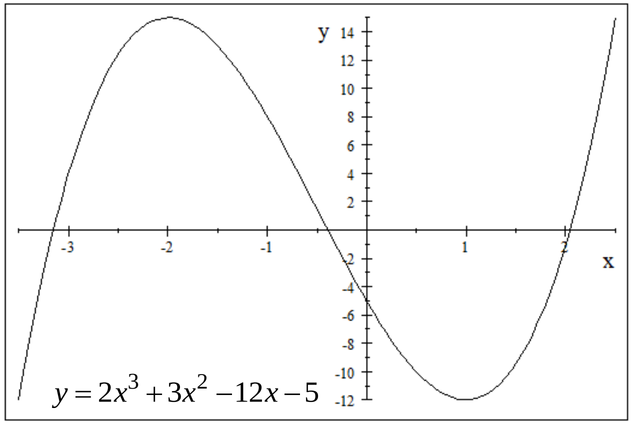
$$f'(x) = \tfrac{d}{dx}(2x^3 + 3x^2 - 12x - 5) = 6x^2 + 6x - 12 = 6(x-1)(x+2)$$

So $f'(x) \geq 0$ if $x \geq 1$ and $x \geq -2$, or if $x \leq 1$ and $x \leq -2$.

That is, f is increasing on the intervals $(-\infty, -2]$ and $[1, +\infty)$.

Similarly, $f'(x) \leq 0$ if $x \leq 1$ and $x \geq -2$.

That is, f is decreasing on the interval $[-2, 1]$.



b. Concavity and inflection points

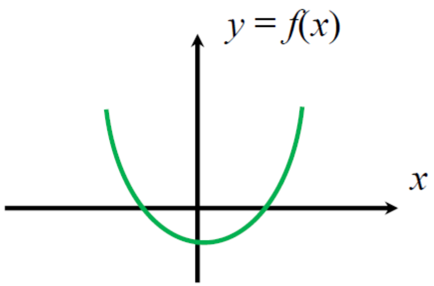
We have seen that a function f is increasing on an interval I if $f'(x) \geq 0$ everywhere on I, and f is decreasing on an interval I if $f'(x) \leq 0$ everywhere on I.

It follows that if f is twice differentiable on an interval I, its first derivative f' is increasing on I if $f''(x) \geq 0$ everywhere on I, and f' is decreasing on I if $f''(x) \leq 0$ everywhere on I.

In connection with this result, we say that

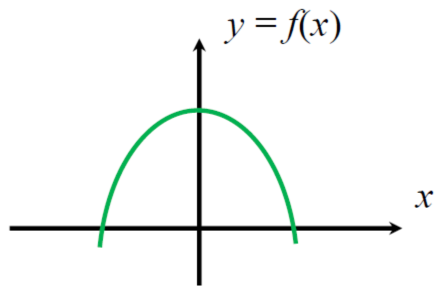
- f is concave up at a point x if $f''(x) > 0$
- f is concave down at a point x if $f''(x) < 0$

[Note that we use the strict inequalities $>$ and $<$ in this definition, not $\geq$ and $\leq$.]



(left) f is concave up everywhere

(right) f is concave down everywhere



We call a point x at which $f''(x) = 0$ a point of inflection of f.

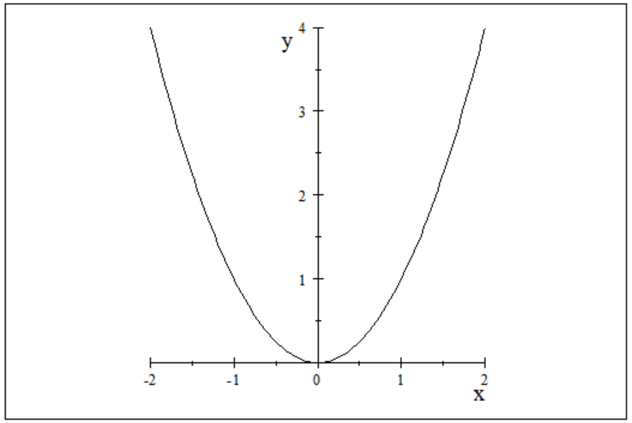
Example 4: (a) The parabolic function $f(x) = x^2$ is concave up everywhere, because

$$f''(x) = \tfrac{d^2}{dx^2}(x^2) = \tfrac{d}{dx}(2x) = 2 > 0$$

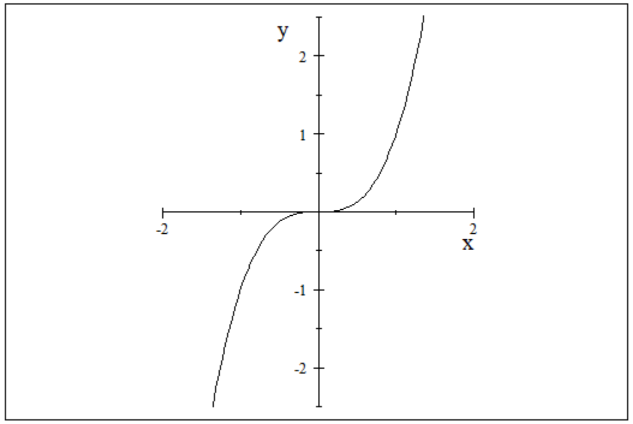
(b) The cubic function $f(x) = x^3$ has the second derivative

$$f''(x) = \tfrac{d^2}{dx^2}(x^3) = \tfrac{d}{dx}(3x^2) = 6x$$

and so is concave down if $x < 0$, is concave up if $x > 0$, and has a point of inflection at $x = 0$.



$y = x^2$



$y = x^3$

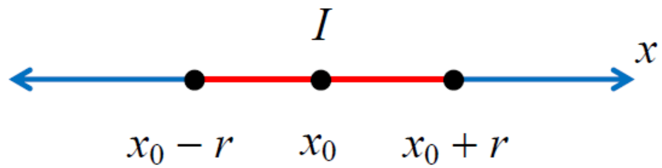
In the case of $f(x) = x^3$, the point $x = 0$ is a point of <u>horizontal inflection</u> of *f*, because $f'(0) = 0$ as well as $f''(0) = 0$. But not all points of inflection are horizontal.

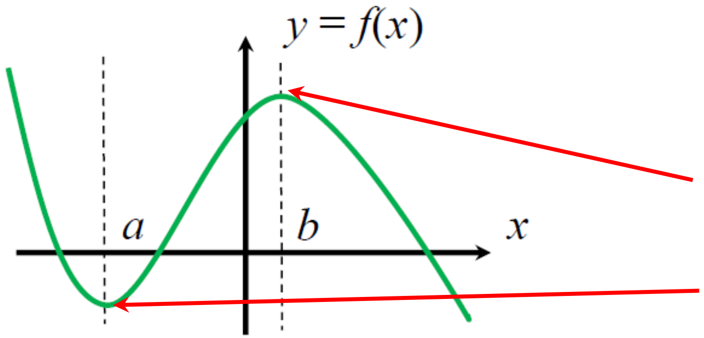
c. Relative maxima and minima

Again, let f be real function of a real continuous variable x, and let x_0 be some fixed value of x. We say that

- f has a <u>relative</u> (or <u>local</u>) <u>maximum</u> at x_0 if there exists an open interval $I=(x_0-r, x_0+r)$ centred on x_0 with the property that $f(x_0) \geq f(x)$ for all x in I.
- f has a <u>relative</u> (or <u>local</u>) <u>minimum</u> at x_0 if there exists an open interval $I=(x_0-r, x_0+r)$ centred on x_0 with the property that $f(x_0) \leq f(x)$ for all x in I.

In the description (x_0-r, x_0+r) of the open interval I, r represents some positive number, no matter how small.





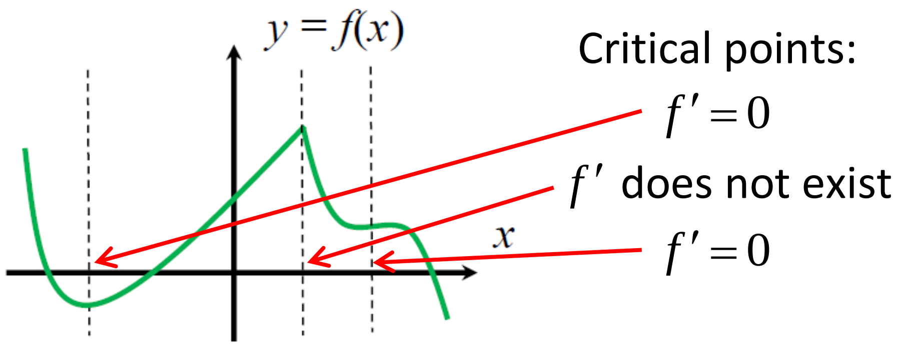
f has a relative or local maximum at $x = b$.

f has a relative or local minimum at $x = a$.

d. Critical points

A point x_0 is said to be a critical point of a function f if one of the two following conditions applies:

- $f'(x_0)$ does not exist; OR
- $f'(x_0) = 0$

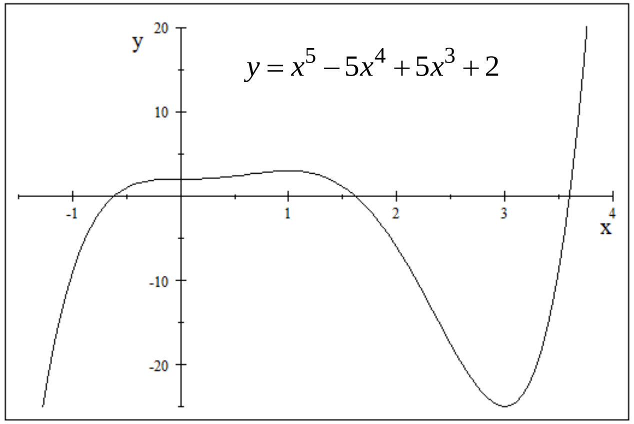


Example 5: Find all the critical points of the function

$$f(x) = x^5 - 5x^4 + 5x^3 + 2$$

Answer: f is a polynomial, so it is differentiable everywhere, and there are no critical points of the first type [$f'(x_0)$ does not exist]. Since

$$f'(x) = \tfrac{d}{dx}(x^5 - 5x^4 + 5x^3 + 2)$$
$$= 5x^4 - 20x^3 + 15x^2 = 5x^2(x-1)(x-3)$$



it follows that f has 3 critical points, at $x = 0$, $x = 1$ and $x = 3$.

Fermat's Theorem states that if a function *f* has a relative maximum or a relative minimum at a point x_0, then x_0 must be a critical point of *f*.

e. First and Second Derivative Tests

Once we have identified all the critical points of a function *f*, there are two tests we can use to determine if *f* has a relative maximum or a relative minimum (or perhaps neither) at each of the critical points.

First Derivative Test: This test can be applied to critical points of either type, those with $f'(x_0)=0$ and those at which $f'(x_0)$ does not exist.

- If $f'(x_0)>0$ in some open interval (x_0-r, x_0) to the left of x_0, and $f'(x_0)<0$ in some open interval (x_0, x_0+r) to the right of x_0, *f* has a relative maximum at x_0.

- If $f'(x_0)<0$ in some open interval (x_0-r, x_0) to the left of x_0, and $f'(x_0)>0$ in some open interval (x_0, x_0+r) to the right of x_0, *f* has a relative minimum at x_0.

A more informal way of stating this test is: If f' changes from positive to negative at x_0, *f* has a local maximum; if f' changes from negative to positive at x_0, *f* has a local minimum.

Example 6: Apply the First Derivative Test to the functions (a) $f(x) = x^3$ and (b) $f(x) = x^4$ at the critical point $x = 0$.

Answers: (a) $f'(x) = \frac{d}{dx}(x^3) = 3x^2$. Since $3x^2 > 0$ for both $x < 0$ and $x > 0$, f' does not change from positive to negative or from negative to positive at $x = 0$. So f <u>does not have</u> a relative maximum or a relative minimum at the critical point $x = 0$.

(b) $f'(x) = \frac{d}{dx}(x^4) = 4x^3$. Since $4x^3 < 0$ when $x < 0$ and $4x^3 > 0$ when $x > 0$, f' changes sign from negative to positive at $x = 0$. So f has a <u>relative minimum</u> at the critical point $x = 0$.

<u>Second Derivative Test</u>: This test can only be applied to critical points x_0 where $f'(x_0) = 0$ and, furthermore, the second derivative $f''(x_0)$ exists.

- If $f''(x_0) < 0$ then f has a relative maximum at x_0.
- If $f''(x_0) > 0$ then f has a relative minimum at x_0.
- If $f''(x_0) = 0$ the test is inconclusive. Use the First Derivative Test instead.

Example 7: Use the Second Derivative Test to classify the critical points $x = 0$, $x = 1$ and $x = 3$ of the function

$$f(x) = x^5 - 5x^4 + 5x^3 + 2$$

as relative maxima or relative minima.

Answer: We know from Example 5 that the first derivative

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

of *f* is 0 at $x = 0$, $x = 1$ and $x = 3$. The second derivative of *f* is:

$$f''(x) = \tfrac{d}{dx}(5x^4 - 20x^3 + 15x^2) = 20x^3 - 60x^2 + 30x$$

So $$f''(0) = 20 \times 0^3 - 60 \times 0^2 + 30 \times 0 = 0$$

$$f''(1) = 20 \times 1^3 - 60 \times 1^2 + 30 \times 1 = -10 < 0$$

and $$f''(3) = 20 \times 3^3 - 60 \times 3^2 + 30 \times 3 = 90 > 0$$

It follows that *f* has a <u>relative maximum</u> at $x = 1$, and a <u>relative minimum</u> at $x = 3$. But the test is inconclusive when $x = 0$, so we should use the First Derivative Test instead. We know that

$$f'(x) = 5x^2(x-1)(x-3)$$

When $x < 0$, $x^2 > 0$ and $x - 1 < 0$ and $x - 3 < 0$, so $f'(x) > 0$ [as $+ - - = +$].

When $0 < x < 1$, $x^2 > 0$ and $x - 1 < 0$ and $x - 3 < 0$, so $f'(x) > 0$ again.

It follows that f' does not change sign at $x = 0$, so f does not have a relative maximum or a relative minimum at the critical point $x = 0$.

f. Graphs of polynomials

A polynomial of degree $n \geq 1$ has the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the numbers a_n, a_{n-1}, … a_1, a_0, are called the coefficients of the polynomial. In general, the coefficients can be positive, negative or 0. The only restriction is that $a_n \neq 0$.

To graph a polynomial, we can use the following procedure:

1. Find all the <u>roots</u> of *f*, if possible. The roots of *f* are the points x at which $f(x)=0$. The graph of *f* intersects the x-axis at each of the roots. If *f* has degree n, it will have <u>at most</u> n distinct roots. Plot the roots in the x-y plane.

2. Find all the <u>critical</u> points of *f*, if possible. The critical points of *f* are the points x at which $f'(x)=0$. Use the First or Second Derivative Tests to classify the critical points as relative maxima, relative minima or neither. If *f* has degree n, it will have <u>at most</u> $n-1$ distinct critical points. There will be at least one critical point between each pair of roots. Plot the critical points in the x-y plane.

3. Draw a curve joining all the roots and critical points. Continue the curve in the directions $x \to -\infty$ and $x \to +\infty$ by using the following rule:

- If n is even and $a_n > 0$ then $f(x) \to +\infty$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.
- If n is even and $a_n < 0$ then $f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to +\infty$.
- If n is odd and $a_n > 0$ then $f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.
- If n is odd and $a_n < 0$ then $f(x) \to +\infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to +\infty$.

Example 8: Use the above procedure to graph the polynomial

$$f(x) = x^3 - 2x^2 - 5x + 6$$

Answer: It is clear that $f(1) = 0$. So (after using long division if necessary) f factorises as

$$f(x) = (x-1)(x^2 - x - 6) = (x-1)(x+2)(x-3)$$

That is, the roots of f are $x = 1$, $x = -2$ and $x = 3$. Place dots on the x-axis at these values of x.

The first derivative of f is:

$$f'(x) = \tfrac{d}{dx}(x^3 - 2x^2 - 5x + 6) = 3x^2 - 4x - 5$$

Using the quadratic formula, we see that $f'(x) = 0$ when

$$x = -\tfrac{-4}{2\times 3} \pm \tfrac{1}{2\times 3}\sqrt{(-4)^2 - 4\times 3\times(-5)} = \tfrac{2}{3} \pm \tfrac{1}{3}\sqrt{19}$$

That is, the critical points of f are

$$x = \tfrac{2}{3} + \tfrac{1}{3}\sqrt{19} \approx 2.12 \quad \text{and} \quad x = \tfrac{2}{3} - \tfrac{1}{3}\sqrt{19} \approx -0.79$$

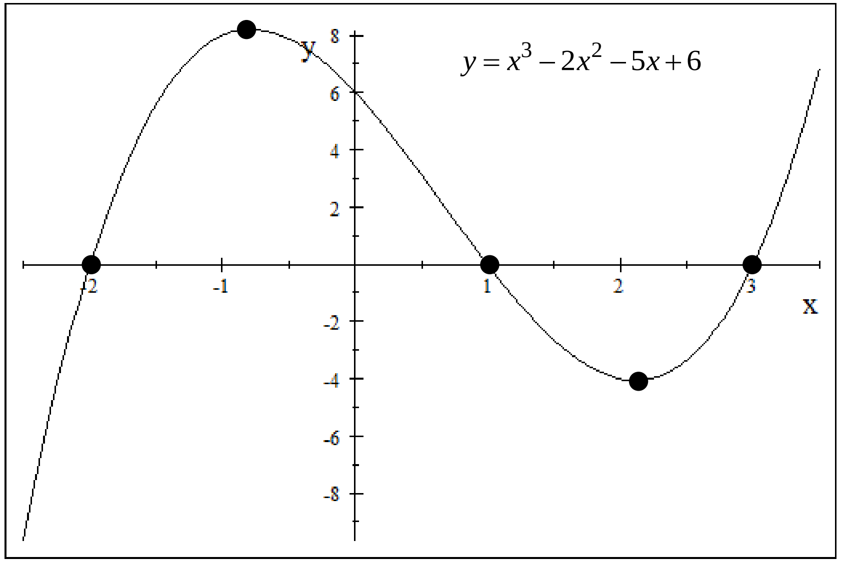
Now calculate $f(2.12) \approx -4.06$ and $f(-0.79) \approx 8.21$, and place dots at the points

$$(x, y) = (2.12, -4.06) \quad \text{and} \quad (x, y) = (-0.79, 8.21)$$

Also, $f''(x) = \frac{d}{dx}(3x^2 - 4x - 5) = 6x - 4$, so $f''(2.12) \approx 8.72 > 0$ and the first critical point is a local minimum, while $f''(-0.79) \approx -8.72 < 0$ and the second critical point is a local maximum.

Finally, the leading term in *f* is $a_n x^n = x^3$, which has $n = 3$ is odd, and $a_n = 1 > 0$. So $f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.

Now join all the dots to form the graph:



g. Graphs of rational functions and asymptotes

A rational function is a function that can be written as the ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are both polynomials, and one is not a constant multiple of the other.

It is also best to ensure that p and q have no factors $x - x_*$ in common (where x_* is any number). This can always be done by cancelling each factor $x - x_*$ that appears in both p and q. [Note that if a factor $x - x_*$ does appear initially in both p and q, then technically f is undefined at $x = x_*$.]

To graph a rational function, we can use the following procedure:

1. Find all the roots of p, if possible, which are the points x at which $p(x) = 0$. The graph of f intersects the x-axis at each of the roots of p. If p has degree m, it will have at most m distinct roots. Plot the roots in the x-y plane.

2. Find all the roots of q, if possible, which are the points x at which $q(x)=0$. The graph of f has a <u>vertical asymptote</u> at each of the roots of q, which means that $f(x)\to+\infty$ or $f(x)\to-\infty$ as x approaches the root. [It is possible, and in fact quite common, that $f(x)\to+\infty$ on one side of the root while $f(x)\to-\infty$ on the other side of the root]. If q has degree n, it will have <u>at most</u> n distinct roots. Indicate the positions of the vertical asymptotes by drawing dashed vertical lines through the roots of q.

3. If $p(x)=a_m x^m + a_{m-1}x^{m-1}+\cdots+a_1 x + a_0$ and $q(x)=b_n x^n + b_{n-1}x^{n-1}+\cdots+b_1 x+b_0$, the <u>asymptotic behaviour</u> of f (that is, its behaviour as $x\to-\infty$ or $x\to+\infty$) is determined by the ratio $(a_m/b_n)x^{m-n}$ of the leading terms.

- If $m=n$, $f(x)\to a_m/b_n$ as $x\to\pm\infty$ and we say that f has a <u>horizontal asymptote</u> at $y=a_m/b_n$. Indicate the position of the horizontal asymptote by drawing a dashed horizontal line with $y=a_m/b_n$.

- If $m<n$, $f(x)\to 0$ as $x\to\pm\infty$ and we say that f has a <u>horizontal asymptote</u> at $y=0$ (the x-axis).

- If $m > n$, f has no horizontal asymptote. Instead, $f(x) \to \pm\infty$ as $x \to \pm\infty$ in the same way as was explained in Rule 3 of Section f above (with n in Section f replaced by $m-n$, and a_n in Section f replaced by a_m/b_n).

4. Now plot the graph so that it passes through the roots of p and approaches all the asymptotes. If it is not clear whether $f(x) \to -\infty$ or $f(x) \to +\infty$ at any of the vertical asymptotes, it is a good idea to calculate the value of f at a point just to the left of the asymptote, and at a second point just to the right of the asymptote.

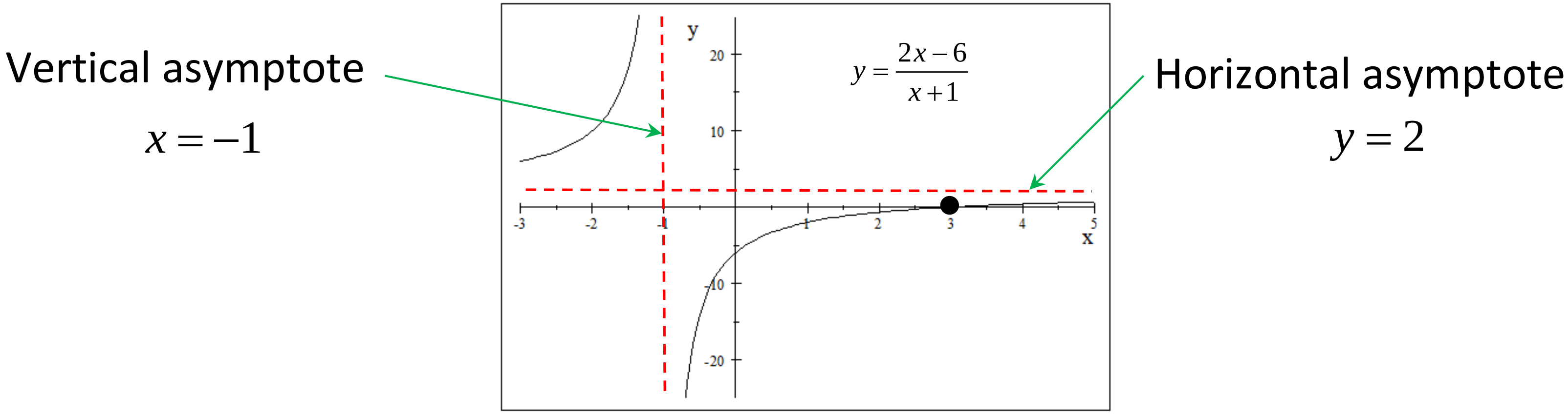
Example 9: Use the above procedure to graph the rational function

$$f(x) = \frac{2x-6}{x+1}$$

Answer: Here, $p(x) = 2x-6 = 2(x-3)$ has one root, at $x=3$, while $q(x) = x+1$ also has one root, at $x=-1$. So place a dot at $x=3$ on the x-axis, and draw a dashed vertical straight line (a vertical asymptote) through $x=-1$.

Also, the ratio of leading terms is $2x/x = 2$. So there is a horizontal asymptote at $y=2$. Draw a dashed horizontal line through $y=2$.

Now draw the graph of *f*. If you are uncertain whether $f(x) \to -\infty$ or $f(x) \to +\infty$ at the vertical asymptote $x = -1$, you can calculate $f(-1.1) = 82$ and $f(-0.9) = -78$. It is then clear that $f(x) \to +\infty$ on the left of the asymptote, and $f(x) \to -\infty$ on the right of the asymptote.



h. Maximum and minimum values of a function

If a function *f* is defined on some subset S of the real line (which might be the whole real line), and x_0 is some point in S, we say that

- *f* has an <u>absolute</u> (or <u>global</u>) <u>maximum</u> on S at x_0 if $f(x_0) \geq f(x)$ for all x in S.
- *f* has an <u>absolute</u> (or <u>global</u>) <u>minimum</u> on S at x_0 if $f(x_0) \leq f(x)$ for all x in S.

In general, there is no guarantee that a given function f will have an absolute maximum or absolute minimum, as it may be unbounded either above (meaning that $f \to +\infty$ in some limit) or below ($f \to -\infty$ in some limit).

But if S is a closed bounded interval $[a, b]$, or a finite union of intervals of this type, and f is continuous on S, then f will have both an absolute maximum and an absolute minimum on S. To find the absolute maximum and the absolute minimum, calculate the value of f at every critical point in S, and at every point on the boundary of S. The largest value of f will be the absolute maximum, and the smallest value will be the absolute minimum.

Example 10: Find the absolute maximum and absolute minimum values of the function

$$f(x) = 3x^4 - 20x^3 + 24x^2 - 7$$

on the set $S = [-1, 5]$.

Answer: First find all the critical points of f, which are the points where its first derivative

$$f'(x) = \tfrac{d}{dx}(3x^4 - 20x^3 + 24x^2 - 7) = 12x^3 - 60x^2 + 48x = 12x(x-1)(x-4)$$

is equal to 0. The critical points are $x = 0$, $x = 1$ and $x = 4$.

Now evaluate *f* at each of the critical points:

$$f(0) = -7,\ f(1) = 0 \text{ and } f(4) = -135$$

The two boundary points of $S = [-1, 5]$ are $x = -1$ and $x = 5$. We evaluate *f* at each of these points as well:

$$f(-1) = 40 \text{ and } f(5) = -32$$

On comparing the 5 values of *f*, it is clear that the absolute maximum value of *f* on S is 40, and occurs at $x = -1$, while the absolute minimum value of *f* on S is -135, and occurs at $x = 4$.

3. APPLIED MAXIMUM AND MINIMUM PROBLEMS

The methods described in the previous section have numerous applications to real-world problems, most importantly in economics, but also in construction, product design and any area where the objective is to find the maximum or minimum of some quantity.

Three simple problems of this type are considered here.

Example 11: An oil refinery produces x barrels of oil per day. The profit P of the refinery is given by the function

$$P(x) = 8x - 0.02x^2$$

How many barrels per day should be produced to maximise the profit, and what is the maximum profit?

Answer: We need to find the maximum value of P on the interval $x \geq 0$. Since

$$P'(x) = \tfrac{d}{dx}(8x - 0.02x^2) = 8 - 0.04x$$

P has only one critical point, at $x = 8/0.04 = 200$.

Furthermore, since

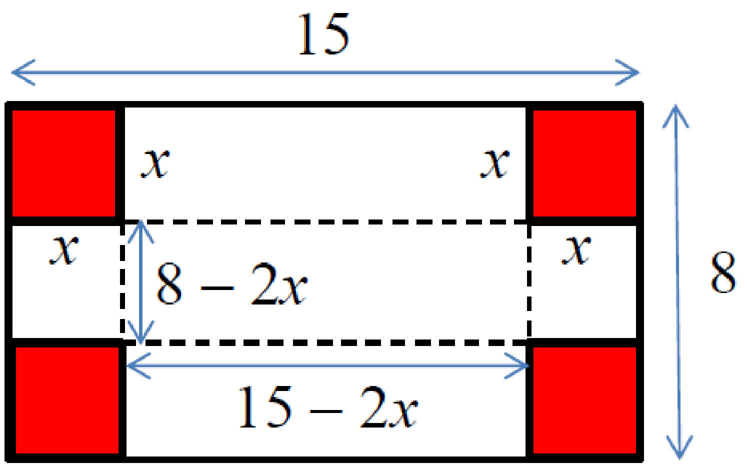
$$P''(x) = \tfrac{d}{dx}(8 - 0.04x) = -0.04 < 0$$

the critical point is a relative maximum.

So 200 barrels per day should be produced to maximise the profit, and the maximum profit is

$$P(200) = 8 \times 200 - 0.02 \times (200^2) = 800$$

Example 12: A rectangular piece of cardboard measures 15 cm by 8 cm. If you decide to use the cardboard to make an open-topped cardboard box by cutting out x cm by x cm squares from each of the corners, as shown in the figure below, what is the maximum possible volume of the box?



Answer: The volume of the box is:

$$V(x) = \text{length} \times \text{width} \times \text{height} = (15-2x)(8-2x)x = 4x^3 - 46x^2 + 120x$$

We need to find the maximum value of V on the interval $(0, 4)$, as the width $8-2x$ of the box goes to 0 when $x=4$ (as does the height x when $x=0$).

The first derivative of V is:

$$V'(x) = \tfrac{d}{dx}(4x^3 - 46x^2 + 120x) = 12x^2 - 92x + 120 = 4(3x-5)(x-6)$$

So the critical points of V are $x = 5/3 \approx 1.67$ and $x = 6$. Since $x = 6$ is outside the interval $(0, 4)$, we can ignore it. Also, the second derivative of V is:

$$V''(x) = \tfrac{d}{dx}(12x^2 - 92x + 120) = 24x - 92$$

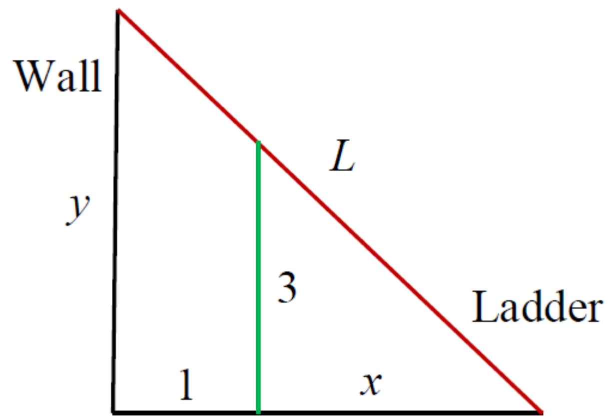
and $V''(5/3) = -52 < 0$, so the critical point at $x = 5/3$ is a relative maximum.

The maximum possible volume of the box is therefore

$$V(5/3) = \frac{2450}{27} \approx 90.74 \text{ cm}^3$$

Example 13: What is the length of the shortest ladder that will reach over a 3-metre high fence to a large wall which is 1 metre behind the fence?

Answer: Suppose that the ladder touches the wall at a distance y metres above the ground, and that the foot of the ladder is x metres from the fence, as shown in the figure to the right.



The two triangles in the figure are similar, so $\dfrac{x+1}{y} = \dfrac{x}{3}$

Solving this equation for y gives: $y = 3\dfrac{x+1}{x} = 3(1+x^{-1})$

If the length of the ladder is L, then according to Pythagoras' Theorem

$$L^2 = (x+1)^2 + y^2 = (x+1)^2 + 9(1+x^{-1})^2 = x^2 + 2x + 10 + 18x^{-1} + 9x^{-2}$$

We need to minimise L, but this is the same as minimising L^2. Now

$$\tfrac{d}{dx}L^2 = \tfrac{d}{dx}(x^2 + 2x + 10 + 18x^{-1} + 9x^{-2}) = 2x + 2 - 18x^{-2} - 18x^{-3},$$

where

$$2x+2-18x^{-2}-18x^{-3}=2x^{-3}(x^4+x^3-18x-18)=2x^{-3}(x+1)(x^3-9)$$

(the quartic $x^4+x^3-18x-18$ can be factorised in this way once we spot that it has a root $x=-1$).

So the critical points of L^2 are $x=-1$ and $x=\sqrt[3]{9}\approx 2.08$. We can ignore $x=-1$ because $x<0$. The only viable critical point is $x=\sqrt[3]{9}\approx 2.08$, which corresponds to

$$y=3(1+1/\sqrt[3]{9})\approx 4.44$$

and the minimum possible length of the ladder is:

$$L=\sqrt{(x+1)^2+y^2}\approx\sqrt{(3.08)^2+(4.44)^2}\approx 5.41\text{ metres}$$

[Note that we have not checked that $x=\sqrt[3]{9}$ is a relative minimum of L^2. This requires a bit more work:

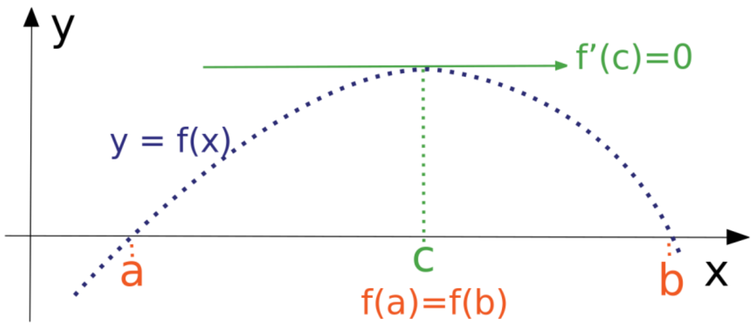
$$\tfrac{d^2}{dx^2}L^2=\tfrac{d}{dx}(2x+2-18x^{-2}-18x^{-3})=2+36x^{-3}+54x^{-4}$$

and so at $x=\sqrt[3]{9}$ we have $\tfrac{d^2}{dx^2}L^2=2+36/9+54/(9\sqrt[3]{9})\approx 8.88>0$. Hence, it is a minimum.]

4. ROLLE'S THEOREM

● **Rolle's Theorem** states the following:

If a function f is continuous at all points on a closed interval $[a, b]$, and is differentiable at all points on the interior (a, b) of the interval, and f has the same value at the two endpoints , so that $f(a) = f(b)$, then there is at least one point c in (a, b) at which $f'(c) = 0$.



Source: https://en.wikipedia.org/wiki/Rolle's_theorem

The **proof** of Rolle's Theorem is straightforward. If f is a constant function, its derivative f' is 0 at all points in (a, b), and the Theorem holds. If f is not a constant function, there is at least one point c in (a, b) where f has a relative maximum or a relative minimum. By Fermat's Theorem, $f'(c) = 0$ or $f'(c)$ does not exist. Since f is differentiable on (a, b), it follows that $f'(c) = 0$.

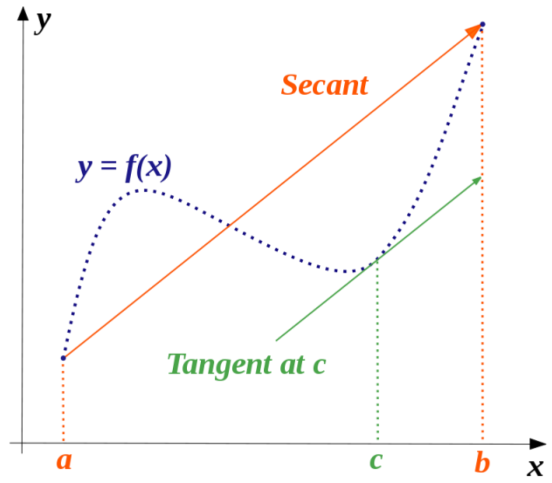
5. MEAN-VALUE THEOREM

● The **Mean-Value Theorem** is a more general version of Rolle's Theorem, and states the following:

If a function f is continuous at all points on a closed interval $[a, b]$, and is differentiable at all points on the interior (a, b) of the interval, then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is, there is at least one point in the interval (a, b) where the slope of f is equal to the average slope of f over the entire interval.



Source: https://en.wikipedia.org/wiki/Mean_value_theorem

The Mean-Value Theorem can be proved using Rolle's Theorem. Given a function f that satisfies the conditions of the Mean-Value Theorem, we define a new function

$$g(x) = f(x) - (x-a)\frac{f(b)-f(a)}{b-a}$$

Clearly, g is continuous on $[a, b]$, and is differentiable on (a, b), with

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

Also, $g(a) = f(a)$ and

$$g(b) = f(b) - (b-a)\frac{f(b)-f(a)}{b-a} = f(b) - [f(b) - f(a)] = f(a)$$

So $g(a) = g(b)$, and g satisfies the conditions of Rolle's Theorem. This means there is at least one point c in (a, b) at which $g'(c) = 0$. But, from above,

$$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \quad \text{so} \quad f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{as the Theorem claims.}$$