Formalización de las matemáticas con Lean. Un caso de estudio: Geometría euclídea plana.

Facultad de Ciencias Matemáticas. Trabajo dirigido por Jorge Carmona Ruber.

Adrián Lattes Grassi

18 de septiembre de 2023

```
lemma neq_lines_have_at_most_one_common_point (Point : Type*) {Line : Type*} [ig : incidence_geometry Point Line] : \forall 1 m : Line, 1 \neq m \rightarrow
```

(∃! A : Point, is_common_point A 1 m)
∨ ¬ have_common_point Point 1 m :=

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
    ∀ 1 m : Line, 1 ≠ m →
    (∃! A : Point, is_common_point A 1 m)
    ∨ ¬ have_common_point Point 1 m :=
```

begin

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  ∀ 1 m : Line, 1 ≠ m →
  (∃! A : Point, is_common_point A 1 m)
```

√ ¬ have_common_point Point 1 m :=

begin

Point Line: Type u ig: incidence_geometry Point Line

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
    ∀ 1 m : Line, 1 ≠ m →
    (∃! A : Point, is_common_point A 1 m)
    ∨ ¬ have_common_point Point 1 m :=
begin
```

intros 1 m

```
Point Line: Type u ig: incidence_geometry Point Line
```

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  ∀ 1 m : Line, 1 ≠ m →
  (∃! A : Point, is_common_point A 1 m)
  ∨ ¬ have_common_point Point 1 m :=
begin
  intros 1 m,
Point Line:
  ig: inciden
  1 m : Line
  1 m : Line
```

```
Point Line: Type u
ig: incidence_geometry Point Line
1 m : Line
```

(∃! A : Point, is_common_point A 1 m) ∨ ¬ have_common_point Point 1 m

contrapose

```
Point Line: Type u
ig: incidence_geometry Point Line
1 m : Line
```

(∃! A : Point, is_common_point A 1 m)
∨ ¬ have_common_point Point 1 m

```
lemma neq_lines_have_at_most_one_common_point
                                                         Point Line: Type u
                                                         ig: incidence_geometry Point Line
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  \forall 1 m : Line, 1 \neq m \rightarrow
    (∃! A : Point, is_common_point A 1 m)

√ ¬ have_common_point Point 1 m :=
begin
```

intros 1 m.

contrapose,

```
lm: Line
```

 \neg 1 \neq m

 $\vdash \neg((\exists! A : Point, is_common_point A 1 m)$

 $\vee \neg$ have common point Point 1 m) \rightarrow

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  \forall 1 m : Line, 1 \neq m \rightarrow
```

begin

intros 1 m.

contrapose,

push_neg

(∃! A : Point, is_common_point A 1 m)

√ ¬ have_common_point Point 1 m :=

lm: Line

 \neg 1 \neq m

Point Line: Type u

ig: incidence_geometry Point Line

 $\vdash \neg((\exists! A : Point, is_common_point A 1 m)$

 $\vee \neg$ have common point Point 1 m) \rightarrow

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  \forall 1 m : Line, 1 \neq m \rightarrow
```

√ ¬ have_common_point Point 1 m :=

begin

intros 1 m.

contrapose,

push_neg,

(∃! A : Point, is_common_point A 1 m)

lm: Line

1 = m

Point Line: Type u

ig: incidence_geometry Point Line

- \vdash ($\neg\exists$! A : Point, is_common_point A 1 m)

 - ∧ have_common_point Point 1 m →

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  \forall 1 m : Line, 1 \neq m \rightarrow
    (∃! A : Point, is_common_point A 1 m)
```

begin intros 1 m.

contrapose, push_neg,

rintro (not_unique, hlm)

√ ¬ have_common_point Point 1 m :=

1 = m

ig: incidence geometry Point Line

Point Line: Type u

lm: Line

 \vdash ($\neg\exists$! A : Point, is_common_point A 1 m)

∧ have_common_point Point 1 m →

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  ∀ 1 m : Line, 1 ≠ m →
    (∃! A : Point, is_common_point A 1 m)
  ∨ ¬ have_common_point Point 1 m :=
  begin
  intros 1 m,
```

contrapose,
push_neg,

rintro (not_unique, hlm),

```
Point Line: Type u
ig: incidence_geometry Point Line
l m : Line
not_unique: ¬∃! A : Point, is_common_point A l m
hlm: have_common_point Point l m
```

push_neg,

rintro (not_unique, hlm),
rw exists_unique at not_unique

```
Point Line: Type u
ig: incidence_geometry Point Line
1 m : Line
not_unique: ¬∃! A : Point, is_common_point A 1 m
hlm: have_common_point Point 1 m
```

```
lemma neq_lines_have_at_most_one_common_point (Point : Type*) {Line : Type*} { [ig : incidence_geometry Point Line] : \forall 1 m : Line, 1 \neq m \rightarrow (\exists! A : Point, is_common_point A 1 m) \vee \neg have_common_point Point 1 m := begin intros 1 m, contrapose.
```

push_neg,

rintro (not_unique, hlm), rw exists unique at not unique.

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
    ∀ 1 m : Line, 1 ≠ m →
        (∃! A : Point, is_common_point A 1 m)
        ∨ ¬ have_common_point Point 1 m :=
begin
   intros 1 m,
   contrapose,
   push_neg,
   rintro (not_unique, hlm),
```

rw exists_unique at not_unique,

push_neg at not_unique

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
    ∀ 1 m : Line, 1 ≠ m →
        (∃! A : Point, is_common_point A 1 m)
        ∨ ¬ have_common_point Point 1 m :=
  begin
   intros 1 m,
   contrapose,
   push_neg,
   rintro (not_unique, hlm),
   rw exists_unique at not_unique,
   push_neg at not_unique,
```

```
Point Line: Type u
ig: incidence_geometry Point Line
l m : Line
not_unique: ∀ A : Point, is_common_point A l m →
(∃ B : Point, is_common_point B l m ∧ B ≠ A)
hlm: have_common_point Point l m
```

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
    ∀ 1 m : Line, 1 ≠ m →
        (∃! A : Point, is_common_point A 1 m)
        ∨ ¬ have_common_point Point 1 m :=
begin
  intros 1 m,
  contrapose,
  push_neg,
  rintro ⟨not_unique, hlm⟩,
  rw exists_unique at not_unique,
```

push_neg at not_unique,
cases hlm with A hA

```
Point Line: Type u
ig: incidence_geometry Point Line
l m : Line
not_unique: ∀ A : Point, is_common_point A l m →
(∃ B : Point, is_common_point B l m ∧ B ≠ A)
hlm: have_common_point Point l m
```

rw exists_unique at not_unique, push_neg at not_unique, cases hlm with A hA.

```
Point Line: Type u
ig: incidence_geometry Point Line
l m : Line
not_unique: ∀ A : Point, is_common_point A l m →
(∃ B : Point, is_common_point B l m ∧ B ≠ A)
```

 \vdash 1 = m

hA: is common point A 1 m

A: Point

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  ∀ l m : Line, l ≠ m →
   (∃! A : Point, is_common_point A l m)
  ∨ ¬ have_common_point Point l m :=

begin
  intros l m,
  contrapose,
  push_neg,
  rintro ⟨not_unique, hlm⟩,
  rw exists_unique at not_unique,
  push_neg at not_unique,
  cases hlm with A hA,
```

rcases not_unique A hA with (B, (hB, hAB))

```
Point Line: Type u
ig: incidence_geometry Point Line
1 m : Line
not_unique: \forall A : Point, is_common_point A 1 m \rightarrow
(\exists B : Point, is_common_point B 1 m \land B \neq A)
A: Point
hA: is_common_point A 1 m
```

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
    ∀ 1 m : Line, 1 ≠ m →
        (∃! A : Point, is_common_point A 1 m)
        ∨ ¬ have_common_point Point 1 m :=
begin
  intros 1 m,
  contrapose,
  push_neg,
```

rcases not_unique A hA with (B, (hB, hAB)),

rintro (not_unique, hlm),
rw exists_unique at not_unique,
push_neg at not_unique,
cases hlm with A hA.

```
Point Line: Type u
ig: incidence_geometry Point Line
l m : Line
A B: Point
hA: is_common_point A l m
hB: is_common_point B l m
```

hAB: $B \neq A$

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  \forall 1 m : Line, 1 \neq m \rightarrow
    (∃! A : Point, is_common_point A 1 m)

√ ¬ have_common_point Point 1 m :=
begin
  intros 1 m.
  contrapose.
  push_neg,
  rintro (not_unique, hlm),
  rw exists_unique at not_unique,
  push_neg at not_unique,
```

rcases not_unique A hA with (B, (hB, hAB)).

cases hlm with A hA.

rw ne comm at hAB

```
Point Line: Type u
ig: incidence geometry Point Line
lm: Line
A B: Point
```

hAB: B \neq A

 \vdash 1 = m

hB: is common point B 1 m

hA: is_common_point A 1 m

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
    ∀ 1 m : Line, 1 ≠ m →
        (∃! A : Point, is_common_point A 1 m)
        ∨ ¬ have_common_point Point 1 m :=
  begin
  intros 1 m,
  contrapose,
  push_neg,
  rintro ⟨not_unique, hlm⟩,
  rw exists_unique at not_unique,
```

rcases not_unique A hA with (B, (hB, hAB)).

push_neg at not_unique, cases hlm with A hA.

rw ne comm at hAB.

```
Point Line: Type u
ig: incidence_geometry Point Line
l m : Line
A B: Point
hA: is_common_point A l m
hB: is_common_point B l m
```

hAB: $A \neq B$

```
lemma neq_lines_have_at_most_one_common_point
                                                        Point Line: Type u
                                                        ig: incidence geometry Point Line
  (Point : Type*) {Line : Type*}
                                                        lm: Line
  [ig : incidence_geometry Point Line] :
                                                        A B: Point
  \forall 1 m : Line, 1 \neq m \rightarrow
                                                        hA: is_common_point A 1 m
    (∃! A : Point, is_common_point A 1 m)
                                                        hB: is common point B 1 m

√ ¬ have_common_point Point 1 m :=
                                                        hAB: A \neq B
begin
  intros 1 m.
                                                        \vdash 1 = m
  contrapose.
  push_neg,
  rintro (not_unique, hlm),
  rw exists unique at not unique.
  push_neg at not_unique,
  cases hlm with A hA.
```

rcases not_unique A hA with (B, (hB, hAB)).

exact unique_of_exists_unique (ig.I1 hAB) (hA.1, hB.1) (hA.2, hB.2)

rw ne comm at hAB.

```
rw exists unique at not unique.
  push neg at not unique.
  cases hlm with A hA.
  rcases not_unique A hA with (B, (hB, hAB)).
  rw ne comm at hAB.
  exact unique_of_exists_unique (ig.I1 hAB) (hA.1, hB.1) (hA.2, hB.2)
ig.I1 {A B : Point} (h : A \neq B) : \exists! 1 : Line, A ~ 1 \land B ~ 1
```

lemma neq_lines_have_at_most_one_common_point

(∃! A : Point, is_common_point A 1 m)

[ig : incidence_geometry Point Line] :

√ ¬ have_common_point Point 1 m :=

(Point : Type*) {Line : Type*}

 \forall 1 m : Line, 1 \neq m \rightarrow

rintro (not_unique, hlm),

begin

intros 1 m.

contrapose. push_neg,

```
ig: incidence geometry Point Line
lm: Line
hB: is common point B 1 m
```

A B: Point hA: is common point A 1 m

Point Line: Type u

 \vdash 1 = m

hAB: $A \neq B$

```
lemma neq_lines_have_at_most_one_common_point
                                                         Point Line: Type u
  (Point : Type*) {Line : Type*}
                                                         lm: Line
  [ig : incidence_geometry Point Line] :
                                                         A B: Point
  \forall 1 m : Line, 1 \neq m \rightarrow
    (∃! A : Point, is_common_point A 1 m)

√ ¬ have_common_point Point 1 m :=
                                                         hAB: A \neq B
begin
  intros 1 m.
                                                         \vdash 1 = m
  contrapose.
  push_neg,
  rintro (not_unique, hlm),
  rw exists unique at not unique.
  push_neg at not_unique,
```

exact unique_of_exists_unique (ig.I1 hAB) (hA.1, hB.1) (hA.2, hB.2)

```
ig: incidence geometry Point Line
hA: is_common_point A 1 m
hB: is common point B 1 m
```

ig.I1 hAB : \exists ! 1 : Line, A ~ 1 \land B ~ 1

rcases not_unique A hA with (B, (hB, hAB)).

cases hlm with A hA.

rw ne comm at hAB.

```
goals accomplished √
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  \forall 1 m : Line, 1 \neq m \rightarrow
    (∃! A : Point, is_common_point A 1 m)

√ ¬ have_common_point Point 1 m :=
begin
  intros 1 m.
  contrapose.
  push_neg,
  rintro (not_unique, hlm),
  rw exists unique at not unique.
  push_neg at not_unique,
  cases hlm with A hA.
  rcases not_unique A hA with (B, (hB, hAB)),
```

exact unique_of_exists_unique (ig.I1 hAB) (hA.1, hB.1) (hA.2, hB.2)

rw ne comm at hAB.

end