

Formalización de las matemáticas con Lean.
Un caso de estudio: Geometría euclídea plana.

Facultad de Ciencias Matemáticas.
Trabajo dirigido por Jorge Carmona Ruber.

Adrián Lattes Grassi

18 de septiembre de 2023

```
lemma neq_lines_have_at_most_one_common_point
  (Point : Type*) {Line : Type*}
  [ig : incidence_geometry Point Line] :
  ∀ l m : Line, l ≠ m →
    (∃! A : Point, is_common_point A l m)
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Point Line: Type u
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⊢ ∀ (l m : Line), l ≠ m →
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begin
  intros l m

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⊢ ∀ (l m : Line), l ≠ m →
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⊢ l ≠ m →
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  ∀ ¬ have_common_point Point l m :=
begin
  intros l m,
  contrapose

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⊢ l ≠ m →
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begin
  intros l m,
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  push_neg

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  rcases not_unique A hA with ⟨B, ⟨hB, hAB⟩⟩,
  rw ne_comm at hAB,
  exact unique_of_exists_unique (ig.I1 hAB) ⟨hA.1, hB.1⟩ ⟨hA.2, hB.2⟩

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l m : Line
A B: Point
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hB: is_common_point B l m
hAB: A ≠ B

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```

ig.I1 {A B : Point} (h : A ≠ B) : ∃! l : Line, A ~ l ∧ B ~ l

```



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ig.I1 hAB : ∃! l : Line, A ~ l ∧ B ~ l

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l m : Line
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lemma neq_lines_have_at_most_one_common_point

goals accomplished ✓

(Point : Type*) {Line : Type*}

[ig : incidence_geometry Point Line] :

∀ l m : Line, l ≠ m →

(∃! A : Point, is_common_point A l m)

∀ ¬ have_common_point Point l m :=

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 rcases not_unique A hA with ⟨B, ⟨hB, hAB⟩⟩,
 rw ne_comm at hAB,
 exact unique_of_exists_unique (ig.I1 hAB) ⟨hA.1, hB.1⟩ ⟨hA.2, hB.2⟩
end