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Course: OLS.

The "Mechanic" behind OLS Let's think about the relationship *schooling* and *earnings*, *controlling* for *experience*. What does it mean "to control for" something in this context?

Now, let's suppose we have the following data,

Name (i)	**Earnings (Y)**	**Education (x1)**	**Experience (x2)**
Alfred	3	2	2
Brandon	5	7	4
Charly	7	3	6

Hypothesis: "The more education, the higher earnings," for the average level of experience (And what do I mean by "for the average level of experience" and *Why does it matter?*)

I. By how much do my earnings rise if my schooling goes up?

The linear model is given by the next formula,

$$\text{Earnings}_i = \beta 0 + \beta_1 \text{Education}_i \;+\; \beta_2 \text{Experience}_i + e_i$$
$$Y_i = \beta 0 + \beta_1 x1_i \;+\; \beta 2 x2_i + e_i$$

- **What we observe:** $\boldsymbol{x}$ and y.
- **What we don't observe, but should estimate**: $\boldsymbol{\beta}$ and ϵ.

Let's revisit the formula, but this time in matrix form:

Let's define what we know:

$$Y \;=\; \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 2 \\ 7 & 4 \\ 3 & 6 \end{bmatrix}$$

...and see how OLS looks like but in matrix form:

$$\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}_y \;=\; \beta 0 + \beta 1 \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}_{x1} \;+\; \beta 2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}_{x2} + e_i$$

It's easy to see that:

- We should multiply $\beta 1$ times $x1$ and $\beta 2$ times $x2$.
- $\beta 0$, $\beta 1$, $\beta 2$ and ϵ_i unknown quantities. Hence, we should infer/estimate that (this is *inferential* statistics!).
- $\beta 0$, $\beta 1$ and $\beta 2$ are scalars (single numbers and constants), where the vector containing all estimations is defined as, $\boldsymbol{\beta} \;=\; \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}_{\boldsymbol{\beta}}$
- The only parameter that's indexed (i.e. one row per observation or "individual," hence the tiny "i") is ϵ_i. We will address this in another class. But basically, it's the "error" or "residual."

  Let's think about the case of "Aldred." If $\beta 0 \;=\; -3$, $\beta 1 \;=\; 1$ and $\beta 2 \;=\; 2$, we have that $y_{\text{Alfred}} - 3 \;+\; 1(2) + 2(2) = 3$. Thus:

$$y_{\text{Alfred}} \;=\; 3 \;=\; -3 \;+1(2) \;+2(2) \;+\; 0$$

Aquí ϵ_{Alfred}=0. En ese sentido, e_i es la diferencia entre lo que estimamos y lo que observamos. "Filosoficamente", significa otra cosa.

Continuemos.

Formula para sacar β (el efecto de X sobre Y):

$\beta \;=\; \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T y$

Recapitulemos lo observado, y hagamos el cálculo:

$$y \;=\; \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 7 & 4 \\ 1 & 3 & 6 \end{bmatrix}$$

$$\boldsymbol{X}^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 7 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\boldsymbol{X}^T \times \boldsymbol{X} \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 2 & 7 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{\boldsymbol{X}^T} \times \begin{bmatrix} 1 & 2 & 2 \\ 1 & 7 & 4 \\ 1 & 3 & 6 \end{bmatrix}_{\boldsymbol{X}} \;=\; \begin{bmatrix} 3 & 12 & 12 \\ 12 & 62 & 50 \\ 12 & 50 & 56 \end{bmatrix}$$

$$\left(\boldsymbol{X}^T \times \boldsymbol{X}\right)^{-1} = \frac{1}{\left(\boldsymbol{X}^T\boldsymbol{X}\right)} \;=\; \frac{1}{\det\left(\boldsymbol{X}^T\boldsymbol{X}\right)} \;\times\; \mathrm{Adj}\left(\boldsymbol{X}^T\boldsymbol{X}\right) \;=\; \begin{bmatrix} 3 & -0.22 & -0.44 \\ -0.22 & 0.074 & -0.0185 \\ -0.44 & -0.0185 & 0.129 \end{bmatrix}$$

$$\boldsymbol{\beta} \;=\; \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T y \;=\; \begin{bmatrix} 3 & -0.22 & -0.44 \\ -0.22 & 0.074 & -0.0185 \\ -0.44 & -0.0185 & 0.129 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 2 & 7 & 3 \\ 2 & 4 & 6 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\boldsymbol{\beta}}$$

Esto quiere decir que $\beta 0 \;=\; 1,\; \beta 1 \;=\; 0$ y $\beta 2 \;=\; 1$. Volvamos a re-escribir nuestra formula:

$$\boldsymbol{\beta} \;=\; \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T y \;=\; \begin{bmatrix}1\\0\\1\end{bmatrix}_{\boldsymbol{\beta}}$$

La formula que teníamos antes: $\begin{bmatrix}3\\5\\7\end{bmatrix}_y \;=\; \beta 0 + \beta 1 \begin{bmatrix}2\\7\\3\end{bmatrix}_{x1} \;+\; \beta 2 \begin{bmatrix}2\\4\\6\end{bmatrix}_{x2} + e_i$

Los resultados que tenemos ahora: $\begin{bmatrix}3\\5\\7\end{bmatrix}_y \;=\; 1 + 0 \begin{bmatrix}2\\7\\3\end{bmatrix}_{x1} \;+\; 1 \begin{bmatrix}2\\4\\6\end{bmatrix}_{x2} + e_i$