Critical Points and Curvature for Embedded Polyhedral Surfaces
Author(s): T. F. Banchoff
Source: *The American Mathematical Monthly*, Vol. 77, No. 5 (May, 1970), pp. 475-485
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2317380
Accessed: 16/05/2013 01:45

CRITICAL POINTS AND CURVATURE FOR EMBEDDED POLYHEDRAL SURFACES

T. F. BANCHOFF, Brown University

The Gauss-Bonnet theorem for a surface M^2 in Euclidean 3-space E^3 and the Critical Point Theorem for height functions on an embedded surface are two of the earliest and most important theorems of "geometry in the large." Both theorems relate geometric properties of the embedded surface (the total curvature of the surface or the sum of a set of geometrically defined indices of singularity) to a topological property of the surface, the Euler-Poincaré characteristic $\chi(M^2)$. Both theorems are very geometric in character despite the fact that the standard definitions of total curvature and index of singularity appear to involve the use of differential calculus and the hypothesis that the surfaces are smooth. In fact both theorems have analogues for polyhedral surfaces embedded in E^3, and the proofs in the polyhedral case are entirely elementary.

Some of the most interesting results in global geometry have exploited the connection between total curvature and critical point theory, as in the work of

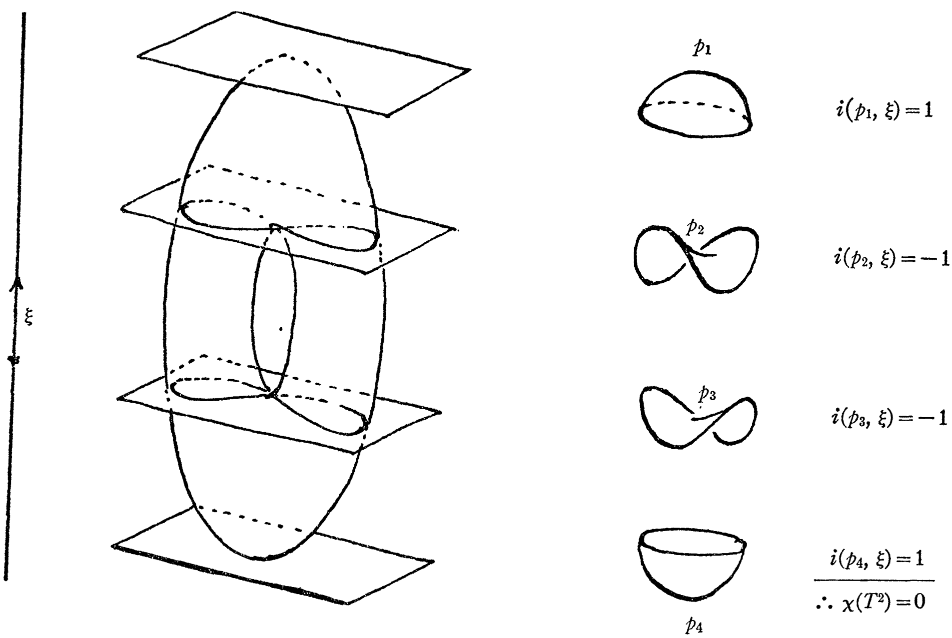


FIG. 1

Thomas Banchoff wrote his Berkeley thesis under Chern in 1964. He was a Peirce Instructor at Harvard, a Fulbright Research Associate at the University of Amsterdam, and presently is an Assistant Professor at Brown. His research is in global differential geometry. *Editor.*

Kuiper [4]. In this paper we shall follow this same procedure to prove the critical point theorem and use it to prove the Gauss-Bonnet theorem. In the polyhedral case, a new feature is an interpretation of the *Theorema Egregium* of Gauss which relates the extrinsic curvature to the intrinsic curvature on a surface.

All of the theorems in this paper have appeared in a generalized (and technical) form in the author's paper [2].

1. The Critical Point Theorem. Consider a closed smooth surface M^2 embedded in E^3 and consider a linear function ξ on E^3 given by projecting all of E^3 to the line determined by a unit vector ξ. A point p of M^2 is said to be a *critical point* for ξ if the tangent plane to M^2 at p is perpendicular to ξ; all other points of M^2 are called *ordinary points* for ξ. In the "standard" example of a height function on a torus of revolution held vertically there are just four critical points, a maximum, a minimum, and two (nondegenerate) saddle points.

The *Critical Point Theorem* for height functions states that if ξ has a finite number of critical points on M^2 and all are of the three types described above, then (number of local maxima) + (number of local minima) − (number of saddle points) $= \chi(M^2)$, where $\chi(M^2)$ is the Euler-Poincaré characteristic of M^2.

We express this theorem more succinctly by indexing each critical point by $i(p, \xi) = 1$ if p is a local maximum or minimum and $i(p, \xi) = -1$ if p is a (nondegenerate) saddle point. The theorem then states

$$\sum_{p \text{ critical for } \xi} i(p, \xi) = \chi(M^2).$$

In classical critical point theory (= Morse Theory) the index is given by considering the sign of the determinant of the matrix of second derivatives, as in [5], but since we are interested in developing the polyhedral analogue of the theorem, we proceed to give a more geometric presentation of this indexing procedure.

If a point q is ordinary for the height function ξ, then the tangent plane at q is not horizontal. Thus the tangent plane divides a "small disc neighborhood" U of q on M^2 into exactly two pieces and it meets a "small circle" about q in precisely two points. This distinguishes an ordinary point from a local maximum or minimum (where a "small circle" about the critical point will not meet the tangent plane at all) and from a nondegenerate saddle point p (where the plane at p perpendicular to q meets a "small circle" about p on M^2 in four distinct points).

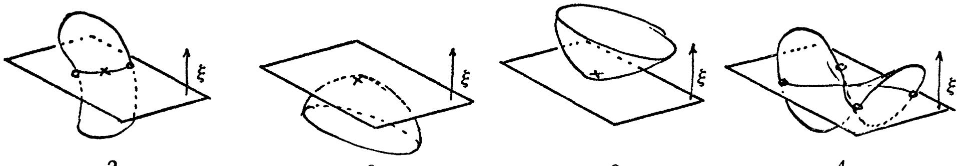


FIG. 2

We may then give an arithmetic definition of the *index* as follows:

$i(p, \xi) = 1 - \frac{1}{2}$ (number of points in which the plane through p perpendicular to ξ meets a "small circle" about p on M^2).

This definition agrees with the previous indexing procedure and has the additional property that $i(q, \xi) = 0$ if q is not a critical point. In the smooth case, however, the definition is somewhat unsatisfactory due to the difficulty of defining precisely the notion of a "small circle." In the polyhedral case, on the other hand, this is exactly the sort of definition which we want.

Consider a polyhedral surface M^2 in E^3 which is expressed as a union of V vertices, E edges, and T triangular faces. The *Euler characteristic* of M^2 is defined to be

$$\chi(M^2) = V - E + T.$$

A height function ξ on E^3 is said to be *general for the polyhedral surface* M^2 if $\xi(v) \neq \xi(w)$ whenever v and w are distinct vertices of M^2. If ξ is general for M^2,

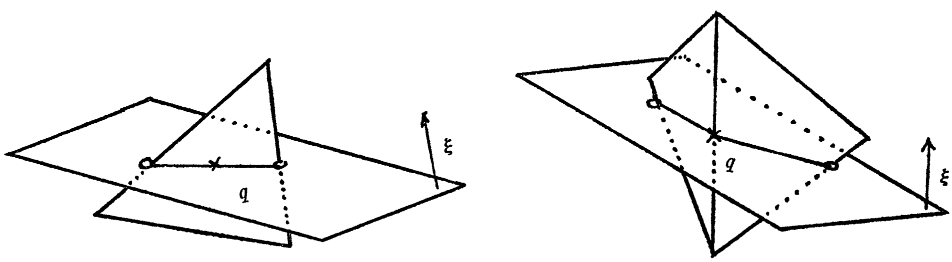


FIG. 3

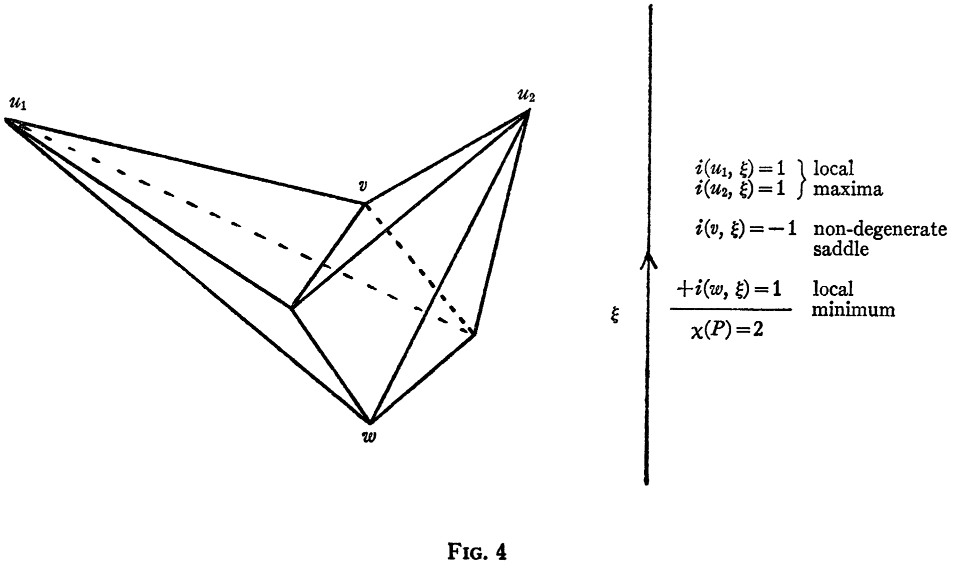
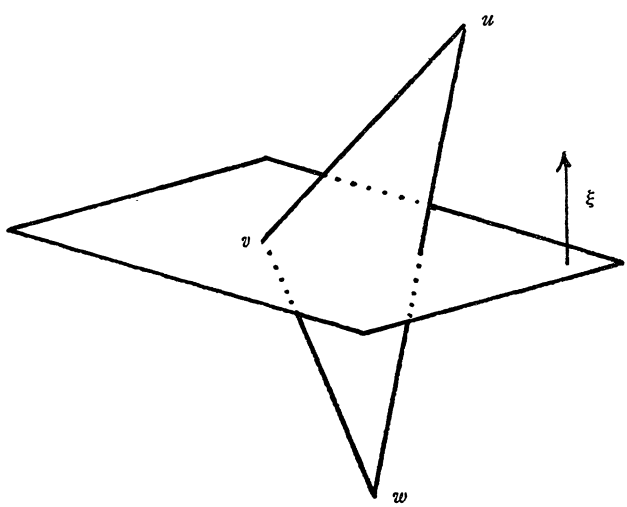


FIG. 4

then the point q is said to be *ordinary* for ξ if the plane through p perpendicular to ξ cuts the disc neighborhood Star(q) into two pieces, where Star(q) is the union of all vertices, edges, and faces which include q. (When we say M^2 is a polyhedral surface, we mean that for each point q, Star(q) is the image of an open disc in the plane under a one-to-one continuous map.) With this definition, any point q in the interior of a face or an edge has to be ordinary since no face or edge can be perpendicular to the vector ξ if ξ is general for M^2.

For vertices, however, there are critical points corresponding to all the types presented for smooth functions, as, for example, in the indicated polyhedron. We may then use the indexing procedure developed for smooth surfaces, where instead of a "small circle" we use the embedded polygon which is the boundary of the star of the vertex v. The number of times the plane through v perpendicular to ξ meets this polygon is then equal to the number of triangles $\triangle$ in Star(v) such that one of the vertices of $\triangle$ lies above the plane and the other lies below.



v is the middle vertex of $\triangle$ for ξ.

FIG. 5

In such a case we say that v is the middle vertex of $\triangle$ for ξ, and we may write the index as follows:

$$i(v, \xi) = 1 - \tfrac{1}{2} \text{ (number of } \triangle \text{ with } v \text{ middle for } \xi).$$

Again this definition corresponds to the definitions given in the smooth case and it gives index 0 for an ordinary point.

The critical point theorem then states:

THEOREM 1. *If ξ is general for M^2, then*

$$\sum_{v \in M^2} i(v, \xi) = \chi(M^2).$$

We require a lemma on polyhedral surfaces:

LEMMA. *For a polyhedral surface,* $3T=2E$.

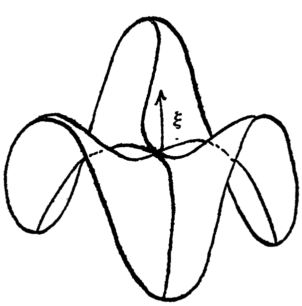
Proof of Lemma. Since an edge in a polyhedral surface has precisely two triangles in its star,

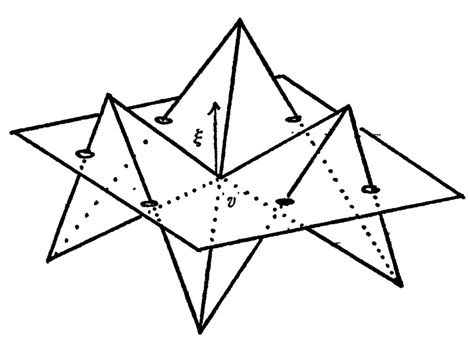
$$3T = \text{number of pairs } (\triangle, \text{edge of } \triangle) = 2E.$$

Proof of Theorem. If ξ is general for M^2,

$$\begin{aligned}\sum_{v\in M} i(v,\xi) &= \sum_{v\in M}\left(1-\tfrac{1}{2}(\text{number of } \triangle \text{ with } v \text{ middle for } \xi)\right),\\ &= V-\tfrac{1}{2}\sum_{v\in M}(\text{number of } \triangle \text{ with } v \text{ middle for } \xi),\\ &= V-\tfrac{1}{2}T \text{ (since each } \triangle \text{ has exactly one middle vertex for } \xi),\\ &= V-\tfrac{1}{2}(2E-2T) \text{ (since } T=2E-2T \text{ by the lemma)},\\ &= V-E+T.\end{aligned}$$

REMARK. For a smooth surface M^2 embedded in E^3 it is a classical result that for almost every unit vector ξ on S^2 (i.e., except for a set of measure zero on S^2), the height function ξ has only finitely many critical points, and furthermore, for almost all ξ, this height function has as critical points only local maxima and minima, and nondegenerate saddle points. In the polyhedral case it is immediate that almost all ξ are general for M^2 (since the nongeneral ξ lie in the finite union of great circles $\{\xi\in S^2 \mid \xi(v)=\xi(w)\}$, one for each pair of distinct vertices v, w). The stronger result, however, is not correct in the polyhedral case. Consider an isolated critical point of a smooth surface which is degenerate—the "monkey saddle" (so called in Hilbert and Cohn-Vossen [3], p. 191, since a monkey riding a bicycle would need three depressions in his saddle, one for





$i(v,\xi)=-2$

FIG. 6

each leg and one for his tail). Although on the smooth monkey saddle, height functions η near ξ on S^2 have only nondegenerate critical points, any η near ξ

has exactly six triangles with v middle, so for each such η, $i(v, \eta) = -2$.

However, in the proof of Theorem 1, we never required that the indices be only 0, 1 or -1. The proof goes through without change for any polyhedral surface M and ξ general for M, regardless of the complexity of the stars of the vertices of M.

2. Total Curvature and the Gauss-Bonnet Theorem. The total curvature or Gaussian curvature of a neighborhood U on a smooth surface M^2 in E^3 has several definitions which appear in texts in differential geometry. The definition which most easily leads to an extrinsic curvature theory for polyhedra is the one originally given by Gauss. We sketch his procedure in the smooth case and then develop the analogous theory for arbitrary embedded polyhedral surfaces in E^3.

Consider a "small" neighborhood U_1 on a convex surface M^2 in E^3. The Gauss map $g: U_1 \rightarrow S^2$ is defined by setting $g(p)$ = outward unit normal vector to M^2 at p. If the mapping g restricted to U_1 is one-to-one, and g is orientation-preserving (outward normals at corresponding points correspond) then U_1 is said to be *strictly convex*. The *total curvature* $\tilde{K}(U_1)$ of U_1 is then defined to be the area of the spherical image $g(U_1)$ on S^2.

If U_2 is a region of a nonconvex surface M^2 on which g is one-to-one and orientation-reversing, then $\tilde{K}(U_2)$ is defined to be the negative of the area of $g(U_2)$.

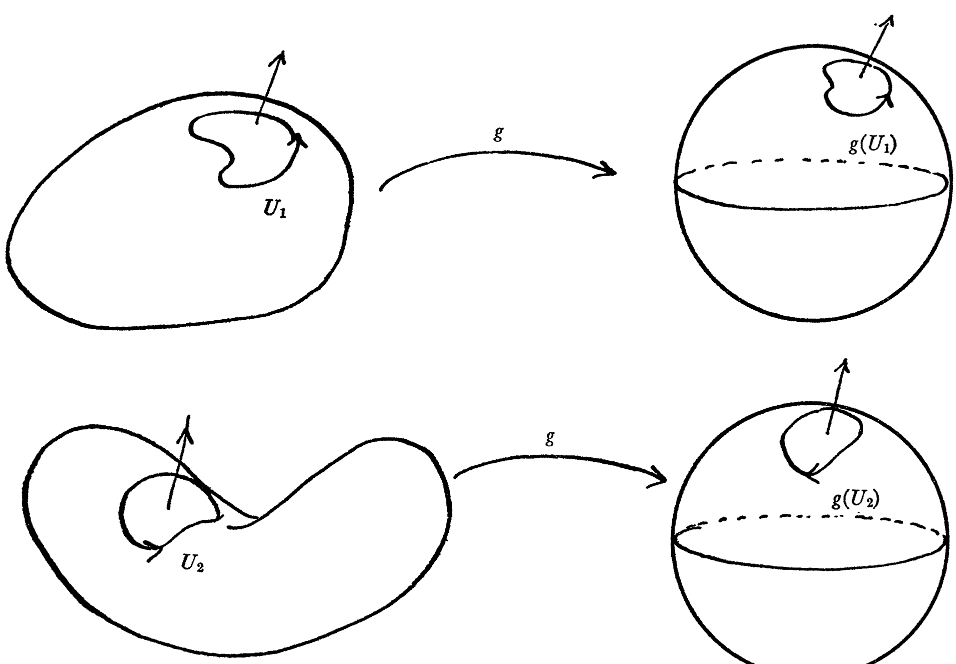


FIG. 7

We may use the index $i(p, \xi)$ to describe the total curvature of U_1 in these cases. Let $d\omega$ denote the area integrand for S^2, so

$$\int_{S^2} d\omega = 4\pi.$$

Then the area of $g(U_1)$ is the integral over S^2 of the characteristic function: $F_{g(U_1)}(\xi) = 1$ if ξ is in $g(U_1)$ and 0 otherwise. But if ξ is in $g(U_1)$, then the height function ξ has a critical point at a point p on U_1. In fact, $i(p, \xi) \neq 0$ for a point p of U_1 if and only if ξ is in $g(U_1)$ or $-\xi$ is in $g(U_1)$. We may assume that U_1 is small enough that $g(U_1)$ contains no pair of antipodal points. The total curvature may then be described as follows:

$$\tilde{K}(U_1) = \int_{S^2} F_{g(U_1)}(\xi) d\omega = \frac{1}{2} \int_{S^2} \sum_{p \in U_1} i(p, \xi) d\omega.$$

But the right-hand expression also serves to define $\tilde{K}(U_2)$, since this expression gives the negative of the area of $g(U_2)$.

The following two paragraphs explain the procedure of defining the total curvature in the smooth case, and this serves to motivate the definition for the polyhedral analogue. In the polyhedral case, however, the technical difficulties concerning the convergence of the integrand do not occur.

In order to define the total curvature of a neighborhood U on which g is not one-to-one, we begin by expressing U as a countable disjoint union of sets U_i on which g is one-to-one together with a set $V = U - \bigcup_{i=1}^{\infty} U_i$ such that $g(V)$ has measure zero on S^2. Then we set $\tilde{K}(U) = \sum_{i=1}^{\infty} \tilde{K}(U_i)$ if this sum converges, and we obtain a totally additive set function on M^2. This definition then coincides with

$$\tilde{K}(U) = \frac{1}{2} \int_{S^2} \sum_{p \in U} i(p, \xi) d\omega,$$

where the integrand is well defined almost everywhere since almost every ξ has only finitely many critical points.

REMARK. In the case that M^2 is sufficiently smooth, the set function $\tilde{K}(U)$ is absolutely continuous with respect to the area measure $A(U)$ on M^2, and we may define the point function $K(p)$ as the limit of $\tilde{K}(U)/A(U)$ for any collection of neighborhoods U_i of p with the limit of the diameters of U_i going to zero. This function is called the *Gaussian curvature* at p, and by integrating this function with respect to the area measure, we obtain

$$\tilde{K}(U) = \int_U K(p) dA.$$

The classical Gauss-Bonnet theorem for embedded closed surfaces states that

$$\tilde{K}(M^2) = \int_{M^2} K(p) dA = 2\pi\chi(M^2).$$

In the case of a polyhedral manifold M^2 in E^3, we may use the same definition as that developed for smooth surfaces. For any open set U of M^2, we set

$$K(U) = \frac{1}{2} \int_{S^2} \sum_{v \in U} i(v, \xi) d\omega.$$

When M^2 is a convex polyhedron, if U is a neighborhood containing only one vertex v, then $\tilde{K}(U)$ gives the measure of the exterior angle at v, i.e., the set of normals to support planes to M^2 at v, and this approach has been used in the classical theory of convex polyhedra, for example in [1].

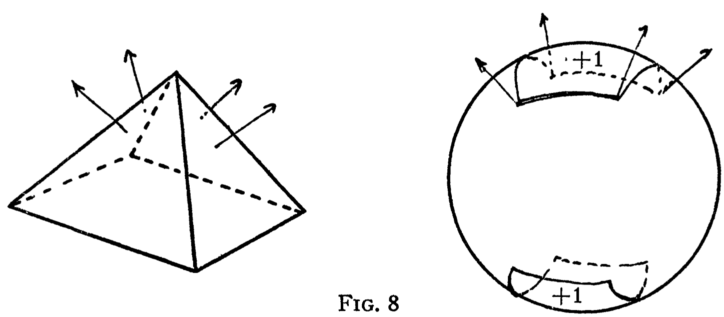


FIG. 8

The expression above also yields a definition of curvature for nonconvex vertices of saddle type as well as for the monkey saddles.

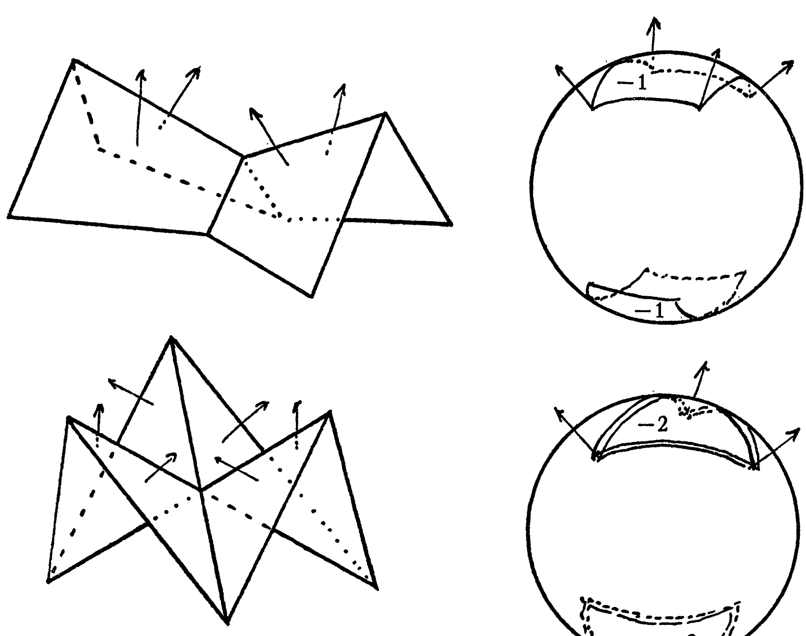


FIG. 9

The Gauss-Bonnet theorem for embedded surfaces, smooth or polyhedral, follows immediately:

THEOREM 2. $\tilde{K}(M^2)=2\pi\chi(M^2)$.

Proof.

$$\tilde{K}(M^2)=\frac{1}{2}\int_{S^2}\sum_{p\in M^2} i(p,\xi)d\omega=\frac{1}{2}\int_{S^2}\chi(M^2)d\omega=\frac{1}{2}\chi(M^2)\int_{S^2}d\omega=2\pi\chi(M^2).$$

3. The Theorema Egregium. Total curvature was defined for a set U on a smooth surface M in a way which involved the extrinsic properties of the surface, that is, the way M is situated in E^3. Gauss, however, proved that in fact $\tilde{K}(U)$ depends only on intrinsic properties of U, i.e., on properties that are determined by measurements made along the surface not taking into account the way the surface is situated in space, and Gauss called this result his *Theorema Egregium.*

We shall prove the analogous theorem for embedded polyhedral surfaces. Observe first of all that for an open set U of M,

$$\tilde{K}(U)=\frac{1}{2}\int_{S^2}\sum_{v\in U} i(v,\xi)d\omega=\sum_{v\in U}\frac{1}{2}\int_{S^2} i(v,\xi)d\omega$$

so we may set

$$\tilde{K}(U)=\sum_{v\in U}\tilde{K}(v),\quad \text{where}\quad \tilde{K}(v)=\frac{1}{2}\int_{S^2} i(v,\xi)d\omega.$$

THEOREM 3. *$\tilde{K}(v)$ is intrinsic, in fact, $\tilde{K}(v)=2\pi-$(sum of interior angles at v of triangles containing v).*

Proof. Let $m(v,\triangle,\xi)$ be the function on E^3 defined by $m(v,\triangle,\xi)=1$ if v is the middle vertex of $\triangle$ for ξ and $m(v,\triangle,\xi)=0$ otherwise. Then $i(v,\xi)=1-\frac{1}{2}\sum_{\triangle\in M} m(v,\triangle,\xi)$ and

$$\begin{aligned}\tilde{K}(v)&=\frac{1}{2}\int_{S^2} i(v,\xi)d\omega=\frac{1}{2}\int_{S^2}\left(1-\frac{1}{2}\sum_{\triangle\in M} m(v,\triangle,\xi)\right)d\omega\\&=\frac{1}{2}\int_{S^2}d\omega-\frac{1}{4}\int_{S^2}\sum_{\triangle\in M} m(v,\triangle,\xi)d\omega=2\pi-\sum_{\triangle\in M}\frac{1}{4}\int_{S^2} m(v,\triangle,\xi)d\omega.\end{aligned}$$

The proof is then complete once we establish the following lemma:

LEMMA. $\int_{S^2} m(v,\triangle,\xi)d\omega=4$ *(interior angle of $\triangle$ at v).*

Proof. First of all, observe that for vectors $\mathbf{n}$ at v in the plane E^2 containing $\triangle$, $m(v,\triangle,\eta)=1$ if and only if $\mathbf{n}$ lies in the region between the lines perpendicular to the edges $u-v$ and $w-v$ and the angle which determines this region is equal to the interior angle of $\triangle$ at v. Any vector $\boldsymbol{\xi}$ of S^2 may be written uni-

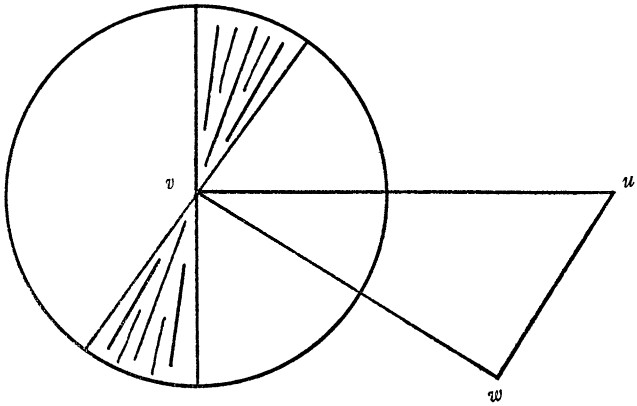


FIG. 10

quely as $\xi=\mathbf{n}+\zeta$ where $\mathbf{n}$ lies in the plane E^2 containing $\triangle$ and ζ is perpendicular to E^2. Then $m(v, \triangle, \xi)=1$ if and only if $\xi\cdot(u-v)>0>\xi\cdot(v-w)$ or $\xi\cdot(u-v)<0<\xi\cdot(v-w)$ so $m(v, \triangle, \xi)=1$ if and only if $m(v, \triangle, \eta)=1$. Thus the set of ξ on S^2 centered at v for which $m(v, \triangle, \xi)=1$ forms a double lune with axis perpendicular to E^2 and with each angle equal to the interior angle of $\triangle$ at v. But the area of a lune is twice the angle of the lune, so

$$\int_{S^2} m(v, \triangle, \xi)d\omega = 4 \text{ (interior angle of } \triangle \text{ at } v).$$

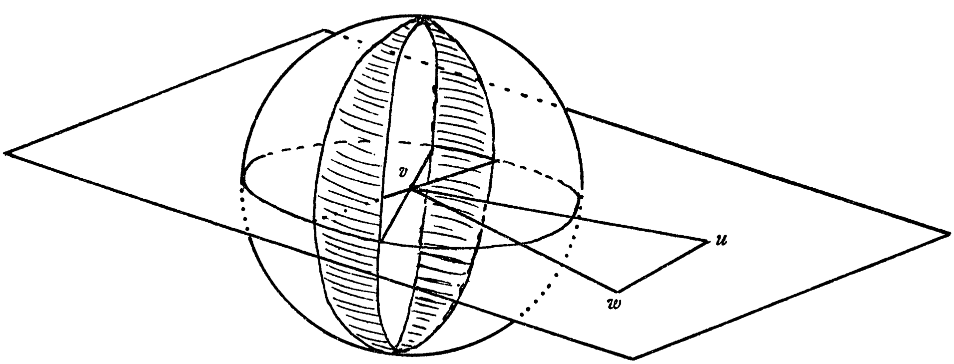


FIG. 11

This completes the proof of the *Theorema Egregium* for polyhedra, no matter how complicated the vertex stars are. Compare Hilbert and Cohn-Vossen ([3], p. 195) for a similar argument for vertices corresponding to nondegenerate saddle points.

REMARK. The theorems of section 2 may be considered as a generalization of the approach of G. Pólya for embedded convex polyhedral discs [6].

References

1. A. D. Alexandroff, Konvexe Polyeder, Akademie Verlag, Berlin, 1958 (Russian original: Moscow, 1950).
2. T. F. Banchoff, Critical Points and Curvature for Embedded Polyhedra, J. of Differential Geometry, number 3, 1 (1967) 257–268.
3. D. Hilbert and S. Cohn-Vossen, Geometry and the Imagination, Chelsea, New York, 1952 (German original: Springer, Berlin, 1932).
4. N. H. Kuiper, Der Satz von Gauss-Bonnet und damit verwandte Probleme, Jber. Deutsch. Math.-Verein., 69 (1967) 77–88.
5. J. Milnor, Morse Theory, Princeton University Press, 1963.
6. G. Pólya, An elementary analogue of the Gauss-Bonnet theorem, this MONTHLY, 61 (1954) 601–603.

FUNDAMENTA MATHEMATICAE: AN EXAMINATION OF ITS FOUNDING AND SIGNIFICANCE

SISTER MARY GRACE KUZAWA, Holy Family College, Philadelphia

Fifty years ago in Poland, a small but determined group of mathematicians launched a daring project—a specialized mathematical journal, FUNDAMENTA MATHEMATICAE. Intuitively aware that the mathematical works of the late nineteenth, and the early years of the twentieth centuries, were but a forecast of tremendous and triumphant achievements to come, many scholars immediately recognized the value of the mathematical research incorporated in the pages of a journal.

One of the first scholars to grasp and to acknowledge the value of the journal issuing from Poland was the greatly respected editor of the AMERICAN MATHEMATICAL MONTHLY, Raymond Clare Archibald. As early as September 1921 he wrote: "Of the ten mathematical periodicals started since January 1919, none are of such notable importance for mathematical research as FUNDAMENTA MATHEMATICAE of which two volumes have been published: the first (224 pages) in 1920, the second (287 pages) in 1921, before May 1" [1].

Growing awareness of the Journal. Consequently, during the years from 1920 to 1939, the publication FUNDAMENTA MATHEMATICAE was widely read and generally acclaimed to be a major achievement by the mathematical experts. When at the end of 1935 the twenty-fifth volume of the journal came off the presses, both Polish and foreign scholars greeted the journal with considerable enthusiasm. The Polish newspaper, WIADOMOŚCI LITERACKIE (Literary News) hailed the appearance of this volume as a remarkable event, "heralding a holiday for Polish mathematics" [2]. THE BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY observed that "the twenty-fifth volume of FUNDAMENTA MATHEMATICAE represents a notable event in the mathematical life of the whole world" [3].